As per the Revised Syllabus Effective August 2007

# Electrical Circuit Analysis

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# Electrical Circuit Analysis

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Dedicated To Our Parents and Students L

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### Foreword

It gives me great pleasure to introduce *Electrical Circuit Analysis* by *Dr A*. *Sudhakar* and *Dr S*. *Shyammohan Palli*, publication of which heralds the completion of a book that caters completely and effectively to the students of JNTU.

The need for a good textbook for this subject can be easily understood. Numerous books are available to the students for the subject, but almost none of them have the right combination of simplicity, rigour, pedagogy and syllabus compatibility. These books usually do not address one or more of the specific problems faced by students of this subject in JNTU. There has always been a need for a good book relevant to the requirements of the students and dealing with all aspects of the course. I am sure that the present book will be able to fill this void.

The book has been organized and executed with lot of care and dedication. The authors have been outstanding teachers with vast experience and expertise in their chosen fields of interest. A conscious attempt has been made to simplify concepts to facilitate better understanding of the subject.

Dr Sudhakar and Dr Shyammohan deserve our praise and thanks for accomplishing this trying task. McGraw-Hill Education, a prestigious publishing house, also deserves a pat on the back for doing an excellent job.

#### DR K. RAJAGOPAL

Vice-Chancellor Jawarharlal Nehru Technological University Hyderabad

## Preface

This book is exclusively designed for use as a text for the course on Electrical Circuit Analysis offered to first year undergraduate engineering students of Jawaharlal Nehru Technological University (JNTU), Hyderabad. The primary goal of this text is to establish a firm understanding of the basic laws of electric circuits which develop a working knowledge of the methods of analysis used most frequently in further topics of electrical engineering. This book also provides a comprehensive insight into the principal techniques available for characterizing circuits and networks theoretically.

Illustrative examples are interspersed throughout the book at their natural locations. These have been selected carefully from the university question papers. With so many years of teaching, we have found that such illustrations permit a level of understanding otherwise unattainable. As an aid to both, the instructor and the student, objective questions and the tutorial problems provided at the end of each chapter progress from simple to complex. Answers to selected problems have been given to instill confidence in the reader. Due care is taken to see that the reader can easily start learning circuit analysis without prior knowledge of mathematics. As such, a student of first year B.Tech/B.E will be able to follow the book without any difficulty.

All the elements with definitions, basic laws and different configurations of the resistive circuits have been introduced in the first chapter. Analyses of the D.C. resistive circuits have been discussed in Chapter 2. Graph theory has been written in an easy to understand manner. Network theorems on resistive circuits have been presented in Chapter 3. A.C. fundamentals have been introduced in Chapters 4 and 5 which include voltage-current relation of elements, complex impedance. Power and power factor concept is discussed in Chapter 6. Due emphasis has been laid on finding out the average and rms values of different waveforms in Chapter 4. The steady state analysis of A.C. circuits including network theorems have been discussed in Chapter 7. Problems, tutorials and objective questions on dependent sources have been included in Chapters 1 to 7. Resonance phenomena in series and parallel circuits and locus diagrams are presented in Chapter 8. A comprehensive study of polyphase systems and power measurement in both balanced and unbalanced circuits is presented in Chapter 9. A brief study of coupled and tuned circuits is introduced in Chapter 10. Magnetic circuits are also discussed in this chapter.

A brief discussion of differential equations is included in Chapter 11. The necessary mathematical background for transient analysis, the transient behavior of A.C and D.C circuits and their response has been discussed in Chapter 12. Laplace transforms and their application is presented in Chapter 13. Network functions and stability criteria have been discussed in Chapter 14. The parameters of two-port network and their inter-relations have been discussed in Chapter 15. The book also includes brief topics of Fourier series, Fourier Transforms and operator j in appendices. Twelve Model Question Papers, Solved May/June 2006, Apr\May 2007 Question Papers and Apr/May 2008 Question Paper (12 Sets) are provided at the end of the book.

#### Acknowledgements

Many people have helped us in producing this book. We extend our gratitude to them for assisting us in their own individual ways to shape the book into the final form. We would like to express our gratitude to the Management of RVR & JC College of Engineering, particularly to the President, Dr K Basava Punnaiah and Secretary and Correspondent, Dr M Gopalkrishna. We extend our appreciation to the Management of Sir C R Reddy College of Engineering, particularly the president, Sri Kommareddy Rambabu; Secretary, Sri Kakarala Rajendra Vara Prasad; Vice President, Dr K Sriramchandramurthy; Joint Secretaries --- Sri Ch. Arun and Dr V V Balakrishna Rao; and Treasurer, Sri M Raghunadha Rao for providing a conducive atmosphere. We are indebted to Dr P S Sankara Rao, Principal of RVR & JC College of Engineering; Dr Y V S S S V Prasada Rao, Principal of Sir C R Reddy College of Engineering; Prof. K A Gopalarao of Andhra University; Sri B Amarendra Reddy of Andhra University; and Dr K K R of Gowtham Concepts School for their support throughout the work. We are thankful to Prof. G S N Raju, Prof. M Ravindra Reddy, Sri T Sreerama Murthy and many other colleagues for their invaluable suggestions. We are obliged to Ms K Swarna Sree for her help in implementing the P-Spice problems. We also thank the students of the ECE Department, particularly R Divyasri, M Rajani, P Mohini, K Bhargavi, K Rajya Lakshmi, and K Divya Raghavi of RVR & JC College of Engineering, Mr T Jayanth Kumar of IIT Kanpur, Mr T Sumanth Babu of Goka Raju & Ranga Raju College of Engineering, and Alapati Haritha of Raghu Engineering College who were involved directly or indirectly with the writing of this book. We are thankful to Mr D S R Anjaneyulu and K Srinivas for the error free typing of the manuscript.

We wish to express our appreciation to the various members of McGraw-Hill Education who handled the book at different stages. We would like to extend our sincere thanks, particularly to Vibha Mahajan, Shalini Jha, Sandhya Sekhar, Dipika Dey, Michael J Cruz, Anjali Razdan and others for their valuable support. Finally, we thank our family members — Madhavi, Aparna, A V Yashwanth, P Siddartha and P Yudhister — whose invaluable support made the whole project possible.

# Road Map to the Syllabus

#### (Effective from August 2007)

#### Jawaharlal Nehru Technological University, Hyderabad Electrical Circuit Analysis

#### **Objective :**

This course introduces the basic concepts of circuit analysis which is the foundation for all subjects of the Electrical Engineering discipline. The emphasis of this course is laid on the basic analysis of circuits which includes Single phase circuits, magnetic circuits, theorems, transient analysis and network topology.

#### UNIT – I INTRODUCTION TO ELECTRICAL CIRCUITS

Circuit Concept – R-L-C parameters – Voltage and Current sources – Independent and dependent sources –Source transformation – Voltage – Current relationship for passive elements – Kirchhoff's laws – network reduction techniques – series, parallel, series parallel, star-to-delta or delta-to-star transformation.

Go To Chapter 1 — Circuit Elements and Kirchhoff's Laws Chapter 3 — Useful Theorems in Circuit Analysis

#### UNIT – II MAGNETIC CIRCUITS

Magnetic Circuits – Faraday's laws of electromagnetic induction – concept of self and mutual inductance – dot convention – coefficient of coupling – composite magnetic circuit – Analysis of series and parallel magnetic circuits.

Go To Chapter 10 — Coupled Circuits

#### UNIT – III SINGLE PHASE A.C CIRCUITS

R.M.S and Average values and form factor for different periodic wave forms, Steady state analysis of R, L and C (in series, parallel and series parallel combinations) with sinusoidal excitation – Concept of Reactance, Impedance, Susceptance and Admittance – Phase and Phase difference – concept of power factor, Real and

Reactive powers – J-notation, Complex and Polar forms of representation, Complex power – Locus diagrams – series R-L, R-C, R-L-C and parallel combination with variation of various parameters – Resonance – series, parallel circuits, concept of band width and Q factor.

Go To Chapter 4 — Introduction to Alternating Currents and Voltages Chapter 5 — Complex Impedance Chapter 6 — Power & Power Factor Chapter 8 — Resonance

#### UNIT – IV THREE PHASE CIRCUITS

Three phase circuits: Phase sequence – Star and delta connection – Relation between line and phase voltages and currents in balanced systems – Analysis of balanced and Unbalanced 3 phase circuits – Measurement of active and reactive power.

**Go To** Chapter 9 — Polyphase Circuits

#### UNIT – V NETWORK TOPOLOGY

Definitions – Graph – Tree, Basic cutset and Basic Tieset matrices for planar networks – Loop and Nodal methods of analysis of Networks with independent voltage and current sources – Duality & Dual networks.

Go To Chapter 2 — Methods of Analysing Circuits Chapter 3 — Useful Theorems in Circuit Analysis

#### UNIT – VI NETWORK THEOREMS (WITHOUT PROOFS)

Tellegen's, Superposition, Reciprocity, Thevenin's, Norton's, Maximum Power Transfer, Millman's and Compensation theorems for d.c. and a.c. excitations.

Go To Chapter 3 — Useful Theorems in Circuit Analysis Chapter 7 — Steady State AC Analysis

#### UNIT – VII TRANSIENT ANALYSIS

Transient response of R-L, R-C, R-L-C circuits (Series combinations only) for d.c. and sinusoidal excitations – Initial conditions – Solution using differential equation approach and Laplace transform methods of solutions.



Chapter 12 Transients Chapter 13 Laplace Transforms

#### UNIT – VIII NETWORK PARAMETERS

Two port network parameters -Z, Y, ABCD and hybrid parameters and their relations - concept of transformed network -2-port network parameters using transformed variables.

Go To Chapter 15 Two Port Networks

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#### **1.1 VOLTAGE**

According to the structure of an atom, we know that there are two types of charges: positive and negative. A force of attraction exists between these positive and negative charges. A certain amount of energy (work) is required to overcome the force and move the charges through a specific distance. All opposite charges possess a certain amount of potential energy because of the separation between them. The difference in potential energy of the charges is called the *potential difference*.

Potential difference in electrical terminology is known as voltage, and is denoted either by V or v. It is expressed in terms of energy (W) per unit charge (Q); i.e.

$$V = \frac{W}{Q}$$
 or  $v = \frac{dw}{dq}$ 

dw is the small change in energy, and

*dq* is the small change in charge.

where energy (W) is expressed in joules (J), charge (Q) in coulombs (C), and voltage (V) in volts (V). One volt is the potential difference between two points when one joule of energy is used to pass one coulomb of charge from one point to the other.

**Example 1.1** If 70 J of energy is available for every 30 C of charge, what is the voltage?

**Solution** 
$$V = \frac{W}{Q} = \frac{70}{30} = 2.33 \text{ V}$$

#### **1.2 CURRENT**

There are free electrons available in all semiconductive and conductive materials. These free electrons move at random in all directions within the structure in the absence of external pressure or voltage. If a certain amount of voltage is applied across the material, all the free electrons move in one direction depending on the polarity of the applied voltage, as shown in Fig. 1.1.



Fig. 1.1

This movement of electrons from one end of the material to the other end constitutes an electric current, denoted by either I or i. The conventional direction of current flow is opposite to the flow of – ve charges, i.e. the electrons.

Current is defined as the rate of flow of electrons in a conductive or semiconductive material. It is measured by the number of electrons that flow past a point in unit time. Expressed mathematically,

$$I = \frac{Q}{t}$$

where I is the current, Q is the charge of electrons, and t is the time, or

$$i = \frac{dq}{dt}$$

where dq is the small change in charge, and dt is the small change in time.

In practice, the unit *ampere* is used to measure current, denoted by A. One ampere is equal to one coulomb per second. One coulomb is the charge carried by  $6.25 \times 10^{18}$  electrons. For example, an ordinary 80 W domestic ceiling fan on 230 V supply takes a current of approximately 0.35 A. This means that electricity is passing through the fan at the rate of 0.35 coulomb every second, i.e.  $2.187 \times 10^{18}$  electrons are passing through the fan in every second; or simply, the current is 0.35 A.

**Example 1.2** Five coulombs of charge flow past a given point in a wire in 2 s. How many amperes of current is flowing?

#### Solution

$$I = \frac{Q}{t} = \frac{5}{2} = 2.5 \text{ A}$$

#### **1.3 POWER AND ENERGY**

Energy is the capacity for doing work, i.e. energy is nothing but stored work. Energy may exist in many forms such as mechanical, chemical, electrical and so on. Power is the rate of change of energy, and is denoted by either P or p. If certain amount of energy is used over a certain length of time, then

Power (P) = 
$$\frac{\text{energy}}{\text{time}} = \frac{W}{t}$$
, or  

$$p = \frac{dw}{dt}$$

where dw is the change in energy and dt is the change in time.

#### We can also write $p = \frac{dw}{dt} = \frac{dw}{dq} \times \frac{dq}{dt}$ = $v \times i = vi$ W

Energy is measured in joules (J), time in seconds (s), and power in watts (W).

By definition, one watt is the amount of power generated when on joule of energy is consumed in one second. Thus, the number of joules consumed in one second is always equal to the number of watts. Amounts of power less than one watt are usually expressed in fraction of watts in the field of electronics; viz. milliwatts (mW) and microwatts ( $\mu$ W). In the electrical field, kilowatts (kW) and megawatts (MW) are common units. Radio and television stations also use large amounts of power to transmit signals.

**Example 1.3** What is the power in watts if energy equal to 50 J is used in 2.5 s?

**Solution** 
$$P = \frac{\text{energy}}{\text{time}} = \frac{50}{2.5} = 20 \text{ W}$$

#### **1.4 THE CIRCUIT**

An electric circuit consists of three parts: (1) energy source, such as battery or generator, (2) the load or sink, such as lamp or motor, and (3) connecting wires as shown in Fig. 1.2. This arrangement represents a simple circuit. A battery is connected to a lamp with two wires. The purpose of the circuit is to transfer energy from source (battery) to the load (lamp). And this is accomplished by the passage of electrons through wires around the circuit.

The current flows through the filament of the lamp, causing it to emit visible light. The current flows through the battery

by chemical action. A closed circuit is defined as a circuit in which the current has a complete path to flow. When the current path is broken so that current cannot flow, the circuit is called an open circuit.

More specifically, interconnection of two or more simple circuit elements (viz. voltage sources, resistors, inductors and capacitors) is called an electric network. If a network



Fig. 1.2

contains at least one closed path, it is called an electric circuit. By definition, a simple circuit element is the mathematical model of two terminal electrical devices, and it can be completely characterised by its voltage and current. Evidently then, a physical circuit must provide means for the transfer of energy.

Broadly, network elements may be classified into four groups, viz.

- 1. Active or passive
- 2. Unilateral or bilateral
- 3. Linear or nonlinear
- 4. Lumped or distributed

#### 1.4.1 Active and Passive

Energy sources (voltage or current sources) are active elements, capable of delivering power to some external device. Passive elements are those which are capable only of receiving power. Some passive elements like inductors and capacitors are capable of storing a finite amount of energy, and return it later to an external element. More specifically, an active element is capable of delivering an average power greater than zero to some external device over an infinite time interval. For example, ideal sources are active elements. A passive element is defined as one that cannot supply average power that is greater than zero over an infinite time interval. Resistors, capacitors and inductors fall into this category.

#### 1.4.2 Bilateral and Unilateral

In the bilateral element, the voltage-current relation is the same for current flowing in either direction. In contrast, a unilateral element has different relations between voltage and current for the two possible directions of current. Examples of bilateral elements are elements made of high conductivity materials in general. Vacuum diodes, silicon diodes, and metal rectifiers are examples of unilateral elements.

#### 1.4.3 Linear and Nonlinear Elements

An element is said to be linear, if its voltage-current characteristic is at all times a straight line through the origin. For example, the current passing through a resistor is proportional to the voltage applied through it, and the relation is expressed as  $V \propto I$  or V = IR. A linear element or network is one which satisfies the principle of superposition, i.e. the principle of homogeneity and additivity. An element which does not satisfy the above principle is called a nonlinear element.

#### 1.4.4 Lumped and Distributed

Lumped elements are those elements which are very small in size and in which simultaneous actions takes place for any given cause at the same instant of time. Typical lumped elements are capacitors, resistors, inductors and transformers. Generally the elements are considered as lumped when their size is very small compared to the wave length of the applied signal. Distributed elements, on the other hand, are those which are not electrically separable for analytical purposes. For example, a transmission line which has distributed resistance, inductance and capacitance along its length may extend for hundreds of miles.

#### **1.5 RESISTANCE PARAMETER**

When a current flows in a material, the free electrons move through the material and collide with other atoms. These collisions cause the electrons to lose some of their energy. This loss of energy per unit charge is the drop in potential across the material. The amount of energy lost by the electrons is related to the physical

property of the material. These collisions restrict the movement of electrons. The property of a material to restrict the flow of electrons is called resistance, denoted by R. The symbol for the resistor is shown in Fig. 1.3.



The unit of resistance is ohm ( $\Omega$ ). Ohm is defined as the resistance offered by the material when a current of one ampere flows between two terminals with one volt applied across it.

According to Ohm's law, the current is directly proportional to the voltage and inversely proportional to the total resistance of the circuit, i.e.

$$I = \frac{V}{R}$$
$$i = \frac{v}{R}$$

or

We can write the above equation in terms of charge as follows.

$$V = R \frac{dq}{dt}$$
, or  $i = \frac{v}{R} = Gv$ 

where G is the conductance of a conductor. The units of resistance and conductance are ohm  $(\Omega)$  and mho  $(\sigma)$  respectively.

When current flows through any resistive material, heat is generated by the collision of electrons with other atomic particles. The power absorbed by the resistor is converted to heat. The power absorbed by the resistor is given by

$$P = vi = (iR)i = i^2 R$$

where i is the current in the resistor in amps, and v is the voltage across the resistor in volts. Energy lost in a resistance in time t is given by

$$W = \int_{0}^{t} pdt = pt = i^{2}Rt = \frac{v^{2}}{R}t$$

where v is the volts R is in ohms t is in seconds and W is in joules **Example 1.4** A 10 W resistor is connected across a 12 V battery. How much current flows through the resistor?

Solution

$$I = \frac{V}{R} = \frac{12}{10} = 1.2 \text{ A}$$

#### **1.6 INDUCTANCE PARAMETER**

V = IR

A wire of certain length, when twisted into a coil becomes a basic inductor. If current is made to pass through an inductor, an electromagnetic field is formed. A change in the magnitude of the current changes the electromagnetic field. Increase in current expands the fields, and decrease in current reduces it. Therefore, a change in current produces change in the electromagnetic field, which induces a voltage across the coil according to Faraday's law of electromagnetic induction.

The unit of inductance is *henry*, denoted by *H*. By definition, the inductance is one henry when current through the coil, changing at the rate of one ampere per second, induces one volt across the coil. The symbol for inductance is shown in Fig. 1.4.

~\_\_\_\_\_C

Fig. 1.4

The current-voltage relation is given by

$$v = L \frac{di}{dt}$$

where v is the voltage across inductor in volts, and i is the current through inductor in amps. We can rewrite the above equations as

$$di = \frac{1}{L} v dt$$

Integrating both sides, we get

$$\int_{0}^{t} di = \frac{1}{L} \int_{0}^{t} v dt$$
$$i(t) - i(0) = \frac{1}{L} \int_{0}^{t} v dt$$
$$i(t) = \frac{1}{L} \int_{0}^{t} v dt + i(0)$$

From the above equation we note that the current in an inductor is dependent upon the integral of the voltage across its terminals and the initial current in the coil, i(0).

The power absorbed by inductor is

$$P = vi = Li \frac{di}{dt}$$
 watts

The energy stored by the inductor is

$$W = \int_{0}^{t} p dt$$
$$= \int_{0}^{t} Li \frac{di}{dt} dt = \frac{Li^{2}}{2}$$

From the above discussion, we can conclude the following.

- 1. The induced voltage across an inductor is zero if the current through it is constant. That means an inductor acts as short circuit to dc.
- 2. A small change in current within zero time through an inductor gives an infinite voltage across the inductor, which is physically impossible. In a fixed inductor the current cannot change abruptly.
- 3. The inductor can store finite amount of energy, even if the voltage across the inductor is zero, and
- 4. A pure inductor never dissipates energy, only stores it. That is why it is also called a non-dissipative passive element. However, physical inductors dissipate power due to internal resistance.

**Example 1.5** The current in a 2 H inductor varies at a rate of 2 A/s. Find the voltage across the inductor and the energy stored in the magnetic field after 2 s.

#### Solution

$$v = L \frac{di}{dt}$$
  
= 2 × 4 = 8 V  
$$W = \frac{1}{2}Li^{2}$$
  
=  $\frac{1}{2} \times 2 \times (4)^{2} = 16$  J

\*\*

#### **1.7 CAPACITANCE PARAMETER**

Any two conducting surfaces separated by an insulating medium exhibit the property of a capacitor. The conducting surfaces are called *electrodes*, and the insulating medium is called *dielectric*. A capacitor stores energy in the form of an electric field that is established by the opposite charges on the two electrodes. The electric field is represented by lines of force between the positive and negative charges, and is concentrated within the dielectric. The amount of charge per unit voltage that is capacitor can store is its capacitance,

denoted by *C*. The unit of capacitance is *Farad* denoted by *F*. By definition, one Farad is the amount of capacitance when one coulomb of charge is stored with one volt across the plates. The symbol for capacitance is shown in Fig. 1.5.

A capacitor is said to have greater capacitance if it can store more charge per unit voltage and the capacitance is given by

$$C = \frac{Q}{V}$$
, or  $C = \frac{q}{v}$ 

We can write the above equation in terms of current as

$$i = C \frac{dv}{dt} \left( \because \quad i = \frac{dq}{dt} \right)$$

where v is the voltage across capacitor, i is the current through it

$$dv = \frac{1}{C} i dt$$

Integrating both sides, we have

$$\int_{0}^{t} dv = \frac{1}{C} \int_{0}^{t} i dt$$
$$v(t) - v(0) = \frac{1}{C} \int_{0}^{t} i dt$$
$$v(t) = \frac{1}{C} \int_{0}^{t} i dt + v(0)$$

where v(0) indicates the initial voltage across the capacitor.

From the above equation, the voltage in a capacitor is dependent upon the integral of the current through it, and the initial voltage across it.

The power absorbed by the capacitor is given by

$$p = vi = vC \frac{dv}{dt}$$

The energy stored by the capacitor is

$$W = \int_{0}^{t} p dt = \int_{0}^{t} vC \frac{dv}{dt} dt$$
$$W = \frac{1}{2} Cv^{2}$$

From the above discussion we can conclude the following

- 1. The current in a capacitor is zero if the voltage across it is constant; that means, the capacitor acts as an open circuit to dc.
- 2. A small change in voltage across a capacitance within zero time gives an infinite current through the capacitor, which is physically impossible. In a fixed capacitance the voltage cannot change abruptly.
- 3. The capacitor can store a finite amount of energy, even if the current through it is zero, and
- 4. A pure capacitor never dissipates energy, but only stores it; that is why it is called *non-dissipative passive element*. However, physical capacitors dissipate power due to internal resistance.

**Example 1.6** A capacitor having a capacitance  $2 \mu F$  is charged to a voltage of 1000 V. Calculate the stored energy in joules.

#### Solution

$$W = \frac{1}{2} Cv^2 = \frac{1}{2} \times 2 \times 10^{-6} \times (1000)^2 = 1 \text{ J.}$$

#### **1.8 ENERGY SOURCES**

According to their terminal voltage-current characteristics, electrical energy sources are categorised into ideal voltage sources and ideal current sources. Further they can be divided into independent and dependent sources.

An ideal voltage source is a two-terminal element in which the voltage  $v_s$  is completely independent of the current  $i_s$  through its terminals. The representation of ideal constant voltage source is shown in Fig. 1.6(a).



Fig. 1.6

If we observe the v - i characteristics for an ideal voltage source as shown in Fig. 1.6(c) at any time, the value of the terminal voltage  $v_s$  is constant with respect to the value of current  $i_s$ . Whenever  $v_s = 0$ , the voltage source is the same as that of a short circuit. Voltage sources need not have constant magnitude; in many cases the specified voltage may be time-dependent like a sinusoidal waveform. This may be represented as shown in Fig. 1.6(b). In many practical voltage sources, the internal resistance is represented in series with the source as shown in Fig. 1.7(a). In this, the voltage across the terminals falls as the current through it increases, as shown in Fig. 1.7 (b).



Fig. 1.7

The terminal voltage  $v_t$  depends on the source current as shown in Fig. 1.7(b), where  $v_t = v_s - i_s R$ .

An ideal constant current source is a two-terminal element in which the current  $i_s$  completely independent of the voltage  $v_s$  across its terminals. Like voltage sources we can have current sources of constant magnitude  $i_s$  or sources whose current varies with time  $i_s(t)$ . The representation of an ideal current source is shown in Fig. 1.8(a).



Fig. 1.8

If we observe the v - i characteristics for an ideal current source as shown in Fig. 1.8(b), at any time the value of the current  $i_s$  is constant with respect to the voltage across it. In many practical current sources, the resistance is in parallel with a source as shown in Fig. 1.9(a). In this the magnitude of the current falls as the voltage across its terminals increases. Its terminal v - i characteristics is shown in Fig. 1.9(b). The terminal current is given by  $i_t = i_s - (v_s/R)$ , where R is the internal resistance of the ideal current source.



Fig. 1.9

The two types of ideal sources we have discussed are independent sources for which voltage and current are independent and are not affected by other parts of the circuit. In the case of dependent sources, the source voltage or current is not fixed, but is dependent on the voltage or current existing at some other location in the circuit.

Dependent or controlled sources are of the following types

- (i) voltage controlled voltage source (VCVS)
- (ii) current controlled voltage source (CCVS)
- (iii) voltage controlled current source (VCCS)
- (iv) current controlled current source (CCCS)

These are represented in a circuit diagram by the symbol shown in Fig. 1.10. These types of sources mainly occur in the analysis of equivalent circuits of transistors.





#### **1.9 KIRCHHOFF'S VOLTAGE LAW**

Kirchhoff's voltage law states that the algebraic sum of all branch voltages around any closed path in a circuit is always zero at all instants of time. When the current passes through a resistor, there is a loss of energy and, therefore, a voltage drop. In any element, the current always flows from higher potential to lower potential. Consider the circuit in Fig. 1.11. It is customary to take the direction of current *I* as indicated in the figure, i.e. it leaves the positive terminal of the voltage source and enters into the negative terminal.



Fig. 1.11

As the current passes through the circuit, the sum of the voltage drop around the loop is equal to the total voltage in that loop. Here the polarities are attributed to the resistors to indicate that the voltages at points a, c and e are more than the voltages at b, d and f, respectively, as the current passes from a to f.

...

$$V_{s} = V_{1} + V_{2} + V_{3}$$

Consider the problem of finding out the current supplied by the source V in the circuit shown in Fig. 1.12.

Our first step is to assume the reference current direction and to indicate the polarities for different elements. (See Fig. 1.13).







By using Ohm's law, we find the voltage across each resistor as follows.

$$V_{R1} = IR_1, V_{R2} = IR_2, V_{R3} = IR_3$$

where  $V_{R1}$ ,  $V_{R2}$  and  $V_{R3}$  are the voltages across  $R_1$ ,  $R_2$  and  $R_3$ , respectively. Finally, by applying Kirchhoff's law, we can form the equation

$$V = V_{R1} + V_{R2} + V_{R3}$$
$$V = IR_1 + IR_2 + IR_3$$

From the above equation the current delivered by the source is given by

$$I = \frac{V}{R_1 + R_2 + R_3}$$

**Example 1.7** For the circuit shown in Fig. 1.14, determine the unknown voltage drop  $V_1$ .



Fig. 1.14

**Solution** According to Kirchhoff's voltage law, the sum of the potential drops is equal to the sum of the potential rises;

Therefore,

 $30 = 2 + 1 + V_1 + 3 + 5$  $V_1 = 30 - 11 = 19 \text{ V}$ 

or

**1 Μ**Ω

10 V

500 kΩ

**Example 1.8** What is the current in the circuit shown in Fig. 1.15? Determine the voltage across each resistor.

**Solution** We assume current *I* in the clockwise direction and indicate polarities (Fig. 1.16). By using Ohm's law, we find the voltage drops across each resistor.

$$V_{\rm IM} = I,$$
  $V_{3.1M} = 3.1I$   
 $V_{500K} = 0.5I,$   $V_{400K} = 0.4I$ 

Now, by applying Kirchhoff's voltage law, we form the equation.

or or

5I = 10 $I = 2 \mu A$ :. Voltage across each resistor is as follows

$$V_{1M} = 1 \times 2 = 2.0 \text{ V}$$

$$V_{3.1M} = 3.1 \times 2 = 6.2 \text{ V}$$

$$V_{400K} = 0.4 \times 2 = 0.8 \text{ V}$$

$$V_{500K} = 0.5 \times 2 = 1.0 \text{ V}$$
500 kΩ  $\geq$ 
400 kΩ
Fig. 1.16

**Example 1.9** In the circuit given in Fig. 1.17, find (a) the current *I*, and (b) the voltage across 30  $\Omega$ .

10 = I + 3.1 I + 0.5 I + 0.4 I



Fig. 1.17

Solution We redraw the circuit as shown in Fig. 1.18 and assume current direction and indicate the assumed polarities of resistors



Fig. 1.18

22

3.1 MΩ

400 kΩ

 $3.1 \text{ M}\Omega$ 

Fig. 1.15

By using Ohm's law, we determine the voltage across each resistor as

 $V_8 = 8I, V_{30} = 30I, V_2 = 2I$ 

By applying Kirchhoff's law, we get

$$100 = 8I + 40 + 30I + 2I$$

40 
$$I = 60$$
 or  $I = \frac{60}{40} = 1.5$  A

\ Voltage drop across 30  $\Omega = V_{30} = 30 \times 1.5 = 45$  V

#### **1.10 VOLTAGE DIVISION**

The series circuit acts as a voltage divider. Since the same current flows through each resistor, the voltage drops are proportional to the values of resistors. Using this principle, different voltages can be obtained from a single source, called a voltage divider. For example, the voltage across a 40  $\Omega$  resistor is twice that of 20  $\Omega$  in a series circuit shown in Fig. 1.19.

In general, if the circuit consists of a number of series resistors, the total current is given by the total voltage divided by equivalent resistance. This is shown in Fig. 1.20.



Fig. 1.20

22

The current in the circuit is given by  $I = V_s/(R_1 + R_2 + ... + R_m)$ . The voltage across any resistor is nothing but the current passing through it, multiplied by that particular resistor.

$$V_{R1} = IR_1$$

$$V_{R2} = IR_2$$

$$V_{R3} = IR_3$$

$$\vdots$$

$$V_{Rm} = IR_m$$

$$V_{Rm} = \frac{V_s(R_m)}{R_1 + R_2 + \dots + R_m}$$

or

From the above equation, we can say that the voltage drop across any resistor, or a combination of resistors, in a series circuit is equal to the ratio of that resistance value to the total resistance, multiplied by the source voltage, i.e.

$$V_m = \frac{R_m}{R_T} V_s$$

where  $V_m$  is the voltage across *m*th resistor,

 $R_m$  is the resistance across which the voltage is to be determined and  $R_T$  is the total series resistance.

**Example 1.10** What is the voltage across the 10  $\Omega$  resistor in Fig. 1.21.



Fig. 1.21

**Solution** Voltage across  $10 \ \Omega = V_{10} = 50 \times \frac{10}{10+5} = \frac{500}{15} = 33.3 \text{ V}$ 

**Example 1.11** Find the voltage between A and B in a voltage divider network shown in Fig. 1.22.



Fig. 1.22

**Solution** Voltage across  $9 \text{ k}\Omega = V_9 = V_{AB} = 100 \times \frac{9}{10} = 90 \text{ V}$ 

#### **1.11 POWER IN SERIES CIRCUIT**

The total power supplied by the source in any series resistive circuit is equal to the sum of the powers in each resistor in series, i.e.

$$P_S = P_1 + P_2 + P_3 + \dots + P_m$$

where *m* is the number of resistors in series,  $P_s$  is the total power supplied by source and  $P_m$  is the power in the last resistor in series. The total power in the series circuit is the total voltage applied to a circuit, multiplied by the total current. Expressed mathematically,

$$P_S = V_s I = I^2 R_T = \frac{V_s^2}{R_T}$$

where  $V_s$  is the total voltage applied,  $R_T$  is the total resistance, and I is the total current.

**Example 1.12** Determine the total amount of power in the series circuit in Fig. 1.23.





**Solution** Total resistance =  $5 + 2 + 1 + 2 = 10 \Omega$ 

$$P_S = \frac{V_s^2}{R_T} = \frac{(50)^2}{10} = 250 \text{ W}$$

Check We find the power absorbed by each resistor

Current = 
$$\frac{50}{10}$$
 = 5 A  
 $P_5 = (5)^2 \times 5 = 125$  W  
 $P_2 = (5)^2 \times 2 = 50$  W  
 $P_1 = (5)^2 \times 1 = 25$  W  
 $P_2 = (5)^2 \times 2 = 50$  W

The sum of these powers gives the total power supplied by the source  $P_s = 250 \text{ W}$ .

#### 1.12 KIRCHHOFF'S CURRENT LAW

Kirchhoff's current law states that the sum of the currents entering into any node is equal to the sum of the currents leaving that node. The node may be an interconnection of two or more branches. In any parallel circuit, the node is a junction point of two or more branches. The total current entering into a node is equal to the current leaving that node. For example, consider the circuit shown in Fig. 1.24, which contains two nodes A and B. The total current  $I_T$  entering node A is divided into  $I_1$ ,  $I_2$  and  $I_3$ . These currents flow out of node A.

According to Kirchhoff's current law, the current into node A is equal to the total current out of node A: that is,  $I_T = I_1 + I_2 + I_3$ . If we consider node B, all three currents  $I_1, I_2, I_3$  are entering B, and the total current  $I_T$  is leaving node B, Kirchhoff's current law formula at this node is therefore the same as at node A.

$$I_1 + I_2 + I_3 = I_T$$



We know

In general, sum of the currents entering any point or node or junction equal to sum of the currents leaving from that point or node or junction as shown in Fig. 1.25.

$$I_1 + I_2 + I_4 + I_7 = I_3 + I_5 + I_6$$

If all of the terms on the right side are brought over to the left side, their signs change to negative and a zero is left on the right side, i.e.

$$I_1 + I_2 + I_4 + I_7 - I_3 - I_5 - I_6 = 0$$

This means that the algebraic sum of all the currents meeting at a junction is equal to zero.

**Example 1.13** Determine the current in all resistors in the circuit shown in Fig. 1.26.



Fig. 1.26

**Solution** The above circuit contains a single node 'A' with reference node 'B'. Our first step is to assume the voltage V at node A. In a parallel circuit the same voltage is applied across each element. According to Ohm's law, the currents passing through each element are  $I_1 = V/2$ ,  $I_2 = V/1$ ,  $I_3 = V/5$ .

By applying Kirchhoff's current law, we have

$$I = I_1 + I_2 + I_3$$

$$I = \frac{V}{2} + \frac{V}{1} + \frac{V}{5}$$

$$50 = V \left[ \frac{1}{2} + \frac{1}{1} + \frac{1}{5} \right] = V [0.5 + 1 + 0.2]$$

$$V = \frac{50}{1.7} = \frac{500}{17} = 29.41 \text{ V}$$

Once we know the voltage V at node A, we can find the current in any element by using Ohm's law.

The current in the 2  $\Omega$  resistor is  $I_1 = 29.41/2 = 14.705$  A.

Similarly 
$$I_2 = \frac{V}{R_2} = \frac{V}{1} = 29.41 \text{ A}$$



 $I_1$ 



$$I_3 = \frac{29.41}{5} = 5.882 \text{ A}$$
  
 $I_1 = 14.7 \text{ A}, I_2 = 29.4 \text{ A}, \text{ and } I_3 = 5.88 \text{ A}$ 

**Example 1.14** For the circuit shown in Fig. 1.27, find the voltage across the 10  $\Omega$  resistor and the current passing through it.



Fig. 1.27

**Solution** The circuit shown above is a parallel circuit, and consists of a single node A. By assuming voltage V at the node A w.r.t. B, we can find out the current in the 10 W branch. (See Fig. 1.28)



Fig. 1.28

22

According to Kirchhoff's current law,

$$I_1 + I_2 + I_3 + I_4 + 5 = 10$$

By using Ohm's law we have

$$I_{1} = \frac{V}{5}; I_{2} = \frac{V}{10}, I_{3} = \frac{V}{2}, I_{4} = \frac{V}{1}$$
$$\frac{V}{5} + \frac{V}{10} + \frac{V}{2} + V + 5 = 10$$
$$V\left[\frac{1}{5} + \frac{1}{10} + \frac{1}{2} + 1\right] = 5$$
$$V\left[0.2 + 0.1 + 0.5 + 1\right] = 5$$
$$V = \frac{5}{18} = 2.78 \text{ V}$$

 $\therefore$  The voltage across the 10  $\Omega$  resistor is 2.78 V and the current passing through it is

*:*.

$$I_2 = \frac{V}{10} = \frac{2.78}{10} = 0.278 \text{ A}$$

**Example 1.15** Determine the current through resistance  $R_3$  in the circuit shown in Fig. 1.29.



Fig. 1.29

Solution According to Kirchhoff's current law,

$$I_T = I_1 + I_2 + I_3$$

where  $I_T$  is the total current and  $I_1$ ,  $I_2$  and  $I_3$  are the currents in resistances  $R_1$ ,  $R_2$  and  $R_3$  respectively.

or

 $50 = 30 + 10 + I_3$  $I_3 = 10 \text{ mA}$ 

#### **1.13 PARALLEL RESISTANCE**

When the circuit is connected in parallel, the total resistance of the circuit decreases as the number of resistors connected in parallel increases. If we consider m parallel branches in a circuit as shown in Fig. 1.30, the current equation is



Fig. 1.30

The same voltage is applied across each resistor. By applying Ohm's law, the current in each branch is given by

$$I_1 = \frac{V_s}{R_1}, I_2 = \frac{V_s}{R_2}, \dots I_m = \frac{V_s}{R_m}$$

\*\*
According to Kirchhoff's current law,

$$I_T = I_1 + I_2 + I_3 + \dots + I_m$$
$$\frac{V_s}{R_T} = \frac{V_s}{R_1} + \frac{V_s}{R_2} + \frac{V_s}{R_3} + \dots + \frac{V_s}{R_m}$$

From the above equation, we have

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_m}$$

**Example 1.16** Determine the parallel resistance between points A and B of the circuit shown in Fig. 1.31.



Fig. 1.31

### $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$ Solution $\frac{1}{R_T} = \frac{1}{10} + \frac{1}{20} + \frac{1}{30} + \frac{1}{40}$ = 0.1 + 0.05 + 0.033 + 0.025 = 0.208 $R_T = 4.8 \ \Omega$ or \*\*

#### 1.14 **CURRENT DIVISION**

In a parallel circuit, the current divides in all branches. Thus, a parallel circuit acts as a current divider. The total current entering into the parallel branches

is divided into the branches currents according to the resistance values. The branch having higher resistance allows lesser current, and the branch with lower resistance allows more current. Let us find the current division in the parallel circuit shown in Fig. 1.32.

The voltage applied across each resistor is  $V_s$ . The current passing through each resistor is given by

$$I_1 = \frac{V_s}{R_1}, I_2 = \frac{V_s}{R_2}$$



Fig. 1.32

If  $R_T$  is the total resistance, which is given by  $R_1R_2/(R_1 + R_2)$ ,

Total current 
$$I_T = \frac{V_s}{R_T} = \frac{V_s}{R_1 R_2} (R_1 + R_2)$$
$$I_T = \frac{I_1 R_1}{R_1 R_2} (R_1 + R_2) \text{ since } V_s = I_1 R_1$$

or

$$I_1 = I_T \cdot \frac{R_2}{R_1 + R_2}$$
$$I_2 = I_T \cdot \frac{R_1}{R_1 + R_2}$$

Similarly,

From the above equations, we can conclude that the current in any branch is equal to the ratio of the opposite branch resistance to the total resistance value, multiplied by the total current in the circuit. In general, if the circuit consists of m branches, the current in any branch can be determined by

$$I_i = \frac{R_T}{R_i + R_T} I_T$$

where  $I_i$  represents the current in the *i*th branch

 $\dot{R}_{i}$  is the resistance in the *i*th branch

 $R_{T}^{i}$  is the total parallel resistance to the *i*th branch and

 $I_{\tau}$  is the total current entering the circuit.

**Example 1.17** Determine the current through each resistor in the circuit shown in Fig. 1.33.



Fig. 1.33

#### Solution

$$I_1 = I_T \times \frac{R_T}{(R_1 + R_T)}$$
$$R_T = \frac{R_2 R_3}{R_2 + R_3} = 2 \Omega$$
$$R_1 = 4 \Omega$$

÷

where

$$I_T = 12 \text{ A}$$

$$I_1 = 12 \times \frac{2}{2+4} = 4 \text{ A}$$
Similarly,
$$I_2 = 12 \times \frac{2}{2+4} = 4 \text{ A}$$

and

$$I_3 = 12 \times \frac{2}{2+4} = 4$$
 A

Since all parallel branches have equal values of resistance, they share current equally.

# 1.15 POWER IN PARALLEL CIRCUIT

The total power supplied by the source in any parallel resistive circuit is equal to the sum of the powers in each resistor in parallel, i.e.

$$P_S = P_1 + P_2 + P_3 + \dots + P_m$$

where *m* is the number of resistors in parallel,  $P_s$  is the total power and  $P_m$  is the power in the last resistor.

ADDITIONAL SOLVED PROBLEMS

**Problem 1.1** A resistor with a current of 3 A through it converts 500 J of electrical energy to heat energy in 12 s. What is the voltage across the resistor?

#### Solution

$$V = \frac{w}{Q}$$

$$Q = I \times t$$

$$= 3 \times 12 = 36 \text{ C}$$

$$V = \frac{500}{36} = 13.88 \text{ V}$$

**Problem 1.2** A 5  $\Omega$  resistor has a voltage rating of 100 V. What is its power rating?

#### Solution

$$P = VI$$
  

$$I = V/R$$
  

$$P = \frac{V^2}{R} = \frac{(100)^2}{5} = 2000 \text{ W} = 2 \text{ kW}$$

**Problem 1.3** Find the inductance of a coil through which flows a current of 0.2 A with an energy of 0.15 J.

#### Solution

$$W = \frac{1}{2} LI^{2}$$
$$L = \frac{2 \times W}{I^{2}} = \frac{2 \times 0.15}{(0.2)^{2}} = 7.5 \text{ H}$$

**Problem 1.4** Find the inductance of a coil in which a current increases linearly from 0 to 0.2 A in 0.3 s, producing a voltage of 15 V.

#### Solution

$$v = L \frac{di}{dt}$$

Current in 1 s =  $\frac{0.2}{0.3}$  = 0.66 A

The current changes at a rate of 0.66 A/s,

```
:.
```

$$L = \frac{v}{\left(\frac{di}{dt}\right)}$$
$$L = \frac{15V}{0.66 \text{ A/s}} = 22.73 \text{ H}$$

**Problem 1.5** When a dc voltage is applied to a capacitor, the voltage across its terminals is found to build up in accordance with  $v_c = 50(1 - e^{-100t})$ . After a lapse of 0.01 s, the current flow is equal to 2 mA.

- (a) Find the value of capacitance in microfarads
- (b) How much energy is stored in the electric field at this time?

#### Solution

(a)  

$$i = C \frac{d_{-C}}{dt}$$
where  $v_C = 50(1 - e^{-100t})$   

$$i = C \frac{d}{dt} 50(1 - e^{-100t})$$

$$= C \times 50 \times 100e^{-100t}$$
At  $t = 0.01$  s,  $i = 2$  mA  

$$C = \frac{2 \times 10^{-3}}{50 \times 100 \times e^{-100 \times 0.01}} = 1.089 \ \mu\text{F}$$
(b)  

$$W = \frac{1}{2} C v_C^2$$
At  $t = 0.01$  s,  $v_C = 50 \ (1 - e^{-100 \times 0.01}) = 31.6 \text{ V}$   

$$W = \frac{1}{2} \times 1.089 \times 10^{-6} \times (31.6)^2$$

$$= 0.000543 \text{ J}$$

**Problem 1.6** Determine the total current in the circuit shown in Fig. 1.34.



Fig. 1.34

**Solution** Resistances  $R_2$ ,  $R_3$  and  $R_4$  are in parallel  $\therefore$  Equivalent resistance  $R_5 = R_2 || R_3 || R_4$ 

$$= \frac{1}{1/R_2 + 1/R_3 + 1/R_4}$$
  
R<sub>5</sub> = 1 Ω

:.

 $R_1$  and  $R_5$  are in series,

: Equivalent resistance  $R_T = R_1 + R_5 = 5 + 1 = 6 \Omega$ 

And the total current  $I_T = \frac{V_s}{R_T} = \frac{30}{6} = 5$  A

**Problem 1.7** Find the current in the 10  $\Omega$  resistance,  $V_1$ , and source voltage  $V_s$  in the circuit shown in Fig. 1.35.



Fig. 1.35

**Solution** Assume voltage at node C = V

By applying Kirchhoff's current law, we get the current in the 10  $\Omega$  resistance

$$I_{10} = I_5 + I_6$$
  
= 4 + 1 = 5 A

The voltage across the 6  $\Omega$  resistor is  $V_6 = 24$  V

 $\therefore$  Voltage at node C is  $V_C = -24$  V.

The voltage across branch CD is the same as the voltage at node C.

Voltage across 10  $\Omega$  only = 10 × 5 = 50 V So  $V_C = V_{10} - V_1$   $-24 = 50 - V_1$   $\therefore$   $V_1 = 74 V$ Now, consider the loop CABD shown in Fig. 1.36 If we apply Kirchhoff's voltage law we get  $V_s = 5 - 30 - 24 = -49 V$ 

....

**Problem 1.8** What is the voltage across *A* and *B* in the circuit shown in Fig. 1.37?



Fig. 1.37

**Solution** The above circuit can be redrawn as shown in Fig. 1.38.



Fig. 1.38

Assume loop currents  $I_1$  and  $I_2$  as shown in Fig. 1.38.

:.

$$I_1 = \frac{6}{10} = 0.6 \text{ A}$$
$$I_2 = \frac{12}{14} = 0.86 \text{ A}$$

 $V_A$  = Voltage drop across 4  $\Omega$  resistor = 0.6 × 4 = 2.4 V

$$V_B$$
 = Voltage drop across 4  $\Omega$  resistor = 0.86 × 4 = 3.44 V

The voltage between points A and B is the sum of voltages as shown in Fig. 1.39.

$$A \circ \underbrace{- \underbrace{24V}_{4\Omega} + \underbrace{12V}_{4\Omega} - \underbrace{3.44V}_{4\Omega} \circ B$$

Fig. 1.39

 $V_{AB} = -2.4 + 12 + 3.44 = 13.04 \text{ V}$ 

....

**Problem 1.9** Determine the current delivered by the source in the circuit shown in Fig. 1.40.



Fig. 1.40

**Solution** The circuit can be modified as shown in Fig. 1.41, where  $R_{10}$  is the series combination of  $R_2$  and  $R_3$ .



Fig. 1.41

 $R_{11}$  is the series combination of  $R_4$  and  $R_5$  $\therefore$   $R_{11} = R_4 + R_5 = 3 \Omega$ 

Further simplification of the circuit leads to Fig 1.42 where  $R_{12}$  is the parallel combination of  $R_{10}$  and  $R_{9}$ .

:.  $R_{12} = (R_{10} || R_9) = (4 || 4) = 2 \Omega$ 

Similarly,  $R_{13}$  is the parallel combination of  $R_{11}$  and  $R_{8}$ 

:.  $R_{13} = (R_{11} || R_8) = (3 || 2) = 1.2 \Omega$ 

In Fig. 1.42 as shown,  $R_{12}$  and  $R_{13}$  are in series, which is in parallel with  $R_7$  forming  $R_{14}$ . This is shown in Fig. 1.43.



Fig. 1.42



:.

$$R_{14} = [(R_{12} + R_{13})//R_7]$$
  
= [(2 + 1.2)//2] = 1.23 \Omega

Further, the resistances  $R_{14}$  and  $R_6$  are in series, which is in parallel with  $R_1$  and gives the total resistance

$$R_T = [(R_{14} + R_6)//R_1]$$
  
= [(1 + 1.23)//(2)] = 1.05 \Omega

The current delivered by the source = 30/1.05 = 28.57 A

**Problem 1.10** Determine the current in the 10  $\Omega$  resistance and find  $V_s$  in the circuit shown in Fig. 1.44.

...

...





**Solution** The current in 10  $\Omega$  resistance

 $I_{10} = \text{total current} \times (R_T)/(R_T + R_{10})$ 

where  $R_{\tau}$  is the total parallel resistance.

$$I_{10} = 4 \times \frac{7}{17} = 1.65 \text{ A}$$

Similarly, the current in resistance  $R_5$  is

$$I_5 = 4 \times \frac{10}{10+7} = 2.35 \text{ A}$$

or

4 – 1.65 = 2.35 A

The same current flows through the 2  $\Omega$  resistance.

:. Voltage across 2  $\Omega$  resistance,  $V_s = I_5 \times 2$ 

$$= 2.35 \times 2 = 4.7 \text{ V}$$

5Ω.

25 Ω

22

**Problem 1.11** Determine the value of resistance *R* and current in each branch when the total current taken by the circuit shown in Fig. 1.45 is 6 A.

Solution The current in branch ADB

 $I_{30} = 50/(25 + 5) = 1.66$  A The current in branch *ACB*  $I_{10+R} = 50/(10 + R)$ . According to Kirchhoff's current law



**Problem 1.12** Find the power delivered by the source in the circuit shown in Fig. 1.46.

 $R = 1.52 \ \Omega$ 





Fig. 1.47 (a, b, c and d)

-

**Problem 1.13** Determine the voltage drop across the 10  $\Omega$  resistance in the circuit as shown in Fig. 1.48.

4Ω D R4 5-5 `5 Ω

**Solution** The circuit is redrawn as shown in Fig. 1.49.





*:*.

Fig. 1.49

This is a single node pair circuit. Assume voltage  $V_A$  at node A. By applying Kirchhoff's current law at node A, we have

$$\frac{V_A}{20} + \frac{V_A}{10} + \frac{V_A}{5} = 10 + 15$$
$$V_A \left[ \frac{1}{20} + \frac{1}{10} + \frac{1}{5} \right] = 25 \text{ A}$$
$$V_A \left[ 0.05 + 0.1 + 0.2 \right] = 25 \text{ A}$$
$$V_A = \frac{25}{0.35} = 71.42 \text{ V}$$

The voltage across 10  $\Omega$  is nothing but the voltage at node A.

$$V_{10} = V_A = 71.42 \text{ V}$$

**Problem 1.14** In the circuit shown in Fig. 1.50 what are the values of  $R_1$  and  $R_2$ , when the current flowing through  $R_1$  is 1 A and  $R_2$  is 5 A? What is the value of  $R_2$  when the current flowing through  $R_1$  is zero?





# **Solution** The current in the 5 $\Omega$ resistance $I_5 = I_1 + I_2 = 1 + 5 = 6$ A

Voltage across resistance 5  $\Omega$  is  $V_5 = 5 \times 6 = 30$  V The voltage at node A,  $V_A = 100 - 30 = 70$  V  $\therefore$   $I_2 = V_A - 30$  70 - 30

$$\frac{R_2}{R_2} = \frac{R_2}{R_2}$$

$$R_2 = \frac{70 - 30}{I_2} = \frac{40}{5} = 8 \Omega$$

$$R_1 = \frac{70 - 50}{I_1} = \frac{20}{1} = 20 \Omega$$

Similarly,

When  $V_A = 50$  V, the current  $I_1$  in resistance  $R_1$  becomes zero.

$$\therefore \qquad I_2 = \frac{50 - 30}{R_2}$$

where  $I_2$  becomes the total current

:. 
$$I_2 = \frac{100 - V_A}{5} = \frac{100 - 50}{5} = 10 \text{ A}$$
  
:.  $R_2 = \frac{20}{I_2} = \frac{20}{10} = 2 \Omega$ 

**Problem 1.15** Determine the output voltage  $V_{out}$  in the circuit shown in Fig. 1.51.



Fig. 1.51

**Solution** The circuit shown in Fig. 1.51 can be redrawn as shown in Fig. 1.52. In Fig. 1.52,  $R_2$  and  $R_3$  are in parallel,  $R_4$  and  $R_5$  are in parallel. The complete circuit is a single node pair circuit. Assuming voltage  $V_A$  at node A and applying Kirchhoff's current law in the circuit, we have

$$10A - \frac{V_A}{4.43} - 5A - \frac{V_A}{2.67} = 0$$
$$V_A \left[ \frac{1}{4.43} + \frac{1}{2.67} \right] = 5 A$$
$$V_A \left[ 0.225 + 0.375 \right] = 5$$

:.







 $V_A = \frac{5}{0.6} = 8.33 \text{ V}$  $V_{\text{out}} = V_A = 8.33 \text{ V}$ 





Fig. 1.53

Solution The circuit in Fig. 1.53 can be redrawn as shown in Fig. 1.54 (a).



Fig. 1.54 (a)

At node 3, the series combination of  $R_7$  and  $R_8$  are in parallel with  $R_6$ , which gives  $R_9 = [(R_7 + R_8)//R_6] = 3 \Omega$ .

At node 2, the series combination of  $R_3$  and  $R_4$  are in parallel with  $R_2$ , which gives  $R_{10} = [(R_3 + R_4)/(R_2)] = 3 \Omega$ .



Fig. 1.54 (b)



Simplifying further we draw it as shown in Fig. 1.54 (c). Total current delivered by the source=  $\frac{100}{R_{\tau}}$ 

$$=\frac{100}{(13/8)}=20.2$$
 A

Current in the 8  $\Omega$  resistor is  $I_8 = 20.2 \times \frac{13}{13+8} = 12.5 \text{ A}$ 

Current in the 13  $\Omega$  resistor is  $I_{13} = 20.2 \times \frac{8}{13+8} = 7.69$  A

So 
$$I_5 = 12.5$$
 A, and  $I_{10} = 7.69$  A  
Current in the 4  $\Omega$  resistance  $I_4 = 3.845$  A  
Current in the 3  $\Omega$  resistance  $I_3 = 6.25$  A  
 $V_{AB} = V_A - V_B$   
where  $V_A = I_3 \times 3 \ \Omega = 6.25 \times 3 = 18.75$  V  
 $V_B = I_4 \times 4 \ \Omega = 3.845 \times 4 = 15.38$  V  
 $\therefore V_{4B} = 18.75 - 15.38 = 3.37$  V

**Problem 1.17** Determine the value of R in the circuit shown in Fig. 1.55, when the current is zero in the branch CD.



Fig. 1.55

**Solution** The current in the branch *CD* is zero, if the potential difference across *CD* is zero.

That means, voltage at point C = voltage at point D.

Since no current is flowing, the branch *CD* is open circuited. So the same voltage is applied across *ACB* and *ADB* 

<sub>V</sub> , 10

т7

... and

$$V_{10} - V_A \times \frac{15}{15}$$

$$V_R = V_A \times \frac{R}{20 + R}$$

$$V_{10} = V_R$$

$$V_A \times \frac{10}{15} = V_A \times \frac{R}{20 + R}$$

$$R = 40 \ \Omega$$

:.

Problem 1.18 Find the power absorbed by each element in the circuit shown in Fig. 1.56.



Fig. 1.56

**Solution** Power absorbed by any element = VI

where V is the voltage across the element and I is the current passing through that element

Here potential rises are taken as (-) sign.

Power absorbed by 10 V source =  $-10 \times 2 = -20$  W

Power absorbed by resistor  $R_1 = 24 \times 2 = 48$  W

Power absorbed by resistor  $R_2 = 14 \times 7 = 98$  W Power absorbed by resistor  $R_3 = -7 \times 9 = -63$  W

Power absorbed by dependent voltage source =  $(1 \times -7) \times 9 = -63$  W 22

**Problem 1.19** Show that the algebraic sum of the five absorbed power values in Fig. 1.57 is zero.



Fig. 1.57

-----

**Solution** Power absorbed by 2 A current source =  $(-4) \times 2 = -8$  W Power absorbed by 4 V voltage source =  $(-4) \times 10 = -4$  W Power absorbed by 2 V voltage source =  $(2) \times 3 = 6$  W Power absorbed by 7 A current source =  $(7) \times 2 = 14$  W Power absorbed by  $2i_x$  dependent current source =  $(-2) \times 2 \times 2 = -8$  W Hence, the algebraic sum of the five absorbed power values is zero.

**Problem 1.20** For the circuit shown in Fig. 1.58, find the power absorbed by each of the elements.



Fig. 1.58





Fig. 1.59

Assume loop current *I* as shown in Fig. 1.59. If we apply Kirchhoff's voltage law, we get

$$-12 + I - 2v_1 + v_1 + 4I = 0$$

The voltage across 3  $\Omega$  resistor is  $v_1 = 3I$ Substituting  $v_1$  in the loop equation, we get I = 6 A Power absorbed by the 12 V source =  $(-12) \times 6 = -72$  W Power absorbed by the 1  $\Omega$  resistor =  $6 \times 6 = 36$  W Power absorbed by  $2v_1$  dependent voltage source =  $(2v_1)I = 2 \times 3 \times 6 \times 6 = -216$  W Power absorbed by 3  $\Omega$  resistor =  $v_1 \times I = 18 \times 6 = 108$  W Power absorbed by 4  $\Omega$  resistor =  $4 \times 6 \times 6 = 144$  W •••

**Problem 1.21** For the circuit shown in Fig. 1.60, find the power absorbed by each element.



Fig. 1.60

**Solution** The circuit shown in Fig. 1.60 is a parallel circuit and consists of a single node *A*. By assuming voltage *V* at node A, we can find the current in each element.

According to Kirchhoff's current law

$$i_3 - 12 - 2i_2 - i_2 = 0$$

By using Ohm's law, we have

$$i_{3} = \frac{V}{3}, i_{2} = \frac{-V}{2}$$

$$V\left[\frac{1}{3} + 1 + \frac{1}{2}\right] = 12$$

$$V = \frac{12}{1.83} = 6.56$$

$$i_{3} = \frac{6.56}{3} = 2.187\text{A}; i_{2} = \frac{-6.56}{2} = -3.28 \text{ A}$$

:.

Power absorbed by the 3 
$$\Omega$$
 resistor = (+ 6.56) (2.187) = 14.35 W  
Power absorbed by 12 A current source = (- 6.56) 12 = - 78.72 W  
Power absorbed by 2*i*, dependent current source

 $= (-6.56) \times 2 \times (-3.28) = 43.03$  W Power absorbed by 2  $\Omega$  resistor = (-6.52) (-3.28) = 21.51 W



- 1.1 (i) Determine the current in each of the following cases
  (a) 75 C in 1 s
  (b) 10 C in 0.5 s
  (c) 5 C in 2 s
  - (ii) How long does it take 10 C to flow past a point if the current is 5 A?
- 1.2 A resistor of 30  $\Omega$  has a voltage rating of 500 V; what is its power rating?
- 1.3 A resistor with a current of 2 A through it converts 1000 J of electrical energy to heat energy in 15 s. What is the voltage across the resistor?
- 1.4 The filament of a light bulb in the circuit has a certain amount of resistance. If the bulb operates with 120 V and 0.8 A of current, what is the resistance of its filament?

- 1.5 Find the capacitance of a circuit in which an applied voltage of 20 V gives an energy store of 0.3 J.
- 1.6 A 6.8 k $\Omega$  resistor has burned out in a circuit. It has to be replaced with another resistor with the same ohmic value. If the resistor carries 10 mA, what should be its power rating?
- 1.7 If you wish to increase the amount of current in a resistor from 100 mA to 150 mA by changing the 20 V source, by how many volts should you change the source? To what new value should you set it?
- 1.8 A 12 V source is connected to a 10  $\Omega$  resistor.
  - (a) How much energy is used in two minutes?
  - (b) If the resistor is disconnected after one minute, does the power absorbed in resistor increase or decrease?
- 1.9 A capacitor is charged to 50  $\mu$ C. The voltage across the capacitor is 150 V. It is then connected to another capacitor four times the capacitance of the first capacitor. Find the loss of energy.
- 1.10 The voltage across two parallel capacitors 5  $\mu$ F and 3  $\mu$ F changes uniformly from 30 to 75 V in 10 ms. Calculate the rate of change of voltage for (i) each capacitor, and (ii) the combination.
- 1.11 The following voltage drops are measured across each of three resistors in series: 5.5 V, 7.2 V and 12.3 V. What is the value of the source voltage to which these resistors are connected? If a fourth resistor is added to the circuit with a source voltage of 30 V. What should be the drop across the fourth resistor?
- 1.12 What is the voltage  $V_{AB}$  across the resistor shown in Fig. 1.61?



Fig. 1.61 (a)

Fig. 1.61 (b)

1.13 The source voltage in the circuit shown in Fig. 1.62 is 100 V. How much voltage does each metre read?



Fig. 1.62

1.14 Using the current divider formula, determine the current in each branch of the circuit shown in Fig. 1.63.



Fig. 1.63

- 1.15 Six light bulbs are connected in parallel across 110 V. Each bulb is rated at 75 W. How much current flows through each bulb, and what is the total current?
- 1.16 For the circuit shown in Fig. 1.64, find the total resistance between terminals A and B; the total current drawn from a 6 V source connected from A to B; and the current through 4.7 k $\Omega$ ; voltage across 3 k $\Omega$ .



Fig. 1.64

1.17 For the circuit shown in Fig. 1.65, find the total resistance.



Fig. 1.65

1.18 The current in the 5  $\Omega$  resistance of the circuit shown in Fig. 1.66 is 5 A. Find the current in the 10  $\Omega$  resistor. Calculate the power consumed by the 5  $\Omega$  resistor.



Fig. 1.66

1.19 A battery of unknown emf is connected across resistances as shown in Fig. 1.67. The voltage drop across the 8  $\Omega$  resistor is 20 V. What will be the current reading in the ammeter? What is the emf of the battery.



Fig. 1.67

- 1.20 An electric circuit has three terminals *A*, *B*, *C*. Between *A* and *B* is connected a 2  $\Omega$  resistor, between *B* and *C* are connected a 7  $\Omega$  resistor and 5  $\Omega$  resistor in parallel and between *A* and *C* is connected a 1  $\Omega$  resistor. A battery of 10 V is then connected between terminals *A* and *C*. Calculate
  - (a) total current drawn from the battery (b) voltage across the 2  $\Omega$  resistor (c) current passing through the 5  $\Omega$  resistor.
- 1.21 Use Ohm's law and Kirchhoff's laws on the circuit given in Fig. 1.68, find  $V_{in}$ ,  $V_s$  and power provided by the dependent source.



Fig. 1.68

1.22 Use Ohm's law and Kirchhoff's laws on the circuit given in Fig. 1.69, find all the voltages and currents.





1.23 Find the power absorbed by each element and show that the algebraic sum of powers is zero in the circuit shown in Fig. 1.70.



Fig. 1.70

1.24 Find the power absorbed by each element in the circuit shown in Fig. 1.71.







- 1.1 How many coulombs of charge do  $50 \times 10^{31}$  electrons possess? (a)  $80 \times 10^{12}$  C (b)  $50 \times 10^{31}$  C (c)  $0.02 \times 10^{-31}$  C (d)  $1/80 \times 10^{12}$  C
- 1.2 Determine the voltage of 100 J/25 C.
  - (a) 100 V (b) 25 V (d) 0.25 V (c) 4 V
- 1.3 What is the voltage of a battery that uses 800 J of energy to move 40 C of charge through a resistor?

L

Γ

			••••					
	(a) 800 V	(b)	40 V					
	(c) 25 V	(d)	20 V					
1.4	Determine the current if a 10 coulomb	chai	ge passes a point in 0.5 seconds.					
	(a) 10 A	(b)	20 A					
	(c) $0.5 \text{ A}$	(d)	2 A					
1.5	If a resistor has 5.5 V across it and 3 mA	flow	ving through it, what is the power?					
	(a) 16.5 mW	(b)	15 mW					
	(c) 1.83 mW	(d)	16.5 W					
1.6	Identify the passive element among the following							
	(a) Voltage source	(b)	Current source					
	(c) Inductor	(d)	Transistor					
1.7	If a resistor is to carry 1 A of current	carry 1 A of current and handle 100 W of power, how						
	many ohms must it be? Assume that	many ohms must it be? Assume that voltage can be adjusted to any re-						
	quired value.							
	(a) 50 $\Omega$	(b)	100 Ω					
	(c) $1 \Omega$	(d)	10 Ω					
1.8	A 100 $\Omega$ resistor is connected across	s th	e terminals of a 2.5 V battery.					
	What is the power dissipation in the	resis	stor?					
	(a) 25 W	(b)	100 W					
	(c) 0.4 W	(d)	6.25 W					
1.9	Determine total inductance of a para	llel c	combination of 100 mH, 50 mH					
	and 10 mH.							
	(a) 7.69 mH	(b)	160 mH					
	(c) 60 mH	(d)	110 mH					
1.10	How much energy is stored by a 100 r	nH i	nductance with a current of 1 A?					
	(a) 100 J	(b)	1 J					
	(c) 0.05 J	(d)	0.01 J					
1.11	Five inductors are connected in serie	es. T	he lowest value is 5 $\mu$ H. If the					
	value of each inductor is twice that of t	he pi	receding one, and if the inductors					
	are connected in order ascending val	ues.	What is the total inductance?					
	(a) 155 $\mu$ H	(b)	155 H					
1 1 2	(c) 155 mH Determine the change when $C = 0.00$	(a)	$25 \mu\text{H}$					
1.12	Determine the charge when $C = 0.00$	(h)	$1 \mu C$					
	(a) $0.001 \text{ C}$	(0)	$1 \mu C$					
1 13	If the voltage across a given canacit	or is	increased does the amount of					
1.15	stored charge	01 15	increased, does the amount of					
	(a) increase	$(\mathbf{b})$	decrease					
	(c) remain constant	(d)	is exactly doubled					
1 14	A 1 $\mu$ F a 2.2 $\mu$ F and a 0.05 $\mu$ F can	acito	are connected in series. The					
1.1.1	total capacitance is less than	aento	is the connected in series. The					
	(a) $0.07$	(b)	3 25					
	(a) $0.07$	(d)	3.2					
1.15	How much energy is stored by $a 0.05 i$	IF ca	pacitor with a voltage of 100 V?					
	(a) 0.025 I	(h)						
	(a) $0.023$ J (c) 5 J	(0)	0.05 J 100 I					
		(a)	100 J					

$\bullet \bullet \bullet$	•					
1 16	Which one of the following is an ide	eal voltage source?				
1.10	(a) voltage independent of current (b) current independent of voltage					
	(c) both (a) and (b)	(d) none of the above				
1.17	The following voltage drops are me	asured across each of three resistors				
	in series: 5.2 V. 8.5 V and 12.3 V. W	hat is the value of the source voltage				
	to which these resistors are connected	ed?				
	(a) 8.2 V	(b) 12.3 V				
	(c) $5.2 V$	(d) $26 \text{ V}$				
1.18	A certain series circuit has a 100 $\Omega$	<b>α</b> . a 270 $\Omega$ , and a 330 $\Omega$ resistor in				
	series. If the 270 $\Omega$ resistor is removed, the current					
	(a) increases	(b) becomes zero				
	(c) decrease	(d) remain constant				
1.19	A series circuit consists of a 4.7 k $\Omega$	$2, 5.6 \text{ k}\Omega, 9 \text{ k}\Omega$ and $10 \text{ k}\Omega$ resistor.				
	Which resistor has the most voltage	across it?				
	(a) 4.7 k $\Omega$	(b) 5.6 kΩ				
	(c) $9 k\Omega$	(d) 10 k $\Omega$				
1.20	The total power in a series circuit i	s 10 W. There are five equal value				
	resistors in the circuit. How much pe	ower does each resistor dissipate?				
	(a) 10 W	(b) 5 W				
	(c) 2 W	(d) 1 W				
1.21	When a 1.2 k $\Omega$ resistor, 100 $\Omega$ resistor	tor, 1 k $\Omega$ resistor and 50 $\Omega$ resistor				
	are in parallel, the total resistance is less than					
	(a) $100 \Omega$	(b) 50 Ω				
	(c) $1 k\Omega$	(d) $1.2 \text{ k}\Omega$				
1.22	If a 10 V battery is connected across the parallel resistors of 3 $\Omega$ , 5 $\Omega$ ,					
	10 $\Omega$ and 20 $\Omega$ , how much voltage is there across 5 $\Omega$ resistor?					
	(a) 10 V	(b) 3 V				
	(c) 5 V	(d) 20 V				
1.23	If one of the resistors in a parallel circ	cuit is removed, what happens to the				
	total resistance?					
	(a) decreases	(b) increases				
	(c) remain constant	(d) exactly doubles				
1.24	The power dissipation in each of three	ee parallel branches is 1 W. What is				
	the total power dissipation of the cir	cuit?				
	(a) 1 W	(b) 4 W				
	(c) 3 W	(d) zero				
1.25	In a four branch parallel circuit, 10 r	nA of current flows in each branch.				
	If one of the branch opens, the curre	ent in each of the other branches				
	(a) increases	(b) decreases				
	(c) remains unaffected	(d) doubles				
1.26	Four aqual value registers are corre-	ated in normalical Fixed walts are are				
1.20	rout equal value resistors are conne	and 2.5 mA are managined from the				
	source What is the value of each resistor?					
	(-) 4 O					

L

Γ

- 1.27 Six light bulbs are connected in parallel across 110 V. Each bulb is related at 75 W. How much current flows through each bulb?(a) 0.682 A(b) 0.7 A
  - (a) 0.082 A (b) 0.7 A(c) 75 A (d) 110 A
- 1.28 A 330  $\Omega$  resistor is in series with the parallel combination of four 1 k $\Omega$ 
  - resistors. A 100 V source is connected to the circuit. Which resistor has the most current through it.
    - (a)  $330 \Omega$  resistor
    - (b) parallel combination of three 1 k $\Omega$  resistors
    - (c) parallel combination of two 1 k $\Omega$  resistors
    - (d)  $1 k\Omega$  resistor

#### 1.29 The current $i_4$ in the circuit shown in Fig. 1.72 is equal to

4 V

- (a) 12 A
- (c) 4 A

(b) - 12 A(d) None of the above





1.30 The voltage V in Fig. 1.73 is equal to

(a) 3 V (c) 5 V







1.31 The voltage V in Fig. 1.74 is always equal to





Fig. 1.75

L

Γ



# 2.1 INTRODUCTION

A division of mathematics called topology or graph theory deals with graphs of networks and provides information that helps in the formulation of network equations. In circuit analysis, all the elements in a network must satisfy Kirchhoff's laws, besides their own characteristics. Based on these laws, we can form a number of equations. These equations can be easily written by converting the network into a graph. Certain aspects of network behaviour are brought into better perspective if a graph of the network is drawn. If each element or a branch of a network is represented on a diagram by a line irrespective of the characteristics of the elements, we get a graph. Hence, network topology is network geometry. A network is an interconnection of elements in various branches at different nodes as shown in Fig. 2.1. The corresponding graph is shown in Fig. 2.2 (a).



The graphs shown in Figs 2.2 (b) and (c) are also graphs of the network in Fig. 2.1.

It is interesting to note that the graphs shown in Fig. 2.2 (a), (b) and (c) may appear to be different but they are topologically equivalent. A branch is represented by a line segment connecting a pair of nodes in the graph of a network. A node is a terminal of a branch, which is represented by a point. Nodes are the end points of branches. All these graphs have identical relationships between branches and nodes.



The three graphs in Fig. 2.2 have six branches and four nodes. These graphs are also called undirected. If every branch of a graph has a *direction* as shown in Fig. 2.3, then the graph is called a *directed graph*.

A node and a branch are incident if the node is a terminal of the branch. Nodes can be incident to one or more elements. The number of branches incident at a node of a graph indicates the degree of the node. For example, in Fig. 2.3 the degree of node 1 is three. Similarly, the degree of node 2 is three. If each element of the connected graph is assigned a direction as shown in Fig. 2.3 it is then said to be oriented. A graph is connected if and only if there is a path between every pair of nodes. A path is said to exist between any two nodes, for example 1 and 4 of the graph in Fig. 2.3, if it is possible to reach node 4 from node 1 by traversing



along any of the branches of the graph. A graph can be drawn if there exists a path between any pair of nodes. A loop exists, if there is more than one path between two nodes.

#### **Planar and Non-Planar Graphs**

A graph is said to be planar if it can be drawn on a plane surface such that no two branches cross each other as shown in Fig. 2.2. On the other hand in a

nonplanar graph there will be branches which are not in the same plane as others, i.e. a non-planar graph cannot be drawn on a plane surface without a crossover. Figure 2.4 illustrates a non-planar graph.

# 2.2 TREE AND CO-TREE

A tree is a connected subgraph of a network which consists of all the nodes of the original graph but no closed paths. The graph of a network may have a number of trees. The number of nodes in a graph is equal to the number nodes in the tree. The number of branches in a tree is less than the number of branches in a graph. A graph is a tree if there is a unique path between any pair of nodes. Consider a graph with four branches and three nodes as shown in Fig. 2.5.

Five open-ended graphs based on Fig. 2.5 are represented by Figs 2.6 (a) to (e). Since each of these open-ended graphs satisfies all the requirements of a tree, each graph in Fig. 2.6 is a tree corresponding to Fig. 2.5.

3

In Fig. 2.6, there is no closed path or loop; the number of nodes n = 3 is the same for the graph and its tree, where as the number of branches in the tree is only two. In general, if a tree contains *n* nodes, then it has (n - 1) branches.

2



Fig. 2.6



Fig. 2.4



2

d

In forming a tree for a given graph, certain branches are removed or opened. The branches thus opened are called links or *link branches*. The links for Fig. 2.6 (a) for example are a and d and for 2.6 (b) are b and c. The set of all links of a given tree is called the co-tree of the graph. Obviously, the branches a, d are a co-tree for Fig. 2.6 (a) and b, c are the co-tree. Similarly, for the tree in Fig. 2.6 (b), the branches b, c are the co-tree. Thus the link branches and the tree branches combine to form the graph of the entire network.

**Example 2.1** For the given graph shown in Fig. 2.7 draw the number of possible trees.



Fig. 2.7

**Solution** The number of possible trees for Fig. 2.7 are represented by Figs 2.8 (a) - (g).



# 2.3 TWIGS AND LINKS

The branches of a tree are called its 'twigs'. For a given graph, the complementary set of branches of the tree is called the co-tree of the graph. The branches of a co-tree are called links, i.e. those elements of the connected graph that are not included in the tree links and form a subgraph. For example, the set of branches (b, d, f) represented by dotted lines in Fig. 2.11 form a co-tree of the graph in Fig. 2.9 with respect to the tree in Fig. 2.10.



The branches a, c and e are the twigs while the branches b, d and f are the links of this tree. It can be seen that for a network with b branches and n nodes, the number of twigs for a selected tree is (n - 1) and the number of links I with respect to this tree is (b - n + 1). The number of twigs (n - 1) is known as the tree value of the graph. It is also called the *rank* of the tree. If a link is added to the tree, the resulting graph contains one closed path, called a loop. The addition of each subsequent link forms one or more additional loops. Loops which contain only one link are independent and are called basic loops.

# Fig. 2.10



# 2.4 INCIDENCE MATRIX (A)

The incidence of elements to nodes in a connected graph is shown by the element node incidence matrix (A). Arrows indicated in the branches of a graph result in an oriented or a directed graph. These arrows are the indication for the current flow or voltage rise in the network. It can be easily identified from an oriented graph regarding the incidence of branches to nodes. It is possible to have an analytical description of an oriented-graph in a matrix form. The dimensions of the matrix A is  $n \times b$  where n is the number of nodes and b is number of branches. For a graph having n nodes and b branches, the complete incidence matrix A is a rectangular matrix of order  $n \times b$ .

In matrix A with n rows and b columns an entry  $a_{ij}$  in the  $i^{th}$  row and  $j^{th}$  column has the following values.

 $a_{ij} = 1$ , if the  $j^{th}$  branch is incident to and oriented away from the  $i^{th}$  node.  $a_{ij} = -1$ , if the  $j^{th}$  branch is incident to and oriented towards the  $i^{th}$  node.  $a_{ij} = 0$ , if the  $j^{th}$  branch is not incident to the  $i^{th}$  node. (2.1)

Figure 2.12 shows a directed graph.



	Notes	Branches $\rightarrow$						
	$\downarrow$	а	b	с	d	е	f	
	1	[ 1	0	1	0	0	1	
1 -	2	-1	-1	0	-1	0	0	
A –	3	0	1	0	0	1	-1	
	4	0	0	-1	1	-1	0	

The entries in the first row indicates that three branches a, c and f are incident to node 1 and they are oriented away from node 1 and therefore the entries

 $a_{11}$ ;  $a_{13}$  and  $a_{16}$  are + 1. Other entries in the 1<sup>st</sup> row are zero as they are not connected to node 1. Likewise, we can complete the incidence matrix for the remaining nodes 2, 3 and 4.

# 2.5 PROPERTIES OF INCIDENCE MATRIX A

Following properties are some of the simple conclusions from incidence matrix A.

- 1. Each column representing a branch contains two non-zero entries + 1 and 1; the rest being zero. The unit entries in a column identify the nodes of the branch between which it is connected.
- 2. The unit entries in a row identify the branches incident at a node. Their number is called the degree of the node.
- 3. A degree of 1 for a row means that there is one branch incident at the node. This is commonly possible in a tree.
- 4. If the degree of a node is two, then it indicates that two branches are incident at the node and these are in series.
- 5. Columns of *A* with unit entries in two identical rows correspond to two branches with same end nodes and hence they are in parallel.
- 6. Given the incidence matrix *A* the corresponding graph can be easily constructed since *A* is a complete mathematical replica of the graph.
- 7. If one row of A is deleted the resulting  $(n 1) \times b$  matrix is called the reduced incidence matrix  $A_1$ . Given  $A_1$ , A is easily obtained by using the first property.

It is possible to find the exact number of trees that can be generated from a given graph if the reduced incidence matrix  $A_1$  is known and the number of possible trees is given by Det  $(A_1A_1^T)$  where  $A_1^T$  is the transpose of the matrix  $A_1$ .

**Example 2.2** Draw the graph corresponding to the given incidence matrix.

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & +1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ -1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Solution** There are five rows and eight columns which indicate that there are five nodes and eight branches. Let us number the columns from *a* to *h* and rows as 1 to 5.

$$A = \begin{bmatrix} A & b & c & d & e & f & g & h \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 3 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 4 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Mark the nodes corresponding to the rows 1, 2, 3, 4 and 5 as dots as shown in Fig. 2.13 (a). Examine each column of A and connect the nodes (unit entries) by a branch; label it after marking an arrow.

For example, examine the first column of A. There are two unit entries one in the first row and  $2^{nd}$  in the last row, hence connect branch a between node 1 and 5. The entry of  $A_{11}$  is – ve and that of  $A_{51}$  is + ve. Hence the orientation of the branch is away from node 5 and towards node 1 as per the convention. Proceeding in this manner we can complete the entire graph as shown in Fig. 2.13 (b).



From the incidence matrix A, it can be verified that branches c and d are in parallel (property 5) and branches e and f are in series (property 4).

**Example 2.3** Obtain the incidence matrix A from the following reduced incidence matrix  $A_1$  and draw its graph.

$$[A_{1}] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

**Solution** There are five rows and seven columns in the given reduced incidence matrix  $[A_1]$ . Therefore, the number of rows in the complete incidence matrix A will

be 5 + 1 = 6. There will be six nodes and seven branches in the graph. The dimensions of matrix A is  $6 \times 7$ . The last row in A, i.e.  $6^{th}$  row for the matrix A can be obtained by using the first property of the incidence matrix. It is seen that the first column of  $[A_1]$  has a single non-zero element – 1. Hence, the first element in the  $6^{th}$  row will be + 1 (– 1 + 1 = 0). Second column of  $A_1$  has two non-zero elements + 1 and – 1, hence the  $2^{nd}$  element in the  $6^{th}$  row will be 0. Proceeding in this manner we can obtain the  $6^{th}$  row. The complete incidence matrix can therefore be written as

$$[A] = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ f \end{bmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ f & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We have seen that any one of the rows of a complete incidence matrix can be obtained from the remaining rows. Thus it is possible to delete any one row from A without loosing any information in  $A_1$ . Now the oriented graph can be constructed from the matrix A. The nodes may be placed arbitrarily. The number of nodes to be marked will be six. Taking node 6 as reference node the graph is drawn as shown in Fig. 2.14.





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# 2.6 INCIDENCE MATRIX AND KCL

Kirchhoff's current law (KCL) of a graph can be expressed in terms of the reduced incidence matrix as  $A_{\perp}I = 0$ .

 $A_1$ , *I* is the matrix representation of KCL, where *I* represents branch current vectors  $I_1$ , as  $I_2$ ,  $\cdots I_6$ .

Consider the graph shown in Fig. 2.15. It has four nodes a, b, c and d.

Let node *d* be taken as the reference node. The positive reference direction of the branch currents corresponds to the orientation of the graph branches. Let the branch currents be  $i_1, i_2, \dots i_6$ . Applying KCL at nodes *a*, *b* and *c*.



Fig. 2.15

$$-i_1 + i_4 = 0$$
  
$$-i_2 - i_4 + i_5 = 0$$
  
$$-i_3 = i_5 - i_6 = 0$$

These equations can be written in the matrix form as follows

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

 $A_1 I_b = 0$ Here,  $I_b$  represents column matrix or a vector of branch currents.

$$I_b = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_b \end{bmatrix}$$

 $A_1$  is the reduced incidence matrix of a graph with *n* nodes and *b* branches. And it is a  $(n-1) \times b$  matrix obtained from the complete incidence matrix of *A* deleting one of its rows. The node corresponding to the deleted row is called the reference node or datum node. It is to be noted that  $A_1 I_b = 0$  gives a set of n - 1 linearly independent equations in branch currents  $I_1, I_2, \cdots I_6$ . Here n = 4. Hence, there are three linearly independent equations.

# 2.7 LINK CURRENTS: TIE-SET MATRIX

For a given tree of a graph, addition of each link between any two nodes forms a loop called the fundamental loop. In a loop there exists a closed path and a circulating current, which is called the link current. The current in any branch of a graph can be found by using link currents.

The fundamental loop formed by one link has a unique path in the tree joining the two nodes of the link. This loop is also called *f*-loop or a tie-set.



Fig. 2.16

(2.2)

Consider a connected graph shown in Fig. 2.16 (a). It has four nodes and six branches. One of its trees is arbitrarily chosen and is shown in Fig. 2.16 (b). The twigs of this tree are branches 4, 5 and 6. The links corresponding to this tree are branches 1, 2 and 3. Every link defines a fundamental loop of the network.

No. of nodes n = 4

No. of branches b = 6

No. of tree branches or twigs = n - 1 = 3

No. of link branches I = b - (n - 1) = 3

Let  $i_1, i_2, \dots, i_6$  be the branch currents with directions as shown in Fig. 2.16 (a). Let us add a link in its proper place to the tree as shown in 2.16 (c). It is seen that a loop  $I_1$  is formed by the branches 1, 5 and 6. There is a formation of link current, let this current be  $I_1$ . This current passes through the branches 1, 5 and 6. By convention a fundamental loop is given the same orientation as its defining link, i.e. the link current  $I_1$  coincides with the branches that forms a closed loop in which the link current flows. By adding the other link branches 2 and 3, we can form two more fundamental loops or *f*-loops with link currents  $I_2$  and  $I_3$  respectively as shown in Figs 2.16 (d) and (e).



Fig. 2.17

There are three fundamental loops  $I_1$ ,  $I_2$  and  $I_3$  corresponding to the link branches 1, 2 and 3 respectively. If  $V_1$ ,  $V_2$ ,  $\cdots$   $V_6$  are the branch voltages the KVL equations for the three f-loops can be written as

$$\begin{array}{c}
V_1 + V_5 - V_6 = 0 \\
V_2 + V_4 - V_5 = 0 \\
V_3 - V_4 = 0
\end{array}$$
(2.3)

In order to apply KVL to each fundamental loop, we take the reference direction of the loop which coincides with the reference direction of the link defining the loop.

The above equation can be written in matrix form as

where B is an  $I \times b$  matrix called the tie-set matrix or fundamental loop matrix and  $V_b$  is a column vector of branch voltages.

The tie set matrix *B* is written in a compact form as  $B[b_{ij}]$  (2.5)

The element  $b_{ii}$  of B is defined as

 $b_{ij} = 1$  when branch  $b_j$  is in the f-loop  $I_i$  (loop current) and their reference directions coincide.

 $b_{ij} = -1$  when branch  $b_j$  is in the f-loop  $I_i$  (loop current) and their reference directions are opposite.

 $b_{ii} = 0$  when branch  $b_i$  is not in the f-loop  $I_i$ .

#### 2.7.2 Tie-set Matrix and Branch Currents

It is possible to express branch currents as a linear combination of link current using matrix *B*.

If  $I_B$  and  $I_I$  represents the branch current matrix and loop current matrix respectively and B is the tie-set matrix, then

$$[I_b] = [B^T] [I_L] \tag{2.6}$$

where  $[B^{T}]$  is the transpose of the matrix [B]. Equation (6) is known as link current transformation equation.

Consider the tie-set matrix of Fig. 2.17

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The branch current vector  $[I_h]$  is a column vector.

$$[I_b] = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}$$

The loop current vector  $[I_L]$  is a column vector

$$\begin{bmatrix} I_L \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

Therefore the link current transformation equation is given by  $[I_b] = [B^T] [I_1]$ 

$$\begin{bmatrix} i_1\\i_2\\i_3\\i_4\\i_5\\i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 1 & -1\\1 & -1 & 0\\-1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1\\I_2\\I_3 \end{bmatrix}$$

The branch currents are

$$\begin{split} i_1 &= I_1 \\ i_2 &= I_2 \\ i_3 &= I_3 \\ i_4 &= I_2 - I_3 \\ i_5 &= I_1 - I_2 \\ i_6 &= -I_1 \end{split}$$

**Example 2.4** For the electrical network shown in Fig. 2.18 (a) draw its topological graph and write its incidence matrix, tie-set matrix, link current transformation equation and branch currents.


#### Solution

Voltage source is short circuited, current source is open circuited, the points which are electrically at same potential are combined to form a single node. The graph is shown in Fig. 2.18 (b).





Fig. 2.18 (c)

Combining the simple nodes and arbitrarily selecting the branch current directions the oriented graph is shown in Fig. 2.18 (c). The simplified consists of three nodes. Let them be x, y and z and five branches 1, 2, 3, 4 and 5. The complete incidence matrix is given by

Nodes branches →  $\downarrow$  1 2 3 4 5 x 1 0 1 0 -1 A = y -1 1 0 1 0 z 0 -1 -1 -1 1

Let us choose node z as the reference or datum node for writing the reduced incidence matrix  $A_1$  or we can obtain  $A_1$  by deleting the last row elements in A.

nodes branches  

$$\downarrow$$
 1 2 3 4 5  
 $A_1 = \begin{array}{c} x \\ y \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 - 1 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix}$ 

For writing the tie-set matrix, consider the tree in the graph in Fig. 2.18 (c).

No. of nodes n = 3No. of branches = 5 No. of tree branches or twigs = n - 1 = 2No. of link branches I = b - (n - 1)= 5 - (3 - 1) = 3The tree shown in Fig. 2.18 (d)

consists of two branches 4 and 5

shown with solid lines and the link





branches of the tree are 1, 2 and 3 shown with dashed lines. The tie-set matrix or fundamental loop matrix is given by

To obtain the link current transformation equation and thereby branch currents the transpose of B should be calculated.

$$B^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
$$] = [BT] [IL]$$

The equation [Ib] = [BT] [IL]

$$\begin{bmatrix} i_1\\i_2\\i_3\\i_4\\i_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\1 & -1 & 0\\1 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1\\I_2\\I_3 \end{bmatrix}$$

The branch currents are given by

$$\begin{split} & i_1 = I_1 \\ & i_2 = I_2 \\ & i_3 = I_3 \\ & i_4 = I_1 - I_2 \\ & i_5 = I_1 + I_3 \end{split}$$

\*\*

### 2.8 CUT-SET AND TREE BRANCH VOLTAGES

A cut-set is a minimal set of branches of a connected graph such that the removal of these branches causes the graph to be cut into exactly two parts. The

important property of a cut-set is that by restoring anyone of the branches of the cut-set the graph should become connected. A cut-set consists of one and only one branch of the network tree, together with any links which must be cut to divide the network into two parts.



Fig. 2.19

Consider the graph shown in Fig. 2.19 (a). If the branches 3, 5 and 8 are removed from the graph, we see that the connected graph of Fig. 2.19 (a) is separated into two distinct parts, each of which is connected as shown in Fig. 2.19 (b). One of the parts is just an isolated node. Now suppose the removed branch 3 is replaced, all others still removed. Figure 2.19 (c) shows the resultant graph. The graph is now connected. Likewise replacing the removed branches 5 and 8 of the set  $\{3, 5, 8\}$  one at a time, all other ones remaining removed, we obtain the resulting graphs as shown in Figs 2.19 (d) and (e). The set formed by the branches 3, 5 and 8 is called the cut-set of the connected graph of Fig. 2.19 (a).

#### 2.8.1 Cut-Set Orientation

A cut-set is oriented by arbitrarily selecting the *a* direction. A cut-set divides a graph into two parts. In the graph shown in Fig. 2.20, the cut-set is  $\{2, 3\}$ . It is represented by a dashed line passing through branches 2 and 3. This cut-set separates the graph into two parts shown as part-1 and part-2. We may take the orientation either from part-1 to part-2 or from part-2 to part-1.



The orientation of some branches of the cut-set may coincide with the orientation of the cut-set while some branches of the cut-set may not coincide. Suppose we choose the orientation of the cut-set  $\{2, 3\}$  from part-1 to part-2 as indicated in Fig. 2.20, then the orientation of branch 2 coincides with the cut-set, whereas the orientation of the branch 3 is opposite.

#### 2.8.2 Cut-Set Matrix and KCL for Cut-Sets

KCL is also applicable to a cut-set of a network. For any lumped electrical network, the algebraic sum of all the cut-set branch currents is equal to zero.

While writing the KCL equation for a cutset, we assign positive sign for the current in a branch if its direction coincides with the orientation of the cut-set and a negative sign to the current in a branch whose direction is opposite to the orientation of the cut-set. Consider the graph shown in Fig. 2.21. It has five branches and four



nodes. The branches have been numbered 1 through 5 and their orientations are also marked. The following six cut-sets are possible as shown in Fig. 2.22 (a)-(f).

Cut-set  $C_1$ : {1, 4}; cut-set  $C_2$ : {4, 2, 3} Cut-set  $C_3$ : {3, 5}; cut-set  $C_4$ : {1, 2, 5} Cut-set  $C_5$ : {4, 2, 5}; cut-set  $C_6$ : {1, 2, 3}





Applying KCL for each of the cut-set we obtain the following equations. Let  $i_1, i_2 \cdots i_6$  be the branch currents.

$$C_{1}: i_{1} - i_{4} = 0$$

$$C_{2}: -i_{2} + i_{3} + i_{4} = 0$$

$$C_{3}: -i_{3} + i_{5} = 0$$

$$C_{4}: i_{1} - i_{2} + i_{5} = 0$$

$$C_{5}: -i_{2} + i_{4} + i_{5} = 0$$

$$C_{6}: i_{1} - i_{2} + i_{3} = 0$$

$$(2.7)$$

These equation can be put into matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$QI_b = 0 \tag{2.8}$$

where the matrix Q is called augmented cut-set matrix of the graph or all cut-set matrix of the graph. The matrix  $I_{h}$  is the branch-current vector.

The all cut-set matrix can be written as  $Q = [q_{ij}]$ . where  $q_{ij}$  is the element in the *i*<sup>th</sup> row and *j*<sup>th</sup> column. The order of Q is number of cut-sets  $\times$  number of branch as in the graph.



Fig. 2.23 (a)

Fig. 2.23 (b)

**Example 2.5** For the network-graph shown in Fig. 2.23 (a) with given orientation obtain the all cut-set (augmented cut-set) matrix.

**Solution** The graph has four nodes and eight branches. There are in all 12 possible cut-sets as shown with dashed lines in Figs 2.23 (b) and (c). The orientation of the cut-sets has been marked arbitrarily. The cut-sets are

 $\mathbf{C}_1$ : {1, 46};  $\mathbf{C}_2$  {1, 2, 3};  $\mathbf{C}_3$ : {2, 5, 8}  $C_4$ : {6, 7, 8};  $C_5$  {1, 3, 5, 8};  $C_6$ : {1, 4, 7, 8}  $C_7$ : {2, 5, 6, 7};  $C_8$ : {2, 3, 4, 6}  $C_9$ : {1, 4, 7, 5, 2}  $\mathbf{C}_{10}$ : {2, 3, 4, 7, 8};  $\mathbf{C}_{11}$ : {6, 4, 3, 5, 8};  $\mathbf{C}_{12}$ : {1, 3, 5, 7, 6}



Fig. 2.23 (c)

Eight cut-sets  $C_1$  to  $C_8$  are shown if Fig. 2.23(b) and four cut-sets  $C_9$  to  $C_{11}$  are shown in Fig. 2.23(c) for clarity.

As explained in section 2.8.2 with the help of equations 2.9, the all cut-set matrix Q is given by

	Cut-sets	Branches $\rightarrow$								
	$\downarrow$	1	2	3	4	5	6	7	8	
Q =	$\mathbf{C}_1$	- 1	0	0	1	0	- 1	0	0 -	
	$\mathbf{C}_2$	1	- 1	- 1	0	0	0	0	0	
	$\mathbf{C}_{3}$	0	1	0	0	1	0	0	- 1	
	$\mathbf{C}_{4}$	0	0	0	0	0	1	1	1	
	<b>C</b> <sub>5</sub>	1	0	- 1	0	1	0	0	- 1	
	$\mathbf{C}_{6}$	- 1	0	0	1	0	0	1	1	(2.10)
	<b>C</b> <sub>7</sub>	0	1	0	0	1	1	1	0	l
	$\mathbf{C}_{8}$	0	- 1	- 1	1	0	- 1	0	0	ļ
	<b>C</b> <sub>9</sub>	1	- 1	0	- 1	- 1	0	- 1	0	
	$\mathbf{C}_{10}$	0	1	1	- 1	0	0	- 1	- 1	
	<b>C</b> <sub>11</sub>	0	0	1	- 1	- 1	1	0	1	l
	$\mathbf{C}_{12}$	- 1	0	1	0	- 1	- 1	- 1	0 _	

Matrix **Q** is a  $12 \times 8$  matrix since there are 12 cut-sets and eight branches in the graph.

#### 2.8.3 Fundamental Cut-Sets

Observe the set of equation 2.7 in Section 2.8.2 with respect to the graph in Fig. 2.22. Only first three equations are linearly independent, remaining equations can be obtained as a linear combination of the first three. The concept of fundamental cut-set (*f*-cut-set) can be used to obtain a set of linearly independent equations in branch current variables. The *f*-cut-sets are defined for a given tree of the graph. From a connected graph, first a tree is selected, and then a twig is selected. Removing this twig from the tree separates the tree into two parts. All

the links which go from one part of the disconnected tree to the other, together with the twig of the selected tree will constitute a cut-set. This cut-set is called a fundamental cut-set or *f*-cut-set or the graph. Thus a fundamental cut-set of a graph with respect to a tree is a cut-set that is formed by one twig and a unique set of links. For each branch of the tree, i.e. for each twig, there will be a *f*-cut-set. So, for a connected graph having *n* nodes, there will be (n - 1) twigs in a tree, the number of *f*-cut-set is also equal to (n - 1).

Fundamental cut-set matrix  $Q_f$  is one in which each row represents a cut-set with respect to a given tree of the graph. The rows of  $Q_1$  correspond to the fundamental cut-sets and the columns correspond to the branches of the graph. The procedure for obtaining a fundamental cut-set matrix is illustrated in Example 2.6.

**Example 2.6** Obtain the fundamental cut-set matrix Qf for the network graph shown in Fig. 2.23 (a).

**Solution** A selected tree of the graph is shown in Fig. 2.24 (a).



The twigs of the tree are  $\{3, 4, 5, 7\}$ . The remaining branches 1, 2, 6 and 8 are the links, corresponding to the selected tree. Let us consider twig 3. The minimum number of links that must be added to twig 3 to form a cut-set  $C_1$  is  $\{1, 2\}$ . This set is unique for  $C_1$ . Thus corresponding to twig 3. The f-cut-set  $C_1$  is  $\{1, 2, 3\}$ . This is shown in Fig. 2.24 (b). As a convention the orientation of a cut-set is chosen to coincide with that of its defining twig. Similarly, corresponding to twig 4, the f-cut-set  $C_2$  is  $\{1, 4, 6\}$  corresponding to twig 5, the f-cut-set  $C_3$  is  $\{2, 5, 8\}$  and corresponding to twig 7, the f-cut-set is  $\{6, 7, 8\}$ . Thus the f-cut-set matrix is given by

f-cut-sets branches  

$$Q_{f} = \begin{array}{c} C_{1} \\ C_{2} \\ C_{3} \\ C_{4} \end{array} \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$
(2.11)

#### 2.8.4 Tree Branch Voltages and *f*-Cut-Set Matrix

From the cut-set matrix the branch voltages can be expressed in terms of tree branch voltages. Since all tree branches are connected to all the nodes in the graph, it is possible to trace a path from one node to any other node by traversing through the tree-branches.

Let us consider Example 2.6, there are eight branches. Let the branch voltages be  $V_1, V_2, \dots V_8$ . There are, four twigs, let the twig voltages be  $V_{13}, V_{14}, V_{15}$  and  $V_{17}$  for twigs 3, 4, 5 and 7 respectively.

We can express each branch voltage in terms of twig voltages as follows.

$$V_{1} = -V_{3} - V_{4} = -V_{t3} - V_{t4}$$

$$V_{2} = +V_{3} + V_{5} = +V_{t3} + V_{t5}$$

$$V_{3} = V_{t3}$$

$$V_{4} = V_{t4}$$

$$V_{5} = V_{t5}$$

$$V_{6} = V_{7} - V_{4} = V_{t7} - V_{t4}$$

$$V_{7} = V_{t7}$$

$$V_{8} = V_{7} - V_{5} = V_{t7} - V_{t5}$$

The above equations can be written in matrix form as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} V_{t_3} \\ V_{t_4} \\ V_{t_5} \\ V_{t_7} \end{bmatrix}$$
(2.12)

The first matrix on the right hand side of Eq. 2.12 is the transpose of the *f*-cut-set matrix  $Q_f$  given in Eq. 2.11 in Ex. 2.6. Hence, Eq. 2.12 can be written as  $V_b = Q_f^T V_t$ . (2.13)

 $V_b = Q_f^T V_t$ . (2.13) Where  $V_b$  is the column matrix of branch-voltages  $V_f$  is the column matrix of twig voltages corresponding to the selected tree and  $Q_f^{T'}$  in the transpose of *f*-cut-set matrix.

Equation 2.13 shows that each branch voltage can be expressed as a linear combination of the tree-branch voltages. For this purpose fundamental cut-set (f-cut-set) matrix can be used without writing loop equations.

### 2.9 MESH ANALYSIS

Mesh and nodal analysis are two basic important techniques used in finding solutions for a network. The suitability of either mesh or nodal analysis to a particular problem depends mainly on the number of voltage sources or current sources. If a network has a large number of voltage sources, it is useful to use mesh analysis; as this analysis requires that all the sources in a circuit be voltage sources. Therefore, if there are any current sources in a circuit they are to be converted into equivalent voltage sources, if, on the other hand, the network has more current sources, nodal analysis is more useful.

Mesh analysis is applicable only for planar networks. For non-planar circuits mesh analysis is not applicable. A circuit is said to be planar, if it can be drawn on a plane surface without crossovers. A non-planar circuit cannot be drawn on a plane surface without a crossover.

Figure 2.25 (a) is a planar circuit. Figure 2.25 (b) is a non-planar circuit and Fig. 2.25 (c) is a planar circuit which looks like a non-planar circuit. It has already been discussed that a loop is a closed path. A mesh is defined as a loop which does not contain any other loops within it. To apply mesh analysis, our first step is to check whether the circuit is planar or not and the second is to select mesh currents. Finally, writing Kirchhoff's voltage law equations in terms of unknowns and solving them leads to the final solution.





Observation of the Fig. 2.26 indicates that there are two loops *abefa*, and *bcdeb* in the network. Let us assume loop currents  $I_1$  and  $I_2$  with directions as indicated in the figure. Considering the loop *abefa* alone, we observe that current  $I_1$  is passing through  $R_1$ , and  $(I_1-I_2)$ is passing through  $R_2$ . By applying Kirchhoff's voltage law, we can write

$$V_s = I_1 R_1 + R_2 (I_1 - I_2)$$



Similarly, if we consider the second mesh *bcdeb*, the current  $I_2$  is passing through  $R_3$  and  $R_4$ , and  $(I_2 - I_1)$  is passing through  $R_2$ . By applying Kirchhoff's voltage law around the second mesh, we have

$$R_2 (I_2 - I_1) + R_3 I_2 + R_4 I_2 = 0$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(R_1 + R_2) - I_2 R_2 = V_s$$
  
+ I\_1 R\_2 + (R\_2 + R\_3 + R\_4)I\_2 = 0

By solving the above equations, we can find the currents  $I_1$  and  $I_2$ . If we observe Fig. 2.26, the circuit consists of five branches and four nodes, including the reference node. The number of mesh currents is equal to the number of mesh equations.

And the number of equations = branches – (nodes – 1). In Fig. 2.26, the required number of mesh currents would be 5 - (4 - 1) = 2.

In general, if we have B number of branches and N number of nodes including the reference node then the number of linearly independent mesh equations M = B - (N - 1).

**Example 2.7** Write the mesh current equations in the circuit shown in Fig. 2.27, and determine the currents.



**Solution** Assume two mesh currents in the direction as indicated in Fig. 2.28. The mesh current equations are

$$5I_1 + 2(I_1 - I_2) = 10$$
  
$$10I_2 + 2(I_2 - I_1) + 50 = 0$$

We can rearrange the above equations as

$$7I_1 - 2I_2 = 10$$
$$-2I_1 + 12I_2 = -50$$

By solving the above equations, we have

$$I_1 = 0.25$$
 A, and  $I_2 = -4.125$  A

Here the current in the second mesh,  $I_2$ , is negative; that is the actual current  $I_2$  flows opposite to the assumed direction of current in the circuit of Fig. 2.28.

**Example 2.8** Determine the mesh current  $I_1$  in the circuit shown in Fig. 2.29.



Fig. 2.29

**Solution** From the circuit, we can form the following three mesh equations

$$10I_1 + 5(I_1 + I_2) + 3(I_1 - I_3) = 50$$
  

$$2I_2 + 5(I_2 + I_1) + 1(I_2 + I_3) = 10$$
  

$$3(I_3 - I_1) + 1(I_3 + I_2) = -5$$

Rearranging the above equations we get

 $I_1 = 3.3 \text{ A}$ 

$$18I_1 + 5I_2 - 3I_3 = 50$$
  

$$5I_1 + 8I_2 + I_3 = 10$$
  

$$-3I_1 + I_2 + 4I_3 = -5$$

According to Cramer's rule

$$I_{1} = \frac{\begin{vmatrix} 50 & 5 & -3 \\ 10 & 8 & 1 \\ -5 & 1 & 4 \\ \hline 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{1175}{356}$$

or

Similarly,

$$I_2 = \frac{\begin{vmatrix} 18 & 50 & -3 \\ 5 & 10 & 1 \\ -3 & -5 & 4 \\ \hline 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{-355}{356}$$

or

$$I_2 = -0.997 A$$

$$I_{3} = \frac{\begin{vmatrix} 18 & 5 & 50 \\ 5 & 8 & 10 \\ -3 & 1 & -5 \\ \hline 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{525}{356}$$

or ∴

$$I_3 = 1.47 A$$
  
 $I_1 = 3.3 A$ ,  $I_2 = -0.997 A$ ,  $I_3 = 1.47 A$ 

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### 2.10 MESH EQUATIONS BY INSPECTION METHOD

The mesh equations for a general planar network can be written by inspection without going through the detailed steps. Consider a three mesh networks as shown in Fig. 2.30.



Fig. 2.30

The loop equations are

$$I_1 R_1 + R_2 (I_1 - I_2) = V_1 \tag{2.14}$$

$$R_2(I_2 - I_1) + I_2R_3 = -V_2 \tag{2.15}$$

$$R_4 I_3 + R_5 I_3 = V_2 \tag{2.16}$$

Reordering the above equations, we have

$$(R_1 + R_2)I_1 - R_2I_2 = V_1 \tag{2.17}$$

$$-R_2I_1 + (R_2 + R_3)I_2 = -V_2 \tag{2.18}$$

$$(R_4 + R_5)I_3 = V_2 \tag{2.19}$$

The general mesh equations for three mesh resistive network can be written as

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_a \tag{2.20}$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_b \tag{2.21}$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_c \tag{2.22}$$

By comparing the Eqs 2.17, 2.18 and 2.19 with Eqs 2.20, 2.21, and 2.22 respectively, the following observations can be taken into account.

1. The self resistance in each mesh.

2. The mutual resistances between all pairs of meshes and

3. The algebraic sum of the voltages in each mesh.

The self resistance of loop 1,  $R_{11} = R_1 + R_2$ , is the sum of the resistances through which  $I_1$  passes.

The mutual resistance of loop 1,  $R_{12} = -R_2$ , is the sum of the resistances common to loop currents  $I_1$  and  $I_2$ . If the directions of the currents passing through the common resistance are the same, the mutual resistance will have a positive sign; and if the directions of the currents passing through the common resistance are opposite then the mutual resistance will have a negative sign.

 $V_a = V_1$  is the voltage which drives loop one. Here, the positive sign is used if the direction of the current is the same as the direction of the source. If the

current direction is opposite to the direction of the source, then the negative sign is used.

Similarly,  $R_{22} = (R_2 + R_3)$  and  $R_{33} = R_4 + R_5$  are the self resistances of loops two and three, respectively. The mutual resistances  $R_{13} = 0$ ,  $R_{21} = -R_2$ ,  $R_{23} = 0$ ,  $R_{31} = 0$ ,  $R_{32} = 0$  are the sums of the resistances common to the mesh currents indicated in their subscripts.

 $V_b = -V_2$ ,  $V_c = V_2$  are the sum of the voltages driving their respective loops.

**Example 2.9** Write the mesh equations for the circuit shown in Fig. 2.31.



Fig. 2.31

**Solution** The general equations for three mesh network are

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_a \tag{2.23}$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_b \tag{2.24}$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_c \tag{2.25}$$

Consider Eq. 2.23

 $R_{11}$  = self resistance of loop 1 = (1  $\Omega$  + 3  $\Omega$  + 6  $\Omega$ ) = 10  $\Omega$ 

 $R_{12}$  = the mutual resistance common to loop 1 and loop 2 = -3  $\Omega$ 

Here, the negative sign indicates that the currents are in opposite direction

 $R_{13}$  = the mutual resistance common to loop 1 and 3 = -6  $\Omega$ 

 $V_a = +10$  V, the voltage driving the loop 1.

Here, the positive sign indicates the loop current  $I_1$  is in the same direction as the source element.

Therefore, Eq. (2.23) can be written as

$$10I_1 - 3I_2 - 6I_3 = 10 \text{ V}$$
 (2.26)

Consider Eq. (2.24)

 $R_{21}$  = mutual resistance common to loop 1 and loop 2 = -3  $\Omega$ 

 $R_{22}$  = self resistance of loop 2 = (3  $\Omega$  + 2  $\Omega$  + 5  $\Omega$ ) = 10  $\Omega$ 

 $R_{23} = 0$ , there is no common resistance between loop 2 and loop 3.

 $V_{\rm b} = -5$  V, the voltage driving the loop 2.

Therefore, Eq. (2.24) can be written as

$$-3I_1 + 10I_2 = -5 \text{ V}$$
(2.27)

Consider Eq. (2.25)

 $R_{31}$  = mutual resistance common to loop 3 and loop 1 = -6  $\Omega$ 

 $R_{32}$  = mutual resistance common to loop 3 and loop 2 = 0

 $R_{_{33}}$  = self resistance of loop 3 = (6  $\Omega$  + 4  $\Omega$ ) = 10  $\Omega$ 

 $V_c$  = the algebraic sum of the voltages driving loop 3

= (5 V + 20 V) = 25 V

Therefore, Eq. (2.25) can be written as

$$-6I_1 + 10I_3 = 25 \text{ V}$$
(2.28)

The three mesh equation are

$$10I_1 - 3I_2 - 6I_3 = 10 \text{ V}$$
  
- 3I\_1 + 10I\_2 = -5 V  
- 6I\_1 + 10I\_3 = 25 V

### 2.11 SUPERMESH ANALYSIS

Suppose any of the branches in the network has a current source, then it is slightly difficult to apply mesh analysis straight forward because first we should assume an unknown voltage across the current source, writing mesh equations as before, and then relate the source current to the assigned mesh currents. This is generally a difficult approach. One way to overcome this difficulty is by applying the supermesh technique. Here we have to choose the kind of supermesh. A supermesh is constituted by two adjacent loops that have a common current source. As an example, consider the network shown in Fig. 2.32.

Here, the current source I is in the common boundary for the two meshes 1 and 2. This current source creates a supermesh, which is nothing but a combination of meshes 1 and 2.



Fig. 2.32

or

$$R_1I_1 + R_3(I_2 - I_3) = V$$
  
$$R_1I_1 + R_3I_2 - R_3I_3 = V$$

Considering mesh 3, we have

$$R_3(I_3 - I_2) + R_3I_3 = 0$$

Finally, the current *I* from current source is equal to the difference between two mesh currents, i.e.

$$I_1 - I_2 = I$$

We have, thus, formed three mesh equations which we can solve for the three unknown currents in the network.

**Example 2.10** Determine the current in the 5  $\Omega$  resistor in the network given in Fig. 2.33.



Fig. 2.33

**Solution** From the first mesh, i.e. *abcda*, we have

$$50 = 10(I_1 - I_2) + 5(I_1 - I_3)$$
  

$$15I_1 - 10I_2 - 5I_3 = 50$$
(2.29)

From the second and third meshes, we can form a supermesh

$$10(I_2 - I_1) + 2I_2 + I_3 + 5(I_3 - I_1) = 0$$
  
- 15I\_1 + 12I\_2 + 6I\_3 = 0 (2.30)

The current source is equal to the difference between II and III mesh currents, i.e.

$$I_2 - I_3 = 2A$$
 (2.31)

Solving 2.29, 2.30 and 2.31, we have

or

or

$$I_1 = 19.99 \text{ A}, \quad I_2 = 17.33 \text{ A}, \text{ and } I_3 = 15.33 \text{ A}$$

The current in the 5  $\Omega$  resistor =  $I_1 - I_3$ 

$$= 19.99 - 15.33 = 4.66$$
 A

 $\therefore$  The current in the 5  $\Omega$  resistor is 4.66 A.

**Example 2.11** Write the mesh equations for the circuit shown in Fig. 2.34 and determine the currents,  $I_1$ ,  $I_2$  and  $I_3$ .



Fig. 2.34

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**Solution** In Fig. 2.34, the current source lies on the perimeter of the circuit, and the first mesh is ignored. Kirchhoff's voltage law is applied only for second and third meshes.

From the second mesh, we have

$$3(I_2 - I_1) + 2(I_2 - I_3) + 10 = 0$$
  
-3I\_1 + 5I\_2 - 2I\_3 = -10 (2.32)

From the third mesh, we have

$$I_3 + 2(I_3 - I_2) = 10$$
  
- 2I\_2 + 3I\_2 = 10 (2.33)

From the first mesh,

 $I_1 = 10 \text{ A}$  (2.34)

From the above three equations, we get

$$I_1 = 10 \text{ A}, \quad I_2 = 7.27 \text{ A}, \quad I_3 = 8.18 \text{ A}$$

### 2.12 NODAL ANALYSIS

In the Chapter 1 we discussed simple circuits containing only two nodes, including the reference node. In general, in a N node circuit, one of the nodes is choosen as reference or datum node, then it is possible to write N - 1 nodal equations by assuming N - 1 node voltages. For example, a 10 node circuit requires nine unknown voltages and nine equations. Each node in a circuit can be assigned a number or a letter. The node voltage is the voltage of a given node with respect to one particular node, called the reference node, which we assume at zero potential. In the circuit shown in Fig. 2.35, node 3 is assumed as the reference node. The voltage at node 1 is the voltage at that node with respect to node 3. Similarly, the voltage at node 2 is the voltage at that node with respect to node 3. Applying Kirchhoff's current law at node 1; the current entering is equal to the current leaving. (See Fig. 2.36).



$$I_1 = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2}$$

where  $V_1$  and  $V_2$  are the voltages at node 1 and 2, respectively. Similarly, at node 2, the current entering is equal to the current leaving as shown in Fig. 2.37.

or

or



From the above equations, we can find the voltages at each node.

**Example 2.12** Write the node voltage equations and determine the currents in each branch for the network shown in Fig. 2.38.



Fig. 2.38

Solution The first step is to assign voltages at each node as shown in Fig. 2.39.



Fig. 2.39

Applying Kirchhoff's current law at node 1,

we have

$$5 = \frac{V_1}{10} + \frac{V_1 - V_2}{3}$$
$$V_1 \left[ \frac{1}{10} + \frac{1}{3} \right] - V_2 \left[ \frac{1}{3} \right] = 5$$
(2.35)

or

Applying Kirchhoff's current law at node 2,

we have  $\frac{V_2 - V_1}{3} + \frac{V_2}{5} + \frac{V_2 - 10}{1} = 0$ or  $-V_1 \left[\frac{1}{3}\right] + V_2 \left[\frac{1}{3} + \frac{1}{5} + 1\right] = 10$  (2.36) From Eqs 2.35 and 2.36, we can solve for  $V_1$  and  $V_2$  to get

$$V_{1} = 19.85 \text{ V}, \ V_{2} = 10.9 \text{ V}$$

$$I_{10} = \frac{V_{1}}{10} = 1.985 \text{ A}, \ I_{3} = \frac{V_{1} - V_{2}}{3} = \frac{19.85 - 10.9}{3} = 2.98 \text{ A}$$

$$I_{5} = \frac{V_{2}}{5} = \frac{10.9}{5} = 2.18 \text{ A}, \ I_{1} = \frac{V_{2} - 10}{1} = 0.9 \text{ A}$$

---

**Example 2.13** Determine the voltages at each node for the circuit shown in Fig. 2.40.



Fig. 2.40

Solution At node 1, assuming that all currents are leaving, we have

$$\frac{V_1 - 10}{10} + \frac{V_1 - V_2}{3} + \frac{V_1}{5} + \frac{V_1 - V_2}{3} = 0$$
$$V_1 \left[ \frac{1}{10} + \frac{1}{3} + \frac{1}{5} + \frac{1}{3} \right] - V_2 \left[ \frac{1}{3} + \frac{1}{3} \right] = 1$$
$$0.96V_1 - 0.66V_2 = 1$$
(2.37)

or

At node 2, assuming that all currents are leaving except the current from current source, we have

$$\frac{V_2 - V_1}{3} + \frac{V_2 - V_1}{3} + \frac{V_2 - V_3}{2} = 5$$
  
- $V_1 \left[ \frac{2}{3} \right] + V_2 \left[ \frac{1}{3} + \frac{1}{3} + \frac{1}{2} \right] - V_3 \left[ \frac{1}{2} \right] = 5$   
- 0.66  $V_1 + 1.16 V_2 - 0.5 V_3 = 5$  (2.38)

At node 3, assuming all currents are leaving, we have

$$\frac{V_3 - V_2}{2} + \frac{V_3}{1} + \frac{V_3}{6} = 0$$
  
- 0.5 V\_2 + 1.66 V\_3 = 0 (2.39)

Applying Cramer's rule, we get

$$V_{1} = \begin{vmatrix} 1 & -0.66 & 0 \\ 5 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \\ \hline 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix} = \frac{7.154}{0.887} = 8.06 \text{ V}$$

Similarly,

$$V_{2} = \begin{vmatrix} 0.96 & 1 & 0 \\ -0.66 & 5 & -0.5 \\ 0 & 0 & 1.66 \\ \hline 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix} = \frac{9.06}{0.887} = 10.2 \text{ V}$$
$$V_{3} = \begin{vmatrix} 0.96 & -0.66 & 1 \\ -0.66 & 1.16 & 5 \\ 0 & -0.5 & 0 \\ \hline 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix} = \frac{2.73}{0.887} = 3.07 \text{ V}$$

\*\*

## 2.13 NODAL EQUATIONS BY INSPECTION METHOD

The nodal equations for a general planar network can also be written by inspection, without going through the detailed steps. Consider a three node resistive network, including the reference node, as shown in Fig. 2.41.



Fig. 2.41

In Fig. 2.41, the points a and b are the actual nodes and c is the reference node.

Now consider the nodes a and b separately as shown in Figs. 2.42 (a) and (b).





 $I_4 + I_5 = I_3$ 

In Fig. 2.42 (a), according to Kirchhoff's current law, we have

 $I_1 + I_2 + I_3 = 0$ 

...

...

$$\frac{V_a - V_1}{R_1} + \frac{V_a}{R_2} + \frac{V_a - V_b}{R_3} = 0$$
(2.40)

In Fig. 2.42 (b), if we apply Kirchhoff's current law, we get

$$\frac{V_b - V_a}{R_3} + \frac{V_b}{R_4} + \frac{V_b - V_2}{R_5} = 0$$
(2.41)

Rearranging the above equations, we get

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) V_a - \left(\frac{1}{R_3}\right) V_b = \left(\frac{1}{R_1}\right) V_1$$
(2.42)

$$\left(-\frac{1}{R_3}\right)V_a + \left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right)V_b = \frac{V_2}{R_5}$$
(2.43)

In general, the above equations can be written as

 $G_{aa} V_a + G_{ab} V_b = I_1$ (2.44)

$$G_{ba} V_a + G_{bb} V_b = I_2 \tag{2.45}$$

By comparing Eqs 2.42, 2.43 and Eqs 2.44, 2.45 we have the self conductance at node a,  $G_{aa} = (1/R_1 + 1/R_2 + 1/R_3)$  is the sum of the conductances connected to node a. Similarly,  $G_{bb} = (1/R_3 + 1/R_4 + 1/R_5)$ , is the sum of the conductances connected to node b.  $G_{ab} = (-1/R_3)$ , is the sum of the mutual conductances connected to node a and node b. Here all the mutual conductances have negative signs. Similarly,  $G_{ba} = (-1/R_3)$  is also a mutual conductance connected between nodes b and a.  $I_1$  and  $I_2$  are the sum of the source currents at node a and node b, respectively. The current which drives into the node has positive sign, while the current that drives away from the node has negative sign.

**Example 2.14** For the circuit shown in Fig. 2.43, write the node equations by the inspection method.



Fig. 2.43

Solution The general equations are

$$G_{aa} V_{a} + G_{ab} V_{b} = I_{1}$$
(2.46)

$$G_{ba} V_{a} + G_{bb} V_{b} = I_{2}$$
(2.47)

Consider Eq. 2.46

 $G_{aa} = (1 + 1/2 + 1/3)$  mho, the self conductance at node *a* is the sum of the conductances connected to node *a*.

 $G_{bb} = (1/6 + 1/5 + 1/3)$  mho the self conductance at node b is the sum of the conductances connected to node b.

 $G_{ab} = -(1/3)$  mho, the mutual conductance between nodes *a* and *b* is the sum of the conductances connected between nodes *a* and *b*.

Similarly,  $G_{ba} = -(1/3)$ , the sum of the mutual conductances between nodes b and a.

$$I_1 = \frac{10}{1} = 10$$
 A, the source current at node *a*,  
 $I_2 = \left(\frac{2}{5} + \frac{5}{6}\right) = 1.23$  A, the source current at node *b*.

Therefore, the nodal equations are

$$1.83 V_a - 0.33 V_b = 10 \tag{2.48}$$

$$-0.33 V_{\perp} + 0.7 V_{\perp} = 1.23$$
 (2.49)

22

# 2.14 SUPERNODE ANALYSIS

Suppose any of the branches in the network has a voltage source, then it is slightly difficult to apply nodal analysis. One way to overcome this difficulty is to apply the supernode technique. In this method, the two adjacent nodes that are connected by a voltage source are reduced to a single node and then the equations are formed by applying Kirchhoff's current law as usual. This is explained with the help of Fig. 2.44.



Fig. 2.44

It is clear from Fig. 2.44, that node 4 is the reference node. Applying Kirchhoff's current law at node 1, we get

$$I = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2}$$

Due to the presence of voltage source  $V_x$  in between nodes 2 and 3, it is slightly difficult to find out the current. The supernode technique can be conveniently applied in this case.

Accordingly, we can write the combined equation for nodes 2 and 3 as under.

$$\frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} + \frac{V_3 - V_y}{R_4} + \frac{V_3}{R_5} = 0$$

The other equation is

$$V_2 - V_3 = V_x$$

From the above three equations, we can find the three unknown voltages.

**Example 2.15** Determine the current in the 5  $\Omega$  resistor for the circuit shown in Fig. 2.45.



Fig. 2.45

**Solution** At node 1

 $10 = \frac{V_1}{3} + \frac{V_1 - V_2}{2}$  $V_1 \left[ \frac{1}{3} + \frac{1}{2} \right] - \frac{V_2}{2} - 10 = 0$  $0.83 \ V_1 - 0.5 \ V_2 - 10 = 0$ (2.50)

or

At node 2 and 3, the supernode equation is

or

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} + \frac{V_3 - 10}{5} + \frac{V_3}{2} = 0$$
  
$$\frac{-V_1}{2} + V_2 \left[\frac{1}{2} + 1\right] + V_3 \left[\frac{1}{5} + \frac{1}{2}\right] = 2$$
  
$$-0.5 V_1 + 1.5 V_2 + 0.7 V_3 - 2 = 0$$
 (2.51)

The voltage between nodes 2 and 3 is given by

$$V_2 - V_3 = 20 \tag{2.52}$$

The current in the 5 W resistor  $I_5 = \frac{V_3 - 10}{5}$ 

Solving Eqs 2.50, 2.51 and 2.52, we obtain

$$V_3 = -8.42$$
 V

:. Current  $I_5 = \frac{-8.42 - 10}{5} = -3.68$  A (current towards node 3) i.e. the current flows towards node 3.

# 2.15 SOURCE TRANSFORMATION TECHNIQUE

In solving networks to find solutions one may have to deal with energy sources. It has already been discussed in Chapter 1 that basically, energy sources are either voltage sources or current sources. Sometimes it is necessary to convert a voltage source to a current source and vice-versa. Any practical voltage source consists of an ideal voltage source in series with an internal resistance. Similarly, a practical current source consists of an ideal current source in parallel with an internal resistance as shown in Fig. 2.46.  $R_v$  and  $R_i$  represent the internal resistances of the voltage source  $V_s$ , and current source  $I_s$ , respectively.

Any source, be it a current source or a voltage source, drives current through its load resistance, and the magnitude of the current depends on the value of the load resistance. Figure 2.47 represents a practical voltage source and a practical current source connected to the same load resistance  $R_{I}$ .



2.35



Fig. 2.47

From Fig. 2.47 (a), the load voltage can be calculated by using Kirchhoff's voltage law as

 $V_{ab} = V_s - I_L R_v$ The open circuit voltage  $V_{OC} = V_s$ The short circuit current  $I_{sc} = \frac{V_s}{R}$ From Fig. 2.47 (b)

$$I_L = I_S - I = I_S - \frac{V_{ab}}{R_I}$$

The open circuit voltage  $V_{ac} = I_S R_I$ The short circuit current  $I_{SC} = I_S$ 

The above two sources are said to be equal, if they produce equal amounts of current and voltage when they are connected to identical load resistances. Therefore, by equating the open circuit voltages and short circuit currents of the above two sources we obtain

$$V_{ac} = I_s R_I = V_S$$
$$I_{SC} = I_S = \frac{V_s}{R_v}$$

It follows that  $R_1 = R_V = R_s \therefore V_s = I_s R_s$ where  $R_s$  is the internal resistance of the voltage or current source. Therefore, any practical voltage source, having an ideal voltage  $V_s$  and internal series resistance  $R_s$  can be replaced by a current source  $I_s = V_s / \tilde{R}_s$  in parallel with an internal resistance  $R_s$ . The reverse transformation is also possible. Thus, a practical current source in parallel with an internal resistance  $R_s$  can be replaced by a voltage source  $V_s = I_s R_s$  in series with an internal resistance  $R_s$ .

**Example 2.16** Determine the equivalent voltage source for the current source shown in Fig. 2.48.





**Solution** The voltage across terminals *A* and *B* is equal to 25 V. Since the internal resistance for the current source is 5  $\Omega$ , the internal resistance of the voltage source is also 5  $\Omega$ . The equivalent voltage source is shown in Fig. 2.49.



**Example 2.17** Determine the equivalent current source for the voltage source shown in Fig. 2.50.

**Solution** The short circuit current at terminals *A* and *B* is equal to

$$I = \frac{50}{30} = 1.66 \text{ A}$$

Since the internal resistance for the voltage source is  $30 \Omega$ , the internal resistance of the current source is also 30  $\Omega$ . The equivalent current source is shown in Fig. 2.51.



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ADDITIONAL SOLVED PROBLEMS

**Problem 2.1** Determine the power dissipation in the 4  $\Omega$  resistor of the circuit shown in Fig. 2.52 by using mesh analysis.



Fig. 2.52

**Solution** Power dissipated in the 4  $\Omega$  resistor is  $P_4 = 4(I_2 - I_3)^2$ By using mesh analysis, we can find the currents  $I_2$  and  $I_3$ . From Fig. 2.52, we can form three equations.

From the given circuit in Fig. 2.52, we can obtain three mesh equations in terms of  $I_1$ ,  $I_2$  and  $I_3$ 

$$8I_1 + 3I_2 = 50$$
  

$$3I_1 + 9I_2 - 4I_3 = 0$$
  

$$-4I_2 + 10I_3 = 10$$

By solving the above equations we can find  $I_1$ ,  $I_2$  and  $I_3$ .

$$I_{2} = \begin{vmatrix} 8 & 50 & 0 \\ 3 & 0 & -4 \\ 0 & 10 & 10 \\ \hline 8 & +3 & 0 \\ 3 & 9 & -4 \\ 0 & -4 & 10 \end{vmatrix} = \frac{-1180}{502} = -2.35 \text{ A}$$
$$I_{3} = \begin{vmatrix} 8 & 3 & 50 \\ 3 & 9 & 0 \\ 0 & -4 & 10 \\ \hline 8 & 3 & 0 \\ 3 & 9 & -4 \\ 0 & -4 & 10 \end{vmatrix} = \frac{30}{502} = 0.06 \text{ A}$$

The current in the 4  $\Omega$  resistor =  $(I_2 - I_3)$ 

$$= (-2.35 - 0.06) A = -2.41 A$$

Therefore, the power dissipated in the 4  $\Omega$  resistor,  $P_4 = (2.41)^2 \times 4 = 23.23$  W.

\*\*

**Problem 2.2** Using mesh analysis, determine the voltage  $V_s$  which gives a voltage of 50 V across the 10  $\Omega$  resistor as shown in Fig. 2.53.



Fig. 2.53

Since the voltage across the 10  $\Omega$  resistor is 50 V, the current Solution passing through it is  $I_4 = 50/10 = 5$  A. From Fig. 2.53, we can form four equations in terms of the currents  $I_1$ ,  $I_2$ ,  $I_3$ 

and  $I_4$ , as

$$\begin{aligned} &4I_1-I_2=60\\ &-I_1+8I_2-2I_3+5I_4=0\\ &-2I_2+6I_3=50\\ &5I_2+15I_4=V_S\end{aligned}$$

Solving the above equations, using Cramer's rule, we get

$$I_4 = \begin{vmatrix} 4 & -1 & 0 & 60 \\ -1 & 8 & -2 & 0 \\ 0 & -2 & 6 & 50 \\ 0 & 5 & 0 & V_S \\ \hline 4 & -1 & 0 & 0 \\ -1 & 8 & -2 & 5 \\ 0 & -2 & 6 & 0 \\ 0 & 5 & 0 & 15 \end{vmatrix}$$
$$\Delta = 4 \begin{vmatrix} 8 & -2 & 5 \\ -2 & 6 & 0 \\ 5 & 0 & 15 \end{vmatrix} + 1 \begin{vmatrix} -1 & -2 & 5 \\ 0 & 6 & 0 \\ 0 & 0 & 15 \end{vmatrix}$$
$$= 4\{8(90) + 2(-30) + 5(-30)\} + 1\{-1(90)\}$$
$$\Delta = 1950.$$

}

$$\Delta_{4} = 4 \begin{vmatrix} 8 & -2 & 0 \\ -2 & 6 & 50 \\ 5 & 0 & V_{S} \end{vmatrix} + 1 \begin{vmatrix} -1 & -2 & 0 \\ 0 & 6 & 50 \\ 0 & 0 & V_{S} \end{vmatrix} - 60 \begin{vmatrix} -1 & 8 & -2 \\ 0 & -2 & 6 \\ 0 & 5 & 0 \end{vmatrix}$$
$$= 4\{8(6 V_{S}) + 2(-2V_{S} - 250)\} + 1\{-1(6V_{S})\} - 60\{-1(-30)\}\}$$
$$= 170 V_{S} - 3800$$
$$I_{4} = \frac{170V_{S} - 3800}{1950}$$
$$V_{S} = \frac{1950 \times I_{4} + 3800}{170} = 79.7 \text{ V}$$

**Problem 2.3** Determine the voltage V which causes the current  $I_1$  to be zero for the circuit shown in Fig. 2.54. Use Mesh analysis.





**Solution** From Fig. 2.54 we can form three loop equations in terms of  $I_1, I_2, I_3$ and V, as follows

$$13I_1 - 2I_2 - 5I_3 = 20 - V$$
  
- 2I\_1 + 6I\_2 - I\_3 = 0  
- 5I\_1 - I\_2 + 10I\_3 = V

Using Cramer's rule, we get

$$I_{1} = \begin{vmatrix} 20 - V & -2 & -5 \\ 0 & 6 & -1 \\ V & -1 & +10 \\ \hline 13 & -2 & -5 \\ -2 & +6 & -1 \\ -5 & -1 & +10 \end{vmatrix}$$
$$\Delta_{1} = (20 - V) (+ 60 - 1) + 2(V) - 5(-6V)$$
$$= 1180 - 27 V$$
$$\Delta = 557$$
$$I_{1} = \frac{\Delta_{1}}{557}$$
$$\Delta_{1} = 0$$

we have

*:*..

$$Y_1 = \frac{\Delta_1}{557}$$
$$A_1 = 0$$

*.*..

$$-27 V + 1180 = 0$$

*:*..

$$V = 43.7 \text{ V}$$

**Problem 2.4** Determine the loop currents for the circuit shown in Fig. 2.55 by using mesh analysis.



Fig. 2.55

**Solution** The branches *AE*, *DE* and *BC* consists of current sources. Here we have to apply supermesh analysis.

The combined supermesh equation is

Solving the above four equations, we can get the four currents  $I_1$ ,  $I_2$ ,  $I_3$  and  $I_4$  as  $I_1 = 14.65$  A

$$I_2 = 19.65 \text{ A}, \quad I_3 = 15 \text{ A}, \text{ and } I_4 = 9.65 \text{ A}$$

**Problem 2.5** Determine the power delivered by the voltage source and the current in the 10  $\Omega$  resistor for the circuit shown in Fig. 2.56.



Fig. 2.56

**Solution** Since branches AC and BD consist of current sources, we have to use the supermesh technique.

The combined supermesh equation is

$$-50 + 5I_1 + 3I_2 + 2I_2 + 10(I_2 - I_3) + 1(I_1 - I_3) = 0$$
  
or  
$$6I_1 + 15I_2 - 11I_3 = 50$$
  
or  
$$I_1 - I_2 = 3 \text{ A or } I_3 = 10 \text{ A}$$

From the above equations we can solve for  $I_1$ ,  $I_2$  and  $I_3$  follows

$$I_1 = 9.76 \text{ A}, \quad I_2 = 6.76 \text{ A}, \quad I_3 = 10 \text{ A}$$

**Problem 2.6** Determine the voltage ratio  $V_{out}/V_{in}$  for the circuit shown in Fig. 2.57 by using nodal analysis.



Fig. 2.57

**Solution**  $I_{10} + I_3 + I_{11} = 0$  $I_{10} = \frac{V_A - V_{\rm in}}{10}$  $I_3 = \frac{V_A}{3}$  $I_{11} = \frac{V_A}{11}$ , or  $\frac{V_{\text{out}}}{6}$  $\frac{V_A - V_{\text{in}}}{10} + \frac{V_A}{3} + \frac{V_A}{11} = 0$  $\frac{V_A}{11} = \frac{V_{\text{out}}}{6}$ Also

 $V_A = V_{\text{out}} \times 1.83$ ...

From the above equations  $V_{out}/V_{in} = 1/9.53 = 0.105$ 

Problem 2.7 Find the voltages V in the circuit shown in Fig. 2.58 which makes the current in the 10  $\Omega$  resistor zero by using nodal analysis.

\*\*





**Solution** In the circuit shown, assume voltages  $V_1$  and  $V_2$  at nodes 1 and 2. At node 1, the current equation in Fig. 2.59 (a) is



Fig. 2.59 (a)

$$\frac{V_1 - V}{3} + \frac{V_1}{2} + \frac{V_1 - V_2}{10} = 0$$
  
0.93 V<sub>1</sub> - 0.1 V<sub>2</sub> = V/3

or

At node 2, the current equation in Fig. 2.59 (b) is



Fig. 2.59 (b)

$$\frac{V_2 - V_1}{10} + \frac{V_2}{5} + \frac{V_2 - 50}{7} = 0$$
$$- 0.1 V_1 + 0.443 V_2 = 7.143$$

or

Since the current in 10  $\Omega$  resistor is zero, the voltage at node 1 is equal to the voltage at node 2.

$$\therefore \qquad \qquad V_1 - V_2 = 0$$

From the above three equations, we can solve for V

$$V_1 = 20.83$$
 Volts and  $V_2 = 20.83$  volts  
 $V = 51.87$  V

**Problem 2.8** Use nodal analysis to find the power dissipated in the 6  $\Omega$  resistor for the circuit shown in Fig. 2.60.



Fig. 2.60

**Solution** Assume voltage  $V_1$ ,  $V_2$  and  $V_3$  at nodes 1, 2 and 3 as shown in Fig. 2.60.

By applying current law at node 1, we have

$$\frac{V_1 - 20}{3} + \frac{V_1 - V_2}{1} + \frac{V_1 - V_3}{2} = 0$$
  
1.83 $V_1 - V_2 - 0.5V_3 = 6.67$  (2.53)

\*\*

or

$$\frac{V_2 - V_1}{1} + \frac{V_2 - V_3}{6} = 5 \text{ A}$$
$$-V_1 - 1.167V_2 - 0.167V_3 = 5 \tag{2.54}$$

or

At node 3,

$$\frac{V_3 - V_1}{2} + \frac{V_3 - V_2}{6} + \frac{V_3}{5} = 0$$
  
- 0.5 V\_1 - 0.167 V\_2 + 0.867 V\_3 = 0 (2.55)

or

Applying Cramer's rule to Eqs 2.53, 2.54 and 2.55, we have

where 
$$\begin{aligned} V_2 &= \frac{\Delta_2}{\Delta} \\ \Delta &= \begin{vmatrix} 1.83 & -1 & -0.5 \\ -1 & -1.167 & -0.167 \\ -0.5 & -0.167 & 0.867 \end{vmatrix} = -2.64 \end{aligned}$$

:.

$$\Delta_2 = \begin{vmatrix} 1.83 & 6.67 & -0.5 \\ -1 & 5 & -0.167 \\ -0.5 & 0 & 0.867 \end{vmatrix} = 13.02$$
$$V_2 = \frac{13.02}{-2.64} = -4.93 \text{ V}$$

Similarly,

:.

$$V_{3} = \frac{\Delta_{3}}{\Delta}$$

$$\Delta_{3} = \begin{vmatrix} 1.83 & -1 & 6.67 \\ -1 & -1.167 & 5 \\ -0.5 & -0.167 & 0 \end{vmatrix} = 1.25$$

$$\therefore \qquad V_{3} = \frac{1.25}{-2.64} = -0.47 \text{ V}$$

The current in the 6  $\Omega$  resistor is

$$I_{6} = \frac{V_{2} - V_{3}}{6}$$
  
=  $\frac{-4.93 + 0.47}{6} = -0.74 \text{ A}$   
The power absorbed or dissipated =  $I_{6}^{2} R_{6}$   
=  $(0.74)^{2} \times 6$   
=  $3.29 \text{ W}$ 

**Problem 2.9** Determine the power dissipated by 5  $\Omega$  resistor in the circuit shown in Fig. 2.61.



Fig. 2.61

**Solution** In Fig. 2.61, assume voltages  $V_1$ ,  $V_2$  and  $V_3$  at nodes 1, 2 and 3. At node 1, the current law gives

$$\frac{V_1 - 40 - V_3}{4} + \frac{V_1 - V_2}{6} - 3 - 5 = 0$$
  
0.42 V\_1 - 0.167 V\_2 - 0.25 V\_3 = 18

or

Applying the supernode technique between nodes 2 and 3, the combined equation at node 2 and 3 is

 $\frac{V_2 - V_1}{6} + 5 + \frac{V_2}{3} + \frac{V_3}{5} + \frac{V_3 + 40 - V_1}{4} = 0$ - 0.42 V<sub>1</sub> + 0.5 V<sub>2</sub> + 0.45 V<sub>3</sub> = -15 V<sub>3</sub> - V<sub>2</sub> = 20 V

or Also

Solving the above three equations, we get

 $V_1 = 52.89$  V,  $V_2 = -1.89$  V and  $V_3 = 18.11$  V

 $\therefore$  The current in the 5  $\Omega$  resistor  $I_5 = \frac{V_3}{5}$ 

$$=\frac{18.11}{5}=3.62$$
 A

The power absorbed by the 5  $\Omega$  resistor  $P_5 = I_5^2 R_5$ =  $(3.62)^2 \times 5$ 

\_

**Problem 2.10** Find the power delivered by the 5 A current source in the circuit shown in Fig. 2.62 by using the nodal method.



Fig. 2.62

**Solution** Assume the voltages  $V_1$ ,  $V_2$  and  $V_3$  at nodes 1, 2, and 3, respectively. Here, the 10 V source is common between nodes 1 and 2. So applying the supernode technique, the combined equation at node 1 and 2 is



 $V_1 = 13.72 \text{ V}; \quad V_2 = 3.72 \text{ V}$  $V_3 = 4.567 \text{ V}$ 

Hence the power delivered by the source (5 A) =  $V_2 \times 5$ 

$$= 3.72 \times 5 = 18.6 \text{ W}$$

**Problem 2.11** Using source transformation, find the power delivered by the 50 V voltage source in the circuit shown in Fig. 2.63.





**Solution** The current source in the circuit in Fig. 2.63 can be replaced by a voltage source as shown in Fig. 2.64.



Fig. 2.64

$$\frac{V-50}{5} + \frac{V-20}{2} + \frac{V-10}{3} = 0$$
$$V [0.2 + 0.5 + 0.33] = 23.33$$
$$V = \frac{23.33}{1.03} = 22.65 \text{ V}$$

or

:. The current delivered by the 50 V voltage source is (50 - V)/5

$$=\frac{50-22.65}{5}=5.47$$
 A

Hence, the power delivered by the 50 V voltage source =  $50 \times 5.47 = 273.5$  W

**Problem 2.12** By using source transformation, source combination and resistance combination convert the circuit shown in Fig. 2.65 into a single voltage source and single resistance.



Fig. 2.65

**Solution** The voltage source in the circuit of Fig. 2.65 can be replaced by a current source as shown in Fig. 2.66 (a).



Fig. 2.66 (a)

Here the current sources can be combined into a single source. Similarly, all the resistances can be combined into a single resistance, as shown in Fig. 2.66 (b).

Figure 2.66 (b) can be replaced by single voltage source and a series resistance as shown in Fig. 2.66 (c).


Fig. 2.66 (c)

**Problem 2.13** For the circuit shown in Fig. 2.67 find the voltage across the 4  $\Omega$  resistor by using nodal analysis.



Fig. 2.67

**Solution** In the circuit shown, assume voltages  $V_1$  and  $V_2$  at nodes 1 and 2. At node 1, the current equation is

$$5 + \frac{V_1}{3} + \frac{V_1 + 5 - V_2}{4} + \frac{V_1 - V_2}{2} = 0$$
  
1.08 V<sub>1</sub> - 0.75 V<sub>2</sub> = -6.25 (2.56)

or

At node 2, the current equation is

$$\frac{V_2 - V_1 - 5}{4} + \frac{V_2 - V_1}{2} - 4V_x + \frac{V_2}{1} = 0$$
$$V_x = V_1 + 5 - V_2$$

or	$-4.75 V_1 + 5.75 V_2 = 21.25$	(2.57)
A 1	1 I I E 0.5C 10.57 1	

Applying Cramer's rule to Eqs 2.56 and 2.57, we have

$$V_{2} = \frac{\Delta_{2}}{\Delta}$$
where
$$\Delta = \begin{vmatrix} 1.08 & -0.75 \\ -4.75 & 5.75 \end{vmatrix} = 2.65$$

$$\Delta_{2} = \begin{vmatrix} 1.08 & -6.25 \\ -4.75 & 21.25 \end{vmatrix} = -6.74$$

$$\therefore \qquad V_{2} = \frac{\Delta_{2}}{\Delta} = \frac{-6.74}{2.65} = -2.54 \text{ V}$$
Similarly,
$$V_{1} = \frac{\Delta_{1}}{\Delta}$$

$$\Delta_{1} = \begin{vmatrix} -6.25 & -0.75 \\ 21.25 & 5.75 \end{vmatrix} = -20$$

$$V_{1} = \frac{\Delta_{1}}{\Delta} = \frac{-20}{2.65} = -7.55 \text{ V}$$
The voltage across the 4 Q resistor is

i ne voi ige across

$$V_x = V_1 + 5 - V_2$$
  
= -.755 + 5 - (-2.54)  
$$V_x = 0.01 \text{ volts}$$

Problem 2.14 For the circuit shown in Fig. 2.68, find the current passing through the 5  $\Omega$  resistor by using the nodal method.



Fig. 2.68

**Solution** In the circuit shown, assume the voltage *V* at node 1. At node 1, the current equation is

$$\frac{V-30}{5} - 2 + \frac{V-36 - 6I_1}{6} = 0$$
$$I_1 = \frac{V-30}{5}$$

where

From the above equation

 $V = 48 \, V$ 

The current in 5  $\Omega$  resistor is

$$I_1 = \frac{V - 30}{5} = 3.6 \text{ A}$$

**Problem 2.15** In the circuit shown in Fig. 2.69, find the power delivered by 4 V source using mesh analysis and voltage across the 2  $\Omega$  resistor.



Fig. 2.69

**Solution** Since branches *BC* and *DE* consists of current sources, we use the supermesh technique.

The combined supermesh equation is

$$2I_1 + 6I_1 + 4(I_1 - I_3) - 4 + 5I_2 + I_2 - I_3 + 4(I_3 - I_1) + I_3 - I_2 = 0$$

or

$$8I_1 + 5I_2 = 4$$

In branch *BC*,  $I_2 - I_1 = 5$ 

In branch *DE*,

 $I_3 = \frac{V_2}{2}$ 

Solving the above equations

$$I_1 = -1.62 \,\mathrm{A}; \quad I_2 = 3.38 \,\mathrm{A}$$

The voltage across the 2  $\Omega$  resistor  $V_2 = 2I_1 = -3.24$  V Power delivered by 4 V source  $P_4 = 4I_2 = 4(3.38) = 13.52$  W

**Problem 2.16** For the circuit shown in Fig. 2.70, find the current through the 10  $\Omega$  resistor by using mesh analysis.

2.51

\*\*





**Solution** The parallel branches consist of current sources. Here we use supermesh analysis. The combined supermesh equation is.

or  $-15 + 10I_1 + 20 + 5I_2 + 4I_3 - 40 = 0$ and  $10I_1 + 5I_2 + 4I_3 = 35$ 

$$I_1 - I_2 = 2$$
  
 $I_3 - I_2 = 2I_1$ 

Solving the above equations, we get

 $I_1 = 1.96 \text{ A}$ 

The current in the 10  $\Omega$  resistor is  $I_1 = 1.96$  A



## **PRACTICE PROBLEMS**

\*\*

2.1 In the circuit shown in Fig. 2.71, use mesh analysis to find out the power delivered to the 4  $\Omega$  resistor. To what voltage should the 100 V battery be changed so that no power is delivered to the 4  $\Omega$  resistor?



Fig. 2.71

2.2 Find the voltage between *A* and *B* of the circuit shown in Fig. 2.72 by mesh analysis.





2.3 In the circuit shown in Fig. 2.73, use nodal analysis to find out the voltage across 40  $\Omega$  and the power supplied by the 5 A source.



Fig. 2.73

2.4 In the network shown in Fig. 2.74, the resistance R is variable from zero to infinity. The current *I* through *R* can be expressed as I = a + bV, where V is the voltage across R as shown in the figure, and a and b are constants. Determine *a* and *b*.





2.5 Determine the currents in bridge circuit by using mesh analysis in Fig. 2.75.



Fig. 2.75

2.6 Use nodal analysis in the circuit shown in Fig. 2.76 and determine what value of V will cause  $V_{10} = 0$ .



Fig. 2.76

2.7 For the circuit shown in Fig. 2.77, use mesh analysis to find the values of all mesh currents.



Fig. 2.77

2.8 For the circuit shown in Fig. 2.78, use node analysis to find the current delivered by the 24 V source.





2.9 Using mesh analysis, determine the voltage across the 10 k $\Omega$  resistor at terminals *A* and *B* of the circuit shown in Fig. 2.79.



Fig. 2.79

2.10 Determine the current I in the circuit by using loop analysis in Fig. 2.80.



Fig. 2.80

2.11 Write nodal equations for the circuit shown in Fig. 2.81, and find the power supplied by the 10 V source.



2.12 Use nodal analysis to find  $V_2$  in the circuit shown in Fig. 2.82.





2.13 Use mesh analysis to find  $V_x$  in the circuit shown in Fig. 2.83.



Fig. 2.83

2.14 For the circuit shown in Fig. 2.84, find the value of  $V_2$  that will cause the voltage across 20  $\Omega$  to be zero by using mesh analysis.







(b) more than

- (a) less than
- (c) equal to

3. The tie-set schedule gives the relation between (a) branch currents and link currents (b) branch voltages and link currents (c) branch currents and link voltages (d) none of the above 4. The cut-set schedule gives the relation between (a) branch currents and link currents (b) branch voltages and tree branch voltages (c) branch voltages and link voltages (d) branch current and tree currents 5. Mesh analysis is based on (a) Kirchhoff's current law (b) Kirchhoff's voltage law (c) Both (d) None 6. If a network contains B branches, and N nodes, then the number of mesh current equations would be (a) B - (N - 1)(b) N - (B - 1)(c) B - N - 1(d) (B + N) - 17. A network has 10 nodes and 17 branches. The number of different node pair voltages would be (a) 7 (b) 9 (c) 45 (d) 10 8. A practical voltage source consists of (a) an ideal voltage source in series with an internal resistance (b) an ideal voltage source in parallel with an internal resistance (c) both (a) and (b) are correct (d) none of the above 9. A practical current source consists of (a) an ideal current source in series with an impedance (b) an ideal current source in parallel with an impedance (c) both are correct (d) none of the above 10. A circuit consists of two resistances,  $R_1$  and  $R_2$ , in parallel. The total current passing through the circuit is  $I_T$ . The current passing through  $R_1$ is

(a) 
$$\frac{I_T R_1}{R_1 + R_2}$$
 (b)  $\frac{I_T (R_1 + R_2)}{R_1}$   
(c)  $\frac{I_T R_2}{R_1 + R_2}$  (d)  $\frac{I_T R_1 + R_2}{R_2}$ 

11. A network has seven nodes and five independent loops. The number of branches in the network is

(b) 12

(a) 13

- (c) 11 (d) 10
- 12. The nodal method of circuit analysis is based on
  - (a) KVL and Ohm's law
  - (c) KCL and KVL
- (b) KCL and Ohm's law
- (d) KCL, KVL and Ohm's law

- 13. The number of independent loops for a network with n nodes and b branches is
  - (a) n-1 (b) b-n
  - (c) b n + 1
  - (d) independent of the number of nodes
- 14. The two electrical sub networks  $N_1$  and  $N_2$  are connected through three resistors as shown in Fig. 2.85. The voltage across the 5  $\Omega$  resistor and the 1  $\Omega$  resistor are given to be 10 V and 5 V respectively. The voltage across the 15  $\Omega$  resistor is

(a) 
$$-105$$
 V  
(b)  $+105$  V  
(c)  $-15$  V  
(d)  $+15$  V



Fig. 2.85

- 15. Relative to a given fixed tree of a network
  - (a) link currents form an independent set
  - (b) branch currents form an independent set
  - (c) link voltages form an independent set
  - (d) branch voltages form an independent set



# 3.1 STAR-DELTA TRANSFORMATION

In the preceding chapter, a simple technique called the *source transformation technique* has been discussed. The star delta transformation is another technique useful in solving complex networks. Basically, any three circuit elements, i.e. resistive, inductive or capacitive, may be connected in two different ways. One way of connecting these elements is called the star connection, or the *Y* connection. The other way of connecting these elements is called the delta ( $\Delta$ ) connection. The circuit is said to be in star connection, if three elements are connected as shown in Fig. 3.1(a), when it appears like a star (*Y*). Similarly, the circuit is said to be in delta connection, if three elements are connected as shown in Fig. 3.1(b), when it appears like a delta ( $\Delta$ ).



Fig. 3.1

The above two circuits are equal if their respective resistances from the terminals AB, BC and CA are equal. Consider the star connected circuit in Fig. 3.1(a); the resistance from the terminals AB, BC and CA respectively are

$$R_{AB}(Y) = R_A + R_B$$
$$R_{BC}(Y) = R_B + R_C$$
$$R_{CA}(Y) = R_C + R_A$$

Similarly, in the delta connected network in Fig. 3.1(b), the resistances seen from the terminals *AB*, *BC* and *CA*, respectively, are

$$R_{AB}(\Delta) = R_1 \parallel (R_2 + R_3) = \frac{R_1 (R_2 + R_3)}{R_1 + R_2 + R_3}$$
$$R_{BC}(\Delta) = R_3 \parallel (R_1 + R_2) = \frac{R_3 (R_1 + R_2)}{R_1 + R_2 + R_3}$$
$$R_{CA}(\Delta) = R_2 \parallel (R_1 + R_3) = \frac{R_2 (R_1 + R_3)}{R_1 + R_2 + R_3}$$

Now, if we equate the resistances of star and delta circuits, we get

$$R_A + R_B = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$
(3.1)

$$R_B + R_C = \frac{R_3 \left(R_1 + R_2\right)}{R_1 + R_2 + R_3} \tag{3.2}$$

$$R_C + R_A = \frac{R_2 \left(R_1 + R_3\right)}{R_1 + R_2 + R_3} \tag{3.3}$$

Subtracting Eq. 3.2 from Eq. 3.1, and adding Eq. 3.3 to the resultant, we have

$$R_A = \frac{R_1 R_2}{R_1 + R_2 + R_3} \tag{3.4}$$

$$R_B = \frac{R_1 R_3}{R_1 + R_2 + R_3} \tag{3.5}$$

$$R_C = \frac{R_2 R_3}{R_1 + R_2 + R_3} \tag{3.6}$$

Thus, a delta connection of  $R_1$ ,  $R_2$  and  $R_3$  may be replaced by a star connection of  $R_A$ ,  $R_B$  and  $R_C$  as determined from Eqs 3.4, 3.5 and 3.6. Now if we multiply the Eqs 3.4 and 3.5, 3.5 and 3.6, 3.6 and 3.4, and add the three, we get the final equation as under:

$$R_A R_B + R_B R_C + R_C R_A = \frac{R_1^2 R_2 R_3 + R_3^2 R_1 R_2 + R_2^2 R_1 R_3}{(R_1 + R_2 + R_3)^2}$$
(3.7)

In Eq. 3.7 dividing the LHS by  $R_A$ , gives  $R_3$ ; dividing it by  $R_B$  gives  $R_2$ , and doing the same with  $R_C$  gives  $R_1$ .

Similarly,

and



$$R_{1} = \frac{R_{A} R_{B} + R_{B} R_{C} + R_{C} R_{A}}{R_{C}}$$
$$R_{2} = \frac{R_{A} R_{B} + R_{B} R_{C} + R_{C} R_{A}}{R_{B}}$$
$$R_{3} = \frac{R_{A} R_{B} + R_{B} R_{C} + R_{C} R_{A}}{R_{A}}$$

and

From the above results, we can say that a star connected circuit can be transformed into a delta connected circuit and vice-versa.

From Fig. 3.2 and the above results, we can conclude that any resistance of the delta circuit is equal to the sum of the products of all possible pairs of star resistances divided by the opposite resistance of the star circuit. Similarly, any resistance of the star circuit is equal to the product of two adjacent resistances in the delta connected circuit divided by the sum of all resistances in delta connected circuit.







Fig. 3.3

**Solution** The above circuit can be replaced by a star connected circuit as shown in Fig. 3.4 (a).



Fig. 3.4

Performing the  $\Delta$  to *Y* transformation, we obtain

$$R_{1} = \frac{13 \times 12}{14 + 13 + 12}, R_{2} = \frac{13 \times 14}{14 + 13 + 12}$$
$$R_{3} = \frac{14 \times 12}{14 + 13 + 12}$$
$$R_{1} = 4 \Omega, R_{2} = 4.66 \Omega, R_{3} = 4.31 \Omega$$

and

÷.

The star-connected equivalent is shown in Fig. 3.4 (b).

**Example 3.2** Obtain the delta-connected equivalent for the star-connected circuit shown in Fig. 3.5.

\*\*



Fig. 3.5

**Solution** The above circuit can be replaced by a delta-connected circuit as shown in Fig. 3.6 (a).

Performing the Y to  $\Delta$  transformation, we get from the Fig. 3.6 (a)



$$R_{2} = \frac{20 \times 10 + 20 \times 5 + 10 \times 5}{10} = 35 \ \Omega$$
$$R_{3} = \frac{20 \times 10 + 20 \times 5 + 10 \times 5}{5} = 70 \ \Omega$$

and

The equivalent delta circuit is shown in Fig. 3.6 (b).

## **3.2 SUPERPOSITION THEOREM**

The superposition theorem states that in any linear network containing two or more sources, the response in any element is equal to the algebraic sum of the responses caused by individual sources acting alone, while the other sources are non-operative; that is, while considering the effect of individual sources, other ideal voltage sources and ideal current sources in the network are replaced by short circuit and open circuit across their terminals. This theorem is valid only for linear systems. This theorem can be better understood with a numerical example.

Consider the circuit which contains two sources as shown in Fig. 3.7.

Now let us find the current passing through the 3  $\Omega$  resistor in the circuit. According to superposition theorem, the current  $I_2$  due to the 20 V voltage source with 5 A source open circuited = 20/(5 + 3) = 2.5 A. (See Fig. 3.8)



The current  $I_5$  due to 5 A source with 20 V source short circuited is

$$I_5 = 5 \times \frac{5}{(3+5)} = 3.125 \text{ A}$$

The total current passing through the 3  $\Omega$  resistor is

$$(2.5 + 3.125) = 5.625 \text{ A}$$

Let us verify the above result by applying nodal analysis. The current passing in the 3  $\Omega$  resistor due to both sources should be 5.625 A. Applying nodal analysis to Fig. 3.10, we have

$$\frac{V-20}{5} + \frac{V}{3} = 5$$
$$V\left[\frac{1}{5} + \frac{1}{3}\right] = 5 + 4$$

....





$$V = 9 \times \frac{15}{8} = 16.875 \text{ V}$$

The current passing through the 3  $\Omega$  resistor is equal to V/3

i.e.

$$I = \frac{16.875}{3} = 5.625 \text{ A}$$

So the superposition theorem is verified.

Let us now examine the power responses.

Power dissipated in the 3  $\Omega$  resistor due to voltage source acting alone

$$P_{20} = (I_{20})^2 R = (2.5)^2 3 = 18.75 \text{ W}$$

Power dissipated in the 3  $\Omega$  resistor due to current source acting alone

$$P_5 = (I_5)^2 R = (3.125)^2 3 = 29.29 \text{ W}$$

Power dissipated in the  $3\,\Omega$  resistor when both the sources are acting simultaneously is given by

$$P = (5.625)^2 \times 3 = 94.92$$
 W

From the above results, the superposition of  $P_{20}$  and  $P_5$  gives

$$P_{20} + P_5 = 48.04 \text{ W}$$

which is not equal to P = 94.92 W

We can, therefore, state that the superposition theorem is not valid for power responses. It is applicable only for computing voltage and current responses.

**Example 3.3** Find the voltage across the 2  $\Omega$  resistor in Fig. 3.11 by using the super-position theorem.



Fig. 3.11

**Solution** Let us find the voltage across the 2  $\Omega$  resistor due to individual sources. The algebraic sum of these voltages gives the total voltage across the 2  $\Omega$  resistor.

Our first step is to find the voltage across the 2  $\Omega$  resistor due to the 10 V source, while other sources are set equal to zero.

The circuit is redrawn as shown in Fig. 3.12 (a).



Fig. 3.12

Assuming a voltage V at node 'A' as shown in Fig. 3.12 (a), the current equation is

$$\frac{V-10}{10} + \frac{V}{20} + \frac{V}{7} = 0$$
$$V [0.1 + 0.05 + 0.143] = 1$$
$$V = 3.41 \text{ V}$$

or

The voltage across the 2  $\Omega$  resistor due to the 10 V source is

$$V_2 = \frac{V}{7} \times 2 = 0.97 \text{ V}$$

Our second step is to find out the voltage across the 2  $\Omega$  resistor due to the 20 V source, while the other sources are set equal to zero. The circuit is redrawn as shown in Fig. 3.12 (b).

Assuming voltage V at node A as shown in Fig. 3.12 (b), the current equation is

$$\frac{V-20}{7} + \frac{V}{20} + \frac{V}{10} = 0$$
$$V [0.143 + 0.05 + 0.1] = 2.86$$
$$V = \frac{2.86}{0.293} = 9.76 \text{ V}$$

or

The voltage across the 2  $\Omega$  resistor due to the 20 V source is

$$V_2 = \left(\frac{V - 20}{7}\right) \times 2 = -2.92 \text{ V}$$

The last step is to find the voltage across the 2  $\Omega$  resistor due to the 2 A current source, while the other sources are set equal to zero. The circuit is redrawn as shown in Fig. 3.12 (c).



Fig. 3.12

The current in the 2  $\Omega$  resistor = 2  $\times \frac{5}{5+8.67}$ 

$$=\frac{10}{13.67}=0.73$$
 A

The voltage across the 2  $\Omega$  resistor =  $0.73 \times 2 = 1.46$  V

The algebraic sum of these voltages gives the total voltage across the 2  $\Omega$  resistor in the network

$$V = 0.97 - 2.92 - 1.46 = -3.41$$
 V

The negative sign of the voltage indicates that the voltage at 'A' is negative.

## **3.3 THEVENIN'S THEOREM**

In many practical applications, it is always not necessary to analyse the complete circuit; it requires that the voltage, current, or power in only one resistance of a circuit be found. The use of this theorem provides a simple, equivalent circuit which can be substituted for the original network. Thevenin's theorem states that any two terminal linear network having a number of voltage current sources and resistances can be replaced by a simple equivalent circuit consisting of a single voltage source in series with a resistance, where the value of the voltage source is

equal to the open circuit voltage across the two terminals of the network, and resistance is equal to the equivalent resistance measured between the terminals with all the energy sources are replaced by their internal resistances. According to Thevenin's theorem, an equivalent circuit can be found to replace the circuit in Fig. 3.13.



In the circuit, if the load resistance  $24 \Omega$  is connected to Thevenin's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experienced in the original circuit. To verify this, let us find the current passing through the  $24 \Omega$  resistance due to the original circuit.

$$I_{24} = I_T \times \frac{12}{12 + 24}$$

$$I_T = \frac{10}{2 + (12 || 24)} = \frac{10}{10} = 1 \text{ A}$$

$$I_{24} = 1 \times \frac{12}{12 + 24} = 0.33 \text{ A}$$

÷

where

The voltage across the 24  $\Omega$  resistor = 0.33 × 24 = 7.92 V. Now let us find Thevenin's equivalent circuit.

The Thevenin voltage is equal to the open circuit voltage across the terminals 'AB', i.e. the voltage across the 12  $\Omega$  resistor. When the load resistance is disconnected from the circuit, the Thevenin voltage

$$V_{\rm Th} = 10 \times \frac{12}{14} = 8.57 \, \rm V$$

The resistance into the open circuit terminals is equal to the Thevenin resistance

$$R_{\rm Th} = \frac{12 \times 2}{14} = 1.71 \ \Omega$$



Fig. 3.14

Thevenin's equivalent circuit is shown in Fig. 3.14.

Now let us find the current passing through the 24  $\Omega$  resistance and voltage across it due to Thevenin's equivalent circuit.

$$I_{24} = \frac{8.57}{24 + 1.71} = 0.33 \text{ A}$$

The voltage across the 24  $\Omega$  resistance is equal to 7.92 V. Thus, it is proved that  $R_L$  (= 24  $\Omega$ ) has the same values of current and voltage in both the original circuit and Thevenin's equivalent circuit.

**Example 3.4** Determine the Thevenin's equivalent circuit across 'AB' for the given circuit shown in Fig. 3.15.



Fig. 3.15

**Solution** The complete circuit can be replaced by a voltage source in series with a resistance as shown in Fig. 3.16 (a)

where  $V_{\text{Th}}$  is the voltage across terminals AB and

 $R_{\rm Th}$  is the resistance seen into the terminals AB.

To solve for  $V_{\rm Th}$ , we have to find the voltage drops around the closed path as shown in Fig. 3.16 (b).

We have 50 - 25 = 10I + 5Ior 15I = 25 $\therefore \qquad I = \frac{25}{15} = 1.67 \text{ A}$ 

Voltage across  $10 \Omega = 16.7 V$ Voltage drop across  $5 \Omega = 8.35 V$ 







or

$$= 50 - 16.7 = 33.3$$
 V

 $V_{\rm Th} = V_{AB} = 50 - V_{10}$ 

To find  $R_{\rm Th}$ , the two voltage sources are removed and replaced with short circuit. The resistance at terminals *AB* then is the parallel combination of the 10  $\Omega$  resistor and 5  $\Omega$  resistor; or

$$R_{\rm Th} = \frac{10 \times 5}{15} = 3.33 \ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.16 (c).

## **3.4 NORTON'S THEOREM**

Another method of analysing the circuit is given by *Norton's theorem*, which states that any two terminal linear network with current sources, voltage sources and resistances can be replaced by an equivalent circuit consisting of a current source in parallel with a resistance. The value of the current source is the short circuit current between the two terminals of the network and the resistance is the equivalent resistance measured between the terminals of the network with all the energy sources are replaced by their internal resistance.

According to Norton's theorem, an equivalent circuit can be found to replace the circuit in Fig. 3.17.





Fig. 3.16

----

In the circuit if the load resistance 6  $\Omega$  is connected to Norton's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experiences in the original circuit. To verify this, let us find the current passing through the 6  $\Omega$  resistor due to the original circuit.

$$I_6 = I_T \times \frac{10}{10+6}$$

where

...

$$I_T = \frac{20}{5 + (10 \parallel 6)} = 2.285 \text{ A}$$

$$I_6 = 2.285 \times \frac{10}{16} = 1.43$$
 A

i.e. the voltage across the 6  $\Omega$  resistor is 8.58 V. Now let us find Norton's equivalent circuit. The magnitude of the current in the Norton's equivalent circuit is equal to the current passing through short circuited terminals as shown in Fig. 3.18.



Fig. 3.18

Fig. 3.19

Here

$$I_N = \frac{20}{5} = 4$$
 A

Norton's resistance is equal to the parallel combination of both the 5  $\Omega$  and 10  $\Omega$  resistors

$$R_N = \frac{5 \times 10}{15} = 3.33 \ \Omega$$

The Norton's equivalent source is shown in Fig. 3.19.

Now let us find the current passing through the 6  $\Omega$  resistor and the voltage across it due to Norton's equivalent circuit.

$$I_6 = 4 \times \frac{3.33}{6+3.33} = 1.43$$
 A

The voltage across the 6  $\Omega$  resistor = 1.43 × 6 = 8.58 V

Thus, it is proved that  $R_L (= 6 \Omega)$  has the same values of current and voltage in both the original circuit and Norton's equivalent circuit.

**Example 3.5** Determine Norton's equivalent circuit at terminals *AB* for the circuit shown in Fig. 3.20.





**Solution** The complete circuit can be replaced by a current source in parallel with a single resistor as shown in Fig. 3.21 (a), where  $I_N$  is the current passing through the short circuited output terminals *AB* and  $R_N$  is the resistance as seen into the output terminals.



To solve for  $I_N$ , we have to find the current ssing through the terminals AB as shown in

passing through the terminals AB as shown in Fig. 3.21 Fig. 3.21 (b). From Fig. 3.21 (b), the current passing through the terminals AB is 4 A. The resistance at terminals AB is the parallel combination of the 10  $\Omega$  resistor and the

5  $\Omega$  resistor,

or

$$R_N = \frac{10 \times 5}{10 + 5} = 3.33 \ \Omega$$

Norton's equivalent circuit is shown in Fig. 3.21 (c).



## **3.5 RECIPROCITY THEOREM**

In any linear bilateral network, if a single voltage source  $V_a$  in branch 'a' produces a current  $I_b$  in branch 'b', then if the voltage source  $V_a$  is removed and inserted in branch 'b' will produce a current  $I_b$  in branch 'a'. The ratio of response to excitation is same for the two conditions mentioned above. This is called the *reciprocity theorem*.

Consider the network shown in Fig. 3.22. AA' denotes input terminals and BB' denotes output terminals.



Fig. 3.22

The application of voltage V across AA' produces current I at BB'. Now if the positions of the source and responses are interchanged, by connecting the voltage source across BB', the resultant current I will be at terminals AA'. According to the reciprocity theorem, the ratio of response to excitation is the same in both cases.

**Example 3.6** Verify the reciprocity theorem for the network shown in Fig. 3.23.



Fig. 3.23

**Solution** Total resistance in the circuit =  $2 + [3 || (2 + 2 || 2)] = 3.5 \Omega$ . The current drawn by the circuit (See Fig. 3.24 (a))



Fig. 3.24

The current in the 2  $\Omega$  branch *cd* is I = 1.43 A.

Applying the reciprocity theorem, by interchanging the source and response we get (See Fig. 3.24 (b)).



Fig. 3.24

Total resistance in the circuit =  $3.23 \Omega$ .

Total current drawn by the circuit =  $\frac{20}{3.23}$  = 6.19 A

The current in the branch *ab* is I = 1.43 A

If we compare the results in both cases, the ratio of input to response is the same, i.e. (20/1.43) = 13.99.

#### **3.6 COMPENSATION THEOREM**

The *compensation theorem* states that any element in the linear, bilateral network, may be replaced by a voltage source of magnitude equal to the current passing through the element multiplied by the value of the element, provided the currents and voltages in other parts of the circuit remain unaltered. Consider the circuit shown in Fig. 3.25 (a). The element *R* can be replaced by voltage source *V*, which is equal to the current *I* passing through *R* multiplied by *R* as shown in Fig. 3.25 (b).



Fig. 3.25

This theorem is useful in finding the changes in current or voltage when the value of resistance is changed in the circuit. Consider the network containing a resistance *R* shown in Fig. 3.26 (a). A small change in resistance *R*, that is  $(R + \Delta R)$ , as shown in Fig. 3.26 (b) causes a change in current in all branches. This current increment in other branches is equal to the current produced by the voltage source of voltage *I*.  $\Delta R$  which is placed in series with altered resistance as shown in Fig. 3.26 (c).



Fig. 3.26



**Solution** The current flowing through the 3  $\Omega$  branch is  $I_3 = 1.11$  A. If we connect the ammeter having 1  $\Omega$  resistance to the 3  $\Omega$  branch, there is a change in resistance. The changes in currents in other branches then result as if a voltage source of voltage  $I_3 \Delta R$ = 1.11 × 1 = 1.11 V is inserted in the 3  $\Omega$  branch as shown in Fig. 3.28.

Current due to this 1.11 V source is calculated as follows.

Current 
$$I'_{3} = 0.17$$
 A

This current is opposite to the current  $I_3$  in the 3  $\Omega$  branch.

Hence the ammeter reading = (1.11 - 0.17) = 0.94 A.

# 3.7 MAXIMUM POWER TRANSFER THEOREM

Many circuits basically consist of sources, supplying voltage, current, or power to the load; for example, a radio speaker system, or a microphone supplying the

input signals to voltage pre-amplifiers. Sometimes it is necessary to transfer maximum voltage, current or power from the source to the load. In the simple resistive circuit shown in Fig. 3.29,  $R_s$  is the source resistance. Our aim is to find the necessary conditions so that the power delivered by the source to the load is maximum.



Fig. 3.29

It is a fact that more voltage is delivered to the load when the load resistance is high as compared to the resistance of the source. On the other hand, maximum current is transferred to the load when the load resistance is small compared to the source resistance.

For many applications, an important consideration is the maximum power transfer to the load; for example, maximum power transfer is desirable from the output amplifier to the speaker of an audio sound system. The maximum Power Transfer Theorem states that maximum power is delivered from a source to a load when the load resistance is equal to the source resistance. In Fig. 3.29, assume that the load resistance is variable.



4Ω

Current in the circuit is  $I = V_s/(R_s + R_L)$ Power delivered to the load  $R_L$  is  $P = I^2 R_L = V_s^2 R_L/(R_s + R_L)^2$ 

To determine the value of  $R_L$  for maximum power to be transferred to the load, we have to set the first derivative of the above equation with respect to  $R_I$ , i.e.

when  $\frac{dP}{dR_L}$  equals zero.

$$\frac{dP}{dR_L} = \frac{d}{dR_L} \left[ \frac{V_S^2}{(R_S + R_L)^2} R_L \right]$$
$$= \frac{V_S^2 \left\{ (R_S + R_L)^2 - (2R_L) (R_S + R_L) \right\}}{(R_S + R_L)^4}$$
$$\frac{(R_S + R_L)^2 - 2R_L (R_S + R_L) = 0}{R_S^2 + R_L^2 + 2R_S R_L - 2R_L^2 - 2R_S R_L = 0}$$
$$R_S = R_L$$

So, maximum power will be transferred to the load when load resistance is equal to the source resistance.

**Example 3.8** In the circuit shown in Fig. 3.30 determine the value of load resistance when the load resistance draws maximum power. Also find the value of the maximum power.



