Engineering Mechanics

(ME101)

Fifth Edition

WBUT–2015

About the Authors

P K Nag (late) had been associated with the Indian Institute of Technology, Kharagpur, for about 40 years, almost since his graduation. After retiring from IIT, he was an Emeritus Fellow of AICTE, New Delhi, stationed at Jadavpur University, Kolkata, till June 2005. He was Visiting Professor in the Technical University of Nova Scotia (now Dalhousie University), Halifax, Canada, for two years during 1985–86 and 1993–94. He has authored four books, apart from this one, and about 150 research papers in several national and international journals and proceedings. His research areas include circulating fluidized bed boilers, combined-cycle power generation, second-law analysis of thermal systems and waste heat recovery. He was the recipient of the President of India Medal (1995) from the Institution of Engineers (India). He was a Fellow of the National Academy of Engineering (FNAE), Fellow of the Institution of Engineers (India), and a Life Member of the Indian Society for Technical Education, Indian Society for Heat and Mass Transfer, and the Combustion Institute, USA (Indian Section). He was formerly a member of the New York Academy of Sciences, USA. He has published the following works with McGraw Hill Education:

- *Power Plant Engineering, 2/e*
- *Basic and Applied Thermodynamics, 2/e*
- *Heat Transfer and Mass Transfer, 2/e*
- *Engineering Thermodynamics, 4/e*

Sukumar Pati is currently Assistant Professor in the Department of Mechanical Engineering at National Institute of Technology (NIT) Silchar. Prior to joining NIT Silchar, he worked as a Post-Doctoral Fellow for almost one year at the Indian Institute of Technology (IIT) Kharagpur, after completion of his PhD from the Mechanical Engineering Department of the same Institute. He received his BE in Mechanical Engineering in 1999 from the Jalpaiguri Government Engineering College (University of North Bengal) and his ME in Fluid Mechanics in 2001 from Bengal Engineering College (presently, Indian Institute of Engineering Science and Technology, Shibpur). He was awarded the Silver Medal for attaining the second rank in BE in Mechanical Engineering from the University of North Bengal. He joined Haldia Institute of Technology, Haldia, as a Lecturer in the Department of Mechanical Engineering in 2001. He was promoted to the post of Senior Lecturer in 2006 and became Assistant Professor in the Department of Mechanical Engineering in 2007. He is author of the textbook *A Textbook on Fluid Mechanics and Hydraulic Machines* and co-author of the textbook *Engineering Thermodynamics and Fluid Mechanics*, both published by McGraw Hill Education (India). He has published several research papers in international journals of repute. His active areas of interest include evaporation and condensation, micro-scale fluid flow and heat transfer, natural convection and non-Newtonian fluid mechanics.

T K Jana is currently working as Associate Professor and Head, Department of Mechanical Engineering at Haldia Institute of Technology, Haldia. He received his BE in Mechanical Engineering from Bengal Engineering College, Shibpur, Howrah (presently, Indian Institute of Engineering Science and Technology, Shibpur) in 1988 and ME from Jadavpur University, Kolkata, in 1990. He joined Haldia Institute of Technology as Lecturer in the Department of Mechanical Engineering in 2002 and became Assistant Professor in 2005. Before this, Prof. Jana had served the industry under several portfolios. He is a member of Institute of Engineers and is the co-author of the textbook *Engineering Thermodynamics and Fluid Mechanics* published by McGraw Hill Education (India). He has several published papers in national and international journals and conferences to his credit. His active areas of interest include Fluid Power, CNC Machining, and Holonic Manufacturing Systems.

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P K Nag
(Retired) Professor
Department of Mechanical Engineering
Indian Institute of Technology
Kharagpur, West Bengal

Sukumar Pati
Assistant Professor
Department of Mechanical Engineering
National Institute of Technology (NIT)
Silchar, Assam

T K Jana
Associate Professor and Head
Department of Mechanical Engineering
Haldia Institute of Technology
Haldia, West Bengal

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Dedicated to my beloved wife

Samayeeta

And son

Gem

T K Jana

Dedicated to my beloved wife

Munmun

And son

Soumik

Sukumar Pati

Contents

Preface

Engineering Mechanics, one of the oldest branches of physical science, is a subject of immense importance. Although it is taught at an earlier semester of engineering, its foundation is rooted in the two other fundamental subjects—applied mathematics and physics. Generically, Mechanics is a subject that deals with the action of force(s). It is broadly classified under *Statics* and *Dynamics.* While the former deals with the action of forces on the rigid bodies at rest, the latter deals with motion characteristics of the bodies when subjected to force. The present text includes preliminary concepts of Strength of Materials which is basically an extension of Statics wherein the internal behaviours of the materials are analyzed when subjected to load, an area not considered in the purview of Statics. It would not be an exaggeration if Mechanics is envisaged as the ABC of Engineering Education, irrespective of the discipline or specialization. Its wide range of application potential—ranging from nature to modern creations, encapsulating a wide spectrum—rightly justifies its candidature as one of the most proliferated subjects with universal applicability.

The main objective of writing this book is fundamental-concept building combined with strong analytical and problem-solving abilities that would form the backbone of many other subjects in higher semesters. It is, therefore, extremely important to emphasize on the basic issues to develop a solid foundation, particularly at the early-stage/ years of engineering education.

The present text is an endeavour towards this goal by offering the students a textbook on a first course in Mechanics in a most comprehensive manner. Plenty of textbooks on this subject are available in the market. However, these books include several advanced topics, which sometimes become too difficult and confusing for students to understand. Further, in most of the cases, Strength of Materials is normally covered by a separate text. Interestingly, the West Bengal University of Technology (WBUT) has modified the course curriculum and **Engineering Mechanics** is considered at the very first semester that includes mechanics and basics of strength of materials. We, therefore, felt the need to write a textbook that would exclusively conform to the syllabus of WBUT.

The West Bengal University of Technology has emerged as a promising technological university with over 90 odd affiliated institutions to its credit, poised to foster technical education in the state of West Bengal. These institutions follow a common syllabus in the first year, irrespective of specialization; and introductory mechanics is taught at the very first semester. This creates difficulty for the beginners since they need to go through several books to extract what would exactly serve their purpose.

The present approach is to provide the students a textbook that would guarantee concept building and equip them with problem-solving ability. The present text will be highly helpful to understand the subject in its true spirit.

This book is primarily meant for the first-semester students of all disciplines pursuing BTech. under WBUT. However, the national scenario shows that similar types of courses are also offered by several institutions and universities in other states as well. Thus, the present book is not only confined to the students of WBUT; rather its appeal is generic. Further, students pursuing diploma courses can also benefit from this text. Moreover, the topics discussed and associated problems and questions framed would be extremely helpful in the preparation for various competitive examinations.

Salient Features

- Complete coverage of the WBUT syllabus (2010 Regulation)
- Lucid explanation of Kinetics and Kinematics of Rigid Bodies
- Concepts of Vector Algebra have been amply supported by illustrations and vector diagrams
- Practice questions and solved examples provided at the end of each chapter
- Model Question Papers based on WBUT pattern
- Addition of Solved 2013 and 2014 Question Papers
- Pedagogy includes
  - o *315 Illustrations*
  - o *135 Multiple-choice questions*
  - o *140 Short answer type questions*
  - o *225 Solved examples*
  - o *145 Numerical problems*

Chapter Organisation of the Book

The entire content is divided strictly as per the syllabus of WBUT. **Chapter 1** highlights the scope and application potential of the subject, its purview, history, basic idealisation of the situation and the units. The concept of idealisation is very important without which the entire analysis would be in vain.

Chapter 2 discusses fundamentals of vector algebra. Although vector algebra comes under the purview of Applied Mathematics, it is mandatory to recapitulate the same since there is a growing need to follow the vector approach. It is the essence and a prerequisite in the study of Mechanics. Interestingly, the various vector operations are explained in the context of Mechanics to justify their relevance.

In the subsequent chapters, the basic approach is fully explored to study force analysis in two-dimensional (**Chapter 3**) coordinate frames. In addition to the vector approach, scalar approach has been adopted too, so that students can trade off between the two. Clear elucidation is provided to distinguish equilibrium of particles from rigid bodies in **Chapter 4**. The study of Mechanics in majority of the cases is based on the premise of condition of equilibrium.

Chapters 5, **6**, along with **Chapter 7**, discuss the major topics of Statics—Centre of Gravity, Friction and Moment of Inertia. Additional emphasis is given to them with quite a good number of solved problems and almost an equal number of problems provided as an exercise. Moreover, one particular problem is solved by several methods. This is one of the key features of the text.

Utmost care has been taken in other areas, namely, Dynamics and Strength of Materials in **Chapters 8 to 14**. Since the coverage of Strength of Materials is confined only to basic definitions of Stress and Strain and their associated properties, these are rightly described. Moreover, Stress–Strain diagram for ductile material is meticulously investigated and results or trends are illustrated in the right perspective. However, quite a good number of problems have been solved involving basic Stress–Strain.

Similarly, emphasis is laid on Newton's Law of Motions and D'Alembert's Principle in **Chapters 11 and 12**.

Projectiles are covered in **Chapter 13**. And finally, work, power and energy are discussed in **Chapter 14**.

At the end of each chapter, adequate questions (both subjective and multiple-choice questions) along with sufficient numerical problems are provided to accustom students with different question patterns and enable them to practice the same in order to grasp the subject in the right spirit. The answers to multiple-choice questions and numerical problems are provided immediately after the end of chapters.

Acknowledgements

In this journey, we are grateful to many of our faculty colleagues, namely, Prof. Sushanta Banerjee and Dr Bikash Bepari of Haldia Institute of Technology, who have contributed significantly by providing valuable suggestions and expertise on different occasions while preparing the manuscript. We thankfully acknowledge the service provided by two of our very sincere ex-students of Mechanical Engineering—Nandan Saha and Saptarshi Saha. They have prepared most of the drawings and the realisation of the present work would not have been possible without them. The patience and endurance shown by our family members during the manuscript-preparation stage is gratefully acknowledged. We extend our gratitude to the editorial team of McGraw Hill Education (India) for taking praiseworthy initiatives to publish this text. We would also like to convey our heartfelt thanks and regards to the following reviewers who assessed various chapters of the script and contributed with valuable suggestions for improvement.

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Last, but not the least, we thank our revered teachers who have instilled into us the values of education for the well being of the mankind.

Feedback

Despite all our efforts, it is possible that some unintentional errors have crept in the text. We would be extremely thankful to the readers for their constructive suggestions and criticism with a view to enhance the utility of the book. Readers can write to us at *jana_tarun@indiatimes.com* and *sukumarpati@rediffmail.com*

P K Nag
Sukumar Pati
T K Jana

Publisher's Note

Remember to write to us. We look forward to receiving your feedback, comments and ideas to enhance the quality of this book. You can reach us at *info.india@mheducation.com*. Kindly mention the title and the author's name as the subject. In case you spot piracy of this book, please do let us know.

Roadmap to the Success

ENGINEERING MECHANICS

(ME101)

Module I

Importance of mechanics in engineering; Introduction to statics; Concept of particle and rigid body; Types of forces: collinear, concurrent, parallel, concentrated, distributed; Vector and scalar quantities; Force is a vector; Transmissibility of a force (sliding vector)

Introduction to vector algebra; Parallelogram law; Addition and subtraction of vectors; Lami's theorem; Free vector; Bound vector; Representation of forces in terms of *i*, *j*, *k*; Cross product and dot product and their applications

Two-dimensional force system; Resolution of forces; Moment; Varignon's theorem; Couple; Resolution of a coplanar force by its equivalent force-couple system; Resultant of forces

GO TO

CHAPTER 1—INTRODUCTION
CHAPTER 2—FUNDAMENTALS OF VECTOR ALGEBRA
CHAPTER 3—TWO-DIMENSIONAL FORCE SYSTEMS

Module II

Concept and equilibrium of forces in two dimensions; Free-body concept and diagram; Equations of equilibrium

Concept of friction; Laws of Coulomb friction; Angle of repose; Coefficient of friction

GO TO

CHAPTER 4—EQUILIBRIUM OF RIGID BODIES
CHAPTER 6—FRICTION

Module III

Distributed force; Centroid and centre of gravity; Centroids of a triangle, circular sector, quadrilateral, composite areas consisting of above figures

Moments of inertia; MI of plane figure with respect to an axis in its plane, MI of a plane figure with respect to an axis perpendicular to the plane of the figure; Parallel axis theorem; Mass moment of inertia of symmetrical bodies, e.g., cylinder, sphere, cone

Concept of simple stresses and strains: normal stress, shear stress, bearing stress, normal strain, shearing strain; Hooke's law; Poisson's ratio; Stress-strain diagram of ductile and brittle materials; Elastic limit; Ultimate stress; Yielding; Modulus of elasticity; Factor of safety

GO TO

CHAPTER 5—CENTRE OF GRAVITY
CHAPTER 7—MOMENT OF INERTIA
CHAPTER 8—ELASTICITY—CONCEPT OF SIMPLE STRESSES AND STRAIN
CHAPTER 9—STRESS—STRAIN DIAGRAM

Module IV

Introduction to dynamics: kinematics and kinetics; Newton's laws of motion; Law of gravitation and acceleration due to gravity; Rectilinear motion of particles; Determination of position, velocity and acceleration under uniform and non-uniformly accelerated rectilinear motion; Construction of x-t, v-t and a-t graphs

Plane curvilinear motion of particles: rectangular components (projectile motion); Normal and tangential components (circular motion)

GO TO

CHAPTER 10—RECTILINEAR MOTION OF A PARTICLE
CHAPTER 12—CURVILINEAR MOTION OF A PARTICLE
CHAPTER 13—PROJECTILE

Module V

Kinetics of particles: Newton's second law; Equation of motion; D'Alembert's principle and free-body diagram; Principle of work and energy; Principle of conservation of energy; Power and efficiency

GO TO

CHAPTER 11—KINETICS OF A PARTICLE IN A RECTILINEAR MOTION
CHAPTER 14—WORK, POWER, ENERGY

Part A

Statics

CHAPTER 1

Introduction

1.1 WHY DO WE STUDY MECHANICS?

If we turn around our eyes to the several creations of civilizations like that of Pyramid of Egypt to Eiffel Tower of Paris; Tajmahal of Agra to Sydney Harbor; Howrah Bridge to Tirupati Temple, we would hardly find any item that is beyond the purview of mechanics. Astonishing! Let us take few more examples. Consider tables and chairs of our drawing room; a building and a factory shed; an automobile and an aircraft; a bicycle and a three wheeler; a bridge and a TV or mobile tower; a lock gate and a dam; a robot and a human being; a column and a jack and so on and so forth. All these aforesaid applications have had their own intended objectives; still these have commonalities from the mechanics point of view. Although these are only a few, there is no dearth of such examples. It is therefore quite interesting and imperative to study this subject which has immense application potentials and form the backbone of engineering study irrespective of specialization. The authenticity of such credentials can be verified as the readers will navigate through various topics of the text. It would not be an exaggeration to say that even people from all walks of life ought to study a little bit of mechanics which is so versatile and has got immense potential to its credit.

1.2 WHAT IS MECHANICS?

It is one of the oldest branches of physical science and perhaps the most powerful that deals with the action of forces on bodies that are either in rest or in motion.

The subject has proliferated to a great extent to emerge as one of the most prominent and fundamental engineering field with wide diversifications. Essentially, mechanics is classified into *Solid Mechanics* and *Fluid Mechanics.*

Solid mechanics has got two broad domains − namely, *Statics* and *Dynamics*.

While *Statics* deals with bodies under rest, *Dynamics* deals with bodies under motion. Statics is further classified as *rigid body mechanics* and *deformable body mechanics*. While the former deals with physical bodies that do not manifest any changes in its dimension when subjected to external forces, the latter deals with bodies that exhibit some kind of deformation under load. Dynamics is also categorised as *kinematics* and *kinetics*. Kinematics involves various attributes of motion of a particle like position, velocity and acceleration but without regard to the force that causes motion. It is kinetics where forces find place for necessary analysis.

Like solid mechanics, fluid mechanics also has got two wings titled *fluid statics* and *fluid dynamics* depending on whether fluid is at rest or in motion. Similar classification of fluid dynamics is also made as *fluid kinematics* and *fluid kinetics*. Further classification of fluid dynamics stems from the fact whether fluid is viscous or not and hence it falls under the category of *viscous fluid flow* or *non-viscous fluid flow*. Another approach to classify fluid flow resulted from the consideration of whether the fluid density remains unchanged or not and hence it is categorised as *incompressible* or *compressible fluid flow* respectively.

Few other emerging areas of mechanics that have come into existence and got adequate attention in recent years are *fracture mechanics*, *tribology*, *computational fluid dynamics*, etc.

In this book, the focus of discussion is solely confined to solid mechanics and hence any further attempt to deal with fluid mechanics is beyond the scope of this text.

1.3 FUNDAMENTAL IDEALISATION

Whenever it is required to analyse any physical system (such as a structure, an automobile, a power plant), it is imperative to develop a model that would describe the behaviour of the system. This model may be of any type that is suitable for this purpose (a graphical model, a mathematical model, a physical model, etc). The real life situations are so complex that it is almost impossible to model it keeping its each and every aspect identical to that of the actual situation. Further, the actual predictability of behaviour of such systems is too low. Under these circumstances, it is prudent to simplify the scheme while modeling such systems for the purpose of analysis with a pledge that such simplification or idealisation would not lead to inclusion of error in the result of analysis so as to cross reasonable accuracy limits. Nevertheless, such an idealisation would reduce the effort, make it amenable for analysis and form the foundation of design. This idealisation or simplification of the physical systems which essentially comprises several members (rigid bodies) that are constrained suitably and are under the action of external loads and reactions from supports, and maintain equilibrium is the fundamental premise of study of mechanics.

1.3.1 Continuum

A definite quantity of matter can be decomposed to several small elements that can be further divided to atoms and molecules. The behavior of these individual entities varies widely from each other and it is too complicated a situation to measure their various attributes individually. It is the aggregation of all such properties that is manifested by the body as a whole. It is therefore worthwhile to consider the average of different attributes of these entities to represent the gross behavior of the body. This eventually leads to the consideration that the *matter of body is the collection of uniformly distributed identical elements.* Such description of matter is called *continuum.*

1.3.2 Space

Space is a geometric region in three-dimension that extends in all directions and is occupied by bodies. The position of a solid body is described in space by a three-dimensional coordinate system.

1.3.3 Time

Time is the measure of duration between successive events. The unit of time is *second* which is a fraction of the period by the earth's rotation, i.e., $\frac{1}{86400}$ of an average solar day.

1.3.4 Rigid Body

Statics, as defined earlier, deals with *analysis of force* on *rigid bodies* under *rest*. A rigid body is a hypothetical consideration which implies that it will not undergo any deformations under load. It may be a continuous member or collection of several members. A finite body when subjected to external forces must undergo some form of deformation, however small it may be. This deformation is accompanied by stress induced in the body which is manifestation of resistance offered by the body as a consequence of applied load. The amount of deformation and hence the stress induced are the properties of materials that characterise its strength. Such behavior, although a reality, is complex to analyse. However, there are ample situations where the deformation pattern and the amount are too small to affect the results. This led to simplification of the analysis, assuming that bodies do not deform under load. These are what is called rigid bodies and form the basis of statics. Such consideration should only be made without much loss of accuracies. Nevertheless, it is the mechanics of deformable bodies where the strength of the materials are dealt with.

1.3.5 Particle

A body of negligible dimensions but having a definite mass concentrated at a point is called *particle*. Strictly speaking, such a consideration is absurd in the sense that the mass of a body is not concentrated at a point; rather it is distributed over the entire space it occupies and hence definite mass of matter must occupy a finite space. Mathematically, above definition can be considered as when the volume approaches zero, entire mass is concentrated at a point. But when the size of bodies is so small compared to its range, such consideration would lead to simplification of the problem without any gross error and the body is considered as a particle. For example, a bullet or a piece of stone is regarded as a particle following the above logic.

Just like entire statics is based on the analysis of force on rigid bodies, the study of dynamics deals with the particle.

1.4 FORCE

It is an external agency that causes disturbances to the existing status of a body. In other words, it is an action on the body that tends to change the state of rest or the motion of the body. It is a vector quantity and it is completely defined by its ***magnitude, direction*** and ***point of application.***

Sir Isaac Newton developed the fundamental laws of mechanics for the motion of a particle. The concepts of space, time, and mass are absolute, independent of each other in Newtonian Mechanics.

It is the ***Newtonian Mechanics*** that is the cynosure of present study.

It is to be noted that

- Newtonian Mechanics deals with Forces
- Hamiltonian Mechanics deals with Impulse and Momentum
- Langrangian Mechanics deals with Energy

1.4.1 Newton's 3 Fundamental Laws

Newton's First Law states that a particle either remains at rest or continues to move in a straight line with a constant velocity provided that there is no unbalanced force acting on it (resultant force = 0).

Newton's Second Law states that the acceleration of a particle is proportional to the resultant force acting on it and its direction is in the direction of this force.

Thus mathematically

$$F = ma \tag{1.1}$$

Here F is the force that causes an acceleration of a to the body of mass m.

Newton's Third Law states that the forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and act along the same line of action (collinear).

All the three laws in combination provide the fundamental consideration of statics as well as dynamics. More precisely, Newton's first law provides the basis for statics where as it is the second law that governs the dynamics.

Critical investigation to the second law reveals that first law has been derived from the second law. In the absence of any unbalanced force ($F = 0$), the particle will maintain its status quo of motion ($ma = 0$) or it will maintain its status of rest. Thus when there is no external unbalanced force, the body will be in equilibrium.

1.4.2 Newton's Law of Gravitation

Newton's Law of gravitation states that

$$F = G\frac{Mm}{r^2} \tag{1.2}$$

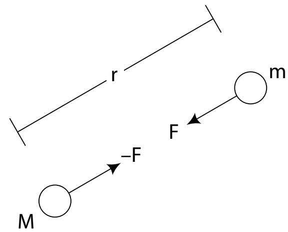
where, F = mutual force of attraction between two particles

G = universal constant known as the constant of gravitation

M, m = masses of the two particles

r = distance between the two particles

If we introduce the constant



$$g = \frac{GM}{r^2} \tag{1.3}$$

and let M = mass of earth

m = mass of a particle

r = radius of earth

g = acceleration of gravity at earth's surface

then from Eq. (1.3) we get

$$G = \frac{gr^2}{M}$$

Substituting in Eq. (1.2), we get $F = \dfrac{gr^2}{M}\left(\dfrac{Mm}{r^2}\right) \Rightarrow F = mg$

$$F = mg$$

Using $F = ma$, at the surface of the earth, we get $a = g$

$$F = mg$$

$$W = mg \tag{1.4}$$

g is dependent upon r.

1.5 DIMENSIONS AND UNITS

A dimension is a physical variable used to specify some characteristic of a system. Mass, length and time are examples of dimension. A unit is a particular amount of a physical quantity. For example, time can be measured in second minute, hour, etc.

The dimension mass (M), length (L) and time (T) are considered to be basic dimensions, from which other dimensions are derived. For example, the dimension of velocity is LT^{-1}, and acceleration is LT^{-2}.

In mentioning a unit, it is recommended that small letter (lower case) be used when abbreviated, and small letters when expanded. For example, the unit of time is s or second, of length and of m or metre. But when a unit is named after a person, capital letter is used when abbreviated, for example, Newton (N), Pascal (Pa), Watt (W). Multiples in powers of 10 are indicated by prefixes, which are also abbreviated. It is internationally accepted to use the prefixes given in Table 1.1.

Dimensions are those terms which are used to characterise physical qualities. Common examples of dimensions include mass M, length L, time t, temperature T, force F, etc. The reason why dimensions are

Table 1.1 *SI Unit prefixes*

Multiple	*Prefix*	*Abbreviation*	*Multiple*	*Prefix*	*Abbreviation*
10	deca	da	10^{-1}	deci	d
10^2	hecto	h	10^{-2}	centi	c
10^3	kilo	k	10^{-3}	milli	m
10^6	mega	M	10^{-6}	micro	μ
10^9	giga	G	10^{-9}	nano	n
10^{12}	tera	T	10^{-12}	pico	p
10^{15}	peta	P	10^{-15}	femto	f
10^{18}	exa	E	10^{-18}	atto	a

important in engineering analysis is that any equation which relates physical quantities must be dimensionally homogeneous. By dimensionally homogeneous we mean that the dimensions of terms on one side of an equation equal to those on the other side. Equations relating physical quantities which do not fulfill the condition of dimensional homogeneity are not correct.

In order to make numerical computation with equations involving physical quantities, there is the additional requirement that units, as well as the dimensions, be homogeneous. Units are those arbitrary magnitudes and names assigned to dimensions which are adopted as standards for measurements. For example, the primary dimension of time may be measured in unit of second, minute, hour, etc.

1.5.1 Base Units

The basic units of mass, length, time and temperature in the SI units are described below:

i) Mass The basic SI unit for mass is kilogram (abbreviated as kg). A standard alloy block of platinum and iridium maintained at the International Bureau of Weights and Measures at Sevres, Paris, is taken as the base unit of mass.

ii) Length The basic SI unit for length is metre (abbreviated as m). The distance between two arks on platinum - iridium bar, kept at the International Bureau of Weights and Measures at Sevres, Paris, France, when measured at 0°C, is taken as the base unit of length.

iii) Time The basic SI unit for time is the second (abbreviated as s). For many years, the accepted basic unit second was defined as $\frac{1}{86400}$ of the mean solar day.

iv) Temperature The basic SI unit for temperature is the Kelvin (abbreviated as K). In 1967, the Thirteenth Conference General des Poids et Measures defined Kelvin as the fraction $\frac{1}{273.16}$ of the thermodynamic temperature of the triple point of water.

1.5.2 Derived Units

The secondary quantities are expressed in term of the derived units, which in turn are formed from the base units. The relation between the derived units and the base units depends on a definition or a law. For example, velocity is defined as $v = \frac{dL}{dt}$; where L is length and t is time. The units of velocity is m/s. The units of force can be derived from the equation $F = ma$ (Newton's second law) as kg-m/s^2. In honour of Newton, the unit of force has been named Newton and is abbreviated as N. The dimensions and units of some of the physical quantities commonly used in thermodynamics are given in Table 1.2.

Table 1.2 *Dimensions and Units of some physical quantities*

Quantity	*Dimensions*	*Units*	*Abbreviation*
Mass	M	Kilogram	Kg
Length	L	Metre	m
Time	T	Second	s
Temperature	θ	Kelvin	K
Velocity	LT^{-1}	m/s	---
Acceleration	LT^{-2}	m/s^2	---
Force	MLT^{-2}	kg m/s^2 (N)	Newton
Pressure	$ML^{-1}T^{-2}$	kg/ms^2 (N/m^2)	Pascal
Energy	ML^2T^{-2}	kg m^2/s^2 (N-m)	joule
Power	ML^2T^{-3}	kg m^2/s^3 (J/s)	watt

SHORT ANSWER TYPE QUESTIONS

1.1 What is mechanics? What is its purview?
1.2 Give few engineering applications that are based on principles of mechanics.
1.3 What is meant by idealisation? Why it is essential in the study of mechanics?
1.4 Distinguish between a *rigid body* and a *particle*.
1.5 What is Force? Is it scalar or vector? What is its unit?

CHAPTER

2

Fundamentals of Vector Algebra

2.1 INTRODUCTION

The various attributes that are encapsulated under the purview of mechanics are broadly classified into two categories – ***scalars*** and ***vectors***. While scalar quantities are represented only by their *magnitude*, the vector quantities require both the *magnitude and direction* to define it completely. *Time, mass, volume, density, energy*, etc., are some of the prominent scalar quantities that appear frequently in the study of mechanics. On the other hand, *force, moment, displacement, velocity, acceleration*, etc., fall under the category of vector quantities.

From the earlier discussion, it is understood that it is the *analysis of force(s)* that form the *nucleus of study of mechanics*. It is therefore of paramount important to discuss and review the fundamentals of vector algebra to understand the mechanics in full vigor.

2.2 TYPES OF VECTOR

A vector quantity is represented by an arrow, where its length represents the magnitude and the head of the arrow indicates its direction. A vector F is shown in Fig. 2.1 (a). *It is conventionally appears in bold face like* ***F***. *The magnitude is represented by* $|F|$.

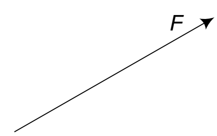


Figure 2.1 (a)

Types of Vectors

(1) *Fixed (or bound) vector* – A vector for which a unique point of application is specified and thus cannot be moved unless the conditions of the problem are modified.

(2) *Free vector* – A vector whose action is not confined to or associated with a unique line in space. This implies that a free vector can be moved anywhere in space so long its magnitude and direction remains unaltered for example, displacement, couple.

(3) *Sliding vector* – A vector for which a unique line in space (line of action) must be maintained. Thus without disturbing its line of action, it can be shifted to any point following the *principle of transmissibility*. **Force** applied on rigid bodies can be regarded as a **sliding vector**.
Three different types of vector are shown in Fig. 2.1 (b).

For two vectors to be equal, they must have same magnitude and direction.
However, their point of application need not necessarily be the same. Figure 2.2 (a) shows two different points A and B where two vectors are applied having same magnitude P and they are parallel, i.e., their directions are same. These two vectors are equal.

A negative vector of a given vector has same magnitude but opposite direction as shown in Fig. 2.2 (b). $\boldsymbol{P}$ and $-\boldsymbol{P}$ are equal and opposite such that $\boldsymbol{P} + (-\boldsymbol{P}) = 0$

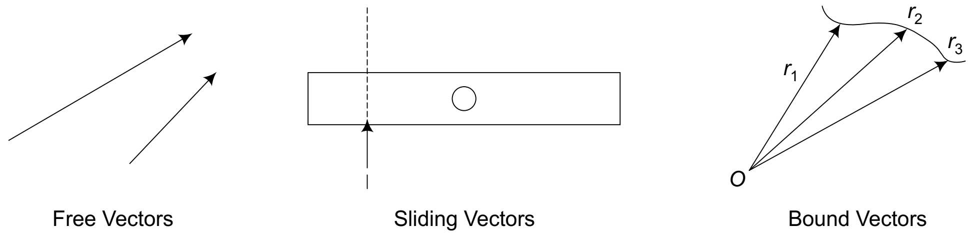


Figure 2.1 (b)

(4) *Unit Vector* – A vector having a magnitude of one unit is called unit vector. If ***P*** is a vector, then its unit vector along ***P*** is $P = \dfrac{P}{|P|}$

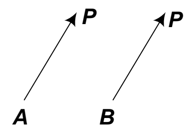


Figure 2.2 (a)

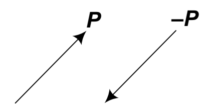


Figure 2.2 (b)

2.3 VECTOR OPERATIONS

2.3.1 Product of a Vector by a Scalar

Product of a scalar and a vector is a vector in the same direction but its magnitude is magnified by a factor equal to that of the scalar.

Example: If ***P*** is a vector, then n***P*** is also a vector, which is n times that of ***P***. Here n is a scalar. Thus

$(+n)$***P*** = vector having same direction as ***P***, but n times long as that of *P*.

$(-n)$***P*** = vector opposite to the direction of ***P***, but n times long as that of *P*.

2.3.2 Vector Addition

If ***P*** and ***Q*** are two vectors, then the vector addition would yield another vector ***R*** such that ***R*** = ***P*** + ***Q***. Nevertheless, this addition is not similar to that of addition of two scalars.

2.3.2.1 Geometrical Significance of Vector Addition

The vector addition can be interpreted as if two vectors ***P*** and **Q** originated from a common point, such that they represent the two adjacent arms of a parallelogram, then the diagonal of the parallelogram having ***P*** and ***Q*** as its adjacent sides will represent the resultant of ***P*** and ***Q*** as shown in Fig. 2.3. This is known as ***Parallelogram Law***.

Thus, resultant ***R*** = ***P*** + ***Q***

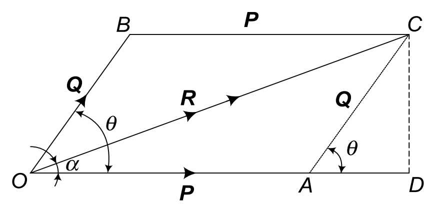


Figure 2.3 *Parallelogram law*

> **Note:** The magnitude of ***P*** + ***Q*** is not usually equal to $|P| + |Q|$.

From the parallelogram *OACB*, two adjacent sides *OA* and *OB* represent the vectors ***P*** and ***Q*** respectively. Thus diagonal *OC* represents the resultant ***R*** = ***P*** + ***Q***.

Since, *OB* and *AC* are equal in magnitude and parallel to each other, *AC* can also represent ***Q***, since it is a free vector. Thus considering the triangle *OAC* as shown in the Fig. 2.3, three arms of the triangle represents ***P***, ***Q*** and ***R***.

Thus vector addition has got a different geometrical interpretation that if the two sides of a triangle can represent two basic vectors in succession such that tail of one vector (***Q***) coincides with the arrow head of another vector (***P***), then the closing side of the triangle (OC) having O as its tail end and C as its arrow end, will represent the resultant (***R***) of ***P*** and ***Q*** as shown in the above Fig. 2.3. This is popularly known as ***Triangle Law***.

Similar logic also holds true if triangle OBC is considered, instead. In this case, $\boldsymbol{R} = \boldsymbol{Q} + \boldsymbol{P}$.

This leads to a very important conclusion that "***addition of vectors is commutative.***"

i.e. $\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{Q} + \boldsymbol{P}$

The magnitude of the resultant ***R*** can be computed geometrically.

Refer to Fig. 2.3. Draw a perpendicular from C so as to meet OA at D.

Thus from the right-angled triangle OCD, $OC^2 = OD^2 + CD^2$

or, $$OC^2 = (OA + AD)^2 + CD^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$\therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

Inclination of the resultant ***R*** with ***P*** can be computed by the relationship $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$.

Polygon Rule This rule is meant for the addition of more than two vectors. To implement this rule, initially two vectors are summed first according to parallelogram law or triangle law, and the resultant so obtained is added to the third vector to get the final resultant. Thus any number of vectors can be added successively to get the final resultant. This is called ***composition of vectors***. Needless to say, such addition is not limited to any specific numbers of vectors.

Let's add three vectors, namely ***P***, ***Q*** and ***S***. First $\boldsymbol{R_1}$ is obtained such that $\boldsymbol{R_1} = \boldsymbol{P} + \boldsymbol{Q}$. Now considering $\boldsymbol{R_1}$ and ***S***, finally ***R*** is obtained such that $\boldsymbol{R} = \boldsymbol{R_1} + \boldsymbol{S} = \boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{S}$. This sequence of operation is presented in Fig. 2.4 (a), (b) and (c) following Parallelogram Law.

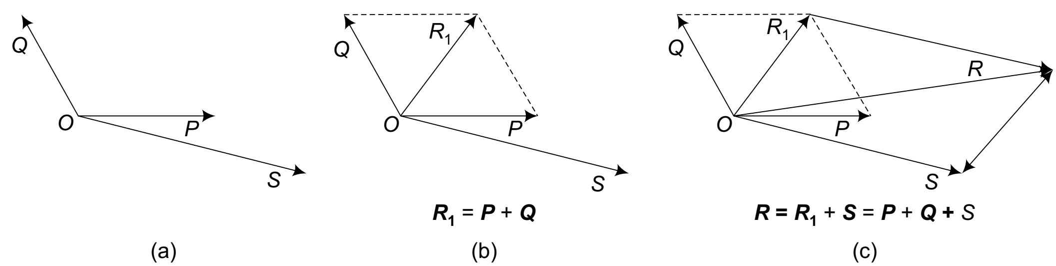


Figure 2.4 *Composition of vectors by Parallelogram law*

Same resultant R can be obtained by following triangle law as shown in Fig. 2.5 from (a) through (c).

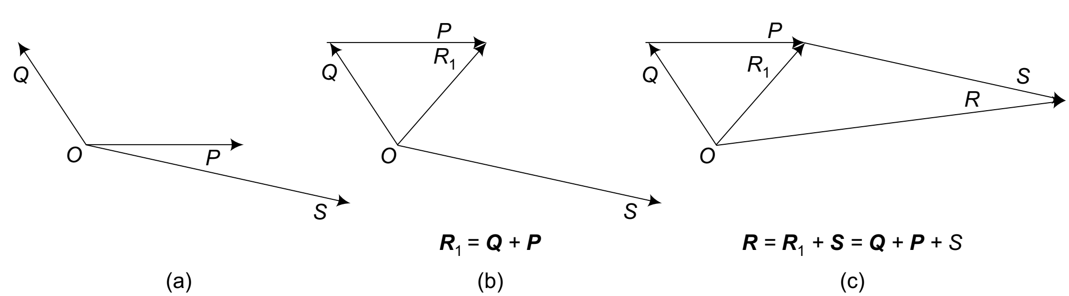


Figure 2.5 *Composition of vectors by Triangle law*

Thus polygon rule is essentially ***successive applications of triangle rule as evident from*** Fig. 2.6.

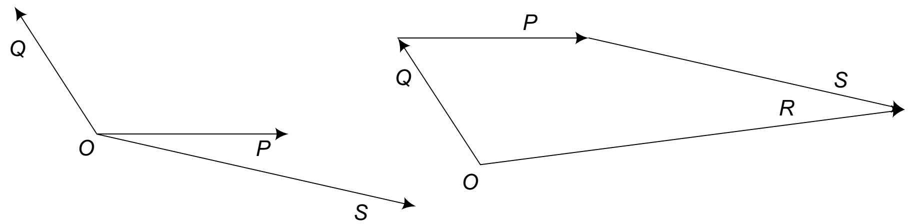


Figure 2.6 *Successive applications of Triangle law*

It is noteworthy to see that in lieu of $\boldsymbol{P}$ and $\boldsymbol{Q}$, $\boldsymbol{Q}$ and $\boldsymbol{S}$ could have been taken first to get their resultant, which subsequently added with $\boldsymbol{P}$ would have given the final resultant, which is essentially identical with the former.

This leads to another important conclusion that ***"addition of vectors is associative"***.

Thus, $\boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{S} = (\boldsymbol{P} + \boldsymbol{Q}) + \boldsymbol{S} = \boldsymbol{P} + (\boldsymbol{Q} + \boldsymbol{S})$.

2.3.3 Vector Subtraction

The vector subtraction can be envisaged as the addition of the corresponding negative vector. Thus $\boldsymbol{P}$ can be added to $(-\boldsymbol{Q})$ to obtain $\boldsymbol{P} - \boldsymbol{Q}$. From the ΔOAB, $OB + BA = OA$. This is shown in Fig. 2.7.

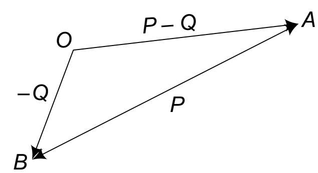


Figure 2.7 *Vector subtraction*

2.3.4 Resolution of Vector into Components

A single vector can be decomposed to two or more vectors. These vectors are components of the original vector. Finding these is called resolving the vector into its components. This is just opposite to that of composition of vectors.

There are mainly two methods to accomplish this.

Case I: ***When one of the two components, P is known***: The second component $\boldsymbol{Q}$ is obtained using the triangle rule. Refer to Fig. 2.8; join the tip of $\boldsymbol{P}$ to the tip of $\boldsymbol{R}$. The magnitude and direction of $\boldsymbol{Q}$ are determined graphically or by trigonometry.

By using properties of triangle, we have

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}.$$

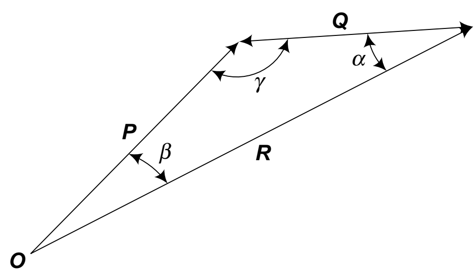


Figure 2.8 *Resolution of vector*

Case II: ***When the lines of action of two components are known:*** The force $\boldsymbol{R}$ can be resolved into two components $\boldsymbol{P}$ and $\boldsymbol{Q}$ provided their line of actions are known. Let us consider $\boldsymbol{a}$ and $\boldsymbol{b}$ are two lines, originated from point O, so as to represent the line of actions of $\boldsymbol{P}$ and $\boldsymbol{Q}$ respectively, as shown in the Fig. 2.9.

From the head of the $\boldsymbol{R}$, draw a line parallel to **b** so as to intersect the line $O\boldsymbol{a}$ at $\boldsymbol{A}$. Similarly, draw another line parallel to **a** so that it meets the line Ob at $\boldsymbol{B}$. Now OA and OB will represent the magnitudes of $\boldsymbol{P}$ and $\boldsymbol{Q}$ respectively.

However, it is quite useful to resolve a vector into two components having included angle 90°. Such components are called *rectangular components or orthogonal components as shown in* Fig. 2.10.

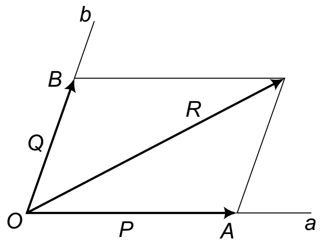


Figure 2.9

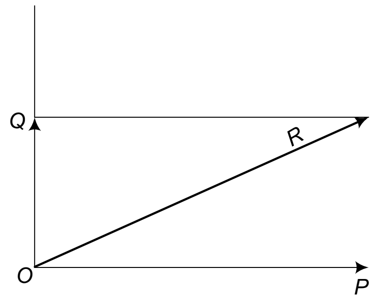


Figure 2.10

2.4 ORTHOGONAL TRIAD OF UNIT VECTORS

A vector ***P*** can be expressed as product of its magnitude $|P|$ and a unit vector **n**, having its direction same as that of ***P***.

Thus ***P*** = $|P|$***n*.** This concept of unit vector can be extended to three-dimensional coordinate system to solve many problems of mechanics.

Let us consider a three-dimensional coordinate system as shown in Fig. 2.11. Three unit vectors ***i***, ***j*** and ***k*** are considered to act along x, y and z axis respectively, called *orthogonal triad* of unit vectors.

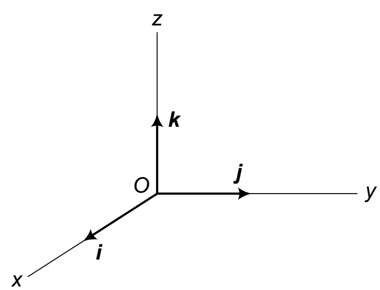


Figure 2.11 *Orthogonal triad of unit vectors*

Therefore, a vector ***P*** can be expressed in terms of the orthogonal triad as $P = P_x i + P_y j + P_z k$, where $P_x i$, $P_y j$ and $P_z k$ are the three vector components of ***P*** as shown in Fig. 2.12.

Note that P_x, P_y and P_x represents the magnitude of the three component vectors, such that $P_x = P \cos \theta_x$, $P_y = P \cos \theta_y$, $P_z = P \cos \theta_z$, where $\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1$.

The magnitude of P can be expressed as $P = \sqrt{P_x^2 + P_y^2 + P_z^2}$.

The three mutually perpendicular scalar components of a vector can also be established by methods of projection as shown in the Fig. 2.13.

The vector ***P*** can have two projections – the vertical one denoted as P_z that aligns with z axis and its projection on xy plane denoted as P_{xy}.

Thus mathematically $P_{xy} = P \cos \theta_1$ and $P_z = P \sin \theta_1$.

Further P_{xy} can now be resolved into two components, namely, P_x and P_y such that $P_x = P_{xy} \cos \theta_2 = P \cos \theta_1 \cos \theta_2$ and $P_y = P_{xy} \sin \theta_2 = P \cos \theta_1 \sin \theta_2$.

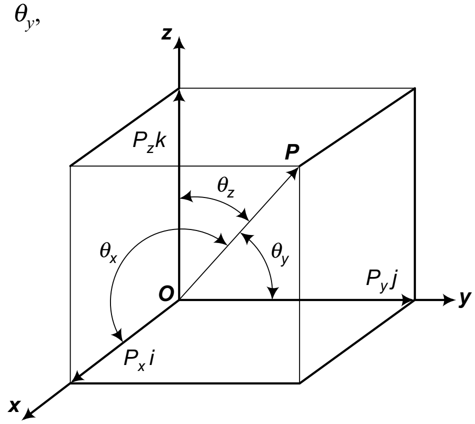


Figure 2.12 *Scalar components of a vector*

A vector is often represented by a line AB that passes through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as shown in Fig. 2.14.

Thus a vector ***P*** can be expressed as

$$\boldsymbol{P} = \boldsymbol{P}n = P\frac{(x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

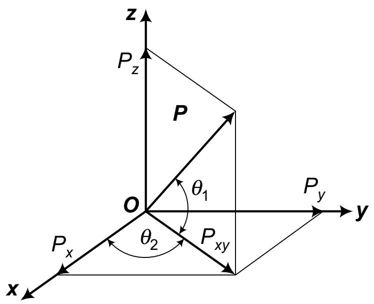


Figure 2.13 *Method of projection*

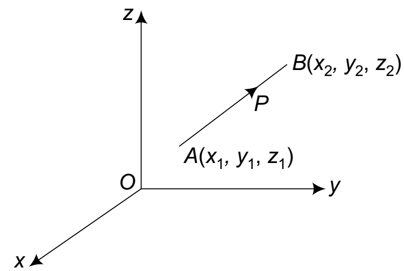


Figure 2.14

This expression is quite useful for solving problems.

Comparing with the previous expression, the inclination of the vector $\boldsymbol{P}$ with three mutually perpendicular axes [Refer Fig. 2.12] can be calculated as

$$\cos\theta_x = \frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$\cos\theta_y = \frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$\cos\theta_z = \frac{(z_2 - z_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

2.5 POSITION VECTOR

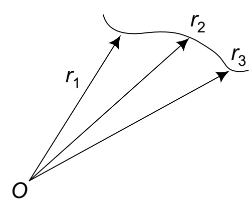


Figure 2.15

As discussed earlier, displacement, which is fundamentally the position of a point w. r. t. a coordinate frame, is a vector. Thus position vector $\boldsymbol{r}$ [Refer Fig. 2.15] of any point $A(x, y, z)$ can also be described as $r = xi + yj + zk$.

2.6 IDENTITY OF VECTORS

Two vectors $P = P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k}$ and $Q = Q_x\boldsymbol{i} + Q_y\boldsymbol{j} + Q_z\boldsymbol{k}$ are said to be identical provided their magnitude and direction are both same [Refer 2.1]. It therefore follows that $P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k} = Q_x\boldsymbol{i} + Q_y\boldsymbol{j} + Q_z\boldsymbol{k}$, which eventually leads to

$$P_x = Q_x;\ P_y = Q_y;\ P_z = Q_z.$$

Let $\boldsymbol{P} = (P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k}),\ \boldsymbol{Q} = (Q_x\boldsymbol{i} + Q_y\boldsymbol{j} + Q_z\boldsymbol{k})$

Then by definition $\boldsymbol{P} + \boldsymbol{Q} = (P_x + Q_x)\boldsymbol{i} + (P_y + Q_y)\boldsymbol{j} + (P_z + Q_z)\boldsymbol{k}$

Similarly, $\boldsymbol{P} - \boldsymbol{Q} = (P_x - Q_x)\boldsymbol{i} + (P_y - Q_y)\boldsymbol{j} + (P_z - Q_z)\boldsymbol{k}$

2.7 SCALAR OR DOT PRODUCT OF VECTORS

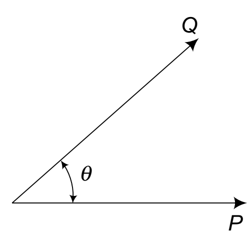


Figure 2.16

Refer to the Fig. 2.16, the *dot or scalar product* of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$, expressed as $\boldsymbol{P.Q}$, is defined as the product of the magnitudes of the two vectors and the cosine of their included angle θ. Thus $P.Q = |P||Q| \cos\theta \Rightarrow \cos\theta = P.Q/PQ$.

2.7.1 Physical Significance of Scalar or Dot Product of Vectors

2.7.1.1 Projection of One Vector on to the Other

$\boldsymbol{P}.\boldsymbol{Q} = PQ \cos \theta = P(Q \cos \theta) = (P \cos \theta)Q$. It therefore follows that $Q \cos \theta$ is the projection of Q in the direction of P or $P \cos \theta$ is the projection of P along Q.

Thus, dot product of two vectors is the product of one vector and the projection or component of the other along the former.

This interpretation of dot product can be related with mechanics as follows.

From the basic definition of work done by a force, *it is quantified by either the product of force multiplied by the component of the displacement along the line of action of the force or displacement multiplied by component of the force along the displacement.*

Thus, refering to the Fig. 2.17, if a force $\boldsymbol{F}$ applied on the block at an inclination θ with the horizontal causes a displacement ds along the horizontal plane, the component of ds along F is $ds \cos \theta$. Thus work done associated with this displacement would be $dW = F(ds \cos \theta) = (F \cos \theta)ds$ = component of the force along the displacement multiplied by the displacement.

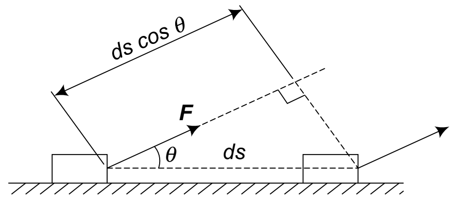


Figure 2.17 *Significance of dot product in mechanics (Work done)*

2.7.1.2 Geometrical Interpretation of Scalar Product

From the ΔOAB, $OA = \boldsymbol{P}$; $OB = \boldsymbol{Q}$. Thus AB becomes $\boldsymbol{P} - \boldsymbol{Q}$.

$$|P - Q|^2 = (P - Q).(P - Q) = P.P + Q.Q - 2(P.Q) = P^2 + Q^2 - 2(P.Q) \quad (2.1)$$

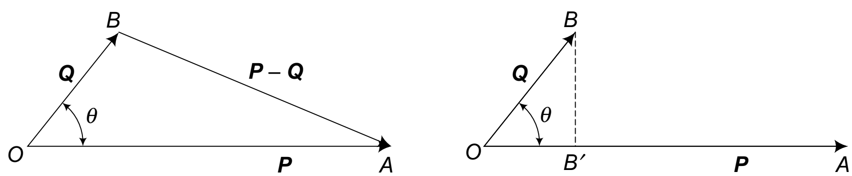


Figure 2.18 *Scalar or Dot product of vectors*

Further from the ΔOAB, cosine law yields $|P - Q|^2 = P^2 + Q^2 - 2PQ \cos \theta$ (2.2)

From Eqs (2.1) and (2.2), it follows that $\boldsymbol{P}.\boldsymbol{Q} = PQ \cos \theta$

Thus, dot product of two vectors is tantamount to cosine law of triangle.

2.7.2 Laws of Scalar or Dot Product of Vectors

The Following laws of multiplication can be derived from the definition of scalar or dot product.

(a) ***P.Q = Q.P*** **(*Commutative Law*)**

From the very definition of the dot product, $\boldsymbol{i}.\boldsymbol{i} = \boldsymbol{j}.\boldsymbol{j} = \boldsymbol{k}.\boldsymbol{k} = 1$, since, $\cos 0° = 1$ and

$$\boldsymbol{i}.\boldsymbol{j} = \boldsymbol{j}.\boldsymbol{i} = \boldsymbol{i}.\boldsymbol{k} = \boldsymbol{k}.\boldsymbol{i} = \boldsymbol{j}.\boldsymbol{k} = \boldsymbol{k}.\boldsymbol{j} = 0 \text{ since } \cos 90° = 0$$

Thus $\boldsymbol{P}.\boldsymbol{Q} = (P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k}).(Q_x\boldsymbol{i} + Q_y\boldsymbol{j} + Q_z\boldsymbol{k}) = P_xQ_x + P_yQ_y + P_zQ_z$

As a special case, when $\boldsymbol{P} = \boldsymbol{Q}$, we get $\boldsymbol{P}.\boldsymbol{P} = \boldsymbol{Q}.\boldsymbol{Q} = P_x^2 + P_y^2 + P_z^2 = Q_x^2 + Q_y^2 + Q_z^2$

> **Note:** When $\boldsymbol{P}$ and $\boldsymbol{Q}$ are perpendicular i.e. $\cos 90 = 0$, then $\boldsymbol{P}.\boldsymbol{Q}$ also becomes zero.

(b) ***P*.(*Q* + *R*) = *P*.*Q* + *P*.*R*** (***Distributive Law***)

Let $\boldsymbol{P} = (P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k})$,

$Q = (Q_x\boldsymbol{i} + Q_y\boldsymbol{j} + Q_z\boldsymbol{k})$ and

$R = (R_x\boldsymbol{i} + R_y\boldsymbol{j} + R_z\boldsymbol{k})$

$\therefore$ $\boldsymbol{Q} + \boldsymbol{R} = (Q_x + R_x)\boldsymbol{i} + (Q_y + R_y)\boldsymbol{j} + (Q_z + R_z)\boldsymbol{k}$

Hence

$$\begin{aligned}\boldsymbol{P}.(\boldsymbol{Q} + \boldsymbol{R}) &= (P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k}).(Q_x + R_x)\boldsymbol{i} + (Q_y + R_y)\boldsymbol{j} + (Q_z + R_z)\boldsymbol{k} \\ &= P_x(Q_x + R_x) + P_y(Q_y + R_y) + P_z(Q_z + R_z) \\ &= P_xQ_x + P_yQ_y + P_zQ_z + P_xR_x + P_yR_y + P_zR_z \\ &= \boldsymbol{P}.\boldsymbol{Q} + \boldsymbol{P}.\boldsymbol{R} \qquad \text{[Proved]}\end{aligned}$$

(c) $(\lambda\boldsymbol{P}).\boldsymbol{Q} = \lambda(\boldsymbol{P}.\boldsymbol{Q}) = \boldsymbol{P}.(\lambda\boldsymbol{Q})$ (Scalar multiple of a scalar product of two vectors)

(d) *Schwarz Inequality* $|P.Q| \le |P||Q|$

$$|P.Q| = |P||Q||\cos\theta| \le |P||Q|$$

2.8 VECTOR OR CROSS PRODUCT OF VECTORS

The *vector or cross product* of two vectors ***P*** and ***Q***, expressed as $\boldsymbol{P} \times \boldsymbol{Q}$, is defined as the product of the magnitudes of the two vectors and the sine of their included angle θ.

Thus $\boldsymbol{R} = \boldsymbol{P} \times \boldsymbol{Q} = P.Q \sin\theta$. This ***R*** is orthogonal to the plane of ***P*** and ***Q*** and pointed in the direction of advance of a right-handed screw when turned in the direction from ***P*** to ***Q*** by an angle θ as shown in the Fig. 2.19 (a) and (b).

The magnitude of the vector product can be obtained by the following relationship.

$$(P \times Q)^2 + (P.Q)^2 = P^2Q^2$$

Note: This expression is the relationship between scalar and vector product.

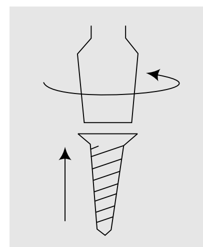


Figure 2.19 (a)

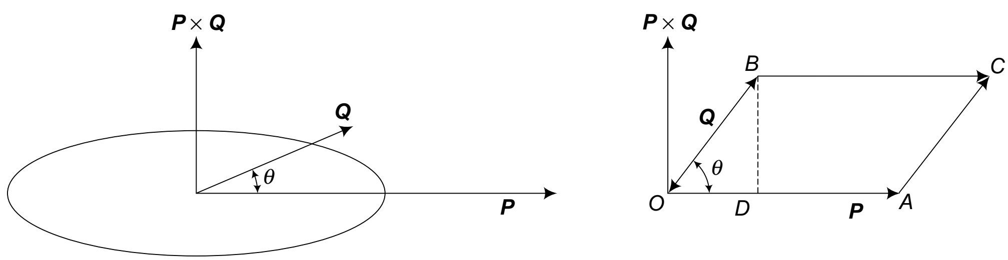


Figure 2.19 (b) *Vector or cross product of vectors*

2.8.1 Geometrical Interpretation of Vector Product

Refer to the parallelogram *OACB*, *OA* and *OB* represents the two vectors ***P*** and ***Q*** respectively.

Thus area of $OACB = OD \times BD + AD \times BD = BD \times (OD + DA) = BD \times OA = PQ \sin\theta$

Hence the magnitude of the vector product is equal to the area of the parallelogram, whose sides are parallel to and have lengths equal to the magnitudes of the vectors and its direction is perpendicular to the parallelogram.

2.8.2 Physical Significance of Vector Product

Refer to Fig. 2.20. A force $\boldsymbol{F}$ is applied on a body at a point A having its position vector $\boldsymbol{r}$. Thus as per definition $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = rF \sin \theta = r(F \sin \theta) = F(r \sin \theta) = Fd$.

This can be illustrated as moment of a force which is the cross product of the position vector $\boldsymbol{r}$ and the force $\boldsymbol{F}$ and it is quantified by *the magnitude of the force and the perpendicular distance between the line of application of the force and the moment center.*

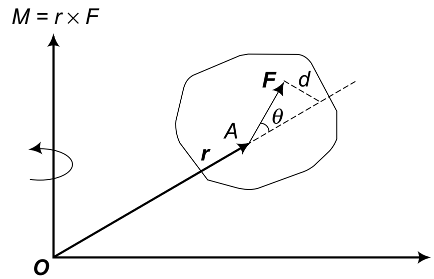


Figure 2.20 *Significance of vector product in mechanics (moment)*

Refer to the Fig. 2.20. $\boldsymbol{F}$ can be resolved into two mutually perpendicular components – one along the position vector $\boldsymbol{r}$ having magnitude $F \cos \theta$ and other perpendicular to $\boldsymbol{r}$ having magnitude $F \sin \theta$.

Since $F \cos \theta$ being directed along $\boldsymbol{r}$, it cannot have any moment about O. However, $F \sin \theta$ can produce moment about O, its magnitude being equal to $r(F \sin \theta)$.

Further, the perpendicular distance between the line of actions of $\boldsymbol{r}$ and $\boldsymbol{F}$ is $r \sin \theta$. The moment therefore can also be quantified as $F(r \sin \theta) = Fd$.

Note: Further illustrations on moment will find its place in chapter 3.

2.8.3 Laws of Vector Product

The following laws of vector product hold true.

(a) $\boldsymbol{P.Q} = -\boldsymbol{Q.P}$ (*It is not commutative*) [since $\sin \theta \neq \sin(-\theta)$]

When $\theta = 0$; $\sin 0 = 0$. This yields $\boldsymbol{P} \times \boldsymbol{P} = \boldsymbol{Q} \times \boldsymbol{Q} = 0$.

(b) $\boldsymbol{P} \times (\boldsymbol{Q} + \boldsymbol{R}) = \boldsymbol{P} \times \boldsymbol{Q} + \boldsymbol{P} \times \boldsymbol{R}$ (*Distributive Law*)

Since $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are orthogonal to each other, $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$, since $\sin 0 = 0$ and $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$; $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$; $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$ and

$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k};\ \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i};\ \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$$

$$\begin{aligned}\boldsymbol{P} \times \boldsymbol{Q} &= (P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k}) \times (Q_x\boldsymbol{i} + Q_y\boldsymbol{j} + Q_z\boldsymbol{k}) \\ &= (P_yQ_z - P_zP_y)\boldsymbol{i} + (P_zQ_x - P_xQ_z)\boldsymbol{j} + (P_xQ_y - P_yQ_x)\boldsymbol{k} \\ &= \begin{vmatrix} i & j & k \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}\end{aligned}$$

Note: Due to presence of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ in the cells, the absolute value of this determinant cannot be computed. That is why this determinant is called *pseudo-determinant*.

Thus $\boldsymbol{P} \times \boldsymbol{P} = \boldsymbol{Q} \times \boldsymbol{Q} = 0$

(c) $(\boldsymbol{P} \times \boldsymbol{Q}).\boldsymbol{P} = (\boldsymbol{P} \times \boldsymbol{Q}).\boldsymbol{Q} = \boldsymbol{0}$ [Since $\boldsymbol{P} \times \boldsymbol{Q}$ is orthogonal to both $\boldsymbol{P}$ and $\boldsymbol{Q}$]

(d) $\lambda(\boldsymbol{P} \times \boldsymbol{Q}) = \lambda\boldsymbol{P} \times \boldsymbol{Q} = \boldsymbol{P} \times \lambda\boldsymbol{Q}$ (Scalar multiple of a vector product of two vectors)

2.9 TRIPLE AND MULTIPLE PRODUCTS

Using mixtures of the pair-wise scalar product and vector product, it is possible to derive *"triple products"* between three vectors, and indeed n number of products between n vectors.

2.9.1 Scalar Triple Product

This is the scalar product of a vector product and a third vector, i.e., $\boldsymbol{P}.(\boldsymbol{Q} \times \boldsymbol{R})$. Unlike the vector product of two vectors, the scalar triple product can be represented by the true determinant.

$$\boldsymbol{P}.(\boldsymbol{Q} \times \boldsymbol{R}) = \begin{vmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{vmatrix}$$

When expanded,

$$\boldsymbol{P}.(\boldsymbol{Q} \times \boldsymbol{R}) = P_x(Q_yR_z - Q_zR_y) + P_y(Q_zR_x - Q_xR_z) + P_z(Q_xR_y - Q_yR_x) = (\boldsymbol{P} \times \boldsymbol{Q}).\boldsymbol{R}$$

Nevertheless, the most generic expression of scalar triple product is

$$\boldsymbol{P}.(\boldsymbol{Q} \times \boldsymbol{R}) = \boldsymbol{Q}.(\boldsymbol{R} \times \boldsymbol{P}) = \boldsymbol{R}.(\boldsymbol{P} \times \boldsymbol{Q})$$

Further

$$(\boldsymbol{P} \times \boldsymbol{Q}).\boldsymbol{R} = \boldsymbol{R}.(\boldsymbol{P} \times \boldsymbol{Q})$$

2.9.1.1 Geometrical Interpretation of Scalar Triple Product

The scalar triple product gives the volume of the parallelepiped whose sides are presented by the vectors $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ as shown in the Fig. 2.21.

We had seen earlier that the vector product $\boldsymbol{P} \times \boldsymbol{Q}$ has magnitude equal to the area of the base and direction is perpendicular to the base. The component of $\boldsymbol{R}$ in this direction is equal to the height of the parallelepiped shown in the Fig. 2.21.

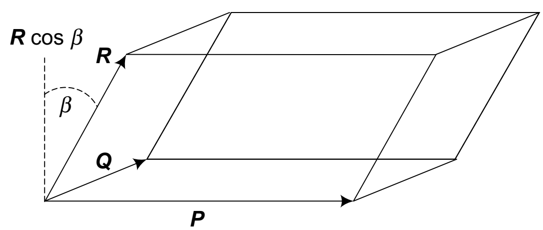


Figure 2.21 *Scalar triple product of vectors*

2.9.2.1 Vector Triple Product

This is defined as the vector product of a vector with a vector product, $\boldsymbol{P} \times (\boldsymbol{Q} \times \boldsymbol{R})$. The vector triple product $\boldsymbol{P} \times (\boldsymbol{Q} \times \boldsymbol{R})$ must be perpendicular to $\boldsymbol{Q} \times \boldsymbol{R}$, which in turn is perpendicular to both $\boldsymbol{Q}$ and $\boldsymbol{R}$. Thus $\boldsymbol{P} \times (\boldsymbol{Q} \times \boldsymbol{R})$ can have no component perpendicular to $\boldsymbol{Q}$ and $\boldsymbol{R}$ and hence should be coplanar with them.

Following the law (a) as mentioned in the topic 2.6.2,

$$(\boldsymbol{P} \times \boldsymbol{Q}) \times \boldsymbol{R} = -\boldsymbol{R} \times (\boldsymbol{P} \times \boldsymbol{Q}) = \boldsymbol{R} \times (\boldsymbol{Q} \times \boldsymbol{P})$$

Further, $$(\boldsymbol{P} \times \boldsymbol{Q}) \times \boldsymbol{R} = \boldsymbol{R.PQ} - \boldsymbol{R.QP}$$

And $$\boldsymbol{P} \times (\boldsymbol{Q} \times \boldsymbol{R}) = \boldsymbol{P.RQ} - \boldsymbol{P.QR}$$

Note: Application of vector operations in mechanics will be discussed in detail in the subsequent chapters.

Here is a question:
What is the significance of vector dot product and cross product in relation to mechanics?

Example 2.1 Given the vectors $\boldsymbol{P} = \boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{Q} = 3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ find $\boldsymbol{P} \times \boldsymbol{Q}$ and $|P \times Q|$.

Solution

$$\boldsymbol{P} \times \boldsymbol{Q} = (P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k}) \times (Q_x\boldsymbol{i} + Q_y\boldsymbol{j} + Q_z\boldsymbol{k})$$
$$= (P_yQ_z - P_zQ_y)\boldsymbol{i} + (P_zQ_x - P_xQ_z)\boldsymbol{j} + (P_xQ_y - P_yQ_x)\boldsymbol{k}$$

$$= \begin{vmatrix} i & j & k \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

$$= \boldsymbol{i}(4 - 4) + \boldsymbol{j}(12 + 2) + \boldsymbol{k}(1 + 6)$$

$$= 14\boldsymbol{j} + 7\boldsymbol{k}$$

$$|P \times Q| = \sqrt{14^2 + 7^2} = \sqrt{196 + 49} = 15.7$$

Example 2.2 A force vector $\boldsymbol{P} = 10\boldsymbol{i} + 9\boldsymbol{j} + 12\boldsymbol{k}$. Find the magnitude of this vector. Also find the unit vector of $\boldsymbol{P}$.

Solution Given vector $\boldsymbol{P} = 10\boldsymbol{i} + 9\boldsymbol{j} + 12\boldsymbol{k}$, which is in the form $P_x\boldsymbol{i} + P_y\boldsymbol{j} + P_z\boldsymbol{k}$

$\therefore$ The magnitude of $P = \sqrt{P_x^2 + P_y^2 + P_z^2} = \sqrt{10^2 + 9^2 + 12^2} = 18.$

Thus the unit vector P along $\boldsymbol{P}$ will be $= 10\boldsymbol{i} + 9\boldsymbol{j} + 12\boldsymbol{k}/18 = 0.55\boldsymbol{i} + 0.5\boldsymbol{j} + 0.67\boldsymbol{k}$

Example 2.3 Determine the magnitude of the resultant force vector $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ and its direction, measured counter-clockwise from the positive x axis. Also find out the resultant in vector expression.

Solution From the figure, we see that included angle of two vectors $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ is $(90° - 30°) + 45° = 105°$.

These two vectors can represent the two adjacent sides of a parallelogram having included angle 105°, and the diagonal of the parallelogram will be the resultant of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$.

Thus the magnitude $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos 105}\ N$

$$= \sqrt{250^2 + 375^2 + 2 \times 250 \times 375 \cos 105} N = 393\ N.$$

The inclination of the resultant makes an angle α with x axis

$$\therefore \quad \tan \alpha = \frac{F_1 \sin 105}{F_2 + F_1 \cos 105} = \frac{250 \sin 105}{375 + 250 \cos 105} = 0.778$$

$$\therefore \quad \alpha = \tan^{-1} 0.778 = 38°$$

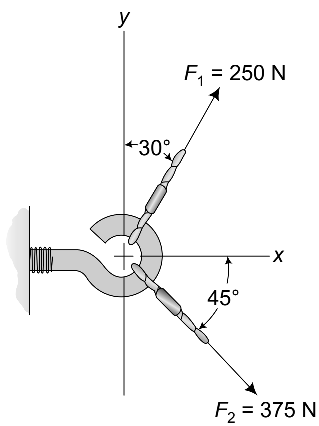


Figure 2.22

Therefore its direction measured counter-clockwise from the positive x axis will be $360° - (45° - 38°) = 353°$.

Both $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ can be resolved into two rectangular components, horizontal components being F_{1x} and F_{2x} and vertical components are F_{1y} and F_{2y} respectively.

Thus $F_{1x} = F_1 \cos 60° = 250 \cos 60° = 125N$; $F_{2x} = F_2 \cos (-45°) = 375 \cos 45° = 265.16N$;

$$F_{1y} = F_1 \sin 60° = 250 \sin 60° = 216.5N;\ F_{2y} = F_2 \sin (-45°) = -375 \sin 45° = -265.16N.$$

Hence $\boldsymbol{R} = \boldsymbol{F_1} + \boldsymbol{F_2} = (F_{1x} + F_{2x})\boldsymbol{i} + (F_{1y} + F_{2y})\boldsymbol{j} = (125 + 265.16)\boldsymbol{i} + (216.5 - 265.16)\,\boldsymbol{j}$

$$= 390.16\boldsymbol{i} - 48.66\boldsymbol{j}.$$

Example 2.4 Express the force $\boldsymbol{F}$ having magnitude 10 kN as a vector in terms of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, as shown in the Fig. 2.13. Assume $\theta_1 = 40°$ and $\theta_2 = 60°$.

Solution Refer to the Fig. 2.14, OP represents the force $\boldsymbol{F}$ having magnitude 10 kN.

Its projection on x–y plane will be represented by $F_{xy} = F \cos \theta_1 = 10 \cos 40° = 7.66$ kN and its projection on z axis will be $F_z = F \sin \theta_1 = 10 \sin 40° = 6.43$ kN.

Thus the component of 10 kN force along x axis becomes

$$F_x = F_{xy} \cos \theta_2 = 7.66 \cos 60° = 3.83 \text{ kN and along } y \text{ axis becomes}$$
$$F_y = F_{xy} \sin \theta_2 = 7.66 \sin 60° = 6.63 \text{ kN.}$$

Thus the vector expression of $\boldsymbol{F}$ in terms of unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ becomes, $F_x\boldsymbol{i} + F_x\boldsymbol{j} + F_x\boldsymbol{k} = 3.83\,\boldsymbol{i} + 6.63\,\boldsymbol{j} + 6.43\boldsymbol{k}$

Example 2.5 Refer to the Fig. 2.23. Find the magnitude of the resultant of vectors AB and AD. Also find the resultant in vector expression.

Solution Let $AB = \boldsymbol{P} = \sqrt{(9-5)^2 + (5-3)^2} = 4.472$ and $AD = \boldsymbol{Q} = \sqrt{(7-5)^2 + (12-3)^2} = 9.22$

$$\tan \theta_1 = \frac{2}{4} = 0.5; \quad \therefore \theta_1 = 26.56°$$

$$\tan \theta_2 = \frac{9}{2} = 4.5; \quad \therefore \theta_2 = 77.47°$$

$\therefore$ $$\theta = \theta_2 - \theta_1 = 50.9°$$

Thus $$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$
$$= \sqrt{4.472^2 + 9.22^2 + 2 \times 4.472 \times 9.22 \cos 50.9}$$
$$= 12.53$$

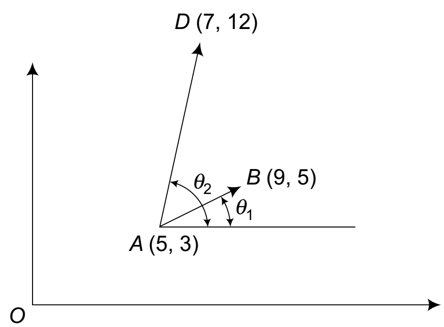


Figure 2.23

Further $\boldsymbol{P} = 4\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{Q} = 2\boldsymbol{i} + 9\boldsymbol{j}$

Thus $\boldsymbol{R} = \boldsymbol{P} + \boldsymbol{Q} = (4 + 2)\boldsymbol{i} + (2 + 9)\boldsymbol{j} = 6\boldsymbol{i} + 11\boldsymbol{j}$

The magnitude of $\boldsymbol{R}$ can also be calculated from $R = \sqrt{6^2 + 11^2} = 12.53$

Example 2.6 A force 150 N originates from the point (2, 4, 6) and passes through the point (4, 9, 15). Express the force in terms of unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$.

Solution

$$\boldsymbol{F} = F\boldsymbol{n} = F\frac{(x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$= 150\frac{(4-2)\boldsymbol{i} + (9-4)\boldsymbol{j} + (15-6)\boldsymbol{k}}{\sqrt{(4-2)^2 + (9-4)^2 + (15-6)^2}}$$

$$= 28.6\boldsymbol{i} + 71.5\boldsymbol{j} + 128.7\boldsymbol{k}$$

Example 2.7 Refer to the Fig. 2.24. The structure shown is subjected to force vectors $\boldsymbol{P}$ and $\boldsymbol{T}$ having magnitude 500 N and 200 N respectively. Combine $\boldsymbol{P}$ and $\boldsymbol{T}$ into a single force $\boldsymbol{R}$.

Solution Given $P = 500$ N
$T = 200$ N

Let the angle between $\boldsymbol{P}$ and $\boldsymbol{T}$ be α. So, $\boldsymbol{P}$ and $\boldsymbol{T}$ can be represented by two adjacent sides of the parallelogram and the resultant $\boldsymbol{R}$ can be represented by the diagonal.

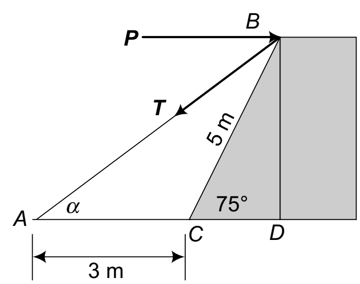
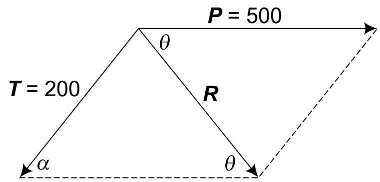


Figure 2.24



From the given geometry,

$$\tan \alpha = \frac{BD}{AD} = \frac{5 \sin 75°}{3 + 5 \cos 75°} \Rightarrow \alpha = 48.4°$$

Using Law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \alpha$$

$$R^2 = 200^2 + 500^2 - 2(200)(500) \cos (48.4°)$$

$$R = 396.5 \text{ N}$$

Using Law of sines, we get $\dfrac{200}{\sin \theta} = \dfrac{396.5}{\sin 48.4°}$

$\therefore$ $\theta = 22.2°$

Thus the magnitude of R is $R = 396.5$ N and its inclination with P is 22.2°.

Example 2.8 A force $\boldsymbol{F}$ of magnitude 200 N is applied to the point O (origin) of a three-dimensional coordinate system and passes through a point A having coordinates (4, 8, 5). Express the force in terms of unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. What is the unit vector in the direction of the 200 N force?

Solution The unit vector $\boldsymbol{n} = \dfrac{4\boldsymbol{i} + 8\boldsymbol{j} + 5\boldsymbol{k}}{\sqrt{4^2 + 8^2 + 5^2}}$ N $= \dfrac{4\boldsymbol{i} + 8\boldsymbol{j} + 5\boldsymbol{k}}{10.25}$ N $= 0.39\boldsymbol{i} + 0.78\boldsymbol{j} + 0.49\boldsymbol{k}$ N

Thus unit vector in the direction of the force $\boldsymbol{F} = F\boldsymbol{n} = 200[0.39\boldsymbol{i} + 0.78\boldsymbol{j} + 0.49\boldsymbol{k}]$ N $= 78\boldsymbol{i} + 156\boldsymbol{j} + 98\boldsymbol{k}$ N

Example 2.9 The ball joint at O is subjected to various loads as shown in the Fig. 2.25. Find out the orthogonal scalar components of the forces on the joint O. The 1000 N force goes through the solid diagonal as shown in the Figure.

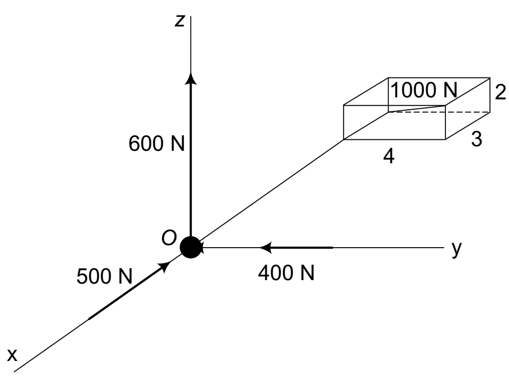


Figure 2.25

Solution The unit vector corresponds to 1000 N force $= \boldsymbol{F}_{FD} = 1000\mathbf{n}_{FD}$

$$= 1000 \frac{3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}}{\sqrt{3^2 + 4^2 + 2}} \text{ N} = 185.7[3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}] \text{ N} = 55.7.1\boldsymbol{i} + 742.8\boldsymbol{j} + 371.4\boldsymbol{k} \text{ N}$$

Thus $F_x = (557.1 - 500)$ N $= 57.1$ N; $F_y = (742.8 - 400)$ N $= 342.8$ N

$F_z = (600 + 371.4)$ N $= 971.4$ N

Example 2.10 A force $\boldsymbol{F}$ of magnitude 400 N is applied to the bracket as shown in the Fig. 2.26. Develop the force vector $\boldsymbol{F}$ and the position vector $\boldsymbol{r}$. Also compute the cross product $\boldsymbol{r} \times \boldsymbol{F}$.

Solution

1st Method: Let Two rectangular scalar components of F will be $F_x = F \cos 75$ and $F_y = F \sin 75$.

Thus the vector $\boldsymbol{F}$ becomes $F \cos 75\boldsymbol{i} + F \sin 75\boldsymbol{j} = 103.52\boldsymbol{i} + 386.37\boldsymbol{j}$.

The position vector $\boldsymbol{r} = 3\boldsymbol{i} + 8\boldsymbol{j}$.

Therefore

$$\boldsymbol{r} \times \boldsymbol{F} = (3\boldsymbol{i} + 8\boldsymbol{j}) \times (103.52\boldsymbol{i} + 386.37\boldsymbol{j}) = 8 \times 103.5(-\boldsymbol{k}) + 3 \times 386.37\boldsymbol{k}$$
$$= 330.95\boldsymbol{k} \text{ N-m.}$$

2nd Method: From the given geometry, $\tan \theta = \dfrac{3}{8} \Rightarrow \theta = 20.5°$

Thus the included angle between $\boldsymbol{r}$ and $\boldsymbol{F}$ measured counter-clockwise becomes $\alpha = 20.5° - 15° = 5.5°$.

The magnitude of the cross product $\boldsymbol{r} \times \boldsymbol{F}$ can be computed by $|rF \sin \alpha|$.

$$= 8.544 \times 400 \times \sin 5.5° \text{ N-m} = 330.6 \text{ N-m.}$$

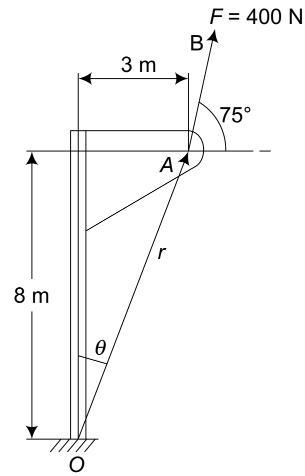


Figure 2.26

Example 2.11 A force $\boldsymbol{F}$ of magnitude 350 N is applied at A that connects two members AB and AC as shown in Fig. 2.27. Find the magnitudes of the two components of $\boldsymbol{F}$ directed along AB and AC.

Solution The vertical force $\boldsymbol{F}$ of 350 N acts downward at A.

Let the induced force in the two members AB and AC are denoted by T_{AB} and T_{AC} respectively.

Knowing the included angles, the three vectors $\boldsymbol{F}$, T_{AB} and T_{AC} can be represented by the three arms of triangle as shown in the figure.

By using Law of sine's;

$$\frac{T_{AB}}{\sin 60°} = \frac{T_{AC}}{\sin 45°} = \frac{350}{\sin 75°}$$

$$T_{AB} = 314 \text{ N}$$
$$T_{AC} = 256 \text{ N}$$

Thus the force in the two members AB and AC become 314 N and 256 N respectively.

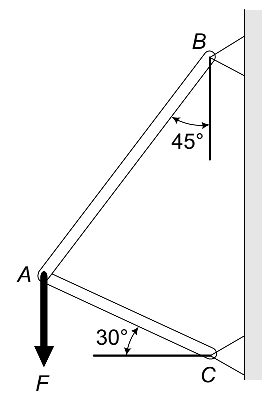


Figure 2.27

Example 2.12 A block is placed on an inclined plane that makes an angle 30º with the horizontal and subjected to a force $\boldsymbol{F} = 5\boldsymbol{i} + 20\boldsymbol{j} + 30\boldsymbol{k}$ as shown in the Fig. 2.28. If the work done associated with the force $\boldsymbol{F}$ is 20 N-m, how far does the block have to move?

Solution Let the displacement of the block along the plane is s.

Therefore the displacement vector $\boldsymbol{s}$ can be written as $0.\boldsymbol{i} + s \cos 30°\boldsymbol{j} + s \sin 30°\boldsymbol{k}$

Work done by the force F corresponds to displacement s

$$= F.s = (5\boldsymbol{i} + 20\boldsymbol{j} + 30\boldsymbol{k}).(0.\boldsymbol{i} + s \cos 30\boldsymbol{j} + s \sin 30\boldsymbol{k})$$
$$= 5 \times 0 + 20 \cos 30°s + 30 \sin 30°s = 32.32s \text{ N-m}$$

Given work done = 20 N-m

$$32.32s = 20$$
$$s = \frac{20}{32.32} = 0.618 \text{ m}$$

Thus displacement becomes $s = 0.618$ m

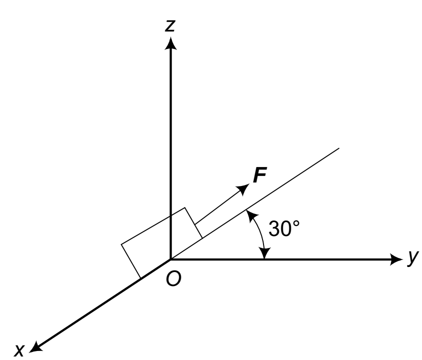


Figure 2.28

MULTIPLE-CHOICE QUESTIONS

2.1 If the resultant of two forces P and Q acting at an angle θ makes an angle α with P, then

(a) $\tan\alpha = \dfrac{P\sin\theta}{Q - P\cos\theta}$ (b) $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$ (c) $\tan\alpha = \dfrac{P\sin\theta}{P + Q\cos\theta}$ (d) $\tan\alpha = \dfrac{P\sin\theta}{P+Q\sin\theta}$

2.2 A force is completely defined when we specify

(a) its magnitude (b) its direction (c) point of application (d) all of the above

2.3 If two equal forces of magnitude P act at an angle θ, their resultant will be

(a) $2P\cos\dfrac{\theta}{2}$ (b) $2P\sin\dfrac{\theta}{2}$ (c) $2P\tan\dfrac{\theta}{2}$ (d) $P\cos\dfrac{\theta}{2}$

2.4 The resultant of two forces P and Q inclined at angle θ will be inclined at following angle with respect to P

(a) $\tan^{-1}\dfrac{P\sin\theta}{Q - P\cos\theta}$ (b) $\tan^{-1}\dfrac{Q\sin\theta}{P + Q\cos\theta}$ (c) $\tan^{-1}\dfrac{P\sin\theta}{P + Q\cos\theta}$ (d) $\tan^{-1}\dfrac{P\sin\theta}{P + Q\cos\theta}$

2.5 If two forces each equal to P in magnitude act at right angles, their effect may be neutralized by a third force acting along their bisector in opposite direction whose magnitude is equal to

(a) $2P$ (b) $\sqrt{2}P$ (c) $\dfrac{P}{\sqrt{2}}$ (d) $\dfrac{P}{2}$

2.6 Which of the following is a vector quantity?

(a) mass (b) energy (c) momentum (d) angle

2.7 Which of the following is not a vector quantity?

(a) mass (b) velocity (c) force (d) moment

SHORT ANSWER TYPE QUESTIONS

2.1 What do you understand by a vector quantity? Give examples.

2.2 State the parallelogram and triangle laws in relation to vector addition.

2.3 What do you mean by composition and resolution of a vector?

2.4 State and explain the principle of transmissibility of vector. Does it affect its behavior?

2.5 Show that cosine of two vector P and Q is equal to $m_Pm_Q + n_Pn_Q + o_Po_Q$ where m_P, n_P, o_P, and m_Q, n_Q and o_Q are the direction cosines of P and Q respectively.

2.6 State the laws of vector dot and cross products. Why is the cross product of two vectors a vector whereas a dot product is a scalar?

2.7 What is unit vector? What is meant by orthogonal triad of unit vectors?

2.8 What is the geometrical interpretations of dot and cross product of two vectors?

2.9 What is a position vector? What type of vector is it?

2.10 Prove that a force vector $\boldsymbol{F}$ can be expressed as $F\boldsymbol{n}$.

NUMERICAL PROBLEMS

2.1 If $\boldsymbol{P} = 5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{Q} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, find

(a) $\boldsymbol{P} \times \boldsymbol{Q}$ and $\boldsymbol{Q} \times \boldsymbol{P}$

(b) $|P \times Q|$

(c) $\sin\theta$ and θ where θ is the smaller angle between $\boldsymbol{P}$ and $\boldsymbol{Q}$.

2.2 If $\mathbf{A} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $B = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, find $A \times B$, $(A \times B)\cdot B$, and $(A \times \mathrm{B})\cdot A$.

2.3 If $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ (b) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

2.4 Evaluate

(a) $2\mathbf{i} \times (3\mathbf{j} - 4\mathbf{k})$ (b) $(\mathbf{i} + 2\mathbf{j}) \times \mathbf{k}$ (c) $(2\mathbf{i} - 4\mathbf{j}) \times (\mathbf{i} + \mathbf{k})$

2.5 Find $\mathbf{P}.\mathbf{Q}$ of two vectors $\mathbf{P} = 10\mathbf{i} + 20\mathbf{j} + 25\mathbf{k}$ and $\mathbf{Q} = 5\mathbf{i} - 10\mathbf{j} + 12\mathbf{k}$. What is cos $(\mathbf{P}.\mathbf{Q})$? What is the projection of $\mathbf{P}$ on $\mathbf{Q}$?

2.6 Given the vectors $\mathbf{P} = 6\mathbf{i} + 3\mathbf{j} + 10\mathbf{k}$, $\mathbf{Q} = 2\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ and $\mathbf{R} = 5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$. Which vector $\mathbf{S}$ gives the following results?

$$\mathbf{S}.\mathbf{P} = 20;\ \mathbf{S}.\mathbf{Q} = 5;\ \mathbf{S}.\mathbf{i} = 10.$$

2.7 A 500 N force is acting along the solid diagonal from O to C i.e. $OC = 500$ N. What is the rectangular component of this force along the other diagonal B to A as shown in the Fig. 2.29?

2.8 If vectors $\mathbf{P}$ and $\mathbf{Q}$ in the xy plane have a dot product of 50 units and if the magnitudes of these vectors are 10 units and 8 units respectively, what is $\mathbf{P} \times \mathbf{Q}$?

2.9 What is the component of the cross product $\mathbf{P} \times \mathbf{Q}$ along the direction n where

$$\mathbf{P} = 10\mathbf{i} + 16\mathbf{j} + 3\mathbf{k};\ \mathbf{Q} = 5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k};\ \mathbf{n} = 0.8\mathbf{i} + 0.6\mathbf{j}$$

2.10 What is the sum of the following three vectors?

$$\mathbf{P} = 6\mathbf{i} + 10\mathbf{j} + 16\mathbf{k};\ \mathbf{Q} = 2\mathbf{i} - 3\mathbf{j}$$

$\mathbf{R}$ is a vector in the xy plane at an inclination of 45º to the positive x axis and directed away from the origin of magnitude 25 N.

2.11 Given a force $\mathbf{F} = 10\mathbf{i} + 5\mathbf{j} + A\mathbf{k}$ N. If this force is to have a rectangular component of 8 N along a line having unit vector $\mathbf{r} = 0.6\mathbf{i} + 0.8\mathbf{k}$. What should be the value of A? What is the angle between F and r?

2.12 Two forces are applied at the end of a screw eye as shown in the Fig. 2.30 in order to remove the post. Determine the angle $\theta(0° \le \theta \le 90°)$ and the magnitude of the force F so that the resultant force acting on the post is directed vertically upward and has a magnitude of 750 N.

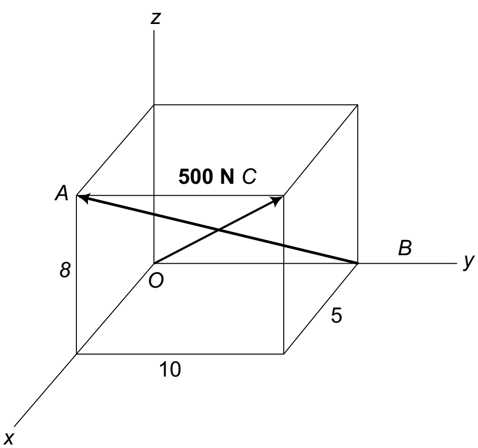


Figure 2.29

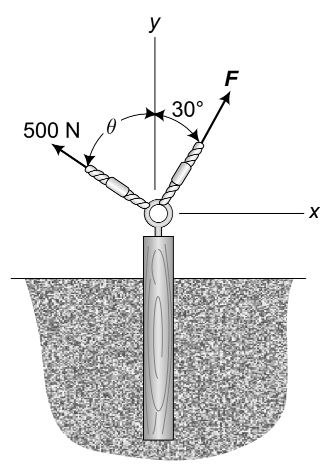


Figure 2.30

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

2.1 (b)	2.3 (a)	2.5 (b)	2.7 (a)
2.2 (d)	2.4 (b)	2.6 (c)	

CHAPTER

3

Two-Dimensional Force Systems

3.1 INTRODUCTION

From the foregoing discussions, it is evident that study of mechanics involves detailed analysis of force(s) on rigid bodies, supported suitably by varieties of constraints to ensure equilibrium of the entire system. It is, therefore, quite reasonable to study the force(s) in full vigor that would form the basis of further analysis.

3.2 FORCE

It has already been explained that force is any action on bodies that tends to change its status. It is a vector quantity and hence it is completely defined primarily by its ***magnitude*** and ***direction***, similar to that of any other vectors.

Let us consider a cantilever beam as shown in Fig. 3.1, loaded by a transverse force $\boldsymbol{P}$ applied at the free end A. This force $\boldsymbol{P}$ will cause deflection δ at the free end A of the beam as shown by the dotted line. Now if the force $\boldsymbol{P}$ is moved to act at a new position B keeping its magnitude and direction unchanged, the beam will be deflected similarly but present deflection λ will be smaller than that of δ. Thus it can be concluded that ***point of application*** of the load, i.e., the force has got important bearings as regard to its effects.

Hence in addition to its ***magnitude,*** and ***direction***, ***point of application*** of the load has to be considered for its complete description.

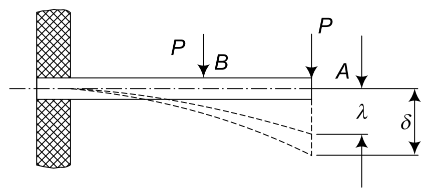


Figure 3.1

3.2.1 Types of Forces

The action of forces on bodies can broadly be classified into two categories: External and Internal.

Refer to Fig. 3.2 (a), a slender bar of weight W is hung from the ceiling and subjected to a load P. The force P which is applied together with W is called *external force.* However, this force P and self weight W are called *active force* since these cause different effects on the bar like reactions, deformations, deflections, stress, tension, compression, etc. Following Newton's third law, every action is accompanied by an equal and opposite reaction. Thus the ceiling, by virtue of its reaction will apply a pull force to the slender bar to counteract the effect of P and W. This is what is called *reactive force* or simply reaction.

Note: The self weight W is called body force and also treated as external force.

The *internal forces* on the other hand are those which are induced in the body as a consequence of resistance that it offers to balance the external force. Under the actions of W and P, the bar will be elongated. This

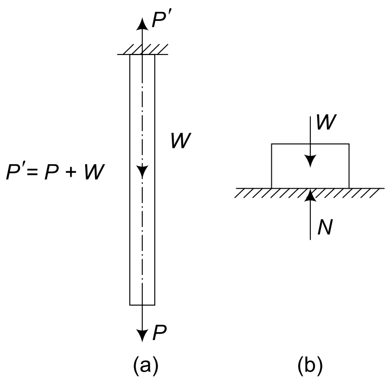


Figure 3.2

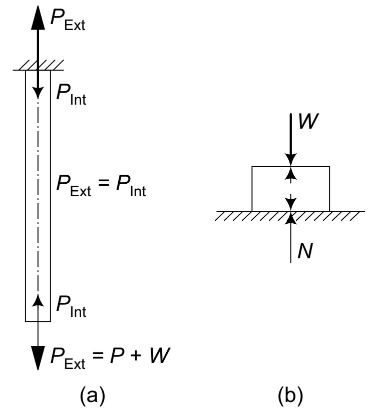


Figure 3.3

elongation is accompanied by stress induced in the bar, the magnitude of which is a property of materials. Analysis of stress and other associated parameters find their place in strength of materials.

Refer to the Fig. 3.3 (a). The load P_{Ext} applied at the lower end of the bar will try to stretch it. Thus, internal force induced by virtue of its tendency to oppose the external load will be directed as shown by P_{int}. So the gross effects of P_{Ext} is to elongate the bar. The internal forces associated with such situations are called *tensile force* or *tension*.

If the direction of externally applied load is reversed, the internal force will also be modified accordingly.

Refer to the Fig. 3.3 (b), the external force will try to shorten or compress the block and internal force so developed, called *compressive force,* will try to oppose it.

3.2.2 Principle of Transmissibility

Refer to the Fig. 3.4. The pull force ***P*** applied to the block can be shifted to any point along its line of action without changing its effect on the system. This is in agreement with the consideration of force as a sliding vector. This is a very useful characteristic of a force.

Following the same logic, the block in Fig. 3.4 (a) which is subjected to a horizontal pull force ***P*** applied at the front side may be transmitted to the rear side maintaining its same line of action to consider as a push force.

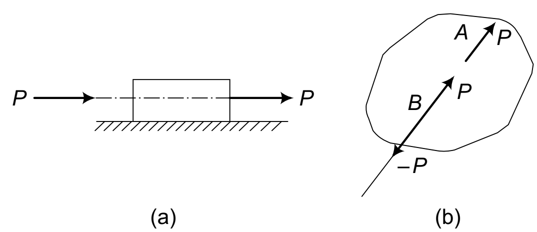


Figure 3.4

The principle by virtue of which a force can be envisaged to act at any position without violating its line of action with same consequences is called *principle of transmissibility*.

This can also be explained by the principle of vector addition.

Let a rigid body is subjected to a force vector ***P***, $P = P_x i + P_y j + P_z k$ applied at point A as shown in Fig. 3.4 (b).

Since $\boldsymbol{P} + (-\boldsymbol{P}) = 0$, presence of any such combination will not alter the situation. Let such a combination exist at point B as shown in Fig. 3.4 (b). Now ***P*** at A and $(-\boldsymbol{P})$ at B if cancelled out result in a system equivalent to an applied force ***P*** at B maintaining same line of action as was earlier.

Thus we can conclude that two situations are equivalent.

3.2.3 Superimposition of Forces

A null vector can be assumed to be a combination of $\boldsymbol{P} + (-\boldsymbol{P})$. If such a combination is added to a system without producing any effect, it is called *superimposition of forces*.

3.3 TYPES OF LOAD

Force can also be classified depending on nature of its loading pattern.

If the entire magnitude of force is assumed to be applied through a point, it is called *concentrated load*; and on the other hand, if it is distributed over a finite area, it is called *distributed load* as shown in the Fig. 3.5 (a) and Fig. 3.5 (b) respectively.

The concept of concentrated load is hypothetical since a definite amount of force can only be transmitted through a definite area. Nevertheless, such an idealisation will help simplify the problem to a great extent without appreciable compromise on accuracy provided the area over which the force is acting is too small compared to other related dimensions. In majority of the analysis, we come across concentrated loading to the structural or machine members.

The distributed load normally follows a definite pattern of loading over the entire area. Unlike concentrated load, it is customary to express it as its intensity in terms of load/unit length.