Fig. 3.30

**Solution** In Fig. 3.30, the source delivers the maximum power when load resistance is equal to the source resistance.

 $R_L = 25 \ \Omega$ 

The current  $I = 50/(25 + R_L) = 50/50 = 1$  A

The maximum power delivered to the load  $P = I^2 R_L$ 

 $= 1 \times 25 = 25$  W

# **3.8 DUALS AND DUALITY**

In an electrical circuit itself there are pairs of terms which can be interchanged to get new circuits. Such pair of dual terms are given below.

....

*.*..



Consider a network containing R—L—C elements connected in series, and excited by a voltage source as shown in Fig. 3.31.



Fig. 3.31

Fig. 3.32

The integrodifferential equation for the above network is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int idt = V$$

Similarly, consider a network containing R—L—C elements connected in parallel and driven by a current source as shown in Fig. 3.32.

The integrodifferential equation for the network in Fig. 3.32 is

$$i = Gv + C\frac{dv}{dt} + \frac{1}{L}\int vdt$$

If we observe both the equations, the solutions of these two equations are the same. These two networks are called *duals*.

To draw the dual of any network, the following steps are to be followed.

- 1. In each loop of a network place a node; and place an extra node, called the *reference node*, outside the network.
- 2. Draw the lines connecting adjacent nodes passing through each element, and also to the reference node, by placing the dual of each element in the line passing through original elements.

For example, consider the network shown in Fig. 3.33.



Fig. 3.33

Our first step is to place the nodes in each loop and a reference node outside the network.

Drawing the lines connecting the nodes passing through each element, and placing the dual of each element as shown in Fig. 3.34 (a) we get a new circuit as shown in Fig. 3.34 (b).



Fig. 3.34

**Example 9.9** Draw the dual network for the given network shown in Fig. 3.35.



Fig. 3.35

**Solution** Place nodes in each loop and one reference node outside the circuit. Joining the nodes through each element, and placing the dual of each element in the line, we get the dual circuit as shown in Fig. 3.36 (a).



Fig. 3.36 (a)

The dual circuit is redrawn as shown in Fig. 3.36 (b)



Fig. 3.36 (b)

#### **3.9 TELLEGEN'S THEOREM**

Tellegen's theorem is valid for any lumped network which may be linear or nonlinear, passive or active, time-varying or time-invarient. This theorem states that in an arbitrary lumped network, the algebraic sum of the powers in all branches at any instant is zero. All branch currents and voltages in that network must satisfy Kirchhoff's laws. Otherwise, in a given network, the algebraic sum of the powers delivered by all sources is equal to the algebraic sum of the powers absorbed by all elements. This theorem is based on Kirchhoff's two laws, but not on the type of circuit elements.

Consider two networks  $N_1$  and  $N_2$ , having the same graph with different types of elements between the corresponding nodes.

$$\sum_{k=1}^{b} v_{1K} i_{2K} = 0$$
$$\sum_{k=1}^{b} v_{2K} i_{1K} = 0$$

and

To verify Tellegen's theorem, consider two circuits having same graphs as shown in Fig. 3.37.



Fig. 3.37

In Fig. 3.37 (a)

d 
$$i_1 = i_2 = 2 \text{ A}; i_3 = 2 \text{ A}$$
  
 $v_1 = -2 \text{ V}, v_2 = -8 \text{ V}, v_3 = 10 \text{ V}$ 

and

....

In Fig. 3.37 (b)

$$i_1^1 = i_2^1 = 4 \text{ A}; i_3^1 = 4 \text{ A}$$
  
 $v_1^1 = -20 \text{ V}; v_2^1 = 0 \text{ V}; v_3^1 = 20 \text{ V}$ 

Now

and

$$\sum_{K=1}^{n} v_{K} i_{K}^{1} = v_{1} i_{1}^{1} + v_{2} i_{2}^{1} + v_{3} i_{3}^{1}$$
$$= (-2) (4) + (-8) (4) + (10) (4) = 0$$

and

$${}^{1}_{K}i_{K} = v_{1}^{1}i_{1} + v_{2}^{1}i_{2} + v_{3}^{1}i_{3}$$
$$= (-20)(2) + (0)(2) + (20)(2) = 0$$

Similarly,

$$\sum_{K=1}^{3} v_K i_K = v_1 i_1 + v_2 i_2 + v_3 i_3$$
  
= (-2) (2) + (-8) (2) + (10) (2) = 0  
$$\sum_{K=1}^{3} v_K^1 i_K^1 = (-20) (4) + (0) (4) + (20) (4) = 0$$

and

This verifies Tellegen's theorem.

3

## 3.10 MILLMAN'S THEOREM

Millman's Theorem states that in any network, if the voltage sources  $V_1, V_2, \cdots V_n$  in series with internal resistances  $R_1, R_2, \cdots R_n$ , respectively, are in parallel, then these sources may be replaced by a single voltage source V'in series with R' as shown in Fig. 3.38.



Fig. 3.38

where

$$V = \frac{V_1 G_1 + V_2 G_2 + \dots + V_n G_n}{G_1 + G_2 + \dots + G_n}$$

Here  $G_n$  is the conductance of the *n*th branch,

$$R' = \frac{1}{G_1 + G_2 + \dots + G_n}$$

and

A similar theorem can be stated for n current sources having internal conductances which can be replaced by a single current source I' in parallel with an equivalent conductance.



Fig. 3.39

where

$$I' = \frac{I_1 R_1 + I_2 R_2 + \dots + I_n R_n}{R_1 + R_2 + \dots + R_n}$$

and

$$G' = \frac{1}{R_1 + R_2 + \dots + R_n}$$

**Example 3.10** Calculate the current *I* shown in Fig. 3.40 using Millman's Theorem.



Fig. 3.40

**Solution** According to Millman's Theorem, the two voltage sources can be replaced by a single voltage source in series with resistance as shown in Fig. 3.41.

We have 
$$V' = \frac{V_1 G_1 + V_2 G_2}{G_1 + G_2}$$
  
=  $\frac{[10(1/2) + 20(1/5)]}{1/2 + 1/5} = 12.86 \text{ V}$  V'  
and  $R' = \frac{1}{G_1 + G_2} = \frac{1}{1/2 + 1/5} = 1.43 \Omega$   
Fig. 3.41

Therefore, the current passing through the 3  $\boldsymbol{\Omega}$  resistor is

$$I = \frac{12.86}{3 + 1.43} = 2.9 \text{ A}$$



**Problem 3.1** Determine the current drawn by the circuit shown in Fig. 3.42.



Fig. 3.42

**Solution** To simplify the network, the star circuit in Fig. 3.42 is converted into a delta circuit as shown under.



Fig. 3.43

$$R_1 = \frac{4 \times 3 + 4 \times 2 + 3 \times 2}{2} = 13 \Omega$$

$$R_2 = \frac{4 \times 3 + 4 \times 2 + 3 \times 2}{4} = 6.5 \Omega$$

$$R_3 = \frac{4 \times 3 + 4 \times 2 + 3 \times 2}{3} = 8.7 \Omega$$

The original circuit is redrawn as shown in Fig. 3.43 (b).



Fig. 3.43

It is further simplified as shown in Fig. 3.43 (c). Here the resistors 5  $\Omega$  and 13  $\Omega$  are in parallel, 6  $\Omega$  and 6.5  $\Omega$  are in parallel, and 8.7  $\Omega$  and 2  $\Omega$  are in parallel.



In the above circuit the resistors 6  $\Omega$  and 1.6  $\Omega$  are in parallel, the resultant of which is in series with 3.6  $\Omega$  resistor and is equal to  $\left[3.6 + \frac{6 \times 1.6}{7.6}\right] = 4.9 \Omega$  as shown in Fig. 3.43 (d).



In the above circuit 4.9  $\Omega$  and 3.1  $\Omega$  resistors are in parallel, the resultant of which is in series with 3  $\Omega$  resistor.

Therefore, the total resistance  $R_T = 3 + \frac{3.1 \times 4.9}{8} = 4.9 \Omega$ 

The current drawn by the circuit  $I_r = 50/4.9 = 10.2$  A (See Fig. 3.43 (e)).

**Problem 3.2** In Fig. 3.44 determine the equivalent resistance by using stardelta transformation.



Fig. 3.44

**Solution** In Fig. 3.44, we have two star circuits, one consisting of 5  $\Omega$ , 3  $\Omega$  and 4  $\Omega$  resistors, and the other consisting of 6  $\Omega$ , 4  $\Omega$  and 8  $\Omega$  resistors. We convert the star circuits into delta circuits, so that the two delta circuits are in parallel.

In Fig. 3.45 (a)

$$R_{1} = \frac{5 \times 3 + 4 \times 3 + 5 \times 4}{4} = 11.75 \ \Omega$$
$$R_{2} = \frac{5 \times 3 + 4 \times 3 + 5 \times 4}{3} = 15.67 \ \Omega$$
$$R_{3} = \frac{5 \times 3 + 4 \times 3 + 5 \times 4}{5} = 9.4 \ \Omega$$

Similarly, in Fig. 3.45 (b)



(a)



Fig. 3.45

The simplified circuit is shown in Fig. 3.45 (c)





In the above circuit, the three resistors  $10 \Omega$ , 9.4  $\Omega$  and 17.3  $\Omega$  are in parallel. Equivalent resistance =  $(10 \parallel 9.4 \parallel 17.3) = 3.78 \Omega$ 

Resistors 13  $\Omega$  and 11.75  $\Omega$  are in parallel

Equivalent resistance =  $(13 \parallel 11.75) = 6.17 \Omega$ 

Resistors 26  $\Omega$  and 15.67  $\Omega$  are in parallel

Equivalent resistance =  $(26 \parallel 15.67) = 9.78 \Omega$ 

The simplified circuit is shown in Fig. 3.45 (d)





From the above circuit, the equivalent resistance is given by

$$R_{eq} = (9.78) \parallel (6.17 + 3.78) \\= (9.87) \parallel (9.95) = 4.93 \ \Omega$$

**Problem 3.3** For the resistive network shown in Fig. 3.46, find the current in each resistor, using the superposition principle.



Fig. 3.46

**Solution** The current due to the 50 V source can be found in the circuit shown in Fig. 3.47 (a).



Total resistance 
$$R_T = 10 + \frac{5 \times 3}{8} = 11.9 \Omega$$
  
Current in the 10  $\Omega$  resistor  $I_{10} = \frac{50}{11.9} = 4.2 \text{ A}$   
Current in the 3  $\Omega$  resistor  $I_3 = 4.2 \times \frac{5}{8} = 2.63 \text{ A}$   
Current in the 5  $\Omega$  resistor  $I_5 = 4.2 \times \frac{3}{8} = 1.58 \text{ A}$ 

The current due to the 25 V source can be found from the circuit shown in Fig. 3.47 (b).

Total resistance 
$$R_T = 5 + \frac{10 \times 3}{13} = 7.31 \Omega$$
  
Current in the 5  $\Omega$  resistor  $I'_5 = \frac{25}{7.31} = 3.42 \text{ A}$ 

Current in the 3  $\Omega$  resistor  $I'_3 = 3.42 \times \frac{10}{13} = 2.63$  A

Current in the 10  $\Omega$  resistor  $I'_{10} = 3.42 \times \frac{3}{13} = 0.79$  A

According to superposition principle

Current in the 10  $\Omega$  resistor =  $I_{10} - I'_{10} = 4.2 - 0.79 = 3.41$  A Current in the 3  $\Omega$  resistor =  $I_3 + I'_3 = 2.63 + 2.63 = 5.26$  A Current in the 5  $\Omega$  resistor =  $I'_5 - I_5 = 3.42 - 1.58 = 1.84$  A

When both sources are operative, the directions of the currents are shown in Fig. 3.47 (c).



Fig. 3.47

....
**Problem 3.4** Determine the voltage across the terminals *AB* in the circuit shown in Fig. 3.48.



Fig. 3.48

**Solution** Voltage across AB is  $V_{AB} = V_{10} + V_5$ .

To find the voltage across the 5  $\Omega$  resistor, we have to use the superposition theorem.

Voltage across the 5  $\Omega$  resistor  $V_5$  due to the 6 V source, when other sources are set equal to zero, is calculated using Fig. 3.49 (a).





$$V_{5} = 6 \text{ V}$$

Voltage across the 5  $\Omega$  resistor  $V'_5$  due to the 10 V sources, when other sources are set equal to zero, is calculated using Fig. 3.49 (b).

$$V'_{5} = 0$$

Voltage across the 5  $\Omega$  resistor  $V_5''$  due to the 5 A source only, is calculated using Fig. 3.49 (c).

$$V_{5}'' = 0$$

According to the superposition  $_{5A}$  (theorem,

Total voltage across the 5  $\Omega$  resistor

$$= 6 + 0 + 0 = 6 V$$

So the voltage across terminals AB is

$$V_{AB} = 10 + 6 = 16$$
 V



Fig. 3.49

\*\*

**Problem 3.5** Use Thevenin's theorem to find the current in 3  $\Omega$  resistor in Fig. 3.50.

**Solution** Current in the 3  $\Omega$  resistor can be found by using Thevenin's theorem.



Fig. 3.50

In circuit shown in Fig. 3.51 (a) can be replaced by a single voltage source in series with a resistor as shown in Fig. 3.51 (b).



Fig. 3.51

$$V_{\rm Th} = V_{AB} = \frac{50}{15} \times 10 = 33.3 \text{ V}$$

 $R_{\rm Th} = R_{_{AB}}$ , the resistance seen into the terminals AB

$$R_{AB} = 2 + \frac{5 \times 10}{15} = 5.33 \ \Omega$$

The 3  $\Omega$  resistor is connected to the Thevenin equivalent circuit as shown in Fig. 3.51 (c).

Current passing through the 3  $\Omega$  resistor



Fig. 3.51

**Problem 3.6** Use Thevenin's theorem to find the current through the 5  $\Omega$  resistor in Fig. 3.52.



Fig. 3.52

**Solution** Thevenin's equivalent circuit can be formed by obtaining the voltage across terminals *AB* as shown in Fig. 3.53 (a).





Current in the 6  $\Omega$  resistor  $I_6 = \frac{100}{16} = 6.25$  A Voltage across the 6  $\Omega$  resistor  $V_6 = 6 \times 6.25 = 37.5$  V Current in the 8  $\Omega$  resistor  $I_8 = \frac{100}{23} = 4.35$  A Voltage across the 8  $\Omega$  resistor is  $V_8 = 4.35 \times 8 = 34.8$  V Voltage across the terminals *AB* is  $V_{AB} = 37.5 - 34.8 = 2.7$  V The resistance as seen into the terminals  $R_{AB}$ 

$$= \frac{6 \times 10}{6+10} + \frac{8 \times 15}{8+15}$$
$$= 3.75 + 5.22 = 8.97 \ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.53 (b).

Current in the 5  $\Omega$  resistor  $I_5 = \frac{2.7}{5+8.97} = 0.193$  A

**Problem 3.7** Find Thevenin's equivalent circuit for the circuit shown in Fig. 3.54.





**Solution** The venin's voltage is equal to the voltage across the terminals AB.  $\therefore \qquad V_{AB} = V_3 + V_6 + 10$ 

Here the current passing through the 3  $\Omega$  resistor is zero.

Hence  $V_3 = 0$ 

By applying Kirchhoff's law we have



Fig. 3.55

$$50 - 10 = 10I + 6I$$
  
 $I = \frac{40}{16} = 2.5 \text{ A}$ 

The voltage across 6  $\Omega$  is  $V_6$  with polarity as shown in Fig. 3.55 (a), and is given by

$$V_6 = 6 \times 2.5 = 15 \text{ V}$$

The voltage across terminals AB is  $V_{AB} = 0 + 15 + 10 = 25$  V. The resistance as seen into the terminals AB

$$R_{AB} = 3 + \frac{10 \times 6}{10 + 6} = 6.75 \ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.55 (b).

**Problem 3.8** Determine the Thevenin's equivalent circuit across terminals *AB* for the circuit in Fig. 3.56.

22





**Solution** The given circuit is redrawn as shown in Fig. 3.57 (a). Voltage  $V_{AB} = V_2 + V_1$ 

Applying Kirchhoff's voltage law to loop 1 and loop 2, we have the following Voltage across the 2  $\Omega$  resistor  $V_2 = 2 \times \frac{10}{7} = 2.85$  V

Voltage across the 1  $\Omega$  resistor  $V_1 = 1 \times \frac{5}{5} = 1$  V



$$V_{AB} = V_2 + V_1$$
  
= 2.85 - 1 = 1.85 V

*:*..

The resistance seen into the terminals AB

$$R_{AB} = (5 || 2) + (4 || 1)$$

$$= \frac{5 \times 2}{5 + 2} + \frac{4 \times 1}{4 + 1}$$

$$= 1.43 + 0.8 = 2.23 \Omega$$
Fig. 3.57

Thevenins's equivalent circuit is shown in Fig. 3.57 (b).

**Problem 3.9** Determine Norton's equivalent circuit for the circuit shown in Fig. 3.58.



Fig. 3.58

••

*∾* B

• A

2.23 Ω

**Solution** Norton's equivalent circuit is given by Fig. 3.59 (a).

 $I_N$  = Short circuit current at terminals AB $R_N$  = Open circuit resistance at terminals ABwhere

The current  $I_N$  can be found as shown in Fig. 3.59 (b).

$$I_N = \frac{50}{3} = 16.7 \text{ A}$$

Norton's resistance can be found from Fig. 3.59 (c)

$$R_N = R_{AB} = \frac{3 \times 4}{3 + 4} = 1.71 \ \Omega$$

Norton's equivalent circuit for the given circuit is shown in Fig. 3.59 (d).



Fig. 3.59

-----

Problem 3.10 Determine Norton's equivalent circuit for the given circuit shown in Fig. 3.60.





Solution The short circuit current at terminals AB can be found from Fig. 3.61 (a) and Norton's resistance can be found from Fig. 3.61 (b).



Fig. 3.61

The current  $I_{N}$  is same as the current in the 3  $\Omega$  resistor or 4  $\Omega$  resistor.



Norton's equivalent circuit is shown in Fig. 3.61 (c).

**Problem 3.11** Determine the current flowing through the 5  $\Omega$  resistor in the circuit shown in Fig. 3.62 by using Norton's theorem.



Fig. 3.62

Solution The short circuit current at terminals AB can be found from the circuit as shown in Fig. 3.63 (a). Norton's resistance can be found from Fig. 3.63 (b).







Norton's equivalent circuit is shown in Fig. 3.63 (c)

... The current in the 5  $\Omega$  resistor

$$I_5 = 30 \times \frac{1.67}{6.67} = 7.51 \text{ A}$$

**Problem 3.12** Replace the given network shown in Fig 3.64 by a single current source in parallel with a resistance.



Fig. 3.64

**Solution** Here, using superposition technique and Norton's theorem, we can convert the given network.

We have to find a short circuit current at terminals AB in Fig. 3.65 (a) as shown

The current  $I'_N$  is due to the 10 A source.  $I'_N = 10$  A

The current  $I''_{N}$  is due to the 20 V source (See Figs 3.65 (b) and (c))

$$I_N'' = \frac{20}{6} = 3.33 \text{ A}$$

The current  $I_N$  is due to both the sources



Fig. 3.65

$$I_N = I'_N + I''_N$$
  
= 10 + 3.33 = 13.33 A

The resistance as seen from terminals AB

 $R_{AB} = 6 \Omega$  (from the Fig. 3.65 (d))

Hence, the required circuit is as shown in Fig. 3.65 (e).



Fig. 3.65

\*\*

**Problem 3.13** Using the compensation theorem, determine the ammeter reading where it is connected to the 6  $\Omega$  resistor as shown in Fig. 3.66. The internal resistance of the ammeter is 2  $\Omega$ .

**Solution** The current flowing through the 5  $\Omega$  branch



Fig. 3.66

So the current in the 6  $\Omega$  branch

$$I_6 = 6.315 \times \frac{2}{6+2} = 1.58 \text{ A}$$

If we connect the ammeter having 2  $\Omega$  internal resistance to the 6  $\Omega$  branch, there is a change in resistance. The changes in currents in other branches results if a voltage source of voltage  $I_6 \Delta R = 1.58 \times 2 = 3.16$  V is inserted in the 6  $\Omega$  branch as shown in Fig. 3.67.



Fig. 3.67

The current due to this 3.16 V source is calculated. The total impedance in the circuit

$$R_T = \{ [(6 \parallel 3) + 5] \parallel [2] \} + \{6 + 2\} \\= 9.56 \ \Omega$$

The current due to 3.16 V source

$$I_6' = \frac{3.16}{9.56} = 0.33 \text{ A}$$

This current is opposite to the current  $I_6$  in the 6  $\Omega$  branch.

Hence, the ammeter reading = (1.58 - 0.33)

**Problem 3.14** Verify the reciprocity theorem for the given circuit shown in Fig. 3.68.



Fig. 3.68

**Solution** In Fig. 3.68, the current in the 5  $\Omega$  resistor is

 $I_5 = I_2 \times \frac{4}{8+4} = 2.14 \times \frac{4}{12} = 0.71 \text{ A}$  $I_2 = \frac{10}{R_T}$ where  $R_T = 4.67$  $I_2 = \frac{10}{4.67} = 2.14 \text{ A}$ ...

and

We interchange the source and response as shown in Fig. 3.69.



Fig. 3.69

In Fig. 3.69, the current in 2  $\Omega$  resistor is

where

and

Ŀ.

$$R_T = 9.33 \ \Omega$$
  
 $I_3 = \frac{10}{9.33} = 1.07 \ A$ 

 $I_2 = I_3 \times \frac{4}{4+2}$ 

 $I_3 = \frac{10}{R_T}$ 

$$I_2 = 1.07 \times \frac{4}{6} = 0.71 \text{ A}$$

In both cases, the ratio of voltage to current is  $\frac{10}{0.71} = 14.08$ .

Hence the reciprocity theorem is verified.

**Problem 3.15** Verify the reciprocity theorem in the circuit shown in Fig. 3.70.



Fig. 3.70

**Solution** The voltage V across the 3  $\Omega$  resistor is

 $V = I_3 \times R$  $I_3 = 10 \times \frac{2}{2+3} = 4 \text{ A}$  $V = 4 \times 3 = 12 \text{ V}$ 

where

*:*.

We interchange the current source and response as shown in Fig. 3.71.



#### Fig. 3.71

\*\*

To find the response, we have to find the voltage across the 2  $\Omega$  resistor

 $V = I_2 \times R$  $I_2 = 10 \times \frac{3}{5} = 6$  A

where

*:*..

$$V = 6 \times 2 = 12 \text{ V}$$

In both cases, the ratio of current to voltage is the same, i.e. it is equal to 0.833. Hence the reciprocity theorem is verified.

**Problem 3.16** Determine the maximum power delivered to the load in the circuit shown in Fig. 3.72.



Fig. 3.72

Solution For the given circuit, let us find out the Thevenin's equivalent circuit across AB as shown in Fig. 3.73 (a).

The total resistance is

$$R_T = [\{(3+2) \parallel 5\} + 10] \\ = [2.5+10] = 12.5 \ \Omega$$

Total current drawn by the circuit is

$$I_T = \frac{50}{12.5} = 4 \text{ A}$$

The current in the 3  $\Omega$  resistor is

$$I_3 = I_T \times \frac{5}{5+5} = \frac{4 \times 5}{10} = 2$$
 A

The venin's voltage  $V_{AB} = V_3 = 3 \times 2 = 6$  V The venin's resistance  $R_{Th} = R_{AB} = [((10 \parallel 5) + 2) \parallel 3] \Omega = 1.92 \Omega$ Thevenin's equivalent circuit is shown in Fig. 3.73 (b).



Fig. 3.73

From Fig. 3.73 (b), and maximum power transfer theorem

$$R_L = 1.92 \ \Omega$$

 $\therefore$  Current drawn by load resistance  $R_L$ 

$$I_L = \frac{6}{1.92 + 1.92} = 1.56 \text{ A}$$
  
Power delivered to the load  $= I_L^2 R_L$   
 $= (1.56)^2 \times 1.92 = 4.67 \text{ W}$ 

**Problem 3.17** Determine the load resistance to receive maximum power from the source; also find the maximum power delivered to the load in the circuit shown in Fig. 3.74.



Fig. 3.74

**Solution** For the given circuit, we find out the Thevenin's equivalent circuit. Thevenin's voltage across terminals *A* and *B* 



Fig. 3.75

Voltage at point A is  $V_A = 100 \times \frac{30}{30 + 10} = 75 \text{ V}$ 

Voltage at point *B* is  $V_{B} = 100 \times \frac{40}{40 + 20} = 66.67 \text{ V}$ 

*:*..

$$V_{AB} = 75 - 66.67 = 8.33$$
 V

To find Thevenin's resistance the circuit in Fig. 3.75 (a) can be redrawn as shown in Fig. 3.75 (b).





From Fig. 3.75 (b), Thevenin's resistance  $R_{AB} = [(30 || 10) + (20 || 40)]$  $= [7.5 + 13.33] = 20.83 \Omega$ 

Thevenin's equivalent circuit is shown in Fig. 3.75 (c).





According to maximum power transfer theorem

$$R_L = 20.83 \ \Omega$$

Current drawn by the load resistance

$$I_L = \frac{8.33}{20.83 + 20.83} = 0.2 \text{ A}$$

: Maximum power delivered to load =  $I_L^2 R_L$ =  $(0.2)^2 (20.83) = 0.833$  W

Problem 3.18 Draw the dual circuit for the given circuit shown in Fig. 3.76.



Fig. 3.76

**Solution** Our first step is to place nodes in each loop, and a reference node outside the circuit.

Join the nodes with lines passing through each element and connect these lines with dual of each element as shown in Fig. 3.77 (a).

The dual circuit of the given circuit is shown in Fig. 3.77 (b).



Fig. 3.77





Fig. 3.78

**Solution** Our first step is to mark nodes in each of the loop and a reference node outside the circuit.

Join the nodes with lines passing through each element and connect these lines with dual of each element as shown in Fig. 3.79 (a).

The dual circuit of given circuit is shown in Fig. 3.79 (b).

3.41



Fig. 3.79

**Problem 3.20** For the circuit shown in Fig. 3.80, find the current  $i_4$  using the superposition principle.



Fig. 3.80

**Solution** The circuit can be redrawn as shown in Fig. 3.81 (a).

The current  $i'_4$  due to the 20 V source can be found using the circuit shown in Fig. 3.81 (b).

Applying Kirchhoff's voltage law

$$-20 + 4i'_4 + 2i'_4 + 2i'_4 = 0$$
  
 $i'_4 = 2.5 \text{ A}$ 



Fig. 3.81

The current  $i''_{4}$  due to the 5 A source can be found using the circuit shown in Fig. 3.81 (c).

By assuming V'' at node shown in Fig. 3.81 (c) and applying Kirchhoff's current law



**Problem 3.21** Determine the current through the 2  $\Omega$  resistor as shown in the Fig 3.82 by using the superposition theorem.



Fig. 3.82

**Solution** The current I' due to the 5 V source can be found using the circuit shown in Fig. 3.83 (a).



Fig. 3.83

By applying Kirchhoff's voltage law, we have

$$3I' + 5 + 2I' - 4V'_3 = 0$$

we know

From the above equations

I' = -0.294 A

 $V'_{3} = -3I'$ 

The current I'' due to the 4 A source can be found using the circuit shown in Fig. 3.83 (b).

By assuming node voltage  $V''_{3}$ , we find

$$I' = \frac{V_3'' + 4V_3''}{2}$$

By applying Kirchhoff's current law at node we have

$$\frac{V_3''}{3} - 4 + \frac{V_3'' + 4V_3''}{2} = 0$$
$$V_3'' = 1.55 \text{ V}$$
$$I'' = \frac{V_3'' + 4V_3''}{2} = 3.875 \text{ A}$$

÷

Total current in the 2  $\Omega$  resistor I = I' + I'' = -0.294 + 3.875

 $\therefore \qquad I = 3.581 \text{ A}$ 

**Problem 3.22** For the circuit shown in Fig. 3.84, obtain Thevenin's equivalent circuit.



Fig. 3.84

**Solution** The circuit consists of a dependent source. In the presence of dependent source  $R_{Th}$  can be determined by finding  $v_{OC}$  and  $i_{SC}$ 

*.*..

$$R_{\rm Th} = \frac{v_{OC}}{i_{SC}}$$

Open circuit voltage can be found from the circuit shown in Fig. 3.85 (a) Since the output terminals are open, current passes through the 2  $\Omega$  branch only.

$$v_x = 2 \times 0.1 \ v_x + 4$$
$$v_x = \frac{4}{0.8} = 5 \ V$$

Short circuit current can be calculated from the circuit shown in Fig. 3.85 (b).





Since  $v_x = 0$ , dependent current source is opened. 6.25 Ω The current  $i_{SC} = \frac{4}{2+3} = 0.8 \text{ A}$ 5 V ( +  $R_{\rm Th} = \frac{v_{OC}}{i_{\rm SC}} = \frac{5}{0.8} = 6.25 \ \Omega$ ... (c)

The Thevenin's equivalent circuit is shown in Fig. 3.85 (c).

Fig. 3.85

**Problem 3.24** For the circuit shown in Fig. 3.86, find the current  $i_2$  in the 2  $\Omega$ resistor by using Thevenin's theorem.



Fig. 3.86

Solution From the circuit, there is open voltage at terminals *ab* which is

where *:*..

$$V_{OC} = -4 V_i$$
$$V_i = -4V_i - 5$$
$$V_i = -1$$

The venin's voltage  $V_{OC} = 4$  V

From the circuit, short circuit current is determined by shorting terminals a and b. Applying Kirchhoff's voltage law, we have



The Thevenin's equivalent circuit is as shown in Fig. 3.87.

The current in the 2  $\Omega$  resistor  $i_2 = \frac{4}{2.4} = 1.67$  A

**Problem 3.25** For the circuit shown in Fig. 3.88, find Norton's equivalent circuit.



Fig. 3.88

**Solution** In the case of circuit having only dependent sources (without independent sources), both  $V_{oc}$  and  $i_{sc}$  are zero. We apply a 1 A source externally

and determine the resultant voltage across it, and then find  $R_{\text{Th}} = \frac{V}{1}$  or we can

also apply the 1 V source externally and determine the current through it and then we find  $R_{\rm Th} = 1/i$ .

By applying the 1 A source externally as shown in Fig. 3.89 (a).



Fig. 3.89

and application of Kirchhoff's current law, we have

\_

$$\frac{V_x}{5} + \frac{V_x + 4V_x}{2} = 1$$
  
V\_x = 0.37 V

The current in the 4  $\Omega$  branch is

$$\frac{V_x - V}{4} = -1$$

Substituting  $V_x$  in the above equation, we get

$$V = 4.37 \text{ V}$$
$$\therefore \qquad R_{\text{Th}} = \frac{V}{1} = 4.37 \Omega$$

If we short circuit the terminals *a* and *b* we have

$$\frac{V_x - 4V_x}{2} = 0$$
$$V_x = 0$$
$$I_{SC} = \frac{V_x}{4} = 0$$

Therefore, Norton's equivalent circuit is as shown in Fig. 3.89 (b).



- 3.1 For the bridge network shown in Fig. 3.90, determine the total resistance seen from terminals *AB* by using star-delta transformation.
- 3.2 Calculate the voltage across *AB* in the network shown in Fig. 3.91 and indicate the polarity of the voltage using star-delta transformation.



3.3 Find the current *I* in the circuit shown in Fig. 3.92 by using the superposi-



Fig. 3.92

3.4 Determine the current *I* in the circuit shown in Fig. 3.93 using the superposition theorem.



3.5 Calculate the new current in the circuit shown in Fig. 3.94 when the resistor  $R_3$  is increased by 30%.



Fig. 3.94

3.6 Find the Thevenin's and Norton's equivalents for the circuit shown in Fig. 3.95 with respect to terminals *ab*.





3.7 Determine the Thevenin and Norton's equivalent circuits with respect to terminals *ab* for the circuit shown in Fig. 3.96.



Fig. 3.96

3.8 By using source transformation or any other technique, replace the circuit shown in Fig. 3.97 between terminals *ab* with the voltage source in series with a single resistor.



Fig.	3.97	
1 12.	5.71	

3.9 For the circuit shown in Fig. 3.98, what will be the value of  $R_L$  to get the maximum power? What is the maximum power delivered to the load? What is the maximum voltage across the load? What is the maximum current in it?





- 3.10 For the circuit shown in Fig. 3.99 determine the value of  $R_L$  to get the maximum power. Also find the maximum power transferred to the load.
- 3.11 The circuit shown in Fig. 3.100 consists of dependent source. Use the superposition theorem to find the current *I* in the 3  $\Omega$  resistor.



3.12 Obtain the current passing through 2  $\Omega$  resistor in the circuit shown in Fig. 3.101 by using the superposition theorem.



Fig. 3.101

- 3.13 Determine the current passing through 2  $\Omega$  resistor by using Thevenin's theorem in the circuit shown in Fig. 3.102.
- 3.14 Find Thevenin's equivalent circuit for the network shown in Fig. 3.103 and hence find the current passing through the  $10 \Omega$  resistor.



Fig. 3.102



3.15 Obtain Norton's equivalent circuit of the network shown in Fig. 3.104.







- 1. Three equal resistance of 3  $\Omega$  are connected in star. What is the resistance in one of the arms in an equivalent delta circuit?
  - (a)  $10 \Omega$  (b)  $3 \Omega$

(c) 9 Ω	(d) 27 Ω
---------	----------

- 2. Three equal resistances of 5  $\Omega$  are connected in delta. What is the resistance in one of the arms of the equivalent star circuit?
  - (a)  $5 \Omega$  (b)  $1.33 \Omega$
  - (c)  $15 \Omega$  (d)  $10 \Omega$
- 3. Superposition theorem is valid only for
  - (a) linear circuits
  - (b) non-linear circuits
  - (c) both linear and non-linear
  - (d) neither of the two
- 4. Superposition theorem is not valid for
  - (a) voltage responses
- (b) current responses
- (c) power responses (d) all the three

5. Determine the current I in the circuit shown in Fig. 3.105. It is





(a) 2.5 A

(c) 3.5 A



- 6. Reduce the circuit shown in Fig. 3.106 to its Thevenin equivalent circuit as viewed from terminal *A* and *B*.
  - (a) The circuit consists of 15 V battery in series with  $100 \text{ k}\Omega$
  - (b) The circuit consists of 15 V  $^{15 \text{ V}}$  battery in series with 22 k $\Omega$
  - (c) The circuit consists of 15 V battery in series with parallel combination of 100 kΩ and 22 kΩ







- 7. Norton's equivalent circuit consists of
  - (a) voltage source in parallel with impedance
  - (b) voltage source in series with impedance
  - (c) current source in series with impedance
  - (d) current source in parallel with impedance
- 8. The reciprocity theorem is applicable to
  - (a) linear networks only
  - (c) linear/bilateral networks
- 9. Compensation theorem is applicable to
  - (a) linear networks only (b) non-li
  - (c) linear and non-linear networks (d) neither of the two
- 10. Maximum power is transferred when load impedance is
  - (a) equal to source impedance
  - (b) equal to half of the source impedance
  - (c) equal to zero
  - (d) none of the above
- 11. In the circuit shown in Fig. 3.107, what is the maximum power 10 V ( transferred to the load
  - (a) 5 W (b) 2.5 W
  - (c) 10 W (d) 25 W



(b) non-linear networks only



2.5 Ω



3.51

Fig. 3.107

- 12. Indicate the dual of series network consists of voltage source, capacitance, inductance in
  - (a) parallel combination of resistance, capacitance and inductance
  - (b) series combination of current source, capacitance and inductance.
  - (c) parallel combination of current source, inductance and capacitance
  - (d) none of the above
- 13. When the superposition theorem is applied to any circuit, the dependent voltage source in that circuit is always
  - (a) opened

(b) shorted

(c) active

- (d) none of the above
- 14. Superposition theorem is not applicable to networks containing.
  - (a) non-linear elements
  - (b) dependent voltage sources
  - (c) dependent current sources
  - (d) transformers
- 15. Thevenins voltage in the circuit shown in Fig. 3.108 is
  - (a) 3 V
  - (b) 2.5 V
  - (c) 2 V
  - (d) 0.1 V
- 16. Norton's current in the circuit shown in Fig. 3.109 is

(a) 
$$\frac{2i}{5}$$

- (b) zero
- (c) infinite
- (d) None

Fig. 3.108



Fig. 3.109

17. A dc circuit shown in Fig. 3.110 has a voltage V, a current source *I* and several resistors. A particular resistor *R* dissipates a power of 4 W when *V* alone is active. The same resistor dissipates a power of 9 W when *I* alone is active. The power dissipated by *R* when both sources are active will be



Fig. 3.110

(a)	1 W	(b)	5 W
(c)	13 W	(d)	25 W



# 4.1 THE SINE WAVE

Many a time, alternating voltages and currents are represented by a sinusoidal wave, or simply a sinusoid. It is a very common type of alternating current (ac) and alternating voltage. The sinusoidal wave is generally referred to as a sine wave. Basically an alternating voltage (current) waveform is defined as the voltage (current) that fluctuates with time periodically, with change in polarity and direction. In general, the sine wave is more useful than other waveforms, like pulse, sawtooth, square, etc. There are a number of reasons for this. One of the reasons is that if we take any second order system, the response of this system is a sinusoid. Secondly, any periodic waveform can be written in terms of sinusoidal function according to Fourier theorem. Another reason is that its derivatives and integrals are also sinusoids. A sinusoidal function is easy to analyse. Lastly, the sinusoidal function is easy to generate, and it is more useful in the power industry. The shape of a

sinusoidal waveform is shown in Fig. 4.1. The waveform may be either a current waveform, or a voltage waveform. As seen from the Fig. 4.1, the wave changes its magnitude and direction with time. If we start at time t = 0, the wave goes to a maximum value and returns to zero, and then decreases to a negative maximum value before returning to zero. The sine wave changes with time in an orderly manner. During the positive portion of voltage, the



Fig. 4.1

current flows in one direction; and during the negative portion of voltage, the current flows in the opposite direction. The complete positive and negative portion of the wave is one cycle of the sine wave. Time is designated by t. The time taken for any wave to complete one full cycle is called the period (T). In general, any periodic wave constitutes a number of such cycles. For example, one cycle of a sine wave repeats a number of times as shown in Fig. 4.2. Mathematically it can be represented as f(t) = f(t + T) for any t.



Fig. 4.2

The period can be measured in the following different ways (See Fig. 4.3).

- From zero crossing of one cycle to zero crossing of the next cycle.
- 2. From positive peak of one cycle to positive peak of the next cycle, and
- 3. From negative peak of one cycle to negative peak of the next cycle.

The frequency of a wave is defined as the number of cycles that a wave completes in one second. V (volts)



In Fig. 4.4 the sine wave completes three cycles in one second. Frequency is measured in hertz. One hertz is equivalent to one cycle per second, 60 hertz is 60 cycles per second and so on. In Fig. 4.4, the frequency denoted by f is 3 Hz,



Fig. 4.4

that is three cycles per second. The relation between time period and frequency is given by

$$f = \frac{1}{T}$$

A sine wave with a longer period consists of fewer cycles than one with a shorter period.

**Example 4.1** What is the period of sine wave shown in Fig. 4.5?



#### Fig. 4.5

**Solution** From Fig. 4.5, it can be seen the sine wave takes two seconds to complete one period in each cycle

$$T = 2 \mathrm{s}$$

**Example 4.2** The period of a sine wave is 20 milliseconds. What is the frequency.

Solution

$$f = \frac{1}{T}$$
$$= \frac{1}{20 \text{ ms}} = 50 \text{ Hz}$$

**Example 4.3** The frequency of a sine wave is 30 Hz. What is its period.

Solution

$$= \frac{1}{f}$$
  
=  $\frac{1}{30} = 0.03333 \text{ s}$   
= 33.33 ms

# 4.2 ANGULAR RELATION OF A SINE WAVE

Т

A sine wave can be measured along the X-axis on a time base which is frequency-dependent. A sine wave can also be expressed in terms of an angular measurement. This angular measurement is expressed in degrees or radians. A radian is defined as the angular distance measured along the circumference of a circle which is equal to the radius of the circle. One radian is equal to 57.3°. In a 360° revolution, there are  $2\pi$  radians. The angular measurement of a sine wave is based on 360° or  $2\pi$  radians for a complete cycle as shown in Figs. 4.6 (a) and (b).



Fig. 4.6

A sine wave completes a half cycle in 180° or  $\pi$  radians; a quarter cycle in 90° or  $\pi/2$  radians, and so on.

#### Phase of a Sine Wave

The phase of a sine wave is an angular measurement that specifies the position of the sine wave relative to a reference. The wave shown in Fig. 4.7 is taken as the reference wave.

When the sine wave is shifted left or right with reference to the wave shown in Fig. 4.7, there occurs a phase shift. Figure 4.8 shows the phase shifts of a sine wave.

In Fig. 4.8(a), the sine wave is shifted to the right by 90° ( $\pi/2$  rad) shown by the dotted lines. There is a phase angle of



Fig. 4.7

90° between A and B. Here the waveform B is lagging behind waveform A by 90°. In other words, the sine wave A is leading the waveform B by 90°. In Fig. 4.8(b) the sine wave A is lagging behind the waveform B by 90°. In both cases, the phase difference is 90°.



Fig. 4.8

**Example 4.4** What are the phase angles between the two sine waves shown in Figs. 4.9(a) and (b)?

**Solution** In Fig. 4.9(a), sine wave A is in phase with the reference wave; sine wave B is out of phase, which lags behind the reference wave by  $45^{\circ}$ . So we say that sine wave B lags behind sine wave A by  $45^{\circ}$ .

In Fig. 4.9(b), sine wave A leads the reference wave by 90°; sine wave B lags behind the reference wave by 30°. So the phase difference between A and B is 120°, which means that sine wave B lags behind sine wave A by 120°. In other words, sine wave A leads sine wave B by 120°.



#### Fig. 4.9

...

# 4.3 THE SINE WAVE EQUATION

A sine wave is graphically represented as shown in Fig. 4.10(a). The amplitude of a sine wave is represented on vertical axis. The angular measurement (in degrees or radians) is represented on horizontal axis. Amplitude A is the maximum value of the voltage or current on the *Y*-axis.

In general, the sine wave is represented by the equation

$$v(t) = V_m \sin \omega t$$

The above equation states that any point on the sine wave represented by an instantaneous value v(t) is equal to the maximum value times the sine of the angular frequency at that point. For example, if a certain sine wave voltage has peak value of 20 V, the instantaneous voltage at a point  $\pi/4$  radians along the horizontal axis can be calculated as

$$v(t) = V_m \sin \omega t$$
$$= 20 \sin \left(\frac{\pi}{4}\right) = 20 \times 0.707 = 14.14 \text{ V}$$

When a sine wave is shifted to the left of the reference wave by a certain angle  $\phi$ , as shown in Fig. 4.10 (b), the general expression can be written as

$$v(t) = V_m \sin\left(\omega t + \phi\right)$$

When a sine wave is shifted to the right of the reference wave by a certain angle  $\phi$ , as shown in Fig. 4.10(c), the general expression is

$$v(t) = V_m \sin\left(\omega t - \phi\right)$$

4.5





**Example 4.5** Determine the instantaneous value at the 90° point on the *X*-axis for each sine wave shown in Fig. 4.11. y(t)

**Solution** From Fig. 4.11, the equation for the sine wave *A* 

$$v(t) = 10 \sin \omega t$$

The value at  $\pi/2$  in this wave is

$$v(t) = 10 \sin \frac{\pi}{2} = 10 \text{ V}$$

The equation for the sine wave B

$$v(t) = 8 \sin(wt - \pi/4)$$

At

$$\omega t = \pi/2$$

$$v(t) = 8\,\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 8 \sin 45^\circ = 8 (0.707) = 5.66 \text{ V}$$

22

# 4.4 VOLTAGE AND CURRENT VALUES OF A SINE WAVE

As the magnitude of the waveform is not constant, the waveform can be measured in different ways. These are instantaneous, peak, peak to peak, root mean square (rms) and average values.



### 4.4.1 Instantaneous Value

Consider the sine wave shown in Fig. 4.12. At any given time, it has some instantaneous value. This value is different at different points along the waveform. V(t) volts

In Fig. 4.12 during the positive cycle, the instantaneous values are positive and during the negative cycle, the instantaneous values are negative. In Fig. 4.12 shown at time 1 ms, the value is 4.2 V; the value is 10 V at 2.5 ms, -2 V at 6 ms and -10 V at 7.5 and so on.

### 4.4.2 Peak Value

The peak value of the sine wave is the maximum value of the wave during positive half cycle, or maximum value of wave during negative half cycle. Since the value of these two are equal in magnitude, a sine wave is characterised by a single peak value. The peak value of the sine wave is shown in Fig. 4.13; here the peak value of the sine wave is 4 V.

#### 4.4.3 Peak to Peak Value

The peak to peak value of a sine wave is the value from the positive to the negative peak as shown in Fig. 4.14. Here the peak to peak value is 8 V.

#### 4.4.4 Average Value

In general, the average value of any function v(t), with period T is given by

$$v_{\rm av} = \frac{1}{T} \int_0^T v(t) \, dt$$

That means that the average value of a curve in the X-Y plane is the total area under the complete curve divided by the distance of the curve. The average value of a sine wave over one complete cycle is always zero. So the average value of a sine wave is defined over a half-cycle, and not a full cycle period.

The average value of the sine wave is the total area under the half-cycle curve divided by the distance of the curve.

The average value of the sine wave

 $v(t) = V_P \sin \omega t$  is given by



16 7.5910

10

4.2 2.1

- 10

25









t (ms)



The average value of a sine wave is shown by the dotted line in Fig. 4.15.

**Example 4.6** Find the average value of a cosine wave  $f(t) = \cos \omega t$  shown in Fig. 4.16.

**Solution** The average value of a cosine wave

$$v(t) = V_{P} \cos \omega t$$

$$V_{av} = \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} V_{P} \cos \omega t \ d(\omega t)$$

$$= \frac{1}{\pi} V_{P} (-\sin \omega t)^{3\pi/2}_{\pi/2}$$

$$= \frac{-V_{P}}{\pi} [-1-1] = \frac{2V_{P}}{\pi} = 0.637 \ V_{P}$$
Fig. 4.16

### 4.4.5 Root Mean Square Value or Effective Value

The root mean square (rms) value of a sine wave is a measure of the heating effect of the wave. When a resistor is connected across a dc voltage source as shown in Fig. 4.17(a), a certain amount of heat is produced in the resistor in a given time. A similar resistor is connected across an ac voltage source for the same time as shown in Fig. 4.17(b). The value of the ac voltage is adjusted such that the same amount of heat is produced in the resistor as in the case of the dc source. This value is called the rms value.



Fig. 4.17

That means the rms value of a sine wave is equal to the dc voltage that produces the same heating effect. In general, the rms value of any function with period T has an effective value given by

$$V_{\rm rms} = \sqrt{\frac{1}{T} \int_{0}^{T} \overline{v(t)}^2 dt}$$

Consider a function  $v(t) = V_P \sin \omega t$ 

The rms value, 
$$V_{\rm rms} = \sqrt{\frac{1}{T}} \int_{0}^{T} (V_P \sin \omega t)^2 d(\omega t)$$
$$= \sqrt{\frac{1}{T}} \int_{0}^{2\pi} V_P^2 \left[\frac{1 - \cos 2\omega t}{2}\right] d(\omega t)$$
$$= \frac{V_P}{\sqrt{2}} = 0.707 V_P$$

If the function consists of a number of sinusoidal terms, that is

$$v(t) = V_0 + (V_{c1} \cos \omega t + V_{c2} \cos 2 \omega t + \cdots)$$
$$+ (V_{s1} \sin \omega t + V_{s2} \sin 2 \omega t + \cdots)$$

The rms, or effective value is given by

$$V_{\rm rms} = \sqrt{V_0^2 + \frac{1}{2} \left(V_{c1}^2 + V_{c2}^2 + \cdots\right) + \frac{1}{2} \left(V_{s1}^2 + V_{s2}^2 + \cdots\right)}$$

**Example 4.7** A wire is carrying a direct current of 20 A and a sinusoidal alternating current of peak value 20 A. Find the rms value of the resultant current in the wire.

Solution The rms value of the combined wave

$$= \sqrt{20^2 + \frac{20^2}{2}}$$
  
=  $\sqrt{400 + 200} = \sqrt{600} = 24.5 \text{ A}$ 

## 4.4.6 Peak Factor

The peak factor of any waveform is defined as the ratio of the peak value of the wave to the rms value of the wave.

Peak factor = 
$$\frac{V_P}{V_{\rm rms}}$$

Peak factor of the sinusoidal waveform =  $\frac{V_P}{V_P/\sqrt{2}} = \sqrt{2} = 1.414$ 

### 4.4.7 Form Factor

Form factor of a waveform is defined as the ratio of rms value to the average value of the wave.

Form factor = 
$$\frac{V_{\rm rms}}{V_{\rm av}}$$

Form factor of a sinusoidal waveform can be found from the above relation.

For the sinusoidal wave, the form factor =  $\frac{V_P/\sqrt{2}}{0.637 V_P} = 1.11$ 

# 4.5 PHASE RELATION IN PURE RESISTOR

When a sinusoidal voltage of certain magnitude is applied to a resistor, a certain amount of sine wave current passes through it. We know the relation between v(t) and i(t) in the case of a resistor. The voltage/current relation in case of a resistor is linear,

i.e. 
$$v(t) = i(t) R$$

Consider the function

$$i(t) = I_m \sin \omega t = IM [I_m e^{j\omega t}]$$
 or  $I_m \angle 0^{\circ}$ 

If we substitute this in the above equation, we have

$$v(t) = I_m R \sin \omega t = V_m \sin \omega t$$
$$= IM [V_m e^{j\omega t}] \text{ or } V_m \angle 0^\circ$$
$$V_m = I_m R$$

where

If we draw the waveform for both voltage and current as shown in Fig. 4.18, there is no phase difference between these two waveforms. The amplitudes of the waveform may differ according to the value of resistance.

As a result, in pure resistive circuits, the voltages and currents are said to be in phase. Here the term impedance is defined as the ratio of voltage to current function. With ac voltage applied to elements, the ratio of exponential voltage to the corresponding current (impedance) consists of magnitude and phase angles. Since the phase difference is zero in case of a resistor, the phase angle is zero. The impedance in case of resistor consists only of magnitude, i.e.



Fig. 4.18

$$Z = \frac{V_m \,\angle 0^\circ}{I_m \,\angle 0^\circ} = R$$

# 4.6 PHASE RELATION IN A PURE INDUCTOR

As discussed earlier in Chapter 1, the voltage current relation in the case of an inductor is given by
$$v(t) = L\frac{di}{dt}$$

Consider the function  $i(t) = I_m \sin \omega t = IM [I_m e^{j\omega t}]$  or  $I_m \angle 0^\circ$ 

$$v(t) = L \frac{d}{dt} (I_m \sin \omega t)$$
  
=  $L\omega I_m \cos \omega t = \omega L I_m \cos \omega t$   
 $v(t) = V_m \cos \omega t$ , or  $V_m \sin (\omega t + 90^\circ)$   
=  $IM [V_m e^{j(\omega t + 90^\circ)}]$  or  $V_m \angle 90^\circ$ 

where

 $V_m = \omega L I_m = X_L I_m$  $e^{j90^\circ} = j = 1 \ \angle 90^\circ$ and

If we draw the waveforms for both, voltage and current, as shown in Fig. 4.19, we can observe the phase difference between these two waveforms.

As a result, in a pure inductor the voltage and current are out of phase. The current lags behind the voltage by 90° in a pure inductor as shown in Fig. 4.20.

The impedance which is the ratio of exponential voltage to the corresponding current, is given by

$$Z = \frac{V_m \sin(\omega t + 90^\circ)}{I_m \sin \omega t}$$
$$V_m = \omega L I_m$$
$$= \frac{I_m \omega L \sin(\omega t + 90^\circ)}{I_m \sin \omega t} = \frac{\omega L I_m \angle 90^\circ}{I_m \angle 0^\circ}$$
$$Z = i\omega L = iX_t$$

000

*:*..

where

where  $X_{I} = \omega L$  and is called the inductive reactance. Hence, a pure inductor has an impedance whose value is  $\omega L$ .

#### PHASE RELATION IN PURE CAPACITOR 4.7

As discussed in Chapter 1, the relation between voltage and current is given by

$$v(t) = \frac{1}{C} \int i(t) dt$$
  
Consider the function  $i(t) = I_m \sin \omega t = IM [I_m e^{j\omega t}]$  or  $I_m \angle 0^\circ$   
 $v(t) = \frac{1}{C} \int I_m \sin \omega t d(t)$   
 $= \frac{1}{\omega C} I_m [-\cos \omega t]$ 



Fig. 4.19





 $=\frac{I_m}{m}\sin(\omega t-90^\circ)$ 

...

$$\omega C$$

$$v(t) = V_m \sin (\omega t - 90^\circ)$$

$$= IM [I_m e^{j(\omega t - 90^\circ)}] \text{ or } V_m \angle -90^\circ$$

$$V_m = \frac{I_m}{\omega C}$$

where

$$\therefore \qquad \frac{V_m \angle -90^\circ}{I_m \angle 0^\circ} = Z = \frac{-j}{\omega C}$$

Hence, the impedance is  $Z = \frac{-j}{\omega C} = -jX_c$ 

where  $X_C = \frac{1}{\omega C}$  and is called the capacitive reactance.

If we draw the waveform for both, voltage and current, as shown in Fig. 4.21, there is a phase difference between these two waveforms.

As a result, in a pure capacitor, the current leads the voltage by  $90^{\circ}$ . The impedance value of a pure capacitor

$$X_C = \frac{1}{\omega C}$$







**Problem 4.1** Calculate the frequency for each of the following values of time period.

(a) 2 ms (b) 100 ms (c) 5 ms (d) 5 s

Solution The relation between frequency and period is given by

$$f = \frac{1}{T} \text{ Hz}$$
(a) Frequency  $f = \frac{1}{20 \times 10^{-3}} = 50 \text{ Hz}$ 
(b) Frequency  $f = \frac{1}{100 \times 10^{-3}} = 10 \text{ Hz}$ 
(c) Frequency  $f = \frac{1}{5 \times 10^{-6}} = 200 \text{ KHz}$ 
(d) Frequency  $f = \frac{1}{5} = 0.2 \text{ Hz}$ 

**Problem 4.2** Calculate the period for each of the following values of frequency.

(a) 50 Hz (b) 100 KHz (c) 1 Hz (d) 2 MHz

Solution The relation between frequency and period is given by

$$f = \frac{1}{T} \text{ Hz}$$
(a) Time period  $T = \frac{1}{f} = \frac{1}{50} = 0.02 \text{ s}$ 
(b) Time period  $T = \frac{1}{f} = \frac{1}{100 \times 10^3} = 10 \ \mu\text{s}$ 
(c) Time period  $T = \frac{1}{f} = \frac{1}{1} = 1 \text{ s}$ 
(d) Time period  $T = \frac{1}{f} = \frac{1}{2 \times 10^6} = 0.5 \ \mu\text{s}$ 

**Problem 4.3** A sine wave has a frequency of 50 kHz. How many cycles does it complete in 20 ms?

**Solution** The frequency of sine wave is 50 kHz.

That means in 1 second, a sine wave goes through  $50 \times 10^3$  cycles.

In 20 ms the number of cycles =  $20 \times 10^{-3} \times 50 \times 10^{3}$ 

= 1 kHz

That means in 20 ms the sine wave goes through 10' cycles.

**Problem 4.4** A sine wave has a peak value of 25 V. Determine the following values.

(a) rms (b) peak to peak (c) average

**Solution** (a) rms value of the sine wave

$$V_{\rm rms} = 0.707 \ V_P$$
  
= 0.707 × 25 = 17.68 V

(b) peak to peak value of the sine wave  $V_{PP} = 2V_P$ 

$$V_{PP} = 2 \times 25 = 50 \text{ V}$$

(c) average value of the sine wave

$$V_{\rm av} = 0.637 V_P$$
  
= (0.637)25 = 15.93 V

**Problem 4.5** A sine wave has a peak value of 12 V. Determine the following values

(a) rms (b) average (c) crest factor (d) form factor **Solution** (a) rms value of the given sine wave

- (b) average value of the sine wave = (0.637)12 = 7.64 V
- Peak value (c) crest factor of the sine wave =rms value

$$=\frac{12}{8.48}=1.415$$

(d) Form factor = 
$$\frac{\text{rms value}}{\text{average value}} = \frac{8.48}{7.64} = 1.11$$

**Problem 4.6** Sine wave 'A' has a positive going zero crossing at 45°. Sine wave 'B' has a positive going zero crossing at 60°. Determine the phase angle between the signals. Which of the signal lags

behind the other?

**Solution** The two signals drawn are shown in Fig. 4.22.

From Fig. 4.22, the signal *B* lags behind signal A by  $15^{\circ}$ . In other words, signal A leads signal *B* by  $15^{\circ}$ .

Problem 4.7 One sine wave has a positive peak at 75°, and another has a positive peak at 100°. How much is each sine wave shifted in phase from the  $0^{\circ}$  reference? What is the phase angle between them?

Solution The two signals are drawn as shown in Fig. 4.23.

The signal A leads the reference signal by  $15^{\circ}$ The signal *B* lags behind the reference signal by 10°

The phase angle between these two signals is 25° 22



(a) 
$$I_{\rm rms}$$
 (b)  $I_{\rm av}$  (c)  $I_P$ 

Solution The function given to the circuit shown is v

$$v(t) = V_P \sin \omega t = 20 \sin \omega t$$

The current passing through the resistor

$$i(t) = \frac{v(t)}{R}$$
$$i(t) = \frac{20}{2 \times 10^3} \sin \omega t$$
$$= 10 \times 10^{-3} \sin \omega t$$
$$I_P = 10 \times 10^{-3} \text{ A}$$



v(t)B 45° 60' ωt (θ)







(d)  $I_{PP}$ 

The peak value  $I_p = 10 \text{ mA}$ Peak to peak value  $I_{pp} = 20 \text{ mA}$ rms value  $I_{rms} = 0.707 I_p$   $= 0.707 \times 10 \text{ mA} = 7.07 \text{ mA}$ Average value  $I_{av} = (0.637) I_p$  $= 0.637 \times 10 \text{ mA} = 6.37 \text{ mA}$ 

**Problem 4.9** A sinusoidal voltage is applied to a capacitor as shown in Fig. 4.25. The frequency of the sine wave is 2 KHz. Determine the capacitive reactance.  $0.01 \, \mu F$ 

**Solution**  $X_C = \frac{1}{2}$ 

$$= \frac{1}{2\pi fC}$$
$$= \frac{1}{2\pi \times 2 \times 10^3 \times 0.01 \times 10^{-6}}$$
$$= 7.96 \text{ k}\Omega$$



Fig. 4.25

**Problem 4.10** Determine the rms current in the circuit shown in Fig. 4.26.



**Problem 4.11** A sinusoidal voltage is applied to the circuit shown in Fig. 4.27. The frequency is 3 KHz. Determine the inductive reactance.

#### Solution

 $X_{L} = 2\pi fL$   $= 2\pi \times 3 \times 10^{3} \times 2 \times 10^{-3}$   $= 37.69 \Omega$ 

**Problem 4.12** Determine the rms current in the circuit shown in Fig. 4.28.

 $X_{-} = 2 \pi f I$ 

### Solution

$$M_{L} = 2\pi \times 10 \times 10^{3} \times 50 \times 10^{-3}$$
$$X_{L} = 3.141 \text{ k}\Omega$$
$$H_{\text{rms}} = \frac{V_{\text{rms}}}{X_{L}}$$
$$= \frac{10}{3.141 \times 10^{3}} = 3.18 \text{ mA}$$



Fig. 4.27



**Problem 4.13** Find the form factor of the half-wave rectified sine wave shown in Fig. 4.29.





**Solution** v = 1

$$v = V_m \sin \omega t$$
, for  $0 < \omega t < \pi$   
= 0, for  $\pi < \omega t < 2\pi$ 

the period is  $2\pi$ .

Average value  $V_{av} = \frac{1}{2\pi} \left\{ \int_{0}^{\pi} V_{m} \sin \omega t \, d(\omega t) + \int_{\pi}^{2\pi} 0 \, d(\omega t) \right\}$   $= 0.318 \, V_{m}$   $V_{rms}^{2} = \frac{1}{2\pi} \int_{0}^{\pi} (V_{m} \sin \omega t)^{2} \, d(\omega t)$   $= \frac{1}{4} \, V_{m}^{2}$   $V_{rms} = \frac{1}{2} \, V_{m}$ Form factor =  $\frac{V_{rms}}{V_{av}} = \frac{0.5 \, V_{m}}{0.318 \, V_{m}} = 1.572$ 

**Problem 4.14** Find the average and effective values of the saw tooth waveform shown in Fig. 4.30 below.

Solution From Fig. 4.30 shown, the period is T.



Effective value

22



**Problem 4.15** Find the average and rms value of the full wave rectified sine wave shown in Fig. 4.31.



**Problem 4.16** The full wave rectified sine wave shown in Fig. 4.32 has a delay angle of 60°. Calculate  $V_{av}$  and  $V_{rms}$ .



Fig. 4.32

**Solution** Average value  $V_{av} = \frac{1}{\pi} \int_{0}^{\pi} 10 \sin(\omega t) d(\omega t)$  $= \frac{1}{\pi} \int_{60^{\circ}}^{\pi} 10 \sin \omega t d(\omega t)$  $V_{av} = \frac{10}{\pi} (-\cos \omega t)_{60}^{\pi} = 4.78$ Effective value  $V_{rms} = \sqrt{\frac{1}{\pi} \int_{0}^{\pi} (10 \sin \omega t)^{2} d(\omega t)}$   $= \sqrt{\frac{100}{\pi} \int_{0}^{\pi} \left(\frac{1 - \cos 2\omega t}{2}\right) d(\omega t)}$ = 6.33

**Problem 4.17** Find the form factor of the square wave as shown in Fig. 4.33. Solution v = 20 for 0 < t < 0.01= 0for 0.01 < *t* < 0.03 20 The period is 0.03 sec. Average value  $V_{av} = \frac{1}{0.03} \int_{0}^{0.01} 20 \ dt$ 0 0.01 0.03 ωt Fig. 4.33  $=\frac{20(0.01)}{0.03}=6.66$ Effective value  $V_{\text{eff}} = \sqrt{\frac{1}{0.03} \int_{0}^{0.01} (20)^2 dt} = 66.6 = 0.816$ Form factor =  $\frac{0.816}{6.66} = 0.123$ 22

**PRACTICE PROBLEMS** 

4.1 Calculate the frequency of the following values of period.

(a)	0.2 s	(b)	50 ms
>		2.45	

(c)  $500 \ \mu s$  (d)  $10 \ \mu s$ 

- 4.2 Calculate the period for each of the values of frequency.
  - (a) 60 Hz (b) 500 Hz
  - (c) 1 KHz (d) 200 kHz
  - (e) 5 MHz
- 4.3 A certain sine wave has a positive going zero crossing at 0° and an rms value of 20 V. Calculate its instantaneous value at each of the following angles.

(a) $33^{\circ}$ (b)	110°
----------------------	------

- (c)  $145^{\circ}$  (d)  $325^{\circ}$
- 4.4 For a particular 0° reference sinusoidal current, the peak value is 200 mA; determine the instantaneous values at each of the following.
  - (a) 35° (b) 190°
  - (c)  $200^{\circ}$  (d)  $360^{\circ}$
- 4.5 Sine wave *A* lags sine wave *B* by 30°. Both have peak values of 15 V. Sine wave *A* is the reference with a positive going crossing at 0°. Determine the instantaneous value of sine wave *B* at 30°, 90°, 45°, 180° and 300°.

\*\*

- 4.6 Find the average values of the voltages across  $R_1$  and  $R_2$ . In Fig. 4.34 values shown are rms.
- 4.7 A sinusoidal voltage is applied to the circuit shown in Fig. 4.35, determine rms current, average current, peak current, and peak to peak current.
- 4.8 A sinusoidal voltage of  $v(t) = 50 \sin(500t)$  applied to a capacitive circuit. Determine the capacitive reactance, and the current in the circuit.



Fig. 4.35

4.9 A sinusoidal voltage source in series with a dc source as shown in Fig. 4.36.





Sketch the voltage across  $R_L$ . Determine the maximum current through  $R_L$  and the average voltage across  $R_L$ .

- 4.10 Find the effective value of the resultant current in a wire which carries a direct current of 10 A and a sinusoidal current with a peak value of 15 A.
- 4.11 An alternating current varying sinusoidally, with a frequency of 50 Hz, has an rms value of 20 A. Write down the equation for the instantaneous value and find this value at (a) 0.0025 s (b) 0.0125 s after passing through a positive maximum value. At what time, measured from a positive maximum value, will the instantaneous current be 14.14 A?
- 4.12 Determine the rms value of the voltage defined by

$$v = 5 + 5 \sin(314t + \pi/6).$$

- 4.13 Find the effective value of the function  $v = 100 + 50 \sin \omega t$ .
- 4.14 A full wave rectified sine wave is clipped at 0.707 of its maximum value as shown in Fig. 4.37. Find the average and effective values of the function.



Fig. 4.37

4.15 Find the rms value of the function shown in Fig. 4.38 and described as follows



Fig. 4.38

4.16 Calculate average and effective values of the waveform shown in Fig.4.39 and hence find from factor.



Fig. 4.39

4.17 A full wave rectified sine wave is clipped such that the effective value is 0.5  $V_m$  as shown in Fig. 4.40. Determine the angle at which the wave form is clipped.



Fig. 4.40



- 1. One sine wave has a period of 2 ms, another has a period of 5 ms, and other has a period of 10 ms. Which sine wave is changing at a faster rate?
  - (a) sine wave with period 2 ms (b) sine wave with period of 5 ms
  - (c) all are at the same rate (d) sine wave with period of 10 ms
- 2. How many cycles does a sine wave go through in 10 s when its frequency is 60 Hz?
  - (a) 10 cycles
  - (c) 600 cycles

- (b) 60 cycles
- (d) 6 cycles

....

L

Γ

4.21

3	If the neak value of a certain sine wa	ve v	oltage is 10 V what is the neak		
5.	to neak value?				
	(a) $20 \text{ V}$	$(\mathbf{h})$	10 V		
	(a) $20^{\circ}$ (c) $5^{\circ}$ V	(d)	7 07 V		
4	If the peak value of a certain sine w	ave	voltage is 5 V what is the rms		
	value?				
	(a) $0.707 \text{ V}$	$(\mathbf{h})$	3 535 V		
	(c) $5 V$	(d)	1 17 V		
5.	What is the average value of a sine w	vave	over a full cycle?		
0.			V		
	(a) $V_m$	(b)	$\frac{m}{\sqrt{2}}$		
			$\sqrt{2}$		
	(c) zero	(d)	$\sqrt{2} V_m$		
6.	A sinusoidal current has peak value of	of 12	2 A. What is its average value?		
	(a) 7.64 A	(b)	24 A		
	(c) 8.48 A	(d)	12 A		
7.	Sine wave <i>A</i> has a positive going zero	o cro	ssing at $30^{\circ}$ . Sine wave <i>B</i> has a		
	positive going zero crossing at 45°. What is the phase angle between two				
	signals?				
	(a) $30^{\circ}$	(b)	45°		
	(c) 75°	(d)	15°		
8.	A sine wave has a positive going zero	o cro	ossing at 0° and an rms value of		
	20  V. What is its instantaneous value	e at 1	145°.		
	(a) $7.32$ V	(b)	16.22 V		
0	(c) 26.57 V	(d)	21.66 V		
9.	In a pure resistor, the voltage and cur	rrent	are		
	(a) out of phase	(b)	in phase		
10	(c) $90^\circ$ out of phase	(d)	45° out of phase		
10.	The rms current through a 10 ks2 res	istor	is 5 mA. what is the rms volt-		
	age drop across the resistor?	(1)	5 37		
	(a) $10 \text{ v}$	(D)	5 V		
11	(c) $50 \text{ v}$	(a)	zero		
11.	(a) is in phase with the surrent	(h)	is out of phase with the summent		
	(a) Is in phase with the current $(a)$ lass habind the current by $00^{\circ}$	(0)	loads the surrant by 00°		
12	(c) lags behind the current by 90	(u)	reads the current by 90		
12.	A sine wave voltage is applied across the voltage is increased the current	saca	apachor, when the frequency of		
	(a) increases	$(\mathbf{b})$	decreases		
	(a) mercases (c) remains the same	(0)	is zero		
13	The current in a pure inductor	(u)	13 2010		
15.	(a) lags behind the voltage by $90^{\circ}$	(h)	leads the voltage by $90^{\circ}$		
	(a) lags beline the voltage by 90 (c) is in phase with the voltage	(0)	lags behind the voltage by 15°		
14	A sine wave voltage is applied across	(u) an i	nductor: when the frequency of		
17.	voltage is increased the current				
	(a) increases	(h)	decreases		
	(c) remains the same	(d)	is zero		
	(-,	( ~ <i>j</i>			



19. For the half wave rectified sine wave shown in Fig. 4.41, the peak factor is





(a) 1.41

(b) 2.0

(c) 2.82

(d) infinite

20. For the square wave shown in Fig. 4.42, the form factor is





(b) 1.0 (d) zero

- 21. The power consumed in a circuit element will be least when the phase difference between the current and voltage is
  - (a)  $0^{\circ}$  (b)  $30^{\circ}$

(c) 90° (d) 180°

22. The voltage wave consists of two components: a 50 V dc component and a sinusoidal component with a maximum value of 50 volts. The average value of the resultant will be

(c) 0.5

(c) 50

- (b) 86.6 V
- (d) none of the above



# 5.1 IMPEDANCE DIAGRAM

So far our discussion has been confined to resistive circuits. Resistance restricts the flow of current by opposing free electron movement. Each element has some resistance; for example, an inductor has some resistance; a capacitance also has some resistance. In the resistive element, there is no phase difference between the voltage and the current. In the case of pure inductance, the current lags behind the voltage by 90 degrees, whereas in the case of pure capacitance, the current leads the voltage by 90 degrees. Almost all electric circuits offer impedance to the flow of current. Impedance is a complex quantity having real and imaginary parts; where the real part is the resistance and the imaginary part is the reactance of the circuit.

Consider the *RL* series circuit shown in Fig. 5.1. If we apply the real function  $V_m \cos \omega t$  to the circuit, the response may be  $I_m \cos \omega t$ . Similarly, if we apply the imaginary function  $jV_m \sin \omega t$  to the same circuit, the response is  $jI_m \sin \omega t$ . If we apply a complex function, which is a combination of real and imaginary functions, we will get a complex response.





This complex function is  $V_m e^{j\omega t} = V_m (\cos \omega t + j \sin \omega t)$ . Applying Kirchhoff's law to the circuit shown in Fig. 5.1,

we get 
$$V_m e^{j\omega t} = Ri(t) + L \frac{di(t)}{dt}$$

The solution of this differential equation is

$$i(t) = I_m e^{j\omega t}$$

By substituting i(t) in the above equation, we get

$$V_m e^{j\omega t} = R I_m e^{j\omega t} + L \frac{d}{dt} (I_m e^{j\omega t})$$
$$V_m e^{j\omega t} = R I_m e^{j\omega t} + L I_m j\omega e^{j\omega t}$$
$$V_m = (R + j\omega L) I_m$$

Impedance is defined as the ratio of the voltage to current function

$$Z = \frac{V_m e^{j\omega t}}{\frac{V_m}{R + j\omega L} e^{j\omega t}} = R + j\omega L$$

Complex impedance is the total opposition offered by the circuit elements to *ac* current, and can be displayed on the complex plane. The impedance is denoted by *Z*. Here the resistance *R* is the real part of the impedance, and the reactance  $X_L$  is the imaginary part of the impedance. The resistance *R* is located on the real axis. The inductive reactance  $X_L$  is located on the positive *j* axis. The resultant of *R* and  $X_L$  is called the complex impedance.

Figure 5.2 is called the impedance diagram for the *RL* circuit. From Fig. 5.2, the impedance  $Z = \sqrt{R^2 + (\omega L)^2}$ , and angle  $\theta = \tan^{-1} \omega L/R$ . Here, the impedance is the vector sum of the resistance and inductive reactance. The angle between impedance and resistance is the phase angle between the current and voltage applied to the circuit.

Similarly, if we consider the RC series circuit, and apply the complex function  $V_m e^{j\omega t}$  to the circuit in Fig. 5.3, we get a complex response as follows.

Applying Kirchhoff's law to the above circuit, we get

$$V_m e^{j\omega t} = Ri(t) + \frac{1}{C} \int i(t) dt$$

Solving this equation we get,

for we get,  

$$i(t) = I_m e^{j\omega t}$$

$$V_m e^{j\omega t} = R I_m e^{j\omega t} + \frac{1}{C} I_m \left(\frac{\pm 1}{j\omega}\right) e^{j\omega t}$$

$$= \left[ RI_m - \frac{1}{\omega C} I_m \right] e^{j\omega t}$$

$$V_m = \left( R - \frac{j}{\omega C} \right) I_m$$







Fig. 5.3

The impedance

$$Z = \frac{V_m e^{j\omega t}}{V_m / [R - j/\omega C] e^{j\omega t}}$$
  
= [R - (j/\omega C]]  
assists of resistance (R), which  
apacitive reactance (X<sub>c</sub> = 1/

Here impedance Z consists of resistance (R), which is the real part, and capacitive reactance ( $X_c = 1/\omega C$ ), which is the imaginary part of the impedance. The resistance, R, is located on the real axis, and the capacitive reactance  $X_c$  is located on the negative j axis in the impedance diagram in Fig. 5.4. -jFig. 5.4

Form Fig. 5.4, impedance  $Z = \sqrt{R^2 + X_C^2}$  or  $\sqrt{R^2 + (1/\omega C)^2}$  and angle  $\theta = \tan^{-1} (1/\omega CR)$ . Here, the impedance, Z, is the vector sum of resistance and capacitive reactance. The angle between resistance and impedance is the phase angle between the applied voltage and current in the circuit.

# 5.2 PHASOR DIAGRAM

A phasor diagram can be used to represent a sine wave in terms of its magnitude and angular position. Examples of phasor diagrams are shown in Fig. 5.5.





In Fig. 5.5(a), the length of the arrow represents the magnitude of the sine wave; angle  $\theta$  represents the angular position of the sine wave. In Fig. 5.5(b), the magnitude of the sine wave is one and the phase angle is 30°. In Fig. 5.5(c)

and (d), the magnitudes are four and three, and phase angles are  $135^{\circ}$  and  $225^{\circ}$ , respectively. The position of a phasor at any instant can be expressed as a positive or negative angle. Positive angles are measured counterclockwise from  $0^{\circ}$ , whereas negative angles are measured clockwise from  $0^{\circ}$ . For a given positive angle  $\theta$ , the corresponding negative angle is  $\theta - 360^{\circ}$ . This is shown in Fig. 5.6(a). In Fig. 5.6(b), the positive angle  $135^{\circ}$  of vector *A* can be represented by a negative angle  $-225^{\circ}$ ,  $(135^{\circ} - 360^{\circ})$ .



Fig. 5.6

A phasor diagram can be used to represent the relation between two or more sine waves of the same frequency. For example, the sine waves shown in Fig. 5.7(a) can be represented by the phasor diagram shown in Fig. 5.7(b).



Fig. 5.7

In the above figure, sine wave *B* lags behind sine wave *A* by  $45^{\circ}$ ; sine wave *C* leads sine wave *A* by  $30^{\circ}$ . The length of the phasors can be used to represent peak, rms, or average values.

**Example 5.1** Draw the phasor diagram to represent the two sine waves shown in Fig. 5.8.

**Solution** The phasor diagram representing the sine waves is shown in Fig. 5.9. The length of the each phasor represents the peak value of the sine wave.



# 5.3 SERIES CIRCUITS

The impedance diagram is a useful tool for analysing series ac circuits. Basically we can divide the series circuits as RL, RC and RLC circuits. In the analysis of series ac circuits, one must draw the impedance diagram. Although the impedance diagram usually is not drawn to scale, it does represent a clear picture of the phase relationships.

### 5.3.1 Series RL Circuit

If we apply a sinusoidal input to an RL circuit, the current in the circuit and all voltages across the elements are sinusoidal. In the analysis of the RL series circuit, we can find the impedance, current, phase angle and voltage drops. In Fig. 5.10 (a) the resistor voltage  $(V_R)$  and current (I) are in phase with each other, but lag behind the source voltage  $(V_S)$ . The inductor voltage  $(V_L)$  leads the source voltage  $(V_S)$ . The phase angle between current and voltage in a pure inductor is always 90°. The amplitudes of voltages and currents in the circuit are completely dependent on the values of elements (i.e. the resistance and inductive reactance). In the circuit shown, the phase angle is somewhere between zero and 90° because of the series combination of resistance with inductive reactance, which depends on the relative values of R and  $X_I$ .



Fig. 5.10(a)

The phase relation between current and voltages in a series RL circuit is shown in Fig. 5.10(b).



Fig. 5.10(b)

Here  $V_R$  and I are in phase. The amplitudes are arbitrarily chosen. From Kirchhoff's voltage law, the sum of the voltage drops must equal the applied voltage. Therefore, the source voltage  $V_s$  is the phasor sum of  $V_R$  and  $V_L$ .

$$V_S = \sqrt{V_R^2 + V_L^2}$$

The phase angle between resistor voltage and source voltage is

$$\theta = \tan^{-1} (V_L/V_R)$$

where  $\theta$  is also the phase angle between the source voltage and the current. The phasor diagram for the series RL circuit that represents the waveforms in Fig. 5.10(c).



Fig. 5.10(c)

**Example 5.2** To the circuit shown in Fig. 5.11, consisting a 1 k $\Omega$  resistor connected in series with a 50 mH coil, a 10 V rms, 10 kHz signal is applied. Find impedance Z, current I, phase angle q, voltage across resistance  $V_{R}$ , and the voltage across inductance  $V_{L}$ .



Fig. 5.11

**Solution** Inductive reactance  $X_L = \omega L$ 

$$= 2\pi f L = (6.28) \ (10^4) \ (50\times 10^{-3}) = 3140 \ \Omega$$

In rectangular form,

Total impedance  $Z = (1000 + j3140) \Omega$ 

:.

$$= \sqrt{R^2 + X_L^2}$$
$$= \sqrt{(1000)^2 + (3140)^2} = 3295.4 \,\Omega$$

Current  $I = V_s/Z = 10/3295.4 = 3.03 \text{ mA}$ Phase angle  $q = \tan^{-1} (X_L/R) = \tan^{-1} (3140/1000) = 72.33^{\circ}$ Therefore, in polar form total impedance  $Z = 3295.4 \angle 72.33^{\circ}$ Voltage across resistance  $V_R = IR$   $= 3.03 \times 10^{-3} \times 1000 = 3.03 \text{ V}$ Voltage across inductive reactance  $V_L = IX_L$  $= 3.03 \times 10^{-3} \times 3140 = 9.51 \text{ V}$ 

**Example 5.3** Determine the source voltage and the phase angle, if voltage across the resistance is 70 V and voltage across the inductive reactance is 20 V as shown in Fig. 5.12.

**Solution** In Fig. 5.12, the source voltage is given by

$$V_{S} = \sqrt{V_{R}^{2} + V_{L}^{2}}$$
$$= \sqrt{(70)^{2} + (20)^{2}} = 72$$

 $=\sqrt{(70)^2+(20)^2}=72.8$  V

The angle between current and source voltage is

$$q = \tan^{-1} (V_L/V_R) = \tan^{-1} (20/70) = 15.94^{\circ}$$

### 5.3.2 Series RC Circuit

When a sinusoidal voltage is applied to an RC series circuit, the current in the circuit and voltages across each of the elements are sinusoidal. The series RC circuit is shown in Fig. 5.13 (a).

Here the resistor voltage and current are in phase with each other. The capacitor voltage lags behind the source voltage. The phase angle between the current and the capacitor voltage is always 90°. The amplitudes and the phase relations between the voltages and current depend on the

ohmic values of the resistance and the capacitive reactance. The circuit is a series combination of both resistance and capacitance; and the phase angle between the applied voltage and the total current is somewhere between zero and 90°, depending on the relative values of the resistance and reactance. In a series RC circuit, the current is the same through the resistor and the capacitor. Thus, the resistor voltage is in phase with the current, and the capacitor voltage lags behind the current by 90° as shown in Fig. 5.13(b).







20 V

Fig. 5.12

70 V

-



Fig. 5.13 (b)

Here, *I* leads  $V_C$  by 90°.  $V_R$  and *I* are in phase. From Kirchhoff's voltage law, the sum of the voltage drops must be equal to the applied voltage. Therefore, the source voltage is given by

$$V_S = \sqrt{V_R^2 + V_C^2}$$

The phase angle between the resistor voltage and the source voltage is

$$\theta = \tan^{-1} (V_C / V_R)$$

Since the resistor voltage and the current are in phase,  $\theta$  also represents the phase angle between the source voltage and current. The voltage phasor diagram for the series RC circuit, voltage and current phasor diagrams represented by the waveforms in Fig. 5.13(b) are shown in Fig. 5.13(c).



Fig. 5.13(c)

**Example 5.4** A sine wave generator supplies a 500 Hz, 10 V rms signal to a 2 k $\Omega$  resistor in series with a 0.1  $\mu$ F capacitor as shown in Fig. 5.14. Determine the total impedance Z, current I, phase angle  $\mu$ , capacitive voltage  $V_c$ , and resistive voltage  $V_g$ .



Fig. 5.14

**Solution** To find the impedance Z, we first solve for  $X_C$ 

$$X_C = \frac{1}{2\pi f C} = \frac{1}{6.28 \times 500 \times 0.1 \times 10^{-6}}$$
  
= 3184.7 \Omega

In rectangular form,

Total impedance  $Z = (2000 - j3184.7) \Omega$ 

$$Z = \sqrt{(2000)^2 + (3184.7)^2}$$
  
= 3760.6 \Omega

Phase angle  $\theta = \tan^{-1} (-X_c/R) = \tan^{-1} (-3184.7/2000) = -57.87^{\circ}$ Current  $I = V_s/Z = 10/3760.6 = 2.66$  mA Capacitive voltage  $V_c = IX_c$ 

$$= 2.66 \times 10^{-3} \times 3184.7 = 8.47$$
 V

Resistive voltage  $V_{R} = IR$ 

$$= 2.66 \times 10^{-3} \times 2000 = 5.32$$
 V

The arithmetic sum of  $V_c$  and  $V_R$  does not give the applied voltage of 10 volts. In fact, the total applied voltage is a complex quantity. In rectangular form,

Total applied voltage  $V_s = 5.32 - j 8.47$  V

In polar form

5.3.3

$$V_{\rm S} = 10 \ \angle -57.87^{\circ} \ {\rm V}$$

The applied voltage is complex, since it has a phase angle relative to the resistive current.

**Example 5.5** Determine the source voltage and phase angle when the voltage across the resistor is 20 V and the capacitor is 30 V as shown in Fig. 5.15.

**Solution** Since  $V_R$  and  $V_C$  are 90° out of phase, they cannot be added directly. The source voltage is the phasor sum of  $V_R$  and  $V_C$ .

:. 
$$V_S = \sqrt{V_R^2 + V_C^2} = \sqrt{(20)^2 + (30)^2} = 36 \text{ V}$$

The angle between the current and source s voltage is

Series R-L-C Circuit

$$\theta = \tan^{-1} (V_C / V_R) = \tan^{-1} (30/20) = 56.3^{\circ}$$







The magnitude and type of reactance in a series RLC circuit is the difference of the two reactance. The impedance for an RLC series circuit is given by Z =

 $\sqrt{R^2 + (X_L - X_C)^2}$ . Similarly, the phase angle for an RLC circuit is

$$\boldsymbol{\theta} = \tan^{-1} \left( \frac{X_L - X_C}{R} \right)$$

**Example 5.6** In the circuit shown in Fig. 5.17, determine the total impedance, current *I*, phase angle  $\theta$ , and the voltage across each element.

**Solution** To find impedance Z, we first solve for  $X_C$  and  $X_L$ 



Fig. 5.17

$$X_C = \frac{1}{2\pi fC} = \frac{1}{6.28 \times 50 \times 10 \times 10^{-6}}$$
  
= 318.5 \Omega

 $X_L = 2\pi f L = 6.28 \times 0.5 \times 50 = 157 \ \Omega$ 

Total impedance in rectangular form

$$Z = (10 + j157 - j318.5) \Omega$$
  
= 10 + j(157 - 318.5) \Omega = 10 - j161.5 \Omega

Here, the capacitive reactance dominates the inductive reactance.

$$Z = \sqrt{(10)^2 + (161.5)^2}$$
$$= \sqrt{100 + 26082 \cdot 2} = 161.8 \ \Omega$$

$$I = V_S / Z = \frac{50}{161.8} = 0.3$$
 A

Phase angle  $\theta = \tan^{-1} [(X_L - X_C)/R] = \tan^{-1} (-161.5/10) = -86.45^{\circ}$ Voltage across the resistor  $V_R = IR = 0.3 \times 10 = 3$  V Voltage across the capacitive reactance  $= IX_C = 0.3 \times 318.5 = 95.55$  V Voltage across the inductive reactance  $= IX_L = 0.3 \times 157 = 47.1$  V

# 5.4 PARALLEL CIRCUITS

The complex number system simplifies the analysis of parallel ac circuits. In series circuits, the current is the same in all parts of the series circuit. In parallel ac circuits, the voltage is the same across each element.

# 5.4.1 Parallel RC Circuits

The voltages for an RC series circuit can be expressed using complex numbers, where the resistive voltage is the real part of the complex voltage and the capacitive voltage is the imaginary part. For parallel RC circuits, the voltage is the same across each component. Here the total current can be represented by a complex number. The real part of the complex current expression is the resistive current; the capacitive branch current is the imaginary part.

**Example 5.7** A signal generator supplies a sine wave of 20 V, 5 kHz to the circuit shown in Fig. 5.18. Determine the total current  $I_T$ , the phase angle and total impedance in the circuit.



Solution Capacitive reactance

$$X_C = \frac{1}{2\pi f C} = \frac{1}{6.28 \times 5 \times 10^3 \times 0.2 \times 10^{-6}} = 159.2 \ \Omega$$

Since the voltage across each element is the same as the applied voltage, we can solve for the two branch currents.

 $\therefore$  Current in the resistance branch

$$I_R = \frac{V_S}{R} = \frac{20}{100} = 0.2 \text{ A}$$

\*\*

and current in the capacitive branch

$$I_C = \frac{V_S}{X_C} = \frac{20}{159.2} = 0.126 \text{ A}$$

The total current is the vector sum of the two branch currents.

:. Total current 
$$I_T = (I_R + jI_C) A = (0.2 + j0.13) A$$

In polar form  $I_T = 0.24 \angle 33^\circ$ 

So the phase angle  $\theta$  between applied voltage and total current is 33°. It indicates that the total line current is 0.24 A and leads the voltage by 33°. Solving for impedance, we get

$$Z = \frac{V_S}{I_T} = \frac{20 \angle 0^\circ}{0.24 \angle 33^\circ} = 83.3 \angle -33^\circ \Omega$$

### 5.4.2 Parallel RL Circuits

In a parallel RL circuit, the inductive current is imaginary and lies on the -j axis. The current angle is negative when the impedance angle is positive. Here also the total current can be represented by a complex number. The real part of the complex current expression is the resistive current; and inductive branch current is the imaginary part.

**Example 5.8** A 50  $\Omega$  resistor is connected in parallel with an inductive reactance of 30  $\Omega$ . A 20 V signal is applied to the circuit. Find the total impedance and line current in the circuit shown in Fig. 5.19.



Fig. 5.19

**Solution** Since the voltage across each element is the same as the applied voltage, current in the resistive branch,

$$I_R = \frac{V_S}{R} = \frac{20 \angle 0^\circ}{50 \angle 0^\circ} = 0.4 \text{ A}$$

current in the inductive branch

$$I_L = \frac{V_S}{X_L} = \frac{20 \angle 0^\circ}{30 \angle 90^\circ} = 0.66 \angle -90^\circ$$

 $I_T = 0.4 - j0.66$ Total current is  $I_T = 0.77 \angle -58.8^{\circ}$ In polar form, Here the current lags behind the voltage by 58.8°

Total impedance

$$Z = \frac{V_S}{I_T}$$
  
=  $\frac{20 \angle 0^{\circ}}{0.77 \angle -58.8^{\circ}} = 25.97 \angle 58.8^{\circ} \Omega$ 

#### 5.5 COMPOUND CIRCUITS

In many cases, ac circuits to be analysed consist of a combination of series and parallel impedances. Circuits of this type are known as series-parallel, or compound circuits. Compound circuits can be simplified in the same manner as a series-parallel dc circuit consisting of pure resistances.

**Example 5.9** Determine the equivalent impedance of Fig. 5.20.



The total impedance

$$Z_T = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}$$
 Fig. 5.20  

$$= (5 + j10) + \frac{(2 - j4)(1 + j3)}{(2 - j4) + (1 + j3)}$$
  

$$= (5 + j10) + \frac{4.47 \angle -63.4^{\circ} \times 3.16 \angle +71.5^{\circ}}{3 - j1}$$
  

$$= (5 + j10) + \frac{14.12 \angle 81^{\circ}}{3 - j1}$$
  

$$= (5 + j10) + \frac{14.12 \angle 81^{\circ}}{3.16 \angle -18^{\circ}}$$
  

$$= 5 + j10 + 4.46 \angle 26.1^{\circ}$$
  

$$= 5 + j10 + 4 + j1.96$$
  

$$= 9 + j11.96$$

The equivalent circuit for the compound circuit shown in Fig. 5.20 is a series circuit containing 9  $\Omega$  of resistance and 11.96  $\Omega$  of inductive reactance. In polar form,  $Z = 14.96 \angle 53.03^{\circ}$ 

The phase angle between current and applied voltage is  $\theta = 53.03^{\circ}$ 

**Example 5.10** In the circuit of Fig. 5.21, determine the values of the following (a)  $Z_{\tau}$  (b)  $I_{\tau}$  (c)  $\theta$ .

\*\*



Fig. 5.21

**Solution** First, the inductive reactance is calculated.





In Fig. 5.22, the 10  $\Omega$  resistance is in series with the parallel combination of 20  $\Omega$ and *j* 31.42  $\Omega$ 

*:*..

$$Z_T = 10 + \frac{(20) (j31.42)}{(20 + j31.42)}$$
  
= 10 +  $\frac{628.4 \angle 90^{\circ}}{37.24 \angle 57.52^{\circ}} = 10 + 16.87 \angle 32.48^{\circ}$   
= 10 + 14.23 + j 9.06 = 24.23 + j 9.06

In polar form,  $Z_T = 25.87 \angle 20.5^\circ$ 

Here the current lags behind the applied voltage by 20.5° т2

Total current

$$I_T = \frac{\nu_S}{Z_T}$$
  
=  $\frac{20}{25.87 \angle 20.5^\circ} = 0.77 \angle -20.5^\circ$ 

The phase angle between voltage and current is

$$\theta = 20.5^{\circ}$$



 $q = 20.5^{\circ}$ 

**Problem 5.1** A signal generator supplies a 30 V, 100 Hz signal to a series circuit shown in Fig. 5.23. Determine the impedance, the line current and phase angle in the given circuit.



Fig. 5.23

**Solution** In Fig. 5.24, the resistances and inductive reactances can be combined.





First, we find the inductive reactance

Current

$$X_L = 2\pi f L$$
  
=  $2\pi \times 100 \times 70 \times 10^{-3} = 43.98 \ \Omega$ 

In rectangular form, the total impedance is

$$Z_T = (40 + j43.98) \,\Omega$$

 $I = \frac{V_S}{Z_T} = \frac{30 \angle 0^{\circ}}{40 + j43.98}$ 

Here we are taking source voltage as the reference voltage

:. 
$$I = \frac{30 \angle 0^{\circ}}{59.45 \angle +47.7^{\circ}} = 0.5 \angle -47.7^{\circ} \text{ A}$$

The current lags behind the applied voltage by 47.7° Hence, the phase angle between voltage and current

$$\theta = 47.7^{\circ}$$

**Problem 5.2** For the circuit shown in Fig. 5.25, find the effective voltages across resistance and inductance, and also determine the phase angle.



Fig. 5.25

Solution In rectangular form,

Total impedance  $Z_T = R + jX_L$ where  $X_L = 2\pi fL$   $= 2\pi \times 100 \times 50 \times 10^{-3} = 31.42 \Omega$  $\therefore \qquad Z_T = (100 + j31.42) \Omega$ 

Current 
$$I = \frac{V_S}{Z_T} = \frac{10 \angle 0^\circ}{(100 + j31.42)} = \frac{10 \angle 0^\circ}{104.8 \angle 17.44^\circ} = 0.095 \angle -17.44^\circ$$

Therefore, the phase angle between voltage and current  $\theta = 17 44^{\circ}$ 

Voltage across resistance is  $V_R = IR$ 

$$= 0.095 \times 100 = 9.5$$
 V

22

Voltage across inductive reactance is  $V_L = IX_L$ = 0.095 × 31.42 = 2.98 V

**Problem 5.3** For the circuit shown in Fig. 5.26, determine the value of impedance when a voltage of (30 + j50) V is applied to the circuit and the current flowing is (-5 + j15) A. Also determine the phase angle.



Fig. 5.26

Solution Impedance  $Z = \frac{V_S}{I} = \frac{30 + j50}{-5 + j15}$ =  $\frac{58.31 \angle 59^\circ}{15.81 \angle 108.43^\circ} = 3.69 \angle -49.43^\circ$  In rectangular form, the impedance Z = 2.4 - j2.8

Therefore, the circuit has a resistance of 2.4  $\Omega$  in series with capacitive reactance 2.8  $\Omega$ .

Phase angle between voltage and current is  $\theta = 49.43^{\circ}$ . Here, the current leads the voltage by  $49.43^{\circ}$ .

**Problem 5.4** A resistor of 100  $\Omega$  is connected in series with a 50  $\mu$ F capacitor. Find the effective voltage applied to the circuit at a frequency of 50 Hz. The effective voltage across the resistor is 170 V. Also determine voltage across the capacitor and phase angle. (See Fig. 5.27)





**Solution** Capacitive reactance 
$$X_C = \frac{1}{2\pi fC}$$

$$=\frac{1}{2\pi\times50\times50\times10^{-6}}=63.66\ \Omega$$

Total impedance  $Z_T = (100 - j63.66) \Omega$ 

Voltage across 100  $\Omega$  resistor is  $V_{R} = 170$  V

Current in resistor,  $I = \frac{170}{100} = 1.7 \text{ A}$ 

Since the same current passes through capacitive reactance, the effective voltage across the capacitive reactance is

$$V_C = IX_C$$
  
= 1.7 × 63.66 = 108.22 V

=

The effective applied voltage to the circuit

$$V_S = \sqrt{V_R^2 + V_C^2}$$
  
=  $\sqrt{(170)^2 + (108.22)^2} = 201.5 \text{ V}$ 

Total impedance in polar form

$$Z_T = 118.54 \ \angle -32.48^\circ$$

Therefore, the current leads the applied voltage by 32.48°.

**Problem 5.5** For the circuit shown in Fig. 5.28, determine the total current, impedance Z and phase angle.



22

**Solution** Here, the voltage across each element is the same as the applied voltage.

Current in resistive branch  $I_R = \frac{V_S}{R} = \frac{50}{100} = 0.5 \text{ A}$ 

Inductive reactance  $X_L = 2\pi f L$ 

$$=2\pi \times 50 \times 0.5 = 157.06 \ \Omega$$

Current in inductive branch

$$I_L = \frac{V_S}{X_L} = \frac{50}{157.06} = 0.318 \text{ A}$$
$$I_T = \sqrt{I_R^2 + I_L^2}$$

(0.5 - i0.318)A = 0.59  $\angle -32.5^{\circ}$ 

Total current

or

For parallel RL circuits, the inductive susceptance is

$$B_L = \frac{1}{X_L} = \frac{1}{157.06} = 0.0064 \text{ S}$$

Conductance  $G = \frac{1}{100} = 0.01 \text{ S}$   $\therefore$  Admittance  $= \sqrt{G^2 + B_L^2} = \sqrt{(0.01)^2 + (0.0064)^2}$ = 0.0118 S

Converting to impedance, we get

$$Z = \frac{1}{Y} = \frac{1}{0.012} = 83.33 \ \Omega$$

Phase angle  $\theta = \tan^{-1}\left(\frac{R}{X_L}\right) = \tan^{-1}\left(\frac{100}{157.06}\right) = 32.48^{\circ}$ 

**Problem 5.6** Determine the impedance and phase angle in the circuit shown in Fig. 5.29.



Fig. 5.29

Solution Capacitive reactance 
$$X_C = \frac{1}{2\pi fC}$$
  
=  $\frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83 \ \Omega$ 

Capacitive susceptance  $B_c = \frac{1}{X_c}$   $= \frac{1}{31.83} = 0.031 \text{ S}$ Conductance  $G = \frac{1}{R} = \frac{1}{50} = 0.02 \text{ S}$ Total admittance  $Y = \sqrt{G^2 + B_c^2}$   $= \sqrt{(0.02)^2 + (0.031)^2}$  = 0.037 STotal impedance  $Z = \frac{1}{Y} = \frac{1}{0.037} = 27.02 \Omega$ Phase angle  $\theta = \tan^{-1}\left(\frac{R}{X_c}\right)$   $= \tan^{-1}\left(\frac{50}{31.83}\right)$  $\theta = 57.52^\circ$ 

**Problem 5.7** For the parallel circuit in Fig. 5.30, find the magnitude of current in each branch and the total current. What is the phase angle between the applied voltage and total current?



Fig. 5.30

Solution First let us find the capacitive reactances.

$$X_{C1} = \frac{1}{2\pi f C_1}$$
  
=  $\frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83 \ \Omega$   
 $X_{C2} = \frac{1}{2\pi f C_2} = \frac{1}{2\pi \times 50 \times 300 \times 10^{-6}}$   
= 10.61  $\Omega$ 

Here the voltage across each element is the same as the applied voltage.

22

Current in the 100  $\mu$ F capacitor  $I_C = \frac{V_S}{X_{C_1}}$   $= \frac{10 \angle 0^{\circ}}{31.83 \angle -90^{\circ}} = 0.31 \angle 90^{\circ} \text{ A}$ Current in the 300  $\mu$ F capacitor  $I_{C_2} = \frac{V_S}{X_{C_2}}$   $= \frac{10 \angle 0^{\circ}}{10.61 \angle -90^{\circ}} = 0.94 \angle 90^{\circ} \text{ A}$ Current in the 100  $\Omega$  resistor is  $I_{R_1} = \frac{V_S}{R_1} = \frac{10}{100} = 0.1 \text{ A}$ Current in the 200  $\Omega$  resistor is  $I_{R_2} = \frac{V_S}{R_2} = \frac{10}{200} = 0.05 \text{ A}$ Total current  $I_T = I_{R_1} + I_{R_2} + j(I_{C_1} + I_{C_2})$  $= 0.1 + 0.05 + j(0.31 + 0.94) = 1.26 \angle 83.2^{\circ} \text{ A}$ 

The circuit shown in Fig. 5.30 can be simplified into a single parallel RC circuit as shown in Fig. 5.31.



In Fig. 5.30, the two resistances are in parallel and can be combined into a single resistance. Similarly, the two capacitive reactances are in parallel and can be combined into a single capacitive reactance.

$$R = \frac{R_1 R_2}{R_1 + R_2} = 66.67 \ \Omega$$
$$X_C = \frac{X_{C_1} X_{C_2}}{X_{C_1} + X_{C_2}} = 7.96 \ \Omega$$

Phase angle  $\theta$  between voltage and current is

$$\theta = \tan^{-1}\left(\frac{R}{X_C}\right) = \tan^{-1}\left(\frac{66.67}{7.96}\right) = 83.19^{\circ}$$

22

Here the current leads the applied voltage by 83.19°.

**Problem 5.8** For the circuit shown in Fig. 5.32, determine the total impedance, total current and phase angle.



Fig. 5.32

Solution First, we calculate the magnitudes of the capacitive reactances.

$$X_{C_1} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83 \ \Omega$$
$$X_{C_2} = \frac{1}{2\pi \times 50 \times 300 \times 10^{-6}} = 10.61 \ \Omega$$

We find the impedance of the parallel portion by finding the admittance.

$$G_{2} = \frac{1}{R_{2}} = \frac{1}{50} = 0.02 \text{ S}$$

$$B_{C_{2}} = \frac{1}{X_{C_{2}}} = \frac{1}{10.61} = 0.094 \text{ S}$$

$$Y_{2} = \sqrt{G_{2}^{2} + B_{C_{2}}^{2}} = \sqrt{(0.02)^{2} + (0.094)^{2}} = 0.096 \text{ S}$$

$$Z_{2} = \frac{1}{Y_{2}} = \frac{1}{0.096} = 10.42 \Omega$$

The phase angle associated with the parallel portion of the circuit

$$\theta_P = \tan^{-1} (R_2 / X_{C_2}) = \tan^{-1}(50/10.61) = 78.02^{\circ}$$

The series equivalent values for the parallel portion are

$$R_{eq} = Z_2 \cos \theta_P = 10.42 \cos (78.02^\circ) = 2.16 \Omega$$
$$X_{C(eq)} = Z_2 \sin \theta_P = 10.42 \sin (78.02^\circ) = 10.19 \Omega$$

The total resistance

$$R_T = R_1 + R_{eq}$$
  
= (10 + 2.16) = 12.16 \Omega  
$$X_{C_T} = X_{C_1} + X_{C(eq)}$$
  
= (31.83 + 10.19) = 42.02 \Omega

Total impedance

$$Z_T = \sqrt{R_T^2 + X_{C_T}^2}$$
$$= \sqrt{(12.16)^2 + (42.02)^2} = 43.74 \ \Omega$$

We can also find the total current by using Ohm's law

$$I_T = \frac{V_S}{Z_T} = \frac{100}{43.74} = 2.29 \text{ A}$$

The phase angle

$$\theta = \tan^{-1} \left( \frac{X_{C_T}}{R_T} \right)$$
$$= \tan^{-1} \left( \frac{42.02}{12.16} \right) = 73.86^{\circ}$$

**Problem 5.9** Determine the voltage across each element of the circuit shown in Fig. 5.33 and draw the voltage phasor diagram.



Fig. 5.33

**Solution** First we calculate 
$$X_{L_1}$$
 and  $X_{L_2}$   
 $X_{L_1} = 2\pi f L_1 = 2\pi \times 50 \times 0.5 = 157.08 \ \Omega$   
 $X_{L_2} = 2\pi f L_2 = 2\pi \times 50 \times 1.0 = 314.16 \ \Omega$ 

Now we determine the impedance of each branch

$$Z_1 = \sqrt{R_1^2 + X_{L_1}^2} = \sqrt{(100)^2 + (157.08)^2} = 186.2 \ \Omega$$
$$Z_2 = \sqrt{R_2^2 + X_{L_2}^2} = \sqrt{(330)^2 + (314.16)^2} = 455.63 \ \Omega$$

The current in each branch

$$I_1 = \frac{V_S}{Z_1} = \frac{100}{186.2} = 0.537 \text{ A}$$
$$I_2 = \frac{V_S}{Z_2} = \frac{100}{455.63} = 0.219 \text{ A}$$

and

The voltage across each element

$$\begin{split} V_{R_1} &= I_1 R_1 = 0.537 \times 100 = 53.7 \text{ V} \\ V_{L_1} &= I_1 X_{L_1} = 0.537 \times 157.08 = 84.35 \text{ V} \\ V_{R_2} &= I_2 R_2 = 0.219 \times 330 = 72.27 \text{ V} \\ V_{L_2} &= I_2 X_{L_2} = 0.219 \times 314.16 = 68.8 \text{ V} \end{split}$$

The angles associated with each parallel branch are now determined.

$$\theta_1 = \tan^{-1}\left(\frac{X_{L_1}}{R_1}\right) = \tan^{-1}\left(\frac{157.08}{100}\right) = 57.52^\circ$$
$$\theta_2 = \tan^{-1}\left(\frac{X_{L_2}}{R_2}\right) = \tan^{-1}\left(\frac{314.16}{330}\right) = 43.59^\circ$$

i.e.  $I_1$  lags behind  $V_s$  by 57.52° and  $I_2$  lags behind  $V_s$  by 43.59° Here,  $V_{R_1}$  and  $I_1$  are in phase and therefore, lag behind  $V_s$  by 57.52°  $V_{R_2}$  and  $I_2$  are in phase, and therefore lag behind  $V_s$  by 43.59°  $V_{L_1}$  leads  $I_1$  by 90°, so its angle is 90° – 57.52° = 32.48°  $V_{L_2}$  leads  $I_2$  by 90°, so its angle is 90° – 43.59° = 46.41°

The phase relations are shown in Fig. 5.34.



Fig. 5.34

**Problem 5.10** In the series parallel circuit shown in Fig. 5.35, the effective value of voltage across the parallel parts of the circuits is 50 V. Determine the corresponding magnitude of V.



Fig. 5.35

**Solution** Here we can determine the current in each branch of the parallel part.

Current in the *j*3  $\Omega$  branch,  $I_1 = \frac{50}{3} = 16.67$  A Current in (10 + *j*30)  $\Omega$  branch,  $I_2 = \frac{50}{31.62} = 1.58$  A Total current  $I_T = 16.67 + 1.58 = 18.25$  A 5.23

Total impedance  $Z_T = 8.5 \angle 30^\circ + \frac{3 \angle 90^\circ \times (10 + j30)}{(10 + j30) + 3 \angle 90^\circ}$  $= 8.5 \angle 30^{\circ} + \frac{3 \angle 90^{\circ} \times 31.62 \angle 71.57^{\circ}}{10 + j33}$  $= 7.36 + j4.25 + \frac{94.86 \angle 161.57^{\circ}}{34.48 \angle 73.14^{\circ}}$  $= 7.36 + i 4.25 + 2.75 \angle 88.43^{\circ}$ = 7.36 + j4.25 + 0.075 + j2.75 $= (7.435 + i7) \Omega$  $= 10.21 / 43.27^{\circ}$ 

In polar form, total impedance is  $Z_T = 10.21 \angle 43.27^\circ$ The magnitude of applied voltage  $V = I \times Z_T = 18.25 \times 10.21 = 186.33$  V.

**Problem 5.11** For the series parallel circuit shown in Fig. 5.36, determine (a) the total impedance between the terminals a, b and state if it is inductive or capacitive (b) the voltage across in the parallel branch, and (c) the phase angle.



Fig. 5.36

Solution Here the parallel branch can be combined into a single branch

 $Z_P = (3 + j4) \parallel (3 + j4) = (1.5 + j2) \Omega$ 

Total impedance

$$Z_T = 1 + j2 + 1.5 + j2 = (2.5 + j4) \,\Omega$$

Hence the total impedance in the circuit is inductive Total current in the circuit

$$I_T = \frac{V_S}{Z_T} = \frac{10 + j20}{2.5 + j4}$$
$$= \frac{22.36 \angle 63.43^{\circ}}{4.72 \angle 57.99^{\circ}}$$
$$I_T = 4.74 \angle 5.44^{\circ} \text{ A}$$

....

i.e. the current lags behind the voltage by 57.99° Phase angle  $\theta = 57.99^{\circ}$ Voltage across in the parallel branch

$$V_{P} = (1.5 + i2) 4.74 \angle 5.44^{\circ}$$
Complex Impedance 5.25  
= 
$$2.5 \times 4.74 \angle (5.44^{\circ} + 53.13^{\circ})$$
  
=  $11.85 \angle 58.57^{\circ} V$ 

Problem 5.12 In the series parallel circuit shown in Fig. 5.37, the two parallel branches A and B are in series with C. The impedances are  $Z_A = 10 + j8$ ,  $Z_B = 9 - j6$ ,  $Z_C = 3 + j2$  and the voltage across the circuit is (100 + j0)  $\mathring{V}$ . Find the currents  $I_A$ ,  $I_B$  and the phase angle between them.



Fig. 5.37

Solution Total parallel branch impedance,

$$Z_{P} = \frac{Z_{A} Z_{B}}{Z_{A} + Z_{B}}$$
  
=  $\frac{(10 + j8) (9 - j6)}{19 + j2}$   
=  $\frac{12.8 \angle 38.66^{\circ} \times 10.82 \angle -33.7}{19.1 \angle 6^{\circ}} = 7.25 \angle -1.04^{\circ}$ 

In rectangular form,

Total parallel impedance  $Z_p = 7.25 - j0.13$ This parallel impedance is in series with  $Z_c$ Total impedance in the circuit

$$Z_T = Z_P + Z_C = 7.25 - j0.13 + 3 + j2 = (10.25 + j1.87) \Omega$$

Total current

$$I_T = \frac{1}{Z_T}$$

$$= \frac{(100 + j0)}{(10.25 + j1.87)} = \frac{100 \angle 0^\circ}{10.42 \angle 10.34^\circ}$$

$$= 9.6 \angle -10.34^\circ$$

The current lags behind the applied voltage by 10.34° Current in branch A is

 $V_{s}$ 

$$I_A = I_T \frac{Z_B}{Z_A + Z_B}$$
  
= 9.6 \angle - 10.34° \times \frac{(9 - j6)}{19 + j2}

$$=\frac{9.6 \angle -10.34^{\circ} \times 10.82 \angle -33.7^{\circ}}{19.1 \angle 6^{\circ}}$$

= 5.44 ∠– 50.04° A

Current in branch *B* is  $I_{B}$ 

$$I_B = I_T \times \frac{Z_A}{Z_A + Z_B}$$
  
= 9.6 \angle - 10.34° \times \frac{10 + j8}{19 + j2}  
= \frac{9.6 \angle - 10.34° \times 12.8 \angle 38.66°}{19.1 \angle 6°  
= 6.43 \angle 22.32° A

The angle between  $I_A$  and  $I_B$ ,

$$\theta = (50.04^\circ + 22.32^\circ) = 72.36^\circ$$

**Problem 5.13** A series circuit of two pure elements has the following applied voltage and resulting current.

$$V = 15 \cos (200 t - 30^{\circ}) \text{ volts}$$
  
I = 8.5 cos (200 t + 15) volts

Find the elements comprising the circuit.

**Solution** By inspection, the current leads the voltage by  $30^\circ + 15^\circ = 45^\circ$ . Hence the circuit must contain resistance and capacitance.

$$\tan 45 = \frac{1}{\omega CR}$$

$$1 = \frac{1}{\omega CR}, \quad \therefore \frac{1}{\omega C} = R$$

$$\frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \sqrt{R^2 + R^2} = \sqrt{2}R$$

$$R = \frac{15}{8.5 \times \sqrt{2}} = 1.248 \ \Omega$$

$$\frac{1}{\omega C} = 1.248 \ \Omega$$

$$C = \frac{1}{200 \times 1.248} = 4 \times 10^{-3} \ F$$

÷

and

**Problem 5.14** A resistor having a resistance of  $R = 10 \Omega$  and an unknown capacitor are in series. The voltage across the resistor is  $V_R = 50 \sin (1000 t + 45^\circ)$  volts. If the current leads the applied voltage by  $60^\circ$  what is the unknown capacitance *C*?

**Solution** Here, the current leads the applied voltage by  $60^{\circ}$ .

Since

$$\tan 60^{\circ} = \frac{1}{\omega CR}$$

$$R = 10 \ \Omega$$

$$\omega = 1000 \text{ radians}$$

$$\tan 60^{\circ} = \frac{1}{\omega CR}$$

$$C = \frac{1}{\tan 60 \times 1000 \times 10} = 57.7 \ \mu\text{F}$$

**Problem 5.15** A series circuit consists of two pure elements has the following current and voltage.

$$v = 100 \sin (2000 t + 50^{\circ}) V$$
  
 $i = 20 \cos (2000 t + 20^{\circ}) A$ 

Find the elements in the circuit.

**Solution** We can write  $i = 20 \sin (2000 t + 20^\circ + 90^\circ)$ Since  $\cos \theta = \sin (\theta + 90^\circ)$ Current  $i = 20 \sin (2000 t + 110^\circ)$  A The current leads the voltage by  $110^\circ - 50^\circ = 60^\circ$ and the circuit must consist of resistance and capacitance.

$$\tan \theta = \frac{1}{\omega CR}$$

$$\frac{1}{\omega C} = R \tan 60^\circ = 1.73 \text{ R}$$

$$\frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \frac{100}{20}$$

$$R\sqrt{1 + (1.73)^2} = \frac{100}{20}$$

$$R(1.99) = 5$$

$$R = 2.5 \Omega$$

$$C = \frac{1}{\omega (1.73 \text{ R})} = 115.6 \,\mu\text{F}$$

and

**Problem 5.16** A two branch parallel circuit with one branch of  $R = 100 \Omega$  and a single unknown element in the other branch has the following applied voltage and total current.

$$v = 2000 \cos (1000 t + 45^{\circ}) V$$
  
 $I_T = 45 \sin (1000 t + 135^{\circ}) A$ 

Find the unknown element.

**Solution** Here, the voltage applied is same for both elements.

Current passing through resistor is  $i_R = \frac{v}{R}$ 

$$i_R = 20 \cos(1000 t + 45^\circ)$$

Total current  $i_T = i_R + i_X$ 

Where  $I_x$  is the current in unknown element.

$$I_X = i_T - i_R$$
  
= 45 sin (1000 t + 135°) - 20 cos (1000 t + 45°)  
= 45 sin (1000 t + 135°) - 20 sin (1000 t + 135°)

Current passing through the unknown element.

$$I_X = 25 \sin(1000 t + 135^\circ)$$

Since the current and voltage are in phase, the element is a resistor. And the value of resistor

$$R = \frac{v}{i_X} = \frac{2000}{25} = 80 \ \Omega$$

**Problem 5.17** Find the total current to the parallel circuit with L = 0.05 H and  $C = 0.667 \mu$ F with an applied voltage of  $v = 200 \sin 5000t$  V.

Solution Current in the inductor  $i_L = \frac{1}{L} \int v dt$   $\therefore$   $i_L = \frac{1}{0.05} \int 200 \sin 5000 t$   $= \frac{-200 \cos 5000 t}{0.05 \times 5000}$   $i_L = -0.8 \cos 5000 t$ Current in the capacitor  $i_C = C \frac{dv}{dt}$ 

$$i_C = 0.667 \times 10^{-6} \frac{d}{dt} (200 \sin 5000 t)$$

$$i_C = 0.667 \cos 500 t$$

Total current  $i_T = i_L + i_C$ 

*:*..

 $= 0.667 \cos 5000 t - 0.8 \cos 5000 t$  $= -0.133 \cos (5000 t)$ 

Total current  $i_T = 0.133 \sin (5000 t - 90^\circ) \text{ A}$ 

...

\*\*



5.1 For the circuit shown in Fig. 5.38, determine the impedance, phase angle and total current.





5.2 Calculate the total current in the circuit in Fig. 5.39, and determine the voltage across resistor  $V_R$ , and across capacitor  $V_C$ .





5.3 Determine the impedance and phase angle in the circuit shown in Fig. 5.40.



Fig. 5.40

5.4 Calculate the impedance at each of the following frequencies; also determine the current at each frequency in the circuit shown in Fig. 5.41.(a) 100 Hz(b) 3 kHz





5.5 A signal generator supplies a sine wave of 10 V, 10 kHz, to the circuit shown in Fig. 5.42. Calculate the total current in the circuit. Determine the phase angle  $\theta$  for the circuit. If the total inductance in the circuit is doubled, does  $\theta$  increase or decrease, and by how many degrees?





5.6 For the circuit shown in Fig. 5.43, determine the voltage across each element. Is the circuit predominantly resistive or inductive? Find the current in each branch and the total current.



Fig. 5.43

5.7 Determine the total impedance  $Z_T$ , the total current  $I_T$ , phase angle  $\theta$ , voltage across inductor L, and voltage across resistor  $R_3$  in the circuit shown in Fig. 5.44.





5.8 For the circuit shown in Fig. 5.45, determine the value of frequency of supply voltage when a 100 V, 50 A current is supplied to the circuit.



Fig. 5.45

5.9 A sine wave generator supplies a signal of 100 V, 50 Hz to the circuit shown in Fig. 5.46. Find the current in each branch, and total current. Determine the voltage across each element and draw the voltage phasor diagram.



Fig. 5.46

5.10 Determine the voltage across each element in the circuit shown in Fig. 5.47. Convert the circuit into an equivalent series form. Draw the voltage phasor diagram.



Fig. 5.47

5.11 For the circuit shown in Fig. 5.48, determine the total current  $I_T$ , phase angle  $\theta$  and voltage across each element.



Fig. 5.48

5.12 For the circuit shown in Fig. 5.49, the applied voltage  $v = V_m \cos \omega t$ . Determine the current in each branch and obtain the total current in terms of the cosine function.





5.13 For the circuit shown in Fig. 5.50, the voltage across the inductor is  $v_L = 15 \sin 200 t$ . Find the total voltage and the angle by which the current lags the total voltage.



Fig. 5.50

- 5.14 In a parallel circuit having a resistance  $R = 5 \Omega$  and L = 0.02 H, the applied voltage is  $v = 100 \sin (1000 t + 50^{\circ})$  volts. Find the total current and the angle by which the current lags the applied voltage.
- 5.15 In the parallel circuit shown in Fig. 5.51, the current in the inductor is five times greater than the current in the capacitor. Find the element values.





5.16 In the parallel circuit shown in Fig. 5.52, the applied voltage is v = 100sin 5000 t V. Find the currents in each branch and also the total current in the circuit.



Fig. 5.52

5.17 For the circuit shown in Fig. 5.53, find the total current and the magnitude of the impedance.







- 1. A 1 kHz sinusoidal voltage is applied to an RL circuit, what is the frequency of the resulting current?
  - (a) 1 kHz

(b) 0.1 kHz

(c) 100 kHz

(d) 2 kHz

- 2. A series *RL* circuit has a resistance of 33 k $\Omega$ , and an inductive reactance of 50 k $\Omega$ . What is its impedance and phase angle?
  - (a) 56.58 Ω, 59.9°
  - (c) 59.9 Ω, 56.58°
- (b) 59.9 kΩ, 56.58°
- (d) 5.99 Ω, 56.58°

L

Γ

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3.	In a certain <i>RL</i> circuit, $V_R = 2$ V and	$V_L =$	3 V. What is the magnitude of
	the total voltage?		
	(a) 2 V	(b)	3 V
	(c) 5 V	(d)	3.61 V
4.	When the frequency of applied volta	ge in	a series RL circuit is increased
	what happens to the inductive reacta	nce?	
	(a) decreases	(b)	remains the same
	(c) increases	(d)	becomes zero
5.	In a certain parallel $RL$ circuit, $R =$	50 🖸	<b>2</b> , and $X_L = 75 \ \Omega$ . What is the
	admittance?		2
	(a) 0.024 S	(b)	75 S
	(c) 50 S	(d)	1.5 S
6.	What is the phase angle between the	ne in	ductor current and the applied
	voltage in a parallel <i>RL</i> circuit?		
	(a) $0^{\circ}$	(b)	45°
	(c) 90°	(d)	30°
7.	When the resistance in an RC circuit	t is g	reater than the capacitive reac-
	tance, the phase angle between the ar	plied	l voltage and the total current is
	closer to	1	5
	(a) 90°	(b)	0°
	(c) 45°	(d)	120°
8.	A series <i>RC</i> circuit has a resistance of	of 33	$k\Omega$ , and a capacitive reactance
	of 50 k $\Omega$ . What is the value of the ir	nped	ance.
	(a) 50 k $\Omega$	(b)	33 kΩ
	(c) $20 \text{ k}\Omega$	(d)	59.91 Ω
9.	In a certain series RC circuit, $V_R = 4$	V an	d $V_C = 6$ V. What is the magni-
	tude of the total voltage?		c c
	(a) 7.2 V	(b)	4 V
	(c) 6 V	(d)	52 V
10.	When the frequency of the applied	volta	ge in a series RC circuit is in-
	creased what happens to the capacity	ive re	eactance?
	(a) it increases	(b)	it decreases
	(c) it is zero	(d)	remains the same
11.	In a certain parallel <i>RC</i> circuit, $R = \frac{1}{2}$	50 Ω	and $X_C = 75 \Omega$ . What is Y?
	(a) 0.01 S	(b)	0.02 Š
	(c) 50 S	(d)	75 S
12.	The admittance of an RC circuit is (	0.003	5 S, and the applied voltage is
	6 V. What is the total current?		
	(a) 6 mA	(b)	20 mA
	(c) 21 mA	(d)	5 mA
13.	What is the phase angle between th	e caj	pacitor current and the applied
	voltage in a parallel <i>RC</i> circuit?	-	
	(a) 90°	(b)	0°
	(c) 45°	(d)	180°

- 14. In a given series RLC circuit,  $X_C$  is 150  $\Omega$ , and  $X_L$  is 80  $\Omega$ , what is the total reactance? What is the type of reactance?
  - (a) 70  $\Omega$ , inductive (b) 70  $\Omega$ , capacitive
  - (c) 70  $\Omega$ , resistive (d) 150  $\Omega$ , capacitive
- 15. In a certain series RLC circuit  $V_R = 24$  V,  $V_L = 15$  V, and  $V_C = 45$  V. What is the source voltage.
  - (a) 38.42 V (b) 45 V
  - (c) 15 V (d) 24 V

16. When  $R = 10 \Omega$ ,  $X_C = 18 \Omega$  and  $X_L = 12 \Omega$ , the current

- (a) leads the applied voltage
- (b) lags behind the applied voltage
- (c) is in phase with the voltage
- (d) is none of the above
- 17. A current  $i = A \sin 500 t$  A passes through the circuit shown in Fig. 5.54. The total voltage applied will be



Fig. 5.54

- (c)  $B \sin (500 t + \theta^{\circ})$  (d)  $B \cos (200 t + \theta^{\circ})$
- 18. A current of 100 mA through an inductive reactance of 100  $\Omega$  produces a voltage drop of
  - (a) 1 V (b) 6.28 V
  - (c) 10 V (d) 100 V
- 19. When a voltage  $v = 100 \sin 5000 t$  volts is applied to a series circuit of L = 0.05 H and unknown capacitance, the resulting current is  $i = 2 \sin (5000 t + 90^{\circ})$  amperes. The value of capacitance is
  - (a) 66.7 pF (b) 6.67 pF
  - (c)  $0.667 \,\mu\text{F}$  (d)  $6.67 \,\mu\text{F}$
- 20. A series circuit consists of two elements has the following current and applied voltage.

$$i = 4 \cos (2000 t + 11.32^{\circ}) \text{ A}$$
  
 $v = 200 \sin (2000 t + 50^{\circ}) \text{ V}$ 

The circuit elements are

- (a) resistance and capacitance (b) capacitance and inductance
  - (d) both resistances
- (c) inductance and resistance (d) both

21. A pure capacitor of  $C = 35 \ \mu\text{F}$  is in parallel with another signal circuit element. The applied voltage and resulting current are

$$v = 150 \sin 300 t \text{ V}$$
  
 $i = 16.5 \sin (3000 t + 72.4^{\circ}) \text{ A}$ 

The other element is

- (a) capacitor of 30  $\mu$ F
- (b) inductor of 30 mH

....

- (c) resistor of 30  $\Omega$
- (d) none of the above



## 6.1 INSTANTANEOUS POWER

In a purely resistive circuit, all the energy delivered by the source is dissipated in the form of heat by the resistance. In a purely reactive (inductive or capacitive) circuit, all the energy delivered by the source is stored by the inductor or capacitor in its magnetic or electric field during a portion of the voltage cycle, and then is returned to the source during another portion of the cycle, so that no net energy is transferred. When there is complex impedance in a circuit, part of the energy is alternately stored and returned by the reactive part, and part of it is dissipated by the resistance. The amount of energy dissipated is determined by the relative values of resistance and reactance.

Consider a circuit having complex impedance. Let  $v(t) = V_m \cos \omega t$  be the voltage applied to the circuit and let  $i(t) = I_m \cos (\omega t + \theta)$  be the corresponding current flowing through the circuit. Then the power at any instant of time is

$$P(t) = v(t) i(t)$$
  
=  $V_m \cos \omega t I_m \cos (\omega t + \theta)$  (6.1)

From Eq. 6.1, we get

$$P(t) = \frac{V_m I_m}{2} \left[ \cos \left( 2 \,\omega t + \theta \right) + \cos \,\theta \right] \tag{6.2}$$

Equation 6.2 represents *instantaneous power*. It consists of two parts. One is a fixed part, and the other is time-varying which has a frequency twice that of the voltage or current waveforms. The voltage, current and power waveforms are shown in Figs 6.1 and 6.2.

Here, the negative portion (hatched) of the power cycle represents the power returned to the source. Figure 6.2 shows that the instantaneous power is negative



Fig. 6.1



Fig. 6.2

whenever the voltage and current are of opposite sign. In Fig. 6.2, the positive portion of the power is greater than the negative portion of the power; hence the average power is always positive, which is almost equal to the constant part of the instantaneous power (Eq. 6.2). The positive portion of the power cycle varies with the phase angle between the voltage and current waveforms. If the circuit is pure resistive, the phase angle between voltage and current is zero; then there is no negative cycle in the P(t) curve. Hence, all the power delivered by the source is completely dissipated in the resistance.

If  $\theta$  becomes zero in Eq. 6.1, we get

$$P(t) = v(t) i(t)$$
  
=  $V_m I_m \cos^2 \omega t$   
=  $\frac{V_m I_m}{2} (1 + \cos 2\omega t)$  (6.3)

The waveform for Eq. 6.3, is shown in Fig. 6.3, where the power wave has a frequency twice that of the voltage or current. Here the average value of power is  $V_m I_m/2$ .

When phase angle  $\theta$  is increased, the negative portion of the power cycle increases and lesser power is dissipated. When  $\theta$  becomes  $\pi/2$ , the positive and negative portions of the power cycle are equal. At this instant, the power dissipated in the circuit is zero, i.e. the power delivered to the load is returned to the source.



Fig. 6.3

#### 6.2 AVERAGE POWER

To find the average value of any power function, we have to take a particular time interval from  $t_1$  to  $t_2$ ; by integrating the function from  $t_1$  to  $t_2$  and dividing the result by the time interval  $t_2 - t_1$ , we get the average power.

Average power 
$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} P(t) dt$$
 (6.4)

In general, the average value over one cycle is

$$P_{\rm av} = \frac{1}{T} \int_{0}^{T} P(t) dt$$
 (6.5)

By integrating the instantaneous power P(t) in Eq. 6.5 over one cycle, we get average power

$$P_{av} = \frac{1}{T} \int_{0}^{T} \left\{ \frac{V_m I_m}{2} \left[ \cos \left( 2\omega t + \theta \right) + \cos \theta \right] dt \right\}$$
$$= \frac{1}{T} \int_{0}^{T} \frac{V_m I_m}{2} \left[ \cos \left( 2\omega t + \theta \right) \right] dt + \frac{1}{T} \int_{0}^{T} \frac{V_m I_m}{2} \cos \theta dt$$
(6.6)

In Eq. 6.6, the first term becomes zero, and the second term remains. The average power is therefore

$$P_{\rm av} = \frac{V_m I_m}{2} \cos \theta \, \mathrm{W} \tag{6.7}$$

We can write Eq. 6.7 as

$$P_{\rm av} = \left(\frac{V_m}{\sqrt{2}}\right) \left(\frac{I_m}{\sqrt{2}}\right) \cos \theta \tag{6.8}$$

In Eq. 6.8,  $V_m/\sqrt{2}$  and  $I_m/\sqrt{2}$  are the effective values of both voltage and current.

$$P_{\rm av} = V_{\rm eff} I_{\rm eff} \cos \theta$$

To get average power, we have to take the product of the effective values of both voltage and current multiplied by cosine of the phase angle between voltage and the current.

If we consider a purely resistive circuit, the phase angle between voltage and current is zero. Hence, the average power is

$$P_{\rm av} = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R$$

If we consider a purely reactive circuit (i.e. purely capacitive or purely inductive), the phase angle between voltage and current is 90°. Hence, the average power is zero or  $P_{av} = 0$ .

If the circuit contains complex impedance, the average power is the power dissipated in the resistive part only.

**Example 6.1** A voltage of  $v(t) = 100 \sin \omega t$  is applied to a circuit. The current flowing through the circuit is  $i(t) = 15 \sin (\omega t - 30^\circ)$ . Determine the average power delivered to the circuit.

**Solution** The phase angle between voltage and current is 30°.

Effective value of the voltage  $V_{\text{eff}} = \frac{100}{\sqrt{2}}$ Effective value of the current  $I_{\text{eff}} = \frac{15}{\sqrt{2}}$ Average power  $P_{\text{av}} = V_{\text{eff}} I_{\text{eff}} \cos \theta$   $= \frac{100}{\sqrt{2}} \times \frac{15}{\sqrt{2}} \cos 30^{\circ}$  $= \frac{100 \times 15}{2} \times 0.866 = 649.5 \text{ W}$ 

**Example 6.2** Determine the average power delivered to the circuit consisting of an impedance Z = 5 + j8 when the current flowing through the circuit is  $I = 5 \angle 30^{\circ}$ .

**Solution** The average power is the power dissipated in the resistive part only.

or

 $P_{\rm av} = \frac{I_m^2}{2} R$ 

:. 
$$P_{\rm av} = \frac{5^2}{2} \times 5 = 62.5 \text{ W}$$

 $I_m = 5 \text{ A}$ 

*.*•.

# 6.3 APPARENT POWER AND POWER FACTOR

The power factor is useful in determining useful power (true power) transferred to a load. The highest power factor is 1, which indicates that the current to a load is in phase with the voltage across it (i.e. in the case of resistive load). When the power factor is 0, the current to a load is 90° out of phase with the voltage (i.e. in case of reactive load).

Consider the following equation

$$P_{\rm av} = \frac{V_m I_m}{2} \cos \theta \, W \tag{6.9}$$

In terms of effective values

$$P_{\rm av} = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos \theta$$
$$= V_{\rm eff} I_{\rm eff} \cos \theta \, W \tag{6.10}$$

The average power is expressed in watts. It means the useful power transferred from the source to the load, which is also called true power. If we consider a dc source applied to the network, true power is given by the product of the voltage and the current. In case of sinusoidal voltage applied to the circuit, the product of voltage and current is not the true power or average power. This product is called *apparent power*. The apparent power is expressed in volt amperes, or simply VA.

 $\therefore$  Apparent power =  $V_{\text{eff}} I_{\text{eff}}$ 

In Eq. 6.10, the average power depends on the value of  $\cos \theta$ ; this is called the *power factor* of the circuit.

$$\therefore \qquad \text{Power factor (pf)} = \cos \theta = \frac{P_{\text{av}}}{V_{\text{eff}} I_{\text{eff}}}$$

Therefore, power factor is defined as the ratio of average power to the apparent power, whereas apparent power is the product of the effective values of the current and the voltage. Power factor is also defined as the factor with which the volt amperes are to be multiplied to get true power in the circuit.

In the case of sinusoidal sources, the power factor is the cosine of the phase angle between voltage and current

$$pf = \cos \theta$$

As the phase angle between voltage and total current increases, the power factor decreases. The smaller the power factor, the smaller the power dissipation. The power factor varies from 0 to 1. For purely resistive circuits, the phase angle between voltage and current is zero, and hence the power factor is unity. For purely reactive circuits, the phase angle between voltage and current is 90°, and hence the power factor is zero. In an RC circuit, the power factor is referred to as *leading* power factor because the current leads the voltage. In an RL circuit, the power factor is referred to as lagging power factor because the current lags behind the voltage.

**Example 6.3** A sinusoidal voltage  $v = 50 \sin \omega t$  is applied to a series RL circuit. The current in the circuit is given by  $i = 25 \sin (\omega t - 53^{\circ})$ . Determine (a) apparent power (b) power factor and (c) average power.

**Solution** (a) Apparent power  $P = V_{\text{eff}} I_{\text{eff}}$  $= \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$  $= \frac{50 \times 25}{2} = 625 \text{ VA}$ 

> (b) Power factor = cos θ where θ is the angle between voltage and current θ = 53°
> ∴ Power factor = cos θ = cos 53° = 0.6
> (c) Average Power P<sub>av</sub> = V<sub>eff</sub> I<sub>eff</sub> cos θ

> > $= 625 \times 0.6 = 375$  W

## 6.4 REACTIVE POWER

We know that the average power dissipated is

$$P_{\rm av} = V_{\rm eff} \left[ I_{\rm eff} \cos \theta \right] \tag{6.11}$$

From the impedance triangle shown in Fig. 6.4

$$\cos \theta = \frac{R}{|Z|} \tag{6.12}$$

and

If we substitute Eqs. (6.12) and (6.13) in Eq. (6.11), we get

 $V_{\rm eff} = I_{\rm eff} Z$ 

 $P_r = iv_I$ 

$$P_{\rm av} = I_{\rm eff} Z \left[ I_{\rm eff} \frac{R}{Z} \right]$$
$$= I_{\rm eff}^2 R \text{ watts}$$

This gives the average power dissipated in a resistive circuit.

If we consider a circuit consisting of a pure inductor, the power in the inductor

(6.13)

$$= iL \frac{di}{dt}$$
Consider  $i = I_m \sin(\omega t + \theta)$   
Then  $P_r = I_m^2 \sin(\omega t + \theta) L\omega \cos(\omega t + \theta)$   
 $= (\omega L) \sin 2 (\omega t + \theta)$   
 $\therefore P_r = I_{eff}^2 (\omega L) \sin 2(\omega t + \theta)$  (6.16)



(6.14)

(6.15)

...

From the above equation, we can say that the average power delivered to the circuit is zero. This is called *reactive* power. It is expressed in volt-amperes reactive (VAR).

$$P_r = I_{\rm eff}^2 X_L \,\rm VAR \tag{6.17}$$

From Fig. 6.4, we have

$$X_L = Z \sin \theta \tag{6.18}$$

Substituting Eq. 6.18 in Eq. 6.17, we get

$$P_r = I_{\text{eff}}^2 Z \sin \theta$$
$$= (I_{\text{eff}} Z) I_{\text{eff}} \sin \theta$$
$$= V_{\text{eff}} I_{\text{eff}} \sin \theta \text{ VAR}$$

## 6.5 THE POWER TRIANGLE

A generalised impedance phase diagram is shown in Fig. 6.5. A phasor relation for power can also be represented by a similar diagram because of the fact that true power  $P_{av}$  and reactive power  $P_r$  differ from R and X by a factor  $I_{eff}^2$ , as shown in Fig. 6.5.

The resultant power phasor  $I_{\text{eff}}^2 Z$ , represents the apparent power  $P_a$ . At any instant in time,  $P_a$  is the total power that appears to be transferred between the source and reactive circuit. Part of the apparent power is true power and part of it is reactive power.

$$\therefore \qquad P_a = I_{\text{eff}}^2 Z$$

The power triangle is shown in Fig. 6.6.



Fig. 6.5

Fig. 6.6

From Fig. 6.6, we can write

$$P_{\rm true} = P_a \cos \theta$$

or average power  $P_{av} = P_a \cos \theta$ and reactive power  $P_r = P_a \sin \theta$  ADDITIONAL SOLVED PROBLEMS

**Problem 6.1** In the circuit shown in Fig. 6.7, a voltage of  $v(t) = 50 \sin(wt+30^\circ)$  is applied. Determine the true power, reactive power and power factor.

**Solution** The voltage applied to the circuit is  $v(t) = 50 \sin(wt + 30^\circ)$ 

The current in the circuit is

$$I = \frac{V}{Z} = \frac{50 \angle 30^{\circ}}{10 + j30} = \frac{50 \angle 30^{\circ}}{31.6 \angle 71.56^{\circ}}$$
$$= 1.58 \angle -41.56^{\circ} \text{ A}$$

The phasor diagram is shown in Fig. 6.8. The phase angle between voltage and current  $\theta = 71.56^{\circ}$ Power factor =  $\cos \theta = \cos 71.56^{\circ} = 0.32$ 

True power or average power

$$P_{av} = V_{eff} I_{eff} \cos \theta$$

$$= \frac{50 \times 1.58}{\sqrt{2} \times \sqrt{2}} \cos 71.56^{\circ}$$

$$= 12.49 W$$
Reactive power =  $V_{eff} I_{eff} \sin \theta$ 

$$= \frac{50 \times 1.58}{\sqrt{2} \times \sqrt{2}} \sin 71.56^{\circ}$$
Fig. 6.8
$$= 37.47 VAR$$

**Problem 6.2** Determine the circuit constants in the circuit shown in Fig. 6.9, if the applied voltage to the circuit  $v(t) = 100 \sin (50t + 20^\circ)$ . The true power in the circuit is 200 W and the power factor is 0.707 lagging.

**Solution** Power factor =  $\cos \theta = 0.707$   $\therefore$  The phase angle between voltage and current  $\theta = \cos^{-1} 0.707 = 45^{\circ}$ Here the current lags behind the voltage by 45°. Hence, the current equation is  $i(t) = I_m \sin (50t - 25^{\circ})$ True power =  $V_{\text{eff}} I_{\text{eff}} \cos \theta = 200 \text{ W}$ 

$$I_{\rm eff} = \frac{200}{V_{\rm eff} \cos \theta}$$
  
=  $\frac{200}{(100/\sqrt{2}) \times 0.707} = 4 \text{ A}$   
 $I_m = 4 \times \sqrt{2} = 5.66 \text{ A}$ 



ί 30 Ω

m

v(t)

Fig. 6.7



ine the true power, reactive power ed to the circuit is  $10 \Omega$  :. The current equation is  $i(t) = 5.66 \sin (50t - 25^\circ)$ The impedance of the circuit

	$Z = \frac{V}{I} = \frac{(100/\sqrt{2}) \angle 20^{\circ}}{(5.66/\sqrt{2}) \angle -25^{\circ}}$
<i>.</i>	$Z = 17.67 \angle 45^{\circ} = 12.5 + j12.5$
Since	$Z = R + jX_L = 12.5 + j12.5$
.:.	$R = 12.5$ ohms, $X_L = 12.5$ ohms
	$X_L = \omega L = 12.5$
	$L = \frac{12.5}{1000} = 0.25 \text{ H}$

50

**Problem 6.3** A voltage  $v(t) = 150 \sin 250t$  is applied to the circuit shown in Fig. 6.10. Find the power delivered to the circuit and the value of inductance in Henrys. 10  $\Omega$  j 15  $\Omega$ 

000  $Z = 10 + i15 \Omega$ Solution  $Z = 18 \angle 56.3^{\circ}$ The impedance The impedance of the circuit  $Z = \frac{V}{I}$ v(t) $18 \angle 56.3^\circ = \frac{(150/\sqrt{2}) \angle 0^\circ}{I}$ Fig. 6.10  $I = \frac{150/\sqrt{2}}{18 \swarrow 56.3^{\circ}} = 5.89 \angle -56.3^{\circ}$ .: Phasor current  $= 5.89 \sqrt{2} \sin (250t - 56.3^{\circ})$ The current equation is i(t) $= 8.33 \sin (250t - 56.3^{\circ})$ The phase angle between the current and the voltage  $\theta = 56.3^{\circ}$ The power delivered to the circuit  $P_{\rm av} = VI\cos\theta$  $=\frac{150}{\sqrt{2}}\times\frac{8.33}{\sqrt{2}}\cos 56.3^{\circ}$ = 346.6 W

The inductive impedance  $X_{L} = 15 \Omega$ 

 $\therefore \qquad \omega L = 15$ 

:. 
$$L = \frac{15}{250} = 0.06 \text{ H}$$

•••

Problem 6.4 Determine the power factor, true power, reactive power and apparent power in the circuit in Fig. 6.11.



Fig. 6.11

**Solution** The impedance of the circuit

 $Z = \sqrt{R^2 + X_C^2}$  $=\sqrt{(100)^2+(200)^2}=223.6 \,\Omega$ 

The current

$$I = \frac{V_S}{Z} = \frac{50}{223.6} = 0.224 \text{ A}$$

The phase angle

The phase angle  

$$\theta = \tan^{-1} \left( \frac{-X_C}{R} \right)$$

$$= \tan^{-1} \left( \frac{-200}{100} \right) = -63.4^{\circ}$$

$$\therefore \text{ The power factor} \qquad pf = \cos \theta = \cos (63.4^{\circ}) = 0.448$$
The true power  

$$P_{av} = VI \cos \theta$$

$$= 50 \times 0.224 \times 0.448 = 5.01 \text{ W}$$
The reactive power  

$$P_v = I^2 X_C$$

$$= (0.224)^2 \times 200 = 10.03 \text{ VAR}$$
The apparent power  

$$P_a = I^2 Z = (0.224)^2 \times 223.6 = 11.21 \text{ VA}$$

Problem 6.5 In a certain RC circuit, the true power is 300 W and the reactive power is 1000 W. What is the apparent power?

**Solution** The true power  $P_{\text{true}}$  or  $P_{\text{av}} = VI \cos \theta$ = 300 WThe reactive power  $P_r = VI \sin \theta$ = 1000 W

From the above results

$$\tan \theta = \frac{1000}{300} = 3.33$$

The phase angle between voltage and current,  $\theta = \tan^{-1} 3.33 = 73.3^{\circ}$ 

The apparent power  $P_a = VI = \frac{300}{\cos 73.3^\circ} = 1043.9 \text{ VA}$ 

**Problem 6.6** A sine wave of  $v(t) = 200 \sin 50t$  is applied to a 10  $\Omega$  resistor in series with a coil. The reading of a voltmeter across the resistor is 120 V and across the coil, 75 V. Calculate the power and reactive volt-amperes in the coil and the power factor of the circuit.

**Solution** The rms value of the sine wave

	$V = \frac{200}{\sqrt{2}} = 141.4 \text{ V}$
Voltage across the resisto	or, $V_R = 120$ V
Voltage across the coil,	$V_L = 75 \text{ V}$
<i>:</i>	IR = 120  V
The current in resistor,	$I = \frac{120}{10} = 12 \text{ A}$
Since	$IX_L = 75 \text{ V}$
	$X_L = \frac{75}{12} = 6.25 \ \Omega$
Power factor,	$pf = \cos \theta = \frac{R}{Z}$
where	$Z = 10 + j6.25 = 11.8 \angle 32^{\circ}$
	$\cos \theta = \frac{R}{Z} = \frac{10}{11.8} = 0.85$
True power	$P_{\text{true}} = I^2 R = (12)^2 \times 10 = 1440 \text{ W}$
Reactive power	$P_r = I^2 X_L = (12)^2 \times 6.25 = 900 \text{ VAR}$

**Problem 6.7** For the circuit shown in Fig. 6.12, determine the true power, reactive power and apparent power in each branch. What is the power factor of the total circuit?



Fig. 6.12

**Solution** In the circuit shown in Fig. 6.12, we can calculate  $Z_1$  and  $Z_2$ .

Impedance 
$$Z_{1} = \frac{100 \ \angle 15^{\circ}}{50 \ \angle 10^{\circ}} = 2 \ \angle 5^{\circ} = (1.99 + j0.174) \ \Omega$$
  
Impedance 
$$Z_{2} = \frac{100 \ \angle 15^{\circ}}{20 \ \angle 30^{\circ}} = 5 \ \angle -15^{\circ} = (4.83 - j1.29) \ \Omega$$
  
True power in branch 
$$Z_{1} \text{ is } P_{t_{1}} = I_{1}^{2} R = (50)^{2} \times 1.99 = 4975 \ W$$

 $Z_1, P_{r_1} = I_1^2 X_L$ = (50)<sup>2</sup> × 0.174 = 435 VAR

 $Z_2, P_{r_2} = I_2^2 X_C$ = (20)<sup>2</sup> × 1.29 = 516 VAR

R

Reactive power in branch

Apparent power in branch

Apparent power in branch 
$$Z_1, P_{a_1} = I_1^2 Z$$
  
 $= (50)^2 \times 2$   
 $= 2500 \times 2 = 5000 \text{ VA}$   
True power in branch  $Z_2, P_{i_2} = I_2^2 R$   
 $= (20)^2 \times 4.83 = 1932 \text{ W}$ 

Reactive power in branch

Apparent power in branch

Apparent power in branch 
$$Z_{2}, P_{a_{2}} = I_{2}^{2} Z$$
$$= (20)^{2} \times 5 = 2000 \text{ VA}$$
Total impedance of the circuit,  $Z = \frac{Z_{1} Z_{2}}{Z_{1} + Z_{2}}$ 

$$= \frac{2 \angle 5^{\circ} \times 5 \times \angle -15^{\circ}}{1.99 + j0.174 + 4.83 - j1.29}$$
$$= \frac{10 \angle -10^{\circ}}{6.82 - j1.116}$$
$$= \frac{10 \angle -10^{\circ}}{6.9 \angle -9.29^{\circ}} = 1.45 \angle -0.71^{\circ}$$

The phase angle between voltage and current,  $\theta = 0.71^{\circ}$ 

Power factor  $pf = \cos \theta$ *:*..

 $= \cos 0.71^{\circ} = 0.99$  leading

**Problem 6.8** A voltage of  $v(t) = 141.4 \sin \omega t$  is applied to the circuit shown in Fig. 6.13. The circuit dissipates 450 W at a lagging power factor, when the voltmeter and ammeter readings are 100 V and 6 A, respectively. Calculate the circuit constants.



**Solution** The magnitude of the current passing through  $(10 + jX_2) \Omega$  is I = 6 A

The magnitude of the voltage across the  $(10 + jX_2)$  ohms, V = 100 V. The magnitude of impedance  $(10 + jX_2)$  is V/I.

Hence 
$$\sqrt{10^2 + X_2^2} = \frac{100}{6} = 16.67 \ \Omega$$

...

$$X_2 = \sqrt{(16.67)^2 - (10)^2} = 13.33 \ \Omega$$

Total power dissipated in the circuit =  $VI \cos \theta = 450$  W

:.

$$V = \frac{141.4}{\sqrt{2}} = 100 \text{ V}$$
$$I = 6 \text{ A}$$
$$100 \times 6 \times \cos \theta = 450$$

1/1 /

The power factor  $pf = \cos \theta = \frac{450}{600} = 0.75$ 

$$\theta = 41.4^{\circ}$$

The current lags behind the voltage by 41.4°

The current passing through the circuit,  $I = 6 \angle -41.4^{\circ}$ The voltage across  $(10 + j13.33) \Omega$ ,  $V = 6 \angle -41.4^{\circ} \times 16.66 \angle 53.1^{\circ}$   $= 100 \angle 11.7^{\circ}$ The voltage across parallel branch,  $V_1 = 100 \angle 0^{\circ} - 100 \angle 11.7^{\circ}$  = 100 - 97.9 - j20.27 $= (2.1 - j20.27) V = 20.38 \angle -84.08^{\circ}$ 

The current in (-*j*20) branch,  $I_2 = \frac{20.38 \angle -84.08^{\circ}}{20 \angle -90^{\circ}} = 1.02 \angle + 5.92^{\circ}$ 

The current in  $(R_1 - jX_1)$  branch,  $I_1$ 

$$= 6 \angle -41.4^{\circ} - 1.02 \angle 5.92^{\circ} = 4.5 - j3.97 - 1.01 - j0.1$$
$$= 3.49 - j4.07 = 5.36 \angle -49.39^{\circ}$$

The impedance  $Z_{1} = \frac{V_{1}}{I_{1}} = \frac{20.38 \angle -84.08^{\circ}}{5.36 \angle -49.39^{\circ}}$  $= 3.8 \angle -34.69^{\circ} = (3.12 - j2.16) \Omega$ Since  $R_{1} - jX_{1} = (3.12 - j2.16) \Omega$  $R_{1} = 3.12 \Omega$  $X_{1} = 2.16 \Omega$ 

**Problem 6.9** Determine the value of the voltage source and power factor in the following network if it delivers a power of 100 W to the circuit shown in Fig. 6.14. Find also the reactive power drawn from the source.



Solution Total impedance in the circuit,

$$Z_{eq} = 5 + \frac{(2+j2)(-j5)}{2+j2-j5}$$
  
= 5 +  $\frac{10-j10}{2-j3} = 5 + \frac{14.14\angle -45^{\circ}}{3.6\angle -56.3^{\circ}} = 5 + 3.93 \angle 11.3^{\circ}$   
= 5 + 3.85 + j0.77 = 8.85 + j0.77 = 8.88 \angle 4.97^{\circ}

Power delivered to the circuit,  $P_T = I^2 R_T = 100 \text{ W}$  $\therefore \qquad I^2 \times 8.85 = 100$ 

Current in the circuit, 
$$I = \sqrt{\frac{100}{8.85}} = 3.36 \text{ A}$$

Power factor  $pf = \cos \theta = \frac{R}{Z}$ 

$$=\frac{8.85}{8.88}=0.99$$

Since

$$VI \cos \theta = 100 \text{ W}$$
  
 $V \times 3.36 \times 0.99 = 100$   
 $V = \frac{100}{3.36 \times 0.99} = 30.06 \text{ V}$ 

*:*..

The value of the voltage source, V = 30.06 V

Reactive power 
$$P_r = VI \sin \theta$$
  
= 30.06 × 3.36 × sin (4.97°)  
= 30.06 × 3.36 × 0.087 = 8.8 VAR

**Problem 6.10** For the circuit shown in Fig. 6.15, determine the circuit constants when a voltage of 100 V is applied to the circuit, and the total power absorbed is 600 W. The circuit constants are adjusted such that the currents in the parallel branches are equal and the voltage across the inductance is equal and in quadrature with the voltage across the parallel branch.



Fig. 6.15

**Solution** Since the voltages across the parallel branch and the inductance are in quadrature, the total voltage becomes  $100 \angle 45^{\circ}$  as shown in Fig. 6.16.

Total voltage is  $100 \angle 45^\circ = V + i0 + 0 + iV$ V∠90° From the above result, 70.7 + j70.7 = V + jV100 ∠ 45° V = 70.7:. If we take current as the reference, then current passing through the circuit is  $I \angle 0^\circ$ . Total power absorbed by the circuit 45°  $= VI \cos \theta = 600 \text{ W}$ V∠0°  $100 \times I \times \cos 45^\circ = 600 \text{ W}$ or Fig. 6.16 ... I = 8.48 AHence, the inductance,  $X_1 = \frac{V \angle 90^{\circ}}{L \angle 00^{\circ}} = \frac{70.7 \angle 90^{\circ}}{8.48} = 8.33 \angle 90^{\circ}$  $X_1 = 8.33 \ \Omega$ ... Current through the parallel branch,  $R_1$  is I/2 = 4.24 A Resistance,  $R_1 = \frac{V \angle 0}{I/2 \angle 0} = \frac{70.7}{4.24} = 16.67 \ \Omega$ Current through parallel branch  $R_2$  is I/2 = 4.24 A Resistance is  $R_2 = \frac{70.7}{4.24} = 16.67 \,\Omega$ 

**Problem 6.11** Determine the average power delivered by the 500  $\angle 0^{\circ}$ voltage source in Fig. 6.17 and also dependent source.



Fig. 6.17

**Solution** The current *I* can be determined by using Kirchhoff's voltage law.

where

$$7+4$$

$$v_4 = 4I$$

$$I = \frac{500 \angle 0^{\circ}}{11} - \frac{12I}{11}$$

$$I = 21.73 \angle 0^{\circ}$$

 $I = \frac{500 \angle 0^\circ - 3v_4}{1}$ 

Power delivered by the 500  $\angle 0^{\circ}$  voltage source =  $\frac{500 \times 21.73}{2} = 5.432$  kW

Power delivered by the dependent voltage source =  $\frac{3v_4 \times I}{2} = \frac{3 \times 4I \times I}{2} = 2.833 \text{ kW}$ 

**Problem 6.12** Find the average power delivered by the dependent voltage source in the circuit shown in Fig. 6.18.



Fig. 6.18

Solution The circuit is redrawn as shown in Fig. 6.19.



Fig. 6.19

Assume current  $I_1$  flowing in the circuit.

The current  $I_1$  can be determined by using Kirchhoff's voltage law.

$$I_{1} = \frac{100 \angle 20^{\circ} + 10 \times 5I_{1}}{5 + j4}$$
$$I_{1} - \frac{50I_{1}}{5 + j4} = \frac{100 \angle 20^{\circ}}{5 + j4}$$
$$I_{1} = 2.213 \angle -154.9^{\circ}$$

Average power delivered by the dependent source

$$= \frac{V_m I_m}{2} \cos \theta$$
$$= \frac{10V_5 I_1}{2} \cos \theta$$
$$= \frac{50 \times (2.213)^2}{2} = 122.43 \text{ W}$$

**Problem 6.13** For the circuit shown in Fig. 6.20, find the average power delivered by the voltage source.



Fig. 6.20

Solution Applying Kirchhoff's current law at node

$$\frac{V - 100 \ \ 20^{\circ}}{2} + \frac{V}{1 + j3} + \frac{V - 50V_x}{-j4} = 0$$
$$V_x = \frac{V}{1 + j3} \text{ volts}$$

Substituting in the above equation, we get

$$\frac{V - 100 \angle 0^{\circ}}{2} + \frac{V}{1 + j3} + \frac{V}{-j4} - \frac{50 V}{(1 + j3)(-j4)} = 0$$
  

$$V = 14.705 \angle 157.5^{\circ}$$
  

$$I = \frac{V - 100 \angle 0^{\circ}}{2} = \frac{14.705 \angle 157.5^{\circ} - 100 \angle 0^{\circ}}{2}$$
  

$$= 56.865 \angle 177.18^{\circ}$$

Power delivered by the source =  $\frac{100 \times 56.865 \cos 177.18^{\circ}}{2}$ 

**Problem 6.14** For the circuit shown in Fig. 6.21, find the average power delivered by the dependent current source.



Fig. 6.21

Solution Applying Kirchhoff's current law at node

$$\frac{V - 20 \angle 0^{\circ}}{10} - 0.5V_1 + \frac{V}{20} = 0$$
$$V_1 = 20 \angle 0^{\circ} - V$$

where

Substituting  $V_1$  in the above equation, we get

$$V = 18.46 \angle 0^{\circ}$$
  
 $V_1 = 1.54 \angle 0^{\circ}$ 

Average power delivered by the dependent source

$$\frac{V_m I_m \cos \theta}{2} = \frac{18.46 \times 0.5 \times 1.54}{2} = 7.107 \text{ W}$$



# **PRACTICE PROBLEMS**

6.1 For the circuit shown in Fig. 6.22, a voltage of 250 sin  $\omega t$  is applied. Determine the power factor of the circuit, if the voltmeter readings are  $V_1$ = 100 V,  $V_2$  = 125 V,  $V_3$  = 150 V and the ammeter reading is 5 A.





6.2 For the circuit shown in Fig. 6.23, a voltage v(t) is applied and the resulting current in the circuit  $i(t) = 15 \sin (\omega t + 30^\circ)$  amperes. Determine the active power, reactive power, power factor, and the apparent power.



Fig. 6.23

6.3 A series RL circuit draws a current of  $i(t) = 8 \sin (50t + 45^\circ)$  from the source. Determine the circuit constants, if the power delivered by the source is 100 W and there is a lagging power factor of 0.707.

- 6.4 Two impedances,  $Z_1 = 10 \angle -60^\circ$  W and  $Z_2 = 16 \angle 70^\circ \Omega$  are in series and pass an effective current of 5 A. Determine the active power, reactive power, apparent power and power factor.
- 6.5 For the circuit shown in Fig. 6.24, determine the value of the impedance if the source delivers a power of 200 W and there is a lagging power factor of 0.707. Also find the apparent power.



Fig. 6.24

- 6.6 A voltage of  $v(t) = 100 \sin 500 t$  is applied across a series R-L-C circuit where  $R = 10 \Omega$ , L = 0.05 H and  $C = 20 \mu f$ . Determine the power supplied by the source, the reactive power supplied by the source, the reactive power of the capacitor, the reactive power of the inductor, and the power factor of the circuit.
- 6.7 For the circuit shown in Fig. 6.25 determine the power dissipated and the power factor of the circuit.



Fig. 6.25

6.8 For the circuit shown in Fig. 6.26, determine the power factor and the power dissipated in the circuit.



6.19

6.9 For the circuit shown in Fig. 6.27, determine the power factor, active power, reactive power and apparent power.





6.10 In the parallel circuit shown in Fig. 6.28, the power in the 5  $\Omega$  resistor is 600 W and the total circuit takes 3000 VA at a leading power factor of 0.707. Find the value of impedance Z.



Fig. 6.28

6.11 For the parallel circuit shown in Fig. 6.29, the total power dissipated is 1000 W. Determine the apparent power, the reactive power, and the power factor.



Fig. 6.29

6.12 A voltage source  $v(t) = 150 \sin \omega t$  in series with 5  $\Omega$  resistance is supplying two loads in parallel,  $Z_A = 60 \angle 30^\circ$ , and  $Z_B = 50 \angle -25^\circ$ . Find the average power delivered to  $Z_A$ , the average power delivered to  $Z_B$ , the average power dissipated in the circuit, and the power factor of the circuit. 6.13 For the circuit shown in Fig. 6.30, determine the true power, reactive power and apparent power in each branch. What is the power factor of the total circuit?





6.14 Determine the value of the voltage source, and the power factor in the network shown in Fig. 6.31 if it delivers a power of 500 W to the circuit shown in Fig. 6.31. Also find the reactive power drawn from the source.



Fig. 6.31

6.15 Find the average power dissipated by the 500  $\Omega$  resistor shown in Fig. 6.32.



Fig. 6.32

6.16 Find the power dissipated by the voltage source shown in Fig. 6.33.



Fig. 6.33

6.17 Find the power delivered by current source shown in Fig. 6.34.





6.18 For the circuit shown in Fig. 6.35, determine the power factor, active power, reactive power and apparent power.





**OBJECTIVE-TYPE QUESTIONS** 

- 1. The phasor combination of resistive power and reactive power is called
  - (a) true power
  - (c) reactive power
- 2. Apparent power is expressed in
  - (a) volt-amperes
  - (c) volt-amperes or watts
- 3. A power factor of '1' indicates
  - (a) purely resistive circuit,
  - (c) combination of both, (a) and (b) (d) none of these
- 4. A power factor of 0 indicates
  - (a) purely resistive element
  - (c) combination of both (a) and (b) (d) none of the above
- 5. For a certain load, the true power is 100 W and the reactive power is 100 VAR. What is the apparent power? (a) 200 VA
  - (b) 100 VA
  - (c) 141.4 VA (d) 120 VA
- 6. If a load is purely resistive and the true power is 5 W, what is the apparent power?

- (b) watts
- (d) VAR
- (b) purely reactive circuit

(b) purely reactive element

- (b) apparent power (d) average power

	•	
	(a) 10 VA	(b) 5 VA
	(c) 25 VA	(d) $\sqrt{50}$ VA
7.	True power is defined as	
	(a) $VI \cos \theta$	(b) <i>VI</i>
	(c) $VI \sin \theta$	(d) none of these
8.	In a certain series RC circuit, the tr	rue power is 2 W, and the reactive
	power is 3.5 VAR. What is the appar	rent power?
	(a) 3.5 VA	(b) 2 VA
	(c) 4.03 VA	(d) 3 VA
9.	If the phase angle $\theta$ is 45°, what is the	ne power factor?
	(a) $\cos 45^{\circ}$	(b) sin 45°
	(c) $\tan 45^{\circ}$	(d) none of these
10.	To which component in an RC circu	it is the power dissipation due?
	(a) capacitance	(b) resistance
	(c) both	(d) none
11.	A two element series circuit with an	n instantaneous current $I = 4.24 \sin \theta$
	$(5000 \ t + 45^{\circ})$ A has a power of	180 watts and a power factor of
	0.8 lagging. The inductance of the ci	rcuit must have the value.
	(a) 3 H	(b) 0.3 H
	(c) 3 mH	(c) 0.3 mH
12.	In the circuit shown in Fig. 6.36, if br	anch A takes 8 KVAR, the power of

12. In the circuit shown in Fig. 6.36, if branch A takes 8 KVAR, the power of the circuit will be



Fig. 6.36

(a)	2 kW	(b) 4 kW
(c)	6 kW	(d) 8 kW

13. In the circuit shown in Fig. 6.37, the voltage across 30  $\Omega$  resistor is 45 volts. The reading of the ammeter A will be



Fig. 6.37

L

Γ

(a)	10 A	(b)	19.4 A
(c)	22.4 A	(d)	28 A

14. In the circuit shown in Fig. 6.38,  $V_1$  and  $V_2$  are two identical sources of  $10 \ \angle 90^{\circ}$ . The power supplied by  $V_1$  is



Fig. 6.38

(a) 6 W (b) 8.8 W (c) 11 W (d) 16 W


## 7.1 MESH ANALYSIS

We have earlier discussed mesh analysis but have applied it only to resistive circuits. Some of the ac circuits presented in this chapter can also be solved by using mesh analysis. In Chapter 2, the two basic techniques for writing network equations for mesh analysis and node analysis were presented. These concepts can also be used for sinusoidal steady-state condition. In the sinusoidal steady-state analysis, we use voltage phasors, current phasors, impedances and admittances to write branch equations, KVL and KCL equations. For ac circuits, the method of writing loop equations is modified slightly. The voltages and currents in ac circuits change polarity at regular intervals. At a given time,

the instantaneous voltages are driving in either the positive or negative direction. If the impedances are complex, the sum of their voltages is found by vector addition. We shall illustrate the method of writing network mesh equations with the following example.

Consider the circuit shown in Fig. 7.1, containing a voltage source and impedances.



The current in impedance  $Z_1$  is  $I_1$ , and the current in  $Z_2$ , (assuming a positive direction downwards through the impedance) is  $I_1 - I_2$ . Similarly, the current in impedance  $Z_3$  is  $I_2$ . By applying Kirchhoff's voltage law for each loop, we can get two equations. The voltage across any element is the product of the phasor current in the element and the complex impedance.

Equation for loop 1 is

$$I_1 Z_1 + (I_1 - I_2) Z_2 = V_1 \tag{7.1}$$

Equation for loop 2, which contains no source is

$$Z_2(I_2 - I_1) + Z_3I_2 = 0 (7.2)$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(Z_1 + Z_2) - I_2 Z_2 = V_1 \tag{7.3}$$

$$-I_1 Z_2 + I_2 (Z_2 + Z_3) = 0 (7.4)$$

By solving the above equations, we can find out currents  $I_1$  and  $I_2$ . In general, if we have *M* meshes, *B* branches and *N* nodes including the reference node, we assume *M* branch currents and write *M* independent equations; then the number of mesh currents is given by M = B - (N - 1).

**Example 7.1** Write the mesh current equations in the circuit shown in Fig. 7.2, and determine the currents.



Fig. 7.2

**Solution** The equation for loop 1 is

$$I_1(j4) + 6(I_1 - I_2) = 5 \angle 0^{\circ}$$
(7.5)

The equation for loop 2 is

$$6(I_2 - I_1) + (j_3)I_2 + (2)I_2 = 0$$
(7.6)

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(6+j4) - 6I_2 = 5 \ \angle 0^\circ \tag{7.7}$$

$$-6I_1 + (8+j3)I_2 = 0 (7.8)$$

Solving the above equations, we have

$$I_{1} = \left[\frac{(8+j3)}{6}\right]I_{2}$$

$$\left[\frac{(8+j3)(6+j4)}{6}\right]I_{2} - 6I_{2} = 5 \angle 0^{\circ}$$

$$I_{2} \left[\frac{(8+j3)(6+j4)}{6} - 6\right] = 5 \angle 0^{\circ}$$

$$I_{2} [10.26 \angle 54.2^{\circ} - 6 \angle 0^{\circ}] = 5 \angle 0^{\circ}$$

$$I_{2} [(6+j8.32) - 6] = 5 \angle 0^{\circ}$$

$$I_{2} = \frac{5 \angle 0^{\circ}}{8.32 \angle 90^{\circ}} = 0.6 \angle -90^{\circ}$$

$$I_{1} = \frac{8.54 \angle 20.5^{\circ}}{6} \times 0.6 \angle -90^{\circ}$$

$$I_{1} = 0.855 \angle -69.5^{\circ}$$
Current in loop 1,  $I_{1} = 0.855 = \angle -69.5^{\circ}$ 
Current in loop 2,  $I_{2} = 0.6 \angle -90^{\circ}$ 

# 7.2 MESH EQUATIONS BY INSPECTION

In general, mesh equations can be written by observing any network. Consider the three mesh network shown in Fig. 7.3.



Fig. 7.3

The loop equations are

$$I_1 Z_1 + Z_2 (I_1 - I_2) = V_1 \tag{7.9}$$

$$Z_2(I_2 - I_1) + Z_3 I_2 + Z_4(I_2 - I_3) = 0$$
(7.10)

$$Z_4(I_3 - I_2) + Z_5 I_3 = -V_2 \tag{7.11}$$

By rearranging the above equations, we have

$$(Z_1 + Z_2)I_1 - Z_2 I_2 = V_1 \tag{7.12}$$

$$-Z_2I_1 + (Z_2 + Z_3 + Z_4)I_2 - Z_4I_3 = 0 (7.13)$$

$$-Z_4 I_2 + (Z_4 + Z_5) I_3 = -V_2 \tag{7.14}$$

In general, the above equations can be written as

$$Z_{11}I_1 \pm Z_{12}I_2 + Z_{13}I_3 = V_a \tag{7.15}$$

$$\pm Z_{21}I_1 + Z_{22}I_2 \pm Z_{23}I_3 = V_b \tag{7.16}$$

$$\pm Z_{31}I_1 \pm Z_{32}I_2 + Z_{33}I_3 = V_c \tag{7.17}$$

If we compare the general equations with the circuit equations, we get the self impedance of loop 1

$$Z_{11} = Z_1 + Z_2$$

i.e. the sum of the impedances through which  $I_1$  passes. Similarly,  $Z_{22} = (Z_2 + Z_3 + Z_4)$ , and  $Z_{33} = (Z_4 + Z_5)$  are the self impedances of loops 2 and 3. This is equal to the sum of the impedances in their respective loops, through which  $I_2$  and  $I_3$  passes, respectively.

 $Z_{12}$  is the sum of the impedances common to loop currents  $I_1$  and  $I_2$ . Similarly  $Z_{21}$  is the sum of the impedances common to loop currents  $I_2$  and  $I_1$ . In the circuit shown in Fig. 7.3,  $Z_{12} = -Z_2$ , and  $Z_{21} = -Z_2$ . Here, the positive sign is used if both currents passing through the common impedance are in the same direction; and the negative sign is used if the currents are in opposite directions. Similarly,  $Z_{13}$ ,  $Z_{23}$ ,  $Z_{31}$ ,  $Z_{32}$  are the sums of the impedances common to the mesh currents indicated in their subscripts.  $V_a$ ,  $V_b$  and  $V_c$  are sums of the voltages driving their respective loops. Positive sign is used, if the direction of the loop current is the same as the direction of the source current. In Fig. 7.3,  $V_{h} = 0$ because no source is driving loop 2. Since the source,  $V_2$  drives against the loop current  $I_3, V_c = -V_2$ .

**Example 7.2** For the circuit shown in Fig. 7.4, write the mesh equations using the inspection method.



Fig. 7.4

**Solution** The general equations are

$$Z_{11}I_1 \pm Z_{12}I_2 \pm Z_{13}I_3 = V_a \tag{7.18}$$

$$\pm Z_{21}I_1 + Z_{22}I_2 \pm Z_{23}I_3 = V_b \tag{7.19}$$

$$\pm Z_{31}I_1 \pm Z_{32}I_2 + Z_{33}I_3 = V_c \tag{7.20}$$

Consider Eq. 7.18

 $Z_{11}$  = the self impedance of loop 1 = (5 + 3 - j4)  $\Omega$  $Z_{12}$  = the impedance common to both loop 1 and loop 2 = -5  $\Omega$ 

The negative sign is used because the currents are in opposite directions.

 $Z_{13} = 0$ , because there is no common impedance between loop 1 and loop 3.

 $V_a = 0$ , because no source is driving loop 1.

 $\therefore$  Equation 7.18 can be written as

$$(8 - j4)I_1 - 5I_2 = 0 \tag{7.21}$$

Now, consider Eq. 7.19

 $Z_{21} = -5$ , the impedance common to loop 1 and loop 2.  $Z_{22} = (5 + j5 - j6)$ , the self impedance of loop 2.  $Z_{23} = -(-j6)$ , the impedance common to loop 2 and loop 3.  $V_{\rm b} = -10 \angle 30^{\circ}$ , the source driving loop 2.

The negative sign indicates that the source is driving against the loop current,  $I_2$ . Hence, Eq. 7.19 can be written as

$$-5I_1 + (5-j1)I_2 + (j6)I_3 = -10 \angle 30^{\circ}$$
(7.22)

Consider Eq. 7.20

 $Z_{31} = 0$ , there is no common impedance between loop 3 and loop 1  $Z_{32} = -(-j 6)$ , the impedance common to loop 2 and loop 3  $Z_{33} = (4 - j 6)$ , the self impedance of loop 3  $V_b = 20 \angle 50^\circ$ , the source driving loop 3

The positive sign is used because the source is driving in the same direction as the loop current 3. Hence, the equation can be written as

$$(j6)I_2 + (4-j6)I_3 = 20 \angle 50^{\circ}$$
(7.23)

The three mesh equations are

$$(8 - j4)I_1 - 5I_2 = 0$$
  
- 5I\_1 + (5 - j1)I\_2 + (j6)I\_3 = -10 \angle 30°  
(j6)I\_2 + (4 - j6)I\_3 = 20 \angle 50°

## 7.3 NODAL ANALYSIS

The node voltage method can also be used with networks containing complex impedances and excited by sinusoidal voltage sources. In general, in an N node network, we can choose any node as the reference or datum node. In many circuits, this reference is most conveniently choosen as the common terminal or ground terminal. Then it is possible to write (N-1) nodal equations using KCL. We shall illustrate nodal analysis with the following example.

Consider the circuit shown in Fig.7.5.



Fig. 7.5

Let us take a and b as nodes, and c as reference node.  $V_a$  is the voltage between nodes a and c.  $V_b$  is the voltage between nodes b and c. Applying Kirchhoff's current law at each node, the unknowns  $V_a$  and  $V_b$  are obtained.

In Fig. 7.6, node *a* is redrawn with all its branches, assuming that all currents are leaving the node *a*.



Fig. 7.6

In Fig. 7.6, the sum of the currents leaving node a is zero.

$$I_1 + I_2 + I_3 = 0 \tag{7.24}$$

where

:.

$$I_1 = \frac{V_a - V_1}{Z_1}, I_2 = \frac{V_a}{Z_2}, I_3 = \frac{V_a - V_b}{Z_3}$$

Substituting  $I_1$ ,  $I_2$  and  $I_3$  in Eq. 1, we get

$$\frac{V_a - V_1}{Z_1} + \frac{V_a}{Z_2} + \frac{V_a - V_b}{Z_3} = 0$$
(7.25)

Similarly, in Fig. 7.7, node *b* is redrawn with all its branches, assuming that all currents are leaving the node *b*.

In Fig. 7.7, the sum of the currents leaving the node b is zero.

$$I_3 + I_4 + I_5 = 0$$

$$I_3 = \frac{V_b - V_a}{Z_3}, I_4 = \frac{V_b}{Z_4}, I_5 = \frac{V_b}{Z_5 + Z_6}$$
(7.26)

where

:.

Substituting 
$$I_2$$
,  $I_4$  and  $I_5$  in Eq. 7.26



Fig. 7.7

$$\frac{V_b - V_a}{Z_3} + \frac{V_b}{Z_4} + \frac{V_b}{Z_5 + Z_6} = 0$$
(7.27)

Rearranging Eqs. 7.25 and 7.27, we get

$$\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right) V_a - \left(\frac{1}{Z_3}\right) V_b = \left(\frac{1}{Z_1}\right) V_1$$
(7.28)

we get

$$-\left(\frac{1}{Z_3}\right)V_a + \left(\frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5 + Z_6}\right)V_b = 0$$
(7.29)

From Eqs 7.28 and 7.29, we can find the unknown voltages  $V_a$  and  $V_b$ . Example 7.3 In the network shown in Fig. 7.8, determine  $V_a$  and  $V_b$ .



Fig. 7.8

**Solution** To obtain the voltage  $V_a$  at *a*, consider the branch currents leaving the node *a* as shown in Fig. 7.9 (a).



Fig. 7.9(a)

Since the sum of the currents leaving the node a is zero,

$$I_{1} + I_{2} + I_{3} = 0$$

$$\frac{V_{a} - 10 \angle 0^{\circ}}{j6} + \frac{V_{a}}{-j6} + \frac{V_{a} - V_{b}}{3} = 0$$

$$\left(\frac{1}{j6} - \frac{1}{j6} + \frac{1}{3}\right)V_{a} - \frac{1}{3}V_{b} = \frac{10 \angle 0^{\circ}}{j6}$$

$$\frac{1}{3}V_{a} - \frac{1}{3}V_{b} = \frac{10 \angle 0^{\circ}}{j6}$$
(7.31)

:.

To obtain the voltage  $V_b$  at b, consider the branch currents leaving node b as shown in Fig. 7.9 (b).

In Fig. 7.9(b),  $I_3 = \frac{V_b - V_a}{3}$ ,  $I_4 = \frac{V_b}{j4}$ ,  $I_5 = \frac{V_b}{(j5 - j4)}$ Since the sum of the currents leaving node *b* is zero  $I_3 + I_4 + I_5 = 0$ 



### Fig. 7.9(b)

$$\frac{V_b - V_a}{3} + \frac{V_b}{j4} + \frac{V_b}{j1} = 0$$
(7.32)

$$-\frac{1}{3}V_a + \left(\frac{1}{3} + \frac{1}{j4} + \frac{1}{j1}\right)V_b = 0$$
(7.33)

From Eqs 7.31 and 7.33, we can solve for  $V_a$  and  $V_b$ .

$$0.33V_a - 0.33V_b = 1.67 \angle -90^{\circ} \tag{7.34}$$

$$-0.33V_a + (0.33 - 0.25j - j)V_b = 0$$
(7.35)

Adding Eqs 7.34 and 7.35 we get  $(-1.25j)V_b = 1.67 \angle -90^\circ$  $-1.25 \angle 90^{\circ} V_b = 1.67 \angle -90^{\circ}$  $V_b = \frac{1.67 \angle -90^{\circ}}{-1.25 \angle 90^{\circ}}$  $= -1.34 \angle -180^{\circ}$ 

Substituting 
$$V_b$$
 in Eq. (7.34), we get  
 $0.33V_a - (0.33) (-1.34 \angle -180^\circ) = 1.67 \angle -90^\circ$   
 $V_a = \frac{1.67 \angle -90^\circ}{0.33} = -1.31 \text{ V}$   
 $V_a = 5.22 \angle -104.5^\circ \text{ V}$   
pltages  $V_a$  and  $V_b$  are  $5.22 \angle -104.5^\circ \text{ V}$  and  $-1.34 \angle -180^\circ \text{ V}$  respect

Vo tively. \*\*

#### NODAL EQUATIONS BY INSPECTION 7.4

In general, nodal equations can also be written by observing the network. Consider a four node network including a reference node as shown in Fig. 7.10.



Fig. 7.10

Consider nodes a, b and c separately as shown in Figs 7.11(a), (b) and (c).



Fig. 7.11

Assuming that all the currents are leaving the nodes, the nodal equations at a, b and c are

$$I_{1} + I_{2} + I_{3} = 0$$

$$I_{3} + I_{4} + I_{5} = 0$$

$$I_{5} + I_{6} + I_{7} = 0$$

$$\frac{V_{a} - V_{1}}{Z_{1}} + \frac{V_{a}}{Z_{2}} + \frac{V_{a} - V_{b}}{Z_{3}} = 0$$
(7.36)

$$\frac{V_b - V_a}{Z_2} + \frac{V_b}{Z_4} + \frac{V_b - V_c}{Z_5} = 0$$
(7.37)

$$\frac{V_c - V_b}{Z_5} + \frac{V_c}{Z_6} + \frac{V_c - V_2}{Z_7} = 0$$
(7.38)

Rearranging the above equations, we get

$$\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right) V_a - \left(\frac{1}{Z_3}\right) V_a = \left(\frac{1}{Z_1}\right) V_1$$
(7.39)

$$\left(\frac{-1}{Z_3}\right)V_a + \left(\frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5}\right)V_b - \left(\frac{1}{Z_5}\right)V_c = 0$$
(7.40)

$$\left(\frac{-1}{Z_5}\right)V_b + \left(\frac{1}{Z_5} + \frac{1}{Z_6} + \frac{1}{Z_7}\right)V_c = \left(\frac{1}{Z_7}\right)V_2$$
(7.41)

In general, the above equations can be written as

$$\begin{aligned} Y_{aa}V_{a} + Y_{ab}V_{b} + Y_{ac}V_{c} &= I_{1} \\ Y_{ba}V_{a} + Y_{bb}V_{b} + Y_{bc}V_{c} &= I_{2} \\ Y_{ca}V_{a} + Y_{cb}V_{b} + Y_{cc}V_{c} &= I_{3} \end{aligned}$$

If we compare the general equations with the circuit equations, the self admittance at node *a* is

$$Y_{aa} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

which is the sum of the admittances connected to node a.

Similarly, 
$$Y_{bb} = \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5}$$
, and  $Y_{cc} = \frac{1}{Z_5} + \frac{1}{Z_6} + \frac{1}{Z_7}$ 

are the self admittances at node b and node c, respectively.  $Y_{ab}$  is the mutual admittance between nodes a and b, i.e. it is the sum of all the admittances connecting nodes a and b.  $Y_{ab} = -1/Z_3$  has a negative sign. All the mutual admittances have negative signs. Similarly,  $Y_{ac}$ ,  $Y_{ba}$ ,  $Y_{bc}$ ,  $Y_{ca}$  and  $Y_{cb}$  are also mutual admittances. These are equal to the sums of the admittances connecting to nodes indicated in their subscripts.  $I_1$  is the sum of all the source currents at node a. The current which drives into the node has a positive sign, while the current driving away from the node has a negative sign.

**Example 7.4** For the circuit shown in Fig. 7.12, write the node equations by the inspection method.



Fig. 7.12

Solution The general equations are

$$Y_{aa} V_a + Y_{ab} V_b = I_1 \tag{7.42}$$

$$Y_{ba} V_a + Y_{bb} V_b = I_2 \tag{7.43}$$

Consider Eq. 7.42

$$Y_{aa} = \frac{1}{3} + \frac{1}{j4} + \frac{1}{-j6}$$

The self admittance at node *a* is the sum of admittances connected to node *a*.

$$Y_{bb} = \frac{1}{-j6} + \frac{1}{5} + \frac{1}{j5}$$

The self admittance at node b is the sum of admittances connected to node b.

$$Y_{ab} = -\left(\frac{1}{-j6}\right)$$

The mutual admittance between nodes *a* and *b* is the sum of admittances connected between nodes *a* and *b*. Similarly,  $Y_{ba} = -(-1/j 6)$ , the mutual admittance between nodes *b* and *a* is the sum of the admittances connected between nodes *b* and *a*.

$$I_1 = \frac{10 \angle 0^\circ}{3}$$

The source current at node a

$$I_2 = \frac{-10 \angle 30^\circ}{5}$$

the source current leaving at node b.

Therefore, the nodal equations are

$$\left(\frac{1}{3} + \frac{1}{j4} - \frac{1}{j6}\right)V_a - \left(\frac{-1}{j6}\right)V_b = \frac{10 \angle 0^\circ}{3}$$
(7.44)

$$-\left(\frac{-1}{j6}\right)V_a + \left(\frac{1}{5} + \frac{1}{j5} - \frac{1}{j6}\right)V_b = \frac{-10 \angle 30^\circ}{5}$$
(7.45)

7.5 SUPERPOSITION THEOREM

The superposition theorem also can be used to analyse ac circuits containing more than one source. The superposition theorem states that the response in any element in a circuit is the vector sum of the responses that can be expected to flow if each source acts independently of other sources. As each source is considered, all of the other sources are replaced by their internal impedances, which are mostly short circuits in the case of a voltage source, and open circuits in the case of a current source. This theorem is valid only for linear systems. In a network containing complex impedance, all quantities must be treated as complex numbers.

Consider a circuit which contains two sources as shown in Fig. 7.13.



Now let us find the current I passing through the impedance  $Z_2$  in the circuit. According to the superposition theorem, the current due to voltage source  $V \angle 0^\circ V$  is  $I_1$  with current source  $I_a \angle 0^\circ A$  open circuited.

$$I_1 = \frac{V \angle 0^\circ}{Z_1 + Z_2}$$

The current due to  $I_a \angle 0^\circ$  A is  $I_2$  with voltage source  $V \angle 0^\circ$  short circuited.



Fig. 7.14

Fig. 7.15

*:*.

 $I_2 = I_a \angle 0^\circ \times \frac{Z_1}{Z_1 + Z_2}$ 

The total current passing through the impedance  $Z_2$  is

$$I = I_1 + I_2$$

**Example 7.5** Determine the voltage across  $(2 + j5) \Omega$  impedance as shown in Fig. 7.16 by using the superposition theorem.



Fig. 7.16

**Solution** According to the superposition theorem, the current due to the 50  $\angle 0^{\circ}$  V voltage source is  $I_1$  as shown in Fig. 7.17 with current source 20  $\angle 30^{\circ}$  A open circuited.

Current  $I_1 = \frac{50 \angle 0^{\circ}}{2 + j4 + j5} = \frac{50 \angle 0^{\circ}}{(2 + j9)}$  $= \frac{50 \angle 0^{\circ}}{9.22 \angle 77.47^{\circ}} = 5.42 \angle -77.47^{\circ} \text{ A}$ 



Fig. 7.17

Fig. 7.18

Voltage across  $(2 + j5) \Omega$  due to current  $I_1$  is

$$V_1 = 5.42 \angle -77.47^\circ (2+j5)$$
  
= (5.38) (5.42)  $\angle -77.47^\circ + 68.19^\circ$   
= 29.16  $\angle -9.28^\circ$ 

The current due to 20  $\angle 30^{\circ}$  A current source is  $I_2$  as shown in Fig. 7.18, with voltage source 50  $\angle 0^{\circ}$  V short circuited.

Current  $I_2 = 20 \angle 30^\circ \times \frac{(j4) \Omega}{(2+j9) \Omega}$   $= \frac{20 \angle 30^\circ \times 4 \angle 90^\circ}{9.22 \angle 77.47^\circ}$  $\therefore$   $I_2 = 8.68 \angle 120^\circ - 77.47^\circ = 8.68 \angle 42.53^\circ$ 

Voltage across  $(2+j5) \Omega$  due to current  $I_2$  is

$$V_2 = 8.68 \angle 42.53^\circ (2 + j5)$$
  
= (8.68) (5.38) \angle 42.53^\circ + 68.19^\circ  
= 46.69 \angle 110.72^\circ

Voltage across  $(2 + j5) \Omega$  due to both sources is

 $V = V_1 + V_2$ = 29.16  $\angle$ - 9.28° + 46.69  $\angle$ 110.72° = 28.78 - *j*4.7 - 16.52 + *j*43.67 = (12.26 + *j*38.97) V

Voltage across  $(2 + j5) \Omega$  is  $V = 40.85 \angle 72.53^{\circ}$ .

### 7.6 THEVENIN'S THEOREM

Thevenin's theorem gives us a method for simplifying a given circuit. The Thevenin equivalent form of any complex impedance circuit consists of an equivalent voltage source  $V_{\rm Th}$ , and an equivalent impedance  $Z_{\rm Th}$ , arranged as shown in Fig. 7.19. The values of equivalent voltage and impedance depend on the values in the original circuit.

\*\*





Fig. 7.20

Though the Thevenin equivalent circuit is not the same as its original circuit, the output voltage and output current are the same in both cases. Here, the Thevenin voltage is equal to the open circuit voltage across the output terminals, and impedance is equal to the impedance seen into the network across the output terminals.

Consider the circuit shown in Fig. 7.20.

Thevenin equivalent for the circuit shown in Fig. 7.20 between points *A* and *B* is found as follows.

The voltage across points A and B is the Thevenin equivalent voltage. In the circuit shown in Fig. 7.20, the voltage across A and B is the same as the voltage across  $Z_2$  because there is no current through  $Z_3$ .

$$V_{\rm Th} = V\left(\frac{Z_2}{Z_1 + Z_2}\right)$$

The impedance between points A and B with the source replaced by short circuit is the Thevenin equivalent impedance. In Fig. 7.20, the impedance from A to B is  $Z_3$  in series with the parallel combination of  $Z_1$  and  $Z_2$ .

:. 
$$Z_{\text{Th}} = Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2}$$

...



Fig. 7.21

The Thevenin equivalent circuit is shown in Fig. 7.21.

**Example 7.6** For the circuit shown in Fig. 7.22, determine Thevenin's equivalent between the output terminals.





**Solution** The Thevenin voltage,  $V_{\text{Th}}$ , is equal to the voltage across the  $(4 + j6) \Omega$  impedance. The voltage across  $(4 + j6) \Omega$  is

$$V = 50 \ \angle 0^{\circ} \times \frac{(4+j6)}{(4+j6) + (3-j4)}$$
  
= 50 \angle 0^{\circ} \times \frac{4+j6}{7+j2}  
= 50 \angle 0^{\circ} \times \frac{7.21 \angle 56.3^{\circ}}{7.28 \angle 15.95^{\circ}}  
= 50 \angle 0^{\circ} \times 0.99 \angle 40.35^{\circ}  
= 49.5 \angle 40.35^{\circ} \text{ V}

The impedance seen from terminals A and B is

$$Z_{\text{Th}} = (j5 - j4) + \frac{(3 + j4)(4 + j6)}{(3 - j4)(4 + j6)}$$
  
=  $j1 + \frac{5 \angle 53.13^{\circ} \times 7.21 \angle 56.3^{\circ}}{7.28 \angle 15.95^{\circ}}$   
=  $j1 + 4.95 \angle -12.78^{\circ} = j1 + 4.83 - j1.095$   
=  $4.83 - j0.095$ 

....

The Thevenin equivalent circuit is shown in Fig. 7.23.

 $Z_{\rm Th} = 4.83 \angle -1.13^{\circ} \Omega$ 

### 7.7 NORTON'S THEOREM

Another method of analysing a complex impedance circuit is given by Norton's theorem. The Norton equivalent form of any complex impedance circuit consists

of an equivalent current source  $I_N$  and an equivalent impedance  $Z_N$ , arranged as shown in Fig. 7.24. The values of equivalent current and impedance depend on the values in the original circuit.

Though Norton's equivalent circuit is not the same as its original circuit, the output voltage and current are the same in both cases; Norton's current is equal to the current

passing through the short circuited output terminals and the value of impedance is equal to the impedance seen into the network across the output terminals.

Consider the circuit shown in Fig. 7.25.

Norton's equivalent for the circuit shown in Fig. 7.25 between points A and B is found as follows. The current passing through points A and B when it is short-circuited is the Norton's equivalent current, as shown in Fig. 7.26.









Fig. 7.26

Norton's current  $I_N = V/Z_1$ 

The impedance between points A and B, with the source replaced by a short circuit, is Norton's equivalent impedance. -∞ A

In Fig. 7.25, the impedance from A to B,  $Z_2$  is in parallel with  $Z_1$ .  $Z_N = \frac{Z_1 Z_2}{Z_1 + Z_2}$   $I_N$ 

Norton's equivalent circuit is shown in Fig. 7.27

Fig. 7.27

Z<sub>N</sub>

∘ B

**Example 7.7** For the circuit shown in Fig. 7.28, determine Norton's equivalent circuit between the output terminals, AB.



Fig. 7.28

**Solution** Norton's current  $I_N$  is equal to the current passing through the short circuited terminals AB as shown in Fig. 7.29.



Fig. 7.30

The current through terminals AB is

$$I_N = \frac{25 \angle 0^\circ}{3 + j4} = \frac{25 \angle 0^\circ}{5 \angle 53.13^\circ} = 5 \angle -53.13^\circ$$

The impedance seen from terminals AB is

$$Z_N = \frac{(3+j4)(4-j5)}{(3+j4)+(4-j5)}$$
  
=  $\frac{5 \angle 53.13^\circ \times 6.4 \angle -51.34^\circ}{7.07 \angle -8.13^\circ} = 4.53 \angle 9.92^\circ$ 

Norton's equivalent circuit is shown in Fig. 7.30.

### 7.8 MAXIMUM POWER TRANSFER THEOREM

In Chapter 3, the maximum power transfer theorem has been discussed for resistive loads. The maximum power transfer theorem states that the maximum power is delivered from a source to its load when the load resistance is equal to the source resistance. It is for this reason that the ability to obtain impedance matching between circuits is so important. For example, the audio output transformer must match the high impedance of the audio power amplifier output to the low input impedance of the speaker. Maximum power transfer is not always desirable, since the transfer occurs at a 50 per cent efficiency. In many situations, a maximum voltage transfer is desired which means that unmatched impedances are necessary. If maximum power transfer is required, the load resistance should equal the given source resistance. The maximum power transfer theorem can be applied to complex impedance circuits. If the source impedance is complex, then the maximum power transfer occurs when the load impedance is the complex conjugate of the source impedance.

Consider the circuit shown in Fig. 7.31, consisting of a source impedance delivering power to a complex load.



Fig. 7.31

Current passing through the circuit shown

$$I = \frac{V_s}{(R_s + jX_s) + (R_L + jX_L)}$$

Magnitude of current  $I = |I| = \frac{V_s}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$ 

...

Power delivered to the circuit is

$$P = I^2 R_L = \frac{V_s^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

In the above equation, if  $R_L$  is fixed, the value of P is maximum when

$$X_s = -X_L$$
  
wer 
$$P = \frac{V_s^2 R_L}{(R_s + R_L)^2}$$

Then the power

Let us assume that  $R_L$  is variable. In this case, the maximum power is transferred when the load resistance is equal to the source resistance (already discussed in Chapter 3). If  $R_L = R_s$  and  $X_L = -X_s$ , then  $Z_L = Z_s^*$ . This means that the maximum power transfer occurs when the load impedance is equal to the complex conjugate of source impedance  $Z_s$ .

**Example 7.8** For the circuit shown in Fig. 7.32, find the value of load impedance for which the source delivers maximum power. Calculate the value of the maximum power.



Fig. 7.32

**Solution** In the circuit shown in Fig. 7.32, the maximum power transfer occurs when the load impedance is complex conjugate of the source impedance

*.*..

 $Z_L = Z^s = 15 - j20$ 

When  $Z_i = 15 - j20$ , the current passing through circuit is

$$I = \frac{V_s}{R_s + R_L} = \frac{50 \angle 0^\circ}{15 + j20 + 15 - j20} = \frac{50 \angle 0^\circ}{30} = 1.66 \angle 0^\circ$$

The maximum power delivered to the load is

$$P = I^2 R_L = (1.66)^2 \times 15 = 41.33 \text{ W}$$

\*\*

ADDITIONAL SOLVED PROBLEMS

**Problem 7.1** Using mesh analysis, determine the voltage  $V_s$  which gives a voltage of 30  $\angle 0^\circ$  V across the 30  $\Omega$  resistor shown in Fig. 7.33.



Fig. 7.33

**Solution** By the inspection method, we can have four equations from four loops.

$$(-j4)I_1 + (3-j1)I_2 - 3I_3 + (j5)I_4 = 0$$
(7.47)

$$-3I_2 + (7+j8)I_3 = 50 \ \angle 0^\circ \tag{7.48}$$

$$(j5)I_2 + (30 - j5)I_4 = -V_s \tag{7.49}$$

Solving the above equations using Cramer's rule, we get

$$I_{4} = \frac{\begin{vmatrix} (5+j4) & (-j4) & 0 & 60 \angle 30^{\circ} \\ (-j4) & (3-j1) & -3 & 0 \\ 0 & -3 & (7+j8) & 50 \angle 0^{\circ} \\ 0 & (j5) & 0 & -V_{s} \end{vmatrix}}{\begin{vmatrix} (5+j4) & (-j4) & 0 & 0 \\ (-j4) & (3-j1) & -3 & (J5) \\ 0 & -3 & (7+j8) & 0 \\ 0 & (j5) & 0 & (30-J5) \end{vmatrix}}$$
$$\Delta = (5+j4) \begin{vmatrix} (3-j1) & -3 & (j5) \\ -3 & (7+j8) & 0 \\ (j5) & 0 & (30-j5) \end{vmatrix}$$
$$+ (j4) \begin{vmatrix} (-j4) & -3 & (j5) \\ 0 & (7+j8) & 0 \\ 0 & 0 & (30-j5) \end{vmatrix}$$
$$= (5+j4) \{ (3-j1) & (7+j8) & (30-j5) + 3 [(-3) & (30-j5)] \\ + j5 [(-j5) & (7+j8)] \} + (j4) \{ (-j4) & (7+j8) & (30-j5) \} \\ = (5+j4) \{ [3.16 \angle -18.4^{\circ} \times 10.6 \angle 48.8^{\circ} \times 30.4 \angle -9.46^{\circ}] \\ -9 \times 30.4 \angle -9.46^{\circ} + 25 & (10.6 \angle 48.8^{\circ} \end{vmatrix}$$

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$$\begin{split} &= (5+j4) \{1018.27 \angle 20.94^{\circ} - 273.6 \angle - 9.46^{\circ} + 265 \angle 48.8^{\circ}\} \\ &+ j4 \{1288.96 \angle - 50.66^{\circ}\} \\ &= (5+j4) \{951+j363.9-269.8+j44.97+174.55+j199.38\} \\ &+ 4 \angle 90^{\circ} \{1288.96 \angle - 50.66^{\circ}\} \\ &= (5+j4) \{855.75+j608.25\} + 4 \angle 90^{\circ} \{1288.96 \angle - 50.66^{\circ}\} \\ &= 6.4 \angle 38.6^{\circ} \times 1049.9 \angle 35.4^{\circ} + 4 \angle 90^{\circ} \times 1288.96 \angle - 50.66^{\circ} \\ &= 6719.36 \angle 74^{\circ} + 5155.84 \angle 39.34^{\circ} \\ &= 1852.1+j6459+3987.5+j3268.3 \\ &= 5839.6+j9727.3 \\ &= 11345.5 \angle 559^{\circ} \\ \\ &\Delta_4 = (5+j4) \begin{vmatrix} (3-j1) & -3 & 0 \\ -3 & (7+j8) & 50 \angle 0^{\circ} \\ 0 & (7+j8) & 50 \angle 0^{\circ} \end{vmatrix} \\ &-60 \angle 30^{\circ} \end{vmatrix} \begin{vmatrix} (-j4) & (3-j1) & -3 \\ 0 & -3 & 7+j8 \\ 0 & (j5) & 0 & -V_s \end{vmatrix} \\ \\ &+ j4 \begin{vmatrix} -j4 & -3 & 0 \\ -3 & (7+j8) & 50 \angle 0^{\circ} \end{vmatrix} \\ &+ (j4) \{(3-j1)(7+j8)(-V_s)\} + 3[(3V_s) - (j5) 50 \angle 0^{\circ}]\} \\ &+ (j4) \{(-j4)(7+j8)(-V_s)\} - 60 \angle 30^{\circ} \{(-j4)(-j5)(7+j8)\}\} \\ &= 6.4 \angle 38.6^{\circ} \{[3.16 \angle -18.4^{\circ} \times 10.6 \angle 48.8^{\circ} (-V_s)]\} \\ &+ (9V_s - (15j) 50 \angle 0^{\circ}]\} \\ &+ 4 \angle 90^{\circ} \{4 \angle -90^{\circ} \times 5 \angle -90^{\circ} \times 10.6 \angle 48.8^{\circ}\} \\ &= 6.4 \angle 38.6^{\circ} \{-33.49 \angle 30.4^{\circ} V_s\} + 6.4 \angle 38.6^{\circ} \times 9V_s \\ &+ 4 \angle 90^{\circ} \{-42.4 \angle -41.2^{\circ} V_s\} - 60 \angle 30^{\circ} \{212 \angle -131.2^{\circ}\} \\ &- 6.4 \angle 38.6^{\circ} + 750 \angle 90^{\circ}\} \\ &= V_s \{-76.8 - j200 + 45 + j35.93 - 111.7 - j127.6\} \\ &- \{-2470.6 - j12477.75 - 2994.6 + j3751.2\} \\ &= V_s \{-143.5 - j291.67\} V_s + (5465.2 + j8726.5) \\ &11345.5 \angle 59^{\circ} \end{aligned}$$

Since voltage across the 30  $\Omega$  resistor is 30  $\angle 0^\circ$  V. Current passing through it is  $I_4$  = 1  $\angle 0^\circ$  A

:.



$$1 \angle 0^{\circ} = \frac{(-143.5 - j291.67) V_s + (5465.2 + j8726.5)}{11345.5 \angle 59^{\circ}}$$
  

$$11345.5 \angle 59^{\circ} = 325 \angle -116.19^{\circ} V_s + 5465.2 + j8726.5$$
  

$$V_s = \frac{-5465.2 - j8726.5 + 5843.36 + j9724.99}{325 \angle -116.19^{\circ}}$$
  

$$= \frac{378.16 + j998.49}{325 \angle -116.19^{\circ}} = \frac{1067.7 + j69.26^{\circ}}{325 \angle -116.19^{\circ}}$$
  

$$V_s = 3.29 \angle 185.45^{\circ}.$$

**Problem 7.2** For the circuits shown in Fig. 7.34, determine the line currents  $I_R$ ,  $I_Y$  and  $I_B$  using mesh analysis.



Fig. 7.34

**Solution** From Fig. 7.34, the three line currents are

$$I_R = I_1 - I_3$$
$$I_Y = I_2 - I_1$$
$$I_B = I_3 - I_2$$

Using the inspection method, the three loop equations are

$$5 \angle 10^{\circ} I_{1} = 100 \angle 0^{\circ}$$

$$5 \angle 10^{\circ} I_{2} = 100 \angle 120^{\circ}$$

$$5 \angle 10^{\circ} I_{3} = 100 \angle -120^{\circ}$$

$$I_{1} = \frac{100 \angle 0^{\circ}}{5 \angle 10^{\circ}} = 20 \angle -10^{\circ}$$

$$I_{2} = \frac{100 \angle 120^{\circ}}{5 \angle 10^{\circ}} = 20 \angle +110^{\circ}$$

$$I_{3} = \frac{100 \angle -120^{\circ}}{5 \angle 10^{\circ}} = 20 \angle -130^{\circ}$$

:.

$$I_R = I_1 - I_3 = 20 \angle -10^\circ - 20 \angle -130^\circ$$
  
= 19.69 - j3.47 + 12.85 + j15.32  
= 32.54 + j11.85 = 34.63 \angle 20^\circ

$$\begin{split} I_Y &= I_2 - I_1 = 20 \angle 110^\circ - 20 \angle -10^\circ \\ &= -6.84 + j18.79 - 19.69 + j3.47 \\ &= -26.53 + j22.26 = 34.63 \angle 140^\circ \\ I_B &= I_3 - I_2 = 20 \angle -130^\circ - 20 \angle 110^\circ \\ &= -12.85 - j15.32 + 6.84 - j18.79 \\ &= -6.01 - j34.11 = 34.63 \angle -100^\circ \end{split}$$

**Problem 7.3** For the circuit shown in Fig. 7.35, determine the value of  $V_2$  such that the current  $(3 + j4) \Omega$  impedance is zero.



Fig. 7.35

Solution The three loop equations are

$$(4+j3) I_1 - (j3)I_2 = 20 \angle 0^\circ$$
  
$$(-j3)I_1 + (3+j2)I_2 + j5I_3 = 0$$
  
$$(j5)I_2 + (5-j5)I_3 = -V_2$$

Since the current  $I_2$  in  $(3 + j4) \Omega$  is zero

*:*..

where

$$I_{2} = \frac{\Delta_{2}}{\Delta} = 0$$

$$\Delta_{2} = 0$$

$$\Delta_{2} = \begin{vmatrix} (4+j3) & 20 \angle 0^{\circ} & 0 \\ (-j3) & 0 & j5 \\ 0 & -V_{2} & (5-j5) \end{vmatrix} = 0$$

$$(4+j3) V_{2}(j5) - 20 \angle 0^{\circ} \{(-j3) (5-j5)\} = 0$$

$$V_{2} = \frac{20 \angle 0^{\circ} \{(-j3) (5-j5)\}}{(j5) (4+j3)}$$

$$= 20 \angle 0^{\circ} \frac{\{-15-j15\}}{-15+j20} = 20 \angle 0^{\circ} \times \frac{21.21 \angle -135^{\circ}}{25 \angle 126.86^{\circ}}$$

$$V_{2} = 16.97 \angle -261.85^{\circ} V$$

**Problem 7.4** For the circuit shown in Fig. 7.36, write the nodal equations using the inspection method and express them in matrix form.

**Solution** The number of nodes and reference node are selected as shown in Fig. 7.36, by assuming that all currents are leaving at each node.



Fig. 7.36

At node 
$$c$$
,  $-\left(\frac{1}{1+j1}\right)V_a - \left(\frac{1}{j3}\right)V_b + \left(\frac{1}{2} + \frac{1}{j3} + \frac{1}{1+j1}\right)V_c = \frac{50 \angle 0^\circ}{1+j1} - \frac{20 \angle 30^\circ}{j3}$ 

In matrix form, the nodal equations are

$$\begin{bmatrix} \frac{1}{4} + \frac{1}{(1+j1)} - \frac{1}{j1} & + \frac{1}{j1} & -\frac{1}{(1+j1)} \\ \frac{1}{j1} & \frac{1}{3} - \frac{1}{j1} + \frac{1}{j3} & -\frac{1}{j3} \\ -\frac{1}{(1+j1)} & -\frac{1}{(j3)} & \frac{1}{2} + \frac{1}{j3} + \frac{1}{1+j1} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ \\ V_c \end{bmatrix}$$
$$= \begin{bmatrix} \frac{-50 \angle 0^\circ}{(1+j1)} \\ \frac{20 \angle 30^\circ}{j3} \\ \left( \frac{50 \angle 0^\circ}{1+j1} - \frac{20 \angle 30^\circ}{j3} \right) \end{bmatrix}$$

**Problem 7.5** For the circuit shown in Fig. 7.37, determine the voltage  $V_{AB}$ , if the load resistance  $R_L$  is infinite. Use node analysis.

Solution If the load resistance is infinite, no current passes through  $R_L$ . Hence  $R_L$ acts as an open circuit. If we consider A as a node and B as the reference node

$$\frac{V_A - 20 \angle 0^\circ}{3+2} + \frac{V_A - 20 \angle 90^\circ}{j4+3} = 0$$



Fig. 7.37

$$= 4 \ \angle 0^{\circ} + 4 \ \angle 36.87^{\circ} = 4 + 3.19 + j2.4 = 7.19 + j2.4$$
$$V_{A} [0.2 + 0.12 - j0.16] = 7.19 + j2.4$$
$$V_{A} = \frac{7.19 + j2.4}{0.32 - j0.16} = \frac{7.58 \ \angle 18.46^{\circ}}{0.35 \ \angle -26.56^{\circ}}$$

Voltage across AB is  $V_{AB} = V_A = 21.66 \angle 45.02^\circ$  V

**Problem 7.6** For the circuit shown in Fig. 7.38, determine the power output of the source and the power in each resistor of the circuit.



Fig. 7.38

**Solution** Assume that the voltage at node A is  $V_A$  By applying nodal analysis, we have

$$\frac{V_A - 20 \angle 30^\circ}{3} + \frac{V_A}{-j4} + \frac{V_A}{2+j5} = 0$$
$$V_A \left[ \frac{1}{3} + \frac{1}{2+j5} - \frac{1}{j4} \right] = \frac{20 \angle 30^\circ}{3}$$
$$V_A \left[ 0.33 + 0.068 + j0.078 \right] = 6.67 \angle 30^\circ$$

:.

$$V_A = \frac{6.67 \angle 30^{\circ}}{0.41 \angle 11.09^{\circ}} = 16.27 \angle 18.91^{\circ}$$

Current in the 2  $\Omega$  resistor

$$I_2 = \frac{V_A}{2+j5} = \frac{16.27 \angle 18.91^{\circ}}{5.38 \angle 68.19^{\circ}}$$
$$I_2 = 3.02 \angle -49.28^{\circ}$$

*:*..

*:*..

Power dissipated in the 2  $\Omega$  resistor

$$P_2 = I_2^2 R = (3.02)^2 \times 2 = 18.24 \text{ W}$$

Current in the 3  $\Omega$  resistor

$$I_{3} = \frac{-20 \ \angle 30^{\circ} + 16.27 \ \angle 18.91^{\circ}}{3}$$
  
= -6.67 \ \alpha 30^{\circ} + 5.42 \ \alpha 18.91^{\circ}  
= -5.78 - j3.34 + 5.13 + j1.76 = -0.65 - j1.58  
$$I_{2} = 1.71 \ \alpha - 112^{\circ}$$

Power dissipated in the 3  $\Omega$  resistor

$$=(1.71)^2 \times 3 = 8.77 \text{ W}$$

Total power delivered by the source

$$= VI \cos \phi = 20 \times 1.71 \cos 142^\circ = 26.95 \text{ W}$$

**Problem 7.7** For the circuit shown in Fig. 7.39, determine the voltage  $V_{AB}$  using the superposition theorem.



Fig. 7.39

**Solution** Let source  $50 \angle 0^\circ$  V act on the circuit and set the source  $4 \angle 0^\circ A$  equal to zero. If the current source is zero, it becomes open-circuited. Then the voltage across *AB* is  $V_{AB} = 50 \angle 0^\circ$ .

Now set the voltage source 50  $\angle 0^\circ$  V is zero, and is short circuited, or the voltage drop across *AB* is zero.

The total voltage is the sum of the two voltages.

$$V_T = 50 \angle 0^\circ$$

**Problem 7.8** For the circuit shown in Fig. 7.40, determine the current in  $(2 + j3) \Omega$  by using the superposition theorem.



Fig. 7.40

**Solution** The current in  $(2+j3) \Omega$ , when the voltage source  $50 \angle 0^\circ$  acting alone is

$$I_1 = \frac{50 \angle 0^{\circ}}{(6+j3)} = \frac{50 \angle 0^{\circ}}{6.7 \angle 26.56^{\circ}}$$
$$I_1 = 7.46 \angle -26.56^{\circ} \text{ A}$$

*:*..

Current in  $(2 + j3) \Omega$ , when the current source  $20 \angle 90^\circ$  A acting alone is

$$I_2 = 20 \ \angle 90^\circ \times \frac{4}{(6+j3)}$$
$$= \frac{80 \ \angle 90^\circ}{6.7 \ \angle 26.56^\circ} = 11.94 \ \angle 63.44^\circ \ A$$

Total current in  $(2 + j3) \Omega$  due to both sources is

$$I = I_1 + I_2$$
  
= 7.46  $\angle -26.56^\circ + 11.94 \angle 63.44^\circ$   
= 6.67 - j3.33 + 5.34 + j10.68  
= 12.01 + j7.35 = 14.08  $\angle 31.46^\circ$ 

\*\*

Total current in  $(2 + j3) \Omega$  is I = 14.08  $\angle 31.46^{\circ}$ .

**Problem 7.9** For the circuit shown in Fig. 7.41, determine the load current by applying Thevenin's theorem.



Fig. 7.41

**Solution** Let us find the Thevenin equivalent circuit for the circuit shown in Fig. 7.42(a).



Fig. 7.42

Voltage across AB is the voltage across  $(j3) \Omega$ 

*:*..

$$V_{AB} = 100 \ \angle 0^{\circ} \times \frac{(j3)}{(j3) + (j4)}$$
$$= 100 \ \angle 0^{\circ} \ \frac{(j3)}{j7} = 42.86 \ \angle 0^{\circ}$$

Impedance seen from terminals *AB* 

$$Z_{AB} = (j5) + \frac{(j4)(j3)}{j7}$$
  
= j5 + j1.71 = j6.71 Ω

The venin's equivalent circuit is shown in Fig. 7.42(b). If we connect a load to Fig. 7.42(b), the current passing through (j5)  $\Omega$  impedance is

$$I_L = \frac{42.86 \angle 0^{\circ}}{(j6.71 + j5)} = \frac{42.86 \angle 0^{\circ}}{11.71 \angle 90^{\circ}} = 3.66 \angle -90^{\circ}$$

**Problem 7.10** For the circuit shown in Fig. 7.43, determine Thevenin's equivalent circuit.



Fig. 7.43

**Solution** Voltage across  $(-j4) \Omega$  is

$$V_{-j4} = \frac{5 \angle 90^{\circ}}{(2+j2)} (-j4)$$
$$= \frac{20 \angle 0^{\circ}}{2.83 \angle 45^{\circ}} = 7.07 \angle -45^{\circ}$$

Voltage across AB is  $V_{AB} = -V_{10} + V_5 - V_{-j4}$ =  $-10 \angle 0^\circ + 5 \angle 90^\circ - 7.07 \angle -45^\circ$ = j5 - 10 - 4.99 + j4.99= -14.99 + j9.99 $V_{AB} = 18 \angle 146.31^\circ$ 

The impedance seen from terminals *AB*, when all voltage sources are short circuited is

$$Z_{AB} = 4 + \frac{(2+j6)(-j4)}{2+j2}$$

$$= 4 + \frac{6.32 \angle 71.56^{\circ} \times 4 \angle -90^{\circ}}{2.83 \angle 45^{\circ}}$$

$$= 4 + 8.93 \angle -63.44^{\circ}$$

$$= 4 + 4 - j \ 7.98 = (8 - j7.98) \ \Omega$$
Fig. 7.44

Thevenin's equivalent circuit is shown in Fig. 7.44.

**Problem 7.11** For the circuit shown in Fig. 7.11, determine the load current  $I_{t}$  by using Norton's theorem.



Fig. 7.45

**Solution** Norton's impedance seen from terminals *AB* is

$$Z_{AB} = \frac{(j3)(-j2)}{(j3) - (j2)} = \frac{6}{j1}$$
$$Z_{AB} = 6 \angle -90^{\circ}$$

Current passing through *AB*, when it is shorted

*.*..



Norton's equivalent circuit is shown in Fig. 7.46

Fig. 7.46

\*\*

Load current is 
$$I_L = I_N \times \frac{6 \angle -90^{\circ}}{5 + 6 \angle -90^{\circ}}$$
  
= 4.16  $\angle -126.8^{\circ} \times \frac{6 \angle -90^{\circ}}{5 - j6}$   
=  $\frac{4.16 \times 6 \angle -216.8^{\circ}}{7.81 \angle -50.19^{\circ}}$   
= 3.19  $\angle -166.61^{\circ}$ 

**Problem 7.12** For the circuit shown in Fig. 7.47, determine Norton's equivalent circuit.



Fig. 7.47

Fig. 7.48

**Solution** The impedance seen from the terminals when the source is reduced to zero

 $Z_{AB} = (5 + j6) \Omega$ 

Current passing through the short circuited terminals, A and B, is

$$I_N = 30 \angle 30^\circ \text{ A}$$

Norton's equivalent circuit is shown in Fig. 7.48.

**Problem 7.13** Convert the active network shown in Fig. 7.49 by a single voltage source in series with impedance.



Fig. 7.49

**Solution** Using the superposition theorem, we can find Thevenin's equivalent circuit. The voltage across *AB*, with  $20 \angle 0^\circ$  V source acting alone, is  $V'_{AB}$ , and can be calculated from Fig. 7.50(a).

Since no current is passing through the  $(3 + j4) \Omega$  impedance, the voltage

$$V_{AB}' = 20 \angle 0^\circ$$

The voltage across AB, with  $5 \angle 0^\circ$  A source acting alone, is  $V'_{AB}$ , and can be calculated from Fig. 7.50(b).



Fig. 7.50

 $V''_{AB} = 5 \angle 0^{\circ} (3 + j4) = 5 \angle 0^{\circ} \times 5 \angle 53.13^{\circ} = 25 \angle 53.13^{\circ} V$ 

The voltage across AB, with 10  $\angle 90^{\circ}$  A source acting alone, is  $V_{AB}^{'''}$ , and can be calculated from Fig. 7.50 (c).



Fig. 7.50

According to the superposition theorem, the voltage across AB due to all sources is

:.

$$V_{AB} = V_{AB}^{\circ} + V_{AB}^{\circ} + V_{AB}^{\circ}$$
$$V_{AB} = 20 \ \angle 0^{\circ} + 25 \ \angle 53.13^{\circ} = 20 + 15 + j19.99$$
$$= (35 + j19.99) \ V = 40.3 \ \angle 29.73^{\circ} \ V$$

A

The impedance seen from terminals AB

$$Z_{\rm Th} = Z_{AB} = (3+j4) \ \Omega$$

 $\therefore$  The required Thevenin circuit is shown in Fig. 7.50(d).

**Problem 7.14** For the circuit shown in Fig. 7.51, find the value of *Z* that will receive maximum power; also determine this power.

Solution The equivalent impedance at terminals

AB with the source set equal to zero is

$$Z_{AB} = \frac{5(j10)}{5+j10} + \frac{7(-j20)}{(7-j20)}$$

$$= \frac{50 \angle 90^{\circ}}{11.18 \angle 63.43^{\circ}} + \frac{140 \angle -90^{\circ}}{21.19 \angle -70.7^{\circ}}$$

$$= 4.47 \angle 26.57^{\circ} + 6.6 \angle -19.3^{\circ}$$

$$= 3.99 + j1.99 + 6.23 - j2.18$$

$$= 10.22 - j0.19$$
Fig. 7.51

The Thevenin equivalent circuit is shown in Fig. 7.52(a). The circuit in Fig. 7.52(a) is redrawn as shown in Fig. 7.52(b).



Current 
$$I_2 = \frac{100 \angle 0^\circ}{7 - j20} = \frac{100 \angle 0^\circ}{21.19 \angle -70.7^\circ} = 4.72 \angle 70.7^\circ$$

Voltage at A,  $V_A = 8.94 \angle -63.43^{\circ} \times j10 = 89.4 \angle 26.57^{\circ}$ Voltage at *B*,  $V_{B} = 4.72 \angle 70.7^{\circ} \times -j20 = 94.4 \angle -19.3^{\circ}$ Voltage across terminals AB

$$V_{AB} = V_A - V_B$$
  
= 89.4 \angle 26.57^\circ - 94.4 \angle - 19.3^\circ  
= 79.96 + j39.98 - 89.09 + j31.2  
= -9.13 + j71.18  
$$V_{Th} = V_{AB} = 71.76 \angle 97.3^\circ V$$

To get maximum power, the load must be the complex conjugate of the source impedance.

Load 
$$Z = 10.22 + i0.19$$

Current passing through the load Z

$$I = \frac{V_{\rm Th}}{Z_{\rm Th} + Z} = \frac{71.76 \,\angle 97.3^{\circ}}{20.44} = 3.51 \,\angle 97.3^{\circ}$$

Maximum power delivered to the load is

$$= (3.51)^2 \times 10.22 = 125.91 \text{ W}$$

**Problem 7.15** For the circuit shown in Fig. 7.53, the resistance  $R_s$  is variable from 2  $\Omega$  to 50  $\Omega$ . What value of  $R_s$  results in maximum power transfer across the terminals AB?

**Solution** In the circuit shown the resistance  $R_L$  is fixed. Here, the maximum power transfer theorem does not apply. Maximum current flows in the circuit when  $R_{\rm c}$  is minimum. For the maximum current

$$R_s = 2$$
  
But 
$$Z_T = R_s - j5 + R_L = 2 - j5 + 20 = (22 - j5) = 22.56 \angle -12.8^\circ$$

В



Fig. 7.53

 $I = \frac{V_s}{Z_T} = -\frac{50 \angle 0^\circ}{22.56 \angle -12.8^\circ} = 2.22 \angle 12.8^\circ$ 

Maximum power  $P = I^2 R = (2.22)^2 \times 20 = 98.6$  W

**Problem 7.16** Determine the voltage V which results in a zero current through the  $2 + j3 \Omega$  impedance in the circuit shown in Fig. 7.54.



Fig. 7.54

Solution Choosing mesh currents as shown in Fig. 7.54, the three loop equations are

$$(5+j5) I_1 - j5 I_2 = 30 \angle 0^{\circ}$$
$$-j5 I_1 + (2+j8) I_2 = -2V_4$$
$$-2V_4 + V_4 + V = 0$$
$$V_4 = V$$

Since the current in  $(2 + j3) \Omega$  is zero

$$I_{2} = \frac{\Delta_{2}}{\Delta} = 0$$

$$\Delta_{2} = \begin{vmatrix} 5+j5 & 30 \angle 0^{\circ} \\ -j5 & -2V \end{vmatrix} = 0$$

$$(5+j5) (-2V) + (j5) & 30 \angle 0^{\circ} = 0$$

$$V = \frac{30 \angle 0^{\circ}(j5)}{2 & (5+j5)} = \frac{150 \angle 90^{\circ}}{14.14 \angle 45^{\circ}}$$

$$V = 10.608 \angle 45^{\circ} \text{ volts}$$

Where

:.

\*\*

**Problem 7.17** Find the value of voltage V which results in  $V_0 = 5 \angle 0^\circ$  V in the circuit shown in Fig. 7.55.





Solution Assuming all currents are leaving the nodes, the nodal equations are

$$V_{1}\left[\frac{1}{5-j2} + \frac{1}{3} + \frac{1}{j5}\right] - V_{2}\left[\frac{1}{j5}\right] = \frac{V}{5-j2}$$
$$-V_{1}\left[\frac{1}{j5}\right] + V_{2}\left[\frac{1}{j5} + \frac{1}{2-j2}\right] = 2V_{5}$$
$$V_{5} = \left(\frac{V_{1} - V}{5-j2}\right)^{5}$$

where

The second equation becomes

$$V_{1}\left[\frac{-1}{j5} - \frac{10}{5 - j2}\right] + V_{2}\left[\frac{1}{j5} + \frac{1}{2 - j2}\right] = \frac{-10V}{5 - j2}$$
$$V_{0} = V_{2} = \frac{\Delta_{2}}{\Delta} = 5 \angle 0^{\circ}$$
$$\frac{\left|\frac{1}{5 - j2} + \frac{1}{3} + \frac{1}{j5} - \frac{V}{5 - j2}\right|}{\frac{-1}{j5} - \frac{10}{5 - j2} - \frac{-10V}{5 - j2}}\right|$$
$$\frac{\left|\frac{1}{5 - j2} + \frac{1}{3} + \frac{1}{j5} - \frac{-1}{j5}\right|}{\frac{-1}{j5} - \frac{10}{5 - j2} - \frac{1}{j5} + \frac{1}{2 - j2}}\right| = 5 \angle 0^{\circ}$$

The source voltage  $V = 2.428 \angle -88.74^{\circ}$  volts.

**Problem 7.18** For the circuit shown in Fig. 7.56, find the current in the  $j5 \Omega$  inductance by using Theorem.

...



Fig. 7.56

**Solution** From the circuit shown in Fig. 7.56 the open circuit voltage at terminals a and b is

where

$$V_{oc} = -9 V_i$$
  

$$V_i = -9V_i - 100 \angle 0^\circ$$
  

$$10V_i = -100 \angle 0^\circ$$
  

$$V_i = -10 \angle 0^\circ$$

The venin's voltage  $V_{oc} = 90 \angle 0^{\circ}$ 

From the circuit, short circuit current is determined by shorting terminals *a* and *b*. Applying Kirchhoff's voltage law, we have



The Thevenin's equivalent circuit is shown in Fig. 7.57

The current in the j2 
$$\Omega$$
 inductor is  $=\frac{90 \ge 0^{\circ}}{j1}$   
= 90 \angle -90^{\circ}

**Problem 7.19** For the circuit shown in Fig. 7.58, find the value of *Z* that will receive maximum power; also determine this power.



**Solution** The equivalent impedance can be obtained by finding  $V_{oc}$  and  $i_{sc}$  at terminals *a b*. Assume that current *i* is passing in the circuit.

$$i = \frac{100 \angle 0^{\circ} - 5V_{4}}{4 + j10}$$

$$= \frac{100 \angle 0^{\circ}}{4 + j10} - \frac{5 \times 4i}{4 + j10}$$

$$i = 3.85 \angle -22.62^{\circ}$$

$$V_{oc} = 100 \angle 0^{\circ} - 4 \times 3.85 \angle -22.62^{\circ}$$

$$= 86 \angle 3.94^{\circ}$$

$$i_{sc} = \frac{100 \angle 0^{\circ}}{4} = 25 \angle 0^{\circ}$$
The venin's equivalent impedance
$$Z_{Th} = \frac{V_{oc}}{4} = 3.44 \angle 3.94^{\circ}$$

 $Z_{\rm Th} = \frac{i_{oc}}{i_{sc}}$ 

$$= 3.43 + j0.24$$
  
The circuit is drawn as shown in Fig. 7.59.

To get maximum power, the load must be the complex conjugate of the source impedance.

Load Z = 3.43 - i0.24*.*..

Current passing through load Z

$$I = \frac{V_{\rm Th}}{Z_{\rm Th} + Z} = \frac{8.6 \angle 3.94^{\circ}}{6.86} = 1.25 \angle 3.94^{\circ}$$

Maximum power delivered to the load is  $(1.25)^2 \times 3.43 = 5.36$  W.



## **PRACTICE PROBLEMS**

7.1 For the circuit shown in Fig. 7.60, determine the value of current  $I_{\rm x}$  in the impedance Z = 4 + i5 between nodes a and b.



Fig. 7.60

7.2 Determine (i) the equivalent voltage generator and (ii) the equivalent current generator which may be used to represent the given network in Fig. 7.61 at the terminals AB.





7.3 For the circuit shown in Fig. 7.62, find the value of Z that will receive the maximum power. Also determine this power.





7.4 Determine the voltage  $V_{ab}$  and  $V_{bc}$  in the network shown in Fig. 7.63 by loop analysis, where source voltage  $e(t) = \sqrt{2} \times 100 \cos (314 t + 45^{\circ})$ .





7.5 Find the current in the 15  $\Omega$  resistor in the network shown in Fig. 7.64 by Thevenin's theorem.



Fig. 7.64
7.6 Determine the power output of the voltage source by loop analysis for the network shown in Fig. 7.65. Also determine the power extended in the resistors.





7.7 In the circuit shown in Fig. 7.66, determine the power in the impedance  $(2+j5) \Omega$  connected between A and B using Norton's theorem.



Fig. 7.66

7.8 Determine the value of source currents by loop analysis for the circuit shown in Fig. 7.67 and verify the results by using node analysis.



Fig. 7.67

7.9 Convert the active network shown in Fig. 7.68 by a single voltage source in series with an impedance, and also by a single current source in parallel with the impedance.



Fig. 7.68

7.10 Determine the power out of the source in the circuit shown in Fig. 7.69 by nodal analysis and verify the results by using loop analysis.





7.11 For the circuit shown in Fig. 7.70, find the current in each resistor using the superposition theorem.



Fig. 7.70

7.12 Use Thevenin's theorem to find the current through the  $(5+j4) \Omega$  impedance in Fig. 7.71. Verify the results using Norton's theorem.



Fig. 7.71

7.13 Determine Thevenin's and Norton's equivalent circuits across terminals *AB*, in Fig. 7.72.



Fig. 7.72

7.14 Determine Norton's and Thevenin's equivalent circuits for the circuit shown in Fig. 7.73.



Fig. 7.73

7.15 Determine the maximum power delivered to the load in the circuit shown in Fig. 7.74.



Fig. 7.74

7.16 For the circuit shown in Fig. 7.75, find the voltage across the dependent source branch by using mesh analysis.



Fig. 7.75

7.17 Find Thevenin's equivalent for the network shown in Fig. 7.76.



7.18 For the circuit shown in Fig. 7.77, obtain the voltage across 500  $\Omega$  resistor.





7.19 For the circuit shown in Fig. 7.78, obtain the Thevenin's equivalent circuit at terminals *ab*.



Fig. 7.78



- 1. The superposition theorem is valid
  - (a) only for ac circuits
  - (b) only for dc circuits
  - (c) For both, ac and dc circuits
  - (d) neither of the two
- 2. When applying the superposition theorem to any circuit
  - (a) the voltage source is shorted, the current source is opened
  - (b) the voltage source is opened, the current source is shorted
  - (c) both are opened
  - (d) both are shorted
- 3. While applying Thevenin's theorem, the Thevenin's voltage is equal to
  - (a) short circuit voltage at the terminals
  - (b) open circuit voltage at the terminals
  - (c) voltage of the source
  - (d) total voltage available in the circuit
- 4. The venin impedance  $Z_{Th}$  is found
  - (a) by short-circuiting the given two terminals
  - (b) between any two open terminals

- (c) by removing voltage sources along with the internal resistances
- (d) between same open terminals as for  $V_{\rm Th}$
- 5. Thevenin impedance of the circuit at its terminals A and B in Fig. 7.79 is





- (a) 5 H (b) 2  $\Omega$ (c) 1.4  $\Omega$  (d) 7 H
- 6. Norton's equivalent form in any complex impedance circuit consists of
  - (a) an equivalent current source in parallel with an equivalent resistance.
  - (b) an equivalent voltage source in series with an equivalent conductance.
  - (c) an equivalent current source in parallel with an equivalent impedance.
  - (d) None of the above.

## 7. The maximum power transfer theorem can be applied

- (a) only to dc circuits
- (b) only to ac circuits
- (c) to both dc and ac circuits (d) neither of the two
- 8. In a complex impedance circuit, the maximum power transfer occurs when the load impedance is equal to
  - (a) complex conjugate of source impedance
  - (b) source impedance
  - (c) source resistance
  - (d) none of the above
- 9. Maximum power transfer occurs at a
  - (a) 100% efficiency (b)
- (b) 50% efficiency
  - (c) 25% efficiency (d) 75% efficiency
- 10. In the circuit shown in Fig. 7.80, the power supplied by the 10 V source is



Fig. 7.80

(a)	6.6 W	(b)	21.7 W
(c)	30 W	(d)	36.7 W

11. The Thevenin equivalent impedance of the circuit in Fig. 7.81 is





- (a)  $(1+j5) \Omega$ (b)  $(2.5+j25) \Omega$ (c)  $(6.25+j6.25) \Omega$ (d)  $(2.5+j6.25) \Omega$
- 12. A source has an emf of 10 V and an impedance of  $500 + j100 \Omega$ . The amount of maximum power transferred to the load will be

(a)	0.5 mW	(b)	0.05 mW
(c)	0.05 W	(d)	0.5 W

13. For the circuit shown in Fig. 7.82, find the voltage across the dependent source.



Fig. 7.82

(a)	8 ∠0°	(b)	4 ∠0°
(c)	4 ∠90°	(d)	8 ∠- 90°



## 8.1 SERIES RESONANCE

In many electrical circuits, resonance is a very important phenomenon. The study of resonance is very useful, particularly in the area of communications. For example, the ability of a radio receiver to select a certain frequency, transmitted by a station and to eliminate frequencies from other stations is based on the principle of resonance. In a series RLC circuit, the current lags behind, or leads the applied voltage depending upon the values of  $X_L$  and  $X_C$ .  $X_L$  causes the total current to lag behind the applied voltage, while  $X_C$  causes the total current to lead the applied voltage. When  $X_L > X_C$ , the circuit is predominantly

inductive, and when  $X_c > X_L$ , the circuit is predominantly capacitive. However, if one of the parameters of the series RLC circuit is varied in such a way that the current in the circuit is in phase with the applied voltage, then the circuit is said to be in resonance.

Consider the series RLC circuit shown in Fig. 8.1.



Fig. 8.1

The total impedance for the series RLC circuit is

$$Z = R + j(X_L - X_C) = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

It is clear from the circuit that the current  $I = V_s/Z$ 

The circuit is said to be in resonance if the current is in phase with the applied voltage. In a series RLC circuit, series resonance occurs when  $X_L = X_C$ . The frequency at which the resonance occurs is called the *resonant frequency*.

Since  $X_L = X_C$ , the impedance in a series RLC circuit is purely resistive. At the resonant frequency,  $f_r$ , the voltages across capacitance and inductance are equal in magnitude. Since they are 180° out of phase with each other, they cancel each other and, hence zero voltage appears across the LC combination.

At resonance

$$X_L = X_C$$
 i.e.  $\omega L = \frac{1}{\omega C}$ 

Solving for resonant frequency, we get

$$2\pi f_r L = \frac{1}{2\pi f_r C}$$
$$f_r^2 = \frac{1}{4\pi^2 LC}$$
$$f_r = \frac{1}{2\pi \sqrt{LC}}$$

....

In a series RLC circuit, resonance may be produced by varying the frequency, keeping L and C constant; otherwise, resonance may be produced by varying either L or C for a fixed frequency.

**Example 8.1** For the circuit shown in Fig. 8.2, determine the value of capacitive reactance and impedance at resonance.





Solution At resonance

Since

*:*.

$$X_L = X_C$$
  

$$X_L = 25 \Omega$$
  

$$X_C = 25 \Omega \quad \therefore \quad \frac{1}{\omega C} = 25$$

The value of impedance at resonance is

**x** 7

**T** 7

$$Z = R$$
$$Z = 50 \ \Omega$$

**Example 8.2** Determine the resonant frequency for the circuit shown in Fig. 8.3.



Fig. 8.3

Solution The resonant frequency is

$$f_r = \frac{1}{2\pi \sqrt{LC}} = \frac{1}{2\pi \sqrt{10 \times 10^{-6} \times 0.5 \times 10^{-3}}}$$
  
$$f_r = 2.25 \text{ kHz}$$

$$f_r = 2.25 \text{ kHz}$$

\*\*

#### **IMPEDANCE AND PHASE ANGLE OF A** 8.2 SERIES RESONANT CIRCUIT

The impedance of a series RLC circuit is

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The variation of  $X_c$  and  $X_L$  with frequency is shown in Fig. 8.4.



Fig. 8.4

At zero frequency, both  $X_c$  and Z are infinitely large, and  $X_L$  is zero because at zero frequency the capacitor acts as an open circuit and the inductor acts as a short circuit. As the frequency increases,  $X_c$  decreases and  $X_L$  increases. Since  $X_c$  is larger than  $X_L$ , at frequencies below the resonant frequency  $f_r$ , Z decreases along with  $X_c$ . At resonant frequency  $f_r$ ,  $X_c = X_L$ , and Z = R. At frequencies above the resonant frequency  $f_r$ ,  $X_L$  is larger than  $X_c$ , causing Z to increase. The phase angle as a function of frequency is shown in Fig. 8.5.



Fig. 8.5

At a frequency below the resonant frequency, current leads the source voltage because the capacitive reactance is greater than the inductive reactance. The phase angle decreases as the frequency approaches the resonant value, and is  $0^{\circ}$  at resonance. At frequencies above resonance, the current lags behind the source voltage, because the inductive reactance is greater than capacitive reactance. As the frequency goes higher, the phase angle approaches  $90^{\circ}$ .

**Example 8.3** For the circuit shown in Fig. 8.6, determine the impedance at resonant frequency, 10 Hz above resonant frequency, and 10 Hz below resonant frequency.



Fig. 8.6

**Solution** Resonant frequency  $f_r = \frac{1}{2\pi\sqrt{LC}}$ =  $\frac{1}{2\pi\sqrt{0.1 \times 10 \times 10^{-6}}} = 159.2 \text{ Hz}$ 

At 10 Hz below  $f_r = 159.2 - 10 = 149.2$  Hz

At 10 Hz above  $f_r = 159.2 + 10 = 169.2$  Hz

Ζ

Impedance at resonance is equal to R

÷

$$= 10 \Omega$$

Capacitive reactance at 149.2 Hz is

$$X_{C_1} = \frac{1}{\omega_1 C} = \frac{1}{2\pi \times 149.2 \times 10^{-6} \times 10}$$

:.

 $X_{C_1} = 106.6 \ \Omega$ Capacitive reactance at 169.2 Hz is

$$X_{C_2} = \frac{1}{\omega_2 C} = \frac{1}{2\pi \times 169.2 \times 10 \times 10^{-6}}$$
  
 $X_{C_2} = 94.06 \Omega$ 

÷

Inductive reactance at 149.2 Hz is

$$X_{L_1} = \omega_2 L = 2\pi \times 149.2 \times 0.1 = 93.75 \ \Omega$$

Inductive reactance at 169.2 Hz is

$$X_{L_2} = \omega_2 L = 2\pi \times 169.2 \times 0.1 = 106.31 \ \Omega$$

Impedance at 149.2 Hz is

$$|Z| = \sqrt{R^2 + (X_{L_1} - X_{C_1})^2}$$
$$= \sqrt{(10)^2 + (93.75 - 106.6)^2} = 16.28 \Omega$$

Here  $X_{C_1}$  is greater than  $X_{L_1}$ , so Z is capacitive. Impedance at 169.2 Hz is

$$|Z| = \sqrt{R^2 + (X_{L_2} - X_{C_2})^2}$$
$$= \sqrt{(10)^2 + (106.31 - 94.06)^2} = 15.81 \ \Omega$$

Here  $X_{L_1}$  is greater than  $X_{C_1}$ , so Z is inductive.

# 8.3 VOLTAGES AND CURRENTS IN A SERIES RESONANT CIRCUIT

The variation of impedance and current with frequency is shown in Fig. 8.7.

At resonant frequency, the capacitive reactance is equal to inductive reactance, and hence the impedance is minimum. Because of minimum impedance, maximum current flows through the circuit. The current variation with frequency is plotted.



Fig. 8.7

The voltage drop across resistance, inductance and capacitance also varies with frequency. At f = 0, the capacitor acts as an open circuit and blocks current. The complete source voltage appears across the capacitor. As the frequency increases,  $X_c$  decreases and  $X_L$  increases, causing total reactance  $X_c - X_L$  to decrease. As a result, the impedance decreases and the current increases. As the current increases,  $X_R$  also increases, and both  $V_c$  and  $V_L$  increase.

When the frequency reaches its resonant value  $f_r$ , the impedance is equal to R, and hence, the current reaches its maximum value, and  $V_R$  is at its maximum value.

As the frequency is increased above resonance,  $X_L$  continues to increase and  $X_C$  continues to decrease, causing the total reactance,  $X_L - X_C$  to increase. As a result there is an increase in impedance and a decrease in current. As the current decreases,  $V_R$  also decreases, and both  $V_C$  and  $V_L$  decrease. As the frequency becomes very high, the current approaches zero, both  $V_R$  and  $V_C$  approach zero, and  $V_L$  approaches  $V_s$ .

The response of different voltages with frequency is shown in Fig. 8.8.

The drop across the resistance reaches its maximum when  $f = f_r$ . The maximum voltage across the capacitor occurs at  $f = f_c$ . Similarly, the maximum voltage across the inductor occurs at  $f = f_L$ .

The voltage drop across the inductor is

where

$$V_{L} = IX_{L}$$

$$I = \frac{V}{Z}$$

$$V_{L} = \frac{\omega LV}{\sqrt{R^{2} + \left(\omega L - \frac{1}{\omega C}\right)^{2}}}$$

*.*..



To obtain the condition for maximum voltage across the inductor, we have to take the derivative of the above equation with respect to frequency, and make it equal to zero.

$$\therefore \qquad \frac{dV_L}{d\omega} = 0$$

If we solve for  $\omega$ , we obtain the value of  $\omega$  when  $V_L$  is maximum.

$$\frac{dV_L}{d\omega} = \frac{d}{d\omega} \left\{ \omega LV \left[ R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2 \right]^{-1/2} \right\}$$
$$LV \left( R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2} \right)^{-1/2}$$
$$- \frac{\omega LV}{2} \left( R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2} \right) \left( 2\omega L^2 - \frac{2}{\omega^3 C^2} \right)$$
$$R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2} = 0$$

From this

$$R^{2} - \frac{2L}{C} + 2/\omega^{2}C^{2} = 0$$
  

$$\omega L = \sqrt{\frac{2}{2LC - R^{2}C^{2}}} = \frac{1}{\sqrt{LC}} \sqrt{\frac{2}{2 - \frac{R^{2}C}{L}}}$$
  

$$f_{L} = \frac{1}{2\pi\sqrt{LC}} \sqrt{\frac{1}{1 - \frac{R^{2}C}{2L}}}$$

*:*.

Similarly, the voltage across the capacitor is

$$V_{C} = IX_{C} = \frac{I}{\omega C}$$

$$V_{C} = \frac{V}{\sqrt{R^{2} + \left(\omega L - \frac{1}{\omega C}\right)^{2}}} \times \frac{1}{\omega C}$$

$$W_{C} = \frac{V}{\sqrt{R^{2} + \left(\omega L - \frac{1}{\omega C}\right)^{2}}}$$

To get maximum value  $\frac{dV_C}{d\omega} = 0$ 

If we solve for  $\omega$ , we obtain the value of  $\omega$  when  $V_c$  is maximum.

$$\frac{dV_C}{d\omega} = \omega C \frac{1}{2} \left[ R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2 \right]^{-1/2} \left[ 2 \left( \omega L - \frac{1}{\omega C} \right) \left( L + \frac{1}{\omega^2 C} \right) \right] + \sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2 C} = 0$$

From this

$$\omega_C^2 = \frac{1}{LC} - \frac{R^2}{2L}$$
$$\omega_C = \sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$
$$f_C = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

*:*.

The maximum voltage across the capacitor occurs below the resonant frequency; and the maximum voltage across the inductor occurs above the resonant frequency.

**Example 8.4** A series circuit with  $R = 10 \Omega$ , L = 0.1 H and  $C = 50 \mu$ F has an applied voltage  $V = 50 \angle 0^{\circ}$  with a variable frequency. Find the resonant frequency, the value of frequency at which maximum voltage occurs across the inductor and the value of frequency at which maximum voltage occurs across the capacitor.

**Solution** The frequency at which maximum voltage occurs across the inductor is

$$f_L = \frac{1}{2\pi\sqrt{LC}} \sqrt{\frac{1}{\left(1 - \frac{R^2 C}{2L}\right)}}$$
$$= \frac{1}{2\pi\sqrt{0.1 \times 50 \times 10^{-6}}} \sqrt{\frac{1}{1 - \left(\frac{(10)^2 \times 50 \times 10^{-6}}{2 \times 0.1}\right)}}$$

= 72.08 Hz

...

Similarly,

$$f_C = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{\kappa}{2L}}$$
$$= \frac{1}{2\pi} \sqrt{\frac{1}{0.1 \times 50 \times 10^{-6}} - \frac{(10)^2}{2 \times 0.1}}$$
$$= \frac{1}{2\pi} \sqrt{200000 - 500}$$
$$= 71.08 \text{ Hz}$$

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Γ.

Resonant frequency  $f_r = \frac{1}{2\pi \sqrt{LC}} = \frac{1}{2\pi \sqrt{0.1 \times 50 \times 10^{-6}}} = 71.18 \text{ Hz}$ 

It is clear that the maximum voltage across the capacitor occurs below the resonant frequency and the maximum inductor voltage occurs above the resonant frequency.

## 8.4 BANDWIDTH OF AN RLC CIRCUIT

The bandwidth of any system is the range of frequencies for which the current or output voltage is equal to 70.7% of its value at the resonant frequency, and it is denoted by *BW*. Figure 8.9 shows the response of a series RLC circuit.

Here the frequency  $f_1$  is the frequency at which the current is 0.707 times the current at resonant value, and it is called the lower cut-off frequency. The frequency  $f_2$  is the frequency at which the current is 0.707 times the current at resonant value (i.e. maximum value), and is called the *upper cut-off frequency*. The bandwidth, or *BW*, is defind as the frequency difference between  $f_2$  and  $f_1$ .



Fig. 8.9

If the current at  $P_1$  is  $0.707I_{\text{max}}$ , the impedance of the circuit at this point is  $\sqrt{2} R$ , and hence

$$\frac{1}{\omega_1 C} - \omega_1 L = R \tag{8.1}$$

Similarly,

$$\omega_2 L - \frac{1}{\omega_2 C} = R \tag{8.2}$$

If we equate both the above equations, we get

$$\frac{1}{\omega_1 C} - \omega_1 L = \omega_2 L - \frac{1}{\omega_2 C}$$
$$L(\omega_1 + \omega_2) = \frac{1}{C} \left( \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \right)$$
(8.3)

From Eq. 8.3, we get

$$\omega_1 \omega_2 = \frac{1}{LC}$$
  

$$\omega_r^2 = \frac{1}{LC}$$
  

$$\omega_r^2 = \omega_1 \omega_2$$
(8.4)

we have

*:*.

If we add Eqs 8.1 and 8.2, we get

$$\frac{1}{\omega_1 C} - \omega_1 L + \omega_2 L - \frac{1}{\omega_2 C} = 2R$$

$$(\omega_2 - \omega_1)L + \frac{1}{C} \left( \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \right) = 2R$$

$$C = \frac{1}{\omega_r^2 L}$$

$$\omega_1 \omega_2 = \omega_r^2$$
(8.5)

Since

and

$$(\omega_2 - \omega_1)L + \frac{\omega_r^2 L(\omega_2 - \omega_1)}{\omega_r^2} = 2R$$
(8.6)

From Eq. 8.6, we have

$$\omega_2 - \omega_1 = \frac{R}{L} \tag{8.7}$$

:.

$$f_2 - f_1 = \frac{R}{2\pi L}$$
(8.8)

or

$$BW = \frac{R}{2\pi L}$$

From Eq. 8.8, we have

$$f_2 - f_1 = \frac{R}{2\pi L}$$
  
$$\therefore \qquad \qquad f_r - f_1 = \frac{R}{4\pi L}$$

$$f_2 - f_r = \frac{R}{4\pi L}$$

The lower frequency limit

$$f_1 = f_r - \frac{R}{4\pi L} \tag{8.9}$$

 $f_2 = f_r + \frac{R}{4\pi L}$ The upper frequency limit

If we divide the equation on both sides by  $f_r$ , we get

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L}$$
(8.11)

Here an important property of a coil is defined. It is the ratio of the reactance of the coil to its resistance. This ratio is defined as the Q of the coil. Q is known as a figure of merit, it is also called quality factor and is an indication of the quality of a coil.

$$Q = \frac{X_L}{R} = \frac{2\pi f_r L}{R}$$
(8.12)

If we substitute Eq. (8.11) in Eq. (8.12), we get

$$\frac{f_2 - f_1}{f_r} = \frac{1}{Q} \tag{8.13}$$

The upper and lower cut-off frequencies are sometimes called the *half-power* frequencies. At these frequencies the power from the source is half of the power delivered at the resonant frequency.

At resonant frequency, the power is

$$P_{\text{max}} = I_{\text{max}}^2 R$$
  
At frequency  $f_1$ , the power is  $P_1 = \left(\frac{I_{\text{max}}}{\sqrt{2}}\right)^2 R = \frac{I_{\text{max}}^2 R}{2}$   
Similarly at frequency  $f_1$  the power is

Similarly, at frequency  $f_2$ , the power is

$$P_2 = \left(\frac{I_{\max}}{\sqrt{2}}\right)^2 R$$
$$= \frac{I_{\max}^2 R}{2}$$

The response curve in Fig. 8.9 is also called the *selectivity curve* of the circuit. Selectivity indicates how well a resonant circuit responds to a certain frequency and eliminates all other frequencies. The narrower the bandwidth, the greater the selectivity.

**Example 8.5** Determine the quality factor of a coil for the series circuit consisting of  $R = 10 \Omega$ , L = 0.1 H and  $C = 10 \mu$ F.

Solution Quality factor 
$$Q = \frac{f_r}{BW}$$
$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1 \times 10 \times 10^{-6}}} = 159.2 \text{ Hz}$$

(8.10)

\*\*

At lower half power frequency,  $X_C > X_L$ 

$$\frac{1}{2\pi f_1 C} - 2\pi f_1 L = R$$
$$f_1 = \frac{-R + \sqrt{R^2 + 4L/C}}{4\pi L}$$

From which

At upper half power frequency  $X_L > X_C$ 

$$2\pi f_2 L - \frac{1}{2\pi f_2 C} = R$$
$$f_2 = \frac{+R + \sqrt{R^2 + 4L/C}}{4\pi L}$$

From which

Bandwidth

$$BW = f_2 - f_1 = \frac{R}{2\pi L}$$

Hence

$$Q_0 = \frac{f_r}{BW} = \frac{2\pi f_r L}{R} = \frac{2 \times \pi \times 159.2 \times 0.1}{10}$$
$$Q_0 = \frac{f_r}{BW} = 10$$

8.5 THE QUALITY FACTOR (Q) AND ITS EFFECT ON BANDWIDTH

The quality factor, Q, is the ratio of the reactive power in the inductor or capacitor to the true power in the resistance in series with the coil or capacitor.

The quality factor

$$Q = 2\pi \times \frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}}$$

In an inductor, the max energy stored is given by  $\frac{LI^2}{2}$ 

Energy dissipated per cycle =  $\left(\frac{I}{\sqrt{2}}\right)^2 R \times T = \frac{I^2 RT}{2}$ 

:. Quality factor of the coil  $Q = 2\pi \times \frac{\frac{1}{2}LI^2}{\frac{I^2R}{2} \times \frac{1}{f}}$ 

$$=\frac{2\pi f L}{R}=\frac{\omega L}{R}$$

Similarly, in a capacitor, the max energy stored is given by  $\frac{CV^2}{2}$ 

The energy dissipated per cycle =  $(I/\sqrt{2})^2 R \times T$ The quality factor of the capacitance circuit

$$Q = \frac{2\pi \frac{1}{2} C \left(\frac{I}{\omega C}\right)^2}{\frac{I^2}{2} R \times \frac{1}{f}} = \frac{1}{\omega C R}$$

In series circuits, the quality factor  $Q = \frac{\omega L}{R} = \frac{1}{\omega CR}$ 

We have already discussed the relation between bandwidth and quality factor, which is  $Q = \frac{f_r}{BW}$ .

A higher value of circuit Q results in a smaller bandwidth. A lower value of Q causes a larger bandwidth.

**Example 8.6** For the circuit shown in Fig. 8.10, determine the value of Q at resonance and bandwidth of the circuit.



Fig. 8.10

Solution The resonant frequency,

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$
$$= \frac{1}{2\pi\sqrt{5 \times 100 \times 10^{-6}}}$$
$$= 7.12 \text{ Hz}$$
$$Q = X_L/R = 2\pi f_r L/R$$

Quality factor

$$=\frac{2\pi \times 7.12 \times 5}{100}=2.24$$

Bandwidth of the circuit is  $BW = \frac{f_r}{Q} = \frac{7.12}{2.24} = 3.178 \text{ Hz}$ 

#### 8.6 **MAGNIFICATION IN RESONANCE**

If we assume that the voltage applied to the series RLC circuit is V, and the current at resonance is I, then the voltage across L is  $V_L = IX_L = (V/R) \omega_r L$ 

Similarly, the voltage across C

$$V_C = IX_C = \frac{V}{R\omega_r C}$$

 $Q = 1/\omega_r CR = \omega_r L/R$ 

Since

where 
$$\omega_r$$
 is the frequency at resonance.

Therefore

$$V_L = VQ$$
$$V_C = VQ$$

The ratio of voltage across either L or C to the voltage applied at resonance can be defined as magnification.

: Magnification =  $Q = V_I / V$  or  $V_C / V$ 

#### 8.7 PARALLEL RESONANCE

Basically, parallel resonance occurs when  $X_c = X_L$ . The frequency at which resonance occurs is called the *resonant* frequency. When  $X_c = X_i$ , the two branch currents are equal in magnitude and 180° out of phase with each other.

Therefore, the two currents cancel each other out, and the total current is zero. Consider the circuit shown in Fig. 8.11. The condition for resonance occurs when  $X_L = X_C$ . In Fig. 8.11, the total admittance

$$Y = \frac{1}{R_L + j\omega L} + \frac{1}{R_C - (j/\omega C)}$$
$$= \frac{R_L - j\omega L}{R_L^2 + \omega^2 L^2} + \frac{R_C + (j/\omega C)}{R_C^2 + \frac{1}{\omega^2 C^2}}$$

...





$$= \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_C}{R_C^2 + \frac{1}{\omega^2 C^2}} + j \left\{ \left[ \frac{1/\omega C}{R_C^2 + \frac{1}{\omega^2 C^2}} \right] - \left[ \frac{\omega L}{R_L^2 + \omega^2 L^2} \right] \right\} (8.14)$$

At resonance the susceptance part becomes zero

$$\frac{\omega_r L}{R_L^2 + \omega_r^2 L^2} = \frac{\frac{1}{\omega_r C}}{R_C^2 + \frac{1}{\omega_r^2 C^2}}$$
(8.15)

8.14

$$\omega_{r}L\left[R_{C}^{2} + \frac{1}{\omega_{r}^{2}C^{2}}\right] = \frac{1}{\omega_{r}C} \left[R_{L}^{2} + \omega_{r}^{2}L^{2}\right]$$

$$\omega_{r}^{2}\left[R_{C}^{2} + \frac{1}{\omega_{r}^{2}C^{2}}\right] = \frac{1}{LC} \left[R_{L}^{2} + \omega_{r}^{2}L^{2}\right]$$

$$\omega_{r}^{2}R_{C}^{2} - \frac{\omega_{r}^{2}L}{C} = \frac{1}{LC}R_{L}^{2} - \frac{1}{C^{2}}$$

$$\omega_{r}^{2}\left[R_{C}^{2} - \frac{L}{C}\right] = \frac{1}{LC}\left[R_{L}^{2} - \frac{L}{C}\right]$$

$$\omega_{r} = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_{L}^{2} - (L/C)}{R_{C}^{2} - (L/C)}}$$
(8.16)

The condition for resonant frequency is given by Eq. 8.16. As a special case, if  $R_L = R_C$ , then Eq. 8.16 becomes

$$\omega_r = \frac{1}{\sqrt{LC}}$$
$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

Therefore

**Example 8.7** Find the resonant frequency in the ideal parallel *LC* circuit shown in Fig. 8.12.



Fig. 8.12

Solution 
$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{50 \times 10^{-3} \times 0.01 \times 10^{-6}}} = 7117.6 \text{ Hz}$$

# 8.8 RESONANT FREQUENCY FOR A TANK CIRCUIT

The parallel resonant circuit is generally called a tank circuit because of the fact that the circuit stores energy in the magnetic field of the coil and in the electric field of the capacitor. The stored energy is transferred back and forth between the capacitor and coil and vice-versa. The tank circuit is shown in Fig. 8.13. The circuit is said to be in resonant condition when the suscep-tance part of admittance is zero.



Fig. 8.13

The total admittance is  $Y = \frac{1}{R_L + jX_L} + \frac{1}{-jX_C}$  (8.17) Simplifying Eq. 8.17, we have

Simplifying Eq. 8.17, we have

$$Y = \frac{R_L - jX_L}{R_L^2 + X_L^2} + \frac{j}{X_C}$$
$$= \frac{R_L}{R_L^2 + X_L^2} + j \left[ \frac{1}{X_C} - \frac{X_L}{R_L^2 + X_L^2} \right]$$

To satisfy the condition for resonance, the susceptance part is zero.

$$\frac{1}{X_C} = \frac{X}{R_L^2 + X_L^2}$$
(8.18)

$$\omega C = \frac{\omega L}{R_L^2 + \omega^2 L^2} \tag{8.19}$$

From Eq. 8.19, we get

$$R_L^2 + \omega^2 L^2 = \frac{L}{C}$$
  

$$\omega^2 L^2 = \frac{L}{C} - R_L^2$$
  

$$\omega^2 = \frac{1}{LC} - \frac{R_L^2}{L^2}$$
  

$$\omega = \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}$$
(8.20)

:.

:.

The resonant frequency for the tank circuit is

$$f_r = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}$$
(8.21)

**Example 8.8** For the tank circuit shown in Fig. 8.14, find the resonant frequency.  $10 \Omega = 0.1 \text{ H}$ 



Fig. 8.14

Solution The resonant frequency

$$f_r = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{1}{0.1 \times 10 \times 10^{-6}} - \frac{(10)^2}{(0.1)^2}}$$
$$= \frac{1}{2\pi} \sqrt{(10)^6 - (10)^2} = \frac{1}{2\pi} (994.98) = 158.35 \text{ Hz}$$

# 8.9 VARIATION OF IMPEDANCE WITH FREQUENCY

The impedance of a parallel resonant circuit is maximum at the resonant frequency and decreases at lower and higher frequencies as shown in Fig. 8.15.

At very low frequencies,  $X_L$  is very small and  $X_C$  is very large, so the total impedance is essentially inductive. As the frequency increases, the impedance also increases, and the inductive reactance dominates until the resonant frequency is reached. At this point  $X_L = X_C$  and the impedance is at its maximum. As the frequency goes above resonance, capacitive reactance dominates and the impedance decreases.



Fig. 8.15

# 8.10 Q FACTOR OF PARALLEL RESONANCE

Consider the parallel RLC circuit shown in Fig. 8.16.



Fig. 8.16

In the circuit shown, the condition for resonance occurs when the susceptance part is zero.

Admittance

$$Y = G + jB$$

$$= \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

$$= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$
(8.23)

The frequency at which resonance occurs is

$$\omega_r C - \frac{1}{\omega_r L} = 0 \tag{8.24}$$

$$\omega_r = \frac{1}{\sqrt{LC}} \tag{8.25}$$

The voltage and current variation with frequency is shown in Fig. 8.17. At resonant frequency, the current is minimum.





The bandwidth,  $BW = f_2 - f_1$ 

For parallel circuit, to obtain the lower half power frequency,

$$\omega_1 C - \frac{1}{\omega_1 L} = -\frac{1}{R} \tag{8.26}$$

From Eq. 8.26, we have

$$\omega_1^2 + \frac{\omega_1}{RC} - \frac{1}{LC} = 0 \tag{8.27}$$

If we simplify Eq. 8.27, we get

$$\omega_{1} = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^{2} + \frac{1}{LC}}$$
(8.28)

Similarly, to obtain the upper half power frequency

$$\omega_2 C - \frac{1}{\omega_2 L} = \frac{1}{R} \tag{8.29}$$

From Eq. 8.29, we have

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$
(8.30)

Bandwidth  

$$BW = \omega_2 - \omega_1 = \frac{1}{RC}$$
  
The quality factor is defined as  $Q_r = \frac{\omega_r}{\omega_2 - \omega_1}$   
 $Q_r = \frac{\omega_r}{1/RC} = \omega_r RC$   
In other words

In other words,

...

$$Q = 2\pi \times \frac{\text{maximum energy stored}}{\text{Energy dissipated/cycle}}$$

In the case of an inductor,

The maximum energy stored =  $\frac{1}{2} LI^2$ Energy dissipated per cycle =  $\left(\frac{I}{\sqrt{2}}\right)^2 \times R \times T$ 

The quality factor  $Q = 2\pi \times \frac{1/2 (LI^2)}{\frac{I^2}{R} \times \frac{1}{r^2}}$ 

$$Q = 2\pi \times \frac{\frac{1}{2}L\left(\frac{V}{\omega L}\right)^2 R}{\frac{V^2}{2} \times \frac{1}{f}}$$
$$= \frac{2\pi f L R}{\omega^2 L^2} = \frac{R}{\omega L}$$

For a capacitor, maximum energy stored =  $1/2 (CV^2)$ 

Energy dissipated per cycle =  $P \times T = \frac{V^2}{2 \times R} \times \frac{1}{f}$ 

The quality factor  $Q = 2\pi \times \frac{1/2 (CV^2)}{\frac{V^2}{2R} \times \frac{1}{f}}$  $= 2\pi fCR = \omega CR$ 

#### 8.11 MAGNIFICATION

Current magnification occurs in a parallel resonant circuit. The voltage applied to the parallel circuit, V = IR

Since 
$$I_L = \frac{V}{\omega_r L} = \frac{IR}{\omega_r L} = IQ_r$$

For the capacitor,  $I_C = \frac{V}{1/\omega_r C} = IR\omega_r C = IQ_r$ 

Therefore, the quality factor  $Q_r = I_L/I$  or  $I_C/I$ 

# 8.12 REACTANCE CURVES IN PARALLEL RESONANCE

The effect of variation of frequency on the reactance of the parallel circuit is shown in Fig. 8.18.



Fig. 8.18

The effect of inductive susceptance,

$$B_L = \frac{-1}{2\pi f L}$$

Inductive susceptance is inversely proportional to the frequency or  $\omega$ . Hence it is represented by a rectangular hyperbola, MN. It is drawn in fourth quadrant, since  $B_L$  is negative. Capacitive susceptance,  $B_C = 2\pi fC$ . It is directly proportional to the frequency f or  $\omega$ . Hence it is represented by OP, passing through the origin. Net susceptance  $B = B_C - B_L$ . It is represented by the curve JK, which is a hyperbola. At point  $\omega_r$ , the total susceptance is zero, and resonance takes place. The variation of the admittance Y and the current I is represented by curve VW. The current will be minimum at resonant frequency.

# 8.13 LOCUS DIAGRAMS

A phasor diagram may be drawn and is expanded to develop a curve; known as a locus. Locus diagrams are useful in determining the behaviour or response of an RLC circuit when one of its parameters is varied while the frequency and voltage kept constant. The magnitude and phase of the current vector in the circuit depends upon the values of R, L, and C and frequency at the fixed source voltage. The path traced by the terminus of the current vector when the parameters R, L or C are varied while f and v are kept constant is called the current locus.

The term circle diagram identifies locus plots that are either circular or semicircular loci of the terminus (the tip of the arrow) of a current phasor or voltage phasor. Circle diagrams are often employed as aids in analysing the operating characteristics of circuits like equivalent circuit of transmission lines and some types of AC machines.

Locus diagrams can be also drawn for reactance, impedance, susceptance and admittance when frequency is variable. Loci of these parameters furnish important information for use in circuit analysis. Such plots are particularly useful in the design of electric wave filters.

### 8.13.1 Series Circuits

To discuss the basis of representing a series circuit by means of a circle diagram consider the circuit shown in Fig. 8.19(a). The analytical procedure is greatly simplified by assuming that inductance elements have no resistance and that capacitors have no leakage current.







The circuit under consideration has constant reactance but variable resistance. The applied voltage will be assumed with constant rms voltage V.

The power factor angle is designated by  $\theta$ . If R = 0,  $I_L$  is obviously equal to  $\frac{V}{X_L}$ and has maximum value. Also *I* lags *V* by 90°. This is shown in Fig. 8.19(b). If *R* is increased from zero value, the magnitude of *I* becomes less than  $\frac{V}{X_L}$  and  $\theta$ 

becomes less than 90° and finally when the limit is reached, i.e. when *R* equals to infinity, *I* equals to zero and  $\theta$  equals to zero. It is observed that the tip of the current vector represents a semicircle as indicated in Fig. 8.19(b).

In general

$$I_L = \frac{V}{Z}$$

or

$$\sin \theta = \frac{X}{Z}$$

$$Z = \frac{X_L}{\sin \theta}$$

$$I = \frac{V}{X_L} \sin \theta$$
(8.31)

For constant V and X, Eq. 8.31 is the polar equation of a circle with diameter  $\frac{V}{X_L}$ . Figure 8.19(b) shows the plot of Eq. 8.31 with respect to V as reference.

The active component of the current  $I_L$  in Fig. 8.19(b) is  $OI_L \cos \theta$  which is proportional to the power consumed in the RL circuit. In a similar way we can draw the loci of current if the inductive reactance is replaced by a capacitive reactance as shown in Fig. 8.19(c). The current semicircle for the RC circuit with variable R will be on the left-hand side of the voltage vector OV with diameter  $\frac{V}{X_L}$  as shown in Fig. 8.19(d). The current vector  $OI_C$  leads V by  $\theta^\circ$ .

The active component of the current  $I_c X$  in Fig. 8.19(d) is  $OI_c \cos \theta$  which is proportional to the power consumed in the RC circuit.



**Circle Equations for an RL Circuit** (a) Fixed reactance and variable resistance. The X-co-ordinate and Y-co-ordinate of  $I_L$  in Fig. 8.19(b) respectively are  $I_X = I_L \sin \theta$ ;  $I_y = I_L \cos \theta$ .

$$IL = \frac{V}{Z}; \sin \theta = \frac{X_L}{Z}; \cos \theta = \frac{R}{Z}; Z = \sqrt{R^2 + X_L^2}$$
$$I_X = \frac{V}{Z} \cdot \frac{X_L}{Z} = V \cdot \frac{X_L}{Z^2}$$
(8.32)

:.

Where

$$I_Y = \frac{V}{Z} \cdot \frac{R}{Z} = V \cdot \frac{R}{Z^2}$$
(8.33)

Squaring and adding Eqs. 8.32 and 8.33, we obtain

$$I_X^2 + I_Y^2 = \frac{V^2}{R^2 + X_L^2}$$
(8.34)

From Eq. 8.32

$$Z^2 = R^2 + X_L^2 = V \cdot \frac{X_L}{I_X}$$

: Equation 8.34 can be written as  $I_X^2 + I_Y^2 = \frac{V}{X_r} \cdot I_X$ 

$$I_X^2 + I_Y^2 - \frac{V}{X_L} \cdot I_X = 0$$

Adding  $\left(\frac{V}{2X_{T}}\right)^{2}$  to both sides the above equation can be written as

$$\left(I_X - \frac{V}{2X_L}\right)^2 + I_Y^2 = \left(\frac{V}{2X_L}\right)^2$$
(8.35)

Equation 8.35 represents a circle whose radius is  $\frac{V}{2X_I}$  and the co-ordinates

of the centre are  $\frac{V}{2X_I}$ , 0.

In a similar way we can prove that for a series RC circuit as in Fig. 8.19(c), with variable R, the locus of the tip of the current vector is a semi-circle and is given by

$$\left(I_X + \frac{V}{2X_C}\right)^2 + I_Y^2 = \frac{V^2}{4X_C^2}$$
(8.36)

The centre has co-ordinates of  $-\frac{V}{2X_I}$ , 0 and radius as  $\frac{V}{2X_I}$ .

(b) Fixed resistance, variable reactance Consider the series RL circuit with constant resistance R but variable reactance  $X_{L}$  as shown in Fig. 8.20(a).



Fig. 8.20(a)

Fig. 8.20(b)

or

When  $X_L = 0$ ;  $I_L$  assumes maximum value of  $\frac{V}{R}$  and  $\theta = 0$ , the power factor of the circuit becomes unity; as the value  $X_L$  is increased from zero,  $I_L$  is reduced and finally when  $X_L$  is  $\alpha$ , current becomes zero and  $\theta$  will be lagging behind the voltage by 90° as shown in Fig. 8.20(b). The current vector describes a semicircle with diameter  $\frac{V}{R}$  and lies in the right-hand side of voltage vector OV. The active component of the current  $OI_L \cos \theta$  is proportional to the power consumed in the RL circuit.

1/2

### Equation of circle

Consider Eq. 8.34

From Eq. 8.33

$$I_X^2 + I_Y^2 = \frac{v}{R^2 + X_L^2}$$
$$Z^2 = R^2 + X_L^2 = \frac{VR}{I_Y}$$
(8.37)

Substituting Eq. 8.37 in Eq. 8.34

$$I_X^2 + I_Y^2 = \frac{V}{R} I_Y$$
(8.38)
$$I_X^2 + I_Y^2 - \frac{V}{R} I_Y = 0$$

Adding  $\left(\frac{V}{2R}\right)^2$  to both sides the above equation can be written as

Ι

$$I_X^2 + \left(I_Y - \frac{V}{2R}\right)^2 = \left(\frac{V}{2R}\right)^2$$
(8.39)

Equation 8.39 represents a circle whose radius is  $\frac{V}{2R}$  and the co-ordinates

of the centre are 0;  $\frac{V}{2R}$ .

Let the inductive reactance in Fig. 8.20(a) be replaced by a capacitive reactance as shown in Fig. 8.21(a).





Fig. 8.21(a)

The current semicircle of a RC circuit with variable  $X_c$  will be on the lefthand side of the voltage vector OV with diameter  $\frac{V}{R}$ . The current vector  $OI_c$ leads V by  $\theta^\circ$ . As before, it may be proved that the equation of the circle shown in Fig. 8.21(b) is

$$I_X^2 + \left(I_Y - \frac{V}{2R}\right)^2 = \left(\frac{V}{2R}\right)^2$$

**Example 8.9** For the circuit shown in Fig. 8.22(a) plot the locus of the current, mark the range of *I* for maximum and minimum values of *R*, and the maximum power consumed in the circuit. Assume  $X_L = 25 \Omega$  and  $R = 50 \Omega$ . The voltage is 200 V; 50 Hz.



Fig. 8.22(a)

Fig. 8.22(b)

### Solution

Maximum value of current  $I_{\text{max}} = \frac{200}{25} = 8 \text{ A}; q = 90^{\circ}$ Minimum value of current  $I_{\text{min}} = \frac{200}{\sqrt{(50)^2 + (25)^2}} = 3.777 \text{ A}; \theta = 27.76^{\circ}$ 

The locus of the current is shown in Fig. 8.22(b).

Power consumed in the circuit is proportional to  $I \cos \theta$  for constant V. The maximum ordinate possible in the semicircle (*AB* in Fig. 8.22(b)) represents the maximum power consumed in the circuit. This is possible when  $\theta = 45^{\circ}$ , under the

condition power factor  $\cos \theta = \cos 45^\circ = \frac{1}{\sqrt{2}}$ .

Hence, the maximum power consumed in the circuit =  $V \times AB = V \times \frac{I_{\text{max}}}{L}$ 

$$I_{\text{max}} = \frac{V}{X_L} = 84 \text{ A}$$
$$P_{\text{max}} = \frac{V^2}{2X_L} = \frac{(200)^2}{2 \times 25} = 800 \text{ W}$$

**Example 8.10** For the circuit shown in Fig. 8.22(a), if the reactance is variable, plot the range of *I* for maximum and minimum values of  $X_L$  and maximum power consumed in the circuit.

### Solution



**Example 8.11** For the circuit shown in Fig. 8.24(a) draw the locus of the current. Mark the range of *I* for maximum and minimum values. Assume  $X_c = 50 \Omega$ ;  $R = 10 \Omega$ ; V = 400 V.



Fig. 8.24(a)



### Solution

$$I_{\text{max}} = \frac{100}{10} = 40 \text{ A}; \ \theta = 0^{\circ}$$
$$I_{\text{min}} = \frac{400}{\sqrt{(50)^2 + (10)^2}} = 7.716 \text{ A}. \ \theta = \tan^{-1} 5 = 77.8^{\circ}$$

The locus of the current is shown in Fig. 8.24(b).

400

## 8.13.2 Parallel Circuits

**Variable**  $X_L$  Locus plots are drawn for parallel branches containing RLC elements in a similar way as for series circuits. Here we have more than one current locus. Consider the parallel circuit shown in Fig. 8.25(a). The quantities that may be varied are  $X_L$ ,  $X_C$ ,  $R_L$  and  $R_C$  for a given voltage and frequency.



Fig. 8.25(a)

Let us consider the variation of  $X_L$  from zero to  $\infty$ . Let OV shown in Fig. 8.25(b), be the voltage vector, taken as reference. A current,  $I_C$  will flow in the condenser branch whose parameters are held constant and leads V by an

angle  $\theta_C = \tan^{-1}\left(\frac{X_C}{R_C}\right)$ , when  $X_L = 0$ , the current  $I_L$ , through the inductive

branch is maximum and is given by  $\frac{V}{R_L}$  and it is in phase with the applied voltage. When  $X_L$  is increased from zero, the current through the inductive branch  $I_L$  decreases and lags V by  $\theta_L = \tan^{-1} \frac{X_L}{R_L}$  as shown in Fig. 8.25(b). For

any value of  $I_L$ , the  $I_L R_L$  drop and  $I_L X_L$  drop must add at right angles to give the applied voltage. These drops are shown as OA and AV respectively. The locus of  $I_L$  is a semicircle, and the locus of  $I_L R_L$  drop is also a semicircle. When  $X_L = 0$ , i.e.  $I_L$  is maximum,  $I_L$  coincides with the diameter of its semicircle and  $I_L R_L$  drop also coincides with the diameter of its semicircle as shown in the figure; both these semicircles are shown with dotted circles as  $OI_L B$  and OAV respectively.

Since the total current is  $I_c + I_L$ . For example, a particular value of  $I_c$  and  $I_L$  the total current is represented by *OC* on the total current semicircle. As  $X_L$  is varied, the locus of the resultant current is therefore, the circle  $I_c CB$  as shown with thick line in the Fig. 8.25(b).



#### Fig. 8.25(b)

(b) Variable  $X_c$  A similar procedure can be adopted as outlined above to draw the locus plots of  $I_1$  and I when  $X_c$  is varying while  $R_L$ ,  $R_c$ ,  $X_L$ , V and f are held constant. The curves are shown in Fig. 8.25(c).



Fig. 8.25(c)

*OV* presents the voltage vector, *OB* is the maximum current through *RC* branch when  $X_L = 0$ ; *OI*<sub>L</sub> is the current through the  $R_L$  branch lagging *OV* by an angle  $\theta_L = \tan^{-1} \frac{C_L}{R_L}$ . As  $X_C$  is increased from zero, the current through the

capacitive branch  $I_c$  decreases and leads V by  $\theta_c = \tan^{-1} \frac{X_c}{R_c}$ . For a particular  $I_c$ , the resultant current  $I = I_L + I_c$  and is given by OC. The dotted semicircle  $OI_cB$  is the locus of the  $I_c$ , thick circle  $I_LCB$  is the locus of the resultant current.

(c) Variable  $R_L$  The locus of current for the variation of  $R_L$  in Fig. 8.26(a) is shown in Fig. 8.26(b). *OV* represents the reference voltage,  $OI_L B$  represents the locus of  $I_L$  and  $I_C CB$  represents the resultant current locus. Maximum  $I_L = \frac{V}{X_L}$  is represented by *OB*.

(d) Variable  $R_c$  The locus of currents for the variation of  $R_c$  in Fig. 8.27(a) is plotted in Fig. 8.27(b) where OV is the source voltage and semicircle OAB represents the locus of  $I_c$ . The resultant current locus is given by  $BCI_L$ .



Fig. 8.27(a)

Fig. 8.27(b)

**Example 8.12** For the parallel circuit shown in Fig. 8.28(a), draw the locus of  $I_1$  and I. Mark the range of values for  $R_1$  between 10  $\Omega$  and 100  $\Omega$ . Assume  $X_L = 25 \Omega$  and  $R_2 = 25 \Omega$ . The supply voltage is 200 V and frequency is 50 Hz, both held constant.

### Solution

Let us take voltage as reference; on the positive X-axis.  $I_2$  is given by  $I_2 = \frac{200}{25}$ = 8A and is in phase with V.



**Example 8.13** Draw the locus of  $I_2$  and I for the parallel circuit shown in Fig. 8.29(a).



Fig. 8.29(a)
#### Solution

 $I_1$  leads the voltage by a fixed angle  $\theta_1$  given by  $\tan^{-1} \frac{X_C}{R_1}$ 

 $I_2$  varies according to the value of  $X_{C_2}$ 

 $I_2$  is maximum when  $X_{C_2} = 0$  and is in phase with V

 $I_2$  is zero when  $X_{C_2} = \infty$  as shown in Fig. 8.29(b).



Fig. 8.29(b)

**Example 8.14** For a parallel circuit shown in Fig. 8.30(a) plot the locus of currents.



Fig. 8.30(a)

Fig. 8.30(b)

Current  $I_1$  leads the voltage by a fixed angle  $\theta_1$  given by  $\tan^{-1} \frac{X_C}{R_1}$ , current  $I_2$  leads the voltage by 90°.  $I_3$  varies according to the value of  $X_L$ , when  $X_L = 0$ ,  $I_3$  is maximum and is given by  $\frac{V}{R_L}$ ; is in phase with V; when  $X_L = \infty$ ,  $I_3$  is zero. Both these extremities are shown in Fig. 8.30(b). For a particular value of  $I_3$  the total current I is given by  $I_1 + I_2 + I_3 = OA + AB + BC$ .

...

## ADDITIONAL SOLVED PROBLEMS

**Problem 8.1** For the circuit shown in Fig. 8.31, determine the frequency at which the circuit resonates. Also find the voltage across the inductor at resonance and the Q factor of the circuit.

**Solution** The frequency of resonance occurs when  $X_L = X_C$ 

$$\omega L = \frac{1}{\omega C}$$





:.

$$\omega = \frac{1}{\sqrt{LC}} \text{ radians/sec}$$
  
=  $\frac{1}{\sqrt{0.1 \times 50 \times 10^{-6}}} = 447.2 \text{ rad/sec}$   
 $f_r = \frac{1}{2\pi} (447.2) = 71.17 \text{ Hz}$ 

The current passing through the circuit at resonance,

$$I = \frac{V}{R} = \frac{100}{10} = 10A$$

The voltage drop across the inductor

$$V_L = IX_L = I\omega L$$
  
= 10 × 447.2 × 0.1 = 447.2 V  
lity factor  $Q = \frac{\omega L}{\omega} - \frac{447.2 \times 0.1}{\omega} = 4.47$ 

The quality factor 
$$Q = \frac{\omega L}{R} = \frac{447.2 \times 0.1}{10} = 4.47$$

**Problem 8.2** A series RLC circuit has a quality factor of 5 at 50 rad/sec. The current flowing through the circuit at resonance is 10 A and the supply voltage is 100 V. The total impedance of the circuit is  $20 \Omega$ . Find the circuit constants.

O = 5, R = 10

**Solution** The quality factor Q = 5

At resonance the impedance becomes resistance.

The current at resonance is  $I = \frac{V}{R}$   $10 = \frac{100}{R}$   $\therefore \qquad R = 10 \ \Omega$  $Q = \frac{\omega L}{R}$ 

Since

	$\omega L = 50$	
<b>∴</b>	$L = \frac{50}{\omega} = 1 \text{ H}$	
Total impedance is	$Q = \frac{1}{\omega CR}$	
Α.	$C = \frac{1}{Q\omega R}$	
	$=\frac{1}{5\times50\times10}$	
	$C = 400 \ \mu F$	00

**Problem 8.3** A voltage  $v(t) = 10 \sin \omega t$  is applied to a series RLC circuit. At the resonant frequency of the circuit, the maximum voltage across the capacitor is found to be 500 V. Moreover, the bandwidth is known to be 400 rad/sec and the impedance at resonance is 100  $\Omega$ . Find the resonant frequency. Also find the values of L and C of the circuit.

**Solution** The applied voltage to the circuit is

$$V_{\rm max} = 10 \text{ V}$$
  
 $V_{\rm rms} = \frac{10}{\sqrt{2}} = 7.07 \text{ V}$ 

The voltage across capacitor  $V_c = 500$  V

The magnification factor  $Q = \frac{V_C}{V} = \frac{500}{7.07} = 70.7$ 

The bandwidth BW = 400 rad/sec

$$\omega_2 - \omega_1 = 400 \text{ rad/sec}$$

The impedance at resonance  $Z = R = 100 \Omega$ 

#### Since

...

$$Q = \frac{\omega_r}{\omega_2 - \omega_1}$$
  

$$\omega_r = Q(\omega_2 - \omega_1) = 28280 \text{ rad/sec}$$
  

$$f_r = \frac{28280}{2\pi} = 4499 \text{ Hz}$$

The bandwidth  $\omega_2 - \omega_1 = \frac{R}{L}$ 

$$L = \frac{R}{\omega_2 - \omega_1} = \frac{100}{400} = 0.25 \text{ H}$$

Since

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$
$$C = \frac{1}{(2\pi f_r)^2 \times L} = \frac{1}{2\pi \times (4499)^2 \times 0.25} = 5 \ nF$$

ξ5Ω

<sup>↓</sup> \$10 Ω

*–j*12 Ω

\*\*

**Problem 8.4** Find the value of L at which the circuit resonates at a frequency of 1000 rad/sec in the circuit shown in Fig. 8.32.

# **Solution** $Y = \frac{1}{10} + \frac{1}{10}$

$$10 - j12 \quad 5 + jX_L$$
  

$$Y = \frac{10 + j12}{10^2 + 12^2} + \frac{5 - jX_L}{25 + X_L^2}$$
Fig. 8.32

$$= \frac{10}{10^2 + 12^2} + \frac{5}{25 + X_L^2} + j \left[ \frac{12}{10^2 + 12^2} - \frac{X_L}{25 + X_L^2} \right]$$

At resonance the susceptance becomes zero.

Then

$$\frac{X_L}{25 + X_L^2} = \frac{12}{10^2 + 12^2}$$
$$12X_L^2 - 244 X_L + 300 = 0$$

From the above equation

$$X_{L}^{2} - 20.3 X_{L} + 25 = 0$$

$$X_{L} = \frac{+20.3 \pm \sqrt{(20.3)^{2} - 4 \times 25}}{2}$$

$$= \frac{20.3 + \sqrt{412 - 100}}{2} \text{ or } \frac{20.3 - \sqrt{412 - 100}}{2}$$

$$= 18.98 \Omega \text{ or } 1.32 \Omega$$

$$X_{L} = \omega L = 18.98 \text{ or } 1.32 \Omega$$

$$L = \frac{18.98}{1000} \text{ or } \frac{1.32}{1000}$$

$$L = 18.98 \text{ mH or } 1.32 \text{ mH}$$

...

**Problem 8.5** Two impedances  $Z_1 = 20 + j10$  and  $Z_2 = 10 - j30$  are connected in parallel and this combination is connected in series with  $Z_3 = 30 + jX$ . Find the value of X which will produce resonance.

Solution Total impedance is

$$Z = Z_3 + (Z_1 || Z_2)$$
  
= (30 + jX) +  $\left\{ \frac{(20 + j10)(10 - j30)}{20 + j10 + 10 - j30} \right\}$   
= (30 + jX) +  $\frac{200 - j600 + j100 + 300}{30 - j20}$ 

$$= 30 + jX + \left(\frac{500 - j500}{30 - j20}\right)$$
  
=  $30 + jX + \left[\frac{500(1 - j)(30 + 20j)}{(30)^2 + (20)^2}\right]$   
=  $(30 + jX) + \left[\frac{500(30 + 20j - 30j + 20)}{900 + 400}\right]$   
=  $30 + jX + \frac{5}{13}(50 - j10)$   
=  $\left(30 + \frac{5}{13} \times 50\right) + j\left(X - \frac{5}{13} \times 10\right)$ 

At resonance, the imaginary part is zero

:. 
$$X - \frac{50}{13} = 0$$
  
 $X = \frac{50}{13} = 3.85 \ \Omega$ 

**Problem 8.6** A 50  $\Omega$  resistor is connected in series with an inductor having internal resistance, a capacitor and 100 V variable frequency supply as shown in Fig. 8.33. At a frequency of 200 Hz, a maximum current of 0.7 A flows through the circuit and voltage across the capacitor is 200 V. Determine the circuit constants.

#### Solution At resonance, current in the circuit is maximum



ince 
$$X_C = \frac{1}{\omega C} = \frac{200}{0.7} = 285.7 \ \Omega$$

$$X_L = \omega L = 285.7 \ \Omega$$

S

$$L = \frac{285.7}{2\pi \times 200} = 0.23 \text{ H}$$

At resonance, the total impedance

...

...

...

$$Z = R + 50$$
  

$$R + 50 = \frac{V}{I} = \frac{100}{0.7}$$
  

$$R + 50 = 142.86 \ \Omega$$
  

$$R = 92.86 \ \Omega$$

**Problem 8.7** In the circuit shown in Fig. 8.34, a maximum current of 0.1 A flows through the circuit when the capacitor is at 5  $\mu$ F with a fixed frequency and a voltage of 5 V. Determine the frequency at which the circuit resonates, the bandwidth, the quality factor Q and the value of resistance at resonant frequency.

Solution At resonance, the current is maximum in the circuits



а.

Since 
$$\frac{f_r}{BW} = Q$$

The bandwidth 
$$BW = \frac{f_r}{Q} = \frac{225}{2.8} = 80.36 \text{ Hz}$$

**Problem 8.8** In the circuit shown in Fig. 8.35, determine the circuit constants when the circuit draws a maximum current at 10  $\mu$ F with a 10 V, 100 Hz supply. When the capacitance is changed to 12  $\mu$ F, the current that flows through the circuit becomes 0.707 times its maximum value. Determine Q of the coil at 900 rad/sec. Also find the maximum current that flows through the circuit.

22

....



Fig. 8.35

**Solution** At resonant frequency, the circuit draws maximum current. So, the resonant frequency  $f_r = 100 \text{ Hz}$ 

$$f_r = \frac{1}{2\pi \sqrt{LC}}$$

$$L = \frac{1}{C \times (2\pi f_r)^2}$$

$$= \frac{1}{10 \times 10^{-6} (2\pi \times 100)^2} = 0.25 \text{ H}$$

$$\omega L - \frac{1}{\omega C} = R$$

We have

$$900 \times 0.25 - \frac{1}{900 \times 12 \times 10^{-6}} = R$$
$$R = 132.4 \ \Omega$$

*:*..

The quality factor  $Q = \frac{\omega L}{R} = \frac{900 \times 0.25}{132.4} = 1.69$ 

The maximum current in the circuit is  $I = \frac{10}{132.4} = 0.075 \text{ A}$ 

**Problem 8.9** In the circuit shown in Fig. 8.36, the current is at its maximum value with capacitor value  $C = 20 \ \mu\text{F}$  and 0.707 times its maximum value with  $C = 30 \ \mu\text{F}$ . Find the value of Q at  $\omega = 500 \ \text{rad/sec}$ , and circuit constants.



Fig. 8.36

**Solution** The voltage applied to the circuit is V = 20 V. At resonance, the current in the circuit is maximum. The resonant frequency  $\omega_r = 500$  rad/sec.

1

Since

...

$$\omega_r = \frac{1}{\sqrt{LC}}$$
$$L = \frac{1}{\omega_r^2 C} = \frac{1}{(500)^2 \times 20 \times 10^{-6}} = 0.2 \text{ H}$$

Since we have

$$\omega L - \frac{1}{\omega C} = R$$

$$500 \times 0.2 - \frac{1}{500 \times 30 \times 10^{-6}} = R$$
$$R = 100 - 66.6 = 33.4$$

*:*..

The quality factor is  $Q = \frac{\omega L}{R} = \frac{500 \times 0.2}{33.4} = 2.99$ 

**Problem 8.10** In the circuit shown in Fig. 8.37, an inductance of 0.1 H having a Q of 5 is in parallel with a capacitor. Determine the value of capacitance and coil resistance at resonant frequency of 500 rad/sec.

 $\mathsf{R}_\mathsf{L}$ 

Fig. 8.37

<sup>-</sup> C

Solution The quality factor 
$$Q = \frac{\omega_r L}{R}$$
  
Since  $L = 0.1 \text{ H}, Q = 5 \text{ and}$   
 $\omega_r = 500 \text{ rad/sec}$   
 $Q = \frac{500 \times 0.1}{R}$   
 $\therefore \qquad R = \frac{500 \times 0.1}{5} = 10 \Omega$ 

Since

$$\omega_r^2 = \frac{1}{LC}$$
$$(500)^2 = \frac{1}{0.1 \times C}$$

2

1

 $\therefore$  The capacitance value  $C = \frac{1}{0.1 \times (500)^2} = 40 \ \mu\text{F}$ 

**Problem 8.11** A series RLC circuit consists of a 50  $\Omega$  resistance, 0.2 H inductance and 10  $\mu$ F capacitor with an applied voltage of 20 V. Determine the resonant frequency. Find the *Q* factor of the circuit. Compute the lower and upper frequency limits and also find the bandwidth of the circuit.

Solution Resonant frequency

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 10 \times 10^{-6}}} = 112.5 \text{ Hz}$$

Quality factor  $Q = \frac{\omega L}{R} = \frac{2\pi \times 112.5 \times 0.2}{50} = 2.83$ Lower frequency limit

$$f_1 = f_r - \frac{R}{4\pi L} = 112.5 - \frac{50}{4 \times \pi \times 0.2} = 92.6 \text{ Hz}$$

Upper frequency limit

$$f_2 = f_r + \frac{R}{4\pi L} = 112.5 + \frac{50}{4\pi \times 0.2} = 112.5 + 19.89 = 132.39 \text{ Hz}$$

Bandwidth of the circuit

 $BW = f_2 - f_1 = 132.39 - 92.6 = 39.79 \text{ Hz}$ 



8.1 For the circuit shown in Fig. 8.38, determine the frequency at which the circuit resonates. Also find the voltage across the capacitor at resonance, and the Q factor of the circuit.





- 8.2 A series RLC circuit has a quality factor of 10 at 200 rad/sec. The current flowing through the circuit at resonance is 0.5 A and the supply voltage is 10 V. The total impedance of the circuit is 40  $\Omega$ . Find the circuit constants.
- 8.3 The impedance  $Z_1 = (5 + j3) \Omega$  and  $Z_2 = (10 j30) \Omega$  are connected in parallel as shown in Fig. 8.39. Find the value of  $X_3$  which will produce resonance at the terminals *a* and *b*.



Fig. 8.39

- 8.4 A RLC series circuit is to be chosen to produce a magnification of 10 at 100 rad/sec. The source can supply a maximum current of 10 A and the supply voltage is 100 V. The power frequency impedance of the circuit should not be more than 14.14  $\Omega$ . Find the values of *R*, *L* and *C*.
- 8.5 A voltage  $v(t) = 50 \sin \omega t$  is applied to a series RLC circuit. At the resonant frequency of the circuit, the maximum voltage across the capacitor is found to be 400 V. The bandwidth is known to be 500 rad/sec and the impedance at resonance is 100  $\Omega$ . Find the resonant frequency, and compute the upper and lower limits of the bandwidth. Determine the values of *L* and *C* of the circuit.
- 8.6 A current source is applied to the parallel arrangement of *R*, *L* and *C* where  $R = 12 \Omega$ , L = 2 H and  $C = 3 \mu$ F. Compute the resonant frequency in rad/sec. Find the quality factor. Calculate the value of bandwidth. Compute the lower and upper frequency of the bandwidth. Compute the voltage appearing across the parallel elements when the input signal is  $i(t) = 10 \sin 1800 t$ .
- 8.7 For the circuit shown in Fig. 8.40, determine the value of  $R_C$  for which the given circuit resonates.





8.8 For the circuit shown in Fig. 8.41, the applied voltage  $v(t) = 15 \sin 1800t$ . Determine the resonant frequency. Calculate the quality factor and bandwidth. Compute the lower and upper limits of the bandwidth.



Fig. 8.41

8.9 In the circuit shown in Fig. 8.42, the current is at its maximum value with inductor value L = 0.5 H, and 0.707 times of its maximum value with L = 0.2 H. Find the value of Q at  $\omega = 200$  rad/sec and circuit constants.





8.10 The voltage applied to the series RLC circuit is 5 V. The Q of the coil is 25 and the value of the capacitor is 200 PF. The resonant frequency of the circuit is 200 KHz. Find the value of inductance, the circuit current and the voltage across the capacitor.



(c) It is zero (d) It remains the same

- (a) because the circuit is predominantly resistive
- (b) because the circuit is predominantly inductive
- (c) because the circuit is predominantly capacitive
- (d) none of the above
- 9. In order to tune a parallel resonant circuit to a lower frequency, the capacitance must
  - (a) be increased

(b) be decreased

- (c) be zero (d) remain the same
- 10. What is the impedance of an ideal parallel resonant circuit without resistance in either branch?
  - (a) zero(c) capacitive

- (b) inductive
- (d) infinite
- 11. If the lower cut-off frequency is 2400 Hz and the upper cut-off frequency is 2800 Hz, what is the bandwidth?
  - (a) 400 Hz

- (b) 2800 Hz (d) 5200 Hz
- (c) 2400 Hz
- 12. What values of L and C should be used in a tank circuit to obtain a resonant frequency of 8 KHz? The bandwidth must be 800 Hz. The winding resistance of the coil is 10  $\Omega$ .
  - (a) 2 mH, 1  $\mu$ F
  - (c) 1.99 mH, 0.2 μF
- (b) 10 H, 0.2 μF
- (d) 1.99 mH, 10 µF



### 9.1 POLYPHASE SYSTEM

In an ac system it is possible to connect two or more number of individual circuits to a common polyphase source. Though it is possible to have any number of sources in a polyphase system, the increase in the available power is not significant beyond the three-phase system. The power generated by the same machine increases 41.4 per cent from single phase to two-phase, and the increase in the power is 50 per cent from single phase to three-phase. Beyond three-phase, the maximum possible increase is only seven per cent, but the complications are many. So, an increase beyond three-phase does not justify the extra complications. In view of this, it is only in exceptional cases where more than three phases are used. Circuits supplied by six, twelve and more phases are used in high power radio transmitter stations. Two-phase systems are used to supply two-phase servo motors in feedback control systems.

In general, a three-phase system of voltages (currents) is merely a combination of three single phase systems of voltages (currents) of which the



Fig. 9.1

three voltages (currents) differ in phase by 120 electrical degrees from each other in a particular sequence. One such three-phase system of sinusoidal voltages is shown in Fig. 9.1.

## 9.2 ADVANTAGES OF THREE-PHASE SYSTEM

It is observed that the polyphase, especially three-phase, system has many advantages over the single phase system, both from the utility point of view as well as from the consumer point of view. Some of the advantages are as under.

- 1. The power in a single phase circuit is pulsating. When the power factor of the circuit is unity, the power becomes zero 100 times in a second in a 50 Hz supply. Therefore, single phase motors have a pulsating torque. Although the power supplied by each phase is pulsating, the total three-phase power supplied to a balanced three-phase circuit is constant at every instant of time. Because of this, three-phase motors have an absolutely uniform torque.
- 2. To transmit a given amount of power over a given length, a three-phase transmission circuit requires less conductor material than a single-phase circuit.
- 3. In a given frame size, a three-phase motor or a three-phase generator produces more output than its single phase counterpart.
- 4. Three-phase motors are more easily started than single phase motors. Single phase motors are not self starting, whereas three-phase motors are.

In general, we can conclude that the operating characteristics of a threephase apparatus are superior than those of a similar single phase apparatus. All three-phase machines are superior in performance. Their control equipments are smaller, cheaper, lighter in weight and more efficient. Therefore, the study of three phase circuits is of great importance.

### 9.3 GENERATION OF THREE-PHASE VOLTAGES

Three-phase voltages can be generated in a stationary armature with a rotating field structure, or in a rotating armature with a stationary field as shown in Figs. 9.2 (a and b).



Fig. 9.2

Single phase voltages and currents are generated by single phase generators as shown in Fig. 9.2(a). The armature (here a stationary armature) of such a generator has only one winding, or one set of coils. In a two-phase generator the armature has two distinct windings, or two sets of coils that are displaced 90° (electrical degrees) apart, so that the generated voltages in the two phases have 90 degrees phase displacement as shown in Fig. 9.3(b). Similarly, three-phase voltages are generated in three separate but identical sets of windings or coils that are displaced by 120 electrical degrees in the armature, so that the voltages generated in them are  $120^{\circ}$  apart in time phase. This arrangement is shown in Fig. 9.3(c). Here *RR'* constitutes one coil (R-phase); *YY'* another coil (*Y*-phase), and *BB'* constitutes the third phase (B-phase). The field magnets are assumed in clockwise rotation.



9.3

The voltages generated by a three-phase alternator is shown in Fig. 9.3(d). The three voltages are of the same magnitude and frequency, but are displaced from one another by  $120^{\circ}$ . Assuming the voltages to be sinusoidal, we can write the equations for the instantaneous values of the voltages of the three phases. Counting the time from the instant when the voltage in phase *R* is zero. The equations are

$$v_{RR'} = V_m \sin \omega t$$
  

$$v_{YY'} = V_m \sin (\omega t - 120^\circ)$$
  

$$v_{BB'} = V_m \sin (\omega t - 240^\circ)$$

At any given instant, the algebraic sum of the three voltages must be zero.

#### 9.4 PHASE SEQUENCE

Here the sequence of voltages in the three phases are in the order  $v_{RR'} - v_{YY'} - v_{BB''}$  and they undergo changes one after the other in the above mentioned order. This is called the *phase sequence*. It can be observed that this sequence depends on the rotation of the field. If the field system is rotated in the anticlockwise direction, then the sequence of the voltages in the three-phases are in the order  $v_{RR'} - v_{YY'}$ ; briefly we say that the sequence is *RBY*. Now the equations can be written as

$$v_{RR'} = V_m \sin \omega t$$
  

$$v_{BB'} = V_m \sin (\omega t - 120)$$
  

$$v_{YY'} = V_m \sin (\omega t - 240)$$

**Example 9.1** What is the phase sequence of the voltages induced in the three coils of an alternator shown in Fig. 9.4? Write the equations for the three voltages.



Fig. 9.4

**Solution** Here the field system is stationary and the three coils, *RR'*, *YY'* and *BB'*, are rotating in the anticlockwise direction, so the sequence of voltages is *RBY*, and the induced voltages are as shown in Fig. 9.4.



Fig. 9.5

$$\begin{aligned} v_{RR'} &= V_m \sin \omega t \\ v_{BB'} &= V_m \sin (\omega t - 120^\circ) \\ v_{YY'} &= V_m \sin (\omega t - 240^\circ) \text{ or } V_m \sin (\omega t + 120^\circ) \end{aligned}$$

**Example 9.2** What is the possible number of phase sequences in Fig. 9.4. What are they?

**Solution** There are only two possible phase sequences; they are *RBY*, and *RYB*.

9.5 INTER-CONNECTION OF THREE-PHASE SOURCES AND LOADS

#### 9.5.1 Inter-connection of Three-phase Sources

In a three-phase alternator, there are three independent phase windings or coils. Each phase or coil has two terminals, viz. *start* and *finish*. The end connections of the three sets of the coils may be brought out of the machine, to form three separate single phase sources to feed three individual circuits as shown in Figs. 9.6 (a and b).



Fig. 9.6

The coils are inter-connected to form a wye (Y) or delta ( $\Delta$ ) connected threephase system to achieve economy and to reduce the number of conductors, and

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thereby, the complexity in the circuit. The three-phase sources so obtained serve all the functions of the three separate single phase sources.

#### 9.5.2 Wye or Star-Connection

In this connection, similar ends (*start* or *finish*) of the three phases are joined together within the alternator as shown in Fig. 9.7. The common terminal so formed is referred to as the neutral point (N), or neutral terminal. Three lines are run from the other free ends (R, Y, B) to feed power to the three-phase load.

Figure 9.7 represents a three-phase, four-wire, star-connected system. The terminals *R*, *Y* and *B* are called the *line terminals* of the source. The voltage between any line and the neutral point is called the *phase voltage* ( $V_{RN}$ ,  $V_{YN}$  and  $V_{BN}$ ), while the voltage between any two lines is called the *line voltage* ( $V_{RY}$ ,  $V_{YR}$  and  $V_{BN}$ ). The currents flowing through the phases are called the phase currents, while those flowing in the lines are called the line currents. If the neutral wire is not available for external connection, the system is called a three-phase, three-wire, star-connected system. The system so formed will supply equal line voltages displaced 120° from one another and acting simultaneously in the circuit like three independent single phase sources in the same frame of a three-phase alternator.



Fig. 9.7

**Example 9.3** Figure 9.8 represents three phases of an alternator. Arrange the possible number of three-phase star connections and indicate phase voltages and line voltages in each case.  $(V_{RR'} = V_{YY'} = V_{BB'})$ 

**Solution** There are two possible star-connections and they can be arranged as shown in Fig. 9.9(a).

The phase voltages are

 $V_{RN}$ ,  $V_{YN}$ ,  $V_{BN}$  and  $V_{R'N}$ ,  $V_{Y'N}$ ,  $V_{B'N}$ 



Fig. 9.8

The line voltages are

 $V_{RY}$ ,  $V_{YB}$ ,  $V_{BR}$  and  $V_{R'Y'}$ ,  $V_{Y'B'}$ ,  $V_{B'R'}$ 

*Note* The phases can also be arranged as shown in Fig. 9.9(b), in which case they do not look like a star; so the name star or wye-connection is only a convention.



Fig. 9.9

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#### 9.5.3 Delta or Mesh-connection

In this method of connection the dissimilar ends of the windings are joined together, i.e. R' is connected to Y, Y' to B and B' to R as shown in Fig. 9.10.

The three line conductors are taken from the three junctions of the mesh or delta connection to feed the three-phase load. This constitutes a three-phase, three-wire, delta-connected system. Here there is no common terminal; only three line voltages  $V_{RY}$ ,  $V_{YB}$  and  $V_{BR}$  are available. These line voltages are also referred to as *phase voltages* in the delta-

These line voltages are also referred to as *phase voltages* in the deltaconnected system. When the sources are connected in delta, loads can be connected only across the three line terminals, *R*, *Y* and *B*. In general, a threephase source, star or delta, can be either balanced or unbalanced. A balanced



Fig. 9.10

three-phase source is one in which the three individual sources have equal magnitude, with  $120^{\circ}$  phase difference as shown in Fig. 9.3(d).

**Example 9.4** Figure 9.11 represents three phases of an alternator. Arrange the possible number of three-phase, delta connections and indicate phase voltages and line voltages in each case (*Note*  $V_{RR'} = V_{YY'} = V_{BB'}$ ).



Fig. 9.11

**Solution** There are two possible delta connections which are shown as follows.

$$V_{\text{phase}} = V_{\text{line}}$$

The line voltages are

From Fig. 9.12(a)  $V_{RY}$ ,  $V_{YB}$  and  $V_{BR}$  and  $V_{RB}$ ,  $V_{BY}$  and  $V_{YR}$  from Fig. 9.12(b).



Fig. 9.12

#### 9.5.4 Inter-connection of Loads

The primary question in a star or delta-connected three-phase supply is how to apply the load to the three-phase supply. An impedance, or load, connected across any two terminals of an active network (source) will draw power from the source, and is called a single phase load. Like alternator phase windings, load can also be connected across any two terminals, or between line and neutral terminal (if the source is star-connected). Usually the three-phase load impedances are connected in star or delta formation, and then connected to the three-phase source as shown in Fig. 9.13.



Fig. 9.13

Figure 9.13(a) represents the typical inter-connections of loads and sources in a three-phase star system, and is of a three-phase four wire system. A threephase star connected load is connected to a three-phase star-connected source, terminal to terminal, and both the neutrals are joined with a fourth wire. Figure 9.13(b) is a three-phase, three-wire system. A three-phase, delta-connected load is connected to a three-phase star-connected source, terminal to terminal, as shown in Fig. 9.13(b). When either source or load, or both are connected in delta, only three wires will suffice to connect the load to source.

Just as in the case of a three-phase source, a three-phase load can be either balanced or unbalanced. A balanced three-phase load is one in which all the branches have identical impedances, i.e. each impedance has the same magnitude and phase angle. The resistive and reactive components of each phase are equal. Any load which does not satisfy the above requirements is said to be an unbalanced load.

**Example 9.5** Draw the inter-connection between a three-phase, delta-connected source and a star-connected load.

**Solution** When either source or load, or both are connected in delta, only three wires are required to connect, the load to source, and the system is said to be a three-phase, three-wire system. The connection diagram is shown in Fig. 9.14.



Fig. 9.14

The three line voltages are  $V_{RY}$ ,  $V_{YB}$  and  $V_{BR}$ .

**Example 9.6** Draw the inter-connection between a three-phase, delta-connected source and delta-connected load.

**Solution** Since the source and load are connected in delta, it is a three-wire system. The connection diagram is shown in Fig. 9.15.





## 9.6 STAR TO DELTA AND DELTA TO STAR TRANSFORMATION

While dealing with currents and voltages in loads, it is often necessary to convert a star load to delta load, and vice-versa. It has already been shown in Chapter 3 that delta ( $\Delta$ ) connection of resistances can be replaced by an

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equivalent star (Y) connection and vice-versa. Similar methods can be applied in the case of networks containing general impedances in complex form. So also with ac, where the same formulae hold good, except that resistances are replaced by the impedances. These formulae can be applied even if the loads are unbalanced. Thus, considering Fig. 9.16(a), star load can be replaced by an equivalent delta-load with branch impedances as shown.



Fig. 9.16

Delta impedances, in terms of star impedances, are

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B}$$
$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R}$$
$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y}$$

and

The converted network is shown in Fig. 9.16(b). Similarly, we can replace the delta load of Fig. 9.16(b) by an equivalent star load with branch impedances as

$$Z_R = \frac{Z_{RY} Z_{BR}}{Z_{RY} + Z_{YB} + Z_{BR}}$$
$$Z_Y = \frac{Z_{RY} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}$$
$$Z_B = \frac{Z_{BR} Z_{YB}}{Z_{RY} + Z_{YR} + Z_{RR}}$$

and

It should be noted that all impedances are to be expressed in their complex form.

**Example 9.7** A symmetrical three-phase, three-wire 440 V supply is connected to a star-connected load as shown in Fig. 9.17(a). The impedances in each branch are  $Z_R = (2 + J3) \Omega$ ,  $Z_Y = (1 - J2) \Omega$  and  $Z_B = (3 + J4) \Omega$ . Find its equivalent delta-connected load. The phase sequence is *RYB*.



Fig. 9.17

**Solution** The equivalent delta network is shown in Fig. 9.17(b). From Section 9.6, we can write the equations to find  $Z_{RY}$ ,  $Z_{YB}$  and  $Z_{BR}$ . We have

$$\begin{split} Z_{RY} &= \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B} \\ Z_R &= 2 + j3 = 3.61 \angle 56.3^{\circ} \\ Z_Y &= 1 - j2 = 2.23 \angle - 63.4^{\circ} \\ Z_B &= 3 + j4 = 5 \angle 53.13^{\circ} \\ Z_R Z_Y + Z_Y Z_B + Z_B Z_R &= (3.61 \angle 56.3^{\circ}) (2.23 \angle - 63.4^{\circ}) \\ &+ (2.23 \angle - 63.4^{\circ}) (5 \angle 53.13^{\circ}) + (5 \angle 53.13^{\circ}) (3.61 \angle 56.3^{\circ}) \\ &= 8.05 \angle - 7.1^{\circ} + 11.15 \angle - 10.27^{\circ} + 18.05 \angle 109.43^{\circ} \\ &= 12.95 + j14.04 = 19.10 \angle 47.3^{\circ} \\ Z_{RY} &= \frac{19.10 \angle 47.3^{\circ}}{5 \angle 53.13^{\circ}} = 3.82 \angle - 5.83^{\circ} = 3.8 - j0.38 \\ Z_{YB} &= \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R} \\ &= \frac{19.10 \angle 47.3^{\circ}}{3.61 \angle 56.3^{\circ}} = 5.29 \angle - 9^{\circ} = 5.22 - j0.82 \\ Z_{BR} &= \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y} \\ &= \frac{19.10 \angle 47.3^{\circ}}{Z_Y} = 8.56 \angle 110.7^{\circ} = -3.02 + j8 \end{split}$$

The equivalent delta impedances are

$$Z_{RY} = (3.8 - j0.38) \Omega$$
  

$$Z_{YB} = (5.22 - j0.82) \Omega$$
  

$$Z_{BR} = (-3.02 + j8) \Omega$$

**Example 9.8** A symmetrical three-phase, three-wire 400 V, supply is connected to a delta-connected load as shown in Fig. 9.18(a). Impedances in each branch are  $Z_{RY} = 10 \angle 30^{\circ} \Omega$ ;  $Z_{YB} = 10 \angle -45^{\circ} \Omega$  and  $Z_{BR} = 2.5 \angle 60^{\circ} \Omega$ . Find its equivalent star-connected load; the phase sequence is *RYB*.



**Solution** The equivalent star network is shown in Fig. 9.18(b). From Section 9.6, we can write the equations to find  $Z_R$ ,  $Z_Y$  and  $Z_B$  as

$$\begin{split} Z_R &= \frac{Z_{RY} \, Z_{BR}}{Z_{RY} + Z_{YB} + Z_{BR}} \\ Z_{RY} + Z_{YB} + Z_{BR} &= 10 \, \angle 30^\circ + 10 \, \angle -45^\circ + 2.5 \, \angle 60^\circ \\ &= (8.66 + j5) + (7.07 - j7.07) + (1.25 + j2.17) \\ &= 16.98 + j0.1 = 16.98 \, \angle 0.33^\circ \, \Omega \\ Z_R &= \frac{\left(10 \, \angle 30^\circ\right) \left(2.5 \, \angle 60^\circ\right)}{16.98 \, \angle 0.33^\circ} = 1.47 \, \angle 89.67^\circ \\ &= (0.008 + j1.47) \, \Omega \\ Z_Y &= \frac{Z_{RY} \, Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}} \\ &= \frac{\left(10 \, \angle 30^\circ\right) \left(10 \, \angle -45^\circ\right)}{16.98 \, \angle 0.33^\circ} = 5.89 \, \angle -15.33^\circ \, \Omega \\ Z_B &= \frac{Z_{BR} \, Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}} \end{split}$$

$$=\frac{(2.5\angle 60^{\circ})(10\angle -45^{\circ})}{16.98\angle 0.33^{\circ}}=1.47\angle 14.67^{\circ}\ \Omega$$

The equivalent star impedances are

 $Z_R = 1.47 \angle 89.67^\circ \Omega, Z_Y = 5.89 \angle -15.33^\circ \Omega \text{ and } Z_B = 1.47 \angle 14.67^\circ \Omega$ 

#### 9.6.1 Balanced Star-Delta and Delta-Star Conversion

If the three-phase load is balanced, then the conversion formulae in Section 9.6 get simplified. Consider a balanced star-connected load having an impedance  $Z_1$  in each phase as shown in Fig. 9.19(a).



Fig. 9.19

Let the equivalent delta-connected load have an impedance of  $Z_2$  in each phase as shown in Fig. 9.19(b). Applying the conversion formulae from Section 9.6 for delta impedances in terms of star impedances, we have

$$Z_2 = 3Z_1$$

Similarly, we can express star impedances in terms of delta  $Z_1 = Z_2/3$ .

**Example 9.9** Three identical impedances are connected in delta as shown in Fig. 9.20(a). Find an equivalent star network such that the line current is the same when connected to the same supply.

**Solution** The equivalent star network is shown in Fig. 9.20(b). From Section 9.6.1, we can write the equation to find  $Z_1 = Z_2/3$ 

$$Z_2 = 3 + j4 = 5 \angle 53.13^{\circ} \Omega$$
$$Z_1 = \frac{5}{3} \angle 53.13^{\circ} = 1.66 \angle 53.15^{\circ} = (1.0 + j1.33) \Omega$$

:.



#### Fig. 9.20

#### 9.7 **VOLTAGE, CURRENT AND POWER IN A STAR CONNECTED SYSTEM**

#### 9.7.1 **Star-Connected System**

Figure 9.21 shows a balanced three-phase, Y-connected system. The voltage induced in each winding is called the phase voltage  $(V_{Ph})$ . Likewise  $V_{RN}$ ,  $V_{YN}$  and  $V_{BN}$  represent the rms values of the induced voltages in each phase. The voltage available between any pair of terminals is called the *line voltage*  $(V_i)$ . Likewise  $V_{RY}$ ,  $V_{YB}$  and  $V_{BR}$  are known as *line voltages*. The double subscript notation is purposefully used to represent voltages and currents in polyphase circuits. Thus,  $V_{RY}$  indicates a voltage V between points R and Y, with R being positive with respect to point Y during its positive half cycle.



Fig. 9.21

Similarly,  $V_{YB}$  means that Y is positive with respect to point B during its positive half cycle; it also means that  $V_{_{RY}} = -V_{_{YR}}$ .

9.15

#### **Voltage Relation** 9.7.2

The phasors corresponding to the phase voltages constituting a three-phase system can be represented by a phasor diagram as shown in Fig. 9.22.



Fig. 9.22

From Fig. 9.22, considering the lines R, Y and B, the line voltage  $V_{RY}$  is equal to the phasor sum of  $V_{RN}$  and  $V_{NY}$  which is also equal to the phasor difference of  $V_{RN}$  and  $V_{YN}$  ( $V_{NY} = -V_{YN}^{NN}$ ). Hence,  $V_{RY}$  is found by compounding  $V_{RN}$  and  $V_{YN}$  reversed. To subtract  $V_{YN}$  from  $V_{RN}$ , we reverse the phasor  $V_{YN}$  and find its phasor sum with  $V_{RN}$  as shown in Fig. 9.22. The two phasors,  $V_{RN}$  and  $-V_{YN}$ , are equal in length and are 60° apart.

 $V_{RY} = 2V_{Ph} \cos 60/2 = \sqrt{3} V_{Ph}$ Similarly, the line voltage  $V_{y_B}$  is equal to the phasor difference of  $V_{y_N}$  and  $V_{RN}$ , and is equal to  $\sqrt{3} V_{Ph}$ . The line voltage  $V_{BR}$  is equal to the phasor difference of  $V_{BN}$  and  $V_{RN}$ , and is equal to  $\sqrt{3} V_{Ph}$ . Hence, in a balanced starconnected system

 $|V_{RN}| = -|V_{VN}| = V_{Ph}$ 

- (i) Line voltage =  $\sqrt{3} V_{Ph}$
- (ii) All line voltages are equal in magnitude and are displaced by 120°, and
- (iii) All line voltages are 30° ahead of their respective phase voltages (from Fig. 9.22).

**Example 9.10** A symmetrical star-connected system is shown in Fig. 9.23(a). Calculate the three line voltages, given  $V_{RN} = 230 \angle 0^{\circ}$ . The phase sequence is *RYB*.



Fig. 9.23

**Solution** Since the system is a balanced system, all the phase voltages are equal in magnitude, but displaced by 120° as shown in Fig. 9.23(b).

*:*..

 $V_{RN} = 230 \angle 0^{\circ} V$   $V_{YN} = 230 \angle -120^{\circ} V$  $V_{RN} = 230 \angle -240^{\circ} V$ 

Corresponding line voltages are equal to  $\sqrt{3}$  times the phase voltages, and are 30° ahead of the respective phase voltages.

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$$V_{RY} = \sqrt{3} \times 230 \ \angle 0 + 30^{\circ} \ V = 398.37 \ \angle 30^{\circ} \ V$$
$$V_{YB} = \sqrt{3} \times 230 \ \angle -120^{\circ} + 30^{\circ} \ V = 398.37 \ \angle -90^{\circ} \ V$$
$$V_{BR} = \sqrt{3} \times 230 \ \angle -240^{\circ} + 30^{\circ} \ V = 398.37 \ \angle -210^{\circ} \ V$$

#### 9.7.3 Current Relations

Figure 9.24(a) shows a balanced three-phase, wye-connected system indicating phase currents and line currents. The arrows placed alongside the currents  $I_R$ ,  $I_Y$  and  $I_B$  flowing in the three phases indicate the directions of currents when they are assumed to be positive and *not* the directions at that particular instant. The phasor diagram for phase currents with respect to their phase voltages is shown in Fig. 9.24(b). All the phase currents are displaced by 120° with respect to each other, ' $\phi$ ' is the phase angle between phase voltage and phase current (lagging load is assumed). For a balanced load, all the phase currents are equal in magnitude. It can be observed from Fig. 9.24(a) that each line conductor is connected in series with its individual phase winding. Therefore, the current in a line conductor is the same as that in the phase to which the line conductor is connected.

$$\therefore \qquad I_L = I_{Ph} = I_R = I_Y = I_B$$

It can be observed from Fig. 9.24(b) that the angle between the line (phase) current and the corresponding line voltage is  $(30 + \phi)^{\circ}$  for a lagging load. Consequently, if the load is leading, then the angle between the line (phase) current and corresponding line voltage will be  $(30 - \phi)^{\circ}$ .



Fig. 9.24

**Example 9.11** In Fig. 9.24(a), the value of the current in phase *R* is  $I_R = 10 \angle 20^\circ$  A. Calculate the values of the three line currents. Assume an *RYB* phase sequence.

**Solution** In a balanced star-connected system  $I_L = I_{Ph}$ , and is displaced by 120°. Therefore the three line currents are

$$I_R = 10 \ \angle 20^\circ \text{ A}$$
  

$$I_Y = 10 \ \angle 20^\circ - 120^\circ \text{ A} = 10 \ \angle -100^\circ \text{ A}$$
  

$$I_B = 10 \ \angle 20^\circ - 240^\circ \text{ A} = 10 \ \angle -220^\circ \text{ A}$$

#### 9.7.4 Power in the Star-Connected Network

The total active power or true power in the three-phase load is the sum of the powers in the three phases. For a balanced load, the power in each load is the same; hence total power =  $3 \times$ power in each phase

or  $P = 3 \times V_{Ph} \times I_{Ph} \cos \phi$ 

It is the usual practice to express the three-phase power in terms of line quantities as follows.

$$V_L = \sqrt{3} \quad V_{\text{Ph}}, I_L = I_{Ph}$$
$$P = \sqrt{3} \quad V_L I_L \cos \phi W$$

or  $\sqrt{3} V_{I_{L}} \cos \phi$  is the active power in the circuit.

Total reactive power is given by

$$Q = \sqrt{3} V_L I_L \sin \phi \text{ VAR}$$

Total apparent power or volt-amperes

$$=\sqrt{3} V_L I_L VA$$

#### 9.7.5 N-Phase Star System

It is to be noted that star and mesh are general terms applicable to any number of phases; but wye and delta are special cases of star and mesh when the system is a three-phase system. Consider an *n*-phase balanced star system with two adjacent phases as shown in Fig. 9.25(a). Its vector diagram is shown in Fig. 9.25(b).





The angle of phase difference between adjacent phase voltages is  $360^{\circ}/n$ . Let  $E_{ph}$  be the voltage of each phase. The line voltage, i.e. the voltage between A and B is equal to  $E_{AB} = E_L = E_{AO} + E_{OB}$ . The vector addition is shown in Fig. 9.25 (c). It is evident that the line current and phase current are same.



Fig. 9.25

 $E_{AB} = E_{AO} + E_{OB}$ Consider the parallelogram *OABC*.

$$OB = \sqrt{OC^2 + OA^2 + 2 \times OA \times OC \times \cos \theta}$$
$$= \sqrt{E_{ph}^2 + E_{ph}^2 + 2E_{ph}^2 \cos\left(180^\circ - \frac{360^\circ}{n}\right)}$$
$$= \sqrt{2E_{ph}^2 - 2E_{ph}^2 \cos\frac{360^\circ}{n}}$$
$$= \sqrt{2} E_{ph} \sqrt{\left[1 - \cos 2\left(\frac{180^\circ}{n}\right)\right]}$$
$$= \sqrt{2} E_{ph} \sqrt{2\sin^2\left(\frac{180^\circ}{n}\right)}$$
$$E_L = 2E_{ph} \sin\frac{180^\circ}{n}$$

The above equation is a general equation for line voltage, for example, for a three-phase system,  $n = 3 E_L = 2 E_{ph} \sin 60^\circ = \sqrt{3} E_{ph}$ .

**Example 9.12** A balanced star-connected load of  $(4 + J3) \Omega$  per phase is connected to a balanced 3-phase 400 V supply. The phase current is 12 A. Find (i) the total active power (ii) reactive power and (iii) total apparent power.

**Solution** The voltage given in the data is always the rms value of the line voltage unless otherwise specified.

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$$Z_{Ph} = \sqrt{4^2 + 3^2} = 5 \Omega$$

$$PF = \cos \phi = \frac{R_{Ph}}{Z_{Ph}} = \frac{4}{5} = 0.8$$

$$\sin \phi = 0.6$$
(i) Active power  $= \sqrt{3} V_L I_L \cos \phi W$   
 $= \sqrt{3} \times 400 \times 12 \times 0.8 = 6651 W$ 
(ii) Reactive power  $= \sqrt{3} V_L I_L \sin \phi VAR$   
 $= \sqrt{3} \times 400 \times 12 \times 0.6 = 4988.36 VAR$ 
(iii) Apparent power  $= \sqrt{3} V_L I_L$   
 $= \sqrt{3} \times 400 \times 12 = 8313.84 VA$ 

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# 9.8 VOLTAGE, CURRENT AND POWER IN A DELTA CONNECTED SYSTEM

#### 9.8.1 Delta-Connected System

Figure 9.26 shows a balanced three-phase, three-wire, delta-connected system. This arrangement is referred to as mesh connection because it forms a closed circuit. It is also known as delta connection because the three branches in the circuit can also be arranged in the shape of delta ( $\Delta$ ).



Fig. 9.26

From the manner of interconnection of the three phases in the circuit, it may appear that the three phases are short-circuited among themselves. However, this is not the case. Since the system is balanced, the sum of the three voltages round the closed mesh is zero; consequently, no current can flow around the mesh when the terminals are open.

The arrows placed alongside the voltages,  $V_{RY}$ ,  $V_{YB}$  and  $V_{BR}$ , of the three phases indicate that the terminals *R*, *Y* and *B* are positive with respect to *Y*, *B* and *R*, respectively, during their respective positive half cycles.

#### 9.8.2 Voltage Relation

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From Fig. 9.27, we notice that only one phase is connected between any two lines. Hence, the voltage between any two lines  $(V_L)$  is equal to the phase voltage  $(V_{p_k})$ .



Fig. 9.27

Since the system is balanced, all the phase voltages are equal, but displaced by  $120^{\circ}$  from one another as shown in the phasor diagram in Fig. 9.27. The phase sequence *RYB* is assumed.

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$$|V_{RY}| = |V_{YB}| = |V_{BR}| = V_L = V_{Ph}$$

**Example 9.13** In Fig. 9.27, the voltage across the terminals *R* and *Y* is 400  $\angle 0^\circ$ . Calculate the values of the three line voltages. Assume *RYB* phase sequence.

**Solution** In a balanced delta-connected system,  $|V_L| = |V_{Ph}|$ , and is displaced by 120°; therefore the three line voltages are

$$V_{RY} = 400 \angle 0^{\circ} V$$
  
 $V_{YB} = 400 \angle -120^{\circ} V$   
 $V_{BR} = 400 \angle -240^{\circ} V$ 

#### 9.8.3 Current Relation

In Fig. 9.28 we notice that, since the system is balanced, the three phase currents  $(I_{Ph})$ , i.e.  $I_R$ ,  $I_Y$ ,  $I_B$  are equal in magnitude but displaced by 120° from one another as shown in Fig. 9.28(b).  $I_1$ ,  $I_2$  and  $I_3$  are the line currents  $(I_L)$ , i.e.  $I_1$  is the line current in line 1 connected to the common point of R. Similarly,  $I_2$  and  $I_3$  are the line currents in lines 2 and 3, connected to common points Y and B, respectively. Though here all the line currents are directed outwards, at no instant will all the three line currents flow in the same direction, either outwards or inwards. Because the three line currents are displaced 120° from one another, when one is positive, the other two might both be negative, or one positive and one negative. Also it is to be noted that arrows placed alongside phase currents in Fig. 9.28(a), indicate the direction of currents when they are assumed to be positive and not their actual direction at a particular instant. We can easily determine the line currents in Fig. 9.28(a),  $I_1$ ,  $I_2$  and  $I_3$  by applying KCL at the three terminals R, Y and B, respectively. Thus, the current in line 1,  $I_1 = I_R - I_B$ ; i.e. the current in any line is equal to the phasor difference of the currents in the two phases attached to that line. Similarly, the current in line 2,  $I_2 = I_y - I_g$ , and the current in line 3,  $I_3 = I_B - I_{Y}$ . The phasor addition of these currents is shown in Fig. 9.28(b). From the

The phasor addition of these currents is shown in Fig. 9.28(b). From the figure,

$$I_{1} = I_{R} - I_{B}$$

$$I_{1} = \sqrt{I_{R}^{2} + I_{B}^{2} + 2I_{R}I_{B} \cos 60^{\circ}}$$

$$I_{1} = \sqrt{3} \quad I_{\text{ph}}, \text{ since } I_{R} = I_{B} = I_{Ph}$$

Similarly, the remaining two line currents,  $I_2$  and  $I_3$ , are also equal to  $\sqrt{3}$  times the phase currents; i.e.  $I_L = \sqrt{3} I_{Ph}$ .

As can be seen from Fig. 9.28(b), all the line currents are equal in magnitude but displaced by  $120^{\circ}$  from one another; and the line currents are  $30^{\circ}$  behind the respective phase currents.



Fig. 9.28

**Example 9.14** Three identical loads are connected in delta to a three-phase supply of 440  $\angle 0^\circ$  V as shown in Fig. 9.29(a). If the phase current  $I_R$  is 15  $\angle 0^\circ$  A, calculate the three line currents.



(a)



Fig. 9.29

**Solution** All the line currents are equal and 30° behind their respective phase currents, and  $\sqrt{3}$  times their phase values, displaced by 120° from one another, assuming *RYB* phase sequence.

Let the line currents in line 1, 2 and 3 be  $I_1$ ,  $I_2$  and  $I_3$ , respectively.

$$I_{1} = \sqrt{3} \times I_{R} \angle (\phi - 30^{\circ})$$
  
=  $\sqrt{3} \times 15 \angle -30^{\circ} = 25.98 \angle -30^{\circ} \text{ A}$   
$$I_{2} = \sqrt{3} \times 15 \angle (-30 - 120)^{\circ} = 25.98 \angle -150^{\circ} \text{ A}$$
  
$$I_{3} = \sqrt{3} \times 15 \angle (-30 - 240)^{\circ} = 25.98 \angle -270^{\circ} \text{ A}$$

The phasor diagram is shown in Fig. 9.29(b).

#### 9.8.4 Power in the Delta-Connected System

Obviously the total power in the delta circuit is the sum of the powers in the three phases. Since the load is balanced, the power consumed in each phase is the same. Total power is equal to three times the power in each phase.

Power per phase =  $V_{Ph} I_{Ph} \cos \phi$ 

where  $\phi$  is the phase angle between phase voltage and phase current.

Total power  $P = 3 \times V_{Ph} I_{Ph} \cos \phi$ 

In terms of line quantities

$$P = \sqrt{3} \quad V_L I_L \cos \phi W$$
Since

$$V_{\rm Ph} = V_L$$
 and  $I_{\rm Ph} = \frac{I_L}{\sqrt{3}}$ 

for a balanced system, whether star or delta, the expression for the total power is the same.

**Example 9.15** A balanced delta-connected load of  $(2 + j3) \Omega$  per phase is connected to a balanced three-phase 440 V supply. The phase current is 10 A. Find the (i) total active power (ii) reactive power and (iii) apparent power in the circuit.

Solution  

$$Z_{Ph} = \sqrt{(2)^{2} + (3)^{2}} = 3.6 \angle 56.3^{\circ} \Omega$$

$$\cos \phi = \frac{R_{Ph}}{Z_{Ph}} = \frac{2}{3.6} = 0.55$$
So,  

$$\sin \phi = 0.83$$

$$I_{L} = \sqrt{3} \times I_{Ph} = \sqrt{3} \times 10 = 17.32 \text{ A}$$
(i) Active power  

$$= \sqrt{3} \quad V_{L}I_{L} \cos \phi$$

$$= \sqrt{3} \times 440 \times 17.32 \times 0.55 = 7259.78 \text{ W}$$
(ii) Reactive power  

$$= \sqrt{3} \quad V_{L}I_{L} \sin \phi$$

$$= \sqrt{3} \times 440 \times 17.32 \times 0.83 = 10955.67 \text{ VAR}$$
(iii) Apparent power  

$$= \sqrt{3} \quad V_{L}I_{L}$$

$$= \sqrt{3} \times 440 \times 17.32 = 13199.61 \text{ VA}$$

### 9.8.5 N-Phase Mesh System

Figure 9.30(a) shows part of an *n*-phase balanced mesh system. Its vector diagram is shown in Fig. 9.30(b).



Let the current in line *BB'* be  $I_L$ . This is same in all the remaining lines of the *n*-phase system.  $I_{AB}$ ,  $I_{BC}$  are the phase currents in *AB* and *BC* phases respectively. The vector addition of the line current is shown in Fig. 9.30(c). It is evident from the Fig. 9.30(b) that the line and phase voltages are equal.



Fig. 9.30(c)

$$\begin{split} I_{BB} &= I_L = I_{AB} + I_{CB} \\ &= I_{AB} - I_{BC} \end{split}$$

Consider the parallelogram OABC.

$$OB = \sqrt{OA^{2} + OC^{2} + 2 \times OA \times OC \times \cos\left(180 - \frac{360}{n}\right)}$$
  
=  $\sqrt{I_{Ph}^{2} + I_{Ph}^{2} - 2I_{Ph}^{2} \cos\frac{360}{n}}$   
=  $\sqrt{2} I_{Ph} \sqrt{1 - \cos 2\left(\frac{180}{n}\right)}$   
=  $\sqrt{2} I_{Ph} \sqrt{2\sin^{2}\frac{180}{n}}$   
 $I_{L} = 2I_{ph} \sin\frac{180}{n}$ 

The above equation is a general equation for the line current in a balanced *n*-phase mesh system.

### 9.9 THREE-PHASE BALANCED CIRCUITS

The analysis of three-phase balanced systems is presented in this section. It is no way different from the analysis of AC systems in general. The relation between voltages, currents and power in delta-connected and star-connected systems has already been discussed in the previous sections. We can make use of those relations and expressions while solving the circuits.

### 9.9.1 Balanced Three-Phase System-Delta Load

Figure 9.31(a) shows a three-phase, three-wire, balanced system supplying power to a balanced three-phase delta load. The phase sequence is *RYB*. We are required to find out the currents in all branches and lines.

Let us assume the line voltage  $V_{RY} = V \angle 0^\circ$  as the reference phasor. Then the three source voltages are given by

$$V_{RY} = V \angle 0^{\circ} V$$
$$V_{YB} = V \angle -120^{\circ} V$$
$$V_{BR} = V \angle -240^{\circ} V$$

These voltages are represented by phasors in Fig. 9.31(b). Since the load is delta-connected, the line voltage of the source is equal to the phase voltage of the load. The current in phase RY,  $I_R$  will lag (lead) behind (ahead of) the phase voltage  $V_{RY}$  by an angle  $\phi$  as dictated by the nature of the load impedance. The angle of lag of  $I_Y$  with respect to  $V_{YB}$ , as well as the angle of lag of  $I_B$  with respect to  $V_{BR}$  will be  $\phi$  as the load is balanced. All these quantities are represented in Fig. 9.31(b).



Fig. 9.31(b)

If the load impedance is  $Z \angle \phi$ , the current flowing in the three load impedances are then

$$I_R = \frac{V_{RY} \angle 0^\circ}{Z \angle \phi} = \frac{V}{Z} \angle -\phi$$

$$I_Y = \frac{V_{YB} \angle -120^{\circ}}{Z \angle \phi} = \frac{V}{Z} \angle -120^{\circ} - \phi$$
$$I_B = \frac{V_{BR} \angle -240^{\circ}}{Z \angle \phi} = \frac{V}{Z} \angle -240^{\circ} - \phi$$

The line currents are  $\sqrt{3}$  times the phase currents, and are 30° behind their respective phase currents.

: Current in line 1 is given by

$$I_1 = \sqrt{3} \left| \frac{V}{Z} \right| \angle (-\phi - 30^\circ)$$
, or  $I_R - I_B$  (phasor difference)

Similarly, the current in line 2

$$I_2 = \sqrt{3} \left| \frac{V}{Z} \right| \angle (-120 - \phi - 30^\circ),$$

or  $I_{Y} - I_{R}$  (phasor difference) =  $\sqrt{3} \left| \frac{V}{Z} \right| \angle (-\phi - 150)^{\circ}$ , and

$$I_{3} = \sqrt{3} \left| \frac{V}{Z} \right| \angle (-240 - \phi - 30)^{\circ}, \text{ or } I_{B} - I_{Y} \text{ (phasor difference)}$$
$$= \sqrt{3} \left| \frac{V}{Z} \right| \angle (-270 - \phi)^{\circ}$$

To draw all these quantities vectorially,  $V_{RY} = V \angle 0^\circ$  is taken as the reference vector.

**Example 9.16** A three-phase, balanced delta-connected load of  $(4 + j8) \Omega$  is connected across a 400 V,  $3 - \phi$  balanced supply. Determine the phase currents and line currents. Assume the phase sequence to be *RYB*. Also calculate the power drawn by the load.

**Solution** Referring to Fig. 9.31(a), taking the line voltage  $V_{RY} = V \angle 0^\circ$  as reference  $V_{RY} = 400 \angle 0^\circ V$ ;  $V_{YB} = 400 \angle -120^\circ V$ ,  $V_{BR} = 400 \angle -240^\circ V$ Impedance per phase =  $(4 + j8) \Omega = 8.94 \angle 63.4^\circ \Omega$ 

Phase currents are:  $I_R = \frac{400 \angle 0^{\circ}}{8.94 \angle 63.4^{\circ}} = 44.74 \angle -63.4^{\circ} \text{ A}$   $I_Y = \frac{400 \angle -120^{\circ}}{8.94 \angle 63.4^{\circ}} = 44.74 \angle -183.4^{\circ} \text{ A}$  $I_B = \frac{400 \angle -240^{\circ}}{8.94 \angle 63.4^{\circ}} = 44.74 \angle -303.4^{\circ} \text{ A}$ 

The three line currents are

$$I_1 = I_R - I_B = (44.74 \angle -63.4^\circ - 44.74 \angle -303.4^\circ)$$
  
= (20.03 - j40) - (24.62 + j37.35) = (-4.59 - j77.35) A  
= 77.49 \angle 266.6^\circ A

Or the line current  $I_1$  is equal to the  $\sqrt{3}$  times the phase current and 30° behind its respective phase current

 $I_{1} = \sqrt{3} \times 44.74 \angle -63.4^{\circ} - 30^{\circ} = 77.49 \angle -93.4^{\circ}$ or = 77.49 \angle 266.6° A Similarly,  $I_{2} = I_{Y} - I_{R}$  $= \sqrt{3} \times 44.74 \angle -183.4^{\circ} - 30^{\circ} = 77.49 \angle -213.4^{\circ} \text{ A} = 77.49 \angle 146.6^{\circ} \text{ A}$  $I_{3} = I_{B} - I_{Y}$ 

 $= \sqrt{3} \times 44.74 \angle -303.4^{\circ} - 30^{\circ} = 77.49 \angle -333.4^{\circ} \text{ A} = 77.49 \angle 26.6^{\circ} \text{ A}$ Power drawn by the load is  $P = 3V_{Ph}I_{Ph}\cos\phi$ 

or

### $\sqrt{3} \times V_L \times I_L \cos 63.4^\circ = 24.039 \text{ kW}$

### 9.9.2 Balanced Three Phase System-Star Connected Load

Figure 9.32(a) shows a three-phase, three wire system supplying power to a balanced three phase star connected load. The phase sequence RYB is assumed.



Fig. 9.32

In star connection, whatever current is flowing in the phase is also flowing in the line. The three line (phase) currents are  $I_p$ ,  $I_y$  and  $I_p$ .

 $V_{RN}$ ,  $V_{YN}$  and  $V_{BN}$  represent three phase voltages of the network, i.e. the voltage between any line and neutral. Let us assume the voltage  $V_{RN} = V \angle 0^{\circ}$  as the reference phasor. Consequently, the phase voltage

$$V_{RN} = V \angle 0^{\circ}$$
$$V_{YN} = V \angle -120^{\circ}$$
$$V_{BN} = V \angle -240^{\circ}$$

Hence

$$I_{R} = \frac{V_{RN}}{Z \angle \phi} = \frac{V \angle 0^{\circ}}{Z \angle \phi} = \left| \frac{V}{Z} \right| \angle -\phi$$

$$I_{Y} = \frac{V_{YN}}{Z \angle \phi} = \frac{V \angle -120^{\circ}}{Z \angle \phi} = \left| \frac{V}{Z} \right| \angle -120^{\circ} -\phi$$

$$I_{B} = \frac{V_{BN}}{Z \angle \phi} = \frac{V \angle -240^{\circ}}{Z \angle \phi} = \left| \frac{V}{Z} \right| \angle -240^{\circ} -\phi$$

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As seen from the above expressions, the currents,  $I_R$ ,  $I_Y$  and  $I_B$ , are equal in magnitude and have a 120° phase difference. The disposition of these vectors is shown in Fig. 9.32(b). Sometimes, a 4th wire, called neutral wire is run from the neutral point, if the source is also star-connected. This gives three-phase, four-wire star-connected system. However, if the three line currents are balanced, the current in the fourth wire is zero; removing this connecting wire between the source neutral and load neutral is, therefore, not going to make any change in the condition of the system. The availability of the neutral wire makes it possible to use all the three phase voltages, as well as the three line voltages. Usually, the neutral is grounded for safety and for the design of insulation.



Fig. 9.32

It makes no difference to the current flowing in the load phases, as well as to the line currents, whether the sources have been connected in star or in delta, provided the voltage across each phase of the delta connected source is  $\sqrt{3}$  times the voltage across each phase of the star-connected source.

**Example 9.17** A balanced star-connected load having an impedance  $(15 + j20) \Omega$  per phase is connected to a three-phase, 440 V; 50 Hz supply. Find the line currents and the power absorbed by the load. Assume *RYB* phase sequence.

**Solution** Referring to Fig. 9.32 (a), taking  $V_{RN}$  as the reference voltage we have

$$V_{RN} = \frac{440 \angle 0^{\circ}}{\sqrt{3}} = 254 \angle 0^{\circ}$$
$$V_{YN} = 254 \angle -120^{\circ}$$
$$V_{RN} = 254 \angle -240^{\circ}$$

Impedance per phase,  $Z_{Ph} = 15 + j20 = 25 \angle 53.13^{\circ} \Omega$ 

The phase currents are  $I_R = \frac{V_{RN}}{Z_{Ph}} = \frac{254 \angle 0^\circ}{25 \angle 53.13^\circ} = 10.16 \angle -53.13^\circ \text{ A}$ 

$$I_Y = \frac{V_{YN}}{Z_{Ph}} = \frac{254 \angle -120^{\circ}}{25 \angle 53.13^{\circ}} = 10.16 \angle -173.13^{\circ} \text{ A}$$
$$I_B = \frac{V_{BN}}{Z_{Ph}} = \frac{254 \angle -240^{\circ}}{25 \angle 53.13^{\circ}} = 10.16 \angle -293.13^{\circ} \text{ A}$$

The three phase currents are equal in magnitude and displaced by 120° from one another. Since the load is star-connected, these currents also represents line currents.

The power absorbed by the load (P)

or  

$$= 3 \times V_{Ph} \times I_{Ph} \cos \phi$$

$$= \sqrt{3} \times V_L \times I_L \cos \phi$$

$$= \sqrt{3} \times 440 \times 10.16 \times \cos 53.13^\circ = 4645.78 \text{ W}$$

### 9.10 THREE-PHASE UNBALANCED CIRCUITS

### 9.10.1 Types of Unbalanced Loads

An unbalance exists in a circuit when the impedances in one or more phases differ from the impedances of the other phases. In such a case, line or phase currents are different and are displaced from one another by unequal angles. So far, we have considered balanced loads connected to balanced systems. It is enough to solve problems, considering one phase only on balanced loads; the conditions on other two phases being similar. Problems on unbalanced threephase loads are difficult to handle because conditions in the three phases are different. However, the source voltages are assumed to be balanced. If the system is a three-wire system, the currents flowing towards the load in the three lines must add to zero at any given instant. If the system is a four-wire system, the sum of the three outgoing line currents is equal to the return current in the neutral wire. We will now consider different methods to handle unbalanced star-connected and delta-connected loads. In practice, we may come across the following unbalanced loads:

- (i) Unbalanced delta-connected load
- (ii) Unbalanced three-wire star-connected load, and
- (iii) Unbalanced four-wire star-connected load.

### 9.10.2 Unbalanced Delta-connected Load

Figure 9.33 shows an unbalanced delta-load connected to a balanced threephase supply.



Fig. 9.33

The unbalanced delta-connected load supplied from a balanced three-phase supply does not present any new problems because the voltage across the load phase is fixed. It is independent of the nature of the load and is equal to the line voltage of the supply. The current in each load phase is equal to the line voltage divided by the impedance of that phase. The line current will be the phasor difference of the corresponding phase currents, taking  $V_{RY}$  as the reference phasor.

Assuming RYB phase sequence, we have

$$V_{RY} = V \angle 0^{\circ} \text{ V}, V_{YB} = V \angle -120^{\circ} \text{ V}, V_{BR} = V \angle -240^{\circ} \text{ V}$$

Phase currents are

$$I_R = \frac{V_{RY}}{Z_1 \angle \phi} = \frac{V \angle 0^\circ}{Z_1 \angle \phi_1} = \left| \frac{V}{Z_1} \right| \angle -\phi_1 A$$

$$I_Y = \frac{V_{YB}}{Z_2 \angle \phi_2} = \frac{V \angle -120^\circ}{Z_2 \angle \phi_2} = \left| \frac{V}{Z_2} \right| \angle -120^\circ -\phi_2 A$$

$$I_B = \frac{V_{BR}}{Z_3 \angle \phi_3} = \frac{V \angle -240^\circ}{Z_3 \angle \phi_3} = \frac{V}{Z_3} \angle -240^\circ -\phi_3 A$$

The three line currents are

 $I_1 = I_R - I_B$  phasor difference  $I_2 = I_Y - I_R$  phasor difference  $I_3 = I_B - I_Y$  phasor difference

**Example 9.18** Three impedances  $Z_1 = 20 \angle 30^\circ \Omega$ ,  $Z_2 = 40 \angle 60^\circ \Omega$  and  $Z_3 = 10 \angle -90^\circ \Omega$  are delta-connected to a 400 V,  $3 - \phi$  system as shown in Fig. 9.34. Determine the (i) phase currents (ii) line currents, and (iii) total power consumed by the load.



Fig. 9.34

**Solution** The three phase currents are  $I_R$ ,  $I_Y$  and  $I_B$ , and the three line currents are  $I_1$ ,  $I_2$  and  $I_3$ . Taking  $V_{RY} = V \angle 0^\circ$  V as reference phasor, and assuming *RYB* phase sequence, we have

$$V_{RY} = 400 \angle 0^{\circ} \text{ V}, V_{YB} = 400 \angle -120^{\circ} V,$$
  

$$V_{BR} = 400 \angle -240^{\circ} \text{ V}$$
  

$$Z_{1} = 20 \angle 30^{\circ} \Omega = (17.32 + j10) \Omega;$$
  

$$Z_{2} = 40 \angle 60^{\circ} \Omega = (20 + j34.64) \Omega;$$
  

$$Z_{3} = 10 \angle -90^{\circ} \Omega = (0 - j10) \Omega$$
  

$$I_{R} = \frac{V_{RY}}{Z_{1} \angle \phi_{1}} = \frac{400 \angle 0^{\circ}}{20 \angle 30^{\circ}} \text{ A} = 20 \angle -30^{\circ} \text{ A}$$
  

$$= (17.32 - j10) \text{ A}$$
  

$$I_{Y} = \frac{V_{YB}}{Z_{2} \angle \phi_{2}} = \frac{400 \angle -120^{\circ}}{40 \angle 60^{\circ}} \text{ A} = 10 \angle -180^{\circ} \text{ A}$$
  

$$= (-10 + j0) \text{ A}$$
  

$$I_{B} = \frac{V_{BR}}{Z_{3} \angle \phi_{3}} = \frac{400 \angle -240^{\circ}}{10 \angle -90^{\circ}} \text{ A} = 40 \angle -150^{\circ} \text{ A}$$
  

$$= (-34.64 - j20) \text{ A}$$

Now the three line currents are

$$I_1 = I_R - I_B = [(17.32 - j 10) - (-34.64 - j 20)]$$
  
= (51.96 + j 10) A = 52.91 \angle 10.89° A  
$$I_2 = I_Y - I_R = [(-10 + j0) - (17.32 - j10)]$$
  
= (-27.32 + j10) A = 29.09 \angle 159.89° A  
$$I_3 = I_B - I_Y = [(-34.64 - j20) - (-10 + j 0)]$$
  
= (-24.64 - j 20) A = 31.73 \angle - 140.94° A

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(iii) To calculate the total power, first the powers in the individual phases are to be calculated, then they are added up to get the total power in the unbalanced load.

Power in *R* phase 
$$= I_R^2 \times R_R = (20)^2 \times 17.32 = 6928$$
 W  
Power in *Y* phase  $= I_Y^2 \times R_Y = (10)^2 \times 20 = 2000$  W  
Power in *B* phase  $= I_B^2 \times R_B = (40)^2 \times 0 = 0$ 

 $\therefore$  Total power in the load = 6928 + 2000 = 8928 W

### 9.10.3 Unbalanced Four Wire Star-Connected Load

Figure 9.35 shows an unbalanced star load connected to a balanced 3-phase, 4-wire supply.





The star point,  $N_L$ , of the load is connected to the star point,  $N_S$  of the supply. It is the simplest case of an unbalanced load because of the presence of the neutral wire; the star points of the supply  $N_S$  (generator) and the load  $N_L$  are at the same potential. It means that the voltage across each load impedance is equal to the phase voltage of the supply (generator), i.e. the voltages across the three load impedances are equalised even though load impedances are unequal. However, the current in each phase (or line) will be different. Obviously, the vector sum of the currents in the three lines is not zero, but is equal to neutral current. Phase currents can be calculated in similar way as that followed in an unbalanced delta-connected load.

Taking the phase voltage  $V_{RN} = V \angle 0^\circ$  V as reference, and assuming *RYB* phase sequences, we have the three phase voltages as follows

$$V_{RN} = V \angle 0^{\circ} \text{ V}, V_{YN} = V \angle -120^{\circ} \text{ V}, V_{BN} = V \angle -240^{\circ} \text{ V}$$

The phase currents are

$$I_R = \frac{V_{RN}}{Z_1} = \frac{V \angle 0^\circ}{Z_1 \angle \phi_1} A = \left| \frac{V}{Z_1} \right| \angle -\phi_1 A$$

$$I_Y = \frac{V_{YN}}{Z_2} = \frac{V \angle -120^\circ}{Z_2 \angle \phi_2} A = \left| \frac{V}{Z_2} \right| \angle -120^\circ - \phi_2 A$$
$$I_B = \frac{V_{BN}}{Z_3} = \frac{V \angle -240^\circ}{Z_3 \angle \phi_3} A = \left| \frac{V}{Z_3} \right| \angle -240^\circ - \phi_3 A$$

Incidentally,  $I_R$ ,  $I_Y$  and  $I_B$  are also the line currents; the current in the neutral wire is the vector sum of the three line currents.

**Example 9.19** An unbalanced four-wire, star-connected load has a balanced voltage of 400 V, the loads are

$$Z_1 = (4 + j8) \Omega; Z_2 = (3 + j4) \Omega; Z_3 = (15 + j20) \Omega$$

Calculate the (i) line currents (ii) current in the neutral wire and (iii) the total power.

### **Solution** $Z_1 = (4+j8) \Omega; Z_2 = (3+j4) \Omega; Z_3 = (15+j20) \Omega$ $Z_1 = 8.94 \angle 63.40^{\circ} \Omega; Z_2 = 5 \angle 53.1^{\circ} \Omega; Z_3 = 25 \angle 53.13^{\circ} \Omega$

Let us assume RYB phase sequence.

The phase voltage  $V_{RN} = \frac{400}{\sqrt{3}} = 230.94$  V.