Thus entire load therefore is calculated by multiplying its length to its intensity. However, there are ample evidences when the distribution does not follow such linear law. Loading of beams is a very good example of distributed loading.

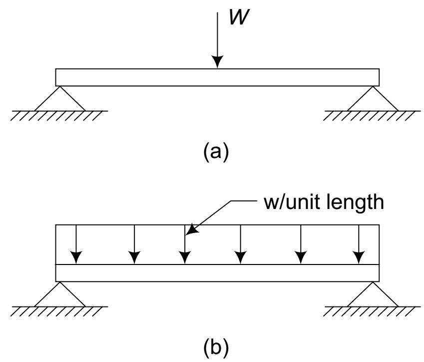


Figure 3.5

3.4 SYSTEM OF FORCES

Most of the engineering problems manifest that systems are subjected to different kinds of forces to constitute what is called a complete system of forces in space. Based on certain similarities, these forces can be grouped together under different titles.

3.4.1 Coplanar Forces

When the lines of action of several forces lie in one plane, the forces are called *coplanar forces* as shown in the Fig. 3.6 (a).

Essentially such forces and their analysis are confined to two-dimensions only.

3.4.2 Non-Coplanar Forces

When the lines of action of several forces are not contained in one plane, the forces are called *non-coplanar forces* as shown in the Fig. 3.6 (b). Force analysis in such a situation requires three-dimensional coordinate systems.

3.4.3 Concurrent Forces

When the lines of action of several forces intersect at a point so that we can consider all these forces are applied at that point, these forces together are called *concurrent forces* as shown in Fig. 3.6 (c).

However, the point of application of all such forces may not be concurrent apparently, but following the principle of transmissibility, these can be made to intersect at a common point and hence they are said to be concurrent forces. The common point of intersection is called point of concurrency. Refer to the Fig. 3.6 (d). Forces $\boldsymbol{F_1}$, $\boldsymbol{F_2}$ and $\boldsymbol{F_3}$ are applied at A, B and C respectively. By extending their lines of action, they do converge at point O and hence considered as concurrent.

If it is not possible to attain any such concurrency, then the force systems are called non-concurrent forces.

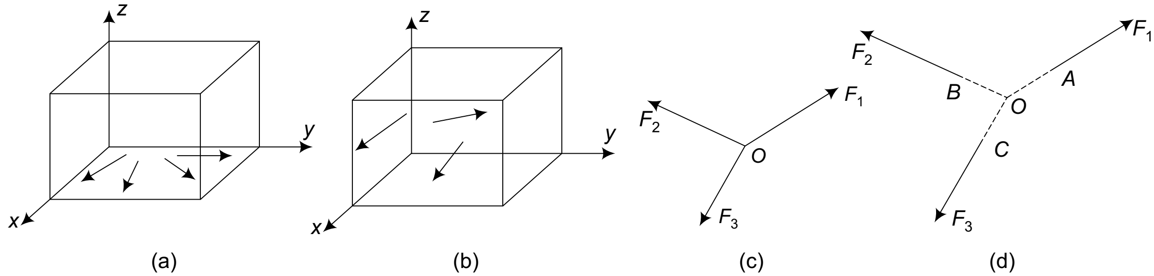


Figure 3.6

3.4.4 Collinear Forces

If the lines of action of several forces are identical, these are called *collinear forces*. For the collinear forces, magnitude as well as directions of different forces may differ.

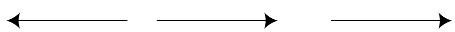


Figure 3.7

3.4.5 Parallel Forces

If the lines of action of several forces are parallel to each other, these are called *parallel forces* as shown in Fig. 3.8.

A system or structure acted upon by a system of forces may exhibit any such combinations.

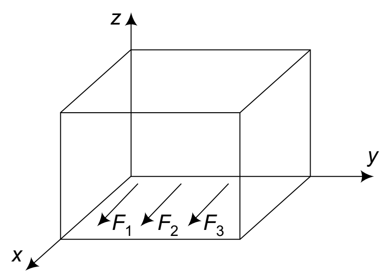


Figure 3.8

3.5 TWO-DIMENSIONAL FORCE SYSTEMS

There are quite a number of occasions, when forces are found to be confined to one plane only, i.e., *coplanar*. Analysis of such system of forces requires two-dimensional coordinate frame only.

If two forces $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are originating from point O, then it will follow parallelogram law (as explained in topic 2.2.1) and their resultant $\boldsymbol{R}$ would be another vector that can be expressed by vector addition of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ such that $\boldsymbol{R} = \boldsymbol{F_1} + \boldsymbol{F_2}$. The term *resultant* implies a single force that would be equivalent to the combined effect(s) of its components.

The magnitude of $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos\theta}$ and its inclination with $\boldsymbol{F_1}$ can be computed by the relationship $\tan\alpha = \dfrac{F_2 \sin\theta}{F_1 + F_2 \cos\theta}$.

It is misnomer to consider that $\boldsymbol{R} = \boldsymbol{F_1} + \boldsymbol{F_2}$ means the magnitude of $R = F_1 + F_2$.

Let us now consider few typical cases.

Case I: When $\theta = 0$, $\cos\theta = 1$.

Thus, $R = F_1 + F_2$.

This implies $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are now collinear having same direction.

It is noteworthy that when $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are parallel having same direction, $\theta = 0$ and $R = F_1 + F_2$.

Case II: When $\theta = 180°$, $\cos\theta = -1$. Thus $R = F_1 - F_2$. Once again $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ become collinear but opposite directed.

When $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are parallel but their directions are just opposite, then $R = F_1 - F_2$.

Case III: When $\theta = 90°$, $\cos\theta = 0$. $R = \sqrt{F_1^2 + F_2^2}$.

This can be interpreted as R being resolved into two mutually perpendicular components – one horizontal ($F_x = F_1 = R\cos\alpha$) and other vertical ($F_y = F_2 = R\sin\alpha$).

Thus in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, $\boldsymbol{R} = F_x\boldsymbol{i} + F_y\boldsymbol{j}$ and $\alpha = \tan^{-1}\dfrac{F_y}{F_x}$.

The above addition of two vectors can also be explained by triangle law as discussed in topic 2.2.1.

3.6 COMPOSITION OF FORCES

Following the discussions of topic 2.4, several such forces can be added successively and vectorically to get the final resultant. While doing so, we can follow either graphical approach or analytical approach.

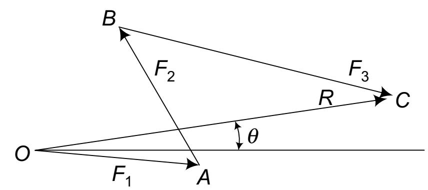


Figure 3.9

A. Graphical Approach: Consider a suitable scale to represent different force vectors.

Say, for example, 50 N force is represented by 1 cm. So scale factor becomes 50.

Therefore, a force of magnitude 200 N will be represented by 4 cm.

Refer to the Fig. 3.9. Vector OA represents force $\boldsymbol{F_1}$. From A, draw a line to represent $\boldsymbol{F_2}$ by AB. From B, again draw BC to represent $\boldsymbol{F_3}$. Now join OC which will represent resultant $\boldsymbol{R}$. Measure the length of OC in cm and multiply it by scale factor 50 to convert it to Newton. Measure the angle θ to obtain its inclination with the horizontal.

Likewise, any number of forces can be added to get their resultant.

This can be stated as *when a system is subjected to several coplanar, concurrent forces, these can be represented by a polygon such that its closing arm directed from origin of first vector to the arrowhead of last vector will represent their resultant. This is popularly known as Polygon of forces.*

B. Analytical Approach: Let us consider similar polygon of forces for this purpose.

Let us consider a two-dimensional coordinate frame, the origin of which coincides with the origin of first vector F_1.

Resolving $\boldsymbol{F_1}$, $\boldsymbol{F_2}$ and $\boldsymbol{F_3}$ and $\boldsymbol{R}$ into x and y directions

We have

$$(F_1)_x = F_1\cos\theta_1;\ (F_1)y = F_1\sin\theta_1$$

$$(F_2)_x = F_2\cos\theta_2;\ (F_2)_y = F_2\sin\theta_2$$

$$(F_3)_x = F_3\cos\theta_3';\ (F_3)_y = F_3\sin\theta_3'$$

$$R_x = R\cos\theta;\ R_y = R\sin\theta$$

From the geometry, it is evident that

$$R\cos\theta = F_1\cos\theta_1 + F_2\cos\theta_2 - F_3\cos\theta_3'$$

$$R\cos\theta = F_1\cos\theta_1 + F_2\cos\theta_2 - F_3\cos(180 - \theta_3)$$

$$R\cos\theta = F_1\cos\theta_1 + F_2\cos\theta_2 + F_3\cos\theta_3'$$

$$= (F_1)_x + (F_2)_x + (F_3)_x = \sum_{i=1}^{i=3}(F_i)_x \qquad (3.1)$$

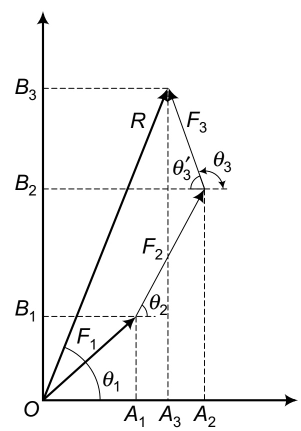


Figure 3.10

Similarly

$$R \sin \theta = F_1 \sin \theta_1 + F_2 \sin \theta_2 + F_3 \sin \theta_3 = (F_1)_y + (F_2)_y + (F_3)_y = \sum_{i=1}^{i=3} (F_i)_y \tag{3.2}$$

In case n number of such forces are added

$$R_x = R \cos \theta = (F_1)_x + (F_2)_x + \ldots + (F_n)_x = \sum_{i=1}^{i=n} (F_i)_x$$

and $R_y = R \sin \theta = (F_1)_y + (F_2)_y + \ldots + (F_n)_y = \sum_{i=1}^{i=n} (F_i)_y$ such that

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{\left[\sum_{i=1}^{i=n} (F_i)_x\right]^2 + \left[\sum_{i=1}^{i=n} (F_i)_y\right]^2} \tag{3.3}$$

and

$$\theta = \tan^{-1} \frac{\sum_{i=1}^{i=n} (F_i)_y}{\sum_{i=1}^{i=n} (F_i)_x} \tag{3.4}$$

C. Vector Approach: Once R_x and R_y are calculated, the resultant $\boldsymbol{R}$ can be expressed in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, so that $\boldsymbol{R} = R_x\boldsymbol{i} + R_y\boldsymbol{j}$.

Following the reverse procedure, $\boldsymbol{R}$ can be decomposed into $\boldsymbol{F_1}, \boldsymbol{F_2} \ldots \boldsymbol{F_n}$ which is called resolution of forces.

Example 3.1 Determine the magnitude of the resultant force and its direction, measured counter-clockwise from the positive x-axis as shown in the Fig. 3.11.

Solution All the three forces F_1, F_2 and F_3 are resolved into two mutually perpendicular components.

$(F_1)_x = F_1 \cos \theta_1 = 600 \cos 45 = 424.26$ N;

$(F_1)_y = F_1 \sin \theta_1 = 600 \sin 45 = 424.26$ N

$(F_2)_x = F_2 \cos \theta_2 = -800 \cos 30 = -692.82$ N;

$(F_2)_y = F_2 \sin \theta_2 = 800 \sin 30 = 400$ N

$(F_3)_x = F_3 \cos \theta_3 = -450 \cos 15 = -434$ N;

$(F_3)_y = F_3 \sin \theta_3 = -450 \sin 15 = -116.47$ N

$$R_x = \sum_{i=1}^{n} (F_i)_x = 424.26 - 692.82 - 434.66 = -703.22 \text{ N};$$

$$R_y = \sum_{i=1}^{n} (F_i)_y = 424.26 + 400 - 116.47\ N = 707.8 \text{ N}$$

Thus $R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-703.22)^2 + (707.8)^2} = 997.75$ N

$$\theta = \tan^{-1} \frac{R_y}{R_x} = \tan^{-1} \frac{707.8}{-703.22} = 134.8°$$

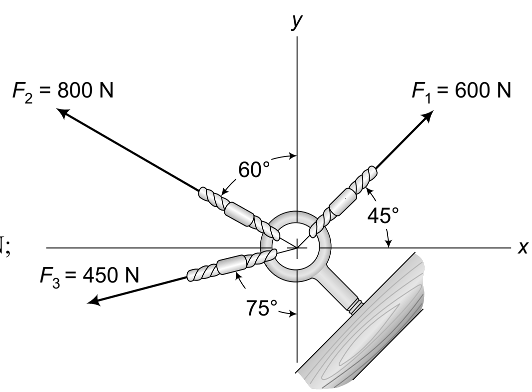


Figure 3.11

Example 3.2 Refer to the Fig. 3.12 the rod AB is subjected to a load of 250 N. Determine the two mutually perpendicular components of this load – one along the crank BC and other perpendicular to it.

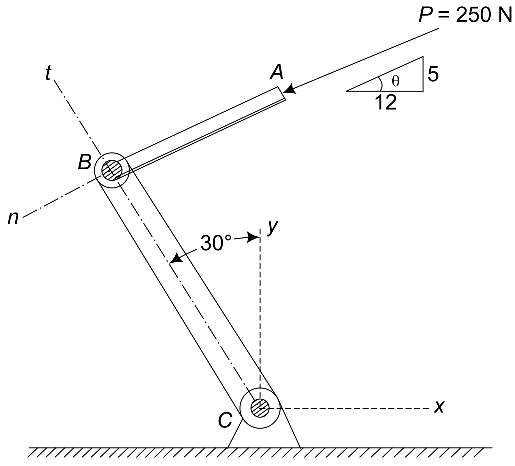


Figure 3.12

Solution Let the inclination of the applied force $\boldsymbol{P}$ with the horizontal be θ.

From the direction of the applied force $\boldsymbol{P}$,

$$\tan\theta = \frac{5}{12}$$

$$\Rightarrow \theta = \tan^{-1}\frac{5}{12} = 22.62°.$$

Refer to the Fig. 3.12, the angle between the force P and a line perpendicular to the crank body BC is $30° - 22.62° = 7.38°$.

Hence the component of $\boldsymbol{P}$ along the direction of the crank becomes $P \sin 7.38$ N = $250 \times \sin 7.38$ N = 32.11 N and its component perpendicular to the crank body becomes $P \cos 7.38$ N = $250 \times \cos 7.38$ N = 247.93 N.

Example 3.3 Refer to the Fig. 3.13. The pin joint at O is subjected to forces 500 N and 400 N that make angles α and β with the horizontal as shown to yield a resultant of 700 N vertically upward. What are the angles α and β?

Solution All the forces are resolved into horizontal and vertical components as shown in the Fig. 3.13.

Thus $$0 = 500 \cos\alpha - 400 \cos\beta \tag{3.5}$$

and $$700 = 500 \sin\alpha + 400 \sin\beta \tag{3.6}$$

Multiplying both sides of Eq. (3.5) by $\cos\alpha$ and (3.6) by $\sin\alpha$ and adding, we have $500 \cos^2\alpha - 400 \cos\alpha \cos\beta + 500 \sin^2\alpha + 400 \sin\alpha \sin\beta = 700 \sin\alpha$

or $$500 - 400 \cos(\alpha + \beta) = 700 \sin\alpha \tag{3.7}$$

Further $$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos\theta}$$

or $$700^2 = 500^2 + 400^2 + 2 \times 500 \times 400 \cos\{180 - (\alpha + \beta)\}$$

or $$\cos(\alpha + \beta) = \frac{-700^2 + 500^2 + 400^2}{2 \times 500 \times 400} = -0.2$$

or $$(\alpha + \beta) = \cos^{-1} - 0.2 = 101.53$$

Putting the value of $\cos(\alpha + \beta)$ in Eq. (3.7), we have

$$500 + 400 \times 0.2 = 700 \sin\alpha \text{ or } \alpha = \sin^{-1}\frac{580}{700} = 56°$$

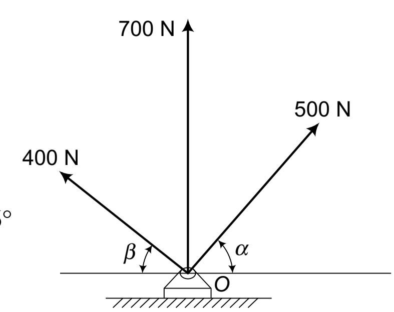


Figure 3.13

$$\therefore \quad \beta = 101.53 - 56 = 45.5°.$$

Therefore α and β are 56° and 45.5° respectively.

Example 3.4 Refer to the Fig. 3.14. Compute the resultant $\boldsymbol{R}$ [in vector expression] of the two forces applied to the bracket.

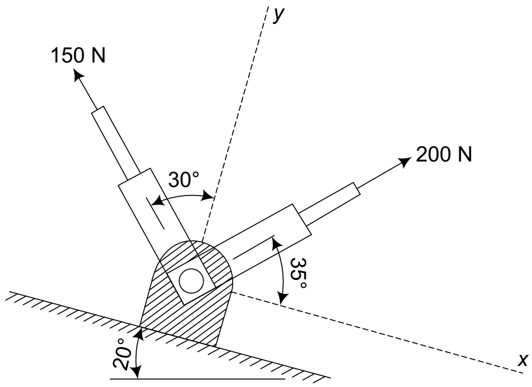


Figure 3.14

Solution Let 200 N and 150 N forces are denoted by $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ respectively.

Resolving these two forces along x and y we have

$$(F_1)_x = F_1 \cos \theta_1 = 200 \cos 35 = 163.83 \text{ N}; \ (F_1)_y = F_1 \sin \theta_1 = 200 \sin 35 = 114.71 \text{ N}$$

$$(F_2)_x = F_2 \cos \theta_2 = 150 \cos 60 = 75 \text{ N}; \ (F_2)_y = F_2 \sin \theta_2 = 150 \sin 60 = 129.9 \text{ N}$$

Thus $\boldsymbol{R} = R_x\boldsymbol{i} + R_y\boldsymbol{j} = \{(F_1)_x - (F_2)_x\}i + \{(F_1)_y + (F_2)_y\}j$

$R = \{163.83 - 75\}i + \{114.71 + 129.9\}j$

Thus $\boldsymbol{R} = 88.83\boldsymbol{i} + 244.6\boldsymbol{j}$ is the required vector expression of the resultant.

Example 3.5 The pulley is subjected to two equal forces T amounts to 400 N applied by means of a cable wrapped around it as shown in the Fig. 3.15. Determine the vector expression of resultant R applied to the pulley.

Solution Tension T which applied at an angle 60 is resolved into two components.

Thus $R_x = (F_1)_x + (F_2)_x = T + T \cos 60 = 1.5T = 1.5 \times 400 \text{ N} =$ 600 N

$R_y = (F_1)_y + (F_2)_y = 0 + T \sin 60 = 0.866T$

$= 0.866 \times 400 \text{ N} = 346 \text{ N}$

Thus $\boldsymbol{R} = 600\boldsymbol{i} + 346\boldsymbol{j}$ is the required vector expression of the resultant [Answer].

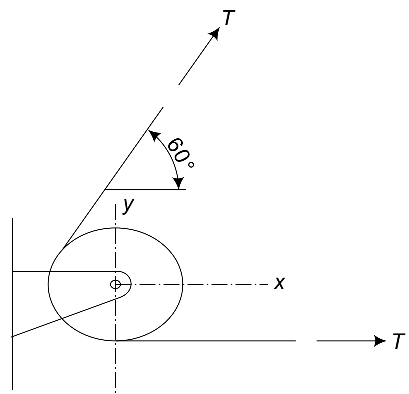


Figure 3.15

The magnitude of R would be $R = \sqrt{R_x^2 + R_y^2} = \sqrt{600^2 + 346^2}$ N = 693 N.

Example 3.6 The '*A*' shaped frame is subjected to two forces as shown in the Fig. 3.16. Combine the two forces into a single force $\boldsymbol{R}$. Express $\boldsymbol{R}$ in vector notation in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. Also compute the magnitude and direction of $\boldsymbol{R}$.

Solution Let 4 kN and 2 kN forces are denoted by $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ respectively.

Resolving these two forces along x and y we have

$$(F_1)_x = F_1 \cos\theta_1 = 4\cos 30 = 3.464 \text{ N};$$

$$(F_1)_y = F_1 \sin\theta_1 = 4\sin 30 = 2 \text{ N}$$

$$(F_2)_x = F_2 \cos\theta_2 = 2\cos 30 = 1.732 \text{ N};$$

$$(F_2)_y = F_2 \sin\theta_2 = 2\sin 30 = 1 \text{ N}$$

Thus $\boldsymbol{R} = R_x\boldsymbol{i} + R_y\boldsymbol{j} = \{(F_1)_x + (F_2)_x\}i + \{(F_1)_y - (F_2)_y\}j$

$$R = \{3.464 + 1.732\}i + \{2 - 1\}j$$

Thus $\boldsymbol{R} = 5.2\boldsymbol{i} + \boldsymbol{j}$ is the required vector expression of the resultant.

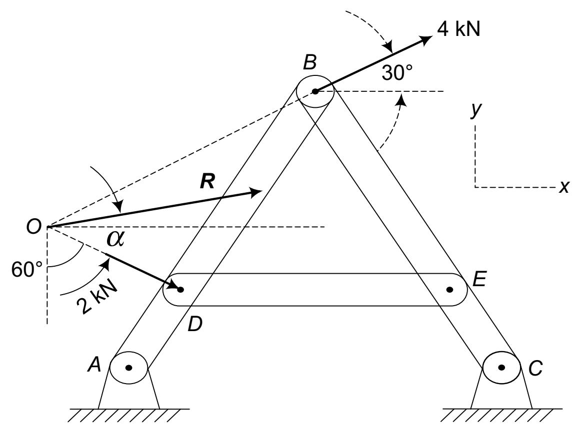


Figure 3.16

The magnitude of $\boldsymbol{R}$ becomes $R = \sqrt{R_x^2 + R_y^2} = \sqrt{5.2^2 + 1^2} = 5.29$ kN

If the two forces are extended following the principle of transmissibility, these would supposed to meet at O as shown with an included angle $\theta = 60°$

$$\text{Thus } \tan\alpha = \frac{F_1 \sin\theta}{F_2 + F_1\cos\theta} = \frac{4\sin 60}{2 + 4\cos 60} \Rightarrow \alpha = 40.89°$$

Therefore the inclination of R with the horizontal is $40.89° - 30° = 10.89°$.

3.7 MOMENT

In earlier discussions, we have seen that the forces try to move the object along its direction which is essentially linear. However, there are instances, when a force may also try to invoke a rotation about an axis (perpendicular to the plane in which the force is applied). Such behavior is desirable in the light of many applications – a common example being tightening a nut or bolt by a spanner as depicted in Fig. 3.17.

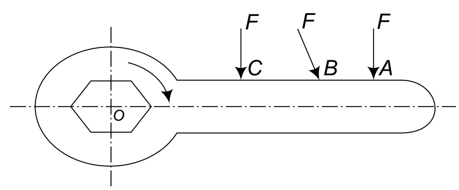


Figure 3.17

From experience of this arrangement, it is evident that success of the rotational tendency - the desired objective- depends not only on the applied force but also on the distance of the point of application of the load from a ***point*** with respect to which body tends to rotate. *Here, point is essentially an axis since an axis becomes a point when viewed perpendicularly.*

It is also observed that force that is applied perpendicular to the axis of the spanner is more effective. Further, increase in the above distance also has an encouraging effect in this endeavor. Thus refer to the Fig. 3.17, the force F applied at A is the most effective as compared to the other two situations when its point of application is shifted to B and C.

Thus, a precise definition of moment would be ***it is the measure of the tendency of the force that tries to produce a rotation with respect to an axis which is neither parallel nor intersecting with the line of application of the applied force.***

It is quantified by the product of the force ($\boldsymbol{F}$) and perpendicular distance measured between line of action of the force and axis of rotation (d).

Thus $M_o = F \times d.$

Since moment is the product of force and distance, its unit following SI system is Newton-metre (N-m).

The moment is a vector and is perpendicular to the plane of the body as shown in the Fig. 3.18.

Since the moment produces rotation – either clockwise or anti-clockwise, it follows the right-hand rule as shown in the Fig. 3.18.

As a sign convention, ***counter-clockwise moment is considered positive (right-hand rule) and clockwise moment is considered negative.***

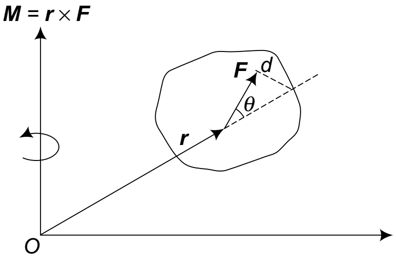


Figure 3.18

The moment of a force ***F*** about a point *O* can be expressed as by the following cross product of vector

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \tag{3.8}$$

where, ***r*** is a position vector that originates from the axis with respect to which moment is computed and directed to meet the line of action of the applied force.

Thus, the basic rules of cross product as mentioned in chapter 2 hold true with moment also.

The direction of ***M*** is defined as the direction which would bring ***r*** in line with ***F*** following *right-hand rule*.

Note: ***r*** is a vector from *O* to any point on the line of action of ***F***.

From the definition of a cross product,

$$M_0 = rF \sin \theta$$
$$M_0 = Fr \sin \theta$$

From the Fig. 3.18, $r \sin \theta = d$, so

$$M_0 = Fd \tag{3.9}$$

where d represents the perpendicular from O to the line of action of ***F***. d is commonly known as the ***moment arm***.

Resultant moment of a system of forces is

$$M_{RO} = \sum (r \times F) \tag{3.10}$$

3.8 PRINCIPLE OF MOMENTS - VARIGNON'S THEOREM

The underlying principle of Varignon's Theorem has a significant contribution in mechanics which can be read as ***"The moment of the resultant of several concurrent forces about a given point O is equal to the sum of the moments of the same individual forces about the same point O".***

Proof: Let's consider several concurrent forces. We can determine its resultant

$$R = F_1 + F_2 \ldots$$

The moment of $\vec{R}$ about O is $\vec{M}_O = \vec{r} \times \vec{R}$ where $\vec{r}$ is the position vector from O to a point on the line of action of $\vec{R}$.

$$\vec{M}_O = \vec{r} \times (\vec{F}_1 + \vec{F}_2 + \ldots)$$
$$\vec{M}_O = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \ldots$$

The authenticity of Varignon's Theorem can also be proved following scalar approach.

Let $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ be the two forces that originate from O and let $\boldsymbol{R}$ represent their resultant as shown in Fig. 3.19.

From any point A which is considered as moment centre on y axis, perpendiculars are dropped.

Thus moment of the resultant R about $A = Rd = R.OA \cos \theta = OA.R \cos \theta = OA.R_x$.

Similarly, moment of the force $\boldsymbol{F_1}$ with respect to the same moment centre $A = F_1 d_1 = F_1.OA \cos \theta_1 = OA.F_1 \cos \theta = OA.(F_1)_x$ and that of $\boldsymbol{F_2} = F_2 d_2 = F_2.OA \cos \theta_2 = OA.F_2 \cos \theta_2 = OA.(F_2)_x$

The sum of moments of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ becomes $= F_1 d_1 + F_2 d_2 = OA[(F_1)_x + (F_2)_x] = OA.R_x = Rd =$ Moment of the resultant with the same moment centre [Proved].

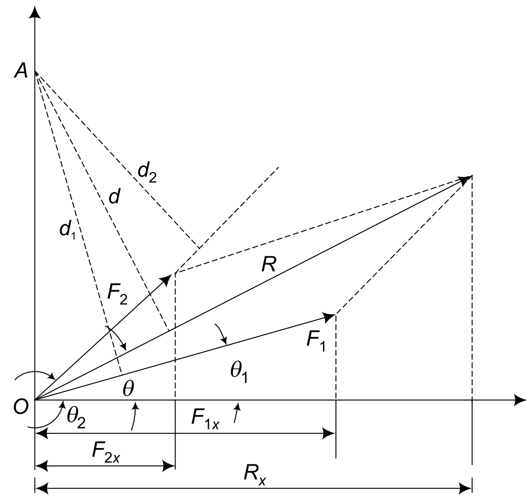


Figure 3.19

Example 3.7 Determine the magnitude and directional sense of the moment of the force at A about point P as shown in the Fig. 3.20.

Solution Let the inclination of the applied force $F = 520$ N be θ with the horizontal.

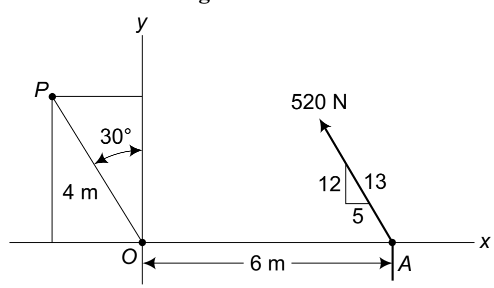


Figure 3.20

$$\therefore \qquad \tan \theta = \frac{12}{5} \Rightarrow \theta = \tan^{-1} \frac{12}{5} = 67.4°$$

Thus horizontal component of $\boldsymbol{F}$ becomes $F_x = F \cos \theta = 520 \cos 67.4 = 199.83$ N; and vertical component becomes $F_y = F \sin \theta = 520 \sin 67.4 = 480$ N.

The perpendicular distance between the line of action of F_x and the moment centre P is $OP \cos 30 = 4 \cos 30 = 3.464$ m and same for F_y is $OP \sin 30 + 6 = 8$ m.

It is evident from the sense of F_x and location of P, its moment with respect to P will be $F_x \times OP \cos 30 = 520 \times 3.464$ m $= 692.21$ N-m and its direction is clockwise and hence negative.

Similarly the moment of F_y with respect to P will be $F_y \times (6 + OP \sin 30) = 480 \times 8 = 3840$ N-m which is clockwise and hence positive.

Thus the algebraic sum of these two moments is the resultant moment about P which is $3840 - 692.21$ N-m $= 3147.8$ N-m.

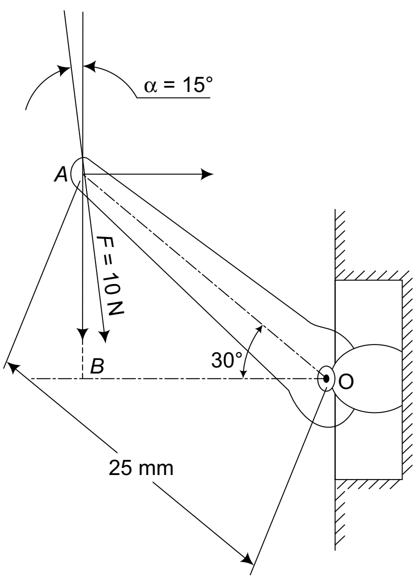


Figure 3.21

Example 3.8 Determine the moment of the 10 N force about the pivot O of the toggle switch as shown in the Fig. 3.21.

Solution The applied force $\boldsymbol{F}$ is resolved into two mutually perpendicular components – the horizontal component is $F \sin \alpha$, vertical component is $F \cos \alpha$.

The inclination of the toggle arm OA with the horizontal is $\theta = 30°$.

$$\sum M_O = F \cos \alpha \times OB - F \sin \alpha \times AB$$
$$= F \cos \alpha \times OA \cos \theta - F \sin \alpha \times OA \sin \theta$$
$$= F \times OA\ [\cos \alpha \cos \theta - \sin \alpha \sin \theta]$$
$$= F \times OA \cos (\alpha + \theta)$$
$$= 10 \times 25 \cos 45°$$
$$= 176.78 \text{ N-mm.}$$

Example 3.9 Compute the moment of the 250 N force on the handle of the monkey wrench about the centre of the bolt as shown in the Fig. 3.22.

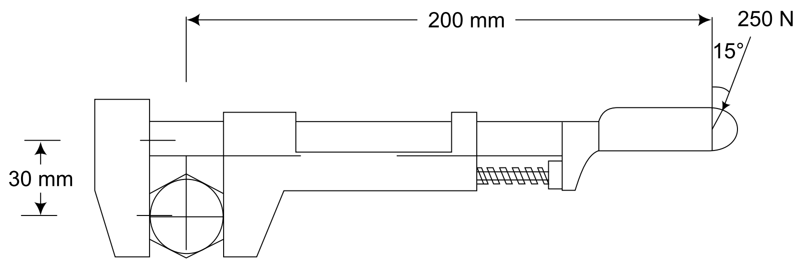


Figure 3.22

Solution The applied force $\boldsymbol{F}$ = 250 N can be resolved into two components. Thus horizontal component of $\boldsymbol{F}$ becomes $F_x = F \sin \theta = 250 \sin 15 = 64.7$ N; and vertical component becomes $F_y = F \cos \theta = 250 \cos 15 = 241.48$ N.

Thus following the sense of moment,

$$\sum M_O = -F_y \times 200 + F_x \times 30 = -241.48 \times 200 + 64.7 \times 30 \text{ N-mm} = -46.36 \text{ N-m}$$

Thus net moment about center of the bolt becomes 46.36 N-m and it is clock wise.

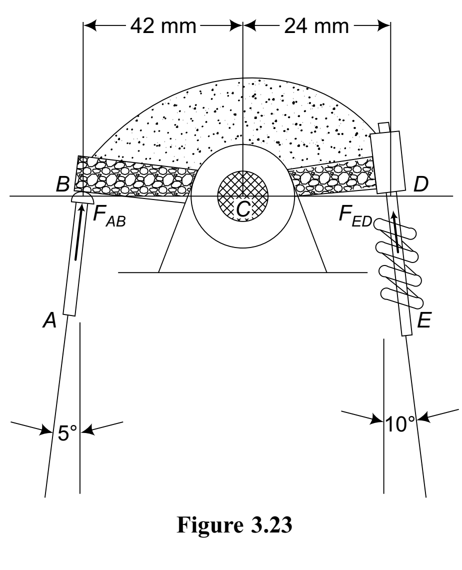


Figure 3.23

Example 3.10 The rocker arm *BD* of an I.C. engine is supported by a non-rotating shaft at *C* as shown in the Fig. 3.23. If the force exerted by the push pin *AB* on the rocker arm is 360 N, determine what force the valve stem *DE* will exert at *D* so that net moment at *C* is zero. Also calculate the resultant of the two forces on the rocker arm on the rocker.

Solution Let the forces exerted on the rocker by pin *AB* and valve stem *DE* are denoted by F_{AB} and F_{DE} respectively. These two forces can be resolved into two components – one horizontal and other vertical.

Thus $(F_{AB})_x = F_{AB} \sin 5$; and $(F_{AB})_y = F_{AB} \cos 5$

Similarly, $(F_{DE})_x = F_{DE} \sin 10$; and $(F_{DE})_y = F_{DE} \cos 10$

$(F_{AB})_x$ and $(F_{DE})_x$ being directed towards point *C*, they cannot produce any moment about *C*.

It is therefore $(F_{AB})_y$ and $(F_{DE})_y$ that will produce moment about C.

Since the net moment about C is zero, clockwise and counter-clockwise moments produced by $(F_{AB})_y$ and $(F_{DE})_y$ respectively will cancel each other.

This implies,

$$\sum M_C = 0 \Rightarrow F_{AB} \cos 5 \times 42 = F_{DE} \cos 10 \times 24$$

or
$$F_{DE} = \frac{F_{AB} \cos 5 \times 42}{24 \cos 10} = \frac{360 \cos 5 \times 42}{24 \cos 10} = 637 \text{ N}$$

If F_{AB} and F_{DE} are extended, they are supposed to meet at a point with an included angle of 10° + 5° = 15°.

Therefore their resultant $R = \sqrt{F_{AB}^2 + F_{DE}^2 + 2F_{AB} F_{DE} \cos 15}$

or
$$R = \sqrt{360^2 + 637^2 + 2 \times 360 \times 637 \cos 15} \text{ N} = 989 \text{ N}$$

$$\tan \alpha = \frac{F_{DE} \sin \theta}{F_{AB} + F_{DE} \cos \theta} = \frac{637 \sin 15}{360 + 637 \cos 15} \Rightarrow \alpha = 9.6°$$

Thus inclination of R with the horizontal becomes 90° − 5° + 9.6° = 94.6°.

Example 3.11 Determine the moment of the 100 N force about point A as shown in Fig. 3.24.

Solution The applied force $P(=100\text{N})$ is resolved into two mutually perpendicular components – the horizontal component being $P \sin \alpha$, vertical component is $P \cos \alpha$,

$$\begin{aligned}
\sum M_A &= P \sin \alpha \times AC + P \cos \alpha \times BC \\
&= P \sin \alpha \times (OA - OC) + P \cos \alpha \times BC \\
&= P \sin \alpha \times (OB - OB \cos \theta) + P \cos \alpha \times OB \sin \theta \\
&= P \times OB (\sin \alpha - \cos \theta \sin \alpha + \sin \theta \cos \alpha) \\
&= P \times OB [\sin \alpha + \sin (\theta - \alpha)] \\
&= 100 \times 0.125 [\sin 20 + \sin (70 - 20)] \text{ N-m} \\
&= 13.85 \text{ N-m}
\end{aligned}$$

Thus the force P will produce a moment of 13.85 N-m and its sense is clockwise.

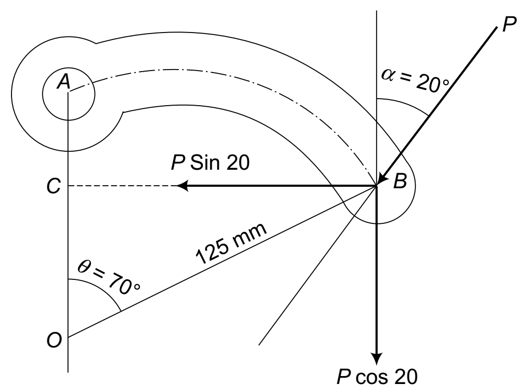


Figure 3.24

Example 3.12 The force exerted by the plunger of cylinder AB on the door is 60 N that acts along AB so as to close the door as shown in Fig. 3.25 [All dimensions are in mm]. Calculate the moment of this force about O. What is the value of the force F_c, normal to the plane of the door, that the door stop at C exerts on the door so that the combined moment about O of the two forces is zero?

Solution Considering the cylinder AB, the force of 60 N is acting along the plunger of the cylinder and is directed towards point A to keep the door in closed condition.

Let the inclination of the plunger AB with the horizontal is θ

$$\tan \theta = \frac{100}{400}$$

$$\theta = \tan^{-1} (0.25) = 14.03°$$

The 60 N force is resolved into horizontal and vertical components having magnitudes 60 cos θ and 60 sin θ respectively.

Both the component of force will produce clockwise moment with respect to point O.

Thus

$$\begin{aligned}\sum M_O &= 60 \sin \theta \times (400 + 25) + 60 \cos \theta \times 75 \\ &= 60[425 \sin 14.03 + 75 \cos 14.03] \text{ N-mm} \\ &= 60[103.03 + 72.762] \text{ N-mm} \\ &= 10.548 \text{ N-m}\end{aligned}$$

If the door stop at C exerts a force F_c, perpendicular to the plane of the door, it must produce same amount of moment but opposite in direction, i.e., counter-clockwise.

Moment arm for F_c would be 400 + 400 + 25 = 825 mm = 0.825 m

$$F_c \times 0.825 = 10.548$$

$$F_c = 12.785 \text{ N}$$

Force exerted by door stop at C is 12.785 N.

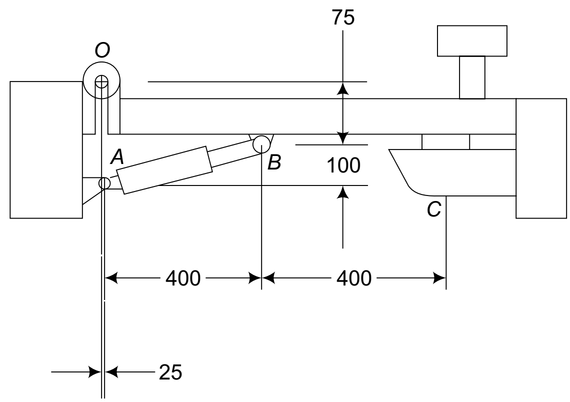


Figure 3.25

3.9 COUPLE

Two equal parallel forces with opposite directions, although do not yield any resultant force, but produce a resultant moment with respect to a moment centre. This moment is called *moment of a couple* and has got a very important role to play in mechanics.

Refer to the Fig. 3.26. Two forces $+\boldsymbol{F}$ and $-\boldsymbol{F}$ have the same magnitude, parallel lines of action and opposite direction.

Although $\Sigma F_y = 0$ but $+\boldsymbol{F}$ and $-\boldsymbol{F}$ together will try to rotate about O.

The moment of these two forces with respect to the point O [Refer Fig. 3.26 (a)] yields $M_O = F \times d + F \times d = F \times 2d$ = *Any one of the force multiplied by perpendicular distance between them.*

Unlike the previous case, if the moment centre lies on any one side, still the above relation will hold true.

Refer to the Fig. 3.26 (b). The moment of the two forces with respect to the point O' now becomes $M_O = F \times (2d + x) - F \times x = F \times 2d$.

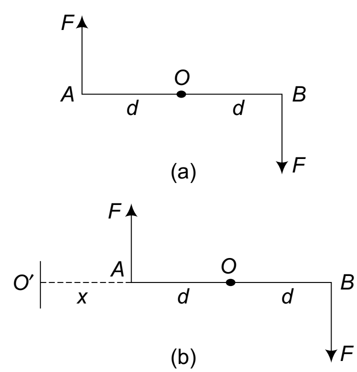


Figure 3.26

The moment of couple can also be computed by vector approach.

Suppose we have two equal and opposite forces $\boldsymbol{F}$ applied at A and B having their position vector $\boldsymbol{r_A}$ and $\boldsymbol{r_B}$ with respect to the origin O as shown in Fig. 3.27.

Thus combined moment of this two equal and opposite forces would be

$$\boldsymbol{M} = \boldsymbol{r_A} \times \boldsymbol{F} + \boldsymbol{r_B} \times (-\boldsymbol{F}) = (\boldsymbol{r_A} - \boldsymbol{r_B}) \times \boldsymbol{F} \tag{3.11}$$

But from the principle of addition of vector, $\boldsymbol{r_B} + \boldsymbol{r} = \boldsymbol{r_A}$ (3.12)

Comparing Eqs (3.11) and (3.12) we have,

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \tag{3.13}$$

It is interesting to note that $\boldsymbol{r}$ has got no relation with the moment centre; rather it is the relative position between two forces and hence moment of a couple is treated as free vector.

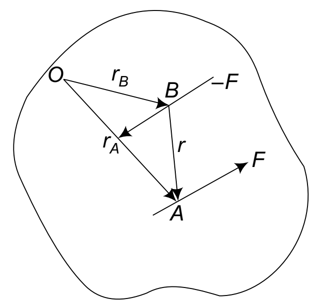


Figure 3.27

Note: Since the moment of a couple is the product of two parameters – the force and the position vector of one with relative to the other – it will remain unchanged even if both $\boldsymbol{r}$ and $\boldsymbol{F}$ are modified in such a way so that their product remains unaltered.

3.10 FORCE – COUPLE SYSTEM

A force applied at any point of a rigid body can be replaced by an equivalent force applied at different point along with a couple.

Refer to the Fig. 3.28. A force $\boldsymbol{F}$ is applied at point A of the body. Following the principle of superimposition of forces, a set of forces $+\boldsymbol{F}$ and $-\boldsymbol{F}$ is applied at point B without changing the status of the body as shown in

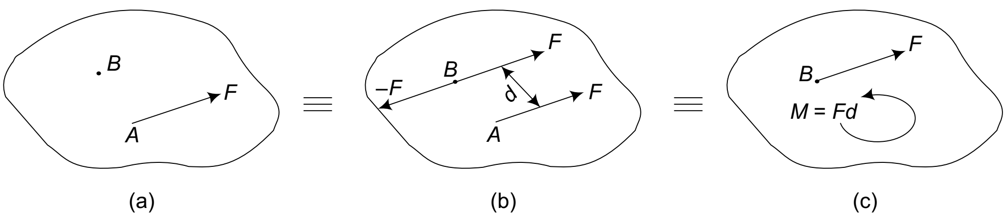


Figure 3.28

the Fig. 3.28 (b). It is clear from the very definition of couple that $+\boldsymbol{F}$ at point A and $-\boldsymbol{F}$ at point B together constitute a couple having its moment $d \times F$ in addition to $+\boldsymbol{F}$ at B without altering its status quo. Thus, initial situation of the body which was acted upon by a force $\boldsymbol{F}$ at A is no way different to the final situation as depicted in Fig. 3.28 (c) when it exhibits presence of a couple with a moment $M = Fd$ along with a force $+\boldsymbol{F}$ at B. This situation is known as *a force can be replaced by same force but at a different location along with a couple.*

Example 3.13 The bracket shown in the Fig. 3.29 is spot welded to the end of the shaft at point O and is subjected to 900 N force. Find out equivalent force and couple to replace 900 N force.

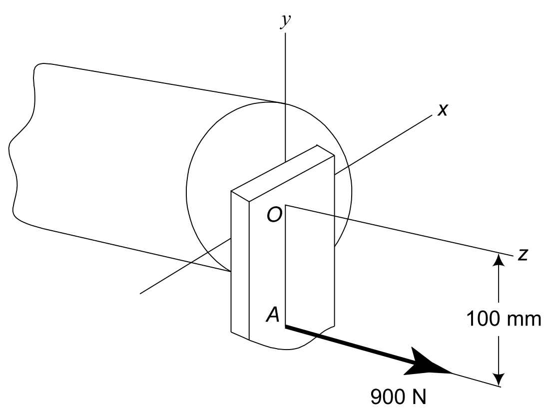


Figure 3.29

Solution Let the 900 N force is applied at point A.

At point O, we can superimpose +900 N and −900 N.

Thus +900 N at A and −900 N at O together constitutes a couple having moment

$$M = 900 \times 100 \text{ N-mm} = 90 \text{ N-m}.$$

In vector notation, applied force vector $\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} = 900\boldsymbol{k}$. Its position vector $\boldsymbol{r} = r_x\boldsymbol{i} + r_y\boldsymbol{j} + r_z\boldsymbol{k} = -100\boldsymbol{j}$.

Therefore $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = -100\boldsymbol{j} \times 900\boldsymbol{k}$ N-mm $= -90\boldsymbol{i}$ N-m.

Example 3.14 Refer to the wrench shown in the Fig. 3.30. It is subjected to the 200 N force and the force $\boldsymbol{P}$. If the equivalent of the two forces is a force $\boldsymbol{R}$ at O and a couple expressed as $\boldsymbol{M} = 20\boldsymbol{k}$ N-m, determine the vector expressions for $\boldsymbol{P}$ and $\boldsymbol{R}$.

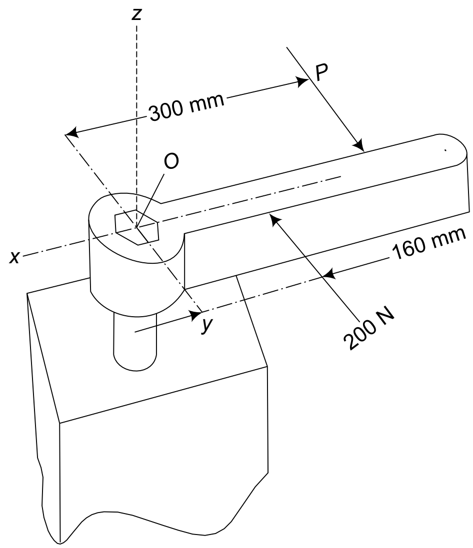


Figure 3.30

The magnitude of the moment of the couple is 20 N-m and it is counter-clockwise.

Thus $200 \times 160 - P \times 300 = 20 \times 1000$; $P = \dfrac{200 \times 160 - 20 \times 1000}{300}$ N = 40 N

Therefore vector expression of P becomes $\boldsymbol{P} = 40\boldsymbol{j}$

The net force $\boldsymbol{R}$ becomes $-200\boldsymbol{j}$ N + $40\boldsymbol{j}$ N = $-160\boldsymbol{j}$ N.

MULTIPLE-CHOICE QUESTIONS

3.1 Force is a
(a) free Vector (b) sliding (c) fixed (d) all of the above

3.2 Force is completely specified by its
(a) magnitude only (b) magnitude and direction
(c) magnitude, direction and point of application (d) none of the above

3.3 Resolution of force means
(a) conversion of several forces to a single force (b) vector addition
(c) Both (i) and (ii) (d) none of the above

3.4 A system of forces are said to be coplanar if they lie in
(a) a single plane (b) two planes
(c) three mutually perpendicular planes (d) none of the above

3.5 Lami's theorem is related to
(a) equilibrium of two coplanar, concurrent forces
(b) equilibrium of three coplanar, concurrent forces
(c) equilibriums of four coplanar, concurrent forces
(d) none of the above

3.6 Transmissibility of force refers to
(a) a force can be shifted any where with in the body
(b) a force can be rotated by 90º

(c) force can be shifted with out altering its line of action and direction
(d) none of the above

3.7 Superimposition of forces
(a) increases the magnitude of the force (b) decreases the magnitude of the force
(c) does not change its status (d) none of the above

3.8 Moment of a force with respect to a moment centre in its plane is the
(a) sum of force and perpendicular distance between the moment centre and the line of action of force.
(b) difference between force and above distance.
(c) ratio of force and the distance.
(d) product of force and the distance.

3.9 Moment is obtained by
(a) vector addition (b) vector subtraction (c) vector dot product (d) vector cross product

3.10 The force or component of a force which is directed towards moment centre will produce
(a) zero moment (b) maximum moment (c) average moment (d) none of the above

3.11 Varignon's theorem is related with
(a) moment of force(s) (b) friction
(c) deformation characteristics of rigid bodies (d) none of the above

3.12 A couple is formed by
(a) two equal and opposite intersecting forces (b) two equal forces that are at 90º
(c) two equal and opposite parallel forces (d) all of the above

3.13 The moment of a couple will remain unchanged if
(a) either the force or the distance remains unchanged.
(b) both are changed arbitrarily.
(c) both are increased and decreased simultaneously by same scale factor.
(d) one is increased and other is deceased by same scale factor.

3.14 The moment of a couple is
(a) independent of the location of the moment centre
(b) independent if it is outside the body
(c) dependent if it is with in the body
(d) none of the above

3.15 $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$ yields
(a) $rF \cos \theta$ (b) $rF \sin \theta$ (c) $rF \tan \theta$ (d) $rF \cot \theta$

SHORT ANSWER TYPE QUESTIONS

3.1 Define force. How it is specified? Is it a vector quantity? If yes, what type of vector is it?
3.2 What is meant by point load and distributed load? Give examples.
3.3 What do you mean by coplanar, concurrent and collinear force? What is the condition of equilibrium for two such forces?
3.4 What is meant by transmissibility and superimposition of forces?
3.5 State Lami's Theorem.
3.6 State and explain Parallelogram Law, Triangle Law and Polygon of Forces. In this context, what is meant by closed polygon and under what situation it is possible to construct a closed polygon?
3.7 Develop a vector expression for resultant of two coplanar, concurrent forces.
3.8 What is meant by composition and resolution of forces?
3.9 Define moment of a force. What types of vector is it? How it is relevant with vector cross product?
3.10 State Varignon's Theorem and prove it.

3.11 What is a couple? Under what circumstances is it formed? Replace a force by an equivalent force-couple system.
3.12 Is couple a vector? Prove that moment of a couple does not depend on the location of the moment center.

NUMERICAL PROBLEMS

3.1 A barge is pulled by 2 tugboats as shown in Fig. 3.31. The resultant of the forces exerted by the tugboats is a 5000 N force directed along the centre axis of the barge. Find tension in each rope if $\alpha = 45^\circ$ and value of α such that the tension in rope 2 is minimum.

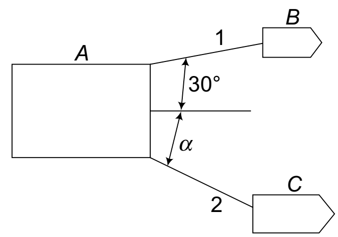


Figure 3.31

3.2 Replace the 6 kN and 4 kN forces as shown in Fig. 3.32 by a single force, expressed in vector notation.
3.3 The resultant of three forces is $\boldsymbol{R}$ = 60 N as shown in the Fig. 3.33. Two of the three forces are also shown as 120 N and 65 N. Determine the third force.
3.4 The pole *OA* is subjected to a force applied at *A* as shown in Fig. 3.34.

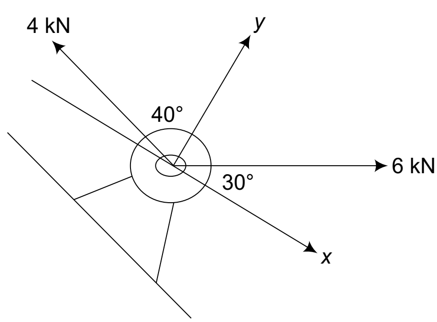


Figure 3.32

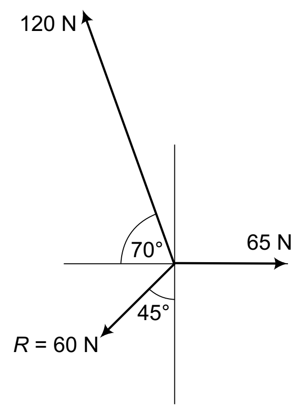


Figure 3.33

Find: (a) Moment of the 100 N force about *O*.
(b) Magnitude of a horizontal force applied at *A* which create the same moment about *O*.
(c) The smallest force applied at *A* which creates the same moment about *O*.
(d) Distance from *O* at which a 240 N vertical force must act to create the same moment about *O*.

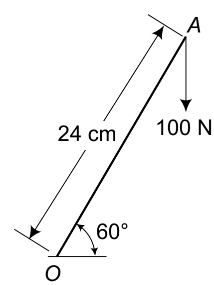


Figure 3.34

3.5 The lever is loaded by various forces and a couple as shown in Fig. 3.35. If the resultant of these forces and couple passes through *O*, calculate $\boldsymbol{M}$.
3.6 The aircraft is subjected to thrust forces *T* as shown in Fig. 3.36. Determine the equivalent force couple system at *O*. Replace this force couple system by a single force and find out its location on *x* axis.

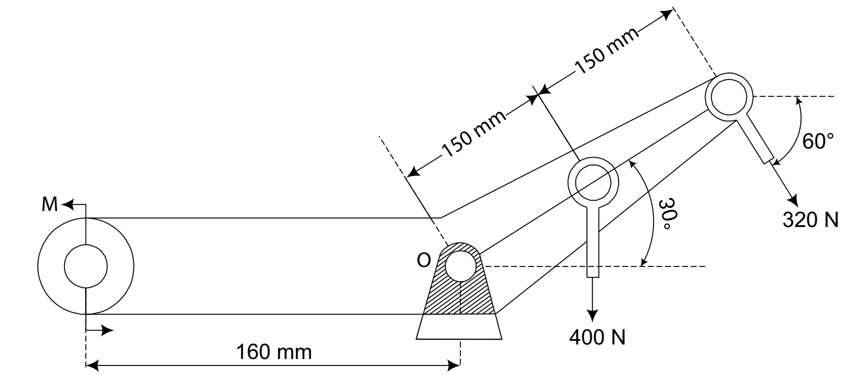


Figure 3.35

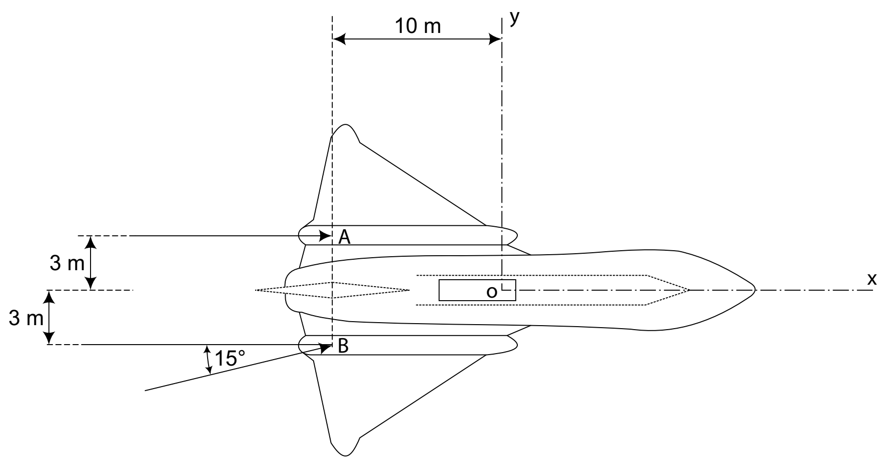


Figure 3.36

3.7 A device called a rolamite is used to replace slipping motion with rolling motion as shown in Fig. 3.37. If the belt, which wraps between the rollers, is subjected to a tension of 15 N, determine the reactive

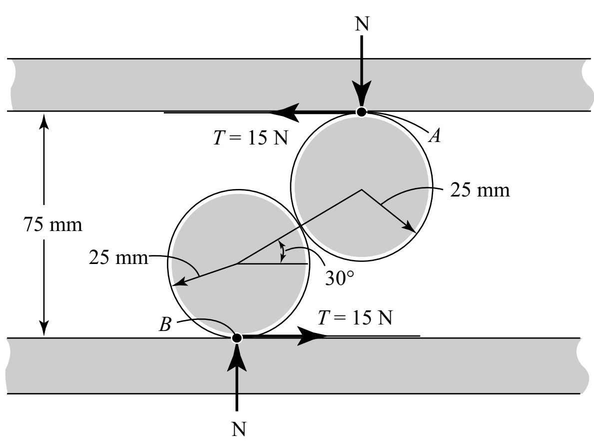


Figure 3.37

forces N of the top and bottom plates on the rollers so that the resultant couple acting on the rollers is equal to zero.

3.8 A roller with a lever of radius 60 mm attached to its centre makes an angle of 60° with the horizontal as shown in Fig. 3.38. Replace the couple and force by an equivalent, single force applied to the lever. Also determine the distance from O to the point of application of this force.

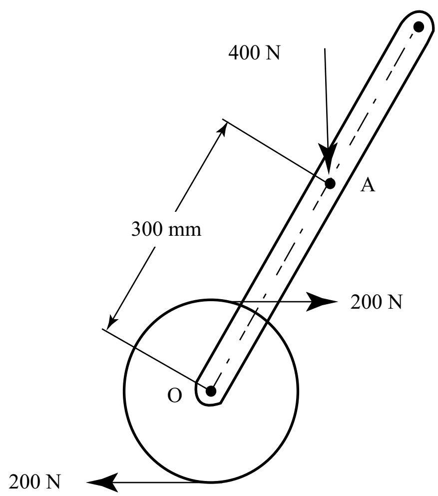


Figure 3.38

3.9 The lever ***ABC*** is hinged at B and subjected to various forces as shown in Fig. 3.39. (a) Replace the three forces with an equivalent force-couple system at B. (b) Determine the single force which is equivalent to the force-couple system obtained in, and (a) locate its point of application on the lever.

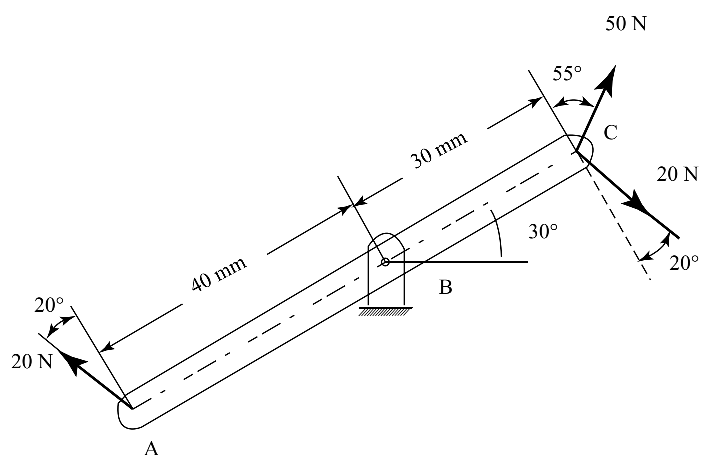


Figure 3.39

3.10 The bracket ***ABCD*** is hinged at B and subjected to a horizontal force $\boldsymbol{P}$ = 80 N applied at A as shown in Fig. 3.40. (a) Replace the force with an equivalent force-couple system at B. (b) What is the magnitude and direction of the vertical applied forces at C and D that would produce equivalent amount of moment at B?

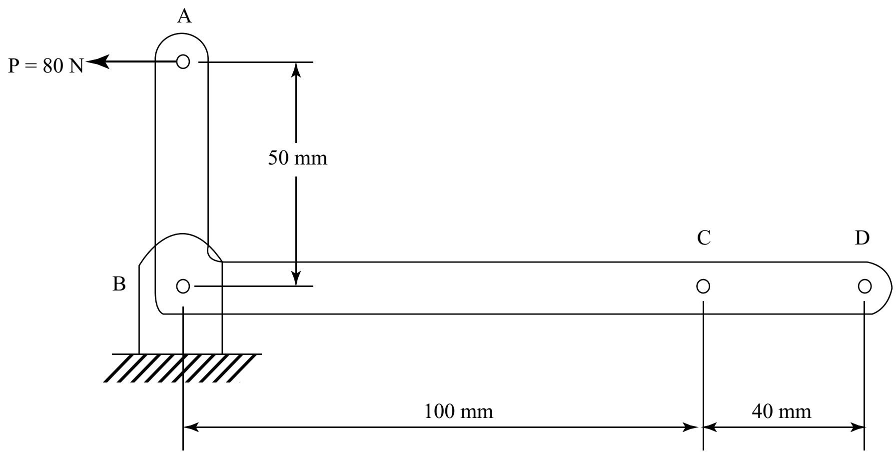


Figure 3.40

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

3.1 (b)	3.4 (a)	3.7 (c)	3.10 (a)	3.13 (d)
3.2 (c)	3.5 (b)	3.8 (d)	3.11 (a)	3.14 (a)
3.3 (d)	3.6 (c)	3.9 (d)	3.12 (c)	3.15 (b)

ANSWERS TO NUMERICAL PROBLEMS

3.1 (a) T_1 = 3660 N, T_2 = 2590 N (b) α = 60°, T_1 = 4330 N, T_2 = 2500 N
3.2 2630**i** + 6060**j**
3.3 168.8 N. 66.84°
3.4 (a) 1200 N-cm (b) 57.7 N (c) 50 N (d) 10 cm
3.5 **M** = 148 N-m
3.6 T = 1.966**i** + 0.259**j**; M_O = 2.69T; $x = -10.4$ m
3.7 26 N
3.8 420 mm
3.9 (a) F = 50 N; 65° with the horizontal, M = 455.21 N-mm, (b) F = 50 N; 65° with the horizontal, 15.86 mm to the left of B
3.10 (a) F = 80 N horizontal at B, towards left; M = 4 N-m (CCW), (b) F_C = 100 N (downward), FD = 100 N (upward)]

CHAPTER

4

Equilibrium of Rigid Bodies

4.1 CONDITIONS OF EQUILIBRIUM

In statics, force analysis is carried out considering the equilibrium of the structures. Thus, establishing condition of equilibrium is utmost important in regard to its study.

These structures quite often comprise several members that are envisaged as rigid bodies. The structure will maintain equilibrium when all the members present in it are in equilibrium, separately.

A body is said to be in equilibrium when it does not have any motion whatsoever, in any direction. The motions essentially are of two types:

(a) Translation
(b) Rotation

Thus, the condition of equilibrium leads to the following requirements that have to be fulfilled simultaneously to prevent both the motions.

- Under the action of several forces – active and reactive – the net force, i.e., the resultant diminishes, and
- The net moment of all the forces with respect to any moment centre should be zero.

4.2 GENERALISED CONDITIONS OF EQUILIBRIUM

A. Graphical Approach: Considering the polygon of forces, *closing arm directed from origin of first vector to the arrowhead of last vector will represent their resultant*.

Thus it can be considered that gross effects of individual forces $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, $\boldsymbol{F_3}$, and so on, can be replaced by a single force $\boldsymbol{R}$ which is the resultant.

Considering the conditions of equilibrium of two forces, if another force $-\boldsymbol{R}$ exists such that it is collinear with that of $\boldsymbol{R}$, the net force will become zero, a necessary condition for equilibrium.

Thus refer to the Fig. 3.9. Following the principle of transmissibility $-\boldsymbol{R}$ can be superimposed over $\boldsymbol{R}$ to obtain equilibrium.

In other words, *when a system is subjected to several coplanar, concurrent forces, these can be represented by a closed polygon under condition of equilibrium.*

A closed polygon implies all the arrowheads follow an order, either clockwise or anti-clockwise to form a closed loop.

If a polygon has n sides, then n^{th} side will represent the magnitude as well as the direction of the resultant of $n - 1$ number of individual forces.

B. Vector Approach: The resultant of several coplanar, concurrent forces can be expressed by $\boldsymbol{R} = R_x\boldsymbol{i} + R_y\boldsymbol{j}$ (Refer to article 3.5)

Thus $\boldsymbol{R}$ becomes zero when both $R_x = 0$ and $R_y = 0$

Mathematically,
$$R_X = \sum_{i=1}^{i=n}(F_i)_x = \sum X = 0 \tag{4.1}$$

and

$$R_Y = \sum_{i=1}^{i=n} (F_i)_y = \sum Y = 0 \tag{4.2}$$

This can be stated as *under condition of equilibrium* ***"Algebraic sum of all the forces along X direction and along Y direction has to be zero simultaneously"***.

These pair of equations is a necessary condition for equilibrium but not sufficient for rigid bodies since zero resultant force implies there is no resultant force, which means that there cannot be any translational motion.

However, to prevent rotary motion, there should not be any resultant moment.

Thus, condition of equilibrium states ***"Algebraic sum of moments of all the forces with respect to any moment centre in its plane should be zero"***.

Thus mathematically
$$\sum M_O = \sum Fd = 0 \tag{4.3}$$

where $\sum M_O$ represents moments of all the forces with respect to any moment centre O.

It can therefore be concluded that the necessary and sufficient conditions of equilibrium of rigid bodies leads to simultaneous satisfaction of Eqs (4.1), (4.2) and (4.3).

However, while dealing with a particle, only Eqs (4.1) and (4.2) become necessary and are sufficient condition of equilibrium. This is in congruence with the Newton's First Law. This is called statics of a particle.

For a particle, the question of moment does not arise since in the absence of moment arm there cannot be any moment.

C. Scalar Approach: From the expression of $R = \sqrt{R_x^2 + R_y^2} = \sqrt{\left[\sum_{i=1}^{i=n}(F_i)_x\right]^2 + \left[\sum_{i=1}^{i=n}(F_i)_y\right]^2}$, it is evident that R is the summation of two square terms.

The necessary condition for R to be zero implies both the terms individually have to be zero, which leads to Eqs (4.1) and (4.2).

Note that $\sum X$ is simplified expression of $\sum_{i=1}^{i=n}(F_i)_x$ and $\sum Y$ is simplified expression of $\sum_{i=1}^{i=n}(F_i)_y$ and hence forth only $\sum X$ and $\sum Y$ will be used.

Let us consider few situations, which are simplified, yet encountered quite often.

Case I: Conditions of equilibrium under two concurrent forces

From the expression of $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos\theta}$, when $\theta = 180°$, and $F_1 = -F_2$, R becomes zero.

This implies net effect of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ on the body is zero and hence equilibrium is restored.

This can be stated as *when a body is subjected to two coplanar, concurrent forces, the equilibrium of the body can lead to the following conditions that must be satisfied simultaneously.*

- *The magnitude of the two forces should be equal.*
- *The direction of the two forces should be opposite.*
- *The forces should be collinear.*

Case II: Conditions of equilibrium under three concurrent forces

When $n = 3$, the polygon is reduced to a triangle as shown in Fig. 2.8.

It can therefore lead to the statement *when a system is subjected to three coplanar, concurrent forces, these can be represented by a closed triangle under condition of equilibrium.*

Thus the expression $\dfrac{P}{\sin\alpha} = \dfrac{Q}{\sin\beta} = \dfrac{R}{\sin\gamma}$ holds true. [Refer to article 2.3.4 and Fig. 2.8]

This is popularly known as ***Lami's Theorem*** and found to be extremely useful in the analysis of structures involving three coplanar, concurrent forces that ensures equilibrium.

4.3 REACTIONS FROM DIFFERENT TYPES OF SUPPORTS

Engineering structure and machine members ensure equilibrium under the influence of active forces and reactions from various supports that constrain their motion. While the magnitude and direction of active forces are known, magnitude or direction of its reactive forces are not known. The force analysis of such systems basically focussed to the computations of reactive forces – both magnitude and directions. Further, depending on the nature of external load to be experienced, wide varieties of supports having their own characteristics are used. It is therefore imperative to study various types of supports before considering force analysis under condition of equilibrium.

Depending on the merits of the structures, the following types of supports are widely used. A particular situation can make use of any combinations of these supports.

4.3.1 Roller or Simple Support

These types of supports are developed by smooth spherical rollers and hence the name. These are called simple supports owing to the fact it is easier to establish the direction of the reaction it offers. In the absence of any frictional forces (since it is smooth), the reactions from such supports are exactly perpendicular to its base on which the rollers are mounted as shown in the Fig. 4.1.

Thus the direction of reactive forces is known with certainty. The contact between the members and the rollers are point contact. Even a smooth wall and a floor may be considered as roller supports when these are smooth, i.e., frictionless. Confirmed direction of reactive forces makes it easy to calculate their magnitudes.

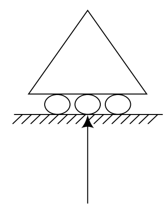


Figure 4.1

4.3.2 Short Inextensible Cable or String

Loads are often applied by means of short inextensible cable as shown in the Fig. 4.2. Such cables are not very strong and hence cannot offer resistance to compression, bending, and torsion and can withstand only tensile force. This tensile force induced in the string acts axially and therefore facilitates to calculate its magnitude only.

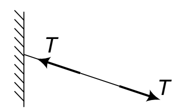


Figure 4.2

4.3.3 Hinge Supports

These supports are developed by inserting a smooth pin through a common hole made between the member to be supported and a bracket of the support as shown in the Fig. 4.3. The bracket is fixed suitably to the intended locations. Such supports, although do not permit any translational movement, but they allow rotation with respect to the centre of the pin. For such supports, complexity arises from the fact that the direction of the reactions it offers is not unique. It is the point of application of external loads together with nature and types of other supports that the system utilizes, govern its direction.

Nevertheless, it would not be very difficult to establish its direction considering the equilibrium of the entire structure as a whole.

To symbolize the uncertainty of directions, it is denoted by a zigzag arrow at the first instance as shown in the Fig. 4.3.

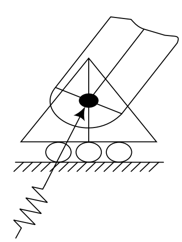


Figure 4.3

Few common examples of hinge joints are tongs, scissors, knifes, etc. Two arms of a tong or of a scissor can be rotated although these cannot be separated.

While dealing with such supports, it is wiser not to ponder too much time over finding its direction. Notwithstanding the fact that its direction is not clear, it can be resolved into two mutually perpendicular components – one horizontal R_X (along $+x$ axis) and other vertical R_Y (along $+y$ axis) as shown in the Fig. 4.4, such that $R = \sqrt{R_X^2 + R_Y^2}$.

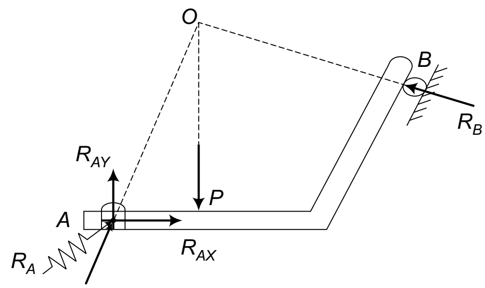


Figure 4.4

Note that this expression does not include the angle of inclination of R.

This does not necessarily mean that R will always make an angle θ with the horizontal such that $0 \le \theta \le 90$, so that R_X and R_Y will always act along positive x axis and positive y axis respectively. If by virtue of calculations, R_X and R_Y are found to be negative, its direction is to be modified accordingly.

Following illustration will help to understand this approach more clearly.

The member AB is subjected to a load P and supported at A and B which are the hinge and roller support respectively.

The reactions from A and B are shown. For roller support, there is no ambiguity. However, being uncertain, the reaction by hinge is shown by a zigzag arrow at the beginning.

This now can be resolved into two components R_{AX} and R_{AY} as shown.

Considering the equilibrium of the member AB under the actions of P, R_A and R_B, these must intersect at a common point. By following principle of transmissibility, P and R_B are made to intersect at O. Thus the third force R_A must intersect no other but the same common point O. Thus, AO represents the exact direction of R_A. Knowing R_A and its inclination, R_{AX} and R_{AY} can be computed.

4.3.4 Cantilever or Fixed Supports

This is another category of supports which is called built-in or fixed, but popularly known as cantilever. It is so named because in essence, one side of the member is fixed and restrains its motion completely and other side is free as shown in the Fig. 4.5. Under such complex loading situation, it can exhibit similar reactions like that of hinge in addition to a couple with moment M to prevent bending.

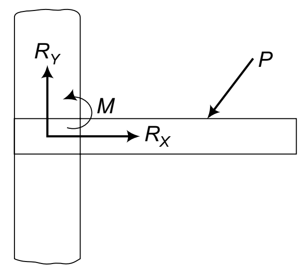


Figure 4.5

4.4 FREE BODY DIAGRAM

Free Body Diagram abbreviated as FBD is a very useful aid to solve the problems of mechanics. The very name implies that a member for which we focus our attention for the purpose of force analysis should be isolated from various constraints and all the forces acting on it – both active and reactive, should be shown without altering its directions. This is a simplified scheme of the actual problem but helps in a bigger way to its solution.

Free-body diagrams now consist of both applied forces and reactions from the other bodies.

Steps for Drawing a FBD

(1) Decide which body to analyse.
(2) Separate this body from all constraints and sketch the contour.
(3) Draw all applied forces. These includes both active forces and reactions from supports.
(4) Include any necessary dimensions and coordinate axis.

Applied forces are forces that try to get the body to move.

Reaction forces are forces that try to prevent motion.

Reactions at supports and connections are of only two types:

1. If a support or connection prevents translation (linear motion) in some direction, then a force may be developed in that direction.
2. If a support or connection prevents rotation about some axis, then a moment may be developed around that axis.

Note: When drawing the forces, if you don't know the direction, assume a direction and let the sign of the answer tell you if the direction is correct or not.

Rules of FBD:

(1) Magnitude and direction of all forces should be clearly indicated.

(2) Indicate the direction of the forces on the body.

4.5 TYPES OF PROBLEMS UNDER EQUILIBRIUM

In most of the 2-D problems, it is found that members ensure equilibrium under the actions of three forces. Given the situations, we have to compute the reactions from various supports – their magnitudes and directions. It may also be required to find out the forces induced in the cables, tie rods, struts, etc, which are also used as supporting members. However, as mentioned earlier, these axial forces are either tensile or compressive.

4.6 SOLUTION STRATEGIES

- Identify the member(s) of interest and draw the free body diagram. The success of solution to equilibrium problem is dependent on this issue.
- Identify the forces (both active and reactive) acting on it along with the clear line of actions.
- Never consider the forces that the member under consideration exerts on the others while drawing FBD.
- Whenever a body is acted upon by three forces, use Lami's Theorem.
- After construction of FBD, resolve the forces into two mutually perpendicular directions, namely x and y. Use Eqs (4.1), (4.2) and (4.3). Note that while taking moment, choose a moment centre scrupulously so that the unknown becomes one. This helps to solve the problem very easily. Further, in one problem, any number of moments can be taken with respect to different moment centres.
- Whenever, a supporting member is stated as tie bar, it implies the induced force will be tensile. On the other hand, if it is termed as strut, the induced force will be compressive.
- While dealing with the hinge supports, do not meddle over finding its direction at the beginning. If the body maintains equilibrium under the actions of three forces involving hinge supports, identify the point of intersection of the other two forces. Thus this point of intersection will be point of concurrency of the three forces. Notwithstanding the direction of the reactions from hinge supports, resolve it into two mutually perpendicular directions.
- You can follow any approach – graphical, analytical or vector.

Example 4.1 A homogeneous prismatic bar AB of weight W is supported by two smooth inclined planes AC and BC that are mutually perpendicular and one (BC) makes an angle α with the horizontal as shown in the Fig. 4.6. If the bar is under equilibrium, what angle (θ) the bar makes with the plane CA?

Solution Since both the planes are perfectly smooth, the reactions offered by them should be just perpendicular to these planes. Thus R_A and R_B meets at point D, implying the other force which is the weight of the bar W will also meet at D. So point D becomes the point of concurrency of the three forces that ensures equilibrium of the bar.

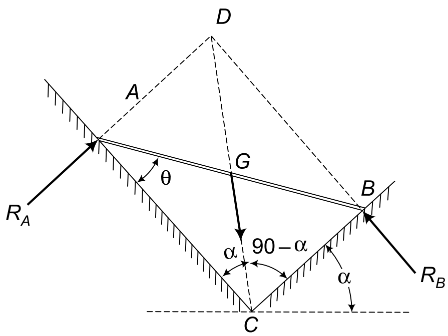


Figure 4.6

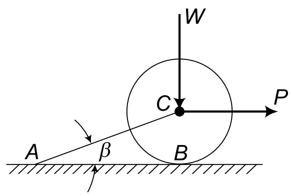


Figure 4.7

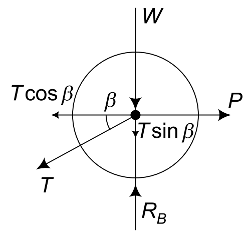


Figure 4.7 (a)

Since the bar is homogeneous; its weight will act at its mid-point (G).
Therefore, $ACBD$ becomes a rectangle.
Thus $\theta = \angle GAC = \angle GCA = \alpha$.
Hence under equilibrium, the bar makes an angle $\theta = \alpha$ with the plane CA.

Example 4.2 A right circular roller of weight W rests on a smooth horizontal plane and is subjected to a pull force P as shown in Fig. 4.7. It is held in position by a string AC. Find the tension T in the string AC and reaction R_B at B.

Solution The free body diagram of the roller is shown in the Fig. 4.7 (a).

Resolving all the forces along x and y direction,

$$\sum X = 0$$

$$P - T \cos \beta = 0$$

$$T = P \sec \beta$$

$$\sum Y = 0$$

$$R_B - T \sin \beta - W = 0$$

$$R_B = P \tan \beta + W$$

Thus tension in the string $AC = T = P \sec \beta$ and $R_B = P \tan \beta + W$

Example 4.3 A load $P = 200$ N is hung by means of a bar AB and a string BC as shown in Fig. 4.8. The inclination of the string with the horizontal is 30º. Calculate the axial forces induced in the bar and in the string.

Solution Let the axial forces induced in the bar and in the string are denoted by T_{AB} and T_{BC} respectively.

From the given configuration of the system, the bar will be under compression and the string will be under tension.

Considering the free body diagram of the point B and using the condition of equilibrium,

$$\sum X = 0$$

$$T_{AB} = T_{BC} \cos \alpha \tag{4.4}$$

$$\sum Y = 0$$

$$P = T_{BC} \sin \alpha \tag{4.5}$$

From (4.5), we get

$$T_{BC} = \frac{P}{\sin \alpha} = \frac{200}{\sin 30} = 400 \text{ N}$$

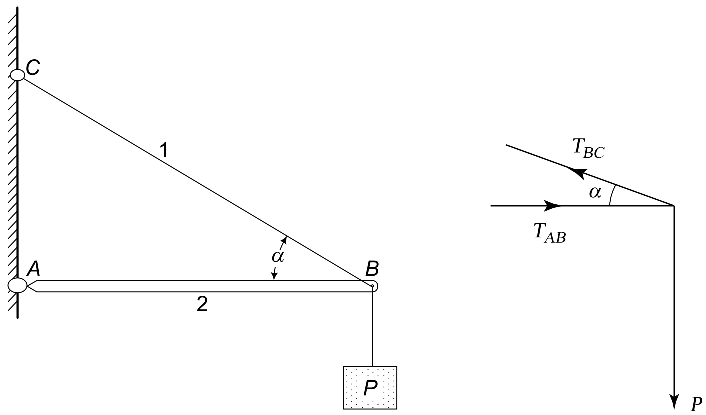


Figure 4.8

Combining Eqs (4.4) and (4.5), $T_{AB} = T_{BC} \cos \alpha = 400 \cos 30 = 346.41$ N

Thus the axial forces induced in the string and the bar are 400 N and 346.41 N respectively.

Example 4.4 A smooth sphere of mass 75 kg is held in a position by means of a vertical wall and an inclined plane as shown in Fig. 4.9. Assuming the supports are frictionless, calculate the reactions exerted by the supports.

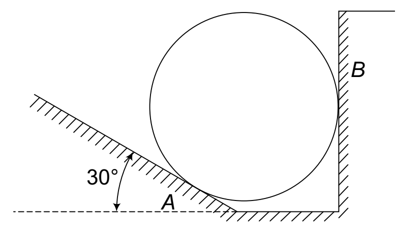


Figure 4.9

Solution The free body diagram of the sphere is shown in Fig. 4.9 (a)

R_A makes an angle 60° with the horizontal.

Thus $(R_A)_X = R_A \cos 60$ and $(R_A)_Y = R_A \sin 60$

Considering the equilibrium of the sphere,

$$\sum X = 0$$

$$R_A \cos 60 = R_B$$

$$\sum Y = 0$$

$$R_A \sin 60 = mg = 75 \times 9.81$$

$$R_A = \frac{75 \times 9.81}{\sin 60} = 849.57 \text{ N}$$

$$R_B = R_A \cos 60 = 849.57 \cos 60 \text{ N} = 424.785 \text{ N}$$

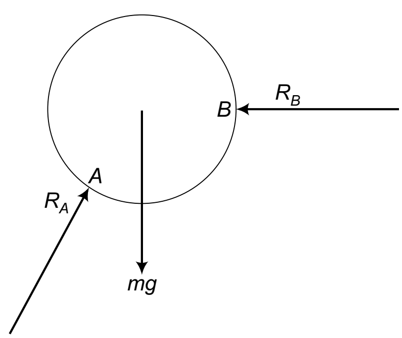


Figure 4.9 (a)

Example 4.5 Two smooth spheres, each of radius $r = 150$ mm and weight $W = 100$ N, rest in a horizontal channel having vertical walls, and the distance between them is $w = 512$ mm, as shown in Fig. 4.10. Find the reactions exerted at their points of contacts by the walls and the floor.

Solution From the free body diagrams of the spheres and geometry,

$$\cos \theta = \frac{C_1E}{C_1C_2}$$

$$C_1E = 2r \cos \theta$$

Thus $w = r + 2r\cos\theta + r = 2r(1 + \cos\theta)$

Taking r = 150 mm and w = 512 mm,

$$\cos\theta = \frac{w}{2r} - 1 = \frac{512}{2\times 150} - 1 = 0.707$$

$$\theta = 45°$$

The force polygons are now drawn for the two spheres as shown in Fig. 4.10 (a) and Fig. 4.10 (b).

Since W = 100 N therefore R_A = 100 N as θ = 45°

Thus $R_{C_1C_2}$ = 100 × $\sqrt{2}$ N = 141.4 N.

From the force polygon of sphere 1; R_B = 100 N and R_D = 200 N.

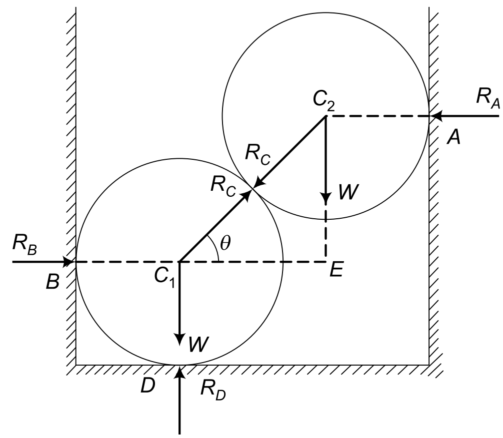


Figure 4.10

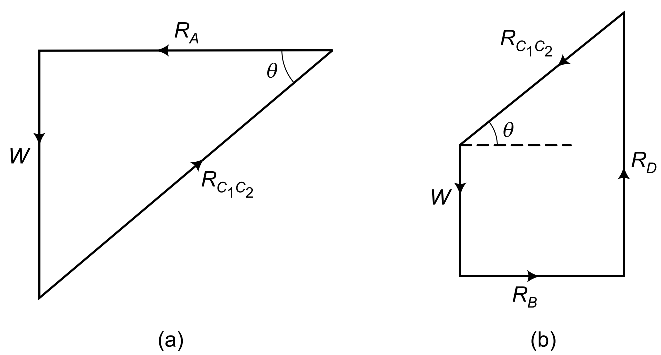


Figure 4.10

Example 4.6 A smooth roller of weight W = 200 N, rests on a smooth inclined plane and is prevented from rolling down by a string as shown in Fig. 4.11. Find the reaction exerted by the plane and the tension in the string.

Solution Force polygon (triangle in this case) is superimposed in the above Fig. 4.11.

By using Lami's theorem,

$$\frac{W}{\sin 75} = \frac{T}{\sin 45} = \frac{R_B}{\sin 60}$$

$$R_B = \frac{\sin 60}{\sin 75} \times W = 179.3 \text{ N}$$

$$T = \frac{\sin 45}{\sin 75} \times W = 146.4 \text{ N}$$

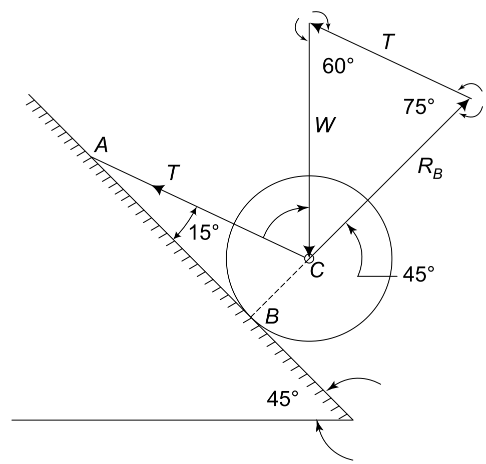


Figure 4.11

Note: Students are advised to solve the same problem by analytical and graphical methods also.

Example 4.7 A smooth right circular cylinder of radius r rests on a horizontal plane and is prevented from rolling by an inclined string AC of length $2r$. A prismatic bar AB of length $3r$ *and* weight W is hinged at point A and leans against the roller as shown in the Fig. 4.12. Find the tension T induced in the string.

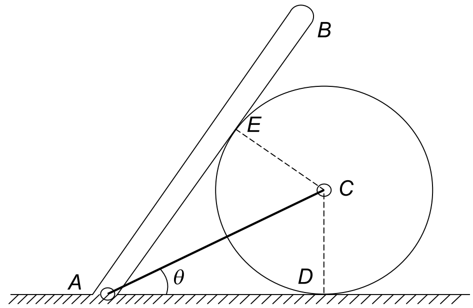


Figure 4.12

Solution From the centre C, perpendiculars CD and CE are drawn on the horizontal plane and AB respectively. ΔACD and ΔACE being similar, $AD = AE$ and $\angle CAD = \angle CAE = \theta$ (say) such that

$$\sin\theta = \frac{r}{2r}$$

or,

$$\theta = 30°$$

Therefore $AD = AE = 2r\cos 30 = \sqrt{3}r.$

Considering the free body of the bar AB [refer Fig. 4.12 (a)] and taking moment at A,

$$\sum M_A = 0$$

$$R_E \times \sqrt{3}r = W\cos 60 \times 1.5r$$

$$R_E = \frac{\sqrt{3}}{4}\ W = 0.433\ W$$

Now considering the free body of the roller [refer Fig. 4.12 (b)] and $\sum X = 0$ we have;

$$R_E\cos\theta = T\cos\theta$$

$$T = R_E = 0.433\ W$$

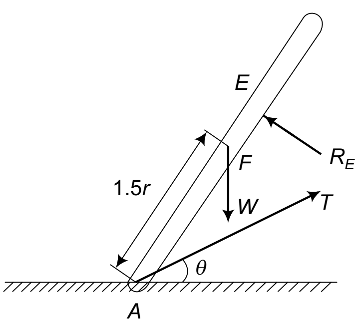


Figure 4.12 (a)

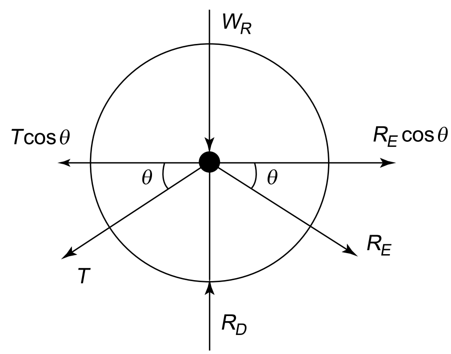


Figure 4.12 (b)

Example 4.8 A vertical prismatic bar AB of negligible weight and length l is hinged to a cylinder of radius r at A and supported at D by an elastic spring CD as shown in Fig. 4.13. The stiffness of the spring is k and the spring is not deformed when $\alpha = 0$. The horizontal force P is applied to the bar AB at B. Find the position of equilibrium, as defined by the angle α.

Solution The force induced in the spring corresponds to deformation amount δ is $F = k\delta$.

In the present situation this deformation δ is tantamount to the arc DG.

Further, from geometry $DG = r\alpha$.

$$\therefore \quad \delta = r\alpha$$

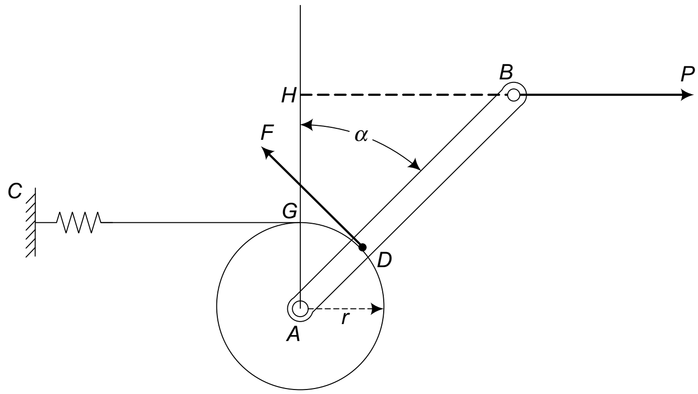


Figure 4.13

The spring force F will act perpendicular to the bar AB.

Considering the equilibrium of the bar AB and having $\sum M_A = 0$

$$F \times r = P \times AH = P \times l \cos \alpha$$

$$k \times r\alpha \times r = Pl \cos \alpha$$

$$\frac{\cos \alpha}{\alpha} = \frac{kr^2}{Pl}$$

Example 4.9 Refer to Fig. 4.14. A mass of weight 30 kg is hung from a bar AB which is hinged at A and is supported by a cable BC. Determine the tension in the cable and the reaction from the hinge at A.

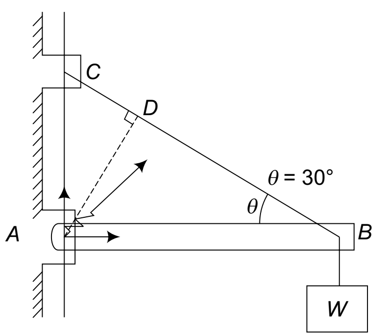


Figure 4.14

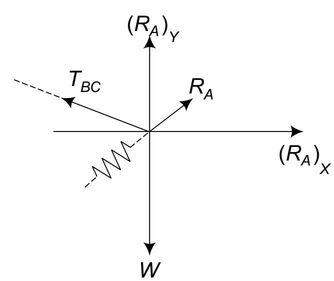


Figure 4.14 (a)

Solution

1st Method: Since A is a hinge point, exact direction of (R_A) cannot be established.

It is therefore convenient to resolve this force into two mutually perpendicular components - along positive x axis and along positive y axis.

However, by virtue of calculation, if $(R_A)_X$ and $(R_A)_Y$ are found to be negative, it implies the directions need to be reversed.

Resolving T_{BC} into two mutually perpendicular components and considering

$$\sum X = 0$$

$$(R_A)_X = T_{BC} \cos \theta \tag{4.6}$$

Since the system is under equilibrium

$$\sum M_A = 0$$

$$W \times AB = T_{BC} \times AD \tag{4.7}$$

$$W \times AB = T_{BC} \times AB \times \sin \theta$$

$$T_{BC} = \frac{W}{\sin\theta}$$

$$= \frac{30 \times 9.81}{\sin 30} = 588.6 \text{ N}$$

$$(R_A)_X = T_{BC} \cos \theta$$

$$= \frac{W}{\tan 30} = \frac{30 \times 9.81}{\tan 30} = 509.74 \text{ N}$$

Considering the bar AB

$$\sum Y = 0$$

$$(R_A)_Y + T_{BC} \sin \theta = W$$

$$(R_A)_Y = 0$$

2nd Method: Consider the bar AB is in equilibrium under the action of three forces, W, T_{BC} and R_A.

So, these three forces constitute a force triangle.

Again, since W and T_{BC} intersect at point B, the third force R_A must pass through point B, implying $(R_A)_X = R_A$ and $(R_A)_Y = 0$.

Considering the force triangle DEF

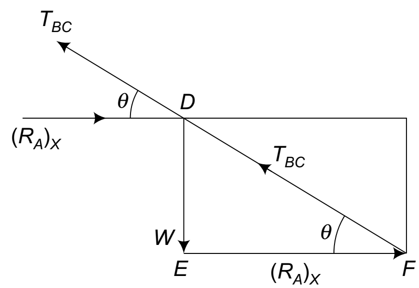


Figure 4.14 (b)

$$\frac{W}{\sin \theta} = \frac{(R_A)_X}{\sin (90 - \theta)} = \frac{T_{BC}}{\sin 90}, \text{ which yields}$$

$$T_{BC} = \frac{W}{\sin \theta} = 588.6 \text{ N}$$

$$(R_A)_X = \frac{W}{\tan \theta} = 509.74 \text{ N}$$

Example 4.10 Two rollers of weight P = 40 N and Q = 80 N are connected by a flexible string AB as shown in Fig. 4.15. These rollers are placed on two inclined planes DE and EF such that $\angle DEF$ = 90° and $\angle EFD = \alpha$ = 30°. Find the tension in the string and the angle θ that it makes with the horizontal when the system is in equilibrium.

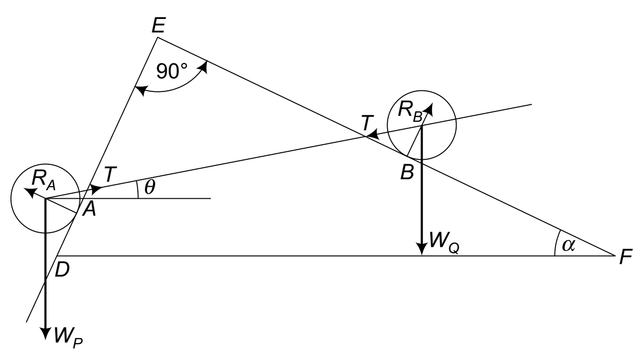


Figure 4.15

Solution From the free body diagram of the two rollers, it is clear that individually they are under equilibrium under three forces and hence these three forces can be represented by closed triangle as shown. Thus the roller at A is in equilibrium under the forces W_P, R_A and T and the roller at B is in equilibrium under the forces W_Q, R_B and T.

Considering force triangle of the roller of weight W_p and using Lami's Theorem,

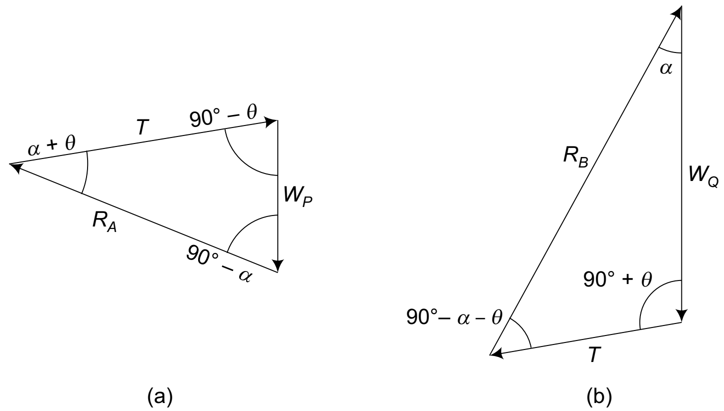


Figure 4.15

$$\frac{W_P}{\sin(\alpha+\theta)} = \frac{T}{\sin(90-\alpha)} = \frac{R_A}{\sin(90-\theta)} \tag{4.8}$$

Similar relation from the force triangle of the roller of weight W_Q yields

$$\frac{W_Q}{\sin\{90-(\alpha+\theta)\}} = \frac{T}{\sin\alpha} = \frac{R_B}{\sin(90+\theta)} \tag{4.9}$$

Comparing Eqs (4.8) and (4.9),

$$T = \frac{\sin(90-\alpha)}{\sin(\alpha+\theta)}W_P = \frac{\sin\alpha}{\sin\{90-(\alpha+\theta)\}}W_Q$$

$$\frac{\cos\alpha}{\sin(\alpha+\theta)}W_P = \frac{\sin\alpha}{\cos(\alpha+\theta)}W_Q$$

$$\tan(\alpha+\theta)\tan\alpha = \frac{W_P}{W_Q}$$

$$\tan(\alpha+\theta) = \frac{W_P}{W_Q\tan\alpha} = \frac{40}{80\tan 30} = 0.866$$

$$(\alpha+\theta) = \tan^{-1} 0.866 = 40.89°$$

$$\theta = 40.89 - 30 = 10.89°$$

Thus, from Eq. (4.8)

$$T = \frac{W_P\cos\alpha}{\sin(\alpha+\theta)} = \frac{40\cos 30}{\sin 40.89} = 52.92 \text{ N}$$

Example 4.11 A prismatic bar AB of weight 50 N and length 3.5 metre is hinged to a vertical wall at A and other end is supported by a horizontal strut BC as shown in Fig. 4.16. Find the axial force induced in the strut and reaction at hinge A.

Solution

A. Graphical Approach Strut is a member which is subjected to compressive load.

From the arrangement of the system, it is evident that self-weight W of the bar will induce axial force in the strut that is directed from B towards C. Thus the strut will exert same amount of force on the bar but in opposite direction.

Thus, the bar AB is under equilibrium under the actions of three forces W, R_A and S_{BC}.

To construct the force triangle, select a scale 1 cm = 10 N. Thus W = 50 N can be represented by 5 cm.

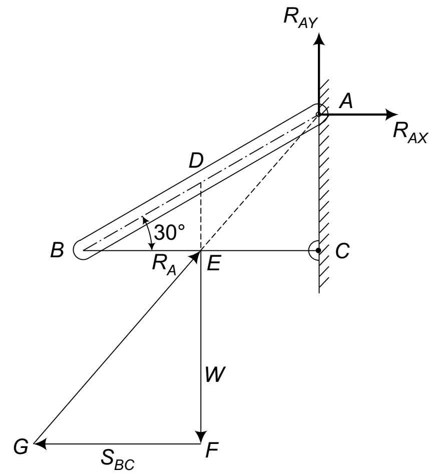


Figure 4.16

From the point D, which is mid-point of AB, a vertical line is drawn. This line will intersect BC at E. Thus point E is the common point of intersection. Thus, reaction R_A will also pass through the point E. From E, a vertical line EF is drawn such that EF = 5 cm. From F, a horizontal line is drawn to represent only the sense of S_{BC} since its magnitude is still not known. Now, AE is extended so as to meet with this horizontal line at G. Now, FG and GE will represent the magnitudes of S_{BC} and R_A respectively. Measure their length in cm and multiply by the scalar factor 10 to convert their magnitudes into N.

B. Analytical Approach Taking moment of all the forces with respect to A,

$$\sum M_A = 0$$

$$S_{BC} \times l \sin 30 = W \times \frac{l}{2} \cos 30$$

$$S_{BC} = \frac{W}{2 \tan 30} = \frac{50}{2 \tan 30} = 43.3 \text{ N}$$

Since A is a hinge, exact direction of R_A is not known. However, it can be resolved into $(R_A)_X$ and $(R_A)_Y$.

From the other conditions of equilibrium,

$$\sum X = 0$$

$$(R_A)_X = S_{BC} = 43.3 \text{ N}$$

and $$\sum Y = 0$$

$$(R_A)_Y = W = 50 \text{ N}$$

Thus $$R_A = \sqrt{(R_A)_X^2 + (R_A)_Y^2} = \sqrt{(43.3)^2 + (50)^2}$$

$$= 66.14 \text{ N}$$

Example 4.12 A nail has to be removed from the ground by the nail diver. If the applied force at A is F = 25 N, what force is applied on the nail?

Solution If the applied force on the nail is R_C, the nail will also apply the same amount of force on the diver as shown in Fig. 4.17.

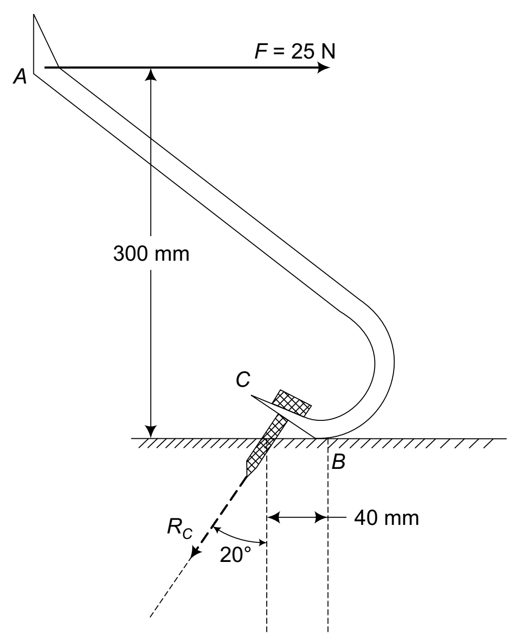


Figure 4.17

Under condition of equilibrium,

$$\sum M_B = 0$$

$$F \times 300 = R_c \cos 20 \times 40$$

$$R_C = \frac{F \times 300}{40 \times \cos 20} = \frac{25 \times 300}{40 \times \cos 20} = 199.53 \text{ N}$$

Thus the force exerted on the nail is 199.53 N.

Note: The horizontal component of R_c (R_c sin 20) will not produce any moment about B since it passes through the moment center.

Example 4.13 A horizontal bar AB is hinged to a vertical wall at A and supported at its mid-point C by a cable CD as shown in Fig. 4.18. The bar is subjected to a vertical load P applied at the free end B. The bar maintains horizontal position. Find the tension T in the cable and the reaction at A. Neglect the weight of the bar.

Solution Let the tension in the cable be T. Thus its horizontal and vertical components become T cos 45 and T sin 45, respectively.

The reaction form hinge A can be resolved into $(R_A)_x$ and $(R_A)_y$ as shown in the free body diagram [Fig. 4.18 (a)].

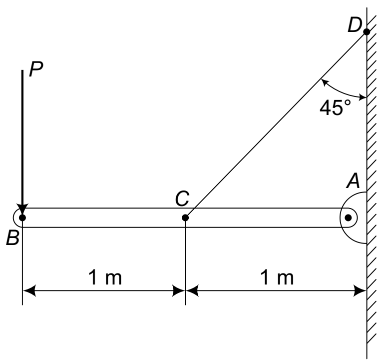


Figure 4.18

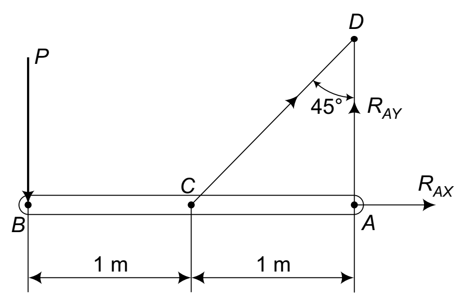


Figure 4.18 (a)

$$\sum M_A = 0$$

$$T \sin 45 \times 1 = P \times 2$$

$$T = \frac{2P}{\sin 45} = 2.828P$$

$$\sum X = 0$$

$$(R_A)_x = -T \cos 45 = -2P$$

$$\sum Y = 0$$

$$(R_A)_y + T \sin 45 = P$$

$$(R_A)_y = P - T \sin 45 = P - 2P = -P$$

Since the sign of $(R_A)_x$ and $(R_A)_y$ is found to be negative, it implies the assumed directions are not correct, rather it has to be reversed to get the correct directions.

Thus $R_A = \sqrt{(-2P)^2 + (-P)^2} = 2.236\ P.$

$\therefore$ The tension in the cable = 2.828 P and the reaction at A due to vertical load P applied at B = 2.236 P.

Example 4.14 A bar AB, 5 m long is hinged to a vertical wall at A and is supported at its other end B by a cable BC. The bar makes an angle 55º with the vertical and the cable makes an angle 30º as shown in the Fig. 4.19. The bar is subjected to a vertical load of 300 N applied at a point D at a distance of 3.5 m along the bar from A. Find the tension T in the cable. Neglect the weight of the bar.

Solution Due to the load applied at D, the bar AB will experience compressive force.

Application of 300 N force will try to rotate the bar with respect to A, but it would be prevented due to tension (T) in the cable BC.

T can be resolved into two mutually perpendicular components as shown in Fig. 4.19.

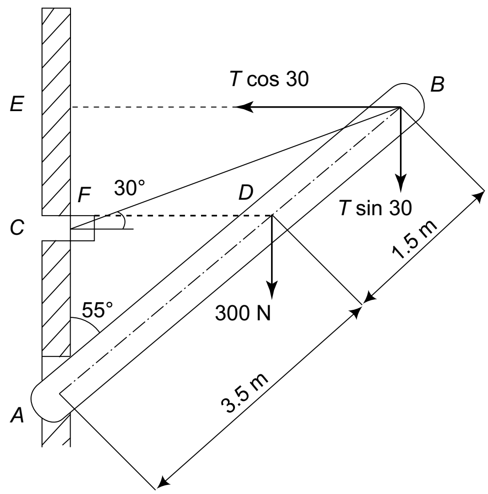


Figure 4.19

Taking moment of all the forces with respect to A and equating it to zero yields

$$\sum M_A = 0$$

$$T \cos 30 \times AE = T \sin 30 \times BE + 300 \times DF$$

From geometry,

$$AE = 5 \cos 55$$

$$BE = 5 \sin 55 \text{ and}$$

$$DF = 3.5 \sin 55$$

$$T[\cos 30 \times 5 \cos 55 - \sin 30 \times 5 \sin 55] = 300 \times 3.5 \sin 55$$

$$T = \frac{300 \times 3.5 \times \sin 55}{5 \times \cos 85} = 1973.73 \text{ N}$$

Thus tension in the cable becomes T = 1973.73 N.

Example 4.15 A man weighing 65 kg stands at the middle rung of a homogeneous ladder that is supported by a smooth vertical wall at B and a stopper at A to prevent slipping. The weight of the ladder is 20 kg and it is 4 m long. Under this configuration, the ladder makes an angle 60º with the horizontal. Find the reactions R_A and R_B at A and B respectively.

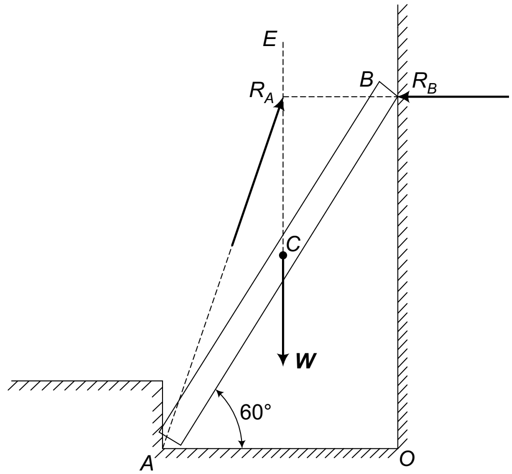


Figure 4.20

Solution The forces acting on the ladder is shown in Fig. 4.20. Since the ladder is homogeneous, its self weight is concentrated at its mid-point C. Thus total vertical downward load on the ladder is $W = W_{man} + W_{ladder} = 65 + 20 = 85$ kg.

Since the wall is smooth, R_B is perpendicular to the wall.

The line of action of W and R_B when extended, meets at point E, implying the third force R_A will also pass through the same point E as shown, since the ladder is under equilibrium under the action of three forces W, R_A and R_B.

Considering equilibrium of the ladder,

$$\sum M_A = 0$$

$$W \times AC \cos\theta = R_B \times AB \sin\theta$$

$$W \times \frac{l}{2} \cos\theta = R_B \times l \sin\theta$$

$$\frac{W}{2} = R_B \tan\theta$$

$$R_B = \frac{W}{2\tan\theta} = \frac{85 \times 9.81}{2 \tan 60} \text{ N} = 240.7 \text{ N}$$

$$\therefore \qquad R_A = \sqrt{(R_B)^2 + (W)^2} = \sqrt{(240.7)^2 + (85 \times 9.81)^2} = 867.9 \text{ N}$$

Thus the reactions R_A and R_B become 867.9 N and 240.7 N, respectively.

Example 4.16 A horizontal prismatic bar AB of negligible weight and length l is hinged at A with the vertical wall at A and supported at B by a tie rod BC that makes an angle θ with the horizontal as shown in Fig. 4.21. A weight W can have any position (x) along the bar. Determine the tension T in the tie bar.

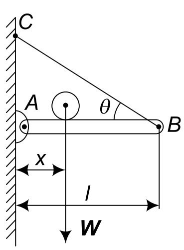


Figure 4.21

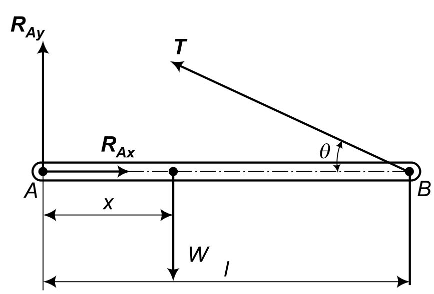


Figure 4.21 (a)

Solution The cable tension T can be resolved into two components – the horizontal component is $T\cos\theta$ and vertical component is $T\sin\theta$.

The reaction form hinge A can be resolved into $(R_A)_x$ and $(R_A)_y$. However, this resolution is not necessary as far as the requirements of the problem is concerned.

Taking moment of all the forces with respect to A,

$$\sum M_A = 0$$

$$T\sin\theta \times l = W \times x$$

$$T = \frac{Wx}{l\sin\theta}$$

Thus tension in the tie bar becomes $T = \dfrac{Wx}{l\sin\theta}$

Example 4.17 A roller of 50 cm diameter weighing 1000 N rests against a rectangular block with a 10 cm height. It is subjected to a pull force P through the centre of the roller so that it will enable the roller to move over the block. Find the minimum value of P and its direction.

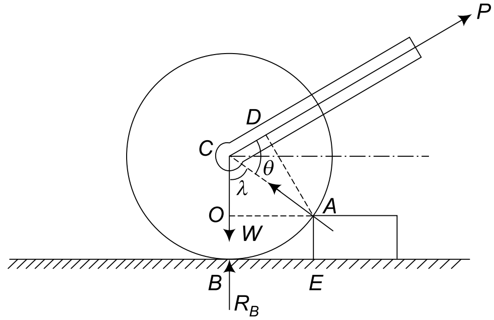


Figure 4.22

Solution The arrangement and the free body diagram of the roller is superimposed and shown in the diagram (Fig. 4.22).

In the situation when the roller is just on the verge of rolling past the block, its contact with the floor ceases. Thus the roller maintains equilibrium under the actions of self weight W, the pull force P and the reaction R_A from the point A of the block.

The radius of the roller = $AC = BC = r$ and height of the block = $AE = OB = h$

From point A, perpendicular AD is drawn on the line of action of P.

Let $\angle ACD = \theta$ and $\angle ACO = \lambda$

$$\sum M_A = 0$$

$$W \times OA = P \times AD = P \times AC\sin\theta$$

$$OA = \sqrt{AC^2 - OC^2} = \sqrt{AC^2 - (BC - AE)^2} = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$$

Thus $$W \times \sqrt{2rh - h^2} = P \times r\sin\theta$$

$$P = \frac{W \times \sqrt{2rh - h^2}}{r\sin\theta}$$

The magnitude of P will be least, when $\sin\theta = 1$ or $\theta = 90°$.

Thus $$P_{min} = \frac{W \times \sqrt{2rh - h^2}}{r} = \frac{1000 \times \sqrt{2 \times 10 \times 25 - 10^2}}{25} = 800 \text{ N}$$

From the ΔACO, $$\cos\lambda = \frac{OC}{AC} = \frac{r-h}{r} = \frac{25-10}{25} = 0.6$$

$$\lambda = 53.13°$$

Thus the direction of P with the horizontal would be $53.13° + 90° - 90° = 53.13°$

Example 4.18 The cylinder of radius r = 25 mm shown in Fig. 4.23 is held by a spanner wrench which is subjected to a vertical load of 80 N applied at free end A. Determine the forces exerted on the cylinder at the points of contact B and C by the spanner wrench. Neglect friction at B and C.

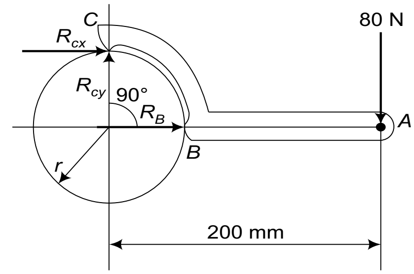


Figure 4.23

Solution Under the action of the vertical load, the cylinder will exert a horizontal reaction force R_B at B as shown. C, being the other point of contact, will act as a hinge, the magnitude of which (R_C) will be equivalent to the resultant of R_B and 80 N.

However, R_C is resolved into two rectangular components $(R_C)_x$ and $(R_C)_y$ as shown.

$$\sum M_C = 0$$

$$R_B \times r = 80 \times 200$$

$$R_B = \frac{80 \times 200}{25} \text{ N} = 640 \text{ N}$$

$$\sum X = 0$$

$$(R_C)_x = -R_B = -640 \text{ N}$$

$$\sum Y = 0$$

$$(R_C)_y = 80 \text{ N}$$

$$\therefore \quad R_C = \sqrt{(-640)^2 + (80)^2} = 645 \text{ N.}$$

Thus reactions at B and C are found to be 640 N and 645 N respectively.

Note: Since $(R_C)_x$ is calculated to be negative, its direction will be opposite to what we have assumed, i.e., it will be directed towards negative x direction.

Example 4.19 A heavy prismatic bar AB of weight W and length l rests at A against a horizontal floor and is pressed against at two intermediate points C and D as shown in Fig. 4.24. Determine the reactions at A, C and at D. Neglect friction at all contact points.

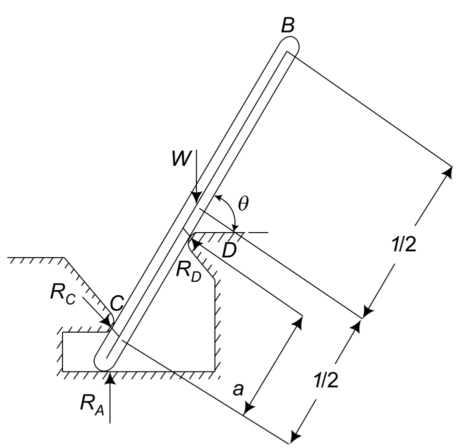


Figure 4.24

Solution The free body diagram of the prismatic bar is shown in Fig. 4.24.

Resolving R_A and W along the axis of the bar (x axis) and its perpendicular direction (y axis) and using the condition of equilibrium,

we have

$$\sum X = 0$$

or

$$R_A \sin \theta = W \sin \theta$$

$$R_A = W$$

$$\sum Y = 0$$

$$R_D + R_A \cos \theta = R_C + W \cos \theta$$

$$R_C = R_D \quad [\text{since } R_A = W]$$

It is interesting to note that R_C and R_D are equal in magnitude, opposite in direction and are parallel. Thus these two forces will form a couple having anti-clockwise moment $M = R_C a = R_D a$.

Likewise R_A and W will also produce a couple having an arm length of $\frac{l}{2}\cos\theta$ that produces clockwise moment, the magnitude of which is $W\frac{l}{2}\cos\theta$.

Equating the clockwise and anti-clockwise moment;

$$R_C \times a = W \cos\,\theta \times \frac{l}{2}$$

$$R_C = \frac{Wl}{2a} \cos\,\theta = R_D$$

Thus $R_A = W$ and $R_C = R_D = \dfrac{Wl}{2a} \cos\,\theta$

Example 4.20 A weight of 200 N is supported by two rigid members AB and BC connected at point B by a pin. Other ends of the two members A and C are hinged to the ground as shown in Fig. 4.25. Calculate the axial forces induced in the two members as a result of 200 N force applied vertically at point B.

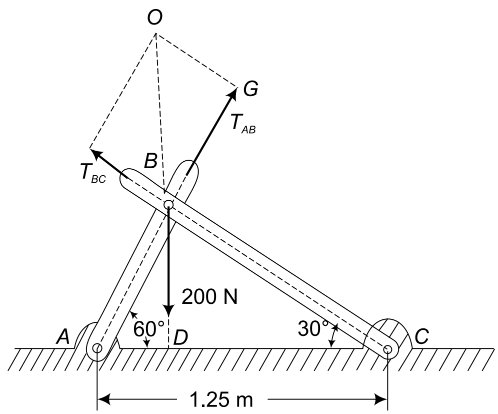


Figure 4.25

Solution Under the action of vertical load, both the members AB and BC will be subjected to axial compressive forces.

The free body diagram of the joint B is shown in Fig. 4.25. It is in equilibrium under the action of three forces namely applied load of 200 N, T_{AB} and T_{BC}.

1st Method: Since the system is under equilibrium, the above three forces will form a closed triangle.

The force triangle OBG is shown in Fig. 4.21.

By using Lami's theorem,

$$\frac{T_{BC}}{\sin 30} = \frac{T_{AB}}{\sin 60} = \frac{200}{\sin 90}$$

$$T_{BC} = 200 \sin 30 = 100 \text{ N}$$

$$T_{AB} = 200 \sin 60 = 173.2 \text{ N}$$

2nd Method: Taking the moments of all the forces with respect to C,

$$\sum M_C = 0$$

$$200 \times CD = T_{AB} \times BC$$

Now,

$$BC = AC \cos 30$$

$$CD = BC \cos 30$$

$$200 \times BC \cos 30 = T_{AB} \times BC$$

$$T_{AB} = 200 \cos 30 = 173.2 \text{ N}$$

Similarly,

$$\sum M_A = 0$$

$$T_{BC} \times AB = 200 \times AD$$

$$T_{BC} \times AB = 200 \times AB \cos 60$$

$$T_{BC} = 200 \cos 60 = 100 \text{ N}$$

Thus axial forces in the members *AB* and *BC* are 173.2 N and 100 N respectively.

Example 4.21 A roller of radius *r* and weight *Q* is to be rolled over a curb of height *h* by a horizontal force *P* applied to the end of a string wound around the circumference of the roller as shown in Fig. 4.26. Find the magnitude of *P* required to start the roller over the curb. There is sufficient friction between the roller surface and the edge of the curb to prevent slip at *A*.

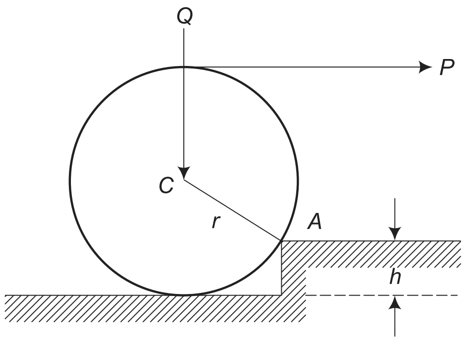


Figure 4.26

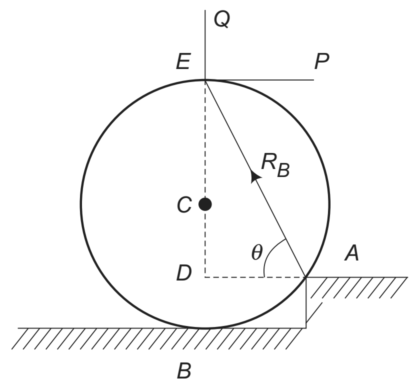


Figure 4.26(a)

Solution The free body diagram of the roller is shown in the Figure 4.26(a). The reactions from the curb R_A, the applied force *P* and the weight of the roller *Q* is represented by a closed triangle (ΔADE) under equilibrium. From the geometry of the Figure [4.26(a)], we get

$$AD = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$$

and $$DE = 2r - h$$

Considering the equilibrium of the roller and taking moment about *A*, we obtain

$$\sum M_A = 0$$

or $$Q \times AD = P \times DE$$

or $$P = \frac{2r - h}{\sqrt{2rh - h^2}} Q$$

Example 4.22 Three identical rollers *A*, *B* and *C* of 350 mm diameter and weight 20 N each are placed in a box which is 760 mm wide as shown in Fig. 4.27. Determine the reactions offered by the floor and the wall on the rollers *B* and *C* and the contact pressures between rollers.

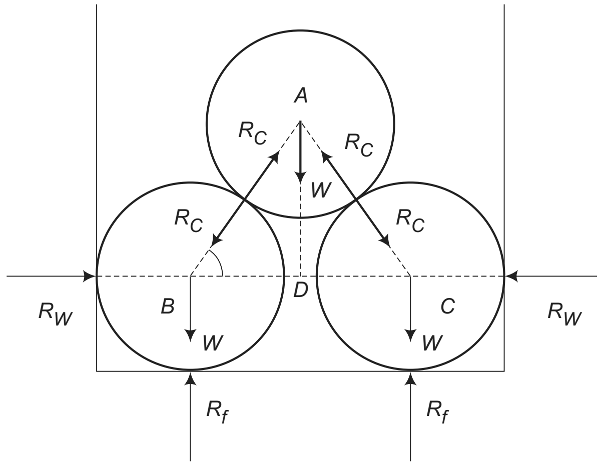


Figure 4.27

Solution From the geometry of the figure [Fig. 4.27], we have

$$\cos\,\theta = \frac{BD}{AB} = \frac{0.38 - 0.175}{0.35} = 0.586$$

$$\sin\,\theta = \sqrt{1-(0.586)^2} = 0.81$$

The reaction exerted by the wall and the floor on B and C as well as contact pressure between A-B and A-C is shown in Fig. 4.27.
Considering the equilibrium of roller A, one can write

$$\sum Y = 0$$

or $$W = 2R_C \sin\,\theta$$

or $$R_C = \frac{W}{2\sin\theta} = \frac{20}{2\times 0.81\sin\theta} = 12.35 \text{ N}$$

Again considering the equilibrium of roller B, we obtain

$$\sum X = 0$$

or $$R_W = R_C \cos\,\theta = 12.35 \times 0.586 = 7.24 \text{ N}$$

and $$\sum Y = 0$$

or $$R_f = W + R_C \sin\,\theta = 20 + 12.35 \times 0.81 = 30 \text{ N}$$

Note: *One can consider the roller C instead of B to get the same results.*

Example 4.23 A uniform roller weighing 100 N is supported by a V-block having included angle 90°. The block is tilted as shown in Fig. 4.28. Compute the angle of tilt θ for which the reaction at B will be one third of that at A. Also compute R_A and R_B.

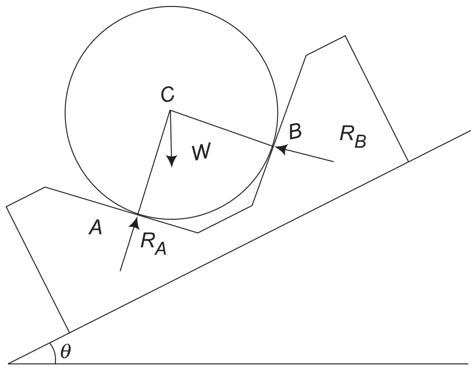


Figure 4.28

Solution The roller is in equilibrium under the action of four coplanner forces, namely W, R_A and R_B [Fig. 4.28]. Using the condition of equilibrium, we have

$$\sum X = 0$$

or $$R_A \cos(45 + \theta) = R_B \cos(45 - \theta)$$

or $$\frac{\cos(45-\theta)}{\cos(45+\theta)} = \frac{R_A}{R_B} = 3 \qquad \left[\because R_B = \frac{1}{3} R_A\right]$$

Solving θ, we get $\theta = 26.56°$

Again, from the other condition of equilibrium, we get

$$\sum Y = 0$$

or $$W = R_A \sin(45 + \theta) + R_B \sin(45 - \theta)$$

Substituting $W = 100$ N, $\theta = 26.56°$ and $R_B = R_A/3$ in the above equation, we obtain $R_A = 94.876$ N and $R_B = 31.625$ N

Example 4.24 A person, weighing 700 N, stands on the middle rung of a ladder of weight 300 N, as shown in Fig. 4.29. Assuming that the floor and the wall are perfectly smooth and slipping is prevented by a string DE; find the tension T in the string and reactions at A and B.

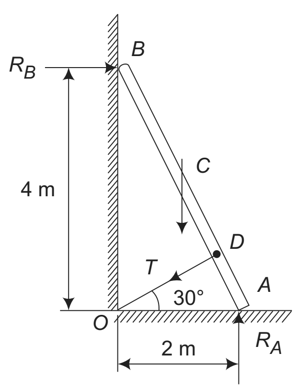


Figure 4.29

Solution The reactive forces R_A and R_B exerted on the ladder by the floor and the wall are vertical and horizontal respectively as shown in the Fig. 4.29. Thus the ladder is in equilibrium under the action of four coplanar forces, namely R_A, R_B, T and total weight W (sum of weight of man and ladder). Using the condition of equilibrium, we have

$$\sum X = 0$$

or
$$T \cos 30° = R_B$$

$$\sum Y = 0$$

or
$$T \sin 30° + 1000 = R_A$$

$$\sum M_O = 0$$

or
$$R_A \times 2 = R_B \times 4 + 1000 \times 1$$

Solving the above three equations, we obtain $R_B = 351.5$ N, $R_A = 1203$ N and $T = 406$ N.

Example 4.25 A uniform ladder 40 meter long and having weight 50 N is held from sliding by a force **P** applied at the lower end as shown in Fig. 4.30. If all surfaces of contact are smooth, determine the force **P**.

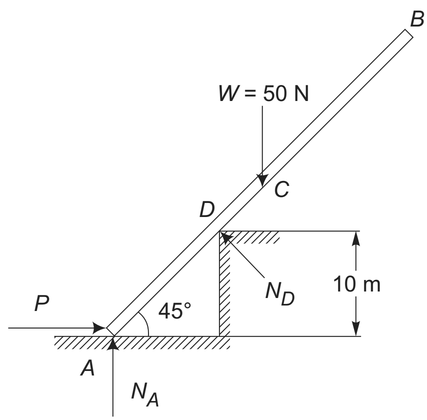


Figure 4.30

Solution Since the ladder is uniform, we have $AC = AB/2 = 20$ m
From geometry of the figure [Fig.4.30], we get

$$AD = \sqrt{2} \times 10 = 14.14 \text{ m}$$

Thus,
$$CD = AC - AD = 5.86 \text{ m}$$

Taking moment about D, we obtain

$$N_A \times 10 + 50 \times 5.86 \cos 45° = P \times 10 \tag{4.4}$$

Further moment about point A yields

$$N_D \times 14.14 = 50 \times 20 \cos 45° \tag{4.5}$$

Again, from the consideration of equilibrium, we get

$$\sum Y = 0$$

or
$$N_A + N_D \sin 45° = 50 \tag{4.6}$$

Solving Eqs. (4.4), (4.5) and (4.6), we obtain $P = 45$ N

Example 4.26 A uniform beam AB of length 1 m and weight 10 N is held in equilibrium against a vertical wall and a string BC attached to it as shown in Fig. 4.31. Find the reaction at the wall and the tension in the string.

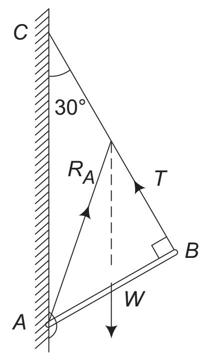


Figure 4.31

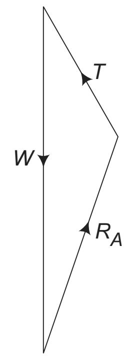


Figure 4.31(a)

Solution The force triangle is shown in Fig. 4.31(a). Following graphical approach T and R_A are measured to have values of 4.33 N and 6.625 N respectively.
Note: *The readers are advised to check the results by analytical method.*

Example 4.27 A homogeneous prismatic bar AB of weight W and length l is supported at one end B by a string BC of length a and rests at A exactly below C by a smooth vertical wall as shown in Fig. 4.32. Determine the position of the bar as defined by the length x that ensures equilibrium.

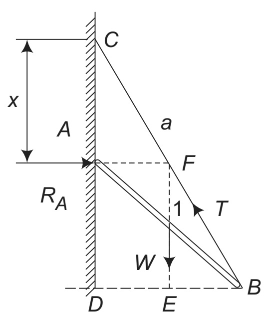


Figure 4.32

Solution The weight of the bar AB acts at the mid of its length, since it is homogeneous. The tensile force T in string BC acts along its length.
The line of action of W and T intersects at F implying that the third force R_A also passes through this common point.

Considering the ΔADB, E becomes the mid point of BD. Further as regard to ΔBCD, F is the midpoint of BC and interestingly this is also the common point of intersection of the three forces W and T and R_A that hold the system under equilibrium. Following the properties of triangle, point A also becomes the midpoint of CD. From $\Delta ABCD$, we get

$$a^2 = (x + x)^2 + BD^2$$

and from ΔADB,

$$l^2 = x^2 + BD^2$$

Comparing the above two equations, we obtain

$$a^2 - 4x^2 = l^2 - x^2,$$

or

$$x = \left[\frac{a^2 - l^2}{3}\right]^{\frac{1}{2}}$$

Example 4.28 A weight W is attached to a pulley that rides on a wire which is attached to a wall at the left and connected to a weight Q that passes over a pulley at the right as shown in Fig. 4.33. The horizontal distance between the left support and the right pulley is L. Determine the sag d at the center in terms of W, Q and L neglecting the dimensions of the pulleys.

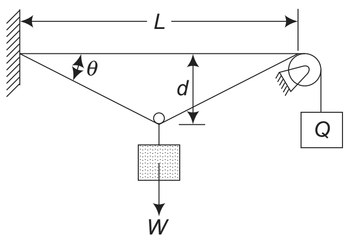


Figure 4.33

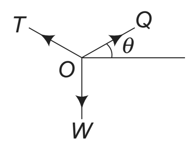


Figure 4.33(a)

Solution Considering equilibrium of the point O [Fig. 4.33 (a)], one can write

$$\sum X = 0$$

or

$$T \cos \theta = Q \cos \theta$$

or

$$T = Q \tag{4.7}$$

and

$$\sum Y = 0$$

or

$$2Q \sin \theta = W \qquad \text{[Since } T = Q\text{]} \tag{4.8}$$

From the geometry of the figure [Fig. 4.33], we get

$$\sin \theta = \frac{d}{\sqrt{d^2 + (L/2)^2}}$$

Substituting the value of $\sin \theta$ into Eq. (4.8), we obtain

$$2Q \times \frac{d}{\sqrt{d^2 + (L/2)^2}} = W$$

Squaring both sides and rearranging; $\dfrac{4Q^2}{W^2} = 1 + \dfrac{(L/2)^2}{d^2}$

from which, we obtain $d = \dfrac{L}{2}\dfrac{1}{\sqrt{\left(\dfrac{2Q}{W}\right)^2 - 1}}$

Example 4.29 A system of levers are loaded and supported as shown in Fig. 4.34. Determine the reaction forces at the supports A, B and C.

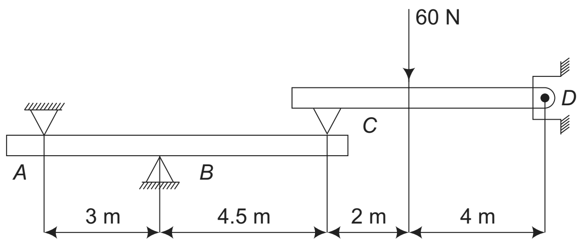


Figure 4.34

Solution The free body diagrams of the two levers are shown in Fig. 4.34(a). Since the entire system of levers is under equilibrium; each lever separately would also maintain equilibrium.
Considering the upper lever and taking moment about D, we obtain

$$\sum M_D = 0$$

or $$R_C \times 6 = 60 \times 4$$

or $$R_C = 40 \text{ N}$$

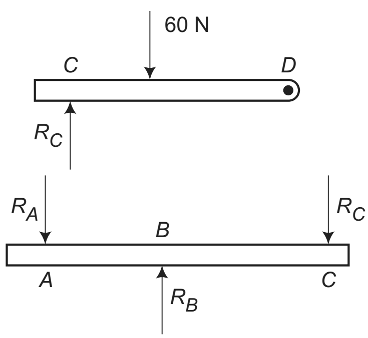


Figure 4.34(a)

When the equilibrium of the lower lever is considered, the same R_C will act downward as evident from the free body diagram. Taking moment about A, we get

$$\sum M_A = 0$$

or $$R_C \times 7.5 = R_B \times 3$$

or $$R_B = 100 \text{ N}$$

Further, from the consideration of $\sum Y = 0$, we obtain

$$R_A + R_C = R_B$$

or $$R_A = 100 - 40 = 60 \text{ N}$$

Example 4.30 The frictionless pulley at A is supported by two rigid bars AB and AC which are hinged at B and C respectively to a vertical wall as shown in Fig 4.35. The flexible cable AD is hinged at D and goes over the pulley to carry a vertical load of 20 kN. Neglecting the size of the pulley, determine the forces in AB and AC.

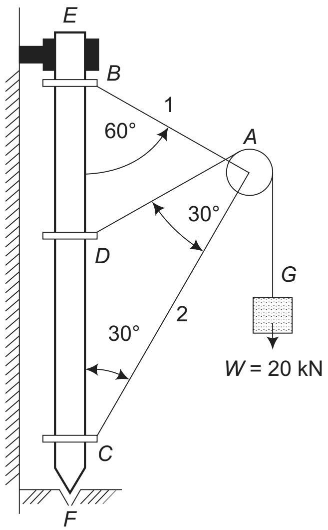


Figure 4.35

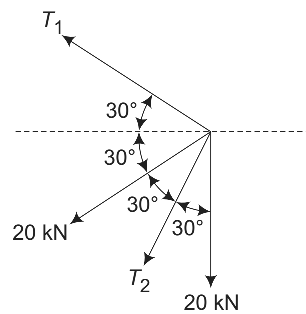


Figure 4.35(a)

Solution The free body diagram is shown in Fig. 4.35(a). From the geometry, it is evident that $\angle BAC = 90°$. Establishing AB and AC as x and y axes respectively, the equilibrium of point A yields

$$\sum X = 0$$

or $$T_1 + 20 \sin 30° - 20 \sin 30° = 0$$

$$T_1 = 0$$

and $$\sum Y = 0$$

or $$T_2 + 20 \cos 30° + 20 \cos 30° = 0$$

or $$T_2 = -34.6 \text{ kN}$$

The negative sign indicates that the axial force in the bar AC will be compressive.

MULTIPLE-CHOICE QUESTIONS

4.1 A number of forces acting at a point will be in equilibrium if
(a) their total sum is zero.
(b) sum of resolved parts in any two perpendicular directions are both zero.
(c) all of them inclined equally.
(d) two resolved parts in two directions at right angles are equal.

4.2 Two non-collinear parallel equal forces acting in opposite direction
(a) balance each other. (b) constitute a moment.
(c) constitute a moment of couple. (d) constitute a couple.

4.3 The necessary and sufficient condition of equilibrium for a two-dimensional force system is
(a) $\sum X = 0$ and $\sum Y = 0$ (b) $\sum M_o = 0$
(c) both (a) and (b) (d) none of the above

4.4 The necessary condition for forces to be in equilibrium is that these should be
(a) coplanar (b) meet at a point (c) both of these (d) none of these

4.5 If three forces acting in different planes can be represented by a triangle, these will be in
(a) equilibrium (b) non-equilibrium (c) unpredictable (d) partial equilibrium

4.6 The algebraic sum of moments of the forces forming couple about any point in their plane is
(a) constant (b) equal to the moment of the couple
(c) both of these (d) none of these

SHORT ANSWER TYPE QUESTIONS

4.1 What is meant by equilibrium of rigid bodies? What are the necessary conditions to ensure the same?
4.2 Mention different types of supports with sketches showing the reactions that are commonly used in structures.
4.3 What is the condition of equilibrium of a rigid body subjected to three coplanar and concurrent forces?
4.4 What is *Free Body Diagram* (FBD)? How it is useful in solving equilibrium problems?
4.5 What do you understand by *statics of a particle*?
4.6 What is a hinge support? What motion does it permit and what motion does it not?
4.7 What types of reactions do a cantilever (fixed support) exhibit?
4.8 For the equilibrium of a particle $\sum X = 0$ and $\sum Y = 0$ are necessary and sufficient conditions but for rigid bodies these are not sufficient. Justify.

NUMERICAL PROBLEMS

4.1 Refer to Fig. 4.36. Determine the magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ so that the particle is in equilibrium.
4.2 Determine the maximum weight of the flowerpot as shown in Fig. 4.37 that can be supported without exceeding a cable tension of 50 N in either cable *AB* or *AC*.

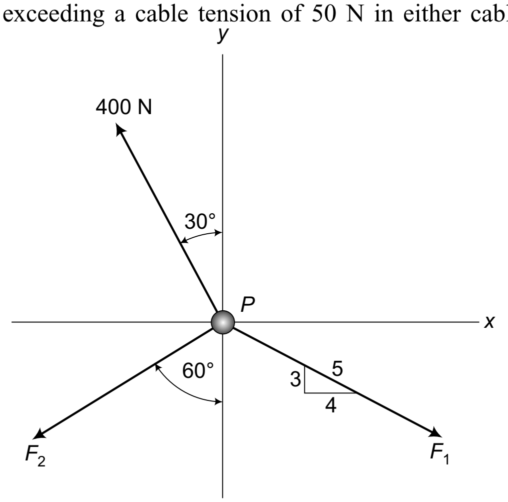


Figure 4.36

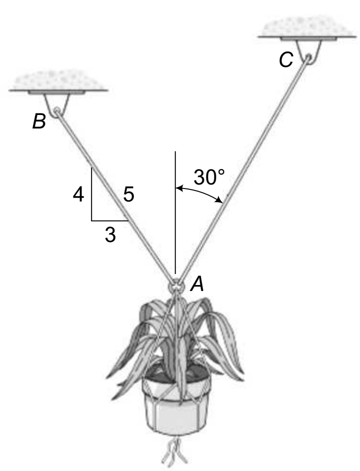


Figure 4.37

4.3 Determine the horizontal force ***P*** and tension in the string that is connected to the ceiling to hold the 80 N force in equilibrium (Fig. 4.38).

4.4 The bar *AB* weighs 250 N is supported by a wall at *C* and a horizontal cable as shown in Fig. 4.39. Assuming all surfaces are smooth, find the cable tension and forces at *A* and *C*.

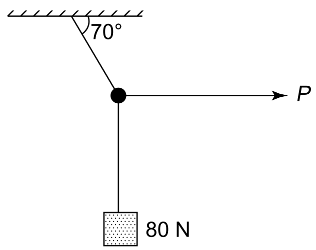


Figure 4.38

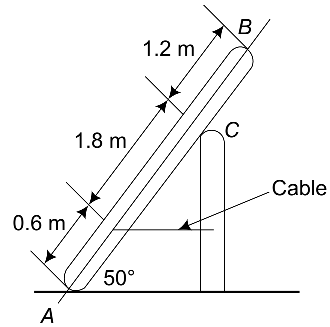


Figure 4.39

4.5 If the wheelbarrow and its contents have a mass of 60 kg and a centre of mass at G as shown in Fig. 4.40, determine the magnitude of the resultant force which the man must exert on each of the two handles in order to hold the wheelbarrow in equilibrium.

4.6 Refer to Fig. 4.41. Determine the tension in the cable and the horizontal and vertical components of the reaction at pin *A*. The pulley at *D* is frictionless and the cylinder weighs 80 N.

4.7 Determine the distance *d* for placement of the load *P* for equilibrium of the smooth bar in the position θ as shown in Fig. 4.42. Neglect the weight of the bar.

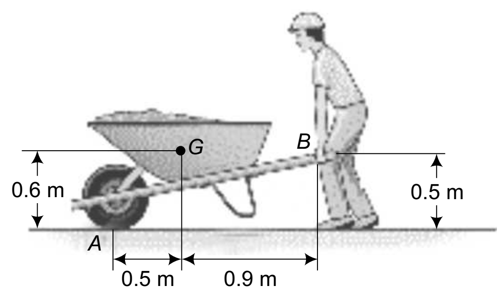


Figure 4.40

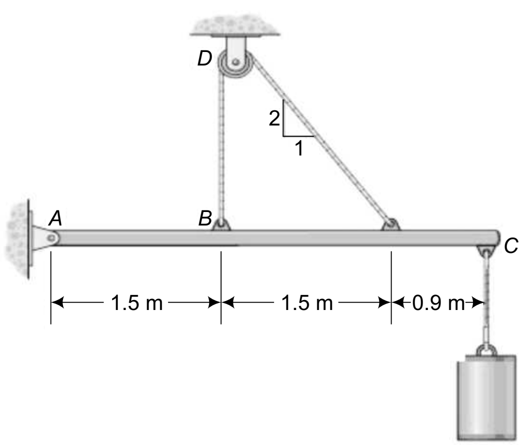


Figure 4.41

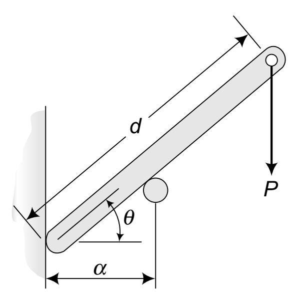


Figure 4.42

4.8 The beam is loaded as shown in Fig. 4.43. Find the reactions at *A*. The reaction includes the reaction force as well as the moment.

4.9 Given the mass of the crate is 50 kg. Find the horizontal force *P* needed to position the crate directly over the wagon as shown in Fig. 4.44.

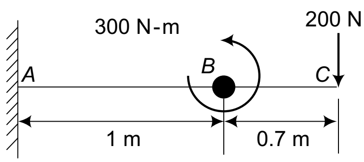


Figure 4.43

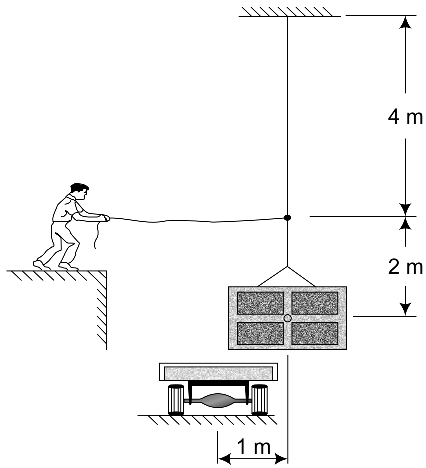


Figure 4.44

4.10 A rigid bar ***AB*** of negligible weight and 4 metres long is hinged at ***A*** and carries a vertical downward load of P = 700 N at ***B*** as shown in the Fig. 4.45. The bar is maintained equilibrium by a horizontal tie bar ***CD***. Assuming *D* as a midpoint of the bar *AB*, determine the tension in the tie bar and reaction at *A*.

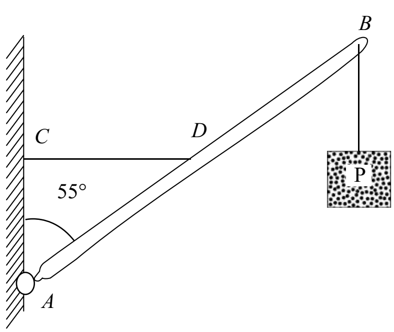


Figure 4.45

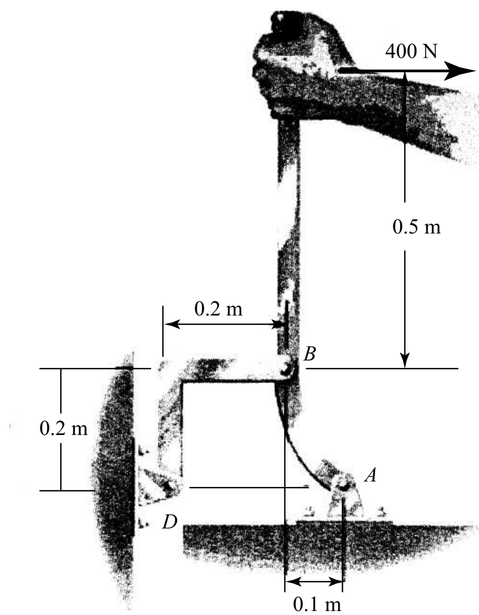


Figure 4.46

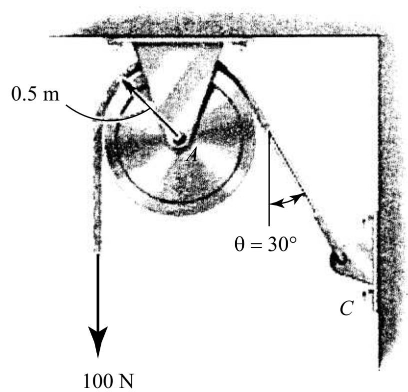


Figure 4.47

4.11 The lever ABC is pin-supported at A and connected to a short link BD as shown in Fig. 4.46. If the weight of the members are negligible, determine the components of the force of the pin on the lever at A.

4.12 The cord shown in Fig. 4.47 supports a force of 100 N and wraps over a frictionless pulley. Determine the tension in the cord at C and the horizontal and vertical components of the reaction at the pin A.

4.13 Two smooth circular cylinders of each of weight W and radius r are connected by a string AB of length l and rest upon a horizontal plane, supporting a third cylinder of weight Q and radius r above them, as shown in Fig. 4.48. Find the tension T in the string AB and the pressure produced by the floor at points of contact D and E.

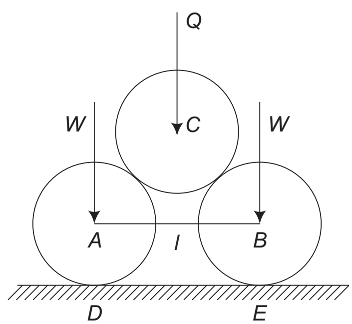


Figure 4.48

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

4.1 (a)	4.3 (c)	4.5 (a)
4.2 (d)	4.4 (c)	4.6 (b)

ANSWERS TO NUMERICAL PROBLEMS

4.1 $F_1 = 434.9$ N, $F_2 = 170.8$ N

4.2 76.72 N

4.3 $T = 85.14$ N, $P = 29.11$ N

4.4 $R_A = 59.2$ N, $R_C = 141.26$ N, $T = 108.2$ N

4.5 105 N

4.6 T = 74.6 N, $A_x = 33.4$ N, $A_y = -61.3$ N

4.7 $\cos^3\theta = \dfrac{a}{d}$

4.8 $(R_A)_X = 0$, $(R_A)_Y = 200$ N, $M_A = 40$ N-m

4.9 126.7 N

4.10 $T_{CD} = 2000$ N, $R_A = 2119$ N

4.11 $A_x = 533$ N, $A_y = 933$ N

4.12 $T = 100$ N, $A_x = 50.0$ N, $A_y = 187$ N

4.13 $T = \dfrac{Ql}{2\sqrt{16r^2 - l^2}}$, $R_D = R_E = W + \dfrac{Q}{2}$

CHAPTER

5

Centre of Gravity

This topic is broadly divided into two. The first part deals with computations of centre of gravity of single geometrical entities while the second part deals with composites.

PART A

5.1 INTRODUCTION

In all our earlier discussions, we had considered that the forces whether active, reactive or any others, are all concentrated at the point of application. But actually such a consideration is of no practical significance since a definite amount of force can be applied only over a finite area. Further, the distribution of the forces over the entire area is also not uniform under all circumstances; rather it varies depending on the geometry of the surfaces and direction of the applied load. Study of the actual distribution pattern of the forces over a finite area is too complex and quite often there is absolutely no need of such study keeping the global objective in mind. In other words, the concept of idealisation of assuming a concentrated load will not be erroneous provided the area is too small compared to other relevant dimensions of the body – a situation which is predominant in the study of mechanics. Thus, the forces that are distributed over the entire area and its equivalence to a single concentrated load are tantamount to a system of parallel forces and its resultant. Such scrupulous approach forms the basis of computation of centre of gravity of rigid bodies.

5.2 IMPORTANCE OF CENTRE OF GRAVITY

The study of mechanics revolves around concept of rigid bodies and consequence of forces applied on it that ensure equilibrium. Such rigid bodies may follow any unique fundamental geometric shape or it may be a composite. While dealing with the forces, when the self-weight of the body is appreciable, we cannot neglect its effect as evident from our preceding discussions. Since the point of application of the load has important bearings to this effect, need for computing the exact point of a body at which the entire weight is concentrated, needs no special mention. *It is the centre of gravity of the rigid body at which the entire weight of the body is concentrated.*

Thus, establishing the centre of gravity is very relevant in the study of mechanics.

5.3 CENTRE OF GRAVITY

The centre of gravity is a point through which the entire body weight acts vertically downwards. The weight of a body is essentially a force by which the body is attracted towards gravity. It is the Newton's well known *Law of Gravitation* that explains this phenomenon.

Physical bodies have a definite mass due to the gross effect of several particles which constitute them and such particles also occupy finite volume in space. All such particles are distributed through out the volume, so their distances from the centre of earth will also be different, and also the gravitational force each of them experiences. Nevertheless, the variations in the aforesaid distances are negligibly small since the various dimensions of the body are too small in comparison to the radius of the earth. Thus, it would be prudent to envisage that these particles are identically located with respect to the earth's centre of attraction. In essence, the gross weight of the body is the resultant gravitational force of its constituent particles and the centre of gravity is a point at which resultant gravitational force is acting. So in the light of mechanics, the situation is nothing special that these gravitational forces (individual and resultant) form a system of parallel forces. Such equivalence helps to compute the location of centre of gravity of rigid bodies.

5.4 COMPUTATION OF CG

In congruence with the above discussion, let us consider a rigid body is divided into infinitesimal small elements of mass *dm* and weight *dW*. Thus *W* is the resultant of all the weights *dW*.

Since the point of applications of these body forces is unique, any change in the orientation of the object will not alter the position of their body forces their resultant. Nevertheless, it is quite useful to describe the location of this point with respect to a coordinate frame.

Thus refer to Fig. 5.1, if *C* is considered the *CG* such that its coordinate is (x_c, y_c, z_c) then by following Varignon's Theorem, the moment of the resultant with respect to the moment centre *O* will be equal to the sum of moments of all the individual forces with respect to the same moment centre *O*.

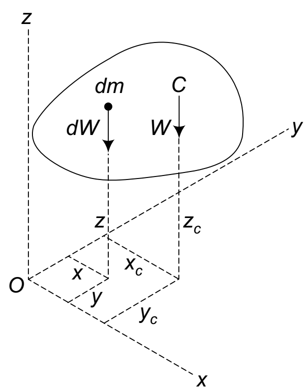


Figure 5.1

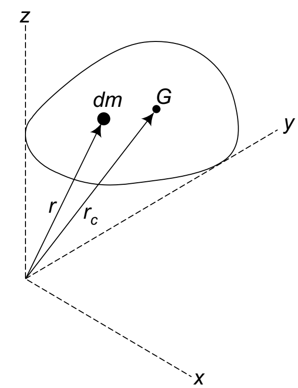


Figure 5.2

Thus mathematically,

$$W \times x_c = \int x dW \text{ or } x_c = \frac{\int x dW}{W} = \frac{\int x dW}{\int dW} \tag{5.1}$$

If all the forces as well as their resultant are rotated by 90°, the situation will remain unaltered and following the same principle, we can compute,

$$y_c = \frac{\int y dW}{\int dW} \text{ and } z_c = \frac{\int z dW}{\int dW}. \tag{5.2}$$

Now $dW = dm.g$

Thus
$$x_c = \frac{\int x \times dm \times g}{\int dm \times g} = \frac{\int xdm}{\int dm};\ y_c = \frac{\int ydm}{\int dm} \text{ and } z_c = \frac{\int zdm}{\int dm} \tag{5.3}$$

If it is considered that $r_c = x_c i + y_c j + z_c k$ is the position vector of the CG and $r = xi + yj + zk$ (Refer Fig. 5.2) is the position vector of a small element of mass dm then

$$r_c = \frac{\int rdm}{\int dm} \tag{5.4}$$

Thus the point C is also called ***the centre of mass*** so long as g remains constant.

Further if the body is homogeneous such that its density ρ remains constant, the expressions of (x_c, y_c, z_c) from Eq. (5.3) become

$$x_c = \frac{\int xdm}{\int dm} = \frac{\int x \times \rho \times dV}{\int \rho \times dV} = \frac{\int xdV}{\int dV};\ y_c = \frac{\int ydV}{\int dV} \text{ and } z_c = \frac{\int zdV}{\int dV} \tag{5.5}$$

5.5 COMPUTATION OF CENTROID

It is noteworthy to see that expressions under Eqs (5.5) include only the geometrical attributes. ***Centroid is a terminology coined with calculations involving geometrical parameters only***. Therefore x_c, y_c and z_c of Eqs (5.5) can be considered as coordinates of centroid of the rigid bodies. *Thus, in case of homogeneous objects, centre of mass and centroid coincide*.

The computation of centroid of various geometrical entities needs no special care since basic concept remains same. Nevertheless, based on few attributes, the entire range of objects can be classified under three categories, namely, ***volumes, areas and lines***. Thus the expressions of Eq. (5.5) get modified and computational efforts also become reduced.

5.5.1 Volumes

This is a feature that is associated with a three-dimensional object. Thus for a solid geometrical object, the Eq. (5.5) will hold true.

5.5.2 Areas

The volume of a three-dimensional object of uniform thickness t can be calculated by multiplying its area with the thickness. Thus for such an object we can write $V = A \times t$ and similarly, the volume of an infinitesimal small element $dV = dA \times t$

Thus from Eq. (5.5),

$$x_c = \frac{\int xdV}{\int dV} = \frac{\int x \times t \times dA}{\int t \times dA} = \frac{\int xdA}{\int dA};\ y_c = \frac{\int ydA}{\int dA} \text{ and } z_c = \frac{\int zdA}{\int dA} \tag{5.6}$$

The numerators on the right-hand side of all the expressions of Eq. (5.6) are called ***first moments of area.***

Thus if the coordinates x_c, y_c and z_c are computed and knowing the area of the object, first moments can be calculated as $\int xdA = x_c \int dA$; $\int ydA = y_c \int dA$ and $\int zdA = z_c \int dA$

5.5.3 Lines or Slender Bars

Quite often, we encounter objects of uniform cross-section, like a slender bar or rod or thin wire. The volume of such objects of uniform cross-section A can be calculated by multiplying its area with the length. Thus for such an object we can write $V = A \times L$ and similarly the volume of an infinitesimal small element $dV = A \times dL$

Thus from Eq. (5.5),

$$x_c = \frac{\int x dV}{\int dV} = \frac{\int x \times A \times dL}{\int A \times dL} = \frac{\int x dL}{\int dL};\ y_c = \frac{\int y dL}{\int dL} \text{ and } z_c = \frac{\int z dL}{\int dL}$$

Note: It is to be borne in mind that centroid of slender bars of uniform cross-section and thin surfaces of uniform thickness may not always lie within the body itself. The results of few case studies in the subsequent discussion will corroborate this statement.

In the context of mechanics, in majority of cases, we are supposed to deal with plane geometrical entities. It is, therefore, imperative to emphasize on computations of centroid. However, centroid and centre of gravity become synonymous and are also found to be identical when the density and acceleration due to gravity remain constant.

Road Map to Solution of Problems

Types of Problems

In problems, we encounter several types of geometrical entities – solid, plane lamina, slender bar or uniform wire for which *CG* is to be computed. However, computations of *CG* based on area are more common. These areas are formed by single geometrical entities or it may be an area formed by intersection of two curves.

Solution Guidelines

- Position of origin with respect to which all the coordinates to be calculated has got significant bearings on overall calculations.
- While selecting an infinitesimal small element, one must be careful about judicious selection of it, such that when integrated over given limits it must cover the entire domain.
- It is to be remembered that in the Eq. (5.5), x, y and z are coordinates of the *CG* of the small element and its value to be correctly computed before integration.
- Uses of polar coordinates is simpler for bodies having circular boundaries.
- If the objects become symmetrical with respect to any particular axis, then its *CG* will lie on that axis. Thus if an area is found to be symmetrical with x axis, its *CG* will lie on x axis and hence y_c will become zero. However, unbiased computational result will justify the authenticity of this statement.
- In case of symmetrical objects, instead of integrating over the entire area, it can be considered just half and results of integration so obtained should be multiplied by 2. At the same time, one must take care of the change of limits of integration.
- For geometrical regions covered by curves for which generic equations are given, the actual equations of the curve are to be obtained first from the given limits.
- It is the sole discretion of an individual to select any type of small infinitesimal element for the purpose of integration provided it serves the intended objectives.

5.6 THEOREMS OF PAPPUS AND GULDINUS

Two theorems developed by **Pappus and Guldinus** are found to be extremely useful in calculating the surface areas and volumes of different geometrical entities and are intertwined with the centre of gravity. These two theorems have got adequate utilities in the ***Computer Aided Drawings***.

5.6.1 Theorem 1

The area of the surface generated by rotating a plane curve about a non-intersecting axis in its plane is equal to the product of the length l of the curve and the distance traveled by its centroid.

Proof: To prove the above theorem, let us consider a plane curve MN of length l having its CG located at C that lies in the xy plane. Let the distance of CG from x axis be $\bar{y}$.

Let's consider an elemental length dl having its CG located at a distance y from x axis.

Now the curve is rotated by an angle θ to assume a new position $M'N'$. Thus the original plane curve of length MN will now generate a surface $MNN'M'$. Consequently the modified position of CG now becomes C'.

Thus the elemental length dl of the curve covers a distance $y\theta$ and the CG of the entire curve covers a distance $\bar{y}\theta$.

The elemental area of the surface generated = $dA = y\theta.dl$

Thus the entire surface area of the curve will be $A = \int dA = \int y\theta.dl = \theta\int ydl.$

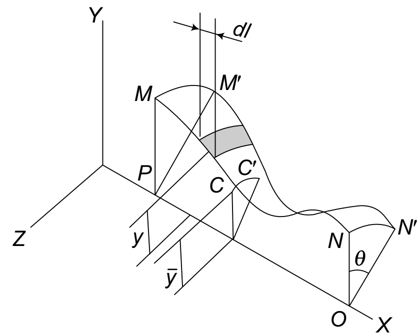


Figure 5.3

But by definition, $\bar{y} = \dfrac{\int ydl}{\int dl}$ $\quad \therefore \int ydl = \bar{y}l$

Thus area of the surface = $A = \theta\int ydl = \theta\,\bar{y}l = (\bar{y}\theta)l$ distance traveled by its CG $(\bar{y}\theta)$ multiplied by the length of the curve (l).

5.6.2 Theorem 2

The volume of the solid generated by rotating a plane surface about a non-intersecting axis in its plane is equal to the product of the area A of the surface and the distance traveled by its centroid.

Proof: To prove the 2nd theorem, let us consider a plane area $MNOP$ of area A having its CG located at C that lies in the xy plane. As before, let the distance of CG from x axis be $\bar{y}$.

Now let's consider an elemental area dA having its CG located at a distance y from x axis.

Now the surface is rotated by an angle θ so as to generate a solid of definite volume. The new position of CG becomes C'.

Thus the elemental surface area dA covers a distance $y\theta$ and the CG of the entire curve covers a distance $\bar{y}\theta$.

The elemental volume of the solid generated = $dV = y\theta.dA$

Hence the entire volume of the solid becomes $V = \int dV = \int y\theta.dA = \theta\int ydA.$

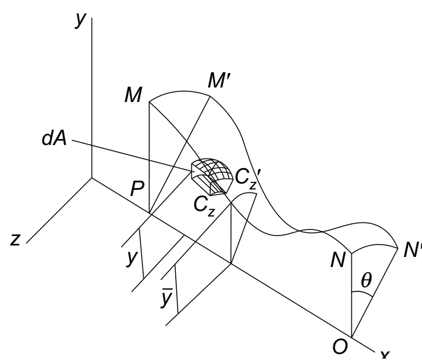


Figure 5.4

But by definition, $\bar{y} = \dfrac{\int ydA}{\int dA}$ $\quad \therefore \int ydA = \bar{y}A$

Thus volume of the solid = $V = \theta\int ydA = \theta\bar{y}A = (\bar{y}\theta)A$ = distance travelled by its CG $(\bar{y}\theta)$ × area of the surface (A) [Proved]

Note: When the curve or surface is rotated by one complete revolution, θ becomes 360°.

Example 5.1 Prove that for a triangle, its centroid lies at a distance *h/3* from its base, where *h* is the height of the triangle.

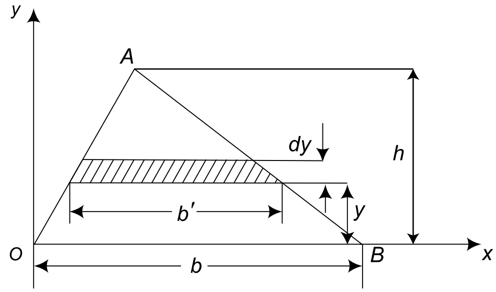


Figure 5.5

Solution Let us consider ΔOAB having height h and base width $OB = b$ as shown in Fig. 5.5.

To compute its centroid, let us consider an elemental rectangular strip of width b' at a distance y from x axis and thickness dy.

$\therefore$ From similarity of triangles,

$$\frac{b'}{b} = \frac{h-y}{h} \Rightarrow b' = \frac{b}{h}(h - y)$$

$\therefore$ Area of the strip $= dA = b' \times dy = \dfrac{b}{h}(h - y)dy$

$$\therefore \qquad y_c = \frac{\int y dA}{\int dA} = \frac{\int_0^h y.\frac{b}{h}(h-y)dy}{\int_0^h \frac{b}{h}(h-y)dy}$$

$$= \frac{h \times \left[\frac{y^2}{2}\right]_0^h - \frac{1}{3}[y^3]_0^h}{h \times [y]_0^h - \frac{1}{2}[y^2]_0^h} = \frac{\frac{h^3}{2} - \frac{h^3}{3}}{h^2 - \frac{h^2}{2}} = \frac{\frac{h^3}{6}}{\frac{h^2}{2}} = \frac{h^3}{6} \times \frac{2}{h^2} = \frac{h}{3} \qquad \text{[Proved]}$$

Example 5.2 Determine the coordinates (x_c, y_c) of the centroid C of the area of one-quarter of an ellipse OAB having major and minor semi-axes a and b respectively.

Solution

1st Method: Refer to Fig. 5.6. OAB is the quarter of an ellipse so that $OA = a$ and $OB = b$.

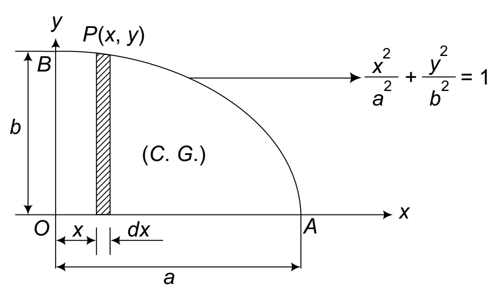


Figure 5.6

To compute its centroid, let us consider a vertical elemental rectangular strip at a distance x from y axis and thickness dx.

Let $P(x, y)$ be a point on the curve.

$\therefore$ Area of the strip $= dA = y \times dx = \dfrac{b}{a}\sqrt{a^2 - x^2}\,dx$ since the equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, from which, $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$

$$\therefore \qquad x_c = \frac{\int x dA}{\int dA} = \frac{\int_0^a x.\frac{b}{a}\sqrt{a^2 - x^2}\,dx}{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx} = \frac{4a}{3\pi}$$

But $y_c = \dfrac{\int \frac{y}{2} dA}{\int dA}$, since the vertical coordinate of the small rectangular strip lies at a distance $\dfrac{y}{2}$ from the x axis.

$$\therefore \qquad y_c = \frac{\int \frac{y}{2} dA}{\int dA} = \frac{\int \frac{y}{2}.ydx}{\int ydx} = \frac{\frac{1}{2}\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx}{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx} = \frac{b}{2a}\,\frac{a^2[x]_0^a - \frac{1}{3}[x^3]_0^a}{\frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \text{ ArcSin } \frac{x}{a}\right]_0^a}$$

$$= \frac{b}{2a}\,\frac{a^3 - \frac{a^3}{3}}{\frac{1}{2}\left[a^2 \times \frac{\pi}{2}\right]} = \frac{b}{2a} \times \frac{\frac{2a^3}{3}}{\frac{\pi a^2}{4}} = \frac{4b}{3\pi}$$

Thus coordinates (x_c, y_c) of the centroid C of the area of one-quarter of an ellipse is $\left(\dfrac{4a}{3\pi}, \dfrac{4b}{3\pi}\right)$

> **Note:** For the special case, when $a = b = r$, the ellipse becomes a circle of radius r and the equation of the circle becomes $x^2 + y^2 = r^2$.

Thus coordinates (x_c, y_c) of the centroid C of the area of one-quarter of a circle become $\left(\dfrac{4r}{3\pi}, \dfrac{4r}{3\pi}\right)$.

2nd Method: Instead of taking the vertical strip, one could have taken similar horizontal strip.

Since $P(x, y)$ is a point on the curve as before, area of the horizontal strip $= dA = x \times dy$ and $x_c = \dfrac{\int \frac{x}{2} dA}{\int dA}$ and $y_c = \dfrac{\int y dA}{\int dA}$. Since $dA = x \times dy$, the variable now being y, its limit will be from 0 to b.

Nevertheless, the end result would be same as computed earlier. Students are advised to check the validity of it.

Example 5.3 Locate the centroid of the circular arc PQ of radius R that subtends an angle ψ as shown in the Fig. 5.7.

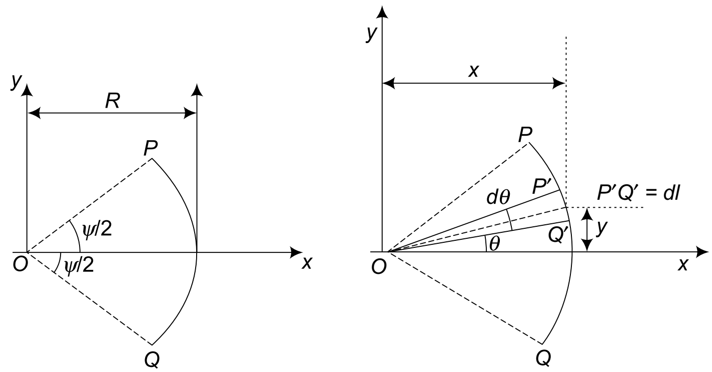


Figure 5.7

Solution Refer to the Fig. 5.7, where $\angle POQ = \psi$ and $OP = OQ = R$.

x axis is considered as axis of symmetry.

To compute its centroid, let us consider a differential length $P'Q' = dl$ and $\angle P'OQ' = d\theta$.

The height of the triangle is R. $\therefore$ $dl = R.d\theta$.

From the geometry, $x = R\cos\theta$ and $y = R\sin\theta$.

$$\therefore \qquad x_c = \frac{\int x dL}{\int dL} = \frac{\int_{-\frac{\psi}{2}}^{+\frac{\psi}{2}} R\cos\theta.Rd\theta}{\int_{-\frac{\psi}{2}}^{+\frac{\psi}{2}} Rd\theta} = R\,\frac{[\sin\theta]_{-\frac{\psi}{2}}^{+\frac{\psi}{2}}}{[\theta]_{-\frac{\psi}{2}}^{+\frac{\psi}{2}}} = R\,\frac{\left[\sin\frac{\psi}{2} - \sin\left(-\frac{\psi}{2}\right)\right]}{\left[\frac{\psi}{2} - \left(-\frac{\psi}{2}\right)\right]} = \frac{2R\sin\frac{\psi}{2}}{\psi}$$

$$\therefore \qquad y_c = \frac{\int y dL}{\int dL} = \frac{\int_{-\frac{\psi}{2}}^{+\frac{\psi}{2}} R\sin\theta.Rd\theta}{\int_{-\frac{\psi}{2}}^{+\frac{\psi}{2}} Rd\theta} = R\,\frac{[-\cos\theta]_{-\frac{\psi}{2}}^{+\frac{\psi}{2}}}{[\theta]_{-\frac{\psi}{2}}^{+\frac{\psi}{2}}} = R\,\frac{\left[-\cos\frac{\psi}{2} - \cos\left(-\frac{\psi}{2}\right)\right]}{\left[\frac{\psi}{2} - \left(-\frac{\psi}{2}\right)\right]} = 0.$$

Thus coordinates (x_c, y_c) of the centroid C of the circular arc PQ are $\left(\dfrac{2R\sin\frac{\psi}{2}}{\psi}, 0\right)$

Note:

1. The value of y_c is found to be zero. This is because that the arc of the circle PQ is symmetrical about x axis. This implies that x axis divides the arc in two equal halves such that portion above x axis is equal to the portion that lies below it. This is why the limits of θ varies from $-\dfrac{\psi}{2}$ to $+\dfrac{\psi}{2}$. However, one can perform the integration over the limits 0 to $+\psi/2$ and results so obtained require to be multiplied by 2 to get the final result.
2. From the coordinates of C, it is clear that for a circular arc, it may not always lie on the element itself.

Example 5.4 Compute the coordinates (x_c, y_c) of the centroid C of the sector of a circle OAB of radius R and included angle 2θ as shown in Fig. 5.8.

Solution

1st Method: Refer to Fig. 5.8 (a), where OAB is the sector of a circle such that x axis is considered as axis of symmetry.

To compute its centroid, let us consider a differential area $OA'B'$. This differential area can be approximated as a triangle having apex angle $\angle A'OB' = d\alpha$. The height of the triangle $= OA' = OB' = R$

The base of the triangle is approximated by arc $A'B' = R.d\alpha$.

$\therefore$ Area of the $\Delta OA'B' = dA = \frac{1}{2} \times R \times R.d\alpha = \frac{1}{2}R^2.d\alpha$

The CG of a triangle lies at a distance $\frac{2}{3}$ of its height from its apex [Refer Problem 5.1].

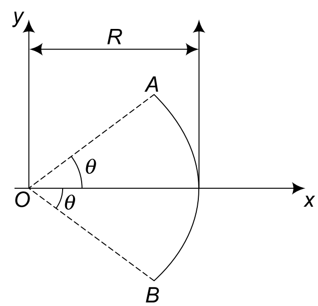


Figure 5.8

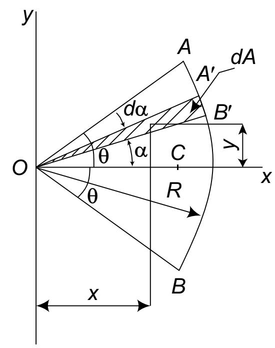


Figure 5.8 (a)

$\therefore$ $OC = \frac{2}{3}R$, from which $x = \frac{2}{3}R\cos\alpha$ and $y = \frac{2}{3}R\sin\alpha$

$$\therefore \qquad x_c = \frac{\int x dA}{\int dA} = \frac{\int_{-\theta}^{+\theta} \frac{2}{3}R\cos\alpha \times \frac{1}{2}R^2 d\alpha}{\int_{-\theta}^{+\theta} \frac{1}{2}R^2 d\alpha} = \frac{2}{3}R\frac{[\sin\alpha]_{-\theta}^{+\theta}}{[\alpha]_{-\theta}^{+\theta}} = \frac{2}{3}R\frac{[\sin\theta - \sin(-\theta)]}{[\theta - (-\theta)]}$$

$$= \frac{2}{3}R\frac{\sin\theta}{\theta}$$

$$\therefore \qquad y_c = \frac{\int y dA}{\int dA} = \frac{\int_{-\theta}^{+\theta} \frac{2}{3}R\sin\alpha \times \frac{1}{2}R^2 d\alpha}{\int_{-\theta}^{+\theta} \frac{1}{2}R^2 d\alpha} = \frac{2}{3}R\frac{[-\cos\alpha]_{-\theta}^{+\theta}}{[\alpha]_{-\theta}^{+\theta}} = \frac{2}{3}R\frac{[\cos\theta - \cos(-\theta)]}{[\theta - (-\theta)]} = 0.$$

Thus coordinates (x_c, y_c) of the centroid C of the sector of a circle are $\left(\frac{2R\sin\theta}{3\theta}, 0\right)$ [Answer].

Note: If the included angle of the sector of the circle is π, it becomes a semi-circle. Thus $x_c = \dfrac{2R\sin\theta}{3\theta} = \dfrac{2R\sin\frac{\pi}{2}}{3\frac{\pi}{2}} = \dfrac{4R}{3\pi}$. However, y_c remains zero since it continues to be remain symmetrical with x axis.

2nd Method: In this method, a differential element as shown in Fig. 5.8 (b) is selected such that its centroid is denoted by $C(x, y)$.

Thus area of the differential element = $dA = (r.d\alpha).dr$.

Since the number of variables being two, we have to resort to multiple integral involving $d\alpha$ and dr for which the limits of α would be from $-\theta$ to $+\theta$ and for r, it would be from 0 to R.

Now, $x = r\cos\alpha$ and $y = r\sin\alpha$

$$\therefore \qquad x_c = \frac{\int x dA}{\int dA} = \frac{\int_{-\theta}^{+\theta}\int_0^R r\cos\alpha.(r.d\alpha).dr}{\int_{-\theta}^{+\theta}\int_0^R (r.d\alpha).dr} = \frac{2\sin\theta\int_0^R r^2 dr}{2\theta\int_0^R r dr} = \frac{\sin\theta}{\theta}\times\frac{R^3/3}{R^2/2}$$

$$\therefore \qquad = \frac{2}{3}R\frac{\sin\theta}{\theta} \qquad [\textit{Same as before}]$$

The computation of y_c is left to the reader.

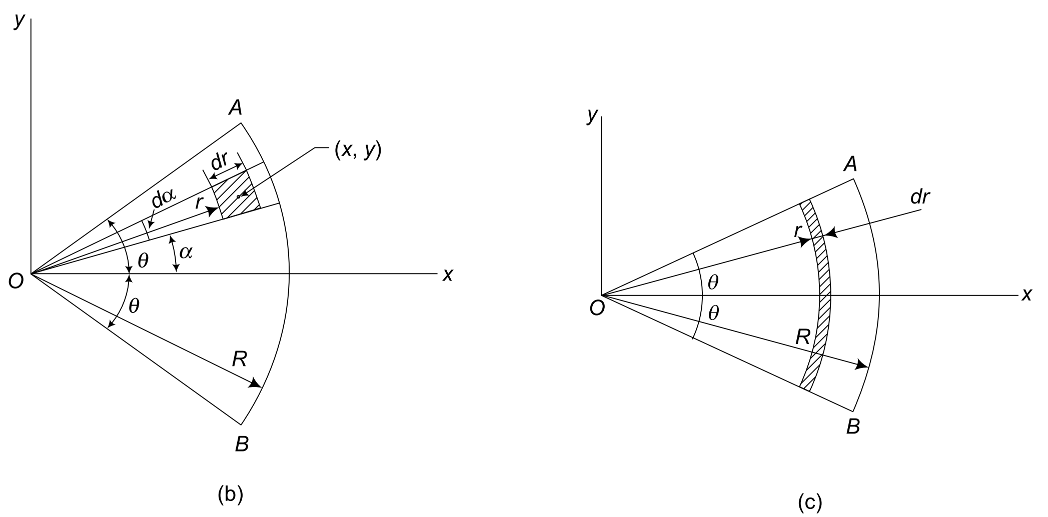


Figure 5.8

3rd Method: This method is based on the results of the calculation of problem 5.3

Here, a uniform circular arc is considered at a distance r from y axis having thickness dr as shown in Fig. 5.8 (c)

$\therefore$ Area of the differential element = $dA = 2\theta.r.dr$.

The x coordinates of the centroid of this differential circular arc = $x = \dfrac{r}{\theta}\sin\theta$ [Refer result of the problem 5.3]

$$\therefore \qquad x_c = \frac{\int x dA}{\int dA} = \frac{\int_0^R \frac{r}{\theta}\sin\theta.2\theta.r.dr}{\int_0^R 2\theta.r.dr} = \frac{2\sin\theta\int_0^R r^2 dr}{2\theta\int_0^R r dr} = \frac{\sin\theta\frac{1}{3}R^3}{\theta\frac{1}{2}R^2} = \frac{2}{3}R\frac{\sin\theta}{\theta}$$

Example 5.5 Compute the coordinates (x_c, y_c) of the centroid C of the sector of the general spandrel OAB as shown in Fig. 5.9.

Solution Let us consider an elemental rectangular strip of thickness dx at a distance x from y axis.

$\therefore$ Area of the strip = $dA = y \times dx$

Let $P(x, y)$ be a point on the curve.

The general equation of the curve is $y = kx^n$.

Since the point $A(a, b)$ lies on the curve, it must satisfy the equation of the curve.

$$\therefore \qquad b = ka^n \Rightarrow k = \frac{b}{a^n}.$$

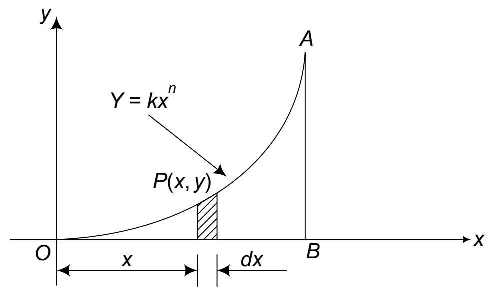


Figure 5.9

Thus equation of the curve becomes $y = \frac{b}{a^n}x^n$.

$\therefore$ Area of the strip = $dA = y \times dx = \frac{b}{a^n}x^n dx$

$$\therefore \qquad x_c = \frac{\int x dA}{\int dA} = \frac{\int_0^a x.\frac{b}{a^n}x^n dx}{\int_0^a \frac{b}{a^n}x^n dx} = \frac{\int_0^a x^{n+1}dx}{\int_0^a x^n dx} = \frac{\frac{1}{n+2}[x^{n+2}]_0^a}{\frac{1}{n+1}[x^{n+1}]_0^a} = \frac{n+1}{n+2}[a^{n+2-n-1}] = \frac{n+1}{n+2}a$$

$$y_c = \frac{\int \frac{y}{2}dA}{\int dA} = y_c = \frac{\int \frac{y}{2}ydx}{\int ydx} = \frac{\int_0^a \frac{1}{2}\frac{b^2}{a^{2n}}x^{2n}dx}{\int_0^a \frac{b}{a^n}x^n dx} = \frac{1}{2}.\frac{b}{a^n}\frac{\frac{1}{2n+1}[x^{2n+1}]_0^a}{\frac{1}{n+1}[x^{n+1}]_0^a}$$

$$= \frac{b}{2a^n}\frac{n+1}{2n+1}[a^{2n+1-n-1}] = \frac{n+1}{4n+2}b.$$

Thus coordinates (x_c, y_c) of the centroid C of the parabolic spandrel is $\left(\frac{n+1}{n+2}a, \frac{n+1}{4n+2}b\right)$

Note: When $n = 2$, it becomes a parabolic spandrel.

Example 5.6 Determine the y-coordinate of the centroid C of the shaded area OAB as shown in Fig. 5.10.

Solution The area of interest, i.e., OAB is the difference of area formed by the curve OA and the straight line OB with the x axis as evident from the figure.

The equation of the curve is $y^2 = bx$.

It is imperative to find out the equation of the straight line AB that passes through the point $\left(b, \frac{b}{2}\right)$.

Since the line passes through the origin, let us consider its generic equation to be $y = mx$.

Putting the coordinates of point B in the above equation, we have $\frac{b}{2} = mb$ or $m = \frac{1}{2}$.

Thus, the equation of the straight line becomes $y = \frac{1}{2}x$.

To calculate the centroid of the given area, we select a differential vertical strip of thickness dx.

The height h of this elemental strip would be the difference between the y coordinate of the curve ($y^2 = bx$.) and the straight line $\left(y = \frac{1}{2}x\right)$.

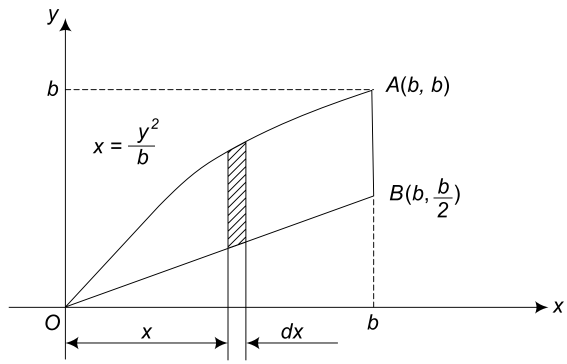


Figure 5.10

$$\therefore \qquad h = \sqrt{bx} - \frac{x}{2}.$$

$$\therefore \text{ Area of the strip} = dA = h \times dx = \left(\sqrt{bx} - \frac{x}{2}\right).dx$$

Let the distance of centroid of the elemental strip be h_c from the x axis.

From geometry, $h_c = \frac{h}{2} + y$ coordinate of the straight line.

$$\therefore \qquad h_c = \frac{h}{2} + \frac{x}{2}$$

$$\therefore \qquad y_c = \frac{\int h_c dA}{\int dA} = \frac{\int_0^b \left(\frac{h}{2} + \frac{x}{2}\right)\left(b^{1/2} x^{1/2} - \frac{x}{2}\right)dx}{\int_0^b b^{1/2} x^{1/2} - \frac{x}{2} dx} = \frac{\frac{b^3}{4} + \frac{b^3}{10} - \frac{b^3}{10} - \frac{b^3}{24}}{\frac{2b^2}{3} - \frac{b^2}{4}} = \frac{5b^3}{24} \times \frac{12}{5b^2} = \frac{b}{2}$$

y coordinate of the centroid of the enclosed area OAB therefore becomes $\frac{b}{2}$

Example 5.7 Determine the coordinates of the centroid C of the shaded area OAB as shown in Fig. 5.11.

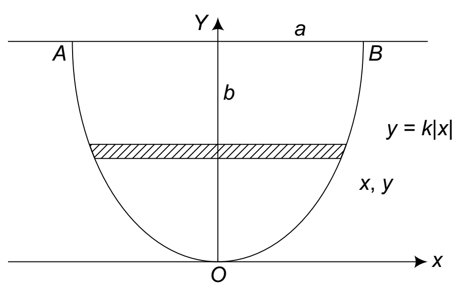


Figure 5.11

Solution The general equation of the curve is $y = k|x|^3$.

Since it passes through the point (a, b), $b = ka^3$ or $k = \frac{b}{a^3}$.

Thus equation of the curve becomes $y = \frac{b}{a^3}|x|^3$.

$\therefore$ Area of the strip $= dA = 2x dy$

$$\therefore \qquad y_c = \frac{\int y dA}{\int dA} = \frac{\int y\, 2x dy}{\int 2x dy} = \frac{\int_0^b y \frac{a}{b^{\frac{1}{3}}} y^{\frac{1}{3}} dy}{\int_0^b \frac{a}{b^{\frac{1}{3}}} y^{\frac{1}{3}} dy} = \frac{\dfrac{b^{\frac{4}{3}+1}}{\frac{4}{3}+1}}{\dfrac{b^{\frac{1}{3}+1}}{\frac{1}{3}+1}} = \frac{3}{7} \times \frac{4}{3} b = \frac{4}{7} b.$$

The given geometry being symmetrical with respect to y axis, $x_c = 0$.

Thus coordinates (x_c, y_c) of the centroid C of the shaded area are $\left(0, \frac{4}{7}b\right)$

Example 5.8 Using Pappus and Guldinus Theorem, compute the volume of a right-circular cone of height h and base radius r.

Solution As per Pappus and Guldinus 2nd Theorem, volume of a solid generated by rotating a plane curve with respect to a non-intersecting axis would be equal to the area of the curve multiplied by the distance travelled by its CG.

To generate a right-circular cone, a plane triangular area ABC is considered. The arm AB is aligned with the vertical axis OO, as shown in the Fig. 5.12.

The height of this triangle is $AB = h$ and base $BC = r$.

Therefore, its CG will lie at a distance $\frac{r}{3}$ from the OO axis.

This triangle, if rotated about OO by 360°, will generate a cone.

The area of the triangle is $= A = \frac{1}{2}rh$.

The distance travelled by its CG when rotated by one complete revolution becomes $2\pi \frac{r}{3}$.

Thus volume of the cone is $V = \frac{1}{2}rh \times 2\pi \frac{r}{3} = \frac{1}{3}\pi r^2 h$

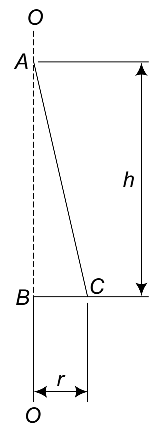


Figure 5.12

Example 5.9 Using Pappus and Guldinus Theorem, compute the surface area and volume of the torus generated by rotating a circle of radius r whose centre is located at a distance R from y axis as shown in Fig. 5.13.

Solution A circle is a plane curve of length = its perimeter $= L = 2\pi r$.

The CG being located at its centre, it covers a distance $= 2\pi R$ when rotated by one complete revolution and consequently generates a torus.

Thus, as per Pappus and Guldinus 1st Theorem, surface area of the torus so generated will be $A = 2\pi r \times 2\pi R = 4\pi^2 Rr$, since the surface area of the circle $A_c = \pi r^2$

By Pappus and Guldinus 2nd Theorem, the volume of the torus becomes

$$V = \pi r^2 \times 2\pi R = 2\pi^2 R r^2.$$

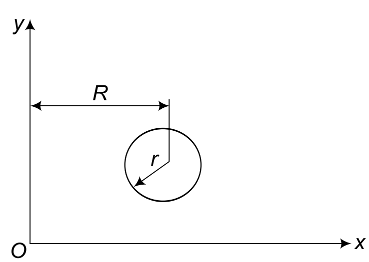


Figure 5.13

Example 5.10 Locate the *CG* of a solid right-circular cone of height h and base radius r as shown in Fig. 5.14.

Solution Let us consider the cone as shown in Fig. 5.10 having height $OA = h$ and base radius $OB = r$.

To compute its centroid, let us consider an elemental circular strip of thickness dy at a distance y from x axis.

Let the radius of the circular strip be r'.

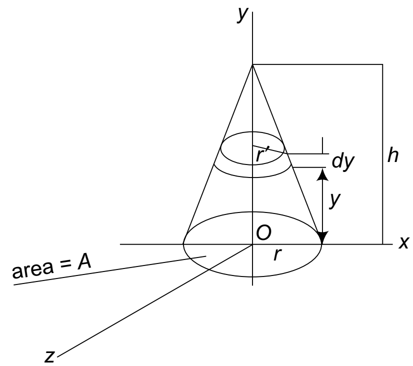


Figure 5.14

$\therefore$ From similarity of triangles, $\dfrac{r'}{r} = \dfrac{h-y}{h} \Rightarrow r' = \dfrac{r}{h}(h - y)$

$\therefore$ Cross-section area of the strip $= A = \pi r'^2 = \pi \dfrac{r^2}{h^2}(h - y)^2$.

$\therefore$ Differential volume of the strip becomes

$$dV = A \times dy = \pi \frac{r^2}{h^2}(h - y)^2 dy$$

$$\therefore \quad y_c = \frac{\int y dV}{\int dV} = \frac{\int_0^h y.\pi \dfrac{r^2}{h^2}(h-y)^2\, dy}{\int_0^h \pi \dfrac{r^2}{h^2}(h-y)^2\, dy} = \frac{h^2 \times \left[\dfrac{y^2}{2}\right]_0^h - 2h\left[\dfrac{y^3}{3}\right]_0^h + \left[\dfrac{y^4}{4}\right]_0^h}{h^2 \times [y]_0^h - 2h\left[\dfrac{y^2}{2}\right]_0^h + \left[\dfrac{y^3}{3}\right]_0^h}$$

$$= \frac{\dfrac{h^4}{2} - \dfrac{2h^4}{3} + \dfrac{h^4}{4}}{h^3 - h^3 + \dfrac{h^3}{3}} = \frac{\dfrac{h^3}{6}}{\dfrac{h^2}{2}} = \frac{h^4}{12} \times \frac{3}{h^3} = \frac{h}{4} \quad \text{[Answer]}$$

Example 5.11 Determine the *CG* of a hemisphere of radius r as shown in Fig. 5.15.

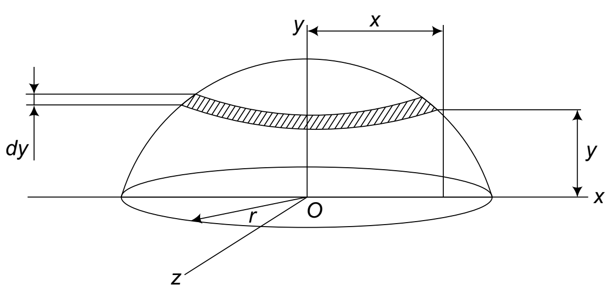


Figure 5.15

Solution To compute its *CG*, let us consider a differential element of radius x and having thickness dy at a distance y from x axis.

Thus the equation of the circular strip becomes $x^2 + y^2 = r^2$

$\therefore$ Cross-section area of the strip $= A = \pi x^2$

$\therefore$ Differential volume of the strip becomes $dV = A \times dy = \pi x^2 dy$.

$$\therefore \qquad y_c = \frac{\int y dV}{\int dV} = \frac{\int_0^r y.\pi x^2 dy}{\int_0^h \pi x^2 dy} = \frac{\int_0^r y(r^2 - y^2)dy}{\int_0^r (r^2 - y^2)dy}$$

$$= \frac{\frac{r^4}{2} - \frac{r^4}{4}}{r^3 - \frac{r^3}{3}} = \frac{\frac{r^4}{4}}{\frac{2r^3}{3}} = \frac{r^4}{4} \times \frac{3}{2r^3} = \frac{3r}{8}$$

By symmetry, $x_c = 0$ and $z_c = 0$

PART B

5.7 COMPUTATIONS OF CG AND CENTROIDS FOR COMPOSITES

In mechanics, one has to confront with several objects which do not conform to any unique geometry. Rather a close look at such objects reveals that although they seem very complicated but basically they are the aggregation of several fundamental geometric shapes. Such objects which are combination of basic geometrical shapes are called *composites* and find wide application in real-life situations. They have several distinct advantages over their simplified counterparts in view of material savings, enhanced strength, reduced weight and cost-effectiveness. It is therefore indeed very relevant to compute centre of gravity and centroids for such composites. However, it is not at all a difficult task to calculate the CG of such composites provided the locations of the centroids of the elementary geometrical shapes, present in it, are known.

Under such situations, it is not prudent to start from scratch; rather one must make use of results that are already obtained for basic geometrical entities.

Refer to Fig. 5.16 for which we are interested to compute the *CG* of the composite. The composite can be divided into three basic finite shapes having masses m_1, m_2 and m_3 and x coordinates of their *CG* being x_1, x_2 and x_3 respectively.

Then by using the principle of Varignon's Theorem, we can find out the *CG* of the composite.

Thus, we have $(m_1 + m_2 + m_3)X_c = m_1x_1 + m_2x_2 + m_3x_3$

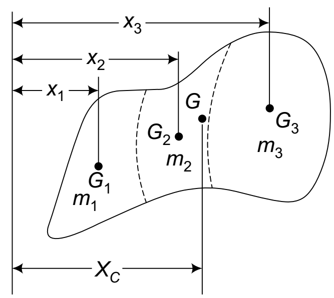


Figure 5.16

$$X_c = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i}$$

Similarly, other two coordinates can be calculated as $Y_c = \dfrac{\sum_{i=1}^{3} m_i y_i}{\sum_{i=1}^{3} m_i}$ and $Z_c = \dfrac{\sum_{i=1}^{3} m_i z_i}{\sum_{i=1}^{3} m_i}$

It is worth mentioning that there is no limit regarding the choice of numbers as well as types of individual finite elements.