Taking  $V_{RN}$  as the reference phasor, we have

$$V_{RN} = 230.94 \angle 0^{\circ} \text{ V}, V_{YN} = 230.94 \angle -120^{\circ} \text{ V}$$
  
 $V_{BN} = 230.94 \angle -240^{\circ} \text{ V}$ 

The three line currents are

(i) 
$$I_R = \frac{V_{RN}}{Z_1} = \frac{230.94 \angle 0^{\circ}}{8.94 \angle 63.4^{\circ}} \text{ A} = 25.83 \angle -63.4^{\circ} \text{ A}$$
  
 $I_Y = \frac{V_{YN}}{Z_2} = \frac{230.94 \angle -120^{\circ}}{5 \angle 53.1^{\circ}} \text{ A} = 46.188 \angle -173.1^{\circ} \text{ A}$   
 $I_B = \frac{V_{BN}}{Z_3} = \frac{230.94 \angle -240^{\circ}}{25 \angle 53.13^{\circ}} \text{ A} = 9.23 \angle -293.13^{\circ} \text{ A}$ 

(ii) To find the neutral current, we must add the three line currents. The neutral current must then be equal and opposite to this sum.

Thus,

$$\begin{split} I_N &= -\left(I_R + I_Y + I_B\right) \\ &= -\left(25.83 \ \angle -\ 63.4^\circ + 46.188 \ \angle -\ 173.1^\circ + 9.23 \ \angle -\ 293.13^\circ\right) \, \mathrm{A} \\ I_N &= -\left[\left(11.56 - j23.09\right) + \left(-\ 45.85 - j5.54\right) + \left(3.62 + j8.48\right)\right] \, \mathrm{A}. \\ I_N &= -\left[\left(-\ 30.67 - j20.15\right)\right] \, \mathrm{A} = \left(30.67 + j20.15\right) \, \mathrm{A} \\ I_N &= 36.69 \ \angle 33.30 \ \circ \, \mathrm{A} \end{split}$$

Its phase with respect to  $V_{RN}$  is 33.3°, the disposition of all the currents is shown in Fig. 9.36.



Fig. 9.36

(iii) Power in *R* phase  $= I_R^2 \times R_R = (25.83)^2 \times 4 = 2668.75 \text{ W}$ Power in *Y* phase  $= I_Y^2 \times R_Y = (46.18)^2 \times 3 = 6397.77 \text{ W}$ Power in *B* phase  $= I_B^2 \times R_B = (9.23)^2 \times 15 = 1277.89 \text{ W}$ 

Total power absorbed by the load

= 2668.75 + 6397.77 + 1277.89 = 10344.41 W

### 9.10.4 Unbalanced Three Wire Star-Connected Load

In a three-phase, four-wire system if the connection between supply neutral and load neutral is broken, it would result in an unbalanced three-wire star-load. This type of load is rarely found in practice, because all the three wire star loads are balanced. Such as system is shown in Fig. 9.37. Note that the supply star point  $(N_s)$  is isolated from the load star point  $(N_t)$ . The potential of the load star point is different from that of the supply star point. The result is that the load phase voltages is not equal to the supply phase voltage; and they are not only unequal in magnitude, but also subtend angles other than 120° with one another. The magnitude of each phase voltage depends upon the individual phase loads. The potential of the load neutral point changes according to changes in the impedances of the phases, that is why sometimes the load neutral is also called a floating neutral point. All star-connected, unbalanced loads supplied from polyphase systems without a neutral wire have floating neutral point. The phasor sum of the three unbalanced line currents is zero. The phase voltage of the load is not  $1/\sqrt{3}$  of the line voltage. The unbalanced three-wire star load is difficult to deal with. It is because load phase voltages cannot be determined directly from the given supply line voltages. There are many methods to solve such unbalanced Y-connected loads. Two frequently used methods are presented here. They are

- (i) Star-delta conversion method, and
- (ii) The application of Millman's theorem





### 9.10.5 Star-Delta Method to Solve Unbalanced Load

Figure 9.38(a) shows an unbalanced wye-connected load. It has already been shown in Section 9.6 that a three phase star-connected load can be replaced by an equivalent delta-connected load. Thus, the star load of Fig. 9.38(a) can be replaced by equivalent delta as shown in Fig. 9.38(b), where the impedances in each phase is given by

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B}$$
$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R}$$
$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y}$$

The problem is then solved as an unbalanced delta-connected system. The line currents so calculated are equal in magnitude and phase to those taken by the original unbalanced wye (Y) connected load.



Fig. 9.38

9.37

**Example 9.20** A 400 V, three-phase supply feeds an unbalanced three-wire, starconnected load. The branch impedances of the load are  $Z_R = (4 + j 8) \Omega$ ;  $Z_y = (3 + j4) \Omega$  and  $Z_B = (15 + j 20) \Omega$ . Find the line currents and voltage across each phase impedance. Assume *RYB* phase sequence.

**Solution** The unbalanced star load and its equivalent delta ( $\Delta$ ) is shown in Fig. 9.39(a) and (b) respectively.





$$Z_{R} = (4 + j8) \Omega = 8.944 \angle 63.4^{\circ} \Omega$$
$$Z_{Y} = (3 + j4) \Omega = 5 \angle 53.1^{\circ} \Omega$$
$$Z_{B} = (15 + j20) \Omega = 25 \angle 53.1 \Omega$$

Using the expression in Section 9.10.5, we can calculate  $Z_{RI}$ ,  $Z_{YB}$  and  $Z_{BR}$ 

$$Z_R Z_Y + Z_Y Z_B + Z_B Z_R$$
  
= (8.94 \angle 63.4°) (5 \angle 53.1°) + (5 \angle 53.1°) (25 \angle 53.1°)  
+ (25 \angle 53.1°) (8.94 \angle 63.4°)

$$= 391.80 \angle 113.23^{\circ}$$

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B} = \frac{391.80 \angle 113.23^{\circ}}{25 \angle 53.1^{\circ}} = 15.67 \angle 60.13^{\circ}$$

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R} = \frac{391.80 \angle 113.23^{\circ}}{8.94 \angle 63.4^{\circ}} = 43.83 \angle 49.83^{\circ}$$

$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y} = \frac{391.80 \angle 113.23^{\circ}}{5 \angle 53.1^{\circ}} = 78.36 \angle 60.13^{\circ}$$

Taking  $V_{RY}$  as reference,  $V_{RY} = 400 \angle 0$ 

$$V_{YB} = 400 \angle -120^{\circ}; V_{BR} = 400 \angle -240^{\circ}$$
$$I_R = \frac{V_{RY}}{Z_{RY}} = \frac{400 \angle 0}{15.67 \angle 60.13^{\circ}} = 25.52 \angle -60.13^{\circ}$$

$$I_Y = \frac{V_{YB}}{Z_{YB}} = \frac{400 \angle -120^{\circ}}{43.83 \angle 49.83^{\circ}} = 9.12 \angle -169.83^{\circ}$$
$$I_B = \frac{V_{BR}}{Z_{RR}} = \frac{400 \angle -240^{\circ}}{78.36 \angle 60.13^{\circ}} = 5.10 \angle -300.13^{\circ}$$

The various line currents in the delta load are

$$\begin{split} I_1 &= I_R - I_B = 25.52 \ \angle -\ 60.13^\circ - 5.1 \ \angle -\ 300.13^\circ \\ &= 28.41 \ \angle -\ 69.07^\circ \ \mathrm{A} \\ I_2 &= I_Y - I_R = 9.12 \ \angle -\ 169.83^\circ - 25.52 \ \angle -\ 60.13^\circ \\ &= 29.85 \ \angle 136.58^\circ \ \mathrm{A} \\ I_3 &= I_B - I_Y = 5.1 \ \angle -\ 300.13^\circ - 9.12 \ \angle -\ 169.83^\circ \\ &= 13 \ \angle 27.60^\circ \ \mathrm{A} \end{split}$$

These line currents are also equal to the line (phase) currents of the original starconnected load. The voltage drop across each star-connected load will be as follows.

Voltage drop across  $Z_R = I_1 Z_R$ 

= 
$$(28.41 \angle -69.070^{\circ}) (8.94 \angle 63.4^{\circ}) = 253.89 \angle -5.67^{\circ} V$$

Voltage drop across  $Z_{y} = I_{2}Z_{y}$ 

$$= (29.85 \angle 136.58^{\circ}) (5 \angle 53.1^{\circ}) = 149.2 \angle 189.68^{\circ} V$$

Voltage drop across  $Z_{R} = I_{3}Z_{R}$ 

$$=(13 \angle 27.60^{\circ})(25 \angle 53.1^{\circ})=325 \angle 80.70^{\circ} V$$

### 9.10.6 Millman's Method of Solving Unbalanced Load

One method of solving an unbalanced three-wire star-connected load by stardelta conversion is described in Section 9.10.5. But this method is laborious and involves lengthy calculations. By using Millman's theorem, we can solve this type of problems in a much easier way. Consider an unbalanced wye (Y)load connected to a balanced three-phase supply as shown in Fig. 9.40(a).  $V_{_{RO}}$ ,  $V_{YO}$  and  $V_{BO}$  are the phase voltages of the supply. They are equal in magnitude, but displaced by 120° from one another.  $V_{RO}$ ,  $V_{YO}$  and  $V_{BO}$  are the load phase voltages; they are unequal in magnitude as well as differ in phase by unequal angles.  $Z_p$ ,  $Z_v$  and  $Z_p$  are the impedances of the branches of the unbalanced wye (Y) connected load. Figure 9.40(b) shows the triangular phasor diagram of the complete system. Distances RY, YB and BR represent the line voltages of the supply as well as load. They are equal in magnitude, but displaced by  $120^{\circ}$ . Here O is the star-point of the supply and is located at the centre of the equilateral triangle RYB. O' is the load star point. The star point of the supply which is at the zero potential is different from that of the star point at the load, due to the load being unbalanced. O' has some potential with respect to O and is shifted away from the centre of the triangle. Distance O'O represents the voltage of the load star point with respect to the star point of the supply Vo'o.





 $V_{o'o}$  is calculated using Millman's theorem. If  $V_{o'o}$  is known, the load phase voltages and corresponding currents in the unbalanced wye load can be easily determined.



Fig. 9.40

According to Millman's theorem,  $V_{o'o}$  is given by

$$V_{o'o} = \frac{V_{Ro} \ Y_R \ + V_{Yo} \ Y_Y + V_{Bo} \ Y_B}{Y_R + Y_Y + Y_R}$$

where the parameters  $Y_R$ ,  $Y_Y$  and  $Y_B$  are the admittances of the branches of the unbalanced wye connected load. From Fig. 9.40(a), we can write the equation

$$V_{Ro} = V_{Ro'} + V_{o'o}$$

or the load phase voltage

$$V_{Ro'} = V_{Ro} - V_{o'o}$$

Similarly,  $V_{Y_{o'}} = V_{Y_o} - V_{o'o}$  and  $V_{Bo'} = V_{Bo} - V_{o'o}$  can be calculated. The line currents in the load are

$$I_{R} = \frac{V_{Ro'}}{Z_{R}} = (V_{Ro} - V_{o'o}) Y_{R}$$

$$I_{Y} = \frac{V_{Yo'}}{Z_{Y}} = (V_{Yo} - V_{o'o}) Y_{Y}$$
$$I_{B} = \frac{V_{Bo'}}{Z_{B}} = (V_{Bo} - V_{o'o}) Y_{B}$$

The unbalanced three-wire star-connected loads can also be determined by using Kirchhoff's laws, and Maxwells mesh or loop equation. In general, any method which gives quick results in a particular case should be used.

**Example 9.21** To illustrate the application of Millmans method to unbalanced loads, let us take the problem in example given in Section 9.10.5.

**Solution** The circuit diagram is shown in Fig. 9.41.





Taking  $V_{RY}$  as reference line voltage = 400  $\angle 0^\circ$ , phase voltages lag 30° behind their respective line voltages. Therefore, the three phase voltages are

$$V_{Ro} = \frac{400}{\sqrt{3}} \angle -30^{\circ} \text{ V}$$
$$V_{Yo} = \frac{400}{\sqrt{3}} \angle -150^{\circ} \text{ V}$$
$$V_{Bo} = \frac{400}{\sqrt{3}} \angle -270^{\circ} \text{ V}$$

The admittances of the branches of the wye load are

$$Y_{R} = \frac{1}{Z_{R}} = \frac{1}{8.94 \angle 63.4^{\circ}} = 0.11 \angle -63.40^{\circ} \ \mho$$
$$Y_{Y} = \frac{1}{Z_{Y}} = \frac{1}{5 \angle 53.1^{\circ}} = 0.2 \angle -53.1^{\circ} \ \mho$$
$$Y_{B} = \frac{1}{Z_{R}} = \frac{1}{25 \angle 53.1^{\circ}} = 0.04 \angle -53.1^{\circ} \ \mho$$

$$V_{R_o}Y_R + V_{Y_o}Y_Y + V_{B_o}Y_B = (230.94 \angle -30^\circ) (0.11 \angle -63.40^\circ) + (230.94 \angle -150^\circ) (0.2 \angle -53.1^\circ) + (230.94 \angle -270^\circ) (0.04 \angle -53.1^\circ) = 36.68 \angle 182.66^\circ Y_R + Y_Y + Y_B = 0.11 \angle -63.4^\circ + 0.2 \angle -53.1^\circ + 0.04 \angle -53.1^\circ = 0.35 \angle -56.2^\circ \mho$$

Substituting the above values in the Millmans theorem, we have

$$V_{o'o} = \frac{V_{Ro} Y_R + V_{Yo} Y_Y + V_{Bo} Y_B}{Y_R + Y_Y + Y_B}$$
$$= \frac{36.68 \angle 182.66^{\circ}}{0.35 \angle -56.2^{\circ}} = 104.8 \angle 238.86^{\circ}$$

The three load phase voltages are

$$V_{Ro'} = V_{Ro} - V_{o'o}$$
  
= 230.94  $\angle$ - 30° - 104.8  $\angle$ 238.86° = 253.89  $\angle$ - 5.67° V  
 $V_{Yo'} = V_{Yo} - V_{o'o}$   
= 230.94  $\angle$ - 150° - 104.8  $\angle$ 238.86° = 149.2  $\angle$ 189.68° V  
 $V_{Bo'} = V_{Bo} - V_{o'o}$   
= 230.94  $\angle$ - 270° - 104.8  $\angle$ 238.86° = 325  $\angle$ 80.7° V

# 9.11 POWER MEASUREMENT IN THREE-PHASE CIRCUITS

## 9.11.1 Power Measurement in a Single Phase Circuit by Wattmeter

Wattmeters are generally used to measure power in the circuits. A wattmeter principally consists of two coils, one coil is called the current coil, and the other the pressure or voltage coil. A diagramatric representation of a wattmeter connected to measure power in a single phase circuit is shown in Fig. 9.42.



The coil represented with less number of turns between M and L is the current coil, which carries the current in the load and has very low impedance. The coil with more number of turns between the common terminal (comn) and V is the pressure coil, which is connected across the load and has high impedance. The load voltage is impressed across the pressure coil. The terminal M denotes the mains side, L denotes load side, common denotes the common point of current coil and pressure coil, and V denotes the second terminal of the pressure coil, usually selected as per the range of the load voltage in the circuit. From the figure, it is clear that a wattmeter has four terminals, two for current coil and two for potential coil. When the current flow through the two coils, they set up magnetic fields in space. An electromagnetic torque is produced by the interaction of the two magnetic fields. Under the influence of the torque, one of the coils (which is movable) moves on a calibrated scale against the action of a spring. The instantaneous torque produced by electromagnetic action is proportional to the product of the instantaneous values of the currents in the two coils. The small current in the pressure coil is equal to the input voltage divided by the impedance of the pressure coil. The inertia of the moving system does not permit it to follow the instantaneous fluctuations in torque. The wattmeter deflection is therefore, proportional to the average power (VI  $\cos \phi$ ) delivered to the circuit. Sometimes, a wattmeter connected in the circuit to measure power gives downscale reading or backward deflection. This is due to improper connection of the current coil and pressure coil.

To obtain up scale reading, the terminal marked as 'Comn' of the pressure coil is connected to one of the terminals of the current coil as shown in Fig. 9.43. Note that the connection between the current coil terminal and pressure coil terminal is not inherent, but has to be made externally. Even with proper connections, sometimes the wattmeter will give downscale reading whenever the phase angle between the voltage across the pressure coil and the current



Fig. 9.43

through the current coil is more than  $90^{\circ}$ . In such a case, connection of either the current coil or the pressure coil must be reversed.

### 9.11.2 Power in Three-Phase Circuits

Measurement of power by a wattmeter in a single phase circuit can be extended to measure power in a three-phase circuit. From Section 9.11.1, it is clear that we require three wattmeters, one in each phase to measure the power consumed in a three-phase system. Obviously, the total power is the algebraic sum of the readings of the three wattmeters. In this way we can measure power in balanced and unbalanced loads. In a balanced case it would be necessary to measure power only in one phase and the reading is multiplied by three to get the total power in all the three phases. This is true in principle, but presents a few difficulties in practice. To verify this fact let us examine the circuit diagram in Figs. 9.44(a) and (b).

Observation of Figs 9.44(a) and (b) reveals that for a star-connected load, the neutral must be available for connecting the pressure coil terminals. The current coils must be inserted in each phase for a delta-connected load. Such connections sometimes may not be practicable, because the neutral terminal is not available all the time in a star-connected load, and the phases of the delta-connected load are not accessible for connecting the current coils of the wattmeter. In most of the commercially available practical three-phase loads, only three line terminals are available. We, therefore, require a method where we can measure power in the three-phases with an access to the three lines connecting the source to the load. Two such methods are discussed here.







Fig. 9.44

#### 9.11.3 **Three Wattmeter and Two Wattmeter Method**

In this method, the three wattmeters are connected in the three lines as shown in Fig. 9.45, i.e. the current coils of the three wattmeters are introduced in the three lines, and one terminal of each potential coil is connected to one terminal of the corresponding current coil, the other three being connected to some common point which forms an effective neutral *n*.





The load may be either star-connected or delta-connected. Let us assume a star-connected load, and let the neutral of this load be denoted by N. Now the reading on the wattmeter  $W_{R}$  will correspond to the average value of the product

9.45

of the instantaneous value of the current  $I_p$  flowing in line 1, with the voltage drop  $V_{Rn}$ , where  $V_{Rn}$  is the voltage between points R and n. This can be written as  $V_{Rn} = V_{RN} + V_{Nn}$ , where  $V_{RN}$  is the load phase voltage and  $V_{Nn}$  is the voltage between load neutral, N, and the common point, n. Similarly,  $V_{Nn} = V_{NN} + V_{Nn}$ , and  $V_{Bn} = V_{BN} + V_{Nn}$ . Therefore, the average power,  $W_R$  indicated by the wattmeter is given by

$$W_R = \frac{1}{T} \int_{o}^{T} V_{Rn} I_R dt$$

where T is the time period of the voltage wave

 $W_R = \frac{1}{T} \int_{-\infty}^{T} (V_{RN} + V_{Nn})I_R dt$  $W_{Y} = \frac{1}{T} \int_{-\infty}^{T} V_{Yn} I_{Y} dt$ Similarly,  $=\frac{1}{T}\int_{-\infty}^{T}(V_{YN}+V_{Nn})I_{Y}\,dt$  $W_B = \frac{1}{T} \int_{-\infty}^{T} V_{Bn} I_B dt$ 

and

$$Total average power = W_R + W_Y + W_E$$

$$= \frac{1}{T} \int_{0}^{T} (V_{RN}I_{R} + V_{YN}I_{Y} + V_{BN}I_{B}) dt + \frac{1}{T} \int_{0}^{T} V_{Nn}(I_{R} + I_{Y} + I_{B}) dt$$

Since the system in the problem is a three-wire system, the sum of the three currents  $I_{R}$ ,  $I_{Y}$  and  $I_{R}$  at any given instant is zero. Hence, the power read by the three wattmeters is given by

 $= \frac{1}{T} \int_{-\infty}^{T} (V_{BN} + V_{Nn}) I_B dt$ 

$$W_{R} + W_{Y} + W_{B} = \frac{1}{T} \int_{0}^{T} (V_{RN}I_{R} + V_{YN}I_{Y} + V_{BN}I_{B}) dt$$

If the system has a fourth wire, i.e. if the neutral wire is available, then the common point, *n* is to be connected to the system neutral, *N*. In that case,  $V_{Nn}$ would be zero, and the above equation for power would still be valid. In other words, whatever be the value of  $V_{Nn}$ , the algebraic sum of the three currents  $I_R$ ,  $I_Y$  and  $I_B$  is zero. Hence, the term  $V_{Nn}(I_R + I_Y + I_B)$  would be zero. Keeping this advantage in mind, suppose the common point, n, in Fig. 9.45 is connected to line B. In such case,  $V_{NR} = V_{NB}$ ; then the voltage across the potential coil of wattmeter  $W_B$  will be zero and this wattmeter will read zero. Hence, this can be removed from the circuit. The total power is read by the remaining two wattmeters,  $W_B$  and  $W_Y$ .

Total power =  $W_R + W_Y$ 

Let us verify this fact from Fig. 9.46. The average power indicated by wattmeter  $W_p$  is

$$W_R = \frac{1}{T} \int_{o}^{T} V_{RB} I_R dt$$

and that by

 $W_Y = \frac{1}{T} \int_{o}^{T} V_{YB} \cdot I_Y dt$  $V_{RB} = V_{RN} + V_{NB}$ 

Also

:.





$$\begin{split} W_{R} + W_{Y} &= \frac{1}{T} \int_{o}^{T} (V_{RB} \cdot I_{R} + V_{YB} \cdot I_{Y}) dt \\ &= \frac{1}{T} \int_{o}^{T} \{ (V_{RN} + V_{NB}) I_{R} + (V_{YN} + V_{NB}) I_{Y} \} dt \\ &= \frac{1}{T} \int_{o}^{T} \{ (V_{RN} I_{R} + V_{YN} I_{Y}) + (I_{R} + I_{Y}) V_{NB} \} dt \\ t &= I_{R} + I_{Y} + I_{B} = 0 \\ &= I_{R} + I_{Y} = -I_{B} \end{split}$$

We know that

Substituting this value in the above equation, we get

$$\begin{split} W_{R} + W_{Y} &= \frac{1}{T} \int_{o}^{I} \{ (V_{RN} I_{R} + V_{YN} I_{Y}) + (-I_{B}) V_{NB} \} dt \\ V_{NB} &= -V_{BN} \\ W_{R} + W_{Y} &= \frac{1}{T} \int_{o}^{T} \{ (V_{RN} I_{R} + V_{YN} \cdot I_{Y} + V_{BN} \cdot I_{B}) \} dt \end{split}$$

which indicates the total power in the load.

From the above discussion it is clear that the power in a three-phase load, whether balanced or unbalanced, star-connected or delta-connected, three-wire or four wire, can be measured with only two wattmeters as shown in Fig. 9.46. In fact, the two wattmeter method of measuring power in three-phase loads has become a universal method. If neutral wire is available in this method it should not carry any current, or the neutral of the load should be isolated from the neutral of the source.

The current flowing through the current coil of each wattmeter is the line current, and the voltage across the pressure coil is the line voltage. In case the phase angle between line voltage and current is greater than  $90^{\circ}$ , the corresponding wattmeter would indicate downscale reading. To obtain upscale reading, the connections of either the current coil, or the pressure coil has to be interchanged. Reading obtained after reversal of coil connection should be taken as negative. Then, the algebraic sum of the two wattmeter readings gives the total power.

### 9.11.4 Power Factor by Two Wattmeter Method

When we talk about the power factor in three-phase circuits, it applies only to balanced circuits, since the power factor in a balanced load is the power factor of any phase. We cannot strictly define the power factor in three-phase unbalanced circuits, as every phase has a separate power factor. The two wattmeter method, when applied to measure power in a three-phase balanced circuits, provides information that help us to calculate the power factor of the load.

Figure 9.47 shows the vector diagram of the circuit shown in Fig. 9.46. Since the load is assumed to be balanced, we can take  $Z_1 \angle \phi_1 = Z_2 \angle \phi_2 = Z_3 \angle \phi_3 = Z$  $\angle \phi$  for the star-connected load. Assuming *RYB* phase sequence, the three rms load phase voltages are  $V_{RN}$ ,  $V_{YN}$  and  $V_{BN} \cdot I_R$ ,  $I_Y$  and  $I_B$  are the rms line (phase) currents. These currents will lag behind their respective phase voltages by an angle  $\phi$ . (An inductive load is considered).

Now consider the readings of the two wattmeters in Fig. 9.46.  $W_R$  measures the product of effective value of the current through its current coil  $I_R$ , effective value of the voltage across its pressure coil  $V_{RB}$  and the cosine of the angle between the phasors  $I_R$  and  $V_{RB}$ . The voltage across the pressure coil of  $W_R$  is given as follows.

$$V_{RB} = V_{RN} - V_{BN}$$
 phasor difference



Fig. 9.47

It is clear from the phasor diagram that the phase angle between

 $V_{RR}$  and  $I_R$  is  $(30^\circ - \phi)$  $W_R = V_{RR} \cdot I_R \cos\left(30 - \phi\right)$ 

Similarly,  $W_v$  measures the product of effective value of the current through its current coil  $I_y$ , the effective value of the voltage across its pressure coil,  $V_{y_R}$ and the cosine of the angle between the phasors  $V_{y_R}$  and  $I_{y'}$ 

 $V_{YB} = V_{YN} - V_{BN}$ 

...

From Fig. 9.47, it is clear that the phase angle between  $V_{yB}$  and  $I_y$  is  $(30^\circ + \phi)$ .  $W_Y = V_{YR} \cdot I_Y \cos(30^\circ + \phi)$ :.

Since the load is balanced, the line voltage  $V_{RB} = V_{YB} = V_L$  and the line current  $I_R = I_Y = I_L$ 

$$W_R = V_L \cdot I_L \cos (30^\circ - \phi)$$
$$W_Y = V_L I_L \cos (30^\circ + \phi)$$

Adding  $W_{R}$  and  $W_{Y}$  gives total power in the circuit, thus

$$W_R + W_Y = \sqrt{3} V_L I_L \cos \phi$$

From the two wattmeter readings, it is clear that for the same load angle  $\phi$ , wattmeter  $W_{R}$  registers more power when the load is inductive. It is also connected in the leading phase as the phase sequence is RYB. Therefore,  $W_{R}$  is higher reading wattmeter in the circuit of Fig. 9.46. In other words, if the load is capacitive, the wattmeter connected in the leading phase reads less for the same load angle. So, if we know the nature of the load, we can easily identify the phase sequence of the system. The higher reading wattmeter always reads positive. By proper manipulation of two wattmeter readings, we can obtain the power factor of the load.

9.49

Taking the ratio of the above two values, we get

$$\frac{W_R - W_Y}{W_R + W_Y} = \frac{\tan \phi}{\sqrt{3}}$$
$$\tan \phi = \sqrt{3} \left[ \frac{W_R - W_Y}{W_R + W_Y} \right]$$
$$\phi = \tan^{-1} \sqrt{3} \left[ \frac{W_R - W_Y}{W_R + W_Y} \right]$$

or

Thereafter, we can find  $\cos \phi$ 

---

**Example 9.22** The two wattmeter method is used to measure power in a threephase load. The wattmeter readings are 400 W and -35 W. Calculate (i) total active power (ii) power factor, and (iii) reactive power.

**Solution** From the given data, the two wattmeter readings  $W_R = 400$  W (Higher reading wattmeter)  $W_Y = -35$  W (Lower reading wattmeter).

(i) Total active power =  $W_1 + W_2$ 

=400 + (-35) = 365 W

(ii)  $\tan \phi = \sqrt{3} \frac{W_R - W_Y}{W_R + W_Y} = \sqrt{3} \frac{400 - (-35)}{400 + (-35)} = \sqrt{3} \times \frac{435}{365} = 2.064$ 

$$\phi = \tan^{-1} 2.064 = 64.15^{\circ}; P.F = 0.43$$

Reactive power =  $\sqrt{3} \times 435 = 753.44$  VAR

**Example 9.23** The input power to a three-phase load is 10 kW at 0.8 Pf. Two wattmeters are connected to measure the power, find the individual readings of the wattmeters.

**Solution** Let  $W_R$  be the higher reading wattmeter and  $W_Y$  the lower reading wattmeter

$$W_{R} + W_{Y} = 10 \text{ kW}$$
(9.1)  

$$\phi = \cos^{-1} 0.8 = 36.8^{\circ}$$

$$\tan \phi = 0.75 = \sqrt{3} \frac{W_{R} - W_{Y}}{W_{R} + W_{Y}}$$

or

$$W_R - W_Y = \frac{(0.75)}{\sqrt{3}} (W_R + W_Y) = \frac{0.75}{\sqrt{3}} \times 10 \text{ kW}$$
  
= 4.33 kW (9.2)

From Eqs (9.1) and (9.2) we get

$$W_R + W_Y = 10 \text{ kW}$$

$$W_R - W_Y = 4.33 \text{ kW}$$

$$2 W_R = 14.33 \text{ kW}$$

$$W_R = 7.165 \text{ kW}$$

$$W_Y = 2.835 \text{ kW}$$

or

### 9.11.5 Variation in Wattmeter Readings with Load Power Factor

It is useful to study the effect of the power factor on the readings of the wattmeter. In Section 9.11.4, we have proved that the readings of the two wattmeters depend on the load power factor angle  $\phi$ , such that

$$W_R = V_L I_L \cos (30 - \phi)^{\circ}$$
$$W_Y = V_L I_L \cos (30 + \phi)^{\circ}$$

We can, therefore, make the following deductions

- (i) When  $\phi$  is zero, i.e. power factor is unity, from the above expressions we can conclude that the two wattmeters indicate equal and positive values.
- (ii) When  $\phi$  rises from 0 to 60°, i.e. upto power factor 0.5, wattmeter  $W_R$  reads positive (since it is connected in the leading phase); whereas wattmeter  $W_Y$  reads positive, but less than  $W_R$ . When  $\phi = 60^\circ$ ,  $W_Y = 0$  and the total power is being measured only by wattmeter  $W_R$ .
- (iii) If the power factor is further reduced from 0.5, i.e. when  $\phi$  is greater than 60°,  $W_R$  indicates positive value, whereas  $W_Y$  reads down scale reading in such case. As already explained in Section 9.11.3 the connections of either the current coil, or the pressure coil of the corresponding wattmeter have to be interchanged to obtain an upscale reading, and the reading thus obtained must be given a negative sign. Then the total power in the circuit would be  $W_R + (-W_Y) = W_R W_Y$ . Wattmeter  $W_Y$  reads downscale for the phase angle between 60° and 90°. When the power factor is zero (i.e.  $\phi = 90^\circ$ ), the two wattmeters will read equal and opposite values.

i.e. 
$$W_R = V_L I_L \cos (30 - 90)^\circ = 0.5 V_L I_L$$
  
 $W_Y = V_I I_L \cos (30 + 90)^\circ = -0.5 V_I I_L$ 

### 9.11.6 Leading Power Factor Load

The above observations are made considering the lagging power factor. Suppose the load in Fig. 9.46(a) is capacitive, the wattmeter connected in the leading phase would read less value. In that case,  $W_R$  will be the lower reading

wattmeter, and  $W_y$  will be the higher reading wattmeter. Figure 9.48 shows the phasor diagram for the leading power factor.



Fig. 9.48

As the power factor is leading, the phase currents,  $I_R$ ,  $I_Y$  and  $I_B$  are leading their respective phase voltage by an angle  $\phi$ . From Fig. 9.48, the reading of the wattmeter connected in the leading phase is given by

$$W_R = V_{RB} \cdot I_R \cos (30 + \phi)^\circ$$
  
=  $V_L I_L \cos (30 + \phi)^\circ$  (lower reading wattmeter)

Similarly, the reading of the wattmeter connected in the lagging phase is given by

$$W_Y = V_{YB} I_Y \cos (30 - \phi)^\circ$$
  
=  $V_L I_L \cos (30 - \phi)^\circ$  (higher reading wattmeter)

Again the total power is given by

$$W_R + W_Y = \sqrt{3} \quad V_L I_L \cos \phi$$
$$W_Y - W_R = V_L I_L \sin \phi$$
$$\tan \phi = \sqrt{3} \quad \frac{W_Y - W_R}{W_Y + W_R}$$

Hence

A comparison of this expression with that of lagging power factor reveals  
the fact that the two wattmeter readings are interchanged, i.e. for lagging power  
factor, 
$$W_R$$
 is the higher reading wattmeter, and  $W_Y$  is the lower reading  
wattmeter; where as for leading power factor,  $W_R$  is the lower reading wattmeter  
and  $W_Y$  is the higher reading wattmeter. While using the expression for power  
factor, whatever may be the nature of the load, the lower reading is to be  
subtracted from the higher reading in the numerator. The variation in the  
wattmeter reading with the capacitive load follows the same sequence as in  
inductive load, with a change in the roles of wattmeters.

**Example 9.24** The readings of the two wattmeters used to measure power in a capacitive load are -3000 W and 8000 W, respectively. Calculate (i) the input power, and (ii) the power factor at the load. Assume *RYB* sequence.

### Solution

- (i) Total power =  $W_R + W_Y = -3000 + 8000 = 5000$  W
- (ii) As the load is capacitive, the wattmeter connected in the leading phase gives less value.
- ÷

*.*..

$$W_R = -3000$$
$$W_V = 8000$$

Consequently

$$\tan \phi = \sqrt{3} \frac{W_Y - W_R}{W_Y + W_R} = \sqrt{3} \frac{(8000 - (-3000))}{5000} = 3.81$$
  
$$\phi = 75.29^\circ \text{ (lead)}; \cos \phi = 0.25$$

### 9.11.7 Reactive Power with Wattmeter

We have already seen in the preceding section that the difference between higher reading wattmeter and lower reading wattmeter yields  $V_L I_L \sin \phi$ . So, the total reactive power =  $\sqrt{3} V_L I_L \sin \phi$ . Reactive power in a balanced three-phase load can also be calculated by using a single wattmeter.

As shown in Fig. 9.49(a), the current coil of the wattmeters is connected in any one line (*R* in this case), and the pressure coil across the other two lines (between *Y* and *B* in this case). Assuming phase sequence *RYB* and an inductive load of angle  $\phi$ , the phasor diagram for the circuit in Fig. 9.49(a) is shown in Fig. 9.49(b).





From Fig. 9.49(a), it is clear that the wattmeter power is proportional to the product of current through its current coil,  $I_R$ , voltage across its pressure coil,  $V_{YR}$ , and cosine of the angle between  $V_{YR}$  and  $I_R$ .

or



Fig. 9.49

From the vector diagram the angle between  $V_{y_R}$  and  $I_R$  is  $(90 - \phi)^\circ$ 

: Wattmeter reading =  $V_{YB}I_R \cos{(90-\phi)^\circ}$ 

 $= V_I I_I \sin \phi \text{VAR}$ 

If the above expression is multiplied by  $\sqrt{3}$ , we get the total reactive power in the load.

**Example 9.25** A single wattmeter is connected to measure reactive power of a three-phase, three-wire balanced load as shown in Fig. 9.49(a). The line current is 17 A and the line voltage is 440 V. Calculate the power factor of the load if the reading of the wattmeter is 4488 VAR.

**Solution** We know that wattmeter reading is equal to  $V_L I_L \sin \phi$ 

*:*..

 $4488 = 440 \times 17 \sin \phi$ sin  $\phi = 0.6$ Power factor = cos  $\phi = 0.8$ 

### 9.12 EFFECTS OF HARMONICS

The relationship between line and phase quantities for wye and delta connections as derived earlier are strictly valid only if the source voltage is purely sinusoidal. Such a waveform is an ideal one. Modern alternations are designed to give a terminal voltage which is almost sinusoidal. But it is nearly impossible to realise an ideal waveform in practice. All sinusoidally varying alternating waveforms deviate to a greater or lesser degree, from an ideal sinusoidal shape. Due to non-uniform distribution of the field flux and armature reaction in a.c. machines, the current and voltage waves may get distorted. Such waveforms are referred to as non-sinusoidal or complex waveforms. In modern machines this distortion is relatively small. All non-sinusoidal waves can be broken up into a series of sinusoidal waves whose frequencies are integral multiples of the frequency of the fundamental wave. The sinusoidal components of a complex wave are called *harmonics*. It is therefore necessary to consider the effect of certain harmonics on currents and voltages in the phase of three-phase wye and delta systems in effecting the line and phase quantities.

The fundamental wave is called the basic wave or first harmonic. The second harmonic has a frequency of twice the fundamental, the third harmonic frequency is three times the fundamental frequency and so on. Each harmonic is a pure sinusoid. Waves having 2f, 4f, 6f, etc. are called even harmonics and those having frequencies 3f, 5f, 7f, etc. are called odd harmonics. Since the negative half of the wave is a reproduction of the positive half, the even harmonics are absent. Therefore, a complex wave can be represented as a sum of fundamental and odd harmonics.

### 9.12.1 Harmonic Effect in Wye

Let us consider a wye connected generator winding, whose arrangement is shown diagrammatically in Fig. 9.50. The voltage induced in phase a of the three-phase symmetrical system, including odd harmonics is given by



Fig. 9.50

 $V_{na} = E_{m_1} \sin(\omega t + \theta_1) + E_{m_3} \sin(3\omega t + \theta_3) + E_{m_5} \sin(5\omega t + \theta_5) + \cdots$  (9.3) Where  $E_{m_1}, E_{m_3}, E_{m_5}$ , etc. are the peak values of the fundamental and other harmonics and  $\theta_1, \theta_3, \theta_5$ , etc. are phase angles. Assuming *abc* phase sequence. The voltage in phase *b* will be

$$v_{nb} = E_{m_1} \sin(\omega t + \theta_1 - 120^\circ) + E_{m_3} \sin(3\omega t - 360^\circ + \theta_3) + E_{m_5} \sin(5\omega t - 600^\circ + \theta_5)$$

$$= E_{m_1} \sin (\omega t + \theta_1 - 120^\circ) + E_{m_3} \sin (3\omega t + \theta_3) + E_{m_3} \sin (5\omega t + \theta_5 - 240^\circ)$$
(9.4)

The voltage in phase *c* will be

$$v_{nc} = E_{m_1} \sin (\omega t + \theta_1 - 240^\circ) + E_{m_3} \sin (3\omega t + \theta_3 - 720^\circ) + E_{m_5} \sin (5\omega t + \theta_5 - 1200^\circ) = E_{m_1} \sin (\omega t + \theta_1 - 240^\circ) + E_{m_3} \sin (3\omega t + \theta_3) + E_{m_5} \sin (5\omega t + \theta_5 - 120^\circ)$$
(9.5)

Equations 9.3, 9.4, and 9.5 show that all third harmonics are in time phase with each other in all the three phases as shown in Fig. 9.51(a). The same applies to the, ninth, fifteenth, twenty first... harmonics, i.e. all odd multiples of 3. Other than odd multiples of 3, the fifth, seventh, eleventh... and all other harmonics are displaced  $120^{\circ}$  in time phase mutually with either the same phase sequence or opposite phase sequence compared with that of the fundamentals. Fifth harmonics and seventh harmonic sequences are shown in Figs. 9.5(c) and 9.5(d) respectively.



Fig. 9.51

Summarising the above facts, the fundamental and all those harmonics obtained by adding a multiple of 6, i.e. 1, 7, 13, 19... etc. will have the same sequence. Similarly, the fifths and all harmonics obtained by adding a multiple of 6, i.e. 5, 11, 17, 23... etc. will have sequence opposite to that of the fundamental.

*Voltage Relations* The voltage between lines *ab* in the wye connected winding in Fig. 9.50 may be written as

$$e_{ab} = e_{an} + e_{nb}$$

Adding of each harmonic separately is shown in Fig. 9.52.





Fig. 9.52

It is seen from the vector diagrams of Fig. 9.52 that there is no third harmonic component in the line voltage. Hence, the rms value of the line voltage is given by

$$E_{ab} = \sqrt{3} \sqrt{\frac{E_{m_1}^2 + E_{m_5}^2 + E_{m_7}^2 + \dots}{2}}$$
(9.6)

From equation (1) the rms value of the phase voltage is

 $E_{m5}$ 

$$E_{na} = \sqrt{\frac{E_{m_1}^2 + E_{m_3}^2 + E_{m_5}^2 + E_{m_7}^2 + \dots}{2}}$$
(9.7)

It is seen from the above equations that in a wye connected system, the line voltage is not equal to  $\sqrt{3}$  times the phase voltage if harmonics are present. This is true only when the third harmonics are absent.

*Current Relations* Similar to the complex voltage wave, the instantaneous value of the complex current wave can be written

$$i = I_{m_1} \sin(\omega t + \phi_1) + I_{m_3} \sin(3\omega t + \phi_3) + I_{m_5} \sin(5\omega t + \phi_5)$$
(9.8)

where  $I_{m_1}$ ,  $I_{m_3}$ ,  $I_{m_5}$ , etc. are the peak values of fundamental and other harmonics;  $(\phi_1 - \phi_1)$  is the phase difference between fundamental component of the harmonic voltage and current and  $(\phi_3 - \phi_3)$  is the phase difference between 3rd harmonics and so on. Applying *KCL* for the three phase wye connected winding in Fig. 9.50.

$$I_{na} + i_{nb} + i_{nc} = 0 \tag{9.9}$$

The equations for  $i_{na}$ ,  $i_{nb}$  and  $i_{nc}$  can be obtained by replacing currents in the place of voltages in equations 9.3, 94 and 9.5 under balanced conditions the sum of the three currents is equal to zero, only when they have equal magnitudes and displaced by 120° apart in time phase in the three phases. All harmonics fulfil the above condition except the third harmonics and their odd multiples as they are in the same phase as shown in Fig. 9.51(a) or 9.52(b). Hence, the resultant of  $i_{na} + i_{nb} + i_{nc}$  consists of the arithmetic sum of the third harmonics in the three phases. Hence, there must be a neutral wire or fourth wire to provide a return path for the third harmonic. We can summarise the above facts as follows. In a balanced three-wire wye connection, all harmonics are present except third and its odd multiples. In a four-wire wye connection, i.e. with a neutral wire, all harmonics will exist.

### 9.12.2 Harmonic Effect in Delta

Let the three windings of the generator be delta-connected as shown in Fig. 9.53. Let  $v_{na}$ ,  $v_{nb}$  and  $v_{nc}$  be the phase emfs and  $v_{na}$ ,  $v_{nb}$  and  $v_{nc}$  be the terminal voltages of the three phases *a*, *b* and *c* respectively. According to *KVL* the algebraic sum of the three terminal voltages in the closed loop is given by

$$v_{na} + v_{nb} + v_{nc} = v_{ca} + v_{ab} + v_{bc} = 0 ag{9.10}$$

There will be a circulating current in the closed loop due to the resultant third harmonic and their multiple induced emfs. This resultant emf is dropped in the closed loop impedance. Hence, the third harmonic voltage does not appear across the terminals of the delta. Hence, the terminal voltages in delta connection  $v_{ca}$ ,  $v_{ab}$  and  $v_{bc}$  are given by equations 1, 2 and 3 respectively without third harmonic terms.

*Current Relations* The three phase windings in Fig. 9.53, carry all the harmonics internally and are given by

$$i_{na} = i_{ca} = I_{m_1} \sin (\omega t + \theta_1) + I_{m_2} \sin (3\omega t + \theta_3) + I_{m_5} \sin (5\omega t + \theta_5) + \dots$$
9.11)  
$$i_{nb} = i_{ab} = I_{m_1} \sin (\omega t + \theta_1 - 120^\circ) + I_{m_3} \sin (3\omega t + \theta_3 - 360^\circ) + I_{m_5} \sin (5\omega t + \theta_5 - 600^\circ) + \dots$$
9.11)  
$$= I_{m_1} \sin (\omega t + \theta_1 - 120^\circ) + I_{m_3} \sin (3\omega t + \theta_3) + I_{m_5} \sin (5\omega t + \theta_5 - 240^\circ) \dots$$
(9.12)  
$$i_{nc} = i_{bc} = I_{m_1} \sin (\omega t + \theta_1 - 240^\circ) + I_{m_5} \sin (3\omega t + \theta_3 - 720^\circ)$$

$$+ I_{m_{5}} \sin (5\omega t + \theta_{5} - 1200^{\circ})$$

$$= I_{m1} \sin (\omega t + \theta_{1} - 24\theta_{3}) + I_{m_{3}} \sin (3\omega t + \theta_{3})$$

$$+ I_{m_{5}} \sin (5\omega t + \theta_{5} - 120^{\circ})$$

$$B \xrightarrow{b_{m}}$$

$$I_{Bb}$$

$$A \xrightarrow{I_{Aa}} n$$

$$A \xrightarrow{d_{Aa}} n$$

$$A \xrightarrow{d_{Aa}} c$$

$$C \xrightarrow{I_{Cc}} c$$

$$C \xrightarrow{I_{Cc}} c$$

$$C \xrightarrow{d_{Cc}} c$$

Fig. 9.53

Equations 9.11, 9.12 and 9.13 represent the phase currents in the delta connection. The line currents  $I_{Aa} I_{Bb}$  and  $I_{Cc}$  can be obtained by applying KCL at the three junctions of the delta in Fig. 9.53. The current vector diagrams are similar to the voltage vector diagrams shown in Fig. 9.52 except that the voltages are to be replaced with currents. The line currents can be obtained in terms of phase currents by applying KCL at three junctions as follows

$$I_{Aa} = i_{ab} - i_{Ca}$$
  
=  $I_{m_1} \sin (\omega t + \theta_1 - 120^\circ) + I_{m_5} \sin (5\omega t + \theta_5 - 240^\circ)$   
 $- I_{m_1} \sin (\omega t + \theta_1) - I_{m_5} \sin (5\omega t + \theta_5)$  (9.14)  
 $i_{Bb} = i_{bc} - i_{ab}$   
=  $I_{m_1} \sin (\omega t + \theta_1 - 240^\circ) + I_m \sin (5\omega t + \theta_5 - 120^\circ)$ 

$$-I_{m_{1}}\sin(\omega t + \theta_{1} - 120^{\circ}) - I_{m_{5}}\sin((\omega t + \theta_{5} - 240^{\circ}))$$
(9.15)  
$$i_{Cc} = i_{ca} - i_{bc}$$
$$= I_{m_{1}}\sin(\omega t + \theta_{1}) + I_{m_{5}}\sin((5\omega t + \theta_{5}) - I_{m_{1}}\sin((\omega t + \theta_{1} - 240^{\circ})))$$
(9.16)

The rms value of the line current from the above equation is

$$I_L = \sqrt{3} \sqrt{\frac{I_{m_1}^2 + I_{m_5}^2 + \dots}{2}}$$
(9.17)

The rms value of the phase current from Equations 9.11, 9.12 and 9.13 is given by

$$I_{\rm ph} = \sqrt{\frac{I_{m_1}^2 + I_{m_3}^2 + I_{m_5}^2 + \dots}{2}}$$
(9.18)

It is seen from Equations 9.17 and 9.18, that in a delta-connected system the line current is not equal to  $\sqrt{3}$  times the phase current. It is only true when there are no third harmonic currents in the system.

### 9.13 EFFECTS OF PHASE-SEQUENCE

The effects of phase sequence of the source voltages are not of considerable importance for applications like lighting, heating, etc. but in case of a threephase induction motor, reversal of sequence results in the reversal of its direction. In the case of an unbalanced polyphase system, a reversal of the voltage phase sequence will, in general, cause certain branch currents to change in magnitude as well as in phase position. Even though the system is balanced, the readings of the two wattmeters in the two wattmeter method of measuring power interchange when subjected to a reversal of phase sequence when two or more three phase generators are running parallel to supply a common load, reversing the phase sequence of any one machine cause severe damage to the entire system. Hence, when working on such systems, it is very important to consider the phase sequence of the system. Unless otherwise stated, the term "phase sequence" refers to voltage phase sequence. The line currents and phase currents follow the same sequence as the system voltage. The phase sequence of a given system, is a small meter with three long connecting leads in side which it has a circular disc. The rotation of which previously been checked against a known phase sequence. In three-phase systems, only two different phase sequences are possible. The three leads are connected to the three lines whose sequence is to be determined, the rotation of the disc can be used as an indicator of phase sequence.

### 9.14 POWER FACTOR OF AN UNBALANCED SYSTEM

The concept of power factor in three-phase balanced circuits has been discussed in Section 9.11.4. It is the ratio of the phase watts to the phase volt-amperes of any one of the three phases. We cannot strictly define the power factor in threephase unbalanced circuits, as each phase has a separate power factor. Generally for three-phase unbalanced loads, the ratio of total watts ( $\sqrt{3} V_I I_I \cos \theta$ ) to

total volt-amperes  $(\sqrt{3} V_L I_L)$  is a good general indication of the power factor.

Another recognised definition for an unbalanced polyphase system is called the vector power factor, given by

Power factor = 
$$\frac{\sum VI \cos\theta}{\sum VI}$$

Where  $\sum VI \cos \theta$  is the algebraic sum of the active powers of all individual phases given by
$\sum VI \cos \theta = V_a I_a \cos \theta_a + V_b I_b \cos \theta_b + V_c I_c \cos \theta_c + \dots$ 

and

 $\sum VI \sin \theta$  is the algebraic sum of the individual phase reactive voltamperes. The following example illustrates the application of vector power factor for unbalanced loads.

 $\sum VI = \sqrt{\left(\sum VI \cos \theta\right)^2 + \left(\sum VI \sin \theta\right)^2}$ 

Consider Example 9.19 where the phase voltage and currents have been already calculated. Here  $V_{RN} = 230.94 \angle 0^{\circ} \text{ V}, V_{YN} = 230.94 \angle -120^{\circ} \text{V},$  $V_{_{BN}} = 230.94 \angle -240^{\circ} V$ 

 $I_{R} = 24.83 \angle -63.4^{\circ} \text{ A}; I_{V} = 46.188 \angle -173.1^{\circ} \text{ A}; I_{R} = 9.23 \angle -293.13^{\circ} \text{ A}.$ Active power of phase  $R = 230.94 \times 25.83 \times \cos 63.4^\circ$  = 2.6709 kW Active power of phase  $Y = 230.94 \times 46.188 \times \cos 53.1^\circ = 6.4044$  kW Active power of phase  $B = 230.94 \times 9.23 \times \cos 53.13^\circ$  = 1.2789 kW 10.3542 kW

 $\sum VI \cos \theta = 10.3542 \text{ kW}$ 

Reactive power of phase  $R = 230.94 \times 25.83 \times \sin 63.4^\circ = 5.3197$  KVAR Reactive power of phase  $Y = 230.94 \times 46.188 \times \sin 53.1^\circ = 8.5299$  KVAR Reactive power of phase  $B = 230.94 \times 9.23 \times \sin 53.13^\circ = 1.7052$  KVAR 15.5548 KVAR

$$\sum VI \sin \theta = 15.5548 \text{ KVAR}$$
Power factor =  $\frac{10.3542}{\sqrt{(15.5548)^2 + (10.3542)^2}} = 0.5541$ 

# ADDITIONAL SOLVED PROBLEMS

**Problem 9.1** The phase voltage of a star-connected three-phase ac generator is 230 V. Calculate the (i) line voltage (ii) active power output if the line current of the system is 15 A at a power factor of 0.7 and (iii) active and reactive components of the phase currents.

**Solution** The supply voltage (generator) is always assumed to be balanced *.*..

$$V_{Ph} = 230 \text{ V}; I_L = I_{Ph} = 15 \text{ A}, \cos \phi = 0.7, \sin \phi = 0.71$$

(i) In a star-connected system  $V_{I} = \sqrt{3} V_{Ph} = 398.37 \text{ V}$ 

(ii) Power output = 
$$\sqrt{3} V_L I_L \cos \phi$$
  
=  $\sqrt{3} \times 398.37 \times 15 \times 0.7 = 7244.96$  W

(iii) Active component of the current  $=I_{Ph}\cos\phi$  $= 15 \times 0.7 = 10.5$  A Reactive component of the current= $I_{Ph} \sin \phi$ = 15 × 0.71 = 10.65 A

**Problem 9.2** A three-phase delta-connected RYB system with an effective voltage of 400 V, has a balanced load with impedances  $3 + j4 \Omega$ . Calculate the (i) phase currents (ii) line currents and (iii) power in each phase.

### Solution



Fig. 9.54

$$V_L = V_{Ph} = 400 \text{ V}$$

Assuming RYB phase sequence, we have

$$V_{RY} = 400 \angle 0^{\circ}; V_{YB} = 400 \angle -120^{\circ}; V_{BR} = 400 \angle -240^{\circ}$$
  
 $Z = 3 + j4 = 5 \angle 53.1^{\circ}$ 

The three phase currents

The three line currents are

$$I_1 = I_R - I_B = 138.55 \angle -83.09^\circ$$
$$I_2 = I_Y - I_R = 138.55 \angle 156.9^\circ$$
$$I_3 = I_B - I_Y = 138.55 \angle 36.89^\circ$$
$$\cos \phi = \frac{R_{Ph}}{Z_{Ph}} = \frac{3}{5} = 0.6$$

(iii) Power consumed in each phase =  $V_{Ph} I_{Ph} \cos \phi$ = 400 × 80 × 0.6 = 19200 W Total power = 3 × 19200 = 57600 W **Problem 9.3** The load in Problem 9.2 is connected in star with the same phase sequence across the same system. Calculate (i) the phase and line currents (ii) the total power in the circuit, and (iii) phasor sum of the three line currents.

**Solution** The circuit is shown in Fig. 9.55.



Fig. 9.55

Assuming RYB phase sequence, since

$$V_L = 400 \text{ V}$$
  
 $V_{Ph} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$ 

Taking  $V_{RN}$  as reference, the three phase voltages are  $V_{RN} = 230.94 \angle 0^{\circ}$ ;  $V_{YN} = 230.94 \angle -120^{\circ}$ ; and  $V_{BN} = 230.9 \angle -240^{\circ}$ .

The three line voltages,  $V_{RY}$ ,  $V_{YB}$  and  $V_{BR}$  are 30° ahead of their respective phase voltages.

$$I_{Ph} = I_L; Z_{Ph} = 3 + j4 = 5 \angle 53.1^\circ$$

The three phase currents are

$$I_{R} = \frac{V_{RN}}{Z_{Ph}} = \frac{230.94 \angle 0^{\circ}}{5 \angle 53.1^{\circ}} = 46.18 \angle -53.1^{\circ}$$
$$I_{Y} = \frac{V_{YN}}{Z_{Ph}} = \frac{230.09 \angle -120^{\circ}}{5 \angle 53.1^{\circ}} = 46.18 \angle -173.1^{\circ}$$
$$I_{B} = \frac{V_{BN}}{Z_{Ph}} = \frac{230.09 \angle -240^{\circ}}{5 \angle 53.1^{\circ}} = 46.18 \angle -293.1^{\circ}$$

(ii) Total power = 
$$\sqrt{3} V_L I_L \cos \phi$$
  
=  $\sqrt{3} \times 400 \times 46.18 \times 0.6 = 19196.6 \text{ W}$ 

Thus, it can be observed that the power consumed in a delta load will be three times more than that in the star connection (iii) Phasor sum of the three line currents

$$= I_R + I_Y + I_B$$
  
= 46.18  $\angle$  - 53.1° + 46.18  $\angle$  - 173.1° + 46.18  $\angle$  - 293.1° = 0.

**Problem 9.4** A three-phase balanced delta-connected load with line voltage of 200 V, has line currents as  $I_1 = 10 \angle 90^\circ$ ;  $I_2 = 10 \angle -150^\circ$  and  $I_3 = 10 \angle -30^\circ$ . (i) What is the phase sequence? (ii) What are the impedances?

**Solution** Figure 9.56(a) represents all the three line currents in the phasor diagram.



Fig. 9.56(a)

(i) It can be observed from Figs 9.56(a) and (b) that the current flowing in line B, i.e. I<sub>3</sub> lags behind I<sub>1</sub> by 120°, and the current flowing in line Y, i.e. I<sub>2</sub> lags behind i<sub>3</sub> by 120°. ∴ The phase sequence is RBY.



Fig. 9.56(b)

(ii)  

$$V_{Ph} = V_L = 200$$

$$I_{Ph} = \frac{I_L}{\sqrt{3}} = \frac{10}{\sqrt{3}}$$

$$Z_{Ph} = \frac{V_{Ph}}{I_{Ph}} = \frac{200\sqrt{3}}{10} = 34.64 \ \Omega$$

**Problem 9.5** Three equal inductors connected in star take 5 kW at 0.7 Pf when connected to a 400 V, 50 Hz three-phase, three-wire supply. Calculate the line currents (i) if one of the inductors is disconnected, and (ii) If one of the inductors is short circuited.

### Solution



Fig. 9.57(a)

Total power when they are connected to 400 V supply

$$P = \sqrt{3} V_L I_L \cos \phi = 5000 \text{ W}$$
$$I_{Ph} = I_L = \frac{5000}{\sqrt{3} \times 400 \times 0.7} = 10.31 \text{ A}$$

Impedance/phase = 
$$\frac{V_{Ph}}{I_{Ph}} = \frac{400}{\sqrt{3} \times 10.31} = 22.4 \Omega$$
  
 $R_{Ph} = Z_{Ph} \cos \phi = 22.4 \times 0.7 = 15.68 \Omega$   
 $X_{Ph} = Z_{Ph} \sin \phi = 22.4 \times 0.714 = 16 \Omega$ 

 (i) If phase Y is disconnected from the circuit, the other two inductors are connected in series across the line voltage of 400 V as shown in Fig. 9.57(a).

$$I_R = I_B = \frac{400}{2 \times Z_{Ph}} = 8.928 \text{ A}$$
  
 $I_Y = 0$ 

(ii) If phase Y and N are short circuited as shown in Fig. 9.57(b), the phase voltages  $V_{RN}$  and  $V_{BN}$  will be equal to the line voltage 400 V.



Fig. 9.57(b)

$$I_{Ph} = I_R = I_B = \frac{400}{Z_{Ph}} = \frac{400}{22.4} = 17.85 \text{ A}$$

The current in the *Y* phase is equal to the phasor sum of the *R* and *B*.

$$I_Y = 2 \times I_{Ph} \cos\left(\frac{60}{2}\right) = 30.91 \text{ A}$$

**Problem 9.6** For the circuit shown in Fig. 9.58, calculate the line current, the power and the power factor. The value of R, L and C in each phase are 10 ohms, 1 H and 100  $\mu$ F, respectively.

Solution Let us assume RYB sequence.

$$V_{RN} = \frac{400}{\sqrt{3}} \angle 0^{\circ} = 231 \angle 0^{\circ}; V_{YN} = 231 \angle -120^{\circ}; V_{BN} = 231 \angle -240^{\circ}$$
  
dmittance of each phase  $Y_{ph} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$   
 $= \frac{1}{10} + \frac{1}{j314} + j314 \times 100 \times 10^{-6}$   
 $Y_{Ph} = 0.1 + j28.22 \times 10^{-3}$   
 $= 0.103 \angle 15.75^{\circ}$   $\mho$   
 $I_{Ph} = V_{Ph} Y_{Ph}$   
 $= (231 \angle 0^{\circ}) (0.103 \angle 15.75^{\circ})$   
 $= 23.8 \angle 15.75^{\circ} A$   
Power  $= \sqrt{3} V_L I_L \cos \phi$   
 $= \sqrt{3} \times 400 \times 23.8 \cos 15.75^{\circ}$   
 $= 15869.57 W$   
Power factor = cos 15.75° = 0.96 leading  
 $R \xrightarrow{I_1}$   
 $400 V; 50 Hz$   
 $V_1 = \frac{I_3}{I_2}$ 

Fig. 9.58

**Problem 9.7** For the circuit shown in Fig. 9.59, an impedance is connected across *YB*, and a coil of resistance 3  $\Omega$  and inductive reactance of 4  $\Omega$  is connected across *RY*. Find the value of *R* and *X* of the impedance across *YB* such that  $I_2 = 0$ . Assume a balanced three-phase supply with RYB sequence.

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**Solution** As usual  $I_R$ ,  $I_Y$  and  $I_B$  are phase currents, and  $I_1$ ,  $I_2$  and  $I_3$  are line currents.

Applying KCL at node *Y*, we have

 $I_2 = I_Y - I_R$ 

Since

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*:*..

$$I_{2} = 0$$

$$I_{Y} = I_{R}$$

$$I_{R} = \frac{V_{RY}}{3 + j4}, I_{Y} = \frac{V_{YB}}{Z_{YB}}$$

$$V_{RY} = V \angle 0^{\circ}, V_{YB} = V \angle -120^{\circ}$$

$$\frac{V \angle 0^{\circ}}{3 + j4} = \frac{V \angle -120^{\circ}}{Z_{YB}}$$

$$Z_{YB} = \frac{V \angle -120^{\circ}}{V \angle 0^{\circ}} (3 + j4)$$

$$= 1.96 - j4.6$$

$$R = 1.96 \ \Omega; X = 4.6 \ \Omega \text{ (capacitive reactance)}$$

**Problem 9.8** A symmetrical three-phase 440 V system supplies a balanced delta-connected load. The branch current is 10 A at a phase angle of 30°, lagging. Find (i) the line current (ii) the total active power, and (iii) the total reactive power. Draw the phasor diagram.

Solution (i) In a balanced delta-connected system

$$I_L = \sqrt{3} I_{Ph} = \sqrt{3} \times 10 = 17.32 \text{ A}$$

(ii) Total active power

$$= \sqrt{3} V_L I_L \cos \phi$$
  
=  $\sqrt{3} \times 440 \times 17.32 \times \cos 30^\circ = 11.431 \text{ kW}$ 

(iii) Total reactive power

$$=\sqrt{3} V_L I_L \sin \phi$$

9.67

 $=\sqrt{3} \times 440 \times 17.32 \times \sin 30^\circ = 6.5998 \text{ KVAR}$ 

The phasor diagram is as under



Fig. 9.60

 $V_R$ ,  $V_Y$  and  $V_B$  are phase voltages, and are equal to the line values.  $I_R$ ,  $I_Y$  and  $I_B$  are the phase currents, and lag behind their respective phase voltages by 30°. Line currents  $(I_R - I_B)$ ,  $(I_Y - I_R)$  and  $(I_B - I_Y)$  lag behind their respective phase currents by 30°.

**Problem 9.9** Find the line currents and the total power consumed by the unbalanced delta-connected load shown in Fig. 9.61.



Fig. 9.61

Solution Assuming RYB phase sequence, from the given data

$$I_R = 10 \angle -36.88^\circ; I_Y = 5 \angle 45.57^\circ; I_R = 7 \angle 0^\circ$$

Line currents are

$$\begin{split} I_1 &= I_R - I_B = 6.08 \ \angle -80^{\circ} \\ I_2 &= I_Y - I_R = 10.57 \ \angle 11.518^{\circ} \\ I_3 &= I_R - I_Y = 5 \ \angle -45.56^{\circ} \end{split}$$

Total power is the sum of the powers consumed in all the three phases.

Power in 
$$RY = V_{RY} \times I_R \times 0.8$$
  
= 400 × 10 × 0.8 = 3200 W  
Power in  $YB = V_{YB} \times I_Y \times 0.7$   
= 400 × 5 × 0.7 = 1400 W  
Power in  $BY = V_{BR} \times I_B \times 1$   
= 400 × 7 × 1 = 2800 W  
Total power = 3200 + 1400 + 2800 = 7400 W

**Problem 9.10** A delta-connected three-phase load has 10  $\Omega$  between *R* and *Y*, 6.36 mH between *Y* and *B*, and 636  $\mu$ F between *B* and *R*. The supply voltage is 400 V, 50 Hz. Calculate the line currents for RBY phase sequence.

#### Solution

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$$Z_{RY} = 10 + j0 = 10 \angle 0^{\circ}; Z_{YB} = 0 + jX_L = 0 + jX_L$$
  
= 0 + j2 \pi fL = 2 \angle 90^{\circ}  
$$Z_{BR} = 0 - jX_C = 0 - \frac{j}{2\pi fC} = 5 \angle - 90^{\circ}$$

Since the phase sequence is RBY, taking  $V_{RY}$  as reference voltage, we have



#### Fig. 9.62

$$I_Y = \frac{V_{YB}}{2 \angle 90^\circ} = \frac{400 \angle -240^\circ}{2 \angle 90^\circ} = 200 \angle -330^\circ$$
$$I_B = \frac{V_{BR}}{5 \angle -90^\circ} = \frac{400 \angle -120^\circ}{5 \angle -90^\circ} = 80 \angle -30^\circ$$

The three line currents are

$$I_{1} = I_{R} - I_{B} = 40 \angle 0^{\circ} - 80 \angle -30^{\circ} = 49.57 \angle 126.2^{\circ}$$

$$I_{2} = I_{Y} - I_{R} = 200 \angle -300^{\circ} \angle -40 \angle 0^{\circ} = 166.56 \angle 36.89^{\circ}$$

$$I_{3} = I_{B} - I_{Y} = 80 \angle -30^{\circ} - 200 \angle -330^{\circ} = 174.35 \angle 233.41^{\circ}$$

**Problem 9.11** The power consumed in a three phase balanced star-connected load is 2 kW at a power factor of 0.8 lagging. The supply voltage is 400 V, 50 Hz. Calculate the resistance and reactance of each phase.

**Solution** Phase voltage =  $\frac{400}{\sqrt{3}}$ Power consumed = 2000 W =  $\sqrt{3} V_L I_L \cos \phi$ Phase current or line current  $I_L = \frac{2000}{\sqrt{3} \times 400 \times 0.8} = 3.6 \text{ A}$ 

Impedance of each phase

$$Z_{Ph} = \frac{V_{Ph}}{I_{Ph}} = \frac{400}{\sqrt{3} \times 3.6} = 64.15 \text{ A}$$

Since the power-factor of the load is lagging, the reactance is inductive reactance. From the impedance triangle shown in Fig. 9.63, we have



Fig. 9.63

Resistance of each phase 
$$R_{Ph} = Z_{Ph} \cos \phi$$
  
= 64.15 × 0.8 = 51.32  $\Omega$   
Reactance of each phase  $X_{Ph} = Z_{Ph} \sin \phi$   
= 64.15 × 0.6 = 38.5  $\Omega$ 

**Problem 9.12** A symmetrical three-phase 100 V; three-wire supply feeds an unbalanced star-connected load, with impedances of the load as  $Z_R = 5 \angle 0^\circ \Omega$ ,  $Z_Y = 2 \angle 90^\circ \Omega$  and  $Z_B = 4 \angle -90^\circ \Omega$ . Find the (i) line currents, (ii) voltage across the impedances and (iii) the displacement neutral voltage.

**Solution** As explained earlier, this type of unbalanced *Y*-connected threewire load can be solved either by star-delta conversion method or by applying Millman's theorem.

(a) Star-Delta Conversion Method

The unbalanced star-connected load and its equivalent delta load are shown in Figs. 9.64 (a) and (b).



Fig. 9.64

$$Z_{R}Z_{Y} + Z_{Y}Z_{B} + Z_{B}Z_{R} = (5 \angle 0^{\circ}) (2 \angle 90^{\circ}) + (2 \angle 90^{\circ}) (4 \angle -90^{\circ}) + (4 \angle -90^{\circ}) (5 \angle 0^{\circ}) = 8 - j10 = 12.8 \angle -51.34^{\circ}$$
$$Z_{RY} = \frac{Z_{R}Z_{Y} + Z_{Y}Z_{B} + Z_{B}Z_{R}}{Z_{B}} = \frac{12.8 \angle -51.34^{\circ}}{4 \angle -90^{\circ}} = 3.2 \angle 38.66^{\circ}$$
$$Z_{YB} = \frac{Z_{R}Z_{Y} + Z_{Y}Z_{B} + Z_{B}Z_{R}}{Z_{R}} = \frac{12.8 \angle -51.34^{\circ}}{5 \angle 0^{\circ}} = 2.56 \angle -51.34^{\circ}$$
$$Z_{BR} = \frac{Z_{R}Z_{Y} + Z_{Y}Z_{B} + Z_{B}Z_{R}}{Z_{Y}} = \frac{12.8 \angle -51.34^{\circ}}{2 \angle 90^{\circ}} = 6.4 \angle -141.34^{\circ}$$

Taking  $V_{RY}$  as the reference, we have

 $V_{RY} = 100 \angle 0^{\circ}, V_{YB} = 100 \angle -120^{\circ}; V_{BR} = 100 \angle -240^{\circ}$ The three phase currents in the equivalent delta load are

$$I_{R} = \frac{V_{RY}}{Z_{RY}} = \frac{100 \angle 0^{\circ}}{3.2 \angle 38.66^{\circ}} = 31.25 \angle -38.66^{\circ}$$
$$I_{Y} = \frac{V_{YB}}{Z_{YB}} = \frac{100 \angle -120^{\circ}}{2.56 \angle -51.34^{\circ}} = 39.06 \angle -68.66^{\circ}$$
$$I_{B} = \frac{V_{BR}}{Z_{BR}} = \frac{100 \angle -240^{\circ}}{6.4 \angle -141.34^{\circ}} = 15.62 \angle -98.66^{\circ}$$

The line currents are

$$I_1 = I_R - I_B = 31.25 \angle -38.66^\circ - 15.62 \angle -98.66^\circ$$
  
= (24.4 - j19.52) - (-2.35 + j15.44) = (26.75 - j4.08)  
= 27.06  $\angle -8.671^\circ$   
$$I_2 = I_Y - I_R = 39.06 \angle -68.66^\circ - 31.25 \angle -38.66^\circ$$
  
= (14.21 - j36.38) - (24.4 - j19.52) = (-10.19 - j16.86)

$$= 19.7 \angle 238.85^{\circ}$$
  

$$I_3 = I_B - I_Y = 15.62 \angle -98.66^{\circ} - 39.06 \angle -98.66^{\circ}$$
  

$$= (-2.35 - j15.44) - (14.21 - j36.38) = (-16.56 + j20.94)$$
  

$$= 26.7 \angle 128.33^{\circ}$$

These line currents are also equal to the line (phase) currents of the original star connected load.

(ii) Voltage drop across each star-connected load will be as under.

Voltage across 
$$Z_R = I_1 \times Z_R$$
  
= (27.06  $\angle - 8.671^\circ$ ) (5  $\angle 0^\circ$ ) = 135.3  $\angle - 8.67^\circ$   
Voltage across  $Z_Y = I_2 \times Z_Y$   
= (19.7  $\angle 238.85^\circ$ ) (2  $\angle 90^\circ$ ) = 39.4  $\angle 328.85^\circ$   
Voltage across  $Z_B = I_3 \times Z_B$   
= (26.7  $\angle 128.33^\circ$ ) (4  $\angle - 90^\circ$ ) = 106.8°  $\angle 38.33^\circ$ 

(b) BY Applying Millman's Theorem
 Consider Fig. 9.64(c), taking V<sub>RY</sub> as reference line voltage = 100 ∠0°.



Fig. 9.64 (c)

Phase voltages log  $30^{\circ}$  behind their respective line voltages. Therefore, the three phase voltages are

$$V_{RO} = \frac{100}{\sqrt{3}} \ \angle -30^{\circ}$$
$$V_{YO} = \frac{100}{\sqrt{3}} \ \angle -150^{\circ}$$
$$V_{BO} = \frac{100}{\sqrt{3}} \ \angle -270^{\circ}$$
$$Y_R = \frac{1}{Z_R} = \frac{1}{5 \ \angle 0^{\circ}} = 0.2 \ \angle 0^{\circ}$$

$$\begin{split} Y_Y &= \frac{1}{Z_Y} = \frac{1}{2\ \angle 90^\circ} = 0.5\ \angle -90^\circ \\ Y_B &= \frac{1}{Z_B} = \frac{1}{4\ \angle -90^\circ} = 0.25\ \angle 90^\circ \\ V_{RO}Y_R &+ V_{YO}Y_Y + Y_{BO}Y_B = (57.73\ \angle -30^\circ)\ (0.2\ \angle 0^\circ) \\ &+ (57.73\ \angle -150^\circ)\ (0.5\ \angle -90^\circ) \\ &+ (57.73\ \angle -270^\circ)\ (0.25\ \angle 90^\circ) \\ &= 11.54\ \angle -30^\circ + 28.86\ \angle -240^\circ + 14.43\ \angle -180^\circ \\ &= (10\ -j5.77)\ + (-14.43\ +j25)\ + (-14.43\ +j0) \\ &= -18.86\ +j19.23\ = 26.93\ \angle 134.44^\circ \\ Y_R + Y_Y + Y_B &= 0.2\ + 0.5\ \angle -90^\circ + 0.25\ \angle 90^\circ \\ &= 0.32\ \angle -51.34^\circ \\ V_{O'O} &= \frac{V_{RO}\ Y_R + V_{YO}\ Y_Y + V_{BO}\ Y_B}{Y_R + Y_Y + Y_B} = \frac{26.93\ \angle 134.44^\circ}{0.32\ \angle -51.34^\circ} \\ &= 84.15\ \angle 185.78^\circ \end{split}$$

The three load phase voltages are

$$\begin{split} V_{RO'} &= V_{RO} - V_{O'O} \\ &= 57.73 \ \angle -30^{\circ} - 84.15 \ \angle 185.78^{\circ} \\ &= (50 - j28.86) - (-83.72 - j8.47) \\ &= (133.72 - j20.4) = 135.26 \ \angle -8.67^{\circ} \\ V_{YO'} &= V_{YO} - V_{O'O} \\ &= 57.73 \ \angle -150^{\circ} - 84.15 \ \angle 185.78^{\circ} \\ &= (-50 - j28.86) - (-83.72 - j8.47) \\ &= 33.72 - j20.4 = 39.4 \ \angle -31.17^{\circ} \text{ or } 39.4 \ \angle 328.8^{\circ} \\ V_{BO'} &= V_{BO} - V_{O'O} \\ &= 57.73 \ \angle -270^{\circ} - 84.15 \ \angle 185.78^{\circ} \\ &= 0 + j57.73 + 83.72 + j8.47 \\ &= 83.72 + j66.2 = 106.73 \ \angle 38.33^{\circ} \\ I_R &= \frac{135.26 \ \angle -8.67^{\circ}}{5 \ \angle 0^{\circ}} = 20.06 \ \angle -8.67^{\circ} \\ I_Y &= \frac{39.4 \ \angle 328.80^{\circ}}{2 \ \angle 90^{\circ}} = 19.7 \ \angle 238.8^{\circ} \\ I_B &= \frac{106.73 \ \angle 38.33^{\circ}}{4 \ \angle -90^{\circ}} \\ &= 26.68 \ \angle 128.33^{\circ} \end{split}$$

**Problem 9.13** A three phase three-wire unbalanced load is star-connected. The phase voltages of two of the arms are

$$V_R = 100 \ \angle -10^\circ; V_Y = 150 \ \angle 100^\circ$$

Calculate voltage between star point of the load and the supply neutral.

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Fig. 9.65

Subtracting Eq. (9.24) from Eq. (9.21), we get O = [(V + jO) - (98.48 - j17.36)] - [(0.5V + j0.866V) - (-26.04 + j147.72)] O = 1.5V - j0.8666V - 124.52 + j165.08 = V (1.5 - j0.866) = 124.52 - j165.08  $V = \frac{124.52 - j165.08^{\circ}}{1.5 - j0.866} = \frac{206.77 \angle -52.97^{\circ}}{1.732 \angle -30^{\circ}}$   $V = 119.38 \angle -22.97^{\circ}$ Voltage between  $O'O = V_{RO} - V_{RO'}$  $V_{O'O} = 119.38 \angle -22.97^{\circ} - 100 \angle -10^{\circ}$  = 109.91 - j46.58 - 98.48 + j17.36  $= 11.43 - j29.22 = 31.37 \angle -68.63^{\circ}$ 

**Problem 9.14** Find the reading of a wattmeter in the circuit shown in Fig. 9.66(a). Assume a symmetrical 400 V supply with RYB phase sequence and draw the vector diagram.



#### Fig. 9.66(a)

Solution The reading in the wattmeter is equal to the product of the current through the current coil  $I_1$  voltage across its pressure coil  $V_{YB}$  and cos of the angle between the  $V_{YB}$  and  $I_1$ .

$$I_R = \frac{V_{RY}}{-j50} = \frac{400 \ \angle 0^\circ}{50 \ \angle -90^\circ} = 8 \ \angle 90^\circ$$
$$I_B = \frac{V_{BR}}{30 + j40} = \frac{400 \ \angle -240^\circ}{50 \ \angle 53.13^\circ} = 8 \ \angle -293.13^\circ \text{ or } 8 \ \angle 66.87^\circ$$

Line current

$$I_{1} = I_{R} - I_{B}$$

$$= 8 \angle 90^{\circ} - 8 \angle -293.13^{\circ}$$

$$= 0 + j8 - 3.14 - j7.35 = -3.14 + j0.65 = 3.2 \angle 168.3^{\circ}$$

$$V_{BR}$$

$$I_{R}$$

$$I_{B}$$

$$I_{R}$$

$$I_{B}$$

$$I_{R}$$

Fig. 9.66(b)

From the vector diagram in Fig. 9.66(b), it is clear that the angle between  $V_{YB}$  and  $I_1$  is 71.7°.

:. Wattmeter reading is equal to  $V_{YB} \times I_1 \cos 71.7^\circ$ = 400 × 3.2 × cos 71.7 = 402 W

. . . 4

0.0

**Problem 9.15** Calculate the total power input and readings of the two wattmeters connected to measure power in a three-phase balanced load, if the reactive power input is 15 KVAR, and the load pf is 0.8.

**Solution** Let  $W_1$  be the lower reading wattmeter and  $W_2$  the higher reading wattmeter

$$cos \phi = 0.8$$
  

$$\phi = 36.86^{\circ}$$
  

$$tan \phi = \frac{\text{Reactive power}}{\text{Active power}}$$
  

$$= \sqrt{3} \frac{W_2 - W_1}{W_2 + W_1}$$
  

$$power = \sqrt{3} (W_2 - W_1) = 15000$$
  

$$= W_2 - W_1 = 8660.508 \text{ W}$$
(9.23)  

$$0.75 = \sqrt{3} \frac{15000}{W_2 + W_1}$$
  
with  $W_1 + W_2 = 24641 \text{ or } W_2$   
(0.24)

\*\*

or

*:*..

Reactive

or Total power input  $W_2 + W_1 = 34641.01$  W (9.24) From Eqs. (9.23) and (9.24) we get  $W_2 = 21650.76$  W

$$V_1 = 12990.24 \text{ W}$$

**Problem 9.16** Two wattmeters are connected to measure power in a threephase circuit. The reading of the one of the meter is 5 kW when the load power factor is unity. If the power factor of the load is changed to 0.707 lagging, without changing the total input power, calculate the readings of the two wattmeters.

**Solution** Both wattmeters indicate equal values when the power factor is unity

$$W_1 + W_2 = 10 \text{ kW} \text{ (Total power input)}$$
 (9.25)

Let  $W_2$  be the higher reading wattmeter Then  $W_1$  is the lower reading wattmeter

$$\cos \phi = 0.707 \quad \therefore \phi = 45^{\circ}$$
$$\tan \phi = \sqrt{3} \ \frac{W_2 - W_1}{W_2 + W_1} \ 1 = \sqrt{3} \ \frac{W_2 - W_1}{10}$$
$$W_2 - W_1 = \frac{10}{\sqrt{3}} = 5.773 \text{ kW}$$
(9.26)

...

From Eqs. (9.25) and (9.26),

$$W_2 = 7.886 \text{ kW}$$
  
 $W_1 = 2.113 \text{ kW}$ 

**Problem 9.17** The line currents in a balanced six-phase mesh connected generator are 35.35 A. What is the magnitude of the phase current?

Solution From Section 9.8.5

$$I_L = 2I_{Ph} \sin \frac{180^{\circ}}{n}$$
$$I_{Ph} = \frac{35.35}{2 \sin \frac{180}{6}} = 35.35$$

**Problem 9.18** Find the voltage between the adjacent lines of a balanced sixphase star-connected system with a phase voltage of 132.8 volts.

Solution From Section 9.7.5  $E_L = 2E_{Ph} \sin \frac{180^\circ}{n}$  $E_L = 2 \times 132.8 \times \sin \frac{180^\circ}{6} = 132.8 \text{ V}$ 

**Problem 9.19** In the wye connected system shown in Fig. 9.67, it is assumed that only fundamental and third harmonic voltages are present when the voltages are measured with a voltmeter between *na* and *ba*. They are given by 230 and 340 volts respectively. Calculate the magnitude of the third harmonic voltages in the system.



**Solution** Only phase voltage  $V_{na}$  of the system shown in Fig. 9.67 contains 3rd harmonic whereas line voltage  $V_{ba}$  contains only 1st harmonic. Hence,

Fundamental component of the phase =  $\frac{340}{\sqrt{3}}$ 

9.77

Third harmonic component = 
$$\sqrt{220^2 - \left(\frac{340}{\sqrt{3}}\right)^2} = 99.33 \text{ V}$$

**Problem 9.20** Illustrate the effect of reversal of voltage sequence up on the magnitudes of the currents in the system shown in Example 9.20.

**Solution** The line currents for RYB sequence have already been calculated.  $I_1 = 28.41 \angle -69.07^\circ$ ,  $I_2 = 29.85 \angle 136.58^\circ$  and  $I_3 = 13 \angle 27.60^\circ$  A.

If the phase sequence is reversed by RBY then

$$I_R = \frac{V_{RY}}{Z_{RY}} = \frac{400 \angle 0^{\circ}}{15.67 \angle 60.13^{\circ}} = 25.52 \angle -60.13^{\circ} \text{ A}$$
$$I_Y = \frac{V_{YB}}{Z_{YB}} = \frac{400 \angle -240^{\circ}}{43.83 \angle 49.83^{\circ}} = 9.12 \angle -289.83^{\circ} \text{ A}$$
$$I_B = \frac{V_{BR}}{Z_{BR}} = \frac{400 \angle -120^{\circ}}{78.36 \angle 60.13^{\circ}} = 5.1 \angle -180.13^{\circ} \text{ A}$$

Various line currents are given by

$$I_1 = I_R - I_B = 25.52 \angle -60.13^\circ - 5.1 \angle -180.13^\circ = 28.41 \angle -51.189^\circ \text{ A}$$
  

$$I_2 = I_Y - I_R = 9.12 \angle -289.83^\circ -25.52 \angle -60.13^\circ = 32.175 \angle 107.37^\circ \text{ A}$$
  

$$I_3 = I_B - I_Y = 5.1 \angle -180.13^\circ - 9.12 \angle -289.83^\circ = 11.85 \angle 46.26^\circ \text{ A}$$

From the above calculations, it can be verified that the magnitudes of the line currents are not same when the phase sequence is changed.



- 9.1 Three non-reactive resistors of 5, 10 and 15  $\Omega$  are star-connected to *R*, *Y* and *B* phase of a 440 V symmetrical system. Determine the current and power in each resistor and the voltage between star point and neutral; assume the phase sequence RYB.
- 9.2 A three-phase, three-wire symmetrical 440 V source is supplying power to an unbalanced, delta-connected load in which  $Z_{RY} = 20 \angle 30^{\circ} \Omega$ ,  $Z_{YB} = 20 \angle 0^{\circ} \Omega$  and  $Z_{BR} = 20 \angle - 30^{\circ} \Omega$ . If the phase sequence is RYB, calculate the line currents.
- 9.3 Three equal resistances connected in star across a three-phase balanced supply consume 1000 W. If the same three resistors were reconnected in delta across the same supply, determine the power consumed.
- 9.4 The currents in  $R_Y$ ,  $Y_B$  and  $B_R$  branches of a mesh connected system with symmetrical voltages are 20 A at 0.7 lagging power factor, 20 A at 0.8 leading power factor, and 10 A at UPF respectively. Determine the current in each line. Phase sequence is RYB. Draw a phasor diagram.

- 9.5 A three-phase, four-wire symmetrical 440 V; RYB system supplies a starconnected load in which  $Z_R = 10 \angle 0^\circ \Omega$ ,  $Z_Y = 10 \angle 26.8^\circ \Omega$  and  $Z_B = 10 \angle -26.8^\circ \Omega$ . Find the line currents, the neutral current and the load power.
- 9.6 A balanced three-phase, star-connected voltage source has  $V_{RN} = 230 \ \angle 60^{\circ} \Omega V_{rms}$  with *RYB* phase sequence, and it supplies a balanced deltaconnected three-phase load. The total power drawn by the load is 15 kW at 0.8 lagging power factor. Find the line currents, load and phase currents.
- 9.7 Three identical impedances  $10 \angle 30^{\circ} \Omega$  in a delta-connection, and three identical impedances  $5 \angle 35^{\circ} \Omega$  in a star-connection are on the same three-phase, three-wire 173 V system. Find the line currents and the total power.
- 9.8 Three impedances of  $(7 + j4) \Omega$ ;  $(3 + j2) \Omega$  and  $(9 + j2) \Omega$  are connected between neutral and the red, yellow and blue phases, respectively of a three-phase, four-wire system; the line voltage is 440 V. Calculate (i) the current in each line, and (ii) the current in the neutral wire.
- 9.9 Three capacitors, each of  $100 \,\mu\text{F}$  are connected in delta to a 440 V, threephase, 50 Hz supply. What will be the capacitance of each of the three capacitors if the same three capacitors are connected in star across the same supply to draw the same line current.
- 9.10 A 400 V, three-phase supply feeds an unbalanced three-wire, star-connected load, consisting of impedances  $Z_R = 7 \angle 10^\circ \Omega$ ,  $Z_Y = 8 \angle 30^\circ \Omega$  and  $Z_B = 8 \angle 50^\circ \Omega$ . The phase sequence is RYB. Determine the line currents and total power taken by the load.
- 9.11 The power taken by a 440 V, 50 Hz, three-phase induction motor on full load is measured by two wattmeters, which indicate 250 W and 1000 W, respectively. Calculate (i) the input (ii) the power factor (iii) the current, and (iv) the motor output, if the efficiency is 80%.
- 9.12 In the two wattmeter method of power measurement, the power registered by one wattmeter is 3500 W, while the other reads down scale. After reversing the later, it reads 300 W. Determine the total power in the circuit and the power factor.
- 9.13 Three non-inductive resistances of 25  $\Omega$ , 10  $\Omega$  and 15  $\Omega$  are connected in star to a 400 V symmetrical supply. Calculate the line currents and the voltage across the each load phase.
- 9.14 Three impedances,  $Z_R = (3 + j2) \Omega$ ;  $Z_Y = j9 \Omega$  and  $Z_B = 3 \Omega$  are connected in star across a 400 V, 3-wire system. Find the loads on the equivalent delta-connected system phase-sequence RYB.
- **OBJECTIVE-TYPE QUESTIONS** 
  - The resultant voltage in a closed balanced delta circuit is given by

     (a) three times the phase voltage

(b)  $\sqrt{3}$  times the phase voltage

- (c) zero
- 2. Three coils A, B, C, displaced by 120° from each other are mounted on the same axis and rotated in a uniform magnetic field in clockwise direction. If the instantaneous value of emf in coil A is  $E_{\text{max}} \sin \omega t$ , the instantaneous value of emf in B and C coils will be

(a) 
$$E_{\max} \sin\left(\omega t - \frac{2\pi}{3}\right); E_{\max} \sin\left(\omega t - \frac{4\pi}{3}\right)$$
  
(b)  $E_{\max} \sin\left(\omega t + \frac{2\pi}{3}\right); E_{\max} \sin\left(\omega t + \frac{4\pi}{3}\right)$   
(c)  $E_{\max} \sin\left(\omega t - \frac{2\pi}{3}\right); E_{\max} \sin\left(\omega t + \frac{4\pi}{3}\right)$ 

- 3. The current in the neutral wire of a balanced three-phase, four-wire star connected load is given by
  - (a) zero
  - (b)  $\sqrt{3}$  times the current in each phase
  - (c) 3 times the current in each phase
- 4. In a three-phase system, the volt ampere rating is given by
  - (a)  $3V_L I_L$
  - (b)  $\sqrt{3} V_L I_L$
  - (c)  $V_L I_L^{-}$
- 5. In a three-phase balanced star connected system, the phase relation between the line voltages and their respective phase voltage is given as under
  - (a) the line voltages lead their respective phase voltages by  $30^{\circ}$ .
  - (b) the phase voltages lead their respective line voltage by  $30^{\circ}$ .
  - (c) the line voltages and their respective phase voltages are in phase.
- 6. In a three-phase balanced delta connected system, the phase relation between the line currents and their respective phase currents is given by
  - (a) the line currents lag behind their respective phase currents by  $30^{\circ}$ .
  - (b) the phase currents lag behind their respective line currents by  $30^{\circ}$ .
  - (c) the line currents and their respective phase currents are in phase.
- 7. In a three-phase unbalanced, four-wire star-connected system, the current in the neutral wire is given by
  - (a) zero
  - (b) three times the current in individual phases
  - (c) the vector sum of the currents in the three lines
- 8. In a three-phase unbalanced star-connected system, the vector sum of the currents in the three lines is
  - (a) zero
  - (b) not zero
  - (c) three times the current in the each phase

- 9. Wattmeter deflection in ac circuit is proportional to the
  - (a) maximum power in the circuit
  - (b) instantaneous power in the circuit
  - (c) average power in the circuit
- 10. Three wattmeter method of power measurement can be used to measure power in
  - (a) balanced circuits

- (b) unbalanced circuits
- (c) both balanced and unbalanced circuits
- 11. Two wattmeter method of power measurement can be used to measure power in
  - (a) balanced circuits
  - (b) unbalanced circuits
  - (c) both balanced and unbalanced circuits
- 12. In two wattmeter methods of power measurements, when the pf is 0.5
  - (a) the readings of the two wattmeters are equal and positive
  - (b) the readings of the two wattmeters are equal and opposite
  - (c) the total power is measured by only one wattmeter
- 13. The reading of the wattmeter connected to measure the reactive power in a three phase circuit is given by zero, the line voltage is 400 V and line current 15 A; then the pf of the circuit is
  - (a) zero
  - (b) unity
  - (c) 0.8



# **10.1 INTRODUCTION**

Two circuits are said to be 'coupled' when energy transfer takes place from one circuit to the other when one of the circuits is energised. There are many types of couplings like conductive coupling as shown by the potential divider in Fig. 10.1(a), inductive or magnetic coupling as shown by a two winding transformer in Fig. 10.1(b) or conductive and inductive coupling as shown by an auto transformer in Fig. 10.1(c). A majority of the electrical circuits in practice are conductively or electromagnetically coupled. Certain coupled elements are frequently used in network analysis and synthesis. Transformer, transistor and electronic pots, etc. are some among these circuits. Each of these elements may be represented as a two port network as shown in Fig. 10.1(d).



Fig. 10.1

# 10.2 CONDUCTIVITY COUPLED CIRCUIT AND MUTUAL IMPEDANCE

A conductively coupled circuit which does not involve magnetic coupling is shown in Fig. 10.2(a).

In the circuit shown the impedance  $Z_{12}$  or  $Z_{21}$  common to loop 1 and loop 2 is called *mutual impedance*. It may consists of a pure resistance, a pure inductance, a pure capacitance or a combination of any of these elements. Mesh analysis, nodal analysis or Kirchhoff's laws can be used to solve these type of circuits as described in Chapter 7.

The general definition of mutual impedance is explained with the help of Fig. 10.2 (b).



Fig. 10.2 (a)



The network in the box may be of any configuration of circuit elements with two ports having two pairs of terminals 1-1' and 2-2' available for measurement. The mutual impedance between port 1 and 2 can be measured at 1-1' or 2-2'. If it is measured at 2-2'. It can be defined as the voltage developed  $(V_2)$  at 2–2' per unit current  $(I_1)$  at port 1-1'. If the box contains linear bilateral elements, then the mutual impedance measured at 2-2' is same as the impedance measured at 1-1' and is defined as the voltage developed  $(V_1)$  at 1-1' per unit current  $(I_2)$  at port 2-2'.

**Example 10.1** Find the mutual impedance for the circuit shown in Fig. 10.3.

**Solution** Mutual impedance is given by

$$\frac{V_2}{I_1} \text{ or } \frac{V_1}{I_2}$$

$$V_2 = \frac{3}{2} I_1 \text{ or } \frac{V_2}{I_1} = 1.5 \Omega$$

$$V_1 = 5 \times I_2 \times \frac{3}{10} \text{ or } \frac{V_2}{I_2} = 1.5 \Omega$$
Fig. 10.3
Fig. 10.3

1

2Ω

2

or

### **10.3 MUTUAL INDUCTANCE**

The property of inductance of a coil was introduced in Section 1.6. A voltage is induced in a coil when there is a time rate of change of current through it. The

inductance parameter L, is defined in terms of the voltage across it and the time rate of change of current through it  $v(t) = L \frac{di(t)}{dt}$ , where v(t) is the voltage across the coil, I(t) is the current through the coil and L is the inductance of the coil. Strictly speaking, this definition is of self-inductance and this is considered as a circuit element with a pair of terminals. Whereas a circuit element "mutual inductor" does not exist. Mutual inductance is a property associated with two or more coils or inductors which are in close proximity and the presence of common magnetic flux which links the coils. A transformer is such a device whose operation is based on mutual inductance.

Let us consider two coils,  $L_1$  and  $L_2$  as shown in Fig. 10.4(a), which are sufficiently close together, so that the flux produced by  $i_1$  in coil  $L_1$  also link coil  $L_2$ . We assume that the coils do not move with respect to one another, and the medium in which the flux is established has a constant permeability. The two coils may be also arranged on a common magnetic core, as shown in Fig. 10.4(b). The two coils are said to be magnetically coupled, but act as a separate circuits. It is possible to relate the voltage induced in one coil to the time rate of change of current in the other coil. When a voltage  $v_1$  is applied across  $L_1$ , a current  $i_1$  will start flowing in this coil, and produce a flux  $\phi$ . This flux also links coil  $L_2$ . If  $i_1$  were to change with respect to time, the flux ' $\phi$ ' would also change with respect to time. The time-varying flux surrounding the second coil,  $L_2$  induces an emf, or voltage, across the terminals of  $L_2$ ; this voltage is proportional to the time rate of change of current flowing through the first coil  $L_1$ . The two coils, or circuits, are said to be inductively coupled, because of this property they are called coupled elements or coupled circuits and the induced voltage, or emf is called the voltage/emf of mutual induction and is given by

 $v_2(t) = M_1 \frac{di_1(t)}{dt}$  volts, where  $v_2$  is the voltage induced in coil  $L_2$  and  $M_1$  is the coefficient of proportionality, and is called the coefficient of mutual inductance

coefficient of proportionality, and is called the coefficient of mutual inductance, or simple mutual inductance.





If current  $i_2$  is made to pass through coil  $L_2$  as shown in Fig. 10.4(c) with coil  $L_1$  open, a change of  $i_2$  would cause a voltage  $v_1$  in coil  $L_1$ , given by  $v_1(t) = M_2 \frac{di_2(t)}{dt}$ .



Fig. 10.4 (c)

In the above equation, another coefficient of proportionality  $M_2$  is involved. Though it appears that two mutual inductances are involved in determining the mutually induced voltages in the two coils, it can be shown from energy considerations that the two coefficients are equal and, therefore, need not be represented by two different letters. Thus  $M_1 = M_2 = M$ .

$$\therefore \qquad v_2(t) = M \frac{di_1(t)}{dt} \text{ Volts}$$
$$v_1(t) = M \frac{di_2(t)}{dt} \text{ Volts}$$

In general, in a pair of linear time invariant coupled coils or inductors, a non-zero current in each of the two coils produces a mutual voltage in each coil due to the flow of current in the other coil. This mutual voltage is present independently of, and in addition to, the voltage due to self induction. Mutual inductance is also

measured in Henrys and is positive, but the mutually induced voltage,  $M \frac{di}{dt}$  may

be either positive or negative, depending on the physical construction of the coil and reference directions. To determine the polarity of the mutually induced voltage (i.e. the sign to be used for the mutual inductance), the dot convention is used.

## **10.4 DOT CONVENTION**

Dot convention is used to establish the choice of correct sign for the mutually induced voltages in coupled circuits.

Circular dot marks and/or special symbols are placed at one end of each of two coils which are mutually coupled to simplify the diagrammatic representation of the windings around its core.

Let us consider Fig. 10.5, which shows a pair of linear, time invariant, coupled inductors with self inductances  $L_1$  and  $L_2$  and a mutual inductance M. If these inductions form a portion of a network, currents  $i_1$  and  $i_2$  are shown, each arbitrarily assumed entering at the dotted terminals, and voltages  $v_1$  and  $v_2$  are developed across the inductors. The voltage across  $L_1$  is, thus composed of two parts and is given by



$$v_1(t) = L_1 \frac{di_1(t)}{dt} \pm M \frac{di_2(t)}{dt}$$

The first term on the RHS of the above equation is the self induced voltage due to  $i_1$ , and the second term represents the mutually induced voltage due to  $i_2$ .

Similarly, 
$$v_2(t) = L_2 \frac{di_2(t)}{dt} \pm M \frac{di_1(t)}{dt}$$

Although the self-induced voltages are designated with positive sign, mutually induced voltages can be either positive or negative depending on the direction of the winding of the coil and can be decided by the presence of the *dots* placed at one end of each of the two coils. The convention is as follows.

If two terminals belonging to different coils in a coupled circuit are marked identically with dots then for the same direction of current relative to like terminals, the magnetic flux of self and mutual induction in each coil add together. The physical basis of the dot convention can be verified by examining Fig. 10.6. Two coils *ab* and *cd* are shown wound on a common iron core.



Fig. 10.6

It is evident from Fig.10.6 that the direction of the winding of the coil *ab* is clock-wise around the core as viewed at *X*, and that of *cd* is anti-clockwise as viewed at *Y*. Let the direction of current  $i_1$  in the first coil be from *a* to *b*, and increasing with time. The flux produced by  $i_1$  in the core has a direction which may be found by right hand rule, and which is downwards in the left limb of the core. The flux also increases with time in the direction shown at *X*. Now suppose that the current  $i_2$  in the second coil is from *c* to *d*, and increasing with

time. The application of the right hand rule indicates that the flux produced by  $i_2$  in the core has an upward direction in the right limb of the core. The flux also increases with time in the direction shown at Y. The assumed currents  $i_1$  and  $i_2$  produce flux in the core that are additive. The terminals a and c of the two coils attain similar polarities simultaneously. The two simultaneously positive terminals are identified by two dots by the side of the terminals as shown in Fig. 10.7.

Fig. 10.7

The other possible location of the dots is the other ends of the coil to get additive fluxes in the core, i.e. at *b* and *d*. It can be concluded that the mutually induced voltage is positive when currents  $i_1$  and  $i_2$  both enter (or leave) the windings by the dotted terminals. If the current in one winding enters at the dot-marked terminals and the current in the other winding leaves at the dot-marked terminal, the voltages due to self and mutual induction in any coil have opposite signs.

**Example 10.2** Using dot convention, write voltage equations for the coils shown in Fig. 10.8.



Fig. 10.8

**Solution** Since the currents are entering at the dot marked terminals the mutually induced voltages or the sign of the mutual inductance is positive; using the sign convention for the self inductance, the equations for the voltages are

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$
$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

**Example 10.3** Write the equation for voltage  $v_0$  for the circuits shown in Fig. 10.9.

**Solution**  $v_0$  is assumed positive with respect to terminal *C* and the equation is given by







(a) 
$$v_0 = M \frac{di}{dt}$$
  
(b)  $v_0 = -M \frac{di}{dt}$   
(c)  $v_0 = -M \frac{di}{dt}$   
(d)  $v_0 = M \frac{di}{dt}$ 

### **10.5 COEFFICIENT OF COUPLING**

The amount of coupling between the inductively coupled coils is expressed in terms of the coefficient of coupling, which is defined as  $K = M/\sqrt{L_1L_2}$ 

where M = mutual inductance between the coils

 $L_1$  = self inductance of the first coil, and

 $L_2$  = self inductance of the second coil

Coefficient of coupling is always less than unity, and has a maximum value of 1 (or 100%). This case, for which K = 1, is called perfect coupling, when the entire flux of one coil links the other. The greater the coefficient of coupling between the two coils, the greater the mutual inductance between them, and vice-versa. It can be expressed as the fraction of the magnetic flux produced by the current in one coil that links the other coil.

For a pair of mutually coupled circuits shown in Fig. 10.10, let us assume initially that  $i_1$ ,  $i_2$  are zero at t = 0.



Fig. 10.10

then

$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$

 $v_2(t) = L_2 \frac{di_2(t)}{dt} + M \frac{di_1(t)}{dt}$ 

and

Initial energy in the coupled circuit at t = 0 is also zero. The net energy input to the system shown in Fig. 10.10 at time t is given by

$$W(t) = \int_{0}^{t} \left[ v_{1}(t) \, i_{1}(t) + v_{2}(t) \, i_{2}(t) \right] dt$$

Substituting the values of  $v_1(t)$  and  $v_2(t)$  in the above equation yields

$$W(t) = \int_{0}^{t} \left[ L_{1}i_{1}(t) \frac{di_{1}(t)}{dt} + L_{2}i_{2}(t) \frac{di_{2}(t)}{dt} + M(i_{1}(t)) \frac{di_{2}(t)}{dt} + i_{2}(t) \frac{di_{1}(t)}{dt} \right] dt$$

From which we get

$$W(t) = \frac{1}{2} L_1[i_1(t)]^2 + \frac{1}{2} L_2[i_2(t)]^2 + M[i_1(t)i_2(t)]$$

If one current enters a dot-marked terminal while the other leaves a dot marked terminal, the above equation becomes

$$W(t) = \frac{1}{2} L_1[i_1(t)]^2 + \frac{1}{2} L_2[i_2(t)]^2 - M[i_1(t)i_2(t)]$$

According to the definition of passivity, the net electrical energy input to the system is non-negative. W(t) represents the energy stored within a passive network, it cannot be negative.

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 $W(t) \ge 0$  for all values of  $i_1, i_2; L_1, L_2$  or M

The statement can be proved in the following way. If  $i_1$  and  $i_2$  are both positive or negative, W(t) is positive. The other condition where the energy equation could be negative is

$$W(t) = \frac{1}{2} L_1[i_1(t)]^2 + \frac{1}{2} L_2[i_2(t)]^2 - M[i_1(t)i_2(t)]$$

The above equation can be rearranged as

$$W(t) = \frac{1}{2} \left( \sqrt{L_1} i_1 - \frac{M}{\sqrt{L_1}} i_2 \right)^2 + \frac{1}{2} \left( L_2 - \frac{M^2}{L_1} \right) i_2^2$$

The first term in the parenthesis of the right side of the above equation is positive for all values of  $i_1$  and  $i_2$ , and, thus, the last term cannot be negative; hence

$$L_2 - \frac{M^2}{L_1} \ge 0$$

 $\frac{L_1 L_2 - M^2}{L_1} \ge 0$  $L_1 L_2 - M^2 \ge 0$  $\sqrt{L_1 L_2} \ge M$  $M \le \sqrt{L_1 L_2}$ 

Obviously the maximum value of the mutual inductance is  $\sqrt{L_1 L_2}$ . Thus, we define the coefficient of coupling for the coupled circuit as

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

The coefficient, K, is a non negative number and is independent of the reference directions of the currents in the coils. If the two coils are a great distance apart in space, the mutual inductance is very small, and K is also very small. For iron-core coupled circuits, the value of K may be as high as 0.99, for air-core coupled circuits, K varies between 0.4 to 0.8.

**Example 10.4** Two inductively coupled coils have self inductances  $L_1 = 50$  mH and  $L_2 = 200$  mH. If the coefficient of coupling is 0.5 (i), find the value of mutual inductance between the coils, and (ii) what is the maximum possible mutual inductance?

**Solution** (i)  $M = K \sqrt{L_1 L_2}$ 

 $= 0.5 \sqrt{50 \times 10^{-3} \times 200 \times 10^{-3}} = 50 \times 10^{-3} \text{ H}$ 

(ii) Maximum value of the inductance when K = 1,

$$M = \sqrt{L_1 L_2} = 100 \text{ mH}$$

### **10.6 IDEAL TRANSFORMER**

Transfer of energy from one circuit to another circuit through mutual induction is widely utilised in power systems. This purpose is served by transformers. Most often, they transform energy at one voltage (or current) into energy at some other voltage (or current).

A transformer is a static piece of apparatus, having two or more windings or coils arranged on a common magnetic core. The transformer winding to which the supply source is connected is called the *primary*, while the winding connected to load is called the *secondary*. Accordingly, the voltage across the primary is called the primary voltage, and that across the secondary, the secondary voltage. Correspondingly  $i_1$  and  $i_2$  are the currents in the primary and secondary windings. One such transformer is shown in Fig. 10.11(a). In circuit diagrams, ideal transformers are represented by Fig. 10.11(b). The vertical lines between the coils represent the iron core; the currents are assumed such that the

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mutual inductance is positive. An ideal transformer is characterised by assuming (i) zero power dissipation in the primary and secondary windings, i.e. resistances in the coils are assumed to be zero, (ii) the self inductances of the primary and secondary are extremely large in comparison with the load impedance, and (iii) the coefficient of coupling is equal to unity, i.e. the coils are tightly coupled without having any leakage flux. If the flux produced by the current flowing in a coil links all the turns, the self inductance of either the primary or secondary coil is proportional to the square of the number of turns of the coil. This can be verified from the following results.



Fig. 10.11

The magnitude of the self induced emf is given by

$$v = L \frac{di}{dt}$$

If the flux linkages of the coil with N turns and current are known, then the self induced emf can be expressed as

$$v = N \frac{d\phi}{dt}$$

$$L \frac{di}{dt} = N \frac{d\phi}{dt}$$

$$L = N \frac{d\phi}{di}$$

$$\phi = \frac{Ni}{\text{reluctance}}$$

$$L = N \frac{d}{di} \left(\frac{Ni}{\text{reluctance}}\right)$$

$$L = \frac{N^2}{\text{reluctance}}$$

$$L = \frac{N^2}{\text{reluctance}}$$

But

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From the above relation it follows that

$$\frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = a^2$$

where  $a = N_2/N_1$  is called the *turns ratio* of the transformer. The turns ratio, *a*, can also be expressed in terms of primary and secondary voltages. If the magnetic permeability of the core is infinitely large then the flux would be confined to the core. If  $\phi$  is the flux through a single turn coil on the core and  $N_1$ ,  $N_2$  are the number of turns of the primary and secondary, respectively, then the total flux through windings 1 and 2, respectively, are

Also we have  $v_{.} =$ 

$$v_1 = \frac{d\phi_1}{dt}$$
, and  $v_2 = \frac{d\phi_2}{dt}$ 

 $\phi_1 = N_1 \phi; \phi_2 = N_2 \phi$ 

so that

$$\frac{v_2}{v_1} = \frac{N_2 \frac{d\phi}{dt}}{N_1 \frac{d\phi}{dt}} = \frac{N_2}{N_1}$$

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Figure 10.12 shows an ideal transformer to which the secondary is connected to a

load impedance  $Z_L$ . The turns ratio  $\frac{N_2}{N_1} = a$ .

The ideal transformer is a very useful model for circuit calculations, because with few additional elements like R, L and C, the actual behaviour of the physical transformer can be accurately represented. Let us analyse this transformer with

sinusoidal excitations. When the excitations are sinusoidal voltages or currents, the steady state response will also be sinusoidal. We can use phasors for representing these voltages and currents. The input impedance of the transformer can be determined by writing mesh equations for the circuit shown in Fig. 10.12.

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 \tag{10.1}$$

$$0 = -j\omega M I_1 + (Z_L + j\omega L_2) I_2$$
(10.2)

where  $V_1$ ,  $V_2$  are the voltage phasors, and  $I_1$ ,  $I_2$  are the current phasors in the two windings.  $j\omega L_1$  and  $j\omega L_2$  are the impedances of the self inductances and  $j\omega M$  is the impedance of the mutual inductance,  $\omega$  is the angular frequency.

from Eq. 10.2 
$$I_2 = \frac{j\omega M I_1}{(Z_L + j\omega L_2)}$$

Substituting in Eq. 10.1, we have

$$V_1 = I_1 j \omega L_1 + \frac{I_1 \omega^2 M^2}{Z_L + j \omega L_2}$$





The input impedance  $Z_{in} = \frac{V_1}{I_1}$ 

$$\therefore \qquad \qquad Z_{\rm in} = j\omega L_1 + \frac{\omega^2 M^2}{(Z_L + j\omega L_2)}$$

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When the coefficient of coupling is assumed to be equal to unity,

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$$M = \sqrt{L_1 L_2}$$
  
$$\therefore \qquad Z_{in} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{(Z_L + j\omega L_2)}$$

We have already established that  $\frac{L_2}{L_1} = a^2$ 

where *a* is the turns ratio  $N_2/N_1$ 

$$\therefore \qquad \qquad Z_{\rm in} = j\omega L_1 + \frac{\omega^2 L_1^2 a^2}{(Z_L + j\omega L_2)}$$

Further simplication leads to

$$Z_{\rm in} = \frac{(Z_L + j\omega L_2) j\omega L_1 + \omega^2 L_1^2 a^2}{(Z_L + j\omega L_2)}$$
$$Z_{\rm in} = \frac{j\omega L_1 Z_L}{(Z_L + j\omega L_2)}$$

As  $L_2$  is assumed to be infinitely large compared to  $Z_L$ 

$$Z_{\rm in} = \frac{j\omega L_1 Z_L}{j\omega a^2 L_1} = \frac{Z_L}{a^2} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

The above result has an interesting interpretation, that is the ideal transformers change the impedance of a load, and can be used to match circuits with different impedances in order to achieve

maximum power transfer. For example, the input impedance of a loudspeaker is usually very small, say 3 to 12  $\Omega$ , for connecting directly to an amplifier. The transformer with proper turns ratio can be placed between the output of the amplifier and the input of the loudspeaker to match the impedances as shown in Fig. 10.13.



Fig. 10.13

**Example 10.5** An ideal transformer has  $N_1 = 10$  turns, and  $N_2 = 100$  turns. What is the value of the impedance referred to as the primary, if a 1000  $\Omega$  resistor is placed across the secondary?

**Solution** The turns ratio  $a = \frac{100}{10} = 10$  $Z_{in} = \frac{Z_L}{a^2} = \frac{1000}{100} = 10 \Omega$ 

The primary and secondary currents can also be expressed in terms of turns ratio. From Eq. 10.2, we have

$$I_1 jwM = I_2(Z_L + jwL_2)$$
$$\frac{I_1}{I_2} = \frac{Z_L + j\omega L_2}{j\omega M}$$

When  $L_2$  is very large compared to  $Z_L$ ,

$$\frac{I_1}{I_2} = \frac{j\omega L_2}{j\omega M} = \frac{L_2}{M}$$

Substituting the value of  $M = \sqrt{L_1 L_2}$  in the above equation  $\frac{I_1}{I_2} = \frac{L_2}{M}$ 

$$\frac{I_1}{I_2} = \frac{L_2}{\sqrt{L_1 L_2}} = \sqrt{\frac{L_2}{L_1}}$$
$$\frac{I_1}{I_2} = \sqrt{\frac{L_2}{L_1}} = a = \frac{N_2}{N_1}$$

**Example 10.6** An amplifier with an output impedance of 1936  $\Omega$  is to feed a loudspeaker with an impedance of 4  $\Omega$ .

- (a) Calculate the desired turns ratio for an ideal transformer to connect the two systems.
- (b) An rms current of 20 mA at 500 Hz is flowing in the primary. Calculate the rms value of current in the secondary at 500 Hz.
- (c) What is the power delivered to the load?

**Solution** (a) To have maximum power transfer the output impedance of the

amplifier = 
$$\frac{\text{Load impedance}}{a^2}$$
  
1936 =  $\frac{4}{a^2}$ 

÷

....

$$a^{2}$$

$$a = \sqrt{\frac{4}{1936}} = \frac{1}{22}$$

$$\frac{N_{2}}{N_{1}} = \frac{1}{22}$$

or

(b) 
$$I_1 = 20 \text{ mA}$$
  
We have  $\frac{I_1}{I_2} = a$ 

RMS value of the current in the secondary winding

$$= \frac{I_1}{a} = \frac{20 \times 10^{-3}}{1/22} = 0.44 \text{ A}$$

(c) The power delivered to the load (speaker)

 $= (0.44)^2 \times 4 = 0.774 \text{ W}$ 

The impedance changing properties of an ideal transformer may be utilised to simplify circuits. Using this property, we can transfer all the parameters of the primary side of the transformer to the secondary side, and *vice-versa*. Consider the coupled circuit shown in Fig. 10.14 (a).



Fig. 10.14

To transfer the secondary side load and voltage to the primary side, the secondary voltage is to be divided by the ratio, a, and the load impedance is to be divided by  $a^2$ . The simplified equivalent circuits is shown in Fig. 10.14 (b).

**Example 10.7** For the circuit shown in Fig. 10.15 with turns ratio, a = 5, draw the equivalent circuit referring (a) to primary and (b) secondary. Take source resistance as 10  $\Omega$ .



Fig. 10.15

**Solution** (a) Equivalent circuit referred to primary is as shown in Fig. 10.16(a).



Fig. 10.16 (a)

(b) Equivalent circuit referred to secondary is as shown in Fig. 10.16(b)



Fig. 10.16 (b)

\*\*

# 10.7 ANALYSIS OF MULTI-WINDING COUPLED CIRCUITS

Inductively coupled multi-mesh circuits can be analysed using Kirchhoff's laws and by loop current methods. Consider Fig. 10.17, where three coils are inductively coupled. For such a system of inductors we can define a inductance matrix L as

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

where  $L_{11}$ ,  $L_{22}$  and  $L_{33}$  are self inductances of the coupled circuits, and  $L_{12} = L_{21}$ ;  $L_{23} = L_{32}$  and  $L_{13} = L_{31}$  are mutual inductances. More precisely,  $L_{12}$  is the mutual inductance between coils 1 and 2,  $L_{13}$  is the mutual inductance between coils 1 and 3, and  $L_{23}$  is the mutual inductance between coils 2 and 3. The inductance matrix has its order equal to the number of inductors and is symmetric. In terms of voltages across the coils, we have a voltage vector related to *i* by


Fig. 10.17

$$[v] = \left[L\right] \left[\frac{di}{dt}\right]$$

where v and i are the vectors of the branch voltages and currents, respectively. Thus, the branch volt-ampere relationships of the three inductors are given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix}$$

Using KVL and KCL, the effective inductances can be calculated. The polarity for the inductances can be determined by using passivity criteria, whereas the signs of the mutual inductances can be determined by using the dot convention.

**Example 10.8** For the circuit shown in Fig. 10.18, write the inductance matrix.

**Solution** Let  $L_1$ ,  $L_2$  and  $L_3$  be the self inductances, and  $L_{12} = L_{21}$ ,  $L_{23} = L_{32}$  and  $L_{13} = L_{31}$  be the mutual inductances between coils, 1, 2, 2, 3 and 1, 3, respectively.

 $L_{12} = L_{21}$  is positive, as both the currents are entering at dot marked terminals, whereas  $L_{13} = L_{31}$  and  $L_{23} = L_{32}$  are negative.

:. The inductance matrix is 
$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$



Fig. 10.18

# 10.8 SERIES CONNECTION OF COUPLED INDUCTORS

Let there be two inductors connected in series, with self inductances  $L_1$  and  $L_2$  and mutual inductance of M. Two kinds of series connections are possible; series aiding as in Fig. 10.19(a), and series opposition as in Fig. 10.19(b).



Fig. 10.19

In the case of series aiding connection, the currents in both inductors at any instant of time are in the same direction relative to like terminals as shown in Fig. 10.19(a). For this reason, the magnetic fluxes of self induction and of mutual induction linking with each element add together.

In the case of series opposition connection, the currents in the two inductors at any instant of time are in opposite direction relative to like terminals as shown in Fig. 10.19(b). The inductance of an element is given by  $L = \phi/i$ , where  $\phi$  is the flux produced by the inductor.

For the series aiding circuit, if  $\phi_1$  and  $\phi_2$  are the flux produced by the coils 1 and 2, respectively, then the total flux

	$\phi=\phi_1+\phi_2$
where	$\phi_1 = L_1 i_1 + M i_2$
	$\phi_2 = L_2 i_2 + M i_1$
<b>.</b>	$\phi = Li = L_1i_1 + Mi_2 + L_2i_2 + Mi_1$
Since	$i_1 = i_2 = i$
	$L = L_1 + L_2 + 2M$

Similarly, for the series opposition

	$\phi = \phi_1 + \phi_2$
where	$\phi_1 = L_1 i_1 - M i_2$
	$\phi_2 = L_2 i_2 - M i_1$
	$\phi = Li = L_1 i_1 - M i_2 + L_2 i_2 - M i_1$
Since	$i_1 = i_2 = i$
	$L = L_1 + L_2 - 2M$

In general, the inductance of two inductively coupled elements in series is given by  $L = L_1 + L_2 \pm 2M$ .

Positive sign is applied to the series aiding connection, and negative sign to the series opposition connection.

**Example 10.9** Two coils connected in series have an equivalent inductance of 0.4 H when connected in aiding, and an equivalent inductance 0.2 H when the connection is opposing. Calculate the mutual inductance of the coils.

**Solution** When the coils are arranged in aiding connection, the inductance of the combination is  $L_1 + L_2 + 2M = 0.4$ ; and for opposing connection, it is  $L_1 + L_2 - 2M = 0.2$ . Solving the two equations, we get

$$4M = 0.2 \text{ H}$$
  
 $M = 0.05 \text{ H}$ 

## **10.9 PARALLEL CONNECTION OF COUPLED COILS**

Consider two inductors with self inductances  $L_1$  and  $L_2$  connected parallel which are mutually coupled with mutual inductance M as shown in Fig. 10.20.



Fig. 10.20

Let us consider Fig. 10.20(a) where the self induced emf in each coil assists the mutually induced emf as shown by the dot convention.

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$
(10.3)

The voltage across the parallel branch is given by

$$v = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \text{ or } L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$
  
also  
$$L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$
  
$$\frac{di_1}{dt} (L_1 - M) = \frac{di_2}{dt} (L_2 - M)$$
  
$$\therefore \qquad \frac{di_1}{dt} = \frac{di_2}{dt} \frac{(L_2 - M)}{(L_1 - M)}$$
(10.4)

Substituting Eq. 10.4 in Eq. 10.3, we get

$$\frac{di}{dt} = \frac{di_2}{dt} \frac{(L_2 - M)}{(L_1 - M)} + \frac{di_2}{dt} = \frac{di_2}{dt} \left[ \frac{(L_2 - M)}{L_1 - M} + 1 \right]$$
(10.5)

If  $L_{\rm eq}$  is the equivalent inductance of the parallel circuit in Fig. 10.20 (a) then v is given by

$$v = L_{eq} \frac{di}{dt}$$
$$L_{eq} \frac{di}{dt} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$
$$\frac{di}{dt} = \frac{1}{L_{eq}} \left[ L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \right]$$

Substituting Eq. 10.4 in the above equation we get

$$\frac{di}{dt} = \frac{1}{L_{eq}} \left[ L_1 \frac{di_2 (L_2 - M)}{dt (L_2 - M)} + M \frac{di_2}{dt} \right]$$
$$= \frac{1}{L_{eq}} \left[ L_1 \frac{(L_2 - M)}{(L_1 - M)} + M \right] \frac{di_2}{dt}$$
(10.6)

Equating Eq. 10.6 and Eq. 10.5, we get

$$\frac{L_2 - M}{L_2 - M} + 1 = \frac{1}{L_{eq}} \left[ L_1 \left( \frac{L_2 - M}{L_1 - M} \right) + M \right]$$

Rearranging and simplifying the above equation results in

$$L_{\rm eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

If the voltage induced due to mutual inductance oppose the self induced emf in each coil as shown by the dot convention in Fig. 10.20(b), the equivalent inductance is given by

$$L_{\rm eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

### **10.10 TUNED CIRCUITS**

Tuned circuits are, in general, single tuned and double tuned. Double tuned circuits are used in radio receivers to produce uniform response to modulated signals over a specified bandwidth; double tuned circuits are very useful in communication system.

### 10.10.1 Single Tuned Circuit

Consider the circuit in Fig. 10.21. A tank circuit (i.e. a parallel resonant circuit) on the secondary side is inductively coupled to coil (1) which is excited by a source,  $v_i$ . Let  $R_s$  be the source resistance and  $R_1$ ,  $R_2$  be the resistances of coils, 1 and 2, respectively. Also let  $L_1$ ,  $L_2$  be the self inductances of the coils, 1 and 2, respectively.

Let  $R_s + R_1 + j\omega L_1 = R_s$ 

with the assumption that  $R_s \gg R_1 \gg j\omega L_1$ The mesh equations for the circuit shown in Fig. 10.21 are



Fig. 10.21

$$i_1 R_s - j\omega M i_2 = v_i$$
$$-j\omega M i_1 + \left(R_2 + j\omega L_2 - \frac{j}{\omega C}\right) i_2 = 0$$

$$i_{2} = \begin{vmatrix} R_{s} & v_{i} \\ -j\omega M & 0 \end{vmatrix} / \begin{vmatrix} R_{s} & (-j\omega M) \\ (-j\omega M) & \left( R_{2} + j\omega L_{2} - \frac{j}{\omega C} \right) \end{vmatrix}$$
$$i_{2} = \frac{j_{-i}\omega M}{R_{s} \left( R_{2} + j\omega L_{2} - \frac{j}{\omega C} \right) + \omega^{2} M^{2}}$$

The output voltage  $v_0 = i_2 \cdot \frac{1}{j\omega C}$ 

or

$$v_o = \frac{j_{-i} \omega M}{j \omega C \left\{ R_s \left[ R_2 + \left( j \omega L_2 - \frac{1}{\omega C} \right) \right] + \omega^2 M^2 \right\}}$$

The voltage transfer function, or voltage amplification, is given by

$$\frac{v_o}{v_i} = A = \frac{M}{C\left\{R_s\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C}\right)\right] + \omega^2 M^2\right\}}$$

When the secondary side is tuned, i.e. when the value of the frequency  $\omega$  is such that  $\omega L_2 = 1/\omega C$ , or at resonance frequency  $\omega_c$ , the amplification is given by

$$A = \frac{v_o}{v_i} = \frac{M}{C[R_s R_2 + \omega_r^2 M^2]}$$

the current  $i_2$  at resonance  $i_2 = \frac{j_{-i} \omega_r M}{R_s R_2 + \omega_r^2 M^2}$ 

Thus, it can be observed that the output voltage, current and amplification depends on the mutual inductance M at resonance frequency, when  $M = K\sqrt{L_1 L_2}$ . The maximum output voltage or the maximum amplification depends on M. To get the condition for maximum output voltage, make  $dv_a/dM = 0$ .

$$\frac{dv_o}{dM} = \frac{d}{dM} \left[ \frac{v_i M}{C \left[ R_s R_2 + \omega_r^2 M^2 \right]} \right]$$
$$= 1 - 2M^2 \omega_r^2 \left[ R_s R_2 + \omega_r^2 M^2 \right]^{-1} = 0$$
$$R_s R_2 = \omega_r^2 M^2$$

From which,

 $M = \sqrt{\frac{R_s R_2}{\omega_r}}$ 

or

From the above value of M, we can calculate the maximum output voltage. Thus

$$v_{oM} = \frac{v_i}{2\omega_r C \sqrt{R_s R_2}},$$

or the maximum amplification is given by

$$A_m = \frac{1}{2\omega_r C \sqrt{R_s R_2}}$$
 and  $i_2 = \frac{j v_i}{2 \sqrt{R_s R_2}}$ 

The variation of the amplification factor or output voltage with the coefficient of coupling is shown in Fig. 10.22.



Fig. 10.22

**Example 10.10** Consider the single tuned circuit shown in Fig. 10.23 and determine (i) the resonant frequency (ii) the output voltage at resonance (iii) and the maximum output voltage. Assume  $R_s \gg \omega_r L_1$ , and K = 0.9.



Fig. 10.23

Solution  $M = K \sqrt{L_1 L_2}$ =  $0.9 \sqrt{1 \times 10^{-6} \times 100 \times 10^{-6}}$ =  $9 \ \mu \text{H}$ 

(i) Resonance frequency

$$\omega_r = \frac{1}{\sqrt{L_2 C}} = \frac{1}{\sqrt{100 \times 10^{-6} \times 0.1 \times 10^{-6}}}$$

or

The value of 
$$\omega_r L_1 = \frac{10^6}{\sqrt{10}} 1 \times 10^{-6} = 0.316$$

 $=\frac{10^6}{\sqrt{10}}$  rad/sec.

 $f_r = 50.292 \text{ KHz}$ 

Thus the assumption that  $\omega_r L_1 \ll R_1$  is justified.

(ii) Output voltage

$$v_o = \frac{Mv_i}{C[R_s R_2 + \omega_r^2 M]}$$
  
=  $\frac{9 \times 10^{-6} \times 15}{0.1 \times 10^{-6} \left[10 \times 10 + \left(\frac{10^6}{\sqrt{10}}\right)^2 \times 9 \times 10^{-6}\right]}$   
= 1.5 mV

(iii) Maximum value of output voltage

$$v_{oM} = \frac{v_i}{2\omega_r C \sqrt{R_s R_2}}$$
  
=  $\frac{15}{2 \times \frac{10^6}{\sqrt{10}} \times 0.1 \times 10^{-6} \sqrt{100}}$   
 $v_{oM} = 23.7 \text{ V}$ 

### 10.10.2 Double Tuned Coupled Circuits

Figure 10.24 shows a double tuned transformer circuit involving two series resonant circuits.





For the circuit shown in the figure, a special case where the primary and secondary resonate at the same frequency  $\omega_r$ , is considered here,

i.e 
$$\omega_r^2 = \frac{1}{L_1 C_1} = \frac{1}{L_2 C_2}$$

The two mesh equations for the circuit are

$$v_{\rm in} = i_1 \left( R_s + R_1 + j\omega L_1 - \frac{j}{\omega C_1} \right) - i_2 j\omega M$$
$$0 = -j\omega M i_1 + i_2 \left( R_2 + j\omega L_2 - \frac{j}{\omega C_2} \right)$$

From which

$$i_{2} = \frac{v_{in} \, j \, \omega M}{\left[ \left( R_{s} + R_{1} \right) + j \left( \omega L_{1} - \frac{1}{\omega C_{1}} \right) \right] \left[ R_{2} + j \left( \omega L_{2} - \frac{1}{\omega C_{2}} \right) \right] + \omega^{2} M^{2}}$$

also  $\omega_r = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}}$  at resonance

$$v_o = \frac{v_{\rm in} M}{C_2 \left[ (R_s + R_1) R_2 + \omega_r^2 M^2 \right]}$$
$$v_o = A v_{\rm in}$$

or

where A is the amplification factor given by

$$A = \frac{M}{C_2 \left[ (R_1 + R_s) R_2 + \omega_r^2 M^2 \right]}$$

The maximum amplification or the maximum output voltage can be obtained by taking the first derivative of  $v_0$  with respect to M, and equating it to zero.

:.

$$\frac{dv_o}{dM} = 0, \text{ or } \frac{dA}{dM} = 0$$
$$\frac{dA}{dM} = (R_1 + R_s)R_2 + \omega_r^2 M^2 - 2M^2 \omega_r^2 = 0$$
$$\omega_r^2 M^2 = R_2(R_1 + R_s)$$
$$M_c = \frac{\sqrt{R_2 (R_1 + R_s)}}{\omega_r}$$

where  $M_c$  is the critical value of mutual inductance. Substituting the value of  $M_c$  in the equation of  $v_o$ , we obtain the maximum output voltage as

$$|v_{o}| = \frac{v_{in}}{2\omega_{r}^{2}C_{2}M_{c}} = \frac{v_{in}}{2\omega_{r}C_{2}\sqrt{R_{2}(R_{1}+R_{s})}}$$

$$|i_{2}| = \frac{v_{in}}{2\omega_{r}M_{c}} = \frac{v_{in}}{2\sqrt{R_{2}(R_{1}+R_{s})}}$$

and

By definition,  $M = K \sqrt{L_1 L_2}$ , the coefficient of coupling, K at  $M = M_c$  is called the critical coefficient of coupling, and is given by  $K_c = M_c / \sqrt{L_2 L_1}$ .

The critical coupling causes the secondary current to have the maximum possible value. At resonance, the maximum value of amplification is obtained by changing M, or by changing the coupling coefficient for a given value of  $L_1$  and  $L_2$ . The variation of output voltage with frequency for different coupling coefficients is shown in Fig. 10.25.



Fig. 10.25

## 10.11 ANALYSIS OF MAGNETIC CIRCUITS

The presence of charges in space or in a medium creates an electric field, similarly the flow of current in a conductor sets up a magnetic field. Electric field is represented by electric flux lines, magnetic flux lines are used to describe the magnetic field. The path of the magnetic flux lines is called the magnetic circuit. Just as a flow of current in the electric circuit requires the presence of an electromotive force, so the production of magnetic flux requires the presence of magneto-motive force (mmf). We now discuss some properties related to magnetic flux.

(i) Flux density (B) The magnetic flux lines start and end in such a way that they form closed loops. Weber (Wb) is the unit of magnetic flux ( $\phi$ ). Flux density (B) is the flux per unit area. Tesla (T) or Wb/m<sup>2</sup> is the unit of flux density.

$$B = \frac{\Phi}{A}$$
 Wb/m<sup>2</sup> or Tesla

where *B* is a quantity called magnetic flux density in teslas,  $\phi$  is the total flux in webers and *A* is the area perpendicular to the lines in m<sup>2</sup>.

(ii) Magneto-motive force MMF (J) A measure of the ability of a coil to produce a flux is called the *magneto-motive force*. It may be considered as a magnetic pressure, just as emf is considered as an electric pressure. A coil with

*N* turns which is carrying a current of *I* amperes constitutes a magnetic circuit and produces an mmf of *NI* ampere turns. The source of flux ( $\phi$ ) in the magnetic circuit is the mmf. The flux produced in the circuit depends on mmf and the length of the circuit.

*(iii) Magnetic field strength (H)* The magnetic field strength of a circuit is given by the mmf per unit length.

$$H = \frac{\Im}{l} = \frac{NI}{l} \text{ AT/m}$$

The magnetic flux density (B) and its intensity (field strength) in a medium can be related by the following equation

$$B = \mu H$$

where  $\mu = \mu_0 \mu_r$  is the permeability of the medium in Henrys/metre (H/m),

 $\mu_0$  = absolute permeability of free space and is equal to  $4\pi \times 10^{-7}$  H/m and  $\mu_r$  = relative permeability of the medium.

Relative permeability is a non-dimensional numeric which indicates the degree to which the medium is a better conductor of magnetic flux as compared to free space. The value of  $\mu_r = 1$  for air and non-magnetic materials. It varies from 1,000 to 10,000 for some types of ferro-magnetic materials.

(iv) **Reluctance**  $(\Re)$  It is the property of the medium which opposes the passage of magnetic flux. The magnetic reluctance is analogous to resistance in the electric circuit. Its unit is AT/Wb. Air has a much higher reluctance than does iron or steel. For this reason, magnetic circuits used in electrical machines are designed with very small air gaps.

According to definition, reluctance =  $\frac{\text{mmf}}{1}$ 

The reciprocal of reluctance is known as permeance  $\frac{1}{\Re} = \frac{\phi}{\Im}$ 

Thus reluctance is a measure of the opposition offered by a magnetic circuit to the setting up of the flux. The reluctance of the magnetic circuit is given by 1 l

$$\Re = \frac{1}{\mu} \frac{i}{a}$$

where I is the length, a is the cross-sectional area of the magnetic circuit and  $\mu$  is the permeability of the medium.

From the above equations

$$\frac{\neg \mu}{\mu} \frac{a}{a} = \frac{\neg}{\phi}$$
$$\frac{\Im}{1} = \frac{1}{\mu} \cdot \frac{\phi}{a}$$
$$\frac{M}{l} = \frac{1}{\mu} \cdot B$$
$$H = \frac{1}{\mu} \cdot B$$
$$B = \mu H$$

1

1

3

or

or

# **10.12 SERIES MAGNETIC CIRCUIT**

A series magnetic circuit is analogous to a series electric circuit. Kirchhoff's laws are applicable to magnetic circuits also. Consider a ring specimen having a magnetic path of l meters, area of cross-section  $(A)m^2$  with a mean radius of R meters having a coil of N turns carrying I amperes wound uniformly as shown in Fig. 10.26. MMF is responsible for the establishment of flux in the magnetic medium. This mmf acts along the magnetic lines of force. The flux produced by the circuit is given by



Fig. 10.26

The magnetic field intensity of the ring is given by  $H = \frac{\text{mmf}}{l} = \frac{NI}{l} = \text{AT/m}$ Where *l* is the mean length of the magnetic path and is given by  $2\pi R$ .

Flux density  $B = \mu_o \mu_r H = \mu_o \mu_r \frac{NI}{l}$  Wb/m<sup>2</sup>

Flux 
$$\phi = \mu$$
HA Webers  
 $= \mu_0 \mu_r \frac{NI}{l} \times A$  Wb  
 $\phi = \frac{NI}{l/\mu_0 \mu_r A}$  Wb

*NI* is the mmf of the magnetic circuit, which is analogous to emf in electric circuit.  $l/\mu_0 \mu_r A$  is the reluctance of the magnetic circuit which is analogous to resistance in electric circuit.

# 10.13 COMPARISON OF ELECTRIC AND MAGNETIC CIRCUITS

A series electric and magnetic circuits are shown in Figs. 10.27(a) and (b) respectively.



Fig. 10.27

Figure 10.27(a) represents an electric circuit with three resistances connected in series, the dc source *E* drives the current *I* through all the three resistances whose voltage drops are  $V_1$ ,  $V_2$  and  $V_3$ . Hence,  $E = V_1 + V_2 + V_3$ , also

 $E = I(R_1 + R_2 + R_3)$ . We also know that  $R = \frac{\rho l}{a}$ , where  $\rho$  is the specific resistance of the material, *l* is the length of the wire of the resistive material and *a* is the area of cross-section of the wire.

The drop across each resistor  $V = RI = \rho l \frac{I}{a}$ 

or

$$\frac{V}{l} = \rho \frac{I}{a}$$

Voltage drop per unit length = specific resistance × current density.

Let us consider the magnetic circuit in Fig. 10.27(b). The MMF of the circuit is given by  $\Im = NI$ , drives the flux  $\phi$  around the three parts of the circuit which are in series. Each part has a reluctance  $\Re = \frac{1}{\mu} \cdot \frac{l}{a}$ , where *l* is the length and *a* is

the area of cross-section of each arm. The mmf of the magnetic circuit is given by  $\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3$ .  $\mathfrak{T} = \phi(\mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3)$  where  $\mathfrak{R}_1 \mathfrak{R}_2$  and  $\mathfrak{R}_3$  are the reluctances of the portion 1, 2 and 3 respectively.

Also

$$\Im = \frac{1}{\mu} \cdot \frac{\iota}{a} \cdot \phi$$
$$\frac{\Im}{l} = \frac{1}{\mu} \cdot \frac{\phi}{a}$$
$$H = \frac{1}{\mu} \cdot B.$$

 $\frac{1}{\mu}$  can be termed as *reluctance* of a cubic metre of magnetic material from

which, the above equation gives the mmf per unit length (intensity) which is analogous to the voltage per unit length. Parallels between electric-circuit and magnetic-circuit quantities are shown in Table 10.1.

Electric circuit	Magnetic circuit
Exciting force = emf in volts	mmf in AT
Response = current in amps	flux in webers
Voltage drop = $VI$ volts	mmf drop = $\Re \phi$ AT
Electric field density = $\frac{V}{l}$ volt/m	Magnetic field Intensity = $\frac{\Im}{1}$ AT/m
$\operatorname{Current}(I) = \frac{E}{R} \operatorname{A}$	Flux $(\phi) = \frac{\Im}{R}$ Web
Current density( $J$ ) = $\frac{I}{a}$ Amp/m <sup>2</sup>	Flux density $(B) = \frac{\phi}{A}$ Web/m <sup>2</sup>
Resistance $(R) = \frac{\rho_l}{a}$ ohm	Reluctance $(\Re) = \frac{1}{\mu} \cdot \frac{l}{a}$ AT/Web
Conductance (G) = $\frac{1}{R}$ Mho	Permeance = $\frac{1}{\Re} = \frac{\mu a}{\mu} \cdot \frac{l}{a}$ Web/AT

 Table 10.1
 Analogy between magnetic and electric circuit

Thus, it is seen that the magnetic reluctance is analogous to resistance, mmf is analogous to emf and flux is analogous to current. These analogies are useful in magnetic circuit calculations. Though we can draw many parallels between the two circuits, the following differences do exists.

The electric current is a true flow but there is no flow in a magnetic flux. For a given temperature,  $\rho$  is independent of the strength of the current, but  $\mu$  is not independent of the flux.

In an electric circuit energy is expended so long as the current flows, but in a magnetic circuit energy is expended only in creating the flux, and not in maintaining it. Parallels between the quantities are shown in Table 10.1.

### **10.14 MAGNETIC LEAKAGE AND FRINGING**

Figure 10.28 shows a magnetised iron ring with a narrow air gap, and the flux which crosses the gap can be regarded as useful flux. Some of the total flux produced by the ring does not cross the air gap, but instead takes a shorter route as shown in Fig. 10.28 and is known as *leakage flux*. The flux while crossing the air gap bulges outwards due to variation in reluctance. This is known as *fringing*. This is because the lines of force repel each other when passing through the air as a result the flux density in the air gap decreases. For the purpose of calculation it is assumed that the iron carries the whole of the total flux throughout its length. The ratio of total flux to useful flux is called the *leakage coefficient* or leakage factor.



Fig. 10.28

Leakage factor = Total flux/useful flux.

**Example 10.11** A coil of 100 turns is wound uniformly over a insulator ring with a mean circumference of 2 m and a uniform sectional area of  $0.025 \text{ cm}^2$ . If the coil is carrying a current of 2 A. Calculate (a) the mmf of the circuit, (b) magnetic field intensity (c) flux density (d) the total flux.

#### Solution

- (a)  $mmf = NI = 100 \times 2 = 2000 \text{ AT}$
- (b)  $H = \frac{\text{mmf}}{l} = \frac{2000}{2} = 1000 \text{ AT/m}$
- (c)  $B = m_0 H = 4p \times 10^{-7} \times 1000 = 1.2565 \text{ mWb/m}^2$ .
- (d)  $f = B \times A = 1.2565 \times 10^{-3} \times 0.025 \times 10^{-4} = 0.00314 \times 10^{-6}$  Wb

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**Example 10.12** Calculate the mmf required to produce a flux of 5 mWb across an air gap of 2.5 mm of length having an effective area of  $100 \text{ cm}^2$  of a cast steel ring of mean iron path of 0.5 m and cross-sectional area of 150 cm<sup>2</sup> as shown in Fig. 10.29. The relative permeability of the cast steel is 800. Neglect leakage flux.

#### Solution



Length of the cast steel path = 0.5 m

The required mmf for the cast steel to produce the necessary flux =  $0.5 \times 332 = 166$  AT

Therefore total mmf = 975 + 166 = 1141 AT

# **10.15 COMPOSITE SERIES CIRCUIT**

Consider a toroid composed of three different magnetic materials of different permeabilities, areas and lengths excited by a coil of *N* turns.



#### Fig. 10.30

With a current of *I* amperes as shown in Fig. 10.30. The lengths of sections *AB*, *BC* and *CA* are  $I_1$ ,  $I_2$  and  $I_3$  respectively. Each section will have its own reluctance and permeability. Since all of them are joined in series, the total reluctance of the combined magnetic circuit is given by

$$\Re_{\text{Total}} = \frac{1}{\mu A}$$
$$= \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} + \frac{l_3}{\mu_3 A_3}$$

The flux produced in the circuit is given by  $\phi = \frac{\text{mmf}}{\text{Total reluctance}}$  Wb

$$\phi = \left[\frac{NI}{\frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} + \frac{l_3}{\mu_3 A_3}}\right] Wb$$

*:*.

### **10.16 PARALLEL MAGNETIC CIRCUIT**

We have seen that a series magnetic circuit carries the same flux and the total mmf required to produce a given quantity of flux is the sum of the mmf's for the separate parts. In a parallel magnetic circuit, different parts of the circuit are in parallel. For such circuits the Kirchhoff's laws, in their analogous magnetic form can be applied for the analysis. Consider an iron core having three limbs

A, B and C as shown in Fig. 10.31(a). A Coil with N turns is arranged around limb A which carries a current I amperes. The flux produced by the coil in limb A.  $\phi_A$  is divided between limbs B and C and each equal to  $\phi_A/2$ . The reluctance offered by the two parallel paths is equal to the half the reluctance of each path (Assuming equal lengths and cross sectional areas). Similar to Kirchhoff's current law in an electric circuit, the total magnetic flux directed towards a junction in a magnetic circuit is equal to the sum of the magnetic fluxes directed away from that junction. Accordingly  $\phi_A = \phi_B + \phi_C$  or  $\phi_A - \phi_B - \phi_C = 0$ . The electrical equivalent of the above circuit is shown in Fig. 10.31(b). Similar to Kirchhoff's second law, in a closed magnetic circuit, the resultant mmf is equal to the algebraic sum of the products of field strength and the length of each part in the closed path. Thus applying the law to the first loop in Fig. 10.31(a), we get  $NI = H_{\perp}l_{\perp} + H_{\rm p}l_{\perp}$ 

or

$$NI = \phi_A \,\mathfrak{R}_A + \phi_B \,\mathfrak{R}_B$$
$$NI = \phi_A \,\mathfrak{R}_A + \phi_B \,\mathfrak{R}_B$$

The mmf across the two parallel paths is identical. Therefore NI is also equal to

$$NI = \phi_A \, \Re_A + \phi_C \, \Re_C$$







Problem 10.1 In the circuit shown in Fig. 10.32, write the equation for the voltages across the coils *ab* and *cd*; also mention the polarities of the terminals.



Fig. 10.32

**Solution** Current  $i_1$  is only flowing in coil *ab*, whereas coil *cd* is open. Therefore, there is no current in coil *cd*. The emf due to self induction is zero on coil *cd*.

$$\therefore \qquad v_2(t) = M \frac{di_1(t)}{dt} \text{ with } C \text{ being positive}$$

Similarly the emf due to mutual induction in coil *ab* is zero.

**Problem 10.2** In the circuit shown in Fig. 10.33, write the equation for the voltages  $v_1$  and  $v_2$ .  $L_1$  and  $L_2$  are the coefficients of self inductances of coils 1 and 2, respectively, and *M* is the mutual inductance.



Fig. 10.33

**Solution** In the figure, *a* and *d* are like terminals. Currents  $i_1$  and  $i_2$  are entering at dot marked terminals.

$$v_1 = L_1 \frac{di_1(t)}{dt} + \frac{M di_2(t)}{dt}$$
$$v_2 = L_2 \frac{di_2(t)}{dt} + \frac{M di_1(t)}{dt}$$

**Problem 10.3** In Fig. 10.34,  $L_1 = 4$  H;  $L_2 = 9$  H, K = 0.5,  $i_1 = 5 \cos (50t - 30^\circ)$  A,  $i_2 = 2 \cos (50t - 30^\circ)$  A. Find the values of (a)  $v_1$ ; (b)  $v_2$ , and (c) the total energy stored in the system at t = 0.



Fig. 10.34

**Solution** Since the current in coil *ab* is entering at the dot marked terminal, whereas in coil *cd* the current is leaving, we can write the equations as

$$v_{1} = L_{1} \frac{di_{1}}{dt} - M \frac{di_{2}}{dt}$$

$$v_{2} = -M \frac{di_{1}}{dt} + L_{2} \frac{di_{2}}{dt}$$

$$M = K \sqrt{L_{1} L_{2}} = 0.5\sqrt{36} = 3$$
(a)  $v_{1} = 4 \frac{d}{dt} [5 \cos (50t - 30^{\circ}) - 3 \frac{d}{dt} [2 \cos (50t - 30^{\circ})]$ 

$$v_{1} = 20 [-\sin (50t - 30^{\circ}) \times 50] - 6 [-\sin (50t - 30^{\circ}) 50]$$
at  $t = 0$ 

$$v_{1} = 500 - 150 = 350 \text{ V}$$
(b)  $v_{2} = -3 \frac{d}{dt} [5 \cos (50t - 30^{\circ})] + 9 \frac{d}{dt} [2 \cos (50t - 30^{\circ})]$ 

$$= -15 [-\sin (50t - 30^{\circ}) \times 50] + 18 [-\sin (50t - 30^{\circ}) 50]$$

$$v_2 = -375 + 450 = 75$$
 V

at t = 0

(c) The total energy stored in the system

$$W(t) = \frac{1}{2} L_1[i_1(t)]^2 + \frac{1}{2} L_2[i_2(t)]^2 - M[i_1(t)i_2(t)]$$
  
=  $\frac{1}{2} \times 4[5 \cos (50t - 30^\circ)]^2 + \frac{1}{2} \times 9[2 \cos (50t - 30^\circ)]^2$   
-  $3 [5 \cos (50t - 30^\circ) \times 2 \cos (50t - 30^\circ)]$   
at  $t = 0 W(t) = 28.5 j$ 

**Problem 10.4** For the circuit shown in Fig. 10.35, write the mesh equations. **Solution** There exists mutual coupling between coil 1 and 3, and 2 and 3. Assuming branch currents  $i_1$ ,  $i_2$  and  $i_3$  in coils 1, 2 and 3, respectively, the equation for mesh 1 is



Fig. 10.35

$$v = v_1 + v_2$$
  

$$v = i_1 j_2 - i_3 j_4 + i_2 j_4 - i_3 j_6$$
(10.7)

 $j_4 i_3$  is the mutual inductance drop between coils (1) and (3), and is considered negative according to dot convention and  $i_3 j_6$  is the mutual inductance drop between coils 2 and 3.

For the 2nd mesh

$$0 = -v_2 + v_3$$
  
= -( $j_4 i_2 - j_6 i_3$ ) +  $j_3 i_3 - j_6 i_2 - j_4 i_1$  (10.8)

$$= -j_4 i_1 - j_{10} i_2 + j_9 i_3 \tag{10.9}$$

$$i_1 = i_3 + i_2$$

**Problem 10.5** Calculate the effective inductance of the circuit shown in Fig. 10.36 across terminals *a* and *b*.



Fig. 10.36

**Solution** Let the current in the circuit be *i* 

$$v = 8\frac{di}{dt} - 4\frac{di}{dt} + 10\frac{di}{dt} - 4\frac{di}{dt} + 5\frac{di}{dt} + 6\frac{di}{dt} + 5\frac{di}{dt}$$
$$\frac{di}{dt}[34 - 8] = 26\frac{di}{dt} = v$$

or

Let *L* be the effective inductance of the circuit across *ab*. Then the voltage across  $ab = v = L \frac{di}{dt} = 26 \frac{di}{dt}$ .

Hence, the equivalent inductance of the circuit is given by 26 H.

**Problem 10.6** For the circuit shown in Fig. 10.37, find the ratio of output voltage to the source voltage.



Fig. 10.37

**Solution** Let us consider  $i_1$  and  $i_2$  as mesh currents in the primary and secondary windings.

As the current  $i_1$  is entering at the dot marked terminal, and current  $i_2$  is leaving the dot marked terminal, the sign of the mutual inductance is to be negative. Using Kirchhoff's voltage law, the voltage equation for the first mesh is

$$i_1(R_1 + j\omega L_1) - i_2 j\omega M = v_1$$
  

$$i_1(10 + j500) - i_2 j250 = 10$$
(10.10)

Similarly, for the 2nd mesh

$$i_{2}(R_{2} + j\omega L_{2}) - i_{1}j\omega M = 0$$

$$i_{2}(400 + j5000) - i_{1}j250 = 0$$
(10.11)
$$i_{2} = \frac{\begin{vmatrix} (10 + j500) & 10 \\ - j250 & 0 \end{vmatrix}}{\begin{vmatrix} (10 + j500) & -j250 \\ - j250 & (400 + j5000) \end{vmatrix}$$

$$i_{2} = 0.00102 \angle - 84.13^{\circ}$$

$$v_{2} = i_{2} \times R_{2}$$

$$= 0.00102 \angle - 84.13^{\circ} \times 400$$

$$= 0.408 \angle - 84.13^{\circ}$$

$$\frac{v_{2}}{v_{1}} = \frac{0.408}{10} \angle - 84.13^{\circ}$$

$$\frac{v_{2}}{v_{1}} = 40.8 \times 10^{-3} \angle - 84.13^{\circ}$$

**Problem 10.7** Calculate the effective inductance of the circuit shown in Fig. 10.38 across *AB*.



Fig. 10.38

Solution The inductance matrix is

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 17 \end{bmatrix}$$

From KVL  $v = v_1 + v_2$ 

 $v_2 = v_3$ 

and From KCL

$$i_{1} = i_{2} + i_{3}$$
(10.14)  
$$\begin{bmatrix} v_{1} \\ v_{2} \\ v_{3} \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 17 \end{bmatrix} \begin{bmatrix} di_{1}/dt \\ di_{2}/dt \\ di_{3}/dt \end{bmatrix}$$
$$v_{1} = 5\frac{di_{1}}{dt} - 2\frac{di_{3}}{dt}$$
(10.15)

$$v_1 = 5\frac{di_1}{dt} - 2\frac{di_3}{dt}$$
(10.15)

 $v_2 = 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$ (10.16)

$$v_3 = -2\frac{di_1}{dt} - 3\frac{di_2}{dt} + 17\frac{di_3}{dt}$$
(10.17)

From Eq. 10.12, we have

$$v = v_1 + v_2$$
  
=  $5\frac{di_1}{dt} - 2\frac{di_3}{dt} + 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$   
 $v = 5\frac{di_1}{dt} + 6\frac{di_2}{dt} - 5\frac{di_3}{dt}$  (10.18)

From Eq. 10.14,

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt}$$
(10.19)

Substituting Eq. 10.19 in Eq. 10.17, we have

$$v_{3} = -2\left[\frac{di_{2}}{dt} + \frac{di_{3}}{dt}\right] - 3\left[\frac{di_{2}}{dt}\right] + 17\left[\frac{di_{3}}{dt}\right]$$
$$-5\frac{di_{2}}{dt} + 15\frac{di_{3}}{dt} = v_{3}$$
(10.20)

or

Multiplying Eq. 10.16 by 5, we get

$$30\frac{di_2}{dt} - 15\frac{di_3}{dt} = 5v_2 \tag{10.21}$$

Adding Eqs. (10.20) and (10.21), we get

$$25\frac{di_2}{dt} = v_3 + 5v_2$$
$$25\frac{di_2}{dt} = 6v_2$$
$$= 6v_3, \text{ since } v_2 = v_3$$
$$v_2 = \frac{25}{6}\frac{di_2}{dt}$$

or

(10.12)

(10.13)

L

From Eq. 10.16

$$\frac{25}{6}\frac{di_2}{dt} = 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$$
$$\frac{di_2}{dt} = \frac{18}{11}\frac{di_3}{dt}$$

from which

From Eq. 10.19

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{11}{18}\frac{di_2}{dt} = \frac{29}{18}\frac{di_2}{dt}$$

Substituting the values of  $\frac{di_2}{dt}$  and  $\frac{di_3}{dt}$  in Eq. 10.18 yields

$$v = 5\frac{di_1}{dt} + 6\frac{18}{29}\frac{di_1}{dt} - 5\frac{11}{18}\frac{di_2}{dt}$$
$$= 5\frac{di_1}{dt} + \frac{108}{29}\frac{di_1}{dt} - \frac{55}{18}\frac{18}{29}\frac{di_1}{dt}$$
$$v = \frac{198}{29}\frac{di_1}{dt} = 6.827\frac{di_1}{dt}$$

 $\therefore$  equivalent inductance across AB = 6.827 H

**Problem 10.8** Write the mesh equations for the network shown in Fig. 10.39.



Fig. 10.39

**Solution** The circuit contains three meshes. Let us assume three loop currents  $i_1$ ,  $i_2$  and  $i_3$ .

For the first mesh

$$5i_1 + j3(i_1 - i_2) + j4(i_3 - i_2) = v_1$$
(10.22)

The drop due to self inductance is  $j3(i_1 - i_2)$  is written by considering the current  $(i_1 - i_2)$  entering at dot marked terminal in the first coil,  $j4(i_3 - i_2)$  is the mutually induced voltage in coil 1 due to current  $(i_3 - i_2)$  entering at dot marked terminal of coil 2. Similarly, for the 2nd mesh,

$$j3(i_2 - i_1) + j5(i_2 - i_3) - j2i_2 + j_4(i_2 - i_3) + j4(i_2 - i_1) = 0$$
(10.23)

...

 $j4(i_2 - i_1)$  is the mutually induced voltage in coil 2 due to the current in coil 1, and  $j4(i_2 - i_3)$  is the mutually induced voltage in coil 1 due to the current in coil 2.

For the third mesh,

$$3i_3 + j5(i_3 - i_2) + j4(i_1 - i_2) = 0$$
(10.24)

Further simplification of Eqs. 10.22, 10.23 and 10.24 leads to

$$(5+j3)i_1 - j7i_2 + j4i_3 = v_1 \tag{10.25}$$

$$-j7i_1 + j14i_2 - j9i_3 = 0 \tag{10.26}$$

$$j4i_1 - j9i_2 + (3 + j5)i_3 = 0 (10.27)$$

**Problem 10.9** The inductance matrix for the circuit of three series connected coupled coils is given in Fig. 10.40. Find the inductances, and indicate the dots for the coils.



#### Fig. 10.40

$$L = \begin{bmatrix} 4 & -4 & 1 \\ -4 & 2 & -3 \\ 1 & -3 & 6 \end{bmatrix}$$

All elements are in henrys

**Solution** The diagonal elements (4, 2, 6) in the matrix represent the self inductances of the three coils 1, 2 and 3, respectively. The second element in the 1st row (-4) is the mutual inductance between coil 1 and 2, the negative sign indicates that the current in the first coil enters the dotted terminal, and the current in the second coil enters at the undotted terminal. Similarly, the remaining elements are fixed. The values of inductances and the dot convention is shown in Fig. 10.41.



Fig. 10.41

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**Problem 10.10** Find the voltage across the 10  $\Omega$  resistor for the network shown in Fig. 10.42.





**Solution** From Fig. 10.42, it is clear that  $v_2 = i_2 10$  (10.28) Mesh equation for the first mesh is  $j4i_1 - j15 (i_1 - i_2) + j3i_2 = 10 \angle 0^\circ$ 

Mesh equation for the 2nd mesh is

$$j2i_{2} + 10i_{2} - j15(i_{2} - i_{1}) + j3i_{1} = 0$$
  

$$j18i_{1} - j13i_{2} + 10i_{2} = 0$$
  

$$j18i_{1} + i_{2}(10 - j13) = 0$$
(10.30)

Solving for  $i_2$  from Eqs. 10.29 and 10.30, we get

$$i_{2} = \begin{bmatrix} -j11 & 10 \angle 0^{\circ} \\ j18 & 0 \end{bmatrix} / \begin{bmatrix} -j11 & j18 \\ j18 & 10 - j3 \end{bmatrix}$$
$$= \frac{-180 \angle 90^{\circ}}{291 - j110}$$
$$= \frac{-180 \angle 90^{\circ}}{311 \angle 20.70^{\circ}} = -0.578 \angle 110.7^{\circ}$$
$$v_{2} = i_{2} 10 = -5.78 \angle 110.7^{\circ}$$
$$|v_{2}| = 5.78$$

**Problem 10.11** The resonant frequency of the tuned circuit shown in Fig. 10.43 is 1000 rad/sec. Calculate the self inductances of the two coils and the optimum value of the mutual inductance.



*:*.

**Solution** From Section 10.7, we know that

$$\omega_r^2 = \frac{1}{L_1 C_1} = \frac{1}{L_2 C_2}$$
$$L_1 = \frac{1}{\omega_r^2 C_1} = \frac{1}{(1000)^2 1 \times 10^{-6}} = 1 \text{ H}$$
$$L_2 = \frac{1}{\omega_r^2 C_2} = \frac{1}{(1000)^2 \times 2 \times 10^{-6}} = 0.5 \text{ H}$$

Optimum value of the mutual inductance is given by

$$M_{\text{optimum}} = \frac{\sqrt{R_1 R_2}}{\omega_r}$$

where  $R_1$  and  $R_2$  are the resistances of the primary and secondary coils

$$M = \frac{\sqrt{15}}{1000} = 3.87 \text{ mH}$$



Fig. 10.44

**Problem 10.12** The tuned frequency of a double tuned circuit shown in Fig. 10.44 is  $10^4$  rad/sec. If the source voltage is 2 V and has a resistance of 0.1  $\Omega$ , calculate the maximum output voltage at resonance if  $R_1 = 0.01 \Omega$ ,  $L_1 = 2 \mu$ H;  $R_2 = 0.1 \Omega$ , and  $L_2 = 25 \mu$ H.

#### Solution

The maximum output voltage  $v_0 = \frac{v_i}{2\omega_r^2 C_2 M_c}$ where  $M_c$  is the critical value of the mutual inductance given by

$$M_c = \frac{\sqrt{R_2 (R_1 + R_s)}}{\omega_r}$$
$$M_c = \frac{\sqrt{0.1 (0.01 + 0.1)}}{10^4} = 10.48 \,\mu\text{H}$$

1

#### At resonance

$$\omega_r^2 = \frac{1}{L_2 C_2}$$

$$C_2 = \frac{1}{\omega_r^2 L_2} = \frac{1}{(10^4)^2 \times 25 \times 10^{-6}} = 0.4 \times 10^{-3} \text{ F}$$

$$v_0 = \frac{2}{2(10^4)^2 \times 0.4 \times 10^{-3} \times 10.48 \times 10^{-6}}$$

$$= 2.385 \text{ V}$$

**Problem 10.13** An iron ring 10 cm dia and 15 cm<sup>2</sup> in cross-section is wound with 250 turns of wire for a flux density of 1.5 Web/m<sup>2</sup> and permeability 500. Find the exciting current, the inductance and stored energy. Find corresponding quantities when there is a 2 mm air gap.

#### Solution

(a) Without air gap Length of the flux path =  $\pi D = \pi \times 10 = 31.41$  cm = 0.3141 mArea of flux path =  $15 \text{ cm}^2 = 15 \times 10^{-4} \text{ m}^2$ mmf = A.T $A = \frac{\text{mmf}}{T}$  $H = \frac{B}{\mu_0 \mu_0} = \frac{1.5}{4\pi \times 10^{-7} \times 500} = 2387$  $mmf = H \times l = 2387 \times 0.3141 = 750 \text{ AT}$ Exciting current =  $\frac{\text{mmf}}{T} = \frac{750}{250} = 3 \text{ A}$ Reluctance =  $\frac{l}{\mu_0 \mu_r A} = \frac{0.3141}{4\pi 10^{-7} \times 500 \times 15 \times 10^{-4}}$ = 333270 Self Inductance =  $\frac{N^2}{\text{Reluctance}} = \frac{(250)^2}{333270} = 0.1875 \text{ H}$ Energy =  $\frac{1}{2}LI^2 = \frac{1}{2} \times 0.1875 \times (3)^2$ = 0.843 Joules (b) With air gap Reluctance of the gap =  $\frac{l}{\mu_0 A} = \frac{2 \times 10^{-3}}{4\pi \times 10^{-7} \times 15 \times 10^{-4}}$ 

 $= 1.06 \times 10^6 \text{ A/Wb}$ 

Total reluctance =  $(0.333 + 1.06) 10^6 = 1.393 \times 10^6 \text{ A/Wb}$ 

$$= 1.5 \times 15 \times 10^{-4} \times 1.393 \times 10^{6}$$
  
= 3134 AT  
Exciting current =  $\frac{3134}{250} = 12.536$  A  
 $L = \frac{N^{2}}{\Re} = \frac{(250)^{2}}{1.393 \times 10^{6}} = 44.8$  mH  
Energy =  $\frac{1}{2}LI^{2}$   
=  $\frac{1}{2} \times 44.8 \times 10^{-3} \times (12.536)^{2}$   
= 3.52 Joules

**Problem 10.14** A 700 turn coil is wound on the central limb of the cast steel frame as shown in Fig. 10.45. A total flux of 1.8 m Wb is required in the gap. What is the current required? Assume that the gap density is uniform and that all lines pass straight across the gap. All dimensions are in centimeters. Assume  $\mu_{\rm r}$  as 600.



Fig. 10.45

**Solution** Each of the side limbs carry half the total flux as their reluctances are equal.

Total mmf required is equal to the sum of the mmf required for gap, central limb and side limb.

Reluctance of gap and central limb are in series and they carry the same flux.

Air gap

$$\phi_g = 1.8 \times 10^{-3} \text{ Wb}$$

$$A_g = 4 \times 4 \times 10^{-4} \text{ m}^2$$

$$B_g = \frac{1.8 \times 10^{-3}}{16 \times 10^{-4}} = 1.125 \text{ Wb/m}^2$$

$$H_g = \frac{B_g}{\mu_0} = \frac{1.125}{4\pi \times 10^{-7}} = 8.95 \times 10^5 \text{ AT/m}$$
Required mmf for the gap =  $H_g l_g$ 

$$= 8.95 \times 10^5 \times 0.001 = 895 \text{ AT}$$

\*\*

Central Limb

$$\phi_c = 1.8 \times 10^{-3} \text{ Wb}$$

$$A_c = 4 \times 4 \times 10^{-4} \text{ m}^2$$

$$B_c = 1.125 \text{ Wb/m}^2$$

$$H_c = \frac{B_c}{\mu_0 \mu_r} = \frac{1.125}{4\pi \times 10^{-7} \times 600} = 1492 \text{ AT/m}$$

Required mmf for central limb =  $H_c l_c$ 

$$= 1492 \times 0.24 = 358 \text{ AT}$$

Side Limb:

$$\phi_s = \frac{1}{2} \times \text{flux in central limb}$$
  
=  $\frac{1}{2} \times 1.8 \times 10^{-3} = 0.9 \times 10^{-3} \text{ Wb}$   
$$B_s = \frac{0.9 \times 10^{-3}}{16 \times 10^{-4}} = 0.5625 \text{ Wb/m}^2$$
  
$$H_s = \frac{B_s}{\mu_0 \mu_r} = \frac{0.5625}{4\pi \times 10^{-7} \times 600} = 746 \text{ AT/m}$$

Required mmf for side limb =  $H_s l_s$ 

$$= 746 \times 0.6 = 447.6 \cong 448$$

\*\*

Total mmf = 895 + 358 + 448 = 1701 AT

Required current = 
$$\frac{1701}{700}$$
 = 2.43 A

# **PRACTICE PROBLEMS**

10.1 Using the dot convention, write the voltage equations for the coils shown in Fig. 10.46.



Fig. 10.46

10.2 Two inductively coupled coils have self inductances  $L_1 = 40$  mH and  $L_2 = 150$  mH. If the coefficient of coupling is 0.7, (i) find the value of mutual inductance between the coils, and (ii) the maximum possible mutual inductance.

10.3 For the circuit shown in Fig. 10.47 write the inductance matrix.



Fig. 10.47

- 10.4 Two coils connected in series have an equivalent inductance of 0.8 H when connected in aiding, and an equivalent inductance of 0.5 H when the connection is opposing. Calculate the mutual inductance of the coils.
- 10.5 In Fig. 10.48,  $L_1 = 2$  H;  $L_2 = 6$  H; K = 0.5;  $i_1 = 4 \sin (40t 30^\circ)$  A;  $i_2 = 2 \sin (40t 30^\circ)$  A. Find the values of (i)  $v_1$  and (ii)  $v_2$ .



Fig. 10.48

10.6 For the circuit shown in Fig. 10.49, write the mesh equations.



Fig. 10.49

10.7 Calculate the effective inductance of the circuit shown in Fig. 10.50 across XY.



Fig. 10.50

10.8 For the circuit shown in Fig. 10.51, find the ratio of output voltage to the input voltage.



Fig. 10.51

10.9 Calculate the effective inductance of the circuit shown in Fig. 10.52.



Fig. 10.52

10.10 Write the mesh equations for the network shown in Fig. 10.53.





10.11 Find the source voltage if the voltage across the 100 ohms is 50 V for the network in the Fig. 10.54.





10.12 The inductance matrix for the circuit of a three series connected coupled coils is given below. Find the inductances and indicate the dots for the coils.

$$L = \begin{bmatrix} 8 & -2 & 1 \\ -2 & 4 & -6 \\ 1 & -6 & 6 \end{bmatrix}$$

# OBJECTIVE-TYPE -QUESTIONS

- 1. Mutual inductance is a property associated with
  - (a) only one coil
  - (b) two or more coils
  - (c) two or more coils with magnetic coupling
- 2. Dot convention in coupled circuits is used
  - (a) to measure the mutual inductance
  - (b) to determine the polarity of the mutually induced voltage in coils
  - (c) to determine the polarity of the self induced voltage in coils

- 3. Mutually induced voltage is present independently of, and in addition to, the voltage due to self induction. (b) false
  - (a) true
- 4. Two terminals belonging to different coils are marked identically with dots, if for the different direction of current relative to like terminals the magnetic flux of self and mutual induction in each circuit add together. (a) true (b) false
- 5. The maximum value of the coefficient of coupling is
  - (a) 100% (b) more than 100%
  - (c) 90%
- 6. The case for which the coefficient of coupling K = 1 is called perfect coupling

(a) true (b) false

- 7. The maximum possible mutual inductance of two inductively coupled coils with self inductances  $L_1 = 25$  mH and  $L_2 = 100$  mH is given by
  - (a) 125 mH (b) 75 mH
  - (c) 50 mH
- 8. The value of the coefficient of coupling is more for aircored coupled circuits compared to the iron core coupled circuits.

(a) true (b) false

9. Two inductors are connected as shown in Fig. 10.55. What is the value of the effective inductance of the combination.





- (a) 8 H (b) 10 H
- (c) 4 H
- 10. Two coils connected in series have an equivalent inductance of 3 H when connected in aiding. If the self inductance of the first coil is 1 H, what is the self inductance of the second coil (Assume M = 0.5 H)
  - (a) 1 H (b) 2 H
  - (c) 3 H

11. For Fig. 10.56 shown below, the inductance matrix is given by



Fig. 10.56

(a)	2 3 1	3 1 2	1 2 3					(b)	$\begin{bmatrix} 2\\ -3\\ 1 \end{bmatrix}$	3	-3 1 -2	1 -2 3	2
(c)	2 3 1	-3 1 2	;	1 -2 3									

L



### **11.1 BASIC CONCEPTS**

Differential equations which denote rates of change, occur in various branches of science and engineering. We make use of differential equations, for example, to determine the motion of a rocket or a satellite, to determine the charge or current in an electric circuit, or to determine the vibrations of a wire or membrane. The mathematical formulation of the above problems gives rise to differential equations.

A differential equation is one which involves derivatives of one or more dependent variables with respect to one or more independent variables. Differential equations are classified according to the variables and derivatives involved in them. Ordinary differential equations are those which involve ordinary derivatives of one or more dependent variables with respect to a single independent variable. For example,

$$dy = \sin x \, dx \tag{11.1}$$

$$\frac{d^3x}{dt^4} + 3\frac{d^2x}{dt^2} + 5x = \cos t \tag{11.2}$$

In Eq. 11.1, x is an independent variable, and y is a dependent variable. In Eq. 11.2, variable t is an independent variable and x is a dependent variable. Partial differential equations are those which involve partial derivatives of one or more dependent variables with respect to more than one independent variables. For example,

$$\frac{\partial v}{\partial u} + \frac{\partial v}{\partial t} = v \tag{11.3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$
(11.4)

In Eq. 11.3, variables u and t are independent, and v is a dependent variable.

In Eq. 11.4, variables x, y, and z are independent, whereas v is a dependent variable.

The order of differential equation is the order of the highest derivative in it. Equation 11.1 is a first order differential equation, since the highest derivative involved is the first order. Similarly, Eq. 11.2 is of the 3rd order. Equations 11.3 and 11.4 are of the first and second order, respectively.

The degree of a differential equation is the degree of the derivative of the highest order; for example,

$$\frac{d^2x}{dt^2} = \left[1 + \left(\frac{dx}{dt}\right)^2\right]^{1/2}$$
(11.5)

Equation 11.5 is of the second degree, since when the radical is removed, it becomes

$$\left(\frac{d^2x}{dt^2}\right)^2 = \left[1 + \left(\frac{dx}{dt}\right)^2\right]$$
(11.6)

Differential equations are further classified as linear and non-linear.

A linear ordinary differential equation of the order n, in the dependent variable x and the independent variable t, is given in the form

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = c(t) \quad (11.7)$$

where  $a_0$  is not identically zero. The order of the equation is *n*. The term c(t) is the *forcing function* and is independent of x(t). When c(t) is zero, the equation is said to homogeneous; otherwise, it is non-homogeneous. A differential equation is said to be linear, when the dependent variable *x* and its derivatives occur in the first degree only, and no products of *x* and its derivatives are present in the equation.

For example,

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 5x = 0$$
(11.8)

In Eq. 11.8, the dependent variable, x, and its derivatives are of the first degree only. A non-linear ordinary differential equation is defined as an equation which is not linear.

For example,

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 5x^2 = 0$$
(11.9)

$$\frac{d^2x}{dt^2} + 5\left(\frac{dx}{dt}\right)^2 + 7x = 0$$
(11.10)
In Eq. 11.7, if  $a_0(t)$ ,  $a_1(t) \dots a_n(t)$  are constants, the equation is said to be linear with constant coefficients; otherwise, the equation is said to be linear with variable coefficients.

# 11.2 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Consider an *n*th order homogeneous linear differential equation with constant coefficients,

$$a_0 \frac{d^n x}{dt} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_{n-1} \frac{dx}{dt} + a_n x = 0$$
(11.11)

where  $a_0, a_1 \dots a_n$  are real constants.

Now we shall find the solution of Eq. 11.11 of the form  $x = e^{mt}$ . By assuming that  $x = e^{mt}$  is a solution for certain *m*, we have

$$\frac{dx}{dt} = me^{mt}$$
$$\frac{d^2x}{dt^2} = m^2 e^{mt}$$
$$\vdots$$
$$\frac{d^n x}{dt^n} = m^n e^{mt}$$

Substituting in Eq. 11.11, we get

$$a_{0}m^{n}e^{mt} + a_{1}m^{n-1}e^{mt} + \dots + a_{n}e^{mt} = 0$$
  

$$e^{mt}(a_{0}m^{n} + a_{1}m^{n-1} + \dots + a_{n}) = 0$$
  

$$a_{0}m^{n} + a_{1}m^{n-1} + \dots + a_{n} = 0$$
(11.12)

or where

This is called the *auxiliary*, or the characteristic equation of the given differential equation. Three cases might occur in the auxiliary equation which are, subject to the roots of Eq. 11.12 being real and distinct, real and repeated, or complex.

#### Case 1 Distinct real roots

If the roots of the Eq. 11.12,  $m_1, m_2 \dots m_n$  are real and distinct, the general solution of Eq. 11.11 is

$$x = k_1 e^{m_1 t} + k_2 e^{m_2 t} + \dots + k_n e^{m_n t}$$

where  $k_1, k_2 \dots k_n$  are arbitrary constants.

 $k_1, k_2 \dots k_n$  values can be determined by using initial conditions.

**Example 11.1** Find the solution for the following equation

$$\frac{d^{3}x}{dt^{3}} + 2\frac{d^{2}x}{dt^{2}} - \frac{dx}{dt} - 2x = 0$$

given the initial conditions

$$x''(0) = 0, x'(0) = 2, x(0) = 1$$

Solution The characteristic equation is

$$m^3 + 2m^2 - m - 2 = 0$$

By taking factors, we have

$$(m+1) (m2 + m - 2) = 0$$
  
(m+1) (m-1) (m+2) = 0

Thus, the roots are distinct, real numbers

 $m_1 = 1, m_2 = -1, m_3 = -2$ 

The general solution is

At 
$$t = 0$$
,  
 $x = k_1 e^{-t} + k_2 e^t + k_3 e^{-2t}$   
 $k_1 + k_2 + k_3 = 1$   
 $x' = -k_1 e^{-t} + k_2 e^t - 2k_3 e^{-2t}$ 
(11.13)

At 
$$t = 0$$
,  $-k_1 + k_2 - 2k_3 = 2$  (11.14)  
 $x'' = k_1 e^{-t} + k_2 e^t + 4k_3 e^{-2t}$ 

At 
$$t = 0$$
,  $k_1 + k_2 + 4k_3 = 0$  (11.15)

Solving Eqs. 11.13, 11.14 and 11.15 we get

$$k_3 = -\frac{1}{3}$$
  
 $k_2 = \frac{4}{3}$  and  $k_1 = 0$ 

The solution for the differential equation is therefore

$$x = \frac{4}{3}e^t - \frac{1}{3}e^{-2t}$$

#### Case 2 Roots are real and repeated

If the roots of Eq. 11.12 are the double real root m, and (n - 2) distinct real roots.

$$m_1, m_2 \ldots m_{n-2}$$

then the linearly independent solutions of Eq. 11.11 are

$$e^{mt}, te^{mt}, e^{m_1 t}, e^{m_2 t} \dots e^{m_{n-2} t}$$

And the general solution may be written as

$$x = k_1 e^{mt} + k_2 t e^{mt} + k_3 e^{m_1 t} + k_4 e^{m_2 t} + \dots + k_n e^{m_{n-2} t}$$

Similarly, if Eq. 11.12 has a triple real root *m*, the general solution is  $(c_1 + c_2 t + c_3 t^2) e^{mt}$ .

**Example 11.2** Find the general solution for the differential equation

$$\frac{d^3x}{dt^3} + 11\frac{d^2x}{dt^2} + 35\frac{dx}{dt} + 25x = 0$$

or

**Solution** The auxiliary equation is

 $m^3 + 11m^2 + 35m + 25 = 0$ 

By taking factors, we have

$$(m+1)(m+5)^2 = 0$$

or

$$(m + 1) (m + 5) (m + 5) = 0$$

... The general solution is

$$x(t) = (k_1 + k_2 t) e^{-5t} + k_3 e^{-5t}$$

#### Case 3 Roots are Complex Conjugate

Consider the auxiliary equation which has the complex number a + jb as a non-repeated root. The corresponding part of the general solution is  $p_1 e^{(a+jb)t} + p_2 e^{(a-jb)t}$ , where  $p_1$  and  $p_2$  are arbitrary constants.

$$p_1 e^{(a+jb)t} + p_2 e^{(a-jb)t} = p_1 e^{at} e^{jbt} + p_2 e^{at} e^{-jbt}$$
  

$$= e^{at} [p_1 e^{jbt} + p_2 e^{-jbt}]$$
  

$$= e^{at} [p_1(\cos bt + j \sin bt) + p_2 (\cos bt - j \sin bt)]$$
  

$$= e^{at} [(p_1 + p_2) \cos bt + j(p_1 - p_2) \sin bt]$$
  

$$= e^{at} [k_1 \sin bt + k_2 \cos bt]$$
  

$$k_1 = j(p_1 - p_2) \text{ and } k_2 = (p_1 + p_2)$$

where

are the new arbitrary constants.

If however, (a + jb) and (a - jb) are each *n* roots of the auxiliary equation, the corresponding general solution may be written as

$$x = e^{at} [(k_1 + k_2t + k_3t^2 + \dots + k_nt^{n-1}) \sin bt + (k_{n+1} + k_{n+2}t + k_{n+3}t^2 + \dots + k_{2n}t^{n-1}) \cos bt]$$

**Example 11.3** Find the general solution of

$$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 20x = 0$$

Solution The auxiliary equation is

$$m^{2} - 5m + 20 = 0$$
$$m = \frac{5 \pm \sqrt{25 - 80}}{2}$$

The roots are

$$m = 2.5 \pm j3.7$$

Here the roots are conjugate complex numbers a + jbwhere a = 2.5, b = 3.7

The general solution may be written as

$$x = e^{2.5t} \left( c_1 \sin 3.7t + c_2 \cos 3.7t \right)$$

## 11.3 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

Now let us consider the following non-homogeneous differential equation,

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_{n-1} \frac{dx}{dt} + a_n x = f(t)$$

where the coefficients  $a_0, a_1, ..., a_n$  are constants, and f(t) is a function of time.

The general solution may be written

$$x = x_c + x_p$$

where  $x_c$  is the complementary function, and  $x_p$  is the particular integral. Since  $x_c$  is the general solution of the corresponding homogeneous equation with f(t) replaced by zero, we have to find out the particular integral  $x_p$ . The particular integral can be calculated by the method of undetermined coefficients. This method is useful to equations

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_n x = c(t)$$

when c(t) is such that the form of a particular solution  $x_p$  of the above equation may be guessed. For example, c(t) may be a single power of t, a polynomial, an exponential, a sinusoidal function, or a sum of such functions. The method consists in assuming for  $x_p$  an expression similar to that of c(t), containing unknown coefficients which are to be determined by inserting  $x_p$  and its derivatives in the original equation.

**Example 11.4** Find the particular integral for the differential equation

$$\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 5x = 10e^{-3t}$$

**Solution** By assuming  $x_p = ke^{-3t}$ , and substituting  $x''_p$ ,  $x'_p$ , and  $x_p$  into the differential equation

$$9ke^{-3t} + 30ke^{-3t} + 5ke^{-3t} = 10e^{-3}$$
$$9k + 30k + 5k = 10$$

∴ or

$$k = \frac{10}{44} = 0.23$$

Therefore, the particular integral is  $x_p = 0.23e^{-3t}$ 

**Example 11.5** Find the particular integral for the differential equation

$$\frac{d^2x}{dt^2} + 2x = 5t^2$$

**Solution** If the driving function is the power of *t*, then we have to assume the particular solution as

$$x_p = k_1 t^2 + k_2 t + k_3$$

11.6

•••

Then

$$x_{p}'' = 2k_{1}$$

Substituting  $x_n''$  and  $x_n$  in the given differential equation, we have

 $2k_1 + 2k_1t^2 + 2k_2t + 2k_3 = 5t^2$ 

Comparing the coefficients

$$2k_1 = 5$$
 and  $2k_2 = 0$ ,  $2k_1 + 2k_3 = 0$   
 $k_1 = 2.5$  and  $k_3 = -2.5$ ,  $k_2 = 0$ 

:. The particular integral

$$x_p = 2.5t^2 - 2.5$$

**Example 11.6** Find the particular integral for the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x = 20\sin t$$

**Solution** If the driving function is sine or cosine function, the particular solution is to be assumed as

Then

$$x_p = k_1 \cos t + k_2 \sin t$$
$$x'_p = -k_1 \sin t + k_2 \cos t$$
$$x''_p = -k_1 \cos t - k_2 \sin t$$

 $-k_1 \cos t - k_2 \sin t - k_1 \sin t + k_2 \cos t + 3k_1 \cos t + 3k_2 \sin t = 20 \sin t$ Comparing the cosine terms and sine terms in the above equation, we have

$$2k_1 + k_2 = 0$$
  
-  $k_1 + 2k_2 = 20$ 

From which

 $k_1 = -4, k_2 = 8$ 

Substituting the values of  $k_1$  and  $k_2$  in particular integral

:.

$$x_p = -4\cos t + 8\sin t$$

This method of undetermined coefficients may be applied to forcing functions of the following.

1. c(t) = A

2. 
$$c(t) = A(k_1t^n + k_2t^{n-1} + k_3t^{n-3} + \dots + k_n)$$

- 3.  $c(t) = e^{mt}$ ; *m* is real or complex
- 4. Any function formed by multiplying terms of type 1, 2, or 3.

# 11.4 APPLICATIONS TO ELECTRICAL CIRCUITS

In this section, we consider the application of differential equations to circuits containing a source, resistors, inductors and capacitors. Before discussing the formation of differential equation for the circuits, let us discuss the v-i relationships for basic network elements.

*Resistor* The resistor shown in Fig. 11.1(a) has the following relation between voltage and current.



Fig. 11.1

v(t) = Ri(t)

where R is given in ohms.

Capacitor For the capacitor shown in Fig. 11.1(b), the v-i relationships are

$$i(t) = C \frac{dv(t)}{dt}$$
$$v(t) = \frac{1}{C} \int_{0}^{t} i(t) dt + v_{C}(0)$$

or

where  $v_c(0)$  is the initial voltage across the capacitor. The capacitor can be represented as shown in Fig. 11.1(c).

*Inductor* For the inductor shown in Fig. 11.2(a), the v-i relationships are

$$i(t) = \frac{1}{L} \int_{0}^{t} v(t)dt + i_{L}(0)$$
$$v(t) = L \frac{di}{dt}$$

or

where  $i_L(0)$  is the initial current passing through the circuit. The inductor can be represented as shown in Fig. 11.2(b).



Fig. 11.2

We now consider the circuit shown in Fig. 11.3



Fig. 11.3

By applying Kirchhoff's law to the circuit in Fig. 11.3, we have

$$v = Ri + L\frac{di}{dt} + \frac{1}{C}\int_{0}^{t} i \, dt + v_C(0)$$

If the capacitor has no initial charge, the above equation becomes

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt = v$$

Differentiating the above equation, we get

$$L\frac{d^{2}i}{dt^{2}} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv}{dt}$$

This is a second order linear differential equation in the single dependent variable, *i*.

**Example 11.7** The circuit shown in Fig. 11.4 consists of series R, L elements which are 5  $\Omega$  and 0.1 H, respectively. If the initial current is zero, find the current at time t > 0.



Fig. 11.4

**Solution** By applying Kirchhoff's laws, we have

$$\frac{1}{10}\frac{di}{dt} + 5i = 50 \sin 20t$$

or

$$\frac{di}{dt} + 50i = 500 \sin 20$$

and

...

$$i = i_c + i_p$$
  

$$i_c = ce^{-50t}$$
  

$$i_p = A \cos 20t + B \sin 20t$$
  

$$i'_p = -20 A \sin 20t + 20B \cos 20t$$

Substituting in the differential equation, we get

 $(D + 50)i = 500 \sin 20t$ 

 $-20A\sin 20t + 20B\cos 20t + 50A\cos 20t + 50B\sin 20t = 500\sin 20t$ 

Comparing the coefficients, we have

$$-20A + 50B = 500$$
  
 $50A + 20B = 0$ 

From which, A = -3.45 and B = 8.62

$$i_n = -3.45 \cos 20t + 8.62 \sin 20t$$

The complete solution is

$$i = i_c + i_p$$
  
=  $ce^{-50t} - 3.45 \cos 20t + 8.62 \sin 20t$ 

Applying the condition i = 0 when t = 0, we find

$$c = 3.45$$

Thus the solution is

$$i = 3.45 e^{-50t} - 3.45 \cos 20t + 8.62 \sin 20t$$

**Example 11.8** The circuit shown in Fig. 11.5 has series *R*, *L*, *C* elements which are  $2\Omega$ ,  $\frac{1}{10}$  H and  $\frac{1}{260}$  F respectively. If the initial current and initial charge on the capacitor are both zero, find the charge on the capacitor at any time t > 0.



Fig. 11.5

Solution By applying Kirchhoff's laws, we have

$$\frac{1}{10}\frac{di}{dt} + 2i + 260\int i dt = 100\sin 60t$$

Since  $i = \frac{dq}{dt}$ , this reduces to

$$\frac{1}{10}\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 260q = 100\sin 60t$$

$$\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 2600q = 1000\sin 60t$$

Since the charge on the capacitor is zero,

$$q(0) = 0$$

Since the initial current is zero and  $i = \frac{dq}{dt}$ 

$$q'(0) = 0$$

The complete solution is  $q = q_c + q_p$ The roots of characteristic equation are  $-10 \pm j50$ 

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... The complementary function becomes

$$q_c = e^{-10t} (c_1 \sin 50t + c_2 \cos 50t)$$

By assuming a particular integral, we have

$$q_p = A \sin 60t + B \cos 60t$$

Differentiating and substituting in the differential equation, we get

$$A = \frac{-25}{61}$$
 and  $B = \frac{-30}{61}$ 

The general solution is

$$q = e^{-10t} \left( c_1 \sin 50t + c_2 \cos 50t \right) - \frac{25}{61} \sin 60t \frac{-30}{61} \cos 60t$$

Differentiating once, and substituting initial conditions, we get

$$c_1 = \frac{36}{61}$$
 and  $c_2 = \frac{30}{61}$ 

: The complete solution is

$$q = e^{-10t} \left( \frac{36}{61} \sin 50t + \frac{30}{61} \cos 50t \right) - \frac{25}{61} \sin 60t - \frac{30}{61} \cos 60t$$



**Problem 11.1** Determine the general solution for the differential equation.

$$\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} - \frac{dx}{dt} + 3x = 0$$

given the initial conditions

$$x''(0) = 3; x'(0) = 1, x(0) = 0$$

or

Solution The auxiliary equation is

$$m^3 - 3m^2 - m + 3 = 0$$

By taking factors, we have

$$(m + 1) (m^2 - 4m + 3) = 0$$
  
 $(m + 1) (m - 1) (m - 3) = 0$ 

Thus, the roots are distinct and real numbers

 $m_1 = -1, m_2 = 1, m_3 = 3$ 

The general solution is

At 
$$t = 0$$
,  
 $x = k_1 e^{-t} + k_2 e^t + k_3 e^{3t}$   
 $k_1 + k_2 + k_3 = 0$   
 $x' = -k_1 e^{-t} + k_2 e^t + 3k_3 e^{3t}$ 
(11.16)

At 
$$t = 0$$
,  $-k_1 + k_2 + 3k_3 = 1$  (11.17)  
 $x'' = k_1 e^{-t} + k_2 e^t + 9k_3 e^{3t}$ 

At 
$$t = 0$$
  $k_1 + k_2 + 9k_3 = 3$  (11.18)

Solving Eqs 11.16, 11.17 and 11.18, we get

$$k_1 = \frac{-1}{8}, k_2 = \frac{-1}{4}, k_3 = \frac{3}{8}$$

Thus, the solution for the differential equation is

$$x = \frac{-1}{8}e^{-t} - \frac{1}{4}e^{t} + \frac{3}{8}e^{3t}$$

Problem 11.2 Find the general solution for the differential equation

$$\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + 32x = 0$$

Solution The auxiliary equation is

$$m^3 - 6m^2 + 32 = 0$$

By taking factors, we have

$$(m+2) (m-4)^2 = 0$$
$$(m+2) (m-4) (m-4) = 0$$

or

Thus, the roots are real and repeated

$$m_1 = -2, m_2 = 4, m_3 = 4$$

The general solution is

$$x(t) = (k_1 + k_2 t)e^{+4t} + k_3 e^{-2t}$$

Problem 11.3 Find the general solution for the differential equation

$$\frac{d^4x}{dt^4} - 4\frac{d^3x}{dt^3} + 14\frac{d^2x}{dt^2} - 20\frac{dx}{dt} + 25x = 0$$

or

Solution The auxiliary equation is

$$m^4 - 4m^3 + 14m^2 - 20m + 25 = 0$$

 $m^4 - 4m^3 + 14m^2 - 20m^2$ The roots of the characteristic equation are

$$(1+j2), (1-j2), (1+j2)(1-j2)$$

Since each pair of conjugate complex roots is double, the general solution is

$$x(t) = e^{t} \left[ (k_1 + k_2 t) \sin 2t + (k_3 + k_4 t) \cos 2t \right]$$

or

**Problem 11.4** Determine the general solution for the differential equation

 $x(t) = k_1 e^t \sin 2t + k_2 t e^t \sin 2t + k_3 e^t \cos 2t + k_4 t e^t \cos 2t$ 

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 25x = 0$$
  
x(0) = 0, x'(0) = -1

 $m = \frac{6 \pm \sqrt{36 - 100}}{2} = 3 \pm j4$ 

**Solution** The auxiliary equation is

$$m^2 - 6m + 25 = 0$$

The roots are

Here the roots are the conjugate complex numbers. The general solution of the differential equations

$$x(t) = e^{3t}(k_1 \sin 4t + k_2 \cos 4t)$$
(11.19)

Differentiating once, we get

$$x'(t) = e^{3t} [(3k_1 - 4k_2) \sin 4t + (4k_1 + 3k_2) \cos 4t]$$
(11.20)  
At  $t = 0, x(0) = 0$ 

Substituting in Eq. 11.19, we get

$$(k_1 \sin 0 + k_2 \cos 0) e^0 = 0$$
  
 $k_2 = 0$  (11.21)

Similarly, at t = 0 x'(0) = -1Substituting in Eq. 11.20, we get

$${}^{0}\left[4k_{1}+3k_{2}\right]=-1 \tag{11.22}$$

Solving Eqs 11.21 and 11.22, we get

$$4k_1 = -1$$
 i.e.,  $k_1 = -\frac{1}{4}$ 

The solution for the differential equation is

$$x(t) = -\frac{1}{4}e^{3t}\sin 4t$$

Problem 11.5 Find the general solution for the differential equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 20e^t - 50\cos t$$

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Solution The corresponding homogeneous equation is

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0$$

The complementary function is

$$x_c = k_1 e^{-t} + k_2 e^{3t}$$

If the driving function is  $20e^{t} - 50 \cos t$ , then we assume

$$x_n = Ae^t + B\sin t + C\cos t$$

as a particular solution, or

$$x'_{p} = Ae^{t} + B\cos t - C\sin t$$
$$x''_{p} = Ae^{t} - B\sin t - C\cos t$$

Substituting the above in differential equation, we get

$$(Ae^t - B\sin t - C\cos t) - 2(Ae^t + B\cos t - C\sin t)$$

$$-3(Ae^{t} + B\sin t + C\cos t) = 20 e^{t} - 50\cos t$$

Comparing exponential, sine and cosine terms on both sides

$$A - 2A - 3A = 20 \tag{11.23}$$

$$-B + 2C - 3B = 0 \tag{11.24}$$

$$-C - 2B - 3C = -50 \tag{11.25}$$

From the above equations, we get

$$A = -5, B = 5, C = 10$$

Thus, the particular solution is

 $x_p = -5e^t + 5\sin t + 10\cos t$ 

Therefore, the complete solution is

$$x = x_c + x_p$$
  
=  $k_1 e^{-t} + k_2 e^{3t} - 5e^t + 5 \sin t + 10 \cos t$ 

**Problem 11.6** Find the general solution of the differential equation,

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 2t^2 + e^t + 2te^t + 4e^{3t}$$

Solution The corresponding homogeneous equation is

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 0$$

The auxiliary equation is

$$(m^2-m-6)=0$$

The roots of the equation are

$$(m-3)(m+2) = 0$$

Thus, the complementary function is

$$x_c = k_1 e^{3t} + k_2 e^{-2t}$$

To find the particular solution, we assume

$$x_p = p_1 t^2 + p_2 t + p_3 + p_4 e^{3t} + p_5 t^2 e^t + p_6 t e^{4t}$$

From this, we have

$$\begin{array}{l} x'_p = 2tp_1 + p_2 + 3p_4 e^{3t} + 2p_5 t e^t + p_5 t^2 e^t + p_6 t e^t + p_6 e^t \\ x''_p = 2p_1 + 9p_4 e^{3t} + 2p_5 e^t + 2p_5 t e^t + p_5 t^2 e^t + 2p_5 t e^t \\ + p_6 t e^t + p_6 e^t + p_6 e^t \end{array}$$

Substituting  $x_p, x'_p$  and  $x''_p$  into differential equation and equating coefficients of like terms, we get

$$p_1 = 1, p_2 = 3, p_3 = 3.5, p_4 = 2, p_5 = -1, p_6 = -3$$

Thus, the particular integral is

$$x_p = t^2 + 3t + 3.5 + 2e^{3t} - t^2e^t - 3te^t$$

Therefore, the general solution is

$$x = x_c + x_p = k_1 e^{3t} + k_2 e^{-2t} + t^2 + 3t + 3.5 + 2e^{3t} - t^2 e^t - 3te^t$$

Problem 11.7 Find the general solution of differential equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 35x = t\sin t$$

where x(0) = 5; x'(0) = 3

Solution The corresponding homogeneous equation is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 35x = 0$$

The auxiliary equation is

 $(m^2 + 2m - 35) = 0$ 

The roots of the equation are

$$(m+7)(m-5) = 0$$

Thus the complementary function is

$$x_{c} = k_{1}e^{-7t} + k_{2}e^{+5t}$$

$$x_{c}' = -7k_{1}e^{-7t} + 5k_{2}e^{+5t}$$
At  $t = 0$ ,  $x_{c}(0) = 5$ 

$$k_{1} + k_{2} = 5$$
At  $t = 0$ ,  $x_{c}'(0) = 3$ 

$$-7k_{1} + 5k_{2} = 3$$
Solving the above equations, we get

Solving the above equations, we get

$$k_1 = 1.83, k_2 = 3.17$$

Therefore, the complementary function is

$$x_c = 1.83 \ e^{-7t} + 3.17 \ e^{-5t}$$

To find the particular solution, we assume

$$x_p = p_1 t \sin t + p_2 t \cos t + p_3 \sin t + p_4 \cos t$$

Then

$$\begin{aligned} x'_{p} &= p_{1} \sin t + p_{1} t \cos t + p_{2} \cos t - p_{2} t \sin t + p_{3} \cos t - p_{4} \sin t \\ x''_{p} &= p_{1} \cos t - p_{1} t \sin t + p_{1} \cos t - p_{2} \sin t - p_{2} \sin t \\ &- p_{2} t \cos t - p_{3} \sin t - p_{4} \cos t \end{aligned}$$

Substituting  $x_p, x'_p$  and  $x''_p$  into differential equation and equating coefficients of like terms, we get

$$p_1 = 1.01; p_2 = 0.056; p_3 = 0.05; p_4 = 0.062$$

Thus the particular integral is

$$x_n = 1.01t \sin t + 0.056t \cos t + 0.05 \sin t + 0.062 \cos t$$

Therefore, the complete solution is

$$x = x_c + x_p = 1.83e^{-7t} + 3.17e^{-5t} + 1.01t \sin t + 0.056t \cos t + 0.05 \sin t + 0.062 \cos t$$

**Problem 11.8** For the series RL circuit shown in Fig. 11.6, find the current at time t > 0. The switch is closed at t = 0. Assume the initial current in the circuit is zero.



Fig. 11.6

**Solution** By applying Kirchhoff's law to the circuit, we have

$$0.5\frac{di}{dt} + 10i = 20$$
$$\frac{di}{dt} + 20i = 40$$

or

The auxiliary equation is

(m + 20) = 0

Therefore, the complementary function

$$i_c = k_1 e^{-20t}$$

The particular integral is

$$i_p = 20e^{-20t} \int e^{20t} dt = \frac{20}{20} = 1$$

Therefore, the complete solution is

$$i = i_c + i_p = k_1 e^{-20t} + 1$$
At  $t = 0$ ,  $i(0) = 0$   
 $\therefore$   $k_1 = -1$   
The complete solution is

$$i = (1 - e^{-20t}) A$$

**Problem 11.9** For the circuit shown in Fig. 11.7, find the current at t > 0. The switch is closed at t = 0. Assume no initial charge on the capacitor.



Solution By applying Kirchhoff's law to the circuit, we have

$$10i + \frac{1}{2 \times 10^{-4}} idt = 50$$

Differentiating the above equation, we get

$$10\frac{di}{dt} + \frac{i}{2 \times 10^{-4}} = 0$$
$$\frac{di}{dt} + 0.5 \times 10^{3}i = 0$$

The auxiliary equation is

$$(m + 500) = 0$$

Since, the equation is a linear homogeneous one, there is no particular integral.

Therefore, the complementary function is

$$i = k_1 e^{-500t}$$

At t = 0, the current passing through the circuit is  $i = \frac{V}{R} = \frac{50}{10} = 5$ A

:. i(0) = 5At t = 0,  $k_1 = 5$ 

The current equation becomes

$$i = 5e^{-500t}$$

\*\*

**Problem 11.10** For the circuit shown in Fig. 11.8, determine the current at any time t > 0. The switch is closed at t = 0. Assume that initial current and initial charge on the capacitor are zero.



Fig. 11.8

Solution By applying Kirchhoff's law, we have

$$30i + 0.2\frac{di}{dt} + \frac{1}{40 \times 10^{-6}} \int i dt = 100$$

Differentiating the above equation, we have

$$\frac{d^2i}{dt^2} + 150\frac{di}{dt} + \frac{1}{8 \times 10^{-6}}i = 0$$

The roots of the auxiliary equation are

$$m_1 = -75 + j345.5$$
$$m_2 = -75 - j345.5$$

Hence, the current is

$$i = e^{-75t} (c_1 \cos 345.5t + c_2 \sin 345.5t) A$$

At t = 0,

*:*..

$$i(0) = 0$$
  

$$c_1 = 0$$

$$i' = c_2 \{ e^{-75t} (345.5) \cos 345.5t + e^{-75t} (-75) \sin 345.5t \}$$

At t = 0, the complete voltage appears across inductor

$$0.2\frac{di}{dt} = 100$$
$$\frac{di}{dt} = 500$$

÷

:. At 
$$t = 0$$
,  $i'(0) = 500$ 

$$b00 = c_2 (345.5)$$
  
 $c_2 = 1.45$ 

Thus the required current is

$$i = 1.45e^{-75t} \sin 345.5t$$
 A



11.4

PRACTICE PROBLEMS

11.1 Find the general solution of each of the following differential equations.

(a) 
$$4 \frac{d^2 x}{dt^2} - 12 \frac{dx}{dt} + 5x = 0$$
  
(b)  $4 \frac{d^2 x}{dt^2} + x = 0$ 

11.2 Find the general solution of each of the following differential equations.

(a) 
$$\frac{d^5x}{dt^5} - 2\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} = 0$$
  
(b)  $\frac{d^4x}{dt^4} + 6\frac{d^3x}{dt^3} + 15\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 12x = 0$   
(c)  $\frac{d^4x}{dt^4} = 0$ 

11.3 Find the general solution of each of the following differential equations.

(a) 
$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 8x = 0$$
  
where  $x(0) = 2x'(0) = 4$   
(b)  $9\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + x = 0$   
 $x(0) = 4; x'(0) = -1$   
(c)  $4\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 37x = 0$   
where  $x(0) = 3, x'(0) = -2$   
(d)  $\frac{d^3x}{dt^3} - 5\frac{d^2x}{dt^2} + 9\frac{dx}{dt} - 5x = 0$   
 $x(0) = 0, x'(0) = 1, x'(0) = 3$   
Solve the following differential equations

(a) 
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = \sin 2t$$
  
(b)  $\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = e^{-t}$   
(c)  $\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 2x = e^t + \cos t$ 

 L

(d) 
$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3t^2 e^{2t} \sin 2t$$
  
(e)  $\frac{d^4x}{dt^4} + 2\frac{d^2x}{dt^2} + x = t^2 \cos^2 t$ 

11.5 Solve the following differential equations

(a) 
$$\frac{d^{2}x}{dt^{2}} - 4 \frac{dx}{dt} + 3x = 4te^{-3t}$$
$$x(0) = 6x'(0) = 3$$
  
(b) 
$$\frac{d^{2}x}{dt^{2}} + 4x = 3te^{t} + 2e^{t} - \sin t$$
$$x(0) = 1, x'(0) = 0, x'(0) = 2$$
  
(c) 
$$\frac{d^{2}x}{dt^{2}} - 6 \frac{dx}{dt} + 9x = 8t^{2} + 3 - 6e^{2t}$$
$$x''(0) = 3, x'(0) = 0, x(0) = 3$$
  
(d) 
$$\frac{d^{3}x}{dt^{3}} - 6 \frac{d^{2}x}{dt^{2}} + 9 \frac{dx}{dt} - 4x = 2te^{2t} + 6e^{t}$$
$$x(0) = 1, x'(0) = 0$$

11.6 For the circuit shown in Fig. 11.9, determine the current at any time t > 0. The switch is closed at t = 0. Assume no initial charge on the capacitor.





11.7 For the circuit shown in Fig. 11.10, determine the current at any time t > 0. The switch is closed at t = 0. Assume no initial current in the circuit.



Fig. 11.10

11.8 For the circuit shown in Fig. 11.11, determine the current at any time t > 0. The switch is closed at t = 0. Assume no initial charge on the capacitor.





11.9 For the circuit shown in Fig. 11.12, determine the current at any time t > 0. The switch is closed at t = 0.





11.10 For the circuit shown in Fig. 11.13, determine the current at any time t > 0. The switch is closed at t = 0. Assume no initial charge on the capacitor.



Fig. 11.13



2. The particular integral of the equation  $\frac{d^4y}{dx^4} + 4y = x^4$  will be

(a) 
$$\frac{1}{4}(x^4-6)$$
  
(b)  $\frac{1}{4}(x^6-6)$   
(c)  $\frac{1}{4}(x^6+6)$   
(d)  $\frac{1}{4}(x+6)$ 

- 3. The differential equation  $\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + k^2y = 0$ , where *a* is any constant, can be expressed as  $y(x) = x^n [c_1 J_n(kx) + c_2 J_{-n}(kx)]$  where *n* is
  - (b) an integer(d) a fraction (a) an odd integer
  - (c) an even integer
- 4. The Bessal's differential equation xy'' + xy' + xy = 0 is a
  - (a) linear non-homogeneous equation
  - (b) non-linear equation
  - (c) non-linear homogeneous equation with constant coefficients
  - (d) linear homogeneous with variable coefficients
- 5. The general solution of  $(D^2 + 4)y = 0$  is
  - (b)  $v = Ae^{2x} + Be^{-2x}$ (a)  $v = A \cos(2x + B)$ (d)  $y = e^{2x} (A - Bx)$ (c)  $y = A \cos 2x + B \sin 2x$

6. The complementary function of  $(D^2 + 9)y = 1$  is

- (b)  $\frac{-1}{9}$ (a)  $\frac{1}{9}$ (c)  $c_1 e^{3x} + c_2 e^{-3x}$ (d)  $c_1 \sin 3x + c_2 \cos 3x$
- 7. The solution of the differential equation

$$y''(t) - 2y'(t) + y(t) = 1 \text{ is}$$
(a)  $y(t) = c_1 e^t + c_2 e^{-t} + 1$  (b)  $y(t) = (c_1 + c_2 t) e^t + 1$   
(c)  $y(t) = (c_1 + c_2 t) e^{-t} + 1$  (d)  $y(t) = (c_1 + c_2) t e^t - 1$ 

8. The differential equation of an electric current containing resistance Rand a capacitor C in series with the voltage source V is

(a) 
$$\frac{dV}{dt} = Ri + \int \frac{1}{C}idt$$
  
(b)  $\frac{dV}{dt} = R\frac{di}{dt} + \int \frac{1}{C}idt$   
(c)  $\frac{dV}{dt} = R\frac{di}{dt} + \frac{i}{C}$   
(d)  $V = R\frac{di}{dt} + \frac{i}{C}$ 

9. The particular integral of differential equation

$$3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x \text{ is}$$

(a) <i>x</i>	(b) $\frac{x}{2}$
(c) $\frac{x}{3}$	(d) $x^4$

••••

10. The differential equation of an electric current containing resistance R and an inductor L in series with a constant voltage source V is

(a) 
$$V = R \int i dt + Li$$
  
(b)  $V = Ri + L \int \left(\frac{di}{dt}\right) dt$   
(c)  $V = Ri + L \int i dt$   
(d)  $\frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} = 0$ 



## 12.1 STEADY STATE AND TRANSIENT REPONSE

A circuit having constant sources is said to be in steady state if the currents and voltages do not change with time. Thus, circuits with currents and voltages having constant amplitude and constant frequency sinusoidal functions are also considered to be in a steady state. That means that the amplitude or frequency of a sinusoid never changes in a steady state circuit.

In a network containing energy storage elements, with change in excitation, the currents and voltages change from one state to other state. The behaviour of the voltage or current when it is changed from one state to another is called the transient state. The time taken for the circuit to change from one steady state to another steady state is called the transient time. The application of KVL and KCL to circuits containing energy storage elements results in differential, rather than algebraic, equations. When we consider a circuit containing storage elements which are independent of the sources, the response depends upon the nature of the circuit and is called the natural response. Storage elements deliver their energy to the resistances. Hence the response changes with time, gets saturated after some time, and is referred to as the transient response. When we consider sources acting on a circuit, the response depends on the nature of the source or sources. This response is called *forced response*. In other words, the complete response of a circuit consists of two parts: the forced response and the transient response. When we consider a differential equation, the complete solution consists of two parts: the complementary function and the particular solution. The complementary function dies out after short interval, and is referred to as the transient response or source free response. The particular solution is the steady state response, or the forced response. The first step in

finding the complete solution of a circuit is to form a differential equation for the circuit. By obtaining the differential equation, several methods can be used to find out the complete solution.

## 12.2 DC RESPONSE OF AN R-L CIRCUIT

Consider a circuit consisting of a resistance and inductance as shown in Fig. 12.1. The inductor in the circuit is initially uncharged and is in series with the resistor. When the switch S is closed, we can find the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.



Fig. 12.1

$$V = Ri + L \frac{di}{dt}$$
(12.1)

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$
(12.2)

In the above equation, the current i is the solution to be found and V is the applied constant voltage. The voltage V is applied to the circuit only when the switch S is closed. The above equation is a linear differential equation of first order. Comparing it with a non-homogeneous differential equation

$$\frac{dx}{dt} + Px = K \tag{12.3}$$

whose solution is

$$x = e^{-pt} \int K e^{+Pt} dt + c e^{-Pt}$$
(12.4)

where c is an arbitrary constant. In a similar way, we can write the current equation as

$$i = ce^{-(R/L)t} + e^{-(R/L)t} \int \frac{V}{L} e^{(R/L)t} dt$$
  

$$i = ce^{-(R/L)t} + \frac{V}{R}$$
(12.5)

:.

To determine the value of *c* in Eq. 12.5, we use the initial conditions. In the circuit shown in Fig. 12.1, the switch *S* is closed at t = 0. At  $t = 0^-$ , i.e. just before closing the switch *S*, the current in the inductor is zero. Since the inductor does not allow sudden changes in currents, at  $t = 0^+$  just after the switch is closed, the current remains zero.

or

Thus at

t = 0, i = 0

Substituting the above condition in Eq. 12.5, we have

 $0 = c + \frac{V}{R}$  $c = -\frac{V}{R}$ 

Hence

Substituting the value of c in Eq. 5, we get

$$i = \frac{V}{R} - \frac{V}{R} \exp\left(-\frac{R}{L}t\right)$$
$$i = \frac{V}{R} \left(1 - \exp\left(-\frac{R}{L}t\right)\right)$$
(12.6)

Equation 12.6 consists of two parts, the steady state part V/R, and the transient part  $(V/R)e^{-(R/L)t}$ . When switch S is closed, the response reaches a steady state value after a time interval as shown in Fig. 12.2.

Here the transition period is defined as the time taken for the current to reach its final or steady state value from its initial value. In the transient part of the solution, the quantity L/R is important in describing the curve since L/R is the time required for the current to reach from its initial value of zero to the final value V/R. The time constant of a function  $\frac{V}{R} e^{-\binom{R}{L}t}$  is the time at which the exponent of e is unity, where e is t



the exponent of *e* is unity, where *e* is the base of the natural logarithms. The term L/R is called the *time constant* and is denoted by  $\tau$ 

$$\tau = \frac{L}{R} \sec \theta$$

 $\therefore$  The transient part of the solution is

$$i = -\frac{V}{R} \exp\left(-\frac{R}{L}t\right) = -\frac{V}{R}e^{-t/\tau}$$

At one TC, i.e. at one time constant, the transient term reaches 36.8 percent of its initial value.

$$i(\tau) = -\frac{V}{R}e^{-t/\tau} = -\frac{V}{R}e^{-1} = -0.368\frac{V}{R}$$

Similarly,

:.

$$i(2\tau) = -\frac{V}{R}e^{-2} = -0.135 \frac{V}{R}$$

$$i(3\tau) = -\frac{V}{R}e^{-3} = -0.0498 \frac{V}{R}$$
$$i(5\tau) = -\frac{V}{R}e^{-5} = -0.0067 \frac{V}{R}$$

After 5 TC, the transient part reaches more than 99 percent of its final value.

In Fig. 12.1, we can find out the voltages and powers across each element by using the current.

Voltage across the resistor is

$$v_{R} = Ri = R \times \frac{V}{R} \left[ 1 - \exp\left(-\frac{R}{L}t\right) \right]$$
$$v_{R} = V \left[ 1 - \exp\left(-\frac{R}{L}t\right) \right]$$

*:*.

Similarly, the voltage across the inductance is

$$v_L = L \frac{di}{dt}$$
$$= L \frac{V}{R} \times \frac{R}{L} \exp\left(-\frac{R}{L}t\right) = V \exp\left(-\frac{R}{L}t\right)$$

The response are shown in Fig. 12.3 Power in the resistor is

$$p_R = v_R i = V \left( 1 - \exp\left(-\frac{R}{L}t\right) \right) \left( 1 - \exp\left(-\frac{R}{L}t\right) \right) \frac{V}{R}$$
$$= \frac{V^2}{R} \left( 1 - 2\exp\left(-\frac{R}{L}t\right) + \exp\left(-\frac{2R}{L}t\right) \right)$$

Power in the inductor is

$$p_L = v_L i = V \exp\left(-\frac{R}{L}t\right) \times \frac{V}{R} \left(1 - \exp\left(-\frac{R}{L}t\right)\right)$$
$$= \frac{V^2}{R} \left(\exp\left(-\frac{R}{L}t\right) - \exp\left(-\frac{2R}{L}t\right)\right)$$

The responses are shown in Fig. 12.4.



Fig. 12.3

Fig. 12.4

**Example 12.1** A series RL circuit with R = 30 W and L = 15 H has a constant voltage V = 60 V applied at t = 0 as shown in Fig. 12.5. Determine the current *i*, the voltage across resistor and the voltage across the inductor.



Solution By applying Kirchhoff's voltage law, we get

$$15\frac{di}{dt} + 30i = 60$$
$$\frac{di}{dt} + 2i = 4$$

The general solution for a linear differential equation is

Dt.

where 
$$P = 2, K = 4$$
  
 $\therefore$   $i = ce^{-Pt} + e^{-Pt} \int Ke^{Pt} dt$   
 $i = ce^{-2t} + e^{-2t} \int 4e^{2t} dt$   
 $i = ce^{-2t} + 2$ 

At t = 0, the switch S is closed.

....

Since the inductor never allows sudden changes in currents. At  $t = 0^+$  the current in the circuit is zero.

Therefore at	$t = 0^+, i = 0$
·:	0 = c + 2
<b>.</b>	c = -2

Substituting the value of c in the current equation, we have

$$i = 2(1 - e^{-2t}) A$$

Voltage across resistor  $v_p = iR$ 

$$= 2(1 - e^{-2t}) \times 30 = 60 (1 - e^{-2t}) V$$

Voltage across inductor  $v_L = L \frac{di}{dt}$ 

$$= 15 \times \frac{d}{dt} 2(1 - e^{-2t}) = 30 \times 2e^{-2t} = 60e^{-2t} V$$

#### 12.3 **DC RESPONSE OF AN R-C CIRCUIT**

Consider a circuit consisting of resistance and capacitance as shown in Fig. 12.6. The capacitor in the circuit is initially uncharged, and is in series with a resistor. When the switch *S* is closed at t = 0, we can determine the complete solution for the current. Application of the Kirchhoff's voltage law to the circuit results in the following differential equation.





$$V = Ri + \frac{1}{C} \int i \, dt \tag{12.7}$$

By differentiating the above equation, we get

$$0 = R \frac{di}{dt} + \frac{i}{C}$$
(12.8)

or

*.*..

$$\frac{di}{dt} + \frac{1}{RC}i = 0 \tag{12.9}$$

Equation 12.9 is a linear differential equation with only the complementary function. The particular solution for the above equation is zero. The solution for this type of differential equation is

$$i = c e^{-t/RC} \tag{12.10}$$

Here, to find the value of c, we use the initial conditions.

In the circuit shown in Fig. 12.6, switch *S* is closed at t = 0. Since the capacitor never allows sudden changes in voltage, it will act as a short circuit at  $t = 0^+$ . So, the current in the circuit at  $t = 0^+$  is V/R

At 
$$t = 0$$
, the current  $i = \frac{V}{R}$ 

Substituting this current in Eq. 12.10, we get

$$\frac{V}{R} =$$

С

: The current equation becomes

$$i = \frac{V}{R} e^{-t/RC} \tag{12.11}$$

When switch *S* is closed, the response decays with time as shown in Fig. 12.7.

In the solution, the quantity *RC* is the time constant, and is denoted by  $\tau$ , where  $\tau = RC \sec \tau$ 

After 5 TC, the curve reaches 99 per cent of its final value. In Fig. 12.6, we can find out the voltage across each element by using the current equation.



Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R} e^{-(1/RC)t}; v_R = V e^{-\frac{t}{RC}}$$

Similarly, voltage across the capacitor is

$$v_{C} = \frac{1}{C} \int i dt$$
  
=  $\frac{1}{C} \int \frac{V}{R} e^{-t/RC} dt$   
=  $-\left(\frac{V}{RC} \times RC e^{-t/RC}\right) + c = -Ve^{-t/RC} + c$ 

At t = 0, voltage across capacitor is zero

$$c = V$$
  
$$v_C = V(1 - e^{-t/RC})$$

The responses are shown in Fig. 12.8.

Power in the resistor

$$p_R = v_R i = V e^{-t/RC} \times \frac{V}{R} e^{-t/RC} = \frac{V^2}{R} e^{-2t/RC}$$

Power in the capacitor

$$p_{C} = v_{C}i = V(1 - e^{-t/RC}) \frac{V}{R} e^{-t/RC}$$
$$= \frac{V^{2}}{R} (e^{-t/RC} - e^{-2t/RC})$$

The responses are shown in Fig. 12.9.



**Example 12.2** A series RC circuit consists of resistor of 10  $\Omega$  and capacitor of 0.1 F as shown in Fig. 12.10. A constant voltage of 20 V is applied to the circuit at t = 0. Obtain the current equation. Determine the voltages across the resistor and the capacitor.



Fig. 12.10

Solution BY applying Kirchhoff's law, we get

$$10i + \frac{1}{0.1} \int i \, dt = 20$$

Differentiating with respect to t we get

$$10 \frac{di}{dt} + \frac{i}{0.1} = 0$$
$$\frac{di}{dt} + i = 0$$

The solution for the above equation is  $i = ce^{-t}$ 

At t = 0, switch S is closed. Since the capacitor does not allow sudden changes in voltage, the current in the circuit is i = V/R = 20/10 = 2 A.

At t = 0, i = 2 A.

... The current equation  $i = 2e^{-t}$ Voltage across the resistor is  $v_R = i \times R = 2e^{-t} \times 10 = 20e^{-t}$  V Voltage across the capacitor is  $v_C = V\left(1 - e^{-\frac{t}{RC}}\right)$  $= 20 (1 - e^{-t})$  V

### 12.4 DC RESPONSE OF AN R-L-C CIRCUIT

Consider a circuit consisting of resistance, inductance and capacitance as shown in Fig. 12.11. The capacitor and inductor are initially uncharged, and are in series with a resistor. When switch *S* is closed at t = 0, we can determine the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.



-

Fig. 12.11

$$V = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$$
(12.12)

By differentiating the above equation, we have

$$D = R \frac{di}{dt} + L \frac{d^2 i}{dt^2} + \frac{1}{C} i$$
 (12.13)

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$
(12.14)

or

The above equation is a second order linear differential equation, with only complementary function. The particular solution for the above equation is zero. Characteristic equation for the above differential equation is

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0$$
(12.15)

*:*..

The roots of Eq. 12.15 are

$$D_1, D_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$K_1 = -\frac{R}{2L} \text{ and } K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$D_1 = K_1 + K_2 \text{ and } D_2 = K_1 - K_2$$

By assuming

Here  $K_2$  may be positive, negative or zero.

 $K_2$  is positive, when  $\left(\frac{R}{2L}\right)^2 > 1/LC$ 

The roots are real and unequal, and give the over damped response as shown in Fig. 12.12. Then Eq. 12.14 becomes

$$[D - (K_1 + K_2)] [D - (K_1 - K_2)] i = 0$$

The solution for the above equation is

$$i = c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$$

i 🛦

The current curve for the overdamped case is shown in Fig. 12.12.

 $K_2$  is negative, when  $(R/2L)^2 < 1/LC$ 

The roots are complex conjugate, and give the underdamped response as shown in Fig. 12.13. Then Eq. 12.14 becomes

 $[D - (K_1 + jK_2)] [D - (K_1 - jK_2)]i = 0$ 

The solution for the above equation is



The current curve for the underdamped case is shown in Fig. 12.13.

 $K_2$  is zero, when  $(R/2L)^2 = 1/LC$ 

The roots are equal, and give the critically damped response as shown in Fig. 12.14. Then Eq. 12.14 becomes

$$(D - K_1) (D - K_1)i = 0$$

The solution for the above equation is

$$i = e^{K_1 t} (c_1 + c_2 t)$$



The current curve for the critically damped case is shown in Fig. 12.14.



Fig. 12.14







**Example 12.3** The circuit shown in Fig. 12.15 consists of resistance, inductance and capacitance in series with a 100 V constant source when the switch is closed at t = 0. Find the current transient.

**Solution** At t = 0, switch S is closed when the 100 V source is applied to the circuit and results in the following differential equation.

....



Differentiating the Eq. 12.16, we get

$$0.05 \frac{d^2 i}{dt^2} + 20 \frac{di}{dt} + \frac{1}{20 \times 10^{-6}} i = 0$$
$$\frac{d^2 i}{dt^2} + 400 \frac{di}{dt} + 10^6 i = 0$$
$$(D^2 + 400D + 10^6)i = 0$$

$$D_1, D_2 = -\frac{400}{2} \pm \sqrt{\left(\frac{400}{2}\right)^2 - 10^6}$$
$$= -200 \pm \sqrt{(200)^2 - 10^6}$$
$$D_1 = -200 + j979.8$$
$$D_2 = -200 - j979.8$$

Therefore the current

$$i = e^{+k_1 t} [c_1 \cos K_2 t + c_2 \sin K_2 t)]$$
  

$$i = e^{-200t} [c_1 \cos 979.8t + c_2 \sin 979.8t)] A$$

[0

At t = 0, the current flowing through the circuit is zero

$$i = 0 = (1) [c_1 \cos 0 + c_2 \sin 0]$$
  
$$\therefore \qquad c_1 = 0$$
  
$$\therefore \qquad i = e^{-200t} c_2 \sin 979.8t \text{ A}$$

Differentiating, we have

$$\frac{di}{dt} = c_2 \left[ e^{-200t} \ 979.8 \cos 979.8t + e^{-200t} \left( -200 \right) \sin 979.8t \right]$$

*:*.

At t = 0, the voltage across inductor is 100 V

$$\therefore \qquad \qquad L \frac{di}{dt} = 100$$

or

At 
$$t = 0$$
  $\frac{di}{dt} = 2000 = c_2 \,979.8 \cos 0$ 

 $\frac{di}{dt} = 2000$ 

*:*.

*.*..

$$c_2 = \frac{2000}{979.8} = 2.04$$

The current equation is

$$i = e^{-200t}$$
 (2.04 sin 979.8t) A

### 12.5 SINUSOIDAL RESPONSE OF R-L CIRCUIT

Consider a circuit consisting of resistance and inductance as shown in Fig. 12.16. The switch, S, is closed at t = 0. At t = 0, a sinusoidal voltage V cos

 $(\omega t + \theta)$  is applied to the series R-L circuit, where V is the amplitude of the wave and  $\theta$  is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.





$$V\cos(\omega t + \theta) = Ri + L\frac{di}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t + \theta)$$

$$V = \frac{1}{2} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t + \theta)$$

$$\frac{1}{2} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t + \theta)$$

The corresponding characteristic equation is

$$\left(D + \frac{R}{L}\right)i = \frac{V}{L}\cos\left(\omega t + \theta\right)$$
 (12.18)

For the above equation, the solution consists of two parts, viz. complementary function and particular integral.

The complementary function of the solution i is

$$i_c = c e^{-t(R/L)}$$
 (12.19)

The particular solution can be obtained by using undetermined co-efficients.

By assuming  $i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta)$  (12.20)

$$i'_{p} = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)$$
(12.21)

Substituting Eqs 12.20 and 12.21 in Eq. 12.18, we have

$$\{-A\omega\sin(\omega t+\theta)+B\omega\cos(\omega t+\theta)+\frac{R}{L} \{A\cos(\omega t+\theta)\}$$

$$+B\sin(\omega t+\theta)\}=\frac{V}{L}\cos(\omega t+\theta)$$

or 
$$\left(-A\omega + \frac{BR}{L}\right)\sin(\omega t + \theta) + \left(B\omega + \frac{AR}{L}\right)\cos(\omega t + \theta) = \frac{V}{L}\cos(\omega t + \theta)$$

Comparing cosine terms and sine terms, we get

$$-A\omega + \frac{BR}{L} = 0$$
$$B\omega + \frac{AR}{L} = \frac{V}{L}$$

From the above equations, we have

$$A = V \frac{R}{R^{2} + (\omega L)^{2}}$$
$$B = V \frac{\omega L}{R^{2} + (\omega L)^{2}}$$

Substituting the values of A and B in Eq. 12.20, we get

$$i_p = V \frac{R}{R^2 + (\omega L)^2} \cos(\omega t + \theta) + V \frac{\omega L}{R^2 + (\omega L)^2} \sin(\omega t + \theta) \quad (12.22)$$

Putting

$$M\cos\phi = \frac{VR}{R^2 + (\omega L)^2}$$

and

 $M\sin\phi = V \frac{\omega L}{R^2 + (\omega L)^2},$ 

to find M and  $\phi$ , we divide one equation by the other

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\omega L}{R}$$

Squaring both equations and adding, we get

$$M^{2}\cos^{2}\phi + M^{2}\sin^{2}\phi = \frac{V^{2}}{R^{2} + (\omega L)^{2}}$$
$$M = \frac{V}{\sqrt{R^{2} + (\omega L)^{2}}}$$

or

: The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$
(12.23)

The complete solution for the current  $i = i_c + i_p$ 

$$i = ce^{-t(R/L)} + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Since the inductor does not allow sudden changes in currents, at t = 0, i = 0

$$c = -\frac{V}{\sqrt{R^{2} + (\omega L)^{2}}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)$$

The complete solution for the current is

$$i = e^{-(R/L)t} \left[ \frac{-V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right) \right]$$
$$+ \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

**Example 12.4** In the circuit shown in Fig. 12.17, determine the complete solution for the current, when switch S is closed at t = 0. Applied voltage is  $v(t) = 100 \cos(10^3 t + \pi/2)$ . Resistance  $R = 20 \Omega$  and inductance L = 0.1 H.



Fig. 12.17

Solution By applying Kirchhoff's voltage law to the circuit, we have

$$20i + 0.1 \frac{di}{dt} = 100 \cos (10^3 t + \pi/2)$$
$$\frac{di}{dt} + 200i = 1000 \cos (1000t + \pi/2)$$
$$(D = 200)i = 1000 \cos (1000t + \pi/2)$$

The complementary function  $i_c = ce^{-200t}$ By assuming particular integral as

$$i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta)$$

we get

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

where  $\omega = 1000 \text{ rad/sec } V = 100 \text{ V}$ 

$$\theta = \pi/2$$
  
L = 0.1 H, R = 20 G

Substituting the values in the above equation, we get

$$i_p = \frac{100}{\sqrt{(20)^2 + (1000 \times 0.1)^2}} \cos\left(1000t + \frac{\pi}{2} - \tan^{-1}\frac{100}{20}\right)$$

$$= \frac{100}{101.9} \cos\left(1000t + \frac{\pi}{2} - 78.6^{\circ}\right)$$
$$= 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6^{\circ}\right)$$

The complete solution is

$$i = ce^{-200t} + 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6^{\circ}\right)$$

At t = 0, the current flowing through the circuit is zero, i.e. i = 0

$$c = -0.98 \cos\left(\frac{\pi}{2} - 78.6^\circ\right)$$

: The complete solution is

$$i = \left[ -0.98 \cos\left(\frac{\pi}{2} - 78.6^{\circ}\right) \right] e^{-200t} + 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6^{\circ}\right)$$

### 12.6 SINUSOIDAL RESPONSE OF R-C CIRCUIT

Consider a circuit consisting of resistance and capacitance in series as shown in Fig. 12.18. The switch, S, is closed at t = 0. At t = 0, a sinusoidal voltage V cos

 $(\omega t + \theta)$  is applied to the R-C circuit, where V is the amplitude of the wave and  $\theta$  is the phase angle. Applying Kirchhoff's voltage law to the circuit results in the following differential equation.



Fig. 12.18  

$$V\cos(\omega t + \theta) = Ri + \frac{1}{C}\int idt \qquad (12.24)$$

$$R\frac{di}{dt} + \frac{i}{C} = -V\omega\sin(\omega t + \theta)$$

$$\left(D + \frac{1}{RC}\right)i = -\frac{V\omega}{R}\sin(\omega t + \theta) \qquad (12.25)$$

The complementary function  $i_c = ce^{-t/RC}$  (12.26)

The particular solution can be obtained by using undetermined coefficients.

$$i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta)$$
 (12.27)

$$E'_{P} = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)$$
(12.28)

Substituting Eqs 12.27 and 12.28 in Eq. 12.25, we get

$$\{-A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)\} + \frac{1}{RC} \{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\}$$

 $=-\frac{V\omega}{R}\sin\left(\omega t+\theta\right)$ 

Comparing both sides,  $-A\omega + \frac{B}{RC} = -\frac{V\omega}{R}$ 

$$B\omega + \frac{A}{RC} = 0$$

From which,

$$A = \frac{VR}{R^2 + \left(\frac{1}{\omega c}\right)^2}$$
$$B = \frac{-V}{\omega C \left[R^2 + \left(\frac{1}{\omega c}\right)^2\right]}$$

and

Substituting the values of A and B in Eq. 12.27, we have

$$i_{p} = \frac{VR}{R^{2} + \left(\frac{1}{\omega c}\right)^{2}} \cos(\omega t + \theta) + \frac{-V}{\omega C \left[R^{2} + \left(\frac{1}{\omega C}\right)^{2}\right]} \sin(\omega t + \theta)$$

Putting

$$M \cos \phi = \frac{VR}{\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$
$$M \sin \phi = \frac{V}{\omega C \left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$

and

To find M and  $\phi$ , we divide one equation by the other,

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{1}{\omega CR}$$

Squaring both equations and adding, we get

$$M^{2}\cos^{2}\phi + M^{2}\sin^{2}\phi = \frac{V^{2}}{\left[R^{2} + \left(\frac{1}{\omega C}\right)^{2}\right]}$$
$$M = \frac{V}{\sqrt{R^{2} + \left(\frac{1}{\omega C}\right)^{2}}}$$

*:*.

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right)$$
(12.29)
The complete solution for the current  $i = i_c + i_p$ 

$$\therefore \qquad i = ce^{-(t/RC)} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \quad (12.30)$$

Since the capacitor does not allow sudden changes in voltages at t = 0,  $i = \frac{V}{R}$ cos  $\theta$ 

$$\therefore \qquad \frac{V}{R}\cos\theta = c + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$
$$c = \frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

The complete solution for the current is

$$i = e^{-(t/RC)} \left[ \frac{V}{R} \cos \theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos \left(\theta + \tan^{-1} \frac{1}{\omega CR}\right) \right] + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos \left(\omega t + \theta + \tan^{-1} \frac{1}{\omega CR}\right)$$
(12.31)

**Example 12.5** In the circuit shown in Fig. 12.19, determine the complete solution for the current when switch S is closed at t = 0. Applied voltage is  $v(t) = 50 \cos\left(10^2 t + \frac{\pi}{4}\right)$ . Resistance  $R = 10 \Omega$  and capacitance  $C = 1 \mu F$ . 50 cos (100 $t + \pi/4$ )

Fig. 12.19

Solution By applying Kirchhoff's voltage law to the circuit, we have

$$10i + \frac{1}{1 \times 10^{-6}} \int i dt = 50 \cos\left(100t + \frac{\pi}{4}\right)$$

$$10\frac{di}{dt} + \frac{i}{1 \times 10^{-6}} = -5(10)^3 \sin\left(100t + \frac{\pi}{4}\right)$$
$$\frac{di}{dt} + \frac{i}{10^{-5}} = -500 \sin\left(100t + \frac{\pi}{4}\right)$$
$$\left(D + \frac{1}{10^{-5}}\right)i = -500 \sin\left(100t + \frac{\pi}{4}\right)$$

The complementary function is  $i_c = ce^{-t/10^{-3}}$ . By assuming particular integral as  $i_p = A \cos{(\omega t + \theta)} + B \sin{(\omega t + \theta)},$ 

 $\theta = \pi/4$ 

 $R = 10 \ \Omega$ 

we get 
$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

 $\omega = 100 \text{ rad/sec}$ 

where

 $C = 1 \mu F$ Substituting the values in the above equation, we have

$$i_p = \frac{50}{\sqrt{(10)^2 + \left(\frac{1}{100 \times 10^{-6}}\right)^2}} \cos\left(\omega t + \frac{\pi}{4} + \tan^{-1}\frac{1}{100 \times 10^{-6} \times 10}\right)$$
$$i_p = 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

At t = 0, the current flowing through the circuit is

$$\frac{V}{R}\cos\theta = \frac{50}{10}\cos\pi/4 = 3.53 \text{ A}$$
$$i = \frac{V}{R}\cos\theta = 3.53 \text{ A}$$
$$i = ce^{-t/10^{-5}} + 4.99 \times 10^{-3}\cos\left(100t + \frac{\pi}{4} + 89.94^{\circ}\right)$$
$$t = 0$$

÷ At

$$c = 3.53 - 4.99 \times 10^{-3} \cos\left(\frac{\pi}{4} + 89.94^{\circ}\right)$$

Hence the complete solution is

$$i = \left[3.53 - 4.99 \times 10^{-3} \cos\left(\frac{\pi}{4} + 89.94^{\circ}\right)\right] e^{-(t/10^{-5})} + 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94^{\circ}\right)$$

## 12.7 SINUSOIDAL RESPONSE OF R-L-C CIRCUIT

Consider a circuit consisting of resistance, inductance and capacitance in series as shown in Fig. 12.20. Switch S is closed at t = 0. At t = 0, a sinusoidal voltage

 $V \cos (\omega t + \theta)$  is applied to the RLC series circuit, where V is the amplitude of the wave and  $\theta$  is the phase angle. Application of V Kirchhoff's voltage law to the circuit results in the following differential equation.



$$V\cos(\omega t + \theta) = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt \qquad (12.32)$$

Differentiating the above equation, we get

$$R \frac{di}{dt} + L \frac{d^2 i}{dt^2} + i/C = -V \omega \sin(\omega t + \theta)$$
$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = -\frac{V\omega}{L}\sin(\omega t + \theta)$$
(12.33)

The particular solution can be obtained by using undetermined coefficients. By assuming

$$i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta)$$
 (12.34)

$$i'_{p} = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)$$
(12.35)

$$i''_p = -A\omega^2 \cos(\omega t + \theta) - B\omega^2 \sin(\omega t + \theta) \qquad (12.36)$$

Substituting *ip*, 
$$t'_p$$
 and  $i''_p$  in Eq. 12.33, we have  
 $\{-A\omega^2 \cos(\omega t + \theta) - B\omega^2 \sin(\omega t + \theta)\}$   
 $+ \frac{R}{L} \{-A\omega \sin(\omega t + \theta) + B\omega \cos(\omega t + \theta)\}$   
 $+ \frac{1}{LC} \{A \cos(\omega t + \theta) + B \sin(\omega t + \theta)\} = -\frac{V\omega}{L} \sin(\omega t + \theta) (12.37)$ 

Comparing both sides, we have Sine coefficients.

$$-B\omega^{2} - A\frac{\omega R}{L} + \frac{B}{LC} = -\frac{V\omega}{L}$$

$$A\left(\frac{\omega R}{L}\right) + B\left(\omega^{2} - \frac{1}{LC}\right) = \frac{V\omega}{L}$$
(12.38)

Cosine coefficients

$$-A\omega^{2} + B\frac{\omega R}{L} + \frac{A}{LC} = 0$$

$$A\left(\omega^{2} - \frac{1}{LC}\right) - B\left(\frac{\omega R}{L}\right) = 0$$
(12.39)

Solving Eqs 12.38 and 12.39, we get

$$A = \frac{V \times \frac{\omega^2 R}{L^2}}{\left[ \left( \frac{\omega R}{L} \right)^2 - \left( \omega^2 - \frac{1}{LC} \right)^2 \right]}$$
$$B = \frac{\left( \omega^2 - \frac{1}{LC} \right) V \omega}{L \left[ \left( \frac{\omega R}{L} \right)^2 - \left( \omega^2 - \frac{1}{LC} \right)^2 \right]}$$

Substituting the values of A and B in Eq. 12.34, we get

$$i_{p} = \frac{V \frac{\omega^{2} R}{L^{2}}}{\left[\left(\frac{\omega R}{L}\right)^{2} - \left(\omega^{2} - \frac{1}{LC}\right)^{2}\right]} \cos(\omega t + \theta)$$
$$+ \frac{\left(\omega^{2} - \frac{1}{LC}\right) V \omega}{L\left[\left(\frac{\omega R}{L}\right)^{2} - \left(\omega^{2} - \frac{1}{LC}\right)^{2}\right]} \sin(\omega t + \theta) \qquad (12.40)$$
$$V \frac{\omega^{2} R}{L}$$

Putting

$$M\cos\phi = \frac{L^2}{\left(\frac{\omega R}{L}\right)^2 - \left(\omega^2 - \frac{1}{LC}\right)^2}$$
$$M\sin\phi = \frac{V\left(\omega^2 - \frac{1}{LC}\right)\omega}{L\left[\left(\frac{\omega R}{L}\right)^2 - \left(\omega^2 - \frac{1}{LC}\right)^2\right]}$$

and

To find M and  $\phi$  we divide one equation by the other

or 
$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$
$$\phi = \tan^{-1}\left[\left(\omega L - \frac{1}{\omega C}\right)/R\right]$$
Squaring both equations and adding, we get

$$M^{2}\cos^{2}\phi + M^{2}\sin^{2}\phi = \frac{V^{2}}{R^{2} + \left(\frac{1}{\omega C} - \omega L\right)^{2}}$$
$$\therefore \qquad M = \frac{V}{\sqrt{R^{2} + \left(\frac{1}{\omega C} - \omega L\right)^{2}}}$$

The particular current becomes

$$i_{p} = \frac{V}{\sqrt{R^{2} + \left(\frac{1}{\omega C} - \omega L\right)^{2}}} \cos \left[\omega t + \theta + \tan^{-1} \frac{\left(\frac{1}{\omega C} - \omega L\right)}{R}\right] (12.41)$$

The complementary function is similar to that of DC series RLC circuit. To find out the complementary function, we have the characteristic equation

$$\left(D^{2} + \frac{R}{L}D + \frac{1}{LC}\right) = 0$$
 (12.42)

The roots of Eq. 12.42, are

$$D_1, D_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming

$$K_1 = -\frac{R}{2L}$$
 and  $K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$   
 $D_1 = K_1 + K_2$  and  $D_2 = K_1 - K_2$ 

...

 $K_2$  becomes positive, when  $(R/2L)^2 > 1/LC$ The roots are real and unequal, which gives an overdamped response. Then Eq. 12.42 becomes

$$[D - (K_1 + K_2)] [D - (K_1 - K_2)]i = 0$$

The complementary function for the above equation is

$$i_c = c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$$

Therefore, the complete solution is

$$i = i_c + i_p$$
  
=  $c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$   
+  $\frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$ 

 $K_2$  becomes negative, when  $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$ 

Then the roots are complex conjugate, which gives an underdamped response. Equation 12.42 becomes

$$[D - (K_1 + jK_2)] [D - (K_1 - jK_2)]i = 0$$

The solution for the above equation is

$$i_c = e^{K_1 t} \left[ c_1 \cos K_2 t + c_2 \sin K_2 t \right]$$

Therefore, the complete solution is

$$i = i_c + i_p$$

 $i = e^{K_1 t} \left[ c_1 \cos K_2 t + c_2 \sin K_2 t \right]$ 

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

 $K_2$  becomes zero, when  $\left(\frac{R}{2L}\right)^2 = 1/LC$ 

Then the roots are equal which gives critically damped response. Then, Eq. 12.42 becomes  $(D - K_1) (D - K_1)i = 0$ .

The complementary function for the above equation is

$$i_c = e^{K_1 t} \left( c_1 + c_2 t \right)$$

Therefore, the complete solution is  $i = i_c + i_p$  $\therefore \qquad i = e^{K_1 t} [c_1 + c_2 t]$ 

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos \left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

**Example 12.6** In the circuit shown in Fig. 12.21, determine the complete solution for the current, when the switch is closed at t = 0. Applied voltage is  $v(t) = 400 \cos \left(500t + \frac{\pi}{4}\right)$ . Resistance  $R = 15 \Omega$ , inductance L = 0.2 H and capacitance  $C = 3\mu$ F.





Solution By applying Kirchhoff's voltage law to the circuit,

$$15i(t) + 0.2\frac{di(t)}{dt} + \frac{1}{3 \times 10^{-6}} \int i(t)dt = 400 \cos\left(500t + \frac{\pi}{4}\right)$$

Differentiating the above equation once, we get

$$15\frac{di}{dt} + 0.2\frac{d^2i}{dt} + \frac{i}{3 \times 10^{-6}} = -2 \times 10^5 \sin\left(500t + \frac{\pi}{4}\right)$$
$$(D^2 + 75D + 16.7 \times 10^5)i = \frac{-2 \times 10^5}{0.2} \sin\left(500t + \frac{\pi}{4}\right)$$

The roots of the characteristic equation are

$$D_1 = -37.5 + j1290$$
 and  $D_2 = -37.5 - j1290$ 

The complementary current

$$i_c = e^{-37.5t} (c_1 \cos 1290t + c_2 \sin 1290t)$$

Particular solution is

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$
$$i_p = 0.71 \cos\left(500t + \frac{\pi}{4} + 88.5^\circ\right)$$

*:*..

The complete solution is

$$i = e^{-37.5t} (c_1 \cos 1290t + c_2 \sin 1290t) + 0.71 \cos (500t + 45^\circ + 88.5^\circ)$$
  
At  $t = 0, i_0 = 0$   
 $\therefore$   $c_1 = -0.71 \cos (133.5^\circ) = +0.49$ 

....

 $c_1$  $0.71 \cos(133.5^{\circ})$ 

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-37.5t} \left( -1290c_1 \sin 1290t + c_2 1290 \cos 1290t \right) -37.5e^{-37.5t} \left( c_1 \cos 1290t + c_2 \sin 1290t \right) -0.71 \times 500 \sin (500t + 45^\circ + 88.5^\circ)$$

At 
$$t = 0$$
,  $\frac{di}{dt} = 1414$   
 $\therefore$  1414 = 1290 $c_2 - 37.5 \times 0.49 - 0.71 \times 500 \sin (133.5^{\circ})$   
1414 = 1290 $c_2 - 18.38 - 257.5$   
 $\therefore$   $c_2 = 1.31$ 

The complete solution is

 $i = e^{-37.5t} (0.49 \cos 1290t + 1.31 \sin 1290t) + 0.71 \cos (500t + 133.5^{\circ})$ 

# **ADDITIONAL SOLVED PROBLEMS**

Problem 12.1 For the circuit shown in Fig. 12.22, find the current equation when the switch is changed from position 1 to position 2 at t = 0.



**Solution** When the switch is at position 2, the current equation can be written by using Kirchhoff's voltage law as

$$30i(t) + 0.2 \frac{di(t)}{dt} = 0$$
$$\left(D + \frac{30}{0.2}\right)i = 0$$
$$(D + 150)i = 0$$
$$i = c_1 e^{-150t}$$

...

At t = 0, the switch is changed to position 2, i.e.  $i(0) = c_1$ .

At t = 0, the initial current passing through the circuit is the same as the current passing through the circuit when the switch is at position 1. At  $t = 0^{-1}$ , the switch is at position 1, and the current passing through the circuit i = 100/50 = 2 A.

At  $t = 0^+$ , the switch is at position 2. Since the inductor does not allow sudden changes in current, the same current passes through the circuit. Hence the initial current passing through the circuit, when the switch is at position 2 is  $i(0^+) = 2A$ .

:.

$$c_1 = 2 \text{ A}$$
  
Therefore the current $i = 2e^{-150t}$ 

**Problem 12.2** For the circuit shown in Fig. 12.23, find the current equation when the switch is opened at t = 0.



Fig. 12.23

**Solution** At t = 0, switch S is opened. By using Kirchhoff's voltage law, the current equation can be written as

$$20i + 20i + 2 \frac{di}{dt} = 0$$
$$40i + 2 \frac{di}{dt} = 0$$
$$D + 20i = 0$$

*.*..

The solution for the above equation is

$$i = c_1 e^{-20t}$$

When the switch has been closed for a time, since the inductor acts as short circuit for dc voltages, the current passing through the inductor is 2.5 A.

22

That means, just before the switch is opened, the current passing through the inductor is 2.5 A. Since the current in the inductor cannot change instantaneously,  $i(0^+)$  is also equal to 2.5 A.

At 
$$t = 0$$
  $c_1 = i(0^+) = 2.5$ 

Therefore, the final solution is  $i(t) = 2.5e^{-20t}$ 

Problem 12.3 For the circuit shown in Fig. 12.24, find the current equation when the switch is opened at t = 0.



Fig. 12.24

**Solution** By using Kirchhoff's voltage law, the current equation is given by

$$\frac{1}{5 \times 10^{-6}} \int i dt + 50i = 0$$

Differentiating the above equation once, we get

$$50 \frac{di}{dt} + \frac{1}{5 \times 10^{-6}} i = 0$$
  

$$\therefore \qquad \left( D + \frac{1}{250 \times 10^{-6}} \right) i = 0$$
  

$$\therefore \qquad i = c_1 \exp\left(\frac{-1}{250 \times 10^{-6}} t\right)$$
(12.43)

At  $t = 0^{-}$ , just before the switch S is opened, the voltage across the capacitor is 200 V. Since the voltage across the capacitor cannot change instantly, it remains equal to 200 V at  $t = 0^+$ . At that instant, the current through the resistor is

$$i(0^+) = \frac{200}{50} = 4A$$
  
In Eq. 12.43, the current is  $i(0^+)$  at  $t = 0$   
 $c_1 = 4 A$ 

*.*..

Therefore, the current equation is

$$i = 4 \exp\left(\frac{-1}{250 \times 10^{-6}} t\right) A$$

Problem 12.4 For the circuit shown in Fig. 12.25, find the current equation when the switch S is opened at t = 0.

\*\*



Fig. 12.25

Solution By using Kirchhoff's voltage law, the current equation is given by

$$\frac{1}{2 \times 10^{-6}} \int i dt + 5i + 10i = 0$$

Differentiating the above equation, we have

$$15 \frac{di}{dt} + \frac{i}{2 \times 10^{-6}} = 0$$
$$\left(D + \frac{1}{30 \times 10^{-6}}\right)i = 0$$
$$i = c_1 \exp\left(\frac{-1}{30 \times 10^{-6}}\right)t$$

:.

At  $t = 0^{-}$ , just before switch S is opened, the current through 10 ohms resistor is 2.5 A. The same current passes through 10  $\Omega$  at  $t = 0^{+}$ 

$$\therefore \qquad i(0^+) = 2.5 \text{ A}$$
  
At  $t = 0 \qquad i(0^+) = 2.5 \text{ A}$   
$$\therefore \qquad c_1 = 2.5$$
  
The complete solution is  $i = 2.5 \exp\left(\frac{-1}{30 \times 10^{-6}}t\right)$ 

**Problem 12.5** For the circuit shown in Fig. 12.26, find the complete expression for the current when the switch is closed at t = 0.

**Solution** By using Kirchhoff's law, the differential equation when the switch is closed at t = 0 is given by



Fig. 12.26

 $i = c_1 e^{-200t} + e^{-200t} \int 1000 e^{200t} dt$  $i = c_1 e^{-200t} + 5$ 

At  $t = 0^-$ , the current passing through the circuit is  $i(0^-) = \frac{100}{50} = 2$  A. Since, the inductor does not allow sudden changes in currents, at  $t = 0^+$ , the same

the inductor does not allow sudden changes in currents, at t = 0, the same current passes through circuit.

$$\therefore \qquad i(0^{+}) = 2 \text{ A}$$
  
At  $t = 0$   
$$\therefore \qquad c_1 = -3$$

The complete solution is  $i = -3e^{-200t} + 5$  A

**Problem 12.6** The circuit shown in Fig. 12.27, consists of series RL elements with  $R = 150 \ \Omega$  and  $L = 0.5 \ H$ . The switch is closed when  $\phi = 30^{\circ}$ . Determine the resultant current when voltage  $V = 50 \cos(100t + \phi)$  is applied to the circuit at  $\phi = 30^{\circ}$ .



Fig. 12.27

**Solution** By using Kirchhoff's laws, the differential equation, when the switch is closed at  $\phi = 30^{\circ}$  is

$$150i + 0.5 \frac{di}{dt} = 50 \cos (100t + \phi)$$
  
$$0.5Di + 150i = 50 \cos (100t + 30^{\circ})$$
  
$$(D + 300)i = 100 \cos (100t + 30^{\circ})$$

The complementary current  $i_c = c e^{-300t}$ 

To determine the particular current, first we assume a particular current

$$i_p = A \cos(100t + 30^\circ) + B \sin(100t + 30^\circ)$$

$$i'_p = -100 A \sin(100t + 30^\circ) + 100 B \cos(100t + 30^\circ)$$

Substituting  $i_p$  and  $i'_p$  in the differential equation and equating the coefficients, we get

$$-100 A \sin (100t + 30^{\circ}) + 100B \cos (100t + 30^{\circ}) + 300 A \cos (100t + 30^{\circ}) + 300B \sin (100t + 30^{\circ}) = 100 \cos (100t + 30^{\circ}) -100 A + 300 B = 0300 A + 100 B = 100$$

\*\*

From the above equation, we get

A = 0.3 and B = 0.1

The particular current is

$$i_p = 0.3 \cos (100t + 30^\circ) + 0.1 \sin (100t + 30^\circ)$$
  
∴  $i_p = 0.316 \cos (100t + 11.57^\circ) A$   
The complete equation for the current is  $i = i_p + i_c$   
∴  $i = ce^{-300t} + 0.316 \cos (100t + 11.57^\circ)$ 

At t = 0, the current  $i_0 = 0$ 

Therefore, the complete solution for the current is

$$i = -0.309e^{-300t} + 0.316 \cos(100t + 11.57^{\circ})$$
 A

**Problem 12.7** The circuit shown in Fig. 12.28, consists of series RC elements with  $R = 15 \Omega$  and  $C = 100 \mu$  F. A sinusoidal voltage  $v = 100 \sin (500t + \phi)$  volts is applied to the circuit at time corresponding to  $\phi = 45^{\circ}$ . Obtain the current transient.

 $c = -0.316 \cos(11.57^{\circ}) = -0.309$ 

Solution By using Kirchhoff's laws, the differential equation is



Differentiating once, we have

$$15\frac{di}{dt} + \frac{1}{100 \times 10^{-6}}i = (100) (500) \cos (500t + \phi)$$
$$\left(D + \frac{1}{1500 \times 10^{-6}}\right)i = 3333.3 \cos (500t + \phi)$$
$$(D + 666.67)i = 3333.3 \cos (500t + \phi)$$

The complementary function  $i_c = ce^{-666.67t}$ 

To determine the particular current, first we assume a particular current

$$\begin{split} i_p &= A \cos (500t + 45^\circ) + B \sin (500t + 45^\circ) \\ i'_p &= -500 \ A \sin (500t + 45^\circ) + 500 \ B \cos (500t + 45^\circ) \end{split}$$

Substituting  $i_n$  and  $i'_n$  in the differential equation, we get

 $-500 A \sin (500t + 45^\circ) + 500 B \cos (500t + 45^\circ)$ 

+ 666.67A cos (500t + 45°) + 666.67B sin (500t + 45°) = 3333.3 cos (500t +  $\phi$ )

By equating coefficients, we get

f

$$500 B + 666.67 A = 3333.3$$

$$566.67B - 500 A = 0$$

From which, the coefficients

$$A = 3.2; B = 2.4$$

Therefore, the particular current is

$$i_p = 3.2 \cos (500t + 45^\circ) + 2.4 \sin (500t + 45^\circ)$$
  
 $i_p = 4 \sin (500t + 98.13^\circ)$ 

The complete equation for the current is

$$i = i_c + i_p$$
  
 $i = ce^{-666.67t} + 4 \sin(500t + 98.13^\circ)$ 

At t = 0, the differential equation becomes

$$15i = 100 \sin 45^{\circ}$$
$$i = \frac{100}{15} \sin 45^{\circ} = 4.71 \text{ A}$$

 $\therefore$  At t = 0

 $4.71 = c + 4 \sin(98.13^\circ)$ c = 0.75

÷

The complete current is

$$i = 0.75 e^{-666.67t} + 4 \sin(500t + 98.13^{\circ})$$

**Problem 12.8** The circuit shown in Fig. 12.29 consisting of series RLC elements with  $R = 10 \Omega$ , L = 0.5 H and  $C = 200 \mu$ F has a sinusoidal voltage  $v = 150 \sin (200t + \phi)$ . If the switch is closed when  $\phi = 30^{\circ}$ , determine the current equation.



Fig. 12.29

**Solution** By using Kirchhoff's laws, the differential equation is

$$10i + 0.5\frac{di}{dt} + \frac{1}{200 \times 10^{-6}}\int idt = 150\sin\left(200t + \phi\right)$$

Differentiating once, we have

 $(D^2 + 20D + 10^4)i = 60000 \cos(200t + \phi)$ 

The roots of the characteristics equation are

$$D_1 = -10 + j99.49$$
 and  $D_2 = -10 - j99.49$ 

The complementary function is

$$i_c = e^{-10t} \left( c_1 \cos 99.49t + c_2 \sin 99.49 \right)$$

We can find the particular current by using the undetermined coefficient method.

Let us assume

 $i_p = A \cos (200t + 30^\circ) + B \sin (200t + 30^\circ)$   $i'_p = -200 A \sin (200t + 30^\circ) + 200 B \cos (200t + 30^\circ)$  $i''_p = -(200)^2 A \cos (200t + 30^\circ) - (200)^2 B \sin (220t + 30^\circ)$ 

Substituting these values in the equation, and equating the coefficients, we get

A = 0.1 B = 0.067

Therefore, the particular current is

$$i_p = 1.98 \cos(200t - 52.4^\circ) \text{ A}$$

The complete current is

$$i = e^{-10t} (c_1 \cos 99.49t + c_2 \sin 99.49t) + 1.98 \cos (200t - 52.4^\circ)$$
 A

From the differential equation at t = 0,  $i_0 = 0$  and  $\frac{di}{dt} = 300$ 

 $\therefore$  At t = 0

$$c_1 = -1.98 \cos(-52.4^\circ) = -1.21$$

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-10t} (-99.49c_1 \sin 99.49t + 99.49c_2 \cos 99.49t)$$
  
- 200 (1.98) sin (200t - 52.4°) - 10e<sup>-10t</sup> (c\_1 cos 99.49t + c\_2 sin 99.49t)  
At t = 0,  $\frac{di}{dt}$  = 300 and  $c_1$  = -1.21  
300 = 99.49  $c_2$  - 396 sin (- 52.4°) - 10 (- 1.21)  
300 = 99.49  $c_2$  + 313.7 + 12.1  
 $c_2$  = -25.8

Therefore, the complete current equation is  $i = e^{-10t} (0.07 \cos 99.49t - 25.8 \sin 99.49t) + 1.98 \cos (200t - 52.4^{\circ}) A$ 

**Problem 12.9** For the circuit shown in Fig. 12.30, determine the transient current when the switch is moved from position 1 to position 2 at t = 0. The circuit is in steady state with the switch in position 1. The voltage applied to the circuit is  $v = 150 \cos (200t + 30^{\circ})$  V.



Fig. 12.30

**Solution** When the switch is at position 2, by applying Kirchhoff's law, the differential equation is

$$200i + 0.5 \frac{di}{dt} = 0$$
$$(D + 400)i = 0$$

... The transient current is

 $i = ce^{-400t}$ 

At t = 0, the switch is moved from position 1 to position 2. Hence the current passing through the circuit is the same as the steady state current passing through the circuit when the switch is in position 1.

When the switch is in position 1, the current passing through the circuit is

$$i = \frac{v}{z} = \frac{150 \angle 30^{\circ}}{R + j\omega L}$$
  
=  $\frac{150 \angle 30^{\circ}}{200 + j(200)(0.5)} = \frac{150 \angle 30^{\circ}}{223.6 \angle 26.56^{\circ}} = 0.67 \angle 3.44^{\circ}$ 

Therefore, the steady state current passing through the circuit when the switch is in position 1 is

$$i = 0.67 \cos (200t + 3.44^{\circ})$$

Now substituting this equation in transient current equation, we get

$$0.67\cos\left(200t + 3.44^\circ\right) = ce^{-400t}$$

At t = 0;  $c = 0.67 \cos(3.44^\circ) = 0.66$ 

Therefore, the current equation is  $i = 0.66e^{-400t}$ 

**Problem 12.10** In the circuit shown in Fig. 12.31, determine the current equations for  $i_1$  and  $i_2$  when the switch is closed at t = 0.



Fig. 12.31

**Solution** By applying Kirchhoff's laws, we get two equations

$$35i_1 + 20i_2 = 100 \tag{12.44}$$

$$20i_1 + 20i_2 + 0.5 \ \frac{di_2}{dt} = 100 \tag{12.45}$$

From Eq. 12.44, we have

$$35i_1 = 100 - 20i_2$$
$$i_1 = \frac{100}{35} - \frac{20}{35}i_2$$

Substituting  $i_1$  in Eq. 12.45, we get

$$20\left(\frac{100}{35} - \frac{20}{35}i_{2}\right) + 20i_{2} + 0.5 \frac{di_{2}}{dt} = 100$$
(12.46)  
$$57.14 - 11.43i_{2} + 20i_{2} + 0.5 \frac{di_{2}}{dt} = 100$$
(D + 17.14) $i_{2} = 85.72$ 

From the above equation,

$$i_2 = ce^{-17.14t} + 5$$

Loop current  $i_2$  passes through inductor and must be zero at t = 0At t = 0,  $i_2 = 0$ 

$$c = -5$$
  

$$i_2 = 5(1 - e^{-17.14t}) A$$
  
and the current  

$$i_1 = 2.86 - \{0.57 \times 5(1 - e^{-17.14t})\}$$
  

$$= (0.01 + 2.85 e^{-17.14t}) A$$

**Problem 12.11** For the circuit shown in Fig. 12.32, find the current equation when the switch is changed from position 1 to position 2 at t = 0.



Fig. 12.32

**Solution** By using Kirchhoff's voltage law, the current equation is given by

$$60i + 0.4 \frac{di}{dt} = 10i$$

At  $t = 0^{-}$ , the switch is at position 1, the current passing through the circuit is

$$i(0^{-}) = \frac{500}{100} = 5 \text{ A}$$
  
0.4  $\frac{di}{dt} + 50i = 0$   
 $\left(D + \frac{50}{0.4}\right)i = 0$   
 $i = ce^{-125t}$ 

At t = 0, the initial current passing through the circuit is same as the current passing through the circuit when the switch is at position 1.

 $t = 0, i(0) = i(0^{-}) = 5 \text{ A}$ At t = 0, c = 5 AAt  $I = 5e^{-125t}$  $\therefore$  The current 

**Problem 12.12** For the circuit shown in Fig. 12.33, find the current equation when the switch S is opened at t = 0.



Fig. 12.33

**Solution** When the switch is closed for a long time,

At 
$$t = 0^-$$
, the current  $i(0^-) = \frac{100}{20} = 5$  A

When the switch is opened at t = 0, the current equation by using Kirchhoff's voltage law is given by

$$\frac{1}{4 \times 10^{-6}} \int i \, dt + 10i = 5i$$
$$\frac{1}{4 \times 10^{-6}} \int i \, dt + 5i = 0$$

Differentiating the above equation

$$5 \frac{di}{dt} + \frac{1}{4 \times 10^{-6}} i = 0$$
$$\left(D + \frac{1}{20 \times 10^{-6}}\right)i = 0$$
$$i = ce^{\frac{-1}{20 \times 10^{-6}}t}$$

:.

At  $t = 0^{-}$ , just before switch S is opened, the current passing through the 10  $\Omega$  resistor is 5 A. The same current passes through 10  $\Omega$  at t = 0.

$$\therefore \text{ At} \qquad t = 0, i(0) = 5 A$$

$$At \qquad t = 0, c_1 = 5 A$$

$$-t$$

The current equation is  $i = 5e^{20 \times 10^{-6}}$ 

**Problem 12.13** For the circuit shown in Fig. 12.34, find the current in the 20  $\Omega$  when the switch is opened at t = 0.



Fig. 12.34

**Solution** When the switch is closed, the loop current  $i_1$  and  $i_2$  are flowing in the circuit.

The loop equations are  $30(i_1 - i_2) + 10i_2 = 50$ 

$$30(i_2 - i_1) + 20i_2 = 10i_2$$

From the above equations, the current in the 20  $\Omega$  resistor  $i_2 = 2.5$  A. The same initial current is flowing when the switch is opened at t = 0. When the switch is opened the current equations

$$30i + 20i + 2 \frac{di}{dt} = 10i$$

$$40i + \frac{2di}{dt} = 0$$

$$(D + 20)i = 0$$

$$i = ce^{-20t}$$
where the integral of the constraints is the constraint of the const

At t = 0, the current i(0) = 2.5 A  $\therefore$  At t = 0, c = 2.5

The current in the 20  $\Omega$  resistor is  $i = 2.5 e^{-20t}$ .

**Problem 12.14** For the circuit shown in Fig. 12.35, find the current equation when the switch is opened at t = 0.



Fig. 12.35

**Solution** When the switch is closed, the current in the 20  $\Omega$  resistor *i* can be obtained using Kirchhoff's voltage law.

$$10i + 20i + 20i = 100$$
  
 $50i = 100, \therefore i = 2$  A

The same initial current passes through the 20  $\Omega$  resistor when the switch is opened at t = 0.

The current equation is

$$20i + 10i + \frac{1}{2 \times 10^{-6}} \int i dt = 20i$$
$$10i + \frac{1}{2 \times 10^{-6}} \int i dt = 0$$

Differentiating the above equation, we get

$$10\frac{di}{dt} + \frac{1}{2 \times 10^{-6}}i = 0$$
$$\left(D + \frac{1}{20 \times 10^{-6}}\right)i = 0$$

-

12.34

The solution for the above equation is  $i = e^{-\frac{1}{20 \times 10^{-6}}t}$ 

At t = 0,  $i(0) = i(0^{-}) = 2$  A  $\therefore$  At t = 0, c = 2 A

The current equation is

$$i = 2e^{\frac{-1}{20 \times 10^{-6}}t}$$

PRACTICE PROBLEMS

- 12.1 (a) What do you understand by transient and steady state parts of response? How can they be identified in a general solution?
  - (b) Obtain an expression for the current i(t) from the differential equation

$$\frac{d^2 i(t)}{dt^2} + 10 \frac{di(t)}{dt} + 25i(t) = 0$$

with initial conditions

$$i(0^+) = 2 \frac{di(0^+)}{dt} = 0$$

12.2 A series circuit shown in Fig. 12.36, comprising of resistance 10  $\Omega$  and inductance 0.5 H, is connected to a 100 V source at t = 0. Determine the complete expression for the current i(t).



Fig. 12.36

12.3 In the network shown in Fig. 12.37, the capacitor  $c_1$  is charged to a voltage of 100 V and the switch S is closed at t = 0. Determine the current expression  $i_1$  and  $i_2$ .



Fig. 12.37

...

12.4 A series RLC circuit shown in Fig. 12.38, comprising  $R = 10 \Omega$ , L = 0.5 H and  $C = 1 \mu$ F, is excited by a constant voltage source of 100 V. Obtain the expression for the current. Assume that the circuit is relaxed initially.



Fig. 12.38

12.5 In the circuit shown in Fig. 12.39, the initial current in the inductance is 2 A and its direction is as shown in the figure. The initial charge on the capacitor is 200 C with polarity as shown when the switch is closed. Determine the current expression in the inductance.





12.6 In the circuit shown in Fig. 12.40, the switch is closed at t = 0 with zero capacitor voltage and zero inductor current. Determine  $V_1$  and  $V_2$  at  $t = 0^+$ .



Fig. 12.40

12.7 In the network shown in Fig. 12.41, the switch is moved from position 1 to position 2 at t = 0. The switch is in position 1 for a long time. Determine the current expression i(t).





12.8 In the network shown in Fig. 12.42, determine the current expression for  $i_1(t)$  and  $i_2(t)$  when the switch is closed at t = 0. The network has no initial energy.



Fig. 12.42

12.9 In the network shown in Fig. 12.43, the switch is closed at t = 0 and there is no initial charge on either of the capacitances. Find the resulting current i(t).



Fig. 12.43

12.10 In the RC circuit shown in Fig. 12.44, the capacitor has an initial charge  $q_0 = 25 \times 10^{-6}$  C with polarity as shown. A sinusoidal voltage v = 100 sin  $(200t + \phi)$  is applied to the circuit at a time corresponding to  $\phi = 30^{\circ}$ . Determine the expression for the current i(t).



12.11 In the network shown in Fig. 12.45, the switch is moved from position 1 to position 2 at t = 0. The switch is in position 1 for a long time. Initial charge on the capacitor is  $7 \times 10^{-4}$  coulombs. Determine the current expression i(t).



Fig. 12.45

12.12 In the network shown in Fig. 12.46, the switch is moved from position 1 to position 2 at t = 0. Determine the current expression.



Fig. 12.46

12.13 In the network shown in Fig. 12.47, find  $i_2(t)$  for t > 0, if  $i_1(0) = 5$  A.



Fig. 12.47

12.14 For the circuit shown in Fig. 12.48, find  $v_5$ , if the switch is opened for t > 0.



12.15 Calculate the voltage  $v_1(t)$  across the inductance for t > 0 in the circuit shown in Fig. 12.49.





12.16 The network shown in Fig. 12.50 is initially under steady state condition with the switch in position 1. The switch is moved from position 1 to position 2 at  $t \neq 0$ . Calculate the current i(t) through  $R_1$  after switching.



Fig. 12.50



- 1. Transient behaviour occurs in any circuit when
  - (a) there are sudden changes of applied voltage.
    - (b) the voltage source is shorted.
    - (c) the circuit is connected or disconnected from the supply.
    - (d) all of the above happen.
- 2. The transient response occurs
  - (a) only in resistive circuits
- (b) only in inductive circuits
- (c) only in capacitive circuits (d
- (d) both in (b) and (c).
  - an daga not allow and an abangag
- 3. Inductor does not allow sudden changes
  - (a) in currents(c) in both (a) and (b)
- (b) in voltages
- (d) in none of the above
- 4. When a series RL circuit is connected to a voltage V at t = 0, the current passing through the inductor L at  $t = 0^+$  is

(a) 
$$\frac{V}{R}$$
 (b) infinite  
(c) zero (d)  $\frac{V}{L}$ 

5. The time constant of a series RL cir	rcuit is
(a) <i>LR</i>	(b) $\frac{L}{P}$
(c) $\frac{R}{r}$	(d) $e^{-R/L}$
<i>L</i> 6 A capacitor does not allow sudden changes	
(a) in currents	(b) in voltages
(c) in both currents and voltages	(d) in neither of the two
7 When a series RC circuit is connect	ted to a constant voltage at $t = 0$ the
current passing through the circuit a	$t = 0^+ $ is
(a) infinite	(b) zero
(c) $\frac{r}{R}$	(d) $\frac{1}{\omega C}$
8 The time constant of a series RC ci	cuit is
	R
(a) $\frac{1}{RC}$	(b) $\frac{\pi}{C}$
(c) RC	(d) $e^{-RC}$
9 The transient current in a loss-free	LC circuit when excited from an ac
source is an sine wave	
(a) undamped	(b) overdamped
(c) under damped	(d) critically damped.
10. Transient current in an RLC circuit	is oscillatory when
(a) $R = 2 \sqrt{I/C}$	(b) $R = 0$
(a) $R = 2\sqrt{L/C}$	$(0) \mathbf{R} = 0$
(c) $R > 2\sqrt{L/C}$	(d) $R < 2\sqrt{L/C}$
11. The initial current in the circuit sho	wh in Fig. 12.51 when the switch is
opened for $t > 0$ is	10.0
<b>→</b> →	
-	<i>i t</i> 2 <i>i</i>
20 V —	
	<b>≍ 0.2 Н</b>
	P
Fig. 12.51	
(a) 1.67 A	(b) 3 A

(c) 0 A (d) 2 A 12. The initial current in the circuit shown in Fig. 12.52 below when the switch is opened for t > 0 is



Fig. 12.52

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- (a) 1.5 A (b) 0 A (c) 2 A (d) 10 A
- 13. For the circuit shown in Fig. 12.53 the current in the 10  $\Omega$  resistor when the switch is changed from 1 to 2 is





(a) 
$$5 e^{+20t}$$
 (b)  $5 e^{-20t}$   
(c)  $20 e^{+5t}$  (d)  $20 e^{-5t}$ 

14. For the circuit shown in Fig. 12.54, the current in the 5  $\Omega$  resistor when the switch is changed from 1 to 2 is



Fig. 12.54





# **13.1 DEFINITION OF LAPLACE TRANSFORM**

The Laplace transform is used to solve differential equations and corresponding initial and final value problems. Laplace transforms are widely used in engineering, particularly when the driving function has discontinuties and appears for a short period only.

In circuit analysis, the input and output functions do not exist forever in time. For casual functions, the function can be defined as f(t) u(t). The integral for the Laplace transform is taken with the lower limit at t = 0 in order to include the effect of any discontinuity at t = 0.

Consider a function f(t) which is to be continuous and defined for values of  $t \ge 0$ . The Laplace transform is then

$$\mathscr{L}[f(t)] = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) u(t) dt = \int_{0}^{\infty} f(t) e^{-st} dt$$

f(t) is a continuous function for  $t \ge 0$  multiplied by  $e^{-st}$  which is integrated with respect to t between the limits 0 and  $\infty$ . The resultant function of the variable s is called Laplace transform of f(t). Laplace transform is a function of independent variable s corresponding to the complex variable in the exponent of  $e^{-st}$ . The complex variable s is, in general, of the form  $s = \sigma + j\omega$  and  $\sigma$  and  $\omega$ being the real and imaginary parts, respectively. For a function to have a

Laplace transform, it must satisfy the condition  $\int_{0}^{\infty} f(t) e^{-st} dt < \infty$ . Laplace transform changes the time domain function f(t) to the frequency domain

transform changes the time domain function f(t) to the frequency domain function F(s). Similarly, inverse Laplace transformation converts frequency domain function F(s) to the time domain function f(t) as shown below.

$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{-j}^{+j} F(s) e^{st} ds$$

Here, the inverse transform involves a complex integration. f(t) can be represented as a weighted integral of complex exponentials. We will denote the transform relationship between f(t) and F(s) as

$$f(t) \xleftarrow{\mathscr{L}} F(s)$$

#### **13.2 PROPERTIES OF LAPLACE TRANSFORMS**

Laplace transforms have the following properties.

(a) *Superposition Property* The Laplace transform of the sum of the two or more functions is equal to the sum of transforms of the individual function,

i.e. if 
$$f_1(t) \xleftarrow{\mathscr{P}} F_1(s)$$
 and  $f_2(t) \xleftarrow{\mathscr{P}} F_2(s)$ , then

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

Consider two functions  $f_1(t)$  and  $f_2(t)$ . The Laplace transform of the sum or difference of these two functions is given by

$$\mathscr{L}{f_1(t) \pm f_2(t)} = \int_0^\infty \{f_1(t) \pm f_2(t)\} e^{-st} dt$$
$$= \int_0^\infty f_1(t) e^{-st} dt \pm \int_0^\infty f_2(t) e^{-st} dt$$
$$= F_1(s) \pm F_2(s)$$

 $\therefore \qquad \mathscr{L}{f_1(t) \pm f_2(t)} = F_1(s) \pm F_2(s)$ 

(b) *Linearity property* If *K* is a constant, then

$$\mathscr{L}[Kf(t)] = K \mathscr{L}[f(t)] = K F(s)$$

Consider a function f(t) multiplied by a constant K. The Laplace transform of this function is given by

$$\mathscr{L}[Kf(t)] = \int_{0}^{\infty} Kf(t)e^{-st} dt$$
$$= K\int_{0}^{\infty} f(t)e^{-st} dt = KF(s)$$

If we can use these two properties jointly, we have

$$\begin{aligned} \mathscr{L}[K_1f_1(t) + K_2f_2(t)] &= K_1\mathscr{L}[f_1(t)] + K_2\mathscr{L}[f_2(t)] \\ &= K_1F_1(s) + K_2F_2(s) \end{aligned}$$

# 13.3 LAPLACE TRANSFORM OF SOME USEFUL FUNCTIONS

(i) The unit step function f(t) = u(t)where u(t) = 1 for t > 0= 0 for t < 0

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$$\mathscr{L}[f(t)] = \int_{0}^{\infty} u(t)e^{-st} dt$$
$$= \int_{0}^{\infty} 1e^{-st} dt = \frac{-1}{s} \left[e^{-st}\right]_{0}^{\infty} = \frac{1}{s}$$
$$\mathscr{L}[u(t)] = \frac{1}{s}$$

(ii) Exponential function  $f(t) = e^{-at}$ 

$$\mathcal{L}(e^{-at}) = \int_{0}^{\infty} e^{-at} e^{-st} dt$$
$$= \int_{0}^{\infty} e^{-(s+a)t} = \frac{-1}{s+a} \left[ e^{-(s+a)t} \right]_{0}^{\infty}$$
$$= \frac{1}{s+a}$$
$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$$

(iii) The cosine function:  $\cos \omega t$ 

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$$\begin{aligned} \mathscr{L}(\cos \omega t) &= \int_{0}^{\infty} \cos \omega t \, e^{-st} \, dt \\ &= \int_{0}^{\infty} e^{-st} \left[ \frac{e^{j\omega t} + e^{-j\omega t}}{2} \right] dt \\ &= \frac{1}{2} \left[ \int_{0}^{\infty} e^{-(s-j\omega)t} \, dt + \int_{0}^{\infty} e^{-(s+j\omega)t} \, dt \right] \\ &= \frac{1}{2} \left[ -\frac{e^{-(s-j\omega)t}}{s-j\omega} \right]_{0}^{\infty} + \frac{1}{2} \left[ -\frac{e^{-(s+j\omega)t}}{s+j\omega} \right]_{0}^{\infty} \\ &= \frac{1}{2} \left[ \frac{1}{s-j\omega} + \frac{1}{s+j\omega} \right] = \frac{s}{s^{2} + \omega^{2}} \end{aligned}$$
$$\begin{aligned} \mathscr{L}(\cos \omega t) &= \frac{s}{s^{2} + \omega^{2}} \end{aligned}$$

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(iv) The sine function:  $\sin \omega t$ 

$$\mathcal{L}(\sin \omega t) = \int_{0}^{\infty} \sin \omega t \ e^{-st} \ dt$$
$$= \int_{0}^{\infty} e^{-st} \frac{1}{2j} \ [e^{j\omega t} - e^{-j\omega t}] dt$$
$$= \frac{1}{2j} \left[ \int_{0}^{\infty} e^{-(s-j\omega)t} \ dt - \int_{0}^{\infty} e^{-(s+j\omega)t} \ dt \right]$$
$$= \frac{1}{2j} \left\{ \left[ -\frac{e^{-(s-j\omega)t}}{(s-j\omega)} \right]_{0}^{\infty} + \left[ \frac{e^{-(s+j\omega)t}}{(s+j\omega)} \right]_{0}^{\infty} \right\}$$
$$= \frac{1}{2j} \left[ \frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right] = \frac{\omega}{s^{2} + \omega^{2}}$$
$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^{2} + \omega^{2}}$$

(v) The function  $t^n$ , where *n* is a positive integer

$$\mathcal{P}(t^n) = \int_0^\infty t^n \ e^{-st} \ dt$$
$$= \left[\frac{t^n \ e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \ \frac{e^{-st}}{-s} \ nt^{n-1} \ dt$$
$$= \frac{n}{s} \int_0^\infty \ e^{-st} \ t^{n-1} \ dt$$
$$= \frac{n}{s} \mathcal{P}(t^{n-1})$$

Similarly,  $\mathscr{L}(t^{n-1}) = \frac{n-1}{s} \mathscr{L}(t^{n-2})$ 

By taking Laplace transformations of  $t^{n-2}$ ,  $t^{n-3}$ ,... and substituting in the above equation, we get

$$\mathscr{L}(t^n) = \frac{n}{s} \frac{n-1}{s} \frac{n-2}{s} \cdots \frac{2}{s} \frac{1}{s} \mathscr{L}(t^{n-n})$$
$$= \frac{\angle n}{s^n} \mathscr{L}(t^0) = \frac{\angle n}{s^n} \times \frac{1}{s} = \frac{\angle n}{s^{n+1}}$$
$$\mathscr{L}(t) = 1/s^2$$

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(vi) The hyperbolic sine and cosine function

$$\mathscr{L}(\cos h \ at) = \int_{0}^{\infty} \cos h \ at \ e^{-st} \ dt$$

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$$= \int_{0}^{\infty} \left[ \frac{e^{at} + e^{-at}}{2} \right] e^{-st} dt$$
$$= \frac{1}{2} \int_{0}^{\infty} e^{-(s-a)t} dt + \frac{1}{2} \int_{0}^{\infty} e^{-(s+a)t} dt$$
$$= \frac{1}{2} \frac{1}{(s-a)} + \frac{1}{2} \frac{1}{(s+a)} = \frac{s}{s^{2} - a^{2}}$$

Similarly,

$$\mathcal{L}(\sin h \, at) = \int_{0}^{\infty} \sin h \, (at)e^{-st} \, dt$$
$$= \int_{0}^{\infty} \left[\frac{e^{at} - e^{-at}}{2}\right]e^{-st} \, dt$$
$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2 - a^2}$$

**Example 13.1** Find the Laplace transform of the function  $f(t) = 4t^3 + t^2 - 6t + 7$ 

Solution

$$\mathcal{L}(4t^{3} + t^{2} - 6t + 7) = 4 \mathcal{L}(t^{3}) + \mathcal{L}(t^{2}) - 6\mathcal{L}(t) + 7\mathcal{L}(1)$$

$$= 4 \times \frac{\angle 3}{s^{4}} + \frac{\angle 2}{s^{3}} - 6\frac{\angle 1}{s^{2}} + 7\frac{1}{s}$$

$$= \frac{24}{s^{4}} + \frac{2}{s^{3}} - \frac{6}{s^{2}} + \frac{7}{s}$$

**Example 13.2** Find the Laplace transform of the function  $f(t) = \cos^2 t$ 

Solution 
$$\mathscr{L}(\cos^2 t) = \mathscr{L}\left(\frac{1+\cos 2t}{2}\right)$$
  
$$= \mathscr{L}\left(\frac{1}{2}\right) + \mathscr{L}\left(\frac{\cos 2t}{2}\right) = \frac{1}{2}\left[\mathscr{L}(1) + \mathscr{L}(\cos 2t)\right]$$
$$= \frac{1}{2s} + \frac{s}{2(s^2 + 4)} = \frac{2s^2 + 4}{2s(s^2 + 4)}$$

**Example 13.3** Find the Laplace transform of the function  $f(t) = 2t^4 - 2t^3 + 4e^{-3t} - 2e^{-3t} + 2e^{-3t} + 2e^{-3t}$ 

Solution 
$$f(t) = 3t - 2t^{2} + 4e^{-2} \sin 5t + 3\cos 2t$$
$$\mathscr{L}(3t^{4} - 2t^{3} + 4e^{-3t} - 2\sin 5t + 3\cos 2t)$$

$$= 3\mathscr{L}(t^{4}) - 2\mathscr{L}(t^{3}) + 4\mathscr{L}(e^{-3t}) - 2\mathscr{L}(\sin 5t) + 3\mathscr{L}(\cos 2t)$$
  
$$= 3\frac{\angle 4}{s^{5}} - 2\frac{\angle 3}{s^{4}} + 4\frac{1}{s+3} - 2 \times \frac{5}{s^{2}+25} + 3 \times \frac{s}{s^{2}+4}$$
  
$$= \frac{72}{s^{5}} - \frac{12}{s^{4}} + \frac{4}{s+3} - \frac{10}{s^{2}+25} + \frac{3s}{s^{2}+4}$$

### **13.4 LAPLACE TRANSFORM THEOREMS**

(a) Differentiation Theorem If a function f(t) is piecewise continuous, then the Laplace transform of its derivative  $\frac{d}{dt} [f(t)]$  is given by

$$\mathscr{L}[f'(t)] = sF(s) - f(0)$$

*Proof* By definition,

$$\mathscr{L}[f'(t)] = \int_{0}^{\infty} f'(t)e^{-st} dt$$
$$= \int_{0}^{\infty} e^{-st} d\{f(t)\}$$

Integrating by parts, we get

$$= \left[e^{-st} f(t)\right]_0^\infty + \int_0^\infty se^{-st} f(t) dt$$
$$= -f(0) + s \int_0^\infty e^{-st} f(t) dt$$
$$= -f(0) + sF(s)$$

Hence we have

$$\mathscr{L}[f'(t)] = sF(s) - f(0)$$

This is applicable to higher order derivatives also. The Laplace transform of second derivative of f(t) is

$$\mathscr{D}[f''(t)] = \mathscr{D}\left[\frac{d}{dt}(f'(t))\right]$$
$$= s \mathscr{D}[f'(t)] - f'(0) = s\{sF(s) - f(0)\} - f'(0)$$
$$= s^2F(s) - sf(0) - f'(0)$$

where f'(0) is initial value of first derivative of f(t)Similarly,

$$\mathscr{L}[f'''(t)] = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

In general, the *n*th order derivative is given by

$$\mathscr{L}(f^{n}(t)] = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{n-1}(0)$$

**Example 13.4** Using the formula for Laplace transform of derivatives, obtain the Laplace transform of (a) sin 3t,  $(b)t^3$ 

Solution (a) Let 
$$f(t) = \sin 3t$$
  
Then  $f'(t) = 3 \cos 3t f''(t) = -9 \sin 3t$   
 $\mathscr{L}[f''(t)] = s^2[\mathscr{L}f(t)] - sf(0) - f'(0)$  (13.1)  
 $f(0) = 0, f'(0) = 3$   
 $\mathscr{L}[f''(t)] = \mathscr{L}[-9 \sin 3t]$   
Substituting in Eq. 13.1, we get  
 $\mathscr{L}[-9 \sin 3t] = s^2 \mathscr{L}[f(t)] - 3$   
 $\mathscr{L}[-9 \sin 3t] - s^2 [\mathscr{L}(\sin 3t)] = -3$   
 $\mathscr{L}[(s^2 + 9) \sin 3t] = 3$   
 $\therefore \mathscr{L}(\sin 3t) = \frac{3}{s^2 + 9}$   
(b) Let  $f(t) = t^3$   
Differentiating successively, we get  
 $f'(t) = 3t^2, f''(t) = 6t, f'''(t) = 6$   
By using differentiation theorem, we get  
 $\mathscr{L}[f'''(t)] = s^3 \mathscr{L}[f(t)] - s^2 f(0) - sf'(0) - f''(0)$   
Substituting all initial conditions, we get  
 $\mathscr{L}[f'''(t)] = s^3 \mathscr{L}[f(t)]$   
 $\mathscr{L}[6] = s^3 \mathscr{L}[f(t)]$   
 $f(s) = \mathscr{L}[f(t)] = \frac{6}{s^4}$   
(b) Integration Theorem If a function  $f(t)$  is continuous, then the Laplace  
transform of its integral  $\int f(t)dt$  is given by  
 $\mathscr{L}\left[\int_{0}^{t} f(t) dt\right] = \frac{1}{s}F(s)$ 

**Proof** By definition

$$\mathscr{L}\left[\int_{0}^{t} f(t) dt\right] = \int_{0}^{\infty} \left[\int_{0}^{t} f(t) dt\right] e^{-st} dt$$

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Integrating by parts, we get

$$=\left[\frac{e^{-st}}{-s}\int_{0}^{t}f(t)\,dt\right]_{0}^{\infty}+\frac{1}{s}\int_{0}^{\infty}e^{-st}\,f(t)\,dt$$

Since, the first term is zero, we have

$$\mathscr{L}\left[\int_{0}^{t} f(t) dt\right] = \frac{1}{s} \mathscr{L}[f(t)] = \frac{F(s)}{s}$$

**Example 13.5** Find the Laplace transform of ramp function r(t) = t.

**Solution** We know that 
$$\int_{0}^{t} u(t) = r(t) = t$$

Integration of unit step function gives the ramp function.

$$\mathscr{D}[\mathbf{r}(\mathbf{t})] = \mathscr{D}\left[\int_{0}^{t} u(t) dt\right]$$

Using the integration theorem, we get

 $\mathscr{L}[u(t)] = \frac{1}{s}$ 

$$\mathscr{D}\left[\int_{0}^{t} u(t) dt\right] = \frac{1}{s} \mathscr{D}\left[u(t)\right] = \frac{1}{s^{2}}$$

since

(c) Differentiation of Transforms If the Laplace transform of the function f(t) exists, then the derivative of the corresponding transform with respect to s in the frequency domain is equal to its multiplication by t in the time domain.

i.e. 
$$\mathscr{L}[tf(t)] = \frac{-d}{ds} F(s)$$

*Proof* By definition,

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_{0}^{\infty} f(t) e^{-st} dt$$

Since *s* and *t* are independent of variables, and the limits  $0, \infty$  are constants not depending on *s*, we can differentiate partially with respect to *s* within the integration and then integrate the function obtained with respect to *t*.

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_{0}^{\infty} [f(t) e^{-st}] dt$$

$$= \int_{0}^{\infty} f(t) \left[-te^{-st}\right] dt = -\int_{0}^{\infty} \{tf(t)\}e^{-st} dt = -\mathcal{L}\left[tf(t)\right]$$
$$\mathcal{L}\left[tf(t)\right] = -\frac{d}{ds} F(s)$$

Hence

**Example 13.6** Find the Laplace transform of function

$$f(t) = t \sin 2t$$

**Solution** Let  $f_1(t) = \sin 2t$  $\mathscr{D}[f_1(t)] = \mathscr{D}[\sin 2t] = F_1(s)$ 

where

...

$$F_1(s) = \frac{2}{s^2 + 4}$$
$$\mathscr{L}(tf_1(t)) = \mathscr{L}(t\sin 2t) = \frac{-d}{ds} \left[\frac{2}{s^2 + 4}\right] = \frac{4s}{\left(s^2 + 4\right)^2}$$

(d) Integration of transforms If the Laplace transform of the function f(t) exists, then the integral of corresponding transform with respect to s in the complex frequency domain is equal to its division by t in the time domain.

i.e. 
$$\mathscr{L}\left[\frac{f(t)}{t}\right] = \int_{s}^{\infty} F(s)ds$$

**Proof** If  $f(t) \leftrightarrow F(s)$ 

$$F(s) = \mathscr{L}[f(t)] = \int_{0}^{\infty} f(t)e^{-st} dt$$

Integrating both sides from s to  $\infty$ 

$$\int_{s}^{\infty} F(s)ds = \int_{s}^{\infty} \left[ \int_{0}^{\infty} f(t)e^{-st} dt \right] ds$$

By changing the order of integration, we get

$$= \int_{0}^{\infty} f(t) \left[ \int_{s}^{\infty} e^{-st} ds \right] dt$$
$$= \int_{0}^{\infty} f(t) \left( \frac{e^{-st}}{t} \right) dt$$
$$= \int_{0}^{\infty} \left[ \frac{f(t)}{t} \right] e^{-st} dt = \mathscr{L} \left[ \frac{f(t)}{t} \right]$$
$$\int_{0}^{\infty} F(s) ds = \mathscr{L} \left[ \frac{f(t)}{t} \right]$$

\*\*

**Example 13.7** Find the Laplace transform of the function

$$f(t) = \frac{2 - 2e^{-t}}{t}$$
Solution Let  $f_1(t) = 2 - 2e^{-2t}$  then
$$\mathscr{L}[f_1(t)] = \mathscr{L}(2 - 2e^{-2t}) = \mathscr{L}(2) - \mathscr{L}(2e^{-2t}) = \frac{2}{s} - \frac{2}{s+2}$$

$$= \frac{2s + 4 - 2s}{s(s+2)} = \frac{4}{s(s+2)}$$
Hence  $\mathscr{L}\left[\frac{2 - 2e^{-2t}}{t}\right] = \int_{s}^{\infty} F_1(s) ds$ 

$$= \int_{s}^{\infty} \frac{4}{s(s+2)} ds$$

By taking partial fraction expansion,

we get 
$$\frac{4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{2}{s} - \frac{2}{s+2}$$
$$\therefore \qquad \mathscr{L}\left[\frac{2-e^{-2t}}{t}\right] = \int_{s}^{\infty} \mathscr{L}\left[2-2e^{-2t}\right] ds = \int_{s}^{\infty} \frac{2}{s} ds - \int_{s}^{\infty} \frac{2}{s+2} ds$$
$$= \left[2\log s - 2\log(s+2)\right]_{s}^{\infty}$$
$$= \left[2\log \frac{1}{1+2/s}\right]_{s}^{\infty} = -2\log\left(\frac{s}{s+2}\right)$$
$$\mathscr{L}\left(\frac{2-2e^{-2t}}{t}\right) = 2\log\left(\frac{s+2}{s}\right)$$

(e) *First Shifting* Theorem If the function f(t) has the transform F(s), then the Laplace transform of  $e^{-at} f(t)$  is F(s + a)

**Proof** By definition, 
$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt$$

and, therefore,

$$F(s+a) = \int_{0}^{\infty} f(t)e^{-(s+a)t} dt$$

$$= \int_{0}^{\infty} e^{-at} f(t) e^{-st} dt = \mathscr{L} \left[ e^{-at} f(t) \right]$$
$$F(s+a) = \mathscr{L} \left[ e^{-at} f(t) \right]$$

*:*..

Similarly, we have

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

**Example 13.8** Find the Laplace transform of  $e^{at} \sin bt$ Solution Let  $f(t) = \sin bt$ 

$$\mathscr{L}[f(t)] = \mathscr{L}[\sin bt] = \frac{b}{s^2 + b^2}$$

since  $\mathscr{L}[e^{at}f(t)] = F(s-a)$ 

$$\mathscr{L}\left[e^{at}\sin bt\right] = \frac{b}{\left(s-a\right)^2 + b^2}$$

**Example 13.9** Find the Laplace transform of  $(t + 2)^2 e^t$ Solution Let  $f(t) = (t + 2)^2 = t^2 + 2t + 4$ 

$$\mathscr{L}[f(t)] = \mathscr{L}[t^2 + 2t + 4] = \frac{2}{s^3} + \frac{2}{s^2} + \frac{4}{s}$$

since

$$\mathscr{L}\left[e^{at}f(t)\right] = F(s-a)$$

$$\mathscr{L}[e^{t}f(t)] = \frac{2}{(s-1)^{3}} + \frac{2}{(s-1)^{2}} + \frac{4}{s-1}$$

(f) Second Shifting Theorem If the function f(t) has the transform F(s), then the Laplace transform of f(t-a)u(t-a) is  $e^{-as} F(s)$ .

*Proof* By definition,

$$\mathscr{L}\left[f(t-a)\ u(t-a)\right]$$
$$= \int_{0}^{\infty} \left[f(t-a)\ u(t-a)\right]e^{-st}\ dt$$

Since f(t-a) u(t-a) = 0 for t < a

$$= f(t-a)$$
 for  $t > a$ 

$$\therefore \quad \mathscr{L}[f(t-a) \ u(t-a)] = \int_{0}^{\infty} f(t-a)e^{-st} \ dt$$

Put

$$t-a = \tau$$
 then  $\tau + a = t$ 

$$dt = d\tau$$

13.11
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Therefore, the above becomes

$$\mathscr{L}[f(t-a) u(t-a)] = \int_{0}^{\infty} f(\tau) e^{-s(\tau+a)} d\tau$$
$$= e^{-as} \int_{0}^{\infty} f(\tau) e^{-s\tau} d\tau = e^{-as} F(s)$$
$$\mathscr{L}[f(t-a) u(t-a)] = e^{-as} F(s)$$

**Example 13.10** If u(t) = 1, for  $t \ge 0$  and u(t) = 0 for t < 0, determine the Laplace transform of [u(t) - u(t - a)].

**Solution** The function f(t) = u(t) - u(t-a) is shown in Fig. 13.1.

$$\mathscr{L}[f(t)] = \mathscr{L}[u(t) - u(t-a)] \qquad f(t)$$

$$= \mathscr{L}[u(t)] - \mathscr{L}[u(t-a)] \qquad 1$$

$$= \frac{1}{s} - e^{-as} \frac{1}{s} = \frac{1}{s} (1 - e^{-as}) \qquad Fig. 13.1$$

(g) *Initial Value Theorem* If the function f(t) and its derivative f'(t) are Laplace transformable then  $\underset{t \to 0}{\text{Lt}} f(t) = \underset{s \to \infty}{\text{Lt}} sF(s)$ 

*Proof* We know that

$$\mathscr{\ell}\left[f'(t)\right] = s[\mathscr{L}\left(f(t)\right)] - f(0)$$

By taking the limit  $s \rightarrow \infty$  on both sides

$$\operatorname{Lt}_{s \to \infty} \mathscr{D} \left[ f'(t) \right] = \operatorname{Lt}_{s \to \infty} \left[ sF(s) - f(0) \right]$$
$$\operatorname{Lt}_{s \to \infty} \int_{0}^{\infty} f'(t) e^{-st} dt = \operatorname{Lt}_{s \to \infty} \left[ sF(s) - f(0) \right]$$

At  $s \to \infty$  the integration of LHS becomes zero

i.e.  

$$\int_{0}^{\infty} \operatorname{Lt}_{s \to \infty} [f'(t) \ e^{-st}] \ dt = 0$$

$$0 = \operatorname{Lt}_{s \to \infty} sF(s) - f(0)$$

$$\therefore \qquad \operatorname{Lt}_{s \to \infty} sF(s) = f(0) = \operatorname{Lt}_{t \to 0} f(t)$$

**Example 13.11** Verify the initial value theorem for the following functions (i)  $5e^{-4t}$  (ii)  $2 - e^{5t}$ 

:.

(i) Let  $f(t) = 5e^{-4t}$ Solution  $F(s) = \frac{5}{s+4}$  $sF(s) = \frac{5s}{s \perp A}$  $\operatorname{Lt}_{s \to \infty} sF(s) = \operatorname{Lt}_{s \to \infty} \frac{5}{1 + 4/s} = 5$  $\lim_{t \to 0} f(t) = \lim_{t \to 0} 5e^{-4t} = 5$ Hence the theorem is proved.  $f(t) = 2 - e^{5t}$ (ii) Let  $F(s) = \mathcal{L}(2 - e^{5t}) = \mathcal{L}(2) - \mathcal{L}[e^{5t}]$  $=\frac{2}{s}-\frac{1}{s-5}=\frac{s-10}{s(s-5)}$  $sF(s) = \frac{s-10}{s-5}$  $\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \left( \frac{1 - 10/s}{1 - 5/s} \right) = 1$  $\lim_{t \to 0} (2 - e^{5t}) = 1$ 

Hence initial value theorem is proved.

(h) *Final Value Thorem* If f(t) and f'(t) are Laplace transformable, then  $\underset{t \to \infty}{\text{Lt}}$ 

$$f(t) = \underset{s \to 0}{\operatorname{Lt}} sF(s)$$

*Proof* We know that

$$\mathscr{L}[f'(t)] = sF(s) - f(0)$$

By taking the limit  $s \rightarrow 0$  on both sides, we have

$$\operatorname{Lt}_{s \to 0} \mathscr{P}[f'(t)] = \operatorname{Lt}_{s \to 0} [sF(s) - f(0)]$$
$$\operatorname{Lt}_{s \to 0} \int_{0}^{\infty} f'(t)e^{-st} dt = \operatorname{Lt}_{s \to 0} [sF(s) - f(0)]$$
$$\therefore \qquad \int_{0}^{\infty} f'(t)dt = \operatorname{Lt}_{s \to 0} [sF(s) - f(0)]$$

-----

$$\left[f(t)\right]_{0}^{\infty} = \operatorname{Lt}_{t \to \infty} f(t) - \operatorname{Lt}_{t \to 0} f(t) = \operatorname{Lt}_{s \to 0} sF(s) - f(0)$$

Since f(0) is not a function of s, it gets cancelled from both sides.

**Example 13.12** Verify the final value theorem for the following functions.

(i) 
$$2 + e^{-3t} \cos 2t$$
 (ii)  $6(1 - e^{-t})$ 

**Solution** (i) Let  $f(t) = 2 + e^{-3t} \cos 2t$ 

$$F(s) = \frac{2}{s} + \frac{(s+3)}{(s+3)^2 + 4}$$

$$sF(s) = 2 + \frac{s^2}{(s+3)^2 + 4^2} + \frac{3s}{(s+3)^2 + 4^2}$$

$$\underset{s \to 0}{\text{Lt}} sF(s) = \underset{s \to 0}{\text{Lt}} \left[ 2 + \frac{(s+3)s}{(s+3)^2 + 4^2} \right] = 2$$

$$\underset{t \to \infty}{\text{Lt}} f(t) = \underset{t \to \infty}{\text{Lt}} (2 + e^{-3t} \cos 2t) = 2$$

Hence the final value theorem is proved.

(ii) Let 
$$f(t) = 6(1 - e^{-t})$$
$$F(s) = \frac{6}{s} - \frac{6}{s+1} = \frac{6}{s(s+1)}$$
$$sF(s) = \frac{6}{s+1}$$
$$Lt_{s \to 0} sF(s) = 6$$
$$Lt_{s \to \infty} f(t) = Lt_{s \to \infty} 6(1 - e^{-t}) = 6$$

Hence the final value theorem is proved.

#### **13.5 THE INVERSE TRANSFORMATION**

So far, we have discussed Laplace transforms of a functions f(t). If the function in frequency domain F(s) is given, the inverse Laplace transform can be determined by taking the partial fraction expansion which will be recognisable as the transform of known functions. **Example 13.13** If  $F(s) = \frac{2}{(s+1)(s+5)}$ , find the function f(t).

Solution First we divide the given function into partial fractions

$$F(s) = \frac{2}{(s+1)(s+5)}$$
$$\frac{2}{(s+1)(s+5)} = \frac{A}{s+1} + \frac{B}{s+5}$$
$$2 = A(s+5) + B(s+1)$$

Comparing both sides

$$A + B = 0$$
$$5A + B = 2$$

From which

$$A = \frac{1}{2}, B = -\frac{1}{2}$$

Hence 
$$\frac{2}{(s+1)(s+5)} = \frac{1}{2(s+1)} + \frac{-1}{2(s+5)}$$

$$\mathscr{Q}^{-1}\left[\frac{2}{\left(s+1\right)\left(s+5\right)}\right] = \mathscr{Q}^{-1}\left[\frac{1}{2\left(s+1\right)}\right] - \mathscr{Q}^{-1}\left[\frac{1}{2\left(s+5\right)}\right]$$

We know that

$$\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$$

and

$$\mathscr{L}^{-1}\left(\frac{1}{s+5}\right) = e^{-5t}$$

:.

#### 13.6 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

Periodic functions appear in many practical problems. Let function f(t) be a periodic function which satisfies the condition f(t) = f(t + T) for all t > 0 where *T* is period of the function.

 $\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-5t}$ 

$$\mathscr{L}[f(t)] = \int_{0}^{T} f(t) e^{-st} dt + \int_{T}^{2T} f(t) e^{-st} dt + \dots + \int_{nT}^{(n+1)T} f(t) e^{-st} dt + \dots$$

$$= \int_{0}^{T} f(t) e^{-st} dt + \int_{0}^{T} f(t) e^{-st} e^{-sT} dt + \dots + \int_{0}^{T} f(t) e^{-st} e^{-nsT} dt + \dots$$
$$= (1 + e^{-sT} + e^{-2sT} + \dots + e^{-nsT} + \dots) \int_{0}^{T} f(t) e^{-st} dt$$
$$= \frac{1}{1 - e^{-sT}} \int_{0}^{T} f(t) e^{-st} dt$$

-----







**Solution** Here the period is 2T

$$\therefore \qquad \mathscr{P}[f(t)] = \frac{1}{1 - e^{-2sT}} \left[ \int_{0}^{2T} f(t) e^{-st} dt \right] \\ = \frac{1}{1 - e^{-2sT}} \left[ \int_{0}^{T} A e^{-st} dt + \int_{T}^{2T} (-A) e^{-st} dt \right] \\ = \frac{1}{1 - e^{-2sT}} \left[ \frac{-A}{s} e^{-st} \right]_{0}^{T} + \frac{A}{s} e^{-st} \Big]_{T}^{2T} \right] \\ = \frac{1}{1 - e^{-2sT}} \left[ -\frac{A}{s} (e^{-sT} - 1) + \frac{A}{s} (e^{-2sT} - e^{-sT}) \right] \\ = \frac{1}{1 - e^{-2sT}} \left[ \frac{A}{s} (1 - e^{-sT})^{2} \right] = \frac{A}{s} \left( \frac{1 - e^{-sT}}{1 + e^{-sT}} \right) \\ \therefore \qquad \mathscr{P}[f(t) = \frac{A}{s} \left( \frac{1 - e^{-sT}}{1 + e^{-sT}} \right)$$

#### 13.7 THE CONVOLUTION INTEGRAL

If F(s) and G(s) are the Laplace transforms of f(t) and g(t), then the product of F(s) G(s) = H(s), where H(s) is the Laplace transform of h(t) given by f(t) \* g(t)and defined by

$$h(t) = f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

**Proof** Let  $\int_{0}^{t} f(\tau) g(t-\tau) d\tau = h(t)$ 

By definition

$$\mathscr{L}[h(t)] = \int_{0}^{\infty} e^{-st} h(t) dt$$
$$= \int_{0}^{\infty} e^{-st} \int_{0}^{t} f(\tau) g(t-\tau) d\tau dt$$
$$= \int_{0}^{\infty} \int_{0}^{t} e^{-st} f(\tau) g(t-\tau) d\tau dt$$

By changing the order of integration of the above equation, we have

$$\mathscr{L}[h(t)] = \int_{0}^{\infty} \int_{\tau}^{\infty} e^{-st} f(\tau) g(t-\tau) dt d\tau$$
$$= \int_{0}^{\infty} f(\tau) \left[ \int_{\tau}^{\infty} e^{-st} g(t-\tau) dt \right] d\tau$$

Put  $t - \tau = y$ , and we get

$$\mathscr{D}[h(t)] = \int_{0}^{\infty} f(\tau) \left[ \int_{0}^{\infty} e^{-s(y+\tau)} g(y) dy \right] d\tau$$
$$= \int_{0}^{\infty} f(\tau) e^{-st} [G(s)] d\tau$$
$$= G(s) \cdot F(s)$$

Therefore,  $\mathscr{L}[h(t)] = H(s) = G(s) \cdot F(s)$ 

$$h(t) = \int_{0}^{t} f(\tau) g(t-\tau) d\tau \text{ defines the convolution of functions } f(t) \text{ and } g(t) \text{ and }$$

is expressed symbolically as

$$h(t) = f(t) * g(t)$$

This theorem is very useful in frequency domain analysis.

**Example 13.15** By using the convolution theorem, determine the inverse Laplace transform of the following functions.

(i) 
$$\frac{1}{s^2(s^2-a^2)}$$
 (ii)  $\frac{1}{s^2(s+1)}$ 

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**Solution** (i) Let  $H(s) = \frac{1}{s^2(s^2 - a^2)}$  and let  $F(s) = \frac{1}{s^2}$  and  $G(s) = \frac{1}{s^2 - a^2}$  $f(t) = \mathscr{Q}^{-1}[F(s)] = \mathscr{Q}^{-1}\left(\frac{1}{s^2}\right) = t$ We know  $g(t) = \mathcal{D}^{-1}[G(s)] = \mathcal{D}^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{1}{s} \sin h (at)$  $L^{-1}\left|\frac{1}{s^2\left(s^2-a^2\right)}\right| = \int_{0}^{t} g(\tau) f(t-\tau)d\tau$ Hence  $=\frac{1}{a}\int (t-\tau)\sin h(a \tau)d\tau$  $\frac{1}{a}\left|(t-\tau)\int_{-\infty}^{t}\sin h(a\tau)d\tau - \int_{-\infty}^{t}(-1)\int \sin h(a\tau)d\tau\right|$  $=\frac{1}{a}\left[\left(t-\tau\right)\frac{\cos ha\tau}{a}\right]_{0}^{t}+\int_{0}^{t}\frac{\cos ha\tau}{a}d\tau$  $=\frac{1}{a}\left|\frac{-t}{a}+\frac{\sin ha\tau}{a}\right|_{a}^{t}$  $=\frac{1}{a^2}\left[\sin h \, at - t\right]$ (ii) Let  $H(s) = \frac{1}{s^2(s+1)}$  and  $F(s) = \frac{1}{s^2} G(s) = \frac{1}{s+1}$  $f(t) = \mathcal{L}^{-1} [F(s)] = t$ We know that  $g(t) = \mathcal{L}^{-1} \left[ G(s) \right] = e^{-t}$  $h(t) = \mathscr{Q}^{-1}[H(s)] = \int_{-\infty}^{t} g(\tau) f(t-\tau) d\tau$ 

t

#### **13.8 PARTIAL FRACTIONS**

Most transform methods depend on the partial fraction of a given transform function. Given any solution of the form N(s) = P(s)/Q(s), the inverse Laplace transform can be determined by expanding it into partial fractions. The partial fractions depend on the type of factor. It is to be assumed that P(s) and Q(s) have real coefficients and contain no common factors. The degree of P(s) is lower than that of Q(s).

Case 1 When roots are real and simple

In this case N(s) = P(s)/Q(s)where Q(s) = (s-a)(s-b)(s-c)

Expanding N(s) into partial fractions, we get

$$N(s) = \frac{A}{(s-a)} + \frac{B}{(s-b)} + \frac{C}{(s-c)}$$
(13.2)

To obtain the constant A, multiplying Eq. 13.2 with (s - a) and putting s = a, we get

$$N(s)(s-a)|_{s=a} = A$$

Similarly, we can get the other constants

$$B = (s - b)N(s) \mid_{s = b}$$
  
$$C = (s - c)N(s) \mid_{s = c}$$

**Example 13.16** Determine the partial fraction expansion for  $N(s) = \frac{s^2 + s + 1}{s(s+5)(s+3)}$ .

$$N(s) = \frac{s^2 + s + 1}{s(s+5)(s+3)}$$

$$\frac{s^2 + s + 1}{s(s+5)(s+3)} = \frac{A}{s} + \frac{B}{s+5} + \frac{C}{s+3}$$

$$= \int_{0}^{t} e^{-\tau} (t - \tau) d\tau$$
  
=  $(t - \tau) (-e^{-\tau})_{0}^{t} - \int_{0}^{t} (-1) (-e^{-\tau}) d\tau$   
=  $t - \int_{0}^{t} e^{-\tau} d\tau$   
=  $t - (-e^{-\tau})_{0}^{t} = t + e^{-t} - 1$ 

$$A = sN(s)|_{s=0} = \frac{s^2 + s + 1}{(s+5)(s+3)}\Big|_{s=0} = \frac{1}{15}$$
$$B = (s+5) N(s)|_{s=-5} = \frac{s^2 + s + 1}{s(s+3)}\Big|_{s=-5}$$
$$= \frac{(25) + (-5) + 1}{(-5)(-5+3)} = \frac{21}{10} = 2.1$$
$$C = (s+3) N(s)|_{s=-3} = \frac{s^2 + s + 1}{s(s+5)}\Big|_{s=-3}$$
$$= \frac{9 - 3 + 1}{(-3)(-3+5)} = \frac{7}{-6} = -1.17$$

**Case 2** When roots are real and multiple In this case N(s) = P(s)/Q(s)

where  $Q(s) = (s - a)^n Q_1(s)$ The partial fraction expansion of N(s) is

 $N(s) = \frac{A_0}{(s-a)^n} + \frac{A_1}{(s-a)^{n-1}} + \dots + \frac{A_{n-1}}{(s-a)} + \frac{P_1(s)}{Q_1(s)}$ 

where  $\frac{P_1(s)}{Q_1(s)} = R(s)$  represents the remainder terms of expansion. To obtain the

constant  $A_0, A_1, ..., A_{n-1}$ , let us multiply both sides of Eq. 13.3 by  $(s-a)^n$ Thus

$$(s-a)^{n} N(s) = N_{1}(s) = A_{0} + A_{1}(s-a) + A_{2}(s-a)^{2} + \dots + A_{n-1} (s-a)^{n-1} + R(s) (s-a)^{n}$$
(13.4)

where R(s) indicates the remainder terms.

Putting s = a, we get

$$A_0 = (s-a)^n N(s)|_{s=a}$$

Differentiating Eq. 13.4 with respect to s, and putting s = a, we get

 $A_{1} = \frac{d}{ds} N_{1}(s) \Big|_{s=a}$  $A_{2} = \frac{1}{2!} \frac{d^{2}}{ds^{2}} N_{1}(s) \Big|_{s=a}$ 

Similarly,

In general,

$$A_n = \frac{1}{n!} \left. \frac{d^n N_1(s)}{ds^n} \right|_{s=a}$$

(13.3)

**Example 13.17** Determine the partial fraction expansion for

$$N(s) = \frac{s-5}{s(s+2)^2}$$

Solution

$$N(s) = \frac{s-5}{s(s+2)^2}$$

$$N(s) = \frac{s-5}{s(s+2)^2} = \frac{A}{s} + \frac{B_0}{(s+2)^2} + \frac{B_1}{s+2}$$

$$A = N(s)s|_{s=0} = \frac{s-5}{(s+2)^2} \Big|_{s=0} = \frac{-5}{4} = -1.25$$

$$N_1(s) = (s+2)^2 N(s) = \frac{s-5}{2}$$

$$B_0 = N(s) (s+2)^2|_{s=-2} = \frac{s-5}{2} \Big|_{s=-2}$$

$$= \frac{-7}{-2} = 3.5$$

$$B_1 = \frac{d}{ds} N_1(s) \Big|_{s=-2}$$

$$= \frac{d}{ds} \left(1 - \frac{5}{s}\right) \Big|_{s=-2}$$

$$= \frac{5}{4} = 1.25$$

*Case 3* When roots are complex

Consider a function  $N(s) = \frac{P(s)}{Q_1(s)(s - \alpha + j\beta)(s - \alpha - j\beta)}$ The partial fraction expansion of N(s) is

$$N(s) = \frac{A}{s - \alpha - j\beta} + \frac{b}{s - \alpha + j\beta} + \frac{P_1(s)}{Q_1(s)}$$
(13.5)

where  $P_1(s)/Q_1(s)$  is the remainder term.

Multiplying Eq. 13.5 by  $(s - \alpha - j\beta)$  and putting  $s = \alpha + j\beta$ , we get

$$A = \frac{P(\alpha + j\beta)}{Q_1(\alpha + j\beta)(+2j\beta)}$$
$$B = \frac{P(\alpha - j\beta)}{(-2j\beta)Q_1(\alpha - j\beta)}$$

Similarly,

In general,  $B = A^*$  where  $A^*$  is complex conjugate of A. If we denote the inverse transform of the complex conjugate terms as f(t)

$$f(t) = \mathscr{L}^{-1} \left[ \frac{A}{s - \alpha - j\beta} + \frac{B}{s - \alpha + j\beta} \right]$$
$$= \mathscr{L}^{-1} \left[ \frac{A}{s - \alpha - j\beta} + \frac{A^*}{s - \alpha + j\beta} \right]$$

where A and  $A^*$  are conjugate terms.

If we denote A = C + jD, then

$$B = C - jD = A^*$$
  

$$\therefore \qquad f(t) = e^{\alpha t} \left( A e^{j\beta t} + A^* e^{-j\beta t} \right)$$

**Example 13.18** Find the inverse transform of the function

$$F(s) = \frac{s+5}{s\left(s^2+2s+5\right)}$$

Solution

$$F(s) = \frac{s+5}{s(s^2+2s+5)}$$

By taking partial fractions, we have

$$F(s) = \frac{s+5}{s(s^2+2s+5)} = \frac{A}{s} + \frac{B}{s+1-j2} + \frac{B^*}{s+1+j2}$$

$$A = F(s)s|_{s=0} = \frac{s+5}{(s^2+2s+5)} = 1$$

$$B = F(s)(s+1-j2)|_{s=-1+j2} = \frac{s+5}{s(s+1+j2)}|_{s=-1+j2}$$

$$= \frac{4+j2}{(-1+j2)j4}$$

$$= \frac{2+j}{2j(-1+j2)} = \frac{2+j}{-2j-4} = \frac{-1}{2}$$

$$B^* = F(s)(s+1+j2)|_{s=-1+j2}$$

$$= \frac{s+5}{s(s+1-j2)} \bigg|_{s=-1-j2}$$

$$= \frac{-1-j2+5}{(-1-j2)(-1-j2+1-j2)}$$

$$= \frac{4-j2}{+(1+j2)(j4)} = \frac{4-j2}{4j-8} = \frac{2(2-j)}{-4(2-j)} = \frac{-1}{2}$$

$$F(s) = \frac{1}{s} - \frac{1}{2(s+1-j2)} - \frac{1}{2(s+1+j2)}$$

*:*.

The inverse transform of F(s) is f(t)

$$f(t) = \mathscr{L}^{-1} [F(s)] = \mathscr{L}^{-1} \left[ \frac{1}{s} - \frac{1}{2(s+1-j2)} - \frac{1}{2(s+1+j2)} \right]$$
$$= \mathscr{L}^{-1} \left[ \frac{1}{s} \right] - \frac{1}{2} \mathscr{L}^{-1} \left[ \frac{1}{(s+1-j2)} \right] - \frac{1}{2} \mathscr{L}^{-1} \left[ \frac{1}{s+1+j2} \right]$$
$$= 1 - \frac{1}{2} e^{(-1+j2)t} - \frac{1}{2} e^{(-1-j2)t}$$

#### **13.9 APPLICATIONS OF LAPLACE TRANSFORMS**

Laplace transform methods are used to find out transient currents in circuits containing energy storage elements. To find these currents, first the differential equations are formed by applying Kirchhoff's laws to the circuit, then these differential equations can be easily solved by using Laplace transformation methods.  $\times s$ 

Consider a series RL circuit shown in Fig. 13.3.

When the switch is closed at t = 0, the voltage V is applied to the circuit.

By applying Kirchhoff's laws, we get



i (t)

R

$$Ri(t) + L \frac{di}{dt} = V \tag{13.6}$$

Now, application of Laplace transform to each term gives,

$$RI(s) + L[sI(s) - i(0)] = \frac{V}{s}$$
  

$$RI(s) + sL I(s) - Li(0) = \frac{V}{s}$$
(13.7)

i(0) is the current passing through the circuit just before the switch is closed. When i(0) = 0, Eq. 13.7, becomes

$$RI(s) + sLI(s) = \frac{V}{s}$$
$$I(s) = \frac{V/L}{s\left(s + \frac{R}{L}\right)}$$

The current i(t) can be determined by taking inverse Laplace transform.

$$i(t) = \mathscr{L}^{-1} \left[ I(s) \right] = \frac{V}{L} \mathscr{L}^{-1} \left[ \frac{1}{s \left( s + R/L \right)} \right]$$

To find the constants, let

$$\frac{1}{s(s+R/L)} = \frac{A}{s} + \frac{B}{s+R/L}$$

$$A = \frac{1}{s(s+R/L)} \times s \Big|_{s=0} = \frac{L}{R}$$

$$B = \frac{1}{s(s+R/L)} \times \left(s + \frac{R}{L}\right) \Big|_{s=-R/L} = \frac{-L}{R}$$

$$i(t) = \mathcal{L}^{-1}[I(s)] = \frac{V}{L} \mathcal{L}^{-1}\left[\frac{L}{Rs} - \frac{L}{R(s+R/L)}\right]$$

$$= \frac{V}{L} \times \left[\frac{L}{R}(1) - \frac{L}{R} e^{-(R/L)t}\right]$$

$$= \frac{V}{L} \times \frac{L}{R} \left[1 - e^{-(R/L)t}\right]$$
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$$i(t) = \frac{V}{L} \left[1 - e^{-(R/L)t}\right]$$

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**Example 13.19** In the circuit shown in Fig. 13.4, determine the current i(t) when the switch is changed from position 1 to 2. The switch is moved from position 1 to 2 at time t = 0.



Fig. 13.4

Solution When the switch is at position 2, application of Kirchhoff's law gives

$$10i(t) + 0.5 \frac{di}{dt} = 50 \tag{13.8}$$

Taking Laplace transforms on both sides

$$10I(s) + 0.5[sI(s) - i(0)] = \frac{50}{s}$$
(13.9)

Where i(0) is the current passing through *RL* circuit when switch is at position 1.

Therefore, the initial current is 10/10 = 1 A

$$i(0) = 1 A$$

Then Eq. 13.9, becomes

$$10I(s) + 0.5[sI(s) - 1] = \frac{50}{s}$$

$$I(s)[10 + 0.5s] - 0.5 = \frac{50}{s}$$

$$I(s) = \frac{50/s + 0.5}{10 + 0.5s} = \frac{0.5}{s} \frac{(s + 100)}{0.5(s + 20)}$$

$$= \frac{s + 100}{s(s + 20)}$$

$$i(t) = \mathscr{L}^{-1}[I(s)] = \mathscr{L}^{-1}\left[\frac{s + 100}{s(s + 20)}\right]$$

$$\frac{s + 100}{s(s + 20)} = \frac{A}{s} + \frac{B}{s + 20}$$

$$A + B = 1$$

$$20A = 100$$

$$A = 5, B = -4$$

$$i(t) = \mathscr{L}^{-1}\left[\frac{5}{8}\right] + \mathscr{L}^{-1}\left[\frac{-4}{s + 20}\right]$$

$$i(t) = 5 - 4e^{-20t}$$

**Example 13.20** In the circuit shown in Fig. 13.5, obtain the equations for  $i_1(t)$  and  $i_2(t)$  when the switch is closed at t = 0.



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**Solution** When the switch is closed, 50 V source is applied to the circuit. By applying Kirchhoff's law, we have

$$20 i_1(t) - 20 i_2(t) = 50 \tag{13.10}$$

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$$30i_2(t) + 1\frac{di_2}{dt} - 20i_1(t) = 0$$
(13.11)

Taking Laplace transform on both sides, we get

$$20I_1(s) - 20I_2(s) = \frac{50}{s}$$
$$20I_1(s) + (30 + s) I_2(s) = i_2(0)$$

Since the current passing through the inductance just after the switch closed is zero,  $i_2(0) = 0$ 

$$\begin{bmatrix} 20 & -20 \\ -20 & (30+s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{50}{s} \\ 0 \end{bmatrix}$$
$$I_1(s) = \frac{\begin{vmatrix} \frac{50}{s} & -20 \\ 0 & (30+s) \\ \hline 20 & -20 \\ -20 & 30+s \end{vmatrix}}{\begin{vmatrix} 20 & -20 \\ -20 & 30+s \end{vmatrix}} = \frac{50/s (30+s)}{20(s+10)}$$
$$= \frac{2.5(s+30)}{s(s+10)}$$
$$I_2(s) = \frac{\begin{vmatrix} 20 & \frac{50}{s} \\ -20 & 0 \\ \hline 20 & -20 \\ -20 & 30+s \end{vmatrix}}{\begin{vmatrix} 20 & \frac{50}{s} \\ 20(s+10) \end{vmatrix}} = \frac{\frac{50}{s} \times 20}{20(s+10)}$$

$$=\frac{50}{s(s+10)}$$

Taking partial fractions, we get

$$I_1(s) = \frac{2.5(s+30)}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$
$$= \frac{+7.5}{s} - \frac{5}{s+10}$$

Taking inverse transform, we get

$$i_{1}(t) = \mathscr{L}[I_{1}(s)] = \mathscr{L}^{-1}\left[\frac{+7.5}{s}\right] - \mathscr{L}^{-1}\left[\frac{5}{s+10}\right]$$
$$i_{1}(t) = +7.5 - 5e^{-10t}$$
$$I_{2}(s) = \frac{50}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$
$$I_{2}(s) = \frac{+5}{s} - \frac{5}{s+10}$$

Similarly,

Taking inverse transform, we have

$$i_{2}(t) = \mathscr{L}^{-1}[I_{2}(s)] = \mathscr{L}^{-1}\left(\frac{+5}{s}\right) - \mathscr{L}^{-1}\left(\frac{5}{s+10}\right)$$
$$i_{2}(t) = +5 - 5e^{-10t}$$



**Problem 13.1** For the waveform shown in Fig. 13.6, find the Laplace transform.

 $f(t) = A \sin t \quad \text{for } 0 < t < \pi$ = 0  $t > \pi$  f(t)By definition, we have  $\mathscr{P}[f(t)] = \int_{0}^{\infty} f(t)e^{-st} dt \qquad A$ =  $\int_{0}^{\pi} f(t)e^{-st} dt + \int_{\pi}^{\infty} f(t)e^{-st} dt$  Fig. 13.6

Since f(t) = 0 for  $t > \pi$ , the second term becomes zero

**Problem 13.2** Find the Laplace transform of f(t) = t for 0 < t < 1= 0 for t > 1

Solution By definition,

$$\mathscr{L}[f(t)] = \int_{0}^{\infty} e^{-st} f(t) dt$$
$$= \int_{0}^{1} f(t) e^{-st} dt + \int_{1}^{\infty} f(t) e^{-st} dt$$



Since f(t) = 0 for t > 1, the second term becomes zero

$$\mathscr{L}[f(t)] = \int_{0}^{1} f(t)e^{-st} dt$$
  
=  $\int_{0}^{1} te^{-st} dt$   
=  $t \int_{0}^{1} e^{-st} dt - \int_{0}^{1} \frac{e^{-st}}{-s} dt$   
=  $t \frac{e^{-st}}{-s} \int_{0}^{1} - \frac{e^{-st}}{s^{2}} \int_{0}^{1}$   
=  $\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^{2}} + \frac{1}{s^{2}}$   
=  $\frac{1}{s^{2}} - e^{-s} \left[ \frac{1}{s} + \frac{1}{s^{2}} \right]$ 

**Problem 13.3** Verify the initial and final value theorems for the function  $f(t) = e^{-t} (\sin 3t + \cos 5t).$ 

**Solution** 
$$f(t) = e^{-t} (\sin 3t + \cos 5t)$$
$$F(s) = \mathscr{L}[f(t)] = \mathscr{L}[e^{-t} (\sin 3t + \cos 5t)]$$

Since

$$\mathscr{L}(e^{-t}\sin 3t) = \frac{3}{(s+1)^2 + 3^2}$$

and

$$\mathscr{L}(e^{-t}\cos 5t) = \frac{s+1}{(s+1)^2 + 5^2}$$

$$F(s) = \mathscr{L}[f(t)] = \frac{3}{(s+1)^2 + 3^2} + \frac{s+1}{(s+1)^2 + 5^2}$$

...

$$\operatorname{Lt}_{t \to 0} f(t) = \operatorname{Lt}_{s \to \infty} sF(s)$$

$$F(s) = \frac{3}{s^2 + 2s + 10} + \frac{s + 1}{s^2 + 2s + 26}$$

$$sF(s) = \frac{3s}{s^2 \left(1 + \frac{2}{s} + \frac{10}{s^2}\right)} + \frac{s^2 + s}{s^2 \left(1 + \frac{2}{s} + \frac{26}{s^2}\right)}$$

$$= \frac{3}{s \left(1 + \frac{2}{s} + \frac{10}{s^2}\right)} + \frac{1}{1 + \frac{2}{s} + \frac{26}{s^2}} + \frac{1}{s \left(1 + \frac{2}{s} + \frac{26}{s^2}\right)}$$
Lt  $sF(s) = 1$ 

$$f(t) = e^{-t} (\sin 3t + \cos 5t)$$
Lt  $f(t) = 1$ 

Hence the initial value theorem is proved. According to the final value theorem,

$$Lt_{t \to \infty} f(t) = Lt_{s \to 0} sF(s)$$

$$Lt_{s \to 0} sF(s) = 0$$

$$Lt_{t \to \infty} f(t) = 0$$

Hence the final value theorem is proved.

Problem 13.4 Determine the inverse Laplace transform of the function

$$F(s) = \frac{s-3}{s^2+4s+13}$$
$$F(s) = \frac{s-3}{s^2+4s+13} = \frac{s-3}{(s+2)^2+9} = \frac{(s+2)-5}{(s+2)^2+9}$$

#### Solution

$$\frac{s+2}{(s+2)^2+9} - \frac{5}{(s+2)^2+9}$$

By taking the inverse Laplace transforms, we get

$$\mathscr{L}^{-1}F(s) = \mathscr{L}^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - \mathscr{L}^{-1}\left[\frac{5}{(s+2)^2+9}\right]$$
$$= e^{-2t}\cos 3t - \frac{5}{3} e^{-2t}\sin 3t = \frac{e^{-2t}}{3} [3\cos 3t - 5\sin 3t]$$

Problem 13.5 Find the inverse transform of the following

(a)  $\log\left(\frac{s+5}{s+6}\right)$ (b)  $\frac{1}{\left(s^2+5^2\right)^2}$  \*\*

**Solution** (a) Let  $F(s) = \log\left(\frac{s+5}{s+6}\right)$  $\frac{d}{ds}\left[F(s)\right] = \frac{d}{ds}\left|\log\left(\frac{s+5}{s+6}\right)\right| = \frac{1}{s+5} - \frac{1}{s+6}$ Then  $\mathscr{Q}^{-1}\left[\frac{d}{ds}F(s)\right] = -tf(t)$ We know that  $\mathscr{Q}^{-1}\left[\frac{d}{ds}F(s)\right] = \mathscr{Q}^{-1}\left[\frac{1}{s+5} - \frac{1}{s+6}\right] = e^{-5t} - e^{-6t}$ ...  $-tf(t) = e^{-5t} - e^{-6t}$ Hence  $f(t) = \frac{e^{-6t} - e^{-5t}}{t}$  $F(s) = \frac{1}{\left(s^2 + 5^2\right)^2}$ (b) Let  $\frac{1}{\left(s^{2}+5^{2}\right)^{2}} = \frac{1}{s} \frac{s}{\left(s^{2}+5^{2}\right)^{2}}$  $\mathscr{Q}^{-1}\left|\frac{1}{\left(s^{2}+5^{2}\right)^{2}}\right| = \mathscr{Q}^{-1}\left|\frac{1}{s}\frac{s}{\left(s^{2}+5^{2}\right)^{2}}\right|$ Therefore According to the integration theorem,  $\mathscr{L}^{-1}\left|\frac{1}{s}\frac{s}{\left(s^{2}+5^{2}\right)^{2}}\right| = \int_{0}^{t}\left|\mathscr{L}^{-1}\frac{s}{\left(s^{2}+5^{2}\right)^{2}}\right| dt$  $\mathscr{L}[f(t)] = F(s)$ , then  $\mathscr{L}\left[\frac{f(t)}{t}\right] = \int_{0}^{\infty} F(s) ds$ If  $\int_{s}^{\infty} \frac{s}{\left(s^{2}+5^{2}\right)^{2}} ds = \frac{-1}{2} \left[\frac{1}{s^{2}+5^{2}}\right]_{s}^{\infty} = \frac{1}{2} \frac{1}{s^{2}+5^{2}}$ Here  $\frac{f(t)}{t} = \mathcal{G}^{-1}\left(\frac{1}{2} \cdot \frac{1}{s^2 + 5^2}\right) = \frac{1}{10} \sin 5t$ Therefore  $f(t) = \frac{t \sin 5t}{10}$ ...  $\mathscr{L}^{-1}\left|\frac{1}{s}\frac{s}{\left(s^{2}+5^{2}\right)^{2}}\right| = \int_{-1}^{t}\frac{t\sin 5t}{10}\,dt$ or  $=\frac{1}{10}\left[t\left(\frac{-\cos 5t}{5}\right)+\frac{\sin 5t}{25}\right]^{t}$  $=\frac{1}{250} [\sin 5t - 5t \cos 5t]$ 

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**Problem 13.6** Find the Laplace transform of the full wave rectified output as shown in Fig. 13.8.



**Problem 13.7** Find the Laplace transform of the square wave shown in A Fig. 13.9.

# **Solution** We have $f(t) = A \ 0 < t < a$



$$= -A \ a < t < 2a$$
Fig. 13.9
$$\mathscr{L}[f(t)] = \frac{1}{1 - e^{-2as}} \left[ \int_{0}^{a} Ae^{-st} \ dt + \int_{a}^{2a} (-A)e^{-st} \ dt \right]$$

$$= \frac{A}{s} \frac{\left(1 - 2e^{-as} + e^{-2as}\right)}{1 - e^{-2as}}$$

$$= \frac{A}{s} \frac{\left(1 - e^{-as}\right)^{2}}{\left(1 + e^{-as}\right)\left(1 - e^{-as}\right)} = \frac{A}{s} \tanh\left(\frac{as}{2}\right)$$

\*\*

**Problem 13.8** Obtain the inverse transform of  $F(s) = \frac{1}{s(s+2)}$  by using the convolution integral.

**Solution** Let  $F_1(s) = \frac{1}{s}$  and  $F_2(s) = \frac{1}{s+2}$ 

We have

$$f_1(t) = \mathcal{L}^{-1}\left[F_1(s)\right] = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

Similarly,  $f_2(t) = \mathcal{G}^{-1}[F_2(s)] = \mathcal{G}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$ 

According to the convolution integral,

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau) d\tau$$
  
Since  $f_1(t-\tau) = 1$  and  $f_2(t) = e^{-2\tau}$ 

$$\therefore \qquad f_1(t) * f_2(t) = \int_0^t 1 \cdot e^{-2\tau} d\tau$$
$$= \frac{e^{-2\tau}}{-2} \int_0^t = \frac{e^{-2t}}{-2} + \frac{1}{2} = \frac{1}{2} [1 - e^{-2t}]$$
$$\therefore \qquad \mathscr{L}^{-1} \left[ \frac{1}{s(s+2)} \right] = \frac{1}{2} [1 - e^{-2t}]$$

**Problem 13.9** Determine the convolution integral when  $f_1(t) = e^{-2t}$  and  $f_2(t) = 2t$ .

Solution We have

Then

$$f_{1}(t) * f_{2}(t) = \int_{0}^{t} f_{1}(\tau) f_{2}(t-\tau) d\tau$$

$$f_{1}(t) * f_{2}(t) = \int_{0}^{t} 2\tau e^{-2(t-\tau)} d\tau = e^{-2t} \int_{0}^{t} 2\tau e^{2\tau} d\tau$$

$$= 2e^{-2t} \left[ \tau \frac{e^{2\tau}}{2} - \int 1 \cdot \frac{e^{2\tau}}{2} d\tau \right]_{0}^{t}$$

$$= 2e^{-2t} \left[ \frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4} \right]$$

$$= \left[ t - \frac{1}{2} + \frac{e^{-2t}}{2} \right]$$

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**Problem 13.10** The circuit shown in Fig. 13.10 consists of series R-L elements. The sine wave is applied to the circuit when the switch *s* is closed at t = 0. Determine the current i(t).



Fig. 13.10

**Solution** In the circuit, the current i(t) can be determined by using Kirchhoff's law.

$$5 \frac{di}{dt} + 10i = 50 \sin 25t$$

Applying Laplace transform on both sides

$$5[sI(s) - i(0)] + 10I(s) = 50 \times \frac{25}{s^2 + (25)^2}$$

where i(0) is the initial current passing through the circuit. Since the inductor does not allow sudden changes in currents, the current i(0) = 0.

$$\therefore \quad 5sI(s) + 10I(s) = \frac{50 \times 25}{s^2 + (25)^2}$$
$$I(s) = \frac{1250}{\left(s^2 + 625\right)(5s + 10)} = \frac{250}{\left(s^2 + 625\right)(s + 2)}$$

By taking partial fractions, we have

$$I(s) = \frac{250}{(s+2)(s+j25)(s-j25)}$$
$$I(s) = \left[\frac{A}{s+2} + \frac{B}{s+j25} + \frac{C}{(s-j25)}\right]$$

where  $A = (s + 2) I(s) |_{s = -2}$ 

$$= (s+2) \frac{250}{(s+2)[s^2 + (25)^2]_{s=-2}}$$
  
=  $\frac{250}{629} = 0.397$   
 $B = (s+j25) I(s)|_{s=-j25}$   
=  $(s+j25) \frac{250}{(s+2)(s+j25)(s-j25)}\Big|_{s=-j25}$   
=  $\frac{250}{(2-j25)(-j50)} = \frac{-5}{(25+j2)}$   
 $C = (s-j25) I(s)|_{s=j25}$ 

 $= (s - j25) \frac{250}{(s+2)(s+j25)(s-j25)} \bigg|_{s=j25}$  $=\frac{250}{(2+j25)(j50)}=\frac{5}{(25-j2)}$ 

Substituting the values of A, B, C in I(s), we get

$$I(s) = \frac{0.397}{s+2} - \frac{5}{(25+j2)(s+j25)} + \frac{5}{(25-j2)(s-j25)}$$

By taking the inverse transform on both sides, we get

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Fig. 13.11

**Problem 13.11** For the circuit shown in Fig. 13.11, determine the current i(t)when the switch is at position 2. The switch s is moved from position 1 to position 2 at time t = 0. The switch has been in position 1 for a long time.

**Solution** When the switch *s* is at position 2, by applying Kirchhoff's voltage law, we get

$$2 \frac{di}{dt} + 50i = 0$$

$$\frac{di}{dt} + 25i = 0$$
50 V
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Taking Laplace transform on both sides

$$s I(s) - i(0) + 25 I(s) = 0$$

where i(0) is the initial current passing through circuit just after the switch is at position 2. Since the inductor does not allow sudden changes in currents, i(0) is the same as the steady state current when the switch is at position 1.

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$$i(0) = \frac{50}{50} = 1$$
A  
Hence  $s I(s) - 1 + 25 I(s) = 0$ 

Hence

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$$I(s) = \frac{1}{s+25}$$

By taking inverse transform of the above equation, we have the current  $i(t) = e^{-25t}$ 22

Problem 13.12 For the circuit shown in Fig. 13.12, find the voltage across the 0.5  $\Omega$  resistor when the switch, s, is opened at t = 0. Assume there is no charge on the capacitor and no current in the inductor before switching.



Solution By applying Kirchhoff's current law to the circuit, we have

$$2v+1\int_{-\infty}^{t} vdt + \frac{dv}{dt} = 5$$
$$2v+1\int_{-\infty}^{0} vdt + 1\int_{0}^{t} vdt + \frac{dv}{dt} = 5$$

Taking Laplace transforms on both sides, we get

$$2V(s) + \mathscr{L}\left[\int_{-\infty}^{0} v dt\right] + \frac{V(s)}{s} + [sV(s) - v(0)] = \frac{5}{s}$$

Since the initial voltage across the capacitor and the initial current in the inductor is zero, the above equation becomes

$$2V(s) + \frac{V(s)}{s} + sV(s) = \frac{5}{s}$$
$$V(s) [2s + s^{2} + 1] = 5$$
$$V(s) = \frac{5}{s^{2} + 2s + 1}$$
$$V(s) = \frac{5}{(s + 1)^{2}}$$

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Taking inverse transforms on both sides, we have

$$v(t) = +5te^{-t}$$

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**Problem 13.13** For the circuit shown in Fig. 13.13, determine the current in the 10  $\Omega$  resistor when the switch is closed at t = 0. Assume initial current through the inductor is zero.



Fig. 13.13

**Solution** By taking mesh currents when the switch is closed at t = 0, we have  $20 = 5i_1(t) - 5i_2(t)$ 

and 
$$-5i_1(t) + 15i_2(t) + 2 \frac{di_2}{dt} = 0$$

Taking Laplace transforms on both sides, we have

$$5I_1(s) - 5I_2(s) = \frac{20}{s}$$
$$-5I_1(s) + 15I_2(s) + 2[sI_2(s) - i(0)] = 0$$

Since the initial current through the inductor is zero i(0) = 0

$$\therefore \qquad \begin{bmatrix} 5 & -5 \\ -5 & (2s+15) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 20/s \\ 0 \end{bmatrix}$$
$$\therefore \qquad I_2(s) = \frac{\begin{vmatrix} 5 & 20/s \\ -5 & 0 \\ 5 & -5 \\ -5 & (2s+15) \end{vmatrix}}{I_2(s) = \frac{20/s \times 5}{5(2s+15) - 25}$$
$$I_2(s) = \frac{100}{5s[2s+10]}$$

Taking partial fractions, we get

$$\frac{10}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

Solving for the constants

$$A = \frac{10}{s(s+5)} s \Big|_{s=0} = 2$$
$$B = \frac{10}{s(s+5)} (s+5) \Big|_{s=-5} = -2$$
$$I_2(s) = \frac{2}{s} - \frac{2}{s+5}$$

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Taking inverse transform on both sides, we have

$$i_2(t) = 2 - 2e^{-5t}$$

Therefore, the current passing through the 10  $\Omega$  resistor is  $(2 - 2e^{-5t})$  A **Problem 13.14** For the circuit shown in Fig. 13.14, determine the current when the switch is moved from position 1 to position 2 at t = 0. The switch has been in position 1 for a long time to get steady state values.



Fig. 13.14

**Solution** When the switch is at position 2, by applying Kirchhoff's law, the current equation is

$$0.1 \ \frac{di}{dt} + 2i = 20$$

Taking Laplace transform on both sides, we get

$$0.1[sI(s) - i(0)] + 2I(s) = \frac{20}{s}$$

i(0) is the current passing through the circuit just after the switch is at position 2. Since the inductor does not allow sudden changes in currents, this current is equal to the steady state current when the switch was at position 1.

## Therefore $i(0) = \frac{10}{2} = 5 \text{ A}$

Substituting i(0), in the equation, we get

$$0.1[sI(s) - 5] + 2I(s) = \frac{20}{s}$$
$$I(s)[0.1s + 2] = \frac{20}{s} + 0.5$$
$$I(s) = \frac{5(s + 40)}{s(s + 20)}$$

By taking partial fractions, we have

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$$\frac{5(s+40)}{s(s+20)} = \frac{A}{s} + \frac{B}{s+20}$$
$$A = \frac{5(s+40)}{s(s+20)} \times s \Big|_{s=0} = 10$$
$$B = \frac{5(s+40)}{s(s+20)} \times (s+20) \Big|_{s=-20} = -5$$
$$I(s) = \frac{10}{s} - \frac{5}{s+20}$$

Taking inverse transforms on both sides, we have

$$i(t) = 10 - 5e^{-20t} A$$

**Problem 13.15** For the circuit shown in Fig. 13.15, determine the current when the switch is closed at a time corresponding to  $\phi = 0$ . Assume initial charge on the capacitor is  $q_0 = 2$  coulombs with polarity shown.



Fig. 13.15

Solution By applying Kirchhoff's voltage law, we have

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$$i(t) + \frac{1}{1} \int_{-\infty}^{t} idt = 50 \cos(50t)$$
$$i(t) + \int_{-\infty}^{0} \frac{dq}{dt} dt + \int_{0}^{t} idt = 50 \cos(50t)$$

Taking Laplace transforms on both sides, we have

$$I(s) + \frac{I(s)}{s} + \frac{q_0}{s} = \frac{50s}{s^2 + 50^2}$$
$$I(s) \left[ 1 + \frac{1}{s} \right] + \frac{2}{s} = \frac{50s}{s^2 + 50^2}$$
$$I(s) = \left[ \frac{50s}{s^2 + 50^2} - \frac{2}{s} \right] \frac{s}{s+1}$$
$$= \frac{\left[ 50s^2 - 2s^2 - 2(50)^2 \right]}{\left[ s^2 + (50)^2 \right] [s+1]}$$
$$= \frac{48s^2 - 2(50)^2}{\left[ s^2 + (50)^2 \right] [s+1]}$$

By taking partial fractions, we have

$$I(s) = \frac{A}{(s+j50)} + \frac{B}{(s-j50)} + \frac{C}{s+1}$$
$$A = I(s) (s+j50)|_{s=-j50}$$
$$= \frac{48s^2 - 2(50)^2}{(s-j50) (s+1)}\Big|_{s=-j50}$$
$$= \frac{1250}{(j+50)}$$
Similarly,  $B = I(s) (s-j50) |_{s=j50}$ 

 $C = I(s)(s+1)|_{s=-1}$ 

$$= \frac{48s^2 - 2(50)^2}{(s+j50)(s+1)}\bigg|_{s=j50} = \frac{1250}{50-j}$$

 $\quad \text{and} \quad$ 

$$= \frac{48s^2 - 2(50)^2}{s^2 + (50)^2} \bigg|_{s=-1} = -1.98$$

Substituting the values of A, B, C, we get

$$I(s) = \frac{1250}{(50+j)(s+j50)} + \frac{1250}{(50-j)(s-j50)} - \frac{1.98}{s+1}$$

Taking inverse transforms

$$i(t) = \left[\frac{1250}{(50+j)}e^{-j50t} + \frac{1250}{50-j}e^{+j50t} - 1.98e^{-t}\right] A \quad \blacksquare$$

**Problem 13.16** For the circuit shown in Fig. 13.16, determine the current in the circuit when the switch is closed at t = 0. Assume that there is no initial charge on the capacitor or current in the inductor.

**Solution** When the switch is closed, by applying Kirchhoff's voltage law, we have



Fig. 13.16

Taking Laplace transforms on both sides

$$2I(s) + [sI(s) - i(0)] + \frac{I(s)}{s} + \frac{q_0}{s} = \frac{100}{s}$$

Since the initial current in the inductor and initial charge on the capacitor is zero, the above equation reduces to

$$2I(s) + sI(s) + \frac{I(s)}{s} = \frac{100}{s}$$
$$I(s) \left[2 + s + \frac{1}{s}\right] = \frac{100}{s}$$
$$I(s) = \frac{100}{s^2 + 2s + 1}$$
$$I(s) = \frac{100}{(s+1)^2}$$

:.

Taking inverse transforms on both sides, we get

$$i(t) = 100 t e^{-t} A$$

**Problem 13.17** For the circuit shown in Fig. 13.17, determine the total current delivered by the source when the switch is closed at t = 0. Assume no initial charge on the capacitor.



Fig. 13.17

Solution By applying Kirchhoff's law, the two mesh equations are

$$5i_1 + \frac{1}{1} \int_{-\infty}^{t} i_1 dt + 5i_2 = 10e^{-t}$$
  
$$5i_1 + 5i_2 + 10i_2 = 10e^{-t}$$

Taking Laplace transforms on both sides, we get

$$5I_1(s) + \frac{I_1(s)}{s} + \frac{q_0}{s} + 5I_2(s) = \frac{10}{s+1}$$

Since the initial charge on the capacitor is zero, the equation becomes

$$5I_1(s) + \frac{I_1(s)}{s} + 5I_2(s) = \frac{10}{s+1}$$

Similarly,  $5I_1(s) + 15I_2(s) = \frac{10}{s+1}$ 

By forming a matrix, we have

$$\begin{bmatrix} (5+1/s) & 5\\ 5 & 15 \end{bmatrix} \begin{bmatrix} I_1(s)\\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{10}{s+1}\\ \frac{10}{s+1} \end{bmatrix}$$
$$I_1(s) = \frac{\begin{vmatrix} \frac{10}{s+1} & 5\\ \frac{10}{s+1} & 15\\ \frac{5+1}{5} & 5\\ 5 & 15 \end{vmatrix} = \frac{\left(\frac{150}{s+1} - \frac{50}{s+1}\right)}{15\left(5+\frac{1}{s}\right) - 25}$$
$$I_1(s) = \frac{\begin{bmatrix} 100/(s+1) \end{bmatrix} s}{50(s+0.3)} = \frac{2s}{(s+0.3)(s+1)}$$

By taking partial fractions, we have

$$I_1(s) = \frac{A}{s+0.3} + \frac{B}{s+1}$$

$$A = I_{1}(s)(s+0.3)|_{s=-0.3}$$

$$= \frac{2s}{s+1}\Big|_{s=-0.3} = \frac{-(0.6)}{0.7} = -0.86$$
by,
$$B = I_{1}(s) (s+1)|_{s=-1}$$

$$= \frac{2s}{s+0.3}\Big|_{s=-1} = \frac{-2}{-(0.7)} = 2.86$$

$$I_{1}(s) = \frac{-0.86}{s+0.3} + \frac{2.86}{s+1}$$
ing inverse transforms on both sides, we have
$$i_{1}(s) = (2.86 e^{-t} - 0.86 e^{-0.3t}) \Lambda$$

Similar

*:*..

Taki

$$i_{1}(t) = (2.86 \ e^{-t} - 0.86 \ e^{-0.3t}) A$$
$$I_{2}(s) = \frac{\begin{vmatrix} \left(5 + \frac{1}{s}\right) & \frac{10}{s+1} \\ 5 & \frac{10}{s+1} \\ \left(5 + \frac{1}{s}\right) & 5 \\ 5 & 15 \end{vmatrix}} = \frac{0.2}{(s+1)(s+0.3)}$$

Similarly

By taking partial fractions, we have

$$I_2(s) = \frac{A}{s+0.3} + \frac{B}{s+1}$$

To get

$$A = I_{2}(s) (s + 0.3)|_{s = -0.3} = \frac{0.2}{s + 1} \Big|_{s = -0.3}$$

$$A = 0.286$$

$$B = I_{2}(s) (s + 1)|_{s = -1}$$

$$= \frac{0.2}{s + 0.3} \Big|_{s = -1} = \frac{0.2}{-0.7} = -0.286$$

$$I_{2}(s) = \frac{0.286}{s + 0.3} - \frac{0.286}{s + 1}$$

*:*..

*.*..

By taking inverse transforms, we have

$$i_2(t) = (0.286e^{-0.3t} - 0.286e^{-t})A$$

Hence, the total current delivered by the source

$$i(t) = i_1(t) + i_2(t)$$
  
 $i(t) = 2.574e^{-t} - 0.574e^{-0.3t})A$ 

Problem 13.18 For the circuit shown in Fig. 13.18, determine the current delivered by the source when the switch is closed at t = 0. Assume that there is no initial charge on the capacitor and no initial current through the inductor.