Thus above three expressions can be expressed generically when the composite is divided into n number elements

$$X_c = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}; \; Y_c = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}; \; Z_c = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

Following the article 5.4, the above sets of expressions of centre of gravity can be extended for computation of volumes, areas and lines.

Thus $$X_c = \frac{\sum_{i=1}^{n} V_i x_i}{\sum_{i=1}^{n} V_i}; \; Y_c = \frac{\sum_{i=1}^{n} V_i y_i}{\sum_{i=1}^{n} V_i}; \; Z_c = \frac{\sum_{i=1}^{n} V_i z_i}{\sum_{i=1}^{n} V_i} \text{ for volumes}$$

$$X_c = \frac{\sum_{i=1}^{n} A_i x_i}{\sum_{i=1}^{n} A_i}; \; Y_c = \frac{\sum_{i=1}^{n} A_i y_i}{\sum_{i=1}^{n} A_i}; \; Z_c = \frac{\sum_{i=1}^{n} A_i z_i}{\sum_{i=1}^{n} A_i} \text{ for areas}$$

$$X_c = \frac{\sum_{i=1}^{n} L_i x_i}{\sum_{i=1}^{n} L_i}; \; Y_c = \frac{\sum_{i=1}^{n} L_i y_i}{\sum_{i=1}^{n} L_i}; \; Z_c = \frac{\sum_{i=1}^{n} L_i z_i}{\sum_{i=1}^{n} L_i} \text{ for slender bars}$$

It may so happen that it may be easier to consider the composites as a difference of two or more basic entities. Thus depending on the merits of the problem, it can be any combination – only additive, only subtractive or hybrid.

In such cases, the *summation is to be interpreted as algebraic sum* both in the numerator and in the denominator. A few worked out problems will help understand this approach in regard to its implementation.

Here is a question for you!

"*While computation of centre of gravity of composites, the summation formula is used, whereas for basic entities, integration method is followed" – Justify with reasons.*

Example 5.12 Determine the coordinates of centroid $C(x_c, y_c)$ of an L section as shown in Fig. 5.17.

Solution Let the given L-section be divided into two rectangles $OABC$ and $CDEF$ and let these be denoted by 1 and 2 respectively.

$$A_1 = 100 \times 10 = 1000 \text{ mm}^2; \qquad A_2 = (80 - 10) \times 10 = 700 \text{ mm}^2$$

$$x_1 = \frac{10}{2} = 5 \text{ mm} \qquad x_2 = \frac{(80-10)}{2} + 10 = 45 \text{ mm}$$

$$y_1 = \frac{100}{2} = 50 \text{ mm} \qquad y_2 = \frac{10}{2} = 5 \text{ mm}$$

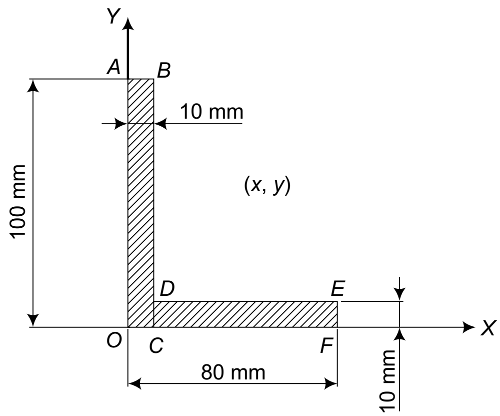


Figure 5.17

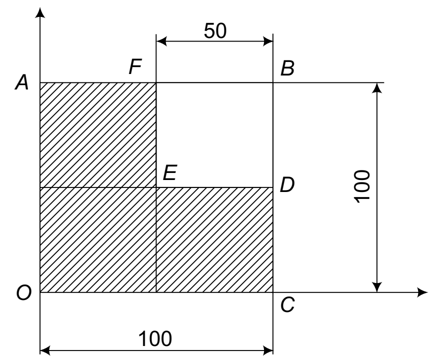


Figure 5.18

$$\therefore \quad x_c = \frac{A_1x_1 + A_2x_2}{A_1 + A_2} = \frac{1000 \times 5 + 700 \times 45}{1000 + 700} \text{ mm} = 21.47 \text{ mm and}$$

$$y_c = \frac{A_1y_1 + A_2y_2}{A_1 + A_2} = \frac{1000 \times 50 + 700 \times 5}{1000 + 700} \text{ mm} = 31.47 \text{ mm.}$$

Thus, coordinates of centroid $C(x_c, y_c)$ of a L-section become (21.47 mm, 31.47 mm)

Example 5.13 Determine the coordinates of centroid $C(x_c, y_c)$ of the shaded area formed by subtracting a square with a 50 mm side from a bigger square with a 100 mm side as shown in Fig. 5.18.

Solution Let the larger square with 100 mm sides be represented by *OABC* and smaller square with 50 mm sides be represented by *BDEF* and let these be denoted by 1 and 2 respectively.

$$A_1 = 100 \times 100 = 10000 \text{ mm}^2; \quad A_2 = 50 \times 50 = 2500 \text{ mm}^2$$

$$x_1 = \frac{100}{2} = 50 \text{ mm} \qquad x_2 = 50 + 25 = 75 \text{ mm}$$

$$y_1 = \frac{100}{2} = 50 \text{ mm} \qquad y_2 = 50 + 25 = 75 \text{ mm}$$

Since $x_1 = y_1$ and $x_2 = y_2$

It therefore follows that $x_c = y_c$

$$\therefore \quad x_c = \frac{A_1x_1 - A_2x_2}{A_1 - A_2} = \frac{10000 \times 50 - 2500 \times 75}{10000 - 2500} \text{ mm} = 41.67 \text{ mm} = y_c$$

Example 5.14 Determine the coordinates of centroid $C(x_c, y_c)$ of the composite developed by combining a sector of a circle and a triangle as shown in Fig. 5.19.

Solution Let the sector of the circle *OAB* and the triangle *OBC* be denoted by 1 and 2 respectively.

$$A_1 = \frac{1}{6} \times \pi \times 60^2 = 600\pi \text{ mm}^2;$$

$$A_2 = \frac{1}{2} \times (60 \cos 30) \times 30 = 900 \cos 30 \text{ mm}^2$$

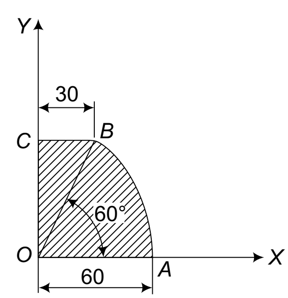


Figure 5.19

$$x_1 = \frac{2r}{3\theta}\sin\theta = \frac{2\times 60}{3\times \pi/3}\sin 60 \text{ mm} = \frac{120}{\pi}\sin 60 \text{ mm} \qquad x_2 = \frac{1}{3}\times 30 = 10 \text{ mm}$$

$$y_1 = \frac{2r}{3\theta}[1-\cos\theta] = \frac{2\times 60}{3\times \pi/3}[1-\cos 60] \text{ mm} = \frac{60}{\pi} \text{ mm} \quad y_2 = \frac{2}{3}(60\cos 30) = 40\cos 30 \text{ mm}$$

$$\therefore \qquad x_c = \frac{A_1x_1 + A_2x_2}{A_1 + A_2} = \frac{600\pi\times\frac{120}{\pi}\sin 60 + 900\cos 30\times 10}{600\pi + 900\cos 30} \text{ mm} = 26.33 \text{ mm.}$$

$$\therefore \qquad y_c = \frac{A_1y_1 + A_2y_2}{A_1 + A_2} = \frac{600\pi\times\frac{60}{\pi} + 900\cos 30\times 40\cos 30}{600\pi + 900\cos 30} \text{ mm} = 23.64 \text{ mm.}$$

Example 5.15 Determine the coordinates of centroid $C(x_c, y_c)$ of the area remaining after a semi-circle is removed from the trapezoid as shown in Fig. 5.20.

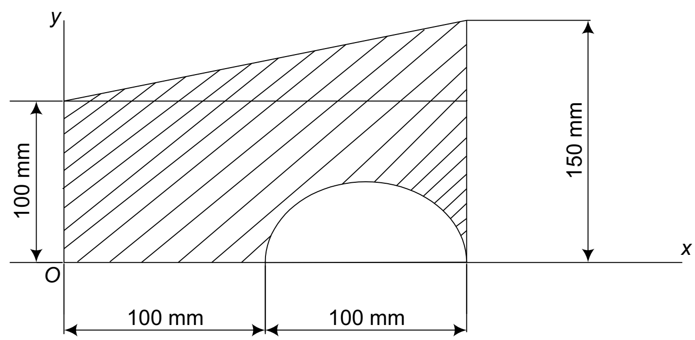


Figure 5.20

Solution The shaded area can be considered as area of the rectangle (size 200 mm × 100 mm) + area of the triangle (base 200 mm × height 50 mm) – area of the semi-circle with a 100 mm diameter.

From the given dimensions,

$$A_1 = 200\times 100 = 20000 \text{ mm}^2; \quad A_2 = \frac{1}{2}\times 200\times 50 = 5000 \text{ mm}^2; \qquad A_3 = \frac{1}{2}\times\pi\times 50^2 = 1250\pi \text{ mm}^2$$

$$x_1 = \frac{200}{2} = 100 \text{ mm} \qquad x_2 = \frac{2}{3}\times 200 \text{ mm} = 133.33 \text{ mm} \qquad x_3 = 100 + \frac{50}{2} \text{ mm} = 125 \text{ mm}$$

$$y_1 = \frac{100}{2} = 50 \text{ mm} \qquad y_2 = 100 + \frac{1}{3}\times 50 \text{ mm} = 116.678 \text{ mm} \quad y_3 = \frac{4\times 50}{3\pi} \text{ mm} = 21.22 \text{ mm}$$

$$\therefore \qquad x_c = \frac{A_1x_1 + A_2x_2 - A_3x_3}{A_1 + A_2 - A_3} = \frac{20000\times 100 + 5000\times 133.33 - 1250\pi\times 125}{20000 + 5000 - 1250\pi} \text{ mm} = 103.25 \text{ mm}$$

$$\therefore \qquad y_c = \frac{A_1y_1 + A_2y_2 - A_3y_3}{A_1 + A_2 - A_3} = \frac{20000\times 50 + 5000\times 116.67 - 1250\pi\times 21.22}{20000 + 5000 - 1250\pi} \text{ mm} = 71.2 \text{ mm}$$

Example 5.16 Locate the coordinates of centroid $C(x_c, y_c)$ of the shaded obtained by removing a semi-circle of diameter a from a quadrant of a circle of radius a, as shown in Fig. 5.21.

Solution Let the quadrant of the circle of radius a be denoted as 1 and semi-circle of diameter a as 2.

$$A_1 = \frac{\pi}{4}a^2 \text{ mm}^2; \qquad A_2 = \frac{1}{2} \times \frac{\pi}{4}a^2 \text{ mm}^2 = \frac{\pi}{8}a^2 \text{ mm}^2$$

$$x_1 = \frac{4a}{3\pi} \text{ mm} \qquad x_2 = \frac{a}{2} \text{ mm}$$

$$y_1 = \frac{4a}{3\pi} \text{ mm} \qquad y_2 = \frac{4}{3\pi}\left(\frac{a}{2}\right) \text{ mm} = \frac{2a}{3\pi} \text{ mm}$$

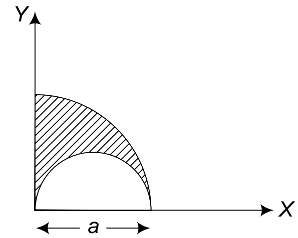


Figure 5.21

$$\therefore \qquad x_c = \frac{A_1x_1 - A_2x_2}{A_1 - A_2} = \frac{\frac{\pi}{4}a^2 \times \frac{4a}{3\pi} - \frac{\pi}{8}a^2 \times \frac{a}{2}}{\frac{\pi}{4}a^2 - \frac{\pi}{8}a^2} \text{ mm} == 0.349a \text{ mm.}$$

$$\therefore \qquad y_c = \frac{A_1y_1 - A_2y_2}{A_1 - A_2} = \frac{\frac{\pi}{4}a^2 \times \frac{4a}{3\pi} - \frac{\pi}{8}a^2 \times \frac{2a}{3\pi}}{\frac{\pi}{4}a^2 - \frac{\pi}{8}a^2} \text{ mm} = 0.636a \text{ mm}$$

Example 5.17 Locate the centroid $C(x_c, y_c)$ of the bent wire of uniform cross-section as shown in Fig. 5.22.

Solution The bent wire has got four distinct parts into which it can be decomposed. These are denoted as 1 through 4 as shown.

$$L_1 = 50 \text{ mm}; \qquad L_2 = 100 \text{ mm};$$

$$L_3 = \pi \times 100 \text{ mm}; \qquad L_4 = 50 \text{ mm}$$

$$x_1 = -\left\{100 + \frac{50}{2}\right\} = -125 \text{ mm}; \qquad x_2 = -100 \text{ mm};$$

$$x_3 = 0; \qquad x_4 = \left\{100 + \frac{50}{2}\right\} = 125 \text{ mm}$$

$$y_1 = 0; \qquad y_2 = \frac{100}{2} \text{ mm} = 50 \text{ mm};$$

$$y_3 = 100 + \frac{2 \times 100}{\pi} \text{ mm}; \qquad y_4 = 100 \text{ mm}$$

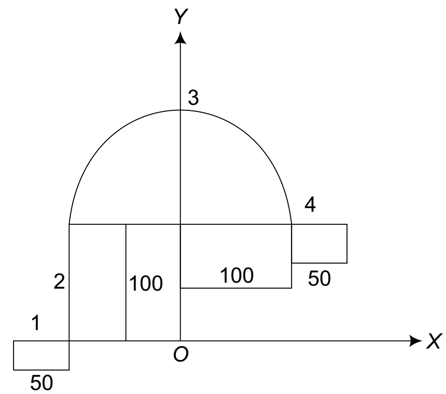


Figure 5.22

$$\therefore \qquad x_c = \frac{\sum_{i=1}^{4} L_i x_i}{\sum_{i=1}^{4} L_i} = \frac{50 \times (-125) + 100 \times (-100) + 100\pi \times 0 + 50 \times 125}{50 + 100 + 100\pi + 50} \text{ mm} = -19.45 \text{ mm}$$

$$\therefore \qquad y_c = \frac{\sum_{i=1}^{4} L_i y_i}{\sum_{i=1}^{4} L_i} = \frac{50 \times 0 + 100 \times 50 + 100\pi \times \left[100 + \frac{200}{\pi}\right] + 50 \times 100}{50 + 100 + 100\pi + 50} \text{ mm} = 119.45 \text{ mm}$$

Example 5.18 An isosceles triangle is to be cut from one edge of a square plate of side 1 metre such that the remaining part of the plate remains in equilibrium in any position when suspended from the vertex of the triangle, as shown in Fig. 5.23.

Solution The remaining part of the plate remains in equilibrium when suspended from the vertex of the triangle, which implies that centroid of the remaining part coincides with the vertex of the triangle.

Let h be the height of the triangle from x axis.

The square plate and the triangle is denoted as 1 and 2 respectively.

$$A_1 = 1 \times 1 \text{ m}^2 = 1 \text{ m}^2; \qquad A_2 = \frac{1}{2} \times 1 \times h\text{m}^2 = \frac{h}{2} \text{ m}^2$$

$$y_1 = \frac{1}{2} = 0.50 \text{ m}$$

$$y_2 = \frac{h}{3} \text{ m}$$

$$\therefore \quad y_c = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{1 \times 0.5 - \frac{h}{2} \times \frac{h}{3}}{1 - \frac{h}{2}} = \frac{0.5 - 0.167h^2}{1 - 0.5h}$$

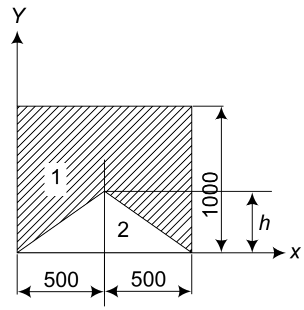


Figure 5.23

By definition $y_c = h$

$$\therefore \quad h = \frac{0.5 - 0.167h^2}{1 - 0.5h} \Rightarrow 0.33h^2 - h + 0.5 = 0$$

$$h = \frac{1 \pm \sqrt{1 - 4 \times 0.33 \times 0.5}}{2 \times 0.33} = \frac{1 \pm 0.583}{0.66} = 2.4 \text{ m or } 0.631 \text{ m}$$

Since the height of the triangle cannot be 2.4 m, it should be 0.631 m.

Thus the area of the triangle becomes $A_2 = \frac{1}{2} \times 1 \times 0.631 \text{ m}^2 = 0.3155 \text{ m}^2$.

Example 5.19 A square hole has to be punched out from a circular lamina as shown in Fig. 5.24. The diagonal of the square which is punched out is equal to the radius of circle. Find the centroid of the remaining lamina.

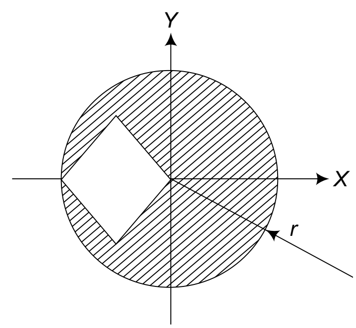


Figure 5.24

Solution Let the circle be denoted by 1 and the square be denoted by 2.

The radius of the circle is = r = diagonal of the square.

Thus sides of the square become $\frac{r}{\sqrt{2}}$.

$$A_1 = \pi r^2 \text{ mm}^2; \qquad A_2 = \frac{r}{\sqrt{2}} \times \frac{r}{\sqrt{2}} \text{ mm}^2 = \frac{r^2}{2} \text{ mm}^2$$

$$x_1 = 0, \quad x_2 = -\frac{r}{2} \text{ mm}$$

$$y_1 = 0, \quad y_2 = 0$$

$$\therefore \quad x_c = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2} = \frac{\pi r^2 \times 0 - \frac{r^2}{2} \times \left(-\frac{r}{2}\right)}{\pi r^2 - \frac{r^2}{2}} \text{ mm} = 0.095\text{r rmm.}$$

The shaded area being symmetrical about x axis y_c will be zero.

Example 5.20 Identify the mass centre of the slender bar bent to the shape as shown in Fig. 5.25.

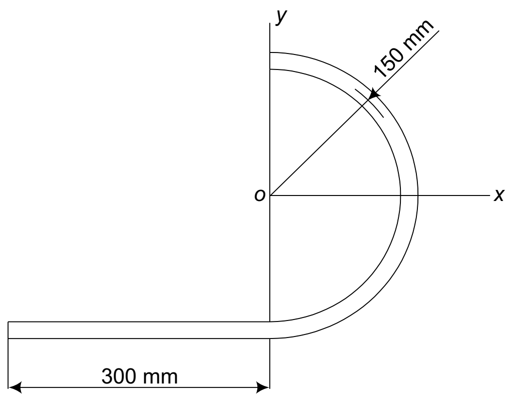


Figure 5.25

Solution Let the circular arc be denoted by 1 and the linear bar be denoted by 2.

The radius of the arc = r = 150 mm.

$$L_1 = \pi r = 150\pi \text{ mm}; \qquad L_2 = 300 \text{ mm}$$

$$x_1 = \frac{2 \times 150}{\pi} \sin 90 = \frac{300}{\pi} \text{ mm} \qquad x_2 = -150 \text{ mm}$$

$$y_1 = 0 \qquad y_2 = -150 \text{ mm}$$

$$\therefore \qquad x_c = \frac{L_1 x_1 + L_2 x_2}{L_1 + L_2} = \frac{150\pi \times \frac{300}{\pi} + 300 \times (-150)}{150\pi + 300} \text{ mm} = 0$$

$$\therefore \qquad y_c = \frac{L_1 y_1 + L_2 y_2}{L_1 + L_2} = \frac{150\pi \times 0 + 300 \times (-150)}{150\pi + 300} \text{ mm} = -58.3 \text{ mm}$$

Thus, the location of the mass centre of the bent bar becomes (0, −58.3 mm)

MULTIPLE-CHOICE QUESTIONS

5.1 Centre of gravity is a point at which
- (a) resultant of all the external force acts
- (b) resultant of all the reaction force acts
- (c) entire body force is concentrated
- (d) none of the above

5.2 For homogeneous rigid bodies, centre of gravity depends on
- (a) shape of the object
- (b) materials of the object
- (c) both shape and materials of the object
- (d) none of the above

5.3 For homogeneous rigid bodies, centre of mass depends on
- (a) position of the body
- (b) orientation of the body
- (c) both Position and Orientation of the body
- (d) none of the above

5.4 For a semi-circular arc of radius r, centre of mass lies at a distance

(a) $\frac{r}{\pi}$ from its centre (b) $\frac{2r}{\pi}$ from its centre (c) $\frac{3r}{2\pi}$ from its centre (d) $\frac{4r}{3\pi}$ from its centre

5.5 For a quarter circular sector of radius r, centre of gravity lies at a distance

(a) $\frac{r}{\pi}$ from its centre (b) $\frac{2r}{\pi}$ from its centre (c) $\frac{3r}{2\pi}$ from its centre (d) $\frac{4r}{3\pi}$ from its centre

5.6 Pappus and Guldinus Theorem is relevant with

(a) friction (b) moment of Inertia
(c) centre of gravity (d) moment of a force with respect to a point

5.7 Computation of centre of gravity by method of integration is valid only for

(a) solid objects (b) 2D lamina
(c) slender bar of uniform cross section (d) all of the above

5.8 The center of gravity of a uniform lamina lies at

(a) the centre of heavy portion (b) the bottom surface
(c) the mid-point of its axis (d) all of the above

5.9 To generate a solid right-circular cone by using Pappus and Guldinus Theorem, the surface to be considered should conform to

(a) triangle (b) circle (c) semicircle (d) trapezoidal

5.10 An object which is symmetrical about y axis, its

(a) $x_c = 0$ (b) $y_c = 0$ (c) $z_c = 0$ (d) none of the above

5.11 For solid right-circular cone of height h, its CG lies at a distance

(a) $\frac{h}{2}$ from the base (b) $\frac{h}{3}$ from the base (c) $\frac{h}{4}$ from the base (d) none of the above

5.12 For a hemisphere of radius r, its CG lies at a distance

(a) $\frac{2r}{3}$ from the base (b) $\frac{3r}{4}$ from the base (c) $\frac{3r}{8}$ from the base (d) $\frac{r}{2}$ from the base

SHORT ANSWER TYPE QUESTIONS

5.1 What is centre of gravity? Does it differ from centre of mass? If yes, under what condition? When did the term centroid get more relevance?

5.2 Why is computation of centre of gravity important in connection with mechanics?

5.3 Is centre of gravity a unique point? If yes, why is it so?

5.4 If method of integration is replaced by summation technique for basic geometrical shapes, what would be the difficulties?

5.5 Why is method of integration not valid for irregular objects?

5.6 For slender rods of uniform cross section, prove that $x_c = \frac{\int x dm}{\int dm} = \frac{\int x dL}{\int dL}$

5.7 What is meant by symmetrical objects? For such objects, why does computations of centre of gravity sometimes become easier and under what conditions?

5.8 Briefly outline the method of calculation of centre of gravity of composites?

5.9 State Pappus and Guldinus 1st and 2nd theorems and prove them.

5.10 For a circle, prove that its CG lies at its centre.

NUMERICAL PROBLEMS

5.1 Determine the location of the centroid of the circular arc of radius R having included angle 90º as shown in Fig. 5.26.

5.2 Determine the location of the centroid of the shaded area formed by the curve $y = k(x - a)^2$ with the x axis as shown in Fig. 5.27.

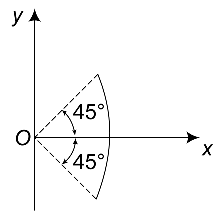


Figure 5.26

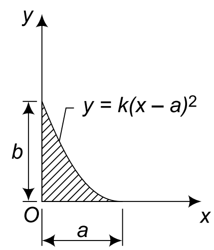


Figure 5.27

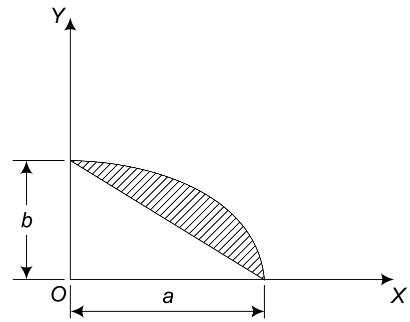


Figure 5.28

5.3 Starting from the fundamentals, compute the location of the centre of gravity of the area formed by subtracting a triangle of base a and height b from a quarter of an ellipse having semi major and minor axes as a and b respectively, as shown in Fig. 5.28.

5.4 Determine by direct integration, the y coordinate y_c of the homogeneous wire bent into the shape as shown in Fig. 5.29.

5.5 Calculate the CG (x_c, y_c) of a semi-circular area of radius r such that centre of the circle coincides with the origin and the object is symmetrical about positive x axis.

5.6 Find out the CG (x_c, y_c) of an **'I'** section as shown in Fig. 5.30.

5.7 Locate the centroid of a composite as shown in Fig. 5.31.

5.8 Locate the centroid of the hatched area formed by the intersection of straight line $y = x$ and a parabola $y = \dfrac{x^2}{a}$ as shown in Fig. 5.32.

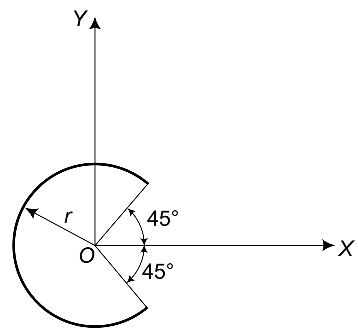


Figure 5.29

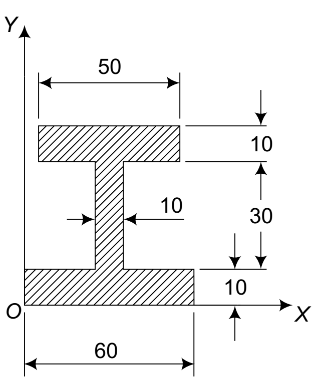


Figure 5.30

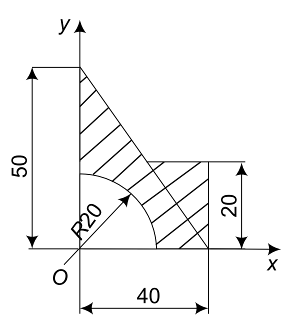


Figure 5.31

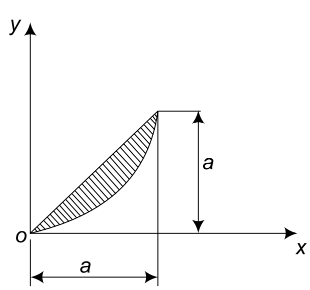


Figure 5.32

5.9 Find out the *CG* of the hatched area formed by the eliminating a square with 40 mm sides from that of a quarter of a circle with 80 mm radius as shown in Fig. 5.33.

5.10 Using Pappus and Guldinus Theorem, compute the surface area and volume of a solid sphere of radius r.

5.11 Using Pappus and Guldinus Theorem, calculate the surface area and volume of a right circular cylinder of radius r and height h.

5.12 Find out the *CG* of the composite as shown in Fig. 5.34.

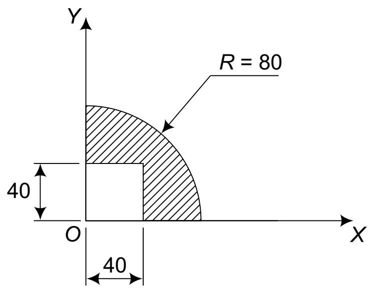


Figure 5.33

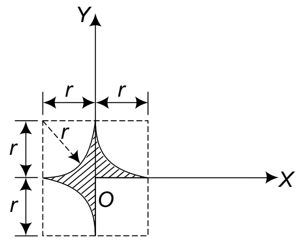


Figure 5.34

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

5.1 (c) 5.2 (a) 5.3 (d) 5.4 (b) 5.5 (d) 5.6 (c) 5.7 (d) 5.8 (c) 5.9 (a) 5.10 (a) 5.11 (c) 5.12 (c)

ANSWERS TO NUMERICAL PROBLEMS

5.1 $x_c = \dfrac{2\sqrt{2R}}{\pi}, y_c = 0$

5.2 $x_c = \dfrac{a}{4}, y_c = \dfrac{3b}{5}$

5.3 $x_c = \dfrac{2a}{3(\pi - 2)}, y_c = \dfrac{2b}{3(\pi - 2)}$

5.4 $y_c = 0$

5.5 $x_c = \dfrac{2R}{\pi}, y_c = 0$

5.6 $x_c = 30$, $y_c = 23.57$

5.7 $x_c = 19.16$, $y_c = 19.07$

5.8 $x_c = \dfrac{a}{2}, y_c = \dfrac{2a}{5}$

5.9 $x_c = 40.5 = y_c$

5.12 $x_c = 0.744a = y_c$

CHAPTER 6

Friction

6.1 INTRODUCTION

As discussed earlier, statics involve primarily the analysis of forces on rigid bodies such that it forms a system of forces that ensures equilibrium of the system. These forces are basically the active forces, i.e., external applied force(s) and or the body force(s), the reactions from different supports that constrain the motion and the induced forces (tension and compression) in several members as a consequence of applied load. Such force analysis would form the basis for subsequent design. While dealing with such force analysis, the reaction forces from various supports are assumed to act normally from the supports. The reason for such consideration is attributed to the premise that these surfaces are *perfectly smooth*, although in reality it is not so. Such idealistic approach helps simplify the complexity of the problem to a great extent without any significant gross error and well accepted in many cases. Nevertheless, there are plenty of situations, when such considerations would be misleading. In fact, all the real life surfaces exhibit some kind of *roughness,* however small it may be. It is therefore found that while dealing with two mating surfaces (similar or dissimilar), these not only exert normal reactions but also offer mutually ***tangential resisting forces that oppose their motion***. Thus, whenever there is a contact between two mating surfaces, a tangential force is developed by virtue of the roughness of the surfaces that always act opposite to the direction of motion. This resisting force is called ***friction force*** and often plays a very dominant role in many engineering problems. Thus the concept of perfectly smooth surface, popularly known as *ideal* surface, is imaginary and therefore, all real life surfaces that manifest some degree of roughness are called *real* surface.

6.2 APPLICATION DOMAIN

Friction force, basically being a resisting force, seems to be undesirable. There are ample situations in engineering applications like *power transmission systems involving gears, shafts, keys, bearings (e.g. gearbox), mechanics of metal cutting (chip flows over tool rake face and rubbing of flank surface of the tool over machined surface), machine tool slides, hydraulic or pneumatic actuators* where presence of friction cause wear and tear of components, undue heat generation, increase in power consumption and encourage premature failure. Notwithstanding this fact, this opposing force are very much welcome in the applications like *brakes, clutches, belt drives, screw jack, wedge block* etc. Even a simple ladder that is used for the purpose of climbing, utilises friction to maintain equilibrium, when a person stands on it. It is therefore very much essential to study friction to incorporate its effects in the analysis of forces in true spirit.

6.3 CLASSIFICATION OF FRICTION

Friction is a natural phenomenon that is exhibited in the wide range of areas. Based on certain attributes and characteristics, it is broadly classified under three categories that help to analyse the problems of engineering.

(a) ***Dry friction*** This type of friction is encountered between the surfaces of two rigid bodies when there exists a sliding motion or there is a tendency of motion in the absence of any oil or lubrication in between. The dry friction is present even if there is no motion, under the condition of impending motion and when there is motion. It is also called *Coulomb* friction, named after the scientist Coulomb who had carried out several experiments in this domain to realise the theory in its present form. The actual mechanism of friction is too complicated to be reasoned. In the present text, considering the relevance, this category of friction will be discussed in detail.

(b) ***Fluid friction*** Fluid does not have any definite shape; rather it conforms to the shape of the conduit or vessel in which it is contained for the purpose of fluid flow to take place or simply for its storage. When fluid flow takes place, the inner wall of the conduit offers resistance to the motion by virtue of friction. Further, the entire fluid column in a conduit can be envisaged as aggregation of several layers and there is a relative motion between the layers of fluid during fluid flow and these layers also offer resistance to the neighboring layer. Such type of friction associated with fluid flow is termed as *fluid friction*. This flow phenomenon and the friction characteristics depend on several parameters like velocity gradient, surface roughness of the conduit, viscosity of the fluid, cross-section of the conduit, etc. This fluid friction being predominant in different types of fluid flow problems, finds its discussion and analysis in the study of a separate subject titled ***fluid mechanics*** and will not be discussed here.

(c) ***Internal friction*** This type of friction occurs in solids undergoing cyclic loading. Nevertheless it is not uniform in all types of solids. Internal friction is considerable in those materials that manifest adequate plastic deformation compared to those that deform less plastically. Whenever a solid is subjected to tangential external force, the material undergoes shear deformation and internal friction comes into existence. The mechanics of internal friction is associated with the subject ***strength of materials*** and hence any further illustration is not under the purview of present discussion.

6.4 MECHANICS OF FRICTION

A surface that apparently seems very smooth is actually not so. A microscopic look at such a surfaces will reveal presence of lots of asperities on it that arise from the intrinsic material qualities or the consequences of the procedures that are followed during manufacturing. Presence of these irregularities put hindrance to its motion over the other surfaces.

To understand the mechanics of friction, consider a body of mass m resting on a rough horizontal floor and subjected to a horizontal pull force $\boldsymbol{P}$ as shown in Fig. 6.1.

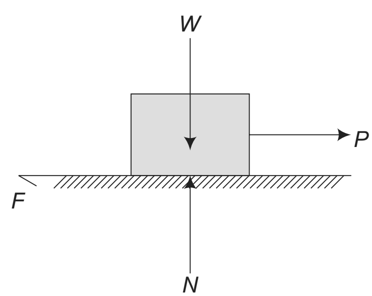


Figure 6.1

The magnitude of $\boldsymbol{P}$ is increased gradually from zero to assume a value till we get a motion of the body with sufficient velocity. Figure 6.1 shows that the body force $W = mg$ of the block will act vertically downward and in turn, the floor on which the block rests will exert a normal reaction N on it. The frictional force F will act at the mating surface and will be tangential to it–its direction being just opposite to that of external applied force $\boldsymbol{P}$.

The manifestation of frictional force is the gross behavior of the surface that resulted from asperities present on it. Meticulous inspection of the surfaces under investigation shows that lots of *hills and valleys* are present and their depth and distribution is not uniform throughout the surface areas, as evident from Fig. 6.2 (a).

Whenever, two surfaces come in contact with each other, irregularities of both the surfaces stick together (temporary locking) and offer resistance to their motions. It is therefore misnomer to consider that higher contact surface area would eventually enhance the frictional resistance; rather *it is the degree of irregularities that present on the surface which contributes to friction.*

Figure 6.2 *(a) Surfaces showing asperities, (b) Conditions during motion*

Since under the action of all these forces, the block is under static equilibrium, it leads to

$$\sum X = 0$$

$$P = F \tag{6.1}$$

$$\sum Y = 0$$

$$W = N \tag{6.2}$$

Thus from Eq. (6.1), when $P = 0$; $F = 0$.

Increase in magnitude of P eventually increases the magnitude of F so as to hold the block under equilibrium.

Thus a plot of friction force (F) against the applied force (P) [Refer Fig. 6.3] shows a linear curve that passes through the origin and having a slope of 45º. While increasing the value of P, an *optimum or threshold value* is reached beyond which the block will start moving. At this point, when the block is just on the verge of motion, the friction force attains its maximum value. Further increase of P enables the block to move. However, friction force now drops almost instantaneously and remains fairly constant despite increase in P. Substantial increase in P will induce appreciable velocity to the block, when F drops slightly. This can be explained by the fact that temporary bonding that took place between the asperities of mating surface that cause friction is *self-adjusting* since it increases linearly from 0 to F_{max} with the increase in value of P. The maximum value of friction force, which comes into play when the motion is impending, is known as *limiting friction*. When the block starts moving, the bonding becomes weak and *it is prevalent only along the humps* as shown in Fig. 6.2 (b) that results in reduction of the frictional resistance.

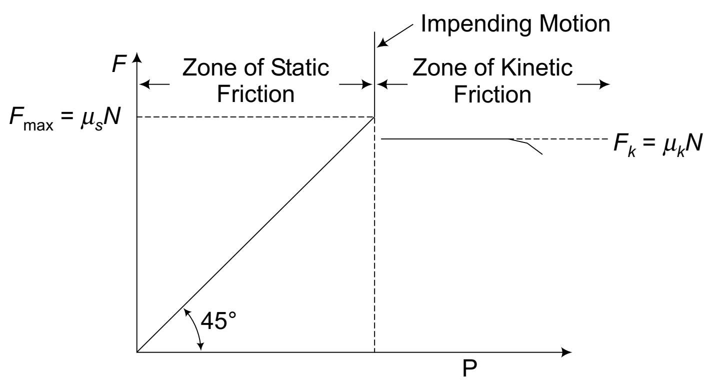


Figure 6.3

Refer to Fig. 6.3; it is clear that there are two distinct regions as divided by the condition of impending motion marked by vertical dotted line. On the left side of this line, it is the region that portrays the behavior of friction force when the body is under rest. When the applied force is less than the limiting friction, the body remains at rest and such frictional force is called *static friction*, which may have any value between zero and the limiting friction. If the value of the applied force exceeds the limiting friction, the body starts moving over the other body and the frictional resistance experienced by the body while moving is known as dynamic friction (or kinetic friction). Kinetic friction is found to be less than limiting value of static friction. To distinguish between static friction and kinetic friction, henceforth F_s and F_k will be used respectively.

6.5 COEFFICIENTS OF FRICTION

Refer to Fig. 6.4 (a), using the concept of composition of forces, the normal reaction N and the frictional force F can be replaced by their resultant R. The block is therefore in equilibrium under the action of three intersecting coplanar forces W, P and R. Thus the number of forces acting on the block is now reduced [from 4 to 3]. Hence these forces will form a closed triangle. This composition of forces is very much useful in solving problems involving friction.

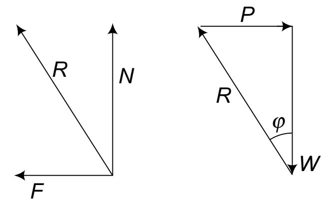


Figure 6.4 (a)

It is interesting to note that experimental results show that the magnitude of limiting friction $F_{\max}$ bears a constant ratio to the normal reaction N between the two surfaces and this ratio is called *coefficient of Friction* (μ).

Thus mathematically, $\mu = \dfrac{F_{\max}}{N}$.

Further from Fig. 6.4 (b), $\tan \varphi_s = \dfrac{F_{\max}}{N}$. Thus $\tan \varphi_s = \mu$.

So, ***the coefficient of friction is equal to the tangent of the angle between the normal reaction and the resultant.*** This angle φ_s is called *angle of limiting friction or angle of static friction.*

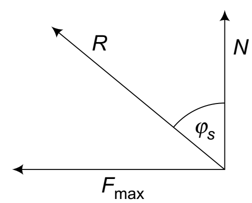


Figure 6.4 (b)

When the value of P is less than that required to cause impending motion say, P', we have a different force triangle as shown in Fig. 6.4 (c). Since, W is constant and $P' < P$, R' will also be less than R. Thus $\tan \beta = \dfrac{F}{N} = \dfrac{P'}{N} < \tan\varphi$ or $\beta < \varphi$.

This observation leads to the conclusion that when β assumes its maximum value = φ, F becomes $F_{\max}$, which corresponds to the condition of limiting friction.

Thus $F_{\max} = \mu_s N$. μ_s is called ***coefficient of static friction***. Its value is less than 1 and for common materials its range varies between 0.2 and 0.5.

Once the block attains its motion due to further increase in P, the friction force is found to be lower than that of its static counterpart. However, it still maintains proportionality with the normal reaction.

Under such condition, friction force is quantified by $F_k = \mu_k N$, where μ_k is called ***coefficient of kinetic friction*** and $\mu_k = \tan \varphi_k$. Angle φ_k is called *angle of kinetic friction.*

The value of friction force F_k remains more or less constant over a wide range until the block attains reasonably high velocity, when it drops marginally.

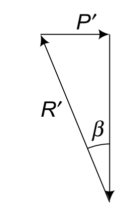


Figure 6.4 (c)

Since $F_k < F_{\max}$, it follows that μ_k is less than μ_s.

Based on the findings of friction characteristics in dry condition, certain laws, called laws of friction were developed. These are popularly known as Coulomb' Laws of Friction, which are mentioned underneath.

6.6 COULOMB'S LAWS OF FRICTION

(1) The friction force always acts in a direction opposite to that in which the body tends to move.
(2) Till the condition of limiting friction is satisfied, the magnitude of friction is exactly equal to the force which tends to move the body.
(3) The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two surfaces.
(4) The force of friction depends upon the materials and the degree of roughness/smoothness of the surfaces.
(5) The friction is independent of the area of contact between the two mating surfaces.

6.7 CONE OF FRICTION

When a body is on the verge of impending motion in the direction of applied force, the frictional force encountered is the limiting friction and the resultant reaction force $\boldsymbol{R}$ ($\boldsymbol{R} = \boldsymbol{F} + \boldsymbol{N}$) will make limiting friction angle φ_s with the normal reaction as shown in the Fig. 6.5. If the direction of applied force is changed in the same plane, the body will have similar impending motion along the modified direction of the applied force. The situation is no different from the former and hence the resultant reaction force makes same angle φ_s with the normal reaction. Likewise, if the direction of applied force is changed continuously from 0º to 360º, the resultant $\boldsymbol{R}$ generates a right circular cone with semi-central angle equal to φ_s, its height and base radius being equal to that of $\boldsymbol{N}$ and F_{max} respectively as shown in Fig. 6.5. If $\boldsymbol{R}$ is on the surface of this right circular cone having semi-central angle as limiting frictional angle φ_s, the body is under the condition of impending motion. From this figure, it is clear that when the base radius of the cone is lower than that of F_{max}, $\boldsymbol{N}$ being unaltered, R will be reduced. Consequently, semi-central angle assumes a value θ lower than that of φ_s, and the situation corresponds to the no motion or static condition. This cone with semi-central angle equal to limiting angle φ_s is called ***Cone of Friction or Cone of Static Friction*** in particular.

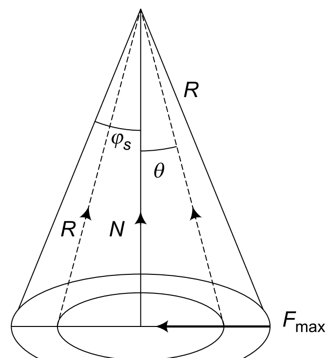


Figure 6.5

A similar cone of friction can also be developed having semi-central angle equal to φ_k, height being same as that of N but base radius equal to $F_k = \mu_k N$. Needless to say, base of such a cone will be smaller than the previous one, since μ_k is less than μ_s. Such a cone is called ***Cone of Kinetic Friction***.

> **Note:** It must be borne in mind that $F_s = \mu_s N$ is valid only under the condition of impending motion. Thus when applied force is not adequately high to initiate any movement, F is to be calculated by using the equations of static equilibrium only. Frictional phenomenon associated with situation upto the condition of impending motion is considered as static friction. Further, when there is motion, the condition of static equilibrium is not valid. Under such a situation, friction force that comes into existence is kinetic friction the magnitude of which is quantified by $F_k = \mu_k N$.

6.8 ANGLE OF REPOSE

Quite often a situation is encountered when objects are placed on an inclined plane. But, if the inclination of plane exceeds a certain limiting value, the object will slide down along the plane in the absence of any external force. This is owing to the fact that a component of the self weight will act along the plane which is directed downwards and is a function of the inclination angle of the plane.

As shown in Fig. 6.6, the object having weight $W = mg$ is resting on an inclined plane having its inclination θ with the horizontal. W can be resolved into two mutually perpendicular components – one along the plane having magnitude of $mg \sin \theta$ and the other, perpendicular to the plane of magnitude $mg \cos \theta$, as is evident from the free body diagram. The component $mg \sin \theta$ will act along the plane, so as to cause a downward movement of the block along the plane.

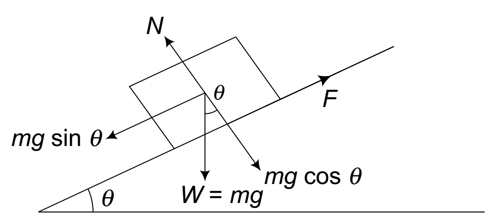


Figure 6.6 *Angle of repose*

Since the tendency of the block is to slide down, the frictional force will act upward along the plane.

Considering the static equilibrium of the block, force balance along the plane and perpendicular to the plane yields

$$\sum X = 0$$

$$F_s = mg \sin \theta \tag{6.3}$$

$$\sum Y = 0$$

$$N = mg \cos \theta \tag{6.4}$$

Dividing Eq. (6.3) by Eq. (6.4), we have

$$\frac{F_s}{N} = \tan \theta \tag{6.5}$$

Further, under the condition of limiting friction

$$\frac{F_s}{N} = \mu_s = \tan \varphi_s \tag{6.6}$$

Comparing Eq. (6.5) and Eq. (6.6) $\tan \theta = \tan \varphi, \Rightarrow \theta = \varphi_s$

Thus under limiting condition,

The inclination of the plane = Angle of static friction.

Hence to ensure stability of the block $\theta \leq \varphi_s$ is the criteria.

This optimum (maximum) inclination of the plane for which a body resting on it would not slide down due to self weight when free from external forces is called ***angle of repose***. The etymological meaning of the term repose means *rest.* Thus angle of repose implies that it is the inclination of a plane such that a person on it can sleep or can take rest comfortably without slipping.

Example 6.1 Two blocks A and B having weights W_A and W_B respectively are attached by an inextensible string and rest on two different horizontal planes as shown in Fig. 6.7. A force $P_{\min}$ is applied to block A so as to induce motion to the system. Find the magnitude and direction of $P_{\min}$. Assume that co-efficient of friction for both the blocks are μ.

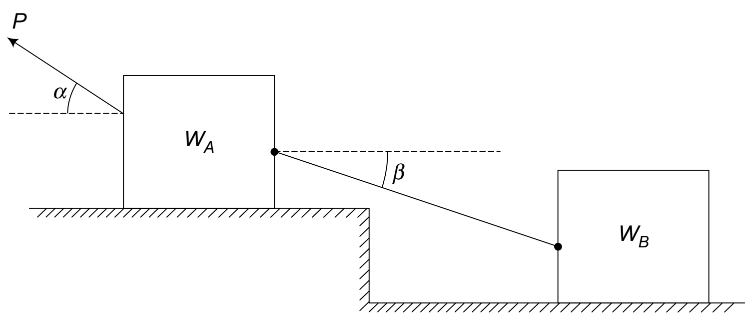


Figure 6.7

Solution Let the force P makes an angle α with the horizontal. Also consider the string has got an inclination β with the horizontal and experience a tensile force T.

Introducing A and B as suffices for the blocks A and B respectively, we have two different free body diagrams of block A [Fig. 6.7 (a)] and B [Fig. 6.7 (b)].

Considering equilibrium of block A when motion impends,

$$\sum X = 0$$

$$P \cos \alpha = T \cos \beta + F_A \tag{6.7}$$

$$\sum Y = 0$$

$$P \sin \alpha + N_A = W_A + T \sin \beta \tag{6.8}$$

$$F_A = \mu N_A \tag{6.9}$$

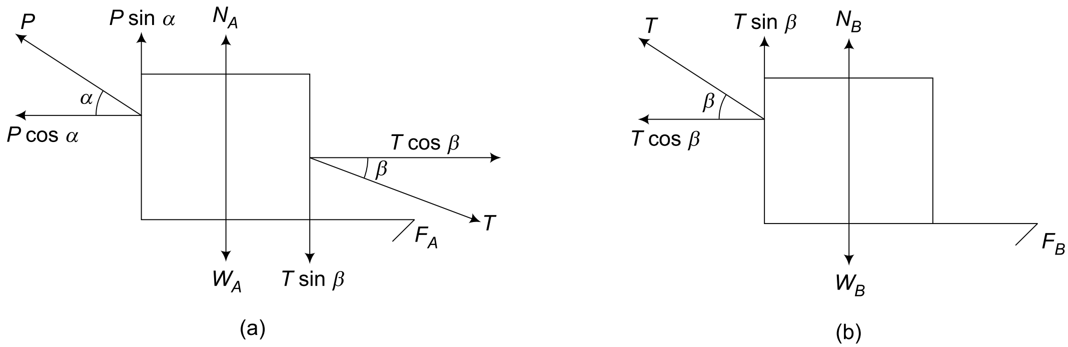


Figure 6.7

From Eqs (6.7) and (6.9),

$$P \cos \alpha = T \cos \beta + \mu N_A \tag{6.10}$$

Similar equilibrium condition of block B leads to

$$\sum X = 0$$

$$T \cos \beta = F_B \tag{6.11}$$

$$\sum Y = 0$$

$$W_B = N_B + T \sin \beta \tag{6.12}$$

$$F_B = \mu N_B \tag{6.13}$$

From Eq. (6.11) and Eq. (6.13),

$$T \cos \beta = \mu N_B \tag{6.14}$$

Equations (6.10) and (6.14) together yield

$$P \cos \alpha = \mu N_A + \mu N_B \tag{6.15}$$

Equations (6.8) and (6.12) together yield

$$P \sin \alpha = (W_A + W_B) - (N_A + N_B) \tag{6.16}$$

From Eqs (6.15) and (6.16),

$$P \sin \alpha = (W_A + W_B) - \frac{P \cos \alpha}{\mu}$$

$$P\left[\sin \alpha + \frac{\cos \alpha}{\mu}\right] = W_A + W_B$$

$$P\left[\sin \alpha + \frac{\cos \alpha}{\tan \varphi}\right] = W_A + W_B; \text{ Since } \tan \varphi = \mu.$$

$$P \cos (\alpha - \varphi) = (W_A + W_B) \sin \varphi$$

$$P = \frac{(W_A + W_B) \sin \varphi}{\cos (\alpha - \varphi)}$$

From the above expression of P, it is evident that P will be minimum, when the denominator of the above expression, is maximum.

Thus $\cos (\alpha - \varphi) = 1 = \cos 0$; this leads to $\alpha = \varphi$

Hence $P_{\min} = (W_A + W_B) \sin \varphi$ and inclination of P is $\alpha = \varphi$

Example 6.2 Two identical blocks A and B each having weight W are connected by rigid link and supported by a vertical wall and a horizontal plane having same co-efficient of friction (μ) as shown in Fig. 6.8. If sliding impends for $\theta = 45°$, calculate μ.

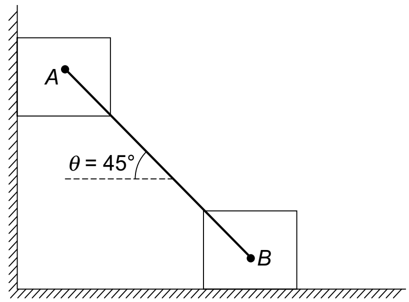


Figure 6.8

Solution The weight of each block is W. These two blocks will experience normal reaction from the supports on which they rest. The friction force offered by the wall and the floor will oppose their motion. Further, both the blocks will exert mutual thrust on each other of magnitude T along the link. The free body diagram of blocks A and B are shown separately in Fig. 6.8 (a) and Fig. 6.8 (b) respectively.

For block A, when motion impends,

$$\sum X = 0 \quad N_A = T\cos\theta \tag{6.17}$$

$$\sum Y = 0 \quad T\sin\theta + F_A = W \tag{6.18}$$

$$F_A = \mu N_A \tag{6.19}$$

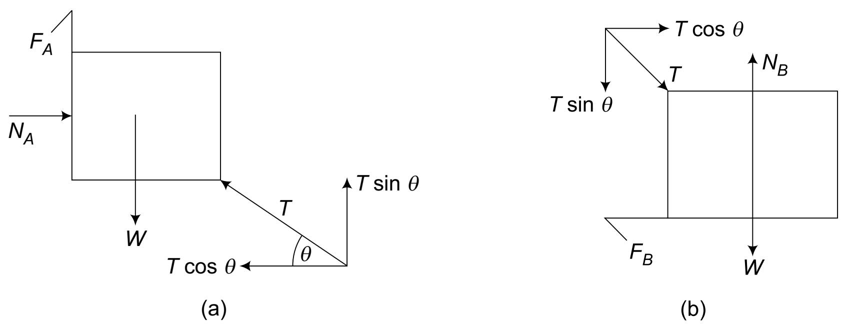


Figure 6.8

From Eqs (6.18) and (6.19),

$$T\sin\theta = W - \mu N_A \tag{6.20}$$

Similar equilibrium condition of block B leads to

$$\sum X = 0 \quad T\cos\theta = F_B = \mu N_B \tag{6.21}$$

$$\sum Y = 0 \quad T\sin\theta + W = N_B \tag{6.22}$$

From Eqs (6.17) and (6.21), $\quad N_A = \mu N_B \qquad (6.23)$

From Eqs (6.20) and (6.22), $W - \mu N_A = N_B - W$

or

$$2W = N_B + \mu^2 N_B = (1 + \mu^2)N_B \tag{6.24}$$

Again, from Eqs (6.21) and (6.22), $\mu N_B = N_B - W$ or $N_B = \dfrac{W}{1-\mu} \qquad (6.25)$

From Eqs (6.24) and (6.25),

$2W = (1 + \mu^2) \times \dfrac{W}{1-\mu} \Rightarrow \mu^2 + 2\mu - 1 = 0$, from which $\mu = \sqrt{2} - 1 = 0.414$.

Thus the common co-efficient of friction μ becomes 0.414.

Example 6.3 Two blocks connected by a horizontal link, are placed on two rough planes as shown in Fig. 6.9. The co-efficient of friction for block A and B are μ_A and μ_B respectively. What is the smallest weight of block A for which the system would be under the condition of equilibrium? Given the weight of the block B is W_B.

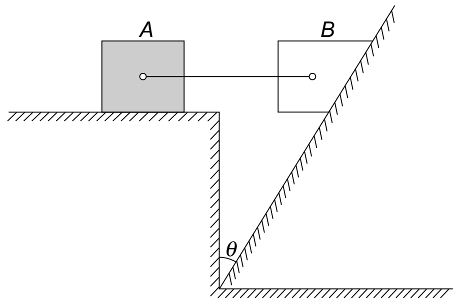


Figure 6.9

Solution The block B is in the form of a wedge that can slide along the inclined plane on which it rests. Since both the blocks are tied together by a rigid link, the downward movements of the block B will cause block A to move towards left along the horizontal floor. As a consequence, an axial load S will be induced in the link.

From free body diagram of block A [Refer Fig. 6.9 (a)],

$$\sum X = 0 \text{ or } F_A = S \tag{6.26}$$

$$\sum Y = 0 \text{ or } N_A = W_A \tag{6.27}$$

From the condition of limiting friction,

$$F_A = \mu_A N_A = \mu_A W_A = S \tag{6.28}$$

Similarly, from free body diagram of block B [Refer Fig. 6.9 (b)],

$$\sum X = 0$$

$$F_B \sin\theta + S = N_B \cos\theta \tag{6.29}$$

$$\sum Y = 0$$

$$F_B \cos\theta + N_B \sin\theta = W_B \tag{6.30}$$

$$F_B = \mu_B N_B \tag{6.31}$$

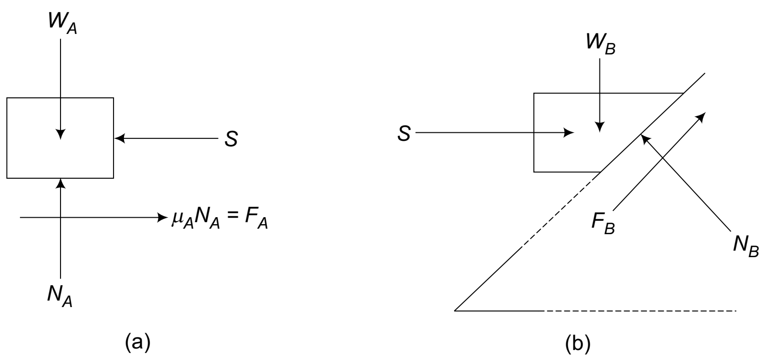


Figure 6.9

From Eqs (6.28), (6.29) and (6.31),

$$\mu_B N_B \sin\theta + \mu_A W_A = N_B \cos\theta$$

$$N_B = \frac{\mu_A W_A}{\cos\theta - \mu_B \sin\theta} \tag{6.32}$$

Further from Eqs (6.30) and (6.31),

$$N_B = \frac{W_B}{\mu_B \cos\theta + \sin\theta} \tag{6.33}$$

Comparing Eqs (6.32) and (6.33),

$$\frac{\mu_A W_A}{\cos\theta - \mu_B \sin\theta} = \frac{W_B}{\mu_B \cos\theta + \sin\theta}$$

or $$\frac{W_A}{W_B} = \frac{1}{\mu_A}\left[\frac{\cos\theta - \tan\varphi_B \sin\theta}{\sin\theta + \tan\varphi_B \cos\theta}\right] = \frac{1}{\mu_A}\left[\frac{\cos\theta - \tan\varphi_B \sin\theta}{\sin\theta + \tan\varphi_B \cos\theta}\right]$$

$\therefore$ $$W_A = \frac{\cot(\theta + \varphi_B)}{\tan\varphi_A} W_B$$

Example 6.4 A ladder $AB = 7$ metre long of negligible weight is supported at A by a smooth vertical wall and a rough horizontal plane at B. A person weighing 60 kg is standing on the ladder at a distance 3 metre from the bottom along its length as shown in Fig. 6.10. If the co-efficient of friction between the floor and the ladder is 0.3, what is the angle (θ) the ladder should make with the horizontal to prevent slipping?

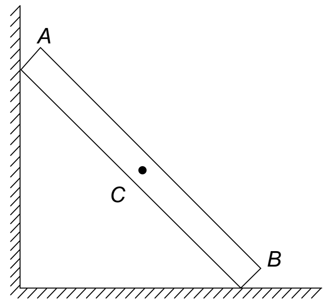


Figure 6.10

Solution Let the weight of the person be W and it acts vertically downward through point C. Let $AB = l = 7$ metre and $BC = x = 3$ metre.

Since the wall is smooth, i.e., frictionless, the reaction R_A would be perpendicular to the wall. However, in presence of friction, the resultant of normal force and friction force from floor will make an angle (φ) with the vertical such that $\tan\varphi = \mu$.

Considering the free body of the ladder, the equilibrium condition leads to

$$\sum X = 0$$

$$R_A = F_B \tag{6.34}$$

$$\sum Y = 0$$

$$N_B = W \tag{6.35}$$

$$\sum M_B = 0$$

$$R_A \times l \sin\theta = W \times x \cos\theta \tag{6.36}$$

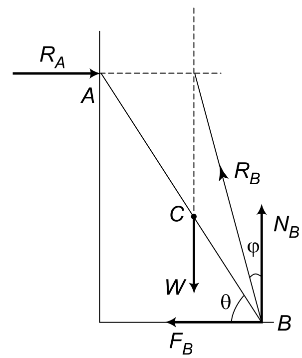


Figure 6.10 (a)

Further under limiting condition,

$$F_B = \mu N_B \tag{6.37}$$

Combining Eqs (6.34), (6.35), (6.36) and (6.37), we have

$$\frac{Wx\cos\theta}{l\sin\theta} = \mu W \text{ which yields } \theta = Arc\tan\frac{x}{\mu l}$$

For $l = 7$ m, $x = 3$ m and $\mu = 0.3$, $\theta = Arc\tan\dfrac{3}{0.3\times 7} = 55°$

[**Note:** Refer to the Fig. 6.10 (a), R_B is the resultant of N_B and F_B]

Example 6.5 A ladder AB of length l carries a person of weight W and is supported by a vertical wall at A and a horizontal floor at B and makes an angle θ with the horizontal as shown in Fig. 6.11. If the co-efficient of friction between all the mating surfaces is μ, what is the location of the person along the length of the ladder as defined by the position x, for which there would not be any slippage?

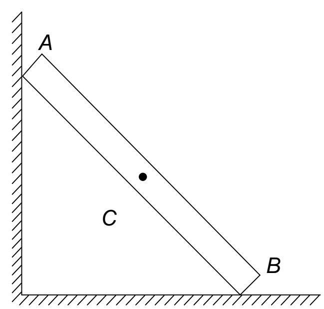


Figure 6.11

Solution

1st Method: Let the weight of the person be W and let it act vertically downward. However, exact location of the person and hence the point of application of W cannot be ascertained at the very beginning.

Since both the wall and the floor offer friction, the resultant of normal force (N) and friction force (F) from the wall as well as from that of floor will make an angle (φ) with the horizontal and vertical respectively as evident from Fig. 6.11 (a), such that $\tan\varphi = \mu$.

Thus R_A is the resultant of N_A and F_A and R_B is the resultant of N_B and F_B respectively.

These two resultants meet at point D. Since the ladder is in equilibrium under the action of R_A, R_B and W, they must have a common point of intersection, which is D.

Thus point C on the ladder is the unique point through which the weight of the person should act vertically downwards. So it is $BC = x$, the location of the person along the ladder from the floor end, that governs the equilibrium of the system.

From the ΔABD, $\angle BAD = \theta + \varphi$ and $\angle ABD = 90° - (\theta + \varphi)$

$$\therefore \quad \angle BDA = 180° - (\angle BAD + \angle ABD) = 90°$$

Further, $AB = l$ and $BC = x$

$$\therefore \quad \sin\{90° - (\theta + \varphi)\} = \frac{AD}{AB}$$

$$AD = l\cos(\theta + \varphi)$$

Again from ΔADE, $\cos\varphi = \dfrac{AE}{AD}$

$$AE = l\cos(\theta + \varphi).\cos\varphi$$

From the ΔAEC, $\cos\theta = \dfrac{AE}{AD}$

$$AC = l\cos(\theta + \varphi).\cos\varphi.\sec\theta$$

$$\therefore \quad BC = x = AB - AC = l[1 - \cos(\theta + \varphi)\cos\varphi.\sec\theta]$$

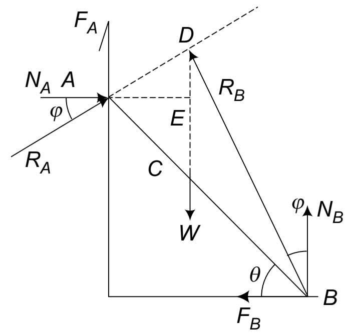


Figure 6.11 (a)

2nd Method: Considering the equilibrium of the ladder,

$$\sum X = 0$$

$$N_A = F_B = \mu N_B \tag{6.38}$$

$$\sum Y = 0$$

$$N_B = W - F_A = W - \mu N_A \tag{6.39}$$

$$\sum M_B = 0$$

$$W \times x\cos\theta = N_A \times l\sin\theta + \mu N_A \times \cos\theta = N_A\, l[\sin\theta + \mu\cos\theta] \tag{6.40}$$

Comparing Eqs (6.38) and (6.39),

$$N_A = \frac{\mu W}{1+\mu^2}$$

Replacing N_A in Eq. (6.40),

$$W \times x\cos\theta = \frac{\mu W}{1+\mu^2}\, l[\sin\theta + \mu\cos\theta]$$

$$x\cos\theta(1 + \mu^2) = l[\mu\sin\theta + \mu^2\cos\theta]$$

$$x\cos\theta\,(1 + \tan^2\varphi) = l[\tan\varphi\sin\theta + \tan^2\varphi\cos\theta]$$

$$x\cos\theta \times \frac{1}{\cos^2\varphi} = l\left[\frac{\sin\varphi}{\cos\varphi}\sin\theta + \frac{\sin^2\varphi}{\cos^2\varphi}\cos\theta\right]$$

$$= l\left[\frac{\sin\theta\sin\varphi\cos\varphi + \sin^2\varphi\cos\theta}{\cos^2\varphi}\right]$$

$$x\cos\theta = l\sin\varphi[\cos\varphi\sin\theta + \sin\varphi\cos\theta] = l\sin\varphi\sin(\theta + \varphi)$$

Note: This expression looks different from the expression of x obtained by 1st method. But, little bit of trigonometry will help achieve the same expression.

$$x \cos\theta = l \sin\varphi \sin(\theta + \varphi)$$

Since $\quad \cos\varphi \cos(\theta+\varphi) + \sin\varphi \sin(\theta+\varphi) = \cos\{\varphi - (\theta+\varphi)\} = \cos\theta$

$$\sin\varphi \sin(\theta+\varphi) = \cos\theta - \cos\varphi \cos(\theta+\varphi)$$

Thus $\quad x\cos\theta = l \sin\varphi \sin(\theta+\varphi) = l[\cos\theta - \cos\varphi\cos(\theta+\varphi)]$

$$x = \frac{l}{\cos\theta}[\cos\theta - \cos\varphi\cos(\theta+\varphi)] = l[1 - \cos(\theta+\varphi)\cos\varphi\sec\theta]$$

Note: From the expression of x, it is evident when $\varphi = 0$, i.e., in the absence of friction, x becomes zero. This implies without friction it is not possible to climb the ladder. Thus presence of friction is desirable in an attempt to climb the ladder.

Example 6.6 Determine the range of values of the weight of block B for which the system as shown in Fig. 6.12 would be on the verge of impending motion.

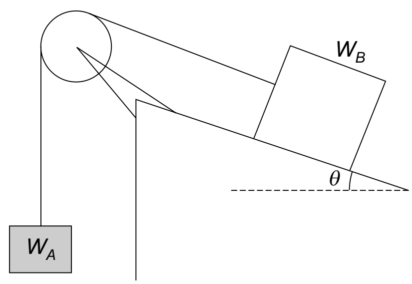


Figure 6.12

Solution Let the weight of the blocks A and B are W_A and W_B respectively.

As regards to the impending motion of the system, two different situations may arise.

(1) When the block *B* slides up along the inclined plane If the weight of the block is less than certain minimum value $(W_B)_{\min}$, W_B will slide up along the plane, under the action of W_A.

Since both the blocks are tied by the string, both will experience common tension T, as evident from the free body diagram.

From the free body diagram of the block B, equilibrium condition leads to

$$\sum X = 0$$

$$W_B \sin\theta + F_B = T = W_A \tag{6.41}$$

$$\sum Y = 0$$

$$N_B = W_B \cos\theta \tag{6.42}$$

From the condition of impending motion of block B, we have

$$F_B = \mu_s N_B \tag{6.43}$$

Combination of Eqs (6.41), (6.42) and (6.43) yields

$$W_B = \frac{W_A}{\sin\theta + \mu_s \cos\theta} = (W_B)_{\min}$$

(2) When the block *B* slides down along the inclined plane If the weight of the block B is greater than certain maximum value $(W_B)_{\max}$, it will slide down along the plane. This will cause block A to move up.

Since the block B changes its direction as compared to the earlier case, the friction force will also change its direction.

Proceeding as before, from the free body diagram of the block B, equilibrium condition leads to

$$\sum X = 0$$

$$W_B \sin\theta = F_B + T = F_B + W_A \tag{6.44}$$

$$\sum Y = 0$$

$$N_B = W_B \cos\theta \tag{6.45}$$

$$F_B = \mu_s N_B \tag{6.46}$$

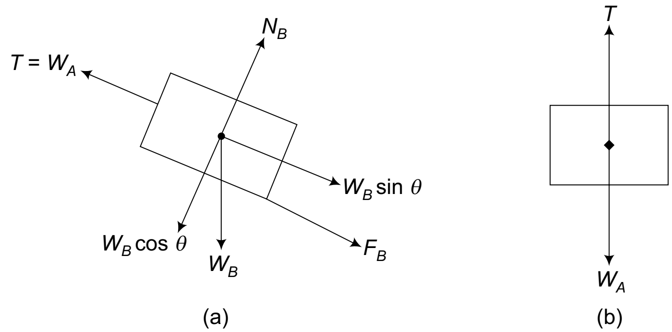


Figure 6.12

Combination of Eqs (6.44), (6.45) and (6.46) yields

$$W_B = \frac{W_A}{\sin\theta - \mu_s \cos\theta} = (W_B)_{max}$$

Example 6.7 Block A of mass 25 kg rests on another block B of mass 35 kg. The two blocks together are placed on an inclined plane that makes an angle 30° with the horizontal as shown in Fig. 6.13. Block A is tied by a horizontal rope that is connected to the vertical wall at C. What should be the magnitude of P applied on block B parallel to the inclined plane so that motion impends? Assume the co-efficient of friction for all the contiguous surfaces to be 0.3.

Solution Since two blocks are placed one over the other, as per Newton's third law, there would be mutual normal reactions.

Further, the lower block B being placed over the inclined plane, it will also experience normal reaction from the plane.

As regard to friction forces, the lower block B having two contact surfaces (the lower one with the inclined plane and the top one with that of block A), will be subjected to two different friction forces that oppose the motion.

Free body diagram of the block A is presented in Fig. 6.13 (a).

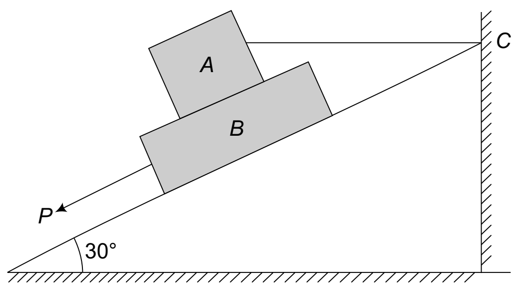


Figure 6.13

$$\sum X = 0$$

$$T\cos\theta = W_A \sin\theta + F_A = W_A \sin\theta + \mu N_A \tag{6.47}$$

$$\sum Y = 0$$

$$T\sin\theta = N_A - W_A\cos\theta \tag{6.48}$$

Dividing Eqs (6.48) by (6.47) yields $= \dfrac{\sin\theta}{\cos\theta} = \dfrac{N_A - W_A\cos\theta}{W_A\sin\theta + \mu N_A}$

Rearranging above expression leads to

$$N_A = \frac{W_A}{\cos\theta - \mu\sin\theta} \tag{6.49}$$

For block B [Refer Fig. 6.13 (b)],

$$\sum X = 0$$

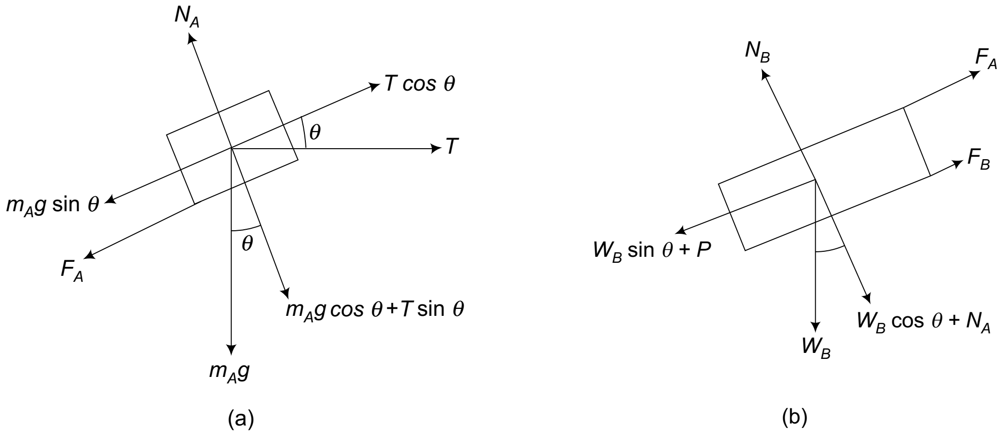


Figure 6.13

$$P + W_B \sin \theta = F_A + F_B = \mu(N_A + N_B) \tag{6.50}$$

$$\sum Y = 0$$

$$W_B \cos \theta + N_A = N_B \tag{6.51}$$

Combining Eqs (6.50) and (6.51),

$$\begin{aligned} P &= \mu[N_A + N_A + W_B \cos \theta] - W_B \sin \theta \\ &= 2\mu N_A - W_B[\sin \theta - \mu \cos \theta] \\ &= 2\mu \frac{W_A}{[\cos \theta - \mu \sin \theta]} - W_B[\sin \theta - \mu \cos \theta] \end{aligned}$$

Putting the values of $\mu = 0.3$, $W_A = 25 \times 9.81$ N, $W_B = 40 \times 9.81$ N and $\theta = 30°$

$$\begin{aligned} P &= 2 \times 0.3 \times \frac{25 \times 9.81}{[\cos 30 - 0.3 \times \sin 30]} - 40 \times 9.81[\sin 30 - 0.3 \times \cos 30] \text{ N} \\ &= 111.26 \text{ N} \end{aligned}$$

Example 6.8 A 50 kg crate is placed on an inclined plane that makes an angle 15° with the horizontal as shown in Fig. 6.14. Assuming the co-efficient of static friction (μ_s) and kinetic friction (μ_k) between the crate and the block are 0.23 and 0.17 respectively, determine the magnitude and direction of friction force offered by the surface on the crate for (a) $P = 0$, (b) $P = 195$ N, (c) $P = 260$ N (d) What is the optimum value of P for which the motion of the crate up along the plane is impending?

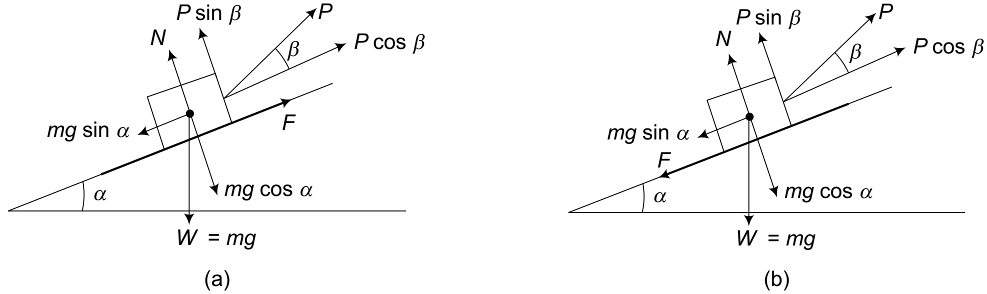


Figure 6.14 *(a) Block is sliding down, (b) Block is moving up*

Solution

(a) When P = 0.

Since the crate is placed on an inclined plane freely, it is worthwhile to investigate whether it will slide down along the inclined plane due to its self weight in the absence of any external force, i.e., when $P = 0$.

Under this circumstance, the condition for stable equilibrium implies

Angle of inclination of the plane (α) $\leq$ Friction Angle (φ_s)

Given $\mu_s = 0.23 = \tan \varphi_s$; $\varphi_s = Arc \tan (0.23) = 13°$

But $\alpha = 15°$, which is greater than φ_s. Hence, the crate will slide down when $P = 0$.

So, the force balance along the plane considering static equilibrium will not hold true.

But force balance normal to the inclined plane yields $N = mg \cos \alpha$ [Refer Fig. 6.11 (a)].

Since the crate is under motion, the friction offered by the inclined plane is kinetic friction.

$\therefore$ Friction force $= F_k = \mu_k N = \mu_k . mg \cos \alpha = 0.17 \times 50 \times 9.81 \cos 15$ N $= 80.53$ N

Since the crate will slide down, the *friction force will act upward along the plane.*

(b) When P = 195 N.

P is resolved into two mutually perpendicular components – one along the plane with a magnitude of $P \cos \beta$ and the other, perpendicular to the plane of magnitude $P \sin \beta$ as evident from the free body diagram [Refer Fig. 6.11 (b)].

As before, $mg \sin \alpha$ would also act along the plane. Since $P \cos \beta$ and $mg \sin \alpha$ are directed opposite, it is imperative to calculate numerical values to establish the direction of friction force.

From the given data, $P \cos \beta = 195 \cos 20$ N $= 183.25$ N and $mg \sin \alpha = 50 \times 9.81 \times \sin 15$ N $= 126.95$ N.

Thus $P \cos \beta > mg \sin \alpha$; this implies that the crate has a tendency to go up.

Friction force would therefore act downwards along the plane.

Considering the static equilibrium, the force balance of the crate along the plane gives

$P \cos \beta = mg \sin \alpha + F \Rightarrow F = P \cos \beta - mg \sin \alpha = 183.25 - 126.95$ N $= 56.3$ N.

Note: It would be a blunder to apply $F_s = \mu_s N$ since the block is not under the condition of impending motion.

(d) Condition of impending motion.

The condition of impending motion leads to

$$F_s = \mu_s N \tag{6.52}$$

Let the P assumes an optimum value $P = P_{opt}$ under this condition.

Considering the static equilibrium of the crate,

$$\sum X = 0$$

$$P \cos \beta = mg \sin \alpha + F_s \tag{6.53}$$

$$\sum Y = 0$$

$$N + P \sin \beta = mg \cos \alpha \tag{6.54}$$

Combining Eqs (6.52), (6.53) and (6.54), we have

$$P \cos \beta = mg \sin \alpha + \mu_s [mg \cos \alpha - P \sin \beta]$$

$$P_{opt} = \frac{mg[\sin \alpha + \mu_s \cos \alpha]}{[\cos \beta + \mu_s \sin \beta]} = \frac{mg[\sin \alpha + \tan \varphi_s \cos \alpha]}{[\cos \beta + \tan \varphi_s \sin \beta]} = \frac{mg \sin(\alpha + \varphi_s)}{\cos(\beta - \varphi_s)}$$

For $\alpha = 15°$, $\varphi_s = 13°$, $\beta = 20°$, $P_{opt} = \dfrac{50 \times 9.81 \sin(15+13)}{\cos(20-13)}$ N $= 232$ N

Thus P should have a value of 232 N to cause motion to impend.

(c) When P = 260 N.

Since $P = 260$ N $> P_{opt}$ (232 N), the crate will slide up. Hence the crate is under kinetic friction and it will act downwards along the plane.

So, the force balance along the plane considering static equilibrium is no longer valid. However, force balance normal to the inclined plane yields $N + p \sin \beta = mg \cos \alpha \Rightarrow N = mg \cos \alpha - P \sin \beta$

$$\therefore \quad \text{Friction force} = F_k = \mu_k N = \mu_k.[mg \cos \alpha - P \sin \beta]$$

$$= 0.17 \times [50 \times 9.81 \times \cos 15 - 260 \times \sin 20] \text{ N} = 65.43 \text{ N}$$

Example 6.9 A block ABCD having height h and base width b rests on an inclined plane that makes an angle θ with the horizontal and subjected to a force P applied at the top and parallel to the plane as shown in Fig. 6.15. Assuming the co-efficient of static friction (μ_s) determine the maximum $\frac{h}{b}$ ratio for which the block will slide along the plane without tipping.

Solution The free body diagram of the block is shown in Fig. 6.15 (a).

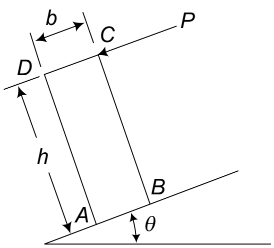


Figure 6.15

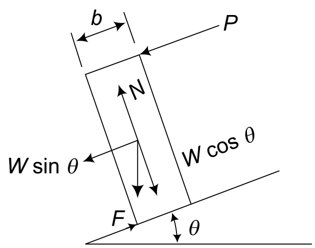


Figure 6.15 (a)

When tipping occurs, the contact between the block and the inclined plane will be only at the lower left edge.

Thus under optimum condition, there are two forces that are held responsible for its equilibrium. These are P and W. P will try to produce a *counter-clockwise moment* with respect to A so that it can tip. On the other hand, W that acts at the CG of the block will produce a *clockwise moment* with respect to A so as to prevent tipping.

Under equilibrium, these two moments must be equal. Further, since the contact between the block and the plane is only at A, the normal reaction N and frictional force F necessarily will act at A.

Thus using the conditions of equilibrium,

$$\sum X = 0$$

$$P + W \sin \theta = F_s = \mu_s N \tag{6.55}$$

$$\sum Y = 0$$

$$N = W \cos \theta \tag{6.56}$$

$$\sum M_B = 0$$

$$P \times h + W \sin \theta \times \frac{h}{2} = W \cos \theta \times \frac{b}{2} \tag{6.57}$$

Combining Eqs (6.55) and (6.56), we have

$$P = W[\mu_s \cos \theta - \sin \theta] \tag{6.58}$$

Now eliminating P from Eqs (6.57) and (6.58), we have

$$Wh[\mu_s \cos \theta - \sin \theta] + W \sin \theta \frac{h}{2} = W \cos \theta \frac{b}{2}$$

$$h\left[\mu_s \cos \theta - \sin \theta + \frac{\sin \theta}{2}\right] = \cos \theta \frac{b}{2}$$

$$h[2\mu_s \cos \theta - \sin \theta] = b \cos \theta$$

$$\frac{h}{b} = \frac{\cos\theta}{2\mu_S\cos\theta - \sin\theta} = \frac{1}{2\mu_S - \tan\theta}$$

Hence
$$\left[\frac{h}{b}\right]_{\max} = \frac{1}{2\mu_S - \tan\theta}$$

Example 6.10 A wedge block A weighing 1000 N is to be raised by another block B weighing 600 N as shown in Fig. 6.16. The angle of wedge for both the block is set at 20º. Determine the minimum horizontal force P to be applied horizontally to the block B so as to enable the block A to rise. Assume the co-efficient of static friction (μ_s) for all the surfaces in contact to be 0.25.

Solution

1st Method: Free body diagram of both the blocks are portrayed in the 6.16 (a).

Under the action of force P, the lower block B will be pushed towards left. Since there is a common taper provided on both the blocks along their surfaces of contact, A will be raised. Thus the block B will experience two different friction forces – one between the block and the floor and other along the contact surface with the block A.

Similarly, block A will experience two different friction forces – one between the block and the wall and the other along the contact surface with the block B.

Note that the friction force along their mutual contact surfaces will be equal but their direction will be just opposite to each other.

Further, both the blocks will be subjected to normal reactions that arise from the supports, i.e., the floor, wall and their common surface of contact.

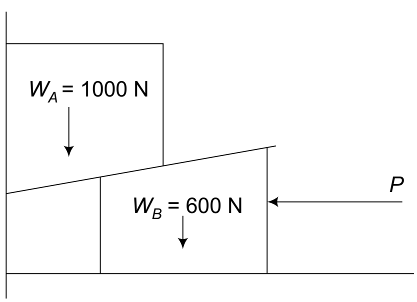


Figure 6.16

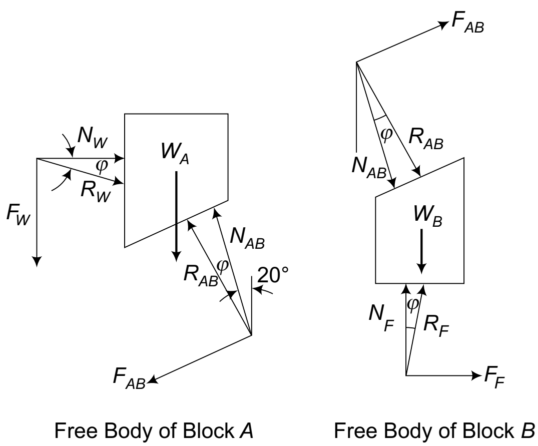


Figure 6.16 (a)

The details of force that are acting on individual block are shown in free body diagrams.

Frictional forces for all the contiguous surfaces are listed first.

Introducing suffix W for wall; F for floor and AB for common contact surface, i.e., $F_w = \mu{\cdot}N_w$; $F_F = \mu{\cdot}N_F$ and $F_{AB} = \mu{\cdot}N_{AB}$

Considering the equilibrium of upper block A,

$$\sum X = 0;$$

$$N_w - F_{AB}\cos 20° - N_{AB}\sin 20° = 0$$

$$N_w - \mu{\cdot}N_{AB}\cos 20° - N_{AB}\sin 20° = 0 \qquad (6.59)$$

This yields

$$N_w = 0.5777.N_{AB} \tag{6.60}$$

$$\sum Y = 0$$

$$N_{AB} \cos 20° - \mu N_{AB} \sin 20° - \mu N_w - 1000 = 0$$

$$0.8545 N_{AB} - 0.25 N_w = W_A = 1000 \tag{6.61}$$

Solving Eqs (6.60) and (6.61), N_{AB} = 1407.95 N and N_W = 812.4 N

Now considering the equilibrium of lower block B

$$\sum X = 0$$

$$\mu N_F + \mu N_{AB} \cos 20° + N_{AB} \sin 20° = P$$

$$0.25 N_F = P - 812.4 \tag{6.62}$$

$$\sum Y = 0$$

$$N_F + \mu N_{AB} \sin 20° = N_{AB} \cos 20° + 600$$

Substituting for N_{AB},

$$N_F = 600 + 1407.95 \cos 20° - 0.25 \times 1407.95 \sin 20° = 1802.6 \text{ N}$$

$$\therefore \quad P = 0.25 \times 1802.6 + 812.4 \text{ N} = 1263.04 \text{ N}. \quad \text{[From Eq. (6.62)]}$$

Thus the value of P should be at least 1263.04 N so that block A can be raised.

2nd Method: This is a graphical method based on *Polygon of forces*. Consider the forces acting on block A followed B. Once N_w is calculated, we can proceed to construct vector diagram.

$$R_W = \sqrt{N_W^2 + F_W^2} = \sqrt{1+\mu^2} \cdot N_W = \sqrt{1+(0.25)^2} \times 811.4 \text{ N} = 837.4 \text{ N}$$

Refer to Fig. 6.16 (b). Select a suitable scale to represent R_W by OL. While doing so, consider a point O from which draw a line OL with an inclination φ with the horizontal. Thus OL is a vector to represent the magnitude and direction of R_W. From point L, draw a vertical line LM equal to the magnitude of W_A = 1000 N. Now LM represents 1000 N.

Since the block A is in equilibrium under the action of three forces namely W_A, R_W and R_{AB}, they must form a closed triangle. Thus MO represents R_{AB}.

Since R_{AB} is common to both the blocks, it will act on block B but its direction will be reversed. Thus OM represents R_{AB} as regard to the block B. From M, draw a vertical line MN to represent 600 N – the weight of the block B.

From N, we draw a horizontal line NT following the sense of $\boldsymbol{P}$. Further from O, a line OT is drawn so that it makes an angle φ with the vertical. NT and OT intersect at T. Thus $\overrightarrow{NT}$ and $\overrightarrow{TO}$ represent the vectors correspond to $\boldsymbol{P}$ and R_F respectively. Thus $OLMNTO$ is the polygon (closed) of forces.

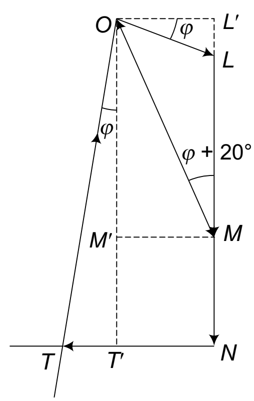


Figure 6.16 (b)

Measure the length of NT and multiply it by the scale factor to express its magnitude in Newton.

3rd Method: This is an analytical method based on *Trigonometry*. Based on Fig. 6.16 (b), from $\Delta OL'L$, $OL' = OL \cos \varphi = R_W \cos \varphi = T'N$ and $LL' = R_W \sin \varphi$

$$\therefore \quad OM' = L'M = L'L + LM = R_W \sin \varphi + 1000$$

Further, $MN = 600 = M'T'$

$$\therefore \quad OT' = OM' + M'T' = R_W \sin \varphi + 1000 + 600 = R_W \sin \varphi + 1600$$

From ΔOTT, $\tan \varphi = \dfrac{TT'}{OT'}$ or $TT' = OT' \tan \varphi$

$\therefore$ $P = NT = NT' + T'T = R_W \cos \varphi + (R_W \sin \varphi + 1600) \tan \varphi = 837.4 \cos 14 + (837.4 \sin 14 + 1600) \times 0.25$

$P = 1263.1$ N

Thus the value of P should be at least 1263.1 N.

Note: Readers are advised to follow any techniques. Nevertheless, 2nd method (graphical) is less time consuming and easier. However, its accuracy depends on the scale factor and the human errors associated while drawing vector diagram.

Example 6.11 Two blocks having weights $W_1 = W_2 = 20$ N are attached by a short string and rest on an inclined plane as shown in Fig. 6.17. If the co-efficient of friction for the blocks are $\mu_1 = 0.2$ and $\mu_2 = 0.3$ respectively, find the angle of inclination of the plane for which motion impends.

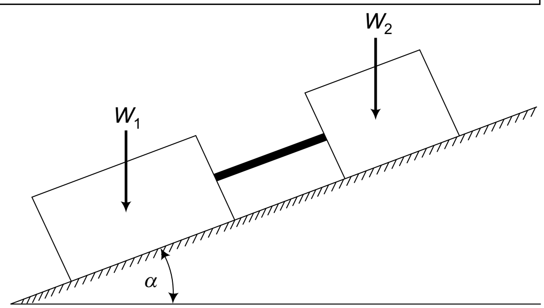


Figure 6.17

Solution From the free body diagrams of block 1 [Fig. 6.17 (a)] and considering equilibrium of block 1 when motion impends,

$$\sum X = 0$$

$$W_1 \sin \alpha = T + F_1 = T + \mu_1 N_1 \qquad (6.63)$$

$$\sum Y = 0$$

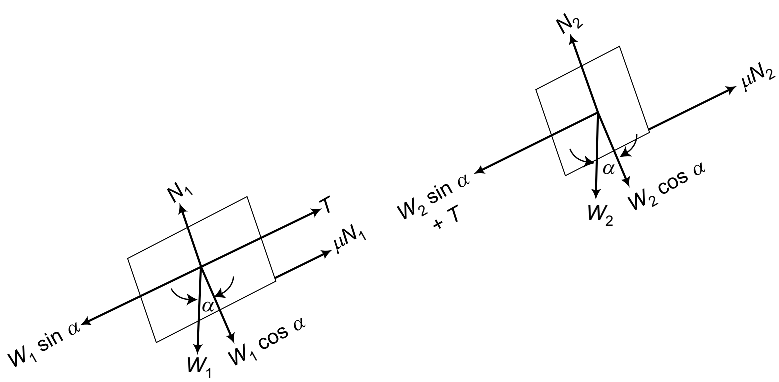


Figure 6.17 (a)

$$W_1 \cos \alpha = N_1 \qquad (6.64)$$

From Eqs (6.63) and (6.64),

$$W_1 \sin \alpha = T + \mu_1 W_1 \cos \alpha_1 \qquad (6.65)$$

Similar equilibrium condition of block 2 leads to

$$\sum X = 0$$

$$W_2 \sin \alpha + T = F_2 = \mu_2 N_2 \qquad (6.66)$$

$$\sum Y = 0$$

$$W_2 \cos \alpha = N_2 \qquad (6.67)$$

From Eqs (6.66) and (6.67),

$$W_2 \sin \alpha = \mu_2 N_2 - T = \mu_2 W_2 \cos \alpha - T \quad (6.68)$$

Comparing Eqs (6.65) and (6.68),

$$W_1 \sin \alpha - \mu_1 W_1 \cos \alpha = \mu_2 W_2 \cos \alpha - W_2 \sin \alpha$$

Since $W_1 = W_2$, simplification of above relationship yields

$$\tan \alpha = \frac{\mu_1 + \mu_2}{2} = \frac{0.2 + 0.3}{2} = 0.25$$

$$\alpha = \tan^{-1} (0.25) = 14.04°$$

Example 6.12 Two blocks having weights $W_1 = W_2 = 10$ N are attached by a flexible string and rest on a horizontal and an inclined plane respectively as shown in Fig. 6.18. The string passes over a frictionless pulley. If the co-efficient of friction for both the blocks is μ, prove that the angle of inclination of the plane should be at least twice the angle of friction so as to have impending motion of the system.

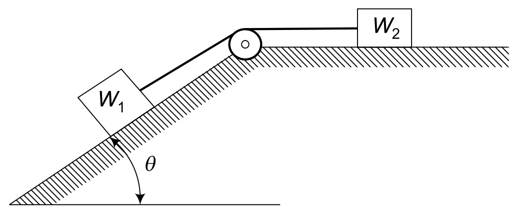


Figure 6.18

Solution From the free body diagram of block 1 that rests on inclined plane (Fig. 6.18) and considering equilibrium of block 1 when motion impends, we have

$$\sum X = 0$$

$$W_1 \sin \theta = T + F_1 = T + \mu N_1 \quad (6.69)$$

$$\sum Y = 0$$

$$W_1 \cos \theta = N_1 \quad (6.70)$$

From eqs (6.69) and (6.70),

$$W_1 \sin \theta = T + \mu W_1 \cos \theta \quad (6.71)$$

From the free body diagram of block 2 that rests on horizontal plane (Fig. 6.8 (b)), we obtain

$$\sum X = 0$$

$$T = F_2 = \mu N_2 \quad (6.72)$$

$$\sum Y = 0$$

$$W_2 = N_2 \quad (6.73)$$

Thus $$T = \mu W_2 \quad (6.74)$$

Eliminating $T = \mu W_2$ from Eqs (6.71) and (6.74),

$$W_1 \sin \theta - \mu W_1 \cos \theta = \mu W_2$$

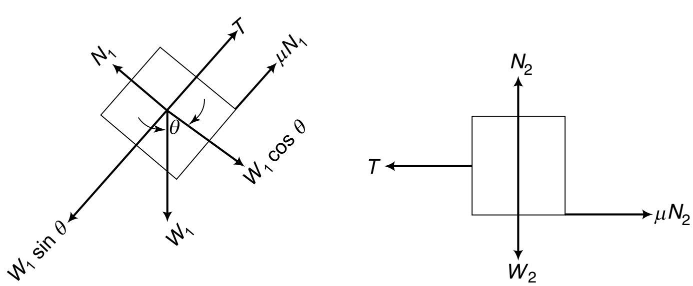


Figure 6.18 (a)

Since $W_1 = W_2$; $\sin\theta - \mu\cos\theta = \mu$

$$\sin\theta = (1 + \cos\theta)\mu$$

$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\cos^2\frac{\theta}{2}\mu$$

$$\tan\frac{\theta}{2} = \mu = \tan\varphi$$

$$\theta = 2\varphi$$

Thus inclination of plane (θ) should be twice of angle of friction (φ).

Example 6.13 A short right circular cylinder of weight W rests on a **'*V*'** block having included angle 2α as shown in Fig. 6.19. If the co-efficient of friction for the cylinder and the **'*V*'** block is μ, find the minimum value of applied horizontal pull force P so that motion of the cylinder impends.

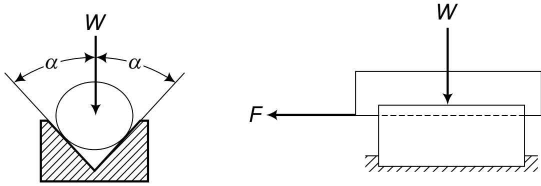


Figure 6.19

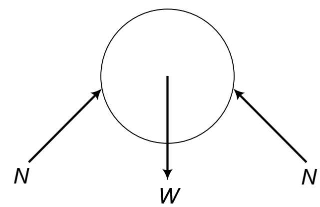


Figure 6.19 (a)

Solution From the front view of the block, $\sum Y = 0$ gives

$$2N\sin\alpha = W$$

$$N = \frac{W}{2\sin\alpha}$$

From the side view of the block, considering the equilibrium, we get

$$P = 2F = 2 \times \mu N = 2\mu \times \frac{W}{2\sin\alpha} = \frac{\mu W}{\sin\alpha}$$

Example 6.14 A right circular cylinder of mass m = 10 kg rests on a **'*V*'** block having included angle 90°. The '***V***' block is now inclined by 30° with the horizontal as shown in Fig. 6.20. If the co-efficient of friction between the cylinder and the **'*V*'** block is μ = 0.5, determine (a) the friction force F acting on each sides of the cylinder before the application of force P and (b) the magnitude of P so that cylinder is on the verge of sliding up the plane.

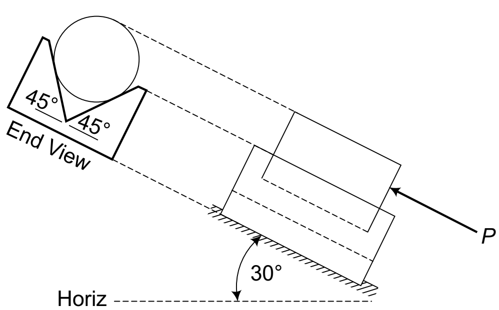


Figure 6.20

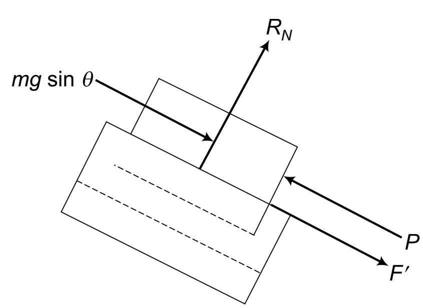


Figure 6.20 (a)

Solution

(a) Since before application of force P, the cylinder is not under the condition of impending motion; the rules of limiting friction i.e. $F = \mu N$ can not be applied. Under this condition the F to be calculated by using the condition of static equilibrium only.

From the free body of the cylinder and considering direction along the plane as x axis;

$$\sum X = 0$$

$$2F = mg \sin \theta$$

$$F = \frac{1}{2} mg \sin \theta = \frac{1}{2} \times 10 \times 9.81 \times \sin 30 = 24.525 \text{ N}$$

(b) When the P is applied so that the cylinder is about to move up along the plane; condition of limiting friction will hold true.

From the front view of the cylinder;

$$2 N \sin \alpha = W = mg$$

$$N = \frac{mg}{2 \sin \alpha} \tag{6.75}$$

Now from the side view of the cylinder; force balance along the plane yields

$$2F' + mg \sin \theta = P \tag{6.76}$$

where F' is the friction force when P is applied

***The interesting fact about this problem is that N computed above is not that so as to have $F' = \mu N$. The normal reactions perpendicular to the direction of motion R_N to be considered so that $F' = \mu R_N$* [refer Fig. 6.20 (a)]**

$\therefore$
$$2\mu R_N + mg \sin \theta = P \tag{6.77}$$

But
$$R_N = N \cos \theta \tag{6.78}$$

Comparing Eqs (6.77) and (6.78)

$\therefore$
$$2\mu \times \frac{mg}{2 \sin \alpha} \times \cos \theta + mg \sin \theta = P$$

$$mg\left[\mu \times \frac{\cos \theta}{\sin \alpha} + \sin \theta\right] = P$$

$$P = 10 \times 9.81\left[0.5 \times \frac{\cos 30}{\sin 45} + \sin 30\right] \text{N} = 109.12 \text{ N}$$

Thus the minimum value of P should be 109.12 N to enable the cylinder to move up along the incline.

Example 6.15 A wedge 'A' having weight 50 N is to be driven between an inclined support and a block 'B' of weight 1500 N as shown in Fig. 6.21. Determine the magnitude of the vertically applied force P on the wedge so as to initiate movement of the blocks. Assume that the co-efficient of friction between all the contiguous surfaces to be $\mu = 0.3$.

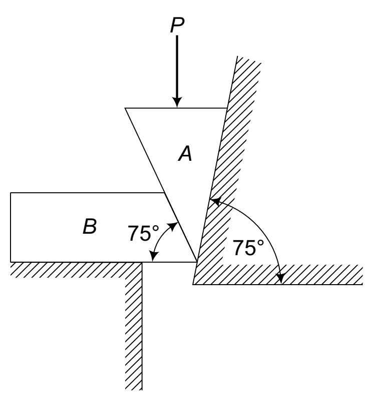


Figure 6.21

Solution The following suffices are introduced as given below

W for wall; F for floor and AB for common contact surface.

Therefore $F_w = \mu \cdot N_w$; $F_F = \mu \cdot N_F$ and $F_{AB} = \mu \cdot N_{AB}$

Considering the equilibrium of the block '***B***',

$$\sum X = 0$$

$$0.3N_{AB} \cos 75 + 0.3N_F - N_{AB} \sin 75 = 0 \tag{6.79}$$

$$\sum Y = 0$$

$$N_F - 1500 - 0.3N_{AB} \sin 75 - N_{AB} \cos 75 = 0 \tag{6.80}$$

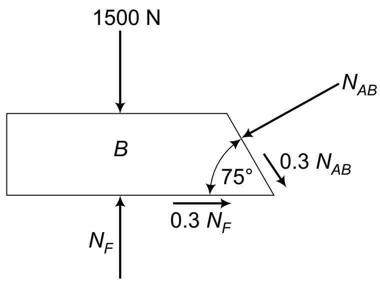


Figure 6.21 (a)

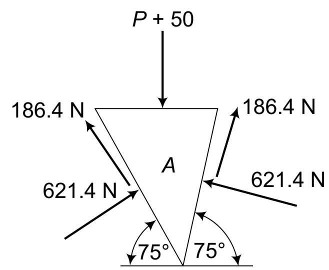


Figure 6.21 (b)

Solving Eqs (6.79) and (6.80),

$$N_{AB} = 621.4 \text{ N}$$

$$\therefore \quad F_{AB} = \mu N_{AB} = 0.3 \times 621.4 \text{ N} = 186.4 \text{ N}$$

Now considering free body of the block ***'A'*** and having

$$\sum Y = 0$$

$$P + 50 = 2(621.4 \cos 75 + 186.4 \sin 75)$$

$$P = 632 \text{ N}$$

Thus the magnitude of P should be 632 N.

Example 6.16 A block of weight $W_1 = 400$ N rests on the horizontal surface and supports on top of it, another block of weight $W_2 = 100$ N. The block W_2 is attached to a vertical wall by the inclined string AB. Find the magnitude of the horizontal force P applied to the lower block as shown in Fig. 6.22, which will be necessary to cause slipping to impend. Take coefficient of static friction for all contiguous surfaces is $\mu = 0.3$.

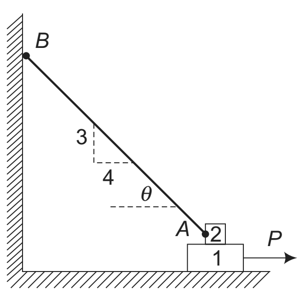


Figure 6.22

Solution From the geometry of the figure, one can write

$$\tan\theta = \frac{3}{4}$$

The free body diagram of the two blocks is shown in Fig. 6.22(a). For limiting equilibrium, one can write

$$F_1 = \mu_1 R_1 \text{ and } F_1' = \mu R_1'$$

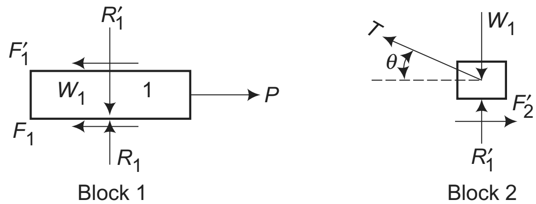


Figure 6.22(a)

From the consideration of equilibrium of block 1, we obtain

$$\sum F_x = 0$$

or $$P = F_1 + F_1' \qquad (6.81)$$

$$\sum F_y = 0$$

or $$R_1 = W_1 + R_1' \qquad (6.82)$$

Substituting $F_1 = \mu_1 R_1$ and $F_1' = \mu R_1'$ into Eq. (6.82), we get

$$\frac{F_1}{\mu} = W_1 + \frac{F_1'}{\mu} \qquad (6.83)$$

From the consideration of equilibrium of block 2, we get

$$\sum F_x = 0$$

or $$F_1' = T\cos\theta$$

or $$T = \frac{F_1'}{\cos\theta} \qquad (6.84)$$

$$\sum F_y = 0$$

or $$R_1' + T\sin\theta = W_2 \qquad (6.85)$$

Substituting the value of T from Eq. (6.84) into Eq. (6.85), we obtain

$$\frac{F_1'}{\mu} + \frac{F_1'}{\cos\theta} \times \sin\theta = W_2$$

or $$F_1' = \frac{W_2}{\frac{1}{\mu} + \tan\theta} = \frac{100}{\frac{1}{0.3} + \frac{3}{4}} = 24.49 \text{ N}$$

From Eq. (6.83), we have

$$F_1 = 0.3 \times 400 + 24.49 = 144.49 \text{ N}$$

From Eq. (6.81), we obtain

$$P = F_1 + F_1' = 144.49 + 24.49 = 168.98 \text{ N}$$

Example 6.17 Two rollers – one large and another small having diameters D and d respectively are placed side by side on a rough floor as shown in Fig. 1. The larger roller is pulled horizontally by a force $\boldsymbol{P}$. Assuming the coefficient of friction, μ is same for all the contiguous surfaces, determine the necessary condition for which the larger roller can be pulled over the smaller one.

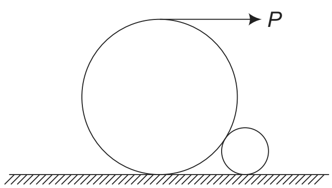


Figure 6.23

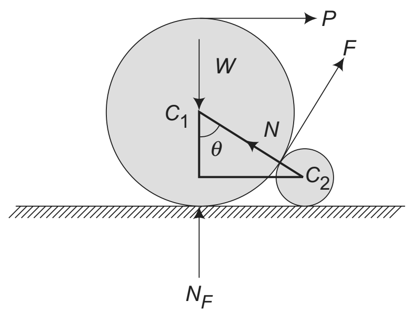


Figure 6.23(a)

Solution The forces acting on the larger roller is shown in Fig. 6.23(a). Considering equilibrium when the motion impends, we obtain

$$\sum X = 0$$

or $$P + F\cos\theta = N\sin\theta$$

or $$P + \mu N\cos\theta = N\sin\theta$$

or $$N = \frac{P}{\sin\theta - \mu\cos\theta} \tag{6.86}$$

Further, taking moment about C_1, we get

$$\sum M_{c_1} = 0$$

or $$P \times \frac{D}{2} = F \times \frac{D}{2}$$

or $$P = F = \mu N$$

or $$N = \frac{P}{\mu} \tag{6.87}$$

Comparing Eqs. (6.86) and (6.87), we have

$$\mu = \sin\theta - \mu\cos\theta \Rightarrow \mu = \frac{\sin\theta}{1+\cos\theta}$$

or $$\mu = \frac{\sin\theta}{1+\cos\theta} \tag{6.88}$$

From the geometry of the figure [6.23(a)], one can write

$$\cos\theta = \frac{D-d}{D+d}$$

$$\sin\theta = \sqrt{1-\left(\frac{D-d}{D+d}\right)^2} = 2\frac{\sqrt{Dd}}{D+d}$$

Substituting the values of cos θ and sin θ into Eq. (6.23), we obtain

$$\mu = \frac{2\sqrt{Dd}\Big/D+d}{1+\dfrac{D-d}{D+d}} = \sqrt{\frac{d}{D}}$$

The necessary condition for which the larger roller can be pulled over the smaller one is $\mu \geq \sqrt{\dfrac{d}{D}}$.

Example 6.18 A smooth circular cylinder of weight W and radius r is placed above two smooth semicircular cylinders each of radius r and weight $W/2$ as shown in the Fig. 6.24. Find the maximum value of distance b for which motion will impend. Consider $\mu = 0.5$ between the semicircular cylinders and the horizontal plane.

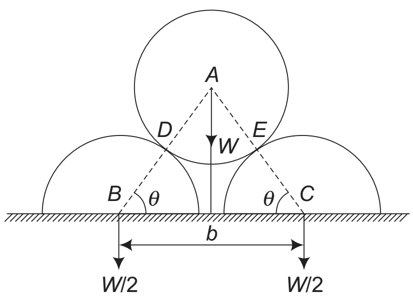


Figure 6.24

Solution The free body diagram of the upper cylinder is shown in Fig. 6.24(a). From the consideration of equilibrium of the upper cylinder and symmetrical configuration, one can write

$$R_D = R_E = \frac{W}{2\sin\theta} \tag{6.89}$$

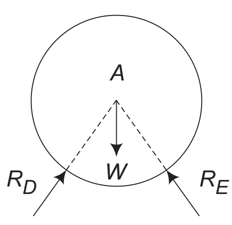


Figure 6.24(a)

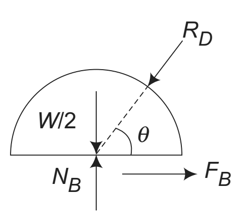


Figure 6.24(b)

The free body diagram of the lower left semicircular cylinder is shown in Fig. 6.24(b). Considering the equilibrium of the cylinder, we get

$$\sum X = 0$$

or

$$F_B = \mu N_B = R_D \cos\theta \tag{6.90}$$

$$\sum Y = 0$$

or $$N_B = \frac{W}{2} + R_D \sin\theta \tag{6.91}$$

Substituting the value of R_D from Eqs. (6.89) into Eq. (6.91), we obtain

$$N_B = \frac{W}{2} + \frac{W}{2\sin\theta}\sin\theta = W$$

Substituting $\mu = 0.5$ and $N_B = W$ into Eqs. (6.90) and (6.91), one can write

$$R_D \cos\theta = \mu N_B = \frac{W}{2}$$

and $$R_D \sin\theta = N_B - \frac{W}{2} = W - \frac{W}{2} = \frac{W}{2}$$

From the above two equations, we have

$$\tan\theta = 1$$

From the geometry of the figure (Fig. 6.24), one can write

$$\tan\theta = \frac{\sqrt{(2r)^2 - \left(\frac{b}{2}\right)^2}}{b/2}$$

or $$1 = \frac{\sqrt{(2r)^2 - \left(\frac{b}{2}\right)^2}}{b/2}$$

or $$b = 2\sqrt{2}r$$

Example 6.19 Two ends of a heavy prismatic bar AB are supported by a circular ring in a vertical plane, as shown in Fig. 6.25. The length of the bar is such that it subtends an angle 90 degree in the ring. If the angle of friction at A and B are each ϕ, what is the greatest angle of inclination that the bar can make with the horizontal in a condition of equilibrium?

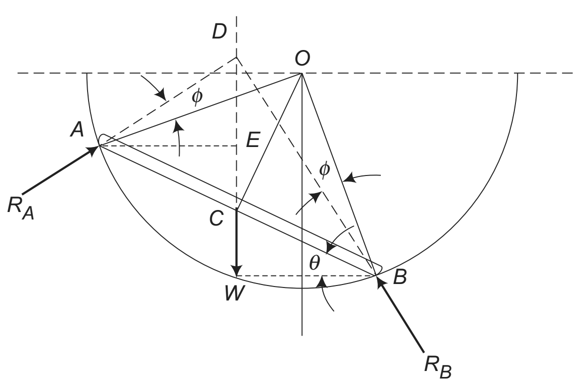


Figure 6.25

Solution The bar AB is under equilibrium under the actions of three forces W, R_A and R_B. Therefore, these three forces meet at common point D.

Let the length of the bar $AB = l$. C being the mid-point of AB, $AC = l/2$.

It is given that $\angle AOB = 90°$ and $OA = OB$, therefore from ΔAOB, we get $\angle OAB = \angle OBA = 45°$

From ΔDAB, one can write, $\angle DAB = 45° + \phi$ and $\angle DBA = 45° - \phi$.

$$\therefore \qquad \angle ADB = 180 - (45° + \phi + 45° - \phi) = 90°$$

From ΔAEC,
$$AE = \frac{l}{2}\cos\theta$$

From ΔDAE,
$$\angle DAE = 45° - \theta + \phi$$

$$\therefore \qquad \cos(45° - \theta + \phi) = \frac{AE}{AD} = \frac{l/2\cos\theta}{AD}$$

or
$$AD = \frac{l/2\cos\theta}{\cos(45° - \theta + \phi)} \qquad (6.92)$$

Again from ΔDAB, we obtain

$$\cos(45° + \phi) = \frac{AD}{AB} = \frac{AD}{l}$$

or
$$AD = l\cos(45° + \phi) \qquad (6.93)$$

Comparing Eqs. (6.92) and (6.93), one can write

$$\frac{l/2\cos\theta}{\cos(45° - \theta + \phi)} = l\cos(45° + \phi)$$

or
$$2\cos(45° - \theta + \phi)\cos(45° + \phi) = \cos\theta$$

Simplifying the above expression, we obtain

$$\cos\{90 - (\theta - 2\phi)\} = 0 \Rightarrow \sin(\theta - 2\phi) = \sin 0 \Rightarrow \theta = 2\phi$$

or
$$\sin(\theta - 2\phi) = \sin 0$$

or
$$\theta = 2\phi$$

MULTIPLE-CHOICE QUESTIONS

6.1 If φ is the angle of friction, then the co-efficient of friction (μ) is
(a) $\cos\varphi$ (b) $\sin\varphi$ (c) $\tan\varphi$ (d) $\sec\varphi$

6.2 The maximum frictional force that acts when a body is about to slide over a surface
(a) sliding friction (b) rolling friction (c) kinetic friction (d) limiting friction

6.3 Under static condition when motion is not impending, the friction force F is the
(a) $F < \mu_s N$ (b) $F = \mu_s N$ (c) $F > \mu_s N$ (d) unpredictable

6.4 The coefficient of friction depends on
(a) strength of surface (b) nature of surface
(c) contact area of surface (d) none of the above

6.5 Strike the incorrect statement.
(a) The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two surfaces.

(b) The frictional force depends on the degree of roughness of the surface.
(c) The frictional force depends on the surface area of contact.
(d) The frictional force acts in a direction opposite to the impending motion of the body.

6.6 One of the laws of kinetic friction is
(a) the frictional force is independent of the normal reaction.
(b) the frictional force is always opposite to the direction of motion.
(c) the frictional force is constant for moderate speeds.
(d) kinetic friction is always less than the static friction.

6.7 The optimum inclination angle of a surface with the horizontal when a body placed on it is on the verge of sliding down in absence of any external force is called
(a) the angle of equilibrium (b) the angle of repose
(c) the angle of sliding (d) the angle of friction.

6.8 A block of 70 N is placed on a horizontal plane. A pull force of P is applied at an angle θ with the horizontal. The magnitude of P will be minimum when
(a) $\theta < \varphi$ (b) $\theta = \varphi$ (c) $\theta > \varphi$ (d) $\theta = 0$

6.9 A block of 40 N is placed on a horizontal plane having $\mu = 0.4$. A pull force of P is applied at an angle θ with the horizontal. The minimum value of P will be
(a) 10.86 N (b) 12.37 N (c) 14.85 N (d) 20 N

6.10 A body of weight W is kept on a rough inclined plane having an angle θ with the horizontal. The minimum force P required to be applied parallel to the plane to slide the body up is
(a) $\dfrac{W\sin(\theta-\varphi)}{\cos\varphi}$ (b) $\dfrac{W\sin(\theta+\varphi)}{\sin\varphi}$ (c) $\dfrac{W\cos(\theta+\varphi)}{\sin\varphi}$ (d) $\dfrac{W\sin(\theta+\varphi)}{\cos\varphi}$

6.11 For a particular surface,
(a) $\mu_s < \mu_k$ (b) $\mu_s = \mu_k$ (c) $\mu_s > \mu_k$ (d) none of the above

SHORT ANSWER TYPE QUESTIONS

6.1 What is friction? Where does it come from? Is it desirable or not?
6.2 Explain the physics behind friction.
6.3 Classify friction and explain its concept.
6.4 Give some application examples where friction is desirable.
6.5 Plot friction versus applied force and explain its nature.
6.6 What is meant be angle of friction and coefficient of friction? What are the parameters that influence its value?
6.7 Write the laws of friction.
6.8 What is meant by angle of repose? What should be its value for a particular surface?
6.9 Explain cone of friction.
6.10 What is meant by limiting value of static friction?
6.11 "Force required to maintain motion of a body is lesser than that to initiate movement" – Justify the statement with reasons.
6.12 Derive an expression for the force P that is required to slide a body of weight W up along an inclined plane when P is applied horizontally. Assume inclination of the plane with the horizontal is θ and angle of friction is φ.

NUMERICAL PROBLEMS

6.1 An inclined force P is applied to a block resting on a rough floor as shown in the Fig. 6.26. The inclination of the force is θ with the horizontal. For P to be least, prove that $\theta = \varphi$, where φ is the angle of friction between the block and the floor.

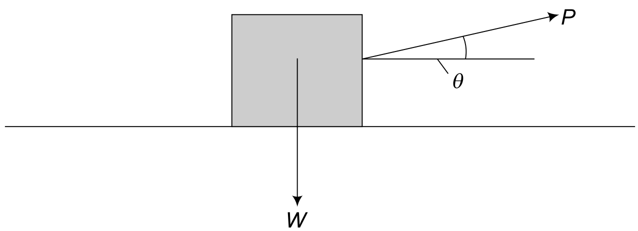


Figure 6.26

6.2 A uniform ladder weighting 28 N rests against a smooth vertical wall with its lower end 5 cm from the wall. The friction co-efficient between the ladder and the floor is 0.32. The inclination of the ladder is 25° with the vertical. Show that the ladder remains in equilibrium in this position.

6.3 A body of certain weight is placed on an inclined plane that is inclined at angle θ with the horizontal. Prove that when $\theta = \varphi$, the angle of friction, the sliding of the body down the plane impends.

6.4 A block weighing 900 N is held on an inclined plane by a horizontal force of 1500 N as shown in Fig. 6.27 is the block in equilibrium? The co-efficient of static friction is 0.3.

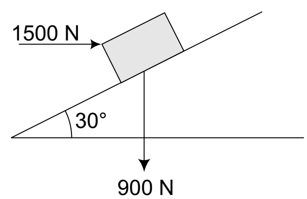


Figure 6.27

6.5 A block of weight 800 N is held on an inclined plane by a force P applied to it at angle of 40° with the plane as shown in Fig. 6.28. The angle of friction between the plane and the body is 15°. Find the minimum value of P when the motion of the block is impeding (a) up the plane. (b) down the plane.

6.6 A 100 kg block is on an inclined plane of 30° as shown in Fig. 6.29. The coefficient of static friction between the block and the plane is 0.4. Determine the range of m for which the system will be in equilibrium.

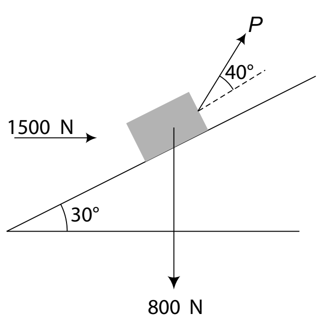


Figure 6.28

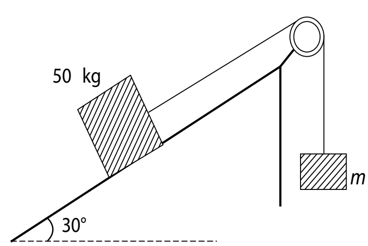


Figure 6.29

6.7 Figure 6.30 shows a braking system to stop the rotation of drum. If a moment M is applied on the drum, find out the force P to keep the drum in equilibrium.

6.8 An equivalent uniform triangular lamina ABC rests vertically with the corner A on a rough horizontal floor and the corner B against a smooth vertical wall as shown in Fig. 6.31. Show that the least inclination θ of the side AB with the horizontal is given by $\cot\theta = \frac{1}{\sqrt{3}} + 2\mu$, where μ is the coefficient of friction at the floor.

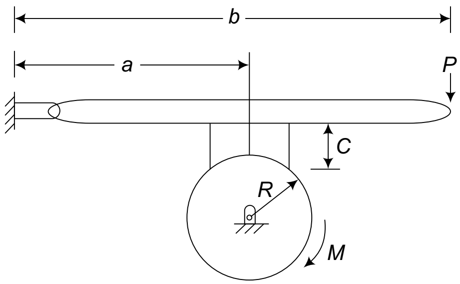


Figure 6.30

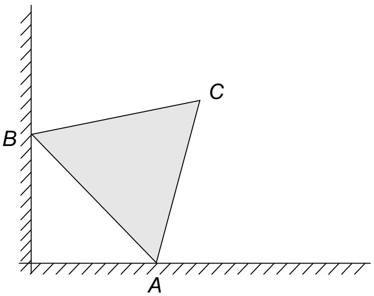


Figure 6.31

6.9 A solid right circular cone of height h and base radius r rests on an inclined plane that makes an angle θ with the horizontal and subjected to a force P applied at the top and parallel to the plane as shown in Fig. 6.32. Assuming the coefficient of static friction (μ_s), determine the maximum $\frac{h}{r}$ ratio for which the cone will slide along the plane without tipping.

6.10 A wedge block weighing 500 N is to be lifted along the wall by applying a force P horizontally to another small wedge as shown in Fig. 6.33. The coefficient of friction between all surfaces of contact is 0.35. Determine the magnitude of the force P required that can enable the wedge to move.

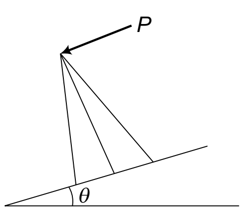


Figure 6.32

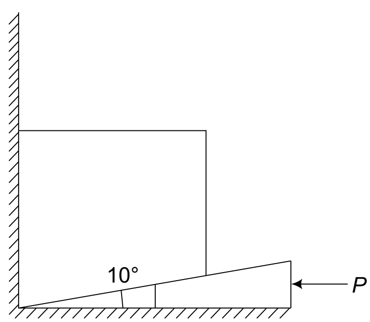


Figure 6.33

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

6.1 (c)
6.2 (d)
6.3 (a)
6.4 (b)
6.5 (c)
6.6 (d)
6.7 (b)
6.8 (b)
6.9 (c)
6.10 (d)
6.11 (c)

ANSWERS TO NUMERICAL PROBLEMS

6.4 No.

6.5 (a) 625 N (b) 335 N

6.6 7.68 kg $\leq m \leq$ 42.32 kg

6.7 $\dfrac{Ma}{\mu bR}$

6.9 $\dfrac{h}{r} = \dfrac{1}{2\mu_s - \dfrac{3}{2}\tan\theta}$

6.10 1133.5 N

CHAPTER

7

Moment of Inertia

7.1 INTRODUCTION

In chapter 5, we have discussed the procedures to calculate centroid and centre of gravity of regular geometrical entities which either conform to basic geometrical primitives (those that can be defined mathematically) or composition of more than one such primitives that are called composites. This computation is fundamentally the identification of a point where the entire body force is assumed to be concentrated, i. e., the location of the resultant of the distributed gravity force. A close look at the pertinent equations shows that these involve a term which is the integration of the product of the infinitesimal small area and its distance from a reference axis [$y_c \times A = \int y.dA$ or $x_c \times A = \int x.dA$].

Although this expression is the simplified form that manifests presence of only the geometrical parameters, original expression is based on the moment of the distributed gravity force with respect to an axis, following Varignon's Theorem. Nevertheless, there are ample occasions when the force is found to be a linear function of aforesaid distance and hence moment equation of such forces generically can be computed by an expression which is *the integration of the product of the infinitesimal small area and square of its distance from a reference axis*. Mathematically, such expression is in the form $\int y^2.dA$ or $\int x^2.dA$. This integral is called ***Moment of Inertia*** of the plane figures with respect to an axis in its plane. In mechanics, there are plenty of situations which are quite often encountered that involve above integrals and hence a detailed study of these integrations is found to be useful to realise the merits of computations. Stresses in beams, deflection of beams, buckling of columns, and torsion of shafts are few such areas, where the moment of inertia appears quite significantly.

It is, however, worth mentioning that apart from mathematical computations, moment of inertia hardly has any physical interpretation whatsoever.

7.2 DEFINITION OF MOMENT OF INERTIA WITH RESPECT TO AN AXIS IN ITS PLANE

Consider an area A in the x–y plane as shown in Fig. 7.1. Let dA be any element of the area at a distance (x, y) from the axes.

The moment of inertia of the area A with respect to the x axis $I_x = \int y^2 dA$ (7.1)

The moment of inertia of the area A with respect to the y axis $I_y = \int x^2 dA$ (7.2)

The moment of inertia of the area A is also called the ***second moment of the area***.

The moment of inertia of an area = (Area) × (Distance)^2 = (Length)^4

Thus, in SI system, the moment of inertia has unit of m^4.

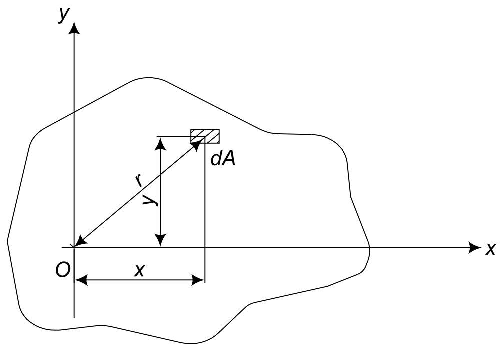


Figure 7.1

7.2.1 Polar Moment of Inertia

The moment of inertia of an area with respect to an axis perpendicular to the x–y plane (z axis) and passing through a point "O" is called the polar moment of inertia and is denoted by J_o.

$$J_o = I_z = \int r^2 dA \tag{7.3}$$

From the above figure, $x^2 + y^2 = r^2$

It therefore follows that $J_o = \int r^2 dA = \int (x^2 + y^2) dA = I_x + I_y$

$$J_o = I_x + I_y \tag{7.4}$$

7.2.2 Radius of Gyration

Consider an area A which has a moment of inertia I_x with respect to x axis.

Let us imagine this area A to be concentrated into a thin strip parallel to the x axis. If this area A (concentrated strip) is to have the same moment of inertia (I_x) with respect to the x axis, the strip should be placed at a distance k_x from the x axis, as given by the relation

$$I_x = k_x^2 A$$

$$k_x = \sqrt{\frac{I_x}{A}} \tag{7.5}$$

k_x is known as the ***radius of gyration*** of the area with respect to the x axis and has the unit of length.

Radius of gyration with respect to the y axis,

$$k_y = \sqrt{\frac{I_y}{A}} \tag{7.6}$$

Radius of gyration with respect to the polar axis,

$$k_o = \sqrt{\frac{J_o}{A}} \tag{7.7}$$

$$J_o = I_x + I_y$$

$$k_o^2 A = k_x^2 A + k_y^2 A$$

$$k_o^2 = k_x^2 + k_y^2 \tag{7.8}$$

7.3 PARALLEL AXIS THEOREM

Let x, y be the rectangular coordinate axes through any point O in the plane of the figure of area A as shown in Fig. 7.2.

x_c, y_c be the corresponding parallel axes through the centroid C of the area. The axes through the centroid of an area are also called the *centroidal axes*.

The moment of inertia of the area A about the x axis is $I_x = \int y^2 dA$, where dA is an element of area at a distance y from x_c axis.

Therefore, the distance of the element from x axis now becomes $y + d_x$, d_x being the perpendicular distance between axes x and x_c.

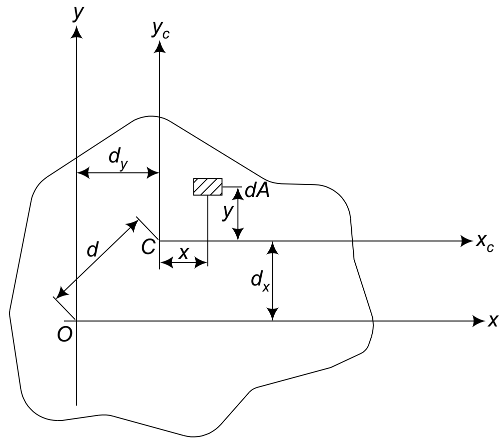


Figure 7.2

$$\therefore \quad I_x = \int (y + d_x)^2 dA = \int (y^2 + 2yd_x + d_x^2) dA$$

$$= \int y^2 dA + 2d_x \int y dA + \int d_x^2 dA = \int y^2 dA + 2d_x.0 + \int d_x^2 dA = \int y^2 dA + \int d_x^2 dA$$

The term $\int y^2 dA$ represents the moment of inertia of the area A about the axis x_c and the term $\int y^2 dA$ represents the first moment of the area A about its own centroidal axis x_c.

$$I_x = I_{xc} + Ad_x^2 \tag{7.9}$$

Similarly,

$$I_y = I_{yc} + Ad_y^2 \tag{7.10}$$

$$I_x + I_y = I_{xc} + I_{yc} + A(d_x^2 + d_y^2)$$

The polar moment of inertia now becomes

$$J_o = I_x + I_y = J_{oc} + Ad^2 \tag{7.11}$$

Example 7.1 Determine the moment of inertia of a rectangle of base b and height h with respect to its centroidal axes as shown in Fig. 7.3. Also find its moment of inertia about its base.

Solution The centroid of rectangular area is at C. Centroidal axes, $x_c - y_c$ is shown in Fig. 7.3.

Consider an element of thickness dy situated at a distance y from the x_c axis.

Area of the element $dA = bdy$

Moment of inertia of the elemental area about x_c axis is $dI_{xc} = y^2 bdy$

Moment of inertia of the rectangular cross-section about x_c axis is

$$I_{xc} = \int dI_{xc} = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 bdy = b\left[\frac{y^3}{3}\right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{bh^3}{12}$$

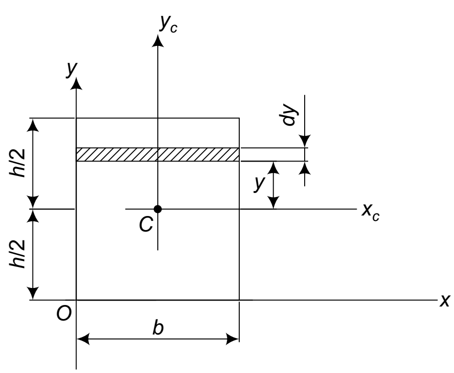


Figure 7.3

Similarly, moment of inertia about y_c axis is

$$I_{yc} = \frac{hb^3}{12}$$

Moment of inertia with respect to base

The time elemental area remains same. But since the axes are modified, the limits of integration will change.

Thus $$I_x = \int dI_x = \int_0^h y^2 b\,dy = b\left[\frac{y^3}{3}\right]_0^h = \frac{bh^3}{3}$$

The same result can be obtained by using parallel axis theorem.

By definition; $I_x = I_{xc} + A\left(\frac{h}{2}\right)^2$, where $\frac{h}{2}$ is the perpendicular distance of the centroid from the base.

$$\therefore \qquad I_x = \frac{bh^3}{12} + bh\left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$

Example 7.2 Find the moment of inertia of a triangle of base b and height h with respect to its base.

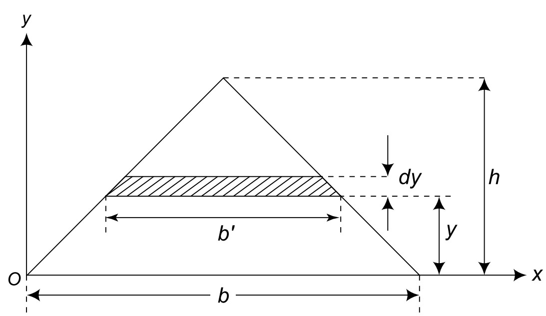


Figure 7.4

Solution Choose x axis to coincide with the base as shown in Fig. 7.4.

Consider an element of thickness dy at a distance y from the x axis.

Area of the element $dA = b'dy$

From similar triangles, we get $\frac{b'}{b} = \frac{h-y}{h}$ or $b' = b\frac{h-y}{h}$

Thus area of the elemental strip becomes $dA = b'.dy = \frac{b}{h}(h-y)dy$

Moment of inertia of the elemental area about x axis is $dI_x = y^2 dA = y^2 b\,\frac{h-y}{h}\,dy$

Moment of inertia of the triangular cross-section about x axis is

$$I_x = \int dI_x = \int_0^h y^2 b\,\frac{h-y}{h}dy = \frac{b}{h}\int_0^h (hy^2 - y^3)dy = \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h = \frac{bh^3}{12}$$

Example 7.3 Determine the moments of inertia of a circular area of diameter D about the centroidal axes. Also calculate its polar moment of inertia and radius of gyration.

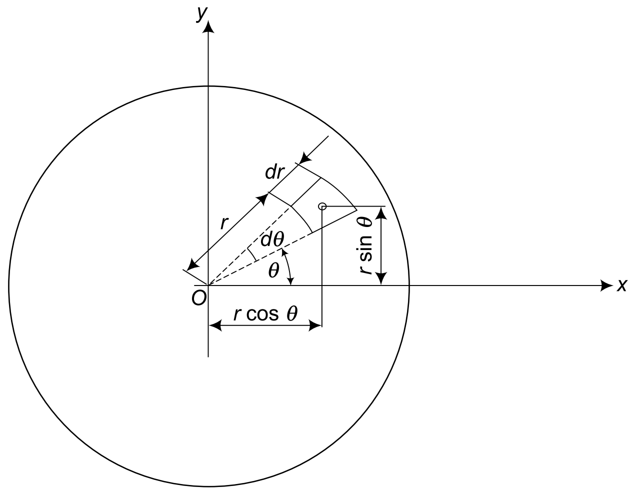


Figure 7.5

Solution The centroid of circular area is its centre. Centroidal axes x–y are shown in Fig. 7.5.

Consider an element of thickness dr having an included angle $d\theta$ situated at a radius r and angle θ from the x axis.

Area of the element $dA = rd\theta dr$

Moment of inertia of the elemental area about x axis

$$dI_x = (r \sin \theta)^2 rd\theta dr$$

Moment of inertia of the circular area about x axis

$$I_x = \int dI_x = \int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} (r \sin \theta)^2 rd\theta dr = \int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3 \sin^2 \theta d\theta dr = \int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3 \frac{1-\cos 2\theta}{2} d\theta dr$$

$$= \int_{r=0}^{r=R} r^3 \frac{1}{2}[2\pi - 0]dr = \pi \int_{r=0}^{r=R} r^3 dr = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$$

Because of symmetry of circular area, $I_x = I_y$

Since $J_o = I_x + I_y$

$$J_o = 2I_x = 2I_y = 2 \times \frac{\pi D^4}{64} = \frac{\pi D^4}{32}$$

However, polar moment of inertia can also be computed directly.

By definition, radius of gyration is

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{\pi D^4}{64}}{\frac{\pi D^2}{4}}} = \frac{D}{4} = k_y \text{ and } k_o = \sqrt{k_x^2 + k_y^2} = k_x \times \sqrt{2} = \frac{D}{4} \times \sqrt{2} = \frac{D}{2\sqrt{2}}.$$

Example 7.4 Determine the moment of inertia of a hollow circular section about its centroidal axes.

Solution This problem is similar to that of Example 7.3.

The limits of r will be from $\frac{D_i}{2}$ to $\frac{D_0}{2}$

Example 7.5 Determine the polar moment of inertia of the shaded area with respect to the point O as shown in Fig. 7.6.

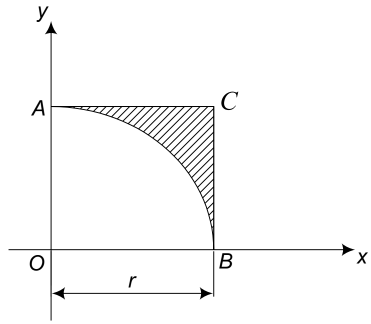


Figure 7.6

Solution The shaded area can be considered as subtraction of quarter circle OAB of radius r from the square $OACB$ of sides r.

Using previous results, the polar moment of inertia of the square is $J_o = 2I_x = 2I_y = 2\frac{r^4}{3}$ and the for one quarter circle it is $J_o' = \frac{1}{4} \times \frac{\pi r^4}{2} = \frac{\pi r^4}{8}$.

Thus the requisite polar moment of inertia for the shaded area is

$$J_o - J_o' = \frac{2}{3} r^4 - \frac{\pi}{8} r^4 = 0.274 r^4$$

Example 7.6 Determine the moment of inertia of the shaded area with respect to the point O as shown in Fig. 7.7.

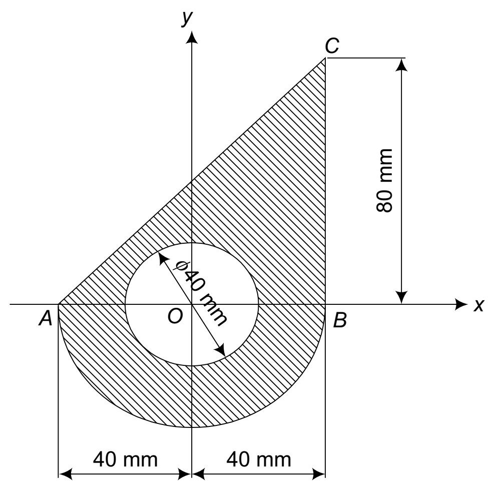


Figure 7.7

Solution The hatch area can be obtained by the following relationship.

Hatch area = Area of the triangle ABC + Area of the semicircle (with a 80 mm diameter) – Area of the hole (with a 40 mm diameter).

Let us denote these geometrical entities by suffices 1, 2 and 3 respectively.

Thus $I_{x1} = \frac{1}{12} bh^3 = \frac{1}{12} \times 80 \times 80^3 \text{ mm}^4 = 3413333.3 \text{ mm}^4$

$$I_{x2} = \frac{1}{2} \times \frac{\pi d^4}{64} = \frac{\pi}{128} \times 80^4 \text{ mm}^4 = 1005309.6 \text{ mm}^4 \text{ and}$$

$$I_{x3} = \frac{\pi d'^4}{64} = \frac{\pi}{64} \times 40^4 \text{ mm}^4 = 125663.7 \text{ mm}^4$$

Therefore the moment of inertia for the shaded area is

$$I_x = I_{x1} + I_{x2} - I_{x3} = 3413333.3 + 1005309.6 - 125663.7 \text{ mm}^4 = 429.3 \text{ cm}^4$$

Example 7.7 Determine the moment of inertia of the angle with respect to a centroidal axis parallel to the x axis as shown in Fig. 7.8.

Solution The angle can be decomposed to two rectangles – one is $ABCF$ and the other is $OFDE$.

All the attributes of these two rectangles are denoted by suffices 1 and 2.

Thus $A_1 = 90 \times 10 \text{ mm}^2 = 900 \text{ mm}^2$ and $A_2 = 100 \times 10 \text{ mm}^2 = 1000 \text{ mm}^2$.

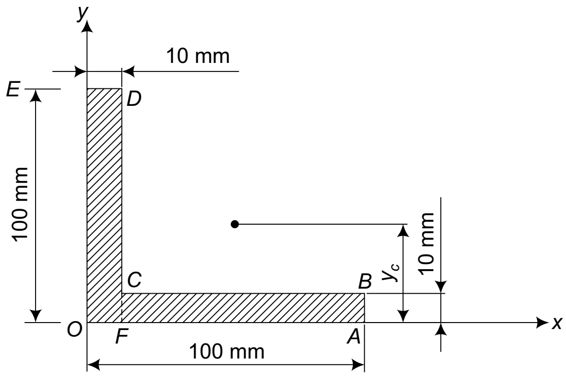


Figure 7.8

$$\therefore \qquad I_{x1} = \frac{bh^3}{3} = \frac{90 \times 10^3}{3} \text{ mm}^4 = 30000 \text{ mm}^4 \text{ and } I_{x2} = \frac{10 \times 100^3}{3} \text{ mm}^4 = 3333333.33 \text{ mm}^4$$

The y coordinate of the centroid of the composite is

$$y_c = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{900 \times 5 + 1000 \times 50}{900 + 1000} \text{ mm} = 28.68 \text{ mm}$$

The moment of inertia of the composite with respect to the base would be

$$\sum_{i=1}^{2} I_{xi} = I_{x1} + I_{x2} = 30000 + 3333333.33 \text{ mm}^4 = 3363333.33 \text{ mm}^4$$

Now by using parallel axes theorem; $I_{xc} = \sum_{i=1}^{2} I_{xi} - d_x^2 \sum_{i=1}^{2} A_i$

$$I_{xc} = 3363333.33 - 28.68^2[900 + 1000] \text{ mm}^4 = 1800502.77 \text{ mm}^4$$

Example 7.8 Determine the moment of inertia of the **"*T*"** section with respect to a centroidal axis parallel to the x axis as shown in Fig. 7.9. Consider all the dimensions are in mm.

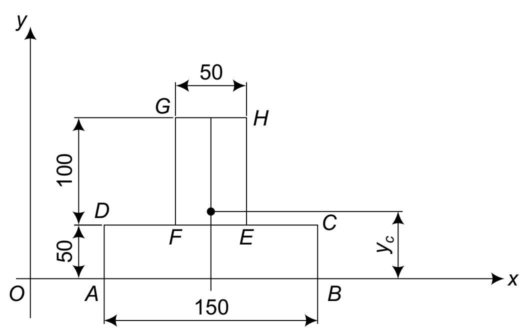


Figure 7.9

Solution For composites, it is imperative to calculate various parameters of the individual areas into which the entire composite is decomposed and present these in a tabular form to improve legibility and to reduce errors.

Following the previous problem,

Identification	*Area (A)*	I_{xi}	y
ABCD	$A_1 = 150 \times 50 = 7500$	$I_{x1} = \dfrac{150\times50^3}{3} = 6250000$	$y_1 = 25$
EFGH	$A_2 = 100 \times 50 = 5000$	$I_{x2} = \dfrac{50\times100^3}{12} + 5000 \times 100^2 = 54170000$	$y_2 = 50 + 50 = 100$
Composite	$A_1 + A_2 = 12500$	$\sum_{i=1}^{2} I_{xi}\ \ I_{x1} + I_{x2} = 60420000$	$y_c = 55$

Thus by using parallel axes theorem, $I_{xc} = \sum_{i=1}^{2} I_{xi} - d_x^2 \sum_{i=1}^{2} A_i$

$$I_{xc} = 60420000 - 55^2 \times 12500 \text{ mm}^4 = 22607500 \text{ mm}^4$$

7.4 PRODUCT OF INERTIA

Consider a plane figure of area A in the x–y plane as shown in Fig. 7.1.

The product of moment of inertia is computed by multiplying the infinitesimal small area with it's coordinates and integrating the product over the entire area.

Thus by definition, the integral $I_{xy} = \int xy dA$ is known as *the product of moment of inertia*.

There are few interesting features of product moment of inertia that need to be discussed.

Although the moments of inertia I_x and I_y are always positive for positive areas (a void, hole or area removed is a negative area), the product of inertia I_{xy} for a positive area may be either positive or negative since any one of x or y, having negative values, will yield negative I_{xy}.

Another interesting characteristic of product of moment of inertia is that it may have a zero value also. When either one or both of the x and y axes are treated as axes of symmetry, I_{xy} can be zero. This implies that either x or y axis divides the given area into two identical areas so that one is treated as mirror image of the other. Under this situation, x and y value will have negative signs so that the product moment of inertia of the entire area is the sum of product moment of inertia of two equal areas having exactly equal and opposite magnitudes so that their combined effect becomes zero.

The product of moment of inertia also varies with the change in reference systems. If the reference system $x_c - C - y_c$ is considered parallel to that of $x - O - y$ such that distance between two horizontal and vertical axes are d_x and d_y respectively [Refer Fig. 7.2], then by using the parallel axes theorem, we get

$$I_{xy} = \int (x + d_x)(y + d_y)dA = \int xy dA + d_x \int y dA + d_y \int x dA + d_x d_y \int dA$$

Since C is the centroid of the plane figure, the 2nd and 3rd term becomes zero.

Thus above relation reduces to

$$I_{xy} = \int xy dA + d_x d_y \int dA = (I_{xy})_c + A d_x d_y \tag{7.12}$$

This expression is extremely useful to the solution of the product of moment of inertia of the composites.

7.5 PRINCIPAL AXES AND PRINCIPAL MOMENT OF INERTIA

From the very definition of the product of inertia, it can be concluded that I_{xy} may have negative value, since it is the product of two different coordinates. Further, if the reference axes are rotated by a definite angle, it may have a zero value. *The reference coordinate system for which the product moment of inertia of an area diminishes is called principle axes*. The two principal axes are perpendicular to each other and are such that the product of inertia of the given area with respect to these axes is zero.

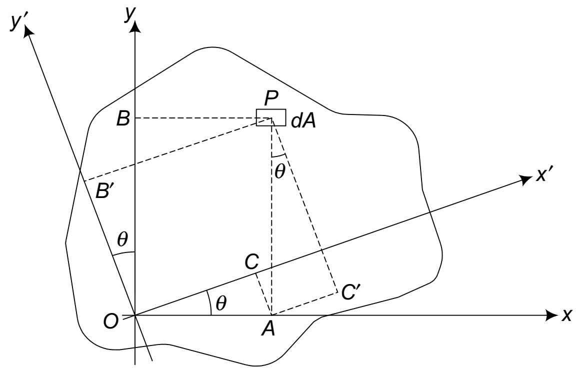


Figure 7.10

Consider a plane figure of area A. Let the moments of inertia I_x, I_y and the product of inertia I_{xy} with respect to the axes x and y passing through any point O are known.

Let the axes be rotated anticlockwise about O by an angle θ to new position x' and y' as shown in Fig. 7.10.

$$x' = OA' = OC + CA' = OA \cos\theta + AC' = OA\cos\theta + PA\sin\theta = x\cos\theta + y\sin\theta \tag{7.13}$$

$$y' = OB' = PA' = PC' - A'C' = PA\cos\theta - AC = PA\cos\theta - OA\sin\theta = y\cos\theta - x\sin\theta \tag{7.14}$$

Thus $I_{x'} = \int (y\cos\theta - x\sin\theta)^2 dA = \int y^2\cos^2\theta dA + \int x^2 \sin^2 dA - \int 2xy\sin\theta\cos\theta dA$

$$= \cos^2\theta . I_x + \sin^2\theta . I_y - \sin 2\theta . I_{xy} \tag{7.15}$$

Since $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$I_{x'} = \frac{1}{2}(1 + \cos 2\theta).I_x + \frac{1}{2}(1 - \cos 2\theta).I_y - \sin 2\theta.I_{xy}$$

Rearranging,

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \tag{7.16}$$

Similarly, $I_{y'} = \int (x\cos\theta + y\sin\theta)^2 dA = \int x^2\cos^2\theta dA + \int y^2\sin^2 dA + \int 2xy\sin\theta\cos\theta dA$

$$= \cos^2\theta . I_y + \sin^2\theta . I_x + \sin 2\theta . I_{xy} \tag{7.17}$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \tag{7.18}$$

Adding Eqs (7.16) and (7.18),

$$I_{x'} + I_{y'} = I_x + I_y$$

This implies in view of rotation of the axes (reference system), the sum of moments of inertia remains constant.

Subtracting Eq. (7.18) from Eq. (7.16).

$$I_{x'} - I_{y'} = (\cos^2\theta - \sin^2\theta)I_x - (\cos^2\theta - \sin^2\theta)I_y - 2\sin 2\theta I_{xy} = (I_x - I_y)\cos 2\theta - 2\sin 2\theta I_{xy} \tag{7.19}$$

The product moment of inertia with respect to new axes is

$$I_{x'y'} = \int (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta)dA$$

$$= \int xy \cos^2 \theta dA + \int y^2 \sin \theta \cos \theta dA - \int x^2 \sin \theta \cos \theta dA - \int xy \sin^2 \theta dA$$

$$= \frac{1}{2}(I_x - I_y) \sin 2\theta + I_{xy} \cos 2\theta \tag{7.20}$$

If x'–y' is considered as principle axes then by definition, $I_{x'y'} = 0$

$$\therefore \qquad \frac{1}{2}(I_x - I_y) \sin 2\theta + I_{xy} \cos 2\theta = 0$$

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \tag{7.21}$$

Example 7.9 Determine the product moment of inertia of the rectangle with respect to x and y axis that coincides with two adjacent sides of the rectangle.

Solution By definition, $I_{xy} = \int xydA = \int_0^h \frac{b}{2} .y.bdy = \frac{b^2}{2} \int_0^h ydy\, ydy = \frac{b^2}{2}\left[\frac{y^2}{2}\right]_0^h = \frac{b^2h^2}{4}$

Example 7.10 Determine the product moment of inertia of the shaded spandrel area as shown in Fig. 7.6.

Solution By using the result of Example 7.9, we have $(I_{xy})_1 = \dfrac{r^4}{4}$

Now

$$(I_{xy})_2 = \int xydA = \int_0^{\frac{\pi}{2}}\int_0^R (r \cos \theta).(r \sin \theta).(rd\theta)dr = \frac{R^4}{4}\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2}\, d\theta = \frac{R^4}{8} \times \frac{-1}{2}[\cos 2\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{-R^4}{16}[\cos \pi - \cos 0] = \frac{-R^4}{16}[-1 - 1] = \frac{R^4}{8}$$

Therefore the product moment of inertia of the shaded spandrel area is

$$(I_{xy})_{Com} = (I_{xy})_1 - (I_{xy})_2 = \frac{R^4}{4} - \frac{R^4}{8} = \frac{R^4}{8}$$

Example 7.11 Determine the product moment of inertia of the **"*L*"** section as shown in Fig. 7.11.

Solution The given L section is divided into two rectangles as shown

$$(I_{xy})_1 = \frac{80^2 \times 10^2}{4} \text{ mm}^4 = 160000 \text{ mm}^4$$

$$(I_{xy})_2 = 10 \times 70 \times \frac{10^2}{2} + \frac{70^2 \times 10^2}{4} \text{ mm}^4 = 157500 \text{ mm}^4$$

$$(I_{xy})_{Com} = (I_{xy})_1 + (I_{xy})_2 = 160000 + 157500 \text{ mm}^4 = 317500 \text{ mm}^4$$

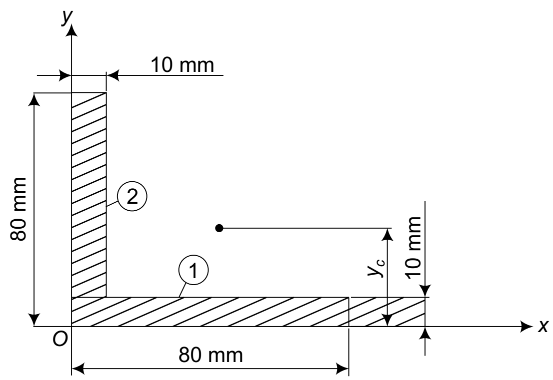


Figure 7.11

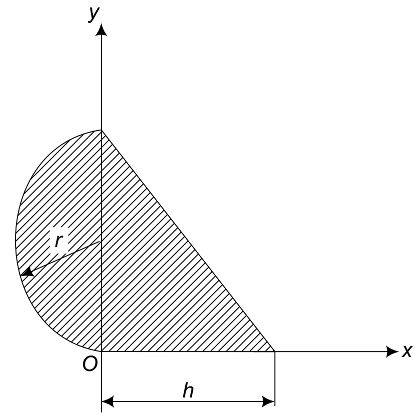


Figure 7.12

Example 7.12 Referring to Fig. 7.12, find the relation between r and h so that x and y will be principle axes for the composite area.

Solution The given composite is divided into one triangle having base h and height $2r$ and a semicircle of radius r. Let us denote them as 1 and 2.

$$\therefore \qquad (I_{xy})_1 = \frac{h^2 \times (2r)^2}{24} = \frac{h^2 r^2}{6}$$

Since the centre of the semicircle does not coincide with the origin O, it is imperative to use parallel axes theorem.

Thus
$$(I_{xy})_2 = (I_{xy})_{2c} + A d_x d_y = 0 + \frac{1}{2} \times \pi r^2 \times r \times \left(\frac{-4r}{3\pi}\right) = -\frac{2}{3} r^4$$

Note: The semicircle being symmetrical with respect to x_c axis, $(I_{xy})_{2c} = 0$.

Therefore, product moment of inertia of the composite is

$$(I_{xy})_{\text{Com}} = (I_{xy})_1 + (I_{xy})_2 = \frac{h^2 r^2}{6} - \frac{2}{3} r^4$$

When x and y will be principle axes for the composite area, its product moment of inertia will be zero,

implying
$$(I_{xy})_{\text{Com}} = \frac{h^2 r^2}{6} \quad \frac{2}{3} r^4 = 0$$

$$\frac{h^2 r^2}{6} = \frac{2r^4}{3}$$

$$h = 2r$$

Thus the required relationship is $h = 2r$.

Note: The readers are advised to compute the product moment of inertia of a triangle from the basics.

Example 7.13 Referring to Fig. 7.13, find the angle θ that will define the principle axes through point O for the right-angled triangle having base b = 30 mm and height h = 40 mm.

Solution For the triangle $I_x = \frac{1}{12}bh^3$; $I_y = \frac{1}{12}b^3h$ and $I_{xy} = \frac{1}{24}b^2h^2$

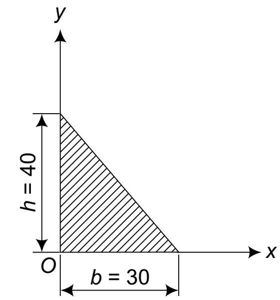


Figure 7.13

The necessary condition for present (rotated) axes will be principle axes if

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{2\times\frac{1}{24}b^2h^2}{\frac{1}{12}b^3h - \frac{1}{12}bh^3} = \frac{b^2h^2}{bh\times(b^2-h^2)} = \frac{bh}{b^2-h^2}$$

$$= \frac{40\times 30}{30^2 - 40^2} = -1.714$$

$$2\theta = \tan^{-1}(-1.714) = -59.74;\ 120.256$$

$$\theta = -29°52';\ 60°7'$$

Thus considering θ being positive; $\theta = 60°7'$

7.6 MASS MOMENT OF INERTIA

The mass moment of inertia of a body about an axis is defined as *the product of the mass and the square of the distance between the mass centre of the body and the axis, with respect to which it is computed.*

Consider a body of mass m. The moment of inertia of the body with respect to the axis OO' is defined by integral

$$I = \int r^2 dm \tag{7.22}$$

where dm is the mass of the element of the body situated at a distance r from the axis OO' and the integration is extended over the entire volume of the body.

From the above definition, it follows that the mass moment of inertia of a body has the dimension of mass × (length)^2

The radius of gyration k of the body with respect to the axis is expressed by the relation

$$I = k^2m$$

$$k = \sqrt{\frac{I}{m}} \tag{7.23}$$

The radius of gyration gives the measurement of the distance of a point from the axis at which the entire mass is assumed to be concentrated.

Following the parallel axes theorem,

$$I_x = I_{xc} + md^2 \tag{7.24}$$

Example 7.14 Determine the mass moment of inertia of a uniform slender rod of length l about its centroidal axis normal to its length.

Solution Refer to the Fig. 7.14, and let us consider a small length element dx located at a distance x from the centroidal axes CC'.

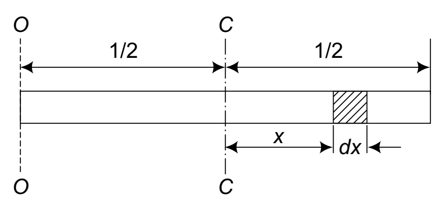


Figure 7.14

Thus the mass of this elemental length is $dM = m.dx.$, where m is the mass per unit length.

Therefore, mass moment of inertia of the entire rod about the axis CC' is $\int_{\frac{-1}{2}}^{\frac{1}{2}} x^2.m.dx = m \times \frac{1}{3}[x^3]_{\frac{-1}{2}}^{\frac{1}{2}} =$

$$m\frac{1}{3}\left[\frac{l^3}{8}+\frac{l^3}{8}\right]=\frac{ml^3}{12}=\frac{(ml)l^2}{12}=\frac{Ml^2}{12}.$$

Example 7.15 Determine the mass moment of inertia of a solid right circular cone of base radius R and height h about its own axis.

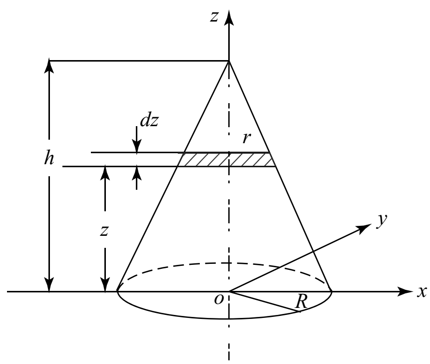


Fig. 7.15

Solution Refer to Fig. 7.15; let us consider a small elecmental thin disk of radius r and thickness dz located at a distance z from the base.
Then

$$r=\left(\frac{h-z}{h}\right)R$$

Thus the mass of this elemental length is $dm = \rho\pi r^2 dz$, where, ρ is the density.

$$\therefore \quad dm=\rho\pi r^2 dz=\rho\pi\left(\frac{h-z}{h}\right)^2 R^2\, dz$$

Since the mass moment of inertia of a thin circular disk = $\frac{1}{2}\times \text{mass}\times \text{radius}^2$
It therefore follows that

$$dI_z=\frac{1}{2}\times\left(\frac{h-z}{h}\right)^2 R^2\times\rho\pi\left(\frac{h-z}{h}\right)^2 R^2\, dz=\frac{1}{2}\rho\pi\frac{R^4}{h^4}(h-z)^4\, dz$$

$$\therefore \quad I_z=\frac{\rho\pi R^4}{2h^4}\int_0^h (h-z)^4\, dz=\frac{\rho\pi R^4 h}{10}$$

But mass of the cone is $M = \dfrac{\rho \pi R^2 h}{3}$

Hence $I_z = \dfrac{3}{10} MR^2.$

MULTIPLE-CHOICE QUESTIONS

7.1 Moment of inertia is called
(a) first moment of the area (b) second moment of the area
(c) third moment of the area (d) none of the above

7.2 Moment of inertia of a rectangle with respect to its base having base b and height h is
(a) $\frac{1}{3} b^2h^2$ (b) $\frac{1}{3} b^3h$ (c) $\frac{1}{3} bh^3$ (d) $\frac{1}{12} b^2h^2$

7.3 Moment of inertia of a rectangle with respect to its centroidal axis parallel to its base having base b and height h is
(a) $\frac{1}{3} bh^3$ (b) $\frac{1}{3} b^3h$ (c) $\frac{1}{12} bh^3$ (d) $\frac{1}{12} b^2h^2$

7.4 Moment of inertia of a circular area of diameter d about an axis perpendicular to the area and passing through its center is given by
(a) $\frac{\pi d^4}{16}$ (b) $\frac{\pi d^4}{32}$ (c) $\frac{\pi d^4}{64}$ (d) $\frac{\pi d^4}{128}$

7.5 Moment of inertia of a triangle with respect to its base having base b and height h is
(a) $\frac{1}{3} bh^3$ (b) $\frac{1}{3} b^3h$ (c) $\frac{1}{12} bh^3$ (d) $\frac{1}{12} b^2h^2$

7.6 The polar moment of inertia of a square of sides a with respect to its centroid is
(a) $\frac{1}{3} a^4$ (b) $\frac{1}{4} a^4$ (c) $\frac{1}{6} a^4$ (d) $\frac{1}{12} a^4$

7.7 The polar moment of inertia of a one quarter circular sector of radius r with respect to its centre is
(a) $\frac{\pi r^4}{2}$ (b) $\frac{\pi r^4}{4}$ (c) $\frac{\pi r^4}{8}$ (d) $\frac{\pi r^4}{16}$

7.8 The moment of inertia of an ellipse with respect to x axis having semi-major and semi-minor axis as a and b respectively is
(a) $I_x = \frac{\pi ab^3}{2}$ (b) $I_x = \frac{\pi ab^3}{3}$ (c) $I_x = \frac{\pi ab^3}{4}$ (d) $I_x = \frac{\pi ab^3}{8}$

7.9 The product of moment of inertia of a geometrical figure which is symmetrical about one or both the axes is
(a) maximum (b) minimum (c) zero (d) unpredictable

7.10 The axes system for which product of moment of inertia of a geometrical figure is zero is called
(a) principle axes (b) major axes (c) minor axes (d) none of the above

7.11 The moment of inertia of a circular area of diameter d about an axis perpendicular to the area as compared to its moment of inertia with respect to a horizontal axis, when the origin coincides with the geometrical centre is

(a) same (b) double (c) half (d) one-third

7.12 As per parallel axes theorem, axial moment of inertia (I_x) is related with centroidal moment of inertia $(I_x)_c$ by the following relationships.

(a) $I_x = I_{xc} + A.d_x^2$ (b) $I_x = I_{xc} - A.d_x^2$

(c) any one of the above (d) none of the above

7.13 The radius of gyration of a circular area of diameter d about x-axis and passing through its centre is given by

(a) $\frac{d}{2}$ (b) $\frac{d}{4}$ (c) $\frac{d}{8}$ (d) $\frac{d}{16}$

SHORT ANSWER TYPE QUESTIONS

7.1 What is moment of inertia? What is its unit? Can it be negative?

7.2 What is meant by polar moment of inertia? What is radius of gyration? What is its unit?

7.3 Does moment of inertia have any physical significance? Justify it.

7.4 State and prove parallel axes theorem.

7.5 What is the product of moment of inertia? Can it be negative or zero?

7.6 Derive an expression for moment of inertia when reference axes system is rotated by an angle ϕ in anti-clockwise direction.

7.7 Prove that sum of axial moments of inertia always remains constant.

7.8 What is principle axes and principle moment of inertia? Derive an expression for angle ϕ for which, the present axes can be regarded as principle axes.

7.9 What is mass moment of inertia?

7.10 Prove that $J_O = I_x + I_y$. Validate the same for a circle.

NUMERICAL PROBLEMS

7.1 Find the moment of inertia of the parallelogram as shown in Fig. 7.16 with respect to its base.

7.2 Find the moment of inertia of a triangle of base b and height h with respect to an axis that passes through its vertex and parallel to its.

7.3 Find the moment of inertia of a square of sides α with respect to its diagonal by using the concept of rotation of the axes system.

7.4 Determine the polar moment of inertia of an isosceles triangle having base b and height h with respect to is vertex.

7.5 Calculate the polar moment of inertia of a circular sector of radius r and included angle θ with respect to its centre.

7.6 Determine the moment of inertia of the **"*I*"** section as shown in Fig. 7.17.

7.7 Determine the axial and polar moment of inertia of an ellipse having semi-major and semi-minor axes as a and b respectively, as shown in Fig. 7.18.

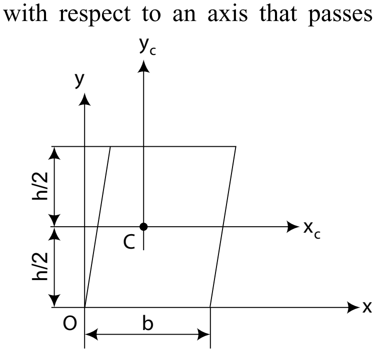


Figure 7.16

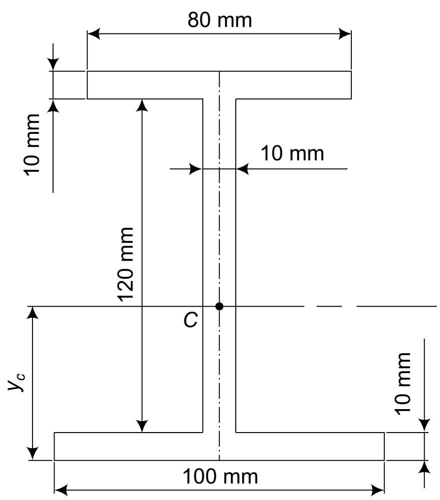


Figure 7.17

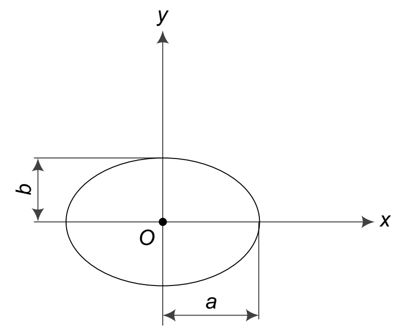


Figure 7.18

7.8 Calculate the axial moment of inertia I_x, for the channel about a centroidal axis that runs parallel to its base, as shown in Fig. 7.19. All the dimensions are in mm.

7.9 Calculate the product of moment of inertia I_{xy} of a three-quarter circular sector of radius r as shown in Fig. 7.20.

7.10 Calculate the product of moment of inertia I_{xy} for the angle section as shown in Fig. 7.21.

7.11 Calculate the moment of inertia of a solid sphere of radius R about its diametral axis.

7.12 Prove that the moment of inertia of a rectangle having dimensions a and b with respect to a diagonal is $I_D = \dfrac{a^3b^3}{6(a^2+b^2)}$.

7.13 Determine the mass moment of inertia of a solid right circular cyclinder of base radius R and height h about its own axis.

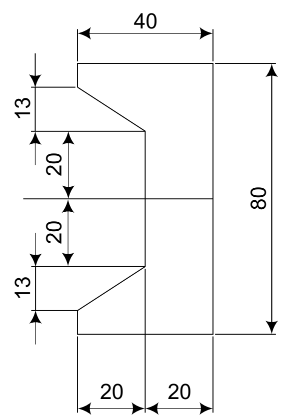


Figure 7.19

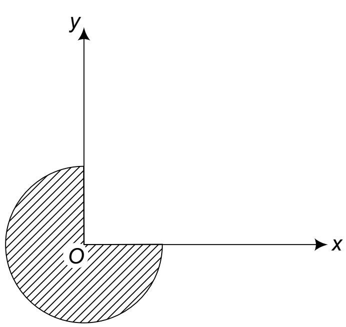


Figure 7.20

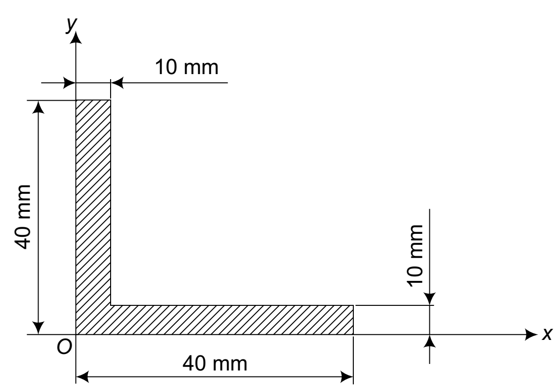


Figure 7.21

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

7.1 (b)	7.3 (c)	7.5 (c)	7.7 (c)	7.9 (c)	7.11 (b)	7.13 (b)
7.2 (c)	7.4 (b)	7.6 (c)	7.8 (c)	7.10 (a)	7.12 (a)	

ANSWERS TO NUMERICAL PROBLEMS

7.1 $\frac{bh^3}{3}$

7.2 $\frac{bh^3}{4}$

7.3 $\frac{a^4}{12}$

7.4 $\frac{bh^3}{4}+\frac{hb^3}{48}$

7.5 $\frac{r^4\theta}{4}$

7.6 9002353.3 mm^4

7.7 $\frac{\pi ab^3}{4}+\frac{\pi a^3 b}{4}$

7.8 1443610 mm^4

7.9 $I_{xy}=\frac{-r^4}{8}$

7.10 $I_{xy} = 7.75 \times 10^4$ mm^4

7.11 $I_x = I_y = I_z = \frac{2}{5}MR^2$

7.13 $I_Z = \frac{1}{2}MR^2$

Part B

Strength of Materials

CHAPTER

8

Elasticity – Concept of Simple Stresses and Strains

8.1 INTRODUCTION

In our earlier discussions on statics, we had contemplated on analysis of forces that are exhibited from various supports and other load bearing members. However, while doing so, we have not considered whether a particular member having a specific geometry and material quality will be able to withstand the stipulated force. Further, it is also pointed out in chapter 1 that statics deals with force analysis of rigid bodies, which are seldom found in reality since all real-life materials undergo some amount of deformation under load, however small it may be. It is of immense relevance to emphasise on the two most important attributes of the materials – how much load it can sustain and how much deformation is permissible considering its intended objectives. Further, it is also desirable that deformation that takes place during loading must not be permanent, implying that immediately after removal of load, the material should regain its original size to enable it for further use (loading). Determination of these parameters forms the backbone of engineering design of structural and machine members.

It is the internal behavior of the materials that answer the above two questions. Strength of Materials is the subject concerned where this does find its places for its study. All materials are not similar, rather their properties and behavior under loading varies widely. Thus success of any design lies on how meticulously these aspects are investigated, and how scrupulous is the selection of one particular category of materials and its geometry for a specific purpose. In contrast to statics, strength of materials comprehensively deals with the effects of internal forces that are induced in the materials to make the system viable.

8.2 INTERNAL FORCE – STRESS

In chapter 3 (Refer article 3.2), we have studied how internal forces are induced in a member which is a consequence of external loading. These internal forces (tensile or compressive) although apparently appear

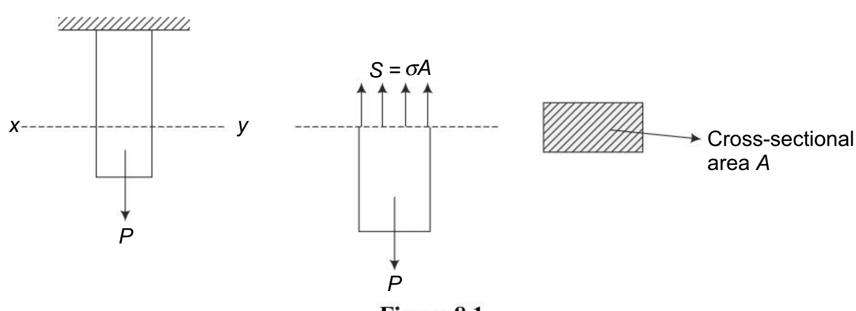


Figure 8.1

to be concentrated at a point, but are actually not so, rather they are distributed all over the cross-section, as shown in Fig. 8.1. It will not be misleading to envisage that in the absence of any non-uniformity in cross-section, the internal force (S) will be evenly distributed over the cross-section as shown in Fig. 8.1. Thus entire force, if divided by the area, will give a measure of the intensity of force that acts on the member and plays a very crucial role in analysis. This force per unit area is called stress. Unit of stress in S.I. system is N/m^2.

However, depending on nature of loading and area that held responsible for this purpose, stresses can be categorised. In other words, under loading, which area is responsible to resist the external load, is considered as basis for classifications.

8.2.1 Normal Stress

The intensity of the force perpendicular to or normal to the section is called the normal stress. Normal stress that causes tension on the surface of a section is called tensile stress. On the other hand, if it causes compression it is called compressive stress. It is denoted by σ. Thus by definition,

$$\text{Normal stress } (\sigma) = \frac{\text{Load (F)}}{\text{Area (A)}}.$$

8.2.2 Bearing Stress

Two plates having thickness t are held together with the help of nut and bolt as shown in Fig. 8.2. When the system is subjected to equal and opposite force P, a highly irregular pressure develops between bolt and plates. The average intensity of this nominal pressure is obtained by dividing the force transmitted by the projected area of the bolt onto the plate. This is called bearing stress (σ_b).

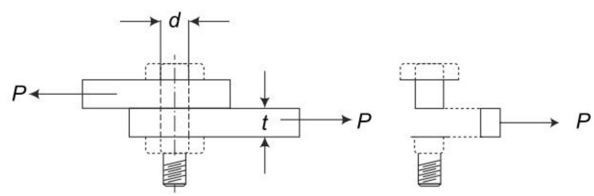


Figure 8.2

Thus $$\sigma_b = \frac{P}{d \times t}$$

8.2.3 Shear Stress

Refer to Fig. 8.2, if we consider the bolt, it can fail due to shear. In view of single force P, the failure of bolt would have been similar to that of sliding of one segment over the other. This failure is due to the shearing action of one over the other.

The intensity of the force which acts parallel to the plane of the area is called shear stress. This acts when one area has a tendency to move past over the other. The shear stress acts over the mating surface. It is denoted by τ.

$$\tau_{av} = \frac{P}{A_s}$$

where A_s is the shear area.

Refer to Fig. 8.2, there are two identical areas of the bolt that held responsible to offer shear resistance.

Thus in this case $$\tau_{av} = \frac{P}{A_s} = \frac{P}{2 \times \frac{\pi}{4} d^2}$$

8.3 ELASTICITY

Let us consider a circular or prismatic bar of initial length l and initial cross-section A subjected to axial load P as shown in Fig. 8.3. As the magnitude of P is increased from zero to P, its length will also increase from zero to δ and it is said that the bar is elongated by the amount δ.

When the load is withdrawn, it is found that the bar returns back to original length, provided the magnitude of P does not exceed a particular limit *(threshold value*). It may also so happen that when magnitude of P is larger than the threshold value, the bar cannot fully recover its original length, i.e., some amount of permanent deformation will take place. It is due to the property by virtue of which the material can regain its original size when the applied load is removed, is called *elasticity*. In view of above two situations, if it can recover entire length, it called *perfectly elastic* and if it is partial, it is called *partially elastic*.

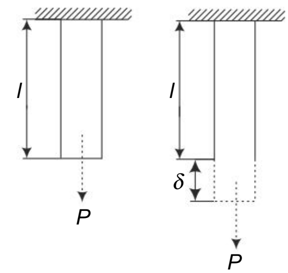


Figure 8.3

8.3.1 Strain

In case of elastic materials, the applied load will cause deformation, which may be temporary or permanent. Strain is defined as the ratio of elongation to original length. It is denoted by ε.

$$\varepsilon = \frac{\delta}{l}$$

where δ = deformation or elongation

l = original length

ε being the ratio of two lengths, it is dimensionless. Depending on nature of load, i.e, tensile or compressive, strain will be tensile strain or compressive strain. Tensile strain is considered positive while compressive strain is considered negative.

Normal Strain

Depending on the nature of loading, deformation and strain can be of varying natures.

In case of axially loaded bar, the strain manifested is called normal strain.

Shearing Strain

Let us consider a square block *ABCD* resting on floor as shown in Fig. 8.4. The upper surface of the block is subjected to a horizontal force of magnitude P that acts tangentially to the top surface *AB*. Since the bottom surface is fixed with the floor, the action of P will cause the block to deform so that initial square shape of the block will now assume the shape of a rhombus *DEFC*.

The shear strain is defined by the ratio of deformation to the distance between two parallel surfaces across of which deformation takes place.

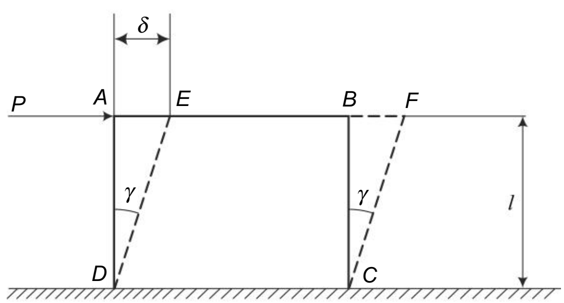


Figure 8.4

Thus shear strain $\dfrac{AE}{AD} = \dfrac{\delta}{l}$

If δ is very small compared to l, then from the ΔADE,

$$\tan\gamma \approx \gamma = \frac{AE}{AD}$$

Thus shear strain becomes $\gamma = \dfrac{\delta}{l}$

8.4 HOOKE'S LAW AND MODULUS OF ELASTICITY

In case of axial loading of the bar, experiments show that up to a certain limit, stress induced in the bar is proportional to the strain. Thus a plot of stress versus strain, as shown in Fig. 8.5, for ductile material shows a linear behavior upto a limit called *proportional limit*.

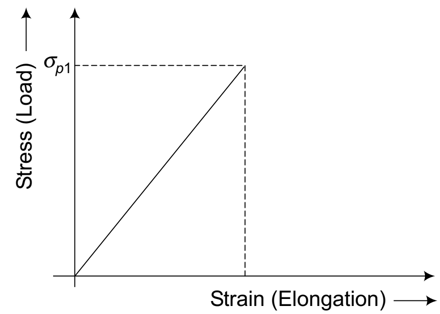


Figure 8.5

Thus, $\sigma \propto \varepsilon$

$$\sigma = E\varepsilon$$

where E is known as Modulus of Elasticity.

By definition, $\sigma = \dfrac{P}{A}$ and $\varepsilon = \dfrac{\delta}{l}$

Thus, $\dfrac{P}{A} = E\,\dfrac{\delta}{l}$

$$\delta = \frac{Pl}{AE}$$

This can be interpreted as that deformation of the bar under axial loading is proportional to the load P, initial length l and inversely proportional to the cross-section A.

8.5 LATERAL STRAIN AND POISSON'S RATIO

Referring to the uni-axially loaded members (elastic) in addition to longitudinal deformation, a certain amount of lateral (transverse) expansion or contraction takes place. To illustrate this, if a solid body is subjected to an axial tension, it contracts laterally; on the other hand, if it is compressed, the material squashes out sideways. Therefore, a longitudinal strain is always accompanied by transverse strain. It is experimentally found that this lateral strain bears a constant relationship to the longitudinal (or axial) strain caused by an axial force, provided the material remains elastic and is homogeneous and isotropic. This constant is a definite property of a material and is called *Poisson's ratio (μ).*

$$\mu = \left|\frac{\text{lateral strain}}{\text{axial strain}}\right| = -\frac{\text{lateral strain}}{\text{axial strain}}$$

The negative sign indicates that when axial strain is positive, lateral strain is negative and vice-versa.

For common engineering materials, μ varies between "0.25 to 0.35".

8.6 VOLUMETRIC STRAIN AND MAXIMUM VALUE OF POISSON'S RATIO

Refer to Fig. 8.1 (a). Let the initial length of the bar be l_0 and initial diameter be d_0.

Thus, initial cross-section of the bar is $A_0 = \dfrac{\pi}{4}\,d_0^2$.

The bar is subjected to axial load F as shown in Fig. 8.3.

Thus, after deformation, final length $l_f = l_0 + \delta = l_0 + \varepsilon l_0 = l_0(l + \varepsilon)$

Since the above deformation is also accompanied by lateral contraction, Poisson's ratio

$$\mu = \frac{\dfrac{\Delta d_0}{d_0}}{\dfrac{\delta}{l_0}}$$

$$\frac{\Delta d_0}{d_0} = \mu\varepsilon$$

Thus present cross-section

$$A_f = \frac{\pi}{4} d_f^2 = \frac{\pi}{4}[d_0 - \Delta d_0]^2 = \frac{\pi}{4} d_0^2\left[1 - \left(\frac{\Delta d_0}{d_0}\right)^2\right] = A_0(1 - \mu\varepsilon)^2.$$

Initial volume of the bar = $V_0 = A_0 l_0$

Final volume of the bar = $V_f = A_f l_f = A_0(1 - \mu\varepsilon)^2\, l_0(1 + \varepsilon) = A_0 l_0[1 - 2\mu\varepsilon + \mu^2\varepsilon^2](1 + \varepsilon) = \approx A_0 l_0$ $[1 - 2\mu\varepsilon + \varepsilon]$ [neglecting higher order terms involving μ and ε].

Therefore, change in volume

$$\Delta V = V_f - V_0 = A_0 l_0[1 - 2\mu\varepsilon + \varepsilon] - A_0 l_0 = A_0 l_0 \varepsilon(1 - 2\mu) = V_0\varepsilon(1 - 2\mu)$$

The volumetric strain is defined by the ratio of change in volume to the original volume.

Therefore, volumetric strain = $\dfrac{\Delta V}{V_0} = \varepsilon(1 - 2\mu)$

The above expression is the volumetric strain in terms of ε and μ.

The limiting condition of ΔV is zero, implying the volume of the bar diminishes.

Thus,

$$\frac{\Delta V}{V_0} = 0 = \varepsilon(1 - 2\mu)$$

$$\mu = \frac{1}{2}$$

This implies that the value of Poisson's ratio cannot exceed $\dfrac{1}{2}$.

8.7 ELONGATION OF A BAR OF CONSTANT CROSS-SECTION DUE TO SELF WEIGHT

Refer to Fig. 8.1. Let the bar hang vertically from a support freely. Thus the load on the bar is its self weight $W = \rho Al$.

A small elemental length dx of the bar is considered at a distance x from the support.

The force acting on it is $F_x = \rho Ax$.

Thus, its elongation $d\delta = \dfrac{F_x dx}{AE}$

Therefore, total elongation of the bar will be

$$\delta = \int d\delta = \int_0^l \frac{F_x dx}{AE} = \int_0^l \frac{\rho Ax dx}{AE} = \frac{\rho}{E}\left[\frac{x^2}{2}\right]_0^l = \frac{\rho l^2}{2E} = \frac{\rho Al.l}{2AE} = \frac{Wl}{2AE}$$

If the bar was subjected to a load of W, elongation would have been $\frac{Wl}{AE}$.

Thus it can be concluded the elongation of the bar due to its own weight is just half that of the elongation resulted from externally applied weight when the magnitude of the applied load is equal to that of the self weight of the bar.

Example 8.1 A short cylinder with 1 cm wall thickness is subjected to a compressive load of 10 ton. Calculate the required outside diameter, if the working stress in compression is 800 kg/cm^2.

Solution Load P = 10 ton = 10 × 1000 kg,

Let the outer diameter = d_0 cm; wall thickness = t = 1 cm

∴ Inner diameter $d_i = (d_0 - 2t)$ cm = $(d_0 - 2)$ cm

$$\sigma_w = 800 \text{ kg/cm}^2$$

$$\sigma_w \times A = P$$

$$800 \times \frac{\pi}{4}\{d_0^2 - (d_0 - 2)^2\} = 10 \times 1000$$

$$d_0 = 4.98 \text{ cm}$$

Thus the required outside diameter is d_0 = 4.98 cm.

Example 8.2 A steel wire hangs vertically under its self-weight. What is the greatest possible length that it can have if the allowable tensile stress is 2000 kg/cm^2? The specific weight of the steel is 8000 kg/cm^3.

Solution The weight of the wire $W = \rho V = \rho A l$

$$\therefore \qquad \sigma_t = \frac{W}{A} = \rho \times l$$

$$\sigma_t = 2000 \text{ kg/cm}^2$$

$$\rho = 8000 \text{ kg/m}^3 = 8000 \times 10^{-6} \text{ kg/cm}^3$$

$$l = \frac{\sigma_t}{\rho} = \frac{2000}{8000 \times 10^{-6}} = 2500 \text{ m}$$

Thus the length of the bar becomes l = 2500 m

Example 8.3 Three pieces of wood having 3.75 cm × 3.75 cm square cross-section are glued together and to the floor as shown in Fig. 8.6. If a horizontal force P = 3000 kg is applied to the middle member, what is the average shear stress in each of the glued joints?

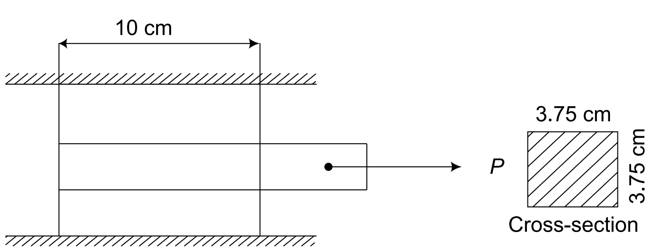


Figure 8.6

Solution $P = 3000$ kg

The area that resists shear stress corresponds to the plan area. Since there are two such areas; area to shear $A_s = 2 \times 10 \times 3.75$ cm^2

$$\therefore \qquad \tau_{av} = \frac{P}{A_s} = \frac{3000}{2 \times 10 \times 3.75} = 40 \text{ kg/cm}^2$$

Thus the average shear stress in each of the glued joints is $\tau_{av} = 40$ kg/cm^2.

Example 8.4 A lever is attached to a spindle by means of a square key 6 mm × 6 mm × 2.5 cm long as shown in Fig. 8.7. If the average shear stress in the key is not to exceed 700 kg/cm^2, what is the safe value of the load P applied at the free end of the lever?

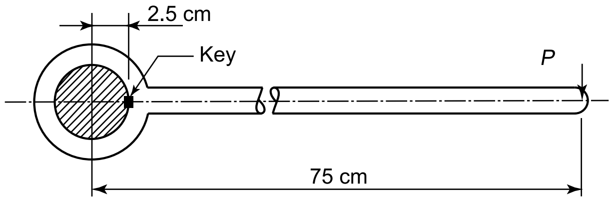


Figure 8.7

Solution $\tau_{av} = 700$ kg/cm^2

Taking moment at the centre C

$$\sum M_C = 0$$

Force at the key is $F = \dfrac{P \times 75}{2.5} = 30\ P$ kg

Area to shear $A_s = 0.6 \times 2.5$ cm^2

$$\tau_{av} \times A_s = F$$

$$700 \times 0.6 \times 2.5 = 30\ P$$

$$P = 35 \text{ kg}$$

Therefore the safe value of the load P is 35 kg.

Example 8.5 A 100 mm diameter shaft has a projected collar of diameter 130 mm over a length of 20 mm and supported by a hollow structure as shown in Fig. 8.8. The shaft is subjected to an axial load of 500 kN. Find the shear stress induced in the collar?

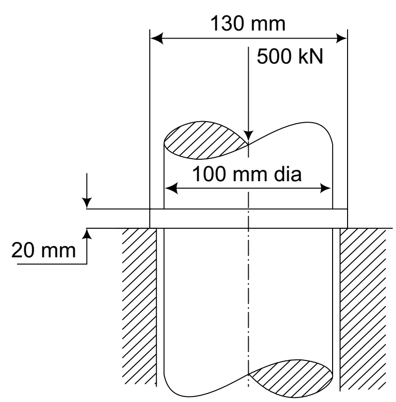


Figure 8.8

Solution In view of the configuration, the shaft can be sheared of owing to failure of the collar. The area that offers shear resistance = $A_s = \pi Dt = \pi \times 100 \times 20$ mm^2 = 2000π mm^2

The load to be withstand = P = 500 kN = 500 × 1000 N

Therefore induced shear stress in the shaft = $\tau = \dfrac{P}{A_s} = \dfrac{500 \times 1000}{2000\pi}$ N/mm^2 = 79.57 N/mm^2

Example 8.6 An aluminum bar 1.8 m long has a 2.5 cm square cross-section along 0.6 m of its length and 2.5 cm diameter circular cross-section over rest 1.2 m. If the bar is subjected to a tensile load of $P = 1750$ kg, what will be the elongation of the bar? Take $E = 75 \times 10^4$ kg/cm^2.

Solution $l_1 = 0.6 \text{ m} = 60 \text{ cm}$

$$l_2 = 1.2 \text{ m} = 120 \text{ cm}$$

$$A_1 = 2.5 \text{ cm} \times 2.5 \text{ cm}$$

$$A_2 = \frac{\pi}{4} \times 2.5^2 \text{ cm}^2$$

Tensile load $P = 1750$ kg

Modulus of elasticity $E = 75 \times 10^4$ kg/cm^2

$$\delta = \delta_1 + \delta_2 = \frac{Pl_1}{A_1E} + \frac{Pl_2}{A_2E} = \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2}\right)$$

$$= \frac{1750}{75 \times 10^4}\left(\frac{60}{2.5 \times 2.5} + \frac{120}{\frac{\pi}{4} \times 2.5^2}\right) \text{cm}$$

$$= 0.794 \text{ cm}$$

Thus the elongation of the bar = 0.794 cm.

Example 8.7 A rigid bar AB is hinged to a vertical wall at A and supported horizontally by a tie bar CD shown in Fig. 8.9 (a). The tie-bar has cross-section area of 0.5 sq cm and its allowable stress in tension is 1500 kg/cm^2. Find the safe value of the load P and the corresponding vertical deflection of point B. The modulus of elasticity of the tie-bar is $E = 2 \times 10^6$ kg/cm^2.

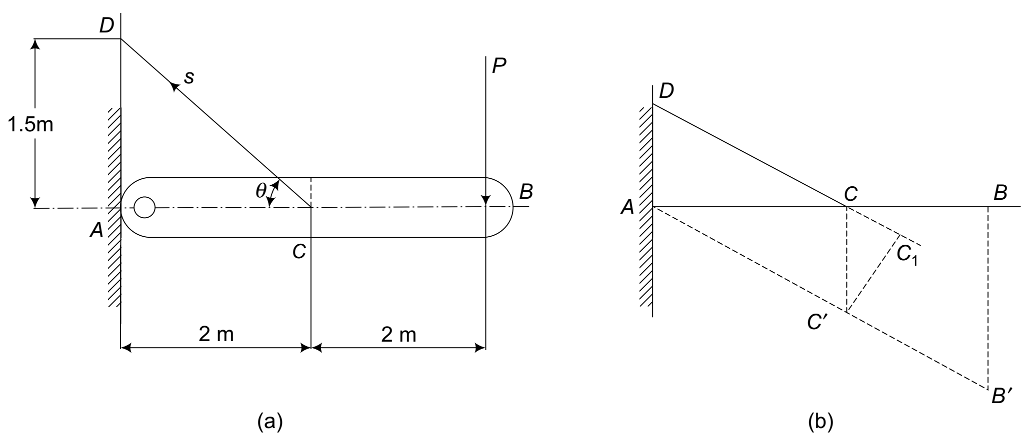


Figure 8.9

Solution The cross-section area of the tie-bar is $A_t = 0.5$ cm^2

$$E_t = 2 \times 10^6 \text{ kg/cm}^2$$

$$\sigma_{\text{all}} = 1500 \text{ kg/cm}^2$$

Let the tension induced in the tie bar be S kg.

$$S = \sigma_{au} \times A_t = 1500 \times 0.5 = 750 \text{ kg}$$

Taking moment about A,

$$S \sin \theta \times 2 = P \times 4$$

$$P = \frac{S \sin \theta}{2} = \frac{750}{2} \times \frac{1.5}{2.5} = 225 \text{ kg}$$

Refer to Fig. 8.9 (b). Elongation of tie rod $CD = CC_1 = \dfrac{S \times l_t}{A_t \times E_t}$

$$= \frac{750 \times 250}{0.5 \times 2 \times 10^6}$$

$$= 1875 \times 10^{-4} \text{ cm}$$

Vertical deflection of point C will be = CC' and vertical deflection of point B will be = BB'.

From $\Delta CC_1C'$,

$$\sin \theta = \frac{CC_1}{CC'}$$

$$CC' = \frac{CC_1}{\sin \theta} = 1875 \times 10^{-4} \times \frac{2.5}{1.5} = 0.3125 \text{ cm}$$

From $\Delta ACC'$ and $\Delta ABB'$,

$$\frac{CC'}{BB'} = \frac{AC}{AB} = \frac{2}{4} = \frac{1}{2}$$

$$BB' = 2CC' = 2 \times 0.3125 = 0.625 \text{ cm} = 6.25 \text{ mm}$$

Example 8.8 A rigid bar AB of length 9 m is suspended by two vertical rods at its ends and hangs in a horizontal position under its own weight as shown in Fig. 8.10. The rod at A is made of brass having 3 m length and cross-section area 10 cm^2. Take modulus of elasticity $E_{Br} = 10 \times 10^5$ kg/cm^2. The rod at B is steel having a length of 5 m, cross-section area 4.45 cm^2, modulus of elasticity $E_{st} = 2 \times 10^6$ kg/cm^2. At what distance x from A may a vertical load $\boldsymbol{P}$ be applied if the bar is to remain horizontal after the load is applied?

Solution Let us introduce "*Br*" and "*St*" as suffices for brass and steel respectively.

Thus $l_{Br} = 3$ m = 300 cm; $A_{Br} = 10$ cm^2 and $E_{Br} = 10 \times 10^5$ kg/cm^2.

Likewise, $l_{st} = 5$ m = 500 cm ; $A_{St} = 4.45$ cm^2 and $E_{st} = 2 \times 10^6$ kg/cm^2.

Let the tension induced in A and B be S_{Br} and S_{St} respectively.

Length of the bar AB = 9 metre.

Taking moment of all the forces about B, equilibrium of the bar AB yields

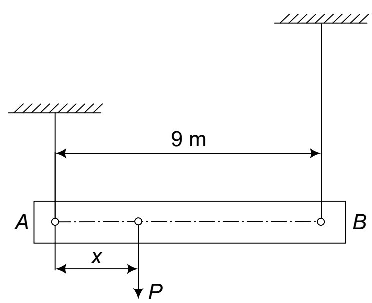


Figure 8.10

$$M_B = 0$$

$$S_{Br} \times 9 = P \times (9 - x)$$

$$S_{Br} = \frac{P(9 - x)}{9}$$

Therefore $$S_{St} = P - \frac{P(9 - x)}{9} = \frac{Px}{9}$$

The deflection/elongation of the two bars are $\delta_{Br} = \dfrac{S_{Br}\, l_{Br}}{A_{Br}\, E_{Br}} = \dfrac{P(9-x)}{9} \times \dfrac{300}{10 \times 10 \times 10^5}$ and $\delta_{St} = \dfrac{S_{St}\, l_{St}}{A_{St}\, E_{St}}$ $= \dfrac{Px}{9} \times \dfrac{500}{4.45 \times 2 \times 10^6}$ respectively.