Fig. 13.18

**Solution** The circuit is redrawn in the *s* domain in impedance form as shown in Fig. 13.19.



The equivalent impedance in the s domain

$$Z(s) = \frac{\left(2 + \frac{1}{s}\right)s}{\left(2 + \frac{1}{s} + s\right)} = \frac{2s(s+0.5)}{s^2 + 2s + 1}$$
$$I(s) = \frac{V(s)}{Z(s)}$$
$$= \frac{\frac{20}{s}(s^2 + 2s + 1)}{2s(s+0.5)} = \frac{10(s^2 + 2s + 1)}{s^2(s+0.5)}$$

The current

$$2s(s+0.5)$$

By taking partial fractions, we have

$$I(s) = \frac{A}{s^2} + \frac{A'}{s} + \frac{B}{s+0.5}$$

The constant *B* for the simple root at s = -0.5 is

$$B = (s + 0.5) I(s)|_{s = -0.5} = 10$$

To obtain the constants of multiple roots, we first find  $I_1(s)$ .

$$I_1(s) = s^2 I(s) = \frac{10(s^2 + 2s + 1)}{(s + 0.5)}$$

Using the general formula for multiple root expansion, we get

$$A = \frac{1}{0!} \frac{d^0}{ds^0} \left[ \frac{10(s^2 + 2s + 1)}{s + 0.5} \right]_{s=0} = 20$$

 $A' = \frac{1}{1!} \frac{d'}{ds'} \left[ \frac{10(s^2 + 2s + 1)}{s + 0.5} \right]_{s=0} = 0$  $I(s) = \frac{20}{s^2} + \frac{10}{s + 0.5}$ 

Therefore,

By taking inverse transform on both sides, we have

$$i(t) = (20t + 10 e^{-0.5t}) A$$

**Problem 13.19** Find the value of  $i(0^+)$  using the initial value theorem for the Laplace transform given below.

$$I(s) = \frac{2s+3}{(s+1)(s+3)}$$

Verify the result by solving it for i(t).

Solution The initial value theorem is given by

$$Lt_{t \to 0} i(t) = Lt_{s \to \infty} SI(s)$$
$$= Lt_{s \to \infty} \frac{s(2s+3)}{(s+1)(s+3)}$$

Bringing *s* in the denominator and putting  $s = \infty$ , we get

$$\operatorname{Lt}_{s \to \infty} \frac{s^2 \left(2 + \frac{3}{s}\right)}{s^2 \left(1 + \frac{1}{s}\right) \left(1 + \frac{3}{s}\right)} = 2$$

To verify the result, we solve for i(t) and put  $t \to \infty$ . Taking partial fractions

$$I(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = (s+1) \frac{2s+3}{(s+1)(s+3)} \Big|_{s=-1} = \frac{1}{2}$$

$$B = (s+3) \frac{2s+3}{(s+1)(s+3)} \Big|_{s=-3} = \frac{3}{2}$$

Taking inverse transform, we get

$$i(t) = \frac{1}{2} e^{-t} + \frac{3}{2} e^{-3t}$$

By putting t = 0, we have i(0) = 2. The result is verified.

**Problem 13.20** Find  $\mathscr{L}^{-1}$  { $F_1(s)F_2(s)$ } by using the convolution of the following functions.

$$F_1(s) = \frac{1}{s+1}$$
 and  $F_2(s) = \frac{1}{s+2}$ 

where,

\*\*

Solution Taking inverse transforms

$$f_1(t) = 5 e^{-t}$$
  
 $f_2(t) = e^{-2t}$ 

Convolution theorem is given by

$$f_{1}(t) * f_{2}(t) = \int_{0}^{t} f_{1}(t-\tau)f_{2}(\tau) d\tau$$
$$= \int_{0}^{t} 5e^{-(t-\tau)} e^{-2\tau} d\tau$$
$$= 5e^{-t} \int_{0}^{t} e^{\tau} \cdot e^{-2\tau} d\tau$$
$$= 5e^{-t} \int_{0}^{t} e^{-\tau} d\tau$$
$$= 5e^{-t} [1-e^{-t}]$$

**Problem 13.21** In the circuit shown in Fig. 13.20, determined the voltage v(t). The capacitor and inductor are initially de-energised.



Fig. 13.20

**Solution** The transform of the given circuit will be as shown in Fig. 13.21.



Fig. 13.21

Applying Kirchhoff's voltage law, we get

$$1 = I_1(s) \frac{1}{s} + \{I_1(s) - I_2(s)\}$$

and

 $O = I_2(s) (1 + 4s) + [I_2(s) - I_1(s)] \times 1$ 

or

$$I = I_1(s)\left(\frac{1}{s} + 1\right) - I_2(s)$$

 $O = 2 I_2(s) (1 + 2s) - I_1(s)$ 

and

Solving the above equations for  $I_1(s)$  and  $I_2(s)$ , we get

$$I_2(s) = \frac{s}{(s+1)\left(s - \frac{1}{2}\right)}$$
$$I_1(s) = 2 - \frac{2}{s - \frac{1}{2}}$$

Taking inverse transform, we get

$$i_{2}(t) = \frac{2}{3} e^{-t} + \frac{1}{3} e^{\frac{1}{2}t}$$

$$i_{1}(t) = 2\delta(t) - 2e^{1/2t}$$

$$v(t) = [i_{1}(t) - i_{2}(t)] \times 1$$

$$= 2\delta(t) - \frac{2}{3} e^{-t} + \frac{7}{3} e^{\frac{7}{2}t}$$

*:*.

**Problem 13.22** Find the current in the circuit shown in Fig. 13.22 at an instant t, after opening the switch if a current of 1 A had been passing through the circuit at the instant of opening.





Solution Applying Kirchhoff's voltage law in the circuit, we get

$$6i(t) + 5 \ \frac{di(t)}{dt} = 12 + 24$$

Taking Laplace transform both sides

$$6I(s) + 5[sI(s) - i(0)] = \frac{36}{s}$$

where, i(0) = 1 A

$$I(s) [6+5s] = \frac{36}{s} + 5$$
$$I(s) = \frac{36+5s}{s(6+5s)}$$

Taking partial fractions

$$I(s) = \frac{6}{8} - \frac{5}{s + \frac{6}{5}}$$

Taking inverse transform, we have

$$i(t) = 6 - 5e^{-\frac{6}{5}t}$$



### **PRACTICE PROBLEMS**

13.1 Find the Laplace transforms of the following functions.

(a) 
$$t^3 + at^2 + bt + 3$$
  
(b)  $\sin^2 5t$   
(c)  $e^{5t+6}$   
(d)  $\cos h^2 3t$ 

13.2 Find the inverse transforms of the following functions

(a) 
$$\frac{1}{s^2 + 9}$$
 (b)  $\frac{2\pi}{s + \pi}$   
(c)  $\frac{8}{(s+3)(s+5)}$  (d)  $\frac{5}{s^2 + 9}$   
(e)  $\frac{k_1}{s} + \frac{k_2}{s^2} + \frac{k_3}{s^3}$ 

13.3 Find the inverse transforms of the following functions.

(a) 
$$\frac{5s+4}{(s-1)(s^2+2s+5)}$$
 (b)  $\frac{4s+2}{s^2+2s+5}$   
(c)  $\frac{s}{s^2-2s+5}$  (d)  $\frac{s(s+1)}{s^2+4s+5}$ 

13.4 Find the transforms of the following functions.

(a) 
$$te^{-2t} \sin 2t + \frac{\cos 2t}{t}$$
 (b)  $\log \left[\frac{s^2 - 1}{s(s+1)}\right]$   
(c)  $(1 + 2t e^{-5t})^3$  (d)  $\frac{s+4}{(s^2 + 5s + 12)^2}$ 

13.5 Using the convolution theorem, determine the inverse transform of the following functions.

(a) 
$$\frac{5}{s^2(s+2)^2}$$
 (b)  $\frac{s}{(s^2+25)^2}$   
(c)  $\frac{s}{(s^2+9)(s^2+25)}$ 

13.6 Find the Laplace transform of the periodic square wave shown in Fig. 13.23.





- 13.7 Find the Laplace transform of a sawtooth waveform f(t) which is periodic, with period equal to unity, and is given by  $f(t) = \alpha t$  for 0 < t < 1.
- 13.8 Find the Laplace transform of the periodic wave form shown in Fig. 13.24.





13.9 For the circuit shown in Fig. 13.25, determine the current when the switch is closed at t = 0. Assume zero charge on the capacitor initially.



Fig. 13.25

13.10 For the circuit shown in Fig. 13.26 , determine the current when the switch is closed at t = 0.


Fig. 13.26

13.11 For the circuit shown in Fig. 13.27 determine the total current when the switch *S* is closed at t = 0.



Fig. 13.27

13.12 For the circuit shown in Fig. 13.28, determine the voltage across the output terminals when the input is unit step function. Assume no initial charge on the capacitor.



Fig. 13.28

13.13 For the circuit shown in Fig. 13.29, determine the current through the circuit, when the switch is moved from position 1 to position 2.



13.14 For the circuit shown in Fig. 13.30, determine the current through the resistor when the switch is moved from position 1 to position 2. Assume that initial charge on the capacitor is 5 C.



Fig. 13.30

13.15 For the circuit shown in Fig. 13.31, determine the current when the switch is closed at t = 0.





- 13.16 For the given function  $f(t) = 3u(t) + 2e^{-t}$ , find its final value  $f(\infty)$  using final value theorem.
- 13.17 An exponential voltage  $v(t) = 10e^{-t}$  is suddenly applied at t = 0 to the circuit shown in Fig. 13.32 obtain the particular solution for current i(t) through the circuit.



13.18 For the circuit shown in Fig. 13.33, the switch is closed at t = 0. Determine  $i_1(t)$  and  $i_2(t)$ . The initial currents  $i_1(0) = 1$  A and  $i_2(0) = 2$  A.





13.19 In the circuit shown in Fig. 13.34, the switch is changed from position 1 to 2 at t = 0. A steady state position is existing in position 1 before t = 0. Determine the current i(t) using Laplace transform method.







- 1. Laplace transform analysis gives
  - (a) time domain response only
  - (b) frequency domain response only
  - (c) both (a) and (b)
  - (d) none
- 2. The Laplace transform of a unit step function is

(a) 
$$\frac{1}{s}$$
 (b) 1  
(c)  $\frac{1}{s^2}$  (d)  $\frac{1}{s+a}$ 

....

3. The Laplace transform of the first de	rivative of a function $f(t)$ is	
(a) $F(s)/s$	(b) $sF(s) - f(0)$	
(c) $F(s) - f(0)$	(d) f(0)	
4. The Laplace transform of the integral of function $f(t)$ is		
(a) $\frac{1}{s} F(s)$	(b) $sF(s) - f(0)$	
(c) $F(s) - f(0)$ 5. The Laplace transform of $e^{5t} f(t)$ is	(d) $f'(0)$	
(a) $F(s)$	(b) $F(s-1)$	
(c) $F\left(\frac{s}{5}\right)$	(d) $F(s-5)$	
6. The initial value of $20 - 10t - e^{25t}$ is		
(a) 20	(b) 19	
(c) 10	(d) 25	
7. The final value of $\frac{2s+1}{s^4+8s^3+16s^2+1$	$\frac{1}{s}$ is	
(a) 2	(b) infinite	
(c) zero	(d) 1	
8. The inverse Laplace transform of $\frac{1}{s}(1-e^{-as})$ is		
(a) $u(t) - u(t - a)$	(b) $u(t)$	
(c) $u(t-a)$	(d) zero	
9. The inverse transform of $\frac{6}{s^4}$ is		
(a) $\frac{3}{2}$	(b) $t^2$	
(c) $t^{3}$	(d) $3t$	
10. The inverse transform of $2 \log \left(\frac{s+2}{s}\right)$ is		
(a) $\frac{2 - e^{-2t}}{t}$	(b) $\frac{e^{-2t}}{t}$	
(c) $\frac{2}{t}$	(d) $\frac{2 + e^{-2t}}{t}$	
11. The Laplace transform of a square wave with amplitude of peak value $A$ and period $T$ is		

(a) 
$$\frac{1+e^{-sT}}{1-e^{-sT}}$$
(b) 
$$\frac{A}{s} \left(\frac{1-e^{-sT}}{1+e^{-sT}}\right)$$
(c) 
$$\frac{A}{s} \left(\frac{1+e^{sT}}{1-e^{sT}}\right)$$
(d) 
$$\frac{A}{s} \left(\frac{1-e^{+sT}}{1+e^{sT}}\right)$$

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12. The convolution of f(t) \* g(t) is

(a) 
$$\int_{0}^{\infty} f(t) g(t-\tau) d\tau$$
  
(b) 
$$\int_{0}^{t} f(\tau) g(t-\tau) d\tau$$
  
(c) 
$$\int_{0}^{t} f(t-\tau) g(t) dt$$
  
(d) 
$$\int_{0}^{t} f(t) g(t-\tau) dt$$

13. The inverse Laplace transform of the function  $\frac{s+5}{(s+1)(s+3)}$  is

(a)  $2e^{t} - e^{-3t}$ (b)  $2e^{-t} + e^{-3t}$ (c)  $e^{-t} - 2e^{-3t}$ (d)  $e^{-t} + 2e^{-3t}$ 

14. If 
$$\mathscr{L}[f(t)] = \frac{2(s+1)}{s^2 + 2s + 5}$$
, then  $f(0^+)$  and  $f(\infty)$  are given by

- (a) 0, 2 respectively (b) 2, 0 respectively
- (c) 0, 1 respectively (d) 2/5, 0 respectively
- 15. The final value theorem is used to find the
  - (a) steady state value of the system output
  - (b) initial value of the system output
  - (c) transient behaviour of the system output
  - (d) none of these

16. The Lapalce transform of a unit ramp function at t = a is

(a) 
$$\frac{1}{(s+a)^2}$$
 (b)  $\frac{e^{-as}}{(s+a)^2}$   
(c)  $\frac{e^{-as}}{s^2}$  (d)  $\frac{a}{s^2}$ 

- 17. A ramp voltage, v(t) = 100 volts, is applied to an RC series circuit with  $R = 5 \text{ k}\Omega$  and  $C = 4\mu\text{F}$ . The maximum output voltage across capacitor is
  - (a) 0.2 volt (b) 2.0 volts
  - (c) 10.0 volts (d) 50.0 volts



## 14.1 SINGULARITY FUNCTIONS

So far we have discussed the response of networks to simple waveforms, such as dc, exponential or sinusoidal. Another class of signals is defined by singularity functions. These are step, ramp and impulse functions. These functions are divided into the following two groups.

- 1. *Non-recurring type* These functions appear for a particular time interval and become zero for all other times, and
- 2. *Recurring type* These functions appear for all time, that is, the wave-form exists for t > 0.

Singularity functions are continuous time functions, and their derivatives, except one, are also continuous. Singularity functions can be obtained from one another by successive differentiation or integration. Our analysis of general networks can be enhanced by the utilisation of singularity functions.

## 14.2 UNIT FUNCTIONS

(a) *Unit step function* This function has already been discussed in the preceding chapter. It is defined as one that has magnitude of one for time greater than zero, and has zero magnitude for time less than zero.

A unit step function is defined mathematically as

```
u(t) = 0 for t < 0
= 1 for t > 0
```

The function is represented as shown in Fig. 14.1

The Laplace transform of the unit step function is





$$= \int_{0}^{\infty} 1e^{-st} dt$$
$$= \left[ -\frac{e^{-st}}{s} \right]_{0}^{\infty} = \frac{1}{s}$$

(b) Unit ramp function If the unit step function is integrated with respect to time t, then the unit ramp function results. It is symbolised by r(t). A unit ramp function increases linearly with time. A unit ramp function may be defined mathematically as

$$r(t) = \int_{-\infty}^{t} u(t) dt$$
$$= \int_{-\infty}^{0} u(t) + \int_{0}^{t} u(t) dt$$
$$= 0 + \int_{0}^{t} u(t) dt = t$$
$$r(t) = 0 \text{ for } t < 0$$
$$= t \text{ for } t > 0$$
etion is represented as shown in Fig.

*:*..

The function is represented as shown in Fig. 14.2.



Fig. 14.2

The Laplace transform of the unit ramp function is

$$\mathscr{L}[f(t)] = \mathscr{L}[r(t)] = \mathscr{L}\left[\int_{0}^{t} u(t) dt\right]$$
$$= \int_{0}^{\infty} t e^{-st} dt$$
$$\mathscr{L}[r(t)] = \frac{1}{s^{2}}$$

(c) Unit impulse function If a unit step function u(t) is differentiated with respect to t, the derivative is zero for time t greater than zero, and is infinite for time t equal to zero. Mathematically, the function is defined as

$$\delta(t) = 0 \text{ for } t \neq 0$$
$$(t) dt = 1$$

 $\int \delta$ 

and

where the symbol  $\delta(t)$  (delta) is used to represent the unit impulse. An impulse of unity amplitude occurring at t = 0 gives that it has an area ' $\delta$ ' equal to unity. The unit impulse function is represented as shown in Fig. 14.3.





The Laplace transform of the unit impulse function is

$$\mathscr{L}[f(t)] = \mathscr{L}[\delta(t)] = \mathscr{L}\left[\frac{d}{dt}u(t)\right] = s \mathscr{L}[u(t)] = s \times \frac{1}{s} = 1$$
$$\mathscr{L}[\delta(t)] = 1$$

Therefore

(d) Unit doublet function If a unit impulse function  $\delta(t)$  is differentiated with respect to t, we get

$$\delta'(t) = \frac{d}{dt} [\delta(t)] = +\infty \text{ and } -\infty \text{ for } t = 0$$
$$= 0 \text{ for } t \neq 0$$

This function is called unit doublet, where  $\delta'(t)$  is the symbol used to represent the unit doublet.

The unit doublet is shown in Fig. 14.4.

The Laplace transform of the unit doublet is

$$\mathscr{L}\left[\boldsymbol{\delta}'(t)\right] = \mathscr{L}\left[\frac{d}{dt}\,\boldsymbol{\delta}(t)\right]$$

where  $\delta(t)$  is a unit impulse occurring at t = 0.

$$\begin{array}{c} f(t) \\ +\infty \\ 0 \\ -\infty \end{array}$$



$$\mathscr{L}\left[\frac{d}{dt}\,\delta(t)\right] = s\left\{\mathscr{L}\left[\delta(t)\right]\right\}$$
$$= s \times 1 = s$$
$$\mathscr{L}\left[\delta'(t)\right] = s$$

*.*..

## **14.3 SHIFTER FUNCTIONS**

Consider unit functions such as unit step, ramp and impulse functions as discussed in Section 14.2. If these functions are displaced by 'a' second or delayed by 'a' second then these functions are said to be delayed functions. These are represented as shown in Fig. 14.5.



Fig. 14.5

The delayed unit step function shown in Fig. 14.5(a) is defined as u(t-a) = 0 for t < a

$$= 1$$
 for  $t > a$ 

The delayed unit ramp function shown in Fig. 14.5(b) is defined as

$$r(t-a) = 0 \text{ for } t < a$$
$$= t \text{ for } t > a$$

The delayed unit impulse function is defined as

$$\delta(t-a) = 0$$
 for  $t \neq a$ 

and

$$\int_{-\infty} \delta(t-a) \, dt = 1$$

## 14.4 GATE FUNCTION

 $\infty$ 

By the use of step functions, any pulse of unit height can be realised. The pulse of width a can be generated by combining unit step function u(t) and delayed inverted unit step function by a time interval a as shown in Fig. 14.6.



Fig. 14.6

In Fig. 14.6(a), the unit step function u(t) combined with -u(t-a), the inverted unit step function, delayed by a results in the waveform shown in Fig. 14.6(c).

$$G(T) = u(t) - u(t-a)$$

The gate function is only for 0 < t < a.

A periodic pulse train with pulse width *a* and pulse repetition period  $T_1$  may be generated by combining a sequence of positive unit step functions u(t),  $u(t - T_1)$ ,  $u(t - 2T_1)$ ..., with negative unit step functions u(t - a),  $u(t - T_1 - a)$ ,  $u(t - 2T_1 - a)$ ..., as shown in Fig. 14.7.



Fig. 14.7

Therefore, the periodic pulses may be defined as,

$$f(t) = u(t) - u(t - a) + u(t - T_1) - u(t - T_1 - a) + \dots$$

## 14.5 NETWORK FUNCTIONS

Network functions give the relation between the transform of the excitation to the transform of the response. Consider the network shown in Fig. 14.8.



Fig. 14.8

For the network shown in Fig. 14.8(a), only one voltage and one current exist and only one network function is defined. It constitutes of one pair of terminals called a port. Generally, a driving source is connected to the pair of terminals. For the two terminal pair network shown in Fig. 14.8(b), two currents and two voltages must exist. Normally in Fig. 14.8(b), 1-1' and 2-2' are called ports. Hence, it is called two-port network. If the driving source is connected across 1-1', the load is connected across 2-2'. Otherwise, if the source is connected across 2-2', the output is taken across 1-1'.

## 14.6 TRANSFER FUNCTIONS OF TWO-PORT NETWORK

For a one-port network, the driving point impedance or impedance of the network is defined as

$$Z(s) = \frac{V(s)}{I(s)}$$

The reciprocal of the impedance function is the driving point admittance function, and is denoted by Y(s).

For the two-port network without internal sources, the driving point impedance function at port 1-1' is the ratio of the transform voltage at port 1-1' to the transform current at the same port.

 $\therefore \qquad \qquad Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$ 

Similarly, the driving point impedance at port 2-2' is the ratio of transform voltage at port 2-2' to the transform current at the same port.

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

For the two-port network, the driving point admittance is defined as the ratio of the transform current at any port to the transform voltage at the same port.

re 
$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$$

or

and

$$Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$$
, which is the driving point admittance.  
other network functions are called transfer functions. These

The four other network functions are called transfer functions. These functions give the relation between voltage or current at one port to the voltage or current at the other port as shown hereunder.

(i) Voltage transfer ratio This is the ratio of voltage transform at one port to the voltage transform at the other port, and is denoted by G(s)

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)}$$
$$G_{12}(s) = \frac{V_1(s)}{V_2(s)}$$

(ii) *Current transfer ratio* This is the ratio of current transform at one port to current transform at other port, and is denoted by  $\alpha(s)$ 

and 
$$\alpha_{12}(s) = \frac{I_1(s)}{I_2(s)}$$
$$\alpha_{21}(s) = \frac{I_2(s)}{I_1(s)}$$

- (iii) *Transfer impedance* It is defined as the ratio of voltage transform at one port to the current transform at the other port, and is denoted by Z(s).
  - $\therefore \qquad Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$ and  $Z_{12}(s) = \frac{V_1(s)}{I_2(s)}$
- (iv) *Transfer admittance* It is defined as the ratio of current transform at one port to the current transform at the other port, and is denoted by Y(s).

and 
$$Y_{21}(s) = \frac{I_2(s)}{V_1(s)}$$
$$Y_{12}(s) = \frac{I_1(s)}{V_2(s)}$$

The above network functions are found by forming the system of equations using node or mesh analysis, and taking the transforms of equations by setting the initial conditions to zero and solving for ratio of the response to excitation.

#### POLES AND ZEROS 14.7

In general, the network function N(s) may be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}$$

where  $a_0, a_1, ..., a_n$  and  $b_0, b_1, ..., b_m$  are the coefficients of the polynomials P(s)and Q(s); they are real and positive for a passive network. If the numerator and denominator of polynomial N(s) are factorised, the network function can be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 (s - z_1) (s - z_2) \dots (s - z_n)}{b_0 (s - p_1) (s - p_2) \dots (s - p_m)}$$

where  $z_1, z_2, ..., z_n$  are the *n* roots for P(s) = 0and  $p_1, p_2, ..., P_m$  are the *m* roots for Q(s) = 0and  $a_0/b_0 = H$  is a constant called the *scale factor*.

 $z_1, z_2, ..., z_n$  in the transfer function are called zeros, and are denoted by 0. Similarly,  $p_1, p_2, ..., p_m$  are called poles, and are denoted by  $\times$ . The network function N(s) becomes zero when s is equal to anyone of the zeros. N(s)becomes infinite when s is equal to any one of the poles. The network function is completely defined by its poles and zeros. If the poles or zeros are not repeated, then the function is said to be having simple poles or simple zeros. If the poles or zeros are repeated, then the function is said to be having multiple poles multiple zeros. When n > m, then (n - m) zeros are at  $s = \infty$ , and for m > mn, (m-n) poles are at  $s = \infty$ .

Consider, the network function

s = -2, s = -3 + j2,shown in Fig. 14.9. The network fun

$$N(s) = \frac{(s+1)^2 (s+5)}{(s+2) (s+3+j2) (s+3-j2)}$$
  
that has double zeros at  $s = -1$  and a  
zero at  $s = -5$ ; and three finite poles at  $-3+j2$   
 $s = -2, s = -3+j2$ , and  $s = -3-j2$  as  
shown in Fig. 14.9.  
The network function is said to be  
stable when the real parts of the poles  $-3-j2$   
 $-3-j2$ 

stable when the real and zeros are negative. Otherwise, the poles and zeros must lie within the left half of the s-plane.



### **NECESSARY CONDITIONS FOR DRIVING** 14.8 POINT FUNCTION

The restrictions on pole and zero locations in the driving point function with common factors in P(s) and Q(s) cancelled are listed below.

1. The coefficients in the polynomials P(s) and Q(s) of network function N(s) = P(s)/Q(s) must be real and positive.

- 2. Complex or imaginary poles and zeros must occur in conjugate pairs.
- 3. (a) The real parts of all poles and zeros must be zero, or negative.(b) If the real part is zero, then the pole and zero must be simple.
- 4. The polynomials P(s) and Q(s) may not have any missing terms between the highest and the lowest degrees, unless all even or all odd terms are missing.
- 5. The degree of P(s) and Q(s) may differ by zero, or one only.
- 6. The lowest degree in P(s) and Q(s) may differ in degree by at the most one.

## 14.9 NECESSARY CONDITIONS FOR TRANSFER FUNCTIONS

The restrictions on pole and zero location in transfer functions with common factors in P(s) and Q(s) cancelled are listed below.

- 1. (a) The coefficients in the polynomials P(s) and Q(s) of N(s) = P(s)/Q(s) must be real.
  - (b) The coefficients in Q(s) must be positive, but some of the coefficients in P(s) may be negative.
- 2. Complex or imaginary poles and zeros must occur in conjugate pairs.
- 3. The real part of poles must be negative, or zero. If the real part is zero, then the pole must be simple.
- 4. The polynomial Q(s) may not have any missing terms between the highest and the lowest degree, unless all even or all odd terms are missing.
- 5. The polynomial P(s) may have missing terms between the lowest and the highest degree.
- 6. The degree of P(s) may be as small as zero, independent of the degree of Q(s).
- 7. (a) For the voltage transfer ratio and the current transfer ratio, the maximum degree of P(s) must equal the degree of Q(s).
  - (b) For transfer impedance and transfer admittance, the maximum degree of P(s) must equal the degree of Q(s) plus one.

## 14.10 TIME DOMAIN RESPONSE FROM POLE ZERO PLOT

For the given network function, a pole zero plot can be drawn which gives useful information regarding the critical frequencies. The time domain response can also be obtained from pole zero plot of a network function. Consider an array of poles shown in Fig. 14.10.



Fig. 14.10

In Fig. 14.10  $s_1$  and  $s_3$  are complex conjugate poles, whereas  $s_2$  and  $s_4$  are real poles. If the poles are real, the quadratic function is

$$s^2 + 2\delta\omega_n s + \omega_n^2$$
 for  $\delta > 1$ 

where  $\delta$  is the damping ratio and  $\omega_n$  is the undamped natural frequency.

The roots of the equation are

$$s_2, s_4 = -\delta\omega_n \pm \omega_n \sqrt{\delta^2 - 1}; \delta > 1$$

For these poles, the time domain response is given by

$$(t) = k_2 e^{s_2 t} + k_4 e^{s_4 t}$$

The response due to pole  $s_4$  dies faster compared to that of  $s_2$  as shown in Fig. 14.11.





 $s_1$  and  $s_3$  constitute complex conjugate poles. If the poles are complex conjugate, then the quadratic function is

$$s^2 + 2\delta\omega_n s + \omega_n^2$$
 for  $\delta < 1$ 

The roots are  $s_1, s_1^* = -\delta \omega_n \pm j\omega_n \sqrt{1-\delta^2}$  for  $\delta < 1$ For these poles, the time domain response is given by

$$i(t) = k_1 e^{-\delta \omega_n t + j \left(\omega_n \sqrt{1 - \delta^2}\right) t} + k_1^* e^{-\delta \omega_n t - j \left(\omega_n \sqrt{1 - \delta^2}\right) t}$$
$$= k e^{-\delta \omega_n t} \sin \left(\omega_n \sqrt{1 - \delta^2}\right) t$$

From the above equation, we can conclude that the response for the conjugate poles is damped sinusoid. Similarly,  $s_3$ ,  $s_3^*$  are also a complex conjugate pair. Here the response due to  $s_3$  dies down faster than that due to  $s_1$  as shown in Fig. 14.12.

Consider a network having transfer admittance Y(s). If the input voltage V(s) is applied to the network, the corresponding current is given by



Fig. 14.12

This may be taken as

$$I(s) = H \frac{(s - s_a)(s - s_b)\dots(s - s_n)}{(s - s_1)(s - s_2)\dots(s - s_m)}$$

where H is the scale factor.

By taking the partial fractions, we get

$$I(s) = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2} + \dots + \frac{k_m}{s - s_m}$$

The time domain response can be obtained by taking the inverse transform

$$i(t) = \mathscr{L}^{-1}\left[\frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \dots + \frac{k_m}{s-s_m}\right]$$

Any of the above coefficients can be obtained by using Heavisides method.

To find the coefficient  $k_i$ 

$$k_{l} = H\left[\frac{(s - s_{a})(s - s_{b})\dots(s - s_{n})}{(s - s_{1})(s - s_{2})\dots(s - s_{m})}\right](s - s_{l})\Big|_{s = s_{l}}$$

Here  $s_i$ ,  $s_m$ ,  $s_n$  are all complex numbers, the difference of  $(s_i - s_n)$  is also a complex number.

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$$(s_l - s_n) = M_{ln} e^{j\phi_{ln}}$$

Hence 
$$k_l = H \frac{M_{la} \ M_{lb} \ \dots \ M_{ln}}{M_{l1} \ M_{l2} \ \dots \ M_{lm}} \times e^{j(\phi_{la} + \phi_{lb} + \ \dots + \phi_{ln}) - (\phi_{l1} + \phi_{l2} + \ \dots + \phi_{lm})}$$

Similarly, all coefficients  $k_1, k_2, ..., k_m$  may be obtained, which constitute the magnitude and phase angle.

The residues may also be obtained by pole zero plot in the following way.

- 1. Obtain the pole zero plot for the given network function.
- 2. Measure the distances  $M_{la}, M_{lb}, ..., M_{ln}$  of a given pole from each of the other zeros.
- 3. Measure the distances  $M_{l1}, M_{l2}, ..., M_{lm}$  of a given pole from each of the other poles.
- 4. Measure the angle  $\phi_{la}$ ,  $\phi_{lb}$ , ...,  $\phi_{ln}$  of the line joining that pole to each of the other zeros.
- 5. Measure the angle  $\phi_{l1}, \phi_{l2}, ..., \phi_{lm}$  of the line joining that pole to each of the other poles.
- 6. Substitute these values in required residue equation.

## 14.11 AMPLITUDE AND PHASE RESPONSE FROM POLE ZERO PLOT

The steady state response can be obtained from the pole zero plot, and it is given by

$$N(j\omega) = M(\omega)e^{j\phi(\omega)}$$

where  $M(\omega)$  is the amplitude

 $\phi(\omega)$  is the phase

These amplitude and phase responses are useful in the design and analysis of network functions. For different values of  $\omega$ , corresponding values of  $M(\omega)$  and  $\phi(\omega)$  can be obtained and these are plotted to get amplitude and phase response of the given network.

# 14.12 STABILITY CRITERION FOR ACTIVE NETWORK

Passive networks are said to be stable only when all the poles lie in the left half of the *s*-plane. Active networks (containing controlled sources) are not always stable. Consider transformed active network shown in Fig. 14.13.



Fig. 14.13

By applying Millman Theorem, we get

$$V_{2}(s) = \frac{V_{1}(s) + k V_{2}(s)}{6 + 5/s + s}$$
  
=  $\frac{s [V_{1}(s) + k V_{2}(s)]}{s^{2} + 6s + 5}$   
 $V_{2}(s) [s^{2} + 6s + 5] - ksV_{2}(s) = sV_{1}(s)$   
 $V_{2}(s) [s^{2} + (6 - k)s + 5] = sV_{1}(s)$   
 $\frac{V_{2}(s)}{V_{1}(s)} = \frac{s}{s^{2} + (6 - k)s + 5}$ 

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From the above transformed equation, the poles are dependent upon the value of k.

The roots of the equation are

$$s = \frac{-(6-k) \pm \sqrt{(6-k)^2 - 4 \times 5}}{2}$$

For k = 0, the poles are at -1, -5, which lie on the left half of the *s*-plane. As k increases, the poles move towards each other and meet at a point  $\sqrt{(6-k)^2 - 20} = 0$ , when k = 1.53 or 10.47. The root locus plot is shown in Fig. 14.14.



Fig. 14.14

The root locus is obtained from the characteristic equation  $s^2 + (6 - k)s + 5 = 0$ . As the value of k increases beyond 1.53, the locus of root is a circle. The poles are located on the imaginary axis at  $\pm j2.24$  for k = 6. At -2.24, poles are coincident for k = 1.53 while at +2.24, poles are coincident for k = 10.47. When k > 10.47, the poles again lie on the real axis but remain on the right half of the s-plane, one pole moving towards the origin and the other moving towards infinity. From this we can conclude, as long as k is less than 6, the poles lie on the left half of the s-plane and the system is said to be stable. For k = 6, the poles lie on the imaginary axis and the system is oscillatory in nature. For values of k greater than 6, the poles lie on the right half of the s-plane. Then the system is said to be unstable.

## 14.13 ROUTH CRITERIA

The locations of the poles gives us an idea about stability of the active network. Consider the denominator polynomial

$$Q(s) = b_0 s^m + b_1 s^{m-1} + \dots + b_m$$
(14.1)

To get a stable system, all the roots must have negative real parts. There should not be any positive or zero real parts. This condition is not sufficient.

Let us consider the polynomial

$$s^{3} + 4s^{2} + 15s + 100 = (s + 5)(s^{2} - s + 20)$$

In the above polynomial, though the coefficients are positive and real, the two roots have positive real parts. From this we conclude that the coefficients of Q(s) being positive and real is not a sufficient condition to get a stable system. Therefore, we have to seek the condition for stability which is necessary and sufficient.

Consider the polynomial Q(s) = 0. After factorisation, we get

$$b_0(s-s_1)(s-s_2)\dots(s-s_m) = 0$$
(14.2)

On multiplication of these factors, we get

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$$Q(s) = b_0 s^m - b_0 (s_1 + s_2 + \dots + s_m) s^{m-1} + b_0 (s_1 s_2 + s_2 s_3 + \dots) s^{m-2} + b_0 (-1)^m (s_1 s_2 \dots s_m) = 0$$
(14.3)

Equating the coefficients of Eqs 14.1 and 14.3, we have

$$\frac{b_1}{b_0} = -(s_1 + s_2 + \dots + s_m) \tag{14.4}$$

= - sum of the roots

$$\frac{b_2}{b_0} = 1(s_1s_2 + s_2s_3 + \dots) \tag{14.5}$$

= sum of the products of the roots taken two at a time

$$\frac{b_3}{b_0} = -(s_1 s_2 s_3 + s_2 s_3 s_4 + \dots) \tag{14.6}$$

= - sum of the products of the roots taken three at a time.

$$(-1)^m \frac{b_m}{b_0} = (s_1 s_2 s_3 \dots s_m) =$$
product of the roots (14.7)

If all the roots have negative real parts, then from the above equations it is clear that all the coefficients must have the same sign. This condition is not sufficient due to the fact that the zero value of a coefficient involves cancellation, which requires some root to have positive real parts.

The Routh criterion for stability is discussed below. Consider a polynomial

$$Q(s) = b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_m$$

Taking first row coefficients and second row coefficients separately, we have

Now we complete the Routh array as follows.

For m = 5

where  $c_1, c_2, d_1, d_2, e_1, f_1$  are determined by the algorithm given below.

$$c_1 = \frac{b_0 \qquad b_2}{b_1 \qquad b_3} = \frac{b_1 b_2 - b_0 b_3}{b_1}$$

$$b_{0} \qquad b_{4}$$

$$c_{2} = \frac{b_{1} \qquad b_{5}}{b_{1}} = \frac{b_{1}b_{4} - b_{0}b_{5}}{b_{1}}$$

$$b_{1} \qquad b_{3}$$

$$d_{1} = \frac{c_{1} \qquad c_{2}}{c_{1}} = \frac{c_{1}b_{3} - b_{1}c_{2}}{c_{1}}$$

$$b_{1} \qquad b_{5}$$

$$d_{2} = \frac{c_{1} \qquad 0}{c_{1}} = \frac{b_{5}c_{1} - 0}{c_{1}}$$

$$e_{1} = \frac{d_{1} \qquad d_{2}}{d_{1}} = \frac{c_{2}d_{1} - c_{1}d_{2}}{d_{1}}$$

$$d_{1} \qquad d_{2}$$

$$f_{1} = \frac{e_{1} \qquad 0}{e_{1}} = \frac{d_{2}e_{1} - 0}{e_{1}}$$

In order to find out the element in *k*th row and *j*th column, it is required to know the four elements. These elements in the row (k - 1) and row (k - 2) just above the elements are in column 1 of the array and (J + 1) column of the array. The product of the elements joined by a line with positive slope has positive sign while the product of elements joined with a line with negative slope has a negative sign. The difference of these products is divided by the element of column 1 and row (k - 1). The above process is repeated till m + 1 rows are found in the Routh array.

According to the Routh-Hurwitz theorem, the number of changes in the sign of the first column to the right of the vertical line in an array moving from top to bottom is equal to the number of roots of Q(s) = 0 with positive real parts. To get a stable system, the roots must have negative real parts.

According to the Routh-Hurwitz criterion, the system is stable, if and only if, there are no changes in signs of the first column of the array. This requirement is, both the necessary and sufficient condition for stability.



**Problem 14.1** For the circuit shown in Fig. 14.15, determine the curent i(t) when the switch is closed at t = 0. Assume that the initial current in the inductor is zero.



Fig. 14.15

Solution By applying Kirchhoff's laws to the circuit

$$2i(t) + 1 \frac{di}{dt} = 2\delta(t-3)$$

Taking Laplace transform on both sides, we get

$$2I(s) + 1[sI(s) - i(0)] = 2e^{-3s}$$

Since the initial current through inductor is zero,

$$i(0) = 0$$

The equation becomes

$$2I(s) + 2I(s) = 2e^{-3s}$$
$$I(s) [s+2] = 2e^{-3s}$$
$$I(s) = \frac{2e^{-3s}}{s+2}$$

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Taking inverse transform, we get

$$i(t) = 2e^{-2(t-3)}u(t-3)$$

**Problem 14.2** For the circuit shown in Fig. 14.16, determine the current i(t) when the switch is closed at t = 0. Assume that the initial charge on the capacitor is zero.



Fig. 14.16

Solution By applying Kirchhoff's law to the circuit, we have

$$5i(t) + 1\frac{di}{dt} + 6\int idt = 5r(t-1)$$

Taking Laplace transforms on both sides, we get

$$5I(s) + 1[sI(s) - i(0)] + 6\left[\frac{I(s)}{s} + \frac{q(0)}{s}\right] = \frac{5e^{-s}}{s^2}$$

Since the initial current in the inductor and initial charge on the capacitor is zero

$$i(0) = 0, q(0) = 0$$

Therefore, the above equation becomes

$$I(s)\left[s+5+\frac{6}{s}\right] = \frac{5e^{-s}}{s^2}$$
$$I(s) = \frac{5e^{-s}}{s(s^2+5s+6)} = \frac{5e^{-s}}{s(s+3)(s+2)}$$

By taking partial fraction, we have

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$$\frac{1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

Applying Heavyside rule, we get the coefficients

$$I(s) = 5e^{-s} \left[ \frac{1}{6s} + \frac{1}{3(s+3)} - \frac{1}{2(s+2)} \right]$$
$$I(s) = 5 \left[ \frac{e^{-s}}{6s} + \frac{e^{-s}}{3(s+3)} - \frac{e^{-s}}{2(s+2)} \right]$$

Taking inverse transform on both sides, we have

$$i(t) = \left[\frac{5}{6}u(t-1) + \frac{5}{3}e^{-3(t-1)}u(t-1) - \frac{5}{2}e^{-2(t-1)}u(t-1)\right]A$$

**Problem 14.3** A rectangular voltage pulse of unit height and *T* seconds duration is applied to a series R-C combination at t = 0, as shown in Fig. 14.11. Determine the current in the capacitor as a function of time. Assume the capacitor to be initially uncharged.



Fig. 14.17

**Solution** The input voltage can be written as a combination of two steps, i.e.

$$v(t) = u(t) - u(t - T)$$

Applying Kirchhoff's law to the circuit, we get

$$Ri(t) + \frac{1}{C} \int i(t) dt = [u(t) - u(t - T)]$$

Taking Laplace transforms on both sides, we get

$$RI(s) + \frac{1}{C} \left[ \frac{I(s)}{s} + \frac{q(0)}{s} \right] = \frac{1}{s} (1 - e^{-sT})$$

Since the initial charge on the capacitor is zero

$$q(0) = 0$$
  
Therefore,  
$$I(s) \left[ R + \frac{1}{Cs} \right] = \frac{1}{s} (1 - e^{-sT})$$
$$I(s) = \frac{1 - e^{-sT}}{R \left( s + \frac{1}{RC} \right)}$$
$$= \frac{1}{R} \left[ \frac{1}{s + 1/RC} - \frac{e^{-sT}}{s + 1/RC} \right]$$

or

Taking inverse transform on both sides, we get

$$i(t) = \frac{1}{R} \{ u(t)e^{-t/RC} - u(t-T) e^{-(1/RC)(t-T)} \}$$

**Problem 14.4** For the network shown in Fig. 14.18, determine the transform impedance Z(s).



Fig. 14.18

Solution The transform network for the network shown in Fig. 14.18 is shown in Fig. 14.19.



Fig. 14.19

From Fig. 14.19, the equivalent impedance at port 1-1' is

$$Z(s) = \left\{ 10 + \left[ 2s || \left( 20 + \frac{1}{5s} \right) \right] \right\}$$
$$= 10 + \frac{2s(20 + 1/5s)}{2s + 20 + 1/5s}$$
$$= \frac{20s + 200 + 2/s + 40s + 2/5}{\frac{10s^2 + 100s + 1}{5s}}$$
$$= \frac{100s^2 + 1000s + 10 + 200s^2 + 2s}{10s^2 + 100s + 1}$$

Therefore, the network transform impedance is

$$Z(s) = \frac{300s^2 + 1002s + 10}{10s^2 + 100s + 1}$$

**Problem 14.5** For the two port network shown in Fig. 14.20, determine the driving point impedance  $Z_{11}(s)$  and the driving point admittance  $Y_{11}(s)$ . Also find the transfer impedance  $Z_{21}(s)$ .



Fig. 14.20

Solution By applying Kirchhoff's law to the circuit, we have

$$V_1(s) = 10I_1(s) + 2s I_1(s)$$
(14.8)

The voltage across port 2-2' is

$$V_2(s) = I_1(s) \times (2s)$$
(14.9)

From Eq. 14.8, the driving point impedance is

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = (2s + 10)$$

Similarly, the driving point admittance is

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)} = \frac{1}{2s+10}$$

From Eq. 14.9, the transfer impedance is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = 2s$$

**Problem 14.6** For the network shown in Fig. 14.21, determine the transfer functions  $G_{21}(s)$  and  $Z_{21}(s)$  and the driving point admittance  $Y_{11}(s)$ .



Fig. 14.21

Solution By applying Kirchhoff's voltage law at the ports, we get

$$V_1(s) = I_1(s) \left[ 5s + \frac{1}{2s} \right]$$
$$V_2(s) = \frac{1}{2s} I_1(s)$$

Therefore, the voltage transfer ratio

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{2s(5s+1/2s)}$$
$$G_{21}(s) = \frac{1}{10s^2 + 1}$$

The transform impedance is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{1}{2s}$$

The driving point admittance is

$$Y_{11}(s) = \frac{I_2(s)}{V_1(s)} = \frac{1}{5s + 1/2s}$$
$$Y_{11}(s) = \frac{2s}{(10s^2 + 1)}$$

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**Problem 14.7** For the network shown in Fig. 14.22, determine the transfer functions  $G_{21}(s)$  and  $Z_{21}(s)$ . Also find the driving point impedance  $Z_{11}(s)$ .



**Solution** From Fig. 14.23, by application of Kirchhoff's laws, we get the following equations

The driving point impedance

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = [20 \parallel (10 + 1/2s)] = \frac{20 \times (10 + 1/2s)}{20 + 10 + 1/2s}$$
$$Z_{11}(s) = \frac{20(10 + 1/2s)}{30 + 1/2s}$$
$$Z_{11}(s) = \frac{400s + 20}{60s + 1}$$

Fig. 14.23

From the above figure, by application of Kirchhoff's laws, we get

$$V_1(s) = 20I_1(s) - 20I_3(s) \tag{14.10}$$

$$10I_3(s) + 20[I_3(s) - I_1(s)] + \frac{1}{2s}[I_3(s) + I_2(s)] = 0$$
(14.11)

$$V_2(s) = [I_2(s) + I_3(s)] \frac{1}{2s}$$
(14.12)

From Eq. 14.11, we get

$$\left(30 + \frac{1}{2s}\right)I_3(s) - 20 I_1(s) = 0$$
  
$$I_3(s) = \frac{40s}{60s+1} I_1(s)$$
(14.13)

From Eq. 14.12, since  $I_2 = 0$  we get

$$V_2(s) = +I_3(s)\left(\frac{1}{2s}\right)$$
 (14.14)

The transfer impedance at port 2 is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{40s}{(60s+1)} \times \frac{1}{2s} = \frac{20}{(60s+1)}$$

The voltage transfer ratio

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{I_3(s)(1/2s)}{20I_1(s) - 20I_3(s)} = \frac{(1/2s)}{\frac{60s + 1 - 40s}{2s}} = \frac{1}{20s + 1}$$

**Problem 14.8** Draw the pole zero diagram for the given network function I(s) and hence obtain i(t).

$$I(s) = \frac{20s}{(s+5)(s+2)}$$

Solution In the network function

and

Q(s) = (s+2)(s+5) = 0

By taking partial fractions, I(s) can be written as

P(s) = 20s

$$I(s) = \frac{k_1}{s+2} + \frac{k_2}{s+5}$$

Therefore, the time domain response is

$$i(t) = k_1 e^{-2t} + k_2 e^{-5t}$$

Here, the coefficients  $k_1$  and  $k_2$  are determined by using the pole zero plot as shown in Fig. 14.24.



Fig. 14.24

Consider a pole at -2

The distance between zero to pole at -2 is

$$M_{02} = 2$$

The angle between the line joining to the pole at -2 to the zero is

$$\phi_{02} = 180^{\circ}$$

Similarly, the distance between pole at -5 to pole at -2 is

$$M_{52} = 3$$

The angle between the line joining the pole at -2 to the pole at -5 is

$$\phi_{52} = 0^{\circ}$$

Hence

$$k_1 = H \frac{M_{02} e^{j\phi_{02}}}{M_{52} e^{j\phi_{52}}}$$
$$= 20 \times \frac{2e^{j180}}{3e^{j0}} = 13.33 e^{j180} = -13.33$$

Similarly,

where

 $k_2 = H \frac{M_{05} e^{j\phi_{05}}}{M_{25} e^{j\phi_{25}}}$  $M_{05} = 5, \, \phi_{05} = 180^{\circ}$  $M_{25} = 3, \phi_{25} = 180^{\circ}$  $k_2 = \frac{20 \times 5}{20} e^{j(180 - 180)}$ 

Hence

$$=\frac{100}{3}=33.3$$

Substituting these values, we get

$$i(t) = (-13.33e^{-2t} + 33.3 e^{-5t}) A$$

Problem 14.9 Draw the pole zero diagram for the given network function and hence obtain v(t)

$$V(s) = \frac{4(s+2)s}{(s+1)(s+3)}$$

**Solution** In the network function

and

p(s) = 4s(s+2)O(s) = (s + 1) (s + 3) = 0

By taking partial fractions, we have

$$V(s) = \frac{k_1}{s+1} + \frac{k_2}{s+3}$$

The time domain response can be obtained by taking the inverse transform

$$v(t) = k_1 e^{-t} + k_2 e^{-3}$$

Here, the coefficients  $k_1$  and  $k_2$  may be determined by using the pole zero plot as shown in Fig. 14.25.

To determine  $k_1$ , we have to find out the distances and phase angles from other zeros and poles to that particular pole.



Fig. 14.25

 $k_1 = H \frac{M_{01}M_{21} e^{j(\phi_{01} + \phi_{21})}}{M_{31} e^{j(\phi_{31})}}$ Hence

where  $M_{01}$  and  $M_{21}$  are the distances between the zeros at 0 and - 2 to the pole at - 1,  $\phi_{01}$ ,  $\phi_{21}$  are the phase angle between the corresponding zeros to the pole.

Similarly,  $M_{31}$  and  $\phi_{31}$  are the distance and phase angle, respectively, from pole at -3 to pole at -1.

*:*.

....

$$M_{01} = 1; \phi_{01} = 180^{\circ}$$
$$M_{21} = 1; \phi_{21} = 0$$
$$M_{31} = 2; \phi_{31} = 0^{\circ}$$
$$k_{1} = 4 \times \frac{1 \times 1}{2} e^{j(180^{\circ})}$$
$$k_{1} = -2$$

Similarly,

where

$$k_{2} = H \frac{M_{03} M_{23}}{M_{13}} e^{+j(\phi_{03} + \phi_{23} - \phi_{13})}$$
$$M_{03} = 3, \phi_{03} = 180^{\circ}$$
$$M_{23} = 1, \phi_{23} = 180^{\circ}$$
$$M_{13} = 2, \phi_{13} = 180^{\circ}$$
$$k_{2} = \frac{4 \times 3 \times 1}{2} e^{j(180 + 180 - 180)}$$
$$k_{2} = -6$$

÷.

Substituting the values, we get

$$v(t) = (-2e^{-t} - 6e^{-3t})$$
V

**Problem 14.10** For the given network function, draw the pole zero diagram and hence obtain the time domain response i(t).

$$I(s) = \frac{5s}{(s+1)(s^2 + 4s + 8)}$$

Solution In the network function

$$P(s) = 5s$$
  
 $Q(s) = (s + 1) (s^2 + 4s + 8) = 0$ 

By taking the partial fraction expansion of I(s), we get

$$I(s) = \frac{k_1}{s+1} + \frac{k_2}{(s+2+j2)} + \frac{k_3}{(s+2-j2)}$$
(14.15)

The time domain response can be obtained by taking the inverse transform as under,

$$i(t) = k_1 e^{-t} + k_2 e^{-(2+j2)t} + k_2 e^{-(2-j2)t}$$
(14.16)

To find the value of  $k_1$ , we have to find out the distances, and phase angles from other zeros and poles to that particular pole as shown in Fig. 14.26.

Hence



Fig. 14.26

 $\therefore \qquad k_{1} = \frac{5 \times 1e^{j180^{\circ}}}{\sqrt{5} \times \sqrt{5} e^{j(-63.44^{\circ} + 63.44^{\circ})}}$   $k_{1} = -1$ Similarly  $k_{2} = \frac{HM_{0p_{1}} e^{j\phi_{0p_{1}}}}{M_{1p_{1}} M_{p_{2}p_{1}} e^{j(\phi_{1p_{1}} + \phi_{p_{2}p_{1}})}}$   $M_{0p_{1}} = \sqrt{8}; \phi_{0p_{1}} = 135^{\circ}$   $M_{1p_{1}} = \sqrt{5}; \phi_{1p_{1}} = 116.56^{\circ}$   $M_{p_{1}p_{2}} = 4; \phi_{p_{2}p_{1}} = 90^{\circ}$ Hence  $k_{2} = \frac{5 \times \sqrt{8}}{\sqrt{5} \times 4} e^{j(135^{\circ} - 116.56^{\circ} - 90^{\circ})}$   $= 1.58 e^{-j(71.56^{\circ})}$ 

$$k_2^* = \frac{H M_{0p_2} e^{j \cdot t_{0p_2}}}{M_{1p_2} M_{p_1p_2} e^{j(\phi_{1p_2} + \phi_{p_1p_2})}}$$
$$= \frac{5 \times \sqrt{8} e^{-j(135^\circ)}}{\sqrt{5} \times 4 e^{j(-116.56^\circ - 90^\circ)}}$$
$$= 1.58 e^{j71.56^\circ}$$

If we substitute the values in Eq. 14.16, we get

$$i(t) = [-1e^{-t} + 1.58 e^{-j(71.56^{\circ})} e^{-(2+j2)t} + 1.58e^{j(71.56^{\circ})} e^{-(2-j2)t}]$$
A

**Problem 14.11** For the given denominator polynomial of a network function, verify the stability of the network by using the Routh criterion.

$$Q(s) = s^3 + 2s^2 + 8s + 10$$

Solution Routh array for this polynomial is given below

There is no change in sign in the first column of the array. Hence, there are no roots with positive real parts. Therefore, the network is stable.

**Problem 14.12** For the given denominator polynomial of a network function, verify the stability of the network using the Routh criterion.

$$Q(s) = s^3 + s^2 + 3s + 8$$

**Solution** Routh array for this polynomial is given below.

There are two changes in sign of the first column, one from 1 to -5 and the other from -5 to +8. Therefore, the two roots have positive real parts. Hence the network is not stable.

**Problem 14.13** For the given denominator polynomial of a network function, determine the value of k for which the network to stable.

$$Q(s) = s^3 + 2s^2 + 4s + k$$

**Solution** Routh array for the given polynomial is given below.

When k < 8, all the terms in the first column are positive. Therefore, there is no sign change in the first column. Hence, the network is stable. When k > 8, the 8 - k/2 is negative. Therefore, there are two sign changes in the first column. There are two roots which have positive real parts. Hence, the network is unstable.

When k = 8, the Routh array becomes

$s^3$	1	4
$s^2$	2	8
$s^1$	α	
$s^0$	8	

The element in the first column and third row is zero. But we can take it as a small number. In this case there are no changes in the sign of the first column. Hence, the network is stable.

**Problem 14.14** Apply Routh criterion to the given polynomial and determine the number of roots (i) with positive real parts (ii) with zero real parts (iii) with negative real parts.

$$Q(s) = s^4 + 4s^3 + 8s^2 + 12s + 15$$

**Solution** The Routh array for the polynomial is

In this case, all the elements in the 4th row have become zero and the array cannot be completed.

The given equation is reduced by taking the new polynomial from the 3rd row

$$5s^2 + 15 = 0$$
  
 $5(s^2 + 3) = 0$ 

Hence the other polynomial  $e^4 + 4e^3 + 8e^2 + 12e + 15$ 

$$Q_2(s) = \frac{s^4 + 4s^3 + 8s^2 + 12s + 15}{5(s^2 + 3)}$$

The equation reduces to the following polynomial

$$(s^2+3)(s^2+4s+5) = 0$$

The roots of the equation  $s^2 + 3 = 0$  are  $s = \pm j\sqrt{3}$ There two roots have zero real parts.

Again forming Routh array for the polynomial

There are no changes in the sign of the first column. Hence, all the two roots have negative real parts. Therefore, out of four roots, two roots have negative real parts and two roots have zero real parts.



**PRACTICE PROBLEMS** 

14.1 For the circuit shown in Fig. 14.27, determine the current i(t), when the switch is closed at t = 0. Assume that there is no initial charge on the capacitor.



Fig. 14.27

14.2 For the circuit shown in Fig. 14.28, determine the voltage across capacitor, when the switch is closed at t = 0. Assume that there is no initial charge on the capacitor.



Fig. 14.28

14.3 For the circuit shown in Fig. 14.29(b), determine the current when the switch is closed at t = 0. The waveform shown in Fig. 14.29(a) is applied to the circuit. Assume that there is no initial charge on the capacitor.





14.4 The waveform shown in Fig. 14.30(a) is applied to the circuit in Fig. 14.30(b) when the switch is closed at t = 0. Assume no initial current in the circuit. Determine the current i(t) in the circuit.





14.5 For the two-port network shown in Fig. 14.31, determine the driving point impedance  $Z_{11}(s)$ , the transfer impedance  $Z_{21}(s)$  and the voltage transfer ratio  $G_{21}(s)$ .



Fig. 14.31

14.6 For the network shown in Fig. 14.32, determine the following transfer functions. (a) G<sub>21</sub> (s), (b) Y<sub>21</sub> (s) and (c) α<sub>21</sub>(s).





14.7 For the network shown in Fig. 14.33, determine the following transfer functions (a)  $G_{21}(s)$ , (b)  $Z_{21}(s)$ .



14.8 For the network shown in Fig. 14.34, determine the following functions (a)  $Z_{11}(s)$ , (b)  $Y_{11}(s)$ , (c)  $G_{21}(s)$  and (d)  $\alpha_{21}(s)$ .





14.9 For the network shown in Fig. 14.35, determine transfer impedance  $Z_{21}(s)$  and  $Y_{21}(s)$ . Also find the transfer voltage ratio  $G_{21}(s)$  and the transfer current ratio  $\alpha_{21}(s)$ .



Fig. 14.35

14.10 For the given network function, draw the pole zero diagram and hence obtain the time domain response. Verify the result analytically.

$$V(s) = \frac{5(s+5)}{(s+2)(s+7)}$$

14.11 For the given network function draw the pole zero diagram and hence obtain the time domain response. Verify this result analytically.

$$I(s) = \frac{3s}{(s+1)(s+3)}$$

14.12 For the given network function, draw the pole zero diagram and hence obtain the time domain response. Verify the result analytically.

$$I(s) = \frac{5s}{(s+3)(s^2+2s+2)}$$

14.13 For the given denominator polynomial of a network function, verify the stability of the network using Routh criteria.

$$Q(s) = s^5 + 3s^4 + 4s^3 + 5s^2 + 6s + 1$$

14.14 For the given denominator polynomial of a network function, verify the stability of the network using Routh criteria.

$$Q(s) = s^4 + s^3 + 2s^2 + 2s + 12$$

- 14.15 Apply Routh criterion to the following equations and determine the number of roots (i) with positive real parts (ii) with zero real parts (iii) with negative real parts
  - (a)  $6s^3 + 2s^2 + 5s + 2 = 0$

- (b)  $s^6 + 5s^5 + 13s^4 + 21s^3 + 20s^2 + 16s + 8 = 0$
- (c)  $s^6 s^5 2s^4 + 4s^3 5s^2 + 21s + 30 = 0$



- 1. The function is said to be non-recurring when it
  - (a) appears for a particular time interval
  - (b) appears for all time
  - (c) both a and b
  - (d) neither of the two
- 2. The inverse transform of 1/S is
  - (a)  $\delta(t)$  (b) u(t)
  - (c) u(t-a) (d) t

3. The Laplace transform of a ramp function is

- (a) 1 (b) 1/s(c)  $1/s^2$  (d)  $1/s^3$
- 4. The inverse transform of *S* is
  - (a) impulse (b) ramp
  - (c) step (d) unit doublet
- 5. The driving point impedance is defined as
  - (a) the ratio of transform voltage to transform current at the same port
  - (b) the ratio of transform voltage at one port to the transform current at the other port
  - (c) both (a) and (b)
  - (d) none of the above
- 6. The transfer impedance is defined as
  - (a) the ratio of transform voltage to transform current at the same port
  - (b) the ratio of transform voltage at one port to the current transform at the other port

  - (d) none of the above
- 7. The function is said to be having simple poles and zeros and only if
  - (a) the poles are not repeated
  - (b) the zeros are not repeated
- (c) both poles and zeros are not repeated
- (d) none of the above
- 8. The necessary condition for a driving point function is
  - (a) the real part of all poles and zeros must not be zero or negative
  - (b) the polynomials P(s) and Q(s) may not have any missing terms between the highest and lowest degree unless all even or all odd terms are missing.

- (c) the degree of P(s) and Q(s) may differ by more than one
- (d) the lowest degree in P(s) and Q(s) may differ in degree by more than two.
- 9. The necessary condition for the transfer functions is that
  - (a) the coefficients in the polynomials P(s) and Q(s) must be real
  - (b) coefficients in Q(s) may be negative
  - (c) complex or imaginary poles and zeros may not conjugate
  - (d) if the real part of pole is zero, then that pole must be multiple
- 10. The system is said to be stable, if and only if
  - (a) all the poles lie on right half of the s-plane
  - (b) some poles lie on the right half of the *s*-plane
  - (c) all the poles does not lie on the right half of the s-plane
  - (d) none of the above.



# **15.1 TWO-PORT NETWORK**

Generally any network may be represented schematically by a rectangular box. A network may be used for representing either source or load, or for a variety of purposes. A pair of terminals at which a signal may enter or leave a network is called a port. A *port* is defined as any pair of terminals into which energy is supplied, or from which energy is withdrawn, or where the network variables may be measured. One such network having only one pair of terminals (1-1') is shown in Fig. 15.1.





A two-port network is simply a network inside a black box, and the network has only two pairs of accessible terminals; usually one pair represents the input and the other represents the output. Such a building block is very common in electronic systems, communication systems, transmission and distribution systems. Figure 15.1 (b) shows a two-port network, or two terminal pair network, in which the four terminals have been paired into ports 1-1' and 2-2'. The terminals 1-1' together constitute a port. Similarly, the terminals 2-2'

constitute another port. Two ports containing no sources in their branches are called *passive ports*; among them are power transmission lines and transformers. Two ports containing sources in their branches are called *active ports*. A voltage and current assigned to each of the two ports. The voltage and current at the input terminals are  $V_1$  and  $I_1$ ; whereas  $V_2$  and  $I_2$  are specified at the output port. It is also assumed that the currents  $I_1$  and  $I_2$  are entering into the network at the upper terminals 1 and 2, respectively. The variables of the two-port network are  $V_1$ ,  $V_2$ , and  $I_1$ ,  $I_2$ . Two of these are dependent variables, the other two are independent variables. The number of possible combinations generated by the four variables, taken two at a time, is six. Thus, there are six possible sets of equations describing a two-port network.

## **15.2 OPEN CIRCUIT IMPEDANCE (Z) PARAMETERS**

A general linear two-port network defined in Section 15.1 which does not contain any independent sources is shown in Fig. 15.2.



Fig. 15.2

The Z parameters of a two-port for the positive directions of voltages and currents may be defined by expressing the port voltages  $V_1$  and  $V_2$  in terms of the currents  $I_1$  and  $I_2$ . Here  $V_1$  and  $V_2$  are dependent variables, and  $I_1$ ,  $I_2$  are independent variables. The voltage at port 1–1' is the response produced by the two currents  $I_1$  and  $I_2$ . Thus

 $V_1 = Z_{11} I_1 + Z_{12} I_2 \tag{15.1}$ 

$$V_2 = Z_{21} I_1 + Z_{22} I_2 \tag{15.2}$$

 $Z_{11}$ ,  $Z_{12}$ ,  $Z_{21}$  and  $Z_{22}$  are the network functions, and are called impedance (Z) parameters, and are defined by Eqs. 15.1 and 15.2. These parameters can be represented by matrices.

We may write the matrix equation [V] = [Z] [I]

where V is the column matrix = 
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
  
Z is the square matrix =  $\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ 

and we may write |I| in the column matrix =  $\begin{vmatrix} I_1 \\ I_2 \end{vmatrix}$ 

Similarly,

Thus,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The individual Z parameters for a given network can be defined by setting each of the port currents equal to zero. Suppose port 2-2' is left open-circuited, then  $I_2 = 0$ 

Thus

$$Z_{11} = \frac{V_1}{I_1} \bigg|_{I_2 = 0}$$

where  $Z_{11}$  is the driving-point impedance at port 1–1' with port 2–2' open circuited. It is called the open circuit input impedance

Similarly, 
$$Z_{21} = \frac{V_2}{I_1} \Big|_{I_2 = 0}$$

where  $Z_{21}$  is the transfer impedance at port 1–1' with port 2–2' open circuited. It is also called the open circuit forward transfer impedance. Suppose port 1–1' is left open circuited, then  $I_1 = 0$ 

$$Z_{12} = \frac{V_1}{I_2} \bigg|_{I_1 = 0}$$

where  $Z_{12}$  is the transfer impedance at port 2–2', with port 1–1' open circuited. It is also called the open circuit reverse transfer impedance.

$$Z_{22} = \frac{V_2}{I_2} \bigg|_{I_1 = 0}$$

where  $Z_{22}$  is the open circuit driving point impedance at port 2–2' with port 1–1' open circuited. It is also called the open circuit output impedance. The equivalent circuit of the two-port networks governed by the Eqs. 15.1 and 15.2, i.e. open circuit impedance parameters is shown in Fig. 15.3.



Fig. 15.3

If the network under study is reciprocal or bilateral, then in accordance with the reciprocity principle

$$\frac{V_2}{I_1}\Big|_{I_2 = 0} = \frac{V_1}{I_2}\Big|_{I_1 = 0}$$
$$Z_{21} = Z_{12}$$

or

It is observed that all the parameters have the dimensions of impedance. Moreover, individual parameters are specified only when the current in one of the ports is zero. This corresponds to one of the ports being open circuited from which the *Z* parameters also derive the name *open circuit impedance parameters*.

**Example 15.1** Find the *Z* parameters for the circuit shown in Fig. 15.4.





**Solution** The circuit in the problem is a *T* network. From Eqs. 15.1 and 15.2 we have

$$V_{1} = Z_{11} I_{1} + Z_{12} I_{2}$$
$$V_{2} = Z_{21} I_{1} + Z_{22} I_{2}$$

When port *b-b'* is open circuited,  $Z_{11} = \frac{V_1}{I_1}$ 

÷

where

$$V_{1} = I_{1}(Z_{a} + Z_{b})$$

$$Z_{11} = (Z_{a} + Z_{b})$$

$$Z_{21} = \frac{V_{2}}{I_{1}} \bigg|_{I_{2} = 0}$$

$$V_{2} = I_{1}Z_{b}$$

where

*:*.

$$V_2 = I_1$$
$$Z_{21} = Z_b$$

When port *a*-*a*' is open circuited,  $I_1 = 0$ 

$$Z_{22} = \frac{V_2}{I_2} \bigg|_{I_1 = 0}$$

 $V_{\gamma} = I_{\gamma}(Z_{b} + Z_{c})$ 

 $Z_{22} = (Z_h + Z_r)$ 

where

$$Z_{12} = \frac{V_1}{I_2} \bigg|_{I_1 = 0}$$

where  $V_1 = I_2 Z_b$ 

It can be observed that  $Z_{12} = Z_{21}$ , so the network is a bilateral network which satisfies the principle of reciprocity.

# 15.3 SHORT CIRCUIT ADMITTANCE (Y) PARAMETERS

 $Z_{12} = Z_{12}$ 

A general two-port network which is considered in Section 15.2 is shown in Fig. 15.5.



Fig. 15.5

The Y parameters of a two-port for the positive directions of voltages and currents may be defined by expressing the port currents  $I_1$  and  $I_2$  in terms of the voltages  $V_1$  and  $V_2$ . Here  $I_1$ ,  $I_2$  are dependent variables and  $V_1$  and  $V_2$  are independent variables.  $I_1$  may be considered to be the superposition of two components, one caused by  $V_1$  and the other by  $V_2$ . Thus, Thus,

$$I_1 = Y_{11} V_1 + Y_{12} V_2 \tag{15.3}$$

Similarly,

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$
(15.4)

 $Y_{11}$ ,  $Y_{12}$ ,  $Y_{21}$  and  $Y_{22}$  are the network functions and are also called the admittance (Y) parameters. They are defined by Eqs 15.3 and 15.4. These parameters can be represented by matrices as follows

$$\begin{bmatrix} I \end{bmatrix} = \begin{bmatrix} Y \end{bmatrix} \begin{bmatrix} V \end{bmatrix}$$
$$I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}; Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$
$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$I_1 = \begin{bmatrix} Y_{11} & Y_{12} \end{bmatrix} \begin{bmatrix} V_1 \end{bmatrix}$$

and

where

Thus,  $\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ 

The individual Y parameters for a given network can be defined by setting each port voltage to zero. If we let  $V_2$  be zero by short circuiting port 2–2', then

$$Y_{11} = \frac{I_1}{V_1} \bigg|_{V_2 = 0}$$

 $Y_{11}$  is the driving point admittance at port 1–1', with port 2–2' short circuited. It is also called the short circuit input admittance.

$$Y_{21} = \frac{I_2}{V_1} \bigg|_{V_2 = 0}$$

 $Y_{21}$  is the transfer admittance at port 1–1 with port 2–2' short circuited. It is also called short circuited forward transfer admittance. If we let  $V_1$  be zero by short circuiting port 1–1', then

$$Y_{12} = \frac{I_1}{V_2} \Big|_{V_1 = 0}$$

 $Y_{12}$  is the transfer admittance at port 2–2' with port 1–1' short circuited. It is also called the short circuit reverse transfer admittance.

$$Y_{22} = \frac{I_2}{V_2} \Big|_{V_1} =$$

 $Y_{22}$  is the short circuit driving point admittance at port 2–2' with port 1–1' short circuited. It is also called the short circuit output admittance. The equivalent circuit of the network governed by Eqs. 15.3 and 15.4 is shown in Fig. 15.6.



Fig. 15.6

If the network under study is reciprocal, or bilateral, then

$$\frac{I_1}{V_2}\Big|_{V_1 = 0} = \frac{I_2}{V_1}\Big|_{V_2 = 0}$$
$$Y_{12} = Y_{21}$$

or

It is observed that all the parameters have the dimensions of admittance which are obtained by short circuiting either the output or the input port from which the parameters also derive their name, i.e. the *short circuit admittance parameters*.

**Example 15.2** Find the *Y* parameters for the network shown in Fig. 15.7.





**Solution** 
$$Y_{11} = \frac{I_1}{V_1}\Big|_{V_2 = 0}$$

When b-b' is short circuited,  $V_2 = 0$  and the network looks as shown in Fig. 15.8(a)





 $V_{1} = I_{1} Z_{eq}$   $Z_{eq} = 2 \Omega$   $V_{1} = I_{1} 2$   $Y_{11} = \frac{I_{1}}{V_{1}} = \frac{1}{2} \nabla$   $Y_{21} = \frac{I_{2}}{V_{2}} \Big|_{V_{2} = 0}$ 

With port *b-b'* short circuited,  $-I_2 = I_1 \times \frac{2}{4} = \frac{I_1}{2}$ 

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$$-I_{2} = \frac{V_{1}}{4}$$
$$Y_{21} = \frac{I_{2}}{V_{1}}\Big|_{V_{2} = 0} = -\frac{1}{4} \ \mho$$

Similarly, when port *a*-*a'* is short circuited,  $V_1 = 0$  and the network looks as shown in Fig. 15.8(b).



Fig. 15.8(b)

$$Y_{22} = \frac{I_2}{V_2} \bigg|_{V_1 = 0}$$

$$V_{2} = I_{2} Z$$

 $V_{_2} = I_{_2} \, Z_{_{eq}}$  where  $Z_{_{eq}}$  is the equivalent impedance as viewed from  $b\text{-}b^{\,\prime}.$ 

$$Z_{eq} = \frac{8}{5} \Omega$$

$$V_{2} = I_{2} \times \frac{8}{5}$$

$$Y_{22} = \frac{I_{2}}{V_{2}} \Big|_{V_{1} = 0} = \frac{5}{8} \nabla$$

$$Y_{12} = \frac{I_{1}}{V_{2}} \Big|_{V_{1} = 0}$$

With *a-a'* short circuited,  $-I_1 = \frac{2}{5} I_2$ 

Since

$$I_2 = \frac{5V_2}{8}$$

$$-I_{1} = \frac{2}{5} \times \frac{5}{8}V_{2} = \frac{V_{2}}{4}$$
$$Y_{12} = \frac{I_{1}}{V_{2}} = -\frac{1}{4} \ \mho$$

:.

The describing equations in terms of the admittance parameters are

$$I_1 = 0.5 V_1 - 0.25 V_2$$
  

$$I_2 = -0.25 V_1 + 0.625 V_2$$

# 15.4 TRANSMISSION (ABCD) PARAMETERS

Transmission parameters, or *ABCD* parameters, are widely used in transmission line theory and cascade networks. In describing the transmission parameters, the input variables  $V_1$  and  $I_1$  at port 1-1', usually called the *sending end*, are expressed in terms of the output variables  $V_2$  and  $I_2$  at port 2-2', called the *receiving end*. The transmission parameters provide a direct relationship between input and output. Transmission parameters are also called general circuit parameters, or chain parameters. They are defined by

$$V_1 = AV_2 - BI_2$$
(15.5)  

$$I_1 = CV_2 - DI_2$$
(15.6)

The negative sign is used with  $I_2$ , and not for the parameter *B* and *D*. Both the port currents  $I_1$  and  $-I_2$  are directed to the right, i.e. with a negative sign in Eqs 15.5 and 15.6 the current at port 2-2' which leaves the port is designated as positive. The parameters *A*, *B*, *C* and *D* are called the *transmission parameters*. In the matrix form, Eqs 15.5 and 15.6 are expressed as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The matrix  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is called the *transmission matrix*.



Fig. 15.9

For a given network, these parameters can be determined as follows. With port 2-2' open, i.e.  $I_2 = 0$ ; applying a voltage  $V_1$  at the port 1-1', using Eq. 15.5, we have

$$A = \frac{V_1}{V_2} \bigg|_{I_2 = 0} \text{ and } C = \frac{I_1}{V_2} \bigg|_{I_2 = 0}$$
$$\frac{1}{A} = \frac{V_2}{V_1} \bigg|_{I_2 = 0} = g_{21} \bigg|_{I_2 = 0}$$

1/A is called the open circuit voltage gain, a dimensionless parameter. And  $\frac{1}{C} = \frac{V_2}{I_1}\Big|_{I_2=0} = Z_{21}$ , which is the open circuit transfer impedance. With port

2-2' short circuited, i.e. with  $V_2 = 0$ , applying voltage  $V_1$  at port 1-1', from Eq. 15.6, we have

$$-B = \frac{V_1}{I_2}\Big|_{V_2 = 0}$$
 and  $-D = \frac{I_1}{I_2}\Big|_{V_2 = 0}$ 

 $-\frac{1}{B} = \frac{I_2}{V_1} \bigg|_{V_2 = 0} = Y_{21},$  which is the short circuit transfer admittance

 $-\frac{1}{D} = \frac{I_2}{I_1}\Big|_{V_2 = 0} = \alpha_{21} \Big|_{V_2 = 0}, \text{ which is the short circuit current gain, a}$ 

dimensionless parameter.

## 15.4.1 Cascade Connection

The main use of the transmission matrix is in dealing with a cascade connection of two-port networks as shown in Fig. 15.10.



Fig. 15.10

Let us consider two two-port networks  $N_x$  and  $N_y$  connected in cascade with port voltages and currents as indicated in Fig. 15.10. The matrix representation of *ABCD* parameters for the network X is as under.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} V_{2x} \\ -I_{2x} \end{bmatrix}$$

And for the network Y, the matrix representation is

$$\begin{bmatrix} V_{1y} \\ I_{1y} \end{bmatrix} = \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix} \begin{bmatrix} V_{2y} \\ -I_{2y} \end{bmatrix}$$

It can also be observed that at for 2-2'

$$V_{2x} = V_{1y}$$
 and  $I_{2x} = -I_{1y}$ .

Combining the results, we have

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix} \begin{bmatrix} V_2 \\ -I_1 \end{bmatrix}$$
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

....

where  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is the transmission parameters matrix for the overall network.

Thus, the transmission matrix of a cascade of a two-port networks is the product of transmission matrices of the individual two-port networks. This property is used in the design of telephone systems, microwave networks, radars, etc.

**Example 15.3** Find the transmission or general circuit parameters for the circuit shown in Fig. 15.11.





Solution From Eqs 15.5 and 15.6 in Section 15.4, we have

 $V_1 = AV_2 - BI_2$   $I_1 = CV_2 - DI_2$ When b-b' is open,  $I_2 = 0$ ;  $A = \frac{V_1}{V_2}\Big|_{I_2 = 0}$ where  $V_1 = 6I_1$  and  $V_2 = 5I_1$   $A = \frac{6}{5}$ 

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$$C = \frac{I_1}{V_2} \bigg|_{I_2 = 0} = \frac{1}{5} \ \mho$$

When *b-b* ' is short circuited;  $V_2 = 0$  (See Fig. 15.12)



Fig. 15.12

 $B = \frac{-V}{I} |_{V} = ; D = \frac{-I}{I} |_{V} =$ 

In the circuit,

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*:*.

$$B = \frac{17}{5} \Omega$$
  
 $I_1 = \frac{7}{17} V_1 \text{ and } -I_2 = \frac{5}{17} V_1$ 

 $D = \frac{7}{5}$ 

 $-I_2 = \frac{5}{17} V_1$ 

Similarly,

\*\*

# 15.5 INVERSE TRANSMISSION (A' B' C' D') PARAMETERS

In the preceding section, the input port voltage and current are expressed in terms of output port voltage and current to describe the transmission parameters. While defining the transmission parameters, it is customary to designate the input port as the sending end and output port as receiving end. The voltage and current at the receiving end can also be expressed in terms of the sending end voltage and current. If the voltage and current at port 2-2' is expressed in terms of voltage and current at port 1-1', we may write the following equations.

$$V_2 = A'V_1 - B'I_1 \tag{15.7}$$

$$I_2 = C'V_1 - D'I_1 \tag{15.8}$$

The coefficients A', B', C' and D' in the above equations are called inverse transmission parameters. Because of the similarities of Eqs. 15.7 and 15.8 with Eqs. 15.5 and 15.6 in Section 15.4, the A', B', C', D' parameters have properties similar to *ABCD* parameters. Thus when port 1-1' is open,  $I_1 = 0$ .



Fig. 15.13

$$A' = \frac{V_2}{V_1} \bigg|_{I_1 = 0}; C' = \frac{I_2}{V_1} \bigg|_{I_1 = 0}$$

If port 1-1' is short circuited,  $V_1 = 0$ 

/

$$B' = \frac{-V_2}{I_1} \bigg|_{V_1 = 0}; D = \frac{-I_2}{I_1} \bigg|_{V_1 = 0}$$

# 15.6 HYBRID (h) PARAMETERS

Hybrid parameters, or h parameters find extensive use in transistor circuits. They are well suited to transistor circuits as these parameters can be most conveniently measured. The hybrid matrices describe a two-port, when the voltage of one port and the current of other port are taken as the independent variables. Consider the network in Fig. 15.14.





If the voltage at port 1-1' and current at port 2-2' are taken as dependent variables, we can express them in terms of  $I_1$  and  $V_2$ .

$$V_1 = h_{11} I_1 + h_{12} V_2 \tag{15.9}$$

$$I_2 = h_{21} I_1 + h_{22} V_2 \tag{15.10}$$

The coefficients in the above equations are called hybrid parameters. In matrix notation

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

From Eqs. 15.9 and 15.10, the individual *h* parameters may be defined by letting  $I_1 = 0$  and  $V_2 = 0$ .

When  $V_2^2 = 0$ , the port 2-2' is short circuited.

Then 
$$h_{11} = \frac{V_1}{I_1}\Big|_{V_2 = 0}$$
 Short circuit input impedance  $\left(\frac{1}{Y_{11}}\right)$ 

$$h_{21} = \frac{I_2}{I_1}\Big|_{V_2 = 0}$$
 Short circuit forward current gain  $\left(\frac{Y_{21}}{Y_{11}}\right)$ 

Similarly, by letting port 1-1' open,  $I_1 = 0$ 

$$h_{12} = \frac{V_1}{V_2} \bigg|_{I_1 = 0}$$
 Open circuit reverse voltage gain  $\left(\frac{Z_{12}}{Z_{22}}\right)$ 
$$h_{22} = \frac{I_2}{V_2} \bigg|_{I_1 = 0}$$
 Open circuit output admittance  $\left(\frac{1}{Z_{22}}\right)$ 

Since the h parameters represent dimensionally an impedance, an admittance, a voltage gain and a current gain, these are called hybrid parameters. An equivalent circuit of a two-port network in terms of hybrid parameters is shown in Fig. 15.15.



Fig. 15.15

## **Example 15.4** Find the *h* parameters of the network shown in Fig. 15.16.



Fig. 15.16

### Solution From Eqs. 15.9 and 15.10, we have

$$h_{11} = \frac{V_1}{I_1}\Big|_{V_2 = 0}; h_{21} = \frac{I_2}{I_1}\Big|_{V_2 = 0}; h_{12} = \frac{V_1}{V_2}\Big|_{I_1 = 0}; h_{22} = \frac{I_2}{V_2}\Big|_{I_1 = 0}$$

If port *b-b'* is short circuited,  $V_2 = 0$ . The circuit is shown in Fig. 15.17(a).



Fig. 15.17(a)

$$h_{11} = \frac{V_1}{I_1}\Big|_{V_2=0}; V_1 = I_1 Z_{eq}$$

 $Z_{eq}$  the equivalent impedance as viewed from the port *a-a'* is 2  $\Omega$ ...  $V_1 = I_1 2 V$ 

$$h_{11} = \frac{V_1}{I_1} = 2 \Omega$$

$$h_{21} = \frac{I_2}{I_1} \Big|_{V_2 = 0} \text{ when } V_2 = 0; -I_2 = \frac{I_1}{2}$$

$$h_{21} = -\frac{1}{2}$$

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If port *a*-*a'* is let open,  $I_1 = 0$ . The circuit is shown in Fig. 15.17(b). Then

$$h_{12} = \frac{V_1}{V_2} \bigg|_{I_1 = 0}$$



### Fig. 15.17(b)

 $V_{1} = I_{Y} 2; I_{Y} = \frac{I_{2}}{2}$   $V_{2} = I_{X} 4; I_{X} = \frac{I_{2}}{2}$   $h_{12} = \frac{V_{1}}{V_{2}} \Big|_{I_{1} = 0} = \frac{1}{2}$   $h_{22} = \frac{I_{2}}{V_{2}} \Big|_{I_{1} = 0} = \frac{1}{2} \nabla$ 

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# 15.7 INVERSE HYBRID (g) PARAMETERS

Another set of hybrid matrix parameters can be defined in a similar way as was done in Section 15.6. This time the current at the input port  $I_1$  and the voltage at the output port  $V_2$  can be expressed in terms of  $I_2$  and  $V_1$ . The equations are as follows.

$$I_1 = g_{11} V_1 + g_{12} I_2 \tag{15.11}$$

$$V_2 = g_{21} V_1 + g_{22} I_2 \tag{15.12}$$

The coefficients in the above equations are called the inverse hybrid parameters. In matrix notation

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$
  
It can be verified that 
$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}^{-1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

The individual g parameters may be defined by letting  $I_2 = 0$  and  $V_1 = 0$  in Eqs 15.11 and 15.12. Thus, when  $I_2 = 0$ 

$$g_{11} = \frac{I_1}{V_1} \Big|_{I_2 = 0} = \text{Open circuit input admittance} \left(\frac{1}{Z_{11}}\right)$$
$$g_{21} = \frac{V_2}{V_1} \Big|_{I_2 = 0} = \text{Open circuit voltage gain}$$
$$n V_1 = 0$$

Whe

$$g_{12} = \frac{I_1}{I_2} \bigg|_{V_1 = 0} = \text{Short circuit reverse current gain}$$
$$g_{22} = \frac{V_2}{I_2} \bigg|_{V_1 = 0} = \text{Short circuit output impedance} \left(\frac{1}{Y_{22}}\right)$$

### 15.8 **INTER RELATIONSHIPS OF DIFFERENT PARAMETERS**

### 15.8.1 Expression of Z-parameters in Terms of Y-parameters and Vice-versa

From Eqs 15.1, 15.2, 15.3 and 15.4, it is easy to derive the relation between the open circuit impedance parameters and the short circuit admittance parameters by means of two matrix equations of the respective parameters. By solving Eqs 15.1 and 15.2 for  $I_1$  and  $I_2$ , we get

$$I_{1} = \begin{vmatrix} V_{1} & Z_{12} \\ V_{2} & Z_{22} \end{vmatrix} / \Delta_{z} \text{ ; and } I_{2} = \begin{vmatrix} Z_{11} & V_{1} \\ V_{21} & V_{2} \end{vmatrix} / \Delta_{z}$$

where  $\Delta_z$  is the determinant of Z matrix

$$\Delta_z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$I_1 = \frac{Z_{22}}{\Delta_z} V_1 - \frac{Z_{12}}{\Delta_z} V_2$$
(15.13)

$$I_2 = \frac{-Z_{21}}{\Delta_z} V_1 + \frac{Z_{11}}{\Delta_z} V_2$$
(15.14)

Comparing Eqs. 15.13 and 15.14 with Eqs. 15.3 and 15.4 we have

$$Y_{11} = \frac{Z_{22}}{\Delta_z}; Y_{12} = \frac{-Z_{12}}{\Delta_z}$$
$$Y_{21} = \frac{Z_{21}}{\Delta_z}; Y_{22} = \frac{Z_{11}}{\Delta_z}$$

In a similar manner, the Z parameters may be expressed in terms of the admittance parameters by solving Eqs. 15.3 and 15.4 for  $V_1$  and  $V_2$ 

$$V_1 = \begin{vmatrix} I_1 & Y_{12} \\ I_2 & Y_{22} \end{vmatrix} / \Delta_y \text{ and } V_2 = \begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & I_2 \end{vmatrix} / \Delta_y$$

where  $\Delta_{y}$  is the determinant of the *Y* matrix

$$\Delta_{y} = \begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix}$$

$$V_{1} = \frac{Y_{22}}{\Delta_{y}} I_{1} - \frac{Y_{12}}{\Delta_{y}} I_{2} \qquad (15.15)$$

$$V_{2} = \frac{-Y_{21}}{\Delta_{y}} I_{1} + \frac{Y_{11}}{\Delta_{y}} I_{2} \qquad (15.16)$$

Comparing Eqs. 15.15 and 15.16 with Eqs. 15.1 and 15.2, we obtain

$$Z_{11} = \frac{Y_{22}}{\Delta_y}; Z_{12} = \frac{-Y_{12}}{\Delta_y}$$
$$Z_{21} = \frac{-Y_{21}}{\Delta_y}; Z_{22} = \frac{Y_{11}}{\Delta_y}$$

**Example 15.5** For a given,  $Z_{11} = 3 \Omega$ ,  $Z_{12} = 1 \Omega$ ;  $Z_{21} = 2 \Omega$  and  $Z_{22} = 1 \Omega$ , find the admittance matrix, and the product of  $\Delta_y$  and  $\Delta_z$ .

Solution The admittance matrix = 
$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{Z_{22}}{\Delta_z} & -\frac{Z_{12}}{\Delta_z} \\ -\frac{Z_{21}}{\Delta_z} & \frac{Z_{11}}{\Delta_z} \end{bmatrix}$$
  
given  $Z = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$   
 $\therefore \qquad \Delta_z = 3 - 2 = 1$   
 $\therefore \qquad \Delta_y = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} = 1$   
 $(\Delta_y) (\Delta_z) = 1$ 

# **15.8.2** General Circuit Parameters or ABCD Parameters in Terms of Z Parameters and Y Parameters

We know that

$$V_{1} = AV_{2} - BI_{2}; V_{1} = Z_{11}I_{1} + Z_{12}I_{2}; I_{1} = Y_{11}V_{1} + Y_{12}V_{2}$$

$$I_{1} = CV_{2} - DI_{2}; V_{2} = Z_{21}I_{1} + Z_{22}I_{2}; I_{2} = Y_{21}V_{1} + Y_{22}V_{2}$$

$$A = \frac{V_{1}}{V_{2}}\Big|_{I_{2}=0}; C = \frac{I_{1}}{V_{2}}\Big|_{I_{2}=0}; B = \frac{-V_{1}}{I_{2}}\Big|_{V_{2}=0}; D = \frac{-I_{1}}{I_{2}}\Big|_{V_{2}=0}$$

Substituting the condition  $I_2 = 0$  in Eqs 15.1 and 15.2 we get

$$\frac{V_1}{V_2}\Big|_{I_2=0} = \frac{Z_{11}}{Z_{21}} = A$$

Substituting the condition  $I_2 = 0$  in Eq. 15.4 we get,

$$\frac{V_1}{V_2}\Big|_{I_2 = 0} = \frac{-Y_{22}}{Y_{21}} = A$$

Substituting the condition  $I_2 = 0$  in Eq. 15.2

we get

$$\frac{I_1}{V_2}\Big|_{I_2 = 0} = \frac{1}{Z_{21}} = C$$

Substituting the condition  $I_2 = 0$  in Eqs 15.3 and 15.4, and solving for  $V_2$  gives  $\frac{-I_1 Y_{21}}{\Delta v}$ 

where  $\Delta y$  is the determinant of the admittance matrix

$$\frac{I_1}{V_2}\Big|_{I_2 = 0} = \frac{-\Delta y}{Y_{21}} = C$$
  
in  $V_2 = 0$  in Eq. 15.4, we

Substituting the condition  $V_2 = 0$  in Eq. 15.4, we get  $V_1 \mid 1 = R$ 

$$\frac{V_1}{I_2}\Big|_{V_2=0} = -\frac{1}{Y_{21}} = B$$

Substituting the condition  $V_2 = 0$  in Eqs. 15.1 and 15.2 and solving for  $I_2 = \frac{-V_1 Z_{21}}{\Delta_z}$ 

$$-\frac{V_1}{I_2}\Big|_{V_2 = 0} = \frac{\Delta_z}{Z_{21}} = B$$

where  $\Delta_z$  is the determinant of the impedance matrix.

Substituting  $V_2 = 0$  in Eq. 15.2

we get 
$$-\frac{I_1}{I_2}\Big|_{V_2=0} = \frac{Z_{22}}{Z_{21}} = D$$

Substituting  $V_2 = 0$  in Eqs. 15.3 and 15.4, we get

$$\frac{-I_1}{I_2}\Big|_{V_2=0} = \frac{-Y_{11}}{Y_{21}} = D$$

The determinant of the transmission matrix is given by

$$-AD + BC$$

Substituting the impedance parameters in A, B, C and D, we have

$$BC - AD = \frac{\Delta z}{Z_{21}} \frac{1}{Z_{21}} - \frac{Z_{11}}{Z_{21}} \frac{Z_{22}}{Z_{21}}$$
$$= \frac{\Delta z}{(Z_{21})^2} - \frac{Z_{11}Z_{22}}{(Z_{21})^2}$$
$$BC - AD = \frac{-Z_{12}}{Z_{21}}$$

For a bilateral network,  $Z_{12} = Z_{21}$ 

BC - AD = -1

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or AD - BC = 1

Therefore, in a two-port bilateral network, if three transmission parameters are known, the fourth may be found from equation AD - BC = 1.

In a similar manner the h parameters may be expressed in terms of the admittance parameters, impedance parameters or transmission parameters. Transformations of this nature are possible between any of the various parameters. Each parameters has its own utility. However, we often find that it is necessary to convert from one set of parameters to another. Transformations between different parameters, and the condition under which the two-port network is reciprocal are given in Table 15.1.

**Example 15.6** The impedance parameters of a two port network are  $Z_{11} = 6\Omega$ ;  $Z_{22} = 4 \Omega$ ;  $Z_{12} = Z_{21} = 3 \Omega$ . Compute the *Y* parameters and *ABCD* parameters and write the describing equations.

**Solution** *ABCD* parameters are given by

$$A = \frac{Z_{11}}{Z_{21}} = \frac{6}{3} = 2; \ B = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}} = 5 \ \Omega$$
$$C = \frac{1}{Z_{21}} = \frac{1}{3} \ \mho; \ D = \frac{Z_{22}}{Z_{21}} = \frac{4}{3}$$

*Y* parameters are given by

$$Y_{11} = \frac{Z_{22}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{4}{15} \ \text{c}; \ Y_{12} = \frac{-Z_{12}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{-1}{5} \ \text{c};$$
$$Y_{21} = Y_{12} = \frac{-Z_{12}}{\Delta z} = \frac{-1}{5} \ \text{c}; \ Y_{22} = \frac{Z_{11}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{2}{5} \ \text{c};$$

The equations, using Z parameters are

$$V_1 = 6I_1 + 3I_2$$
  
 $V_2 = 3I_1 + 4I_2$ 

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Using Y Parameters.

$11 15'^{1}$	5 <sup>2</sup>
$I_2 = \frac{-1}{5}V_1 +$	$\frac{2}{5}V_2$
	Table 15.1

	Ζ	Y	ABCD	A'B'C'D'	h	g
z	$Z_{11} Z_{12}$	$\frac{Y_{22}}{\Delta_y} \frac{-Y_{12}}{\Delta_y}$	$\frac{A}{C} \frac{\Delta_T}{C}$	$\frac{D'}{C'} \frac{1}{C'}$	$\frac{\Delta_h}{h_{22}} \frac{h_{22}}{h_{22}}$	$\frac{1}{g_{11}} \frac{-g_{12}}{g_{11}}$
	$Z_{21} Z_{22}$	$\frac{-Y_{21}}{\Delta_y} \frac{Y_{11}}{\Delta_y}$	$\frac{1}{C}\frac{D}{C}$	$\frac{\Delta_{T'}}{C'}\frac{A'}{C'}$	$\frac{-h_{21}}{h_{22}} \frac{1}{h_{22}}$	$rac{g_{21}}{g_{11}}rac{\Delta_g}{g_{11}}$
Y	$\frac{Z_{22}}{\Delta_z} \frac{-Z_{12}}{\Delta_z}$	$Y_{11} \ Y_{12}$	$\frac{D}{B} \frac{-\Delta_T}{B}$	$\frac{A'}{B'}\frac{-1}{B'}$	$\frac{1}{h_{11}} \frac{-h_{12}}{h_{11}}$	$\frac{\Delta_g}{g_{22}} \frac{g_{12}}{g_{22}}$
	$\frac{-Z_{21}}{\Delta z} \frac{Z_{11}}{\Delta z}$	Y <sub>21</sub> Y <sub>22</sub>	$\frac{-1}{B}\frac{A}{B}$	$\frac{-\Delta_{T'}}{B'} \frac{D'}{B'}$	$\frac{h_{21}}{h_{11}}\frac{\Delta_h}{h_{11}}$	$\frac{-g_{21}}{g_{22}} \frac{1}{g_{22}}$
AB	$\frac{Z_{11}}{Z_{21}} \frac{\Delta z}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}} \frac{-1}{Y_{21}}$	A B	$\frac{D'}{\Delta_{T'}} \frac{B'}{\Delta_{T'}}$	$\frac{\Delta_h}{h_{21}} \frac{h_{11}}{h_{21}}$	$\frac{1}{g_{21}} \frac{g_{22}}{g_{21}}$
CD	$\frac{1}{Z_{21}} \frac{Z_{22}}{Z_{21}}$	$\frac{\Delta Y}{Y_{21}} \frac{-Y_{11}}{Y_{21}}$	C D	$\frac{C'}{\Delta_{T'}} \frac{A'}{\Delta_{T'}}$	$\frac{-h_{22}}{h_{21}} \frac{-1}{h_{21}}$	$\frac{g_{11}}{g_{21}}\frac{\Delta_g}{g_{21}}$
A' B'	$\frac{Z_{22}}{Z_{12}} \frac{\Delta z}{Z_{12}}$	$\frac{-Y_{11}}{Y_{12}} \frac{-1}{Y_{12}}$	$\frac{D}{\Delta_T} \frac{B}{\Delta_T}$	A' B'	$\frac{1}{h_{12}} \frac{h_{11}}{h_{12}}$	$\frac{-\Delta_g}{g_{12}} \frac{-g_{22}}{g_{12}}$
C' D'	$\frac{1}{Z_{12}} \frac{Z_{11}}{Z_{12}}$	$\frac{-\Delta_Y}{Y_{12}} \frac{-Y_{22}}{Y_{12}}$	$\frac{C}{\Delta_T} \frac{A}{\Delta_T}$	C' D'	$\frac{h_{22}}{h_{12}}\frac{\Delta_h}{h_{12}}$	$\frac{-g_{11}}{g_{12}} \frac{-1}{g_{12}}$
h	$\frac{\Delta_z}{Z_{22}} \frac{Z_{12}}{Z_{22}}$	$\frac{1}{Y_{11}} \frac{-Y_{12}}{Y_{11}}$	$\frac{B}{D} \frac{\Delta_T}{D}$	$\frac{B'}{A'}\frac{1}{A'}$	$h_{11} h_{12}$	$rac{g_{22}}{\Delta_g} rac{-g_{12}}{\Delta_g}$
	$\frac{-Z_{21}}{Z_{22}} \frac{1}{Z_{22}}$	$\frac{Y_{21}}{Y_{11}} \frac{\Delta_Y}{Y_{11}}$	$\frac{-1}{D}\frac{C}{D}$	$\frac{\Delta_{T'}}{A'}\frac{C'}{A'}$	$h_{21} h_{22}$	$\frac{-g_{21}}{\Delta_g}\frac{g_{11}}{\Delta_g}$
g	$\frac{1}{Z_{11}} \frac{-Z_{12}}{Z_{11}}$	$\frac{\Delta_Y}{Y_{22}} \frac{Y_{12}}{Y_{22}}$	$\frac{C}{A} \frac{-\Delta_T}{A}$	$\frac{C'}{D'} \frac{-1}{D'}$	$\frac{h_{22}}{\Delta_h} \frac{-h_{12}}{\Delta_h}$	$g_{11} \ g_{12}$
	$\frac{Z_{21}}{Z_{11}}\frac{\Delta_Z}{Z_{11}}$	$\frac{-Y_{21}}{Y_{22}} \frac{1}{Y_{22}}$	$\frac{1}{A} \frac{B}{A}$	$\frac{\Delta_{T'}}{D'}\frac{B'}{D'}$	$\frac{-h_{21}}{\Delta_h}\frac{h_{11}}{\Delta_h}$	g <sub>21</sub> g <sub>22</sub>
The two port is reci- procal If	$Z_{12} = Z_{21}$	$Y_{12} = Y_{21}$	The deter- minant of the trans- mission matrix = 1 $(\Delta_T = 1)$	The deter- minant of the inverse trans- mission matrix = 1	$h_{12} = -h_{21}$	$g_{12} = -g_{21}$

Using ABCD parameters

$$V_1 = 2V_2 - 5I_2$$
$$I_1 = \frac{1}{3}V_2 - \frac{4}{3}I_2$$

### **INTER-CONNECTION OF TWO-PORT** 15.9 **NETWORKS**

#### Series Connection of Two-port Network 15.9.1

It has already been shown in Section 15.4.1 that when two-port networks are connected in cascade, the parameters of the interconnected network can be conveniently expressed with the help of ABCD parameters. In a similar way, the Z-parameters can be used to describe the parameters of series connected two-port networks; and Y parameters can be used to describe parameters of parallel connected two-port networks. A series connection of two-port networks is shown in Fig. 15.18.





Let us consider two two-port networks, connected in series as shown. If each port has a common reference node for its input and output, and if these references are connected together then the equations of the networks X and Y in terms of Z parameters are

$$\begin{split} V_{1X} &= Z_{11X} I_{1X} + Z_{12X} I_{2X} \\ V_{2X} &= Z_{21X} I_{1X} + Z_{22X} I_{2X} \\ V_{1Y} &= Z_{11Y} I_{1Y} + Z_{12Y} I_{2Y} \\ V_{2Y} &= Z_{21Y} I_{1Y} + Z_{22Y} I_{2Y} \end{split}$$

From the inter-connection of the networks, it is clear that

and  

$$I_1 = I_{1X} = I_{1Y}; I_2 = I_{2X} = I_{2Y}$$
  
 $V_1 = V_{1X} + V_{1Y}; V_2 = V_{2X} + V_{2Y}$   
 $\therefore$   $V_1 = Z_{11X}I_1 + Z_{12X}I_2 + Z_{11Y}I_1 + Z_{12Y}I_2$ 

*.*..

$$= (Z_{11X} + Z_{11Y})I_1 + (Z_{12X} + Z_{12Y})I_2$$
  

$$V_2 = Z_{21X}I_1 + Z_{22X}I_2 + Z_{21Y}I_1 + Z_{22Y}I_2$$
  

$$= (Z_{21X} + X_{21Y})I_1 + (Z_{22X} + Z_{22Y})I_2$$

The describing equations for the series connected two-port network are

$$\begin{split} V_1 &= Z_{11} I_1 + Z_{12} I_2 \\ V_2 &= Z_{21} I_1 + Z_{22} I_2 \\ Z_{11} &= Z_{11X} + Z_{11Y}; \ Z_{12} &= Z_{12X} + Z_{12Y} \\ Z_{21} &= Z_{21X} + Z_{21Y}; \ Z_{22} &= Z_{22X} + Z_{22Y} \end{split}$$

where

Thus, we see that each Z parameter of the series network is given as the sum of the corresponding parameters of the individual networks.

### 15.9.2 Parallel Connection of Two Two-port Networks

Let us consider two two-port networks connected in parallel as shown in Fig. 15.19. If each two-port has a reference node that is common to its input and output port, and if the two ports are connected so that they have a common reference node, then the equations of the networks X and Y in terms of Y parameters are given by





$$I_{1X} = Y_{11X} V_{1X} + Y_{12X} V_{2X}$$
  

$$I_{2X} = Y_{21X} V_{1X} + Y_{22X} V_{2X}$$
  

$$I_{1Y} = Y_{11Y} V_{1Y} + Y_{12Y} V_{2Y}$$
  

$$I_{2Y} = Y_{21Y} V_{1Y} + Y_{22Y} V_{2Y}$$

From the interconnection of the networks, it is clear that

and ∴

$$I_{1} = I_{1X} + I_{1Y}; I_{2} = I_{2X} + I_{2Y}$$

$$I_{1} = Y_{11X} + I_{1Y}; I_{2} = I_{2X} + I_{2Y}$$

$$I_{1} = Y_{11X} + Y_{12X} + Y_{2Y} + Y_{11Y} + Y_{12Y} + Y_{2Y}$$

$$= (Y_{11X} + Y_{11Y}) + (Y_{12X} + Y_{12Y}) + Y_{2Y}$$

$$I_{2} = Y_{21X} + Y_{22X} + Y_{21Y} + Y_{22Y} + Y_{22Y} + Y_{22Y}$$

$$= (Y_{21X} + Y_{21Y}) + (Y_{22X} + Y_{22Y}) + Y_{22Y}$$

 $V_1 = V_{1V} = V_{1V}; V_2 = V_{2V} = V_{2V}$ 

The describing equations for the parallel connected two-port networks are

$$\begin{split} I_1 &= Y_{11} \ V_1 + Y_{12} \ V_2 \\ I_2 &= Y_{21} \ V_1 + Y_{22} \ V_2 \\ Y_{11} &= Y_{11X} + Y_{11Y}; \ Y_{12} &= Y_{12X} + Y_{12Y} \\ Y_{21} &= Y_{21X} + Y_{21Y}; \ Y_{22} &= Y_{22X} + Y_{22Y} \end{split}$$

Thus we see that each *Y* parameter of the parallel network is given as the sum of the corresponding parameters of the individual networks.

**Example 15.7** Two networks shown in Figs. 15.20(a) and (b) are connected in series. Obtain the *Z* parameters of the combination. Also verify by direct calculation.



Fig. 15.20

**Solution** The *Z* parameters of the network in Fig. 15.20(a) are

$$Z_{11X} = 3 \ \Omega \ Z_{12X} = Z_{21X} = 2 \ \Omega \ Z_{22X} = 3 \ \Omega$$

The Z parameters of the network in Fig. 15.20 (b) are

$$Z_{11y} = 15 \ \Omega Z_{21y} = 5 \ \Omega Z_{22y} = 25 \ \Omega Z_{12y} = 5 \ \Omega$$

The Z parameters of the combined network are

$$Z_{11} = Z_{11X} + Z_{11Y} = 18 \Omega$$
$$Z_{12} = Z_{12X} + Z_{12Y} = 7 \Omega$$
$$Z_{21} = Z_{21X} + Z_{21Y} = 7 \Omega$$
$$Z_{22} = Z_{22X} + Z_{22Y} = 28 \Omega$$

*Check* If the two networks are connected in series as shown in Fig. 15.20(c), the *Z* parameters are



Fig. 15.20(c)

where

$$Z_{11} = \frac{V_1}{I_1} \Big|_{I_2 = 0} = 18 \Omega$$

$$Z_{21} = \frac{V_2}{I_1} \Big|_{I_2 = 0} = 7 \Omega$$

$$Z_{22} = \frac{V_2}{I_2} \Big|_{I_1 = 0} = 28 \Omega$$

$$Z_{12} = \frac{V_1}{I_2} \Big|_{I_1 = 0} = 7 \Omega$$

**Example 15.8** Two identical sections of the network shown in Fig. 15.21 are connected in parallel. Obtain the *Y* parameters of the combination.



### Fig. 15.21

**Solution** The *Y* parameters of the network in Fig. 15.21 are (See Ex. 15.2).

$$Y_{11} = \frac{1}{2} \ \mbox{$\nabla$} \ Y_{21} = \frac{-1}{4} \ \mbox{$\nabla$} \ Y_{22} = \frac{5}{8} \ \mbox{$\nabla$} \ Y_{12} = \frac{-1}{4} \ \mbox{$\nabla$}$$

If two such networks are connected in parallel then the *Y* parameters of the combined network are

# 15.10 T AND $\Pi$ REPRESENTATION

A two-port network with any number of elements may be converted into a two-port three-element network. Thus, a two-port network may be represented by an equivalent T network, i.e. three impedances are connected together in the form of a T as shown in Fig. 15.22.

It is possible to express the elements of  $1'^{-}$ the *T*-network in terms of *Z* parameters, or *ABCD* parameters as explained below. *Z* parameters of the network



Fig. 15.22

$$Z_{11} = \frac{V_1}{I_1} \Big|_{I_2 = 0} = Z_a + Z_c$$
$$Z_{21} = \frac{V_2}{I_1} \Big|_{I_2 = 0} = Z_c$$
$$Z_{22} = \frac{V_2}{I_2} \Big|_{I_1 = 0} = Z_b + Z_c$$
$$Z_{12} = \frac{V_1}{I_2} \Big|_{I_1 = 0} = Z_c$$

From the above relations, it is clear that

$$\begin{split} Z_a &= Z_{11} - Z_{21} \\ Z_b &= Z_{22} - Z_{12} \\ Z_c &= Z_{12} = Z_{21} \end{split}$$

ABCD parameters of the network

$$A = \frac{V_1}{V_2}\Big|_{I_2 = 0} = \frac{Z_a + Z_c}{Z_c}$$
$$B = \frac{-V_1}{I_2}\Big|_{V_2 = 0}$$

When 2-2' is short circuited

$$-I_{2} = \frac{V_{1} Z_{c}}{Z_{b} Z_{c} + Z_{a} (Z_{b} + Z_{c})}$$
$$B = (Z_{a} + Z_{b}) + \frac{Z_{a} Z_{b}}{Z_{c}}$$
$$C = \frac{I_{1}}{V_{2}} \Big|_{I_{2} = 0} = \frac{1}{Z_{c}}$$

$$D = \frac{-I_1}{I_2} \bigg|_{V_2 = 0}$$

When 2-2' is short circuited

$$-I_2 = I_1 \frac{Z_c}{Z_b + Z_c}$$
$$D = \frac{Z_b + Z_c}{Z_c}$$

From the above relations we can obtain

$$Z_a = \frac{A-1}{C}; \quad Z_b = \frac{D-1}{C}; \quad Z_c = \frac{1}{C}$$

**Example 15.9** The *Z* parameters of a two-port network are  $Z_{11} = 10 \Omega$ ;  $Z_{22} = 15 \Omega$ ;  $Z_{12} = Z_{21} = 5 \Omega$ . Find the equivalent *T* network and *ABCD* parameters.

**Solution** The equivalent *T* network is shown in Fig. 15.23,

 $Z_a = Z_{11} - Z_{21} = 5 \Omega$ 

 $Z_{h} = Z_{22} - Z_{12} = 10 \ \Omega$ 

where

and

 $Z_{1} = 5 \Omega$ The ABCD parameters of the network are

$$A = \frac{Z_a}{Z_c} + 1 = 2; B = (Z_a + Z_b) + \frac{Z_a Z_b}{Z_c} = 25 \Omega$$

$$C = \frac{1}{Z_c} = 0.2 \ \text{o} \ D = 1 + \frac{Z_b}{Z_c} = 3$$

In a similar way, a two-port network may be represented by an equivalent  $\pi$ -network, i.e. three impedances or admittances are connected together in the form of  $\pi$  as shown in Fig. 15.24.

It is possible to express the elements of the  $\pi$ -network in terms of Y parameters or ABCD parameters as explained below. *Y* parameters of the network

$$Y_{11} = \frac{I_1}{V_1}\Big|_{V_2 = 0} = Y_1 + Y_2$$
$$Y_{21} = \frac{I_2}{V_1}\Big|_{V_2 = 0} = -Y_2$$



Fig. 15.24



Fig. 15.23

$$Y_{22} = \frac{I_2}{V_2} \bigg|_{V_1 = 0} = Y_3 + Y_2$$
$$Y_{12} = \frac{I_1}{V_2} \bigg|_{V_1 = 0} = -Y_2$$

From the above relations, it is clear that

$$\begin{aligned} Y_1 &= Y_{11} + Y_{21} \\ Y_2 &= -Y_{12} \\ Y_3 &= Y_{22} + Y_{21} \end{aligned}$$

Writing ABCD parameters in terms of Y parameters yields the following results.

$$A = \frac{-Y_{22}}{Y_{21}} = \frac{Y_3 + Y_2}{Y_2}$$
$$B = \frac{-1}{Y_{21}} = \frac{1}{Y_2}$$
$$C = \frac{-\Delta y}{Y_{21}} = Y_1 + Y_3 + \frac{Y_1 Y_3}{Y_2}$$
$$D = \frac{-Y_{11}}{Y_{21}} = \frac{Y_1 + Y_2}{Y_2}$$

From the above results, we can obtain

$$Y_{1} = \frac{D-1}{B}$$

$$Y_{2} = \frac{1}{B}$$

$$Y_{3} = \frac{A-1}{B}$$

**Example 15.10** The port currents of a two-port network are given by

$$I_1 = 2.5V_1 - V_2$$
$$I_2 = -V_1 + 5V_2$$

Find the equivalent  $\pi$ -network.

**Solution** Let us first find the *Y* parameters of the network

$$Y_{11} = \frac{I_1}{V_1}\Big|_{V_2 = 0} = 2.5 \ \text{c}; \ Y_{21} = \frac{I_2}{V_1}\Big|_{V_2 = 0} = -1 \ \text{c}$$



# **15.11 TERMINATED TWO-PORT NETWORK**

# **15.11.1 Driving Point Impedance at the Input Port of a Load Terminated Network**

Figure 15.26 shows a two-port network connected to an ideal generator at the input port and to a load impedance at the output port. The input impedance of this network can be expressed in terms of parameters of the two port network.



Fig. 15.26

(*i*) In Terms of Z Parameters The load at the output port 2-2' impose the following constraint on the port voltage and current,

i.e.,  $V_2 = -Z_L I_2$ 

Recalling Eqs 15.1 and 15.2, we have

 $I_{2} =$ 

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Substituting the value of  $V_2$  in Eq. 15.2, we have

$$-Z_L I_2 = Z_{21} I_1 + Z_{22} I_2$$
$$-I_L Z_{2L}$$

from which

$$\frac{-I_1 Z_{21}}{Z_L + Z_{22}}$$

Substituting the value of  $I_2$  in Eq. 15.1 gives

$$V_{1} = Z_{11} I_{1} - \frac{Z_{12} Z_{21} I_{1}}{Z_{L} + Z_{22}}$$
$$V_{1} = I_{1} \left( Z_{11} - \frac{Z_{12} Z_{21}}{Z_{L} + Z_{22}} \right)$$

Hence the driving point impedance at 1-1' is

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12} \ Z_{21}}{Z_L + Z_{22}}$$

If the output port is open, i.e.  $Z_L \rightarrow \infty$ , the input impedance is given by  $V_1/I_1 = Z_{11}$ 

If the output port is short circuited, i.e.  $Z_{L} \rightarrow 0$ ,

The short circuit driving point impedance is given by

$$\frac{Z_{11} \ Z_{22} - Z_{12} \ Z_{21}}{Z_{22}} = \frac{1}{Y_{11}}$$

(ii) In Terms of Y Parameters If a load admittance  $Y_L$  is connected across the output port. The constraint imposed on the output port voltage and current is

$$-I_2 = V_2 Y_L$$
, where  $Y_L = \frac{1}{Z_L}$ 

Recalling Eqs 15.3 and 15.4 we have

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
  
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Substituting the value of  $I_2$  in Eq. 15.4, we have

$$\begin{split} V_2 \; Y_L &= Y_{21} \; V_1 + Y_{22} \; V_2 \\ V_2 &= - \left( \frac{Y_{21}}{Y_L + Y_{22}} \right) \; V_1 \end{split}$$

Substituting  $V_2$  value in Eq. 15.3, we have

$$I_1 = Y_{11} V_1 - \frac{Y_{12} Y_{21} V_1}{Y_L + Y_{22}}$$

From which  $\frac{I_1}{V_1} = Y_{11} - \frac{Y_{12} Y_{21}}{Y_1 + Y_{22}}$ 

Hence the driving point impedance is given by

$$\frac{V_1}{I_1} = \frac{Y_{22} + Y_L}{Y_{11}(Y_1 + Y_{22}) - Y_{12} Y_{21}}$$

If the output port is open, i.e.,  $Y_L \rightarrow 0$ 

$$\frac{V_1}{I_1} = \frac{Y_{22}}{\Delta_y} = Z_{11}$$

If the output port is short circuited, i.e.  $Y_L \rightarrow \infty$ 

Then  $Y_{in} = Y_{11}$ 

In a similar way, the input impedance of the load terminated two port network may be expressed in terms of other parameters by simple mathematical manipulations. The results are given in Table 15.2.

# **15.11.2 Driving Point Impedance at the Output Port** with Source Impedance at the Input Port

Let us consider a two-port network connected to a generator at input port with a source impedance  $Z_s$  as shown in Fig. 15.27. The output impedance, or the driving point impedance, at the output port can be evaluated in terms of the parameters of two-port network.



Fig. 15.27

## (i) In terms of Z parameters

If  $I_1$  is the current due to  $V_s$  at port 1-1' From Eqs. 15.1 and 15.2, we have

$$V_{2} = Z_{21}I_{1} + Z_{22}I_{2}$$

$$V_{1} = V_{s} - I_{1}Z_{s}$$

$$= Z_{11}I_{1} + Z_{12}I_{2} - (I_{1}) (Z_{s} + Z_{11}) = Z_{12}I_{2} - V_{s}$$

$$-I_{1} = \frac{Z_{12}I_{2} - V_{s}}{Z_{s} + Z_{11}}$$

Substituting  $I_1$  in Eq. 15.2, we get

$$V_2 = -Z_{21} \frac{(Z_{12} I_2 - V_s)}{Z_s + Z_{11}} + Z_{22} I_2$$

With no source voltage at port 1-1', i.e. if the source  $V_s$  is short circuited

$$V_2 = \frac{-Z_{21} \ Z_{12}}{Z_s + Z_{11}} \ I_2 + Z_{22}I_2$$

Hence the driving point impedance at port  $2-2' = \frac{V_2}{r}$ 

$$\frac{V_2}{I_2} = \frac{Z_{22} Z_s + Z_{22} Z_{11} - Z_{21} Z_{12}}{Z_s + Z_{11}} \text{ or } \frac{\Delta_z + Z_{22} Z_s}{Z_s + Z_{11}}$$

If the input port is open, i.e.  $Z_s \rightarrow \infty$ 

Then

$$\frac{V_2}{I_2} = \left[\frac{\frac{\Delta_Z}{Z_s} + Z_{22}}{1 + \frac{Z_{11}}{Z_s}}\right]_{Z_s = \infty} = Z_{22}$$

If the source impedance is zero with a short circuited input port, the driving point impedance at output port is given by

$$\frac{V_2}{I_2} = \frac{\Delta_Z}{Z_{11}} = \frac{1}{Y_{22}}$$

(ii) In terms of Y parameters Let us consider a two-port network connected to a current source at input port with a source admittance  $Y_s$  as shown in Fig. 15.28.



Fig. 15.28

At port 1-1'  $I_1 = I_s - V_1 Y_s$ 

Recalling Eqs. 15.3 and 15.4, we have

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Substituting  $I_1$  in Eq. 15.3, we get

$$\begin{split} I_s - V_1 Y_s &= Y_{11} V_1 + Y_{12} V_2 \\ - V_1 (Y_s + Y_{11}) &= Y_{12} V_2 - I_s \\ - V_1 &= \frac{Y_{12} V_2 - I_s}{Y_s + Y_{11}} \end{split}$$

Substituting  $V_1$  in Eq. 15.4, we get

$$I_2 = -Y_{21} \left( \frac{Y_{12} V_2 - I_s}{Y_s + Y_{11}} \right) + Y_{22} V_2$$

With no source current at 1-1', i.e. if the current source is open circuited

$$I_2 = \frac{-Y_{21}Y_{12}V_2}{Y_s + Y_{11}} + Y_{22}V_2$$

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			In terms of		
Driving point impedance at input port, or input impedance	Z parameters	<i>Y</i> parameters	ABCD	A'B'C'D'	h parameter
$\left(\frac{V_1}{I_1}\right)$	$\frac{\Delta_z + Z_{11} Z_L}{Z_{22} + Z_L}$	$\frac{Y_{22} + Y_L}{\Delta_y + Y_{11} Y_L}$	$\frac{AZ_L + B}{CZ_L + D}$	$\frac{B' - D'Z_L}{C'Z_L - A'}$	$\frac{\Delta_h Z_L + h_{11}}{1 + h_{22} Z_L}$
Driving point impedance at output port, or output impedance					
$\left(\frac{V_2}{I_2}\right)$	$\frac{\Delta_z + Z_{22} Z_s}{Z_1 + Z_{11}}$	$\frac{Y_{11} + Y_s}{\Delta_y + Y_s Y_{22}}$	$\frac{DZ_s + B}{CZ_s + A}$	$\frac{A'Z_s + B'}{C'Z_s + D'}$	$\frac{h_{11} + Z_s}{\Delta_h + h_{22} Z_s}$

**Table 15.2** 

*Note* The above relations are obtained, when  $V_s = 0$  and  $I_s = 0$  at the input port.

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g parameter

 $\frac{1+g_{22} Y_L}{\Delta_{gYL}+g_{11}}$ 

 $\frac{g_{22} + \Delta_s}{1 + g_{11} Z_s}$ 

Hence the driving point admittance at the output port is given by

$$\frac{I_2}{V_2} = \frac{Y_{22} Y_s + Y_{22} Y_{11} - Y_{21} Y_{12}}{Y_s + Y_{11}} \text{ or } \frac{\Delta_y + Y_{22} Y_s}{Y_s + Y_{11}}$$

If the source admittance is zero, with an open circuited input port, the driving point admittance at the output port is given by

$$\frac{I_2}{V_2} = \frac{\Delta_y}{Y_{11}} = \frac{1}{Z_{22}} = Y_{22}$$

In a similar way, the output impedance may be expressed in terms of the other two port parameters by simple mathematical manipulations. The results are given in Table 15.2.

Example 15.11 Calculate the input impedance of the network shown in Fig. 15.29.



Fig. 15.29

**Solution** Let us calculate the input impedance in terms of Z parameters. The Zparameters of the given network (see Solved Problem 15.1) are  $Z_{11} = 2.5 \Omega$ ;  $Z_{21} =$  $1 \Omega; Z_{22} = 2 \Omega; Z_{12} = 1 \Omega$ 

From Section 15.11.1 we have the relation

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}}$$

where  $Z_{L}$  is the load impedance = 2  $\Omega$ 

$$\frac{V_1}{I_1} = 2.5 - \frac{1}{2+2} = 2.25 \ \Omega$$

The source resistance is 1  $\Omega$ 

:.

$$Z_{\rm in} = 1 + 2.25 = 3.25 \ \Omega$$

**Example 15.12** Calculate the output impedance of the network shown in Fig. 15.30 with a source admittance of 1 T at the input port.



Fig. 15.30

**Solution** Let us calculate the output impedance in terms of *Y* parameters. The *Y* parameters of the given network (see Ex. 15.2) are

$$Y_{11} = \frac{1}{2} \ \mbox{$\mathcal{G}$}; \ Y_{22} = \frac{5}{8} \ \mbox{$\mathcal{G}$}, \ Y_{21} = Y_{12} = \frac{-1}{4} \ \mbox{$\mathcal{G}$}$$

From Section 15.11.2, we have the relation

$$\frac{I_2}{V_2} = \frac{Y_{22} Y_s + Y_{22} Y_{11} - Y_{21} Y_{12}}{Y_s + Y_{11}}$$

where  $Y_s$  is the source admittance = 1 mho

$$Y_{22} = \frac{I_2}{V_2} = \frac{\frac{5}{8} \times 1 + \frac{5}{8} \times \frac{1}{2} - \frac{1}{16}}{1 + \frac{1}{2}} = \frac{7}{12} \ \sigma$$
$$Z_{22} = \frac{12}{7} \ \sigma$$

\*\*

or

## **15.12 LATTICE NETWORKS**

One of the common four-terminal two-port network is the lattice, or bridge network shown in Fig. 15.31(a). Lattice networks are used in filter sections and are also used as attenuaters. Lattice structures are sometimes used in preference to ladder structures in some special applications.  $Z_a$  and  $Z_d$  are called series arms,  $Z_b$  and  $Z_c$  are called the diagonal arms. It can be observed that, if  $Z_d$  is zero, the lattice structure becomes a  $\pi$ -section. The lattice network is redrawn as a bridge network as shown in Fig. 15.31(b).



Fig. 15.31

Z Parameters

$$Z_{11} = \frac{V_1}{I_1} \Big|_{I_2 = 0}$$
When  $I_2 = 0$ ;  $V_1 = I_1 \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d}$ 

$$\therefore \qquad Z_{11} = \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d}$$
If the network is symmetric, then  $Z_a = Z_d$  and  $Z_b = Z_c$ 
(15.17)

$$Z_{11} = \frac{Z_a + Z_b}{2}$$
$$Z_{21} = \frac{V_2}{I_1} \Big|_{I_2 = 0}$$

When  $I_2 = 0$ ,  $V_2$  is the voltage across 2–2'

$$V_2 = V_1 \left[ \frac{Z_b}{Z_a + Z_b} - \frac{Z_d}{Z_c + Z_d} \right]$$

Substituting the value of  $V_1$  from Eq. 15.17, we have

$$V_{2} = \left[\frac{I_{1}(Z_{a} + Z_{b})(Z_{d} + Z_{c})}{Z_{a} + Z_{b} + Z_{c} + Z_{d}}\right] \left[\frac{Z_{b}(Z_{c} + Z_{d}) - Z_{d}(Z_{a} + Z_{b})}{(Z_{a} + Z_{b})(Z_{c} + Z_{d})}\right]$$
$$\frac{V_{2}}{I_{1}} = \frac{Z_{b}(Z_{c} + Z_{d}) - Z_{d}(Z_{a} + Z_{b})}{Z_{a} + Z_{b} + Z_{c} + Z_{d}} = \frac{Z_{b}Z_{c} - Z_{a}Z_{d}}{Z_{a} + Z_{b} + Z_{c} + Z_{d}}$$
$$Z_{21} = \frac{Z_{b}Z_{c} - Z_{a}Z_{d}}{Z_{a} + Z_{b} + Z_{c} + Z_{d}}$$

:.

If the network is symmetric,  $Z_a = Z_d$ ,  $Z_b = Z_c$ 

$$Z_{21} = \frac{Z_b - Z_a}{2}$$

When the input port is open,  $I_1 = 0$ 

$$Z_{12} = \frac{V_1}{I_2} \Big|_{I_1 = 0}$$

The network can be redrawn as shown in Fig. 15.31(c).



Fig. 15.31(c)
$$V_1 = V_2 \left[ \frac{Z_c}{Z_a + Z_c} - \frac{Z_d}{Z_b + Z_d} \right]$$
(15.18)

$$V_2 = I_2 \left[ \frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d} \right]$$
(15.19)

Substituting the value of  $V_2$  in Eq. 15.18, we get

$$V_{1} = I_{2} \left[ \frac{Z_{c} (Z_{b} + Z_{d}) - Z_{d} (Z_{a} + Z_{c})}{Z_{a} + Z_{b} + Z_{c} + Z_{d}} \right]$$

$$\frac{V_1}{I_2} = \frac{Z_c \, Z_b - Z_a \, Z_d}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric,  $Z_a = Z_d$ ;  $Z_b = Z_c$ 

$$\frac{V_1}{I_2} = \frac{Z_b^2 - Z_a^2}{2(Z_a + Z_b)}$$
$$Z_{12} = \frac{Z_b - Z_a}{2}$$

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$$Z_{22} = \frac{V_2}{I_2} \bigg|_{I_2 = 0}$$

From Eq. 15.19, we have

$$\frac{V_2}{I_2} = \frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric,

$$Z_a = Z_d; Z_b = Z_c$$
$$Z_{22} = \frac{Z_a + Z_b}{2} = Z_{11}$$

From the above equations, 
$$Z_{11} = Z_{22} = \frac{Z_a + Z_b}{2}$$

 $Z_{12} = Z_{21} = \frac{Z_b - Z_a}{2}$ 

and ∴

$$Z_b = Z_{11} + Z_{12}$$
$$Z_a = Z_{11} - Z_{12}.$$

**Example 15.13** Obtain the lattice equivalent of a symmetrical *T* network shown in Fig. 15.32.



Fig. 15.32

**Solution** A two two-port network can be realised as a symmetric lattice if it is reciprocal and symmetric. The *Z* parameters of the network are (see Ex. 15.1).  $Z_{11} = 3 \Omega$ ;  $Z_{12} = Z_{21} = 2 \Omega$ ;  $Z_{22} = 3 \Omega$ .

Since  $Z_{11} = Z_{22}$ ;  $Z_{12} = Z_{21}$ , the given network is symmetrical and reciprocal  $\therefore$  The parameters of the lattice network are

> $Z_a = Z_{11} - Z_{12} = 1 \ \Omega$  $Z_b = Z_{11} + Z_{12} = 5 \ \Omega$

The lattice network is shown in Fig. 15.33.



Fig. 15.33

**Example 15.14** Obtain the lattice equivalent of a symmetric  $\pi$ -network shown in Fig. 15.34.

Solution The Z parameters of the given network are

$$Z_{11} = 6 \ \Omega = Z_{22}; \ Z_{12} = Z_{21} = 4 \ \Omega$$

Hence the parameters of the lattice network are

$$Z_a = Z_{11} - Z_{12} = 2 \Omega$$
  
 $Z_b = Z_{11} + Z_{12} = 10 \Omega$ 

The lattice network is shown in Fig.15.35



## **15.13 IMAGE PARAMETERS**

The image impedance  $Z_{I1}$  and  $Z_{I2}$  of a two-port network shown in Fig. 15.36 are two values of impedance such that, if port 1–1' of the network is terminated in  $Z_{I1}$ , the input impedance of port 2-2' is  $Z_{I2}$ ; and if port 2-2' is terminated in  $Z_{I2}$ , the input impedance at port 1-1' is  $Z_{I1}$ .



Fig. 15.36

Then,  $Z_{I1}$  and  $Z_{I2}$  are called image impedances of the two port network shown in Fig. 15.36. These parameters can be obtained in terms of two-port parameters. Recalling Eqs 15.5 and 15.6 in Section 15.4, we have

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

If the network is terminated in  $Z_{12}$  at 2-2' as shown in Fig. 15.37.



Fig. 15.37

$$V_{2} = -I_{2} Z_{I2}$$

$$\frac{V_{1}}{I_{1}} = \frac{AV_{2} - BI_{2}}{CV_{2} - DI_{2}} = Z_{I1}$$

$$Z_{I1} = \frac{-AI_{2} Z_{I2} - BI_{2}}{-CI_{2} Z_{I2} - DI_{2}}$$

$$Z_{I1} = \frac{-AZ_{I2} - B}{-CZ_{I2} - D}$$

$$Z_{I1} = \frac{AZ_{I2} + B}{CZ_{I2} + D}$$

or

Similarly, if the network is terminated in  $Z_{I1}$  at port 1-1' as shown in Fig. 15.38, then





$$V_{1} = -I_{1}Z_{I1}$$

$$\frac{V_{2}}{I_{2}} = Z_{I2}$$

$$-Z_{I1} = \frac{V_{1}}{I_{1}} = \frac{AV_{2} - BI_{2}}{CV_{2} - DI_{2}}$$

$$-Z_{I1} = \frac{AI_{2}Z_{I2} - BI_{2}}{CI_{2}Z_{I2} - DI_{2}}$$

$$-Z_{I1} = \frac{AZ_{I2} - B}{CZ_{I2} - D}$$

$$Z_{I2} = \frac{DZ_{I1} + B}{CZ_{I1} + A}$$

From which

From

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Substituting the value of  $Z_{I1} + A$ 

$$Z_{I2}\left[C\frac{(-AZ_{I2}+B)}{(CZ_{I2}-D)}+A\right] = D\left[\frac{-AZ_{I2}+B}{CZ_{I2}-D}\right] + B$$
  
which  $Z_{I2} = \sqrt{\frac{BD}{AC}}$ 

Similarly, we can find  $Z_{I1} = \sqrt{\frac{AB}{CD}}$ If the network is symmetrical, then A = D

$$Z_{I1} = Z_{I2} = \sqrt{\frac{B}{C}}$$

If the network is symmetrical, the image impedances  $Z_{I1}$  and  $Z_{I2}$  are equal to each other; the image impedance is then called the *characteristic* impedance, or the *iterative* impedance, i.e. if a symmetrical network is terminated in  $Z_L$ , its input impedance will also be  $Z_L$ , or its impedance transformation ratio is unity. Since a reciprocal symmetric network can be described by two independent parameters, the image parameters  $Z_{I1}$  and  $Z_{I2}$  are sufficient to characterise reciprocal symmetric networks.  $Z_{I1}$  and  $Z_{I2}$  the two image parameters do not completely define a network. A third parameter called *image transfer constant*  $\phi$  is also used to describe reciprocal networks. This parameter may be obtained from the voltage and current ratios.

If the image impedance  $Z_{12}$  is connected across port 2-2', then

$$V_1 = AV_2 - BI_2 \tag{15.20}$$

$$V_2 = -I_2 Z_{I2} \tag{15.21}$$

$$V_1 = \left[A + \frac{B}{Z_{12}}\right] V_2 \tag{15.22}$$

$$I_1 = CV_2 - DI_2 \tag{15.23}$$

$$I_1 = -\left[CZ_{I2} + D\right]I_2 \tag{15.24}$$

From Eq. 15.22

$$\frac{V_1}{V_2} = \left[A + \frac{B}{Z_{I2}}\right] = A + B \sqrt{\frac{AC}{BD}}$$

$$\frac{V_1}{V_2} = A + \sqrt{\frac{ABCD}{D}}$$
(15.25)

From Eq. 15.24

$$\frac{-I_1}{I_2} = [CZ_{I2} + D] = D + C \sqrt{\frac{BD}{AC}}$$
$$\frac{-I_1}{I_2} = D + \sqrt{\frac{ABCD}{A}}$$
(15.26)

Multiplying Eqs. 15.25 and 15.26 we have

$$\frac{-V_1}{V_2} \times \frac{I_1}{I_2} = \left(\frac{AD + \sqrt{ABCD}}{D}\right) \left(\frac{AD + \sqrt{ABCD}}{A}\right)$$
$$\frac{-V_1}{V_2} \times \frac{I_1}{I_2} = \left(\sqrt{AD} + \sqrt{BC}\right)^2$$

...

...

or 
$$\sqrt{AD} + \sqrt{BC} = \sqrt{\frac{-V_1}{V_2} \times \frac{I_1}{I_2}}$$
  
 $\sqrt{AD} + \sqrt{AD - 1} = \sqrt{\frac{-V_1}{V_2} \times \frac{I_1}{I_2}}$  (::  $AD - BC = 1$ )

Let

$$\tan h \phi = \frac{\sqrt{AD - 1}}{\sqrt{AD}} = \sqrt{\frac{BC}{AD}}$$
$$\phi = \tan h^{-1} \sqrt{\frac{BC}{AD}}$$

 $\cos h \phi = \sqrt{AD}$ ;  $\sin h \phi = \sqrt{AD - 1}$ 

Also

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$$e^{\phi} = \cos h \phi + \sin h \phi = \sqrt{-\frac{V_1 I_1}{V_2 I_2}}$$

$$\phi = \log_e \sqrt{\left(-\frac{V_1 I_1}{V_2 I_2}\right)} = \frac{1}{2} \log_e \left(\frac{V_1}{V_2} \frac{I_1}{I_2}\right)$$
$$V_1 = Z_{I1} I_1; V_2 = -I_2 Z_{I2}$$

Since

$$\phi = \frac{1}{2} \log_e \left[ \frac{Z_{I1}}{Z_{I2}} \right] + \log \left[ \frac{I_1}{I_2} \right]$$

For symmetrical reciprocal networks,  $Z_{I1} = Z_{I2}$ 

$$\phi = \log_e \left[ \frac{I_1}{I_2} \right] = \gamma$$

where  $\gamma$  is called the *propagation constant*.

**Example 15.15** Determine the image parameters of the *T* network shown in Fig.15.39.



Fig. 15.39

Solution The ABCD parameters of the network are

$$A = \frac{6}{5}; B = \frac{17}{5}; C = \frac{1}{5}; D = \frac{7}{5}$$
 (See Ex. 15.3)

Since the network is not symmetrical,  $\phi$ ,  $Z_{I1}$  and  $Z_{I2}$  are to be evaluated to describe the network.

$$Z_{I1} = \sqrt{\frac{AB}{CD}} = \sqrt{\frac{\frac{6}{5} \times \frac{17}{5}}{\frac{1}{5} \times \frac{7}{5}}} = 3.817 \ \Omega$$
$$Z_{I2} = \sqrt{\frac{BC}{AC}} = \sqrt{\frac{\frac{17}{5} \times \frac{7}{5}}{\frac{6}{5} \times \frac{1}{5}}} = 4.453 \ \Omega$$
$$\phi = \tan h^{-1} \sqrt{\frac{BC}{AD}} = \tan h^{-1} \sqrt{\frac{17}{42}}$$
$$\phi = \ln \left[\sqrt{AD} + \sqrt{AD - 1}\right]$$

or



**Problem 15.1** Find the *Z* parameters for the circuit shown in Fig. 15.40.

 $\phi = 0.75$ 



## Solution

$$Z_{11} = \frac{V_1}{I_1} \Big|_{I_2 = 0}$$

When  $I_2 = 0$ ;  $V_1$  can be expressed in terms of  $I_1$  and the equivalent impedance of the circuit looking from the terminal *a-a'* as shown in Fig. 15.41(a).



Fig. 15.41(a)

$$Z_{eq} = 1 + \frac{6 \times 2}{6 + 2} = 2.5 \Omega$$
$$V_1 = I_1 Z_{eq} = I_1 2.5$$
$$Z_{11} = \frac{V_1}{I_1} \Big|_{I_2 = 0} = 2.5 \Omega$$
$$Z_{21} = \frac{V_2}{I_1} \Big|_{I_2 = 0}$$

 $V_2$  is the voltage across the 4  $\Omega$  impedance as shown in Fig. 15.41(b).

$\rightarrow$ $I_1$	 1 Ω	I <sub>x</sub>	2 Ω	← I <sub>2</sub>	= 0
<i>V</i> <sub>1</sub>		$\gtrsim 2 \Omega$		$\begin{cases} 4 \Omega \end{cases}$	V <sub>2</sub>
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#### Fig. 15.41(b)

Let the current in the 4  $\Omega$  impedance be  $I_x$ 

$$I_{x} = I_{1} \times \frac{2}{8} = \frac{I_{1}}{4}$$

$$V_{2} = I_{x}4 = \frac{I_{1}}{4} \times 4 = I_{1}$$

$$Z_{21} = \frac{V_{2}}{I_{2}}\Big|_{I_{2} = 0} = 1 \Omega$$

$$Z_{22} = \frac{V_{2}}{I_{2}}\Big|_{I_{1} = 0}$$

When port a - a' is open circuited the voltage at port b - b' can be expressed in terms of  $I_2$ , and the equivalent impedance of the circuit viewed from b - b' as shown in Fig. 15.41(c).



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$$V_{2} = I_{2} \times 2$$
$$Z_{22} = \frac{V_{2}}{I_{2}} \Big|_{I_{1} = 0} = 2 \Omega$$
$$Z_{12} = \frac{V_{1}}{I_{2}} \Big|_{I_{1} = 0}$$

 $V_1$  is the voltage across the 2  $\Omega$  (parallel) impedance, let the current in the 2  $\Omega$  impedance is  $I_y$  as shown in Fig. 15.41(d).



Fig. 15.41(d)

$$I_{Y} = \frac{I_{2}}{2}$$

$$V_{1} = 2 I_{Y}$$

$$V_{1} = 2 \frac{I_{2}}{2}$$

$$Z_{12} = \frac{V_{1}}{I_{2}}\Big|_{I_{1} = 0} = 1 \Omega$$

Here  $Z_{12} = Z_{21}$ , which indicates the bilateral property of the network. The describing equations for this two-port network in terms of impedance parameters are

$$V_1 = 2.5I_1 + I_2 V_2 = I_1 + 2I_2$$

**Problem 15.2** Find the short circuit admittance parameters for the circuit shown in Fig. 15.42.



Fig. 15.42

**Solution** The elements in the branches of the given two-port network are admittances. The admittance parameters can be determined by short circuiting the two-ports.

When port *b-b'* is short circuited,  $V_2 = 0$ . This circuit is shown in Fig. 15.43(a).



Fig. 15.43(a)

$$V_1 = I_1 Z_{ec}$$

where  $Z_{ea}$  is the equivalent impedance as viewed from *a-a'*.

$$\begin{aligned} Z_{eq} &= \frac{1}{Y_{eq}} \\ Y_{eq} &= Y_A + Y_B \\ V_1 &= \frac{I_1}{Y_A + Y_B} \\ Y_{11} &= \frac{I_1}{V_1} \bigg|_{V_2 = 0} = (Y_A + Y_B) \end{aligned}$$

With port b-b' short circuited, the nodal equation at node 1 gives

$$-I_2 = V_1 Y_B$$
$$Y_{21} = \frac{I_2}{V_1} \Big|_{V_2 = 0} = -Y_B$$

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when port *a-a*' is short circuited;  $V_1 = 0$  this circuit is shown in Fig. 15.43(b).



Fig. 15.43(b)

$$V_2 = I_2 Z_{eq}$$

where  $Z_{ea}$  is the equivalent impedance as viewed from b-b'

$$Z_{eq} = \frac{1}{Y_{eq}}$$

$$Y_{eq} = Y_b + Y_c$$

$$V_2 = \frac{I_2}{Y_B + Y_C}$$

$$Y_{22} = \frac{I_2}{V_2} \Big|_{V_1 = 0} = (Y_B + Y_C)$$

With port a-a' short circuited, the nodal equation at node 2 gives

$$-I_1 = V_2 Y_B$$
$$Y_{12} = \frac{I_1}{V_2} \Big|_{V_1 = 0} = -Y_B$$

The describing equations in terms of the admittance parameters are

$$I_1 = (Y_A + Y_B)V_1 - Y_BV_2$$
  

$$I_2 = -Y_BV_1 + (Y_C + Y_B)V_2$$

**Problem 15.3** Find the Z parameters of the RC ladder network shown in Fig. 15.44.



Fig. 15.44

**Solution** With port *b-b'* open circuited and assuming mesh currents with  $V_1(S)$  as the voltage at *a-a'*, the corresponding network is shown in Fig. 15.45(a).

The KVL equations are as follows

$$V_2(S) = I_3(S) \tag{15.27}$$

$$I_3(S) \times \left(2 + \frac{1}{S}\right) = I_1(S)$$
 (15.28)

$$\left(1 + \frac{1}{S}\right)I_1(S) - I_3(S) = V_1(S)$$
(15.29)

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From Eq. 15.28,  $I_3(S) = I_1(S) \left(\frac{S}{1+2S}\right)$  $\left(\frac{S+1}{S}\right)I_1(S) - I_1(S)\frac{S}{1+2S} = V_1(S)$ From Eq. 15.29  $I_1(S)\left(\frac{1+S}{S} - \frac{S}{1+2S}\right) = V_1(S)$  $I_1(S)\left(\frac{S^2 + 3S + 1}{S(1 + 2S)}\right) = V_1(S)$  $Z_{11} = \frac{V_1(S)}{I_1(S)} \bigg|_{I_2 = 0} = \frac{\left(S^2 + 3S + 1\right)}{S(1 + 2S)}$  $V_2(S) = I_3(S) = I_1(S) \frac{S}{1+2S}$ Also

$$Z_{21} = \frac{V_2(S)}{I_1(S)}\Big|_{I_2 = 0} = \frac{S}{1 + 2S}$$

With port *a-a'* open circuited and assuming mesh currents with  $V_2(S)$  as the voltage as b-b', the corresponding network is shown in Fig. 15.45(b).



Fig. 15.45(b)

The KVL equations are as follows

$$V_1(S) = I_3(S) \tag{15.30}$$

$$\left(2 + \frac{1}{S}\right)I_3(S) = I_2(S) \tag{15.31}$$

$$V_2(S) = I_2(S) - I_3(S)$$
(15.32)

From Eq. 15.31  $I_3(S) = I_2(S) \left( \frac{S}{2S+1} \right)$ From Eq. 15.32  $V_2(S) = I_2(S) - I_2(S) \left(\frac{S}{2S+1}\right)$  $V_2(S) = I_2(S) \left( 1 - \frac{S}{2S+1} \right)$  $Z_{22} = \frac{V_2(S)}{I_2(S)} \bigg|_{I_1(S) = 0} = \frac{S+1}{2S+1}$  $V_1(S) = I_3(S) = I_2(S) \left(\frac{S}{2S+1}\right)$ 

Also

$$Z_{12} = \frac{V_1(S)}{I_2(S)} \bigg|_{I_1(S) = 0} = \left(\frac{S}{2S + 1}\right)$$

The describing equations are

$$V_{1}(S) = \left[\frac{S^{2} + 3S + 1}{3(2S + 1)}\right] I_{1} + \left[\frac{S}{2S + 1}\right] I_{2}$$
$$V_{2}(S) = \left[\frac{S}{2S + 1}\right] I_{1} + \left[\frac{S + 1}{2S + 1}\right] I_{2}$$

Problem 15.4 Find the transmission parameters for the circuit shown in Fig. 15.46.



Fig. 15.46

**Solution** Recalling Eqs 15.5 and 15.6, we have

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

When port *b-b'* is short circuited with  $V_1$  across *a-a'*,  $V_2 = 0$   $B = \frac{-V_1}{I_2}$  and the circuit is as shown in Fig. 15.47(a)



### Fig. 15.47(a)

$$-I_2 = \frac{V_1}{2} I_1 = V_1$$
$$B = 2 \Omega$$
$$D = \frac{-I_1}{I_2} = 2$$

...

When port *b-b'* is open with  $V_1$  across *a-a'*,  $I_2 = 0$   $A = V_1/V_2$  and the circuit is as shown in Fig. 15.47(b), where  $V_1$  is the voltage across the 2  $\Omega$  resistor across port *a-a'* and  $V_2$  is the voltage across the 2  $\Omega$  resistor across port *b-b'* when  $I_2 = 0$ .



Fig. 15.47(b)

From Fig. 15.47(b),  $I_Y = \frac{V_1}{4}$  $V_2 = 2 \times I_Y = \frac{V_1}{2}$ A = 2From Fig. 15.47(b)  $I_x = \frac{V_1}{2}$  $C = \frac{I_1}{V_2}$ 

where  $I_1 = \frac{3V_1}{4}$ 

Therefore

**Problem 15.5** Find *h* parameters for the network in Fig. 15.48.

 $C = \frac{3}{2} \nabla$ 





**Solution** When  $V_2 = 0$  the network is as shown in Fig. 15.49.



Fig. 15.49

$$h_{11} = \frac{V_1}{I_1} \Big|_{V_2 = 0} = 2 \Omega$$
$$h_{21} = \frac{I_2}{I_1} \Big|_{V_2 = 0}; I_2 = -I_1$$
$$h_{21} = -1$$

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When

 $I_{1} = 0; h_{12} = \frac{V_{1}}{V_{2}} \Big|_{I_{1} = 0}; h_{22} = \frac{I_{2}}{V_{2}} \Big|_{I_{1} = 0}$  $V_{1} = I_{2} 4, V_{2} = I_{2} 4$  $h_{12} = 1 h_{22} = \frac{1}{4} \nabla$ 

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**Problem 15.6** For the hybrid equivalent circuit shown in Fig. 15.50, (a) determine the current gain, and (b) determine the voltage gain.



Fig. 15.50

**Solution** From port 2-2' we can find

$$I_2 = \frac{(25I_1)(0.05 \times 10^6)}{(1500 + 0.05 \times 10^6)}$$

(a) current gain 
$$\frac{I_2}{I_1} = \frac{1.25 \times 10^6}{0.0515 \times 10^6} = 24.3$$

(b) applying KVL at port 1-1'

$$V_{1} = 500 I_{1} + 2 \times 10^{-4} V_{2}$$

$$I_{1} = \frac{V_{1} - 2 \times 10^{-4} V_{2}}{500}$$
(15.33)

Applying KCL at port 2-2'

$$I_2 = 25I_1 + \frac{V_2}{0.05} \times 10^{-6}$$
$$I_2 = \frac{-V_2}{1500}$$

also

$$\frac{-V_2}{1500} = 25I_1 + \frac{V_2}{0.05} \times 10^{-6}$$

Substituting the value of  $I_1$  from Eq. 15.33, in the above equation, we get

$$\frac{-V_2}{1500} = 25 \left( \frac{V_1 - 2 \times 10^{-4} V_2}{500} \right) + \frac{V_2}{0.05} \times 10^{-6}$$
$$- 6.6 \times 10^{-4} V_2 = 0.05 V_1 - 0.1 \times 10^{-4} V_2 + 0.2 \times 10^{-4} V_2$$
$$\frac{V_2}{V_1} = -73.89$$

...

The negative sign indicates that there is a 180° phase shift between input and output voltage.

**Problem 15.7** The hybrid parameters of a two-port network shown in Fig. 15.51 are  $h_{11} = 1$  K;  $h_{12} = 0.003$ ;  $h_{21} = 100$ ;  $h_{22} = 50 \ \mu$ T. Find  $V_2$  and Z parameters of the network.





Solution  $V_1 = h_{11} I_1 + h_{12} V_2$ (15.34) $I_2 = h_{21} I_1 + h_{22} V_2$ (15.35) $V_2 = -I_2 2000$ At port 2-2' Substituting in Eq. 15.35, we have  $I_2 = h_{21}I_1 - h_{22}I_2$  2000 
$$\begin{split} I_2 \left( 1 + h_{22} \ 2000 \right) &= h_{21} \ I_1 \\ I_2 (1 + 50 \times 10^{-6} \times 2000) &= 100 \ I_1 \end{split}$$
 $I_2 = \frac{100 I_1}{1.1}$ Substituting the value of  $V_2$  in Eq. 15.34, we have  $V_1 = h_{11} I_1 - h_{12} I_2 2000$ Also at port 1-1',  $V_1 = V_S - I_1 500$  $V_S - I_1 500 = h_{11} I_1 - h_{12} \frac{100 I_1}{1 1} \times 2000$ *.*..  $(10 \times 10^{-3}) - 500 I_1 = 1000 I_1 - 0.003 \times \frac{100}{1.1} I_1 \times 2000$  $954.54I_1 = 10 \times 10^{-3}$  $I_1 = 10.05 \times 10^{-6} \text{ A}$  $V_1 = V_S - I_1 \times 500$  $= 10 \times 10^{-3} - 10.5 \times 10^{-6} \times 500 = 4.75 \times 10^{-3} \text{ V}$  $V_2 = \frac{V_1 - h_{11} I_1}{h_{12}}$  $V_2 = \frac{4.75 \times 10^{-3} - 1000 \times 10.5 \times 10^{-6}}{0.003} = -1.916 \text{ V}$ 

(b) Z parameters of the network can be found from Table 15.1.

$$Z_{11} = \frac{\Delta_h}{h_{22}} = \frac{h_{11}h_{22} - h_{21}h_{12}}{h_{22}} = \frac{1 \times 10^3 \times 50 \times 10^{-6} - 100 \times 0.003}{50 \times 10^{-6}}$$

$$= -5000 \Omega$$

$$Z_{12} = \frac{h_{12}}{h_{22}} = \frac{0.003}{50 \times 10^{-6}} = 60 \Omega$$

$$Z_{21} = \frac{-h_{21}}{h_{22}} = \frac{-100}{50 \times 10^{-6}} = -2 \times 10^{6} \Omega$$

$$Z_{22} = \frac{1}{h_{22}} = 20 \times 10^{3} \Omega$$

**Problem 15.8** The Z parameters of a two port network shown in Fig. 15.52 are  $Z_{11} = Z_{22} = 10 \Omega$ ;  $Z_{21} = Z_{12} = 4 \Omega$ . If the source voltage is 20 V, determine  $I_1$ ,  $V_2$ ,  $I_2$  and input impedance.



Fig. 15.52

**Solution** Given  $V_1 = V_s = 20 \text{ V}$ From Section 15.11.1,  $V_1 = I_1 \left( Z_{11} - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}} \right)$ where  $Z_L = 20 \Omega$ 

:.

$$20 = I_1 \left( 10 - \frac{4 \times 4}{20 + 10} \right)$$
  

$$I_1 = 2.11 \text{ A}$$
  

$$I_2 = -I_1 \frac{Z_{21}}{Z_L + Z_{22}} = -2.11 \times \frac{4}{20 + 10} = -0.281 \text{ A}$$

At port 2-2'

Input impedance

$$V_2 = -I_2 \times 20 = 0.281 \times 20 = 5.626 \text{ V}$$
$$= \frac{V_1}{I_1} = \frac{20}{2.11} = 9.478 \Omega$$

**Problem 15.9** The *Y* parameters of the two-port network shown in Fig. 15.53 are  $Y_{11} = Y_{22} = 6 \ rmodel{eq:Y11}$ ;  $Y_{12} = Y_{21} = 4 \ rmodel{eq:Y11}$ 

(a) determine the driving point admittance at port 2-2' if the source voltage is 100 V and has an impedance of 1 ohm.



Fig. 15.53

**Solution** From Section 15.11.2,

$$\frac{I_2}{V_2} = \frac{Y_{22} Y_S + Y_{22} Y_{11} - Y_{21} Y_{12}}{Y_S + Y_{11}}$$

where  $Y_s$  is the source admittance = 1  $\mho$ 

:. The driving point admittance =  $\frac{6 \times 1 + 6 \times 6 - 4 \times 4}{1 + 6} = 3.714$   $_{\odot}$ 

Or the driving point impedance at port 2-2' =  $\frac{1}{3.714} \Omega$ 

**Problem 15.10** Obtain the *Z* parameters for the two-port unsymmetrical lattice network shown in Fig. 15.54.

\*\*



Fig. 15.54

**Solution** From Section 15.12, we have

$$Z_{11} = \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d} = \frac{(1+3)(2+5)}{1+3+5+2} = 2.545 \,\Omega$$
$$Z_{21} = \frac{Z_b Z_c - Z_a Z_d}{Z_a + Z_b + Z_c + Z_d} = \frac{3 \times 5 - 1 \times 2}{11} = 1.181 \,\Omega$$
$$Z_{21} = Z_{12}$$

$$Z_{22} = \frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d} = \frac{(1+5)(2+3)}{11} = 2.727 \,\Omega$$

**Problem 15.11** For the ladder two-port network shown in Fig. 15.55, find the open circuit driving point impedance at port 1-2.











Then the open circuit driving point impedance at port 1-2 is given by

$$Z_{11} = (s+1) + \frac{1}{s + \frac{1}{(s+1) + \frac{1}{s + \frac{1}{(s+1) + \frac{1}{s}}}}}$$
$$= \frac{s^{6} + 3s^{5} + 8s^{4} + 11s^{3} + 11s^{2} + 6s + 1}{s^{5} + 2s^{4} + 5s^{3} + 4s^{2} + 3s}$$

**Problem 15.12** For the bridged *T* network shown in Fig. 15.57, find the driving point admittance  $y_{11}$  and transfer admittance  $y_{21}$  with a 2  $\Omega$  load resistor connected across port 2.

....



Fig. 15.57

**Solution** The corresponding Laplace transform network is shown in Fig. 15.58.



Fig. 15.58

The loop equations are

$$I_{1}\left(1+\frac{1}{s}\right)+I_{2}\left(\frac{1}{s}\right)-I_{3}=V_{1}$$
$$I_{1}\left(\frac{1}{s}\right)+I_{2}\left(1+\frac{1}{s}\right)+I_{3}=0$$
$$I_{1}\left(-1\right)+I_{2}+I_{3}\left(2+\frac{1}{s}\right)=0$$

Therefore,

$$\Delta = \begin{vmatrix} \left(1 + \frac{1}{s}\right) & \frac{1}{s} & -1 \\ \frac{1}{s} & 1 + \frac{1}{s} & 1 \\ -1 & 1 & 2 + \frac{1}{s} \end{vmatrix} = \frac{s+2}{s^2}$$

1

Т

Similarly,

$$\Delta_{11} = \begin{vmatrix} \left(1 + \frac{1}{s}\right) & \frac{1}{s} \\ 1 & \left(2 + \frac{1}{s}\right) \end{vmatrix} = \frac{s^2 + 3s + 1}{s^2}$$
$$\Delta_{12} = \begin{vmatrix} \frac{1}{s} & +1 \\ +1 & \left(2 + \frac{1}{s}\right) \end{vmatrix} = \frac{s^2 + 2s + 1}{s^2}$$

1 )

and

$$y_{11} = \frac{\Delta_{11}}{\Delta} = \frac{s^2 + 3s + 1}{s + 2}$$

Hence,

 $y_{21} = \frac{\Delta_{12}}{\Delta} = \frac{-(s^2 + 2s + 1)}{s + 2}$ 

and

Problem 15.13 For the two port network shown in Fig. 15.59, determine the *h*-parameters. Using these parameters calculate the output (Port 2) voltage,  $V_2$ , when the output port is terminated in a 3  $\Omega$  resistance and a 1V(dc) is applied at the input port ( $V_1 = 1$  V).



Fig. 15.59

**Solution** The *h* parameters are defined as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For  $V_2 = 0$ , the circuit is redrawn as shown in Fig. 15.60(a)



Fig. 15.60(a)

15.57

$$\begin{aligned} h_{11} &= \frac{V_1}{I_1} \Big|_{V_2 = 0} = \frac{i_1 \times 1 + 3i_1}{i_1} = 4 \\ h_{21} &= \frac{I_2}{I_1} \Big|_{V_2 = 0} = \frac{i_2}{i_1} = \frac{2i_1 - i_1}{i_1} = 1 \end{aligned}$$

For  $I_1 = 0$ , the circuit is redrawn as shown in Fig. 15.60(b).



Fig. 15.60(b)

$$h_{12} = \frac{V_1}{V_2} = 1; h_{22} = \frac{I_2}{V_2} = \frac{1}{2} = 0.5$$
$$h = \begin{bmatrix} 4 & 1 \\ 1 & 0.5 \end{bmatrix}$$
$$V_1 = 1 V$$
$$V_1 = 4I_1 + V_2$$
$$I_2 = I_1 + 0.5 V_2$$

Hence,

Eliminating  $I_1$  from the above equations and putting

$$V_1 = 1 \text{ and } I_2 = \frac{-V_2}{3} \text{ we get, } V_2 = \frac{-3}{7} \text{ V}$$

**Problem 15.14** Find the current transfer ratio  $\frac{I_2}{I_1}$  for the network shown in Fig. 15.61.



**Solution** By transforming the current source into voltage source, the given circuit can be redrawn as shown in Fig. 15.62.



Fig. 15.62

Applying Kirchhoff's nodal analysis

$$\frac{V_1 - (I_1 + 2I_3)}{1} + \frac{V_1}{1} + \frac{V_1 - V_2}{2} = 0$$
$$\frac{V_2 - V_1}{2} - \frac{I_1}{2} - I_2 = 0$$

and

Putting  $V_1 = -I_3$  and  $V_2 = -I_2$ 

The above equations become

$$-I_3 - I_1 - 2I_3 - I_3 + \frac{I_2 - I_3}{2} = 0$$
$$\frac{I_2 - I_3}{2} - \frac{I_1}{2} - I_2 = 0$$
$$I_1 0.5I_2 - 4.5I_2 = 0$$

and

or and

$$-0.5 I_1 - 1.5 I_2 + 0.5 I_3 = 0$$

By eliminating  $I_3$ , we get

$$\frac{I_2}{I_1} = \frac{-5.5}{13} = -0.42$$



# **PRACTICE PROBLEMS**



15.1 Find the Z parameters of the network shown in Fig. 15.63.



Fig. 15.63

15.2 Find the transmission parameters for the R-C network shown in Fig. 15.64.





15.3 Find the inverse transmission parameters for the network in Fig. 15.65.



Fig. 15.65

15.4 Calculate the overall transmission parameters for the cascaded network shown in Fig. 15.66.





15.5 For the two-port network shown in Fig. 15.67, find the *h* parameters and the inverse *h* parameters.



Fig. 15.67

15.6 Determine the impedance parameters for the T network shown in Fig. 15.68 and draw the Z parameter equivalent circuit.



Fig. 15.68

15.7 Determine the admittance parameters for the  $\pi$ -network shown in Fig. 15.69 and draw the Y parameter equivalent circuit.





15.8 Determine the impedance parameters and the transmission parameters for the network in Fig. 15.70.



Fig. 15.70

15.9 For the hybrid equivalent circuit shown in Fig. 15.71, determine (a) the input impedance (b) the output impedance.



Fig. 15.71

15.10 Determine the input and output impedances for the Z parameter equivalent circuit shown in Fig. 15.72



Fig. 15.72

15.11 The hybrid parameters of a two-port network shown in Fig. 15.73 are  $h_{11} = 1.5$  K;  $h_{12} = 2 \times 10^{-3}$ ;  $h_{21} = 250$ ;  $h_{22} = 150 \times 10^{-6}$   $\heartsuit$  (a) Find  $V_2$  (b). Draw the Z parameter equivalent circuit.





15.12 The Z parameters of a two-port network shown in Fig. 15.74 are  $Z_{11} = 5$  $\Omega$ ;  $Z_{12} = 4 \Omega$ ;  $Z_{22} = 10 \Omega$ ;  $Z_{21} = 5 \Omega$ . If the source voltage is 25 V, determine  $I_1$ ,  $V_2 I_2$ , and the driving point impedance at the input port.





15.13 Obtain the image parameters of the symmetric lattice network given in Fig. 15.75.



Fig. 15.75

- 15.14 Determine the Z parameters and image parameters of a symmetric lattice network whose series arm impedance is 10  $\Omega$  and diagonal arm impedance is 20  $\Omega$ .
- 15.15 For the network shown in Fig. 15.76, determine all four open circuit impedance parameters.



15.16 For the network shown in Fig. 15.77, determine  $y_{12}$  and  $y_{21}$ 





15.17 For the network shown in Fig.15.78, determine *h* parameters at  $\omega = 10^8$  rad/sec.



Fig. 15.78

15.18 For the network shown in Fig. 15.79, determine y parameters.



Fig. 15.79



1. A two-port network is simply a network inside a black box, and the network has only

- (a) two terminals
- (b) two pairs of accessible terminals
- (c) two pairs of ports
- 2. The number of possible combinations generated by four variables taken two at a time in a two-port network is (c) six
  - (a) four (b) two
- 3. What is the driving-point impedance at port one with port two open circuited for the network in Fig. 15.80?



Fig. 15.80

- (b) 5 Ω (a)  $4 \Omega$ (c)  $3 \Omega$ 4. What is the transfer impedance of the two-port network shown in Fig. 15.80?
  - (a)  $1 \Omega$ (b)  $2 \Omega$ (c)  $3 \Omega$
- 5. If the two-port network in Fig. 15.80 is reciprocal or bilateral then (a)  $Z_{11} = Z_{22}$ (b)  $Z_{12} = Z_{21}$  (c)  $Z_{11} = Z_{12}$
- 6. What is the transfer admittance of the network shown in Fig. 15.81.





- (a) -2 75 (b) - 3 75 (c)  $-4 \ O$ 7. If the two-port network in Fig. 15.81 is reciprocal then
  - (a)  $Y_{11} = Y_{22}$ (b)  $Y_{12} = Y_{22}$ (c)  $Y_{12} = Y_{11}$
- 8. In describing the transmission parameters
  - (a) the input voltage and current are expressed in terms of output voltage and current
  - (b) the input voltage and output voltage are expressed in terms of output current and input current.

- (c) the input voltage and output current are expressed in terms of input current and output voltage.
- 9. If  $Z_{11} = 2 \Omega$ ;  $Z_{12} = 1 \Omega$ ;  $Z_{21} = 1 \Omega$  and  $Z_{22} = 3 \Omega$ , what is the determinant of admittance matrix.

(a) 5 (b) 
$$1/5$$
 (c) 1

10. For a two-port bilateral network, the three transmission parameters are

given by 
$$A = \frac{6}{5}$$
;  $B = \frac{17}{5}$  and  $C = \frac{1}{5}$ , what is the value of *D*?  
(a) 1 (b)  $\frac{1}{5}$  (c)  $\frac{7}{5}$ 

11. The impedance matrices of two, two-port networks are given by  $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ 

and  $\begin{bmatrix} 15 & 5\\ 5 & 25 \end{bmatrix}$ . If the two networks are connected in series. What is the impedance matrix of the combination?

(a) 
$$\begin{bmatrix} 3 & 5\\ 2 & 25 \end{bmatrix}$$
 (b)  $\begin{bmatrix} 18 & 7\\ 7 & 28 \end{bmatrix}$  (c)  $\begin{bmatrix} 15 & 2\\ 5 & 3 \end{bmatrix}$ 

12. The admittance matrices of two two-port networks are given by  $\begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 5/8 \end{bmatrix}$  and  $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$ . If the two networks are connected in parallel, what is the admittance matrix of the combination?

(a) 
$$\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$$
 (b)  $\begin{bmatrix} 2 & -1 \\ -1 & 5/2 \end{bmatrix}$  (c)  $\begin{bmatrix} 3/2 & -3/4 \\ -3/4 & 15/8 \end{bmatrix}$ 

13. If the Z parameters of a two-port network are  $Z_{11} = 5 \Omega Z_{22} = 7 \Omega$ ;  $Z_{12} = Z_{21} = 3 \Omega$  then the A, B, C, D parameters are respectively given by

(a) 
$$\frac{5}{3}; \frac{26}{3}; \frac{1}{3}; \frac{7}{3}$$
 (b)  $\frac{10}{3}; \frac{52}{3}; \frac{2}{3}; \frac{14}{3}$  (c)  $\frac{15}{3}; \frac{78}{3}; \frac{3}{3}; \frac{21}{3}$ 

14. For a symmetric lattice network the value of the series impedance is 3  $\Omega$  and that of the diagonal impedance is 5  $\Omega$ , then the *Z* parameters of the network are given by

(a) 
$$Z_{11} = Z_{22} = 2 \Omega$$
  
 $Z_{12} = Z_{21} = 1/2 \Omega$   
(b)  $Z_{11} = Z_{22} = 4 \Omega$   
 $Z_{12} = Z_{21} = 1 \Omega$   
(c)  $Z_{11} = Z_{22} = 8 \Omega$   
 $Z_{12} = Z_{21} = 2 \Omega$ 

15. For a two-port network to be reciprocal.

(a) 
$$Z_{11} = Z_{22}$$
  
(b)  $y_{21} = y_{22}$   
(c)  $h_{21} = -h_{12}$   
(d)  $AD - BC = 0$ 

- 16. Two-port networks are connected in cascade. The combination is to be represented as a single two port network. The parameters of the network are obtained by adding the individual
  - (a) Z parameter matrix (c)  $A^1 B^1 C^1 D^1$  matrix

- (b) *h* parameter matrix

- (d) *ABCD* parameter matrix
- 17. The *h* parameters  $h_{11}$  and  $h_{12}$  are obtained
  - (a) By shorting output terminals
  - (c) By shorting input terminals
- (b) By opening input terminals
- (d) By opening output terminals
- 18. Which parameters are widely used in transmission line theory
  - (a) Z parameters

- (b) Y parameters
- (c) *ABCD* parameters
- (d) *h* parameters



## A.1 INTRODUCTION

In most of the cases, the response of linear circuits to sinusoidal excitations can be found easily. A function f(t) is said to be periodic, if the process repeats itself every *T* sec, so that we have

$$f(t+T) = f(t)$$

If a periodic function f(t) is to have a Fourier series, it must satisfy the following Dirichlet conditions.

(i) f(t) must be bounded and possess a finite number of discontinuities.

(ii) f(t) must have a finite number of maxima and minima, and

(iii) f(t) must have a finite average value.

The function f(t) can be represented over a complete period from  $t = -\infty$  to  $t = +\infty$ , except at the discontinuities, by a series of simple harmonic functions, the frequencies of which are integral multiples of the fundamental frequency. A series in this form is called a Fourier Series.

## A.2 DEFINITIONS AND DERIVATIONS

A periodic function f(t) can be expressed in the complex form

$$f(t) = a_{0} + a_{1} e^{j\omega t} + a_{2} e^{2j\omega t} + \dots + a_{n} e^{nj\omega t} + \dots + a_{-1} e^{-j\omega t} + a_{-2} e^{-2j\omega t} + \dots + a_{-n} e^{-nj\omega t} + \dots$$

$$f(t) = \sum_{n=-\infty}^{\infty} a_{n} e^{jn\omega t}$$
(1)

or

where

$$\omega = \frac{2\pi}{T}$$

To determine  $a_0$ , integrating both sides of Eq. 1 over one complete period, we get

$$\int_{0}^{2\pi/\omega} f(t)dt = \int_{0}^{2\pi/\omega} \left( \sum_{n=-\infty}^{\infty} a_n e^{jn\omega t} \right) dt$$
$$= \sum_{n=-\infty}^{\infty} a_n \int_{0}^{2\pi/\omega} e^{jn\omega t} dt$$
(2)

$$\int_{0}^{2\pi/\omega} f(t)dt = \int_{0}^{2\pi/\omega} a_0 dt = a_0 \frac{2\pi}{\omega} = a_0 T$$
(3)

or

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$
 (4)

To determine the other term,  $a_n$ , we multiply both sides of Eq. 1 by  $e^{-jn\omega t}$ , and integrate from 0 to  $2\pi/\omega$  to obtain

$$\int_{0}^{T} f(t) e^{-jn\omega t} dt = a_n T$$
(5)

$$a_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega t} dt$$
(6)

Similarly, we have from Eq. 6, the relation

$$a_{-n} = \frac{1}{T} \int_{0}^{T} f(t) e^{jn\omega t} dt$$
 (7)

Then Eq. 1 may be written in the form,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n e^{jn\omega t} + a_{-n} e^{-jn\omega t})$$
(8)

By using Euler's relation, the function f(t) may be written in the form

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n + a_{-n}) \cos n\omega t + \sum_{n=1}^{\infty} j(a_n - a_{-n}) \sin n\omega t$$
(9)

Now let

$$A_n = a_n + a_{-n}; \quad B_n = j(a_n - a_{-n}); \quad \frac{A_0}{2} = a_0$$

We get

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega t + \sum_{n=1}^{\infty} B_n \sin n\omega t$$
(10)

Therefore, we have

$$A_n = a_n + a_{-n} = \frac{1}{T} \int_0^T f(t) \left( e^{jn\omega t} + e^{-jn\omega t} \right) dt$$
  

$$= \frac{2}{T} \int_0^T f(t) \cos n\omega t \, dt \qquad (11)$$
  

$$B_n = j(a_n - a_{-n})$$
  

$$= \frac{1}{T} \int_0^T f(t) j(e^{jn\omega t} - e^{-jn\omega t}) \, dt$$
  

$$= \frac{2}{T} \int_0^T f(t) \sin n\omega t \, dt \qquad (12)$$

## **Example of Fourier Series**

To determine the Fourier series for the square wave shown in Fig. A.1.





The function f(t) is represented as

$$f(t) = 20, 0 < \omega t < \pi$$
  
= -20,  $\pi < \omega t < 2\pi$ 

Since the average value of the wave is zero, the term  $A_0/2 = 0$ The cosine coefficients are obtained as follows.

$$a_n = \frac{1}{\pi} \left\{ \int_0^{\pi} 20 \cos n\omega t \, d(\omega t) + \int_{\pi}^{2\pi} (-20) \cos n\omega t \, d(\omega t) \right.$$
$$= \frac{20}{\pi} \left\{ \left[ \frac{1}{n} \sin n\omega t \right]_0^{\pi} - \left[ \frac{1}{n} \sin n\omega t \right]_{\pi}^{2\pi} \right\} = 0 \text{ for all } n$$

Thus, the series contains no cosine terms.

----

To determine the sine terms

$$b_n = \frac{1}{\pi} \left\{ \int_0^{\pi} 20 \cos n\omega t \, d(\omega t) + \int_{\pi}^{2\pi} (-20) \cos n\omega t \, d(\omega t) \right.$$
$$= \frac{20}{\pi} \left\{ \left[ \frac{-1}{n} \cos n\omega t \right]_0^{\pi} + \left[ \frac{1}{n} \cos n\omega t \right]_{\pi}^{2\pi} \right\}$$
$$= \frac{20}{\pi n} \left[ -\cos n\pi + \cos 0 + \cos n2\pi - \cos n\pi \right] = \frac{40}{\pi n} \left( 1 - \cos n\pi \right)$$

Then

$$b_n = \frac{80}{\pi n}$$
 for  $n = 1, 3, 5, \cdots$   
= 0 for  $n = 2, 4, 6, \cdots$ 

The series for the square wave is

$$f(t) = \frac{80}{\pi} \sin \omega t + \frac{80}{3\pi} \sin 3\omega t + \frac{80}{5} \sin 5\omega t + \cdots$$

The Fourier series contains only odd harmonic sine terms.


## **B.1 FOURIER INTEGRAL**

In this section, the limiting form of the Fourier series as the period T is made to approach infinity. Then the resulting function is called the Fourier integral representation, or simply, the Fourier integral of f(t).