In order to maintain horizontal position of the bar,

$$\delta_{Br} = \delta_{St}$$

$$\frac{P(9-x)}{9} \times \frac{300}{10 \times 10 \times 10^5} = \frac{Px}{9} \times \frac{500}{4.45 \times 2 \times 10^6}$$

Solving; $x = 3.13$ m

Thus the force P has to be applied at a distance of $x = 3.13$ m from the end A to maintain horizontal position of the bar AB.

Example 8.9 A prismatic steel bar having cross-section of 3 sq. cm is subjected to axial loading as shown in Fig. 8.11. Determine net deformation of the bar. Take $E_s = 2 \times 10^6$ kg/cm^2.

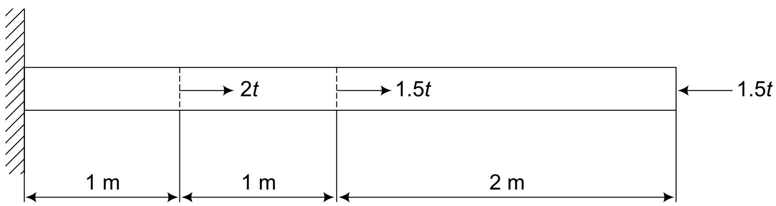


Figure 8.11

Solution To solve this problem; method of superimposition is followed, i.e., initial one metre length of the bar from the support is subjected to 2 T load (tensile) only. For this condition, deformation is calculated.

Similarly, next 2 metres of length is subjected to 1.5 T load (tensile) and finally we consider the entire bar of 4 metres which is subjected to 1.5 T load (compressive). For these two situations, deformations are calculated.

Once the deformations are calculated individually for different loads and of different length of the bar, we take the algebraic sum of the deformations to get the final deformation of the bar.

Introducing suffices for above three situations by 1, 2 and 3,

$$A = 3 \text{ cm}^2$$

Modulus of elasticity $E = 2 \times 10^6$ kg/cm^2

$$P_1 = 2 \text{ ton} = 2000 \text{ kg}$$

$$l_1 = 1 \text{ m} = 100 \text{ cm}$$

$$\therefore \quad \delta_1 = \frac{P_1 l_1}{AE} = \frac{2000 \times 100}{3 \times 2 \times 10^6} = 0.033 \text{ mm}$$

$$P_2 = 1.5 \text{ ton} = 1500 \text{ kg}$$

$$l_2 = 2 \text{ m} = 200 \text{ cm}$$

$$\therefore \quad \delta_2 = \frac{P_2 l_2}{AE} = \frac{1500 \times 200}{3 \times 2 \times 10^6} = 0.05 \text{ mm}$$

$$P_3 = 1.5 \text{ ton} = 1500 \text{ kg}$$

$$l_3 = 4 \text{ m} = 400 \text{ cm}$$

$$\therefore \qquad \delta_3 = -\frac{P_3 l_3}{AE} = -\frac{1500 \times 400}{3 \times 2 \times 10^6} = -0.1 \text{ mm}$$

$$\delta = \delta_1 + \delta_2 + \delta_3 = 0.033 + 0.05 - 0.1 = -0.17 \text{ mm}$$

Thus net deformation of the bar is –0.17 mm.

Example 8.10 A vertical load $P = 2100$ kg is supported by two inclined steel wires AC and BC as shown in Fig. 8.12. Determine the required cross-section area A of each wire if the allowable working stress in tension is 700 kg/cm^2. Take angle $\theta = 30°$ and $AB = 10$ m. Also calculate vertical deflection of the point C.

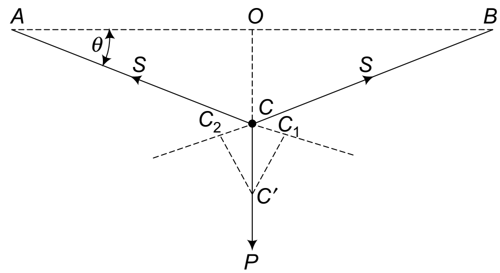


Figure 8.12

Solution $P = 2100$ kg; $\theta = 30°$; $AB = 10$ m

$\therefore$ $OA = 5$ m $\therefore$ $OC = 5 \tan 30° = 2.886$ m

Therefore $AC = \sqrt{5^2 + 2.886^2} = 5.773$ m

From symmetry of the configuration, it is evident that the tension induced in the two wires, namely, AC and BC will be same.

Let this tension is S. The allowable working stress in tension is $\sigma_{all} = 700$ kg/cm^2

From the force balance considering equilibrium of the point C,

$$2S \sin \theta = P$$

$$S = P, \text{ since } \theta = 30°$$

Thus $$\sigma_{all} \times A = S = P = 2100$$

$$A = \frac{2100}{700} \text{ cm}^2 = 3 \text{ cm}^2$$

The required cross-section area of the wires becomes $A = 3$ cm^2

Let CC_1 is the elongation of the wire AC.

$$\therefore \qquad CC_1 = \frac{S \times l_{AC}}{A_{AC} \times E}$$

$$= \frac{P \times l_{AC}}{A_{AC} \times E}$$

$$= \frac{2100 \times 5.773 \times 100}{3 \times 2 \times 10^6} \text{ cm}$$

$$= 0.\ 202 \text{ cm}$$

Vertical deflection of point C will be

$$\Delta = CC' = \frac{CC_1}{\sin \theta} = \frac{0.202}{\sin 30°} = 0.404 \text{ cm} = 4.04 \text{ mm}$$

Therefore vertical deflection of the point C is 4.04 mm.

Example 8.11 Refer to Fig. 8.13. The structure is subjected to a vertical load $P = 200$ kg applied at B. Member BC is steel wire with a 3 mm diameter and member AB is a wood strut of 2.5 cm square cross-section. Determine the vertical and horizontal components of the deflection of point B.

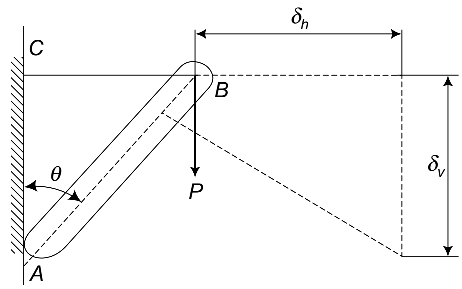
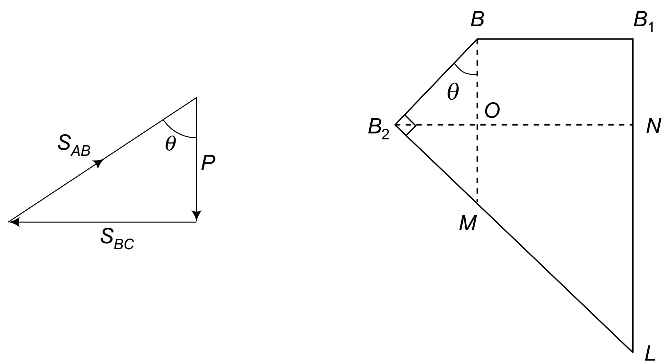


Figure 8.13

Solution



(a) (b)

Figure 8.13

$$A_{BC} = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ cm}^2$$

$$A_{AB} = 2.5 \times 2.5 = 6.25 \text{ cm}^2$$

$$P = 200 \text{ kg}$$

$$E_S = 2 \times 10^6 \text{ kg/cm}^2$$

$$E_w = 10 \times 10^4 \text{ kg/cm}^2$$

Under the action of load P, member BC will be subjected to tension and strut AB will be compressed. Length of BC will be increased and AB will be shortened.

$$\delta_{BC} = \frac{S_{BC} \times l_{BC}}{A_{BC} E_S}$$

$$\delta_{AB} = \frac{S_{AB} \times l_{AB}}{A_{AB} E_W}$$

To calculate the force induced in BC and AB, we construct the triangle of force as shown in Fig. 8.13 (a).

$$AC = 2 \text{ m}$$

$$BC = 1.5 \text{ m}$$

$$AB = \sqrt{AC^2 + BC^2} = \sqrt{2^2 + 1.5^2} = 2.5 \text{ m}$$

$$\sin \theta = \frac{1.5}{2.5}$$

$$\cos \theta = \frac{2}{2.5}$$

$$\frac{S_{AB}}{\sin 90^\circ} = \frac{S_{BC}}{\sin \theta} = \frac{P}{\sin (90^\circ - \theta)}$$

$$S_{AB} = \frac{P}{\cos \theta} = \frac{200}{\frac{2}{2.5}} = 250 \text{ kg}$$

$$S_{BC} = \frac{P \sin \theta}{\cos \theta} = P \tan \theta = 200 \times \frac{1.5}{2} = 150 \text{ kg}$$

$$\delta_{BC} = \frac{S_{BC} \times l_{BC}}{A_{BC} E_S} = \frac{150 \times 150}{0.0707 \times 2 \times 10^6} = 0.159 \text{ cm}$$

Horizontal deflection $\delta_H = \delta_{BC} = 0.159$ cm

$$\delta_{AB} = \frac{S_{AB} \times l_{AB}}{A_{AB} E_W} = \frac{250 \times 250}{6.25 \times 10 \times 10^4} = 0.1 \text{ cm}$$

From Fig. 8.13 (b)

Vertical deflection

$$\delta_V = B_1 L = B_1 N + NL = OB + NL$$

$$BO = BB_2 \cos \theta = \delta_{AB} \cos \theta = 0.1 \times \frac{2}{2.5} = 0.08 \text{ cm}$$

$$\frac{OM}{NL} = \frac{OB_2}{B_2 N} = \frac{OB_2}{OB_2 + ON}$$

$$OM = BM - OB = \frac{BB_2}{\cos \theta} - OB = \frac{0.1}{\frac{2}{2.5}} - 0.08 = 0.045 \text{ cm}$$

$$OB_2 = BB_2 \sin \theta = 0.1 \times \frac{1.5}{2.5} = 0.06 \text{ cm}$$

$$NL = 0.164 \text{ cm}$$

$$B_1 L = B_1 N + NL = OB + NL = 0.08 + 0.164 = 0.244 \text{ cm} = \delta_V$$

Thus vertical deflection of point B is $\delta_V = 0.244$ cm.

Example 8.12 Prove that volumetric strain of a rectangular bar is the algebraic sum of strains of length, width and height.

Solution Consider a rectangular bar having initial length, width and height as l_0, w_0 and h_0 respectively.

Thus, initial volume $V_0 = l_0 w_0 h_0$

In view of deformation, the final corresponding dimensions are, say, l_f, w_f and h_f.

Thus $l_f = l_0 + \delta l_0$; $w_f = w_0 + \delta w_0$ $h_f = h_0 + \delta h_0$

Therefore, final volume

$$V_f = l_f w_f h_f = (l_0 + \delta l_0)(w_0 + \delta w_0)(h_0 + \delta h_0) \approx l_0 w_0 h_0 + w_0 h_0 \delta l_0 + l_0 h_0 \delta w_0 + l_0 w_0 \delta h_0$$

Thus volumetric strain =

$$\frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0} = \frac{w_0 h_0 \delta l_0 + l_0 h_0 \delta w_0 + l_0 w_0 \delta h_0}{l_0 w_0 h_0} = \frac{\delta l_0}{l_0} + \frac{\delta w_0}{w_0} + \frac{\delta h_0}{h_0} = \varepsilon_l + \varepsilon_w + \varepsilon_h \text{ [Proved]}$$

Example 8.13 Refer to Fig. 8.12. Prove that the vertical deflection of point C, is $\Delta = \frac{l}{2}\sqrt[3]{\frac{P}{AE}}$

Solution $AB = l$

$$AO = BO = \frac{l}{2} = L \text{ (say)}$$

From static equilibrium,

$$P = 2T \sin \theta$$

Let $AC = L'$

Again,

$$\frac{TL'}{AE} = L' - L = L' - L' \cos \theta = L'(1 - \cos \theta)$$

$$T = AE(1 - \cos \theta)$$

$$P = 2AE(1 - \cos \theta) \sin \theta$$

$$= 2AE\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + ---\right)\left(\theta - \frac{\theta 3}{3!} + ---\right)$$

Keeping only first one term of each series,

$$P \approx 2AE \frac{\theta^2}{2} \theta = AE\theta^3$$

Since the deflection is very small compared to length,

$$\tan \theta \approx \theta = \frac{\Delta}{L}$$

$$P = AE\frac{\Delta^3}{L^3}$$

$$\Delta^3 = L^3 \frac{P}{AE}$$

$$\Delta = L\sqrt[3]{\frac{P}{AE}} = \frac{l}{2}\sqrt[3]{\frac{P}{AE}}$$

Example 8.14 The frame shown in Fig. 8.14 is made of 10 cm × 10 cm square wooden post, for which allowable stress in shear parallel to the grain is τ_w = 7 kg/cm^2, while that in compression perpendicular to the grain is σ_w = 28 kg/cm^2. Calculate the minimum safe values of the dimensions a, b and c. The vertical post is pinned at its lower end to a foundation plate.

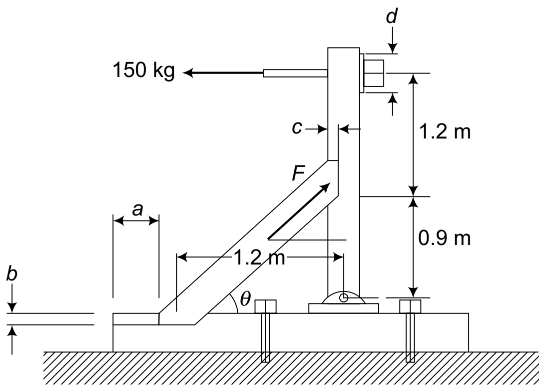


Figure 8.14

Solution From the given geometry, $\tan\theta = \dfrac{0.9}{1.2}$

$$\theta = 37°$$

Considering the equilibrium of vertical post and taking moment at its base,

$$\sum M_{Base} = 0$$

$$150 \times 2.1 - F\cos\theta \times 0.9 = 0$$

$$F\cos\theta = 350 \text{ kg}$$

$\therefore$ $$F\sin\theta = 263 \text{ kg}$$

Now $$F\sin\theta = \sigma_w \times (c \times w)$$

$$c = \frac{F\sin\theta}{\sigma_w \times w} = \frac{263}{28 \times 10} \text{ cm} = 0.9393 \text{ cm}$$

Similarly, a compressive force $F\cos\theta$ is exerted on the area amounts to $b \times w$.

$\therefore$ $$F\cos\theta = \sigma_w \times (b \times w)$$

$$b = \frac{F\cos\theta}{\sigma_w \times w} = \frac{350}{28 \times 10} \text{ cm} = 1.25 \text{ cm}$$

Further $F\cos\theta$ will cause a shear stress over the area $a \times w$

$\therefore$ $$F\cos\theta = \tau_w \times (a \times w)$$

$$a = \frac{F\cos\theta}{\tau_w \times w} = \frac{350}{7 \times 10} \text{ cm} = 5 \text{ cm}$$

MULTIPLE-CHOICE QUESTIONS

8.1 Stress is defined by
- (a) force/unit area
- (b) force × unit area
- (c) (force/unit length) × area
- (d) none of the above

8.2 Strain is defined by
(a) original length/final length (b) final length/original length
(c) elongation/original length (d) none of the above

8.3 According to the Hook's law,
(a) stress ∝ (1/strain) (b) stress ∝ strain (c) stress ∝ $(\text{strain})^2$ (d) none of the above

8.4 Poisson's ratio is defined by
(a) ratio of lateral strain and longitudinal strain (b) ratio of longitudinal strain and lateral strain
(c) ratio of lateral stress and longitudinal stress (d) ratio of longitudinal stress and lateral stress

8.5 Shear stress is quantified by
(a) unit angular deformation (b) unit lateral deformation
(c) unit longitudinal deformation (d) none of the above

8.6 Modulus of Elasticity (E) can be found out by following relationship
(a) $E = \sigma + \varepsilon$ (b) $E = \sigma - \varepsilon$ (c) $E = \sigma \times \varepsilon$ (d) $E = \dfrac{\sigma}{\varepsilon}$

8.7 The stress produced in the members so to prevent sliding of a section over other is called
(a) nominal stress (b) bearing stress (c) shear stress (d) none of the above

8.8 The maximum value Poisson's ratio is
(a) 0.25 (b) 0.50 (c) 0.75 (d) 1.0

SHORT ANSWER TYPE QUESTIONS

8.1 Define stress and strain. What is meant by elasticity of materials?
8.2 What is normal stress, bearing stress and shear stress? Explain with sketches and quantify them.
8.3 What is Poisson's ratio? Prove that Poisson's ratio of materials can never exceed 0.5.
8.4 State and explain Hooke's law with necessary diagram. What is modulus of elasticity? What is its unit?
8.5 What are longitudinal strain, lateral strain and volumetric strain?
8.6 What is shear strain? How it is quantified?
8.7 Obtain an expression for the elongation of a bar of uniform cross-section subjected to self-weight only.
8.8 Prove that the volumetric strain of a sphere is three times of diametrical strain.

NUMERICAL PROBLEMS

8.1 Prove that volumetric strain of a cylindrical bar is sum of longitudinal strain and twice that of lateral strain.
8.2 Find the stresses induced in two different sections of the stepped bar as shown in Fig. 8.15, subjected to a compressive load of 20 kN.
8.3 The cylinder has inside diameter D = 25 cm and is subjected to internal gas pressure of intensity P = 20 kg/cm^2 gauge. How many 12.5 mm-diameter steel bolts will be required to fasten the top cover plate to the cylinder if the working stress for the bolt is 700 kg/cm^2?

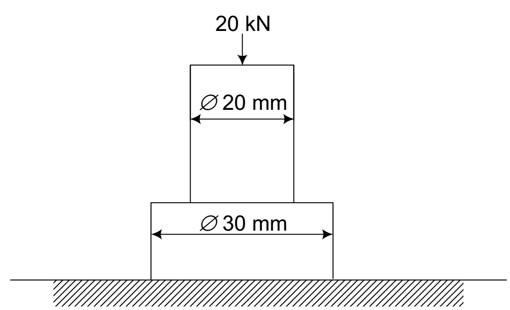


Figure 8.15

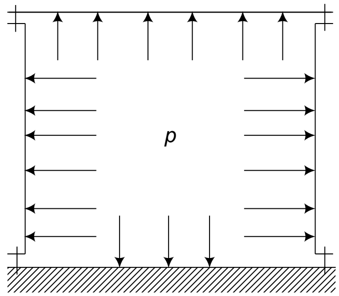


Figure 8.16

8.4 A stepped steel bar of varying cross-section (rectangular) is loaded as shown in Fig. 8.17. Calculate the total deformation. Take $E = 2 \times 10^5$ N/mm^2. The magnitude of different forces are $P_1 = 100$ kN, $P_2 = 80$ kN and $P_3 = 50$ kN.

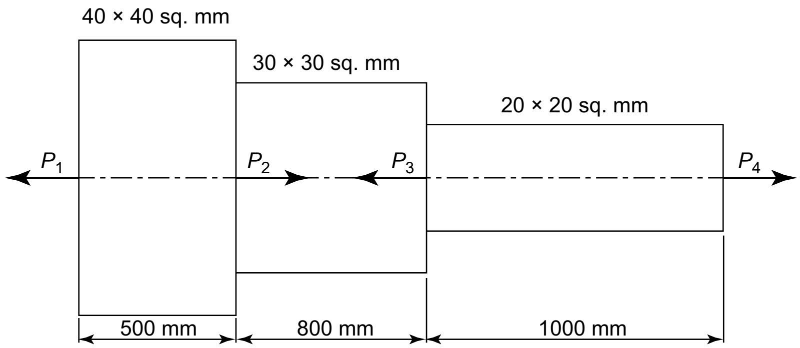


Figure 8.17

8.5 Find the change in length of the bar loaded as shown in Fig. 8.18. Take $E = 2 \times 10^6$ kg/cm^2.

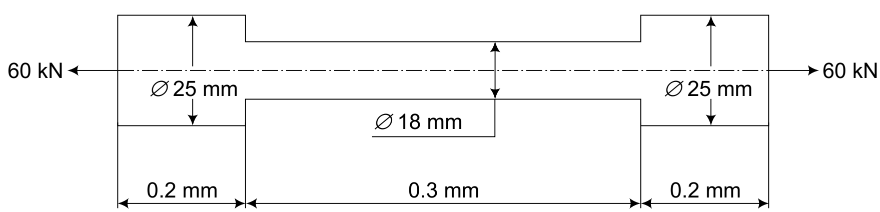


Figure 8.18

8.6 A rod of steel 60 mm wide and 15 mm thick is 8 m long. It extends by 6 mm when an axial pull of 100 kN is applied. Find the modulus of elasticity of steel.

8.7 A steel column is 100 mm in diameter and is 2.8 m long. Find the intensity of stress and the strain when it carries an axial compressive load of 800 kN. Take $E_s = 2 \times 10^5$ N/mm^2.

8.8 A rod circular in section tapers from 20 mm diameter at one end to 10 mm diameter at other end and is 200 mm long. On applying an axial pull of 10 kN, it was found to extend by 0.068 mm. Find the Young's modulus of the material of the rod.

8.9 A copper rod 5 mm in diameter when subjected to a pull of 600 N extends by 0.125 mm over a gauge length of 350 mm. Find the Young's modulus for copper.

8.10 A hollow steel column of external diameter 300 mm has to support an axial load of 3500 kN. If the ultimate stress for the steel column is 480 N/mm^2, find internal diameter of the column. Allow a load factor of 2.50.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

8.1 (a)	8.3 (b)	8.5 (a)	8.7 (c)
8.2 (c)	8.4 (a)	8.6 (d)	8.8 (b)

ANSWERS TO NUMERICAL PROBLEMS

8.2 63.7 N/mm^2, 28.3 N/mm^2
8.3 12
8.4 1.12 mm
8.5 0.6 mm
8.6 1.48×10^5 N/mm^2
8.7 $\sigma = 102$ N/mm^2, $\varepsilon = 5.1 \times 10^{-4}$
8.8 1.87×10^5 N/mm^2
8.9 8.55×10^4 N/mm^2
8.10 284 mm

CHAPTER

9

Stress – Strain Diagram

9.1 VARIATION OF STRESS IN REGARD TO CROSS-SECTION

In chapter 8, we have found that the prismatic bar is loaded uniaxially by application of tensile force P, as shown in Fig. 8.1.

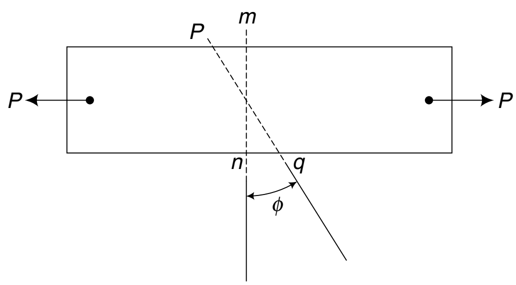


Figure 9.1

Under the action of force P, normal stress induced $\sigma = \dfrac{P}{A}$

To investigate the stress characteristics, let us consider an oblique section ***p-q*** having inclination ϕ with the vertical as shown in Fig. 9.1.

Considering equilibrium of the left side of the section, the internal force $S = P$

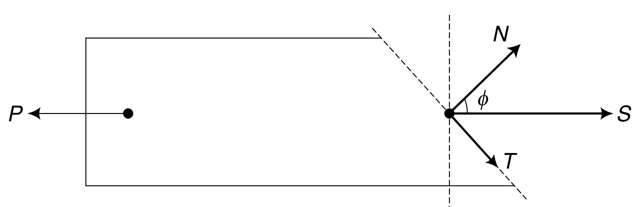


Figure 9.2

Refer to Fig. 9.2. The two mutually perpendicular components of S - one perpendicular to the oblique plane and the other along the plane denoted by N and T respectively will have values

$$N = P \cos \phi$$

$$T = P \sin \phi$$

The cross-section of the oblique plane

$$A' = \frac{A}{\cos \phi}$$

Therefore, stress corresponds to N is $\sigma_n = \dfrac{P\cos\phi}{A/\cos\phi} = \dfrac{P}{A}\cos^2\phi$

And stress corresponds to T is $\tau = \dfrac{P\sin\phi}{A/\cos\phi} = \dfrac{P}{2A}\sin 2\phi$

Let us consider few typical cases based on the values of ϕ.

(i) When $\phi = 0$, $\sigma_n = \dfrac{P}{A} = (\sigma_n)_{max}$ and $\tau = 0$

(ii) When $\phi = 90°$, $\sigma_n = 0$ and $\tau = 0$

(iii) When $\phi = 45°$, $\sigma_n = \tau = \dfrac{P}{2A} = \tau_{max}$

Further, when $\phi = 90° + \phi$,

$$\sigma_n' = \frac{P}{A}\cos^2(90° + \phi) = \frac{P}{A}\sin^2\phi$$

$$\tau' = \frac{P}{2A}\sin(180° + 2\phi) = -\frac{P}{2A}\sin 2\phi$$

$$\sigma_n + \sigma_{n'} = \frac{P}{A}(\cos^2\phi + \sin^2\phi) = \frac{P}{A}$$

$$\tau + \tau' = \frac{P}{2A}(\sin 2\phi - \sin 2\phi) = 0$$

$$\tau = -\tau'$$

9.2 CONCEPT OF STRAIN ENERGY

In case of an axially loaded bar when load P causes a deformation δ, the work done by the force P is $\dfrac{1}{2}P\delta$ as evident from Fig. 9.3.

This energy is stored in the body.

Thus total energy $U = \dfrac{P\delta}{2}$

$$\delta = \frac{Pl}{AE}$$

$$U = \frac{P^2 l}{2AE} = \frac{AE\delta^2}{2l}$$

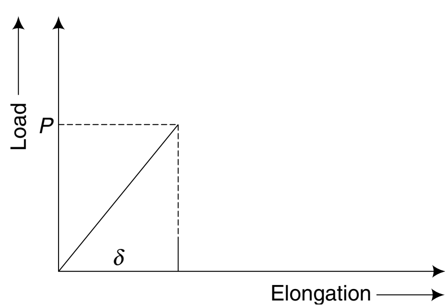


Figure 9.3

This is known as *strain energy*.

When it is expressed per unit volume,

$$u = \frac{U}{Al} = \frac{AE\delta^2}{2l \times Al} = \frac{E}{2} \times \left(\frac{\delta}{l}\right)^2 = \frac{E}{2}\varepsilon^2$$

$$= \frac{E}{2} \times \frac{\sigma^2}{E^2} = \frac{\sigma^2}{2E}$$

9.3 PROPERTIES OF MATERIALS

In regard to different behaviors exhibited by materials under different types of load, it is imperative to discuss few salient properties of materials that are found to be extremely useful in justifying their candidature for intended use. Needless to say, all do not respond equally in varieties of loading conditions.

9.3.1 Ductility

It is the property of the material by virtue of which it can be drawn into thin wire, i.e., the material undergoes sufficient elongation before failure when subjected to tensile load. Mild steel and structural steel are more ductile than cast iron.

9.3.2 Brittleness

This property is just opposite to that of ductility implying the property of the material by which the material does not manifest adequate deformation before failure when subjected to axial load. Failure of such types of materials under loading takes place without any appreciable changes in its dimensions. Examples are cast iron and glass.

9.3.3 Hardness

It is the property of the material by virtue of which it offers resistance to indentation or scratch. It is quantified by depth and distribution of the indentation caused by a standard diamond ball. There are various scales by which it is measured and expressed such as Brinnell Hardness number, Rockwell hardness, Vickers Pyramid Number, etc. It is desirable that hardness of the materials should be evenly distributed all over the surface.

9.3.4 Malleability

It is the property of the material that makes it amenable to be rolled into thin sheet. Aluminum, magnesium are highly malleable materials.

9.3.5 Toughness

The toughness is the amount of energy a material can absorb before actual fracture.

Once various properties of materials are discussed, it is imperative to study ***stress-strain*** behavior of ductile materials such as mild steel to investigate its behavior under loading. Mild steel is widely used materials in engineering applications. Thus, its recommendation for multifarious use calls for gathering of data's pertaining to its strength which can be made available from experiments. Thus a plot of *stress versus strain* based on experimental data is of paramount importance.

9.4 STRESS-STRAIN DIAGRAM OF DUCTILE MATERIALS SUBJECTED TO TENSILE LOADING

To carry out the test, a standard specimen having a definite gauge length and diameter with collars at two ends is prepared by suitable machining. Experiment is carried out in equipment called Universal Testing Machine (UTM).

The sample or specimen is gripped firmly between two jaws of the machine and subjected to axial loading. Before start of the experiment, the initial length (l_0) and initial diameter (d_0) of the specimen is measured and recorded.

The load is applied slowly from zero to a magnitude such that the specimen breaks. The load and the corresponding deformations are recorded and plotted automatically that is similar to one as shown in Fig. 9.4.

From 0 to 1, the curve is linear following Hooke's law. This implies that stress is proportional to the strain. The point 1 in the curve is the last point along straight line behavior of the curve, known as *proportional limit*.

Since $\sigma = E\varepsilon$, it therefore follows that E can be interpreted as the slope of the curve up to point 1. Withdrawal of load brings the material to original state, i.e., the behavior of the material is *elastic*. Further loading beyond proportional limit shows somewhat different characteristics as evident from the curve. At point 2, material starts *yielding*. Without appreciable increase in load, higher deformation takes place. This phenomenon continues till it reaches point 3. The point 2 is called *upper yield point* whereas point 3 is called *lower yield point*. After point 2 material becomes plastic and the deformation is permanent, i.e., even after removal of load, this deformation will be present in the material. Thus yield point is a transition point from elastic to plastic.

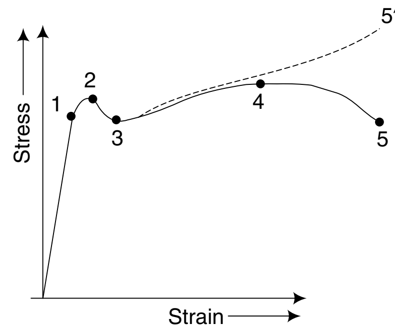


Figure 9.4

Further loading will cause deformation and this continues upto point 4 when stress reaches its peak after which it reduces. The maximum value of stress, i.e., stress at point 4 is known as *Ultimate stress*. After point 4, further loading of the material is not associated with increased stress value. This can be attributed to the formation of a *neck* which causes rapid yielding. Finally, the material breaks after reaching point 5. The stress value corresponds to point 5 is known as *breaking stress*, which is lower than ultimate stress. It is interesting to note that the breaking stress has got some what lower value than the ultimate stress. This can be explained as the stresses are based on original cross-sectional area. However, the dotted line which is *true stress-strain diagram* shows that the breaking stress (actual) is higher than that at point 5, since the cross-section area of the specimen between point 4 and 5 reduces significantly to form a neck prior to failure as shown in Fig. 9.5. This is why, in actual stress-strain diagram, stress is calculated as *load divided by instantaneous cross-sectional area, which shows continuous rise in stress value till it fails*. During breakage, area reduces sharply. Thus actual breaking stress 5′ is higher than that of point 5.

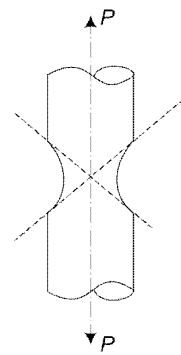


Figure 9.5

Meticulous investigation shows that failure takes place due to shear rather than tensile along a 45° plane. This is in congruence with the theory as explained earlier.

The above nature of curve of stress-versus-strain is valid for ductile material.

From the linear segment of the above curve, it is evident that area under the curve is equivalent to the area of a triangle, which is the measure of the strain energy up to proportional limit. Thus the total area under the curve represents the energy absorbed by the material before failure, which is measure of toughness. After yielding takes place, material enters into the plastic zone implying withdrawal of load will not allow the material to regain its original size. Thus, total energy is the sum of energy absorbed elastically and plastically.

9.4.1 Modulus of Resilience and Resilience

Resilience is the property of a material which means the amount of energy it can absorb elastically.

The maximum energy which can be stored in a body up to elastic limit or more precisely up to proportional limit is called the *Proof resilience* and the proof resilience per unit volume is called *modulus of resilience*.

$$\text{Modulus of resilience } u = \frac{\sigma_{\text{p.l.}}^2}{2E}$$

9.4.2 Measure of Ductility

After the test is over, the final diameter (d_f) and final length (l_f) are measured.

Thus % elongation of the specimen is calculated by $\dfrac{\text{increase in length}}{\text{original length}} \times 100$

$$\frac{\delta}{l_0} \times 100\% = \frac{l_f - l_0}{l_0} \times 100\%$$

Similarly % reduction in area is calculated by $\dfrac{\text{decrease in area}}{\text{original area}} \times 100$

$$\frac{A_0 - A_f}{A_0} \times 100\% = \frac{d_0^2 - d_i^2}{d_0^2} \times 100\%$$

These two parameters are used to measure the ductility of the material.

9.5 STRESS-STRAIN DIAGRAM OF BRITTLE MATERIALS

Similar experiments, when carried out for brittle materials like cast iron, a curve similar to that shown in Fig. 9.6 is obtained.

In contrast to the ductile materials, no distinct yield point is obtained. Further amount of deformation before failure is also not pronounced as compared to that of ductile materials. This leads to the conclusion, that responsiveness of cast iron to withstand tensile loading is poor. However, to consider a stress which would be tantamount to yield point that forms the basis for design, the concept of *proof stress* is introduced.

A line parallel to the straight line portion of the curve is drawn to intersect x axis so that corresponding elongation is 0.2%. Thus, a stress that corresponds to above strain is called *proof stress* and considered as basis for design calculations.

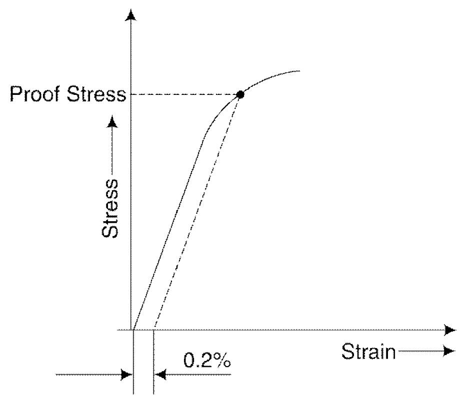


Figure 9.6

9.6 WORKING STRESS AND FACTOR OF SAFETY

For ductile material, proportional limit, yield point and ultimate stress can be determined experimentally. These values provides basis for design.

Nevertheless, there are certain practical reasons as mentioned below for which data obtained from laboratory experiments cannot be used directly.

- During actual design, exact load cannot be estimated, which is normally found to be higher that it is assumed.
- Static load considerations sometimes become misleading. Load may also come from unforeseen circumstances, which is not hitherto considered and of varying natures.
- Materials used in manufacturing often exhibits lower strength due to non-availability of requisite materials as recommended in design.

To cope with such situations, it is imperative to keep the stress value below that which has been found experimentally so as to be remain safe.

The design stress, commonly known as *working stress*, is therefore reduced by a factor (>1). If the design is based on yield stress, working stress becomes

$$\sigma_{\text{working}} = \frac{\sigma_{\text{yield point}}}{n}, \text{ where } n > 1.$$

The factor **"*n*"** is called *factor of safety*.

Instead of yield stress, design may be based on ultimate stress. In such a case,

$$\sigma_{\text{working}} = \frac{\sigma_{\text{ultimate}}}{n'}, \text{ where } n' > 1$$

Sometimes working stress is also called as *allowable stress*.
Both n and n' are called factor of safety.

> **Note:** While dealing with the stress, we come across two categories. The applied force divided by the area is the stress induced in the materials as a consequence of load. The other one is the characteristics of materials, which is the limiting or maximum stress that a particular materials can withstand before rupture. This is what is obtained from experiments. Thus while designing a load bearing member, one must bear in mind that induced stress should never exceed the stress that a material can endure safely.

Example 9.1 A mild steel specimen of circular cross-section of diameter $d = 1.25$ cm shows an elongation of 0.005 cm over a gauge length of 5 cm. Calculate the maximum shear stress in the material. Assume $E = 2 \times 10^6$ kg cm^2.

Solution $d = 1.25$ cm; elongation $\delta = 0.005$ cm; initial length $l = 5$ cm

$$\varepsilon = \frac{\delta}{l} = \frac{0.005}{5} = 0.001$$

$$E = 2 \times 10^6 \text{ kg/cm}^2 \quad \therefore \ \sigma = E\varepsilon = 2 \times 10^6 \times 0.001 = 2 \times 10^3 \text{ kg/cm}^2$$

$$\therefore \quad \tau_{\max} = \frac{\sigma}{2} = 1000 \text{ kg/cm}^2$$

The maximum shear stress in the material is $\tau_{\max} = 1000$ kg/cm^2

Example 9.2 A prismatic bar carrying an axial tensile stress σ_x is cut by an oblique section $p - q$ as shown in Fig. 9.2. If the normal and shear stress on this section are $\sigma_n = 825$ kg/cm^2 and $\tau = 275$ kg/cm^2 respectively; find the values σ_x and the angle ϕ.

Solution

$$\sigma_n = 825 \text{ kg/cm}^2 \text{ and}$$

$$\tau = 275 \text{ kg/cm}^2$$

$$\sigma_n = \frac{P}{A}\cos^2\phi = \sigma_x \cos^2\phi$$

$$\sigma_x \cos^2\phi = 825 \tag{9.1}$$

$$\tau = \frac{P}{2A}\sin 2\phi = \frac{\sigma_x}{2}\, 2\sin\phi\cos\phi$$

$$\sigma_x \sin\phi\cos\phi = 275 \tag{9.2}$$

Dividing Eqs (9.2) by (9.1),

$$\tan\phi = \frac{1}{3}$$

$$\phi = 18°26'$$

$$\sigma_x = \frac{825}{\cos^2 18°26'} = 917 \text{ kg/cm}^2$$

Example 9.3 A prismatic bar is subjected to an axial tensile force. Find the aspect angle ϕ that defines an oblique section on which the normal and shear stress on this section are equal.

Solution

$$\sigma_n = \frac{P\cos\phi}{A/\cos\phi} = \frac{P}{A}\cos^2\phi \text{ and}$$

$$\tau = \frac{P\sin\phi}{A/\cos\phi} = \frac{P}{2A}\sin 2\phi$$

Since $\sigma_n = \tau$

$$\frac{P}{A}\cos^2\phi = \frac{P}{2A}\sin 2\phi = 2\sin\phi\cos\phi$$

$$\tan\phi = 1$$

$$\phi = 45°$$

Thus aspect angle ϕ becomes 45°.

Example 9.4 While carrying out experiment (tensile test) in the laboratory; following observations were made.

Diameter of the specimen is 12.5 mm.
Length of the specimen (gauge length) is 50 mm.
Load at proportional limit is 3000 kg.
Load at yield point is 3100 kg.
Maximum load is 5250 kg.
Strain at proportional limit is 0.11%.
Final length is 64 mm.
Diameter over neck is measured as 9.72 mm.
Calculate the following:

(a) Modulus of elasticity E.
(b) Proportional limit
(c) Ultimate stress
(d) % elongation
(e) % reduction in area
(f) Allowable stress based on yield point, considering factor of safety as 1.75.

Solution Initial diameter (d_0) of the specimen is 12.5 mm.

$$\text{Initial area } (A_0) = \frac{\pi}{4}d_0^2 = \frac{\pi}{4} \times 1.25 \times 1.25 \text{ cm}^2 = 1.227 \text{ cm}^2$$

Strain at proportional limit (ε) is 0.11% = 0.0011
Load at proportional limit is 3000 kg

$$\text{Therefore, proportional limit} = \sigma_{Pl} = \frac{3000}{1.227} = 2445 \text{ kg/cm}^2$$

From $\sigma_{Pl} = E_\varepsilon$

$$E = \frac{\sigma_{Pl}}{\varepsilon} = \frac{2445}{0.0011} = 2.2 \times 10^6 \text{ kg/cm}^2$$

Maximum load is 5250 kg

$$\therefore \text{ Ultimate stress} = \sigma_{Ult} = \frac{5250}{1.227} = 4278 \text{ kg/cm}^2$$

Initial length of the specimen (gauge length) (l_0) is 50 mm and final length (l_f) is 64 mm.

$$\text{Thus \% elongation} = \frac{l_f - l_0}{l_0} \times 100\% = \frac{64-50}{50} \times 100\% = 28\%$$

% reduction in area =

$$\frac{A_0 - A_f}{A_0} \times 100\% = \frac{d_0^2 - d_i^2}{d_0^2} \times 100\% = \left[1 - \left(\frac{9.72}{12.5}\right)^2\right] \times 100\% = 39.54\%$$

$$\text{Yield stress} = \sigma_{yp} = \frac{3100}{1.227} = 2526.5 \text{ kg/cm}^2$$

$$\therefore \qquad \sigma_{\text{all}} = \frac{\sigma_{yp}}{n} = \frac{2526.5}{1.75} = 1443.6 \text{ kg/cm}^2$$

Example 9.5 A prismatic steel bar, 25 cm long and having cross-section of 12.5 cm^2, is subjected to an axial compressive force P = 2000 kg. Find the strain energy in the bar. Assume $E = 2 \times 10^6$ kg/cm^2.

Solution Strain energy $U = \dfrac{P^2 l}{2AE} = \dfrac{2000 \times 2000 \times 25}{2 \times 12.5 \times 2 \times 10^6} = 2$ kg/cm

Example 9.6 Compute the modulus of resilience for structural steel. Assume $E = 2 \times 10^6$ kg/cm^2 and proportional limit is 2000 kg/cm^2.

Solution Modulus of resilience $u = \dfrac{\sigma_{p.l.}^2}{2E} = \dfrac{2000 \times 2000}{2 \times 10^6} = 2$ kg/cm^2

Example 9.7 A prismatic steel bar of length l and cross-section area A is hanging vertically under its own weight. How much strain energy is stored in the bar if its density is and the modulus of elasticity is E.

Solution Refer to article 8.7.

The tensile force in the elemental strip dx is ρAx that causes an infinitesimal small elongation $d\delta = \dfrac{(\rho Ax).dx}{AE}$

Thus strain energy induced in this small element $= du = \dfrac{1}{2} \times (\rho Ax) \dfrac{(\rho Ax).dx}{AE}$

Therefore, strain energy induced in the entire bar is $U = \int du = \dfrac{\rho^2 A}{2E} \int_0^l x^2 dx = \dfrac{\rho^2 A l^3}{6E}$

Example 9.8 A wooden block with a 5 cm × 5 cm square cross-section has a glued joint at its mid-section similar to that as shown in Fig. 9.1. If the allowable working stresses for the glue in tension and shear are $\sigma_w = 70$ kg/cm^2 and $\tau_w = 42$ kg/cm^2 respectively, what is the optimum angle $\phi < 45°$ for the joint? What is the corresponding safe load P_w for the stick?

Solution

$$\sigma_w = \frac{P\cos\phi}{A/\cos\phi} = \frac{P}{A}\cos^2\phi$$

$$P\cos^2\phi = A.\sigma_w$$

$$\tau_w = \frac{P\sin\phi}{A/\cos\phi} = \frac{P}{2A}\sin 2\phi \tag{9.3}$$

$$P\sin 2 = 2A.\tau_w \tag{9.4}$$

Dividing Eqs (9.4) by (9.3) yields

$$\tan\phi = \frac{\tau_w}{\sigma_w} = \frac{42}{70} = 0.6$$

$$\phi = \tan^{-1} 0.6 = 31°$$

Thus $$P_w = \frac{A\sigma_w}{\cos^2\phi} = \frac{5\times5\times70}{\cos^2(31)} = 2382 \text{ kg}$$

MULTIPLE-CHOICE QUESTIONS

9.1 The limit up to which stress varies linearly with strain is called
(a) proportional limit (b) yield point (c) ultimate stress (d) breaking stress

9.2 The values of normal stress and shear stress becomes equal in a plane that makes
(a) zero angles (b) 30° (c) 45° (d) 90°

9.3 The ductility is a property by virtue of which materials
(a) undergo adequate elongation before failure. (b) can be rolled into thin sheets
(c) offer resistance against indentation. (d) none of the above

9.4 The hardness is a property of materials that exhibits its response to
(a) undergo adequate elongation before failure. (b) be rolled into thin sheets
(c) offer resistance against indentation. (d) none of the above

9.5 The following materials have very good malleability.
(a) Mild Steel (b) Cast Iron (c) Aluminum (d) Silver

9.6 Energy stored in the body by virtue of deformation under load is called
(a) potential energy (b) internal energy (c) strain energy (d) heat energy

9.7 The factor by which allowable stress is calculated is called
(a) power factor (b) factor of safety (c) scale factor (d) none of the above

9.8 Toughness is the measure of
(a) resistance against abrasion (b) ductility
(c) brittleness (d) energy absorbed before failure

9.9 In case of an axially loaded bar when load P causes a deformation s, the strain energy absorbed is

(a) $P\delta$ (b) $\frac{P\delta}{2}$ (c) $\frac{P\delta}{3}$ (d) $\frac{P\delta}{2E}$

9.10 For brittle materials, following stress is considered as a basis for design

(a) yield stress (b) proof stress (c) ultimate stress (d) breaking stress

9.11 Energy absorbed by materials per unit volume up to proportional limit is called

(a) resilience (b) proof resilience
(c) modulus of resilience (d) toughness

9.12 The maximum values of normal stress $(\sigma)_{max}$ and shear stress $(\tau)_{max}$ can be related by following expression.

(a) $(\sigma)_{max} = (\tau)_{max}$ (b) $(\sigma)_{max} = \frac{1}{2}(\tau)_{max}$ (c) $(\tau)_{max} = \frac{1}{2}(\sigma)_{max}$ (d) $(\tau)_{max} = \frac{1}{3}(\sigma)_{max}$

9.13 The following material is found to be extremely brittle.

(a) Cast iron (b) Copper (c) Gold (d) Glass

9.14 Area under stress-strain diagram represents

(a) resistance (force) against deformation (b) energy absorbed before failure
(c) depth and distribution of hardness (d) % elongation

9.15 For ductile materials loaded axially

(a) ultimate stress < breaking stress (b) ultimate stress = breaking stress
(c) ultimate stress > breaking stress (d) none of the above

9.16 The failure of ductile materials during tensile test is actually due to

(a) normal stress (b) fatigue (c) shear (d) strain hardening

SHORT ANSWER TYPE QUESTIONS

9.1 Derive expressions for normal stress and shear stress for an axial loaded (tensile) bar in a plane that makes an angle 90° + ϕ with the direction of the load.

9.2 Refer to the $Q1$; what are the values of for which these two stresses will be maximum, minimum and equal?

9.3 What is strain energy? Derive an expression for the same in terms of geometric attributes and material properties when a prismatic bar is subjected to tensile loading.

9.4 Explain the following properties of materials: Ductility, Malleability, Brittleness, Hardness and Toughness.

9.5 Draw a stress-strain diagram of a mild steel specimen for tensile test, clearly stating the implications of important points.

9.6 Define proof resilience and modulus of resilience.

9.7 What is proof stress? Name the category of materials to which it is associated? What purpose does it serve?

9.8 Explain the need of introducing factor of safety for design purpose.

9.9 What is the interpretation of the area under stress-strain diagram?

9.10 Why is the breaking stress less than that of ultimate stress?

9.11 How is ductility of materials quantified?

NUMERICAL PROBLEMS

9.1 Referring to the case of axial tension of a prismatic bar as shown in Fig. 9.1, consider following numerical data. A = 5 sq cm, P = 5,000 kg, ϕ = 20°. Calculate the stresses σ_n, σ'_n, τ, τ' for section p–q and p'–q', where the later is inclined at +90° with the former.

9.2 Referring to Fig. 9.1, assume that the angle ϕ = 30° and that σ_n = 700 kg/cm^2. What is the shear stress (τ)?

9.3 Refer to the problem 2. Here, σ_n = –1000 kg/cm^2, σ_n = –655 kg/cm^2. What is the aspect angle defining the orientation of the section and what is the axial stress P/A to which the bar is subjected?

9.4 A concrete test cylinder having length l = 30 cm and diameter d = 15 cm is subjected to axial compressive forces P in a testing machine. If the maximum shear stress in the concrete is not to exceed 140 kg/cm^2, what is the safe value for the axial load P?

9.5 Following observations were made during tensile test of a mild steel sample in UTM.
Diameter of the sample is 20 mm.
Gauge length is 200 mm.
Load at yield point is 66 kN.
Maximum load is 128 kN.
Deformation at yield point is 0.9545 mm.
Final length is 267 mm.
Diameter over neck is measured as 15.67 mm.
Calculate the following:
(a) Modulus of elasticity E
(b) Yield point.
(c) Ultimate stress
(d) % elongation
(e) % reduction in area
(f) Working stress based on yield point, considering factor of safety as 2.25.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

9.1 (a)	9.5 (c)	9.9 (b)	9.13 (d)
9.2 (c)	9.6 (c)	9.10 (b)	9.14 (b)
9.3 (a)	9.7 (b)	9.11 (c)	9.15 (c)
9.4 (c)	9.8 (d)	9.12 (c)	9.16 (c)

ANSWERS TO NUMERICAL PROBLEMS

9.1 σ_n = 884 kg/cm^2, σ'_n = 117 kg/cm^2, $\tau = -\tau'$ = 321.4 kg/cm^2
9.2 τ = 404.15 kg/cm^2
9.3 ϕ = 39°
9.4 49480 kg
9.5 (a) 44 kN/mm^2 (b) 210 N/mm^2 (c) 407.65 N/mm^2
(d) 33.5% (e) 38.6% (f) 93.33 N/mm^2

Part C

Dynamics

CHAPTER 10

Rectilinear Motion of a Particle

10.1 INTRODUCTION

In statics the entire analysis was focused on the rigid bodies that are under rest. In contrast, dynamics deals with bodies under motion. Although the statics is an old branch of mechanics, dynamics is relatively new. Another distinguishing feature of dynamics is that in addition to the force analysis like that of statics, it deals with other attributes that are associated with motion – namely *position, velocity and acceleration*, which are essentially function of time. It is already outlined in the chapter 1 that dynamics has got two wings – *kinematics* and *kinetics*. While the former deals with the study of motion parameters like position, velocity and acceleration without paying any attention to the force which causes motion, the latter deals with the analysis of force on the bodies under motion. In dynamics, the majority of the analysis is revolved around particle (the concept of particle is illustrated in chapter 1) giving rise to the concept of *particle dynamics*. Further, as regards to motion, it can be of two types – *rectilinear* and *curvilinear*.

10.2 RECTILINEAR MOTION

If a particle moves along a straight path, it is said to have undergone a rectilinear motion. Under this situation, its position will change with respect to time if measured from a reference point (origin).

10.2.1 Position or Displacement

The instantaneous position of the particle measured from a reference point is called its *displacement* with respect to the origin during a specified time interval.

Refer to Fig. 10.1. Say a particle P moves along a straight path to occupy a position at A during time interval t such that $OA = x$. This x is called its displacement in time t. The sign of x is used to symbolise whether it is moving along positive or negative x direction. If x is known for every value of t, it is said that the motion of the particle is known.

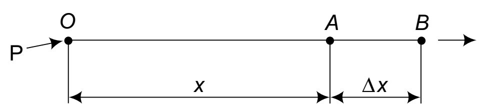


Figure 10.1

Mathematically, it can be expressed as $x = f(t)$. Essentially x is linear length unit and therefore in S.I. units it is expressed in metre. Since $x = f(t)$, a plot of x vs. t will exhibit a complete picture of variations of x with respect to t.

10.2.2 Velocity

The position of P at any instant of time t may be specified by its distance x measured from some convenient reference point O fixed on the line. Further, at time $t + \Delta t$, the particle has moved to B and its displacement becomes $x + \Delta x$. The change in the position during the interval Δt is called the **displacement** Δx of the particle. The average velocity of the particle is defined as the rate of change of displacement with respect to time.

Thus mathematically, average velocity $v = \dfrac{\Delta x}{\Delta t}$. As the time interval t approaches zero, the motion becomes uniform and above ratio assumes the velocity at any particular instance.

Thus the **instantaneous velocity** of the particle is defined as

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x}$$

In S.I. units the velocity is expressed in m/s. The sign of v becomes positive or negative depending on whether x is positive or negative.

10.2.3 Acceleration

Similar to the average velocity, the average acceleration of the particle is defined as the rate of change of velocity with respect to time. Let at the time interval t, the velocity be v and at time $t + \Delta t$, let the velocity be $v + \Delta v$.

Thus mathematically, average acceleration $a = \dfrac{\Delta v}{\Delta t}$.

The **instantaneous acceleration** of the particle is defined as

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 x}{dt^2} = \ddot{x}$$

In S.I. units the velocity is expressed in m/s^2.

The sign of a becomes positive or negative depending on whether x is positive or negative. Further, positive a implies that the velocity increases and negative a implies that the velocity decreases. When a is negative, it is popularly known as **deceleration**.

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$vdv = adx$$

The motion parameters can also be computed following vector approach.

If a particle moves along any arbitrary path in space such that its position vectors at two different locations are given by $\vec{r}_1 = x_1 i + y_1 j + z_1 k$ and $\vec{r}_2 = x_2 i + y_2 j + z_2 k$ as shown in Fig. 10.2, then the displacement during the time interval t is expressed as

$$\Delta \vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

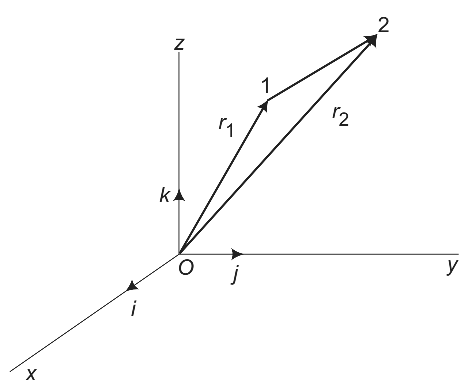


Figure 10.2

Thus average velocity of the particle is $v_{av} = \dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{\Delta x}{\Delta t}i + \dfrac{\Delta y}{\Delta t}j + \dfrac{\Delta z}{\Delta t}k$

The instantaneous velocity can be expressed as $v_{inst} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{d\vec{r}}{dt}$

$$= \lim_{\Delta t\to 0}\left[\frac{\Delta x}{\Delta t}i + \frac{\Delta y}{\Delta t}j + \frac{\Delta z}{\Delta t}k\right] = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k = \dot{x}i + \dot{y}j + \dot{z}k$$

Thus magnitude v of the velocity $\bar{v}$ is expressed as $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

Similarly average acceleration $a_{av} = \dfrac{\Delta\vec{v}}{\Delta t}$ and instantaneous acceleration

$$\vec{a} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{\Delta\dot{x}}{\Delta t}i + \frac{\Delta\dot{y}}{\Delta t}j + \frac{\Delta\dot{z}}{\Delta t}k = \ddot{x}i + \ddot{y}j + \ddot{z}k$$

Thus magnitude a of the acceleration $\vec{a}$ is expressed as $a = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$

10.3 GRAPHICAL REPRESENTATIONS OF POSITION, VELOCITY AND ACCELERATION

The motion of a particle is said to be known if the position x is known for every values of t. Thus, we can have $x = f(t)$.

This forms the basis for x vs. t plot. Considering t as abscissa and x as ordinate, the above curve can be plotted. This is what is called *displacement-time* diagram.

Figure 10.3 shows *displacement-time* diagram for the particle P. By definition $v = \lim\limits_{\Delta t\to 0}\dfrac{\Delta x}{\Delta t} = \dfrac{dx}{dt}$

Thus velocity at any instant is equivalent to the slope of the curve at that point.

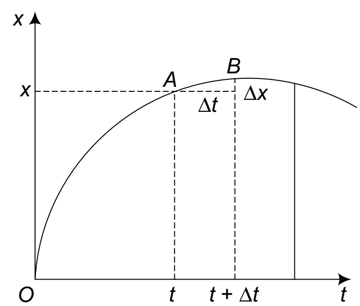


Figure 10.3

Similarly, if the above function is differentiated with respect to time, it gives velocity. Thus $x' = f'(t)$ and it is possible to plot v vs. t.

Figure 10.4 shows *velocity-time* diagram for the particle P. By definition,

$$a = \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Thus, acceleration of the particle at any instant is equivalent to the slope of the velocity-time curve at that point.

Further, the area of the small rectangular strip under the curve = vdt

Thus, the entire area under the curve can be obtained by $\displaystyle\int_{t_1}^{t_2} vdt$

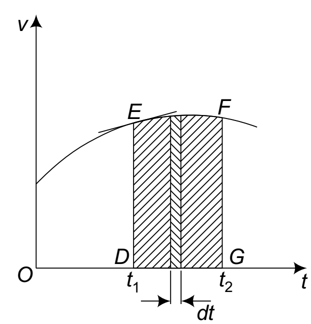


Figure 10.4

Further $$v = \frac{dx}{dt}$$

$$dx = vdt$$

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt$$

$$x_2 - x_1 = \int_{t_1}^{t_2} v\,dt$$

This can be interpreted as *the area under the "v–t" curve considering a finite time interval represents the displacement by the particle during the same time interval.*

Figure 10.5 shows *acceleration-time* diagram for the particle *P*. By definition,

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Further, the area of the small rectangular strip under the curve = adt

Thus the entire area under the curve can be obtained by $\int_{t_1}^{t_2} adt$

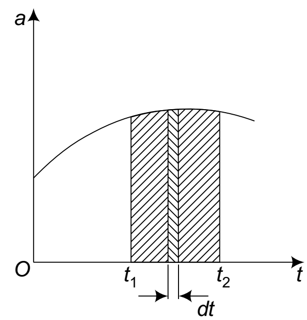


Figure 10.5

Further $$a = \frac{dv}{dt}$$

$$dv = adt$$

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} adt$$

$$v_2 - v_1 = \int_{t_1}^{t_2} adt$$

This can be interpreted as *the area under the "a–t" curve considering a finite time interval represents the change in velocity of the particle during the same time interval.*

There are situations when acceleration is expressed as function of time, displacement and velocity.

1. Acceleration is a given function of time.

Mathematically $a = f(t)$

By definition $$a = \frac{dv}{dt}$$

$$dv = adt = f(t)dt$$

$$\int dv = \int f(t)dt$$

Observing that when $t = 0$, v has an initial value of v_i and after the time t, v has final value of v_f

Thus integrating both sides within given limits

$$\int_{v_i}^{v_f} dv = \int_{o}^{t} f(t)dt$$

$$v_f - v_i = \int_{o}^{t} f(t)dt$$

Thus v is now expressed in terms of t, say $v = g(t)$.

Further
$$v = \frac{dx}{dt}$$

$$dx = vdt = g(t)dt$$

$$\int dx = \int g(t)dt$$

Observing that when $t = 0$ x has an initial value of x_i and after the time t, x has final value of x_f we have

$$\int_{x_i}^{x_f} dx = \int_o^t g(t)\ dt$$

$$x_f - x_i = \int_o^t g(t)dt$$

2. Acceleration is a given function of displacement.

Mathematically,
$$a = f(x)$$

$$vdv = adx = f(x)dx$$

$$\int_{v_0}^{v} vdv = \int_{x_0}^{x} f(x)dx$$

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \int_{x_0}^{x} f(x)dx$$

$$\int vdv = \int f(x)dx$$

$$v = \frac{dx}{dt}$$

$$dt = \frac{dx}{v}$$

3. Acceleration is a given function of velocity.

Mathematically,
$$a = f(v)$$

By definition,
$$\frac{dv}{dt} = f(v)$$

$$dt = \frac{dv}{f(v)}$$

Integrating both sides, $\int dt = \int \frac{dv}{f(v)}$. This is the relationship between v and t.

Further, $vdv = adx$

$$a = f(v) = \frac{vdv}{dx}$$

$$dx = \frac{vdv}{f(v)}$$

Integrating both sides, $\int dx = \int \frac{vdv}{f(v)}$.

This is the relationship between x and v.

10.4 UNIFORM RECTILINEAR MOTION

A motion is considered to be uniform if the velocity of the particle remains unchanged ($v = C$). This implies the acceleration of the particle is zero.

$$v = \frac{dx}{dt}$$

$$\int_{x_0}^{x} dx = v\int_{o}^{t} dt$$

$$x - x_0 = vt$$

$$x = x_0 + vt.$$

10.4.1 Uniform Accelerated Rectilinear Motion

If a particle moves with constant acceleration, its motion is considered to be uniform accelerated rectilinear motion.

$$a = \frac{dv}{dt}$$

$$\int_{v_0}^{v} dv = a\int_{o}^{t} dt$$

$$v - v_0 = at$$

$$v = v_0 + at$$

$$x - x_0 = vt \tag{10.1}$$

Since $v = \dfrac{dx}{dt}$

$$\frac{dx}{dt} = v_0 + at$$

$$\int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at)dt$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$

$$x = x_0 + v_0 t + \frac{1}{2} at^2 \qquad (10.2)$$

Again, $vdv = adx$

$$\int_{v_0}^{v} vdv = \int_{x_0}^{x} adx$$

$$\frac{1}{2}(v^2 - v_0^2) = a(x - x_0)$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad (10.3)$$

Equations (10.1), (10.2) and (10.3) are very useful relationships for rectilinear motion with uniform acceleration.

10.4.2 Non-uniform Accelerated Rectilinear Motion

When the acceleration of the particle is not uniform, the motion is said to be non-uniform accelerated rectilinear motion. Under such conditions, basic equations are used to relate displacement, velocity, acceleration and time.

10.5 RECTILINEAR MOTION UNDER GRAVITY

When a particle is allowed to fall freely, it is subjected to gravity force only. Under this condition, the acceleration of the particle would be $a = g$.

Thus if a particle is released from a height of h, the above equations can be written as

$$v = gt$$

$$h = \frac{1}{2} gt^2$$

$$v^2 = 2gh$$

Here, v represents velocity of the particle with which it touches the ground.

On the other hand, if a particle is thrown vertically upward with an initial velocity of v_0, the distance traveled can be computed by the relationship $v^2 = v_0^2 + 2a(x - x_0)$

$$0 = v_0^2 - 2gh$$

$$h = \frac{v_0^2}{2g}$$

Time required to attain a height h can be calculated from $v = v_0 + at$

$$0 = v_0 - gt$$

$$t = \frac{v_0}{g}$$

10.6 RELATIVE MOTION OF TWO PARTICLES

When more than one particle moves along the same line, their position, velocity and acceleration can be described independently with respect to the same reference frame.

However, it's quite useful to establish various motion attributes of one particle with respect to the other. This is what is called *relative motion*.

Let there be two particles namely P_1 and P_2 are moving along positive x direction so as to occupy a position A and B respectively form reference point O as shown in Fig. 10.6.

Let $OA = x_{P1}$ and $OB = x_{P2}$ which are the distances of P_1 and P_2 respectively from O.

Thus the distance between P_1 and $P_2 = x_{P1P2} = AB = OB - OA = x_{p2} - x_{p1}$

This is the relative distance between the two particles.

From the above relation, $x_{P2} = x_{P1} + x_{P1P2}$

This is the relative position of P_2 with respect to P_1.

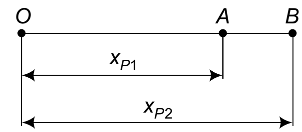


Figure 10.6

Similarly, relative velocity of P_2 with respect to P_1 can be expressed as $v_{P2} = v_{P1} + v_{P1P2}$ and relative acceleration of P_2 with respect to P_1 can be expressed as $a_{P2} = a_{P1} + a_{P1P2}$

10.7 DEPENDENT MOTIONS

There are ample instances when position of one particle is dependent on the other. Lifting/lowering of loads by means of wire ropes wrapped around several pulleys is one of the widely used means in industrial applications. Under such condition, velocity and the acceleration of one particle is dependent on the others.

Refer to Fig. 10.7. The load B is to be lifted by lowering the load A by means of two pulleys D and E. The entire system is hung from the ceiling as shown.

Let x_A = the distance of load A from the ceiling

x_B = the distance of load B from the ceiling

x_D and x_E are the centre distances of the two pulleys at D and E respectively from the ceiling.

h = distance between the load B from the center of the pulley at E

r = radius of the pulleys

Note that r, h and x_D are constant.

From the geometry of the above configuration, we have

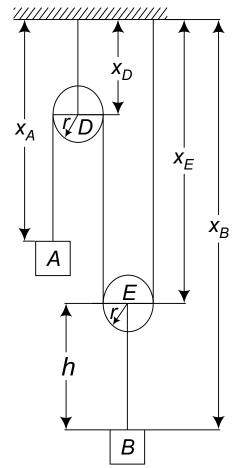


Figure 10.7

$$x_A - x_D + \pi r + x_E - x_D + \pi r + x_E = L$$

$$x_A + 2x_E = L - 2\pi r + 2x_D$$

$$x_A + 2(x_B - h) = L - 2\pi r + 2x_D$$

$$x_A + 2x_B = L - 2\pi r + 2x_D + 2h = \text{Constant}$$

From this expression, $\Delta x_A + 2\Delta x_B = 0$. This implies an infinitesimal small displacement (downward) Δx_A if given to the weight A, corresponding displacement of the weight B will be $\Delta x_B = \frac{-\Delta x_A}{2}$

Thus the displacement of the weight B will be half of that of A. Negative sign implies movement of the weight B is upward.

Differentiating both sides with respect to time t yields

$$\dot{x}_A + 2\dot{x}_B = 0$$

$$\dot{x}_B = \frac{-\dot{x}_A}{2}$$

Differentiating this equation with respect to time t yields

$$\ddot{x}_A + 2\ddot{x}_B = 0$$

$$\ddot{x}_B = \frac{-\ddot{x}_A}{2}$$

Thus the acceleration of the weight B is half of that of A and in the opposite direction.

Example 10.1 The motion of a particle is expressed as $x = x_0 + V_0 t + \frac{1}{2}at^2$. Calculate the displacement and velocity at time $t = 5s$. Given the initial displacement and velocity are 12 m and 5 m/s respectively. Acceleration of the particle is 20 m/s^2.

Solution
$$x = x_0 + V_0 t + \frac{1}{2}at^2$$
$$x_0 = 12 \text{ m}, \ v_0 = 5 \text{ m/s}, \ a = 20 \text{ m/s}^2$$
$$x = 12 + 5t + \frac{1}{2} \times 20t^2$$
$$x_{t=5} = 12 + 5 \times 5 + \frac{1}{2} \times 20 \times (5^2) = 12 + 25 + 250 \text{ m} = 287 \text{ m}$$
$$v_{t=5} = \left.\frac{dx}{dt}\right|_{t=5}$$
$$= V_0 + \left.\frac{1}{2}a \times 2t\right|_{t=5}$$
$$= 5 + 20 \times 5 = 105 \text{ m/s}$$

Example 10.2 The velocity of a particle is described as $\dot{x} = \frac{1}{2}at^2$ where $a = 10$ m/s^2. Calculate displacement of the particle when $t = 5$ s.

Solution
$$\dot{x} = \frac{1}{2}at^2$$
$$\frac{dx}{dt} = \frac{1}{2}at^2$$
$$dx = \frac{1}{2}at^2\,dt$$
$$x = \int dx = \int_0^5 \frac{1}{2}at^2 dt$$
$$= \frac{1}{2}a \times \frac{1}{3}[t^3]_0^5$$
$$= \frac{1}{2} \times 10 \times \frac{1}{3} \times 5^3 = 208.33 \text{ m}$$

Example 10.3 The motion of a particle is defined by the relation $x = t^4 - 3t^3 + 2t^2 - 8$ where x is in metre and t is in second. Determine the velocity and acceleration when $t = 5$ s.

Solution The equation of motion of the particle is $x = t^4 - 3t^3 + 2t^2 - 8$
$$\dot{x} = \frac{dx}{dt} = 4t^3 - 3 \times 3t^2 + 2 \times 2t$$

Thus $\dot{x}_{t=5} = 4 \times 5^3 - 3 \times 3 \times 5^2 + 2 \times 2 \times 5 = 295$ m/s

$$\dot{x} = \frac{dx}{dt} = 4t^3 - 3 \times 3t^2 + 2 \times 2t$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = 4 \times 3t^2 - 3 \times 3 \times 2t + 2 \times 2$$

$$\ddot{x}_{t=5} = 4 \times 3 \times 5^2 - 3 \times 3 \times 2 \times 5 + 2 \times 2 = 214 \text{ m/s}^2$$

Example 10.4 The acceleration of a particle is defined by the relation $a = t^2 - 2t + 2$, where a is in m/s^2 and t is in sec. The displacement and velocity of the particle at $t = 1$ s is found to be 14.75 m and 6.33 m/s. Find the distance traveled, velocity and acceleration when $t = 3$ s.

Solution The acceleration of the particle is $a = \dfrac{dv}{dt} = t^2 - 2t + 2$

$$\int dv = \int a dt = \int (t^2 - 2t + 2) dt$$

$$v = \frac{1}{3} t^3 - 2 \times \frac{1}{2} t^2 + 2t + c_1$$

When $t = 1$ sec; $v = 6.33$ m/s

Thus $$6.33 = \frac{1}{3} 1^3 - 2 \times \frac{1}{2} 1^2 + 2 \times 1 + c_1 \text{ or } c_1 = 5$$

Therefore $$v = \frac{dx}{dt} = \frac{1}{3} t^3 - 2 \times \frac{1}{2} t^2 + 2t + 5$$

$$x = \int dx = \int v dt = \int \frac{1}{3} t^3 dt - \int t^2 dt + 2\int t dt + 5\int dt$$

$$x = \frac{1}{3 \times 4} t^4 - \frac{1}{3} t^3 + 2 \times \frac{1}{2} t^2 + 5t + c_2$$

When $t = 1$ s; $x = 14.75$ m

Thus $$14.75 = \frac{1}{3 \times 4} 1^4 - \frac{1}{3} 1^3 + 2 \times \frac{1}{2} 1^2 + 5 \times 1 + c_2$$

$$c_2 = 9$$

Hence $$x = \frac{1}{12} t^4 - \frac{1}{3} t^3 + t^2 + 5t + 9$$

When $t = 3$ s;

$$x = \frac{1}{12} \times 3^4 - \frac{1}{3} \times 3^3 + 3^2 + 5 \times 3 + 9 = 30.75 \text{ m}$$

Similarly

$$v = \frac{1}{3} \times 3^3 - 3^2 + 2 \times 3 + 5 = 11 \text{ m/s}$$

$$a = 3^2 - 2 \times 3 + 2 = 5 \text{ m/s}^2$$

Example 10.5 A particle starts from rest and moves along a straight line with constant acceleration a. If the particle attains a velocity v = 10 m/s after a distance of 25 m, find the acceleration.

Solution

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

When $u = 0$,

$$v = at$$

$$s = \frac{1}{2}at^2$$

$$s = \frac{1}{2}at \times t = \frac{1}{2} \times v \times \frac{v}{a} = \frac{1}{2} \times \frac{v^2}{a}$$

$$25 = \frac{1}{2} \times \frac{10^2}{a}$$

$$a = 2 \text{ m/s}^2$$

Example 10.6 A particle is moving with constant acceleration a. It covers initial distance of 16 m in 10 seconds. What time it will take to cover entire distance of 400 m? What will be its final velocity? Assume that the particle has started from rest.

Solution

$$s = \frac{1}{2}at^2$$

$$s_1 = \frac{1}{2}at_1^2$$

$$a = \frac{2s_1}{t_1^2}$$

$$s_1 = 16 \text{ m}, \ t_1 = 10 \text{ s}$$

$$a = \frac{2 \times 16}{100} = 0.32 \text{ m/s}^2$$

$$s_2 = \frac{1}{2}at_2^2$$

$$t_2^2 = \frac{2s_2}{a}$$

$$= \frac{2 \times 400}{0.32}$$

$$t_2 = 50 \text{ s}$$

$$V = at_2 = 0.32 \times 50 = 16 \text{ m/s}$$

Example 10.7 A car starts its motion from rest with a maximum permissible acceleration a so as to attain its maximum velocity v and immediately after that retards with the same deceleration rate so as to come to a halt. Find the minimum time in which the car can move from one location to other if the distance between two is S.

Solution Between start and stop, the characteristics of movement will be as stated below.

(1) Acceleration from zero velocity to maximum velocity of v with acceleration a.

(2) Deceleration from velocity v to stoppage at a deceleration rate a.

(3) The distance covered by the train in (1) and (2) are x and y respectively and the corresponding times are t_1 and t_2 respectively.

$$x = \frac{1}{2}at_1^2$$

$$v^2 = 2ax$$

$$v = at_1$$

$$2x = vt_1 \qquad (10.4)$$

Again,

$$y = vt_2 - \frac{1}{2}at_2^2$$

$$v = at_2$$

$$2y = vt_2 \qquad (10.5)$$

Equations (10.4) and (10.5) give

$$2(x + y) = v(t_1 + t_2)$$

$$t_{\min} = \frac{S}{v} + \frac{S}{v}$$

$$v^2 = 2ax$$

$$v^2 = 2ay$$

$$v^2 + v^2 = 2a(x + y)$$

$$2v^2 = 2aS$$

$$\frac{v}{a} = \frac{S}{v}$$

$$t_{\min} = \frac{S}{v} + \frac{S}{v}$$

$$= \frac{S}{v} + \frac{v}{a}$$

Example 10.8 A rope AB is attached at B to a small block and passes over a small pulley C so that its free end A hangs 5 m above the ground as shown in Fig. 10.8. The end A is moved horizontally following a straight line with a uniform velocity v_O. Establish a relationship between velocity of the block with time.

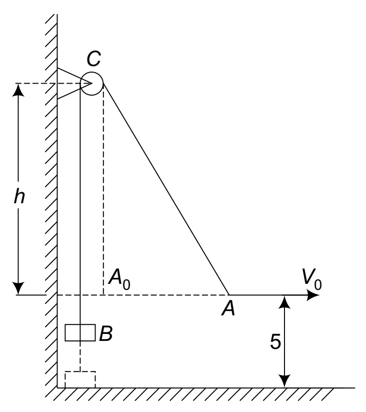


Figure 10.8

Solution Let the vertical distance traveled by the block B is x during the time interval t.

From ΔA_0CA we get $A_0C = h$ and $A_0A = v_0t$

$$\therefore \qquad AC = \sqrt{A_OC^2 + A_OA^2} = \sqrt{h^2 + v_O^2t^2}$$

Length of the rope before movement of the block = $L = h + 5 + \pi r + h = 2h + \pi r + 5$

Further, length of the rope during movement of the block to a position $B = AC + h + \pi r + 5 - x = h + 5 + \pi r - x + \sqrt{h^2 + v_O^2 t^2}$

$$2h + \pi r + 5 = h + 5 + \pi r - x + \sqrt{h^2 + v_O^2 t^2}$$

$$x = \sqrt{h^2 + v_O^2 t^2} - h$$

$v = \dfrac{dx}{dt} = \dfrac{1}{2} \times \dfrac{1}{\sqrt{h^2 + v_O^2 t^2}} \times v_O^2 \times 2t = \dfrac{v_O^2 t}{\sqrt{h^2 + v_O^2 t^2}}$ is the *velocity-time* relationship of the block.

Example 10.9 A particle falls freely from the top of a tower and during the last second of its motion, it falls 5/9 of the entire height. Find the height of the tower.

Solution Let the height of the tower be h and it takes time t second to cover it.

Thus $$h = \frac{1}{2} gt^2$$

Let h' be the distance the particle covers in $(t - 1)$ s.

$$h' = \frac{1}{2} g(t - 1)^2$$

Thus $h - h' = \dfrac{1}{2} gt^2 - \dfrac{1}{2} g(t - 1)^2$ is the distance covered by the particle in last second.

By problem statement, $h - h' = \dfrac{5}{9} h$

$$\frac{1}{2} g[t^2 - t^2 + 2t - 1] = \frac{5}{9} \times \frac{1}{2} gt^2$$

$$(5t - 3)(t - 3) = 0$$

$$t = \frac{3}{5} \text{ s}$$

or $$t = 3 \text{ s}$$

As per problem, $t > 1$ s; hence $t = \dfrac{3}{5}$ s is not acceptable.

Thus $$t = 3 \text{ s}$$

$\therefore$ Total distance covered by the particle = height of the tower = $h = \dfrac{1}{2} gt^2 = \dfrac{1}{2} \times 9.8 \times 3^2 \text{ m} = 44.1 \text{ m}$

Example 10.10 A car starts from rest to attain a speed v with a constant acceleration a_1; it maintains the same speed v for sometime and then comes to rest following a constant deceleration a_2. If the total distance covered is S, find the total time t required to cover this distance.

Solution Let S_1 be the distance during initial part when it moves with constant acceleration a_1;
S_2 be the distance during intermediate part when it moves with constant velocity v;
S_3 be the distance during last part when it moves with constant deceleration a_2 and t_1, t_2 and t_3 are the corresponding time.

$$S_1 = \frac{1}{2} a_1 t_1^2 \text{ and } v = a_1 t_1;$$

Thus $$t_1 = \frac{v}{a_1} \text{ and } S_1 = \frac{1}{2} v t_1$$

$$S_2 = vt_2 \text{ and } t_3 = \frac{v}{a_2} \text{ and } S_3 = \frac{1}{2} vt_3$$

Thus
$$S = S_1 + S_2 + S_3 = \frac{1}{2} vt_1 + vt_2 + \frac{1}{2} vt_3$$

$$t_1 + 2t_2 + t_3 = \frac{2S}{v}$$

$$(t_1 + t_2 + t_3) + t_2 = \frac{2S}{v} \tag{10.6}$$

Further,
$$t_1 + t_3 = \frac{v}{a_1} + \frac{v}{a_2} = v\left[\frac{1}{a_1} + \frac{1}{a_2}\right]$$

$$t_1 + t_2 + t_3 - t_2 = v\left[\frac{1}{a_1} + \frac{1}{a_2}\right]$$

$$t - t_2 = v\left[\frac{1}{a_1} + \frac{1}{a_2}\right] \tag{10.7}$$

Adding Eqs (10.6) and (10.7), we get $2t = \frac{2S}{v} + v\left[\frac{1}{a_1} + \frac{1}{a_2}\right]$

$$t = \frac{S}{v} + \frac{v}{2}\left[\frac{1}{a_1} + \frac{1}{a_2}\right]$$

Example 10.11 The acceleration of a particle at any point A is expressed by the relation $a = 200x(1 + kx^2)$, where a and x are expressed in m/s^2 and metres respectively and k is a constant. If the velocity of the particle at A is $v_A = 2.5$ m/s when $x = 0$ and $v_A = 5$ m/s when $x = 0.15$ m, find the value of k.

Solution Given $a = 200x(1 + kx^2)$

$$\frac{dv}{dt} = 200x(1 + kx^2)$$

$$\frac{dv}{dx}\frac{dx}{dt} = 200x(1 + kx^2)$$

$$v\frac{dv}{dx} = 200x(1 + kx^2)$$

$$vdv = 200x(1 + kx^2)dx$$

$$\Rightarrow \quad \int vdv = \int 200x(1 + kx^2)dx$$

Integrating both sides,

$$\frac{1}{2}v^2 = 200 \times \frac{x^2}{2} + 200k \times \frac{x^4}{4} + C_1$$

when $x = 0$; $v_A = 2.5$ m/s.

Thus $$C_1 = \frac{2.5^2}{2}$$

Further, when $x = 0.15$; $v_A = 5$ m/s

$$\therefore \qquad \frac{1}{2}5^2 = 200 \times \frac{(0.15)^2}{2} + 200k \times \frac{(0.15)^4}{4} + \frac{2.5^2}{2}$$

Solving, we get

$$k = 281 \text{ m}^{-2}$$

Example 10.12 Steep safety ramps are built to enable vehicles with defective brakes to stop safely. A truck enters such a ramp of 240 m at a high speed v_0 and travels 165 m in 6 s at constant deceleration so as to reduce its speed to $\frac{v_0}{2}$. Assuming the same uniform deceleration, determine (a) the additonal time required to stop the truck, and (b) the additional distance traveled by the truck.

Solution Let the deceleration of the truck is a m/s^2

$$\therefore \qquad \frac{v_0}{2} = v_0 - a \times 6$$

or, $$a = \frac{v_0}{12}$$

Following $x = v_0 t - \frac{1}{2}\ at^2$;

$$165 = v_0 \times 6 - \frac{1}{2} \times \frac{v_0}{12} \times 6^2$$

or, $$v_0 = \frac{165}{4.5}\text{m/s}$$

(a) Let the additional time required to stop the truck is t' s
Thus from the relation $v_f - v_o = at'$, we have

$$0 - \frac{v_0}{2} = \frac{v_0}{12}t'$$

or, $$t' = 6 \text{ s}$$

(b) Let the additional distance traveled by the truck is x' m.
It therefore follows from the relationship $v_f^2 - v_o^2 = 2ax'$

$$\frac{v_0^2}{4} = 2 \times \frac{v_0}{12} \times x'$$

or, $$x' = 1.5 v_0 = 1.5 \times \frac{165}{4.5} = 55 \text{ m}$$

Example 10.13 The position vector of a particle moving in the x-y plane at time $t = 4$ s is $5.05\boldsymbol{i} + 3.2\boldsymbol{j}$ m. At $t = 4.5$ s its position vector becomes $6.27\boldsymbol{i} + 4.7\boldsymbol{j}$ m. Determine the magnitude v of its average velocity during this interval and the angle θ made by v with x-axis.

Solution The position vector at $t = 4$ s is $\vec{r}_1 = 5.05\boldsymbol{i} + 3.2\boldsymbol{j}$ and the same at $t = 4.5$ s is $\vec{r}_2 = 6.27\boldsymbol{i} + 4.7\boldsymbol{j}$

Thus, $$\Delta r = \vec{r}_2 - \vec{r}_1 = (6.27\boldsymbol{i} + 4.7\boldsymbol{j}) - (5.05\boldsymbol{i} + 3.2\boldsymbol{j}) = 1.22\boldsymbol{i} + 1.5\boldsymbol{j}$$

$$\therefore \qquad v_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\boldsymbol{i} + \frac{\Delta y}{\Delta t}\boldsymbol{j} = \frac{1.22}{0.5}\boldsymbol{i} + \frac{1.5}{0.5}\boldsymbol{j} = 2.44\boldsymbol{i} + 3\boldsymbol{j} = 2.44\boldsymbol{i} + 3\boldsymbol{j}$$

Therefore the magnitude v of its average velocity during this interval becomes $\sqrt{2.44^2 + 3^2}$ m/s = 3.867 m/s and the angle θ made by v with x-axis is $\tan^{-1}\left(\frac{3}{2.44}\right) = 50.87°$

Example 10.14 A particle undergoing rectilinear translation along x-axis has an acceleration $a_x = -kv_x$, where k is a constant and v_x is the instantaneous velocity. Find the velocity and displacement at any time t, if the initial velocity and initial displacement are v_0 and 0 respectively.

Solution

It is given that
$$a_x = -kv_x$$
or
$$v_x \frac{dv_x}{dx} = -kv_x$$
or
$$dv_x = -kdx$$
Integrating the above equation, we get
$$v_x = -kx + C$$
where C is constant of integration. Applying the initial condition $x = 0$, $v_x = v_0$, we get $C = v_0$

Thus the instantaneous velocity becomes $v_x = -kx + v_0$

Further, one can write
$$v_x = \frac{dx}{dt}$$
or
$$\frac{dx}{v_x} = dt$$
or
$$\frac{dx}{v_0 - kx} = dt$$
Integrating the above equation, we obtain
$$-\frac{1}{k}\ln(v_0 - kx) = t + C_1$$
where C_1 is constant of integration. Applying the initial condition $x = 0$, $v_x = v_0$, we get $C_1 = -\frac{1}{k}\ln v_0$.

Therefore, one can write
$$-\frac{1}{k}\ln(v_0 - kx) = t - \frac{1}{k}\ln v_0$$
or
$$\ln\frac{v_0 - kx}{v_0} = -kt$$
or
$$x = \frac{v_0}{k}(1 - e^{-kt})$$
The velocity is then found to be
$$v_x = \frac{dx}{dt} = v_0\, e^{-kt}$$

Example 10.15 A particle travels along a straight line with a velocity $v_p = \frac{a}{b + x_p}$. Determine its acceleration when $x_p = 2$ m. Given that $a = 6$ m^2/s and $b = 3$ m.

Solution

It is given that
$$v_p = \frac{a}{b + x_p}$$
Differentiating the above expression with respect to x_p, we obtain
$$\frac{dv_p}{dx_p} = \frac{-a}{(b + x_p)^2}$$

The acceleration of the particle becomes

$$v_p \frac{dv_p}{dx_p} = \frac{-a}{(b+x_p)^2} \frac{a}{b+x_p} = \frac{-a^2}{(b+x_p)}$$

For the given condition, the acceleration becomes

$$= \frac{-6^2}{(3+2)^3} = -0.288 \text{ m/s}^2$$

Example 10.16 The velocity of a particle that moves along x-axis is expressed as $v = 2 + 5t^{\frac{3}{2}}$, where t is in s and v is in m/s. Determine the displacement x, the velocity v, and acceleration a when $t = 4$ s. The particle is at the origin when $t = 0$

Solution It is given that

$$v = 2 + 5t^{\frac{3}{2}}$$

or

$$\frac{dx}{dt} = 2 + 5t^{\frac{3}{2}}$$

or

$$dx = \left(2 + 5t^{\frac{3}{2}}\right) dt$$

Integrating the above equation, we obtain

$$x = 2t + 5 \times \frac{3}{2} t^{\frac{5}{2}} + C$$

where C is constant of integration. Applying the condition $x = 0$, when $t = 0$, we get $C = 0$

Thus, the above equation becomes $x = 2t + 2t^{\frac{5}{2}}$

When $t = 4$ s, the displacement becomes $x = 2 \times 4 + 2 \times 4^{\frac{5}{2}} = 72$ m.

Velocity at $t = 4$ s becomes $v = 2 + 5 \times 4^{\frac{3}{2}} = 42$ m/s

Acceleration can be expressed as

$$a = \frac{dv}{dt} = 5 \times \frac{3}{2} \times t^{\frac{1}{2}}$$

Acceleration at $t = 4$ s becomes $a = 5 \times \frac{3}{2} \times t^{\frac{1}{2}} = 15$ m/s^2

Example 10.17 The acceleration a of a particle that moves in $+x$ direction varies with its position as shown in Fig. 10.9. If the velocity of the particle is 0.8 m/s when $x = 0$, determine the velocity v, when $x = 1.4$ m.

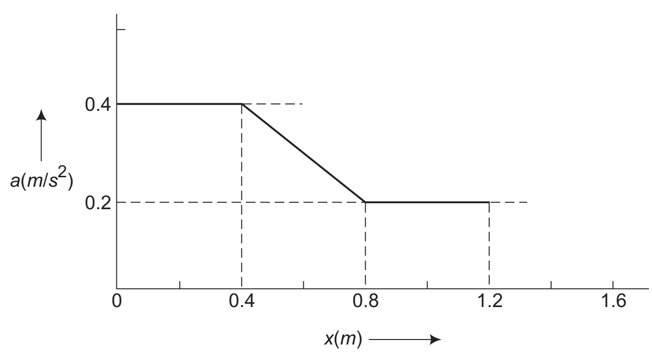


Figure 10.9

Solution

(i) The acceleration remains constant at $a = 0.4$ m/s^2 for $0 < x < 0.4$.

We know that $$a = v\frac{dv}{dx}$$

or $$vdv = adx$$

Integrating both sides, we get

$$\int vdv = \int adx = a\int dx$$

When $x = 0$; $v = 0.8$ m/s

Let the velocity is v_1 when $x = 0.4$ m

Hence above integration is written as $\int_{0.8}^{v_1} vdv = a\int_{0}^{0.4} dx$, from which $v_1 = 0.98$ m/s

(ii) The acceleration follows straight line for $0.4 < x < 0.8$

It is now essential to develop the acceleration equation considering $a = f(x)$

From the graph, it is found that when $x = 0.4$; $a = 0.4$ and when $x = 0.8$; $a = 0.2$

Let the general equation of the curve is in the form $a = mx + k$

Thus $0.4 = m \times 0.4 + k$ and $0.2 = m \times 0.8 + k$

Solving these two equations; the particular equation becomes $a = -0.5x + 0.6$

Let the velocity is v_2 when $x = 0.8$ m

Then $\int_{0.98}^{v_2} vdv = \int_{0.4}^{0.8} (-0.5x + 0.6)dx$ which yields $v_2 = 1.0956$ m/s

(iii) During the period $0.8 < x < 1.2$ acceleration once again remains constant at $a = 0.2$ m/s^2

If the velocity of the particle is v_3 when $x = 1.2$ m, then from the relationship

$v_3^2 = v_2^2 + 2ax$; $v_3 = 1.166$ m/s

Since $a = 0$ and when $x > 1.2$; the velocity of the particle remains constant at $v_3 = 1.166$ m/s.

This implies velocity $v = v_3 = 1.166$ m/s, when $x = 1.4$ m.

Example 10.18 The acceleration a of a particle following rectilinear translation is defined as $a = -k\sqrt{v}$, where k is a constant. Knowing that $x = 0$ and $v = 25$ m/s at $t = 0$, and that $v = 12$ m/s when $x = 6$ m, determine (a) the velocity of the particle when $x = 8$ m and (b) the time required for the particle to come to rest.

Solution It is given that $$a = -k\sqrt{v} = -kv^{\frac{1}{2}}$$

or $$v\frac{dv}{dx} = -kv^{\frac{1}{2}}$$

or $$v^{\frac{1}{2}}dv = -kdx$$

Integrating the above equation, we obtain

$$\frac{2}{3}v^{\frac{3}{2}} = -kx + C$$

where C is constant of integration. Applying the condition $v = 25$ m/s, when $x = 0$, we get $C = 83.33$.

From the above equation, we obtain

$$\frac{2}{3}v^{\frac{3}{2}} = -kx + 83.33$$

Further, when $v = 12$ m/s; $x = 6$ m.

Thus, $$\frac{2}{3}(12)^{\frac{3}{2}} = -k \times 6 + 83.33$$

or $$k = 9.27$$

Finally, one can write

$$\frac{2}{3}v^{\frac{3}{2}} = -9.27x + 83.33$$

(a) The velocity of the particle when $x = 8$ m is found to be

$$\frac{2}{3}(v_{x=8})^{\frac{3}{2}} = -9.27 \times 8 + 83.33$$

or $$v_{x=8} = 5.746 \text{ m/s}$$

(b) Again, $$a = -k\sqrt{v} = -kv^{\frac{1}{2}}$$

or $$\frac{dv}{dt} = -kv^{\frac{1}{2}}$$

or $$v^{-\frac{1}{2}}\, dv = -kdt$$

Integrating the above equation, we obtain

$$2v^{\frac{1}{2}} = -kt + C_1$$

When $t = 0$, $v = 25$ m/s; from which $C_1 = 10$

Thus, $$2v^{\frac{1}{2}} = -9.27t + 10$$

The time required for the particle to come to rest is $t_{v=0} = \dfrac{10}{9.27} = 1.078$ s

NUMERICAL PROBLEMS

10.1 A particle has straight line motion according to the equation $x = t^3 - 3t^2 - 5$, where x is in metre and t is in second. Find the change in position when its velocity changes from 8 m/s to 40 m/s.

10.2 A particle is undergoing a rectilinear motion such that its displacement from a fixed origin can be expressed by $x = 3t^2 + 2t$, where x is in metre and t is in second. Find the displacement, velocity and acceleration at the end of 4 seconds.

10.3 A train **"*A*"** starts from rest from a point O and travels along a straight line with an acceleration of 2 m/s^2. Another train **"*B*"** starts from rest from same point O, but 4 sec later and moves with an acceleration of 3 m/s^2. At what distance from O will the train B overtake the train A?

10.4 The acceleration of a particle is given by the equation $a = \dfrac{10}{v+1}$, where a is expressed in m/s^2 and v in m/s. The particle starts from rest at $x = 0$. What is the position of the particle when $v = 10$ m/s?

10.5 The position of a particle describing rectilinear motion can be described by $x = t^3 - 9t^2 + 15t + 18$, where x is expressed in metre and t is in second. Determine the time, displacement, and acceleration of the particle when its velocity is zero.

10.6 A car moves along a straight line with a constant acceleration 2 m/s^2. How long will it take to change its speed from 6 m/s to 9 m/s? What will be the displacement during this period?

10.7 A train starts from rest from station *O*; it gains speed at the rate of 1 m/s^2 for 5 seconds and then at the rate of 1.5 m/s^2 until it reaches the speed of 10 m/s. The train maintains the same speed until it reaches the station *A* when brakes are applied so that the train has a constant deceleration and comes to rest in 5 seconds. The total duration of travel between the stations *O* and *A* is 40 seconds. Find the distance between the two stations.

10.8 A particle is dropped from the top of a tower *h* metres high and at the same time another particle is thrown upwards from the ground. These two particles meet when the 1st particle has covered $\frac{1}{n}$ times of the total height *h*. Prove that the velocities when they meet are in the ratio 2: (n – 2) and that the initial velocity of projection of the second particle is $\sqrt{\frac{ngh}{2}}$.

10.9 Water drips from a faucet at a uniform rate of *n* drops per second. Find the distance *x* between any two adjacent drops as a function of the time *t* that the trailing drop has been in motion.

10.10 Refer to Fig. 10.10. Determine the velocity and acceleration of the block **3** at the instant considered. Given $\dot{x}_1$ = 4.0 m/s; $\ddot{x}_1$ = 1.5 m/s^2 and $\dot{x}_2$ = 2.5 m/s; $\ddot{x}_2$ = 2.0 m/s^2

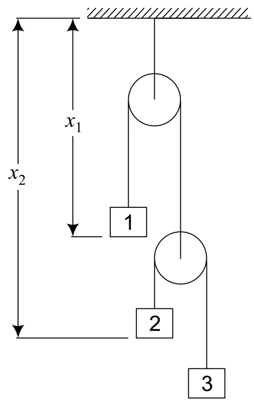


Figure 10.10

10.11 A train starts at station *A* so as to reach another station *B* located along a straight line. The train accelerates in such a way that its velocity increases uniformly from 0 to 20 m/s and then decreases uniformly to 0 at *B*. If the total time taken by the train to cover *AB* is 5 min, draw the v – t diagram and hence determine the distance between the two stations.

10.12 A train accelerates from rest with constant acceleration of a_1 to acquire a maximum velocity of v_{max} and immediately starts decelerating with constant deceleration of a_2 so as to come to rest. If the total duration of the travel is *T*, prove that $v_{max} = \frac{a_1 a_2}{a_1 + a_2} T$.

10.13 The acceleration a of a particle is described as $a = 40 - 160x$, where a and x are expressed in m/s2 and in m respectively. If the velocity of the particle is 0.3 m/s, when x = 0.4 m, determine (a) the maximum velocity of the particle and (b) the positions at which the velocity is zero.

10.14 While traveling a distance of 4 km between points *A* and *D*, a car is driven at 100 km/hr from *A* to *B* for *t s* and at 60 km/hr from *C* to *D* also for *t s*. If the brakes are applied for 5 s between *B* and *C* to ensure uniform deceleration calculate *t* and the distance *s* between *A* and *B*.

10.15 The velocity v of a particle with respect to time t is as shown in Fig. 10.11. Draw the $a - t$ and $x - t$ graphs for the particle during the period $0 < t < 40$, if $x = -14.6$ m at $t = 0$. Also determine (a) the maximum value of its position, (b) the values of t for which $x = 32.6$ m, (c) the total distance traveled by the particle during the period $t = 0$ to $t = 30$ and (d) the two values of t for which the particle passes through the origin.