Consider the complex Fourier-series expansion of the periodic function f(t);

$$f(t) = \sum_{\substack{n = -\infty \\ \pi}}^{\infty} a_n e^{jn\omega t} \quad T = \frac{2\pi}{\omega}$$
(1)

where

$$a_{n} = \frac{1}{T} \int_{0}^{T} f(x) e^{-jn\omega x} dx$$
  
$$a_{n} = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-jn\omega x} dx, \quad \omega = \frac{2\pi}{T}$$
(2)

or

Substituting this into Eq. 1, we get

$$f(t) = \sum_{n = -\infty}^{\infty} \left[ \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-jn\omega x} dx \right] e^{jn\omega t}$$
$$= \sum_{n = -\infty}^{\infty} \left[ \frac{1}{T} \int_{-T/2}^{T/2} f(x) \exp\left[\left(\frac{2\pi nj}{T}\right)(t-x)\right] dx \right]$$
(3)

Let  $1/T = \Delta s$ Then

$$f(t) = \sum_{n = -\infty}^{\infty} \Delta s \int_{-T/2}^{T/2} f(x) e^{2\pi n j(t-x)\Delta s} dx$$
(4)

Now the definite integral  $\int_{0}^{\infty} \tau(s) ds$  may be defined as the limit, as  $\Delta s$  approaches zero, of the sum.

$$\sum_{n=0}^{\infty} \tau(n\,\Delta s)\,\Delta s \tag{5}$$

Also we have

$$\int_{-\infty}^{\infty} \tau(s) \, ds = \int_{-\infty}^{0} \tau(s) \, ds + \int_{0}^{\infty} \tau(s) \, ds$$
$$= \lim_{\Delta s \to 0} \sum_{n = -\infty}^{\infty} \tau(n \, \Delta s) \, \Delta s \tag{6}$$

From this it follows that as T grows beyond all bounds, the expression in Eq. 4 passes over into the Fourier integral, or

$$f(t) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} f(x) e^{2\pi j s(t-x)} dx$$
$$= \int_{-\infty}^{\infty} e^{2\pi j s t} ds \int_{-\infty}^{\infty} f(x) e^{-2\pi j s x} dx$$
(7)

This is the general Fourier integral representation. Another form of the Fourier integral may be obtained from Eq. 7 by using Euler's relation on the complex exponentials. We thus obtain the real form of the Fourier integral.

$$f(t) = 2 \int_{0}^{\infty} ds \int_{-\infty}^{\infty} f(x) \cos 2\pi s (t-x) dx$$
(8)

## **B.2 FOURIER TRANSFORMS**

Equation 7 can be written in slightly different form. Let us introduce another variable

$$\omega = 2\pi s$$

In terms of the variable  $\omega$ , Eq. 7 is transformed to

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$
(1)

If we write

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$
<sup>(2)</sup>

Then Eq. 1 can be written as

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{j\omega t} d\omega$$
(3)

The relations in Eqs. 2 and 3 are known as Fourier transforms. The expression  $g(\omega)$  in Eq. 2 is usually called the Fourier transform of the function f(t).



## C.1 DEFINITION OF *j* FACTOR

*j* is used in all electrical circuits to denote imaginary numbers. Alternate symbol for *j* is  $\sqrt{-1}$ , and is known as *j* factor or *j* operator.

Thus

$$\sqrt{-1} = \sqrt{(-1)(1)} = j(1)$$
  
$$\sqrt{-2} = \sqrt{(-1) 2} = j\sqrt{2}$$
  
$$\sqrt{-4} = \sqrt{(-1) 4} = j2$$
  
$$\sqrt{-5} = \sqrt{(-1) 5} = j\sqrt{5}$$

Since *j* is defined as  $\sqrt{-1}$ , it follows that  $(j)(j) = j^2 = (\sqrt{-1})(\sqrt{-1}) = -1$ 

∴ Since

$$(j3) (j3) = j^2 3^2$$
  
 $j^2 = -1$   
 $(j3) (j3) = -9$ 

(i.e.) the square root of -9 is j3

Therefore *j*3 is a square root of -9

The use of j factor provides a solution to an equation of the form  $x^2 = -4$ 

Thus

$$x = \sqrt{-4} = \sqrt{(-1)4}$$
$$x = (\sqrt{-1})2$$
$$i = \sqrt{-1}, x = i2$$

With

The real number 9 when multiplied three times by *j* becomes -j9.

$$(j) (j) (j) = (j)^2 j = (-1)j = -j$$

Finally when real number 10 is multiplied four times by *j*, it becomes 10

$$j = +j$$
  

$$j^{2} = (j) (j) = -1$$
  

$$j^{3} = (j^{2}) (j) = (-1)j = -j$$
  

$$j^{4} = (j^{2}) (j)^{2} = (-1) (-1) = +1$$

**Example C.1** Express the following imaginary numbers using the *j* factor

(a)  $\sqrt{-13}$  (b)  $\sqrt{-9}$  (c)  $\sqrt{-29}$  (d)  $\sqrt{-49}$ 

Solution

(a) 
$$\sqrt{-13} = \sqrt{(-1)(13)} = j\sqrt{13}$$
  
(b)  $\sqrt{-9} = \sqrt{(-1)9} = i3$ 

(c) 
$$\sqrt{-29} = \sqrt{(-1)29} = j\sqrt{29}$$

(d) 
$$\sqrt{-49} = \sqrt{(-1)}(49) = j7$$

#### C.2 RECTANGULAR AND POLAR FORMS

A complex number (a + jb) can be represented by a point whose coordinates are (a, b). Thus, the complex number 3 + j4 is located on the complex plane at a point having rectangular coordinates (3, 4).



Fig. C.1

This method of representing complex numbers is known as the rectangular form. In ac analysis, impedances, currents and voltages are commonly represented by complex numbers that may be either in the rectangular form or in the polar form. In Fig. C.1 the complex number in the polar form is represented. Here *R* is the magnitude of the complex number and  $\phi$  is the angle of the complex number. Thus, the polar form of the complex number is  $R \angle \phi$ . If the rectangular coordinates (a, b) are known, they can be converted into polar

::

form. Similarly, if the polar coordinates  $(R, \phi)$  are known, they can be converted into rectangular form.

In Fig. C.1, a and b are the horizontal and vertical components of the vector

*R*, respectively. From Fig. C.1, *R* can be found as  $R = \sqrt{a^2 + b^2}$ . Also from Fig. C.1,

$$\sin \phi = \frac{b}{R}$$
$$\cos \phi = \frac{a}{R}$$
$$\tan \phi = \frac{b}{a}$$
$$\phi = \tan^{-1} \frac{b}{a}$$
$$R = \sqrt{a^2 + b^2}$$

**Example C.2** Express 10 ∠53.1° in rectangular form.

Solution

$$a + jb = R (\cos \phi + j \sin \phi)$$

$$R = 10 \angle \phi = 53.1^{\circ}$$

$$a + jb = R \cos \phi + jR \sin \phi$$

$$R \cos \phi = 10 \cos 53.1^{\circ} = 6$$

$$R \sin \phi = 10 \sin 53.1^{\circ} = 8$$

$$a + jb = 6 + j 8$$

**Example C.3** Express 3 + j4 in polar form

#### Solution

$$R\cos\phi = 3\tag{1}$$

$$R\sin\phi = 4\tag{2}$$

Squaring and adding the above equations, we get

$$R^{2} = 3^{2} + 4^{2}$$
$$R = \sqrt{3^{2} + 4^{2}} = 5$$

From (1) and (2),  $\tan \phi = 4/3$ 

$$\phi = \tan^{-1} \frac{4}{3} = 53.13^{\circ}$$

Hence the polar form is  $5 \angle 53.13^{\circ}$ 

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## C.3 OPERATIONS WITH COMPLEX NUMBERS

The basic operations such as addition, subtraction, multiplication and division can be performed using complex numbers.

*Addition* It is very easy to add two complex numbers in the rectangular form. The real parts of the two complex numbers are added and the imaginary parts of the two complex numbers are added. For example,

$$(3+j4) + (4+j5) = (3+4) + j(4+5)$$
  
= 7 + j9

**Subtraction** Subtraction can also be performed by using the rectangular form. To subtract, the sign of the subtrahand is changed and the components are added. For example, subtract 5 + j3 from 10 + j6:

$$10 + j6 - 5 - j3 = 5 + j3$$

**Multiplication** To multiply two complex numbers, it is easy to operate in polar form. Here we multiply the magnitudes of the two numbers and add the angles algebraically. For example, when we multiply  $3 \angle 30^{\circ}$  with  $4 \angle 20^{\circ}$ , it becomes (3) (4)  $\angle 30^{\circ} + 20^{\circ} = 12 \angle 50^{\circ}$ .

*Division* To divide two complex numbers, it is easy to operate in polar form. Here we divide the magnitudes of the two numbers and subtract the angles. For example, the division of

$$9 \angle 50^{\circ}$$
 by  $3 \angle 15^{\circ} = \frac{9 \angle 50^{\circ}}{3 \angle 15^{\circ}} = 3 \angle 50^{\circ} - 15^{\circ} = 3 \angle 35^{\circ}$ 



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# **ANSWERS TO OBJECTIVE-TYPE QUESTIONS**

# Chapter 1

1. (a) 6. (c) 11. (a) 16. (a) 21. (b) 26. (b) 31. (d)	2. (c) 7. (b) 12. (b) 17. (d) 22. (a) 27. (a) 32. (a)	3. (d) 8. (d) 13. (a) 18. (a) 23. (b) 28. (a)	4. (b) 9. (a) 14. (c) 19. (d) 24. (c) 29. (b)	5. (a) 10. (c) 15. (a) 20. (c) 25. (c) 30. (a)
Chapter 2				
1. (b) 6. (a) 11. (c)	2. (a) 7. (c) 12. (b)	3. (a) 8. (a) 13. (c)	4. (b) 9. (b) 14. (a)	5. (b) 10. (c) 15. (a), (d)
Chapter 3				
1. (c) 6. (a) 11. (c) 16. (a)	2. (b) 7. (d) 12. (c) 17. (d)	3. (a) 8. (c) 13. (c)	4. (c) 9. (c) 14. (a)	5. (c) 10. (a) 15. (b)
Chapter 4				
1. (a) 6. (a)	2. (c) 7. (d)	3. (a) 8. (b)	4. (b) 9. (b)	5. (c) 10. (c)

D.2	Ele	ectrical Circuit A	nalysis	
11. (c) 16. (d) 21. (c)	12. (a) 17. (d) 22. (c)	13. (a) 18. (c)	14. (b) 19. (c)	15. (c) 20. (b)
Chapter 5				
1. (a) 6. (c) 11. (a) 16. (a) 21. (c)	2. (b) 7. (b) 12. (c) 17. (c)	3. (d) 8. (d) 13. (a) 18. (c)	4. (c) 9. (a) 14. (b) 19. (c)	5. (a) 10. (b) 15. (a) 20. (d)
Chapter 6				
1. (b) 6. (b) 11. (c)	2. (a) 7. (a) 12. (d)	3. (a) 8. (c) 13. (b)	4. (b) 9. (a) 14. (c)	5. (c) 10. (b)
Chapter 7				
1. (c) 6. (c) 11. (d)	2. (a) 7. (c) 12. (c)	3. (b) 8. (a) 13. (a)	4. (d) 9. (b)	5. (b) 10. (d)
Chapter 8				
1. (d) 6. (b) 11. (a)	2. (a) 7. (a) 12. (c)	3. (c) 8. (b)	4. (a) 9. (a)	5. (b) 10. (d)
Chapter 9				
1. (c) 6. (a) 11. (c)	2. (b) 7. (c) 12. (c)	3. (a) 8. (b) 13. (b)	4. (c) 9. (c)	5. (a) 10. (c)
Chapter 10				
1. (c) 6. (a) 11. (b)	2. (b) 7. (c)	3. (a) 8. (b)	4. (b) 9. (c)	5. (a) 10. (a)
Chapter 11				
1. (a) 6. (d)	2. (b) 7. (b)	3. (b) 8. (c)	4. (a) 9. (b)	5. (c) 10. (d)

		Appendix D		D.3
••••				
Chapter 12				
1. (d)	2. (d)	3. (a)	4. (c)	5. (b)
6. (b)	7. (c)	8. (a)	9. (a)	10. (d)
11. (a)	12. (b)	13. (a)	14. (a)	
Chapter 13				
1. (c)	2. (a)	3. (b)	4. (a)	5. (d)
6. (b)	7. (d)	8. (c)	9. (c)	10. (a)
11. (b)	12. (b)	13. (a)	14. (b)	15. (a)
16. (c)	17. (b)			
Chapter 14				
1. (a)	2. (b)	3. (c)	4. (d)	5. (a)
6. (b)	7. (c)	8. (b)	9. (a)	10. (c)
Chapter 15				
1. (b)	2. (c)	3. (a)	4. (c)	5. (b)
6. (c)	7. (b)	8. (a)	9. (c)	10. (c)
11. (b)	12. (c)	13. (a)	14. (b)	15. (c)
16. (a)	17. (a)	18. (c)		. /

# **ANSWERS TO SELECTED PRACTICE PROBLEMS**

## Chapter 1

1.1	(a) 75 A	(b) 20 A	(c) 2.5 A; 2 S
1.3	3.33 V		
1.5	1.5 mF		
1.7	10 V; 30 V		
1.9	$0.3 \times 10^{-2} \text{ J}$		
1.11	25 V; 5 V		
1.13	$V_1 = V_2 = V_3 = 100 \text{ V}$		
1.15	0.682 A; 4.092 A		
1.17	150 Ω		
1.19	0.7 A; 67.3 V		
1.21	– 4 V, 12 V, 192 W		
1.23	$P_{0.2} = -148.8 \text{ W}, P_{20} =$	$-1090.9 \text{ W}, \text{P}_4 = 743.8 \text{W}, \text{P}_4 = 743.8 \text{W}, \text{P}_4 $	$P_6 = 495.9 V$

# Chapter 2

2.1 2580 W; - 32 V

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2.3	– 60.9 V; 195.7 W
2.5	$I_2 = I_4 = 6.25 \text{ A}; I_3 = 0; I_1 = I_5 = 1.25 \text{ A}; I = 7.5 \text{ A}$
2.9	1.2 A; 4.2 A; 2 A; 3.2 A
2.9	2.65 V
2.11	$1.25 V_1 - 0.72 V_2 = -12.5$
	$-0.75 V_1 + 1.75 V_2 = -2.5 + 4 V_3$
	36.8 W
2.13	18.5 V

### Chapter 3

### **Chapter 4**

- 4.1 5 Hz; 20 Hz; 2 KHz; 100 KHz
- 4.3 15.4 V; 26.57 V; 16.22 V; -16.22 V
- 4.5 12.99 V; 12.99 V; 14.49 V; -7.5 V; -7.5 V

-• **b** 

- 4.7 7.07 mA; 6.37 mA; 10 mA; 20 mA
- 4.9  $V_{RL}$  is 300 V peak to peak sine wave riding on a 200 V dc level.  $I_{max} = 3.5 \text{ A}, V_{av} = 200 \text{ V}$
- 4.11 2.82 cos 100  $\pi t$ ; 20 A; -20 A; 1/300 sec
- 4.13 106.06
- 4.15 27.57
- 4.17 55.25°

## **Chapter 5**

- 5.1 157.4  $\angle -17.6^{\circ}$ ; 17.6° lead, 0.635 A
- 5.3 55.85 ∠ 57.5°; 57.5°

5.5 0.074 A; 41.9°; increases by 19°

- 5.7 944.2  $\Omega$ ; 0.053 A; 3.67°; 16.3 V; 30.7 V
- 5.9 (0.3 j3.15) A; (0.48 + j3.1) A; (0.044 j0.66) A  $V_3 = 9.5$  V;  $V_5 = 15.7$  V;  $V_{10} = 6.61$  V;  $V_{0.1H} = 99.35$  V;  $V_{100 \ \mu F} = 99.8$  V;  $V_{0.5H} = 103.93$  V  $V_{500 \ \mu F} = 4.21$  V



Fig. 5.9

5.11 1.44 A; 7.05°; 
$$V_{100 \ \mu F} = 22.9 \text{ V}; V_{10 \ \Omega} = 14.4 \text{ V}$$
  
 $V_{30 \ \Omega} = 38.93 \text{ V}; V_{0.1\text{H}} = 38.93 \text{ V}$ 

5.13 
$$V_T = \overline{\right) R^2 + (\omega L)^2} l_m \sin\left(\omega t + \tan^{-1}\frac{\omega L}{R}\right)$$

$$\theta = \tan^{-1} \frac{\omega L}{R}$$
, where  $\omega = 200$  rad/sec

5.15 
$$L = 6.67$$
 mH; C = 3.33  $\mu$ F

5.17 
$$i_T = 1.74 \sin (100t + 67.4^\circ) \text{ A}$$
  
 $\theta = 67.4^\circ; Z = 115 \Omega$ 

### **Chapter 6**

6.1 0.97
6.3 3.12 Ω, 9.93 H
6.5 (0.28 + *j*0.78) Ω; 282.7 VA
6.7 486.5; 0.27
6.9 0.891; 1587.7 W; 806.2 VAR; 1781.9 VA
6.11 1136.36 VA; 529.6 VAR; 0.88
6.13 15.396 kW; 3944 VAR; 15.87 KVA; 0.97
6.15 0.0812 mW
6.17 - 0.114 W

## **Chapter 7**

7.1  $3.39 \angle -97.3^{\circ}$ 

7.3  $(3.82 - j1.03) \Omega$ ; 15.11 W

1.5 1.5/11
------------

- 7.7 2.69 W
- 7.9 (20-j5) V in series with  $(2-j) \Omega$ (8.99 + j2) A in parallel with  $(2-j) \Omega$
- 7.11  $I_{10} = 7.34 \angle -21.84^{\circ}; I_5 = 1.65 \angle 33.69^{\circ}; I_3 = 8.39 \angle -12.5^{\circ}$
- 7.13 (1.1 + *j*4.7) V in series with (0.93 + *j*0.75)  $\Omega$ (3.2 + *j*2.4) A in parallel with (0.93 + *j*0.75)  $\Omega$
- 7.15 1874.9 W
- 7.17  $(-0.18 j0.6)V_1$  volts in series with  $(100 j30) \Omega$
- 7.19 0.894  $\angle -63.4^{\circ}$  in series with (0.4 + *j*1.25)  $\Omega$

## **Chapter 8**

- 8.1 50.3 Hz; 63.2 V; 3 (approx.)
- 8.3 2.07 Ω
- 8.5 875.35 Hz; 914.42 Hz; 836.28 Hz; 0.2H; 0.165 μF
- 8.7 1.77
- 8.9  $Q = 1; R = 60 \Omega; C = 50 \mu F$

## Chapter 9

- 9.1  $i_R = 50.8 \angle 0^\circ; i_Y = 25.4 \angle -120^\circ; i_B = 16.936 \angle 120^\circ$   $P_R = 12903.2 \text{ W}; P_Y = 6451.6 \text{ W}; P_B = 4302.4 \text{ W}$ (Taking *R*-phase voltage reference) 9.3 3000 W
- 9.5 Taking  $V_{RN}$  reference  $i_R = 25.4 \angle 0^\circ$ ;  $i_Y = 25.4 \angle -146.8^\circ$ ;  $i_B = 25.4 \angle 146.8^\circ$  $i_N = 17.1 \angle 0^\circ$ ; Power = 17967.7 W
- 9.7  $i_R = 50 \angle -62^\circ$ ;  $i_Y = 50 \angle -182^\circ$ ;  $i_B = 50 \angle 58^\circ$ ; Power = 12705 W
- 9.9 173 μF
- 9.11 1250 W; 0.693; 2.36 A; 1000 W
- 9.13 Taking  $V_{RY}$  reference  $i_R = 11.25 \angle -23.42^\circ$ ;  $i_Y = 18.06 \angle 218.25^\circ$ ;  $i_B = 16.12 \angle 76^\circ$  $281.32 \angle -23.42^\circ$ ;  $180.6 \angle 218.25^\circ$ ;  $241.8 \angle 76^\circ$

## **Chapter 10**

10.1 
$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}; v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

10.3  $\begin{bmatrix} 2 & 5 & -2 \\ 5 & 4 & 0 \\ -2 & 0 & 6 \end{bmatrix}$ 10.5  $v_1 = 181.44 \cos (40t - 30^\circ)$   $v_2 = 202.88 \cos (40t - 30^\circ)$ 10.7 L = 13 H 10.9  $L = \frac{2}{3}$  M 10.11  $1 \angle -90$  V

## Chapter 12

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12.1 
$$i(t) = (2 + 10t)e^{-5t}$$
  
12.3  $i_1(t) = 9.99 - 8.49 e^{-5 \times 10^4 t}; i_2(t) = 5e^{-5 \times 10^4 t}$   
12.5  $i(t) = 101.2 + 30.9 e^{-0.1t} - 52.11 e^{-4.94t}$   
12.7  $i(t) = 5.06 [e^{-0.033t} - e^{-4.966t}]$   
12.9  $i(t) = 3.8 + e^{-0.05t} + 0.12 e^{-0.31t}$   
12.11  $i(t) = -0.35 e^{-500t}$   
12.13  $5e^{-5.71t}$   
12.15  $V_1(t) = -4e^{-0.4t} + 4e^{-4999.8t}$ 

## Chapter 13

13.1 (a) 
$$\frac{6+2as+bs^{2}+3s^{2}}{s^{4}}$$
  
(b) 
$$\frac{100}{2s(s^{2}+100)}$$
  
(c) 
$$\frac{e^{6}}{s-5}$$
  
(d) 
$$\frac{s^{2}-18}{s(s-6)(s+6)}$$
  
13.3 (a) 
$$\frac{9}{8}e^{t} - \frac{9}{8}e^{-t}\cos 2t - \frac{11}{8}e^{-t}\sin 2t$$
  
(b) 
$$e^{-t}[4\cos 2t + \sin 2t]$$
  
(c) 
$$\frac{e^{t}}{2}[2\cos 2t + \sin 2t]$$

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(d) 
$$\delta(t) + e^{-t} [\sin t - 3 \cos t]$$
  
13.5 (a)  $\frac{1}{4} [te^{-2t} + e^{-2t} + t - 1]$   
(b)  $\frac{1}{100} [10 t \sin 5t + \cos 5t - \cos 5t \cos 10t - \sin 5t \ln 10t]$   
(c)  $\frac{1}{16} [\cos 3t - \cos 5t]$   
13.7  $\frac{\alpha}{1 - e^{-s}} \left[ \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right]$   
13.9  $\sin 50t - \frac{\cos 50t}{2500} - \frac{1}{2500} e^{-t/50}$   
13.11  $\frac{1}{5} te^{-t} + \frac{2}{3} e^{-t} - \frac{2}{12} e^{-t/4}$   
13.13  $51e^{-t} - \cos 50t + \frac{1}{50} \sin 50t$   
13.15  $i_1(t) = 0.08e^{-10/3t} + 0.013 \sin 20t = 0.08 \cos 20t$   
 $i_2(t) = 0.24e^{-10/3t} + 0.04 \sin 20t - 0.24 \cos 20t$   
13.17  $i(t) = -20e^{-2t} + 20e^{-t}$   
13.19  $i(t) = \frac{V}{R} \cos\left(\frac{t}{\sqrt{LC}}\right)$ 

# Chapter 14

14.1 
$$i(t) = e^{-2.5(t-5)} [\cos h \ 2.46 \ (t-5) - 1.01 \ \sin h \ 2.46 \ (t-5)]$$

14.2 
$$1 - \frac{1}{3}e^{-t/3} + \frac{2}{15}e^{-(t-3)/5}$$

14.5 
$$Z_{11}(s) = \frac{7s^2 + 7s + 5}{s^2 + s + 1}; Z_{12}(s) = \frac{2s}{s^2 + s + 1};$$

$$G_{21}(s) = \frac{2s}{7s^2 + 7s + 5}$$

14.7 
$$G_{12}(s) = \frac{s^2 + 1}{2s^2 + 1}; Z_{12}(s) = \frac{s^2 + 1}{s(3s^2 + 1)}$$

14.9 
$$Z_{12}(s) = \frac{5s}{s^2 + 1}; Y_{12}(s) = \frac{s^2 + 1}{5s}; G_{12}(s) = G_{12}(s) = 1$$

14.11  $i(t) = 4.5e^{-3t} - 1.5e^{-t}$ 14.13 Unstable 14.15 (a) 2, 0, 1 (b) 0, 2, 4 (c) 2, 2, 2

# Chapter 15

15.1 
$$Z_{11} = \frac{Y_B + Y_C}{\Delta Y}; Z_{12} = Z_{21} = \frac{Y_C}{\Delta Y}; Z_{22} = \frac{Y_A + Y_C}{\Delta Y}$$
  
 $\Delta_Y = Y_A Y_B + Y_B Y_C + Y_C Y_A$   
15.3  $A' = 3; B' = 2; C' = 4; D' = 3$   
15.5  $h_{11} = \frac{4}{3}; h_{21} = \frac{-2}{3}; h_{22} = \frac{1}{6}; h_{12} = \frac{2}{3}$   
 $g_{11} = \frac{1}{4}; g_{12} = -1; g_{21} = 1; g_{22} = 2$   
15.7  $Y_{11} = (0.5 - j0.2)10^{-3}; Y_{12} = Y_{21} = (j0.2 \times 10^{-3})$   
 $Y_{22} = j(0.02 \times 10^{-3})$ 



Fig. 15.7

15.9 
$$Z_i = 1.5 \text{ k} \Omega; Z_0 = 0.033 \times 10^{-3} \Omega$$
  
15.15  $\begin{bmatrix} 5.71 & -4.29 \\ 2.14 & 2.14 \end{bmatrix}$   
15.17  $\begin{bmatrix} 0.857 \angle -31^\circ \text{k}\Omega & 0.17 \angle 59^\circ \\ 8.58 \angle -32.1^\circ & 1.89 \angle 61.1^\circ \text{m} \end{bmatrix}$ 

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# PAPER 1

- 1. (a) Obtain the response of R-L-C series circuit for impulse excitations.
  - (b) Define reluctance of a magnetic circuit and derive an expression for reluctance.

**Solution** Refer section 10.11 in the textbook.

- 2. In an electrical circuit R, L and C are connected in parallel.  $R = 10 \Omega$ , L = 0.1H,  $C=100 \mu$ F. The circuit is energized with a supply at 230 V, 50 Hz. Calculate
  - (a) Impedance
  - (b) Current taken from supply
  - (c) p.f. of the circuit
  - (d) Power consumed by the circuit

**Solution** The circuit is as shown in figure.



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The impedance of 3 branches are

$$Z_{1} = 10 \Omega$$

$$Z_{2} = j2 \pi f L = 2 \times 50 \times 0.1 = j31.41 \Omega$$

$$Z_{3} = \frac{-j}{2\pi f c} = \frac{-j}{2 \times 50 \times 100 \mu} = -j31.84 \Omega$$

(a) Impedance of circuit 
$$Z = \left[\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right]^{-1}$$
  
=  $\left[\frac{1}{10} + \frac{1}{j31.41} + \frac{1}{-j31.84}\right]^{-1}$   
 $\approx 10\Omega$ 

(b) Current taken from supply  $I = \frac{V}{Z} = \frac{230\angle 0^{\circ}}{10} = 23A$ . i.e.  $23\angle 0^{\circ}$  A

- (c) p.f. of the circuit =  $\cos \theta = 1$
- (d) Power consumed by the circuit Real power consumed =  $I^2R = 23^2 \times 10 = 5.3$  kW Reactive power consumed = 0 KVAR
- 3. A constant voltage at a frequency of 1 MHz is applied to an inductor in series with a variable capacitor when the capacitor is set to 500 PF, the current has the max. value, while it is reduced to one half when capacitance (i) 600 PF, find (i) resistance (ii) inductance (iii) *Q* factor of inductor.

Solution Given 
$$f = 1$$
 MHz  
Let the max. current be  $I_{max}$ .  
Given at 1 MHz, for  $C = 500$  Pf  
 $I = I_{max}$   
 $\therefore$  Imaginary part of impedance is  
zero, i.e.  $X_L = X_C$   
 $2\pi fL = \frac{1}{2\pi fc}$   
 $6.283 \times 10^6 \times L = 318.31$   
 $L = 50.66 \ \mu\text{H}$   
Now also given  $I = \frac{I_{max}}{2}$  at  $C = 600$  PF  
 $I = \frac{I_{max}}{2} = \frac{V}{R + j(6.283 \times 10^6 L - 265.25)}$  (1)  
 $\left(\because X_C = \frac{1}{2\pi fc} = \frac{1}{2\pi \times 10^6 \times 600 \times 10^{-12}} = 265.25\right)$ 

and 
$$I_{\text{max}} = \frac{V}{R}$$
  
Dividing Equation (2) by Equation (1)  

$$Z = \frac{R + j (6.283 \times 10^{6} L - 265.25)}{R}$$

$$\Rightarrow 2R = R + j (6.283 \times 10^{6} L - 265.25)$$

$$R = j(318.31 - 265.25)$$

$$R = 53.06 \Omega$$
(i)  $R = 53.06 \Omega$ 
(ii)  $L = 50.66 \mu$ H
(iii)  $G = \frac{\omega L}{R} = 5.999 \approx 6$ 

*.*..

4. For the given graph and tree shown in the figure, write the tie-set matrix and obtain the relation between branch currents and link currents.



- **Solution** Number of link branches = b (n 1)Where *b* is number of branches and *n* is number of nodes  $\therefore$  Link branches = 4 - (3 - 1) = 2The link branches are *a* and *b*. Let the branch currents are  $i_a$ ,  $i_b$ ,  $i_c$  and  $i_d$ 
  - The two link currents are  $i_1$  and  $i_2$  as shown in the figure.



There are two fundamental loops corresponding to the link branches a and b. If  $V_a$  and  $V_b$  are branch voltages, the KVL equations for the two f-loops can be written as

(2)

$$V_a + V_d - V_c = 0$$
$$V_b + V_d - V_c = 0$$

The above equation can be written in matrix form as Loop branches Currents

- Currents  $\downarrow \qquad \rightarrow \qquad a \quad b \quad c \quad d$  $i_1 \qquad \qquad \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & +1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \end{bmatrix} = 0$
- 5. Find the equivalent resistance between *AB* in the circuit shown in the figure. All resistances are equal to *R*.







The equivalent circuit for this is as shown below.



 $\therefore \text{ Resistance between } AB \text{ is } R = \frac{10}{15}R = 0.666R$ 

6. Find the Thevenins equivalent for the circuit in figure



**Solution** The Thevenins equivalent resistance is calculated assuming all voltage sources shorted and as seen from AB, the circuit will be as shown below:



$$R_{\text{Th}} = \left[ \left\{ (5/6) - 7 \right\} / 8 \right] + 5$$
$$\left[ \left\{ \frac{30}{11} + 7 \right\} / 8 \right] + 5 = \left[ \frac{107}{11} \times 8}{\left[ \frac{107}{11} + 8 \right]} \right] + 5 = 4.389 + 5 = 9.389 \ \Omega$$

Let us assure voltages at nodes (1) and (2) be  $V_1$  and  $V_2$ . Now writing node equations.

$$\frac{V_1 - 8}{8} + \frac{V_1 - V_2}{7} = 0$$
  
7V\_1 - 56 + 8V\_1 - 8V\_2 = 0  $\Rightarrow$  15 V\_1 - 8V\_2 = 56 (1)

$$\frac{V_2}{6} + \frac{V_2 - V_1}{7} + \frac{V_2 - 5}{5} = 0 \implies -30V_1 + 107V_2 = 210$$
(2)

on solving equations (1) and (2) we get

$$V_1 = 5.6 \text{ V} \implies V_{\text{OC}} = 5.6$$

: Thevenins equivalent circuit is



7. The switch in the circuit shown in figure is in position (1) for two time constants and then charged to position (2) find transient response.



\*\*

**Solution** When the switch is in position (1) Convert equation in laplace transform is given as

$$I(S) = \frac{V(S)}{R + LS} = \frac{5 / S}{5 + 0.001S} = \frac{5000}{S(5000 + 3)}$$

Assuming initial conditions be zero.

$$I(S) = \frac{1}{S} - \frac{1}{S + 5000}$$
  
Taking inverse Laplace transform

$$i(t) = 1 - e^{-500}$$

the switch is cosed for two time constants  $\therefore i(t)$  after two time constants i

$$= 1 - e^{-2} = 0.864$$
 A

Now when switch is moved to position (2) the mesh equation is given by

i

$$L\frac{di}{dt} + Ri(t) = 0$$
  

$$\Rightarrow i(t) = C_1 e^{-5000t}$$
  
initially  $i(o) = 0.864$  A  
 $C_1 = 0.864$  A  
 $\therefore \quad i(t) = 0.864 e^{-5000t}$ 



The response can be plotted as

8. Derive phase and line voltage, current relations in a balanced star and delta connected loads.

Solution Refer Sections 9.7.1, 9.7.2, 9.7.3, and 9.8.1, 9.8.2, and 9.8.3.

# PAPER 10

1. (a) What are passive and active circuit elements? Explain the voltagecurrent relationships of passive elements with examples.

#### Solution:

Refer Sections 1.4.1; 1.5, 1.6 and 1.7

- (b) Reduce the network of 1A( figure below into an equivalent network across terminals A and B with
  - (i) one equivalent voltage source
  - (ii) one equivalent current source



#### Solution:

Using the source transformation, we get the N/W





By again converting current source, into voltage source



 $\therefore$  One equivalent voltage source is





One equivalent current source is

2. (a) A cast steal iron core has a square cross section of side 3 cm. Assuming 1 the permeability of steel to be 800, find the mmf required to produce a flux  $\phi = 0.2$  mwb 105 the right limb as shown in the figure.



Solution:

$$\phi = 0.2 \text{ mwb}$$

$$\mu_r = 800$$

$$H_g = 3 \times 3 \times 10^{-4} \text{ m}^2$$

$$\phi = 0.2 \times 10^{-3} \text{ wb}$$

$$B = \frac{\phi}{A} = \frac{0.2 \times 10^{-3}}{9 \times 10^{-4}} = 0.22 \text{ wb/m}^2$$

$$H = \frac{B}{\mu_o \mu_r} = \frac{0.22}{4\pi \times 10^{-7} \times 800} = 221.04 \text{ AT/m}$$

mmf required is given by  $HI = 60 \times 10^{-2} \times 221.04$ 

 $(\phi = NI = HI) = 132.62 \text{ AT}$ 

- : mmf required to produce 0.2 mwb in right limb is 132.62 AT
- (b) Define self and mutual inductances. Establish the polarity of two mutually coupled coils on a single magnetic core.

#### Solution:

**Solution** 

*:*..

Refer Section 10.3

3. (c) Find the equivalent inductance of the following circuit figure.

$$V_1 = L_{eq} \frac{di_1}{dt}$$



but by applying mesh analysis

$$O = L_1 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

 $V_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$ 

$$\cdot \qquad \frac{di_2}{dt} = \frac{-M}{L_2} \frac{di_1}{dt}$$

$$\therefore \qquad V_1 = L_1 \frac{di_1}{dt} + M \left[ \frac{-M}{L_2} \frac{di_1}{dt} \right]$$

$$\Rightarrow \quad L_1 \frac{di_1}{dt} - \frac{M^2}{L_2} \frac{di_1}{dt} \Rightarrow \left( L_1 - \frac{M^2}{t_2} \right) \frac{di_1}{dt} \tag{2}$$

Compare Eq. (1) with Eq. (2)

$$L_{\rm eq} = L_1 - \frac{M^2}{L_2}$$

 $\therefore$  L<sub>eq</sub> for the magnetic circuit is obtained.

\*\*

3. (a) Explain about active, reactive and apparent powers. Give expression for the above. Draw the power triangle.

#### Solution:

Refer Sections 6.3; 6.4 and 6.5

(b) Given  $i = 50 \sin(\omega t + 60)$ 

 $V = 200 \sin (\omega t + 30)$  find the elements of the network with their values active, reactive and apparent power.

#### Solution:

 $i = 50 \sin (\omega t + 60)$  $V = 200 \sin (\omega t + 30)$ 

Here the current leads the voltage by  $30^{\circ}$ 

: the elements of the network are resistance and capacitance. and the power factor of the network is,  $\cos 30^\circ = 0.866$  (leading) Active Power:

$$P_{\text{active}} = V_{\text{eff}} I_{\text{eff}} \cos \theta$$
$$= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos \theta$$
$$= \frac{\sqrt{50}}{\sqrt{2}} \cdot \frac{200}{\sqrt{2}} \cos 30 = 4330.12 \text{ W}$$

Reactive Power,

$$P_{\text{reactive}} = V_{\text{eff}} I_{\text{eff}} \sin \theta$$

$$= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \sin \theta$$

$$= -\frac{50}{\sqrt{2}} \cdot \frac{200}{\sqrt{2}} \sin 30 = -2500 \text{ VAR}$$

Apparent Power

$$P_{\text{apparent }P} = \sim V_{\text{eff}} I_{\text{eff}}$$
$$= \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} = \frac{50}{\sqrt{2}} \cdot \frac{200}{\sqrt{2}} = 5000 \text{ VA}$$

Component in n.w. R.c.

 $\therefore \text{ Active Power} = 433012 \text{ W}$ Reactive Power = -2500 VAR
Apparent Power = 5000 VA  $\overline{\theta} = 3.464 \Omega$   $\overline{\theta} = 3.464 \Omega$   $\overline{C} = \frac{1}{2\omega}$   $\overline{C} = \frac{1}{2\omega} \frac{11}{\omega c} = +2J$ 

...

4. (a) Obtain the expression for frequency at which the voltage across the inductance becomes a maximum in a series RLC circuit. Explain what is meant by voltage magnification factor.

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#### Solution:

Refer Sections 8.3; 8.6

(b) Obtain the transmission parameters for the following figure || circuit. Verify your result for reciprocity condition.



When

 $V_{2} = 0$ 

$$B = \frac{-V_1}{I_2}$$
$$-I_2 = \frac{I_1(3-4j)}{6-4j}$$

$$\begin{aligned} -I_2 &= +I_1 \left( 0.65 - 0.23j \right) \\ V_1 &= I_1 \left[ (5+6j) + \left[ (3-4j) \parallel 3 \right] \right] \\ &= I_1 \left[ 6.96 + 5.3j \right] \\ B &= -\frac{V_1}{I_2} = \frac{I_1 (6.96 + 5.3j)}{I_1 (0.65 - 0.23j)} = 6.95 + 10.61j \\ B &= 6.95 + 10.61j \ \Omega \\ D &= -\frac{I_1}{I_2} = \frac{I_1}{I_1 (0.65 - 0.23j)} = 1.367 + 0.48j \\ A &= 0.64 + 1.52j \\ B &= 6.95 + 10.61j \ \Omega \\ C &= 0.12 + 0.16j \ \Im \\ D &= 1.367 + 0.48j \end{aligned}$$

Reciprocity condition:

$$AD - BC = 1$$
  
(0.64 + 1.52*j*) (1.367 + 0.48*j*) - (6.95 + 10.61*j*) (0.12 + 0.16*j*)  
= 1.00 - 1.6 × 10<sup>-4</sup>*j*  
= 1.008 ∠-0.009 ≈ 1

#### Solution:

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...

<u>Refer Sections 9.8.1; 9.8.2 and 9.8.3</u>

(b) A 3 phase 400 V, 4 wire system has a star connected load with  $z_A = (10+j0) \Omega$ ,  $z_B = (15+j10) \Omega$ ,  $z_C = (0+5j) \Omega$ . Find the line currents and current through neutral conductor. Draw the phasor diagram.

### Solution:



\*\*

$$V_{Ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94$$

$$V_{AN} = 230.94 \angle 0 V$$

$$V_{BN} = 230.94 \angle -120 V$$

$$V_{CN} = 230.94 \angle -240 V$$

$$I_A = \frac{V_{AN}}{Z_A} = \frac{230.94 \angle 0}{(10 + 0j)}$$

$$= 23.09 + 0j A$$

$$I_B = \frac{V_{BN}}{Z_B} = \frac{230.94 \angle -120}{(15 + 10j)}$$

$$= -11.48 - 5.67j A$$

$$I_C = \frac{V_{CN}}{Z_C} = \frac{230.94 \angle -240}{(0 + 5j)}$$

$$= 39.99 + 23.094j A$$

$$I_N = -(I_A + I_B + I_C)$$

$$= -[23.094 - 11.48 - 5.67j + 39.99 + 23.094j]$$

$$= -[51.604 + 17.424j]$$

$$= 54.46 \angle -161.34 A$$

 $I_N$  phase with respect to  $V_{AN}$  is -161.34 phasor diagram is



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Γ

6. (a) What is duality? Explain the procedure for obtaining the dual of the given planar network shown below figure below.



#### Solution:

Redrawing the N/W. I2



(b) Construct the incidence matrix for the graph shown in figure below. **Solution:** 



: Incidence matrix for the given graph is constructed.

(c) Use nodal analysis, to determine the voltages  $V_1$  and  $V_2$  in the circuit shown in figure below.

\*\*



Apply nodal analysis,

$$\frac{V_1 - 5}{2} + \frac{V_1}{1} + \frac{V_1 - V_2}{2} = 0$$
(1)

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} = 1 \tag{2}$$

$$V_1 - 5 + 2V_1 + V_1 - V_2 = 5$$
  
4V\_1 - V\_2 = 5 (3)

$$V_2 - V_1 + 2V_1 = 2$$
  

$$3V_2 - V_1 = 2$$
(4)

Solving equations (3) and (4) we get

$$V_1 = 1.545 \text{ V}$$
  
 $V_2 = 1.181 \text{ V}$   
 $V_1 = 1.545 \text{ V}$  and  $V_2 = 1.181 \text{ V}$ 

7. (a) State and explain the Reciprocity theorem? Is this theorem valid for N/W with two sources? Subtantiate your answers.



#### Solution:

 $\Rightarrow$ 

*:*..



By using the current division the value of *I* can be obtained.

then

:.

-

$$I = \frac{-5jI_1}{(10-5j)} = \frac{-5j(20)}{(10-5j)} = 4 - 8i$$

Response to excitation is  $\frac{V}{I} = \frac{100}{4-8i} = 5 + 10j$ 

the N/W can be reduced





$$I_T = \frac{I_T(-5j)}{(3+4j-5j)} = \frac{(5.6+0.8j)(-5j)}{(3-j)} = 5.6+0.8j$$

By using the current division,

$$I = \frac{I_T(-5j)}{(3+4j-5j)} = \frac{(5.6+0.8j)(-5j)}{(3-j)} = 4-8j$$

Response to excitation is  $\frac{V}{I} = \frac{100}{4-8j} = 5 + 10j$ 

: Reciprocity theorem is verified.

\*\*

8. (a) Compare the classical and Laplace transform method of solution of the network.

-----

(2)

#### Solution:

Refer Chapter 13

(b) Draw the network in Laplace domain and find  $i_1(t)$  and  $i_2(t)$  the following figure.



By applying mesh analysis,

$$100 = 15i_1(t) + 2 \frac{di_1}{dt} - 10i_2(t) - \frac{2di_2}{dt}$$
(1)

$$0 = 14i_2(t) + \frac{2di_2}{dt} + \frac{1}{2} \left| i_2dt - 10i_1(t) - \frac{2di_1}{dt} \right|$$
(2)

Applying Laplace transform on both sides, for the two equations

$$15I_{1}(s) + 2SI_{1}(s) - 2SI_{2}(s) - 10I_{2}(s) = \frac{100}{s}$$
  
$$-10I_{1}(s) - 2SI_{1}(s) + 14I_{2}(s) + 2SI_{2}(s) + \frac{I_{2}(s)}{2s} = 0$$
  
$$I_{2}(s) \left[ 14 + 2s + \frac{1}{2s} \right] = I_{1}[10 + 2s]$$
(1)  
$$d \qquad I_{1}(s) [15 + 2s] - I_{2}(s) [10 + 2s] = \frac{100}{s}$$
(2)

and

....

$$I_1(s) [15+2s] - I_1(s) \frac{(10+2s)^2}{\left(14+2s+\frac{1}{2s}\right)} = \frac{100}{s}$$

$$I_1(s)\left[\frac{38s+111}{14+2s+\frac{1}{2s}}\right] = \frac{100}{s}$$

$$I_1(s) = \frac{50}{s^2} \frac{(28s + 4s^2 + 1)}{(38s + 111)}$$
$$= \frac{50 (s + 0.038) (s + 6.96)}{s^2 (38s + 111)}$$

Taking the partial fractions

 $\frac{(28s+4s^2+1)}{s^2(38s+111)} = \frac{A}{s} + \frac{B}{s^2} + \frac{c}{38s+111}$   $28s+4s^2+1 = \text{As } (38s+111) + B (38s+111) + cs^2$ Compare co-efficients of  $s^2$ , s,  $s^\circ$  38A + C = 4 111A + 38B = 28 B111 = 1Solving these three equation: A = 0.249 B = 0.009 C = -5.468  $\therefore \qquad I_1(s) = \frac{50(0.249)}{s} + \frac{50(0.009)}{s^2} - \frac{50(5.468)}{38s+111}$   $I_1(s) = \frac{12.45}{s} + \frac{0.45}{s^2} - \frac{273.4}{383+111}$ 

Applying inverse. Laplace transform

$$I_{1}(t) = 12.45 + 0.45t - \frac{273.4}{38}e^{-\frac{111}{38}t}$$
  

$$= 12.45 + 0.45t - 7.19e^{-2.92t}$$
  

$$\parallel ly I_{2}(s) = I_{1}(s) \frac{10 + 2s}{14 + 2s + \frac{1}{2s}}$$
  

$$= \frac{50}{s^{2}}\frac{(28s + 4s^{2} + 1)}{(38s + 111)}\frac{(10 + 2s)}{(14 + 2s + \frac{1}{2s})}$$
  

$$= \frac{50}{s^{2}}\frac{(10 + 2s)}{(38s + 111)} \cdot \frac{(28s + 4s^{2} + 1)}{(28s + 4s^{2} + 1)}$$
  

$$= \frac{100}{s}\frac{(10 + 2s)}{(38s + 111)}$$

....

Taking partial fractions.

$$\frac{(10+2s)}{s(38s+111)} = \frac{A}{s} + \frac{B}{38s+111}$$

$$10+2s = A(38s+111) + Bs$$

$$38A+B=2 \implies A = 0.09$$

$$A 111 = 10 \qquad B = -1.423$$

$$I_2(s) = \frac{100(0.09)}{s} - \frac{100(1.423)}{38s+111}$$

$$= \frac{9}{8} - \frac{142.3}{38\left[s + \frac{111}{38}\right]}$$

take inverse Laplace transform

$$i_2(t) = 9 - 3.744 \ e^{-2.92t}$$
  
$$\therefore \qquad i_1(t) = 12.45 + 0.45t - 7.19 \ e^{-2.92t}$$
  
$$i_2(t) = 9 - 3.744 \ e^{-2.92t}$$

# PAPER 11

1. (a) Find the equivalent resistance between terminals y and z in the figure shown below.



Solution: The above circuit can be represented as



(b) In the network shown in figure below, determine  $i_x$ .



**Solution:** Apply source transformation for the current source (4A) is  $3I_x$  current source.



In the above circuit voltage sources in series can be added and Eqn resistance is place.

$$\begin{array}{c|c} x \\ \hline 30 \ \Omega \\ \hline i_x \\ \downarrow \\ \hline (60i_x + 40) \end{array} \begin{array}{c} i_x \\ \downarrow \\ \hline 5 \ \Omega \end{array} \begin{array}{c} \bullet \\ \bullet \\ \hline \bullet \\ -3 \ A \end{array}$$

Apply nodal analysis at note *x*.

$$\frac{(v_x - 0)}{5\Omega} + \frac{v_x - (60i_x + 40)}{30\Omega} + 3 = 0$$
$$v_x = 59_x$$

But  $v_x$ replacing  $v_x$  by  $5i_x$ 

$$i_x + \frac{5i_x - 60i_x - 40}{30} + 3 = 0$$
  

$$30i_x + 5i_x - 60i_x - 40 + 90 = 0$$
  

$$- 25i_x + 50 = 0$$
  

$$i_x = \frac{50}{25} = 2A$$

 $\therefore$  The current  $i_x = 2A$ 

\*\*
2. (a) Define mmf, for x and reluctance of  $\omega$  magnetic circuit:

Solution:

#### Refer Section 10.11

(b) An iron ring has a mean diameter of 25 cms, and a cross-sectional area of 4 cms<sup>2</sup>. It is wound with a coil of 1200 turns. An air gap of 1.0 mm width is cut in the ring. Determine the current required in the coil to produce a flux of 0.48 mwb in the air gap. The relative permeability of iron under the condition is 800. Neglect Leakage.



#### **Solution:** *Given data*:

Mean diameter = 25 cms = 
$$D = 0.25$$
 mb  
Cross-sectional area,  $A = 4$  cm<sup>2</sup> =  $4 \times 10^{-4} M^2$   
No. of turns =  $N = 1200$ 

Relative permeability of iron 
$$= 800$$

$$\phi = 0.48 \text{ mwb}$$

Air gap reluctance 
$$Rl_g = \frac{l_g}{\mu_0 A_C} = \frac{1.5 \times 10^{-3} \text{ mt}}{4\pi \times 10^{-7} \times \mu \times 10^{-4}}$$
  
= 10<sup>8</sup> × 0.02981  
= 2.9841 × 10<sup>6</sup> AT/wb

Iron core reluctance

$$R_{LC} = \frac{l_C}{\mu_0 \mu_r A_C} = \frac{(\pi D - l_g)}{4\pi \times 10^{-7} \times 800 \times 4 \times 10^{-4}}$$
$$= \frac{(\pi \times 0.25 - 1.5 \times 10^{-3})}{16\pi \times 8 \times 10^{-9}}$$
$$= 1.949 \times 10^{+6} \text{ AT/wb.}$$
Total reluctance  $= Rl_g + R_{l_c}$ 
$$= 4.933 \times 10^6 \text{ AT/wb}$$
mmf = flux × reluctance  
 $NI = \phi \cdot Rl$ 
$$1200 \times i = 0.48 \times 10^{-3} \times 4.9331 \times 10^{6}$$
$$i = \frac{0.48 \times 10^{-3} \times 4.9331 \times 10^{6}}{1200}$$
current required = **1.973 Amp**

*.*..

3. (a) Get the expression for complex power and sign of the active power. **Solution:** 

Refer Chapter 6.

::

(b) Find  $I_1$ ,  $I_2$ ,  $I_3$  and I find also the power consumed. Draw the phasor diagram (Fig. 19)



:.



4. (a) Obtain the Y parameters for the following figure and network in Laplace transform variable.



### Solution:

Y parameter Equations



$$\begin{split} I_1 &= Y_{11} V_1 + Y_{12} V_2 \\ I_2 &= Y_{21} V_1 + Y_{22} V_2 \\ Y_{11} &= \frac{I_1}{V_1} \bigg|_{V_2 = 0} \\ Y_1 &= I_1 \times 2 \left( \frac{1 \times 1/2s}{1 + 1/2s} \right) \\ V_1 &= \frac{2I_1}{2s + 1} \qquad Z_{eq} = 2 \times \frac{\left( 1 \times \frac{1}{2s} \right)}{\left( 1 + \frac{1}{2s} \right)} \end{split}$$

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$$\begin{array}{c} Y_{11} = \frac{I_1}{V_1} = \frac{(2s+1)}{2} \\ Y_{21} = \frac{I_2}{V_1} \\ I_1 = I_{\omega} - I_b \\ I_{\omega} = \frac{I_1 \times 1}{1 + \frac{1}{2s}} = \frac{2sJ_1}{1 + 2s} \\ I_b = \frac{I_1 \times 1/2s}{1 + 1/2s} = \frac{I_1}{1 + 2s} \\ I_{\omega} - I_b = I_2 = \frac{I_1}{1 + 2s} (2s-1) \\ = \frac{(2s-1)}{(1 + 2s)} \times \frac{1}{2} (2s+1)V_1 \\ I_2 = \left(\frac{2s-1}{2}\right) V_1 \\ I_2 = \left(\frac{2s-1}{2}\right) V_1 \\ V_{22} = \frac{I_2}{V_1} \Big|_{V_{1=0}} = 0 \\ V_{22} = \frac{I_2}{V_2} \Big|_{V_{1=0}} = 0 \\ V_2 = I_2 \times \frac{2}{2s+1} \\ \frac{I_2}{V_2} = \frac{(2s+1)}{2} = Y_{22} \\ \text{and} \\ Y_{12} = \frac{I_1}{I_2} \Big|_{V_{1=0}} \\ From figure \\ I_1 = I_{\omega} - I_b \\ I_{\omega} = \frac{I_2 \times 1}{1 + 1/2s} = \frac{2sI_1}{1 + 2s} \end{array}$$

 $I_b = \frac{I_2 \times 1/2s}{1 + 1/2s} = \frac{I_L}{1 + 2s}$ 

$$I_{1} = \frac{(2s-1)I_{L}}{1+2s} = \frac{(2s-1)}{(1+2s)} \times \frac{(2s+1)V_{2}}{2}$$
$$I_{1} = \frac{2s-1}{2}V_{2}$$
$$\frac{I_{1}}{V_{2}} = \frac{2s-1}{2} = 1/12$$

 $\therefore$  Y matrix

$$Y = \begin{bmatrix} \frac{2s+1}{2} & \frac{2s-1}{2} \\ \frac{2s-1}{2} & \frac{2s+1}{2} \end{bmatrix}$$

(b) A tuned circuit consists of a coil having an inductance of  $200 \,\mu\text{H}$  and a resistance of  $15 \,\Omega$  in parallel with a series combination of a variable capacitance and resistor of  $80 \,\Omega$ . It is supplied by a 60 V source. If the supply frequency is 1 MHz what is the value of *C* to give resonance.



**Solution:** Total admittance, 
$$Y = \frac{1}{R_L + j\omega L} + \frac{1}{R_C - (j/\omega C)}$$

$$Y = \frac{R_L - j\omega L}{R_L^2 + \omega^2 L^2} + \frac{R_C + j/\omega C}{R_C^2 + \frac{1}{\omega^2 C^2}}$$
$$= \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_C}{R_C^2 + \frac{1}{\omega^2 C^2}}$$
$$+ j \left[ \frac{1/\omega C}{R_C^2 + \frac{1}{\omega^2 C^2}} - \frac{\omega L}{R_L^2 + \omega^2 L^2} \right]$$

at resonance, susceptance part becomes zero

$$\frac{\omega_r L}{R_L^2 + \omega_r^2 L^2} = \frac{\frac{1}{\omega_r C}}{R_C^2 + \frac{1}{\omega_r^2 C^2}}$$
$$\omega_r L \left[ R_C^2 + \frac{1}{\omega_r^2 C^2} \right] = \frac{1}{\omega_r C} \left[ R_L^2 + \omega_r^2 L^2 \right]$$

$$\omega_r^2 \left[ R_C^2 + \frac{1}{\omega_r^2 C^2} \right] = \frac{1}{LC} \left[ R_L^2 + \omega_r^2 L^2 \right]$$

$$\omega_r = \frac{1}{\sqrt{LC}} \sqrt{\frac{R_L^2 - 4C}{R_C^2 - 4C}} \quad \omega_r = \frac{1}{\sqrt{LC}} (R_L = R_C)$$

$$\left[ \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{LC}} \sqrt{\frac{R_L^2 - 4C}{R_L^2 - 4C}} \quad \text{resonant frequency}$$

$$R_C^2 - \frac{L}{C} = R_L^2 - \frac{L}{C}$$

$$R_C = R_L \right]$$

$$(15 + j1256) \left( 80 - j \frac{1}{2\pi C \times 10^6} \right)$$

Imaginary part = 0 at resonance

$$256 \times 80 = \frac{15}{2\pi C \times 10^6}$$
  
C = 23.76 pF

5. (a) Show that a balanced star connected load can be transformed in to an equivalent delta connected load and vice-versa.

#### Solution:

Refer Section 3.1

(b)  $3\phi$ , 3 wire, 208 V, CBA, has Y load  $Z_A = 5 \angle 0$ ,  $Z_B = 5 \angle 30^\circ$ ,  $Z_C = 10 \angle -68\Omega$ , find true current voltage on across each Load Impedance.

By converter Y N W into  $\Delta$  form

$$\begin{split} Z_{BC} &= \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_A} \\ &= \frac{105.85 \angle - 31.81}{5 \angle 0} \\ &= 21.17 \angle - 31.81^\circ \, \Omega \\ Z_{CA} &= \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_B} \\ &= \frac{105.85 \angle - 31.81}{5 \angle 30^\circ} \\ &= 21.17 \angle - 61.81^\circ \, \Omega \\ Z_{BA} &= \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_C} \end{split}$$



 $=\frac{105.85\angle -31.81}{2}$  $10 \swarrow - 60^{\circ}$  $= 10.58 \angle 28.19$  $\therefore$  Phase sequence *h* CBA  $V_{CB} = 208 \angle 0^{\circ}, V_{AC}$  $V_{RA} = 208 \angle -120^{\circ}$  $V_{AC} = 208 \angle -240^{\circ}$  $I_C = \frac{V_{CB}}{Z_{PC}} = \frac{208\angle 0}{21.17\angle -31.81} = 9.82 \angle 31.81 \text{ A}$  $I_B = \frac{V_{BA}}{Z_{AB}} = \frac{208\angle -120^{\circ}}{10.58\angle 28.19} = 19.65 \angle -148.19^{\circ}$  $I_A = \frac{V_{AC}}{Z_{AC}} = \frac{208\angle -240}{21.17\angle -61.81} = 9.825 \angle -179.19^{\circ}$ Line currents are  $I_1 = I_c - I_a = 9.82 \ \angle 31.81 - 9.825 \ \angle -179.19^{\circ}$  $I_1 = I_{CL} = 18.93 \angle 16.30$  $I_2 = I_b - I_c = 19.65 \angle -148.19 - 9.82 \angle 31.81$  $I_2 = I_{BL} = 29.47 \angle -148.19$  $I_3 = I_a - I_B = 9.825 \angle -179.19 - 19.65 \angle -148.19$  $I_3 = I_{AL} = 12.31 \angle 56.06$ Voltage across each load Impedance are  $V_{ZC} = (I_{CL})Z_C = (18.93 \angle 16.30) (10 \angle -60^\circ)$  $V_{ZC} = 189.3 \angle -43.7^{\circ}$  $I_{CI}$  $V_{ZB} = (I_{B_I}) (Z_B) = (29.47 \angle -148.19) (5 \angle 30^{\circ})$  $V_{ZB} = 147.35 \angle -118.19^{\circ}$  $V_{ZA} = I_{AL} Z_A = I (12.31 \angle 56.06) (5 \angle 0^\circ)$ IBL

#### **Phasor diagram**

 $V_{ZA} = 61.55 \angle 56.06$ 

6. (a) What is duality? Explain the procedure for obtaining the dual of the given planar network shown below in the figure.

#### Solution:

<u>Refer 3.8</u>

IZC



(b) Construct the incidence matrix for the graph shown in the figure. (2)

**Solution:** Let  $i_1, i_2, i_3, i_4$  be the current in the branches 1, 2, 3, 4.

$$-i_1 + i_2 = 0$$
  
$$-i_4 - i_3 + i_1 = 0$$
  
$$-i_2 + i_3 + i_4 = 0$$



The incidence matrix is

S

Nodes branches  

$$[A_i] = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & -1 & 1 & 1 \end{bmatrix}$$

(c) Use nodal analysis, to determine the voltage  $V_1 \leftarrow V_L$  in the circuit shown in figure below.



Solution:  $\frac{(V_1 - 5)}{2} + \frac{V_1 - 0}{1} + \frac{V_1 - V_2}{2} = 0$  $4V_1 - V_2 - 5 = 0$  (1)

$$1 + \frac{V_1 - V_2}{2} + \frac{-V_2}{1} = 0$$

$$V_1 - 3V_2 + 2 = 0$$
 (2)  
olving (1) and (2)  $V_1 = 1.545$  V  
 $V_L = -1.18$  V

20

7. (a) State and Explain the superposition theorem.

#### Solution:

Refer Section 3.2

(b) Is superposition valied for power? Substantiate your answer.

#### Solution:

Refer section 3.2

(c) Using superposition theorem find  $V_{ab}$  volts shown in figure below



Solution: 2A current source alive.



$$V_{A_2} - V_{B_2} = -\frac{6}{3} = -2V$$

Within 2V voltage source alone.

Current  

$$2 = \frac{2}{6+4+2} = \frac{2}{12} = \frac{1}{6} A$$

$$V_{A_3} + (4+2) \times \frac{1}{6} = V_{B_3}$$

$$\Rightarrow \qquad V_{A_3} - V_{B_3} = V_{ab_3} = -1V$$

$$\therefore \text{ By superposition } V_{ab} = V_{ab_1} + V_{ab_2} + V_{ab_3}$$

$$= -4 + -2 + -1$$

$$V_{ab} = -7V$$

8. (a) For the ckt shown below find the inerted condition of  $q_1, p_2, \frac{dq_1}{dt}, \frac{dq_2}{dt}$  and voltage across capacitor the ckt was in steady state before t = 0.



Solution:  $\mathbf{t} = 0^-$ 

$$i = \frac{100}{10+6} = 6.25A$$

$$V_{C}(0^{-}) = \frac{6 \times 100}{16} = 37.5 \text{ V} \quad i_{1}(0^{-}) = i_{2}(0^{-}) = 6.25A$$

$$t = 0, i_{2}(0^{-}) = 6.25 \text{ A}, i_{1}(0^{-}) = 6.25A$$
1

at

$$100 = (i_1 - i_2) \ 20 \ \frac{1}{2\mu F} \ (i_1 - i_2) \ dt \tag{1}$$

$$100 = 6i_2(t) + 2\frac{ai_2}{dt}$$
(2)  
$$i_1 = i_2 = 6.25$$
10  $\Omega$  P

from Eq. (2)  $100 = 6(6.25) + 2 \frac{di_2}{dt}$  $\frac{di_2(0^+)}{dt} = 31.25 \text{ A sec}$ 



Taking derivative eq. (1)

$$0 = \left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) 20 + \frac{1}{2 \times 10^{-6}} (i_1 - i_2)$$
$$i_1 = i_2 = 0$$
$$\frac{di_1(0^+)}{dt} = \frac{di_2(0^+)}{dt} = 31.25 \text{ A}$$

(b) Switch is opened at t = 0 find the current i(t) for  $t \ge 0$  in the following figure.



*:*..

at



By applying K.V.L to loop



 $10i(t) = 30i(t) + 20i(t) + \frac{2di(t)}{dt}$ 

So

at ∴ ∴

$$\frac{2 di(t)}{dt} + 40i(t) = 0$$

$$(D + 20) i(t) = 0$$

$$i(t) = ke^{-20t}$$

$$t = 0 \quad i(t) = 2A$$

$$k = 2$$

$$10i \quad + 2H \quad 2H \quad 10 = 2A$$

$$i(t) = 2e^{-20t} \quad A$$

# PAPER 12

1. (a) Find the equivalent resistance between terminals. y and z in the figure given below.

Solution:



(b) In the network shown in the following, determine  $i_x$ . Solution: At node (b)



•••

- $\frac{V_1}{10} + \frac{V_1 V_2}{20} + 3\left(\frac{V_2}{5}\right) = 4$ Solving A and B  $V_1\left(\frac{1}{10} + \frac{1}{20}\right) + v_2\left[\frac{-1}{20} + \frac{3}{5}\right] = 4 \quad (1)$   $\frac{V_2 V_1}{20} + \frac{V_2}{5} = 3\left[\frac{V_2}{5}\right] 3$   $\frac{-V_1}{20} + V_2\left[\frac{1}{20} + \frac{1}{5} \frac{-3}{5}\right] = -3 \quad (2)$   $0.15V_1 + 0.55V_2 = 4 \qquad (A)$   $-0.05V_1 0.35V_2 = -3 \qquad (B)$
- (a) State and explain Faradays law of Electromagnetic induction. What are statically and dynamically induced EMFs.

#### Solution:

Refer Section 1.6

(b) An iron ring 15 cms in diameter and 10 cm<sup>2</sup> in area cross section. A wand with a coil of 800 kms. Determine the current in the coil to establish a fix density of 1 wb/m<sup>2</sup> of rela-

tive permeable w 500. In case if an air gap of 2 mm is cut in the ring what is the current in the coil to establish the same feet density.



Solution: Given data:

 $\beta = 1 \text{ wb/m}^2$ Diameter = 150 cm = 0.15 m Core area  $AC = 10 \text{ cm}^2 = 10 \times 10^{-4} \text{ m}^2$  $l_c = \pi D = \pi (0.15) \text{ mb}$ 

B, magnetic flux density =  $\frac{\text{IN} \cdot \mu_0 \mu_v}{l_c}$ 

$$1 = \frac{I \times 200 \times 4\pi \times 10^{-7} \times 500}{\pi \times 0.15}$$

$$\frac{100 \times 0.15}{4} = I \rightarrow I = 3.75 \text{ Amp.}$$
$$B = \frac{\text{mm} f}{\text{reluctance} \times \text{area}} = \frac{\text{NI}}{\text{reluctance} \times \text{area}}$$

22

If 2 mm is cut the reluctance will be sum of reluctance of air gap of core.

e.g. (air gap flux) = 
$$\frac{l_g}{\mu_0 AC}$$
  
=  $\frac{2 \times 10^{-3}}{4\pi \times 10^{-7} \times 10 \times 10^{-4}}$   
= 0.159 × 10<sup>7</sup>  
= 1.59 × 10<sup>6</sup> AT/wb  
 $R_{lc}$  (reluctance of core) =  $\frac{l_c}{\mu_0 \mu_r AC}$   
=  $\frac{(\pi D - l_g)}{4\pi \times 10^{-7} \times 500 \times 10 \times 10^{-4}}$   
=  $\frac{(\pi \times 0.15 - 2 \times 10^{-3})}{4\pi \times 10^{-7} \times 0.5}$   
= 0.746 × 10<sup>6</sup>  
 $R_{lc} + R_g = 2.336 \times 10^6 \text{ AT/wb}$ 

$$B = 1 = \frac{200 \times 1}{2.336 \times 10^{6} \times 10 \times 10^{-4}} = \frac{NI}{Rl \times AC}$$
  
I = 11.684 Amp

3. (a) Explain the significance of J-operator? What are the different forms of expressing the sinusoidal quarter in complex form?

#### Solution:

Refer Appendix C

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(b) Find the components of Z such that the current drawn quantity by the circuit same at all frequencies the following figure.

$$R_{L} = R_{C} = 5 \Omega$$

$$R_{L} = R_{C} = \sqrt{\frac{L}{C}}$$

$$5 = \sqrt{\frac{0.05}{C}}$$

$$C = \frac{0.05}{25} = 2 \times 10^{-3} \text{ F} = 2 \text{ mF}$$

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L

Γ

Solution: (	(c) The condition is that	$t R_L = R_C = \sqrt{\frac{L}{C}} .$	
<i>.</i> .	$R_L = R_C = 5 \Omega$		
from w	which $5 = \sqrt{\frac{0.05}{C}}$	5 Ω Š	ξ5Ω
	$C = 2 \times 10^{-3} \mathrm{F}$	2×10 <sup>-3</sup> F 🕂	ਂ ਡ੍ਰੋ 0.05 ਮ
The con	nponents of $Z$ are sho	own in	
figure.		••	
4. (a) Define t	he following terms		0.2 <i>I</i> <sub>1</sub>
(i) Ban	dwidth		$\longrightarrow$
$\begin{array}{c} (11)  Q-12\\ (11)  balf \end{array}$	notor	$I_1 = R_1$	R3 I2
	power nequencies	$\rightarrow$	
Solution:		+	Ŧ
<u>Refer Se</u>	ections 8.4, 8.5	V <sub>1</sub>	$\sim R_2 V_2$
(b) Obtain a	$\pi$ -equivalent circuit for	or the <	> -
	ig figure of 2 port netw	Ofk	-
Solution:		0	0
$-V_1 + I_1 R_1 + (I_1 + I_2) R_L = 0$			
	$(R_1 + R_2)$	$_{2})I_{1} + R_{2}I_{2} = V_{1}$	(1)
	$R_L(I_2 + I_1) + R_3(I_2 - 0)$	$.2 I_1) - V_1 = 0$	
	$(R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_1 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_2 + (R_2 - 0.2 R_3) I_3 $	$(R_3 + R_2)I_2 = V_2$	(2)
w.r.t.	$V_1 = Z_{11} I_1 + Z_{12} I_2$		
G	$V_2 = Z_{21} I_1 + Z_{22} I_2$		
50	$Z_{11} = R_1 + R_2$ $Z_1 = R_1$		
	$Z_{12} - R_2$ $Z_{21} = R_2 - 0.2 R_2$		
$Z_{21} = R_2 + R_2$			
$\Delta t = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} = \begin{vmatrix} R_1 + R_2 & R_2 \\ R_2 - 0.2R_3 & R_3 + R_2 \end{vmatrix}$			
	$Y_{11} = \frac{Z_{22}}{\Delta Z} = -\frac{R_3 R_2}{\Delta Z}$		
	$Y_{12} = \frac{-Z_{12}}{\Delta Z} = -\frac{R_2}{\Delta Z}$		
	$Y_{21} = \frac{-Z_{21}}{\Delta t} = \frac{R_2 - 0}{\Delta t}$	$\frac{0.2R_3}{Z}$	
	$Y_{22} = \frac{Z_{11}}{\Delta t} = \frac{R_1 + R_2}{\Delta Z}$	<u>.</u>	

••••

$$Y_{1} = Y_{11} + Y_{21}$$

$$= \frac{R_{3} + R_{2} + R_{2} - 0.2 R_{3}}{\Delta Z}$$

$$= \frac{2R_{2} - 0.8 R_{3}}{\Delta Z} + \frac{Y_{2}}{\nabla Z} + Y_{2} + Y_{2} = -Y_{12} = \frac{-R_{2}}{\Delta Z} + V_{1} + Y_{1} + Y_{3} + V_{2}$$

$$Y_{3} = Y_{22} + Y_{12} - Y_{12} - Y_{12} - Y_{12} - Y_{12} - Y_{2} + Y_{2$$

----

5. (a) Derive the relationship between and phase quantities in a balanced star connected system.

#### Solution:

<u>Refer Sections 9.7.1, 9.7.2 and 9.7.3</u>

(b) A 3 phase 4-wire CBA system of phase sequence, with effective line voltage of 100 V has a star-connected impedance given by

 $Z_A = 3.0 \angle 0^{\circ} \Omega$ ,  $Z_B = 4.5 \angle 56.31^{\circ} \Omega$  $Z_C = 2.24 \angle -26.57^{\circ} \Omega$ , obtain the line currents and the current in neutral wire draw the phasor diagram.

#### Solution:

$$V_{ph} = \frac{100}{\sqrt{3}} = 57.735 \text{ V}$$

$$V_{CN} = 57.735 \angle 0^{\circ}$$

$$V_{BN} = 57.735 \angle -120^{\circ}$$

$$V_{AN} = 57.735 \angle -240^{\circ}$$

$$I_{C} = \frac{VCN}{Z_{C}} = \frac{57.735}{2.24\angle -26.57^{\circ}} = 25.77 \angle 26.57^{\circ} = 23.048 + 11.52j$$

$$I_{B} = \frac{V_{BN}}{Z_{B}} = \frac{57.735\angle -120^{\circ}}{4.5\angle 56.31^{\circ}} = 12.83 \angle -176.31^{\circ} = -12.8 - 0.825j$$

$$I_{A} = \frac{V_{AN}}{Z_{A}} = \frac{57.735\angle -240^{\circ}}{3^{\circ}} = 19.245 \angle -240^{\circ} = -9.62 + 16.66j$$

 $I_{\text{Neutral}} = -(I_A + I_B + I_C) = -(0.628 + 27.355j)$ 



- 6. (a) For the circuit shown below in Fig. 31. Find the currents of voltages in all the branches of circuit. Use node voltage method.
  - **Solution:** Let  $V_1$  be the voltage as shown in figure

 $\frac{8-V}{1}+5+$ 

At  $V_1$ 

$$\frac{8-V}{1} + 5 + \frac{V_2 - V_1}{3} = 0$$

$$8 \vee 0$$

$$- 4 \vee 0$$

$$4 \Omega$$

$$39 - 4V_1 + V_2 = 0$$
(1)

1Ω

2Ω

V1

В

3Ω

 $V_2$ 

c

At  $V_2$ 

$$\frac{V_2 - 8}{2} + \frac{V_2 - V_1}{3} + 4V_2 = 0$$
  
-2V<sub>1</sub> + 29V<sub>2</sub> - 24 = 0 (2)

Α

From Eq. (1) and Eq. (2)  $V_1 = 10.13$ ;  $V_2 = 1.52$ 

Current in 
$$2\Omega = \frac{8-1.52}{2} = 3.24$$
 A from A to C

Current in 
$$1\Omega = \frac{8 - 10.13}{1} = -2.13$$
 A from *A* to *B*

Current in 
$$3\Omega = \frac{10.13 - 1.52}{3} = 2.87$$
 A from *B* to *C*

Current in  $4 \mho = 1.52 \times 4 = 6.08$  A downwards

- (b) Draw the dual of the network shown in the following figure. Explain the procedure employed.





(c) Obtain the Expression for characteristic impedance of symmetrical T network.

#### Solution:

Refer Section 15.13.

7. (a) State and explain superposition theorem.

#### Solution:

Refer Section 3.2

\*\*

...

(b) Using superposition theorem find the current in 2  $\Omega$  resistor. Verify the result by any other method in following figure.

#### Solution:

Consider source (1) current source on line.



**Solution:** For procedure refer Section 3.8

Current through  $2\Omega = 0$ as two point ends are shorted. Consider voltage source alive.



Current through  $2\Omega$  resistor =  $\frac{10V}{2}$  = 5A by superposition current = 0 + 5 = 5A*Verification:* Consider the entire network.



By source transfer



Current through 
$$2\Omega = \frac{V}{2}$$
  
=  $\frac{-10}{2} = -54$  Downward  
 $I = 54$  upward. Hence proved.

20

8. (a) What are the Initial conditions? How do you need them?

#### Solution:

Refer Chapters 12 and 13

(b) Explain why the current in a pure Inductance cannot change in zero time?

#### Solution:

Refer Section 1.6

(c) Switch is closed at t = 0. Find initial conditions at  $t(0^+)$  for  $i, i_2, V_C$ 

$$\frac{di_1}{dt}, \frac{di_2}{dt}, \frac{d^2i_1}{dt^2}$$
 and  $\frac{d^2i_1}{dt^2}$  in the following figure.

$$60 \text{ V} \xrightarrow{t=0 2 \text{ HF}}_{t=0 2 \text{ HF}} 10 \Omega$$

#### Solution:

At  $t = 0^-$  the circuit in un energised so all initial condition are zero. at  $t = 0^+$ 

$$i_1(0^+) = \frac{60}{20} = 3A, \quad i_2(0^+) = 0A$$



By writing KVL to loops

$$60 = \frac{1}{2 \times 10^{-6}} \int i_1(t) + 20[i_1(t) - i_2(t)]$$
(1)

$$20 (i_2(t) - i_1(t)) + 10i_2(t) + \frac{2di_2(t)}{dt} = 0$$
<sup>(2)</sup>

By substituting Initial Coils (2)

$$20(0-3) + 30 + 2\frac{di_2(0^+)}{dt} = 0 \quad \frac{di_2(0^+)}{dt} = 30$$

Differentiate (1)

$$0 = \frac{1}{2 \times 10^{-6}} i_1(t) + 20 \left( \frac{di_1(t)}{dt} - \frac{di_2(t)}{dt} \right)$$
(3)

$$\left(\frac{di_1}{dt} - 30\right) = -\frac{3}{40} \times 10^6$$
$$\frac{di_1}{dt} = 30 - \frac{3}{40} \times 10^5 = -74.97 \times 10^3$$
$$\frac{di_1(0^+)}{dt} = -74.97 \times 10^3$$

Differentiate (2)

$$20\left(\frac{di_2(t)}{dt} - \frac{di_1}{dt}\right) + 10\frac{di_2}{dt} + 2\frac{d^{\nu}i_2}{dt^{\nu}}$$
$$20(30 + 74.97 \times 10^3) + 10 \times 30 + 2\frac{d^{\nu}i_2}{dt^{\nu}} = 0$$

$$\frac{d^{\nu}i_2(0^+)}{dt^{\nu}} = -760.15 \times 10^3$$

Differentiate Eq. (3)

$$0 = \frac{1}{2 \times 10^{-6}} \frac{di}{dt} + 20 \left( \frac{d^{\nu}i_{1}}{dt^{\nu}} - \frac{d^{\nu}i_{2}}{dt^{\nu}} \right)$$

$$0 = \frac{1}{2 \times 10^{-6}} (-74.97 \times 10^{3}) + 20 \left( \frac{d^{\nu}i}{dt^{\nu}} + 750.15 \times 10^{3} \right)$$

$$i_{1}(0^{+}) = 3A$$

$$i_{2}(0^{+}) = 0A$$

$$\frac{d^{\nu}i_{i}}{dt^{\nu}} (0^{+}) = 1.873 \times 10^{\circ}$$

$$\frac{di_{1}}{dt} (0^{+}) = -74.97 \times 10^{3}$$

$$\frac{d^{\nu}i_{2}(0^{+})}{dt^{\nu}} = -750.15 \times 10^{3}$$

$$\frac{di_{2}}{dt} (0^{+}) = 30$$

## PAPER 2

- 1. (a) Discuss Kirchhoff's Laws.
- **Solution** Refer Sections 1.9 and 1.12.
  - (b) Derive the expression for self, mutual inductance and coefficient of coupling. \*\*
- Solution Refer Sections 10.3. and 10.5.
  - (c) Explain source transformation with example.

#### **Solution** Refer Section 2.15.

2. (a) What is the use of operator j?

#### Solution Refer Appendix C.

(b) For the circuit shown in figure, find the current I drawn from the source.

-----



The impedance as seen by the source is

$$Z = (10 + j20) // (8 - j15)$$
$$= \frac{380 + j10}{18 + 5j} = 19.742 - j4.928$$

 $\therefore \text{ Current drawn from source } I = \frac{V}{Z} = \frac{100}{19.742 - j4.928}$ 

$$= 4.768 + j1.19$$
  
= 4.914 |14.01°

or

$$I_{1} = \frac{100}{10 + 20j} = 2 - 4j$$
$$I_{2} = \frac{100}{8 - 15j} = 2.768 + 5.1903j$$
$$I = I_{1} + I_{2} = 4.768 + j1.19$$
$$= 4.914 | 14.01^{\circ}$$

3. (a) A series RLC circuit with Q = 250 is resonant at 1.5 MHZ. Find the

frequencies at half power points and also bandwidth.



**Solution** Given Q = 250

$$Q = \frac{\omega_o L}{R}$$
  
250 =  $\frac{2\pi \times f_o \times L}{R} \Rightarrow \frac{R}{L} = \frac{2\pi \times 1.5 \times 10^6}{250} = 37.7 \times 10^3$ 

Lower half power frequency  $f_1 = f_r - \frac{R}{4\pi L}$ 

$$= 1.5 \times 10^{6} - \frac{37.7 \times 10^{5}}{\Delta \pi}$$
$$= 1.5 \times 10^{6} - 3 \times 10^{3}$$
$$= 1.496 \text{ MHz}$$

Upper half power frequency  $f_2 = f_r + \frac{R}{4\pi L}$ 

$$= 1.5 \times 10^{6} + \frac{37.7 \times 10^{3}}{4\pi}$$
$$= 1.5M + 3k = 1.53 \text{ MHz}$$

Bandwidth =  $f_2 - f_1 = 1.53 \text{ M} - 1.496 \text{ M} = 6 \text{ kHz}$ 

(b) Distinguish between the average value and rms value of an alternating current.

**Solution** Refer Section 4.4.

4. Write and solve the equation for Mesh Current in the network shown.



Solution By source transformation technique transform 5A and 4A cur-

rent sources into voltage sources.

5A current source in parallel with 3  $\Omega$  can be transformed to 15V in series with 3  $\Omega$  and 4A current source in parallel with 3  $\Omega$  can be transformed to 12 volts in series with 3  $\Omega$ . The equivalent circuit is as shown below:



The mesh equations are

$$5I_1 + 1(I_1 - I_2) = 15$$
  

$$1(I_2 - I_1) + 4I_2 = 41$$
  

$$-I_1 + 5I_2 = 41$$
 (1)  

$$6I_1 - I_2 = 15$$
 (2)

 $\Rightarrow$ 

$$I_1 - I_2 = 15 \tag{2}$$

on solving equations (1) and (2) we get

 $I_1 = 4$  Amps  $I_2 = 9$  Amps

5. Determine the line currents for the unbalanced delta connected load of the figure given. Assume phase sequence RYB.



 $V_{BR}$ 

$$I_{RY} = \frac{200|0^{\circ}}{30+i40} [-53.13^{\circ}]$$

$$I_{YB} = \frac{200 \left| -120^{\circ} \right|}{8 + j4} = 22.36 \left| -93.43^{\circ} \right|$$
$$I_{BR} = \frac{200 \left| -240^{\circ} \right|}{15 + j12} = 10.41 \left| -278.65^{\circ} \right|$$

The line currents are  $I_R = I_{BR} - I_{RY}$  $I_Y = I_{RY} - I_{YB}$ 

$$I_B = I_{YB} - I_{BR}$$

6. The circuit shown in the figure below has resistance R which absorbs maximum power. Compute the value of R and maximum power.



**Solution** According to maximum power transfer theorem, maximum power can be transferred when load resistance is equal to the interval resistance of the source which can be calculated as the resistance seen from *AB* with source open.



Now the circuit can be drawn as



According to current dividing rule

$$I_1 = \frac{20 \times 5}{(5+3.235)} = 12.14 \text{ A}$$
$$I_2 = \frac{I_1 \times 3}{5.1} = \frac{12.14 \times 3}{5} = 7.14 \text{ A}$$

So the maximum power that can be delivered to resistor R is

 $I^2 R = (7.14)^2 \times 2.1 = 107$  watts.

7. In the figure shown below v(t) = 10 V, find  $i_2(t)$ . Assume all initial conditions to be zero. Use Laplace transform technique.



Solution Writing mesh equation

$$\frac{10}{S} = 0.1S I_1(s) + I_1(s) + \frac{2}{s} (I_1(s) - I_2(s))$$
  

$$0 = \frac{2}{S} (I_2(s) - I_1(s)) + 2I_2(s) = 0$$
  

$$\left(\frac{2}{s} + 0.1s + 1\right) I_1(s) - \frac{2}{s} I_2(s) = \frac{10}{s}$$
(1)

$$-\frac{2}{s}I_1(s) + \left(\frac{2}{s} + 2\right)I_2(s) = 0$$
(2)

on solving equations (1) and (2) we get

$$I_2(s) = \frac{100}{s(s^2 + 11s + 60)}$$
$$\frac{1}{S(S^2 + 11S + 60)} = \frac{AS + B}{S^2 + 11S + 60} + \frac{C}{S}$$
$$A = \frac{-1}{60}; \ B = \frac{-11}{60}; \ C = \frac{1}{60}$$

$$\therefore L^{-1} \left[ \frac{1}{S(S^2 + 11S + 60)} \right] = L^{-1} \left[ \frac{1}{60} \frac{1}{S} - \frac{1}{60} \frac{S}{S^2 + 11S + 60} - \frac{11}{60} \cdot \frac{1}{S^2 + 11S + 60} \right]$$
$$= \frac{1}{60} L^{-1} \left[ \frac{1}{S} - \frac{\left(S + \frac{11}{2}\right)}{\left(S + \frac{11}{2}\right)^2 + \left(\frac{\sqrt{199}}{2}\right)^2} + \frac{319}{\left(S + \frac{11}{2}\right)^2 + \left(\frac{\sqrt{119}}{2}\right)^2} \right]$$
$$\therefore i_2(t) = \frac{100}{60} \left[ 1 - e^{\frac{11}{2}t} \cos \frac{\sqrt{119}}{2} t - 319 \times \frac{2}{\sqrt{119}} e^{\frac{11}{2}t} \cdot \sin \frac{\sqrt{119}}{2} t \right]$$
$$\therefore i_2(t) = \left[ 1.667 - 1.667 e^{-5.5t} \cos 5.45t - 97.47e^{5.5t} \sin 5.45t \right]$$

8. (a) In a two-port bilateral network show that AD - BC = 1. Solution Refer Section 15.8.2.

(b) Derive an expression for DC response in an RC circuit. *Solution* Refer Section 12.3.

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## PAPER 3

1. (a) State the voltage current relationships for (i) resistance (ii) inductance and (iii) capacitance.

**Solution** Refer Sections 1.5, 1.6 and 1.7.

(b) Two coupled coils with self inductances  $L_1=0.8$ H and  $L_2=0.2$ H have a coupling coefficient of 0.6 has 500 turns. If the current in coil 1 is  $I_1(t) = 10 \sin 200t$ ; determine the voltage at coil 2 and the maximum flux set up by the coil 1.

Solution

$$M = K\sqrt{L_1L_2}$$

$$M = 0.6\sqrt{0.8 \times 0.2}$$

$$M = K\sqrt{L_1L_2}$$

$$M = 240 \text{ mH}$$

The voltage across the coil 2  $v_2(t) = \pm M \frac{di_1(t)}{dt}$ 

$$v_2(t) = \frac{d}{dt} (10\sin 200t)$$
$$v_2(t) = 2000 \text{ C is } 200t \text{ volts}$$

(c) A torroid is made of steel rod of 2 cm diameter. The mean radius of torroid is 20 cm relative permeability of steel is 2000. Compute the current required to produce 1 m web of flux and 1000 turns in the torroid.

**Solution** Length of the flux path =  $\pi D = \pi \times 20 = 62.83$  cm = 0.6283 m

Area of flux path 
$$=\frac{\pi}{4}d^2 = \frac{\pi}{4}(2)^2 = 3.141 \text{ cm}^2$$

Magnetic field intensity  $H = \frac{B}{\mu_o \mu_r}$ 

$$B = \frac{\phi}{\text{Area}} = \frac{10^{-3}}{3.141 \times 10^{-4}} 3.1 \text{ web / m}^2$$
$$H = \frac{3.1}{4\pi \times 10^{-7} \times 2000} = 1233.45 \text{ AT / m}$$
$$mmf = H \times l = 1233.45 \times 0.6283$$
$$= 775 \text{ A.T.}$$

Exciting current = 
$$\frac{mmf}{T}$$
  
=  $\frac{775}{1000}$  = 0.775 A  
2. (a) If  $I_1 = 10 | \underline{0^{\circ}}, I_2 = 20 | \underline{60^{\circ}}$  and  $I_3 = 12 | \underline{-30^{\circ}}$  find  $I_1 + I_2 + I_3$ .  
Solution  $I_1 + I_2 + I_3 = 10 | \underline{0^{\circ}} + 20 | \underline{60^{\circ}} + 12 | \underline{-30^{\circ}}$   
=  $30.392 + j11.32$   
=  $32.432 | \underline{20.429}$ 

- (b) Prove that the form factor for a sinusoidal current wave form is 1.11.
- **Solution** Refer Section 4.4.7.
- 3. (a) Derive an expression for resonance frequency of a series R-L-C circuit.

Solution Refer Section 8.1.

(a) A coil of resistance 10*l* and an inductance of 0.1 H is connected in series with a capacitor of capacitance 150  $\mu$ F a cross at 200V, 50Hz supply. Calculate (i) Impedance (ii) Current (iii) Power and power factor of the circuit



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Solution (i) Total impedance

$$Z = R + j\omega L - \frac{j}{\omega c}$$
  
= 10 + j31.45 - j21.22  
= 10 + j10.194  
= 14.279 [45.55]  
(ii) Current  $I = \frac{V}{Z}$   
=  $\frac{200[0^{\circ}]}{14.279[45.55^{\circ}]}$   
=  $14[-45.55^{\circ}]$ 

(iii) Power factor = cos (45.55°) = 0.7 lagging Real power = VI cos  $\phi$ = 200 × 14 × 0.7 = 1.9 kW Reactive power = VI sin  $\phi$ = 200 × 14 × sin (- 45.55) = -1.998 KVAR

"-1" Sign indicates that it absorbs the reactive power.

4. (a) Define cut set and tie set

**Solution** Refer Sections 2.7 and 2.8.

(b) Determine the current in the 10  $\Omega$  resistor in the circuit shown in the figure below.



**Solution** Apply nodal analysis at point (1), we get

$$\frac{V-50|\underline{0}^{\circ}}{4-j5} + \frac{V}{10} + \frac{V-50|\underline{30}^{\circ}}{5+j5} = 0$$

$$V\left[\frac{1}{4-j5} + \frac{1}{10} + \frac{1}{5+j5}\right] = \frac{50|\underline{0}^{\circ}}{4-j5} + \frac{50|\underline{30}^{\circ}}{5+j5}$$

$$V\left[0.297 + j0.0219\right] = 11.708 + j4.267$$

$$V\left[0.298|\underline{4.219^{\circ}}\right] = 12.46|\underline{20.02^{\circ}}$$

$$\Rightarrow V = 41.812 |\underline{15.801^{\circ}}$$

Current through the 10  $\Omega$  resistor  $I_{10} = \frac{V}{R}$ 

$$= \frac{41.812 | 15.801^{\circ}}{10}$$
$$= 4.1812 | 15.80^{\circ} \text{ Amp}$$

5. (a) Draw the dual network for the given network as in the following figure.



(b) A balanced star connected load of  $8 + 6j \Omega$ /phase is connected to a  $3\phi 230V$ , 50Hz supply. Find the line current, power factor, total Active and Reactive powers.



$$I_{R} = \frac{V_{RY}}{\sqrt{3} \times Z}$$
  
=  $\frac{230}{\sqrt{3} \times 10|36.86^{\circ}}$   
= 10.623|-36.86°  
 $I_{Y} = \frac{230|-120^{\circ}}{\sqrt{3} \times 10|36.86^{\circ}}$   
= 10.623|-156.86°  
 $I_{B} = \frac{230|-240^{\circ}}{\sqrt{3} \times 10}$  |36.86°  
= 10.623|276.86°  
P.f. = cos  $\phi$  = cos (36.86°) = 0.8 (lagging)  
Active power =  $3I^{2}R$   
=  $3(10.623)^{2} 8$   
= 2.708 kW  
Reactive power =  $3VI \sin \phi$   
=  $3(320) (10.623) \sin (-36.86)$   
= 4.396 KVAR

6. (a) Obtain the Norton's equivalent circuit at the terminals A, B for the following figure.



For finding the Nortons resistance, replace the voltage sources by the short circuit.



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#### = 2.253 Ω

For finding the  $I_{\rm N}$  short the terminals A and B and find current  $I_{\rm N}$ . Apply superposition

(i) with 100 V source



Fotal current 
$$I = \frac{100}{Z}$$
  
=  $\frac{100}{2.67} = 37.45 \text{ A}$ 

(ii) With 20 V source



$$I_{SN2} = \frac{20}{2.91} = 6.872 \text{ A}$$

$$\therefore I_{SN} = I_{SN_1} + I_{SN_2} \\= 31.21 + 6.872 \\= 38.08 \text{ A}$$

: Nortons equivalent circuit is given by





For the 2 port network, find *h*-parameters.

**Solution** We know that

.:.



7. A series RLC circuit with  $R = 5 \Omega$ , L = 0.1H and  $C = 500 \mu$ F has a D.C. voltage of 100V applied at t = 0 through a switch. Find the resulting current transient.



**Solution**  $5i(t) + 0.1 L \frac{di(t)}{dt} + \frac{1}{500 \times 10^{-6}} \int i(t)dt = 100$ 

$$\frac{d^{2}i}{dt^{2}} + 50\frac{di}{dt} + 2000 = 0$$

$$D^{2} + 50D + 2000 = 0$$

$$D = -25 \pm j37.08$$

$$\therefore i(t) = e^{-25t} [K_{1}\cos 37.08t + K_{2}\sin 37.08t] \qquad (1)$$
1<sup>st</sup> initial condition is that current through the inductor cannot change

1<sup>st</sup> initial condition is that current through the inductor cannot change instantaneously.

Also voltage drop across capacitor cannot change instantaneously Hence at  $t = 0^+$ 

$$\frac{di}{dt}(0^+) = \frac{V_o}{L} = \frac{100}{0.1} = 1000$$

Substituting initial conditions  $i(0^+) = 0$ 

$$\therefore \qquad 0 = K_3$$

On differentiating equation (1), we get

$$\frac{di}{dt} = e^{-25t} \left[ -37.08K_1 \sin 37.08t + 37.08K_2 \cos 37.08 \right]$$
  
$$-25e^{-25t} \left( K_1 \cos 37.08t + K_2 \sin 37.08 \right)$$
  
$$= e^{-25t} \left[ \sin 37.08t \left( -37.08K_1 - 25K_2 \right) + \cos 37.08t (37.08K_2 - 25K_1) \right]$$
  
$$\frac{di}{dt} (0^+) = 37.08K_2 - 25K_1$$
  
$$\Rightarrow K_2 = \frac{1000}{37.08} = 26.96$$
  
$$\therefore i(t) = e^{-25t} (26.96 \sin 37.08t)$$

8. (a) Explain Dot convention.

Solution Refer Section 10.4.

8. (b) Explain briefly about the locus diagrams.Solution Refer Section 8.13.

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## PAPER 4

- 1. (a) Explain about dot convention.
- **Solution** Refer Section 10.4.
  - (b) An iron ring of mean length 50 cm has an air gap of 1 mm and a winding of 200 turns. If the permeability of iron is 400 when a current of 1.25 A flows through the coil. Find the flux density.

**Solution**  $AT_1$  required for iron path in the ring  $= H_i \times l_i = \frac{B}{\mu_o \mu_r} \times l_i$ 

$$=\frac{B}{4\pi\times10^{-7}\times400}\times0.5$$

 $AT_2$  required for air gap of 1 mm =  $H_g l_g = \frac{B}{\mu_o} \times l_g$ 

$$=\frac{B}{4\pi\times10^{-7}}\times1\times10^{-3}$$

Total ampere turns =  $AT_1 + AT_2$ 

$$200 \times 1.25 = \left[\frac{B \times 0.5}{4\pi \times 10^{-7} \times 400} + \frac{B}{4\pi \times 10^{-7}} \times 10^{-3}\right]$$
  
$$250 = \frac{B}{4\pi \times 10^{-7}} \left[1.25 \times 10^{-3} + 10^{-3}\right]$$
  
$$B = 0.314 \text{ web/m}^2.$$

2. Derive the expressions for half power frequencies, Q factor  $\phi$  Bandwidth of a series resonant circuit.

**Solution** Refer Sections 8.4. and 8.5.

3. (a) For the parallel network shown below, determine the value of *R* at 10  $\Omega$  resonance.



Solution

Z = (10 + j10) | | (R - j2) $= \frac{(10 + j10)(R - j2)}{10 + j10 + R - j2}$ 

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$$= \frac{10R - j20 + j10R + 4}{10 + R + 8j}$$

$$= \frac{10R + 4 + j(10R - 20)}{10 + R + 8j}$$

$$= \frac{[(10R + 4) + j(10R - 20)][10 + R - j8]}{(10 + R)^2 + 64}$$

$$= [(10R + 4)(10 + R) + 8 (10R - 20) - j8(10R + 4) + j(10 + R)$$

$$(10R - 20)] \frac{1}{(10 + R)^2 + 64}$$
At resonance imaginary part = 0

(b) Define average value, rms value and form factor in a circuit.

# **Solution** Refer Section 4.4.

4. Determine the current in all branches of the following network and the voltage across for resistor using loop method.



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**Solution** Applying mesh equation to the loops (1), (2) and (3) We get



$$5(I_1 - I_3) + 7(I_1 - I_2) = 5$$
  
12I\_1 - 7I\_2 - 5I\_2 = 5 (1)

- $7(I_2 I_1) + 6(I_2 I_3) + 5I_2 = -25$ 
  - $-7I_1 + 18I_2 6I_3 = -25 \tag{2}$

$$10I_3 + 5(I_3 - I_1) + 6(I_3 - I_2) = 0$$
  
-5I\_1 - 6I\_2 + 21I\_3 = 0 (3)

By solving above 3 equations, we get

$$I_1 = -1.231 \text{ A}$$
  
 $I_2 = -2.172 \text{ A}$   
 $I_3 = -0.9138 \text{ A}$ 

Current in 5  $\Omega$  resistor is -0.3172 A  $\Omega$  resistor is -1.231 A  $\Omega$  resistor is -1.2882 A  $\Omega$  resistor is -0.9138 A  $\Omega$  resistor is -2.172 A

5. A 440V,  $3\phi$ , 3-wire system is connected to an unbalanced star connected load shown in the figure. Determine the line currents and power I/P to the network.

$$V_{RY} = 440$$

$$I_{R} = \frac{440}{\sqrt{3} \times 10} = 25.4 \text{ A}$$

$$I_{Y} = \frac{440 |-120^{\circ}}{\sqrt{3} \times 15}$$

$$= 16.93 |-120^{\circ} \text{ A}$$

$$I_{B} = \frac{440 |-240^{\circ}}{\sqrt{3} \times 20} \text{ Y}$$

$$= 12.7 |-240^{\circ} \text{ A}$$

$$P = I_{R}^{2} R_{R} + I_{Y}^{2} R_{Y} + I_{B}^{2} R_{B}$$

$$= 13.976 \text{ kW}$$

6. (a) Verify reciprocity theorem in circuit shown in the following figure.



E.23

**Solution** Let us find current in 3  $\Omega$  resistor.

$$I_3 = 10 \times \frac{2}{2+3}$$
$$= 4 \text{ A}$$
$$V_{ab} = 3 \times 4 = 12$$

According to reciprocity theorem the voltage across ab  $V_{ab} = 12$ Now connect the current source across *ab* and find the voltage across *m* and *n*.



The voltage across  $mn = 2 \times 6 = 12$  volts, same as  $V_{ab}$ . Hence, the reciprocity theorem is proved.

(b) State and explain compensation theorem.

**Solution** Refer Section 3.6.

7. Find transfer function  $\frac{V_o(S)}{V_i(S)}$  for the circuit shown in the following

figure.



Also  $\frac{V_A - V_i}{1} + V_A S + \frac{V_A - V_O}{2} = 0$  $V_A (1.5 + S) = V_i + \frac{V_o}{2}$  $V_A = \frac{V_i + \frac{V_o}{2}}{S + 1.5}$  \*\*

Also 
$$V_o = \frac{V_A \times \frac{2}{5}}{2 + \frac{2}{5}}$$
  
 $= \frac{V_A}{S+1}$   
 $V_o = \frac{2V_i + V_o}{(2S+3)(S+1)}$   
 $V_o \left[1 - \frac{1}{(2S+3)(S+1)}\right] = \frac{2V_i}{(2S+3)(S+1)}$   
 $V_o [(2S+3)(S+1) - 1] = 2V_i$   
 $\frac{V_o}{V_i} = \frac{2}{2S^2 + 5S + 2}$ 

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8. (a) In the circuit shown find the expression for transient current.

Solution 
$$5i(t) + 3\frac{di(t)}{dt} = 100$$
  
 $5I(S) + 3[SI(S) - i(0)] = \frac{100}{S}$   
 $i(0) = -6$   
 $100 = 5I + 3\frac{di}{dt} + 1^{\circ}(0)$   
 $\frac{100}{S} = (5 + 3S)I(S) + 18$   
 $I(S) = \frac{100 - 18S}{S(3S + 5)}$   
 $= \frac{20}{S} - \frac{78}{3S + 5}$   
 $= \frac{20}{S} - \frac{26}{S + \frac{5}{3}}$   
 $i(t) = 20 - 26 e^{5/3t}$ 

8. (b) Obtain the lattice equivalent of a symmetrical T-network.Solution Refer Example 15.13.

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# PAPER 5

1. (a) Obtain the expressions for star-delta equivalence of resistive networks.

#### Solution

Refer Section 3.1 (Chapter 3).

(b) Determine the voltage appearing across terminals y-z, if a d.c. voltage of 100V is applied across x-y terminals in the figure below.



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#### Solution Converting delta network to star network



Current,  $i = \frac{100}{1 + 3.846 + 0.77 + 2} = \frac{100}{7.616} = 13.13$ A

Voltage across  $y_z^N, V_z = -13.13 \times (2 + 0.77)$ = -36.37 V

- ....
- 2. (a) State and explain Faraday's law of electromagnetic induction. What are statically and dynamically induced emfs.

# Solution

 Refer Section 1.6 (Chapter 1).

 First law :
 It states that whenever the magnetic flux linked with a circuit changes an emf is always induced in it.

 Second law :
 It states that the magnitude of the induced emf is equal to the rate of change of flux linkage.

 Explanation :
 Suppose a coil with 100 turns undergoes a change of flux from zero refers to 2 mwb in one millisec ond.

Initial flux linkages = 0Final flux linkages =  $100 \times 2 \times 10^{-3}$  wb.T Induced emf =  $\frac{100 \times 2 \times 10^{-3} - 0}{1 \times 10^{-3}} = 2000 \text{ V}$ Induced emf can be expressed as  $e = \frac{d}{dt} (NQ) = N \frac{dQ}{dt} v$ 

Generally, a minus sign is associated with the  $N \frac{dQ}{dt}$  to signify the

fact that the induced emf sets up current in a such a direction that the magnetic effect produced by it opposes the very cause producing it. It is called Lenz's law

$$\therefore \qquad e = -N\frac{dQ}{dt}$$

Statically induced EMF

EMF induced in a coil due to the change of its own flux linked with it or emf induced in one coil by the influence of the other coil is known as statically induced emf.

Dynamically Induced EMF:

When a coil with certain number of turns or a conductor is rotated in a magnetic filed (as in d.c. generator's), an emf is induced in it which is known as dynamically induced emf.

2. (b) An iron ring 15 cms in diameter and 10  $\text{cm}^2$  in area of cross section is wound with a coil of 200 turns. Determine the current in the coil to establish a flux density of  $1 \text{ wb/m}^2$  if the relative permeability of iron is 500. In case if an air gap of 2 mm is cut in the ring, what is the current in the coil to establish the same flux density.

#### Solution

Refer Example 10.12, Chapter 10: Refer Problem 10.13 Chapter 10

(i) Without air gap

Diameter of Iron ring =  $15 \text{ (cm)} = 15 \times 10^{-2} \text{ m}$ Area of Iron ring =  $10 \text{ cm}^2 = 10 \times 10^{-4} \text{ m}^2$ Number of turns (N) = 200Reluctance of Iron ring  $(\Re_i) = \frac{l_i}{\mu_0 \ \mu_r.A}$ Length of Iron path  $(l_i)$ 

$$=\pi.d$$

$$=\pi \times 15 \times 10^{-2} \mathrm{m}$$

$$\Re_i = \frac{15\pi \times 10^{-2}}{4\pi \times 10^{-7} \times 500 \times 10 \times 10^{-4}} = 7.5 \times 105 \text{ AT/Wb}$$

 $mmf = Flux \times reluctance$ 

2m.n

$$I = \frac{1 \times 10 \times 10^{-4} \times 7.5 \times 10^{5}}{200} = 3.75 \text{ A}$$

(ii) With 2 mm air gap cut in the iron ring reluctance of air gap

$$(\Re_g) = \frac{2 \times 10^{-3}}{4\pi \times 10^{-7} \times 10 \times 10^{-4}}$$
  
= 15.915 × 10<sup>5</sup> AT/Wb

With 2 mm air gap the length of the Iron path is reduced by 2 mm.  $\therefore \qquad l_i = 15\pi \times 10^{-2} - 2 \times 10^{-3}$ 

But this is negligibly small.

:. Total reluctance =  $\Re_i + \Re_g = 23.415 \times 10^5 \text{ AT/Wb}$ 

$$I = \frac{\phi \cdot \Re}{N} = \frac{B \cdot A \cdot \Re}{N}$$
$$= \frac{1 \times 10 \times 10^{-4} \times 23.415 \times 10^{5}}{200}$$

Required current (I) = 11.707A

If the gap length is taken into consideration:

Total emf = 
$$\frac{B_i l_i}{\mu_0 \mu_r} + \frac{B_i l_g}{\mu_0}$$
  
=  $\frac{1(\pi \times 15 \times 10^{-2} - 2 \times 10^{-3})}{4\pi \times 10^{-7} \times 500} + \frac{1 \times 2 \times 10^{-3}}{4\pi \times 10^{-7}}$  2338.35 AT  
∴  $I = \frac{2338.35}{200} = 11.691$ A

3. (a) Find the form factor for the following waveform.



#### Solution

Refer Section 4.4.7 (Chapter 4)

Form factor =  $\frac{\text{R.M.S. value}}{\text{Average value}}$ 

Average value of the triangular waveform 0 to 2 sec

$$V_{av} = \frac{1}{2} \left[ \int_{0}^{1} V \cdot t \, dt + \int_{1}^{2} -V(t-2) \, dt \right]$$
$$= \frac{1}{2} \left[ V \frac{t^{2}}{2} \Big|_{0}^{1} + -V \frac{t^{2}}{2} \Big|_{1}^{2} + 2V \cdot t \Big|_{1}^{2} \right]$$
$$= \frac{1}{2} \left[ \frac{V}{2} - \frac{3}{2} V + 2V \right] = V/2$$
R.M.S. value,  $(V_{r.m.s.}) = \left[ \frac{1}{2} \left\{ \int_{0}^{1} V^{2} t^{2} \, dt + \int_{1}^{2} V^{2} (t-2)^{2} \, dt \right\}^{1/2}$ 
$$= \left[ \frac{1}{2} \left\{ V^{2} \frac{t^{3}}{3} \Big|_{0}^{1} + V^{2} \frac{t^{3}}{3} \Big|_{1}^{2} + 4V^{2} t \Big|_{1}^{2} - 4V^{2} \frac{t^{2}}{2} \Big|_{1}^{2} \right\} \right]^{1/2}$$
$$= \left[ \frac{1}{2} \left\{ \frac{V^{2}}{3} + \frac{7V^{2}}{3} - 2V^{2} \right\} \right]^{1/2}$$
$$= \left[ \frac{1}{2} \left\{ \frac{8V^{2} - 6V^{2}}{3} \right\} \right]^{1/2} = \frac{V}{\sqrt{3}}$$
Form factor =  $V/\sqrt{3}/V/2 = \frac{2}{\sqrt{3}} = 1.155$ 

3. (b) Find the branch currents, total current and the total power in the circuit shown below:



Solution

Branch currents 
$$I_1 = \frac{100 + j0}{5 - j5} = 10 + j10$$
  
 $I_2 = \frac{100 + j0}{4 - j3} = 16 - j12$ 

$$I_{3} = \frac{100 + j0}{10} = 10 + j0$$
  
Total current (I) =  $I_{1} + I_{2} + I_{3}$   
= 36 - J2  
= 36.055  $\lfloor -3.179^{\circ} \rfloor$   
Total power =  $VI \times \cos Q$   
= 100 × 36.055 × cos 3.179°  
= 3599.95 watts.

4. (a) Obtain the expression for the frequency at which maximum voltage occurs across the capacitance in series resonance circuit in terms of the *Q*-factor and resonance frequency.

# Solution

*:*..

<u>Refer Section 8.3 (Chapter 8)</u> From Section 8.3 we know that The frequency at which  $V_c$  is maximum is given by

$$f_{c} = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^{2}}{2L^{2}}}$$

$$f_{c} = \frac{1}{2\pi} \left[ \sqrt{\frac{1}{LC}} \left[ 1 - \frac{R^{2}C}{2L} \right] \right]$$

$$= \frac{1}{2\pi} \left[ \sqrt{\frac{R^{2}}{LC}} \left( \frac{1}{R^{2}} - \frac{C}{2L} \right) \right]$$

$$= \frac{1}{2\pi} \frac{R}{\sqrt{LC}} \left[ \sqrt{\frac{1}{R^{2}} - \frac{C}{2L}} \right]$$

$$= \frac{1}{2\pi} \frac{R}{\sqrt{LC}} \left[ \sqrt{\frac{C}{L}} \left[ \frac{L}{CR^{2}} - \frac{1}{2} \right] \right]$$

$$= \frac{1}{2\pi} \sqrt{LC} \cdot R \sqrt{\frac{C}{L}} \left[ \frac{L}{CR^{2}} - \frac{1}{2} \right]^{1/2}$$

$$f_{o} = \frac{1}{2\pi} \sqrt{LC}; Q = \frac{1}{R} \sqrt{\frac{L}{C}} \Rightarrow \frac{1}{Q} = R \sqrt{\frac{C}{L}}$$

$$f_{c} = \frac{f_{o}}{Q} \left[ \frac{L}{CR^{2}} - \frac{1}{2} \right]^{1/2}$$

4. (b) In a series RLC circuit if the applied voltage is 10V, and resonance frequency is 1 kHz, and Q factor is 10, what is the maximum voltage across the inductance.

Resonance freq 
$$(fr) = \frac{1}{2\pi\sqrt{LC}} = 1000$$
 (1)

Quality factor 
$$(Q) = \frac{1}{R} \sqrt{\frac{L}{C}} = 10$$
 (2)

$$\sqrt{LC} = \frac{1}{2\pi \times 1000} = 6283.18$$
$$LC = 39.47 \times 10^{6}$$
$$\frac{1}{2\pi} = \sqrt{LC} \ 1000$$
(3)

From 1,

From 2,

$$\frac{1}{R} = \sqrt{\frac{C}{L}} \ 10 \tag{4}$$

From 3 and 4

$$\frac{1}{2\pi R} = 10^4 \sqrt{LC} \sqrt{\frac{C}{L}}$$
$$\frac{1}{2\pi RC} = 10000$$
$$RC = 1.59154 \times 10^{-5} \simeq 1.6 \times 10^{-5}.$$

The maximum voltage across the inductance occurs at frequency greater than the resonance frequency which is given by

$$f_L = \frac{1}{2\pi \sqrt{LC - \frac{(RC)^2}{2}}}$$
$$f_L = \frac{1}{2\pi \sqrt{39.47 \times 10^6 - \frac{(1.6 \times 10^{-5})^2}{2}}} = 1002.5$$

It can be observed that, the above frequency is approximately equal to resonance frequency,

$$f_r = \frac{1}{2\pi \sqrt{LC}} = \frac{1}{2\pi \sqrt{39.47 \times 10^6}}$$

Hence we can take the voltage across the inductor

$$= Q \times V$$
  
= 10 × 10  
= 100 volts

(c) In a parallel resonance circuit shown in figure find the resonance frequency, dynamic resistance and bandwidth.

$$R = 2 \Omega$$

$$L = 1 \text{ mH}$$

$$C = 10 \mu\text{F}$$

The circuit shown in the above figure is the most common form of parallel resonant circuit in practical use and is also called the tank circuit.

The admittance of the circuit is

$$Y = \frac{1}{z} = \frac{1}{z_C} + \frac{1}{Z_L}$$

$$Y = \frac{1}{-jX_C} + \frac{1}{R+jX_L}$$

$$= j\omega C + \frac{1}{R+j\omega L}$$

$$= j\omega C + \frac{R-j\omega L}{R^2 + \omega^2 L^2}$$

$$= \frac{R}{R^2 + \omega^2 L^2} + j\omega \left(C - \frac{L}{R^2 + \omega^2 L^2}\right)$$

At resonance the susceptance part is zero.

Hence at 
$$\omega = \omega_r, C = \frac{L}{R^2 + \omega_r^2 L^2} = 0$$
  
 $R^2 + \omega_r^2 L^2 = \frac{L}{C}$   
 $\omega_r^2 L^2 = \frac{L}{C} - R^2 \Rightarrow \omega_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$ 
(1)

Resonance frequency,  $f_r = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$  (2)

$$f_r = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \frac{1}{2\pi L} \sqrt{\frac{L}{C} - R^2}$$
$$= \frac{1}{2\pi \times 1 \times 10^{-3}} \sqrt{\frac{1 \times 10^{-3}}{10 \times 10^{-6}} - 4}$$
$$= 1559.4 \text{ Hz}$$

Dynamic impedance:

The input admittance at resonance is given by

$$Y_r = \frac{R}{R^2 + \omega_r^2 L^2}$$

The impedance at resonance is

$$Z_r = \frac{1}{y_r} = \frac{R^2 + \omega_r^2 L^2}{R} = R + \frac{\omega_r^2 L^2}{R}$$

Substituting  $\omega r^2 L^2$  from Eq. 1 gives,

$$Z_r = R + \frac{\frac{L}{C} - R^2}{R} = R + \frac{L}{CR} - R$$

 $Z_r = \frac{L}{CR}$  which is called dynamic impedance.

This is a pure resistance because it is independent of the frequency.

Here, dynamic resistance 
$$= \frac{1 \times 10^{-3}}{10 \times 10^{-6} \times 2}$$
$$= 50 \Omega$$

Bandwidth of the parallel resonance circuit =  $\frac{\omega_r}{Q}$ 

$$\omega_r = \frac{1}{L} \sqrt{\frac{L}{C} - R^2}$$
  
= 9797.95  
$$Q_o = \frac{\omega o L}{R} = \frac{9797.95 \times 1 \times 10^{-3}}{2} = 4.898$$
  
Bandwidth =  $\frac{1559.4}{4.898} = 318.311 + Z$ 

5. (a) A symmetrical 440V, 3 phase system supplies a star connected load with the following branch impedances: 
$$Z_r = 10\Omega Z_y = j5\Omega Z_B = j5\Omega$$
 Calculate voltage drop across each branch and the potential difference between neutral and star point. The phase sequence is RYB Draw phasor diagram.

# Solution

Refer Problem 9.12 (Chapter 9)

Applying KVL for the two loops

$$V_{\rm RY} = 440 \angle 0 \text{ V}$$
  
 $V_{\rm YB} = 440 \angle -120 \text{ V}$ 



$$10I_1 - J_5I_2 = 440 \angle 0$$

$$j_5I_2 + (I_1 + I_2) (-J_5) = 440 \angle -120^\circ$$

$$-i_5I_1 + (i_5 - i_5)I_2 = -220 - J \ 381.05$$
(2)

$$I_1 = \frac{-220 - j \, 381.05}{-j \, 5}$$
$$I_1 = (76.21 - j44)$$

Substituting the value of  $I_1$  in Eq.1

10 [76.21 - j44] - j5 
$$I_2 = 440$$
  
-j5  $I_2 = -322.1 + j440$   
 $I_2 = \frac{-322.1 + j440}{-j5}$   
 $I_2 = [-88 - j64.42]$   
Drop in the *R*-phase = 10 [76.21 - j44]  
 $V_{RO'} = 880 \angle -30^{\circ}$   
Drop in the *Y*-phase = j5 [-88 - j64.42]  
 $V_{YO'} = 545.3 \angle -53.7^{\circ}$   
Drop in the *B*-Phase = j5[ $I_1 + I_2$ ]  
= j5 (-11.79 - J108.42]  
= 542.1 - J58.98  
 $V_{BO'} = 545.3 \angle -6.2$ 

....

Γ



Taking  $V_{RY}$  as reference,

$$V_{RY} = 440 \angle 0$$
  

$$V_{RO} = \frac{440}{\sqrt{30}} \angle -30^{\circ} = 254 \angle -30^{\circ}$$
  

$$V_{YO} = 254 \angle -150^{\circ}; V_{BO} = 254 \angle 90^{\circ}$$
  

$$Y_R = \frac{1}{Z_R} = \frac{1}{10 \angle 0} = 0.1 \angle 0^{\circ},$$
  

$$Y_Y = \frac{1}{Z_Y} = \frac{1}{j5} = 0.2 \angle -90^{\circ}$$
  

$$Y_B = \frac{1}{Z_B} = \frac{1}{-j5} = 0.2 \angle 90^{\circ}.$$

Neutral to star point voltage  $V'_{o'o} = \frac{V_{RO} Y_R + V_{YO} Y_y + V_{BO} Y_B}{Y_R + Y_Y + Y_B}$ 

$$V'_{o'o} = 254 \frac{0.1 \angle -30^\circ + 0.2 \angle -240^\circ + 0.2 \angle 180^\circ}{0.1 \angle 0^\circ + 0.2 \angle -90^\circ + 0.2 \angle 90^\circ}$$
  
= 625.8 \angle 150^\circ  
$$V'_{o'o} = 625.8 \ angle 150^\circ.$$

The phasor diagram follows



E.35

5. (b) A balanced star connected load is supplied from a symmetrical 3 phase, 440V, 50Hz supply. The current in each phase is 20A and lags behind its phase voltage by an angle of 40°. Calculate (i) load parameters (ii) total power and (iii) readings of two wattmeters connected in the load circuit to measure total power.

(i) Let the phase sequence be *RYB*.

The line voltage,  $V_{RY} = 440 \angle 0^{\circ} \text{ V}$ Phase voltage,  $V_R = \frac{440 \angle 0}{\sqrt{3}} = 254 \angle 0^{\circ}$  $Z_R = \frac{254 \angle 0}{20 \angle -40} = 12.7 \angle 4^{\circ}$ 

$$= (9.72 + j8.16) \Omega$$

Load parameters are  $R = 9.72 \Omega$ 

$$x_L = 8.16; f = 50 \text{ H}; L = \frac{8.16}{2 \times \pi \times 50} = 25.9 \text{ mH}$$

(ii) Total active power =  $\sqrt{3} V_{\rm L} I_{\rm L} \cos \phi$ 

$$= \sqrt{3} \times 440 \times 20 \times \cos 40^{\circ}$$
$$= 11676.08 \text{ watts}$$

(iii) Reading of first watt meter  $W_1 = V_L I_L \cos (30 + \phi)$ 

$$= 440 \times 20 \cos (30 + 40) = 3009.777 W$$

Reading of second watt meter,  $W_z = V_L I_L \cos (30 - \phi)$ 

 $= 440 \times 20 \cos (30 - 40) = 8666.308 W$ 

Total power,  $w_1 + w_2 = 11676.08$  watts

- (a) Define the following
  - (i) Oriented graph
  - (ii) Tree of a graph
  - (iii) Cut set and basic cut set
  - (iv) Tie set and Basic Tie set

#### Solution

Refer Sections 2.1, 2.2, 2.7 and 2.8 (Chapter 2)

6. (b) For the topological graph shown in figure, obtain the fundamental Tie set matrix choosing the tree containing two elements 5 and 6.



Refer Section 2.7 (Chapter 2)

The tree of the graph is shown with solid lines (5 and 6) and the links are shown with dashed lines (1, 2, 3, 4).



For a given tree of a graph, addition of each link between any two nodes forms a loop called the fundamental loop. In a loop there exists a closed path and a circulating current, which is called the link current.

The fundamental loop formed by one link at a time, has a unique path in the tree rolling the two nodes of the link. This loop is also called f-loop or a tie-set. Every link defines a fundamental loop of the network.

No. of nodes in the graph n = 3 = (A, B, C)No. of branches, b = 6 = (1, 2, 3, 4, 5, 6)No. of tree branches or twigs = n - 1 = 2 = (5, 6)No. of link branches, 1 = b - (n - 1) = 4 (1, 2, 3, 4)The following are the figures of the Tie-sets.



Tie set Matrix can be formed by considering the four fundamental loops. Corresponding to the link branches 1, 2, 3, 4. If  $V_1$ ,  $V_2$ ,  $V_3$ ,  $V_4$ ,  $V_5$  and  $V_6$  are the respective branch voltages. The KVL equations for the three *f*-loops can be written as

$$V_1 + V_5 + V_6 = 0$$
$$V_2 - V_5 = 0$$
$$V_3 - V_6 = 0$$
$$V_4 + V_5 + V_6 = 0$$

In order to apply KVL to each loop, we take the reference direction of the loop which coincides with the reference direction of the link defining the loop.

The above equations can be written in matrix form as  $[B][V_h] = 0$ , where B is a 4 × 6 Tie-set matrix.

Lo	loops Branches									
¥	1	2	3	4	5	6	$V_1$			
$l_1$	1	0	0	0	1	1	$V_2$		$\begin{bmatrix} 0 \end{bmatrix}$	
$l_2$	0	1	0	0	-1	0	$V_3$		0	
$l_3$	0	0	1	0	0	-1	$V_4$	=	0	
$l_4$	0	0	0	1	1	1	$V_5$		0	
							$V_6$			

Therefore, Tie-set Matrix, 
$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

7. (a) State and explain the superposition theorem.

# Solution

Refer Section 3.2 (Chapter 3)

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7. (b) Is superposition valid for power? Explain.

# Solution

Superposition theorem is valid only for linear systems.

Superposition cannot be applied for power because the equation for power is non linear.

Let us consider a network with a voltage source and current source as shown below and find the power consumed in  $9\Omega$  resistor by super position.



When 14V source is acting the current in 9 $\Omega$  is 1A The power =  $i^2 \times 9 = 9$  watts When 14A source is acting the current in 9 $\Omega$  is 5A The power =  $i^2 \times 9 = 225$  watts Total power = 225 + 9 = 234 Watts When both are acting the KVL for loop 1 and 2 are 14 =  $5i_1 + 9(i_1 + i_2)$ 14 $i_1 = -112$   $i_1 = -8A; i_2 = 14A$ Current in 9 $\Omega$  resistor is  $i_1 + i_2 = 6A$ Power =  $(6)^2 \times 9 = 324$  watts

 $Fower = (0) \times 9 = 324$  watts

Since power is not the same in both the cases, the superposition theorem does not hold true.

Consider the circuit shown below.



When  $V_a$  is acting.



 $I^1$  be the current through  $R_L$ ; and Power =  $(I)^2 R_L$ When  $V_b$  is acting  $I^{''}$  be the current



through  $B_L$  and Power =  $(I'')^2 R_L$ Total current through  $R_L$  by superposition I = I' + I'', and power =  $I^2 R_L$ 



7. (c) Using superposition theorem, find  $V_{AB}$ .



#### Solution

When 4V source is acting











Voltage across  $AB = V_{AB1} + V_{AB2} + V_{AB2}$ = -2 + 1 -4 = -5 volts 8. (a) Explain why the voltage across capacitor cannot change instantaneously?

# Solution

Refer Section 1.7 (Chapter 1)

8. (b) What is the significance of time constant for R-L circuit? What are the difficult ways of defining time constant?

# Solution

Refer Section 12.2 (Chapter 12)

8. (c) Switch S is closed at t = 0. Find initial conditions for voltage across capacitor.

*i i*.  $\frac{di_1}{di_1}$  and  $\frac{di_2}{di_2}$ 

$$100 \vee \underbrace{t=0}_{10 \Omega} \underbrace{t=0}_{i_1} \underbrace{t=0}_{i_2} \underbrace{t=0}_{i_1} \underbrace{t=0}_{i_2} \underbrace{t=0}_{i_1} \underbrace{t=0}_{i_2} \underbrace{t=0}_{i$$

Solution



E.41



Applying KVL for the loops at  $t = 0^+$ 

$$5i_{1} + 3\frac{di_{1}}{dt} = 100$$

$$3\frac{di_{1}}{dt} = 100 - 5i_{1}$$

$$\frac{di_{1}}{dt}\Big|_{t=0^{+}} = \frac{100 - 5i_{1}(0^{+})}{3} = \frac{100 - 5 \times 6.667}{3}$$

$$\frac{di_{1}}{dt}(0^{+}) = 22.21 \text{ A/sec.}$$

$$20i_{2} + \frac{1}{C}\int i_{2} dt = 100$$

$$20\frac{di_{2}}{dt} + \frac{i_{2}}{C} = 0$$

$$\frac{di_{2}}{dt}\Big|_{t=0^{+}} = \frac{-i_{2}(0^{+})}{20 \times 10 \times 10^{-6}} = \frac{-3.33}{2 \times 10^{-4}}$$

$$\frac{di_{2}}{dt}(0^{+}) = -16.65 \times 10^{3} \text{ A/sec.}$$

# PAPER 6

1. (a) Explain KCL and KVL. Solution

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Refer Sections 1.12 and 1.9 (Chapter 1)
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- 1. (b) A capacitor is charged to 1 volt at t = 0. A resistor of 1 ohm is connected across its terminals. The current is known to be of the form  $i(t) = e^{-t}$  amperes for t > 0. At a particular time the current drops to 0.37A at that instant determine.
  - (i) At what rate is the voltage across the capacitor changing?
  - (ii) What is the value of the charge on the capacitor?
  - (iii) What is the voltage across the capacitor?
  - (iv) How much energy is stored in the electric field of the capacitor?
  - (v) What is the voltage across the resistor?

#### Solution

Refer Problem 12.3 (Chapter 12).

The current equation is

given as  $i(t) = i(0^+) e^{-t|RC}$ ; given  $i(t) = e^{-t|RC}$   $i(0^+) = 1A$ ; RC=1; C=1FWhen i(t)=0.37 amperes  $i(t) = 0.37 = e^{-t/1}$   $-t \log_e e = \log_e 0.37$  t = 0.9942 sec  $i(t) = C \frac{dV(t)}{dt} \Rightarrow \frac{dV(t)}{dt} = \frac{i(t)}{C} = \frac{0.37}{1} = 0.37$  V/sec or  $V_i(t) = \frac{1}{C} \int_0^t i(t) dt + V_0$   $= -\frac{1}{C} \int_0^t e^{-t} dt + V_0$  [ $\therefore i(t) = -i(t)$ ]  $= \frac{-1}{1} \frac{e^{-t}}{(-1)} + 1 = e^{-t}$ 

$$V_c(t) = e^{-t}$$
 for  $t > 0$   
 $\therefore \frac{dV_C(t)}{dt} = -e^{-t} = -e^{-0.9942} = -0.37$  V/sec

(ii) Charge on the capacitor

 $Q = C V_c = 1.e^{-t} = 0.37$  coulombs

(iii) Voltage across the capacitor

$$V_C(t) = e^{-t} = 0.37$$
 volts

(iv) Energy stored in the capacitor

$$W_C = \frac{1}{2} C V_c^2 = \frac{1}{2} 1(e^{-t})^2 = \frac{e^{-2t}}{2} = 0.06845$$
 joules

(v) Voltage across the resistor at t = 0.9942 sec

$$V_R = i(t).R = e^{-t} = 0.37 \text{ V}$$

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2. (a) Define Magneto Motive Force (MMF); reluctance, and flux density in a magnetic circuit. Specify the units of each of the above quantities.

#### Solution

Refer Section 10.11 (Chapter 10).

(b) Explain "dot convention" for a set of magnetically coupled coils. A cast steel electromagnet has an air gap of length 2 mm and an iron path of length 30 cms. Find the MMF needed to produce a flux density of 0.8T in the air gap. The relative permeability of the steel core at this flux density is 1000. Neglect leakage and fringing.

#### Solution

For "dot convention" refer Section 10.4 (Chapter 10). Refer Example 10.2 (Chapter 10). Air-gap length  $l_g = 2 \text{ mm} = 2 \times 10^{-3} \text{ m}$ Iron path length  $l_i = 30 \text{ cm} = 30 \times 10^{-2} \text{ m}$ Flux density  $B = 0.8\text{T} = 0.8 \text{ Wb/m}^2$   $\mu_r = 1000$ Total A.T = mmf =  $H_i l_i + H_g l_g$   $\frac{B \times l_i}{\mu_0 \ \mu_g} + \frac{B}{\mu_0} \ l_g$   $= \frac{0.8 \times 30 \times 10^{-2}}{4\pi \times 10^{-7} \times 1000} + \frac{0.8 \times 2 \times 10^{-3}}{4\pi \times 10^{-7}}$ = 1464 A.T.

Hence, total MMF required to produce a flux density of 0.8T = 1464 AT.

3. (a) Find R.M.S. and average value of the following waveform.



# Solution

Refer problem 4.13 (Chapter 4)

R.M.S. value, 
$$V_{r.ms.} = \sqrt{\frac{1}{2\pi} \int_{0}^{\pi} V_{m}^{2} \sin^{2} \theta \, d\theta}$$
  
 $= \sqrt{\frac{V_{m}^{2}}{2\pi} \int_{0}^{\pi} \frac{(1 - \cos 2\theta)}{2} \, d\theta}$   
 $= \sqrt{\frac{V_{m}^{2}}{4\pi} \left[ \theta - \frac{\sin 2\theta}{2} \right]_{0}^{\pi}}$   
 $= \frac{V_{m}}{2}$   
Average value,  $V_{ave} = \frac{1}{2\pi} \int_{0}^{2\pi} V_{m} \sin \theta \, d\theta = \frac{V_{m}}{2\pi} \left[ -\cos \theta \right]_{0}^{2\pi}$   
 $= \frac{V_{m}}{\pi}$ 

3. (b) Find the total current and the power consumed by the circuit.



\*\*

Total impedance of the circuit,

$$Z_{\rm T} = (5+j5) \parallel (t-j8) + 10$$
  

$$Z_{T} = 16.15 + j0.769$$
  

$$I = \frac{V}{Z} = \frac{200 \angle 0}{16 + 5 + j0.769} = 12.35 - j0.588A$$
  

$$= 12.36 \angle -2.72^{\circ}$$
  
Power consumed =  $I^{2}R$   

$$= (12.36)^{2} \times 16.15 = 2467W$$
  
or  $VI \cos\theta = 200 \times 12.36 \times \cos(-2.72)$   

$$= 2467 \text{ W.}$$

4. (a) For a series RL circuit obtain the locus of current as inductance is changed from 0 to  $\infty$  when the applied voltage is constant.

# **Solution**

Refer Section 13.1(b) (Chapter 13)

4. (b) Show that for a series resonant circuit  $f_1f_2 = f_r^2$  where  $f_1$  and  $f_2$  are half power frequencies and  $f_r$  is the resonance frequency. Solution

Refer Section 8.4 (Chapter 8)

4. (c) Obtain the z-parameters of the following two-parts network. Two-port network



**Solution** 

$$V_{1} = Z_{11} I_{1} + Z_{12} I_{2}$$

$$V_{2} = Z_{21} I_{1} + Z_{22} I_{2}$$

$$z_{11} = \frac{V_{1}}{I_{1}}\Big|_{I_{2}=0} = \frac{6 \times 2}{6 + 2} + 2 = 3.5 \Omega$$

$$z_{22} = \frac{V_{2}}{I_{2}}\Big|_{I_{1}=0} = \frac{6 \times 2}{6 + 2} + 2 = 3.5 \Omega$$

$$z_{12} = \frac{V_{1}}{I_{1}}\Big|_{I_{1}=0} = \frac{\frac{I_{2} \times 2}{6 + 2} \times 2}{I_{2}} = 0.5 \Omega$$

22

$$z_{21} = \frac{V_2}{I_1}\Big|_{I_2=0} = \frac{\frac{I_1 \times 2}{6+2} \times 2}{I_2} = 0.5 \,\Omega$$

The parameters of the network are

$$z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3.5 & 0.5 \\ 0.5 & 3.5 \end{bmatrix}$$

5. (a) Derive the relationship between phase quantities and line quantities in a 3 phase balanced (i) star connected system and (ii) Delta connected system. Draw phasor diagrams showing voltages and currents.

#### Solution

Refer Sections 9.7 and 9.8 (Chapter 9).

5. (b) A 3 phase supply with line voltage of 250V, has an unbalanced delta connected load as shown in figure. Determine the line currents, total active and reactive powers if the phase sequence is *ABC*.



#### Solution

Refer Problem 9.9 (Chapter 9).

$$I_{AB} = \frac{V_{AB}}{Z_{AB}} = \frac{250 \ \angle 0^{\circ}}{25 \ \angle 90^{\circ}} = 10 \ \angle -90^{\circ}$$
$$I_{BC} = \frac{V_{BC}}{Z_{BC}} = \frac{250 \ \angle -120^{\circ}}{16 \ \angle 20^{\circ}} = 15.625 \ \angle -140^{\circ}$$
$$I_{CA} = \frac{V_{CA}}{Z_{CA}} = \frac{250 \ \angle 120^{\circ}}{20 \ \angle 0^{\circ}} = 12.5 \ \angle 120^{\circ}$$

The line currents are

$$I_A = I_{AB} - I_{CA} = 10\angle -90^\circ - 12.5\angle 120^\circ$$
$$I_B = I_{BC} - I_{AB} = 15.625 \angle -140^\circ - 10\angle -90^\circ$$

$$I_{C} = I_{CA} - I_{BC} = 12.5 \angle 120^{\circ} - 15.625 \angle -140^{\circ}$$

$$Z_{AB} = 0 + j25; Z_{BC} = 15.03 + j5.47; Z_{CA} = 20 + j0$$
Active Powers  $P_{AB} = I_{AB}^{2} R_{AB} = 10^{2} \times 0 = 0$ 

$$P_{BC} = I_{BC}^{2} R_{BC} = (15.625)^{2} \times 15.03 = 3669.4W$$

$$P_{CA} = I_{CA}^{2} \times R_{CA} = (12.5)^{2} \times 20 = 3125W$$

Total active power =  $P_{AB} + P_{BC} + P_{CA} = 6795$ W

Reactive powers

$$Q_{AB} = I_{AB}^2 \times \chi_{AB} = (10^2) \times 25 = 2500 \text{ VAR}$$
$$Q_{BC} = I_{BC}^2 \times \chi_{BC} = (15.625)^2 \times 5.47 = 1335 \text{ VAR}$$
$$Q_{CA} = I_{CA}^2 \chi_{CA} = (12.5)^2 \times 0 = 0$$

Total reactive power =  $Q_{AB} + Q_{BC} + Q_{CA} = 3835$  VAR

Complex power, S = P + jQ

$$= 6795 + j3835$$

6. (a) What is duality? Explain the procedure for obtaining the dual of the given planar network shown below.



# Solution

Refer Section 3.8 (Chapter 3)

*Rule 1* If a voltage source in the original network produces a c.w current in the mesh, the corresponding dual element is a

current source whose direction is towards node representing the corresponding mesh.

*Rule 2* If a current source in the original network produces a current in clockwise direction in the mesh, the voltage source in the dual network will have a polarity such that the node representing the corresponding mesh is positive.



Dual of the planar circuit given in 6(a).

6. (b) Construct the incident matrix for the graph show in figure.



#### Solution

Refer Section 2.4 (Chapter 2)

The dimensions of incidence matrix 'A' is  $n \times b$  where *n* is number of nodes and *b* is number of branches, hence the dimensions of the incidence matrix for the above graph is  $3 \times 4$ .

Incidence matrix

n — nodes

b — branches

	n p	1	2	3	4
4	1	1	0	-1	-1
A =	2	-1	1	0	0
	3	0	-1	1	1

\*\*

The incidence matrix is given by

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

-

6. (c) Use nodal analysis, to determine the voltage  $V_1$  and  $V_2$  in the circuit shown.



#### Solution

<u>Refer Section 2.12 (Chapter 2).</u> The nodal equation for the two nodes are

$$\frac{V_1 - 5}{2} + \frac{V_1}{3} + \frac{V_1 - V_2}{2} = 0 \qquad \dots 1$$

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} = 3 \qquad \dots 2$$
  
1.333  $V_1 - 0.5 V_2 = 2.5$ 

From 2  $-0.5 V_1 + 1.5 V_2 = 3$ 

Solving the above equations for  $V_1$  and  $V_2$  yields

$$V_1 = 3 \text{ V} \text{ and } V_2 = 3 \text{ V}.$$

7. (a) State and explain the Thevenin's theorem? State for what type problems this theorem is useful.

#### Solution

From 1

Refer Section 3.3 (Chapter 3).

7. (b) Find the current through 10  $\Omega$  resistor using Thevenin's theorem.



<u>Refer Problem 3.6 (Chapter 3).</u> Let us redraw the circuit by removing  $10\Omega$ .



....

$$R_{1} = \frac{6 \times 15 + 15 \times 2 + 2 \times 6}{2} = 66$$
$$R_{2} = \frac{132}{15} = 8.8$$
$$R_{3} = \frac{132}{6} = 22$$



$$R_{\rm ab} = R_{\rm th} = \frac{66 \times 6.82}{72.82} = 6.184 \ \Omega$$

Thevenin's equivalent circuit is given by



where 
$$I = \frac{11.538}{16.184} = 0.7129$$
A

8. (a) For R-L-C series circuit with d.c. excitation discuss the underdamped, over-damped and critically damped cases.

#### Solution

Refer Section 12.4 (Chapter 12).

8. (b) Obtain the current i(t) for  $t \ge 0$  using tune domain approach.



Refer Example 12.3 (Chapter 12). Writing KVL for the above circuit.

$$100 = 10i + 0.5 \frac{di}{dt} + \frac{1}{1 \times 10^{-6}} \int i \, dt$$

Differentiating w.r.t. t

$$0 = 10 \frac{di}{dt} + 0.5 \frac{di}{dt^2} + 10^6 i$$
$$\frac{d^2 i}{dt^2} + 20 \frac{di}{dt} + 2 \times 10^6 i = 0$$
$$(D^2 + 20D + 2 \times 10^{-6})i = 0 \text{ where } D = \frac{di}{dt}$$
$$D_1, D_2 = \frac{-20 \pm \sqrt{400 - 4 \times 2 \times 10^6}}{2}$$
$$D_1 = -10 + j1414; D_2 = -10 - j1414$$

1:

12;

The roots are in the form of  $-K_1 \pm jK_2$ Therefore the solution for the current is given by

> $i(t) = e^{-kt} [C_1 \cos k_2 t + C_2 \sin k_2 t]$  $i(t) = e^{-10t} [C_1 \cos 1414 t + C_2 \sin 1414 t]$

Substitute the initial conditions to find  $C_1$  and  $C_2$ At t = 0; the current following through the circuit is zero.

$$i = 0 = 1 [C_1 \cos 0 + C_2 \sin 0]$$
  

$$C_1 = 0$$
  

$$i(t) = e^{-10t} C_2 \sin 1414t.$$
  

$$\frac{di(t)}{dt} = C_2 [e^{-10t} 1414 \cos 1414 t + e^{-10t} (-10) \sin 1414t]$$

At t = 0, the voltage across the inductor

$$L \frac{di(t)}{dt} = 100$$

$$\frac{di(t)}{dt} = 200$$

$$200 = C_2 e^{-(10 \times 0)} 1414$$

$$C_2 = 0.1414$$
The equation for current is given by
$$i(t) = e^{-10t} (0.1414 \sin 1414 t)$$

$$i(t) = 0.1414 e^{-10t} \sin 1414t$$

# PAPER 7

1. (a) Explain how source transformation is achieved.

Solution

Refer Section 2.15 (Chapter 2).

(b) A current of 0.5A is supplied by a source to an inductor of 1H. Calculate the energy stored in the inductor. What happens to this energy if the source is short circuited?

# Solution

Energy stored 
$$\frac{1}{2}$$
 L I<sup>2</sup> =  $\frac{1}{2}$  1 × 1<sup>2</sup> = 0.5 Joules

If the inductor has an internal resistance, the stored energy is dissipated in the resistance after the short circuit as per the time constant (1/r) of the coil.

If the coil is a perfect inductor, the current would circulate through the shorted coil continuously.

(c) A current source  $i = I_m \sin \omega t$  is applied across (i) a 1F capacitor (ii) 1H inductor. Assume initial conditions to be zero, show the voltage waveforms in the above two cases.

# Solution

Refer Sections 4.6 and 4.7 (Chapter 4).

$$V_c(t) = \frac{1}{C} \int i \, dt = \frac{1}{C} \int I_m \sin \omega t \, dt$$
$$= \frac{1}{\omega C} I_m [-\cos \omega t]$$

[: initial conditions assumed to be zero]

$$= \frac{I_m}{\omega C} \sin (\omega t - 90)$$

$$v(t) = V_n \sin (\omega t - 90^\circ)$$

$$v(t) = IM [V_m e^{j(\omega t - 90^\circ)}] \text{ or } V_m \angle -90^\circ$$

where  $V_m = I_m / wC$ .



$$V_L(t) = L \frac{di(t)}{dt}$$
  
=  $L \cdot \frac{d}{dt} (I_m \sin \omega t)$   
=  $LW I_M \cos \omega t = WL I_m \cos \omega t$   
 $V_L(t) = V_m \cos \omega t \text{ or } V_m \sin (\omega t + 90^\circ)$   
=  $IM [V_m e^{t(\omega + 90^\circ)}] \text{ or } V_m \angle 90^\circ$   
where  $V_m = WLI_m$ 

The waveform is shown in the figure above.

2. (a) Define MMF, flux density, magnetising force and permeability specify the merits for each of the above quantities.

# Solution

Refer Section 10.11 (Chapter 10).

2. (b) Two coupled coils have self induction  $\cos L_1 = 50$  mH and  $L_2 = 200$  mH and a coefficient of coupling of 0.7. If coil 2 has 1000 turns and  $i_1 = 5.0 \sin 400t$ . Determine the voltage across coil 2.

# Solution

Refer Problem 10.3 (Chapter 10).

 
$$L_1 = 50 \text{ mH}, L_2 = 200 \text{ mH}; K = 0.7$$
 $M = K\sqrt{L_1 L_2} = 0.7 \sqrt{50 \times 200} \text{ mH}$ 
 $= 70 \text{ mH}$ 
 $V_2 = M \frac{di_1}{dt}$  (Voltage induced in coil 2)

  $= 70 \times 10^{-3} \frac{d}{dt}$  (5 sin 400 t)

  $= 70 \times 10^{-3} \times 2000 \cos 400 t$ 

 Total voltage induced in coil 2 is  $= 140 \cos 400 t$ 

Total voltage induced in coil 2 is =  $140 \cos 400 t$  volts.

2. (c) Write the voltage equation for the following circuit shown.

\*\*



Applying KVL around the loop is given by

$$V(t) = L_{1} \frac{di(t)}{dt} + M_{A} \frac{di(t)}{dt} - M_{C} \frac{di(t)}{dt} + R_{1}i(t)$$
  
+  $L_{2} \frac{di(t)}{dt} + M_{A} \frac{di(t)}{dt} - M_{B} \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt$   
+  $L_{3} \frac{di(t)}{dt} - M_{C} \frac{di(t)}{dt} - M_{B} \frac{di(t)}{dt} + R_{2}i(t)$   
 $v(t) = (L_{1} + L_{2} + L_{3}) \frac{di(t)}{dt} + (R_{1} + R_{2})i(t) + \frac{1}{C} \int i(t) dt$   
+  $(2M_{A} - 2M_{B} - 2M_{C}) \frac{di(t)}{dt}$ 

3. (a) Define rms value, average value, form factor and peak factor.

# Solution

Refer Section 4.4 (Chapter 4).

3. (b) Find the value of R<sub>1</sub> and X<sub>1</sub> when a lagging current in the circuit gives a power of 2kW.



Let us take the voltage across  $(10 + j13.3\Omega)$  impedance as reference and calculate the total current I.

$$I = \frac{200 \ \angle 0}{10 + j13.3} = 7.223 - j9.606 = 12.02 \ \angle -53.06^{\circ} \text{A}$$

Let us assume the phase angle between supply voltage and total current as  $\phi$  which is equal to  $(\theta + 53.06^{\circ})$ .

Hence, real power in the circuit  $2000 = 200 \times 12.02 \cos (\theta + 53.06)$ Therefore,  $\theta = -19.5^{\circ}$  and source voltage  $V = 200 \angle -19.5^{\circ}$ 

Voltage across 
$$R_1 + jX_1 = 200 \ 2 - 195^\circ - 200 \ 20^\circ$$
  
 $= -11.47 - j66.76$   
 $I_2 = \frac{-11.47 - j66.76}{-j20} = 3.338 - j \ 0.5 \ 735$   
 $I_1 = I - I_2$   
 $= 7.223 - J9.606 - 3.338 + J0.5735$   
 $= 9.8325 \ 2 - 66.72^\circ$   
 $Z_1 = \frac{V}{I_1} = \frac{-11.47 - j66.76}{9.8325 \ 2 - 66.72}$   
 $= 5.776 - j3.7543$   
Thus,  $R_1 = 5.776\Omega$  and  $x_1 = 3.7543\Omega$ .

Thus,

4. (a) For the parallel resonant circuit shown in the figure find the value of capacitance at which maximum impedance occurs at a given frequency.



#### **Solution**

Refer Section 8.8 (Chapter 8)

The parallel resonant circuit shown is generally called a tank circuit. The impedance of the parallel resonant circuit is maximum at the resonance frequency. -----
4. (b) Determine the admittance parameters of the symmetrical lattice shown in the figure.

....



### Solution

Refer Section 15.12 (Chapter 15)

The lattice network can be redrawn as a bridge network as shown. Assume  $I_3$  in AD as indicated.



Writing mesh equation ADC 1'1

$$-4 I_3 + 2 (I_2 - I_3) + V_1 = 0$$
  
$$V_1 = -2 I_2 + 6 I_3$$
(1)

Writing mesh equation  $BC D_2 2'$ 

$$-4(I_1 - I_3 + I_2) - 2(I_2 - I_3) + V_2 = 0$$
  

$$V_2 = 4I_1 + 6I_2 - 6I_3$$
(2)
Writing mesh equation *ABCDA*

$$-2(I_1 - I_3) - 4(I_1 - I_3 + I_2) - 2(I_2 - I_3) + 4I_3 = 0$$

$$I_3 = \frac{1}{2}(I_1 + I_2)$$
(3)

 $V_{1} = -2 I_{2} + 3 (I_{1} + I_{2})$   $V_{1} = 3 I_{1} + I_{2}$ Substituting equation 3 in 2  $V_{2} = 4 I_{1} + 6 I_{2} - 3 (I_{1} + I_{2})$   $V_{2} = I_{1} + 3 I_{2}$ From equation 4  $I_{2} = V_{1} - 3I_{1}$ Substituting in equation 5  $V_{2} = I_{1} + 3 (V_{1} - 3I_{1})$   $V_{2} = -8I_{1} + 3V_{1}$ or  $I_{1} = \frac{3}{8} V_{1} - \frac{V_{2}}{8}$ From equations 4 and 5  $V_{1} - 3V_{2} = -8I_{2}$ or,  $I_{2} = -\frac{V_{1}}{8} + \frac{3}{8}V_{2}$ From the equation form

Equation 6 and 7 are of the form

Substituting equation 3 in 1

 $I_1 = Y_{11} V_1 + Y_{12} V_2$   $I_2 = Y_{21} V_1 + Y_{22} V_2$ Therefore,  $Y_{11} = Y_{22} = \frac{3}{8}; y_{12} = y_{21} = -\frac{1}{8}$ 

Also equation 4 and 5 are of the form

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
  

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$
  
Therefore 
$$Z_{11} = Z_{22} = 3; Z_{12} = Z_{21} = 1$$

5. (a) A balanced delta connected load of 5.0 ∠30° Ω and a balanced star connected load of 5.0 ∠45° Ω are supplied by the same balanced 240V, 3 phase *ABC* system. Obtain line currents I<sub>A</sub>, I<sub>B</sub> and I<sub>C</sub>.

Solution



(4)

(5)

(6)

(7)

The two loads are connected parallel across a 240V 3 phase system. Let us convert the star connected load into delta and redraw the circuit is as shown below.



The phase currents are given by

$$I_{AB} = \frac{V_{AB}}{Z_{AB}} = \frac{240 \angle 0}{3.77 \angle 33.73} = 63.5584 \angle -33.73$$
$$I_{BC} = 63.5584 \angle -153.73$$
$$I_{CA} = 6.35584 \angle -273.73$$

The line currents are  $\sqrt{3}$  times the phase currents and lag 30° behind their respective phase currents.

Therefore, 
$$I_A = \sqrt{3} \times 63.5584 \angle -33.73 - 30^\circ$$
  
= 110  $\angle -63.73^\circ$   
Similarly,  $I_B = 110 - 183.73^\circ$  and  $I_C = 110 \angle -303.73^\circ$ 

- 5. (b) Derive phase and line relations in a balanced delta connected load. **Solution** Refer Sections 9.8.1, 9.8.2 and 9.8.3.
- 6. (a) For the given network graph shown below, write down the basic Tie set matrix, taking the tree consisting of edges 2, 4 and 5. Write down the KVL network equations from the matrix.



### Solution

Refer Section 2.7 and Example 2.4 (Chapter 2)

The twigs of the tree are 2, 4 and 5. The links corresponding to the tree are 1, 3 and 6 as shown in the figure.



Number of nodes, n = 4Number of branches, b = 6Number of tree branches or twigs = n - 1 = 3Number of link branches l = b - (n - 1) = 3For writing the tie-set matrix consider the three links one at a time, the tie-set matrix *B* or fundamental loop matrix is given by.

 $B = \begin{bmatrix} lops & Branches \longrightarrow \\ 1 & 2 & 3 & 4 & 5 & 6 \\ l_1 \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ l_2 & 0 & 1 & 1 & 0 & 1 & 0 \\ l_3 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ 

There are three fundamental loops  $l_1$ ,  $l_2$  and  $l_3$  as shown by the tie sets.



From the tie-set matrix we can write KVL network equations as

$$[B][V_{h}] = 0$$

where *B* is an  $l \times b$  tie-set matrix or fundamental loop matrix and  $V_b$  is a column vector of branch voltages of 1, 2, 3, 4, 5 and 6 respectively.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = 0$$

The KVL network Equation for the Three Tie-sets are

$$V_1 - V_4 + V_5 = 0 \tag{1}$$

 $\lceil \nu \rceil$ 

$$V_2 + V_3 + V_5 = 0 \tag{2}$$

$$V_2 + V_4 + V_5 + V_6 = 0 \tag{3}$$

- 6. (b) Find the voltage across the  $5\Omega$  resistor for the coupled network shown in figure.



### Solution

Refer Problem 10.10 (Chapter 10)

Applying KVL for loop 1

$$50 \angle 45^\circ = 5i_1 + j4i_1 + j3 \ (i_1 - i_2) + j5(i_1 - i_2) + j3i_1 \tag{1}$$

Simplifying and rearranging the above equation yields to

$$50 \angle 45^\circ = (5+j15) i_1 - j8i_2$$
 (2)

Applying KVL for loop 2

$$0 = -j8i_2 + j5(i_2 - i_1) - j3i_1$$
(3)

Simplifying the above equation yields to

$$j8i_1 = -j3i_2 \text{ or } i_2 = -\frac{8}{3}i_1$$
 (4)

Substituting equation 4 in 2

$$50 \angle 45^\circ = (5 + j15) i_1 + j8 \left(\frac{8}{3}\right) i_1$$

From which 
$$i_1 = \frac{150 \ \angle 45^\circ}{15 + j \ 109} = 1.363 \ \angle -37.165^\circ \text{A}$$

Therefore voltage across  $5\Omega$  resistor is 5i,

$$= 5 \times 1.363 = 6.815$$
 volts

7. (a) State and explain Millman's theorem. **Solution** 

Refer Section 3.10 (Chapter 3)

(b) Using Millman's theorem find the neutral shift voltage  $V_{ON}$ .



### Solution

<u>Refer Example 9.21 (Chapter 9)</u> Converting load impedances into admittances

$$Y_R = \frac{1}{10} \ \Omega; \ Y_y = \frac{j}{10}; \ Y_B = \frac{1}{3+j4}$$

According to Millmans theorem the neutral shift voltage  $V_{\rm ON}$  due to unbalanced load is given by

$$V_{ON} = \frac{V_{RN} Y_N + V_{YN} Y_Y + V_{BN} Y_B}{Y_R + Y_B + Y_Y}$$
$$V_{ON} = \frac{100 \angle 0^{\circ} \left(\frac{1}{10}\right) + 100 \angle 120^{\circ} \left(\frac{j}{10}\right) + 100 \angle -120^{\circ} \left(\frac{1}{3+j}\right)}{\frac{1}{10} + \frac{j}{10} + \frac{1}{3+j4}}$$

$$V_{ON} = \frac{10 + 10 \angle 210^{\circ} - 19.856 - j 2.392}{0.22 - j 0.06}$$

00

$$= \frac{-18.5166 - j7.3923}{0.22 - j0.06} = -69.81 - j52.64$$
  
$$V_{ON} = 87.43 \angle -142.98 \text{ V}$$

8. (a) Explain initial value theorem of Laplace transform.

For 
$$I(s) = \frac{s+4}{(s+2)(s+3)}$$
, find  $I(0)$ 

### Solution

Refer Section 13.4(g) (Chapter 13) From initial value theorem

$$I(0) = \lim_{s \to \infty} sI(s)$$

$$I(0) = \lim_{s \to \infty} \frac{s(s+4)}{(s+2)(s+3)}$$

$$= \lim_{s \to \infty} \frac{s^2(1+4/s)}{s^2 \left(1+\frac{2}{s}\right) \left(1+\frac{3}{s}\right)}$$

$$I(\infty) = \frac{(1+4/\infty)}{(1+2/\infty)(1+3/\infty)} = 1$$

8. (b) Draw the network in Laplace domain and find I(s).



### Solution

<u>Refer Problems 13.18 and 13.21 (Chapter 13).</u> Before the switch is opened, the voltage across the capacitor is =

voltage drop across  $3\Omega = 10 \times \frac{3}{3+2} = 6V$ 

Therefore,  $v_C(0^+) = 6V$ Initial current in the inductor before the opening of switch is  $i_L(0^+)$ 

$$=\frac{10}{5}=2A$$

...

The transformed circuit in *<s*-domain



Applying KVL for the loop.

$$\frac{10}{s} = 2I(s) + sI(s) - i_L(0^+) + \frac{I(s)}{s} + \frac{v_C(0^+)}{s}$$
$$\frac{10}{s} = I(s) \left[ s + \frac{1}{s} + 2 \right] - 2 + \frac{6}{s}$$
$$I(s) = \frac{2(s+2)}{(s+1)^2}$$

# PAPER 8

1. (a) Differentiate between independent and dependent sources. What is their circuit representation.

### Solution

Refer Section 1.8 (Chapter 1)

1. (b) What is the value of R such that the powers supplied by both the sources are equal?

....



### Solution

Converting current source into voltage source



Applying KVL for both the meshes

$$4R = (R+3)i_1 + i_2 \tag{1}$$

 $50 = i_1 + i_2$  (2)

The power supplied by both the sources are equal

$$ARi_1 = 50i_2$$

$$R = 12.5 \, \frac{i_2}{i_1} \tag{3}$$

From eq 1

$$4R - i_1 R - 3i_1 - i_2 = 0$$
  

$$R(4 - i_1) - 3i_1 - i_2 = 0$$
(4)

Substituting equation 3 in 4

$$12.5 \frac{i_2}{i_1} (4 - i_1) - 3 i_1 - i_2 = 0$$
(5)

$$50\frac{i_2}{i_1} - 13.5 i_2 - 3 i_1 = 0 \tag{6}$$

From equation 2,  $i_2 = 50 - i_1$  (7) Substituting equation 7 in 6

$$50\left(\frac{50-i_1}{i_1}\right) - 13.5(50-i_1) - 3i_1 = 0 \tag{8}$$

$$10.5i_1 - 725\ i_1 + 2500 = 0 \tag{9}$$

from which 
$$i_1 = \frac{725 \pm 648.556}{21} = 65.407 \text{ or } 3.6402 \text{ A}$$
  
If  $i_1 = 65.407 \text{ A}$ ;  
from equation  $2 i_2 = -15.407 \text{ A}$   
and  $R = 12.5 \frac{(-15.407)}{21} = -2.945 \text{ O}$ 

and

If

$$R = 12.5 \frac{(-10.407)}{65.407} = -2.945 \Omega$$
$$i_1 = 3.6402 \text{ A},$$
$$i_2 = 46.3598\text{ A}$$
$$R = 12.5 \times \frac{46.3598}{3.6402} = 159.194 \Omega$$

and

Considering positive value of R = 159.194 W Power supplied by current source

 $= 4 \times 159.194 \times 3.6402 = 2317.99$  W

Power supplied by voltage source

$$= 50 × 46.3598 = 2317.99 Ω$$
  
∴ The value of  $R = 159.194 Ω$ 

2. (a) State and explain Faraday's law of electromagnetic induction. Distinguish between self and mutual induced voltages.

### Solution

Refer Section 1.6 (Chapter 1).

2. (b) Explain "Dot convention" and determine the dotted ends of the set of coils shown in figure.



### Solution

Refer Section 10.4 (Chapter 10).

2. (c) A circular iron ring having a cross section area of 5 cm<sup>2</sup> and a length of  $4\pi$  cm in iron has an air gap of  $0.1\pi$  cm made as a saw cut. The relative permeability of iron is 800. The ring is wound with a coil of 2000 turns and carries a current of 100 mA. Determine the air gap flux. Neglect leakage and fringing.

Solution Refer Example 10.12 (Chapter 10) Cross section area of Iron ring,  $I_i = 5 \times 10^{-4} \text{m}^2$  $l_i = 4\pi \times 10^{-2} \mathrm{m}$ Length of iron ring,  $l_g = 0.1\pi \times 10^{-2} \mathrm{m}$ Length of air gap,  $\mu_{\nu} = 800$ No. of turns, N = 2000i = 100 mA $= N \times i$ Total ampere turns (MMF)  $= 2000 \times 100 \times 10^{-3}$ = 200 AT $R = \frac{l_i}{a_i \,\mu_0 \,\mu_i} + \frac{l_g}{a_g \,\mu_0}$ Total reluctance  $=\frac{4\pi \times 10^{-2}}{5 \times 10^{-4} \times 4\pi \times 10^{-7} \times 800}$  $+\frac{0.1\pi\times10^{-2}}{5\times10^{-4}\times4\pi\times10^{-7}}$  $= 5.25 \times 10^{6} \text{ AT/wb}$  $= \frac{\text{Total MMF}}{\text{Reluctance}} = \frac{200}{5.25 \times 10^6}$ Air gap flux  $\phi_g = 3\delta\mu wb$ 

3. (a) Define power factor, apparent power, active power and reactive power.

### Solution

Refer Sections 6.2 and 6.3 (Chapter 6).	
---	--

3. (b) Find complex power in the following circuit.



### Solution

Taking the source voltage as reference

$$V = 200 \angle 0V$$

Complex power

= 
$$(200 \angle 0)(13.52 \angle -8^{\circ})$$
  
S = VI\* = 2704  $\angle -8^{\circ}$  VA

Complex Power  $(P + jQ) = 2704 \angle -80 = (2677.68 - j376.32)$ 

= 13.52 ∠8°

 $= VI^*$ 

$$P = 2677.68 W; Q = 376.32 VAR$$
 leading.

 $I = \frac{200 \angle 0}{10 + \frac{(6+i8)(3-j4)}{(9+j4)}} = 13.396 + j1.886$ 

4. (a) Obtain the *y*-parameters of the following bridged *T*-networks.



### Solution

Refer Problem 15.12 (Chapter 15).

$$I_1 = y_{11} V_1 + y_{12} V_2$$
$$I_2 = y_{21} V_1 + y_{22} V_2$$

Convert delta to star and redraw the circuit.



$$\begin{aligned} y_{11} &= \frac{I_1}{V_1} \Big|_{V_2 = 0} = \frac{I_1}{\left(\frac{3 \times 0.5}{3.5} + 1\right)I_1} = 0.7\\ y_{12} &= \frac{I_1}{V_2} \Big|_{V_1 = 0} = \frac{-I_2 \times \frac{3}{4}}{\left(\frac{1 \times 0.3}{4} + 0.5\right)}I_2 = -0.6 \end{aligned}$$

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$$y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0} = \frac{-I_1 \times \frac{3}{3.5}}{\left(\frac{3 \times 0.5}{3.5} + 1\right)I_1} = -0.6$$
$$y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0} = \frac{I_2}{\left(\frac{3 \times 1}{3+1} + 0.5\right)I_2} = 0.8$$
$$y = \begin{bmatrix} y_{11} & y_{12} \\ y_{22} & y_{21} \end{bmatrix} = \begin{bmatrix} 0.7 & -0.6 \\ -0.6 & 0.8 \end{bmatrix}$$

4. (b) Obtain the expression for *Y*-parameter in terms of transmission parameters.

### Solution

Refer Section 15.8.2 (Chapter 15)

4. (c) For a series resonance circuit obtain the expression for bandwidth in terms of resonance frequency and *Q*-factor.

### Solution

Refer Section 8.4 (Chapter 8)

5. (a) Each phase of a balanced star connected load consists of  $R = 10 \Omega$ and  $C = 10 \mu$ F. Calculate the line currents and total real and reactive powers when a symmetrical 400V, 50Hz, 3 phase supply is applied to it. If two wattmeters are employed to measure total power, find  $W_1$  and  $W_2$ .

## Solution

...

$$R = 10\Omega; C = 10 \ \mu\text{F}; f = 50 \text{Hz} \ V_L = 400 \text{V}$$
$$Z = (R \pm jX)$$
$$\chi_c = \frac{1}{2\pi \ f.C} = \frac{10^6}{2\pi \times 50 \times 10} = 318.3\Omega$$
$$Z = 10 - j318.3\Omega = 318.466 \ \angle -88.Z^\circ$$

- -

Power factor =  $\cos(-88.2) = 0.0314$  leading.

The line currents which are also equal to phase currents are

$$I_R = \frac{\frac{400 \ \angle 0}{\sqrt{3}}}{318.466 - 88.20} \\= 0.725 \ \angle 88.2^{\circ}$$

Similarly we can write

$$I_Y = \frac{\frac{400 \ \angle -120^{\circ}}{\sqrt{3}}}{318 \cdot 466 \ \angle -88 \cdot 2^{\circ}} = 0.725 \ \angle -31.8^{\circ}$$

 $I_B = \frac{\frac{400 \angle -240^\circ}{\sqrt{3}}}{318 \cdot 466 \angle -88 \cdot 2^\circ}$  $= 0.725 \angle -151.8^{\circ}$ Readings of the two wattmeters  $W_1 = V_L I_L \cos \left( 30 + \phi \right)$  $=400 \times 0.721 \cos (30 + 88.2^{\circ})$ = -136.28 W $W_2 = V_{\rm L} I_{\rm L} \cos \left( 30 - \phi \right)$  $=400 \times 0.721 \cos (30 - 88.2^{\circ})$ = 151.97 WTotal active power =  $W_1 + W_2$  $P_{\rm T} = 15.69 \mu w$ Total reactive power =  $\sqrt{3} (W_1 - W_2)$  $Q = \sqrt{3} (-136.28 - 151.97)$ = -500 VAR  $Q = \sqrt{3} V_L I_L \sin \phi$ or  $=\sqrt{3} \times 400 \times 0.725 \sin(88.2^{\circ}).$ 

5. (b) A 400V, 50 Hz, 3 phase supply of phase sequence ABC is applied to a delta connected load consisting of 100  $\Omega$  between lines A & B, 318 mH inductance between lines B&C and 31.8µF capacitance between lines C&A. Determine phase and line currents.



From the given data

 $R = 100\Omega, X_L = j100\Omega; X_c = -100\Omega$ 

$$Z_{ab} = 100 \angle 0; Z_{bc} = j100\Omega; Z_{ca} = -j100\Omega$$

Phase currents

 $I_{AB} = 400 \angle 0/100 \angle 0 = 4 \angle 0$ 

$$I_{BC} = 400 \ \angle 120^{\circ} / 100 \ \angle 90^{\circ} = 4 \angle -210^{\circ}$$
$$I_{C4} = 400 \ \angle -240^{\circ} / 100 \angle -90^{\circ} = 4 \angle -210^{\circ}$$

Line currents

$$I_{A} = I_{AB} - I_{CA} = 7.72 \ \angle 15^{\circ} \text{A}$$
$$I_{B} = I_{BC} - I_{AB} = 7.727 \ \angle 165^{\circ} \text{A}$$
$$I_{C} = I_{CA} - I_{BC} = 4 \ \angle -90^{\circ} \text{A}$$

6. (a) In the network shown below find current *I* using nodal analysis.



#### Solution

<u>Refer Example 7.3 (Chapter 7).</u> Writing node equations at node 1 and 2

$$\frac{V_1 - 50}{5} + \frac{V}{j_1} + \frac{V_1 - V_2}{4} = 0 \tag{1}$$

$$\frac{V_2 - V_1}{4} + \frac{V_2}{j_1} + \frac{V_2 - 50 \angle 90^\circ}{2} = 0$$
<sup>(2)</sup>

Simplifying equation 1 leads to

$$(0.45 - j) V_1 - 0.25 V_2 = 10$$
(3)

Simplifying equation 2 leads to

$$-0.25 V_1 + (0.75 + j) V_2 = 25 \angle 90^{\circ}$$
(4)

Solving equations 3 and 4

 $V_1 = 2.732 + j13.28$   $V_2 = 18.43 + j13.156.$ Current  $I = \frac{V_1 - V_2}{4}$  = -3.9245 - J0.056 $I = 3.92 \angle -179.18^\circ$ 



6. (b) Obtain the basic cut-set matrix for the given oriented graph, taking 1, 2, 3, 4 as tree branches. Write down KCL network equations from the matrix.

#### Solution

Refer Section 2.8.2 (Chapter2).

The fundamental cut-set or basic cut-set matrix are defined for a given tree of the graph. The procedure is to select a tree and then a twig is selected removing this twig from the tree separates the tree into two-parts. All the links which go from one part of the disconnected tree to the other, together with the twig of the selected tree will costitute a cut-set. The fundamental cut-set matrix  $Q_f$  is one in which each row represents a cut-set with respect to a given tree of the graph, and the columns correspond to the branches of the graph.

For each twig there will be a basic cut-set therefore for a network graph with *n* nodes and *b* branches, there will be (n - 1) number of basic cut-sets.

From the given graph the number of nodes are 5. The twigs of the tree are 1, 2, 3, 4 and the links are 5, 6, 7, 8.

The number of basic cut-sets = (5 - 1) = 4

(5-1) = 4.

The tree is represented by solid lines.

Consider twig. 3 Corresponding to twig 3. The f-cut set is  $\{3, 5, 6\}$ which is cut-set  $C_1$ . Its orientation coincides with the defining twig 3.

Corresponding to twig 4, the *f*-cut set is {4, 6, 7}

Which is cut-set  $C_2$  having the same orientation as 4.



Corresponding to twig 1 the *f*-cut set is  $\{1, 6, 7, 8\}$ 

Which is cut-set  $C_3$ . The orientation of  $C_3$  is coincident with the direction of twig 1.

Corresponding to twig 2, the *f*-cut set is  $\{2, 5, 6, 7, 8\}$  which is cutset  $C_4$ .

The *f*-cut-set matrix is written as follows:

$$mQ_{f} = \begin{array}{c} f\text{-cut-sets branches} \longrightarrow \\ \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ C_{1} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$$

The basic property of the fundamental cut-sets is that they give linearly independent KCL equations.

\*\*

Applying KCL to the *f*-cut-sets of the graph,

$$C_1: i_3 - i_5 - i_6 = 0$$
  

$$C_2: i_4 + i_6 - i_7 = 0$$
  

$$C_3: i_1 + i_6 - i_7 + i_8 = 0$$
  

$$C_4: i_2 - i_5 - i_6 + i_7 - i_8 = 0$$

In the matrix form

$$\begin{array}{c} C_{1} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ C_{2} \\ C_{3} \\ C_{4} \end{bmatrix} \begin{pmatrix} i_{1} \\ i_{2} \\ i_{3} \\ i_{4} \\ i_{5} \\ i_{6} \\ i_{7} \\ i_{8} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## 7. (a) State and explain Millman's theorem. **Solution**

Refer Section 3.10 (Chapter 3).

## 7. (b) Find the current $I_L$ . Use Millman's theorem.



### Solution

*:*..

Refer Example 3.10 (Chapter 3) Millman's theorem states that

$$V' = \frac{V_1 G_1 + V_2 G_2 + \dots + V_n G_n}{G_1 + G_2 + G_3 + \dots + G_n}$$
$$R' = \frac{1}{G_1 + G_2 + \dots + G_n}$$
$$V' = \frac{20 \times \frac{1}{5} + 40 \times \frac{1}{4} + \left(-10 \times \frac{1}{2}\right)}{\frac{1}{5} + \frac{1}{4} + \frac{1}{2}} = 9.47736$$
$$R' = \frac{1}{\frac{1}{5} + \frac{1}{4} + \frac{1}{2}} = 1.0526$$



$$I_L = \frac{9.4736}{1.0526 + 10} = 0.857 \text{A}$$

7. (c) Verify the reciprocity theorem for the network shown.



Solution

$$V_x = \frac{10 \times 5}{5 + 4 - j4} (-j4) = 8.24 - j18.556$$
  
= 20.3 \angle -66°  
Output/input =  $\frac{20.3 \ angle - 66^\circ}{10 \ angle 0^\circ} = 2.03 \ angle -66^\circ.$ 

Exchanging the excitation and response



$$V_x = \frac{10 \angle 0^\circ (-j4)}{5+4-j4} \times 5 = 20.3 \angle -66^\circ$$
  
Output/input =  $\frac{20.3 \angle -66^\circ}{10 \angle 0^\circ}$   
= 2.03  $\angle -66^\circ$ 

8. (a) Explain the final value theorem of Laplace transform. **Solution** 

Refer Section 13.4 (h) Chapter 13.

\*\*

8. (b) Find  $V_{(\infty)}$  given  $V(s) = \frac{S^2 + 2S + 3}{S(S+1)(S^2 + 2S + 2)}$ .

Final value 
$$V_{\infty} = \underset{\beta \to 0}{\text{Lt}} \beta' V(\beta') = \underset{\beta' \to 0}{\text{Lt}} \frac{\beta' (\beta' + 2\beta' + 3)}{\beta' (\beta' + 1) (\beta'^2 + 2\beta' + 2)}$$
  
Therefore,  $V_{\infty} = \frac{3}{2}$  V.

L

....

## PAPER 9

1. (a) Write a note on source transformation.

### Solution

Refer Section 2.15.

...

(b) Using KCL and KVL, find the currents in all the sources of the circuit of the following figure.



### Solution:

Using KVL, the loop equations can be written as

$$5 = 5I_1 - I_2$$
(1)

$$-5 = 7I_2 - I_1 - 2I_3 \tag{2}$$

$$5 = 6I_3 - 2I_2$$
 (3)

Solving Eq.(1), Eq. (2), and Eq. (3), we get

$$V_1 = 0.92 \text{ A}$$
  
 $V_2 = -0.38 \text{ A}$   
 $V_3 = 0.706 \text{ A}$ 

2. (a) Calculate the current to be passed through the coil so that a flux of 1 mwb is produced in the air gap (as shown in the following figure) the case of square cross section over its entire length and has permeability of 800.

$$\phi = 1 \text{ mwb}$$
  
 $u_r = 800$ 



Assuming no. of turns = 500 Flux produced is given by

$$\phi = \frac{\text{mmf}}{\text{total reluctance}}$$
$$\phi = \left[\frac{\text{mmf}}{\frac{l_1}{u_1A_1} + \frac{l_2}{u_1A_2} + \frac{l_3}{u_3A_3}}\right]$$

By dividing the given fig. in no. of section

$$1 \times 10^{-3} = \frac{I \times 500}{\frac{(20+20)10^{-2}}{800 \times 4\pi \times 10^{-7} \times 16 \times 10^{-4}} + \frac{(20+8)10^{-2}}{800+4\pi \times 10^{-7} \times 64 \times 10^{-4}}} + \frac{1 \times 10^{-3}}{4\pi \times 10^{-7} \times 4 \times 10^{-5}}$$
$$1 \times 10^{-3} = \frac{1 \times 500 \times 4\pi \times 10^{-7}}{0.3125 + 0.05459 + 25}$$
$$I = 40.37 \text{ A}$$

- (b) Define the following terms
  - (i) Co-efficient of coupling in coupled coils
  - (ii) Magnetic flux density
  - (iii) Reluctance of magnetic path
  - (iv) Permeability

### Solution:

Refer Sections 10.5; 10.11

3. (a) A series R-C circuit is excited by sinusoidal voltage find the expression for impedance using phasor diagram.



### E.79

### Solution: Refer Section 5.1

(b) Determine the supply voltage and the power factor in the following figure network if the total power delivered is 200 W.

### Solution:

 $z_{\rm eq} = 10 + (-5\,j||(6+2\,j)$ = 13.33 - 3.33 i $z_{eq} = 13.33 - 3.33 j = 13.74 \angle -14.02$ :. but given that total power delivered is 200 W  $200 = I^2 (13.33)$ :. I = 3.87 AV = IZ = 3.87(13.33 - 3.33 j)... = 51.63 - 12.89 i= 53.22 ∠-14.02  $V = 53.22 \angle -14.02 \text{ V}$ *.*•. Power factor =  $\frac{R}{Z} = \frac{13.33}{13.74}$ , 0.97 ∴ Supply voltage is 53.22 ∠-14.02 V and power factor is 0.97

4. (a) For a series RL circuit obtain the locus of current as inductance is changed from 0 to  $\infty$  when the applied voltage is constant.