Note: *Solve (c) by **method of integration** and validate the result by **area** method.*

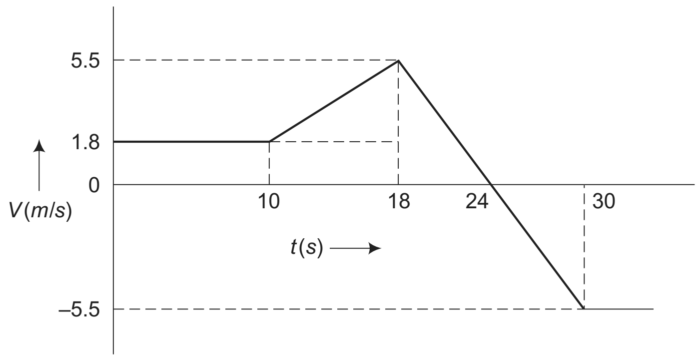


Figure 10.11

10.16 The velocity of a particle having motion in the x-y plane at time $t = 3.5$ s is $4.12\boldsymbol{i} + 3.17\boldsymbol{j}$ m/s. The average acceleration during next 0.05 s becomes $2\boldsymbol{i} + 2.5\boldsymbol{j}$ m/s^2. Determine the velocity at $t = 3.55$ s and the angle a made by average acceleration and the velocity vector at $t = 3.55$ s.

ANSWERS TO NUMERICAL PROBLEMS

10.1 41.6 m

10.2 $x = 56$ m, $\dot{x} = 26$ m/s, $\ddot{x} = 6$ m/s^2

10.3 475 m

10.4 38.33 m

10.5 $t = 1$ s, $x = 25$ cm, $\ddot{x} = -12$ m/s^2 and $t = 5$ s, $x = -7$ m, $\ddot{x} = 12$ m/s^2

10.6 $t = 1.5$ s; $x = 11.25$ m

10.7 $t = 11.6$ s; $x = 50.46$ m (measured from 'O')

10.9 $x = \dfrac{gt}{n} + \dfrac{g}{2n^2}$

10.10 $\dot{x}_3 = 10.5$ m/s $\uparrow$; $\ddot{x}_3 = 5$ m/s^2 $\uparrow$

10.11 AB = 3000 km

10.13 (a) $V_{max} = 1.921$ m/s, (b) 0.09812 m, 0.402 m

10.14 $t = 87.5$ s, $s = 2.43$ km

10.15 (a) 49.1m, (b) 18 s and 30 s, (c) 8.1 s and 36 s

10.16 $v = 0.412\boldsymbol{i} + 0.396\boldsymbol{j}$ m/s; $\alpha = 7.12°$]

CHAPTER

11

Kinetics of a Particle in Rectilinear Motion

11.1 INTRODUCTION

In chapters 1 and 10, it is explained that kinetics is the branch of dynamics that deals with motion attributes along with force that causes motion. It is the force that causes motion of the particle with acceleration and deceleration. The motion parameters and associated force analysis are based on the **Newton's** well known *laws of motion*.

11.2 NEWTON'S LAWS OF MOTIONS

11.2.1 Newton's First Law of Motion

A particle maintains its state of rest or state of motion along a straight line with a constant velocity unless and until it is acted upon by a force.

This implies that it is the inherent nature of a particle to maintain its status quo in regard to its state of rest or motion. Thus, if a particle moves with constant velocity, it will never stop unless some force opposes its motion so as to stop it. If a ball is thrown with an initial velocity over a floor, we find that after moving through a certain distance it comes to rest. This is owing to the fact that frictional resistance between the ball and the floor becomes instrumental to stop the motion of the ball. Thus, it is a resistive force that brings down the velocity of the ball. Similarly, when the speed of a car increases it is due to the tractive force developed by the engine. Further, a particle if at rest will not start its motion from its own. It is therefore can be concluded that without any external agency in the form of a force, change of status is not possible. This characteristic of continuation of original status of rest or motion is called the ***inertia*** of the particle.

11.2.2 Newton's Second Law of Motion

The acceleration of a particle is proportional to the net force acting on it and its direction is in the direction of this force.

Thus mathematically

$$\sum F = ma \tag{11.1}$$

Here F is the force that causes an acceleration of a to the body of mass m.

If the force $\boldsymbol{F}$ is resolved into two mutually perpendicular components, namely, F_x and F_y such that $\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j}$ and acceleration a has also got the same components a_x and a_y, then

$$\sum F_x = ma_x \tag{11.2}$$

and

$$\sum F_y = ma_y \tag{11.3}$$

Closer look at the second law reveals that when $a = 0$, $F = 0$. This implies that in the absence of resultant force, the particle cannot have any acceleration. Further, if it is already under rest, it will remain stationary. This is in congruence with the Newton's first law.

11.2.3 Newton's Third Law of Motion

For every action, there is always an equal and opposite reaction.

This statement does not require any further elaboration since from the very beginning of statics, we have considered the concept of equal and opposite reactive forces in order to restore equilibrium. This is in compliance with the equilibrium of bodies under the actions of two forces. However, this concept of equal and opposite forces is not only confined to the bodies under rest, rather it is equally applicable to the bodies under motion.

Example 11.1 A weight W = 100 N is lifted by means of two pulleys as shown in Fig. 11.1. If the free end of the rope is pulled down vertically with constant acceleration $a = 5$ m/s^2, find the tension in the rope. Neglect friction in the pulleys.

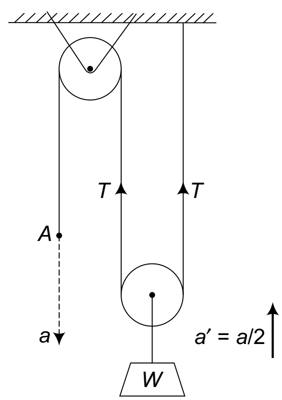


Figure 11.1

Solution Let the tension in the string be T.

Since the free end of the rope is pulled down with an acceleration a, the acceleration of the weight will be $a' = -\dfrac{a}{2}$ [Negative sign implies W is moving opposite to the direction of applied load at the free end of the rope.]

Considering the free body of the weight and considering Newton's second law of motion,

$$\sum F = ma$$

$$\Rightarrow 2T - W = \frac{W}{g}a' = \frac{W}{g}\frac{a}{2}$$

$$T = \frac{W}{2}\left[1 + \frac{a}{2g}\right] = \frac{100}{2}\left[1 + \frac{5}{2 \times 9.81}\right] \text{ N} = 62.74 \text{ N}$$

Thus, the tension in the rope is 62.74 N.

Example 11.2 The driver of a train that moves along a straight line suddenly applies brake so as to stop the train in 3 seconds. The train covers a distance of 9.81 m before it comes to rest. Assuming the train has got a constant deceleration, find the co-efficient of friction between the wheel and the track.

Solution $t = 3$ s; $x = 9.81$ m; $v_f = 0$

Let the velocity of the train at the time of braking be v_0 and the deceleration rate is a.

Then considering linear motion, we have

$$v_f = v_0 - at$$

$$0 = v_0 - 3a$$

$$v_0 = 3a \tag{11.4}$$

and

$$v_f^2 = v_0^2 - 2ax$$

$$0 = v_0^2 - 2 \times a \times 9.81$$

$$v_0^2 = 2 \times 9.81 \times a \tag{11.5}$$

Comparing Eqs (11.4) and (11.5),

$$9 \times a^2 = 2 \times 9.81 \times a$$

$$a = \frac{2 \times 9.81}{9} \text{ m/s}^2$$

The force that causes deceleration of the train is friction.

Thus $$\sum F = ma = F_f = \mu N = \mu.mg$$

$$\mu = \frac{a}{g} = \frac{2 \times 9.81}{9 \times 9.81} = 0.22$$

Thus the co-efficient of friction between the wheel and the track is 0.22.

Example 11.3 Weights W and $2W$ are supported by a string in a vertical plane as shown in Fig. 11.2. Find the magnitude of an additional weight Q applied over W that will give a downward acceleration of $a = 0.1g$. Find the tension in the rope. Neglect friction in the pulleys.

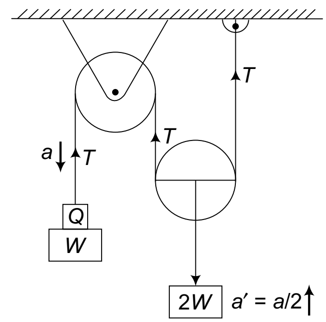


Figure 11.2

Solution Let the tension in the string be T.

The situation being similar to that of the problem 11.1. If the combined W and Q move down with an acceleration a, the acceleration of the weight $2W$ will be $a' = -\dfrac{a}{2}$.

Applying Newton's second law of motion,

$$\sum F = ma$$

$$(W + Q) - T = \frac{W + Q}{g} a$$

$$T = (W + Q)\left[1 - \frac{a}{g}\right] \tag{11.6}$$

Similar consideration of the weight $2W$ yields

$$2T - 2W = \frac{2W}{g}.a' = \frac{2W}{g}\frac{a}{2}$$

$$T = W\left[1 + \frac{a}{2g}\right] \tag{11.7}$$

Comparing Eqs (11.6) and (11.7) and replacing $a = 0.1g$

$$(W + Q)\left[1 - \frac{0.1g}{g}\right] = W\left[1 + \frac{0.1g}{2g}\right]$$

$$(W + Q) \times 0.9 = W \times 1.05$$

$$Q = \frac{0.15}{0.90} W = \frac{W}{6}$$

Thus, weight Q becomes $\dfrac{W}{6}$

Example 11.4 Two blocks of masses m_1 = 15 kg and m_2 = 10 kg are connected by a flexible string as shown in Fig. 11.3. Assuming the coefficient of friction between block of mass m_2 and the horizontal surface on which it rests to be μ = 0.25, find the acceleration of the system of masses and tension in the string. Neglect friction in the pulley.

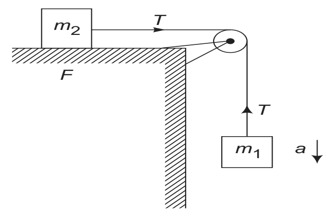


Figure 11.3

Solution Let the block 1 move downwards at an acceleration of a m/s^2 and let the tension in the string be T.

Since both the blocks are connected by a common string, the block 2 that rests on the floor will also have same acceleration as that of block 1.

Considering the free body of the block 1 and applying Newton's second law of motion,

$$\sum F = ma$$

$$m_1 g - T = m_1 a$$

$$T = m_1(g - a) \tag{11.8}$$

For block 2, $T - F_f = m_2 a$

$$T - \mu m_2 g = m_2 a$$

$$T = m_2(a + \mu g) \tag{11.9}$$

Eliminating T from Eqs (11.8) and (11.9), we have

$$m_1(g - a) = m_2(a + \mu g)$$

$$a = \frac{[m_1 - \mu m_2]g}{m_1 + m_2} = \frac{[15 - 0.25 \times 10]}{15 + 10} \times 9.81 \text{ m/s}^2 = 4.905 \text{ m/s}^2$$

From Eq. (11.8), we have $T = m_1(g - a)$ = 15(9.81 – 4.905) N = 73.575 N

Example 11.5 A small block of mass m rests on an inclined plane as shown in Fig. 11.4. Sliding of the block impends when the inclination angle θ = 30°. If the inclination angle is increased to 45°, what would have been the acceleration of the block? Assume $\mu_s = \mu_k$.

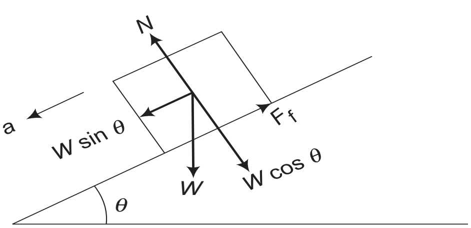


Figure 11.4

Solution From the free body diagram of the block and considering static equilibrium,

We have $N = W \cos\theta$ and

$$F_f = W \sin\theta = \mu N = \mu W \cos\theta$$

$$\mu = \tan\theta = \tan 30°$$

When θ is increased to θ' = 45°; there will be motion and the block is under dynamic equilibrium, implying

$$W \sin\theta' - F_f' = \frac{W}{g} a$$

$$W \sin\theta' - \mu N' = \frac{W}{g} a$$

$$W \sin\theta' - \mu W \cos\theta' = \frac{W}{g} a$$

$$a = g[\sin\theta' - \mu\cos\theta']$$

$$a = g[\sin 45° - \tan 30° \cos 45°] = 0.3g$$

Example 11.6 A small block of mass m rests on an inclined plane, starts from point A and slides down the plane as shown in Fig. 11.5. What distance along the horizontal plane BC will it travel before it comes to rest? The co-efficient of friction between the block and both the planes is $\mu = 0.3$. Assume there is no loss of velocity while changing its motion from inclined plane to that of horizontal plane.

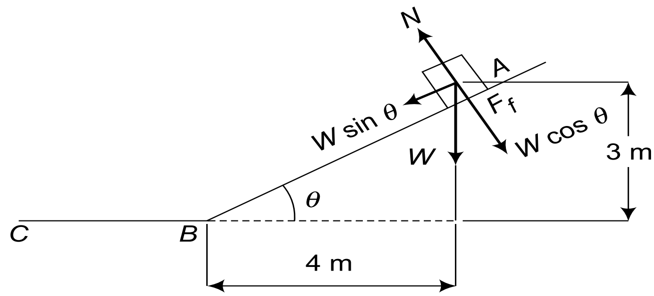


Figure 11.5

Solution $AB = \sqrt{3^2 + 4^2} = 5$ m

Let the inclination of the plane with the horizontal be θ;

Thus $\cos\theta = \dfrac{4}{5}$ and $\sin\theta = \dfrac{3}{5}$

From the free body diagram of the block and considering dynamic equilibrium,

$$W\sin\theta - \mu W\cos\theta = \frac{W}{g}a$$

$$a = g[\sin\theta - \mu\cos\theta] \quad [a = \text{acceleration of the block along the inclined plane}]$$

Since the block starts from rest;

$$v_f^2 = 2as = 2 \times g[\sin\theta - \mu\cos\theta] \times 5 = 2 \times 9.81 \times \left[\frac{3}{5} - \frac{3}{10} \times \frac{4}{5}\right] \times 5$$

$$v_f = 5.943 \text{ m/s}$$

While moving along the plane BC, the block will come to rest due to retarding force friction. If a' is the corresponding deceleration,

$$F = ma' = F_f = \mu mg$$

$$a' = \mu g = 0.3 \times 9.81 \text{ m/s}^2 = 2.943 \text{ m/s}^2$$

Let the distance covered by the block along horizontal plane BC be x.

$$0 = v_f^2 - 2a'x$$

$$x = \frac{v_f^2}{2a'} = \frac{5.943 \times 5.943}{2 \times 2.943} \text{ m}$$

$$x = 6 \text{ m}$$

Example 11.7 Two small cars of weights W_1 = 200 N and W_2 = 100 N are connected by a flexible but inextensible string wrapped around a pulley C and are free to roll on an inclined plane. If the cars are released from rest having the position as shown in Fig. 11.6, find the time required for them so as to exchange their positions. Assume entire system is frictionless.

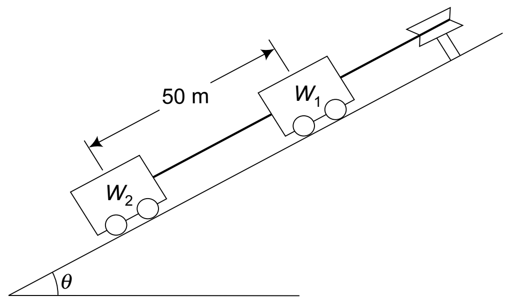


Figure 11.6

Solution For car 1, $W_1 \sin 30° - T = \frac{W_1}{g}a$

$$T = W_1\left[\sin 30° - \frac{a}{g}\right] = \frac{W_1}{2}\left[1 - \frac{2a}{g}\right] = 100\left[1 - \frac{2a}{g}\right] \tag{11.10}$$

For car 2, $T - W_2 \sin 30° = \frac{W_2}{g}a$

$$T = W_2\left[\sin 30° + \frac{a}{g}\right] = W_2\left[\frac{1}{2} + \frac{a}{g}\right] = 100\left[\frac{1}{2} + \frac{a}{g}\right] \tag{11.11}$$

From Eqs (11.10) and (11.11),

$$100\left[1 - \frac{2a}{g}\right] = 100\left[\frac{1}{2} + \frac{a}{g}\right]$$

$$\frac{3a}{g} = \frac{1}{2}$$

$$a = \frac{g}{6}$$

$$x = \frac{1}{2}at^2$$

$$50 = \frac{1}{2} \times \frac{9.81}{6}t^2$$

$$t = \sqrt{\frac{600}{9.81}} \text{ sec} = 7.82 \text{ s}$$

11.3 EQUATIONS OF DYNAMIC EQUILIBRIUM: D'ALEMBERT'S PRINCIPLE

From Newton's Second Law,

$$\sum F = ma$$

$$\sum F - ma = 0$$

$$\sum F + (-ma) = 0 \tag{11.12}$$

The above equation can be interpreted as if it is considered that $\sum F$ is the net force acting along the direction of motion and $-ma$ is the force that acts opposite to the motion so that their combined effect will restore equilibrium, i.e., no unbalance force is acting on the body. This can, however, be envisaged that equation of dynamic equilibrium is tantamount to the equation of static equilibrium. The product of mass and acceleration with a negative sign is called inertia force to assume to act so as to oppose the motion.

The above equation is treated as *equation of dynamic equilibrium of the particle*.

To obtain dynamic equilibrium of a particle, *a fictitious force called inertia force is added opposite to the direction of motion so that resultant force on the particle becomes zero. This concept is known as D' Alembert's Principle and is a very useful approach in the solution of problems in kinetics.*

Example 11.8 A block of weight W, height $2h$ and width $2c$ rests on a flat trailer car that moves horizontally with a constant acceleration a as shown in Fig. 11.7. Determine

(a) the acceleration at which slipping of the block will impend if the coefficient of friction between the block and the car is μ.

(b) the value of the acceleration at which tipping of the block about the rear edge of the block will impend, assuming sufficient friction to prevent slipping.

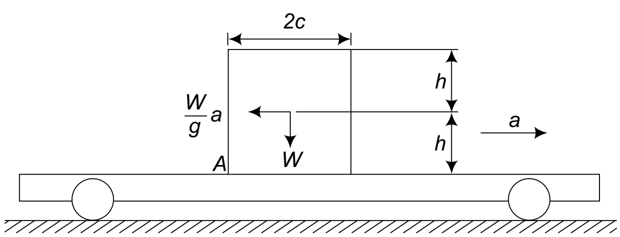


Figure 11.7

Solution Let the acceleration of the car be a_1.

(a) From the free body of the block, it is clear that the motion of the car is governed by friction force F_f and the inertia force ma_1.

Under the condition of impending motion; Friction force = Inertia Force

$$F_f = ma_1$$

$$\mu N = ma_1$$

$$\mu mg = ma_1$$

$$a_1 = \mu g$$

Thus the acceleration at which slipping of the block will impend is $a_1 = \mu g$

(b) When the tipping of the block about the rear edge will impend, there will be no surface contact between the block and the floor of the car; rather the contact is along the edge A.

Thus with respect to A, the inertia force will try to topple the block which will be counteracted by the weight.
From the mechanics point of view, the CCW moment produced by the inertia force should be equal to the CW moment produced by the body force.

$$\sum M_A = 0$$

$$W \times c = \frac{W}{g} a_2 \times h$$

$$a_2 = \frac{cg}{h}.$$

The value of the acceleration at which tipping of the block about the rear edge of the block will impend is $a_2 = \dfrac{cg}{h}$.

Example 11.9 Neglecting friction and inertia of the two step pulley as shown in Fig. 11.8, find the acceleration of the falling weight P. Assume P = 20 N, Q = 30 N and $r_1 = 2r_2$.

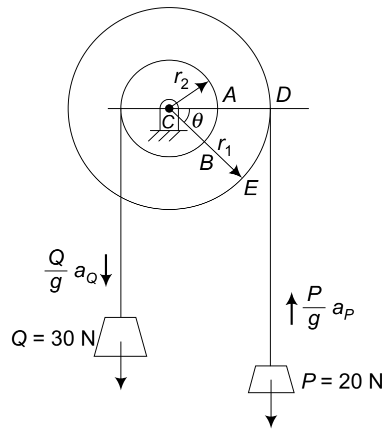


Figure 11.8

Solution Let the acceleration of the weight P on pulley of radius $r_1 = a_p$ and the acceleration of the weight Q on pulley of radius $r_2 = a_Q$.

Let during any time interval t the rotation of the pulleys be θ.

$$\therefore \qquad \theta = \frac{l_1}{r_1} = \frac{l_2}{r_2}$$

$$l_1 r_2 = l_2 r_1$$

Differentiating both sides twice with respect to t yields

$$\frac{d^2}{dt^2}(l_1 r_2) = \frac{d^2}{dt^2}(l_2 r_1)$$

$$a_p r_2 = a_Q r_1$$

$$a_Q = \frac{a_P r_2}{r_1} = \frac{a_P}{2} \quad [\text{since } r_1 = 2r_2]$$

Considering the dynamic equilibrium of the weights, moment about C yields

$$\sum M_C = 0$$

$$\left[W_Q + \frac{W_Q}{g}.a_Q\right] r_2 = \left[W_P - \frac{W_P}{g}.a_P\right] r_1$$

$$W_Q\left[1 + \frac{a_P}{2g}\right] r_2 = W_P\left[1 - \frac{a_P}{g}\right] 2r_2$$

$$30 \times \left[1 + \frac{a_P}{2g}\right] = 20 \times 2\left[1 - \frac{a_P}{g}\right]$$

$$\frac{3a_P}{2g} + \frac{4a_P}{g} = 4 - 3 = 1$$

$$a_P = \frac{2}{11} g$$

$\therefore$ Acceleration of the falling weight $P = \dfrac{2}{11} g$.

Example 11.10 Refer to Fig. 11.9. Find the maximum permissible acceleration that the car can develop without tipping over backward.

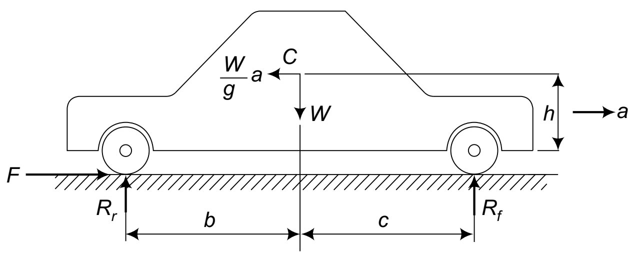


Figure 11.9

Solution When the backward tipping of the car takes place, there would not be any contact between the front wheel and the road.

Following the similar situations as that of problem 11.8.

Taking moment about the point of contact between the rear wheel and the road

$$W \times b = \frac{W}{g} a \times h$$

$$a = \frac{bg}{h}$$

Thus the maximum permissible acceleration that the car can develop with out tipping over backward is $a = \frac{bg}{h}$.

Example 11.11 Two blocks A and B weighing W_A = 45 N and W_B = 90 N respectively are placed side by side on an inclined plane having inclination angle θ = 30° as shown in Fig. 11.10, so that they can slide together. If the coefficient of friction between the blocks and the plane are μ_A = 0.15 and μ_B = 0.30 respectively, find the contact thrust existing between the blocks under motion.

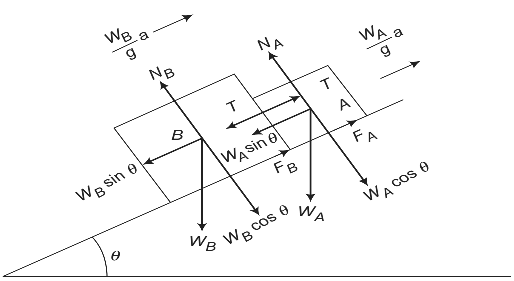


Figure 11.10

Solution Let the mutual thrust existing between the blocks be T and the acceleration of the blocks be a.

Considering the free body of the block A and using dynamic equilibrium,

$$T + \frac{W_A}{g} a + F_A = W_A \sin\theta \tag{11.13}$$

Similar considerations of the block B yields

$$T + W_B \sin\theta = \frac{W_B}{g} a + F_B \tag{11.14}$$

Eqs (11.13) – (11.14) yield

$$\frac{W_A}{g} a + F_A - W_B \sin\theta = W_A \sin\theta - \frac{W_B}{g} a - F_B$$

$$(W_A + W_B)\frac{a}{g} = (W_A + W_B) \sin\theta - (F_A + F_B)$$

$$= (W_A + W_B)\sin\theta - (\mu_A N_A + \mu_B N_B)$$

$$= (W_A + W_B)\sin\theta - (\mu_A \times W_A \cos\theta + 2 \times \mu_A \times W_B \cos\theta)$$

$$(W_A + W_B)\frac{a}{g} = (W_A + W_B)\sin\theta - \mu_A \cos\theta(W_A + 2W_B) \tag{11.15}$$

Using $W_B = 2W_A$, Eq. (11.15) becomes

$$3W_A \times \frac{a}{g} = 3W_A \sin\theta - 5W_A\mu_A \cos\theta$$

$$3\frac{a}{g} = 3\sin\theta - 5\mu_A\cos\theta$$

$$a = g\sin\theta - \frac{g}{3}5\mu_A\cos\theta = 9.81 \times \sin 30 - \frac{9.81}{3} \times 5 \times 0.15 \times \cos 30 = 2.78 \text{ m/s}^2$$

Thus from Eq. (11.13)

$$T = W_A\sin\theta - \frac{W_A}{g}a - \mu_A W_A\cos\theta = 45\sin 30 - \frac{45}{9.81} \times 2.78 - 0.15 \times 45\cos 30 = 3.86 \text{ N}$$

Thus the contact thrust existing between the blocks under motion is 3.86 N.

Example 11.12 A block of weight W_1 = 150 N is placed on an inclined plane having inclination angle θ = 45° and connected to another weight W_2 = 100 N by means of two frictionless pulleys and string as shown in Fig. 11.11. If the coefficient of friction between the block and the plane is μ = 0.15, find the tension in the string.

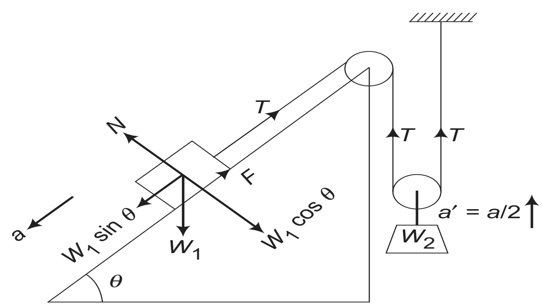


Figure 11.11

Solution Let the tension in the string be T and the acceleration of the weights be a_1 and a_2 respectively.

The most interesting aspects of the given problem is that it is not clearly mentioned whether the block will move up or slide down along the plane. This is to be ascertained first.

Since the motion is dependent, the acceleration of the two weights are related, which is also required to be established.

As regards to the first criteria, let us consider force analysis of the block and assume the weight is in static condition.

Considering static equilibrium of the weight W_2 = 100 N,
we have $2T = W_2 = 100$

$$T = 50 \text{ N}$$

As regards the block, the component of the weight $mg\sin\theta = 150\sin 45 = 106.06$ N is acting down the plane and is greater than the tension T = 50 N in the string that is acting upward along the plane.

Thus the block will slide down with a consequence of rising of weight W_2.

Now considering the motion of the system, if the displacement of the block is x in a time interval t, the upward movement of the weight W_2 is $\frac{x}{2}$.

Thus $x = \frac{1}{2}at^2$ and $\frac{x}{2} = \frac{1}{2}a't^2$

Comparing these two equations, we have $a' = \dfrac{a}{2}$

Now considering dynamic equilibrium of the block,
we have from D'Alembert's principle,

$$T + \mu N + \frac{W_1}{g} a = W_1 \sin \theta$$

$$T = W_1 \left[\sin \theta - \frac{a}{g} - \mu \cos \theta \right] \tag{11.16}$$

Similarly for the weight W_2,
we have from D'Alembert's principle,

$$2T = W_2 + \frac{W_2}{g} a' = W_2 \left[1 + \frac{a'}{g} \right]$$

$$T = \frac{W_2}{2} \left[1 + \frac{a'}{g} \right] \tag{11.17}$$

Comparing Eqs (11.16) and (11.17),

$$W_1 \left[\sin \theta - \frac{a}{g} - \mu \cos \theta \right] = \frac{W_2}{2} \left[1 + \frac{a'}{g} \right] = \frac{W_2}{2} \left[1 + \frac{a}{2g} \right]$$

Putting the values of W_1, W_2, g, μ and θ, we have a = 1.953 m/s^2

Thus from Eq. (11.17), $$T = \frac{W_2}{2} \left[1 + \frac{a'}{g} \right] = \frac{W_2}{2} \left[1 + \frac{a}{2g} \right] = \frac{100}{2} \left[1 + \frac{1.953}{2 \times 9.81} \right] = 55 \text{ N}$$

Therefore the tension in the string is T = 55 N.

Example 11.13 A system of weights and pulleys is arranged in a vertical plane as shown in Fig. 11.12. Find the acceleration of each weight, if their magnitudes are in the ratio W_A:W_B:W_C = 3:2:1. Assume the pulleys are frictionless.

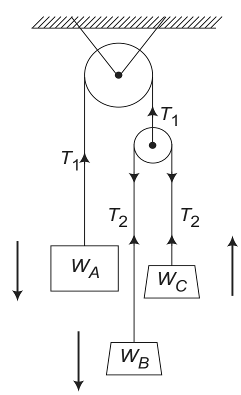


Figure 11.12

Solution Let the accelerations of three weights be denoted by a_A, a_B and a_C respectively.

Let the weight A be coming down. Thus force balance of the weight A leads to

$$T_1 + \frac{W_A}{g} a_A = W_A$$

$$T_1 = W_A \left[1 - \frac{a_A}{g} \right] \tag{11.18}$$

Since the weights B and C are connected with a pulley, lowering of weight A with an acceleration of a_A will cause the pulley (that connects weights B and C) to move up with the same acceleration.

However, this will cause weight B to come down and weight C to move up with the same acceleration say, a.

Thus the weight B will have an actual acceleration $a_B = (a - a_A)$ and the weight C will have an actual acceleration $a_C = (a + a_A)$.

Therefore, the force balance of weights B and C yields

$$T_2 + \frac{W_B}{g}(a - a_A) = W_B$$

$$\Rightarrow \qquad T_2 = W_B\left[1 - \frac{(a - a_A)}{g}\right] \qquad (11.19)$$

and

$$T_2 - \frac{W_C}{g}(a + a_A) = W_c$$

$$\Rightarrow \qquad T_2 = W_C\left[1 + \frac{(a + a_A)}{g}\right] \qquad (11.20)$$

Now considering the equilibrium of the pulley (which is moveable), neglecting friction and inertia,

$$T_1 = 2T_2$$

From Eqs (11.18) and (11.19),

$$W_A\left[1 - \frac{a_A}{g}\right] = 2 \times W_B\left[1 - \frac{(a - a_A)}{g}\right]$$

Since $W_A:W_B = 3:2$;

$$3 \times \left[1 - \frac{a_A}{g}\right] = 2 \times 2\left[1 - \frac{(a - a_A)}{g}\right]$$

$$\Rightarrow \qquad 7a_A - 4a = -g \qquad (11.21)$$

Further equating Eqs (11.19) and (11.20),

$$W_B\left[1 - \frac{(a - a_A)}{g}\right] = W_C\left[1 + \frac{(a + a_A)}{g}\right]$$

Introducing $W_B:W_C = 2:1$

$$2 \times \left[1 - \frac{(a - a_A)}{g}\right] = 1 \times \left[1 + \frac{(a + a_A)}{g}\right]$$

or

$$3a - a_A = g \qquad (11.22)$$

Solving Eqs (11.21) and (11.22) yields $a_A = \dfrac{g}{17}$ and $a = \dfrac{6g}{17}$

Thus the acceleration of weight A is $a_A = \dfrac{g}{17} \downarrow$.

Acceleration of weight B is $a_B = a - a_A = \dfrac{6g}{17} - \dfrac{g}{17} = \dfrac{5g}{17} \downarrow$.

Acceleration of weight C is $a_C = a + a_A = \dfrac{6g}{17} + \dfrac{g}{17} = \dfrac{7g}{17} \uparrow$.

MULTIPLE-CHOICE QUESTIONS

11.1 A particle if in motion will
- (a) continue its status of motion in the absence of any force.
- (b) comes to rest after certain time in the absence of any force.
- (c) accelerate automatically in the absence of any force.
- (d) none of the above.

11.2 According to Newton's Second Law of motion,
- (a) net force on a particle is proportional to its velocity.
- (b) net force on a particle is proportional to the square of the velocity.
- (c) net force on a particle is proportional to its acceleration.
- (d) none of the above.

11.3 D'Alembert's Principle is used
- (a) to solve the problems of friction.
- (b) to solve the problems of kinetics by equivalent statics problem.
- (c) to find the range and time of flight of projectiles.
- (d) none of the above.

11.4 An elevator weighing 1000 N attains an upward velocity of 5 m/s in 3 sec following a uniform acceleration. The tension in the cables that supports the elevator is

(a) 850 N (b) 1000 N (c) 1250 N (d) 1500 N

11.5 The acceleration of a particle placed on an inclined plane of angle θ with the horizontal due to its self weight, when it starts from rest is

(a) $g(\sin\theta + \mu\cos\theta)$ (b) $g(\sin\theta - \mu\cos\theta)$

(c) $g(\cos\theta + \mu\sin\theta)$ (d) $g(\cos\theta - \mu\sin\theta)$

SHORT ANSWER TYPE QUESTIONS

11.1 State Newton's 1st and 2nd laws of motion.

11.2 Newton's 1st law of motion is a special case of 2nd law – Justify this statement.

11.3 State and explain D'Alembert's Principle.

11.4 What is inertia force? What is its role in kinetics of a particle?

11.5 Compute the acceleration of a particle placed on an inclined plane of angle θ with the horizontal due to its self weight, when it starts from rest.

NUMERICAL PROBLEMS

11.1 A train weighing 70 kN is capable of developing a tractive effort of 25000 N. Find the acceleration obtained on a straight path if the coefficient of friction is 0.25 between the wheels and the track.

11.2 A small block of weight 10 N is projected with an initial velocity of 8 m/s along a horizontal plane. If the block travels a distance of 9 m before coming to rest, what is the coefficient of friction between the block and the floor?

11.3 Two blocks A and B of masses 10 kg and 20 kg respectively are connected by an inclined string. A horizontal force P = 80 N is applied to the block B as shown in Fig. 11.13. Determine the tension in the string and the acceleration of the systems. Assume μ_A and μ_B are 0.4 and 0.2 respectively.

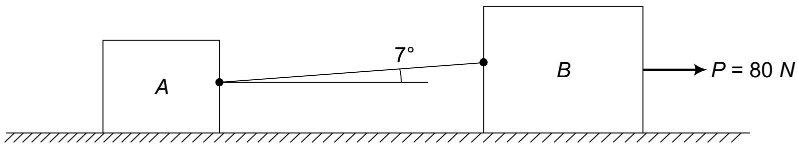


Figure 11.13

11.4 A block of mass 40 kg is resting on a horizontal table at a distance of 2 m from its edge. The block is connected another block weighing 5 kg by means of a string that passes over a frictionless pulley as shown in Fig. 11.14. If the coefficient of friction is 0.05 between the block and the table, find the acceleration of the system and time required for the block to come to the edge of the table.

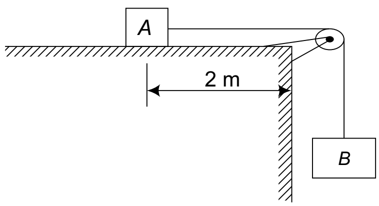


Figure 11.14

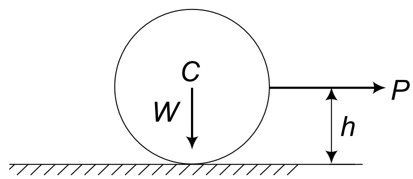


Figure 11.15

11.5 A homogeneous sphere of radius r and weight W slides along the floor under the action of a constant horizontal force P applied to a string as shown in Fig. 11.15. Determine the height h during this motion, if the coefficient of friction is μ between the block and the floor.

11.6 Two weights of masses m_1 = 25 kg and m_2 = 15 kg are connected by a light inextensible string that passes over a small frictionless pulley as shown in Fig. 11.16. Find the acceleration of the system and the tension in the string.

11.7 Two blocks of masses m_1 = 5 kg and m_2 = 10 kg are connected by a bar of negligible weight and rest on an inclined plane having inclination angle θ = 30° as shown in Fig. 11.17. Find the acceleration of the system and induced force in the bar when the system slides down the plane. Assume μ_A and μ_B are 0.15 and 0.3 respectively.

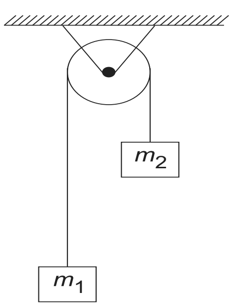


Figure 11.16

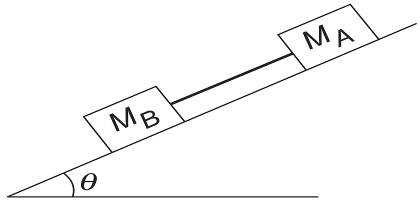


Figure 11.17

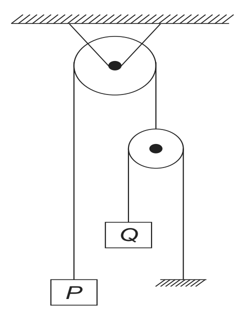


Figure 11.18

11.8 Refer to Fig. 11.18. Find the acceleration of the weight Q, assuming that $P = Q$. Neglect friction and inertia in the pulleys.

11.9 Find the maximum possible acceleration that the rear-wheel drive car, as shown in Fig. 11.9, can develop if the coefficient of friction is μ between the car and the road.

11.10 A mass M resting on a smooth table is connected to masses M_1 and M_2 by strings as shown in Fig. 11.19. Find the acceleration of the system assuming M_1 is moving down.

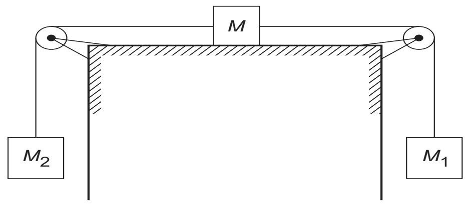


Figure 11.19

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

11.1 (a) 11.2 (c) 11.3 (b) 11.4 (c) 11.5 (b)

ANSWERS TO NUMERICAL PROBLEMS

11.1 1.05 m/s^2

11.2 0.36

11.3 T = 38.5 N, a = 0.8 m/s^2

11.4 $a = \dfrac{g}{15}$ m/s^2, t = 2.473 s

11.5 $h = r\left(1 - \dfrac{\mu W}{P}\right)$

11.6 a = 0.25 g m/s^2, T = 184 N

11.7 a = 0.355 g m/s^2, F = 2.95 N

11.8 0.4 g

11.9 $\dfrac{\mu c g}{b + c - \mu h}$

11.10 $\dfrac{(M_1 - M_2)g}{(M_1 + M_2 + M)}$

CHAPTER

12

Curvilinear Motion of a Particle

12.1 INTRODUCTION

In the preceding articles, we had considered motion of particles undergoing rectilinear motions. Nevertheless, there are situations when a particle moves following a curved path, giving rise to the concept of curvilinear motion. If the curved path lies in a plane, it is termed as plane curvilinear motion. In the situation when the direction of the applied force acting on a particle varies or when the particle has some initial motion in a direction that does not coincide with the direction of the force acting on the particle, the particle moves in a curved path. For example, an object when thrown horizontally with some initial velocity moves in a curved parabolic path, because the force of gravity acting on the object does not coincide with the initial velocity of the object and the object moves in a curved path.

12.2 DISPLACEMENT, VELOCITY AND ACCELERATION

When a particle moves along a curved path, its position at any instant is defined by considering its displacement s along the curved path at any interval of time t. Thus we have $s = f(t)$. However, it is rather convenient to resolve the displacement into two rectangular components, namely, along the x axis and the y axis, where x and y are the two separate functions of t. Mathematically, this can be expressed by $x = f_1(t)$ and $y = f_2(t)$.

However, the motion can also be defined as $y = f(x)$.

Refer to Fig. 12.1; at time t, the displacement of the particle is s having horizontal and vertical components as x and y respectively. After a small time interval Δt, the corresponding values are Δs, Δx and Δy.

Following vector algebra, the position vector $\boldsymbol{r}$ of a point on such a curve in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ along x and y axes respectively can be expressed by $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$.

Refer to Fig. 12.2. Consider now the position P' of the particle, at a later time $t + \Delta t$ as defined by a position vector r'.

The vector Δr joining P and P' represents the change in the position vector $\boldsymbol{r}$ during the time interval Δt. This can be verified by applying the triangle law as

$$r + \Delta r = r'$$

$$\Delta r = r' - r$$

Δr represents the change in the magnitude as well as the change in the direction of the position vector r.

Instantaneous velocity of the particle can be defined as

$$v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}$$

As Δr and Δt become smaller, the points P and P' get closer and the vector v obtained at the limit becomes

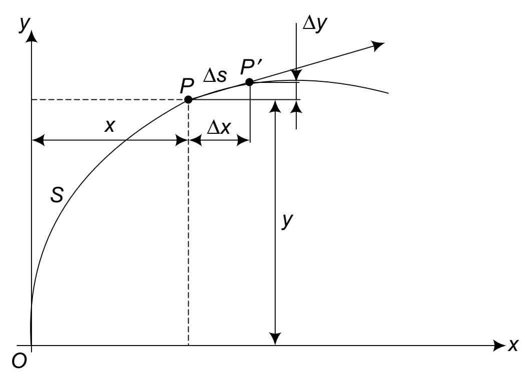


Figure 12.1

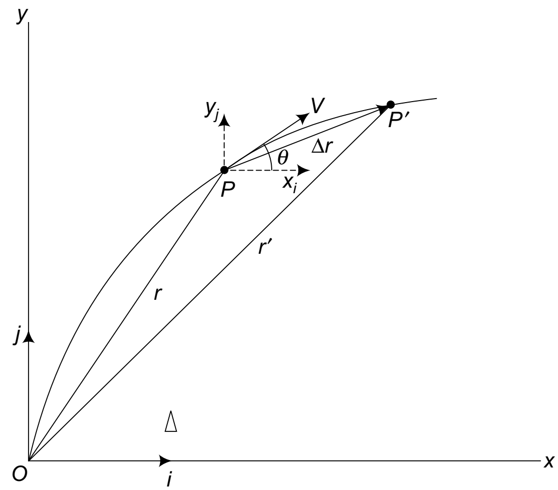


Figure 12.2

tangent to the path at P. The magnitude of Δr is given by the length of the line segment PP'. But as Δt approaches zero, the length of the line segment PP' approaches the length Δs of the arc PP'.

The magnitude of the velocity v is thus obtained as

$$v = \lim_{\Delta t \to 0} \frac{PP'}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

The speed of the particle can thus be obtained by differentiating with respect to the time the length of the arc described by the particle.

$$v = \frac{ds}{dt}$$

12.3 COMPONENTS OF MOTION: RECTANGULAR COMPONENTS

12.3.1 Rectangular Components of Velocity

As the direction of the velocity of a particle in curvilinear motion changes continuously, so it is convenient to deal with its components v_x and v_y.

As the particle moves, $\boldsymbol{r}$ changes and so also the velocity $\boldsymbol{v}$ changes.

Since $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$,

differentiating, we get $v = \dfrac{dr}{dt} = \dot{r} = \dfrac{dx}{dt}\, i + \dfrac{dy}{dt} j = \dot{x}\, i + \dot{y}\, j$

Thus the speed of the particle at any given instance can be expressed by above expression; the magnitude of which is $|v| = \sqrt{\dot{x}^2 + \dot{y}^2}$

If θ is the angle the velocity vector of the particle makes with x axis; $\theta = \tan^{-1}\left(\dfrac{\dot{y}}{\dot{x}}\right)$. It therefore follows that velocity vector is tangent to the curved path at any point P.

12.3.2 Acceleration of the Particle

It may be emphasised here that in general, the acceleration of the particle at any instant is not tangential to the path of the particle. That is, the direction of acceleration and velocity may not be the same in a curvilinear motion.

By definition, acceleration $\boldsymbol{a}$ is also a vector which is the time rate of change of $\boldsymbol{v}$.

Thus acceleration $a = \dfrac{dv}{dt} = \dot{v} = \ddot{r} = \dfrac{d^2x}{dt^2}\, i + \dfrac{d^2y}{dt^2}\, j = \ddot{x}\, i + \ddot{y}\, j$ and its magnitude becomes $|a| = \sqrt{\ddot{x}^2 + \ddot{y}^2}$

12.4 TANGENTIAL AND NORMAL COMPONENTS

In the previous article, the velocity and acceleration were resolved into two mutually perpendicular components along the unit vectors i and j, called rectangular components. Since the trajectory of the particle is a curve, these can also be resolved into two mutually perpendicular components, one tangent to the curve and the other normal to the curve. Such resolution is sometimes found to be extremely useful for computational purpose.

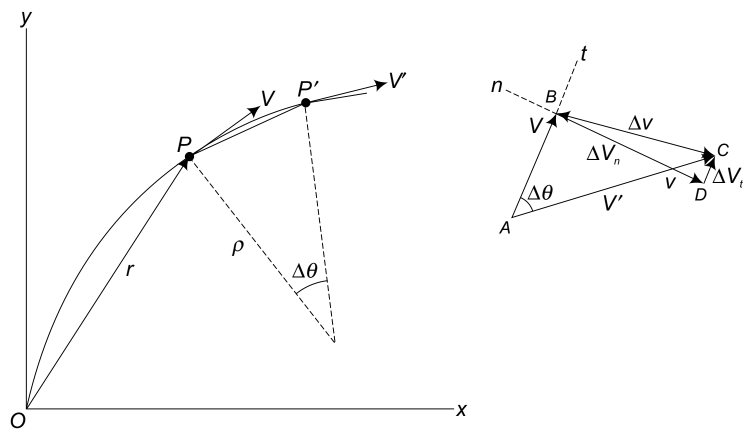


Figure 12.3

Refer to Fig. 12.3. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ represents velocity v and v' respectively. Thus following triangle rule, $\overrightarrow{BC}$ represents change in velocity $\Delta v = v' - v$ in time Δt. This Δv can now be resolved into two components, one tangent to the path Δv_t and other normal to the path Δv_n.

Therefore, $\qquad \Delta v = \Delta v_t + \Delta v_n$

Since by definition; acceleration $\boldsymbol{a} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t}$

$$a = \underset{\Delta t \to 0}{Lt} \frac{\Delta v_t}{\Delta t} + \underset{\Delta t \to 0}{Lt} \frac{\Delta v_n}{\Delta t} = a_t + a_n$$

12.4.1 Tangential Component

Refer to Fig. 12.3; $\Delta v_t \approx v' - v$

$\therefore$
$$a_t = \lim_{\Delta t \to 0} \frac{(v' - v)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Thus the tangential component of acceleration is equal to the rate of change of the speed of the particle.

Tangential acceleration (a_t) is considered to be positive in the direction of the tangent coinciding with the sense of the motion.

12.4.2 Normal Acceleration

Refer to the same figure; $\Delta v_n \approx v\Delta\theta$ for a small change in the angle θ

Therefore,
$$a_n = \lim_{\Delta t \to 0} \frac{v\Delta\theta}{\Delta t}$$

If ρ is the radius of curvature of the curve then,

$$\Delta s = \rho\Delta\theta$$

$$\Delta\theta = \frac{\Delta s}{\rho}$$

$$a_n = \lim_{\Delta t \to 0} \frac{v\Delta\theta}{\Delta t} = \lim_{\Delta t \to 0} \frac{v}{\rho}\frac{\Delta s}{\Delta t}$$

$$a_n = \frac{v}{\rho}\frac{ds}{dt}$$

But
$$\frac{ds}{dt} = v$$

So,
$$a_n = \frac{v^2}{\rho}$$

Thus *Normal acceleration a_n of a particle at a point is equal to the square of its speed divided by the radius of curvature of the path at that point.*

The direction of the normal acceleration is such that it is always directed towards the centre of curvature of the path. This normal acceleration is also called as the *centripetal acceleration.*

If we consider two unit vectors - one $\boldsymbol{e_t}$ tangent to the curved path at point P and the other $\boldsymbol{e_n}$ normal to it, then $\boldsymbol{a}$ can be expressed as $\boldsymbol{a} = a_t\boldsymbol{e_t} + a_n\boldsymbol{e_n}$.

Direction $\theta = \tan^{-1} \dfrac{a_n}{a_t}$

Example 12.1 The motion of a particle is defined by the following equations, which are $x = \dfrac{(t-4)^3}{6} + t^2$ and $y = \dfrac{t^3}{6} - \dfrac{(t-1)^2}{4}$, where x and y are expressed in metres and t is in seconds. Determine the acceleration of the particle when $t = 2$ s. Also calculate the radius of curvature of the path.

Solution

$$x = \frac{(t-4)^3}{6} + t^2$$

$$\dot{x} = \frac{1}{6} \times 3 \times (t-4)^2 + 2t$$

$$\therefore \quad \dot{x}_{t=2} = \frac{1}{2} \times (2-4)^2 + 2 \times 2 = 6 \text{ m/s}$$

$$\ddot{x} = \frac{1}{2} \times 2 \times (t-4) + 2 = t - 2$$

$$\therefore \quad \ddot{x}_{t=2} = 2 - 2 = 0$$

$$y = \frac{t^3}{6} - \frac{(t-1)^2}{4}$$

$$\dot{y} = \frac{1}{6} \times 3t^2 - \frac{2}{4}(t-1) = \frac{t^2}{2} - \frac{1}{2}(t-1)$$

$$\therefore \quad \dot{y}_{t=2} = \frac{1}{2} \times 2^2 - \frac{1}{2}(2-1) = 1.5 \text{ m/s}$$

$$\ddot{y} = \frac{1}{2} \times 2t - \frac{1}{2} = t - \frac{1}{2}$$

$$\therefore \quad \ddot{y}_{t=2} = 2 - \frac{1}{2} = 1.5 \text{ m/s}^2$$

$$|a| = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{0 + (1.5)^2} = 1.5 \text{ m/s}^2$$

By definition, the radius of curvature $\dfrac{1}{\rho} = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} = \dfrac{6 \times 1.5 - 0}{(6^2 + 1.5^2)^{\frac{3}{2}}}$

$$\rho = 26.3 \text{ m}$$

Example 12.2 The motion of a particle is expressed by the following equations: $x = t^2 + 8t + 6$ and $y = t^3 + 3t^2 + 8t + 6$, where x and y are expressed in metres and t is in seconds. Determine (a) the initial velocity of the particle, (b) the velocity at $t = 2$ s and (c) acceleration of the particle at $t = 2$ s.

Solution

$$x = t^2 + 8t + 6$$

$$\dot{x} = 2t + 8$$

$$\ddot{x} = 2$$

$$\therefore \quad \dot{x}_{t=0} = 8 \text{ m/s}; \ \dot{x}_{t=2} = 2 \times 2 + 8 = 12 \text{ m/s and } \ddot{x}_{t=2} = 2 \text{ m/s}^2$$

$$y = t^3 + 3t^2 + 8t + 6$$

$$\dot{y} = 3t^2 + 3 \times 2t + 8 = 3t^2 + 6t + 8$$

$$\ddot{y} = 6t + 6$$

$$\therefore \quad y_{t=0} = 8 \text{ m/s}; \ \dot{y}_{t=2} = 3 \times 2^2 + 6 \times 2 + 8 = 32 \text{ m/s and}$$

$$\ddot{y}_{t=2} = 6 \times 2 + 6 = 18 \text{ m/s}^2$$

The velocity $|v|_{t=0} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{8^2 + 8^2} = 11.31$ m/s and

$$|v|_{t=2} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{12^2 + 32^2} = 34.176 \text{ m/s and } \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{32}{12}\right)$$

$$\theta = 69.45°$$

$$|a|_{t=2} = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{2^2 + (18)^2} = 18.11 \text{ m/s}^2$$

Example 12.3 The distance s travelled by a particle moving along a circular path of radius r is given by the following equation:

$s = kt^2$, where k is a constant.

If the particle starts its motion from rest, find (a) the tangential velocity and acceleration (b) the normal velocity and acceleration.

Solution

$$s = kt^2$$

$$v_t = \frac{ds}{dt} = k \times 2t = 2kt$$

$$a_t = \frac{dv}{dt} = 2k$$

$$v_n = 0 \text{ and}$$

$$a_n = \frac{v^2}{\rho} = \frac{v_t^2}{r} = \frac{4k^2t^2}{r}$$

Example 12.4 Prove that if the ends A and B of a bar of length $2l$ are constrained along the x axis and y axis respectively as shown in Fig. 12.4, its mid-point C describes a circle of radius l with centre at O while any intermediate point D describes an ellipse with semi-major and semi-minor axes as $l + b$ and $l - b$ respectively.

Solution Let the coordinate of the point C is (x, y) considering O as origin.

Thus from the simple geometry $x = l \cos \theta$ and $y = l \sin \theta$, where θ is the inclination of the bar AB with the floor.

Thus $x^2 + y^2 = l^2 (\cos^2 \theta + \sin^2 \theta) = l^2$.

This is the equation of a circle of radius l.

Further, let the coordinate of the point D be (x', y'), considering O as origin.

Now, $x' = (l + b) \cos \theta$ and $y' = (l - b) \sin \theta$

Therefore $x'^2 + y'^2 = (l + b)^2 \cos^2 \theta + (l - b)^2 \sin^2 \theta$

$$\frac{x'^2}{(l+b)^2} + \frac{y'^2}{(l-b)^2} = \cos^2 \theta + \sin^2 \theta = 1;$$ which is the equation of an ellipse with semi-major and semi-minor axes as $l + b$ and $l - b$ respectively.

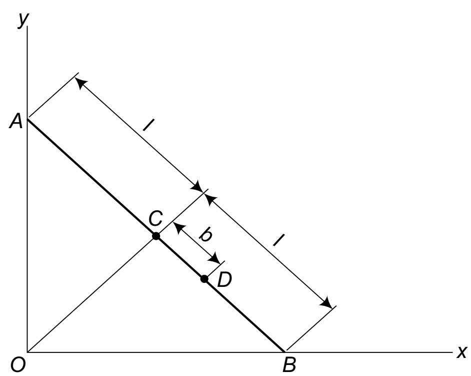


Figure 12.4

Example 12.5 A car starts from rest on a curved road of radius 250 m and attains a speed of 18 km/hour at the end of 60 seconds while travelling with a uniform acceleration. Find the tangential and normal accelerations of the car 30 seconds after it started.

Solution $\rho = r = 250$ m; $v = 18$ km/hr $= \dfrac{18 \times 1000}{60 \times 60}$ m/s $= 5$ m/s $v_0 = 0$, $t = 60$ s

Let the acceleration of the car be a.

Therefore $$v_t = v = v_0 + at$$

$$a = \frac{v}{t} = \frac{5}{60} = 0.083 \text{ m/s}^2$$

Since the car is moving with constant acceleration, $a_t = 0.083$ m/s^2 after 30 seconds.

Let the velocity of the particle be v' after 30 seconds.

$$v' = v_0 + at' = 0 + 0.083 \times 30 \text{ m/s}^2 = 2.5 \text{ m/s}^2$$

$$\therefore \quad (a_n)_{t=30} = \frac{v'^2}{r} = \frac{2.5^2}{250} \text{ m/s}^2 = 0.025 \text{ m/s}^2$$

Thus tangent and normal accelerations of the car after 30 seconds becomes

$$a_t = 0.083 \text{ m/s}^2 \text{ and } a_n = 0.025 \text{ m/s}^2 \text{ respectively}$$

Example 12.6 A car enters a curved section of a road of length equal to the quarter of a circle with a 100 metres radius at a speed of 18 km/hour and leaves at 36 km/hour. If the car is moving with a uniform tangential acceleration, find the magnitude and direction of total acceleration when (a) it enters the curve and (b) it leaves the curve.

Solution

$$v_1 = 18 \text{ km/hr} = \frac{18 \times 1000}{60 \times 60} \text{ m/s} = 5 \text{ m/s};$$

$$v_2 = 36 \text{ km/hr} = \frac{36 \times 1000}{60 \times 60} \text{ m/s} = 10 \text{ m/s}.$$

$$\rho = r = 100 \text{ m}$$

Length of the curved path = distance covered = $\frac{1}{4} \times 2\pi r = \frac{\pi r}{2}$ m

Let the uniform tangential acceleration be a_t.

$$v_2^2 = v_1^2 + 2 \times a_t \times \frac{\pi r}{2}$$

$$a_t = \frac{v_2^2 - v_1^2}{\pi r} = \frac{10^2 - 5^2}{\pi \times 100} = 0.238 \text{ m/s}^2$$

$$(a_n)_1 = \frac{v_1^2}{r} = \frac{5^2}{100} = 0.25 \text{ m/s}^2 \text{ and}$$

$$(a_n)_2 = \frac{v_2^2}{r} = \frac{10^2}{100} = 1 \text{ m/s}^2$$

Therefore, total velocity at the entry is

$$a_1 = \sqrt{(a_t)^2 + (a_n)_1^2} = \sqrt{(0.238^2) + (0.25)^2} = 0.345 \text{ m/s}^2 \text{ and its direction is}$$

$$\theta_1 = \tan^{-1}\left(\frac{(a_n)_1}{a_t}\right) = \tan^{-1}\left(\frac{0.25}{0.238}\right) = 46.4°$$

The same at the exit is $a_2 = \sqrt{(a_t)^2 + (a_n)_2^2} = \sqrt{(0.238^2) + 1^2} = 1.028$ m/s² and its direction is

$$\theta_2 = \tan^{-1}\left(\frac{(a_n)_2}{a_t}\right) = \tan^{-1}\left(\frac{1}{0.238}\right) = 76.4°$$

Example 12.7 The telescopic rod in Fig. 12.5 forces the pin P to move along the fixed path $y = \frac{x^2}{9}$, where x and y are expressed in centimetres. At any instant t, the x-coordinate of P is given by $x = t^2 - 14t$. Determine the y components of the velocity and acceleration of P at $t = 15$ s.

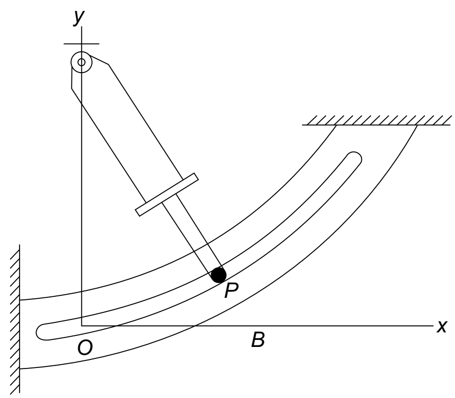


Figure 12.5

Solution

$$x = t^2 - 14t \text{ and}$$

$$y = \frac{x^2}{9}$$

Combining these two equations,

$$y = \frac{(t^2 - 14t)^2}{9}$$

$$\dot{y} = \frac{1}{9} \times 2(t^2 - 14t)(2t - 14)$$

$$\ddot{y} = \frac{2}{9}\,[(t^2 - 14t) \times 2 + (2t - 14)(2t - 14)]$$

$\therefore$

$$\dot{y}_{t=15s} = \frac{1}{9} \times 2(15^2 - 14 \times 15)(2 \times 15 - 14) = 53.33 \text{ cm/s and}$$

$$\ddot{y}_{t=15s} = \frac{2}{9}[(15^2 - 14 \times 15) \times 2 + (2 \times 15 - 14)^2] = 63.55 \text{ cm/s}^2$$

12.5 KINETICS OF CURVILINEAR MOTION

It has already been analysed that the acceleration of a particle moving in a curvilinear path is a vector which can be resolved into two rectangular components a_x and a_y, along the directions of the coordinates axes x and y respectively. The same can also be resolved as a_t and a_n along the directions of the tangent and normal to the curve respectively.

The equations of motion, therefore, can be written following either set of the components of acceleration.

12.6 EQUATION OF MOTION IN RECTANGULAR COMPONENTS

If a particle is acted upon by several forces that cause its motion, then these forces can be resolved in the x and y directions. Let $\sum X$ and $\sum Y$ represent the sum of the components of the forces in the x and y directions respectively.

Applying Newton's second law of motion,

$$\sum X = ma_x$$

and

$$\sum Y = ma_y$$

12.7 EQUATION OF MOTIONS: IN TANGENTIAL AND NORMAL COMPONENTS

Sometimes it is convenient to resolve the forces acting on the particle into components F_t and F_n, the former being in the tangential direction, the latter being along the normal.

Applying Newton's second law once again,

$$\sum F_t = ma_t$$

and

$$\sum F_n = ma_n$$

12.8 EQUATIONS OF DYNAMIC EQUILIBRIUM (D'ALEMBERT'S PRINCIPLE)

Similar to the case of rectilinear motion of a particle, the curvilinear motion can also be expressed by using D'Alembert's Principle, which are known as the equations of dynamic equilibrium.

12.8.1 In Rectangular Components

The equations of motion are

$$\sum X = ma_x$$

and

$$\sum Y = ma_y$$

These can be written as $\sum X + (-ma_x) = 0$ and $\sum Y + (-ma_y) = 0$ and are called the *equations of dynamic equilibrium*, where $(-ma_x)$ and $(-ma_y)$ are the *inertia forces* added to the system of forces acting on the particle, in the directions opposite to the direction of accelerations a_x and a_y. The basic philosophy is that condition of dynamic equilibrium is equivalently converted to the conditions of static equilibrium.

12.8.2 In Normal and Tangential Components

$$\sum F_t = ma_t \text{ and } \sum F_n = ma_n$$

Following D'Alembert's Principle, these can be written as

$$\sum F_t + (-ma_t) = 0$$

$$\sum F_n + (-ma_n) = 0$$

Example 12.8 A locomotive of weight W = 534 kN negotiates a curve of radius r = 300 m at a speed of 72 km/hr. Determine the total lateral thrust on the rails.

Solution The speed of the locomotive is $v = 72 \text{ km/hr} = \dfrac{72 \times 1000}{60 \times 60} \text{ m/s} = 20 \text{ m/s}$

The lateral thrust is due to the force associated with normal acceleration.

Therefore $T = m.\dfrac{v^2}{r} = \dfrac{534}{9.81} \times \dfrac{20^2}{300} \text{ kN} = 72.57 \text{ kN}$

Example 12.9 A motorcycle and rider having a total weight W = 2225 N travels in a vertical plane following a curve AB of radius r = 300 m at a speed of 72 km/hr. Compute the thrust exerted by the road as it passes over the crest C on the curve, as shown in Fig. 12.6.

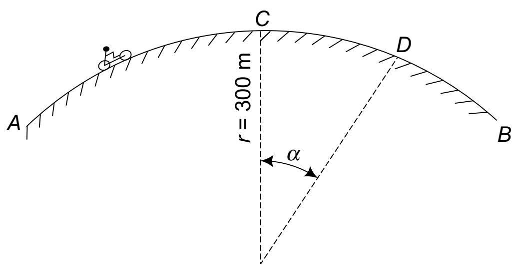


Figure 12.6

Solution The speed of the motorcycle is $v = 72 \text{ km/hr} = \dfrac{72 \times 1000}{60 \times 60} \text{ m/s} = 20 \text{ m/s}$

Let the thrust be R Newton.

The force balance along the vertical direction yields $W - R = \dfrac{W}{g} . \dfrac{v^2}{r}$

$$R = W\left[1 - \frac{v^2}{gr}\right] = 2225\left[1 - \frac{20^2}{9.81 \times 300}\right] \text{N} = 1922.4 \text{ N}$$

Example 12.10 The bob of a conical pendulum of length l and weight W describes a horizontal circle defined by equations $x = a \cos \omega t$, $y = a \sin \omega t$, where a is the radius of the circular path and ω is a constant, as shown in Fig. 12.7. Prove that the tension in the string is constant during such motion and find its magnitude.

Solution Let at any instant t, the bob occupies a position on its circular path in horizontal plane as shown in Fig. 12.7. $x = a \cos \omega t$; $\dot{x} = -a\omega \sin \omega t$; $y = a \sin \omega t$; $\dot{y} = a\omega \cos \omega t$

Therefore velocity of the bob becomes $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \omega a$

The forces on the bob at this position are shown in Fig. 12.7.

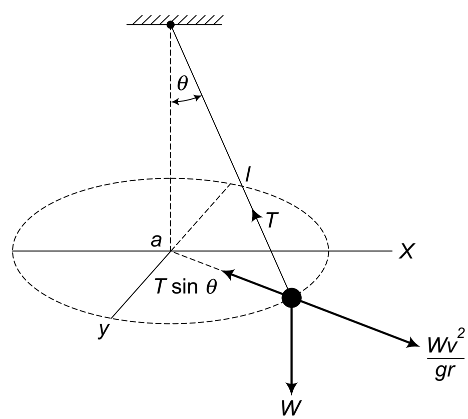


Figure 12.7

The force balance normal to the circular path considering the equilibrium of the bob yields $T \sin\theta = \frac{W}{g}.\frac{v^2}{r} = \frac{W}{g}.\frac{\omega^2 a^2}{a} = \frac{W}{g}.\omega^2 a$

From the geometry, $$\sin\theta = \frac{a}{l}$$

Therefore $$T \times \frac{a}{l} = \frac{W}{g}.\omega^2 a \text{ or } T = \frac{W}{g}\omega^2 l.$$

Since all the parameters on the right side of the expression are constant, so T is also a constant.

Example 12.11 A small block of weight W rests on a horizontal turntable at a distance r from the axis of rotation as shown in Fig. 12.8. If the coefficient of friction between the block and the surface of the turntable is μ, find the maximum uniform speed that the block can have due to rotation of the turntable without slipping off.

Solution The lateral thrust owing to normal acceleration will try to slip off the block outward which will be prevented by frictional forces.

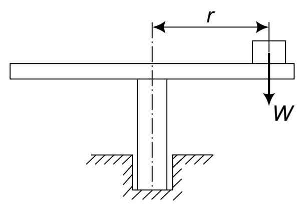


Figure 12.8

Under the condition of equilibrium, $$F_f = m.\frac{v_m^2}{r}$$

$$\mu W = \frac{W}{g}.\frac{v_m^2}{r}$$

$$v_m = \sqrt{\mu g r}$$

Thus maximum velocity at which the turntable can be rotated is $v_m = \sqrt{\mu g r}$

Example 12.12 An automobile of weight W travels with a uniform speed v over a vertical curve ACB which is parabolic, as shown in Fig. 12.9. Determine the total pressure exerted by the wheels of the car as it passes over the crest at C. Given $\delta = 1.2$ m, $l = 60$ m and $v = 96$ km/hr.

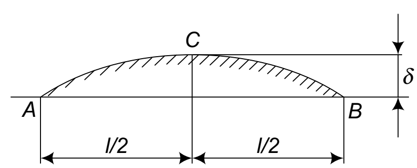


Figure 12.9

Solution Considering the problem as similar to that of example 9, we have $R = W\left[1 - \frac{v^2}{gr}\right]$.

$$v = 96 \text{ km/hr} = \frac{96 \times 1000}{60 \times 60} \text{ m/s} = 26.67 \text{ m/s}$$

However, unlike the previous case, the curve now conforms to a parabola, for which radius of curvature ρ is to be computed from given data.

Let the equation of the parabola be $x^2 = ky$. Since point $C\left(\frac{l}{2}, \delta\right)$ lies on the curve, this point must satisfy the equation, leading to

$$\frac{l^2}{4} = k\delta$$

$$k = \frac{l^2}{4\delta}$$

Thus $$x^2 = \frac{l^2}{4\delta} y$$

$$\frac{dy}{dx} = \frac{4\delta}{l^2} \times 2x \text{ and}$$

$$\frac{d^2y}{dx^2} = \frac{4\delta}{l^2} \times 2$$

Now $\left.\frac{dy}{dx}\right|_{x=0} = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{4\delta}{l^2} \times 2$

The radius of curvature ρ is given by

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{\frac{8\delta}{l^2}}{1} = \frac{8 \times 1.2}{60^2}$$

$$\rho = \frac{60^2}{1.2 \times 8} \text{ m} = 375 \text{ m}$$

Therefore, $R = W\left[1 - \frac{v^2}{gr}\right] = W\left[1 - \frac{26.67^2}{9.81 \times 375}\right] = 0.807W$

Thus the total pressure exerted by the wheels of the car as it passes over the crest at C is $R = 0.807W$

Example 12.13 Find the proper super-elevation e for 7.2 m highway curve of radius $r = 600$ m, in order that a car travelling with a speed of 80 km/hr will have no tendency to skid sideways.

Solution The super-elevation is provided on the outer track so as to experience equal thrust on two wheels– inner and outer.

Refer to Fig. 12.10, the force balance on the car in horizontal plane gives $R \sin \theta = \frac{W}{g} v^2 r$ and perpendicular to the track gives $R \cos \theta = W$.

Thus $$\tan \theta = \frac{v^2}{rg}$$

Further from geometry, $\sin\theta = \frac{e}{b}$, where e is the super-elevation and b is the distance between the wheels.

For small value of θ, $\sin\theta \approx \tan\theta$

$$\frac{e}{b} = \frac{v^2}{rg}$$

$$e = \frac{bv^2}{rg} = \frac{7.2 \times 22.22^2}{600 \times 9.81} = 0.604 \text{ m}$$

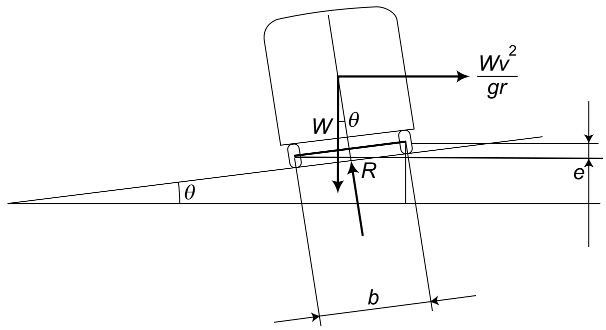


Figure 12.10

Example 12.14 Racing cars travel around a circular track of radius r = 300 m with a speed of 384 km/hr. What angle α should the floor of the track make with the horizontal in order to avoid skidding?

Solution

$$\tan\alpha = \frac{v^2}{rg} = 3.866$$

$$\alpha = \tan^{-1}(3.866) = 75°29'$$

Thus the requisite angle is $\alpha = 75°29'$

MULTIPLE-CHOICE QUESTIONS

12.1 The tangential component of acceleration of a particle in a curvilinear motion is defined by

(a) $a_t = \frac{dv}{dt}$ (b) $a_t = \frac{d\dot{x}}{dt}$ (c) $a_t = \frac{d\dot{y}}{dt}$ (d) $a_t = \frac{v^2}{\rho}$

12.2 The normal component of acceleration of a particle in a curvilinear motion is defined by

(a) $a_n = \frac{dv}{dt}$ (b) $a_n = \frac{d\dot{x}}{dt}$ (c) $a_n = \frac{d\dot{y}}{dt}$ (d) $a_n = \frac{v^2}{\rho}$

12.3 The maximum velocity of a car following a circular motion of radius r and having coefficient of friction as μ to avoid skidding is

(a) μgr (b) $\frac{1}{2}ugr$ (c) $\sqrt{\mu gr}$ (d) $\frac{1}{2}\sqrt{\mu gr}$

12.4 On a curved railway track, the amount by which the outer track is raised is known as

(a) cambering (b) super-elevation (c) deflection (d) none of the above

12.5 On a curved track, the super-elevation is provided for the following purpose.

(a) To reduce centrifugal force (b) To reduce vertical thrust
(c) To equalize thrust on two sets of wheels (d) None of the above

12.6 The amount of super-elevation is quantified by

(a) $e = \frac{bv^3}{gr^2}$ (b) $e = \frac{bv^2}{gr}$ (c) $e = \frac{bvr}{g}$ (d) None of the above

SHORT ANSWER TYPE QUESTIONS

12.1 Define curvilinear motion. How does it differ from a straight line motion?
12.2 Define and quantify tangential and normal acceleration of a particle in relation to curvilinear motion.
12.3 If a particle moves with a constant speed, what would be its tangential acceleration?
12.4 Write the equations of dynamic equilibrium of a particle for curvilinear motion in a plane.
12.5 Explain and apply D'Alembert's principle to the curvilinear motion of a particle.

NUMERICAL PROBLEMS

12.1 A particle moves along the path $y = \frac{1}{3}x^2$ with a constant velocity of 8 m/s. What are the x and y components of the velocity when $x = 3$? What is the corresponding acceleration? Note that x and y are expressed in metres.
12.2 A particle moves with constant speed v along a parabolic path $y = kx^2$, where k is a constant. Find the maximum acceleration of the particle.
12.3 The equations of motions by a particle undergoing curvilinear motion can be described by $x = 2t^2 + 8t$ and $y = 4.9t^2$, where x and y are expressed in metres and t is in seconds. Determine the velocity and the acceleration at the end of 4 seconds.
12.4 A car is moving along a curved path with 150 m radius with a uniform velocity of 90 km/hr. Find the normal and tangential acceleration of the car.
12.5 The bob of a simple pendulum of length l and weight W undergoes oscillation that is defined by $s = s_0 \cos pt$, where s_0 is the amplitude of oscillation and p is a constant such that $p = \sqrt{\frac{g}{l}}$. Determine the maximum value of the tensile force in the string.
15.6 Refer to the Example 12.1. Determine (a) the magnitude of the smallest velocity reached by the particle and (b) the corresponding time, position and direction of the velocity.
12.7 Refer to the Example 12.9. Determine the location D of the motorcycle on the curve as defined by angle α so that the reaction R becomes zero.
12.8 The coefficient of friction between the road and the wheels of a car is found to be 0.2. At what constant velocity should the car move so as to avoid skidding, if the radius of the curve is 240 m. Assume that the road is levelled.
12.9 A car starts from rest on a curved road of 250 m radius and accelerates at a constant tangential acceleration of 0.6 m/s^2. Determine the distance and the time for which the car should travel before the magnitude of the total acceleration attained by the car becomes 0.75 m/s^2.
12.10 A motorist is moving along a curved path with a 300 metre radius at a speed of 72 km/hr. He suddenly applies brake that causes its speed to decrease to 40 km/hr at a constant rate in 10 seconds. Calculate the tangential and normal components of acceleration immediately after the application of brake and 6 seconds after that.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

12.1 (a) 12.2 (d) 12.3 (c) 12.4 (b) 12.5 (a) 12.6 (b)

ANSWERS TO NUMERICAL PROBLEMS

12.1 $\dot{x} = 3.58$ m/s, $\dot{y} = 7.16$ m/s , a = 3.82 m/s^2
12.2 $a_{\max} = 2kv^2$
12.3 $\dot{x} = 24$ m/s, $\dot{y} = 39.2$ m/s, $\ddot{x} = 4\,\text{m/s}^2$, $\ddot{y} = 9.8$ m/s^2
12.4 $a_n = 4.167\,\text{m/s}^2$, $a_t = 0$
12.5 $W\left(1+\dfrac{{s_0}^2}{l_2}\right)$
12.6 (a) $v_{\min} = 6.14$ m/s, (b) $t = 1.758$ s, $x = 1.206$ m, $y = 0.761$ m, $\theta = 10.9°$
12.7 $\alpha = 82°14'$
12.8 27.45 m/s
12.9 $x = 93.65$ m, $t = 17.68$ s
12.10 $a_n = 1.333$ m/s^2, $a_t = -0.88$ m/s^2; $a_n = -0.722$ m/s^2 , $a_t = -0.88$ m/s^2

CHAPTER

13

Projectile

13.1 INTRODUCTION

When an object is thrown with an initial velocity such that further motion is exclusively under the action of gravity, the object is called a projectile and its motion is called motion of a projectile. This motion is quite common in practice. Firing of a bullet from a gun, release of a bomb from a fighter jet, throwing a stone are few such examples that fall under this category. A projectile covers a distance along horizontal as well as along vertical directions.

13.2 TERMINOLOGY OF PROJECTILE MOTION

13.2.1 Velocity of Projection

The initial velocity with which a projectile is thrown is called velocity of projection.

13.2.2 Angle of Projection

The angle at which the projectile is thrown with the horizontal is called the angle of projection of the projectile.

13.2.3 Range of a Projectile

Range of a projectile is the horizontal distance it covers from the point of origin and the point on the ground where it touches.

13.2.4 Trajectory of a Projectile

The path a projectile navigates through space is called its trajectory. The trajectory of a projectile is parabolic.

13.3 EQUATION OF THE PATH

Let us consider a particle that is thrown from the point ***O*** with an initial velocity v_0 that makes an angle θ with the horizontal as shown in Fig. 13.1.

The velocity v_0 has got two components – the horizontal component being $v_0 \cos \theta$, the vertical component is $v_0 \sin \theta$.

Since there is no force acting along the horizontal direction, the particle cannot have any acceleration in this direction. However, along the vertical direction, it is under the action of gravity. But since the particle is moving against the gravity, it will decelerate with constant deceleration g.

Let the time of travel of the particle is t corresponds to position $\boldsymbol{C}(x, y)$.

Thus the distance covered by the projectile along horizontal is

$$x = v_0 \cos \theta t \tag{13.1}$$

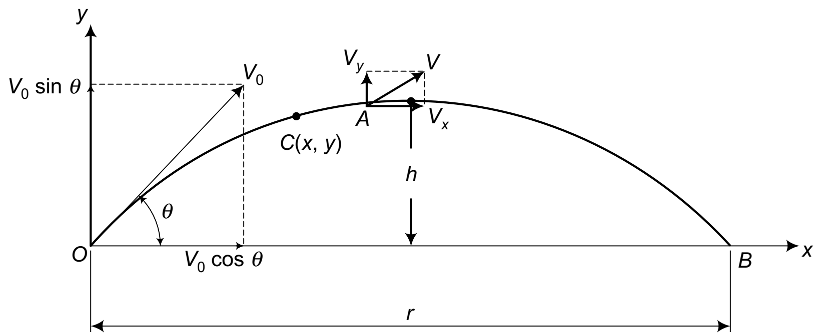


Figure 13.1

and the distance along the vertical direction is

$$y = v_0 \sin \theta t - \frac{1}{2} gt^2 \tag{13.2}$$

Eliminating t from Eqs (13.1) and (13.2) yields

$$y = v_0 \sin \theta \frac{x}{v_0 \cos \theta} - \frac{1}{2} g \left(\frac{x}{v_0 \cos \theta} \right)^2$$

$$y = \tan \theta x - \left(\frac{g}{2v_0^2 \cos^2 \theta} \right) x^2 \tag{13.3}$$

The equation is in the form of $y = Ax + Bx^2$ which is the equation of a parabola.

Thus Eq. (13.3) is the requisite trajectory of a projectile that manifests a parabola.

At any point on the curve, say, at A, the velocity v can be resolved into two components as shown in the above figure.

Time of Flight The time of flight of a projectile is defined as the time interval between the object is thrown and the instant it touches the ground. Thus it is the duration of the particle during which it flies.

From $v = v_0 + at$, we have

$0 = v_0 \sin \theta - gt$

$t = \frac{v_0 \sin \theta}{g}$ [since the particle is moving against the gravity and at the end of its vertical motion, its velocity becomes zero.]

The value of t so calculated is for only upward movement.

Thus by definition *Time of Flight* $t_{Flight} = 2t = \frac{2v_0 \sin \theta}{g}$.

Height The maximum height the projectile can reach is called its height.

From $v^2 = v_0^2 + 2ax$, we have $0 = (v_0 \sin \theta)^2 - 2gh$

$$h = \frac{v_0^2 \sin^2 \theta}{2g}.$$

Thus the height of a projectile $h = \frac{v_0^2 \sin^2\theta}{2g}$.

Range of a Projectile In the absence of any acceleration, the displacement is $x = vt$.

Thus the range $r = (v_0 \cos\theta)2t = (v_0 \cos\theta)\,\frac{2v_0 \sin\theta}{g}$

$$r = \frac{v_0^2 \sin 2\theta}{g}$$

For a given v_0, r is a function of θ only, since g is a constant.
Thus the condition of r to be maxima; $\sin 2\theta = 1 = \sin 90 \Rightarrow \theta = 45°$

Hence $$r_{\max} = \frac{v_0^2}{g}$$

Slope of the Curve From Eq. (16.3) we have $y = \tan\theta\, x - \left(\frac{g}{2v_0^2 \cos^2\theta}\right)x^2$

Thus the slope of the curve is $\frac{dy}{dx} = \tan\theta - \left(\frac{g}{2v_0^2 \cos^2\theta}\right) \times 2x$

Putting $\frac{dy}{dx} = 0$, we have $\tan\theta - \left(\frac{g}{2v_0^2 \cos^2\theta}\right) \times 2x = 0$

$$x = \frac{\tan\theta v_0^2 \cos^2\theta}{g} = \frac{(v_0 \cos\theta)(v_0 \sin\theta)}{g} = \frac{v_{0x}v_{0y}}{g}$$

Further, $$x = \frac{v_0^2 \sin 2\theta}{2g} = \frac{r}{2}$$

This implies the slope of the curve becomes zero when the projectile covers half of its range.

Example 13.1 An aircraft is moving horizontally with a velocity of 500 km/h. It releases a bomb when it is at an altitude of 1800 m above the ground to hit a target on the ground. Determine the location of the target in terms of the horizontal distance from the instantaneous position of the aircraft when the bomb was released.

Solution Let the time of flight of the bomb be t seconds.

Initial velocity $$v_0 = 500 \text{ km/hr} = \frac{500 \times 1000}{60 \times 60} \text{ m/s} = 138.88 \text{ m/s}$$

Thus $$y = h = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1800}{9.8}} \text{ s} = 19.167 \text{ s}$$

$$x = v_0.t = 138.88 \times 19.167 \text{ m} = 2661 \text{ m} \approx 2.662 \text{ km}$$

Example 13.2 Refer to Fig. 13.2. The pilot of an aircraft flying horizontally at a speed of 480 km/hr at an elevation of 600 metre above ground wants to hit the target on the ground. At what angle θ below the horizontal should the pilot see the target at the instant of releasing the bomb in order to hit the same?

Solution From the geometry of the system, $\tan\ \theta = \dfrac{h}{x}$

$$h = \frac{1}{2} g t^2$$

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 600}{9.8}}\ \text{s} = 11.06\ \text{s}$$

$$x = v_0.t = \frac{480}{60 \times 60} \times 11.06\ \text{km} = 1.474\ \text{km} = 1474\ \text{m}$$

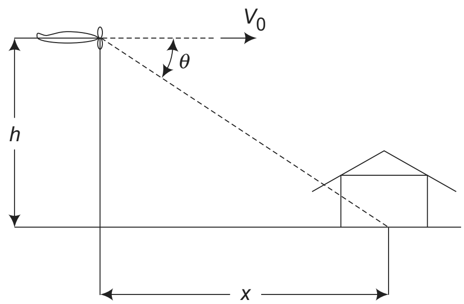


Figure 13.2

$$\therefore \quad \tan\ \theta = \frac{h}{x} = \frac{600}{1474}$$

$$\theta = 22°8'$$

Example 13.3 The maximum range of a projectile is found to be 2000 m. What should be the angle of projection so as to obtain a range of 1500 m if the velocity at which it is thrown, remains same?

Solution

$$r_{\max} = \frac{v_0^2}{g}$$

$$v_0^2 = r_{\max}\ g$$

Let the angle of projection be θ.

$$r = \frac{v_0^2 \sin 2\theta}{g} \sin\ 2\theta = r_{\max}\ \sin\ 2\theta$$

$$\sin 2\theta = \frac{r}{r_{\max}} = \frac{1500}{2000} = \frac{3}{4}$$

$$2\theta = \sin^{-1}\ (0.75) = 48.6°;\ 131.4° \quad [\text{since } \sin\ \theta = \sin\ (180 - \theta)]$$

$$\theta = 24.3°(24°18')\ \text{or}\ 65.7°(65°42')$$

Example 13.4 Two adjacent guns having the same muzzle velocity of 500 m/s, fire simultaneously at angles of projections θ_1 and θ_2 for the same target located at a range of 5000 m. Find the time difference between the two hits.

Solution Initial velocity v_0 = 500 m/s and range r = 5000 m

For 1st gun, $$r = \frac{v_0^2 \sin 2\theta}{g}$$

$$\therefore \quad 5000 = \frac{500^2 \sin 2\theta_1}{g}$$

or $$\sin 2\theta_1 = \frac{5000 \times 9.81}{500 \times 500}$$

$$2\theta_1 = 11.31$$
$$\theta_1 = 5.65°$$

Thus $$2\theta_2 = 180 - 2\theta_1 = 180 - 11.31 = 168.69°$$
$$\theta_2 = 84.345°$$

Considering the 1st gun, let t_1 be the time of flight.

$$\therefore \quad t_1 = \frac{2v_0 \sin\theta_1}{g} = \frac{2 \times 500 \times \sin 5.65}{9.81}\ \text{s} = 10.03\ \text{s}$$

Similarly, for 2nd gun, let t_2 be the time of flight.

$$\therefore \quad t_2 = \frac{2v_0 \sin\theta_2}{g} = \frac{2 \times 500 \times \sin 84.345}{9.81}\ \text{s} = 101.44\ \text{s}$$

Thus time difference between two hits $= t_2 - t_1 = 101.44 - 10.03$ sec $= 91.41$ s

Example 13.5 A person throws a stone so as to clear a wall of height 3.685 m located at a distance 5.25 m from the origin. The stone touches the ground at a distance of 3.58 m from the wall – away from the origin. Find the least initial velocity at which the stone to be thrown along with its direction.

Solution The range of the stone $r = 5.25 + 3.58$ m $= 8.83$ m

The trajectory of the stone can be expressed by $y = \tan\theta\, x - \left(\dfrac{g}{2v_0^2 \cos^2\theta}\right)x^2$

From the given situation, the top of the wall will lie on the trajectory. Thus the coordinate of the top of the wall (5.25 m, 3.685 m) will satisfy the equation.

Thus $$3.685 = \tan\theta \times 5.25 - \left(\frac{g}{2 \times v_0^2 \cos^2\theta}\right) \times 5.25^2$$

Further, range of the stone $= r = \dfrac{v_0^2 \sin 2\theta}{g} = 8.83$

$$\frac{v_0^2}{g} = \frac{8.83}{\sin 2\theta}$$

Comparing the two equations, we get

$$3.685 = \tan\theta \times 5.25 - \left(\frac{\sin 2\theta}{2 \times 8.83 \times \cos^2\theta}\right) \times 5.25^2$$

$$3.685 = \tan\theta \times 5.25 - \left(\frac{2\sin\theta\cos\theta}{2 \times 8.83 \times \cos^2\theta}\right) \times 5.25^2$$

$$3.685 = 5.25 \tan\theta - 3.12 \tan\theta = 2.13 \tan\theta$$

$$\tan\theta = \frac{3.685}{2.13} = 1.73 \Rightarrow \theta = \tan^{-1}(1.73) = 60°$$

Thus $$\frac{v_0^2}{g} = \frac{8.83}{\sin 2\theta} = \frac{8.83}{\sin 120}$$

$$v_0^2 = \frac{8.83 \times 9.81}{\sin 120} = 100$$

$$v_0 = 10 \text{ m/s}$$

Thus the stone is to be thrown with an initial velocity of 10 m/s having its inclination of 60° with the horizontal so as to just cross the wall.

Example 13.6 A projectile is thrown with an initial velocity of 400 m/s at an angle 60º with the horizontal. Find the velocity and the direction of the particle after 20 seconds from start.

Solution Initial velocity v_0 = 400 m/s and the angle of projection is θ.

The horizontal component of the velocity is $(v_{0x}) = v_0 \cos\theta = 400 \cos 60$ m/s = 200 m/s and the vertical component is $(v_{0y}) = v_0 \sin\theta = 400 \sin 60$ m/s = 346.4 m/s.

The horizontal component of the velocity remains unchanged throughout its motion.

To compute vertical component of the velocity after 20 seconds, we have

$$(v_{fy}) = v_{0y} - gt = 346.4 - 9.81 \times 20 \text{ m/s} = 150.2 \text{ m/s}$$

Thus the resultant velocity would be $v' = \sqrt{v_x'^2 + v_y'^2} = \sqrt{200^2 + 150.2^2} = 250.12$ m/s

$$\tan\alpha = \frac{v'_y}{v'_x} = \frac{150.2}{200} = 0.75$$

$$\alpha = \tan^{-1}(0.75) = 37°$$

Example 13.7 A particle is thrown with an initial velocity of 12 m/s at an angle 60º with the horizontal. If another particle is thrown from the same position at an angle 45º with the horizontal, find the velocity of the latter for the following situations.

(a) Both have same horizontal range
(b) Both have same maximum height
(c) Both have same time of flight

Solution Denoting the 1st particle as A and 2nd particle as B,
we have; v_{0A} = 12 m/s, θ_A = 60° and θ_B = 45°

(a) Both have same horizontal range:

$$r_A = r_B$$

$$\frac{v_{0A}^2 \sin 2\theta_A}{g} = \frac{v_{0B}^2 \sin 2\theta_B}{g}$$

$$v_{0B}^2 = \frac{v_{0A}^2 \sin 2\theta_A}{\sin 2\theta_B} = \frac{(12)^2 \times \sin 120}{\sin 90} = 124.7$$

$$v_{0B} = \sqrt{124.7} \text{ m/s} = 11.16 \text{ m/s}$$

(b) Both have same maximum height:

$$h_A = h_B$$

$$\frac{v_{0A}^2 \sin^2\theta_A}{2g} = \frac{v_{0B}^2 \sin^2\theta_B}{2g}$$

$$v_{0B}^2 = \frac{v_{0A}^2 \sin^2 \theta_A}{\sin^2 \theta_B} = \frac{(12)^2 \times \sin^2 60}{\sin^2 45} = 216$$

$$v_{0B} = \sqrt{216} \text{ m/s} = 14.7 \text{ m/s}$$

(c) Both have same time of flight:

$$t_{FA} = t_{FB}$$

$$\frac{2v_{0A} \sin \theta_A}{g} = \frac{2v_{0B} \sin \theta_B}{g}.$$

$$v_{0B} = \frac{v_{0A} \sin \theta_A}{\sin \theta_B} = \frac{12 \times \sin 60}{\sin 45} = 14.7 \text{ m/s}$$

Example 13.8 A projectile is launched with an initial speed of 200 m/s at an angle of 60° with respect to the horizontal. Compute the length of an inclined plane *AB* that makes an angle 22º with the horizontal such that point *B* lies on the trajectory as shown in Fig. 13.3.

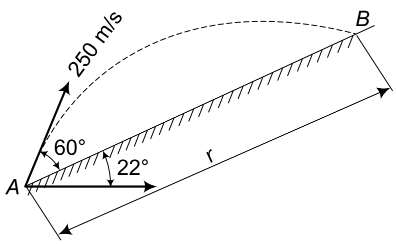


Figure 13.3

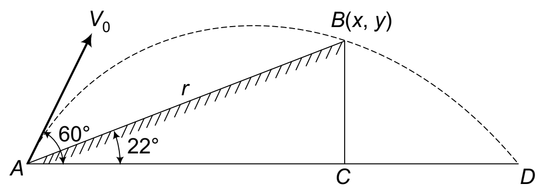


Figure 13.3 (a)

Solution The trajectory of the projectile and location of the inclined plane *AB* is shown in Fig. 13.3 (a)

The trajectory of the projectile can be expressed by $y = \tan \theta\, x - \left(\dfrac{g}{2v_0^2 \cos^2 \theta} \right) x^2$
For $\theta = 60°$, the above equation becomes

$$y = \tan 60\, x - \left(\frac{g}{2v_0^2 \cos^2 60} \right) x^2$$

$$y = 1.732\, x - \left(\frac{g}{2 \times 250^2 \cos^2 60} \right) x^2$$

$$y = 1.732\, x - 0.0003x^2 \tag{13.4}$$

Further from ΔABC, $\tan 22 = \dfrac{y}{x}$

$$y = 0.404x \tag{13.5}$$

Solving Eqs (13.4) and (13.5), we have

$$x = 4426.67 \text{ and } y = 1788.37$$

Thus $AC = 4426.67$ m; $BC = 1788.37$ m

$\therefore$ $$r = AB = r = AB = \sqrt{x^2 + y^2} = \sqrt{AC^2 + BC^2} = \sqrt{4426.67^2 + 1788.37^2} = 4774.3 \text{ m}$$

Example 13.9 A bullet is fired from a rifle with an initial velocity of 50 m/s so as to just clear a vertical wall of 20 m high and located at a distance of 30 m measured horizontally from the point of projection. Find the two angles of projection with horizontal.

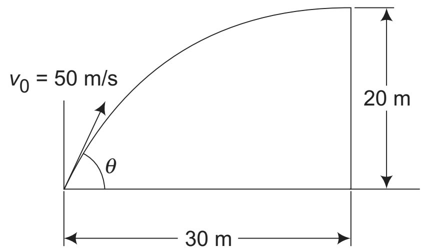


Figure 13.4

Solution The trajectory of the bullet can be expressed by using Eq. (13.3) as

$$y = \tan\theta x - \left(\frac{g}{2v_0^2\cos^2\theta}\right)x^2$$

From the given situation, the top of the wall will lie on the trajectory. Thus the coordinate of the top of the wall (30 m, 20 m) satisfies the equation. Thus, we get

$$20 = \tan\theta \times 30 - \left(\frac{g}{2\times 50^2\cos^2\theta}\right)\times 30^2$$

or
$$1.7658\tan^2\theta + 30\tan\theta - 21.7658 = 0$$

Solving the above equation, we get

$$\theta = 86.76° \text{ or } 34.87°$$

Example 13.10 A bullet is fired at an angle of 30° up the horizontal with velocity 100 m/s from the top of a tower, 50 m high. Determine:

(i) The time of flight
(ii) The horizontal range along the ground
(iii) The maximum height the bullet can attain from the ground
(iv) The velocity of the bullet after 6 sec.

Assume horizontal ground at the foot of the tower.

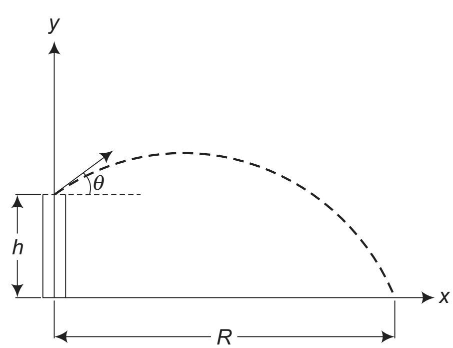


Figure 13.5

Solution The initial velocity v_0 has got two components – the horizontal component being $v_0 \cos \theta$, the vertical component is $v_0 \sin \theta$.

Let the total time of flight is T s. Also consider that the projectile will cover h' m height above the tower to reach its peak and its corresponding time is t' and during fall, it covers h m in t s.

Thus
$$T = 2t' + t$$
$$0 = v_0 \sin \theta - gt'$$
$$2t' = \frac{2v_0 \sin\theta}{g} = \frac{2 \times 100 \sin 30°}{9.81} = 10.194 \text{ s}$$

Further
$$h = v_0 \sin \theta t + \frac{1}{2} gt^2;$$
$$50 = 100 \sin 30° t + \frac{1}{2} \times 9.81 \times t^2$$

or
$$t = 0.92 \text{ s}$$

Thus total time of flight becomes $T = 2t' + t = 10.194 + 0.92 = 11.114$ s

Now,
$$0 = (v_0 \sin \theta)^2 - 2gh'$$

or
$$h' = \frac{(100 \sin 30°)^2}{2 \times 9.81} = 127.42 \text{ m}$$

∴ Total height above the ground becomes $H = h' + h = 127.42 + 50 = 177.42$ m

The range $R = (v_0 \cos \theta)T = 100 \cos 30° \times 11.114 = 962.5$ m

Since $t' = 5.097$ s, after 8 s the projectile on its downward movement from the peak height.