### Solution:

Refer Section 8:13.1



(b) Show that for a series resonant circuit  $f_1f_2 = f_r^2$  where  $f_1$  and  $f_2$  are half power frequencies and  $f_r$  is the resonance frequency.

### Solution:

Refer Section 8.4

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(c) Obtain the z-parameters of the following two-port Networks. 2Ω 2Ω  $\begin{array}{c|c} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ &$  $I_1$ | V1 V<sub>2</sub> 1′ Solution:  $V_1 = Z_{11} I_1 + Z_{12} I_2$  $V_2 = Z_{21} I_1 + Z_{22} I_2$  $\begin{array}{c|c} 4 \Omega & l_2 \\ \hline \\ \hline \\ 1_3 & \\ 2 \Omega & \\ \end{array} \begin{array}{c} 2 \Omega & \\ \end{array} \begin{array}{c} l_4 \\ \hline \\ 2 \Omega & \\ \end{array} \end{array}$ *I*<sub>1</sub> 2Ω → √√√  $V_1$  $I_2 = 0$  $V_1 = I_1(3.5)$ when  $Z_{11} = \frac{V_1}{I_1} = 3.5$ *:*..  $Z_{21} = \frac{V_2}{I_1}$   $V_2 = I_42 \text{ but } I_4 = \frac{I_1(2)}{4+2+2} = \frac{I_1(2)}{\$_4} = \frac{I_1}{4}$  $V_2 = \frac{I_1}{4_2} (2) = \frac{I_1}{2}$ :.

$$Z_{21} = \frac{V_2}{V_1} = \frac{1}{2} = 0.5$$



when

 $I_1 = 0$   $V_2 = I_2 (3.5)$  $Z_{22} = \frac{V_2}{I_2} = 3.5$ 

and

...

$$Z_{12} = \frac{V_1}{I_2}$$

but

$$V_1 = I_6 2$$
 but  $I_6 = \frac{I_2(2)}{8} = \frac{I_2}{4}$ 

$$V_1 = \frac{I_2}{4}(2) = \frac{I_2}{2}$$

$$Z_{12} = \frac{V_1}{I_2} = \frac{I}{2} = 0.5$$

: Z-parameters are

$$Z_{11} = 3.5 \qquad Z_{12} = 0.5$$
$$Z_{21} = 0.5 \qquad Z_{22} = 3.5$$

5. (a) Determine the line currents and total power supplied to a delta connected load of  $z_{ab} = 10\angle 60^\circ$ ,  $z_{bc} = 20\angle 90^\circ$  and  $z_{ca} = 25\angle 30^\circ$ . Assume a 3-phase, 400 V, ABC system.



$$I_C = \frac{V_{CA}}{Z_{CA}} = \frac{400\angle -240}{25\angle 30^\circ} = 16\angle 90^\circ$$

Line currents

$$I_1 = I_A - I_C = (40 \angle -60) - (16 \angle 90) = 54.44 \angle -68.44 \text{ A}$$
  

$$I_2 = I_B - I_A = (20 \angle 150) - (40 \angle -60) = 58.18 \angle 129.89 \text{ A}$$
  

$$I_3 = I_C - I_B = (16 \angle 90) - (20 \angle 150) = 18.33 \angle 19.10 \text{ A}$$

Power

Power in *A* phase =  $I_A^2 R_A = (40)^2 (5) = 8000 \text{ W}$ Power in *B* phase =  $I_B^2 R_B = (20)^2 (0) = 0$ Power in *C* phase =  $I_C^2 R_C = (16)^2 (21.65) = 5542.4 \text{ W}$ Total power consumed by load is 8000 + 5542.4 = 13542.4 W

(b) Derive the Relationship between line and phase voltages in a balanced three phase star connected load.

### Solution:

Refer Sections 9.7.1; 9.7.2 and 9.7.3

6. (a) Explain clearly what you understand by a cutset and tieset. Write down the basic tieset schedule for the network shown in the figure by taking 10Ω resistor branches as free branches.



Solution: From the N/w the graph is to be drawn.

Now select  $10\Omega$ -resistor branches as free branches then tieset matrix.





Tie set (loop)

Branches  $\rightarrow$ 

(b) For the N/W shown in figure determine the ratio of  $I_2/I_1$ .

 $V_A = -I_3 \times 1 = -I_3$ 



 $V_{B} = \left(I_{1} + \frac{3}{2}I_{2} + I_{3}\right) \times 1$   $V_{A} - V_{B} = 2I_{3}$   $\Rightarrow \quad -I_{3} - I_{1} - \frac{3}{2}I_{2} - I_{3} = 2I_{3}$   $\Rightarrow \quad 4I_{3} = -I_{1} - \frac{3}{2}I_{2}$   $V_{C} = -I_{2} \times 1 = -I_{2}$   $V_{C} - V_{A} = -I_{2} + I_{3} = \frac{3I_{2}}{2}$ 

$$I_3 = \frac{5}{2} I_2 = 2.5I_2 \tag{3}$$

From Eqs (1) and (2)

Solution:

$$4(2.5I_2) = -I_1 - \frac{3}{2} I_2$$

$$\left(10 + \frac{3}{2}\right) I_2 = -I_1$$

$$\boxed{\frac{I_2}{I_1} = \frac{-2}{23}}$$

7. (a) State the explain the superposition theorem?

### Solution:

Refer Section 3.2

(b) Using superposition theorem find the current in  $2\Omega$ . Verify your result by any other method.

(1)

(2)

To know the current in the  $2\Omega$  resistor



(i) Only having voltage source i.e. current source is replaced by a open circuit.



$$(6+3) || (4+6) \Rightarrow 9 || 10$$
  
= 4.736  $\Omega$ 

the circuit can be drawn as



then

.

(ii) Only current source is present, and voltage source is replaced by short circuit.

By this short circuit, the current flowing through the  $2\Omega$  resistance is zero.

$$I = 0 A$$



 $\therefore$  total current flowing through 2 $\Omega$  resistor is

$$-6 + 0 = -6$$
 Ar

8. (a) Derive the expression for i(t) for R–L series circuit when excited by a sinusoidal source.

### Solution:

Refer Sections 4.5, 4.6 and 5.1

(b) For R-L-C series circuit with  $R = 10 \Omega$ , L = 0.2 H,  $C = 50 \mu$ F determine the current i(t) when the switch is closed at t = 0. Applied voltage is  $V(t) = 100 \cos (1000t + 60^{\circ})$ 



$$V(t) = 100 \cos(1000t + 60)$$

Loop equation is

$$V(t) = 10 \ i(t) + 0.2 \ \frac{di(t)}{dt} + \frac{1}{50 \times 10^{-6}} \left| idt$$

$$10i(t) + 0.2 \frac{di}{dt^2} + \frac{1}{50 \times 10^{-6}} \left| idt = 100 \cos(1000t + 60) \right|$$

$$10\frac{di}{dt} + 0.2\frac{di}{dt^2} + 2 \times 10^5 i(t) = -100 \sin(1000t + 60) \times 1000 \quad (2)$$
$$[0.20^2 + 100 + 20000]i = -10^5 \sin(1000t + 60)$$

22

Characteristic equation

 $0.2D^2 + 10 D + 20000 = 0$  $D = -25 \pm 315.23j$ 

complementary solution:

$$i_c = e^{-25t} [c_1 \cos 315.23t + c_2 \sin 315.23t]$$

Assume particular solution

$$i_p = A \cos(1000t + 60) + B \sin(1000t + 60)$$

$$\frac{di_p}{dt} = -1000 \text{ A} \sin(1000t + 60) + 1000 \cos(1000t + 60)$$

$$\frac{d^2 ip}{dt^2} = -(1000)^2 A \cos(1000t + 60) - (1000)^2 \sin(1000t + 60)$$

Substituting these values in Eq. (2)

 $10[-1000 A \sin (1000t + 60) + 1000 \cos (1000t + 60)]$  $+ 0.2 [-(1000)^2 A \cos (1000t + 60) - (1000)^2 \sin (1000t + 60)] + 2 \times 10^5 [A \cos (1000t + 60^\circ + B \sin (1000t + 60)] = -100 \sin (1000t + 60) 1000$ 

$$-B(1000)^{2} - \frac{A(1000)(10)}{0.2} + \frac{B}{0.2(50 \times 10^{-6})} = \frac{-100(1000)}{(0.2)}$$
  

$$\Rightarrow \qquad A(50.000) + B[900.000] = 500000 \qquad (3)$$
  

$$-A(1000)^{2} + \frac{B(1000)(10)}{0.2} + \frac{A}{0.2(50 \times 10^{-6})} = 0$$

$$A[-900,000] + B\ 50000 = 0 \tag{4}$$

Solving for (3) and (4)

$$4 = 0.03$$

$$B = 0.55$$

$$i_p = 0.03 \cos(1000t + 60) + 0.55 \sin(1000t + 60)$$

solution is

$$i = e^{-25t} [c_1 \cos 315.23t + c_2 \sin 315.23t] + [0.03 \cos (1000t + 60) + 0.55 \sin (1000t + 60)]$$

to evaluate,  $c_1$  and  $c_2$ 

$$i = 0 \text{ when } t = 0$$
  

$$O = (1) [c_1 + 0] + 0.03 \cos (60) + 0.56 \sin (60)$$
  

$$C_1 = -[0.03 \cos 60 + 0.55 \sin 60] = -0.491$$
  

$$\frac{di}{dt} = e^{-25t} (-25) [c_1 \cos 315.23t + c_2 \sin 315.23t]$$

 $+ e^{-25t} [-c_1 \sin 315.23t (315.23) + c_2 \cos 315.23t (315.23)]$  $-0.03 \sin (1000t + 60) (1000) + 0.55 \cos (1000t + 60) 1000$ when  $t = 0, \frac{di}{dt} = 250$ ∴  $250 = (-25) [c_1] + [c_2 (315.23)] - 0.03 \times 1000 \sin 60$  $+ 0.55 \times 1000 \cos (60)$  $<math>250 = -25 (-0.491) + c_2 (315.23) - 0.03 \times 1000 \sin 60$  $+ 0.55 \times 1000 \cos 60$  $c_2 = 0.035$ ∴ Solution is  $i(t) = e^{-25t} [-0.491 \cos 315.23t + 0.035 \sin 315.23t]$ 

 $+0.03 \cos(1000t+60)+0.55 \sin(1000t+60)$ 



# SET 1

1. (a) Find the voltage to be applied across *AB* in order to drive a current of 5A into the circuit by using star-delta transformation. Refer Fig. 1.1.



Fig. 1.1

Solution





1. (b) Using Kirchhoff's current law, find the values of the currents  $i_1$  and  $i_2$  in the circuit shown in Fig. 1.2.



Fig. 1.2

## Solution



Applying KCL at node V and also  $i_1 = \frac{V}{3}$ 

$$\frac{V}{3} + \frac{V}{2} = 2 + 4i_1 \Longrightarrow \frac{V}{3} + \frac{V}{2} - \frac{4V}{3} = 2$$

from which V = -4 volts

Using star-delta transformer

$$i_1 = -\frac{4}{3}, i_2 = -2$$
 A

2. (a) Define the following:

- (i) Self Inductance
- (ii) Mutual Inductance
- (iii) Static Induced e.m.f.
- (iv) Dynamically induced e.m.f.

Solution Refer Sections 10.3 and 1.6

 (b) Derive the relationship between the self, mutual inductances and coefficient of coupling.

### Solution <u>Refer Section 10.5</u>

2. (c) Two similar coils connected in series gave a total inductance of 600 mH and when one of the coil is reversed, the total inductance is 300 mH. Determine the mutual inductance between the coils and coefficient of coupling?

**Solution** 
$$L_1 + L_2 + 2 \text{ M} = 600 \text{ mH}$$
 (1)

 $L_1 + L_2 - 2 \text{ M} = 390 \text{ mH}$  (2)

(1) - (2) 
$$\Rightarrow 4 \text{ M} = 300 \text{ mH}$$
  
 $M = 75 \text{ mH}$   
 $L_1 = L_2 = L$   
From Eq. (2)  $2L - 2 \times 75 = 300$   
 $L = 225$   
 $K = \frac{M}{\sqrt{L_1 L_2}} = \frac{75}{\sqrt{(225)^2}} = \frac{1}{3}$ .

3. (a) Bring out the differences between series and parallel resonance?

### Solution Refer Sections 8.1 and 8.7

- 3. (b) A series RLC circuit consists of resistance  $R = 20 \Omega$ , inductance, L = 0.01H and capacitance,  $C = 0.04 \mu$ F. Calculate the frequency at resonance. If a 10 Volts of frequency equal to the frequency of resonance is applied to this circuit, calculate the values of  $V_C$  and  $V_L$ across C and L respectively. Find the frequencies at which these voltages  $V_C$  and  $V_L$  are maximum?
- 3. (b) Sol

Solution 
$$R = 20 \ \Omega; L = 0.01 \ \text{H}; C = 0.04 \ \mu\text{F}$$
  
 $f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.01 \times 0.04 \times 10^{-6}}} = 7.957 \ \text{kHz}$   
At resonance  $I = \frac{V}{R} = \frac{10}{20} = 0.5 \ \text{A}$ 

Voltage across C;  $V_C = IX_C$ 

$$= \frac{I}{\omega_C} = \frac{0.5}{2 \times \pi \times 7957 \times 0.04 \times 1} = 250 \text{ volts}$$

Voltage across *L*;  $V_L = IX_L = I\omega_L = 250$  volts

Also refer Example 8.4.

4. (a) Three impedances each of (3-*j*4) Ω is connected in delta connection across a 3-φ, 230 V balanced supply. Calculate the line and phase currents in the Δ connected load and the power delivered to the load?

### Solution

P



hase currents 
$$I_{AB} = \frac{230\angle 0}{3-j4} = 46 \angle 53.13^{\circ}$$
  
 $I_{BC} = 46 \angle -53.13^{\circ} - \angle 120^{\circ} = 46 \angle -66.87^{\circ}$   
 $I_{CA} = 46 \angle -66.87^{\circ} - \angle 120^{\circ} = 46 \angle -186.87^{\circ}$ 

Line currents are  $\sqrt{3}$  times the phase values and lags their respective phase values by 30°.

:.  $I_A = 79.67 \ \angle 23.13^{\circ}$  $I_B = 79.67 \ \angle -96.87^{\circ}$  $I_C = 79.67 \ \angle -216.87^{\circ}$ 

4. (b) In power measurement of 3- $\phi$  load connected by 3- $\phi$  supply by two wattmeter method, prove that  $\tan \theta = \frac{-\sqrt{3} (w_1 - w_2)}{w_1 + w_2}$  for leading power factor loads.

Solution Refer Section 9.11.4

- 5. (a) For the circuit shown in Fig. 3, draw the graph and indicate tree.
  - (i) Branch (ii) Node
  - (iii) Degree of a node (iv) Links
- Solution Refer Sections 1.12; 2.1 and 2.3

5. (b) Using Nodal method, find the current through 5 *W* resistor, in the following circuit.



Fig. 1.3

### Solution



Equation at 
$$V_1$$
;  $\frac{V_1 - V_3}{5} + \frac{V_1 - V_2}{3} = 5$   
 $8V_1 - 5V_2 - 3V_3 = 75$  (1)

Equation at super node

$$\frac{V_2 - V_1}{3} + V_2 + \frac{V_3 - V_1}{5} + \frac{V_3}{2} = 0$$

$$-16V_1 + 40V_2 + 21V_3 = 0$$

$$V_3 - V_2 = 2i$$

$$i = V_{2/1} = V_2$$

$$V_3 - V_2 = 2V_2 \implies V_3 = 3V_2$$
(2)

Solving for  $V_1$ ,  $V_2$  and  $V_3$ 

 $V_1 = 12.87; V_2 = 2; V_3 = 6$  volts

Current through 5  $\Omega$  from  $V_1$  to  $V_3$  is equal to  $\frac{V_1 - V_3}{5} = 11.67$  amps.

6. (a) Explain the steps for solving a network problem using Thevenin's theorem.

Solution Refer Section 3.3

6. (b) Find the current *I* the circuit shown in Fig. 1.4.





## Solution



Applying superposition Open circuit the current source



Short the voltage source



$$I_2 = 1 \times \frac{1}{5} = \frac{1}{5}$$

Total current through  $4 \Omega = \frac{1}{5} - \frac{1}{5} = 0$ 

7. Find  $\vartheta_c(t)$  at t = 0 + while the switching is done from x to y at t = 0 as shown in Fig. 1.5.




Solution



At  $t = 0^{-1}$ 



7

KVL gives 
$$10 = 7I - 4V_C(0^-)$$
  
Also  $V_C(0^+) = -4V_C(0^-) + 5 \cdot \frac{10 + 4V_C(0^+)}{10 + 4V_C(0^+)}$ 

from which  $V_C(0^-) = \frac{50}{15} = 3.333$  volts

$$V_C(0^+) = V_C(0^-) = 3.333$$
 volts.

8. (a) Determine the *Z*-parameter of the network shown in Fig. 1.6. **Solution** 



Substituting  $\Sigma N4$  in 1 and 2

$$V_{1} = \frac{11}{3} I_{1} + \frac{5}{3} I_{2} = Z_{11} I_{1} + Z_{12} I_{2}$$
$$V_{2} = \frac{5}{3} I_{1} + \frac{35}{12} I_{2} = Z_{21} I_{1} + Z_{22} I_{2}$$
$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 11/3 & 5/3 \\ 5/3 & 35/12 \end{bmatrix}$$

- 8. (b) The *y*-parameters of a two port network are  $y_{11} = 0.6$  mho,  $y_{22} = 1.2$  mho and  $y_{12} = -0.3$  mho.
  - (i) Determine the ABCD Parameters and
  - (ii) Equivalent  $\Pi$  network.

*.*..

AC

**Solution** Given *Y*-parameters 
$$Y_{11} = 0.6$$
  $\mho$ ;  $Y_{22} = 1.2$   $\mho$ ;  $Y_{12} = -0.3$   $\mho$ 

For a reciprocal  $n.w Y_{21} = Y_{12}$ 



Equivalent  $\pi$  *n*.*w* is shown in figure

$$Y_{11} = Y_A + Y_C, Y_{12} = Y_{21} = -Y_C; Y_{22} = Y_B + Y_C$$
  
 $0.6 = Y_A + 0.3$ 

 $Y_A = 0.3 \ \mho; \quad Y_C = 0.3 \ \mho; \quad Y_B = 0.9 \ \mho$ 

ABCD parameters can be expressed in terms of Y-parameters.

$$A = \frac{-Y_{22}}{Y_{21}} = 4; B = \frac{-1}{Y_{21}} = 3.33 \ \Omega$$
$$C = \frac{-\Delta Y}{Y_{21}}; \Delta Y = \begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix} = 0.63 = 2.1$$
$$D = \frac{-Y_{11}}{Y_{21}} = 2$$
$$\begin{vmatrix} B \\ D \end{vmatrix} = \begin{vmatrix} 4 & 3.33 \\ 2.1 & 2 \end{vmatrix}$$

# SET 2

1. (a) For the circuit shown in Fig. 2.1, find the current through  $20\Omega$  resistor?





Solution



Applying nodal analysis

$$\frac{V-10}{75} + \frac{V}{20} + \frac{V+15}{50} = 0$$
  
V = -2 volts,  $I = \frac{V}{20} = -0.1$  A

1. (b) Reduce the network shown in Fig. 2.2, to a single loop network by successive source transformation, to obtain the current in the 12  $\Omega$  resistor.



Fig. 2.2

## Solution







2. (a) Write short notes on dot convention used in magnetically coupled coils.

Solution Refer Section 10.4

2. (b) In the network shown in Fig.2.3,  $L_1 = 1$  H,  $L_2 = 2$  H, M = 1.2 H. Assuming the inductance coils to be ideal, find the amount of energy stored after 0.1 see of the circuit connected to a d.c. source of 10 V.



Fig. 2.3

#### Solution Refer Chapter 10

- 3. (a) Explain the concept of
  - (i) Susceptance and
  - (ii) Admittance

Solution <u>Refer Problems 5.5 and 5.6</u>

- (b) An inductive coil takes 10 A and dissipates 1000 watts when connected to a supply of 250 V, 25 Hz. Calculate.
  - (i) the impedance (ii) the effective resistance
  - (iii) reactance
- (iv) the inductance
- (v) power factor. Also, draw the vector diagram.

### Solution



(v) 
$$\cos \phi = \frac{R}{Z} = 0.4.$$

4. (a) A balanced 3-ph star connected load of 150 KW takes a leading current of 100 A with a line voltage of 1100 V, 50 Hz. Find the circuit constants of the load per phase?

**Solution**  $P = 150 \text{ k}\omega; V_L = 110 \text{ V}; f = 50 \text{ Hz}; I = 100 \text{ A}$ 

$$\sqrt{3} \times V_L \times I_L \cos \phi = 150 \times 10^3$$
  

$$\cos \phi = \frac{150 \times 10^3}{\sqrt{3} \times 1100 \times 100} = 0.7873$$
  

$$Z_{Ph} = \frac{V_{Ph}}{I_{Ph}} = \frac{1100}{\sqrt{3} \times 100} = 6.35$$
  

$$R_{Ph} = Z_{Ph} \cos \phi = 6.35 \times 0.7873 = 4.9 \Omega$$
  

$$X_{Ph} = Z_{Ph} \sin \phi = 6.35 \times 0.6168 = 3.917 \Omega$$

 (b) Three equal star connected inductors takes 8 KW at P.f of 0.8 when connected to 460 V, 3-φ, 3 wire supply. Find the line currents, if one conductor is short circuited.

Solution



Power = 8000 watts  $\sqrt{3} V_L I_L \cos \varphi = 8000$   $I_L = 12.55 \text{ A} = I_{Ph}$   $Z_{Ph} = \frac{V_{Ph}}{I_{Ph}} = \frac{460}{\sqrt{3} \times 12.55} = 21.16 \Omega$   $I_R = I_Y = \frac{460}{21.16} = 21.73 \text{ A}$   $I_B = 2 \times I_{Ph} \times \cos 30 = 37.637 \text{ A}$ (a) Define the following and explain by taking an

- 5. (a) Define the following and explain by taking an example.
  - (i) Node(ii) Tree(iii) Sub graph(iv) Loop
  - (v) Links (vi) Directed graph.

Solution Refer Sections 1.12; 2.1 and 2.2

5. (b) Find the fundamental tie-set and cut-set matrix for the graph and for the tree shown in the Fig. 2.4.



Fig. 2.4

## Solution



There are 5 nodes; n = 5There are 7 branches = b = 7No. of twigs or tree branches = n - 1 = 4 (2, 3, 4, 5) No. of black branches = b - (n - 1) = 3 (1, 6, 7) The Tie-sets are shown below.



Tie set matrix loop; Branches  $\rightarrow$ 

г ¬

The required Tie-set matrix is given by

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Cut-set

For the given Tree there are four fundamental cut-sets each for one twig and given by

Twig 2; *f*-cut-set [1, 2, 6] Twig 3; *f*-cut-set [1, 3, 6, 7]

Twig 4; *f*-cut-set [1, 4]

Twig 5; *f*-cut-set [5, 7]

The cut-sets are formed as shown



6. (a) State and explain compensation theorem.

Solution Refer Section 3.6 and Example 3.7

6. (b) In the network shown in Fig. 2.5, find the value of  $Z_L$  so that the power transfer from the source is maximum. Also find  $P_{\text{max}}$ .



Fig. 2.5

Solution



 $V_{\rm TH}$  is the voltage across  $ab = V_a - V_b$ 

$$V_{\rm Th} = 10 \left( \frac{21}{(21+12+j24)} - \frac{50}{50+(30+j60)} \right)$$
  
= 0.162 - j0.027  
= 0.1644 \angle - 9.46°  
$$Z_{\rm Th} \ \text{across} \ ab = [(12+j24)//21 \ \Omega] + [(30+j60)//50]$$
  
= 42.9 + j22.38

To transfer maximum power  $Z_L = Z_{TH}^*$ 

$$\therefore \qquad Z_L = (42.9 - j22.3) \\ P_{\text{max}} = (I_{\text{max}})^2 \times 42.26 \\ = \left(\frac{0.1644}{2 \times 42.9}\right)^2 \times 42.9 \\ = 0.16 \text{ m.w.}$$

7. (a) A dc voltage of 100 V is applied in the circuit shown in Fig. 2.6 and the switch is kept open. The switch K is closed at t = 0. Find the complete expression for the current.



Fig. 2.6

## Solution



For an inductor 
$$i_L(0) = i_L(0^-) = i_L(0^+)$$
  

$$= \frac{100}{30} = 3.333 \text{ A}$$

$$i_L(\infty) = \frac{100}{20} = 5 \text{ A}$$

$$\therefore \qquad i_L(t) = i_L(\infty) + (i_{LCO}) - i_L(\infty))e^{-t/2}$$

$$\tau = \frac{L}{R} = \frac{0.1}{20} = 0.005$$

$$\therefore \qquad i_L(t) = 5 + (3.33 - 5)e^{-t/0.005}$$

$$= 5 - 1.67e^{-200t}.$$

- 7. (b) A dc voltage of 20 V is applied in a *RL* circuit where  $R = 5 \Omega$ and L = 10 H. Find
  - (i) The time constant
  - (ii) The maximum value of stored energy.

Solution  $V_{dc} = 20 \text{ V}; R = 5 \Omega; L = 10 \text{ H}$   $\tau = \frac{L}{R} = \frac{10}{5} = 2 \text{ sec}$   $i_{\text{max}} = \frac{V}{R} = \frac{20}{5} = 4 \text{ A}$ Energy stored =  $\frac{1}{2} Li^{2}_{\text{max}}$   $= \frac{1}{2} \times 10 \times (4)^{2}$  = 80 Joules.

8. (a) Find the Z-parameters for the network shown in Fig. 2.7.



Fig. 2.7

**Solution** *Z*-parameters of the given n.w.

Applying KVL for both the loops

 $V_1 = 6 \times 10^3 I_1 + 15 \times 10^3 (I_1 + I_2) = 21 \times 10^3 I_1 + 15 \times 10^3 I_2$  $V_2 = 4 \times 10^3 I_2 + 15 \times 10^3 (I_1 + I_2) = 15 \times 10^3 I_1 + 19 \times 10^3 I_2$ 



On comparison  $V_1 = Z_{11} I_1 + Z_{12} I_2$ 

$$\begin{split} V_2 &= Z_{21} I_1 + Z_{22} I_2 \\ Z_{11} &= 21 \ \mathrm{k}\Omega; \ Z_{12} &= 16 \ \mathrm{k}\Omega; \ Z_{21} &= 15 \ \mathrm{k}\,\Omega; \ Z_{22} &= 19 \ \mathrm{k}\Omega. \end{split}$$

8. (b) For the *h* parameter equivalent network shown in Fig. 2.8 find the voltage gain load resistance is  $R_L$ .



Fig. 2.8

Solution



$$-I_{22} = -h_{21} I_1 \times \frac{1}{h_{22}} / \frac{1}{h_{22}} + R_L = \frac{-h_{21} \cdot I_1}{1 + h_{22} R_L}$$
$$V_2 = -I_{22} \times R_L = \left(\frac{-h_{21} \cdot I_1}{1 + h_{22} R_L}\right) \cdot R_L$$

KVL in L.H.S. loop  $V_1 = h_{11} I_1 + h_{12} V_2$ from which  $I_1 = \frac{V_1 - h_{12} V_2}{h_{11}}$ 

Substituting in  $V_2$  and simplifying we get

$$V_2 (1 + h_{22} R_L) h_{11} = -h_{21} R_L (V_1 - h_{12} V_2)$$
$$\frac{V_2}{V_1} = \frac{-h_{21} \cdot R_L}{(h_{11} + h_{11} h_{22} R_L) - h_{12} h_{21} R_L}$$

# SET 3

1. (a) For the circuit shown in Fig. 3.1, find the current through 20  $\Omega$  resistor.





Solution



$$I = I_1 - I_2 = 0.16 - 0.26 = -0.1$$
A.

1. (b) Reduce the network shown in Fig. 3.2 to a single loop network by successive source transformation, to obtain current in  $12 \Omega$  resistor.



Fig. 3.2



- 2. (a) Explain
  - (i) Statically induced emf and
  - (ii) Dynamically induced emf

## Solution Refer Section 1.6

2. (b) The combined inductance of two coils connected in series is 0.6 H or 0.1 H, depending upon the relative directions of the currents in the coils. If one of the coils when isolated has a self inductance of

0.2 H, Calculate (i) Mutual inductance, and (ii) The coefficient of coupling. 2. (b) When 2 coils are connected in series,  $L_{\rm eq.} = L_1 + L_2 \pm 2M$  $L_1 + L_2 + 2M = 0.6$ (1) $L_1 + L_2 - 2M = 0.1$ (2)Substitute (2) in (1) $0.1 + 2M + 2M = 0.6 \implies M = 0.125$  $L_1 = 0.2 \text{ H}$  $0.2 + L_2 + 2(0.125) = 0.6$  $L_2 = 0.15 \text{ H}$ Co-efficient of coupling  $k = \frac{M}{\sqrt{L_1 L_2}}$  $k = \frac{0.125}{\sqrt{0.15 \times 0.2}}$ = 0.722. (c) Explain the terms

- (i) MMF (ii) Reluctance
- Solution <u>Refer Section 10:11</u>
- 3. (a) Draw the current, impedance and admittance loci for an *RL* series circuit having fixed resistance but variable reactance.

## Solution



- 3. (b) Figure 3.3 shows a series parallel circuit. Find
  - (i) admittance of each branch
  - (ii) admittance between points b and g.
  - (iii) impedance between points b and g.
  - (iv) total circuit impedance

(v) total current and power factor

(vi) currents in each branch.



Fig. 3.3

Solution



100 V, 50 Hz supply

(i) admittance of  $ab = \frac{1}{1.6 + j7.2}$   $= \frac{1}{7.37\angle 77.47^{\circ}} = 0.135 \angle -77.47^{\circ}$   $= 0.03 - j0.132 \mho$ admittance of  $cd = \frac{1}{4 + j3} = \frac{1}{5\angle 36.87^{\circ}}$   $= 0.16 - j0.12 \mho$ admittance of  $ef = \frac{1}{6 - j8} = \frac{1}{10\angle -53.13^{\circ}}$   $= 0.1 \angle 53.13^{\circ} = 0.06 + j0.08 \mho$ (ii) admittance between points *b* and *g*. = admittance of (cd + ef) = 0.16 - j0.12 + 0.06 + j0.08

$$= 0.22 - i0.04$$
  $\heartsuit$ 

(iii) impedance between points b and g $=\frac{1}{0.22-j0.04}=\frac{1}{0.224\,\angle-10.3^{\circ}}$  $= 4.46 \angle 10.3^{\circ} = 4.38 + j0.79 \Omega$ (iv) total circuit impedance = 1.6 + j7.2 + 4.38 + j0.79 $= 5.98 + i7.99 \Omega$ (v) total current =  $\frac{100 \angle 0^{\circ}}{5.98 + j7.99}$  $= \frac{100 \angle 0^{\circ}}{9.98 + \angle 53.19^{\circ}} = 10.02 \angle -53.19^{\circ} \text{ A}$ = 6 - i8 APower factor =  $\cos(53.19^\circ)$ = 0.599 lagging (vi) Current in branch  $ab = 10.02 \angle -53.19^{\circ}$  A current in branch  $cd = 10.02 \ \angle -53.19^{\circ} \times \frac{4+j3}{10-j5}$  $= 10.02 \angle -53.19^{\circ} \times \frac{5 \angle 53.13^{\circ}}{11.18 \angle -26.56^{\circ}}$ = 4.48 ∠26.5° A

Current in branch  $ef = 10.02 \angle -53.19^{\circ} - 4.48 \angle 26.5^{\circ}$ = 6 - i8 - 4 - i1.999

$$= 2 - j9.999 = 10.19 \angle -78.69^{\circ} \text{ A}$$

4. (a) Three identical resistances are connected in a star fashion against a balanced three phase voltage supply. If one of the resistance is removed, how much power is to be reduced?

## **Solution** 1/3rd power

- 4. (b) A 3-phase load has a resistance of 10  $\Omega$  in each phase and is connected in
  - (i) star and
  - (ii) delta against a 400 V, 3-phase supply. Compare the power consumed in both the cases.

#### Solution

(i) line voltage = 400 V



(ii)



line voltage = 400 V phase voltage = 400 V power factor = 1

$$I_{\rm Ph} = \frac{400}{10} = 40 \text{ A}$$
$$I_L = \sqrt{3} \ I_{\rm Ph} = \sqrt{3} \ \times 40 = 69.28 \text{ A}$$
Power consumed =  $\sqrt{3} \ V_L \ I_L \cos \varphi$ 
$$= \sqrt{3} \ \times 400 \times 69.28 = 48 \text{ kW}$$

Power consumed in delta connected case is three times that of in star connection

$$P_Y = \frac{1}{3} P_\Delta$$

4. (c) What is the difference between *RYB* phase sequence with *RBY* phase sequence?

Solution Refer Section 9.4 and example 9.1

- 5. (a) Define the following and explain by taking an example.
  - (i) Branch (ii) Node
  - (iii) Path (iv) Sub graph
  - (v) Tree (vi) Degree of node.

#### Solution Refer Sections 1.12; 2.1 and 2.2

5. (b) Draw the oriented graph of the network shown in Fig. 3.4 and write the cut set matrix.



**Solution** The oriented graph of the n.w. is shown in figure an Orbitrary Tree is selected to form fundamental cut set (*f*-cut set) matrix. The tree branches (Twigs) are shown with thick lines and the line branches are shown with dashed lines.



No. of branches = 7 No. of nodes (n) = 4Twigs = n - 1 = 3 (2, 3, 6) No. of links (l) = b - (n - 1) = 4 (1, 4, 5, 7) For twig 2; f-cut set  $C_1 \longrightarrow (1, 2, 5)$ For twig 3: f-cut set  $C_2 \longrightarrow (1, 3, 4, 5)$ For twig 6; f-cut set  $C_3 \longrightarrow (4, 5, 6, 7)$ Fundamental cut-set matrix f-cut set  $1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7$  $C_1 \begin{bmatrix} -1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \\ -1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \\ 0 \ 0 \ -1 \ -1 \ 1 \ 1 \end{bmatrix}$ 

6. (a) State the explain the Millmann's theorem.Solution Refer Section 9.10.6 and Example 9.21

6. (b) Find the current in the 6  $\Omega$  resistor using Superposition theorem as shown in Fig. 3.5.



Fig. 3.5

Solution



 $I_1 = \frac{10\angle 60^{\circ}}{6+j6-j8} = \frac{10\angle 60^{\circ}}{6-j2} = \frac{10\angle 60^{\circ}}{6.32\angle -18.43^{\circ}} = 1.58 \angle 78.43^{\circ} \text{ A}$ 



$$I_2 = 2\angle 0^\circ \times \frac{j6}{6+j6-j8} = 2\angle 0^\circ \times \frac{j6}{6-j2}$$
$$= 1\angle 0^\circ \times \frac{j6}{3-j1} = \frac{1\angle 0^\circ \times 6\angle 90^\circ}{3.16\angle -18.43^\circ} = 1.899 \angle 108.43^\circ \text{ A}$$

By superposition theorem, current through

- $$\begin{split} & 6\Omega = I_1 + I_2 \\ & = 1.58 \ \angle 78.43^\circ + 1.899 \ \angle 108.43^\circ \\ & = 0.317 + j1.548 + [-0.6 + j1.8] \\ & = -0.283 + j3.348 = 3.36 \ \angle 94.83^\circ \text{ A} \end{split}$$
- 7. (a) A dc voltage of 100 V is applied in the circuit shown in Fig. 3.6 and the switch is kept open. The switch K is closed at t = 0. Find the

complete expression for the current.





Solution



$$20i + 0.1 \frac{di}{dt} = 100$$
  

$$(D + 200)i = 1000$$
  

$$i = C_1 e^{-200t} + e^{-200t} \int 1000 e^{200t} dt$$
  

$$= C_1 e^{-200t} + 5$$
  
at  $t = 0^{-1}$   $i(0^-) = \frac{100}{20 + 10} = \frac{100}{30} = 3.33$  A  
at  $t = 0^+$ ,  $i = 3.33$  A  
 $i = C_1 e^{-200(0)} + 5 = 3.33$   
 $C_1 = 3.33 - 5 = -1.67$ 

The complete solution is

$$i(t) = -1.67e^{-200t} + 5$$

- 7. (b) A dc voltage of 20 V is applied in a *RL* circuit where  $R = 5 \Omega$ and L = 10 H. Find
  - (i) The time constant

*:*..

(ii) The maximum value of stored energy.

## Solution



time constant = 
$$\frac{L}{R} = \frac{10}{5} = 2 \sec \theta$$

Max. value of stored energy =

$$\frac{1}{2} Li^2 = \frac{1}{2} \times 10 \times \left(\frac{20}{5}\right)^2 = \frac{1}{2} \times 10 \times 16 = 80 \text{ Joules}$$

8. (a) In a T network shown in Fig. 3.7,  $Z_1 = 2 \angle 0^\circ$ ,  $Z_2 = 5 \angle -90^\circ$ ,  $Z_3 = 3 \angle 90^\circ$ , find the Z parameters.



Fig. 3.7

Solution



 $Z_{1} = 2 \ \angle 0^{\circ} \qquad Z_{2} = 5 \ \angle -90^{\circ} \qquad Z_{3} = 3 \ \angle 90^{\circ}$  $Z_{11} = Z_{1} + Z_{3}$  $= 2 \ \angle 0^{\circ} + 3 \ \angle 90^{\circ} = 2 + j3 = 3.6 \ \angle 56.3^{\circ}$  $Z_{12} = Z_{3} = 3 \ \angle 90^{\circ}$  $Z_{21} = Z_{3} = 3 \ \angle 90^{\circ}$  $Z_{22} = Z_{2} + Z_{3}$  $= 5 \ \angle -90^{\circ} + 3 \ \angle 90^{\circ} = -5j + 3j = -2j = +2 \ \angle -90^{\circ}$  $Z = \begin{bmatrix} 3.6 \ \angle 56.3^{\circ} \ 3 \ \angle 90^{\circ} \\ 3 \ \angle 90^{\circ} \ 2 \ \angle -90^{\circ} \end{bmatrix}$ 

8. (b) Z-parameters for a two port network are given as  $Z_{11} = 25 \Omega$ ,  $Z_{12} = Z_{21} = 20 \Omega$ ,  $Z_{22} = 50 \Omega$ . Find the equivalent *T*-network.

---

## Solution



 $Z_{11} = 25 \ \Omega$  $Z_{12} = Z_{21} = 20 \ \Omega$  $Z_{22} = 50 \ \Omega$ 

Consider General *T*-network

$$V_{1} = Z_{1}I_{1} + Z_{3}(I_{1} + I_{2})$$

$$V_{1} = (Z_{1} + Z_{3})I_{1} + Z_{3}I_{2}(1)$$

$$V_{2} = Z_{2}I_{2} + Z_{3}(I_{1} + I_{2})$$

$$V_{2} = Z_{3}I_{1} + (Z_{3} + Z_{2})I_{2}(2)$$

$$\therefore \qquad Z_{11} = Z_{1} + Z_{3} \Longrightarrow Z_{1} + Z_{3} = 25 \ \Omega$$

$$Z_{21} = Z_{12} = Z_{3} = 20 \ \Omega$$
From which we get  $Z_{1} = 5 \ \Omega$ ;  $Z_{22} = Z_{2} + Z_{3} = 50 \ \Omega$ 

$$Z_2 = 45 \ \Omega.$$

## SET 4

- 1. (a) Explain
  - (i) KCL (ii
    - (ii) KVL
  - (iii) Practical current source (iv) Practical voltage source.

Solution Refer Sections 1.8; 1.9 and 1.12

 (b) A 20 V battery with an internal resistance of 5 ohms is connected to a resistor of x ohms. If an additional resistance of 6 Ω is connected across the battery, find the value of x, so that the external power supplied by the battery remain the same.

### Solution



Power supplied to x by battery =  $\left(\frac{20}{5+x}\right)^2 x = P_1$ 



$$I_2 = \frac{20}{5 + \frac{6x}{6 + x}} = \frac{120}{30 + 11x}$$

Power supplied to  $x = \left(\frac{120}{30+11x}\right)^2 x = P_2$   $P_1 = P_2 \implies \frac{20}{5+x} = \frac{120}{11x+30}$ x = 0

- 2. (a) Explain the following terms:
  - (i) Magnetic circuit (ii) Permeability
  - (iii) Magneto motive force (iv) Reluctance.

## Solution Refer Section 10.11

2. (b) A cast steel structure is made of a rod of square section  $2.5 \text{ cm} \times 2.5 \text{ cm}$  as shown in Fig. 4.1. What is the current that should be passed in a 500 turn coil on the left limb, so that a flux of 2.5 mwb is made to pass in the right limb. Assume permeability as 750 and neglect leakage.





## Solution



$$\varphi = \varphi_1 + \varphi_2$$
Also mmf in  $C = \text{mmf in } D$ 

$$\therefore \qquad \varphi_1 \frac{25}{\mu A} = 2.5 \times 10^{-3} \frac{40}{\mu A}$$

$$\varphi_1 = \varphi_1 + \varphi_2 = 6.5 \text{ mmf}$$

Total A.T. for the hole circuit is

- (i) that required for path E and
- (ii) that required for path C or D.

Flux density in 
$$E = \frac{6.5 \times 10^{-3}}{(2.5)^2 \times 10^{-4}} = 10.4 \text{ web/m}^2$$
  
A.T. in  $E = \frac{10.4 \times 0.4}{4\pi \times 10^{-7} \times 750} = 4414 \text{ A.T.}$ 

Flux density in path  $D = \frac{2.5 \times 10^{-3}}{(2.5)^2 \times 10^{-4}} = 40 \text{ web/m}^2$ A.T. in  $D = \frac{40 \times 0.4}{4\pi \times 10^{-7} \times 750} = 1698 \text{ A.T.}$ Total A.T. = 4414 + 1698 = 6112 Current needed =  $\frac{6112}{500} = 12.224 \text{ A}$ 

3. (a) Derive the expression for power in  $a_1$ - $\phi$  A.c. circuits.

Solution Refer Sections 6.1 and 6.2

- 3. (b) In the circuit shown in Fig. 4.2, calculate.
  - (i) The total impedance
  - (ii) The total current
  - (iii) Power factor

- (iv) The total S, P and Q
- (v) The total admittance. Also, draw vector diagram.







40 V, 50 Hz,  $1-\varphi$  supply

Admittance between A and B is

$$\frac{1}{2+j5} + \frac{1}{1-j2} + \frac{1}{2}$$

$$= \frac{1}{5.38 \angle 68.2^{\circ}} + \frac{1}{2.24 \angle -63.4^{\circ}} + 0.5$$

$$= 0.069 - j0.17 + 0.199 + j0.399 + 0.5$$

$$= 0.768 + j0.229 = 0.8 \angle 16.6^{\circ} \ \mho$$
Impedance between A and  $B = \frac{1}{0.8 \angle 16.6^{\circ}} = 1.25 \angle -16.6^{\circ}$ 
Total impedance =  $1 + j1 + 1.198 - j0.36 = 2.29 \angle 16.23^{\circ} \ \Omega$ 
(ii) Total current =  $\frac{40}{2.29 \angle 16.23^{\circ}} = 17.47 \angle -16.23^{\circ} \ A$ 
(iii) Power factor =  $\cos 16.23 = 0.96$  lagging
(iv)  $P = VI \cos \varphi$ 

$$= 40 \times 17.47 \cos 16.23^{\circ} = 670.95 \ W$$
 $Q = VI \sin \varphi$ 

$$= 40 \times 17.47 \sin 16.23^{\circ} = 195.31 \ VAR$$
 $S = P + jQ = 640.95 + j195.31$ 

$$= 698.798 \angle 16.23^{\circ} \ VA.$$
(v) Total admittance =  $\frac{1}{2.29 \angle 16.23^{\circ}} = 0.43 \angle -16.23^{\circ} \ \mho$ 

4. (a) Two resistors each of  $100 \Omega$  are connected in series. The phases *a* and *c* of a three phase 400 V supply are connected to the two ends and phase *b* is connected to the junction of the two resistors. Find the line currents.

## Solution



Current in phase a

$$I_a = \frac{400}{100} = 4 \text{ A}$$

Current in phase C is also 4 A

Current in phase  $b = 2 \times I_P \times \cos \frac{60}{2} = 6.928 \text{ A}$ 

4. (b) Derive the expressions for wattmeter readings in two wattmeter method with balanced star connected load. How do you calculate the power factor of the balanced load from wattmeter readings?

#### Solution



Sum of the instantaneous powers measured by two wattmeters



$$\frac{W_2 - W_1}{W_1 + W_2} = \frac{\sqrt{3} V_{Ph} I_{Ph} \sin \theta}{3 V_{Ph} I_{Ph} \cos \theta} = \frac{\tan \theta}{\sqrt{3}}$$
Power factor angle,  $\theta = \tan^{-1} \frac{\sqrt{3} (W_2 - W_1)}{W_1 + W_2}$ 
Power factor =  $\cos \theta$ 
a) Define the following and explain by taking an

- 5. (a) Define the following and explain by taking (i) Node (ii) Tree
  - (iii) Sub graph (iv) Loop
  - (v) Links (vi) Directed graph.

## Solution Refer Sections 1.12; 2.1 and 2.2

5. (b) Find the fundamental tie-set and cut-set matrix for the graph and for the tree shown in the Fig. 4.3.



Fig. 4.3

Solution



The required Tie set and cut set matrices are given below. The procedure is illustrated in solution to Question 5 (b) of Set No. 2.

 Tie-set

  $\begin{bmatrix}
 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 0 & 1 & 1 & 0 & -1 & -1 & 0 \\
 0 & 0 & 1 & 0 & -1 & 0 & 1
 \end{bmatrix}$ *f* $-cut set

 <math display="block">
 \begin{bmatrix}
 -1 & 1 & 0 & 0 & 0 & 1 & 0 \\
 -1 & 0 & 1 & 0 & 0 & 1 & -1 \\
 -1 & 0 & 0 & 1 & 0 & 0
 \end{bmatrix}$ 

example.

6. (a) State and explain compensation theorem.

## Solution <u>Refer Section 3.6</u>

6. (b) In the network shown in Fig. 4.4, find the value of  $Z_L$  so that the power transfer from the source is maximum. Also find  $P_{max}$ .



Fig. 4.4

10 ∠ 0° V j24 Ω **12** Ω **21** Ω  $I_1$ А 2000  $Z_L$ j60 Ω  $30 \Omega$ 50 Ω 12 m В  $I_1 = \frac{10 \angle 0^{\circ}}{33 + j24} = \frac{10 \angle 0^{\circ}}{40 \angle 36.87^{\circ}} = 0.85 \angle -36.02^{\circ} \text{ A}$  $I_2 = \frac{10\angle 0^{\circ}}{80 + j60} = \frac{10\angle 0^{\circ}}{100\angle 36.87^{\circ}} = 0.1 \angle -36.87^{\circ} \text{ A}$  $V_A = 0.25 \ \angle -36.02^\circ \times 21 = 5.25 \ \angle -36.02^\circ \ V$  $V_B = 0.1 \ \angle -36.87^\circ \times 50 = 5 \ \angle -36.87^\circ \text{ V}$  $V_{Th} = V_A - V_B = 5.25 \ \angle -36.02^\circ - 5 \ \angle -36.87^\circ$ = 4.246 - i3.08 - 3.99 + i3 $= 0.256 - j0.08 = 0.268 \angle -17.35^{\circ} V$  $Z_{Th} = \frac{(30+j60)50}{80+j60} + \frac{(12+j24)21}{33+j24}$ 

Solution

$$\begin{split} Z_{Th} &= \frac{3354.1 \angle 63.43^{\circ}}{100 \angle 36.87^{\circ}} + \frac{563.489 \angle 63.43^{\circ}}{40.8 \angle 36.03^{\circ}} \\ &= 33.541 \angle 26.56^{\circ} + 13.81 \angle 27.4^{\circ} \\ &= 30 + j14.99 + 12.26 + j6.35 \\ &= 42.26 + j21.34 = 47.34 + j26.79 \\ &= 47.34 \angle 26.79^{\circ} \, \Omega \end{split}$$

To get max. power the load must be complex conjugate of source impedance.

: load,  $Z = 42.26 - j21.34 \Omega$ 



Current through the load = 
$$\frac{V_{Th}}{Z_{Th} + Z}$$

$$I = \frac{0.268 \angle -17.35^{\circ}}{(42.26)2} = 0.00317 \angle -17.35^{\circ} \text{ A}$$

Maximum power delivered to the load is

 $= (0.00317)^2 \times 42.26 = 0.000425 \text{ W} = 0.425 \text{ mW}.$ 

7. In the Fig. 4.5, the switch is close at position 1 at t = 0. At t = 0.5 m sec. The switch is moved to position 2. Find the expression for the current in both the conditions and sketch the transient.



Fig. 4.5

## Solution

When switch is in position 1

$$-10 + 50(i) = 0 \implies i = \frac{10}{50} = 0.2 \text{ A}$$

 $I(0^{-}) = 0.2 \text{ A}$ 



When switch is moved to position 2

$$I(0^{+}) = 0.2 \text{ A}$$
  
- 5 + 50 I(5) + 0.5[SI(5) - I(0)] = 0  
I(S) [50 + 0.55] = 5 + 0.5 × 0.2 = 5.01

$$I(5) = \frac{5.01}{50 + 0.55} = \frac{10.02}{5 + 100}$$
$$i(t) = 10.02e^{-100t}$$
$$at t = 0.5 ms$$
$$i(t) = 10.02e^{-100 \times 0.5 \times 10^{-3}} = 9.53 A$$



8. Determine Y-Parameters of the network shown in Fig. 4.6.



 $\bullet \bullet \bullet \bullet$ 

 $-\frac{1}{3}$ 

## Solution



$$-I_{1} + \frac{V_{1}}{3} + \frac{V_{1} - V_{2}}{3} = 0$$
  

$$\Rightarrow \qquad V_{1} \left[ \frac{1}{3} + \frac{1}{3} \right] - V_{2} \left[ \frac{1}{3} \right] = I_{1} \qquad (1)$$

$$\frac{V_2 - V_1}{3} + \frac{V_2}{3} + 3I_1 - I_2 = 0$$
  
$$\Rightarrow -V_1 \left[\frac{1}{3}\right] + V_2 \left[\frac{1}{3} + \frac{1}{3}\right] = -3I_1 + I_2$$
(2)

from (1) 
$$\frac{2}{3} V_1 - \frac{1}{3} V_2 = I_1$$
  
from (2)  $-\frac{1}{3} V_1 + \frac{2}{3} V_2 = -3I_1 + I_2$   
 $-\frac{1}{3} V_1 + \frac{2}{3} V_2 = I_2 - 3\left[\frac{2}{3}V_1 - \frac{1}{3}V_2\right]$   
 $-\frac{1}{3} V_1 + \frac{2}{3} V_2 = I_2 - 2V_1 + V_2$   
 $I_2 = -\frac{1}{3} V_1 + 2V_1 + \frac{2}{3} V_2 - V_2$   
 $= \frac{5}{3} V_1 - \frac{1}{3} V_2$   
 $-\frac{1}{3} V_1 + \frac{2}{3} V_2 = I_2 - 3I_1$   
 $= \frac{5}{3} V_1 - \frac{1}{3} V_2 - 3I_1$   
 $3I_1 = 2V_1 - V_2$   
 $I_1 = \frac{2}{3} V_1 - \frac{1}{3} V_2$   
 $Y_{11} = \frac{2}{3}, Y_{12} = -\frac{1}{3}, Y_{21} = \frac{5}{3}, Y_{22} = -\frac{1}{3}$ 

 $Y = \begin{bmatrix} 2/3 & -1/3 \\ 5/3 & -1/3 \end{bmatrix}$ 



# SET 1

1. (a) Find the value of current  $I_i$  in Fig. 1.1.



Fig. 1.1

Solution Converting current source into equivalent voltage source



By applying KVL

$$10 - 7I_1 - 15 - 5I_1 - 9I_1 + 45 - 4 = 0$$
  
$$36 = 21I_1$$
  
$$I_1 = \frac{36}{21} = 1.714 \text{ A}$$
  
$$\boxed{I_1 = 1.714 \text{ A}}$$

(b) Find the value of E in the network shown in Fig. 1.2





Solution Calculating current through all branches



$$E = 16 V$$

(c) Write short notes on dependent sources.

Solution Refer Section 1.8.

2. (a) What is magnetic coupling? What is its effect? How can you arrange two coils so that they do not have magnetic coupling?

Solution Refer Section 10.1.

- (b) Two coils having 30 and 600 turns are wound side by side on a closed iron circuit of 100 roman cross section and mean length 150 cm. Calculate:
  - (i) The self inductance of the two coils and mutual inductance if relative permeability of iron is 2000. Assume no magnetic leakage.
  - (ii) 0 to 10 A steadily in 0.01 sec

**Solution** 
$$N_1 = 30, a = 100 \text{ cm}^2 = 10 \times 10^{-4} \text{ m}^2, N_2 = 600$$
  
 $l = 150 \text{ cm} = 1.5 \text{ m}, \mu = 2000$ 

Reluctance =  $\frac{l}{\mu_r \mu_r \cdot a}$ =  $\frac{1.5}{4\pi \times 10^{-7} \times 2000 \times 100 \times 10^{-4}}$ = 0.05968 × 10<sup>6</sup>  $L_1 = \frac{N_1^2}{\text{Reluctance}} = \frac{(30)^2}{0.05968 \times 10^6}$ = 15 mH  $L_2 = \frac{N_2^2}{\text{Reluctance}} = \frac{(600)^2}{0.05968 \times 10^6}$ 

If there is no magnetic leakage

$$\mu = \sqrt{L_1 L_2} = \sqrt{15 \times 10^{-3} \times 6} = 0.3 \text{ H}$$

(c) Define reluctance. Give its units.

Solution Refer Section 10.11.

3. (a) Explain the phenomenon of *acceptor resonance* in electrical circuits.

**Solution** Acceptor resonance is nothing but series resonance. Refer Section 8.1.

(b) Proceeding analytically, sketch the resonance curves for a series resonant circuit with variable frequency and constant R, L and C.

**Solution** Refer Section 8.2.

(c) A series circuit comprising *R*, *L* and *C* is supplied at 220 V, 50 Hz. At resonance, the voltage across the capacitor is 550 V. The current at resonance is 1 A. Determine the circuit parameters *R*, *L* and *C*.

Solution At resonance  $X_{L} = X_{C}$ Current at resonance  $= I = \frac{V}{R + g(X_{L} - X_{C})} = \frac{V}{R} = 220$   $1 = \frac{220}{R}$   $\boxed{\therefore R = 220 \Omega}$   $V_{C} = I_{o} X_{C}$   $550 = 1 \times \frac{1}{\omega_{o}c}$   $C = \frac{1}{550 \times 2\pi f} = \frac{1}{550 \times 2 \times \pi \times 50}$   $C = 5.78 \,\mu\text{F}$   $f_{o} = \frac{1}{2\pi \sqrt{LC}}$   $LC = \left(\frac{1}{2\pi f_{o}}\right)^{2}$   $L = \frac{1}{C} \left(\frac{1}{2\pi f_{o}}\right)^{2}$   $= \frac{1}{5.78 \times 10^{-6}} \left(\frac{1}{100\pi}\right) = 1.750 \,\text{H}$ 

: Circuit elements at resonance are

 $R = 220 \Omega, L = 1.75 \text{ H}, C = 5.78 \mu\text{F}$ 

4. (a) For the network shown in Fig. 1.4, calculate the line currents and power consumed if the phase sequence is *ABC*.



Fig. 1.3
**Solution**  $V_{ab} = 100 \ \underline{0}$ ;  $V_{bc} = 100 \ \underline{-120^{\circ}}$ ;  $V_{ca} = 100 \ \underline{-240^{\circ}}$ .

$$I_{ab} = \frac{100|0}{(3+J4)} = \frac{100|0}{5|53.13^{\circ}} = 20 |-53.13^{\circ}$$
$$I_{bc} = \frac{100|-120^{\circ}}{5} = 20 |-120^{\circ}$$
$$I_{ca} = \frac{100|-240^{\circ}}{(2-J2)} = \frac{100|-240^{\circ}}{2.828|-45^{\circ}} = 35.36 |-195^{\circ}$$

Line currents are

$$\begin{split} I_A &= I_{AB} - I_{CA} \\ &= 20 \mid -53.13^{\circ} - 35.36 \mid -195^{\circ} \\ &= 12 - J16 + 34.155 - j9.15 \\ &= 46.155 - J25.15 = 52.56 \mid -28.586^{\circ} \\ I_B &= I_{BC} - I_{AB} \\ &= 20 \mid -0120^{\circ} - 20 \mid -53.13^{\circ} \\ &= -10 - j17.32 - 12 + j16 \\ &= -32 - j1.32 = 32.02^{\circ} \mid 182.36^{\circ} \\ I_C &= I_{CA} - I_{BC} \\ &= 35.36 \mid -195^{\circ} - 20 \mid -120^{\circ} \\ &= -34.155 + j9.15 + 10 + j17.32 \\ &- 24.155 + j26.47 = 55.83 \mid 132.38^{\circ} \\ \end{split}$$

= 1200

or

$$I_{ab}^2 \times R_{ab} = (20)^2 \times 3 = 1200$$
  
In  $BC = 100 \times 20 \times \cos 0 = 200$   
In  $CA = 100 \times 35.36 \times \cos 45 = 2500$ 

Total power = 5700 watts.

(b) An unbalanced star connected load is connected across a  $3-\phi$ , 400 V balanced supply of phase sequence RYB as shown in Fig. 1.3. Two wattmeters are connected to measure the total power supplied as shown in the figure. Find the readings of the wattmeters.



Fig. 1.4

Solution Refer Section 9.10.3 and 9.10.4.



Unbalanced star connected three wire loads can be solved by Kirchhoff's laws, Millman's theorem star delta conversion. Here, KVL method is illustrated.

$$V_{RY} = I_R Z_{R0} + (I_R + I_B) Z_{OY}$$
  
$$V_{BY} = I_B Z_{B0} + (I_B + I_R) Z_{OY}$$

Solve for  $I_R$  and  $I_B$  (line or phase currents) and voltage drops across each phase. And line voltages are specified in data as

 $V_{RY} = 400 \ |\underline{0}; V_{YB} = 400 \ |\underline{-120^{\circ}}; V_{BR} = 400 \ |\underline{-240^{\circ}}$ 

Wattmeter  $W_1$  carries  $I_R$  and has voltage  $V_{RY}$  impressed across the pressure coil. Power can be found by using  $V_{RY}$   $I_R^* - 1$ 

The other wattmeter  $W_2$  carries  $I_B$  and has voltage  $V_{BY}(-V_{YB})$  impressed across the pressure coil. Power consumed by  $W_2$  is given by  $V_B^* - 2$ . Real parts of equation 1 and 2 are the wattmeter readings  $W_1$  and  $W_2$  respectively.

5. (a) Explain the procedure for obtaining fundamental tie-set matrix of a given network.

Solution Refer Section 2.7.

(b) Draw the oriented graph of the network shown in Fig. 1.5 and write the incidence matrix.



Fig. 1.5

Solution Directions of currents are arbitr arily assumed as shown in the current.



Ideal voltage sources and current sources do not appear in the graph of a linear network. Ideal voltage source is represented by short circuit and an ideal current source is replaced by an open circuit. The nodes that appear in the graph are numbered (1) (2) (3) (4) and (5); branches as a, b, c, d, e, f and g. The graph is as shown in the figure.



For a graph with *n* nodes and *b* branches, the order of the incidence matrix is  $(n-1) \times b$ . Choose node (5) as reference (ordatum) node for writing incidence matrix. The required incidence matrix is given by

6. (a) State and explain compensation theorem.

Solution Refer Section 3.6.

(b) In the network shown in Fig. 1.6, find the value of  $Z_L$  so that the power transfer from the source is maximum. Also find  $P_{max}$ .



Fig. 1.6

**Solution** Let us remove  $z_L$ . The Internal impedance of the circuit looking through x-y is given by

$$z_{in} = \frac{(21)(12 + J24)}{21 + 12 J24} + \frac{50(30 + J60)}{50 + 30 + J60}$$
$$= \frac{563.44 \angle 63.43^{\circ}}{40.8 \angle 36^{\circ}} + \frac{3354.10 \angle 63.43^{\circ}}{100 \angle 36.87^{\circ}}$$
$$= 13.81 \angle 27.43^{\circ} + 33.54 \angle 26.56^{\circ}$$
$$z_{in} = 42.19 + J21.49 \ \Omega$$

As per maximum power transfer theorem,  $z_{l}$  should be the complex of  $z_{in}$ 

$$\begin{aligned} z_L &= z_{in}^* = (42.19 - J21.49) \,\Omega \\ V_{O,C} &= V_{xy} \\ V_x &= \frac{12 + J24}{12 + J24 + 21} \times 10 \,\angle 0^\circ \\ &= 6.577 \,\angle 27.43^\circ \,\mathrm{V} \\ V_y &= \frac{30 + J60}{30 + J60 + 50} \times 10 \,\angle 0^\circ = 6.71 \,\angle 26.56^\circ \\ V_{OC} &= V_x - V_y = 6.577 \,\angle 27.43^\circ - 6.71 \,\angle 26.56^\circ \\ &= -0.163 + J0.029 \\ V_{O,C} &= 0.1657 \,\angle 170^\circ \,\mathrm{V} \\ P_{max} &= \frac{V_{oc}^2}{4R_L} = \frac{(0.1657)^2}{4 \times 42.19} = 0.1627 \,\mathrm{mW} \end{aligned}$$

7. (a) A dc voltage of 100 V is applied in the circuit shown in Fig. 1.7 and the switch is kept open. The switch *K* is closed at t = 0. Find the complete expression for the current.



Fig. 1.7

#### **Solution** At $t = 0^+$

$$100 = 20i + 0.1 \frac{di}{dt}$$

$$P + 200)i = 1000.$$

The complete solution is given by

 $i = i_c + i_p$ where  $i_c =$ Complementary function  $= ce^{-200t}$  $i_p =$ Particular function

$$= e^{-(R/L) \cdot t} \int \left(\frac{V}{L}\right) e^{(R/L)t} dt$$

$$i_p = \frac{v}{R} = \frac{100}{20} = 5A$$
  
 $i = ce^{-200t} + 5$ 

However, below switching operation, the study state current in the circuit is

$$\frac{V}{\left(20+10\right)\Omega} = 3.33 \text{ A}$$

*.*..

Due to the presence of inductor

at  $t = 0_1 i = 3.33$  A Then i = c + 5 (8)  $3.33 = c + 5 \rightarrow c = -1.67$ .

Hence complete solution in

$$i = -1.67e^{-200t} + 5A$$

- (b) A dc voltage of 20 V is applied in an RL circuit where  $R = 5 \Omega$  and L = 10 H. Find
  - (i) the time constant
  - (ii) the maximum value of stored energy



8. (a) Find the y-parameters of the network shown in Fig. 1.8.



Fig. 1.8

Solution Y-parameter of the Network shown in the Figure.



**Y-Parameter** 

$$\begin{aligned} Y_{11} &= \left. \frac{I_1}{V_1} \right|_{v_2 = 0} & Y_{12} &= \left. \frac{I_1}{V_2} \right|_{v_1 = 0} \\ Y_{21} &= \left. \frac{I_2}{V_1} \right|_{v_2 = 0} & Y_{22} &= \left. \frac{I_2}{V_2} \right|_{v_1 = 0} \end{aligned}$$

(1)  $Y_{1F}$ 



 $Y_{21}$ 

$$I_{2} = \frac{-I_{1} \cdot 175}{275}$$

$$I_{1} = -I_{2} \ 1.571$$

$$\frac{I_{1}}{V_{1}} = \frac{-I_{2} \times 1.571}{V_{1}} = 0.0157$$

$$Y_{21} = \frac{I_{2}}{V_{1}} = -9.99 \times 10^{-3} \text{ (Mho)}$$

 $Y_{11} = \frac{I_1}{V_1} = 0.0157 \, \mho \, (\text{Mho})$ 

(b) Calculate the Z-parameters for the lattice network shown in Fig. 1.9.



Fig. 1.9



$$V_{1} = I_{1} \left( \frac{1}{2} \right)$$

$$z_{11} = \left. \frac{V_{1}}{I_{1}} \right|_{I_{2}=0} = \frac{z_{1} + z_{2}}{2}$$

$$V_{2} = V_{c} - Vd$$

$$= (V_{1} + I_{3} z_{1}) - (V_{1} - I_{4} z_{2})$$

$$= I_{4} z_{2} - I_{3} z_{1}$$

$$I_{3} = I_{1} \frac{z_{2} + z_{1}}{(z_{1} + z_{2})(z_{2} + z_{1})} = \frac{I_{1}}{2}$$

 $(z_1 + z_2)$ 

 $(z_1 + z_2) \parallel (z_1 + z_2) = \frac{z_1 + z_2}{2}$ 

also,  $I_4 = \frac{I_1}{2}$ 

Hence  $V_2 = \frac{I_1}{2} \times z_2 - z_1 \times \frac{I_1}{2} = \frac{z_2 - z_1}{2} I_1$  $\frac{V_2}{I_1} = z_{21}|_{I_2 = 0} = \frac{z_2 - z_1}{2}$ 

Hence

$$z_{11} = z_{22} = \frac{z_1 - z_2}{2}$$
$$z_{12} = z_{21} = \frac{z_2 - z_1}{2}$$



# SET 2

1. (a) What is the difference between an ideal source and a practical source? Draw the relevant characteristics of the above sources.

**Solution** Refer Section 1.8.

(b) Explain the difference between active elements and passive elements with suitable examples.

**Solution** Refer Section 1.4.1.

(c) Determine the current through the  $6-\Omega$  resistor and the power supplied by the current source for the circuit shown in Fig. 2.1.



Fig. 2.1

**Solution** Current through  $6-\Omega$  resistor and power supplied by the current source



$$I_{1} + I_{2} = 21 \text{ A}$$

$$I_{2} = \frac{21 \times 5}{7} = 15 \text{ A}$$

$$I_{1} = 6 \text{ A}$$

$$I_{1} = I_{3} + I_{4} = 6 \text{ A}$$

$$I_{4} = \frac{6 \times 6}{9} = 4 \text{ A}, I_{3} = 2 \text{ A}$$

Current through 6- $\Omega$  resistor is =  $I_3 = 2$  A

Ι

$$I_3 = 2 \text{ A}$$

Power supplied by the current source.



Power supplied by current source = Power consumed in the resistor.

$$= I^2 R = (21)^2 \times 1.428$$

$$P = 629.748 \omega$$

2. (a) Solve for the currents  $I_1$  and  $I_2$  in the circuit shown in Fig. 2.1. Also, find the ratio of  $V_2/V_1$ .



Fig. 2.2

Solution w = 2 rad/sec  $J \times L_1 = J_2 \Omega$   $J \times L_2 = J(4 \times 2) = J8$ KVL to Loop 1

> $M = J_4$  $I_1 (I + J2) + (J4)I_2 = V_1$  (1)

KVL to Loop 2

$$(J_4) I_1 + (2 + J_8) I_2 = 0 (2)$$

So the mesh equations are

$$(1 + J2)I_{1} + (J4)I_{2} = V_{1} = 10$$

$$(J4)I_{1} + (2 + J8)I_{2} = 0$$

$$\begin{bmatrix} 1 + J2 & J4 \\ J4 & 2 + J8 \end{bmatrix} \begin{bmatrix} I_{1} \\ I_{2} \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$I_{1} = \frac{\begin{vmatrix} 10 & J4 \\ 0 & 2 + J8 \end{vmatrix}}{\Delta} \quad I_{2} = \frac{\begin{vmatrix} 1 + J2 & 10 \\ J4 & 0 \\ \Delta \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1+J2 & J4 \\ J4 & 2+J8 \end{vmatrix} = 2+12i$$

$$I_1 = \frac{20+80i}{2+12i} \qquad I_2 = \frac{-40i}{2+12i}$$

$$I_1 = 6.75 - 0.540i \qquad I_2 = -3.243 - 0.540i$$

$$V_2 = 2I_2 \qquad I_2 = -3.243 - 0.540i$$

$$I_2 = 3.287 \angle -170.53^\circ \text{ A}$$

$$\frac{V_2}{V_1} = \frac{2 \times (3.287 \angle -170.53^\circ)}{10 \angle 0^\circ}$$

$$\frac{V_2}{V_1} = 0.657 \angle -170.537^\circ$$

Ratio

(b) What is magnetic circuit? Compare magnetic circuit with electric circuit in any four aspects.

**Solution** Refer Sections 10.11 and 10.13.

- 3. (a) The voltage of a circuit is  $V = 200 \sin(\omega t + 30^\circ)$  and the current is  $I = 50 \sin(\omega t + 60^\circ)$ . Calculate
  - (i) the average power, reactive volt-amperes and apparent power
  - (ii) the circuit elements if  $\omega = 100 \pi \text{ rad/sec}$

Solution

Reactive volt

$$V = 200 \sin (\omega t + 30^{\circ})$$
  

$$i = 50 \sin (\omega t + 60^{\circ})$$
  
(i) Avg. power =  $V_m I_m \cos \theta$   

$$= \frac{200}{\sqrt{2}} \times \frac{50}{\sqrt{2}} \cos (60 - 30)$$
  

$$P_{av} = 4330.127 W$$
  
ampere =  $V_m I_m \sin \theta$   

$$= \frac{200}{\sqrt{2}} \cdot \frac{50}{\sqrt{2}} \sin (60 - 30)$$
  

$$P_r = 2500 VAR$$
  

$$P_a$$
  

$$P_a$$

Apparent power =  $\frac{P_{av}}{\cos \theta} = \frac{4330.127}{\cos 30^\circ} = 5000 \text{ VA}$ 

(ii) The current leads the voltage by  $30^{\circ}$ . Hence the circuit must contain *R* and *C*.

$$\tan \theta = \frac{1}{\omega RC} \implies \tan 30^\circ = \frac{1}{100\pi \times RC}$$

....

$$\Rightarrow RC = 0.0055 \Rightarrow C = \frac{0.0055}{R}$$

$$Z = \frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega c}\right)^2}$$

$$\frac{200}{50} = \sqrt{R^2 + \left(\frac{R}{100\pi \times 0.0055}\right)^2}$$

$$4 = 1.155 \text{ R} \Rightarrow R = \frac{4}{1.155} = 3.46 \Omega$$

$$0.0055$$

and

$$C = \frac{0.0055}{3.46} = 1.59 \text{ mF}$$

(i) Voltage 
$$V = 200 \sin(\omega t + 30^\circ)$$



1. Average power

$$P_{avg} = VI \cos \phi$$
$$= \frac{200}{\sqrt{2}} \times \frac{50}{\sqrt{2}} \cos (30^{\circ})$$
$$P_{avg} = \frac{8660.25\omega}{2} = 4330.12 \ \omega$$

2. Reactive volt amperes

$$\theta = \frac{V}{\sqrt{2}} \frac{I}{\sqrt{2}} \sin \phi$$
$$\theta = \frac{500\omega}{2} = 2500 \ \omega$$

3. Apparent power

$$= V_{rms} \times I_{rms} = \frac{200}{\sqrt{2}} \times \frac{50}{\sqrt{2}} = 5000 \text{ VA}$$

(ii)  $\omega = 100 \pi r/sec$ 

Average power =  $i_2 R$ 

= 4330.12 watts

$$\therefore \qquad i = \frac{50}{\sqrt{2}}$$
$$\therefore \qquad R = \frac{4330.12 \times 2}{2500} = 3.464 \,\Omega$$

If the circuit is assumed as a series circuit by inspection of voltage and current equations, current leads the voltage by  $30^{\circ}$ . Hence it is an *RC* circuit.

$$\tan 30^\circ = \frac{1}{\omega CR}$$
  
$$\therefore \qquad C = \frac{1}{0.5773 \times 100 \times \pi \times 3.464}$$
$$= 1.59 \text{ mH}$$

(b) Find the form factor of the following waveform shown in Fig. 2.3.



Fig. 2.3

**Solution** From 0 to  $\pi/_3$ ,  $V = \frac{3V_1}{\pi}t$ From  $\pi/_3$  to  $2\pi/_3$ ,  $V = V_1$ From  $2\pi/_3$  to  $\pi$ ,  $V = 3V_1 - \frac{3V_1}{\pi}t$ Form factor  $= \frac{V_{rms}}{V_{avg}}$ 



$$= \sqrt{\frac{5}{9}V_1^2} = \frac{\sqrt{5}}{3}V_1$$
  
from factor =  $\frac{V_{rms}}{V_{avg}}$   
$$= \frac{\frac{\sqrt{5}}{3}V_1}{\frac{2}{3}V_1} = \frac{\sqrt{5}}{2} = 1.12$$

- 4. (a) Two wattmeters are used to measure power in a 3-phase three wire load. Determine the total power, power factor and reactive power, if the two wattmeters read
  - (i) 1000 W each, both positive
  - (ii) 1000 W each, but of opposite sign

Solution Let the wattmeter readings be

$$P_1 = 1000 \text{ W}$$
  
and  $P_2 = 100 \text{ W}$   
(i) Total active power =  $P_1 + P_2$   
= 1000 + 1000  
= 2000 W

Power factor angle be  $\varphi$ 

$$\tan \varphi = \sqrt{3} \quad \frac{P_1 - P_2}{P_1 + P_2}$$
  
=  $\sqrt{3} \quad \frac{1000 - 1000}{1000 + 100}$   
=  $0$   
 $\therefore \cos \varphi = \cos 0 = 1$   
Reactive power =  $\sqrt{3} \quad (P_1 - P_2)$   
=  $\sqrt{3} \quad (1000 - 1000) = 0$   
(ii)  $P_1 = 100 \text{ W} \quad P_2 = -1000 \text{ W}$   
Total power =  $P_1 + P_2$   
=  $1000 - 1000 = 0$   
 $\tan \varphi = \sqrt{3} \quad \frac{P_1 - P_2}{P_1 + P_2}$   
=  $\sqrt{3} \quad \frac{1000 + 1000}{1000 - 100} = \infty$   
 $\varphi = 90^\circ$   
Power factor =  $\cos \varphi = \cos 90 = 0$ 

Reactive power =  $\sqrt{3} (P_1 - P_2)$ 

$$= \sqrt{3} (1000 + 1000) = 3464.1 \text{ VAR}$$

(b) What is phase sequence? Explain its significance.

Solution Refer Section 9.4

(c) What are the advantages of a polyphase system over a single-phase system?

**Solution** Refer Section 9.2.

5. (a) Draw the oriented graph of the network shown in Fig. 2.4.



Fig. 2.4

**Solution** The graph represented in Fig. 5(a) itself represents the oriented graph in which (1)-(5) are nodes and 1-7 are branches.



**Oriental graph** 

(b) Obtain the fundamental loop and fundamental cut-set matrices for the graph shown in Fig. 2.5.



Fig. 2.5

**Solution** For the given graph, an arbitrary tree is chosen for which the no. of nodes n = 5



For a given tree of a graph, addition of each link between any two nodes towns a loop called the fundamental loop. (*f*-loop) or a tie-set. By adding links 1, 3 and 4, we can form three fundamental loops as shown in the figure. By convention, a fundamental loop is marked with the same orientation as its defining link current.



#### **Tie-sets**

#### Tie-set schedule (Fundamental loop matrix)

	Branch No						
Link no	1	2	3	4	5	6	7
1	1	-1	0	0	-1	0	0
3	0	-1	1	0	0	-1	1
4	0	0	0	1	1	-1	0

#### Cut-set

Consider the tree of the graph shown in Fig. 2.4 with 5 nodes 1-5 and four tree branches.



The following are the fundamental cut-sets



*f*-cut set corresponding to twig 2;  $C_1 = \{1, 2, 3\}$ *f*-cut set corresponding to twig 5;  $C_2 = \{1, 4, 5\}$ *f*-cut set corresponding to twig 6;  $C_3 = \{3, 4, 6\}$ *f*-cut set corresponding to twig 7;  $C_4 = \{3, 7\}$ Thus, the *f*-cut set matrix is given by *f*-cut sets.

	1	2	3	4	5	6	7
$C_1$	1	1	1	0	0	0	0]
$C_2$	1	0	0	1	1	0	0
$C_3$	0	0	1	1	0	1	0
$C_4$	0	0	-1	0	0	0	1



By short circuiting voltage source and open circuiting current source, the oriented graph can be drawn as shown.



The number of nodes are 4 and branches are five. An arbitrary tree is chosen as shown, with twig branches as a, c, e and links as d and b.



Tree

f-loop matrixBranches links a b c d e  $b\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & -1 \end{bmatrix}$ 

The cut-sets are given by

$$\begin{array}{ll} C_1 = \; \{a, d\} \\ C_2 = \; \{b, c, d\} \\ C_3 = \; \{b, d, e\} \end{array}$$



6. (a) State and explain compensation theorem.

**Solution** Refer Section 3.6.

(b) In the network shown in Fig. 2.6, find the value of  $Z_L$  so that the power transfer from the source is maximum. Also find  $P_{max}$  [8+8]



Fig. 2.6

**Solution** Let us remove  $z_L$ . The Internal impedence of the circuit looking through x-y is given by

$$z_{in} = \frac{(21)(12+J24)}{21+12+J24} + \frac{50(30+J60)}{50+30+J60}$$
$$= \frac{563.44\angle 63.43^{\circ}}{40.8\angle 36^{\circ}} + \frac{3354.10\angle 63.43^{\circ}}{100\angle 36.87^{\circ}}$$

 $= 13.81 \angle 27.43^{\circ} + 33.54 \angle 26.56^{\circ}$ 

$$z_{in} = 42.19 + J21.49 \,\Omega$$

As per maximum power transfer theorem,  $z_{l}$  should be the complex of  $z_{in}$ 

$$\begin{aligned} z_L &= z_{in}^* = (42.19 - J21.49) \,\Omega \\ V_{O.C} &= V_{xy} \\ V_x &= \frac{12 + J24}{12 + J24 + 21} \times 10 \,\angle 0^\circ \\ &= 6.577 \,\angle 27.43^\circ \,\text{V} \\ V_y &= \frac{30 + J60}{30 + J60 + 50} \times 10 \,\angle 0^\circ = 6.71 \,\angle 26.56^\circ \\ V_{OC} &= V_x - V_y = 6.577 \,\angle 27.43^\circ - 6.71 \,\angle 26.56^\circ \\ &= -0.163 + J0.029 \\ V_{O.C} &= 0.1657 \,\angle 170^\circ \,\text{V} \\ P_{max} &= \frac{V_{oc}^2}{4R_L} = \frac{(0.1657)^2}{4 \times 42.19} = 0.1627 \,\text{mW} \\ \hline P_{max} &= 0.1627 \,\text{mw} \end{aligned}$$

7. Find  $V_c(t)$  at t = 0 + while the switching is done from x to y at t = 0, as shown in Fig. 2.7.



Fig. 2.7

**Solution** The voltage across the capacitor at t > 0 flowing while switching is done at t = 0 from x to y is given by  $V_C = \text{V0} e^{-t/RC} (t > 0)$ 

$$V_{O} = V_{C}(0^{-})$$

R = equivalent resistance.

To find 'R', the capacitance is removed, all independent source are made equal to zero and a diving point current source is applied at the capacitor terminal.

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Nodal analysis

$$\frac{V_C}{2} - 1 + \frac{4V_C + V_C}{5} = 0$$

$$\frac{V_C}{2} + \frac{4}{5}V_C + \frac{V_C}{5} = 1$$

$$1.5V_C = 1$$

$$V_C = \frac{1}{1.5} = 0.67 \text{ V}$$

$$R = \frac{V_C (\text{Volts})}{1A} = 0.67 \Omega$$

$$R = 0.67 \Omega$$
So,  $RC = (0.67 \times 1) = 0.67 \text{ sec.}$ 
To find  $V_C(0-)$ 
Applying KVL
$$5I - 10 + 2I - 4V_C = 0$$
(Capacitor acts as open circuit. So  $I_c = 0$ )
$$7I - 10 - 4V_C = 0(1)$$
Also,  $-V_C - 4V_C + 5I = 0$ 

$$I = V_C \quad (2)$$
From (1) and (2)
$$7V_C - 10 - 4V_C = 0$$
V<sub>C</sub> =  $\frac{10}{3}$  V (at  $t = 0 - ) = V_o$ 
Thus
$$V_C = \frac{10}{3} e^{-t/0.67}$$

$$\therefore \qquad V_C = \frac{10}{3} e^{-1.49t} \text{ V}$$

8. (a) Determine the ABCD parameters of the network shown in Fig. 2.8.





**Solution**  $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$  $A = \frac{V_1}{V_2} \qquad B = \frac{V_1}{-I_2} \\ C = \frac{V_1}{V_2} \\ I_2 = 0 \qquad D = \frac{I_1}{-I_2} \\ V_2 = 0$  $A = \frac{V_1}{V_2}$  $I_2 = 0$  $I_3 = \frac{I_{14}}{13}$  $V_1 = 2I_1 + 9I_3$  $V_1 = \left(\frac{13}{2} + 9\right)I_3$  $V_2 = 6I_3$  (a)  $V_1 = \frac{31}{2} I_3$  (b)  $V_2 = 6 \times \frac{4}{13} I_1$   $\frac{V_2}{6} = I_3$  $C = \frac{I_1}{V_2} = \frac{13}{24}$   $\frac{2V_1}{31} = I_3$  $\frac{2V_1}{31} = \frac{V_1}{6}$  $A = \frac{V_1}{V_2} = \frac{31}{12}$  $V_{2} = 0$ 

When



(b) Determine the ABCD parameters of the network shown in Fig. 2.9.



Fig. 2.9

Solution  $A = \frac{V_1}{V_2}\Big|_{I_2 = 0}$   $B = \frac{V_1}{-I_2}\Big|_{V_2 = 0}$  $C = \frac{I_1}{V_2}\Big|_{I_2 = 0}$   $D = \frac{I_1}{-I_2}\Big|_{V_2 = 0}$  $I_2 = 0$ 

$$\frac{V_1}{I_1} = 6.14$$
(1)  

$$I_1 = I_a + I_b 
I_a = \frac{I_1 6}{12.6} 
I_a = 0.476 I_1$$
(a)  

$$I_{a_2} = \frac{I_a 6}{15} 
I_{a_2} = 0.476 \times \frac{6}{15} I_1$$
(b)  

$$V_2 = 6I_{a_2} 
V_2 = 6 \times 0.1904 I_1 
V_2 = 1.14 I_1$$
(c)

From (a) and (c)

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$B = \frac{V_1}{-I_2} = 15.75$	
A = 5.37	B = 15.75
C = 0.877	D = 2.75

## SET 3

1. (a) Write short notes on source transformation.

**Solution** Refer Section 2.15.

(b) Find the power supplied by 12 V source as shown in Fig.3.1.



Fig. 3.1.

Solution



The nodal equations are

$$1 + \frac{V_1}{6} + \frac{V_1 + V_2}{6} = 0 \tag{1}$$

$$\frac{V_2}{6} + \frac{V_2 - V_1}{6} + \frac{V_2 - V_3}{2} = 0$$
<sup>(2)</sup>

$$\frac{V_3 - V_2}{2} + \frac{V_3}{4} + \frac{V_4}{2} + \frac{V_4}{2} + 2 = 0$$
(3)

 $V_4 - V_3 = 12$  is the supernode equation

$$(1) \Rightarrow V_1 \left[ \frac{1}{3} \right] - V_2 \left[ \frac{1}{6} \right] + 1 = 0$$

$$(2) \Rightarrow V_1 \left[ -\frac{1}{6} \right] + V_2 \left[ \frac{5}{6} \right] - V_3 \left[ \frac{1}{2} \right] = 0$$

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$$(3) \Rightarrow V_{2}\left[-\frac{1}{2}\right] + V_{3}\left[\frac{3}{4}\right] + V_{4}[1] + 2 = 0$$

$$-\frac{1}{2} V_{2} + \frac{7}{4} V_{3} + 12 + V_{3} + 2 = 0$$

$$-\frac{1}{2} V_{2} + \frac{7}{4} V_{3} + 14 = 0$$

$$\Rightarrow V_{3} = \frac{4}{7}\left[\frac{1}{2}V_{2} - 14\right] = \frac{2}{7} V_{2} - 8 \qquad (4)$$
From (2),  $-\frac{1}{6} V_{1} + \frac{5}{6} V_{2} - \frac{1}{2}\left[\frac{2}{7}V_{2} - 8\right] = 0$ 

$$-\frac{1}{6} V_{1} + \frac{5}{6} V_{2} - \frac{1}{7} V_{2} + 4 = 0$$

$$V_{1} = 6\left\{\frac{29}{42}V_{2} + 4\right\} = \frac{29}{7} V_{2} + 24 \quad (5)$$

$$\frac{1}{3} V_{1} - \frac{1}{6} V_{2} + 1 = 0$$

Substitute for  $V_1$ 

$$\frac{1}{3} \left[ \frac{29}{7} V_2 + 24 \right] - \frac{1}{6} V_2 + 1 = 0$$

$$\frac{29}{21} V_2 + 8 - \frac{1}{6} V_2 + 1 - 0 \implies V_2 = \frac{-126}{17} V$$

$$V_3 = \frac{2}{7} V_2 - 8 = \frac{-172}{17} V$$

$$V_4 = V_3 + 12 = \frac{32}{17} V$$

Current through 12 V source is

$$I = \frac{V_4}{2} + \frac{V_4}{2} + 2 = \frac{66}{17} \text{ A}$$

Power  $V_1 = 12 \times \frac{66}{17} = \frac{792}{17}$  W

(c) Draw the volt-current characteristic of practical current source.

**Solution** Refer Section 1.8.



- 2. (a) Define the following:
  - (i) Self inductance
  - (ii) Mutual inductance
  - (iii) Statically induced emf
  - (iv) Dynamically induced emf

**Solution** (i) Refer Section 10.3.

- (ii) Refer Section 10.3.
- (iii) Statically induced emf

Emf induced in a coil due to the change of its own flux linked with it or emf induced in one coil by the influence of the other coil is known as statically induced emf.

(iv) Dynamically induced emf

When a conductor or a coil with certain number of turns is rotated with a uniform speed in a magnetic field (Section 9.3), an emf is induced in it which is known as dynamically induced emf.

(b) Derive the relationship between the self, mutual inductances and coefficient of coupling.

**Solution** Refer Section 10.5. m = 0.075 H; K = 
$$\frac{\pi}{\sqrt{L_1 L_2}}$$
 = 0.1767

(c) Two similar coils connected in series gave a total inductance of 600 mH and when one of the coils is reversed, the total inductance is 300 mH. Determine the mutual inductance between the coils and coefficient of coupling.

**Solution** Refer Example 10.9.

- 3. (a) The voltage of a circuit is  $v = 200 \sin(\omega t + 30^\circ)$  and the current is  $i = 50 \sin(\omega t + 60^\circ)$ . Calculate
  - (i) the average power, reactive volt-amperes and apparent power
  - (ii) find the circuit elements if  $\omega = 100\pi$  rad/sec

### Solution

$$V = 200 \sin \left(\omega t + 30^\circ\right)$$

$$i = 50 \sin(\omega t + 60^\circ)$$

(i) Avg. power = 
$$V_m I_m \cos \theta$$

$$= \frac{200}{\sqrt{2}} \times \frac{50}{\sqrt{2}} \cos(60 - 30)$$

$$P_{av} = 4330.127 \text{ W}$$

Reactive volt amperes =  $V_m I_m \sin \theta$ 

$$= \frac{200}{\sqrt{2}} \cdot \frac{50}{\sqrt{2}} \sin(60 - 30)$$

----



(ii) The current leads the voltage by  $30^{\circ}$ . Hence the circuit must contain *R* and *C*.

$$\tan \theta = \frac{1}{\omega RC} \Rightarrow \tan 30^\circ = \frac{1}{100\pi \times RC}$$
$$\Rightarrow RC = 0.0055 \Rightarrow C = \frac{0.0055}{R}$$
$$Z = \frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega c}\right)^2}$$
$$\frac{200}{50} = \sqrt{R^2 + \left(\frac{R}{100\pi \times 0.0055}\right)^2}$$
$$4 = 1.155 \text{ R} \Rightarrow R = \frac{4}{1.155} = 3.46 \Omega$$
$$C = \frac{0.0055}{3.46} = 1.59 \text{ mF}$$

and

(a) (i) Voltage  $V = 200 \sin (\omega t + 30^\circ)$   $I = 50 \sin (\omega t + 60^\circ)$  $V = \frac{200 \angle 30^\circ}{\sqrt{2}}$ 



1. Average power

$$P_{avg} = VI\cos\phi$$
$$= \frac{200}{\sqrt{2}} \times \frac{50}{\sqrt{2}}\cos(30^{\circ})$$

$$P_{avg} = \frac{8660.25\omega}{2} = 4330.12 \ \omega$$

2. Reactive volt amperes

$$\theta = \frac{V}{\sqrt{2}} \frac{I}{\sqrt{2}} \sin \phi$$
$$\theta = \frac{5000\omega}{2} = 2500 \ \omega$$

3. Apparent power

$$= V_{rms} \times I_{rms} = \frac{200}{\sqrt{2}} \times \frac{50}{\sqrt{2}} = 5000 \text{ VA}$$

(ii)  $\omega = 100 \pi$  r/sec

Average power =  $i^2 R$ 

$$\therefore \qquad i = \frac{50}{\sqrt{2}}$$
$$\therefore \qquad R = \frac{4330.12 \times 2}{2500} = 3.464 \ \Omega$$

If the circuit is assumed as a series circuit by inspection of voltage and current equations, current leads the voltage by  $30^{\circ}$ . Hence it is an RC circuit.

$$\tan 30^\circ = \frac{1}{\omega CR}$$

:. 
$$C = \frac{1}{0.5773 \times 100 \times \pi \times 3.464}$$
  
= 1.59 mH

(b) Find the form factor of the following waveform shown in Fig. 3.2.





**Solution** From 0 to  $\pi/_3$ ,  $V = \frac{3V_1}{\pi} t$ From  $\pi/_3$  to  $2\pi/_3$ ,  $V = V_1$ From  $2\pi/_{3}$  to  $\pi$ ,  $V = 3V_{1} - \frac{3V_{1}}{\pi}t$  $\frac{V_{rms}}{V_{avg}}$ Form factor =  $V_1$ 0  $\pi/3$  $2\pi/3$ π  $V_{avg} = \frac{1}{T} \int V(t) dt$  $= \frac{1}{\pi} \left[ \int_{0}^{\pi/3} \frac{3V_1}{\pi} t \, dt + \int_{\pi/3}^{2\pi/3} V_1 \, dt + \int_{2\pi/3}^{\pi} 3V_1 - \frac{3V_1}{\pi} t \, dt \right]$  $= \frac{1}{\pi} \left[ \frac{3V_1}{\pi} \cdot \left(\frac{\pi}{3}\right) \frac{1}{2} + V_1 \left(\frac{2\pi}{3} - \frac{\pi}{3}\right) + 3V_1 \left(\pi - \frac{2\pi}{3}\right) - \frac{3V_1}{\pi} \cdot \frac{1}{2} \left[\pi^2 - \frac{4\pi^2}{9}\right] \right]$  $= \frac{1}{\pi} \left[ \frac{V_1}{6} \cdot \pi + \frac{V_1}{3} \pi + V_1 \cdot \pi - \frac{5}{6} V_1 \right] = \frac{2}{3} V_1$ 

$$\begin{split} V_{rms} &= \sqrt{\frac{1}{T} \int_{o}^{T} \left[ V(t) \right]^{2} dt} \\ &= \sqrt{\frac{1}{\pi} \left[ \int_{o}^{\pi/3} \left( \frac{3V_{1}}{\pi} t \right)^{2} dt + \int_{\pi/3}^{2\pi/3} (V_{1})^{2} dt + \int_{2\pi/3}^{\pi} \left( 3V_{1} - \frac{3V_{1}}{\pi} t \right)^{2} dt \right] \\ &= \sqrt{\frac{1}{\pi} \left[ \int_{o}^{\pi/3} \frac{9V_{1}^{2}}{\pi^{2}} t^{2} dt + \int_{\pi/3}^{2\pi/3} V_{1}^{2} dt + \int_{2\pi/3}^{\pi} 9V_{1}^{2} + \frac{9V_{1}^{2}}{\pi^{2}} t^{2} - \frac{18V_{1}}{\pi} t dt \right] \\ &= Sqrt \left\{ \frac{1}{\pi} \left[ \frac{9V_{1}^{2}}{\pi^{2}} \cdot \frac{1}{3} \cdot \left( \frac{\pi}{3} \right)^{3} + V_{1}^{2} \left( \frac{2\pi}{3} - \frac{\pi}{3} \right) + 9V_{1}^{2} \left( \pi - \frac{2\pi}{3} \right) \right. \\ &+ \frac{9V_{1}^{2}}{\pi^{2}} \cdot \frac{1}{3} \left( \pi^{3} - \frac{8\pi^{3}}{27} \right) - \frac{18V_{1}}{\pi} \cdot \frac{1}{2} \left( \pi^{2} - \frac{4\pi^{2}}{9} \right) \right] \right\} \\ &= \sqrt{\frac{1}{\pi} \left[ \frac{9V_{1}^{2}}{\pi^{2}} \cdot \frac{1}{3} \cdot \frac{\pi^{3}}{27} + V_{1}^{2} \cdot \frac{\pi}{3} + 9V_{1}^{2} \cdot \frac{\pi}{3} + \frac{3V_{1}^{2}}{\pi^{2}} \cdot \frac{19\pi^{3}}{27} - \frac{9V_{1}^{2}}{\pi} \cdot \frac{5\pi^{2}}{9} \right] \\ &= \sqrt{\frac{1}{\pi} \left[ \frac{\pi}{9} V_{1}^{2} + \frac{\pi}{3} V_{1}^{2} + 3\pi V_{1}^{2} + \frac{19}{9} \pi V_{1}^{2} - 5\pi V_{1}^{2} \right] \\ &= \sqrt{\frac{5}{9}} V_{1}^{2} = \frac{\sqrt{5}}{3} V_{1} \\ &\text{from factor} = \frac{V_{rms}}{V_{avg}} \\ &= \frac{\sqrt{5}}{\frac{2}{3} V_{1}} = \frac{\sqrt{5}}{2} = 1.12 \end{split}$$

4. (a) The power delivered to a balanced delta connected load by a 400 volt 3-phase supply is measured by a two-wattmeter method. If the readings of the two wattmeters are 2000 and 1500 watts respectively, calculate the magnitude of the impedance in each arm of the delta load and its resistive component.

**Solution** Power consumed =  $W_1 + W_2$ 

$$= 2000 + 1500$$
$$= 3500 \text{ W}$$
$$P = \sqrt{3} V_L I_L \cos \varphi$$

$$\tan \varphi = \sqrt{3} \cdot \frac{P_1 - P_2}{P_1 + P_2}$$
$$= \sqrt{3} \cdot \frac{2000 - 1500}{2000 + 1500} = 0.247$$
$$\cos \varphi = 0.97$$
$$P = 3500 = \sqrt{3} \times 400 \times I_L \times 0.97$$
$$I_L = 5.2 \text{ A}$$
Impedance  $Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{400}{\frac{5.2}{\sqrt{3}}} = 133 \Omega$ 
$$R_{Ph} = Z_{ph} \cos \varphi = 133 \times 0.97 = 129 \Omega$$

- (b) A balanced delta connected load of  $(2 + j3) \Omega$  per phase is connected to a balanced three-phase 440 V supply. The phase current is 10 A. Find the
  - [8 + 8]

- (i) total active power
- (ii) reactive power
- (iii) apparent power in the circuit

**Solution** Z = 2 + j3

$$z = \sqrt{2^2 + 3^2} = 3.6 |56.3^\circ \Omega|$$

$$\cos \varphi = \frac{R_{ph}}{Z_{ph}} = \frac{2}{3.6} = 0.55$$

 $\sin \varphi = 0.83$ 

$$I_L = \sqrt{3} \ I_{ph} = \sqrt{3} \times 10 = 17.32 \text{ A}$$

(i) Active power =  $\sqrt{3} V_L I_L \cos \varphi$ 

$$=\sqrt{3} \times 440 \times 17.32 \times 0.55 = 7259.78$$
 W

(ii) Reactive power =  $\sqrt{3} V_L I_L \sin \varphi$ 

$$=\sqrt{3} \times 440 \times 17.32 \times 0.83 = 10955.67$$
 VAR

(iii) Apparent power =  $\sqrt{3} V_L I_L$ 

$$=\sqrt{3} \times 400 \times 17.32 = 13199.61$$
 VA

5. (a) Draw the dual of the network shown in Fig. 3.3.



Fig. 3.3

Solution



(b) Find the current through  $Z_2$  in the network shown in Fig. 3.4 using mesh analysis. [6 + 10]



Fig. 3.4

**Solution** Let the current through  $Z_L$  be  $I_1$  and in the left loop be  $I_2$ , and in the right loop be  $I_3$ .

The mesh equations are

$$I_1(2+j_4) + 10(J_1 - I_3) + 5(I_1 - I_2) = 0$$
(1)

$$5(I_2 - I_1) + 50 + j_2 (I_2) = 0$$
<sup>(2)</sup>

$$10(I_3 - I_1) + j_2(I_3) - 50 = 0$$
(3)

(1) 
$$\Rightarrow I_1(2+10+5+j_4) - I_2(5) - I_3(10) = 0$$
 (6)

(2)  $\Rightarrow I_1(-5) + I_2(5+j_2) + 50 = 0$ 

$$I_2 = \frac{1}{5+j2} \ [5I_1 - 50] \tag{4}$$

(3) 
$$\Rightarrow$$
  $I_1(-10) + I_3(10 + j_2) - 50 = 0$ 

$$I_3 = \frac{1}{10 + j2} [50 + 10I_1]$$
 (5)

Substitute (4) and (5) in (6)

$$I_{1}(17+j_{4}) - \frac{5}{5+j2} (5I_{1}-50) - \frac{10}{10+j2} (50+10I_{1}) = 0$$
$$I_{1} \left[ 17+j4 - \frac{25}{5+j2} - \frac{100}{10+j2} \right] + \frac{250}{5+j2} - \frac{500}{10+j2} = 0$$
$$\Rightarrow I_{1} = 1.104 |\underline{168.79^{\circ}}| A$$

Hence the current through  $Z_L$  is 1.104 A.

6. (a) State and explain reciprocity theorem.

**Solution** Refer Section 3.5.

(b) Find the current *i* in the circuit shown in Fig. 3.5 using superposition theorem.





```
Solution 6i_1 - 2V_x - V_x = 10

i_2 = -2A

V_x = -2(i_1 - i_2)

= -2i_1 - 4

6\Omega

2V_x

V_x \neq 2\Omega

i_1

V_x \neq 2\Omega

i_2

i_2

i_3

i_4

i_5

i_4

i_5

i_5

i_6

2V_x

i_7

i_8

i_1

i_2

i_1

i_2

i_2

i_3

i_4

i_1

i_2

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i
```

 $V_{\rm r} = -2i_1^1$ 

 $-10 + 6i_1 - 2V_x - V_x = 0$


7. (a) A dc voltage of 100 V is applied in the circuit shown in Fig. 3.6 and the witch is kept open. The switch K is closed at t = 0. Find the complete expression for the current.



Fig. 3.6

Solution



The current equation is

$$0.1 \ \frac{di}{dt} + 20i = 100$$
$$\frac{di}{dt} + 200i = 1000$$
$$(D + 200)i = 1000$$

The solution is

$$i = C_1 e^{-200t} + e^{-200t} \int 1000 e^{200t} dt$$
  
=  $C_1 e^{-200t} + 5$   
At t = 0<sup>-</sup> the current is  $i(0^-) = \frac{100}{20 + 10} = 3.33 A$ 

Inductor does not allow sudden changes in current. Hence, i(0+) = 3.33 A

$$3.33 = C_1 e^{-200(0)} + 5 \implies C_1 = -1.67$$

Hence the complete expression for current is

$$i = -1.67 e^{-200t} + 5 A.$$

- (b) A dc voltage of 20 V is applied in an RL circuit where  $R = 5 \Omega$  and L = 10H. Find [8 + 8]
  - (i) the time constant
  - (ii) the maximum value of stored energy

#### Solution



Inductor acts as short circuit to dc

Hence,

$$i = \frac{20}{5} = 4 A$$
  
 $W = \frac{1}{2} Li^2 = \frac{1}{2} \times 10 \times 4^2 = 80$  Joules.

8. (a) In a *T*-network shown in Fig. 3.7,  $Z_1 = 2 \lfloor \underline{0^\circ}, Z^2 = 5 \lfloor \underline{-90^\circ}, Z^3 = 3 \lfloor \underline{90^\circ}, \text{ find the } Z\text{-parameters.}$ 



Fig. 3.7

Solution



 $Z_1 = 2 \ \underline{0^\circ} = 2 \ \Omega$ 

 $Z_2 = 5 \ \underline{-90^\circ} = -j_5 \ \Omega$ 

 $Z_3 = 3 \ \underline{90^\circ} = j_3 \Omega$ 

The mesh equations are

$$-V_1 + I_1 Z_1 + Z_3 (I_1 + I_2) = 0$$
 (1)

$$-V_2 + I_2 Z_2 + Z_3 (I_1 + I_2) = 0$$
<sup>(2)</sup>

(1) 
$$\Rightarrow V_1 = I_1(Z_1 + Z_3) + I_2 Z_3$$

(2) 
$$\Rightarrow V_2 = I_1 Z_3 + I_2 (Z_2 + Z_3)$$

$$Z_{11} = Z_1 + Z_3 = 2 + j_3 \Omega$$
  

$$Z_{12} = Z_3 = j_3 \Omega$$
  

$$Z_{21} = Z_3 = j_3 \Omega$$
  

$$Z_{22} = Z_2 + Z_3 = -j_5 + j_3 = -j_2 \Omega$$

(b) Z-parameters for a two-port network are given as  $Z_{11} = 25 \ \Omega$ ,  $Z_{12} = Z_{21} = 20 \ \Omega$ ,  $Z_{22} = 50 \ \Omega$ . Find the equivalent *T*-network. [8 + 8]

**Solution** Given  $Z_{11} = 25 \Omega$ ;  $Z_{12} = Z_{21} = 20 \Omega$ ;  $Z_{22} = 50 \Omega$ 

The equivalent T-network is shown in the figure.



 $\begin{aligned} &Z_a = \ Z_{11} - Z_{21} = 5 \ \Omega \\ &Z_b = \ Z_{22} - Z_{12} = 30 \ \Omega \\ &Z_c = \ Z_{12} - Z_{21} = 20 \ \Omega \end{aligned}$ 

# SET 4

1. (a) For the circuit shown in Fig. 4.1, find the current through the 20  $\Omega$  resistor.



Fig. 4.1

Solution



#### Let $I_1 I_2 I_3$ be the currents in the meshes. The mesh equations are

$$10 = 75 I_1 + 20(I_1 - I_2) \tag{1}$$

$$15 = 50 I_2 + 20(I_2 - I_1) \tag{2}$$

$$0 = 15 I_3$$
 (3)

From (3), current  $I_3 = 0$ 

From (2),  $20I_1 - 70I_2 + 15 = 0$ 

 $\Rightarrow$ 

 $\Rightarrow$ 

$$I_2 = \frac{1}{70} (20 I_1 + 15) \tag{4}$$

From (1)  $95I_1 - 20I_2 - 10 = 0$ 

Substitute (4) in (5)

$$95 I_1 - 20 \cdot \frac{1}{70} (20 I_1 + 15) - 10 = 0$$
$$95 I_1 - \frac{40}{7} I_1 - \frac{30}{7} - 10 = 0$$
$$I_1 = 0.16 \text{ A}$$
$$I_2 = \frac{1}{70} [20 \times 0.16 + 15] = 0.26 \text{ A}$$
$$I = I_1 - I_2 = 0.16 - 0.26 = -0.1 \text{ A}$$

(b) Reduce the network shown in Fig. 4.2 to a single loop network by successive source transformation to obtain the current in the 12- $\Omega$  resistor. [8 + 8]



Fig. 4.2

Solution

(5)















(1)22.5 A

Γ

 $\leq 12 \Omega$ 

Ι



The current I in the 12- $\Omega$  resistor is

$$I = \frac{108}{12 + 4.8} = 6.428 \text{ A}$$

2. (a) Explain Faraday's Law of electromagnetic induction.

Solution Refer Section 1.6.

**First law** Whenever the magnetic flux linked with a circuit changes, an emf is always induced in it.

- Second law The magnitude of the induced emf is equal to the rate of change of flux linkage.
- (b) A cast steel ring has a circular cross section 3 cm in diameter and a mean circumference of 80 cm. The ring is uniformly wound with 600 turns.
- (i) Estimate the current required to produce a flux of 0.5 mcob in the ring.
- (ii) If a 2-mm wide saw cut is made in the ring, find approximately the flux produced by the current found (i).
- (iii) Find the current value which will give the same flux as in (i). Assume the gap density to be the same as in the iron and neglect fringing.

**Solution** (i) Length of the flux path = Mean circumference

$$= 80 \times 10^{-2} \text{ m}$$

A = area = 
$$\frac{\pi}{4} d^2 = \frac{\pi}{4} (3 \times 10^{-2})^2 = 7.068 \times 10^{-4} m^2$$
  
H =  $\frac{B}{\mu_o \mu_r} = \frac{\phi}{A \times \mu_o \mu_r}$   
 $\phi = flux = 0.5 m Wb$   
 $\mu_o = 4\pi \times 10^{-7}$   
 $\mu_r = 600$  for cast steel iron  
 $H = \frac{0.5 \times 10^{-3}}{7.068 \times 10^{-4} \times \pi \times 10^{-7} \times 60}$   
= 9382.36

 $mmf = H \times l$  $= 9382.36 \times 80 \times 10^{-2}$  $= 7505.89 \,\mathrm{AT}$ N = no. of turns = 600 $\therefore$  exciting current =  $\frac{mmf}{N}$  $=\frac{7505.89}{600}$ = 12.5 A: *i* = 12.5 A1 (ii) Reluctance =  $\frac{l}{\pi_o \pi_r A} = \frac{80 \times 10^{-12}}{4\pi \times 10^{-17} \times 600 \times 7.068 \times 10^{-4}}$  $= 1.500 \times 10^{6} A/Wb$ Reluctance of air gap =  $\frac{1}{\mu_0 A}$  $=\frac{3\times10^{-3}}{4\pi\times10^{-7}\times7.068\times10^{-4}}$  $= 2.25 \times 10^{6} \text{ A/Wb}$ Total reluctance =  $(1.5 + 2.25)10^6$  $= 3.75 \times 10^{6} \text{ A/Wb}$  $mmf = \varphi \times reluctance$  $\varphi = \frac{7505.89}{3.75 \times 10^6} = 2 \text{ mWb}$ (iii) For  $\varphi = 0.5 \ mWb$ Total reluctance =  $3.75 \times 10^6$  A/Wb  $mmf = \phi \times reluctance$  $= 0.5 \times 10^{-3} \times 3.75 \times 10^{6}$  $= 1.875 \times 10^{3}$ = 1875 AT Exciting current =  $\frac{mmf}{no. of turns}$ no. of turns = 600 $\therefore$  exciting current =  $\frac{1875}{600}$ 

= 3.125 *A*.

3. (a) What is the form factor of an alternating quantity? Explain its significance.

**Solution** Refer Section 4.4.6.

It is useful to find the rms values of the alternating quantities from the average values.

(b) In the circuit shown in Fig. 4.3, what 50-Hz voltage is to be applied across A B terminals so that a current of 10 A will flow in the capacitor.



Figure 4.3

Solution



$$Z_1 = 5 + j2\pi \times 50 \times 0.0191 = 5 + j6\Omega$$

$$Z_2 = 7 + \frac{1}{j2\pi \times 50 \times 398\mu} = 7 - j8\,\Omega$$
$$Z_3 = 8 + j2\pi \times 50 \times 0.0318 = 8 + j10\,\Omega$$

Given that current through the capacitor is  $10 \text{ A} = I_2$ . Hence voltage across  $Z_2$  is

$$V_1 = 10 \times Z_2$$
  
= 10 (7 - j8) = 70 - j80 V

The current through the other branch is

$$I_1 = \frac{V_1}{z_1}$$
$$= \frac{70 - j80}{5 + j6} = -2.13 - j13.44 \text{ A}$$

Total current in the network is

$$I = I_1 + I_2$$
  
= -2.13 - j13.44 + 10  
= 7.87 - j13.44 A

Let  $V_2$  be the voltage across  $Z_3$ .

$$V_2 = I Z_3$$
  
= (7.87 - j13.44) (8 + j10)  
= 197.38 - j28.85 V

The voltage to be applied across *AB* terminals so that a current of 10 A will flow in the capacitor is  $V = V_1 + V_2$ 

$$= 70 - j80 + 197.38 - j28.85$$
  
= 267.38 - j108.85  
= 288.68  $- 22.15^{\circ}$  V.

- 4. (a) Two wattmeters are used to measure power in a 3-phase three wire load. Determine the total power, power factor and reactive power if the two wattmeters read
- (i) 1000 W each, both positive
- (ii) 1000 W each, but of opposite sign

**Solution** Let the wattmeter readings be

$$P_1 = 1000 \text{ W}$$
  
nd  $P_2 = 1000 \text{ W}$   
(i) Total active power =  $P_1 + P_2$   
= 1000 + 1000  
= 2000 W

Power factor angle be  $\varphi$ 

а

$$\tan \varphi = \sqrt{3} \cdot \frac{P_1 - P_2}{P_1 + P_2}$$
  
=  $\sqrt{3} \cdot \frac{1000 - 1000}{1000 + 100}$   
= 0  
 $\therefore \cos \varphi = \cos 0 = 1$   
Reactive power =  $\sqrt{3} (P_1 - P_2)$   
=  $\sqrt{3} (1000 - 1000) = 0$   
(ii)  $P_1 = 1000 \text{ W } P_2 = -1000 \text{ W}$   
Total power =  $P_1 + P_2$   
=  $1000 - 1000 = 0$ 

 $\tan \varphi = \sqrt{3} \cdot \frac{P_1 - P_2}{P_1 + P_2}$ =  $\sqrt{3} \cdot \frac{1000 + 1000}{1000 - 100} = \infty$  $\varphi = 90^\circ$ Power factor =  $\cos \varphi = \cos 90 = 0$ Reactive power =  $\sqrt{3} (P_1 - P_2)$ =  $\sqrt{3} (1000 + 1000) = 3464.1$  VAR

(b) What is phase sequence? Explain its significance.

**Solution** Refer Section 9.4.

(c) What are the advantages of a polyphase system over a singlephase system?

**Solution** Refer Section 9.2.

5.(a) Explain the procedure for obtaining fundamental tie-set matrix of a given network.

Solution Refer Section 2.7.

(b) Draw the oriented graph of the network shown in Fig. 4.4 and write the incidence matrix. [6 + 10]



Fig. 4.4

**Solution** Directions of currents are arbitrarily assumed as shown in the circuit.



Ideal voltage sources and current sources do not appear in the graph of a linear network. Ideal voltage source is represented by short circuit and an ideal current

source is replaced by an open circuit. The nodes that appear in the graph are numbered (1) (2) (3) (4) and (5); branches as a, b, c, d, e, f and g. The graph is as shown in the figure.



For a graph with *n* nodes and *b* branches, the order of the incidence matrix is  $(n-1) \times b$ . Choose node (5) as reference (ordatum) node for writing incidence matrix. The required incidence matrix is given by

	а	b	с	a	e	I	g	
A	1	[-1	1	0	0	0	0	0]
	0	0	-1	1	0	0	0	1
	3	0	0	-1	1	0	-1	0
	4	0	0	0	-1	1	0	0

6. (a) State and explain compensation theorem.

**Solution** Refer Section 3.6.

(b) In the network shown in Fig. 4.5, find the value of  $Z_L$  so that the power transfer from the source is maximum. Also find  $P_{max}$ .



Fig. 4.5

**Solution** Let us remove  $z_L$ . The Internal impedence of the circuit looking through x-y is given by



As per maximum power transfer theorem,  $z_L$  should be the complex of  $z_{in}$ 

ſ

$$\begin{split} z_{L} &= z_{in}^{*} = (42.19 - J21.49) \,\Omega \\ V_{O,C} &= V_{xy} \\ V_{x} &= \frac{12 + J24}{12 + J24 + 21} \times 10 \,\angle 0^{\circ} \\ &= 6.577 \,\angle 27.43^{\circ} \, \text{V} \\ V_{y} &= \frac{30 + J60}{30 + J60 + 50} \times 10 \,\angle 0^{\circ} = 6.71 \\ &\angle 26.56^{\circ} \\ &= -0.163 + J0.029 \\ V_{O,C} &= 0.1657 \,\angle 170^{\circ} \, \text{V} \\ P_{max} &= \frac{V_{oc}^{2}}{4R_{L}} = \\ &= \frac{(0.1657)^{2}}{4 \times 42.19} = 0.1627 \, \text{mW} \\ \hline P_{max} &= 0.1627 \, \text{mw} \\ I1 &= \frac{10 \, |0^{\circ}}{12 + 21 + j24} \end{split}$$

$$= 0.198 - j0.144 \text{ A}$$

$$I_2 = \frac{10 | 0^{\circ}}{30 + 50 + j60}$$

$$= 0.08 - j0.06 \text{ A}$$

$$V_A = 21I_1 = 21(0.198 - j0.144) \text{ V}$$

$$V_B = 50I_2 = 50(0.08 - j0.06) \text{ V}$$

$$V_{AB} = V_A - V_B = 0.158 - j0.024 \text{ V}$$

$$R_{th} = \frac{21(12 + j24)}{21 + 12 j24}$$

$$= 42.26 + j21.36 \Omega$$

For max. power transfer,  $Z_L = 42.26 - j21.36 \Omega$ 

$$I = \frac{V_{AB}}{R_{th} + Z_L} = \frac{0.158 - j0.024}{2 \times 42.26} = 0.00187 - 0.000284 \text{ A}$$
$$P_{max} = I_2 R = (0.00187 - j0.000284)^2 \times 42.26 = 0.14 \ mW$$

7. (a) A dc voltage of 100 V is applied in the circuit shown in Fig. 4.6 and the witch is kept open. The switch K is closed at t = 0. Find the complete expression for the current.



Fig. 4.6

Solution



The current equation is

$$0.1 \ \frac{di}{dt} + 20i = 100$$
$$\frac{di}{dt} + 200i = 1000$$
$$(D + 200)i = 1000$$

The solution is

$$\begin{aligned} \vec{e} &= C_1 e^{-200t} + e^{-200t} \int 1000 e^{200t} dt \\ &= C_1 e^{-200t} + 5 \end{aligned}$$

 $t = 0^{-}$  the current is  $i(0^{-}) = \frac{100}{20 + 10} = 3.33$  A

At

Inductor does not allow sudden changes in current. Hence, i(0+) = 3.33 A

$$3.33 = C_1 e^{-200(0)} + 5 \implies C_1 = -1.67$$

Hence the complete expression for current is

$$i = -1.67 e^{-200t} + 5 A.$$

- (b) A dc voltage of 20 V is applied in an RL circuit where  $R = 5 \Omega$  and L = 10 H. Find
- (i) the time constant
- (ii) the maximum value of stored energy

Solution



Inductor acts as short circuit to dc

Hence, 
$$i = \frac{20}{5} = 4$$
 A

$$W = \frac{1}{2} Li^2 = \frac{1}{2} \times 10 \times 4^2 = 80$$
 Joules.

8. (a) Find the *y*-parameters of the network shown in Fig. 4.7.



Fig. 4.7

### Solution



The *y*-parameters are given by

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

The nodal equations are

$$I_1 = \frac{V_1}{175} + \frac{V_1 - V_2}{100} \tag{1}$$

$$I_2 = \frac{V_2}{200} + \frac{V_2 - V_1}{100}$$
(2)

$$(1) \Rightarrow V_1 \left[ \frac{1}{175} + \frac{1}{100} \right] - V_2 \left[ \frac{1}{100} \right] = I_1$$
$$(2) \Rightarrow V_1 \left[ -\frac{1}{100} \right] + V_2 \left[ \frac{1}{200} + \frac{1}{300} \right] = I_2$$

Hence 
$$I_1 = 0.0157 V_1 - 0.01 V_2$$

$$I_2 = -0.01 V_1 + 0.00833 V_2$$

The y – parameters are

 $Y_{11} = 0.0157$   $Y_{12} = -0.01$  $Y_{21} = -0.01$   $Y_{22} = 0.00833$  (b) Calculate the Z-parameters for the lattice network shown in Fig. 4.8



Fig. 4.8

#### Solution

8 (b) Redrawing the given circuit, we get



 $=\frac{Z_2-Z_1}{2}\cdot I_1$  $Z_{21} = \frac{V_2}{I_1} = \frac{Z_2 - Z_1}{2}$ When  $I_1 = 0$  $Z_{22} = \frac{V_2}{I_2}$  $Z_{12} = \frac{V_1}{I_2}$  $V_2 = I_2 \cdot (Z_1 + Z_2) \| (Z_1 + Z_2)$  $= I_2 \cdot \frac{(Z_1 + Z_2) (Z_1 + Z_2)}{Z_1 + Z_2 + Z_1 + Z_2}$  $= I_2 \cdot \frac{(Z_1 + Z_2)^2}{2(Z_1 + Z_2)}$  $=I_2\cdot\frac{Z_1+Z_2}{2}$  $Z_{22} = \frac{V_2}{I_2} = \frac{Z_1 + Z_2}{2}$  $V_1 = V_2 \left[ \frac{Z_2}{Z_1 + Z_2} - \frac{Z_1}{Z_1 + Z_2} \right]$  $= V_2 \cdot \frac{Z_2 - Z_1}{Z_1 + Z_2}$  $= \frac{Z_1 - Z_2}{2} \cdot I_2 \cdot \frac{Z_2 - Z_1}{Z_1 + Z_2}$  $= \frac{Z_2 - Z_1}{2}$ .  $I_2$  $Z_{12} = \frac{V_1}{I_2} = \frac{Z_2 - Z_1}{2}$ 

Hence the Z-parameters are

$$Z_{11} = \frac{Z_1 - Z_2}{2} = Z_{22}$$
$$Z_{12} = \frac{Z_2 - Z_1}{2} = Z_{21}$$

Code No: 07AIEC02

# I B.TECH REGULAR EXAMINATIONS, MAY/JUNE 2008 ELECTRICAL CIRCUIT ANALYSIS (Common to Electrical and Electronic Engineering and Instrumentation and Control Engineering)

#### Time: 3 hours

#### Max Marks: 80

Set No.

#### Answer any FIVE Questions All Questions carry equal marks

#### \*\*\*\*

- 1. (a) Distinguish between passive and active elements with suitable examples.
  - (b) Find the voltage and current source equivalent representation of the following network across AB, as shown in Figure 1b. [6 + 10]



#### Figure 1b

- 2. (a) Derive an expression for the energy stored in an inductor and a capacitor.
  (b) Obtain an expression for the co-efficient of coupling. [10 + 6]
- Obtain the rms value, average value, form factor and peak factor for a voltage of symmetrical square shape whose amplitude is 10 V and time period is 40 seconds. [16]
- 4. (a) Three identical impedances of  $(3 + j4)\Omega$  are connected in delta. Find an equivalent star network such that the line current is the same when connected to the same supply.
  - (b) Three impedances of  $(7+j4)\Omega$ ,  $(3+j2)\Omega$  and  $(9+j2)\Omega$  are connected between neutral and the *R*, *Y* and *B* phases. The line voltage is 440 V, Calculate
    - i. the line currents, and
    - ii. the current in the neutral wire.
    - iii. Find the power consumed in each phase and the total power drawn by the circuit. [4 + 12]
- 5. (a) Obtain the cut-set matrix for the network, as shown in Figure 5a.





(b) For the network shown in Figure 5b. Determine the power dissipated in the 9  $\Omega$  resistor using mesh analysis. [6 + 10]





6. (a) Using Norton's theorem, find the current through the load impedance  $Z_L$  as shown in Figure 6a.





(b) State and explain the reciprocity theorem. [10+6]

- 7. Derive an expression for the current response in R-L series circuit with a sinusoidal source. [16]
- 8. Find the *Y*-parameters for the bridged T-network as shown in Figure 8.

[16]



### SOLUTION

 (a) Passive elements are those which are capable of only receiving power. Some passive elements like inductors and capacitors are capable of storing a finite amount of energy.

An active element is capable of delivering an average power greater than zero to some external device over an infinite time interval. Ideal sources are active elements.



By Thevenin's theorem

$$\frac{V_{1}-4}{2} + \frac{V_{1}}{2} = 2A$$

$$V_{1} = 2 + 2 = 4V$$

$$R_{th} = 2 || 2 = 1 \Omega$$

$$\frac{1 \Omega}{R_{CH}}$$
Voltage equivalent circuit

By source-transformation method,



2. (a) Inductor

The current-voltage relation is given by

$$v = L \frac{di}{dt}$$

where V is the voltage across the inductor in volts, and i is the current through the inductor in amperes. We can rewrite the above equations as

$$\mathrm{d}i = \frac{1}{L} \ v \ \mathrm{d}t$$

Integrating both sides, we get

$$\int_{0}^{t} dt = \frac{1}{L} \int_{0}^{t} v \, dt$$
$$i(t) - i(0) = \frac{1}{L} \int_{0}^{t} v \, dt$$
$$i(t) = \frac{1}{L} \int_{0}^{t} v \, dt + i(0).$$

From the above equation, we note that the current in an inductor is dependent upon the integral of the voltage across its terminals and the initial current in the coil, i(0).

The power absorbed by the inductor is

$$P = vi = Li \frac{\mathrm{d}i}{\mathrm{d}t}$$
 watts

The energy stored by the inductor is

$$W = \int_{0}^{t} P \, \mathrm{d}t$$
$$= \int_{0}^{t} Li \, \frac{\mathrm{d}i}{\mathrm{d}t} \times \mathrm{d}t = \frac{Li^2}{2}$$

#### Capacitor

The current-voltage relation is

$$v(t) = \frac{1}{C} \int_{0}^{t} i \mathrm{d}t + v(o)$$

where v(0) indicates the initial voltage across the capacitor from the above equation. The voltage in a capacitor is dependent upon the integral of the current through it, and the initial voltage across it.

The power absorbed by the capacitor is given by

$$P = vi = vc \ \frac{\mathrm{d}v}{\mathrm{d}t}$$

The energy stored by the capacitor is

$$W = \int_{o}^{t} P \, dt = \int_{o}^{t} vc \, \frac{dv}{dt} \, dt$$
$$W = \frac{1}{2} cv^{2}$$

(b) The amount of coupling between the inductively coupled coils is expressed

in terms of the coefficient of coupling, which is defined as  $K = \frac{M}{\sqrt{L_1 L_2}}$ 

where M = Mutual inductance between the coils

 $L_1$  = Self-inductance between the first coil

 $L_2$  = Self-inductance of the second coil

The coefficient of coupling is always less than unity, and has a maximum value of 1. This case, for which K = 1, is called perfect coupling, when the entire flux of one coil links the other. The greater the coefficient of coupling between two coils, the greater the mutual inductance between them and vice-versa.

For a pair of mutually coupled circuits, let us assume initially that  $i_1, i_2$  are zero at P = 0.



Then

$$v_1(t) = L_1 \frac{\mathrm{d}i_1(t)}{\mathrm{d}t} + M \frac{\mathrm{d}i_2(t)}{\mathrm{d}t}$$
$$v_2(t) = L_2 \frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + M \frac{\mathrm{d}i_1(t)}{\mathrm{d}t}$$

and

The initial energy in the coupled circuit at t = 0 is also zero. The net energy input to the system at time 't' is given by

$$W(t) = \int_{0}^{t} [(v_1(t) \ i_1(t) + v_2(t)i_2(t)]dt$$

substituting the values of  $v_1(t)$  and  $v_2(t)$  in the above equation yields, W(t)

$$= \int_{0}^{t} \left[ (L_{1}i_{1}(t)\frac{di_{1}(t)}{dt} + L_{2}i_{2}(t)\frac{di_{2}(t)}{dt} \right] + \left[ M(i_{1}(t))\frac{di_{2}(t)}{dt} + i_{2}(t)\frac{di_{1}(t)}{dt} \right] dt$$

from which we get

$$W(t) = \frac{1}{2}L_1[i_1(t)]^2 + \frac{1}{2}L_2[i_2(t)]^2 + M[i_1(t)i_2(t)]$$

If one current enters a dot-marked terminal while the other leaves a dotmarked terminal, the above equation becomes

$$W(t) = \frac{1}{2}L_1[i_1(t)]^2 + \frac{1}{2}L_2[i_2(t)]^2 - M[i_1(t)i_2(t)].$$

According to the definition of passivity, the net electrical energy input to the system is non-negative. W(t) represents the energy stored within a passive network, and it cannot be negative.

 $W(t) \ge 0$  for all values of  $i_1, i_2, L_1, L_2$  or M.

The statement can be proved in the following way. If  $i_1$  and  $i_2$  both are both positive or negative, W(t) is positive. The other condition where the energy equation could be negative is

$$W(t) = \frac{1}{2}L_1[i_1(t)]^2 + \frac{1}{2}L_2[i_2(t)]^2 - M[i_1(t)i_2(t)]$$

The above equation can be rearranged as

$$W(t) = \frac{1}{2} \left( \sqrt{L_1 i_1} - \frac{M}{\sqrt{L_1}} i_2 \right)^2 + \frac{1}{2} \left( L_2 - \frac{M^2}{L_1} \right) i_2^2$$
$$L_2 - \frac{M^2}{L_1} \ge 0$$
$$\frac{L_1 L_2 - M^2}{L_1} \ge 0$$
$$L_1 L_2 - M^2 \ge 0$$

 $\Rightarrow$ 

Coefficient of coupling for the coupled circuit

 $M \leq \sqrt{L_1 L_2}$ 

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

The coefficient K is a non-negative number.

Max voltage,  $v_p = 10$  V



**RMS** value

$$V_{\rm rms} = \sqrt{\frac{1}{T} \int_{o}^{T} v(t)^2 dt}$$
$$= \sqrt{\frac{1}{40} \int_{o}^{40} v(t)^2 dt}$$
$$= \sqrt{\frac{1}{40} \left[ \int_{o}^{20} 10^2 dt + \int_{20}^{40} 10^2 dt \right]}$$
$$V_{\rm rms} = \sqrt{\frac{1}{40} [100 \times 20 + 100 \times 20]} = 10V$$

Average value

$$V_{\text{avg}} = \frac{1}{T} \int_{o}^{T} v(t) dt$$
$$= \frac{1}{20} \int_{o}^{20} 10 dt = \frac{1}{20} \times 20 \times 10$$
$$V_{\text{avg}} = 10 \text{V}$$

Form factor

Peak factor

$$F = \frac{V_{\text{rms}}}{V_{\text{avg}}} = \frac{10}{10} = 1$$
$$= \frac{V_p}{V_{\text{rms}}} = \frac{10}{10} = 1$$

4. (a)



$$Z_{RY} = \frac{(3+j4)(3+j4)}{3+j4+3+j4+3+j4}$$

 $= \frac{(3+j4)^2}{3(3+i4)} = \frac{3+j4}{3}$  $Z_{YB} = Z_{BY} = \frac{3+j4}{3}$ (b)  $(3+j2) \Omega$ (i)  $I_1 = \frac{440|0}{7+i4} = \frac{440|0}{8.06|29.74} = 54.59|-29.74$  amps amps = 47.39 - j27.08  $I_2 = \frac{440|120^{\circ}}{3+j2} = \frac{440|120^{\circ}}{3.6|33.69^{\circ}} = 122.22 |86.31^{\circ} \text{ amps} = 7.86 + j121.96$  $I_3 = \frac{440|240^\circ}{9+j2} = \frac{440|240^\circ}{9.21|12.152^\circ} = 47.77 \ \underline{|227.48^\circ|} \ \text{amps} = -32.28 - j35.2$ (ii) The current in the neutral wire is zero. (iii)  $P_1 = \sqrt{3} \times 54.59 \mid -29.74^\circ \times 440$  $= 41603.16 \mid -29.74^{\circ} = 36123.41 - j20637.37$  watt  $P_2 = \sqrt{3} \times 122.22 |86.31 \times 440$ = 93144.14 | 86.31 = 5994.58 + j92951.103 watt  $P_2 = \sqrt{3} 47.77 |227.48^\circ \times 440$  $= 36405.62 | 227.48^{\circ} = -24604.64 - j26832.45$ Total power =  $P_1 + P_2 + P_3 = 17513.35 + j45480.71$ 5. (a)



Cutset,  $C_a : (1, 4)$ Cutset,  $C_b : (1,2,3)$ Cutset,  $C_c : (1,2,3)$ 



$$C_{a}: i_{1} + i_{4} = 0$$

$$C_{b}: i_{2} + i_{3} - i_{4} = 0$$

$$C_{c}: i_{1} + i_{2} + i_{3} = 0$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{1} \\ i_{2} \\ i_{3} \\ i_{4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$





By applying mesh analysis, the equations obtained are

$$\begin{array}{c} 100 = 10(i_1 - i_2) + 90(i_1 - i_3) \\ \Rightarrow & 100i_1 - 10i_2 - 90i_3 = 100 \quad (1) \\ 9i_2 - 10i_1 - 9i_3 = 0 \quad (2) \\ 9i_3 - 90i_1 - 9i_2 = 0 \quad (3) \end{array}$$

On solving the three equations

 $i_1 = 3.3 \text{ A}$  $i_2 = 1.4 \text{ A}$  $i_3 = 2.4 \text{ A}$  Power dissipated in the 9- $\Omega$  resistor is

 $(i_2 - i_3)^2 \times R = (2.4 - 1.4)^2 \times 9 = 9 \text{ W}$ 

6. (a)



By using Norton's theorem, for the short-circuit current



**10** Ω

0

(b) In any linear bilateral network, if a single voltage source  $V_a$  in branch 'a' produces a current  $I_b$  in branch 'b' then if the voltage source  $V_a$  is re-

moved and inserted in branch 'b', it will produces a current  $I_b$  in branch 'a'. The ratio of response to excitation is the same for the two conditions. This is called the reciprocity theorem.



The application of voltage V across A - A' produces current I at B - B', the resultant current I will be at A - A'. According to the reciprocity theorem, the ratio of response to excitation is same in both the cases.



Consider a circuit consisting of resistance and inductance as shown in the figure. The switch *S* is closed at t = 0. At t = 0, a sinusoidal voltage *V* cos ( $\omega t + \theta$ ) is applied to the series R - L circuit, where *V* is the amplitude of the wave and  $\theta$  is the phase angle. Application of Kirchhoff's voltage law to the circuit results,

$$V\cos(\omega t + \theta) = R_{i} + L\frac{di}{dt}$$
  
$$\therefore \qquad \frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t + \theta)$$

The characteristic equation is

$$\left(D+\frac{R}{L}\right)i = \frac{V}{L}\cos\left(\omega t + \theta\right)$$

The complementary function of the solution is

 $i_c = ce^{-t}(R/L)$ 

Particular solution,  $i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta)$  $i'_p = -A\omega \sin(\omega t + \theta) + B\omega \cos(\omega t + \theta)$ 

 $\{-A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)\} + \frac{R}{L}\{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\} = \frac{V}{L}\cos(\omega t + \theta)$ 

$$(\text{or})\left(-A\omega + \frac{BR}{L}\right)\sin\left(\omega t + \theta\right) + \left(B\omega + \frac{AR}{L}\right)\cos\left(\omega t + \theta\right) = \frac{V}{L}\cos\left(\omega t + \theta\right)$$

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Comparing cosine terms and sine terms,

$$-A\omega + \frac{BR}{L} = 0$$
$$B\omega + \frac{AR}{L} = \frac{V}{L}$$

On solving,

$$A = V \cdot \frac{R}{R^2 + (\omega L)^2}$$
$$B = V \cdot \frac{\omega L}{R^2 + (\omega L)^2}.$$

 $\therefore \qquad i_p = V \cdot \frac{R}{R^2 + (\omega L)^2} \cos(\omega t + \theta) + V \cdot \frac{\omega L}{R^2 + (\omega L)^2} \sin(\omega t + \theta)$ 

 $(\omega t + \theta)$ 

Putting  $M \cos \phi = \frac{VR}{R^2 + (\omega L)^2}$  $M \sin \phi = V \cdot \frac{\omega L}{R^2 + (\omega L)^2}$ 

To find M and  $\phi$ ,

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\omega L}{R}$$

Squaring both sides,

$$M^{2} \cos^{2} \phi + M^{2} \sin^{2} \phi = \frac{V^{2}}{R^{2} + (\omega L)^{2}}$$

$$\Rightarrow \qquad M = \frac{V}{\sqrt{R^{2} + (\omega L)^{2}}}$$

$$i_{p} = \frac{V}{\sqrt{R^{2} + (\omega L)^{2}}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

$$i = i_{c} + i_{p}$$

$$i = ce^{-t} (RL) + \frac{V}{\sqrt{R^{2} + (\omega L)^{2}}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

:. The inductor does not allow sudden changes in the current at t = 0, i = 0.

$$C = -\frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Compete solution is

$$i = e^{-(R/L)t} \left[ \frac{-V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right) \right]$$
$$\frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L}{R}\right)\right).$$

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Code No: 07AIEC02

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# B.TECH REGULAR EXAMINATIONS, MAY/JUNE 2008

# ELECTRICAL CIRCUIT ANALYSIS

# (Common to Electrical and Electronic Engineering and Instrumentation and Control Engineering)

Time: 3 hours

#### Max Marks: 80

Set No. 2

# Answer any FIVE Questions All Questions carry equal marks

- 1. (a) A bridge network *ABCD* is arranged as follows:
  - Resistance between terminals *AB*, *BC*, *CD*, *DA* and *BD* are 10 ohms, 30 ohms, 15 ohms, 20 ohms and 40 ohms respectively. A 4 V battery is connected with negligible internal resistance between terminals *A* and *C*. Determine the current through each element in the network using network reduction techniques.
  - (b) Three equal resistance are available. Find
    - i. two ratios of the equivalent resistances when they are connected in parallel, and
    - ii. the ratio of the current through each elements when they are connected in parallel. [10+6]
- 2. A non-magnetic ring having a mean diameter of 30 cm and a cross-sectional area of 4 cm<sup>2</sup> is uniformly wound with two coils A and B one over the other. A has 100 turns and B has 250 turns. Calculate the mutual inductance between the coils. Also, calculate the emf induced in B when a current of 6 A in A is reversed in 0.02 seconds. Derive the formulae used.
- 3. Define form factor and peak factor of an alternating quantity. Calculate the average and rms value, the form factor and peak factor of a periodic current having the following values for equal time intervals, changing suddenly from one value to next: 0, 40, 60, 80, 100, 80, 60, 40, 0, -40, -60, -80 A. [16]
- 4. (a) Three identical impedances of  $(3 + j4)\Omega$  are connected in delta. Find an equivalent star network such that the line current is the same when connected to the same supply.
  - (b) Three impedance of  $(7 + i4)\Omega$ ,  $(3 + j2)\Omega$  and  $(9 + j2)\Omega$  are connected between neutral and the *R*, *Y* and *B* phases. The line voltages is 440 V, Calculate
    - i. the line currents, and
    - ii. the current in the neutral wire.
    - iii. Find the power consumed in each phase and the total power drawn by the circuit. [4 + 2]

5. (a) For the network shown in Figure 5a draw the oriented graph and frame the cut-set matrix.



Figure 5a

(b) Compute node voltages for the circuit as shown in Figure 5 b. [6 + 10]



Figure 5b

6. (a) Find the value of  $R_L$  so that maximum power is delivered to the load resistance  $R_L$  as shown in Figure 6a, and find the maximum power.



Figure 6a

- (b) State and explain Thevenin's theorem. [8+8]
- 7. Derive the expression for the transient response of RLC series circuit with unit step unit. [16]
- 8. Find the *Z* and transmission parameters for the resistances n/w shown in Figure 7. [8 + 8]



Figure 7

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# SOLUTION

1. (a)



$$4 V = 35 i_1 - 20 i_2 - 15 i_3$$
  
(20 + 10 + 40)i\_2 - 40 i\_3 - 20 i\_1 = 0  
(15 + 40 + 30)i\_3 - 40 i\_2 = 0

By solving, we get

$$i_1 = 0.22 \text{ A}$$
  
 $i_2 = 0.117 \text{ A}$   
 $i_3 = 0.094 \text{ A}$ 

Current in the 20  $\Omega$  resistor =  $i_1 - i_2 = 0.103$  A

10 
$$\Omega = i_2 = 0.117$$
 A  
40  $\Omega = i_2 - i_3 = 0.023$  A  
15  $\Omega = i_1 - i_3 = 0.126$  A  
30  $\Omega = i_3 = 0.094$  A

When they are connected in parallel, they have the same resistance value. So the current is divided in equal amounts.

 $\therefore$  the ratio of current through each element is 1:1:1



2. (a) 
$$M = \frac{N_1 N_2}{\text{Reluctance}} = \frac{N_1 N_2}{1/\mu_o \mu_r A}$$

$$M = \frac{N_1 N_2 \mu_o \mu_r A}{l}$$

$$l = 0.3 \ \Pi m \qquad \mu_r = 1 \qquad \mu_0 = 4\Pi \times 10^{-7}$$
$$A = 4 \times 10^{-4}, N_1 = \frac{100 \times 250 \times 4\Pi 10^{-7} \times 4 \times 10^{-4}}{0.3 \pi} = 1.34 \times 10^{-6} = 1.34 \ \mu H$$
$$e_B = N_2 \frac{di}{dt} = 250 \times \frac{6+6}{0.2} = 15 \ \text{kV}$$

# Derivation

Let there be two magnetically coupled coils having  $N_1$  and  $N_2$  turns respectively. The coefficient of mutual inductance between the two coils is defined as the Weber turns in one coil due to one ampere current in the other.

Flux produced by coil one 
$$\varphi_1 = \frac{N_1 I_1}{\text{Reluctance of the path}}$$
$$= \frac{N_1 I_1}{l/\mu_o \mu_r A}$$

Flux/ampere =  $\frac{\varphi_1}{I_1} = \frac{N_1}{l_1 / \mu_o \mu_r A}$ 

Assuming that whole of this flux is linked with the other coil having  $N_2$  turns,

$$M = \frac{N_2 \varphi_1}{I_2} = \frac{N_2 \cdot N_1}{l_1 / \mu_o \mu_r A}$$
$$M = \frac{\mu_o \mu_r A \cdot N_1 N_2}{l} Henry$$
$$M = \frac{N_1 N_2}{l}$$

Reluctance

or



$$Z_{RY} = \frac{(3+j4)(3+j4)}{3+j4+3+j4+3+j4}$$
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$$= \frac{(3+j4)^2}{3(3+j4)} = \frac{3+j4}{3}$$

$$Z_{YB} = Z_{BY} = \frac{3+j4}{3}$$
(b)  
(17+j4) Ω  
(3+j2) Ω  
(9+j2) Ω  
(i)  $I_1 = \frac{440}{7+j4} = 54.59 | -29.74^{\circ} = 47.39 - j 27.08 \text{ A}$   
 $I_2 = \frac{440}{3+j2} = 122.22 | 86.31^{\circ} = 7.86 + j 121.96 \text{ A}$   
 $I_3 = \frac{440}{9+j2} = 47.77 | 227.48^{\circ} = -32.28 - j 325.20 \text{ A}$   
(ii) The current in the neutral wire is zero.  
(iii)  $P_1 = \sqrt{3} \times 54.59 | -29.74^{\circ} \times 440$   
 $= 41603.16 | -29.74^{\circ} = 36123.41 - j 20637.37 \text{ watts}$   
 $P_2 = \sqrt{3} \times 122.22 | 86.31^{\circ} \times 440$   
 $= 93144.14 | 86.31^{\circ} = 5994.58 + j 92951.103 \text{ watts}$   
 $P_3 = \sqrt{3} \times 47.77 | 227.48^{\circ} \times 440$   
 $= 36405.62 | 227.48^{\circ} = -24604.64 - j 26832.45 \text{ watts}$   
Total power  $= P_1 + P_2 + P_3$   
 $= 17513.35 + j 45480.71 \text{ watts}$   
5. (a)  
 $A = \frac{10}{14} = \frac{10}{220} = 20 = 10$ 





On solving,

$$V_1 = 81.176 \text{ V}$$
  
 $V_2 = 21.764 \text{ V}$ 

6. (a)







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$$R_{eq} = [(5/15) + 10]//20 = 12.5//20$$
  
 $R_{eq} = 7.692 \ \Omega$ 



$$i_2 = \frac{30.769}{2(7.692)} = 2$$
 A

Power delivered to the load resistance =  $i^2 R$ 

$$= 2^2 \times 7.692 = 30.768$$
 W

(b) Thevenin's theorem states that any two terminal linear networks having a number of voltage and current sources and a resistance equal to the equivalent circuit consisting of a single voltage source in series is a resistance where the value of the voltage source is equal to the open-circuit voltage across the two terminals of the network and the resistance is equal to the equivalent resistance measured between the terminals. All the energy sources are replaced by their internal resistances.



7.



The capacitor and inductor are initially unchanged, and are in series with a resistor. When the switch *s* is closed at t = 0, we can determine the complete solution for the current. The difference equation is

$$1 V = R_{i} + L\frac{di}{dt} + \frac{1}{c}\int i dt$$

On differentiation,

$$0 = R \frac{di}{dt} + L \frac{d^2i}{dt^2} + \frac{1}{c}i$$
$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{Lc}i = 0$$

 $\Rightarrow$ 

The characteristic equation is

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0$$

(i)  $K_2$  is +ve when  $\left(\frac{R}{2L}\right)^2 > 1/LC$ 

 $i = C_1 e^{(K_1 + K_2)t} + C_2 e^{(K_1 - K_2)t}$ 

The roots are real and unequal and give an overdamped response.  $[D - (K_1 + K_2)] [D - (K_1 - K_2)]i = 0$ Solution for the above equation

Its roots are

$$D_{1}, D_{2} = \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^{2} - \frac{1}{LC}}$$

$$K_{1} = \frac{-R}{2L}, K_{2} = \sqrt{\left(\frac{R}{2L}\right)^{2} - \frac{1}{LC}}$$

$$D_{1} = K_{1} + K_{2} \text{ and } D_{2} = K_{1} - K_{2}$$

$$K_{2} \text{ may be +ve, -ve or zero.}$$

0

(ii)  $K_2$  is -ve when  $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$ . The roots are complex conjugate and give an underdamped response.  $[D - (K_1 + jK_2)] [D - (K_1 - jK_2)]i = 0$ The solution for the above equation is

$$i = e^{K_1 t} \left[ C_1 \cos K_2 t + C_2 \sin K_2 t \right]$$

(iii) 
$$K_2$$
 is zero when  $\left(\frac{R}{2L}\right)^2 = \frac{1}{Lc}$ 

The roots are equal and give a critically damped response.

$$(D - K_1) (D - K_1)i = 0$$

The solution of the above equation is

$$i = e^{K_1 t} (C_1 + C_2 t)$$

$$\begin{split} Y_{11} &= \left. \frac{I_1}{V_1} \right|_{V_2 = 0} \\ V_1 &= \left. \frac{I_1}{Y_1 + Y_2} \implies Y_{11} = Y_1 + Y_2 \\ Y_{11} &= 1 + 2 = 3 \ \Omega \\ Y_{22} &= Y_2 + Y_3 = 2 + 2 = 4 \ \Omega \\ I_1 &= VY_1 + (V_1)Y_2 \\ I_2 &= -V_1Y_2 + Y_3(0) \qquad \{ \text{when } V_2 \text{ is short-circuited} \} \\ Y_{21} &= \left. \frac{I_2}{V_2} \right|_{V_2} = -Y_2 \end{split}$$

$$= -2 \Omega$$

$$I_1 = -V_2 Y_2$$

$$I_2 = +V_2 Y_2 + Y_3 V_2$$
 {when  $V_1$  is short-circuited}  

$$Y_{12} = \frac{I_2}{V_2} = -Y_2 = -2 \Omega$$





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# **Z-parameters**

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{4}{8} = \frac{1}{2} \Omega, \ \Delta Y \begin{vmatrix} 3 & -2 \\ -2 & 4 \end{vmatrix} = 8$$
$$Z_{12} = Z_{21} = \frac{Y_{21}}{-\Delta Y} = \frac{+2}{-8} = \frac{1}{4} \Omega$$
$$Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{3}{8} \Omega$$

# Transmission parameters

$$A = \frac{Y_{22}}{Y_{21}} = \frac{-4}{-2} = 2$$
$$B = -\frac{1}{Y_{21}} = \frac{-1}{-2} = \frac{1}{2} \Omega$$
$$C = \frac{-\Delta Y}{Y_{21}} = \frac{-8}{-2} = 4 \mho$$
$$D = \frac{Y_{11}}{Y_{21}} = \frac{-3}{-2} = \frac{3}{2}$$

Code No: 07A1EC02

# I B.TECH REGULAR EXAMINATIONS, MAY/JUNE 2008 ELECTRICAL CIRCUIT ANALYSIS (Common to Electrical and Electronic Engineering and Instrumentation and Control Engineering)

### Time: 3 hours

## Max Marks: 80

Set No. 3

## Answer any FIVE Questions All Questions carry equal marks

#### \*\*\*\*\*

- 1. (a) A voltage of 60 V d.c. is applied across two capacitors of 100  $\mu$ F. Find the voltage sharing between them if they are connected in series. What is the energy stored in each of the capacitors?
  - (b) Find the equivalent capacitance between the terminals *A* and *B* in the circuit shown in Figure 1b. [8 + 8]



Figure 1 b

- 2. The mean diameter of a steel ring is 40 cm and flux density of  $0.9 \text{ wb/m}^2$  is produced by 3500 ampere turns per meter. If the cross-section of the ring is 15 cm<sup>2</sup> and the number of turns is 440, calculate
  - (a) the exciting current,
  - (b) the self-inductance, and
  - (c) the exciting current and the inductance when an air gap of 2 cm is cut in the ring, the flux density being the same. Ignore leakage and fringing. [16]
- (a) Derive an expression for the current, impedance, average power for a series RC circuit excited by a sinusoidally alternating voltage and also find the power factor of the circuit. Draw the phasor diagram.
  - (b) A series R-L series circuit having a resistance of 4  $\Omega$  and 3 ohms inductive reactance is fed by a 100 V, 50 Hz,  $1 \phi$  supply. Find current, power drawn by the circuit and power factor. [8 + 8]

- 4. (a) Explain how power is measured in a three-phase delta-connected load using two wattmeters.
  - (b) A balanced mesh-connected load of  $(8 + i6)\Omega$  per phase is connected to a 3-phase, 50 Hz, 230 V supply. Calculate
    - i. line current,
    - ii. power factor,
    - iii. reactive volt-ampere, and
    - iv. total volt-ampere
- 5. (a) For the network shown in Figure 5a draw the oriented graph and frame the cut-set matrix.



Figure 5a

(b) Compute node voltages for the circuit as shown in Figure 5b.

[6 + 10]

а

[8 + 8]





6. (a) Find the current through the branch A-B of the network shown in the Figure 6a using Thevenins theorem.



## Figure 6a

	(b) State and explain compensation theorem.	[6 + 10]
7.	Derive an expression for the current response in R-L series cit	rcuit with a
	sinusoidal source.	[16]

8. Find the transformed Z-parameters of the n/w shown in Figure 8. [16]



# **SOLUTION**

1. (a)



The current in a capacitor is zero. If the voltage across it is zero or constant, the capacitor acts as an open circuit. So the total voltage appears across the capacitor as energy stored in a capacitor



 $d = 40 \times 10^{-2}$  m Flux density (B) = 0.9 wb/m<sup>2</sup> 2. Area =  $15 \times 10^{-4}$  m<sup>2</sup>: N = 440 AT/M = 3500a. mmf = NIwhere I is the exciting current and N is number of turns Exciting current =  $\frac{\text{mmf}}{N}$ Length of the flux path =  $\Pi d$  $=\Pi \times 40 \times 10^{-2}$ = 1.2566 m $mmf = \frac{AT}{m \times I}$  $= 3500 \times \Pi d = 4398.22$ Exciting current =  $\frac{439822}{440}$  = 10 A b. Self-inductance =  $\frac{N\varphi}{I}$  $\varphi = BA$  $L = \frac{N.B \times A}{I} = \frac{440 \times 0.9 \times 15 \times 10^{-4}}{10}$ = 59.4 mHc. mmf of air gap =  $\varphi X$  reluctance of the gap Reluctance of gap =  $\frac{l}{\mu A}$  $mmf = B \times A \times \frac{l}{\mu_0 A} = \frac{B \times l}{\mu_0}$  $=\frac{0.9\times2\times10^{-2}}{4\Pi\times10^{-7}}=0.1432\times10^{5}$ = 14320 AT Total AT = 14320 + 4398= 18718 AT  $I = \frac{18718}{440} = 425 \text{A}$  $L = \frac{N\varphi}{I} = \frac{440 \times 0.9 \times 15 \times 10^{-4}}{425} = 13.9 \text{ mH}$ 

3. (a) **Series RC circuit** When a sinusoidal voltage is applied to an RC series circuit, the current in the circuit and voltages across each of the elements are sinusoidal.



Here, the resistor voltage and current are in-phase with each other. The capacitor voltage lags behind the source voltage. The phase angle between the current and capacitor voltage is always 90. The amplitudes and phase relations between the voltages and current depend on the ohmic values of resistance and capacitor reactance. The circuit is a series combination of both resistance and capacitance. The phase between the current and applied voltage is in between zero and 90, depending on the values of R and C.



Current I leads  $V_c$  by 90° and is in phase with  $V_p$ .

$$V_S = \sqrt{V_R^2 + V_C^2}$$

The phase angle between resistor and source voltage

$$\theta = \tan^{-1}(V_C/V_R)$$
$$I = \frac{V_S}{R - j \times C}$$
$$Z = R - j \times C$$

Phasor diagrams

*.*•.





Power drawn by the circuit =  $i^2 R$ 

$$= (20)^{2} \times 4 = 1.6 \text{ k}\omega)$$
$$\theta = \tan^{-1}\left(\frac{XL}{R}\right)$$
$$= \tan^{-1}\left(\frac{3}{4}\right) = 36.87$$
$$= \cos \theta = 0.8$$

Power factor

Phase angle



6

Total power = WR + WY

The average power indicated by the watt-meter wp is

$$W_{R} = \frac{1}{T} \int_{0}^{T} V_{RB} I_{R} dt$$
  
and that by  $W_{Y} = \frac{1}{T} \int_{0}^{T} V_{YB} I_{Y} dt$   
 $V_{RB} = V_{RN} + V_{NB}$   
 $V_{YB} = V_{YN} + V_{NB}$   
 $W_{R} + W_{Y} = \frac{1}{T} \int_{0}^{T} (V_{RB} I_{R} + V_{YB} I_{Y}) dt$   
 $= \frac{1}{T} \int_{0}^{T} [(V_{RN} + V_{NB})I_{R} + (V_{YN} + Y_{NB})I_{Y}] dt$   
 $= \frac{1}{T} \int_{0}^{T} [(V_{RN} I_{R} + V_{YN}I_{Y}) + (I_{R} + I_{Y}) V_{NB}] dt$   
 $I_{R} + I_{Y} + I_{B} = 0$   
 $I_{R} + I_{Y} = -I_{B}.$   
 $= \frac{1}{T} \int_{0}^{T} [V_{RN} I_{R} + V_{YN}I_{Y} - I_{B}V_{NB}] dt$   
 $V_{NB} = -V_{BN}$   
 $W_{R} + W_{Y} = \frac{1}{T} \int_{0}^{T} \{(V_{RN} I_{R} + V_{YN}I_{Y} + V_{BN}I_{B})\} dt$ 

which indicates total power.

(b) Given phase voltage = 230 V

Impedance 
$$Z = [8 + j(6)]\Omega = 10 |36.87^{\circ}|$$
  
phase current  $I_R = \frac{230 |0^{\circ}|}{10 |36.87^{\circ}|} = 23 |-36.87^{\circ}|$   
 $I_Y = 23 |-36.87^{\circ}| + 120 = 23 |83.13^{\circ}|$   
 $I_B = 23 |83.13^{\circ} + 120| = 23 |203.13^{\circ}|$ 

Line current = phase current  $\times \sqrt{3}$  = 39.84 Amp (ii) Power factor = cos  $\theta$  = cos (36.87) = 0.8 (iii) Reactive power =  $V_{\rm ph} I_{\rm ph} \sin \phi$ = 23 × 230 sin 36.87° = 3.174 kw (iv) Total volt ampere =  $V_{\rm Ph} I_{\rm Ph}$  = 5.29 kw.



cutset,  $c_a$ : (1, 4) cutset,  $c_b$ : (4, 2, 3) cutset,  $c_c$ : (1, 2, 3)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_{b}: i_{2} + i_{3} - i_{4} = 0$$
  
$$c_{c}: i_{1} + i_{2} + i_{3} = 0$$



$$\frac{V_1}{10} + \frac{+V_1 - V_2}{5} + 5 = 25$$

$$6 + \frac{V_2 - 10}{2} = \frac{V_1 - V_2}{5} + 5$$

$$\frac{V_1}{10} + \frac{V_1}{5} - \frac{V_2}{5} = 20$$

$$\frac{V_2}{2} + \frac{V_2}{5} - \frac{V_1}{5} = -1$$
$$\frac{V_1}{5} - V_2 \left(\frac{1}{2} + \frac{1}{5}\right) = 1$$

On solving,

$$V_1 = 81.176 \text{ V}$$
  
 $V_2 = 21.764 \text{ V}$ 





By Thevenin's theorem

$$V_{\rm th} = \frac{20|\underline{0} \times 5 + j(4)}{10 + 5 + j(4)} = \frac{20|\underline{0} \times (5 + j(4))}{15 + j(4)}$$
$$= \frac{20|\underline{0} \times 6.4 |\underline{38.66}}{15.52 |\underline{14.93^{\circ}}} = 8.25 |\underline{23.73^{\circ}}$$
$$R_{\rm th} = 10 || (5 + j(4))$$
$$= \frac{10 \times 5 + j(4)}{10 + 5 + j(4)} = \frac{10 \times 6.4 |\underline{38.66^{\circ}}}{15.52 |\underline{14.93^{\circ}}} = 4.124 |\underline{23.73^{\circ}}$$
$$i = \frac{8.25 |\underline{23.73}}{4.124 |\underline{23.73} + 5}$$
$$= \frac{8.25 |\underline{23.73}}{8.775 + j(1.66)} = 0.92 |\underline{13.02}$$

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6. (b) The Compensation theorem states that any element in a linear bilateral network may be replaced by a voltage source of magnitude equal to the current passing through the element multiplied by the value of the element, provided the current and voltages in other parts of the circuit remain unaltered.



This theorem is useful in finding the changes in current or voltage when the value of the resistance is changed in the circuit.

7. The switch S is closed at t = 0. At t = 0, a sinusoidal voltage  $V \cos(\omega t + \theta)$  is applied to the series R-L circuit, where V is the amplitude of the wave and  $\theta$  is the phase angle.



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Differential equation

$$V\cos(\omega t + \theta) = Ri + L\frac{dt}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t + \theta)$$
Characteristic equation is  $\left(D + \frac{R}{L}\right)i = \frac{V}{L}\cos(\omega t + \theta)$ 
Complementary function is  $i_c = c \ e^{-t}(R/L)$ 
Particular solution is  $i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta)$ 

$$= i'_p = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)$$

Substituting,

$$\{-A\sin(\omega t + \theta) + B\cos(\omega t + \theta)\} + \frac{R}{L} \{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\}$$

$$= \frac{V}{L}\cos(\omega t + \theta)$$
(or)  $\left(-A\omega + \frac{BR}{L}\right)\sin(\omega t + \theta) + \left(B\omega + \frac{AR}{L}\right)\cos(\omega t + \theta) = \frac{V}{L}\cos(\omega t + \theta)$ 
Comparing the sine and cosine terms,  
 $-A\omega + \frac{BR}{L} = 0$   
 $B\omega + \frac{AR}{L} = \frac{V}{L}$   
On solving,  $A = V \frac{R}{R^2 + (\omega L)^2}$   
 $B = V \frac{\omega L}{R^2 + (\omega L)^2}$   
 $\therefore \qquad i_p = \frac{VR}{R^2 + (\omega L)^2}\cos(\omega t + \theta) + V \frac{\omega L}{R^2 + (\omega L)^2}\sin(\omega t + \theta)$ 
Putting  $M\cos\phi = \frac{VR}{R^2 + (\omega L)^2}$ ,  $M\sin\phi = V \frac{\omega L}{R^2 + (\omega L)^2}$   
 $\phi_1 \frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\omega L}{R}$   
 $M: M^2\cos^2\phi + M^2\sin^2\phi = \frac{V^2}{R^2 + (\omega L)^2}\cos(\omega t + \phi - \tan^{-1}\frac{\omega L}{R})$   
 $c.s. i = i_c + i_p = ce^{-t(R/L)} + \frac{V}{\sqrt{R^2 + (\omega L)^2}}\cos(\omega t + \theta - \tan^{-1}\frac{WL}{R}).$ 

At t = 0, i = 0

$$\therefore \qquad c = -\frac{-V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\left(\frac{\omega L}{R}\right)\right).$$

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$$i = e^{-R/L^{t}} \left[ \frac{-V}{\sqrt{R^{2} + (\omega L)^{2}}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right) \right] + \frac{V}{\sqrt{R^{2} + (\omega L)^{2}}} \cos\left(\omega L + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$



By applying mesh analysis,

$$V_1 = (600 - j(300) I_1 + j(300) (-I_2)$$
$$V_2 = (400 + j(100) I_2 - j(300) I_1$$

For *Z*-parameters when  $I_1 = 0$ 

$$V_{1} = -j(300)I_{2} \qquad V_{2} = (400 + j(100))I_{2}$$

$$\frac{V_{1}}{I_{2}} = -j(300) \qquad \frac{V_{2}}{I_{2}} = 400 + j(100)$$

$$Z_{12} = -j(300)\Omega \qquad Z_{22} = 400 + j(100)\Omega$$

$$I_{2} = 0$$

$$V_{1} = (800 - j(300)I_{1} \qquad V_{2} = -j(300)I_{1}$$

$$\frac{V_{1}}{I_{1}} = 800 - j(300)\Omega \qquad \frac{V_{2}}{I_{1}} = -j(300)$$

$$Z_{11} = 800 - j(300)\Omega \qquad Z_{21} = -j(300)\Omega$$

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Code No: 07A1EC02

# Set No. 4

# I B.TECH REGULAR EXAMINATIONS, MAY/JUNE 2008 ELECTRICAL CIRCUIT ANALYSIS (Common to Electrical and Electronic Engineering and Instrumentation and Control Engineering)

## Time: 3 hours

#### Max Marks: 80

### Answer any FIVE Questions All Questions carry equal marks

#### \*\*\*\*\*

- (a) Two resistances when they are in series have an equivalent resistance of 9 ohms and when connected in parallel have an equivalent resistance of 2 ohms. Find the resistances and the ratio of the voltage and current sharing between these elements if supply voltage is 50 V.
  - (b) Find the equivalent resistance between the terminals AB in the network as shown in Figure 5b, if each has a resistance of R ohms and hence find the total current, current through each of the element if the total voltage is 45V. [8 + 8]



#### **Figure 5b**

- 2. The number of turns in a coil is 250. When a current of 2 A flows in the coil, the flux in the coil is 0.3 mwb. When the current is reduced to zero in 2 ms, the voltage induced in a coil lying in the vicinity of the coil is 63.75 V. If the co-efficient of coupling between the coils is 0.75, find
  - (a) the self-inductance of two coils
  - (b) mutual inductance
  - (c) number of turns in the second coil Derive the formulae used.
- Derive the formulae used. [16] 3. Why is the rms value of an alternating quantity more important than its average value. Find the rms value of the resultant current in a conductor which carries simultaneously sinusoidal alternating current with a maxi-
- which carries simultaneously sinusoidal alternating current with a maximum value of 15 A and direct current of 15 A, by deriving necessary expressions. [16]
- 4. A symmetrical 3-phase, 3-wire, 440 V supply is connected to a star connected load. The impedances in each branch are  $Z_1 = (2 + j3)\Omega$ ,

 $Z_2 = (1 - j2) \Omega$ ,  $Z_3 = (3 + j4) \Omega$ . Find its equivalent delta-connected load. Hence, find the phase and line currents and the total power consumed in the circuits. [16]

5. (a) Write the tie-set schedule for the network shown in Figure 5a



Figure 5a

(b) Using mesh analysis, determine the voltage V which gives a voltage of 50 V across the 10  $\Omega$  resistor shown in Figure 5b. [6 + 10]



6. (a) Obtain Norton's equivalent across terminals A and B for network shown in Figure 6a.



(b) State and explain the maximum power transfer theorem. [10+6]
7. In the circuit shown in the Figure 7, the switch is put in position – 1 for 1 m sec and then thrown to position – 2. Find the transient current in both intervals. [16]



Figure 7

8. (a) Find the *Y* parameters of the pie shown in Figure 8a.



Figure 8a

(b) Find the Z parameters of the T-network shown in Figure 8b. Verify if the network is reciprocal or not. [4 + 12]



\*\*\*\*\*

# SOLUTION

1. (a)

Ans. Let  $R_1$  and  $R_2$  be the two resistances When they are in series

$$R_1 + R_2 = 9 \ \Omega \tag{1}$$

When they are connected in parallel

$$\frac{R_1 R_2}{R_1 + R_2} = 2 \tag{2}$$

From (1) and (2),

$$R_1 R_2 = 2 \times 9 = 18$$
$$(R_1 - R_2)^2 = (R_1 + R_2)^2 - 4R_1 R_2$$
$$= (9)^2 - 4 \times 8 = 81 - 72$$

$$(R_{1} - R_{2})^{2} = \frac{5 \times 18}{10} = 9.0 = 9$$

$$R_{1} - R_{2} = 3$$

$$R_{1} + R_{2} = 9$$

$$2R_{1} = 12$$

$$R_{1} = 6 \Omega$$

$$R_{2} = 3 \Omega$$

$$i = \frac{50}{9}$$
50

Current in the 6- $\Omega$  resistor =  $\frac{50}{9}$ Voltage ratio for 6- $\Omega$  to 3- $\Omega$  resistors =  $\frac{50}{9} \times 6 : \frac{50}{9} \times 3 = 2 : 1$ 



Current ratio in the 6- $\Omega$  and 3  $\Omega$  resistors =  $\frac{1}{6} : \frac{1}{3}$ 

$$=\frac{\frac{1}{6}}{\frac{1}{3}}=1:2$$



Total equivalent resistance =  $R + \frac{R}{2} + \frac{R}{3}$ 

$$= R \left[ 1 + \frac{1}{2} + \frac{1}{3} \right]$$
$$= R \left[ 1 + \frac{5}{6} \right] = \frac{11R}{6}$$



Total current =  $\frac{45V}{\frac{11R}{6}} = \frac{45 \times 6}{11R}$ 

Current in resistor  $AC(R) = \frac{45 \times 6}{11R}$ Between C and  $D(R) = \frac{45 \times 3}{11R}$ Between D and  $B(R) = \frac{45 \times 2}{11R}$ 

3. The root mean square (rms) value of a sine wave is a measure of the heating effect of the wave. When a resistor is connected across a dc voltage source, a certain amount of heat is produced in the resistor in a given time. If we consider all the values from every time interval, we have

$$V_{\rm rms} = \sqrt{\frac{1}{T} \int_{0}^{T} (V(t))^2 dt}$$
$$V(t) = Vp \sin \omega t$$
$$V_{\rm rms} = \frac{V_{\rm m}}{\sqrt{2}}$$

Consider

$$V_{\rm rms} = \frac{\rm V_m}{\sqrt{2}}$$

If the function consists of a number of sinusoidal terms, that is,

$$V(t) = V_0 + (V_{C_1} \cos \omega t + V_{C_2} \cos^2 \omega t + \dots) + (V_{S_1} \sin \omega t + V_{S_2} \sin^2 \omega t + \dots)$$
$$V_{\rm rms} = \sqrt{V_0^2 + \frac{1}{2} (V_{C_1}^2 + V_{C_2}^2 + \dots) + \frac{1}{2} (V_{S_1}^2 + V_{S_2}^2 + \dots)}$$

Given,

maximum value of sinusoidal current = 15 ADirect current = 15 A

$$V_{\rm rms} = \sqrt{V_0^2 + \frac{1}{2} V_{S_1}^2}$$
$$I_{\rm rms} = \sqrt{15^2 + \frac{1}{2} (15)^2} = 15 \times \sqrt{\frac{3}{2}}$$
$$I_{\rm rms} = 18.37 \text{ V}$$



$$Z_R = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$
  
=  $\frac{(2 + j(3))(1 - j(2)) + (1 - j(2))(3 + j(4)) + (3 + j(4))(2 + j(3))}{3 + j(4)}$   
=  $\frac{3.61 | 56.31 \times 2.236 | -63.4 + 2.36 | -63.4 \times 5 | 53.13 + 5 | 53.13 \times 3.61 | 56.3.13}{5 | 53.13}$ 

$$= \frac{6.708 [-7.09 + 11.18 | 10.27 + 18.05 | 109.44}{5 | 53.13}$$

$$= \frac{6.6567 - j(0.83) + 11 + j(1.99) + (-6.007) + j(17.02)}{5 | 53.13}$$

$$= \frac{11.65 + j18.18}{5 | 53.13} = \frac{21.592 | 57.34}{5 | 53.13}$$

$$= 4.3184 | 4.21$$

$$Z_Y = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$= \frac{(2 + j(3))(1 - j(2)) + (1 - j(2))(3 + j(4)) + (3 + j(4))(2 + j(3))}{1 - j(2)}$$

$$= \frac{21.592 | 57.34}{Z_1} = 9.656 | 120.77^{\circ}$$

$$Z_B = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$= \frac{21.592 | 57.34}{Z_1} = 5.98 | 1.03$$

$$I_R = \frac{V}{Z_R}$$

$$= \frac{440 | 0}{4.3184 | 4.21} = 101.88 | -4.21$$

$$I_Y = \frac{440 | 0}{9.656 | 120.77} = 45.567 | -120.77$$

$$I_B = \frac{440 | 0}{5.98 | 1.03} = 73.58 | -1.03$$



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$$V_{1} + V_{2} + V_{4} = 0$$

$$V_{3} + V_{5} - V_{2} = 0$$

$$-V_{4} - V_{5} + V_{6} = 0$$

$$I_{1} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ I_{3} \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} V_{1} \\ V_{2} \\ V_{3} \\ V_{4} \\ V_{5} \\ V_{6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



By applying mesh analysis

$$50 = 2 i_{1} + (i_{1} - i_{2})$$

$$-5i_{3} + 8i_{2} - 2i_{3} - i_{1} = 0$$

$$8i_{2} - 7i_{3} - i_{1} = 0$$

$$7(i_{3} - i_{2}) + 10i_{3} + V - 50 + 4i_{3} = 0$$

$$21i_{3} - 7i_{2} + V - 50 = 0$$
Given  $10i_{3} = 50 V$ 

$$i_{3} = 5 A$$

$$21 \times 5 - 50 = 7i_{2} - V \Longrightarrow 55$$

$$8i_{2} - i_{1} = 35$$

$$3i_{1} - i_{2} = 50$$

$$i_{1} = 18.91 A$$

$$i_{2} = 6.74 A$$

$$V = 7 \times 6.74 - 55 = -7.82 V$$
(4)



By applying Norton's theorem, for short-circuit current, short-circuit the terminals *A* and *B*.



Current 
$$I_s = \frac{10|0^{\circ} \times j(15)}{3+2+j(15)}$$
  
=  $\frac{10|0^{\circ} \times 15|90}{5+j(15)} = \frac{10 \times 15|90}{15.81|71.56} = 9.487 |18.44|$ 

For Norton resistance,



$$= \frac{5|-90^{\circ} \times 15.81|71.56}{11.18|63.43^{\circ}}$$
$$= 7.07 |-81.87$$

### (b) Maximum power transfer theorem

**Statement** The maximum power transfer theorem states that the maximum power is delivered from a source to a load

when the load resistance is equal to the source resistance.

Current in the circuit  $I = V_S/R_S + R_L$ Power delivered to load  $R_L$  is



$$P_L = I^2 R_L = \frac{V_S^2 R_L}{(R_S + R_L)^2}$$

To determine value of R - L for maximum power to load

$$\frac{dP}{dL} = \frac{d}{dR_L} \left[ \frac{V_S^2 R_L}{(R_S + R_L)^2} \right] = 0$$
$$= \frac{V_S^2 \{ (R_S + R_L)^2 - 2R_L (R_S + R_L) \}}{(R_S + R_L)^4} = 0$$
$$(R_S + R_L)^2 - 2P_L (R_S + R_L) = 0$$
$$R_S + R_L - 2R_L = 0$$
$$R_S = R_L$$

So maximum power is transferred to the load when the load resistance is equal to the source resistance.

7.



When the switch is at Position 2, the current equation is

$$50 = 25i + \frac{1}{1 \times 10^{-3}} \int i \, dt$$

Differentiating both sides

$$0 = 25 \frac{di}{dt} + 10^3 i$$
 (1)

$$(D(25) + 10^{3})i = 0$$
  

$$(D(25) + 10^{3})i = 0$$
  

$$\left(D + \frac{1000}{25}\right)i = 0$$
  

$$(D + 40)i = 0$$
  

$$i = C_{1}e^{-40t}$$

At t = 0, the switch is changed to the position  $2 i(0) = C_1$ At t = 0, the initial current passing through the circuit is the same as the current passing through the circuit and is same as position 1. At t = 0, the switch is at the position 1

$$i = \frac{100}{25} = 4$$
 A  
 $C_1 = 4$ 

Hence

Therefore, current  $i = 4e^{-40t}$ 



When  $V_2 = 0$ ,

$$V_1 = I_1 \frac{(1)}{Y_A + Y_C}$$
  $I_2 = Y_C$ 

$$\begin{array}{ll} \frac{I_1}{V_1} &= Y_A + Y_C = Y_{11} & & \frac{I_2}{V_1} &= Y_C \\ 0, & & & Y_{21} = Y_C \end{array}$$

When  $V_1 = 0$ ,

$$V_2 = I_2 \frac{(1)}{Y_B + Y_C} \qquad I_1 = Y_C V_2$$

$$\frac{I_2}{V_2} = Y_B + Y_C \qquad \qquad \frac{I_1}{V_2} = Y_C \\ Y_{22} = Y_B + Y_C \qquad \qquad Y_{12} = Y_C$$



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When $I_2 = 0$ ,		
	$V_1 = I_1(Z_a + Z_b)$	$V_2 = Z_b I_1,$
	$\frac{V_1}{I_1} = Z_a + Z_b$	$\frac{V_2}{I_1} = Z_b$
when $I = 0$	$Z_{11} = Z_b + Z_b$	$Z_{21} = Z_b$
when $I_1 = 0$ ,	$V_2 = I_2(Z_b + Z_c)$	$V_1 = Z_b I_2$
	$\frac{V_2}{I_2} = Z_b + Z_c$	$\frac{V_1}{I_2} = Z_b$
	$Z_{22} = -Z_b + Zc$	$Z_{12} = Z_b$