Thus the downward component of the velocity after 6 seconds if denoted by $v_{v,6}$, then

$$v_{v,6} = g \times 0.903 = 8.858 \text{ m/s}$$

The horizontal component being unchanged at $v_0 \cos \theta$ implying

$$v_{h,6} = 100 \cos 30° = 86.6 \text{ m/s}$$

Thus resultant velocity after 6 s is

$$v_8 = \sqrt{(v_{h,6})^2 + (v_{v,6})^2} = \sqrt{86.6^2 + 8.858^2} = 87.05 \text{ m/s}$$

Example 13.11 A particle is projected at an angle θ with the horizontal from point O, as shown in Fig. 13.6. If after time t the particle is at A, prove that $\beta = \frac{\tan\theta + \tan\alpha}{2}$, where α is the angle with the horizontal which the particle makes at A and β is the angle of elevation of point A.

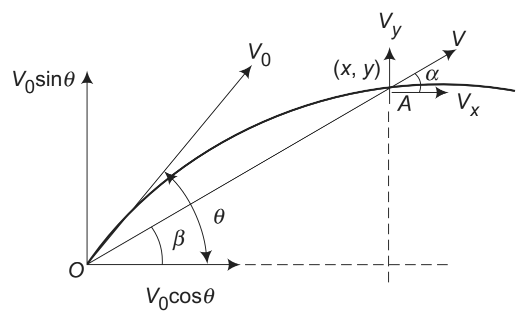


Figure 13.5

Solution Let the position of the particle is at $A(x, y)$ after time t [Fig. 13.5]. Then, using Eqs. (13.1) and (13.2), we obtain

$$x = v_0 \cos \theta t$$

and
$$y = v_0 \sin \theta t - \frac{1}{2} g t^2$$

From the above two equations, we have

$$\tan \beta = \frac{y}{x} = \tan \theta - \frac{gt}{2v_0} \sec \theta \tag{13.6}$$

Let the velocity of the particle at A is v. Then, one can write

$$v_x = v \cos \alpha = v_o \cos \theta$$

and
$$v_y = v \sin \alpha = v_0 \sin \theta - gt$$

From the above two equations, we obtain

$$\tan \alpha = \tan \theta - \frac{gt}{v_0} \sec \theta \tag{13.7}$$

Comparing (13.6) and (13.7), we get

$$\tan \beta - \tan \theta = \frac{1}{2}(\tan \alpha - \tan \theta)$$

or
$$\tan \beta = \frac{1}{2}(\tan \theta + \tan \alpha)$$

Example 13.12 The velocity of a particle when it reaches its maximum height is $\sqrt{\frac{2}{5}}$ times of its velocity when it reaches half of its maximum height. Prove that the angle of projection is 60° for such condition.

Solution Let the angle of projection is θ with an initial velocity v_0. Velocity of the particle when it reaches its maximum height is $v_0 \cos \theta$.

Let the velocity at half of its maximum height is v. Then, one can write

$$v_o \cos \theta = \sqrt{\frac{2}{5}} v$$

or
$$v^2 = \frac{5}{2} v_0^2 \cos^2 \theta$$

The horizontal component of v is $v_x = v_0 \cos \theta$ since there is no acceleration in the horizontal direction.

The maximum height of the projectile is $h = \dfrac{v_0^2 \sin^2 \theta}{2g}$

The half of the maximum height of the projectile therefore is $h' = \dfrac{1}{2} \dfrac{v_0^2 \sin^2 \theta}{2g}$

Let the vertical component of v is v_y. Then, we have

$$v_y^2 = (v_0 \sin \theta)^2 - 2g \frac{v_0^2 \sin^2 \theta}{4g} = \frac{v_0^2 \sin^2 \theta}{2}$$

Since $v^2 = v_x^2 + v_y^2$, we have

$$\frac{5}{2} v_0^2 \cos^2 \theta = (v_0 \cos \theta)^2 + \frac{v_0^2 \sin^2 \theta}{2}$$

or
$$3 \cos^2 \theta = \sin^2 \theta$$

or
$$\tan \theta = \sqrt{3} = \tan 60°$$

or
$$\theta = 60°$$

Example 13.13 A daredevil tries to jump a canyon of width 10 m. To do so, he drives his motorcycle up an incline sloped at an angle of 15° as shown in Fig. 13.6. What minimum speed is necessary to clear the canyon?

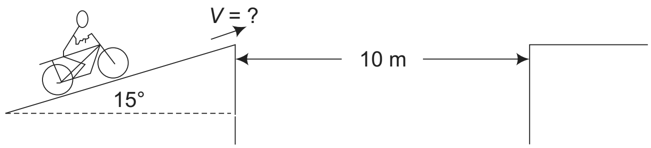


Figure 13.6

Solution Let the required velocity is v m/s, and the time required is t seconds.

The velocity v has got two components – the horizontal component being $v \cos \theta$, the vertical component is $v \sin \theta$.

Thus the distance covered by the cyclist along horizontal = the width of the canyon.

$$(v \cos 15°)t = 10$$

and
$$0 = (v \sin 15°)t = \frac{1}{2} gt^2$$

Combining the above two equations and eliminating t, we get

$$v \sin 15° \frac{10}{v \cos 15°} = \frac{1}{2} g\left(\frac{10}{v \cos 15°}\right)^2$$

or
$$v = 14 \text{ m/s}$$

Example 13.14 A brick is thrown upward from the top of a building at an angle of 25° above the horizontal and with an initial speed of 15 m/s. If the brick is in the air for 3 seconds, how high is the building? At what distance from the base of the building, the brick will touch the ground?

Solution The horizontal component of the velocity is $v_x = v_o \cos \theta = 15 \cos 25° = 13.6$ m/s

The time of flight $t = 3$ seconds

Therefore horizontal distance covered is found to be

$$= 13.6 \times 3 = 40.8 \text{ m}$$

Let the height of the building is h m. Using Eq. (13.2), we obtain

$$h = v_y t - \frac{1}{2} gt^2 = (v \sin 15°)3 - \frac{1}{2} g3^2 = -25.18 \text{ m}$$

The height of the building is 25.18 m.

MULTIPLE-CHOICE QUESTIONS

13.1 The path of a projectile is

(a) straight line (b) circular (c) parabolic (d) any arbitrary

13.2 The range of a projectile is expressed by the following expression:

(a) $r = \dfrac{v_0^2 \sin 2\theta}{g}$ (b) $r = \dfrac{v_0^2 \sin \theta}{g}$ (c) $r = \dfrac{v_0^2 \sin 2\theta}{2g}$ (d) $r = \dfrac{v_0^2 \sin \theta}{2g}$

13.3 The time of flight of a projectile is expressed by the following expression:

(a) $T = \frac{v_0 \sin\theta}{g}$ (b) $T = \frac{2v_0 \sin\theta}{g}$ (c) $T = \frac{v_0 \cos\theta}{g}$ (d) $T = \frac{2v_0 \cos\theta}{g}$

13.4 The slope of the curve (trajectory) generated by a projectile after it covers half of its range is

(a) positive (b) negative (c) zero (d) unpredictable

13.5 The horizontal component of the velocity of a projectile

(a) increases continuously (b) decreases continuously (c) remains constant (d) is zero

13.6 The maximum height attained by a projectile is expressed by the following expression:

(a) $h = \frac{v_0 \sin\theta}{g}$ (b) $h = \frac{v_0^2 \sin\theta}{g}$ (c) $h = \frac{v_0^2 \sin^2\theta}{g}$ (d) $h = \frac{v_0^2 \sin^2\theta}{2g}$

13.7 The range of a projectile is maximum when its angle of projection is

(a) 30° (b) 45° (c) 60° (d) 75°

13.8 The maximum value of the range of a projectile is

(a) $r_{max} = \frac{v_0^2}{g}$ (b) $r_{max} = \frac{v_0^2}{2g}$ (c) $r_{max} = 1.5\frac{v_0^2}{g}$ (d) $r_{max} = \frac{2v_0^2}{g}$

SHORT ANSWER TYPE QUESTIONS

13.1 What is a projectile? Give few real - life examples.
13.2 Prove that the trajectory of a projectile is a parabola.
13.3 Compute the range of a projectile and what is the condition for maximum range?
13.4 Determine the time of flight and maximum height attained by a projectile.
13.5 Prove that the slope of the curve that a projectile generates is zero when it covers half of its range.

NUMERICAL PROBLEMS

13.1 A ball is thrown horizontally from the top of a building having height 20 m. The ball touches the ground at a distance of 25 m from the base of the building. What was the initial velocity of the ball?

13.2 If the initial velocity of a projectile is increased by 20%, what would be the change in maximum range?

13.3 A particle is projected with an initial velocity of 60 m/s at an angle of 70° with the horizontal. Find the maximum height it can reach, the time of flight and the range.

13.4 The maximum range of a particle is r_{max}. Find the range, time of flight and maximum height it can reach when angle of projection is 60° having same initial velocity.

13.5 An object is projected with an initial velocity of 100 m/s at an angle of 65º with the horizontal. Find its position and velocity [both magnitude and direction] after 8 seconds.

13.6 A projectile is aimed at a target situated on the horizontal plane. It is observed that when angle of projection is 15°, it falls 12 m short of the target and it overshoots by 24 m when the angle of projection is 45°. Find the correct angle of projection so as to hit the target.

13.7 If t_1 be the time in which a projectile reaches a point on its path and t_2 be the time in which the projectile reaches the ground from the previous point, prove that the height of the point above the ground is $\frac{1}{2}gt_1t_2$.

13.8 If a particle is projected at an angle θ_1 with the horizontal from the foot of an inclined plane that makes an angle θ_2 with the horizontal, it strikes the plane at right angle. Prove that $\cot\theta_2 = 2\tan(\theta_1 - \theta_2)$.

13.9 The horizontal range of a projectile is found to be 2.3 times of its maximum height. Determine the angle of projection.

13.10 A bullet is fired at 125 m/s from a point on the ground so as to hit a target at a horizontal distance of 1000 m from the origin and at a height of 203 m from the ground. Determine (a) the angle at which the bullet is fired, (b) the maximum height attained by the bullet and (c) time required to hit the target.

13.11 A projectile is shot from the edge of a cliff 125 m above ground level with an initial speed of 65 m/s at an angle of 37° above the horizontal. Determine the magnitude and the direction of the velocity at the maximum height.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

13.1 (c)	13.3 (b)	13.5 (c)	13.7 (b)
13.2 (a)	13.4 (c)	13.6 (d)	13.8 (a)

ANSWERS TO NUMERICAL PROBLEMS

13.1 $v_0 = 12.5$ m/3

13.2 44%

13.3 $h_{\max} = 162$ m, $t_{flight} = 11.5$s, $r = 236$ m

13.4 $r = 0.866r_{\max}$, $t_{flight} = 0.553\sqrt{r_{\max}}$, $h = 0.375r_{\max}$

13.5 $x = 338$ m, $y = 411$ m, $\dot{x} = 42.26$ m/s, $\dot{y} = 12.15$ m/s

13.6 21°

13.9 60°

13.10 (a) 33°, (b) $h_{\max} = 236.23$ m, (c) $t = 9.54$s

13.11 52 m/s [forward] (horizontally)

CHAPTER

14

Work, Power, Energy

14.1 INTRODUCTION

In the chapter 11, we have discussed the effect of force(s) on the motion parameters of a particle by using Newton's second law of motion and using D'Alembert's principle. Since as a consequence to the application of force, there is displacement, it therefore follows that the force produces some work done. This work done by a force is a very important attribute to deal with power and energy analysis pertaining to a mechanical system. For example, if we consider a car or a train, it is essential to consider power developed by the engines to propel the vehicle with definite motion characteristics under load.

14.2 WORK OF A FORCE

From the basic definition of work done by a force, *it is quantified by either the product of force multiplied by the component of the displacement along the line of action of the force or displacement multiplied by component of the force along the displacement.*

Thus refer to Fig. 14.1. If a force $\boldsymbol{F}$ applied on the block at an inclination θ with the horizontal causes a displacement ds along the horizontal plane, the component of ds along F is $ds \cos \theta$. Thus work done associated with this displacement would be $dU = F(ds \cos \theta) = (F \cos \theta)ds$ = component of the force along the displacement multiplied by displacement.

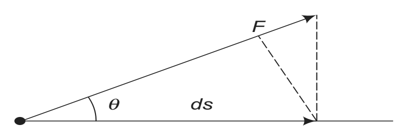


Figure 14.1

Thus work done by a force $\boldsymbol{F}$ during a finite displacement from any position P_1 to another position P_2 can be computed by $U_{1-2} = \int_{s_1}^{s_2} (F \cos \theta)\, ds$, where s_1 and s_2 are displacement along the path AB and measured from a reference frame O.

Following vector algebra, work done by a force $\boldsymbol{F}$ acting on a particle moving along any path is defined as the line integral from position P_1 at time t_1 to position P_2 at time t_2. Thus $U = \int_A F.dr$, where $\boldsymbol{dr}$ is the infinitesimal change in the position vector $\boldsymbol{r}$.

Since $\boldsymbol{F} = (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k})$, and $\boldsymbol{r} = (dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k})$,

$$U = \int_A (F_x i + F_y j + F_z k).(dxi + dyj + dzk) = \int_A (F_x dx + F_y dy + F_z dz)$$

This can be expressed further to a time integral as

$$U = \int_{t_1}^{t_2} \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} \right) dt$$

14.2.1 Graphical Interpretation of Work Done

Figure 14.2 shows the plot of $F \cos \theta$ versus s. This is called *force-displacement* diagram. The area of the infinitesimal rectangular strip of thickness ds becomes $F \cos \theta.\, ds$. Thus $U_{1-2} = \int_{s_1}^{s_2} (F \cos \theta ds)$ = entire shaded area under the curve as shown in the above Fig. 14.2.

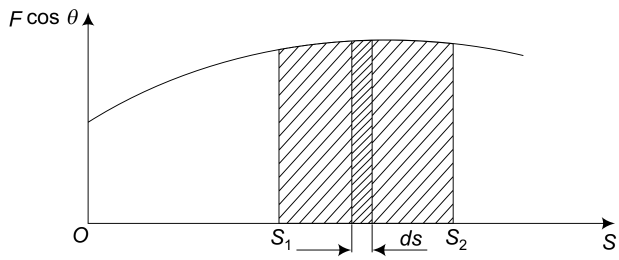


Figure 14.2

Unit: *The unit of work done following S.I. system is Nm or Joule (J). Thus one joule of work corresponds to displacement of a particle by one metre when the magnitude of force along the direction of displacement is one Newton.*

14.3 ENERGY

Energy of a particle implies its ability to do work. Thus it is the stored energy of a particle at the expense of which it can carry out some work to overcome resistance. Thus energy and work is tantamount to each other and hence the unit of energy is same as that of the work. Nevertheless, energy has many forms; thermal or heat energy, electrical energy, mechanical energy, light energy, etc. However, in regard to present discussion, we will focus our attention to only mechanical energy which has got two wings, namely, potential energy and kinetic energy.

14.3.1 Conservative Forces

A force ***F*** acting on a particle is called conservative if work corresponds to it is independent of the path followed by the particle as it changes its position. Such work can be quantified as a change in its potential energy. Gravity force is a conservative force, where as friction is non-conservative force. Other types of conservative forces are spring force and elastic force.

Following the above definition, if a particle undergoes a closed path so that initial position and final position is identical, the work is zero.

Mathematically, $$\int F.dr = 0$$

14.3.2 Work Done by Gravity Force – Potential Energy

The energy gained or expended by a particle while doing some work against the gravity force is called its potential energy.

Refer to Fig. 14.3, let a particle of mass m be moved vertically from its initial position of A to a new position B such that $AB = OB - OA = y_2 - y_1$

The force acting on the particle is gravity force = $W = mg$

Thus corresponding work done would be $U_{A-B} = \int_{y_1}^{y_2} (-mg)dy = -mg(y_2 - y_1) =$ $-mgh$ [Note that (–) sign implies the direction of force and displacement are 180°.]

Thus the potential energy of the particle is quantified by product of weight and vertical displacement.

It is interesting to note that since weight has got no horizontal components, any displacement along horizontal direction will have no effect on potential energy.

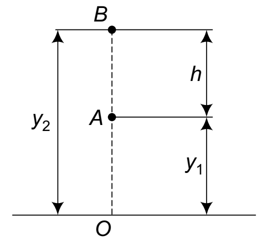


Figure 14.3

14.3.3 Work Done by Net Force – Kinetic Energy

Kinetic energy is the energy of a particle that it gains by virtue of its motion.

From Newton's second law of motion; $F = ma = m\ddot{x}$

Thus
$$F.dx = m\frac{dx}{dt}.d\dot{x} = d\left(m\frac{\dot{x}^2}{2}\right)$$

The left-hand side of the above expression shows the work done by the force ***F*** that causes a displacement of dx. The term with in the bracket in the right hand side is called *kinetic energy which is half of the product of mass and square of the velocity.*

Integrating both sides of the above equation with in two limits such that x changes its value from x_1 to x_2 and v changes its value from v_1 to v_2.

$$F(x_2 - x_1) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

This can be interpreted as *the work done associated with the force for a finite amount of displacement is equal to the change in kinetic energy of the particle between the two points*.

$$U = \int_A F.dr = \int_{r_1}^{r_2} F.dr = \int_{r_1}^{r_2} m\ddot{r}.dr = \int_{r_1}^{r_2} m\ddot{r}\frac{dr}{dt}.dt$$

$$= m\int_{t_1}^{t_2}(\ddot{r}.\dot{r})\,dt = \frac{1}{2}m\int_{t_1}^{t_2}\frac{d}{dt}(\dot{r})^2\,dt = \frac{1}{2}m[v^2]_{v_1}^{v_2}$$

$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Example 14.1 A particle of mass 5 kg is thrown up vertically with an initial velocity of 10 m/s. What is its kinetic energy (a) at the moment of release, (b) after half second, (c) after one second? Assume $g = 10$ m/s^2.

Solution

(a) Initial velocity of the particle is $v_0 = 10$ m/s

Therefore K.E. corresponds to this velocity = $\frac{1}{2}mv_0^2 = \frac{1}{2} \times 5 \times 10^2$ N-m = 250 N-m

(b) Let the velocity after $t_1 = \frac{1}{2}$ sec is v_1. Therefore, $v_1 = v_0 - gt_1 = 10 - 10 \times \frac{1}{2} = 5$ m/s

Therefore K.E. after $t_1 = \frac{1}{2}$ sec = $\frac{1}{2}mv_1^2 = \frac{1}{2} \times 5 \times 5^2$ N-m = 62.5 N-m

(c) Let the velocity after $t_2 = 1$ sec be v_2. Therefore, $v_2 = v_0 - gt_2 = 10 - 10 \times 1 = 0$
Therefore K.E. after $t_2 = 1$ sec is zero.

Example 14.2 A block of weight W is thrown with an initial velocity of v_0 along a rough horizontal plane and is brought to rest by friction in a distance x Determine the coefficient of friction.

Solution Let the coefficient of friction between the block and the floor be μ.

Thus the work done by the frictional force is $U = F_f.x = \mu N.x = \mu W.x = \mu.mg.x$

Change in K.E. $= \frac{1}{2} mv_0^2 - 0 = \frac{1}{2} mv_0{}^2$

Since the work done is equal to the change in K.E., it therefore follows that

$$\mu.mg.x = \frac{1}{2} mv_0^2$$

$$\mu = \frac{v_0^2}{2gx}$$

Thus the coefficient of friction becomes $\mu = \frac{v_0^2}{2gx}$.

Example 14.3 A block of weight W slides down an inclined plane from rest from a height h, as shown in Fig. 14.4. Find the velocity of the block when it reaches the ground.

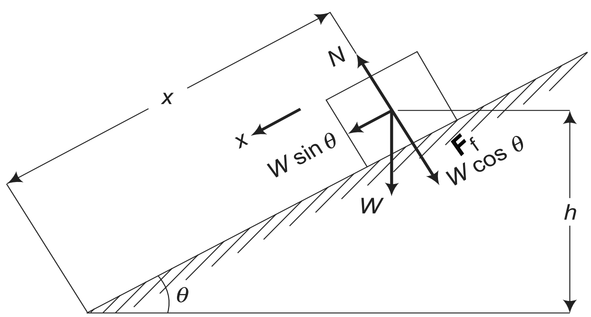


Figure 14.4

Solution Let the distance traveled by the block from initial position to the final position is x and corresponding velocity of the block is v.

Since the block is lying on the inclined plane, the net force that causes displacement along the plane is $F = W \sin \theta - \mu W \cos \theta$.

The net work done corresponds to this force $= U = Fx = [W \sin \theta - \mu W \cos \theta]x$

Thus net work done = change in K.E.

$$U = Fx = [W \sin \theta - \mu W \cos \theta]x = \frac{1}{2} mv^2$$

$$mg[\sin \theta - \mu \cos \theta]x = \frac{1}{2} mv^2$$

From the geometry, $$x = \frac{h}{\sin \theta}$$

Thus, $$g[\sin\,\theta - \mu\,\cos\,\theta]\frac{h}{\sin\theta} = \frac{1}{2}v^2$$

$$v = \sqrt{2gh(1-\mu\cot\theta)}$$

Thus the velocity of the block becomes $v = \sqrt{2gh(1-\mu\cot\theta)}$

Example 14.4 Calculate the work done on a body of mass 5 kg by 60 N force applied to it as shown in Fig. 14.5 when the same is placed on an inclined plane having inclination angle θ = 30° and associate displacement 6 m. Assume α = 15° and μ = 0.25.

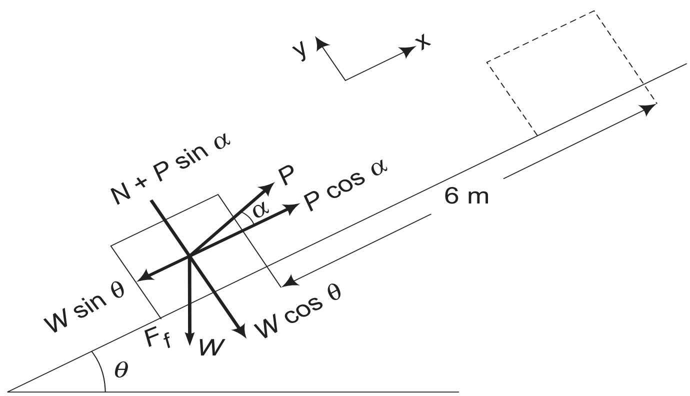


Figure 14.5

Solution From the free body diagram of the block and considering equilibrium perpendicular to the plane.

$$\sum Y = 0$$

$$N + P\,\sin\,\alpha = mg\,\cos\,\theta$$

$$N = mg\,\cos\,\theta - P\,\sin\,\alpha = 5 \times 9.8\,\cos\,30 - 60\,\sin\,15\text{ N} = 26.95\text{ N}$$

Net force on the body along the plane = F_{net}

$$P\,\cos\,\alpha - mg\,\sin\,\theta - \mu N = 60\,\cos\,15 - 5 \times 9.8\,\sin\,30 - 0.25 \times 26.95 = 26.7\text{ N}$$

Thus work done due to F_{net} = 26.7 × 6 = 160.2 Nm

Example 14.5 A bullet loses $\frac{1}{20}$ th of its velocity in passing through a wooden plank. Determine how many such uniform planks it would pass through before coming to rest. Assume that the resistance offered by the planks is uniform.

Solution Let the mass of the bullet be m kg and its velocity be v m/s.

Hence after penetrating one plank, the velocity of the bullet becomes $v' = \left(1-\frac{1}{20}\right)v = \frac{19}{20}v$

Thus loss of K.E. while passing through one wooden plank = $\frac{1}{2}mv^2 - \frac{1}{2}mv'^2 = \frac{1}{2}mv^2 - \frac{1}{2}m\left(\frac{19}{20}v\right)^2 =$ $\frac{1}{2} \times 0.0975\ mv^2$.

Let the number of planks that the bullet can penetrate is n such that after passing through n number of planks, its kinetic energy is completely lost.

Thus
$$n \times \frac{1}{2} \times 0.0975\, mv^2 = \frac{1}{2}mv^2$$
$$n = \frac{1}{0.0975} = 10.25$$

Since number of planks cannot be fraction, $n = 10$.

Therefore, the bullet can penetrate 10 planks.

Example 14.6 A particle of mass m moves linearly along x axis under the action of force $F = kx$, where k is a constant. Find the velocity v as a function of displacement x if the initial conditions of motion are $x_0 = 0$ and $\dot{x}_0 = v_0$.

Solution

$$F = kx = m\dot{x}$$
$$m\frac{d\dot{x}}{dt} = kx$$
$$m\frac{dv}{dt} = kx$$
$$mdv = kxdt = kx\frac{dt}{dx}dx$$
$$mdv = kx\frac{1}{\frac{dx}{dt}}dx = kx\frac{1}{v}dx$$
$$mvdv = kxdx$$
$$m\int_{v_0}^{v} vdv = k\int_{o}^{x} xdx$$
$$\frac{1}{2}m(v^2 - v_0^2) = \frac{1}{2}kx^2$$
$$v^2 - v_0^2 = kx^2/m$$
$$v = \sqrt{v_0^2 + kx^2/m}$$

This is the requisite expression of v.

Example 14.7 Two blocks A and B of masses 20 kg and 25 kg respectively are connected by a string as shown in Fig. 14.6. If the system is released from rest, find the velocity of the blocks after the distance moved by the block A along the table on which it rests is 1 metre. Assume the coefficient of friction between the block and the table is 0.25.

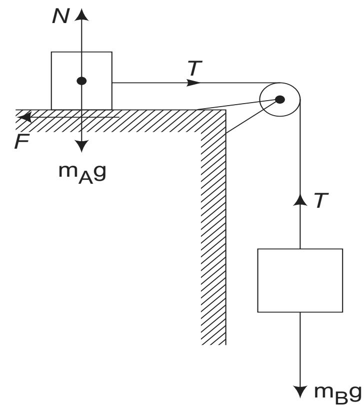


Figure 14.6

Solution m_A = 20 kg; m_B = 25 kg; x = 1 metre μ = 0.25

Let T be the tension in the string and v is the velocity of the blocks at the end.

For the block A, change in K.E. = $\frac{1}{2} m_A v^2 - 0 = \frac{1}{2} \times 20 \times v^2 = 10v^2$

Net force on the block $A = F_A = T - \mu m_A g = T - 0.25 \times 20g = T - 5g$ N

Thus work done on the block $A = F_A \times x = (T - 5g)x = (T - 5g) \times 1 = T - 5g$ N-m

For the block B, change in K.E. = $\frac{1}{2} m_B v^2 - 0 = \frac{1}{2} \times 25 \times v^2 = 12.5v^2$

Net force on the block $B = F_B = m_B g - T = 25g - T$ N

Thus work done on the block $B = F_B\, x = (25g - T)x = 25g - T$ N – m

For the entire system comprising block A and B, we have

$$T - 5g + 25g - T = 10v^2 + 12.5v^2$$

$$22.5v^2 = 20g = 20 \times 9.81$$

$$v = \sqrt{\frac{20 \times 9.81}{22.5}} \text{ m/s} = 2.95 \text{ m/s}$$

Thus the velocity of the blocks becomes v = 2.95 m/s.

14.4 PRINCIPLE OF CONSERVATION OF ENERGY

From the principle of work-energy,

Work done = Change in kinetic energy.

Thus between any two finite states, work done = $(K.E.)_2 - (K.E.)_1$

If a particle moves under the action of gravity force, work done is stored as potential energy.

Thus net work done by a particle when it changes its height between two finite positions = $(P.E.)_1 - (P.E.)_2$

Comparing these two equations, we have $(K.E.)_2 - (K.E.)_1 = (P.E.)_1 - (P.E.)_2$

$$(P.E.)_1 + (K.E.)_1 = (P.E.)_2 + (K.E.)_2$$

Thus "*the sum of the potential energy and the kinetic energy of a particle or a system comprising several particles remain constant during its motion under the action of conservative forces*". However, potential energy can be changed to kinetic energy and vice versa so as to maintain their sum at a constant value. This is popularly known as ***law of conservation of energy.***

14.4.1 Principle of Conservation of Energy – Applied to a Free Falling Particle

Let a particle of mass m be released freely from a height h. This implies initial velocity of the particle $v_0 = 0$

At this stage sum of $P.E.$ and $K.E.$ = $mgh + 0 = mgh$ (14.1)

Let us now consider any intermediate stage when the particle is in motion but not reached the ground. Let its height at present is h' from the ground as shown in Fig. 14.7. Also consider its velocity as v'.

Thus the distance moved by the particle = $h - h'$

Therefore $$v'^2 = 2g(h - h')$$

Hence, sum of P.E. and K.E. $= mgh' + \frac{1}{2}mv'^2 = mgh' + \frac{1}{2}m \times 2g(h - h')$

$$= mgh \qquad (14.2)$$

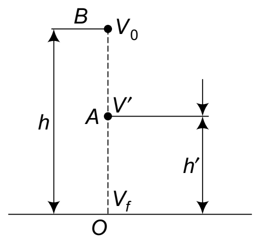


Figure 14.7

Finally, let us consider the particle is just about to touch the ground with a velocity v_f.

Thus the distance moved by the particle = h, implying $v_f^2 = 2gh$

Sum of P.E. and K.E. at this stage $= 0 + \frac{1}{2}mv_f^2 = 0 + \frac{1}{2}m \times 2gh = mgh. \qquad (14.3)$

Comparing Eqs (14.1), (14.2) and (14.3), it can be concluded that the law of conservation of energy holds good for a freely falling particle.

Example 14.8 If the system of weights as shown by solid lines [Refer Fig. 14.8] is released from rest, find the maximum distance h that the weight P will fall. Neglect friction and consider the pulleys are very small.

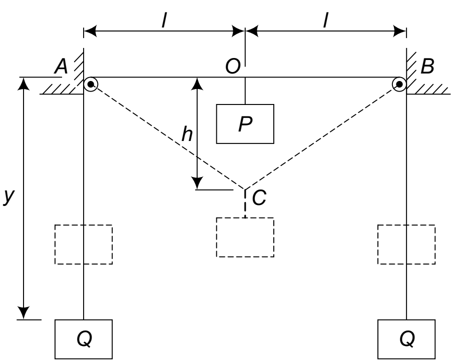


Figure 14.8

Solution Let the location of the weights Q below the line AB be y.

Thus total length of the string following initial configuration is $L_T = 2(l + y)$

From the ΔOAC, $$AC = \sqrt{OA^2 + OC^2} = \sqrt{l^2 + h^2}$$

Let the lift of the weight Q be δ. Thus the present length of the string becomes

$$2 \times [\sqrt{l^2 + h^2} + y - \delta] = L_T = 2(l + y)$$

$$\sqrt{l^2 + h^2} + y - \delta = l + y$$

$$l + \delta = \sqrt{l^2 + h^2}$$

Squaring both sides, we have $$l^2 + \delta^2 + 2l\delta = l^2 + h^2$$

$$\delta^2 + 2l\delta - h^2 = 0$$

$$\delta = \frac{-2l \pm \sqrt{4l^2 + 4h^2}}{2}$$

Since δ cannot be negative, we take $\delta = \sqrt{l^2 + h^2} - l$

The loss of potential energy by the weight P due to lowering by height h should be equal to the gain in P.E. by the weights Q.

Thus $$P.h = Q.2\delta \Rightarrow P.h = 2Q.(\sqrt{l^2 + h^2} - l)$$

$$\frac{Ph}{2Q} + l = \sqrt{l^2 + h^2}$$

Squaring both sides we get, $$\left(\frac{Ph}{2Q} + l\right)^2 = l^2 + h^2$$

$$\frac{P^2h^2}{4Q^2} + l^2 + 2 \times \frac{Ph}{2Q} \times l = l^2 + h^2$$

$$\frac{Pl}{Q} = h\left[1 - \frac{P^2}{4Q^2}\right]$$

$$h = \frac{4PQl}{4Q^2 - P^2}$$

Example 14.9 If a system of two masses $m_A = m_B = 20$ kg are arranged as shown in Fig. 14.9 are released from rest, find the velocity of the mass m_B after it has fallen a vertical distance of 2 m. Neglect the inertia of the pulleys and friction.

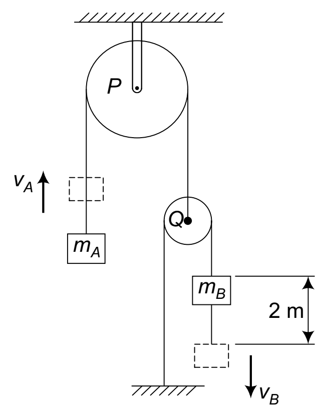


Figure 14.9

Solution From the given configuration, it can be concluded that when mass m_B is lowered by 2 m, the pulley Q is lowered by 1 m and the mass m_A is lifted up by 1 m.

At this stage, if the velocity of the mass m_B is $v_B = v$ m/s; the velocity of the mass m_A would be $v_A = \frac{v}{2}$ m/s;

Before start of the movement, the combined system is considered to coincide with datum, implying (P.E.)$_1$ as zero. Since there is no motion in the system, (K.E.)$_1$ is also zero.

When m_B is lowered by 2 m,

the total P.E. of the system = $(\text{P.E.})_2 = (\text{P.E.})_{2A} + (\text{P.E.})_{2B} = m_A g h_A - m_B g h_B = 20g \times 1 - 20g \times 2 = -20g$

Total K.E. of the system = $(\text{K.E.})_2 = (\text{K.E.})_{2A} + (\text{K.E.})_{2B} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$

$$= \frac{1}{2} \times 20 \times \left(\frac{v}{2}\right)^2 + \frac{1}{2} \times 20 \times v^2 = \frac{25}{2} v^2.$$

Thus total energy at this position is $(\text{P.E.})_2 + (\text{K.E.})_2 = -20g + 12.5v^2$

Applying the principle of conservation of energy, we get $-20g + 12.5v^2 = 0$

$$v^2 = \frac{20g}{12.5}$$

$$v = 3.962 \text{ m/s}$$

Thus the velocity of the mass m_B after it has fallen a vertical distance of 2 m becomes $v_B = 3.96$ m/s.

14.5 POWER

Power is defined as time rate of work. As regard to application, power plays very significant role compared to energy in the sense that a definite work to be delivered in a specified time. Thus in most of the engineering applications, equipments are specified in terms of their power output. Examples are a car, a motor and a railway engine.

14.5.1 Average Power

If ΔU is the work done during an interval Δt, then average power is defined by $\frac{\Delta U}{\Delta t}$.

The instantaneous power is defined as $\underset{\Delta t \to 0}{Lt} \frac{\Delta U}{\Delta t} = \frac{dU}{dt} = \frac{F.dr}{dt} = F.v$

$$dU = F \cos \theta . ds;$$

$$v = \frac{ds}{dt}$$

$$\therefore \qquad dU = F \cos \theta . vdt$$

$$\frac{dU}{dt} = (F \cos \theta).v$$

This implies that power is the product of component of the force along the velocity and the velocity.

When both are having the same direction,

$$\theta = 0;$$

$$\frac{dU}{dt} = Fv$$

Units: *The unit of power in S.I. system is Nm/s or Joule/s or Watt. A larger unit of power is kW, where 1 kW = 1000 W.*

Example 14.10 A train of mass 600 ton starts from rest and accelerates uniformly to attain a speed of 100 km/hr in 55 seconds. The total frictional resistance to motion is 20 kN. Determine (a) the maximum power required by the train, (b) the power required to maintain above speed.

Solution Initial velocity of the train is $v_0 = 0$

Final velocity of the train is $v_f = 100 \text{ km/hr} = \frac{100 \times 1000}{3600} \text{ m/s} = 27.78 \text{ m/s}$.

Time required to reach this velocity is $t = 55$ s.

From $\qquad v_f = v_0 + at$ or $a = \frac{v_f - v_0}{t} = \frac{27.78}{55} = 0.5 \text{ m/s}^2$

Let the force generated by the train be F_T

Therefore $$F_T = F_f + Fa = 20 + \frac{600 \times 1000 \times 0.5}{1000} \text{ kN} = 320 \text{ kN}$$

Thus power = $F_T.v_f$ = 320 × 27.78 kW = 8889.6 kW.

Example 14.11 A train of mass 100 ton is moving uniformly along an incline of 1 in 200 having frictional resistance as 6 N/kN. If the power produced by the engine is 1.2 × 10^5 W, find the speed of the train.

Solution Frictional resistance is @ 6 N/kN = $\frac{6 \times 100 \times 1000 \times 9.81}{1000}$ N = 5886 N

For the inclined plane, $$\tan \theta = \frac{1}{200} \approx \sin \theta$$

In the absence of any acceleration, force balance along the incline gives

$$\sum F = 0$$

$$W \sin \theta + F_f = F_T$$

$$F_T = 100 \times 1000 \times 9.81 \times \frac{1}{200} + 5886 \text{ N} = 10791 \text{ N}$$

∴ $$P = F_T.v.$$

$$1.2 \times 10^5 = 10791 \times v$$

$$v = 11.12 \text{ m/s} = 40 \text{ km/hr}$$

Therefore the speed of the train is 40 km/hr.

14.6 EFFICIENCY

The efficiency is one of the major attributes of a mechanical system or machine that indicates its performance. Machines utilise one form of energy to be converted to other form compatible to its intended application. For example in an automobile, the fuel is burnt to liberate heat energy. This heat energy is converted to mechanical energy so as to propel the vehicle. Similarly, in an electric motor, electrical energy is converted to mechanical energy. However, the output mechanical energy is not same as that of input energy; rather it is always less than the input energy. It is owing to the fact that some amount of input energy is always lost to overcome friction between various mating members. This energy is lost in the form of dissipated heat and not possible to recover for useful purpose. the performance of a machine is said to be good if it can successfully convert the input energy to the useful work. It is the efficiency of the machine that indicates its performance and is defined as the ratio of the output power to the input power.

Thus $$\text{efficiency } \eta = \frac{\text{Output energy}}{\text{Input energy}} = \frac{\text{Output power}}{\text{Input power}}$$

The value of η is always less than 1. The efficiency is usually expressed in terms of percentage (%). While computing η, both the output and input to be expressed in same unit. Higher the value of η, better is the performance. Thus in other words, efficiency signifies the amount of energy loss during conversion.

MULTIPLE-CHOICE QUESTIONS

14.1 Work is quantified by
(a) component of the force along the displacement multiplied by displacement.
(b) component of the displacement along the force multiplied by force.
(c) component of the force perpendicular to the displacement multiplied by displacement.
(d) either (a) or (b).

14.2 Work is a
(a) fixed vector.
(b) sliding vector.
(c) scalar quantity.
(d) may be scalar or vector depending on its nature.

14.3 The unit of work in S.I. system is
(a) lb-ft (b) N-m (c) joule (d) both (b) and (c)

14.4 Following category of energy is associated with conservative force.
(a) Kinetic Energy (b) Potential Energy
(c) Energy lost due to friction (d) None of the above

14.5 Following category of energy is associated with conservative force.
(a) Kinetic Energy (b) Potential Energy
(c) Energy lost due to friction (d) None of the above

14.6 The kinetic energy of a particle of mass m and velocity v is quantified by
(a) mv (b) mv^2 (c) $\frac{1}{2}mv^2$ (d) $2mv^2$

14.7 As per conservation of energy, following statement holds true
(a) $(P.E.)_1 + (K.E.)_1 = (P.E.)_2 + (K.E.)_2$ (b) $(P.E.)_1 + (P.E.)_2 = (K.E.)_1 + (K.E.)_2$
(c) $(P.E.)_1 \times (K.E.)_1 = (P.E.)_2 \times (K.E.)_2$ (d) $(P.E.)_1 \div (K.E.)_1 = (P.E.)_2 \div (K.E.)_2$

14.8 The potential energy of a particle of mass m and density ρ and at an elevation h from a reference datum is quantified by
(a) ρgh (b) mgh (c) $\frac{1}{2}\rho gh$ (d) $2mgh$

14.9 Power is
(a) time rate of force (b) time rate of work
(c) time rate of momentum (d) time rate of impulse

14.10 The unit of power in S.I. system is
(a) hp (b) Joule/s (c) Watt (d) both (b) and (c)

SHORT ANSWER TYPE QUESTIONS

14.1 Define work and energy. What are their units in S.I. system?
14.2 Explain work by vector algebra.
14.3 State and prove the principle of work and energy.
14.4 What do you mean by *conservative force*? Give examples.
14.5 Prove the conservation of energy for a free falling particle.
14.6 What is power? How it is quantified? What is its unit in S.I. system?
14.7 From the Newton's second law of motion, prove that change in kinetic energy is equal to the work done.

14.8 Illustrate work from *force-displacement* diagram.
14.9 State and prove the *law of conservation of energy*.
14.10 Define potential energy? Does its change in magnitude depend on path?

NUMERICAL PROBLEMS

14.1 A force of 500 N is applied on a block of mass 50 kg that is resting on a rough horizontal floor having coefficient of friction 0.4. Determine the velocity of the block after it covers a distance of 8 m.

14.2 A small block of weight W = 50 N is given an initial velocity v_0 = 5 m/s down the inclined plane as shown in Fig. 14.10. If the coefficient of friction between the plane and the block is 0.25, find the velocity v_f of the block at B after it has travelled a distance $AB = x$ = 2 m.

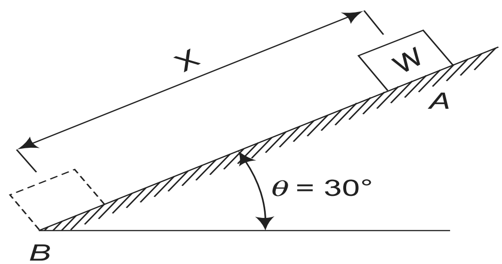


Figure 14.10

14.3 Find the total work done on an 8 kg block that is acted upon by the 100 N force and consequently moves to the right by 4 m, as shown in Fig. 14.11. Assume coefficient of friction is 0.30.

14.4 A car of weight W = 20 kN is driven down a 7° incline at a speed of 95 km/hr when the brakes are applied. The total braking force is estimated as 7kN. Determine the distance travelled by the car before it stops.

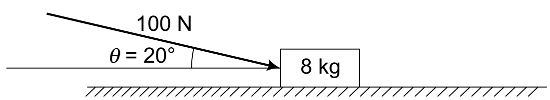


Figure 14.11

14.5 A bullet of mass m can penetrate a thickness t of a fixed plate of mass M. Show that if the plate is free to move, the thickness it can penetrate is $t' = \left(\dfrac{M}{M+m}\right)t$.

14.6 A block of mass 10 kg is released from rest so that it can slide down a distance of 8 m along an incline of slope 30°. If the coefficient of friction between the plane and the block is 0.25, find the time required by the block to cover the inclined path.

14.7 A block "***A***" of mass 20 kg rests on an inclined plane that is connected to another block "***B***" of mass 40 kg by a string that passes over a frictionless pulley as shown in Fig. 14.12. If both the masses are released from rest, calculate their velocities after the block B is lowered by 0.5 m.

14.8 Two weights P and Q are hung and initially at rest as shown in Fig. 14.13. Find the velocity of the falling weight P when it covers a vertical distance of 3 m. Given $P = Q$ = 10 N; r_2 = 100 mm; r_1 = 150 mm.

14.9 A train weighing 1000 kN is accelerated uniformly up an inclined plane of 2% grade. The velocity increases from 9 m/s to 18 m/s in a distance of 600 m. The resistance offered by the track is 50 N per kN weight of the train. Determine the maximum power developed by the train.

14.10 An engine weighing 500 kN, moving on a horizontal track, attains a speed of 40 km/hr in 4 minutes starting from rest following uniform acceleration. Take the resistance to motion as 5 N/kN. Determine the power developed by the engine.

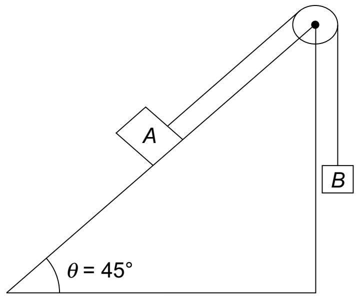


Figure 14.12

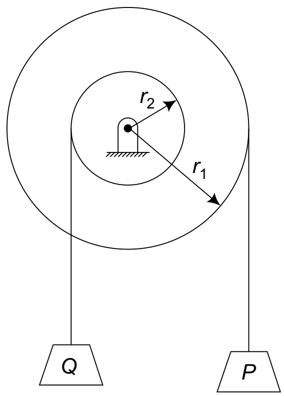


Figure 14.13

14.11 A 5 kg block slides 3 m on a horizontal floor. If the coefficient of friction between the floor and the block is 0.25, determine (a) the work done by the block on the surface, (b) the work done by the surface on the block.

14.12 A 5 kg block slides down by 2 m along an inclined plane that makes an angle 40° with the horizontal. If the coefficient of sliding friction is 0.40, determine the work done by all forces acting on the block. Also calculate its velocity when it touches the ground.

14.13 A particle moves along the path $x = 2t$, $y = t^3$ where t is in seconds and distances are in metres. What is the work done in the interval from $t = 0$ to $t = 3$ s by a force whose components are $F_x = 2 + t$ and $F_y = 2t^2$. Forces are in Newton.

14.14 An automobile weighing 20 kN is driven down a 5° incline at a speed of 85 km/h when the brakes are applied. The total braking force exerted by the road on the tier is 7000 N. Determine the distance covered by the automobile before it comes to a halt.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

14.1 (d)	14.2 (c)	14.3 (b)	14.4 (b)	14.5 (c)
14.6 (c)	14.7 (a)	14.8 (b)	14.9 (b)	14.10 (c)

ANSWERS TO NUMERICAL PROBLEMS

14.1	9.86 m/s	14.2	$v_f = 6$ m/s	14.3	241 N-m
14.4	55.7 m	14.6	2.82 s	14.7	2.9 m/s
14.8	3.685 m/s	14.9	1356 kW	14.10	54 kW
14.11	36.78 N-m	14.12	32.95 N-m, 3.63 m/s	14.13	313 N-m
14.14	108 m				

Model Question Paper
Set-1
ENGINEERING MECHANICS
SEMESTER – 1

Time: 3 Hours | Full Marks: 70

Group – A
(Multiple-Choice Questions)

1. Choose the correct alternatives for the following: $10 \times 1 = 10$

(i) Two non-collinear parallel equal forces acting in opposite directions
(a) balance each other (b) constitute a moment
(c) constitute a couple (d) constitute a moment of a couple
(e) constitute a resultant couple

(ii) The centre of gravity of a uniform lamina lies at the
(a) centre of the heavy portion (b) bottom surface
(c) mid point of its axis (d) all of these
(e) none of these

(iii) The ratio of limiting friction and reaction is known as
(a) coefficient friction (b) angle of friction (c) angle of repose (d) sliding friction
(e) friction resistance

(iv) D' Alembert's principle is applied to solve problems related to
(a) statics (b) strength of structures (c) dynamics (d) none of these

(v) Materials having the same elastic properties in all directions are called
(a) ideal materials (b) isotropic materials (c) elastic materials (d) uniform materials

(vi) For stable equilibrium, the potential energy will be
(a) maximum (b) minimum (c) zero (d) none of these

(vii) When a body slides down an inclined surface of inclination θ, the acceleration 'f' of the body is given by
(a) $f = g$ (b) $f = g \sin \theta$ (c) $f = g \cos \theta$ (d) $f = \tan \theta$
(e) $f = g/\sin \theta$

(viii) According to the principle of transmissibility of forces, the effect of a force upon a body is
(a) maximum when it acts at the centre of gravity of the body
(b) different at different points in its line of action
(c) the same at every point in its line of action
(d) minimum when it acts at the CG of the body
(e) none of the above

(ix) Poisson's ratio is defined as
(a) longitudinal stress and longitudinal strain (b) longitudinal stress and lateral stress
(c) lateral stress and longitudinal stress (d) lateral stress and lateral strain
(e) none of the above

(x) The maximum strain energy that can be stored in a body is known as the
(a) impact energy
(b) resilience
(c) proof resilience
(d) modulus of resilience
(e) toughness

Group – B
(Short-Answer Type Questions)

Answer any *three* questions $3 \times 5 = 15$

2. (i) State and prove Lami's theorem.
(ii) Define free-body diagram.
3. State and prove the perpendicular axis theorem of the area moment of inertia.
4. A circular roller of weight 100 N and 10-cm radius hangs by a tied rod AB = 20 cm and rest against a smooth vartical wall at C as shown in the Fig. 1. Determine the force F in the rod.

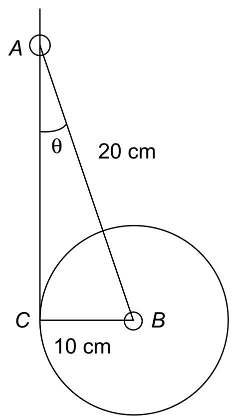


Figure 1

5. A particle moves along the path $y = \frac{1}{3}x^2$ with a constant velocity of 8 m/s. What are the x and y components of the velocity when $x = 3$? What is the corresponding acceleration? Note that x and y are expressed in metres.

Group – C
(Long-Answer Type Questions)

Answer any *three* questions $3 \times 15 = 45$

6. (i) Two blocks (1) and (2) are resting on the horizontal surface as shown in Fig. 2. The block 2 is attached to a vertical wall by an inclined string AB. Find the magnitude of the horizontal force P that will be necessary to cause slipping to impend. Take μ between the surface and block as 0.4 and μ between the blocks as 0.25.

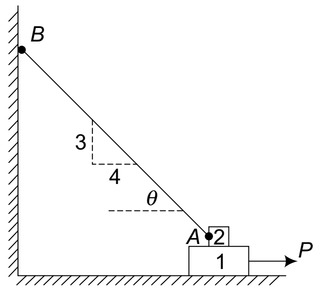


Figure 2

(ii) Two smooth spheres, each of radius r = 150 mm and weight W = 100 N, rest in a horizontal channel having vertical walls, the distance between them is w = 560 mm, as shown in Fig. 3. Find the reactions exerted at their points of contacts by the walls and the floor.

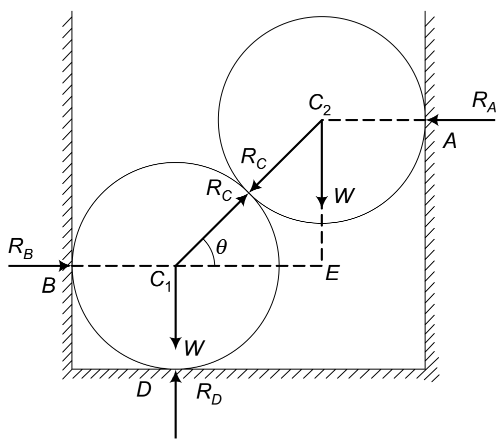


Figure 3

7. (i) Two blocks having weights $W_1 = W_2$ = 20 N are attached by a short string and rest on an inclined plane as shown in Fig. 4. If the coefficients of friction for the blocks are μ_1 = 0.2 and μ_2 = 0.3 respectively, find the angle of inclination of the plane for which motion impends.

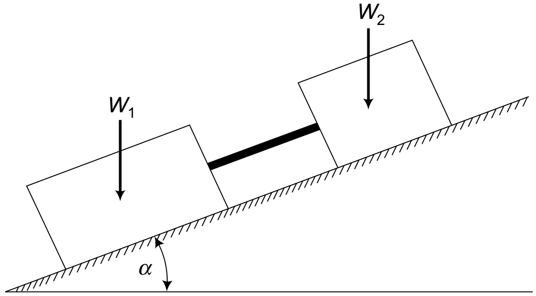


Figure 4

(ii) Determine the moment of inertia of the *T* section with respect to a centroidal axis parallel to the *x* axis as shown in Fig. 5. Consider all the dimensions to be in mm.

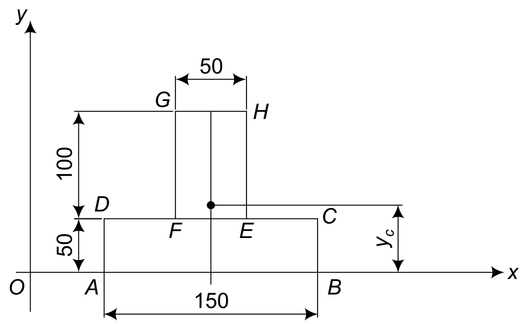


Figure 5

8. (i) Using the Pappus and Guldinus theorem, compute the volume of a right-circular cone of height *h* and base radius *r*.

(ii) The bar *AB* weighs 250 N is supported by a wall at *C* and a horizontal cable as shown in the Fig. 6. Assuming all surfaces are smooth, find the cable tension and forces at *A* and *C*.

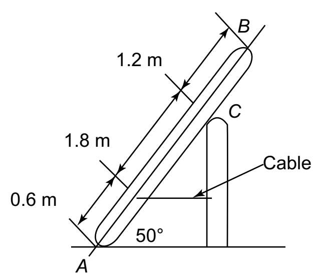


Figure 6

9. (i) A lever is attached to a spindle by means of a square key 6 mm × 6 mm × 2.5 cm long as shown in the Fig. 7. If the average shear stress in the key is not to exceed 700 kg/cm^2, what is the safe value of the load *P* applied at the free end of the lever?

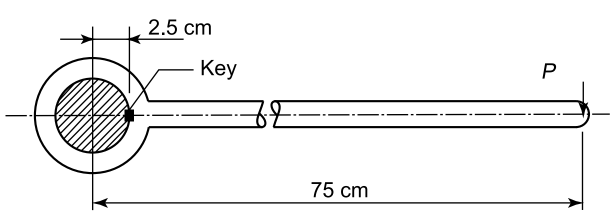


Figure 7

(ii) A body of 5 kg mass drops freely from a height of 60 m and penetrates the ground by one metre. Find the average resistance of penetration and the time of penetration.

10. (i) A train weighing 1000 kN is accelerated uniformly up an inclined plane of 2% grade. The velocity increases from 5 m/s to 15 m/s in a distance of 500 m. The resistance offered by the track is 50N per kN weight of the train. Determine the maximum power developed by the train.

(ii) The motion of a particle is defined by the following equations:

$x = \dfrac{(t-4)^3}{6} + t^2$ and $y = \dfrac{t^3}{6} - \dfrac{(t-1)^2}{4}$, where x and y are expressed in metre and t is in second. Determine (a) the magnitude of the smallest velocity reached by the particle, and (b) the corresponding time, position and direction of the velocity.

Model Question Paper
Set-2
ENGINEERING MECHANICS
SEMESTER – 1

Time: 3 Hours | Full Marks: 70

Group – A
(Multiple-Choice Questions)

1. Choose the correct alternatives for the following: $10 \times 1 = 10$

(i) Lami's theorem is related to
- (a) equilibrium of two coplanar, concurrent forces
- (b) equilibrium of three coplanar, concurrent forces
- (c) equilibriums of four coplanar, concurrent forces
- (d) none of the above

(ii) Moment of a force is obtained by
- (a) vector addition (b) vector subtraction (c) vector dot product (d) vector cross product

(iii) A block of 40 N is placed on a horizontal plane having $\mu = 0.4$. A pull force of P is applied at an angle θ with the horizontal. The minimum value of P will be
- (a) 10.86 N (b) 12.37 N (c) 14.85 N (d) 20 N

(iv) Two non-collinear parallel equal forces acting in opposite directions
- (a) balance each other (b) constitute a moment
- (c) constitute a couple (d) constitute a moment of a couple

(v) An elevator weighing 1000 N attains an upward velocity of 5 m/s in 3 s following a uniform acceleration. The tension in the cables that support the elevator is
- (a) 850 N (b) 1000 N (c) 1250 N (d) 1500 N

(vi) When the temperature increases, the thermal stress induced in a bar is
- (a) tensile (b) compressive (c) shear (d) unpredictable

(vii) The following category of energy is associated with conservative force:
- (a) Kinetic energy (b) Potential energy
- (c) Energy lost due to friction (d) None of the above

(viii) Poisson's ratio is defined by
- (a) ratio of lateral strain and longitudinal strain (b) ratio of longitudinal strain and lateral strain
- (c) ratio of lateral stress and longitudinal stress (d) ratio of longitudinal stress and lateral stress

(ix) The moment of inertia of a rectangle having base b and height h with respect to its base is
- (a) $\frac{1}{3} b^2h^2$ (b) $\frac{1}{3} b^3h$ (c) $\frac{1}{3} bh^3$ (d) $\frac{1}{3} b^2h^2$

(x) The energy absorbed by materials per unit volume up to a proportional limit is called
- (a) resilience (b) proof resilience
- (c) modulus of resilience (d) toughness

Group – B
(Short-Answer Type Questions)

Answer any *three* questions $3 \times 5 = 15$

2. What is a couple? Under what circumstances is it formed? Replace a force by an equivalent force–couple system.
3. Prove that work done is a scalar product of two vectors.
4. The acceleration of a particle at any point A is expressed by the relation $a = 200x(1 + kx^2)$; where a and x are expressed in m/s^2 and metres respectively, and k is a constant. If the velocity of the particle at A is $v_A = 2.5$ m/s when $x = 0$ and $v_A = 5$ m/s when $x = 0.15$ m; find the value of k.
5. Draw the stress–strain diagram of mild steel and explain the implications of the salient points. What is the interpretation of area under such a curve?
6. What is meant by angle of repose? What should be its value for a particular surface? Prove the same.

Group – C
(Long-Answer Type Questions)

Answer any *three* questions $3 \times 15 = 45$

7. (i) A right-circular roller of weight W rests on a smooth horizontal plane and is subjected to a pull force P, as shown in Fig. 1. It is held in position by a string AC. Find the tension T in the string AC and reaction R_B at B.

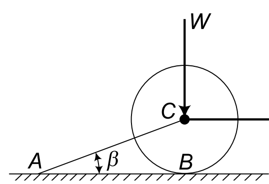


Figure 1

(ii) State Coulomb's laws of friction.

(iii) The bracket shown in Fig. 2 is spot welded to the end of the shaft at the point O and is subjected to a 900 N force. Find out the equivalent force and couple to replace the 900 N force.

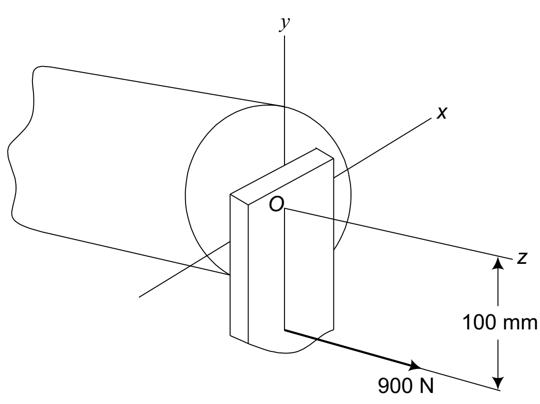


Figure 2

8. (i) A right-circular cylinder of mass m = 10 kg rests on a ***V*** block having an included angle of 90°. The ***V*** block is now inclined by 30° with the horizontal as shown in Fig. 3. If the coefficient of friction between the cylinder and the ***V*** block is μ = 0.5, determine (a) the friction force F acting on each side of the cylinder before the application of force P, and (b) the magnitude of P so that the cylinder is on the verge of sliding up the plane.

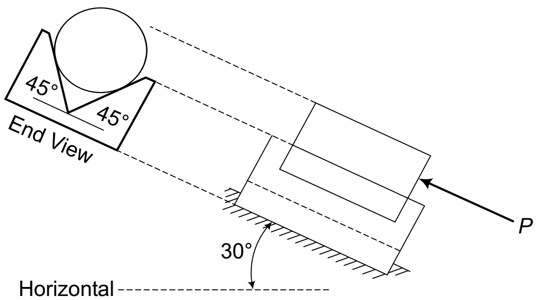


Figure 3

(ii) Determine the moment of inertia of the shaded area with respect to the point O as shown in Fig. 4.

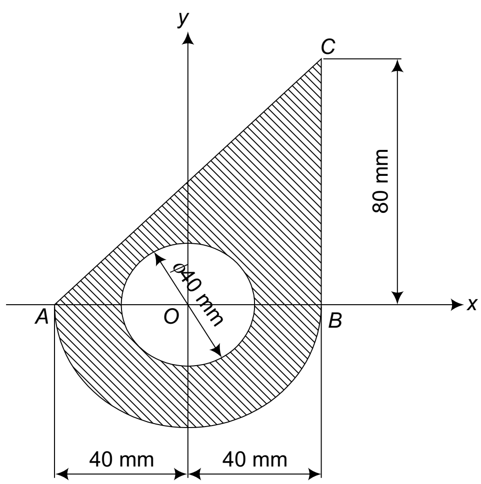


Figure 4

9. (i) A 100 mm diameter shaft has a projected collar of 130 mm diameter over a length of 20 mm and is supported by a hollow structure as shown in Fig. 5. The shaft is subjected to an axial load of 500 kN. Find the shear stress induced in the collar.

(ii) A mass M resting on a smooth table is connected to masses M_1 and M_2 by strings as shown in Fig. 6. Find the acceleration of the system, assuming M_1 is moving down.

10. (i) A lever is loaded by various forces and a couple as shown in Fig. 7. If the resultant of these forces and the couple passes through O, calculate $\boldsymbol{M}$.

(ii) A person throws a stone so as to clear a wall of 3.685 m height located at a distance of 5.25 m from the origin. The stone touches the ground at a distance of 3.58 m from the wall—away from the origin. Find the least initial velocity at which the stone was thrown along with its direction.

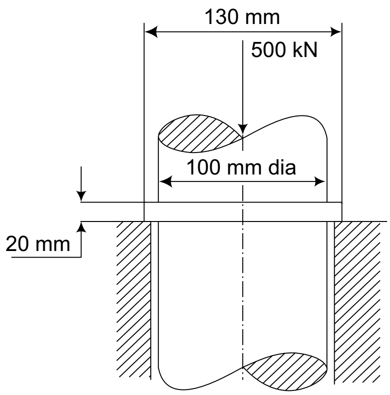


Figure 5

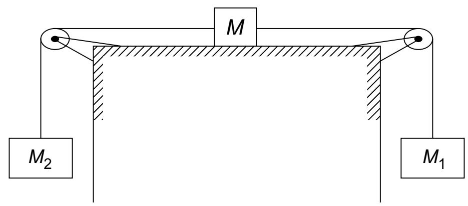


Figure 6

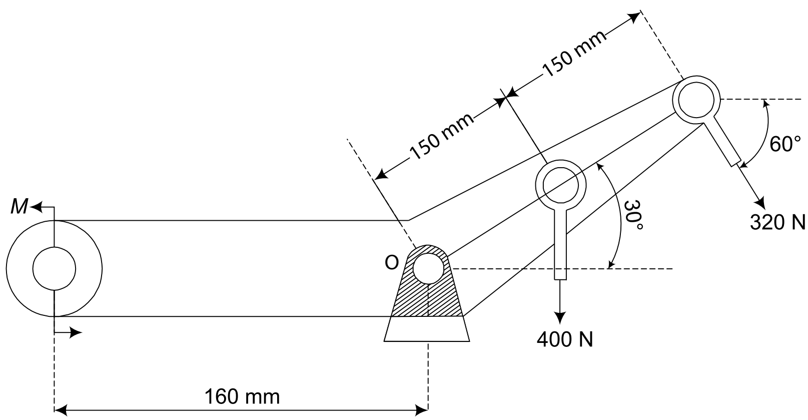


Figure 7

11. (i) A motorcycle and rider having a total weight W = 2225N travels in a vertical plane following a curve AB of radius r = 300 m, at a speed of 72 km/hr. Compute the thrust exerted by the road as it passes over the crest C on the curve, as shown in the Fig. 8.

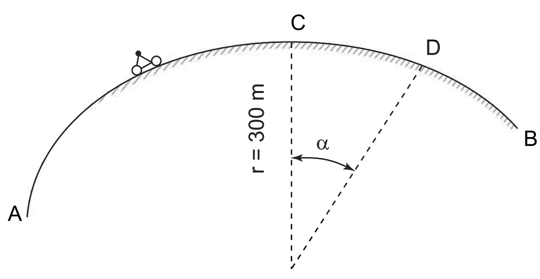


Figure 8

(ii) The frame shown in Fig. 9 is made of a 10 cm × 10 cm square wooden post, for which the allowable stress in shear parallel to the grain is τ_w = 7 kg/cm^2, while that in compression perpendicular to the grain is σ_w = 28 kg/cm^2. Calculate the minimum safe values of the dimensions a, b and c. The vertical post is pinned at its lower end to a foundation plate.

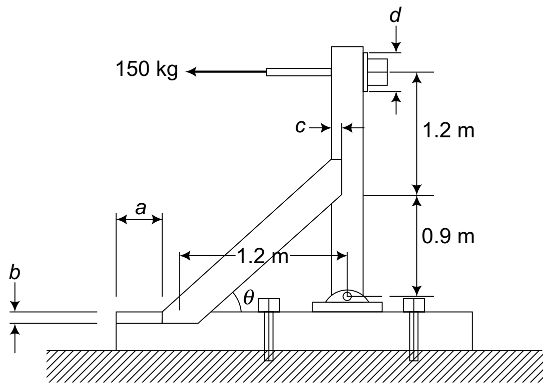


Figure 9

12. (i) If a system of two masses $m_A = m_B = 20$ kg are arranged as shown in Fig. 10 are released from rest, find the velocity of the mass m_B after it has fallen a vertical distance of 2 m. Neglect the inertia of the pulleys and friction.

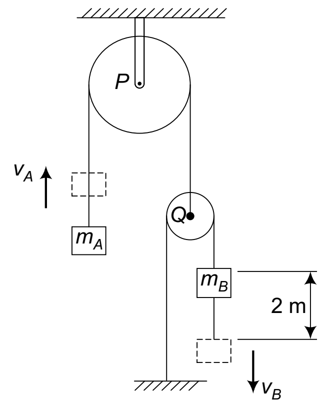


Figure 10

(ii) Compute the coordinates (x_c, y_c) of the centroid C of the sector of the general spandrel OAB as shown in Fig. 11.

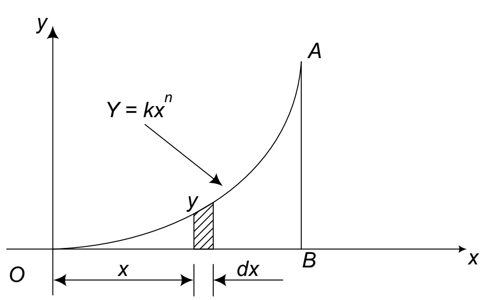


Figure 11

ENGINEERING & MANAGEMENT EXAMINATIONS, DECEMBER-2008
MECHANICAL SCIENCES
SEMESTER - 1

[Time: 3 Hours] [Full Marks: 70]

Group-A

(Multiple-Choice Type Questions)

1. Choose the correct alternatives for the following: $10 \times 1 = 10$

(i) Free body diagram can be applied only in

(a) a dynamic equilibrium problem (b) a static equilibrium problem

(c) both dynamic and static equilibrium problem (d) none of these

Answer (c)

(ii) The condition of equilibrum for coplanar non-concurrent forces are

(a) $\sum F_X = 0; \sum F_Y = 0$ (b) $\sum F_X = 0; \sum M = 0$

(c) $\sum F_Y = 0; \sum M = 0$ (d) $\sum F_X = 0; \sum F_Y = 0; \sum M = 0$

Answer (d)

(iii) The centre of gravity of a solid hemisphere of radius R is

(a) $3R/8$ (b) $R/2$ (c) $3R/4$ (d) none of these

Answer (a)

(iv) The equation of motion of a particle is $s = 2t^3 - t^2 - 2$ where s is the displacement in metres and t is time in seconds. The acceleration of the particle after 1 second will be

(a) 8 m/s^2 (b) 9 m/s^2 (c) 10 m/s^2 (d) 5 m/s^2

Answer (c)

(v) When a body slides down an inclined surface of inclination θ, the acceleration of the body is given by

(a) $f = g$ (b) $f = g \sin \theta$ (c) $f = g \cos \theta$ (d) $f = g/\sin \theta$

Answer (b)

(vi) The maximum strain energy that can be stored in a body is known as

(a) impact energy (b) resilience (c) proof resilience (d) modulus of resilience

Answer (c)

(vii) When two ships are moving along inclined directions then the time when the two ships will be closest together depends upon

(a) velocity of one of the ships
(b) velocity of both the ships
(c) angle between the two directions
(d) all of these

Answer (d)

(viii) The maximum height of a projectile on ahoriontal range is

(a) $\frac{(u^2 \sin 2\alpha)}{2g}$
(b) $\frac{(u^2 \sin \alpha)}{2g}$
(c) $\frac{(u^2 \sin^2 2\alpha)}{2g}$
(d) $\frac{(u^2 \sin^2 \alpha)}{2g}$

Answer (d)

(ix) The differential equation of a falling body under gravity is

(a) $\ddot{x} = 0, \ddot{y} = 0$
(b) $\ddot{x} = 0, \ddot{y} = g$
(c) $\ddot{x} = c, \ddot{y} = g$
(d) $\ddot{x} = 0, \ddot{y} = 0$
(e) none of these

Answer (e)

(x) If a momentum of a body is doubled, its kinetic energy will
(a) increase by two times
(b) increase by four times
(c) remain same
(d) get halved
(e) reduce to four times

Answer (b)

Group-B

(Short-Answer Type Questions)

$3 \times 5 = 15$

2. What do you mean by a free-body diagram? Draw the free-body diagrams of the following as shown below: $1 + 2 + 2$

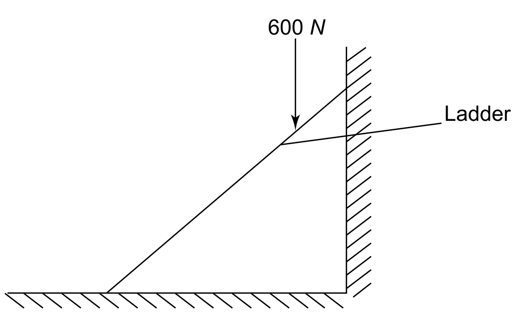


Fig. Q.1

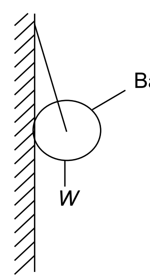


Fig. Q.2

Answer

A free-body diagram, abbrebiated as FBD, is a very useful aid to solve the problems of mechanics. The very name implies that a member for which we focus our attention for the purpose of force analysis should be isolated from various constraints and all the forces acting on it, both active and reactive, should be shown without altering its directions. This is a simplified scheme of the actual problem but helps in a bigger way to arrive at its solution.

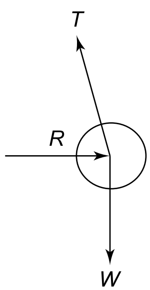


Fig. Q.3 Free-body diagram of the ball

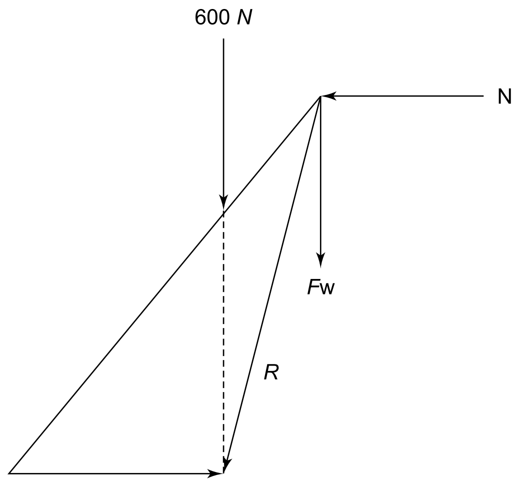


Fig. Q.4 Free-body diagram of the ladder

3. Determine the force P required to intend the motion of the block B shown in the Fig. Q.5. Take $\mu = 0.3$ for all surfaces of contact, where μ = coefficient of friction. 5

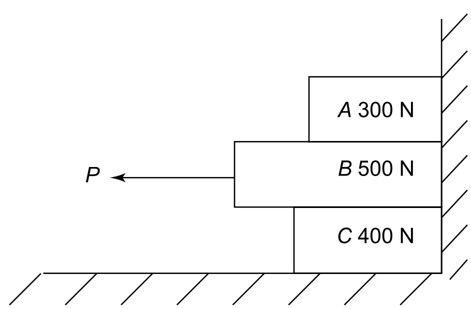


Fig. Q.5

Answer

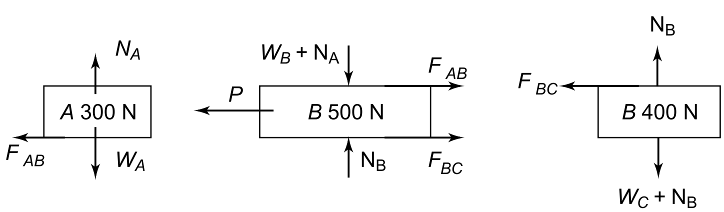


Fig. Q.6

Since all the blocks are placed one over the other, as per Newton's third law, there would be mutual normal reactions.

As regards the middle block B, under the equilibrium condition when there is impending motion, the minimum force P will be just sufficient enough to overcome the frictional forces resulted from two mating surface, namely between A and B and between B and C.

The free-body diagrams of all the blocks are shown above.

Thus for the block A, $N_A = 300$ N and $F_{AB} = \mu$.NA $= 0.3 \times$ N $= 90$ N

For the block B, considering $\sum Y = 0$

$N_A + W_B = N_C$

$N_C = 300 + 500$ N $= 800$ N

Similarly for the block C,

$\sum Y = 0$

$N_B + W_C = N_C$

$N_B = 800 - 400$ N $= 400$ N

$\therefore F_{BC} = \mu N_C = 0.3 \times 800$ N $= 240$ N

For the block B, considering $\sum X = 0$

$P = F_{AB} + F_{BC} = 90 + 240$ N $= 330$ N

4. A force $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$ acts at a point A whose coordinates are (1 − 2, 3) m

   Compute

   (a) moment of force about origin (b) moment of force about the point B(2, 1, 2) m

Answer

(a) The force vector $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$ and the position vector for point of application of force at A is $\mathbf{r_{AO}} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

Thus as per definition, $\mathbf{M_O} = \mathbf{r_{AO}} \times \mathbf{F} = (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}) = (-24 + 12)\mathbf{i} + (9 - 12)\mathbf{j} + (-4 + 6)\mathbf{K} = -12\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(b) The position vector now becomes $\mathbf{r_{AB}} = (1 - 2)\mathbf{i} + (-2 - 1)\mathbf{j} + (3 - 2)\mathbf{j} + (3 - 2)\mathbf{k} = -3\mathbf{j} + \mathbf{k}$ $\mathbf{M_B} = \mathbf{r_{AB}} \times \mathbf{F} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}) = (-36 + 4)\mathbf{i} + (3 + 12)\mathbf{j} + (4 + 9)\mathbf{k} = -32\mathbf{i} + 15\mathbf{j} + 13\mathbf{k}$

5. If the string AB is horizontal, find the angle that the string AC makes with the horizontal when the ball is in a position of equilibrium. Also, find the pressure R between the ball and the plane.

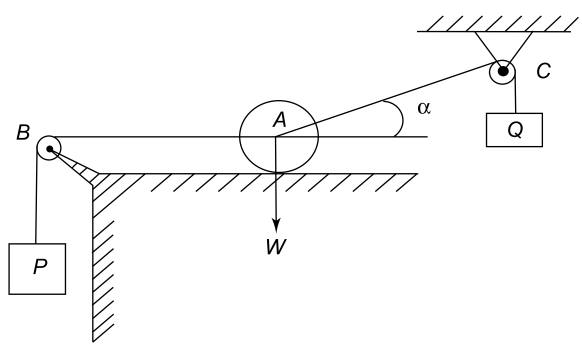


Fig. Q.7

Answer The free-body diagram of the roller is shown in Fig. Q.8

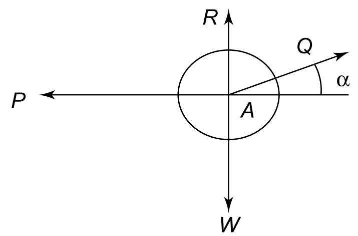


Fig. Q.8

Resolving all the forces alobg the *x* and *y* directions

$$\sum X = 0$$

$$P = Q \cos \alpha$$

$$\cos \alpha = \frac{P}{Q}$$

$$\alpha = \cos^{-1}\left(\frac{P}{Q}\right)$$

$$\sum Y = 0$$

$$R + Q \sin \alpha = \mathrm{W}$$

$$R = W - Q \sin \alpha = W - \sqrt{Q^2 - P^2}$$

Group-C

(Long-Answer Type Questions)

Answer any three of the following questions $3 \times 15 = 45$

6. (a) A short semi-circular right cylinder of radius r and weight W rests on a horizontal surface and is pulled at right angles to its geometric axis by a horizontal force P applied at the middle B of the frontage as shown. Find the angle α that the flat face will make with the horizontal plane just before sliding vbegins if the coefficient of friction at the line of contact A is μ. The gravity force W must be considered as acting at the centre of gravity C as shown in Fig. Q.9.

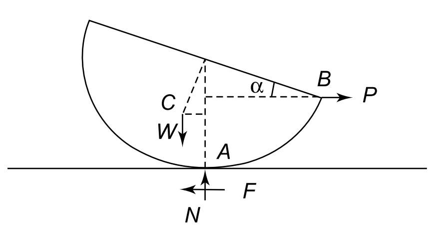


Fig. Q.9

Answer Considering the equilibrium of the semi-circular cylinder during impending motion

$$\sum X = 0;\ P = F_f = \mu N \tag{1}$$

$$\sum Y = 0;\ N = W \tag{2}$$

and

$$\sum M_A = 0; W \times \frac{4r}{3\pi}\sin\alpha = P \times (r - r\sin\alpha) \tag{3}$$

Combining equations (1), (2) and (3);

$$W \times \frac{4r}{3\pi}\sin\alpha = \mu W r(1 - \sin\alpha)$$

$$\frac{4r}{3\mu r} = \frac{1 - \sin\alpha}{\sin\alpha}$$

$$\frac{1}{\sin\alpha} = 1 + \frac{4}{3\mu\pi}$$

$$a = \sin^{-1}\left[\frac{3\mu\pi}{4 + 3\mu\pi}\right]$$

(b) Determine the coordinates of the centroid *C* of the area of the circular sector *OBD* of radius *r* and central angle α. (7)

Answer The problem is worked out in example 5.4 (page 5.9).

7. (a) The following details refer to the bar as shown:

Portion	Length	Cross-section
AB	600 mm	40×40 mm
BC	800 mm	30×30 mm
CD	1000 mm	20×20 mm

If the load P_4 = 80 kN, P_2 = 60 kN and P_3 = 40 kN, find the extension of the bar, where $E = 2 \times 10^5$ N/mm^2.

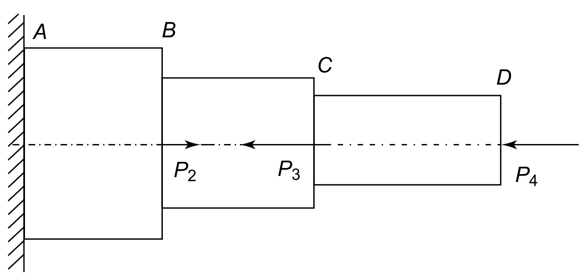


Fig. Q.10

Answer These types of problems are solved following the method of superimposition, i.e., individual sections are in equilibrium under load as well as the entire bar. Thus, force balance of the bar axially yielding that reaction force generated by the support is $R = P_3 + P_4 - P_2 = 80 + 40 - 60$ N = 120 N.

Now the free-body diagrams of the three different sections are shown in Fig. Q.11.

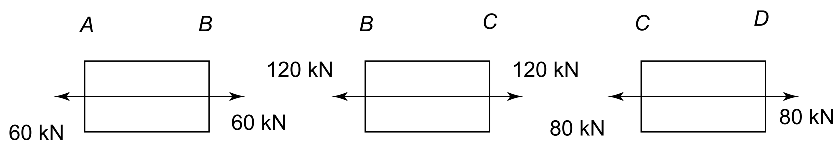


Fig. Q.11

Introducing suffices for the above three situations by AB, BC and CD and having the modulus of elasticity $E = 2 \times 105$ N/mm^2, we have

$$\delta_{AB} = \frac{60 \times 10^3 \times l_{AB}}{A_{AB}\ E} = \frac{60 \times 1000 \times 600}{40 \times 40 \times 2 \times 10^5} \text{ mm} = 0.1125 \text{ mm}$$

$$\delta_{BC} = \frac{120 \times 10^3 \times l_{BC}}{A_{BC}\ E} = \frac{120 \times 1000 \times 800}{30 \times 30 \times 2 \times 10^5} \text{ mm} = 0.5333 \text{ mm and}$$

$$\delta_{CD} = \frac{80 \times 10^3 \times l_{CD}}{A_{CD}\ E} = \frac{80 \times 1000 \times 1000}{20 \times 20 \times 2 \times 10^5} \text{ mm} = 1 \text{ mm}$$

$\delta = \delta_{AB} + \delta_{BC} = \delta_{CD} = 0.1125 + 0.5333 + 1 = 1.6458$ mm

thus, net deformation of the bar is –1.6458.

(b) Calculate the increase in stress for each segment of the compound bar shown in Fig. Q.12, if the temperature increases by 100°F. Assume that the supports are unyielding and that the bar is suitably braced against bckling. Take (8)

$E_{Al} = 10 \times 10^6$ psi, $A_{Al} = 2.0$ in^2, $\alpha_{Al} = 12.8 \times 10^{-6}$/°F and

$E_{St} = 29 \times 10^6$ psi, $A_{St} = 1.5$ in^2, $\alpha_{St} = 6.5 \times 10^{-6}$/°F

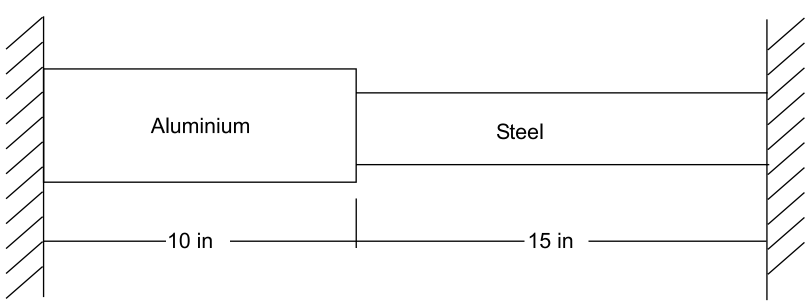


Fig. Q.12

Answer Change (rise) in temperature is $\Delta t = 100°F$

Both the bas will be under compression.

Force balance of the two bars yields $\sigma_{St} \times A_{St} = \sigma_{Al} \times A_{Al}$

$\sigma_{St} \times 1.5 = \sigma_{Al} \times 2$

$\sigma_{St} = 1.33\sigma_{Al}$ (4)

Now, $\sigma = E\varepsilon = E \times \frac{\delta}{1}$

$$\therefore\ \delta_{St} = \frac{\sigma_{St} \times l_{St}}{E_{St}} = \frac{15\sigma_{St}}{19 \times 10^6} = 0.517 \times 10^{-6} \times \sigma_{St}$$

$$\text{Similarly, } \delta_{Al} = \frac{\sigma_{Al} \times l_{Al}}{E_{Al}} = \frac{10\sigma_{Al}}{10 \times 10^6} = 10^{-6} \times \sigma_{Al}$$

Free contraction = $\delta_{St} + \delta_{Al}$

Further free contraction = $l_{St}\ \alpha_{St}\ \Delta t + l_{Al}\ \alpha_{Al}\ \Delta t$

$= 100\ [15 \times 6.5 \times 10^{-4} + 10 \times 12.8 \times 10^{-6}] = 0.2255$ inch

Therefore, $0.2255 = 0.517 \times 10^{-6} \times \sigma_{St} + 10^{-4} \times \sigma_{Al}$ (5)

Solving Eq. (4) and (5)

$\sigma_{St} = 17{,}690$ psi and $\sigma_{Al} = 13{,}352$ psi.

8. (a) A spring normally 150 mm long is connected to the two masses as shown in the figure and is compressed by 50 mm. If the system is released on a smooth horizontal plane, what will be the speed of each block when the spring is again in its normal length? The spring constant is 2100 N/m. (8)

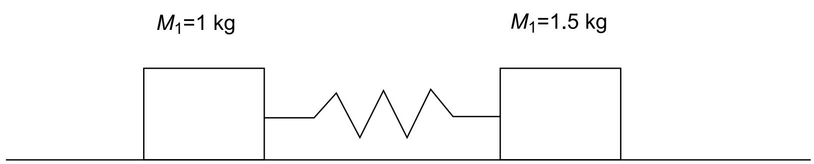


Fig. Q.13

Answer The free-body diagrams of the two masses are shown in Fig. Q.14.

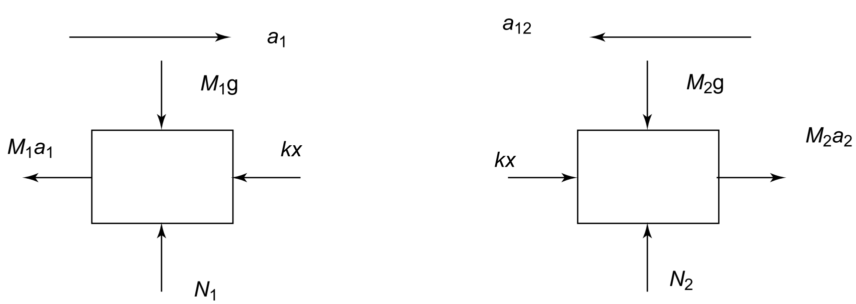


Fig. Q.14

Considering the mass M_1; the dynamic equation becomes

$$M_1 \ddot{x} + kx = 0$$

$$\ddot{x} = \frac{-kx}{M_1}$$

Thus differential equation of mass M_1 becomes $\ddot{x} = \frac{kx}{M_1} = 0$

Angular velocity $\omega_1 = \sqrt{\frac{k}{M_1}}$

The $x - t$ relationship becomes $x = x_0 \sin \omega t$

$\therefore\ \dot{x} = x_0\omega \cos \omega t$

Thus $\dot{x} = x_0\omega = 50 \times 10^{-3} \times \sqrt{\frac{2100}{1}}$ m/s^2 = 2.3 m/s^2

Similarly $\dot{x}$ becomes 1.87 m/s.

(b) A projectline is aimed at a mark on the horizontal plane through the point of projection and falls 12 m short when the angle of projection is 15°, while it overshoots the mark by 24 m when the same angle is 45°. Find the angle of projection to hit the mark. Assume no air resistance. Consider the velocities of projections are constant in all cases.

Answer The range $r = \frac{v_0^2 \sin 2\theta}{g}$

Let the distance of the target from the point of projection along the horizontal direction be x.

Initially, $\theta - \theta_1$ (say) = 15°. The range $r = r_2$ (say) therefore becomes $x - 12$ m

$$\therefore\ x - 12 = \frac{v_0^2 \sin (2 \times 15°)}{g} = \frac{v_0^2}{2g} \tag{6}$$

During the second occasion, $\theta = \theta_2$ (say) = 45°. The range $r = r_2$ (say) therefore becomes $(x + 24)$ m

$$\therefore\ x + 24 = \frac{v_0^2 \sin 2 \times 45}{g} \tag{7}$$

Solving Eqs (6) and (7), $x = 48$ and $\frac{v_0^2}{g} = 72$

Therefore, the range of the projecttile should be 48 m to hit the target.

Let the correct angle of projection be θ.

$$\therefore\ 48 = \frac{v_0^2 \sin 2\theta}{g} = 72 \sin 2\theta.$$

$$\theta = 20.9°$$

9. (a) Two inclined rollers, each weighing $Q = 100$ kgf are supported by an inclined plane and a vertical wall as shown in Fig. Q.14. Assuming smooth surfaces, find the reactions induced at the points *AB* and *C* (7)

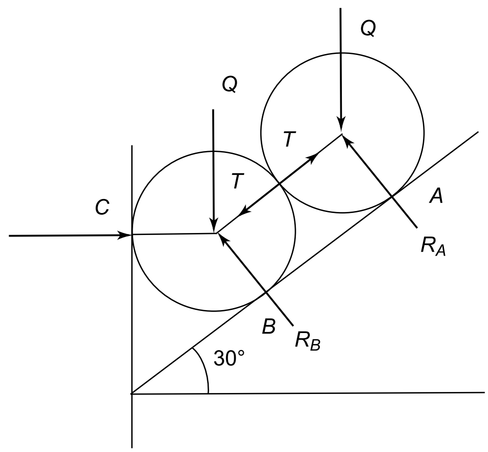


Fig. Q.15

Answer Let the mutual thrust exerted by two rollers from their common point of contact be *T*.

Thus $(R_A)_X = R_A \cos 60°$ and $(R_A)_Y = R_A \sin 60°$

Considering the equilibrium of the lower roller; $\sum X = 0;\ R_B \sin 30° + T \cos 30° = R_C$ (8)

$\sum Y = 0;\ R_B \cos 30° = T \sin 30° + Q$ (9)

Similar considerations of the upper roller yields

$\sum X = 0;\ T \cos 30° = R_A \sin 30°$ (10)

$\sum Y = 0;\ R_A \cos 30° + T \sin 30° = Q$ (11)

From Eqs (10) and (11), $\tan 30° = \dfrac{Q - T_A \cos 30°}{R_A \sin 30°}$

$R_A\,[\tan 30° \sin 30° + \cos 30°] = Q$

$R_A = Q \cos 30° = 100 \cos 30° = 86.6\ kgf$

From Eqs (11), $T \sin 30° = Q - R_A \cos 30° = Q \sin^2 30° = 25\ kgf$

Thus, from Eq. (9), $R_B \cos 30° + T \sin 30° + Q = 25 + 100 = 125$

$R_B = \dfrac{125}{\cos 30} = 144.34\ kgf$

$\therefore\ R_C = R_B \sin 30° + T \cos 30° = 144.34 \sin 30° + 50 \cos 30° = 115.47\ kgf$

(b) Two blocks of weight W_1 and W_2 rest as shown. If the angle of friction of each block is φ, find the magnitude and direction of the least force *P* applied to the upper block that will induce sliding.

Answer The problem is worked out in Example 6.1 (page 6.6).

MECHANICAL SCIENCES

SEMESTER – 1

Time: 3 Hours | Full Marks: 70

Group – A
(Multiple Choice Questions)

$10 \times 1 = 10$

1. Choose the correct alternatives for the following:

(i) Lami's theorem is related to
- (a) equilibrium of two co-planar, concurrent forces
- (b) equilibrium of three co-planar, concurrent forces
- (c) equilibriums of three co-planar, non-concurrent forces
- (d) none of the above

Answer (b)

(ii) Strain energy is the
- (a) maximum energy which can be stored in a body
- (b) energy stored in a body when stresses to the elastic limit
- (c) energy stored in a body when stresses to the breaking point
- (d) none of these

Answer (c)

(iii) Poisson's ratio is defined as
- (a) longitudinal stress by lateral stress
- (b) lateral stress by longitudinal stress
- (c) longitudinal strain by lateral strain
- (d) lateral strain by longitudinal strain

Answer (d)

(iv) Free-body diagram of a body is drawn
- (a) by isolating the body and its surrounding
- (b) by indicating the forces acting on it
- (c) both of these
- (d) none of these

Answer (c)

(v) If a momentum of a body is doubled, its kinetic energy will
- (a) increase by two times
- (b) increase by four times
- (c) remain same
- (d) get halved
- (e) reduced by four times

Answer (a)

(vi) A body falling freely from a height of 10 m rebounds from the floor. If it loses 20% of its energy in the impact, how high will it rebound?
(a) 10 m
(b) 8 m
(c) 12 m
(d) None of these
Answer (b)

(vii) The dot product of two orthogonal vectors is
(a) one
(b) zero
(c) no definite value
(d) none of these
Answer (b)

(viii) The centre of gravity of a uniform lamina lies at
(a) the centre of the heavy portion
(b) the bottom surface
(c) the midpoint of its axis
(d) none of the these
Answer (c)

(ix) If the velocity of a projectile is u and the angle of projection is a, the maximum height attained by a projectile above the horizontal plane is
(a) $u^2 \cos^2 a/2g$
(b) $u^2 \sin^2 a/2g$
(c) $u^2 \tan^2 a/2g$
(d) $u^2 \sin^2 a/g$
Answer (b)

(x) Three forces $\sqrt{3}p,\ p$ and $2p$ acting on a particle are in equilibrium. If the angle between the first and second be 90°, the angle between the second and third will be
(a) 30°
(b) 60°
(c) 120°
(d) 150°
Answer (b)

Group – B

$3 \times 5 = 15$

(Short-Answer Type Questions)

Answer any *three* questions

2. (a) State D' Alembert's principle.

Answer

From Newton's second law,

$$\Sigma F = m\ddot{x}$$

$$\Sigma F - m\ddot{x} = 0$$

$$\Sigma F + (-m\ddot{x}) = 0$$

The above equation can be interpreted as if it is considered that ΣF is the net force acting along the direction of motion and $-m\ddot{x}$ is the force that acts opposite to the motion so that their combined effect will restore equilibrium, i.e., no unbalanced force is acting on the body. This can, however, be envisaged that equation of dynamic equilibrium is tantamount to the equation of static equilibrium. The product of mass and acceleration with a negative sign is called inertia force to assume to act so as to oppose the motion.

The above equation is treated as *equation of dynamic equilibrium of the particle*.

To obtain dynamic equilibrium of a particle, *a fictitious force called inertia force is added opposite to the direction of motion so that resultant force on the particle becomes zero. This concept is known as D' Alembert's principle and is a very useful approach in the solution of problems in kinetics.*

(b) A smooth circular cylinder of radius 1.5-m is lying in a rectangular groove, as shown in Fig. 1. Find the reactions at the surfaces of contact, if there is no friction and the cylinder weighs 1000 N.

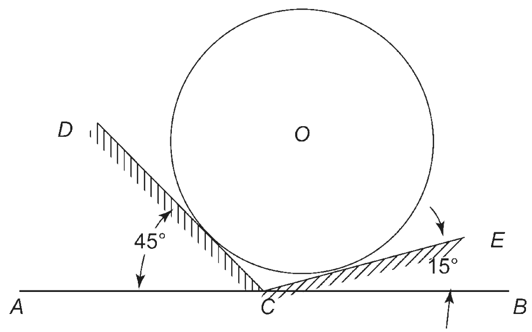


Figure 1

Answer

The FBD is shown in Fig. 1 (a). The reactions from the two supports (R_G and R_H) and the weight of the cylinder (W) will constitute a closed triangle under equilibrium.

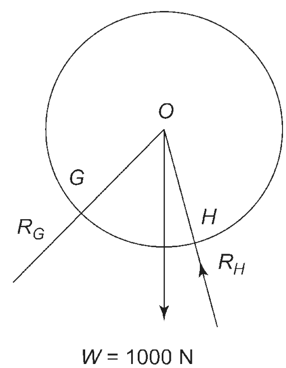


Figure 1 (a)

Therefore, following Lami's theorem, $\dfrac{W}{\sin 120°} = \dfrac{R_G}{\sin 15°} = \dfrac{R_H}{\sin 45°}$ which yields

$$R_G = 298.8 \text{ N and } R_H = 816.5 \text{ N}$$

3. Refer to Fig. 2, and determine the range of values of mass m_0 so that the 100-kg block will neither move up nor slip down the inclined plane. The coefficient of static friction for the surfaces in contact is 0.3.

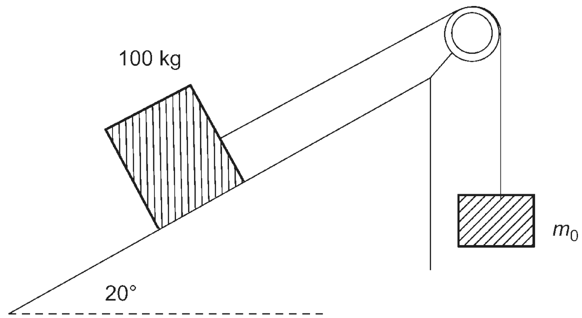


Figure 2

Answer

Denoting the 100-kg block that rests on the inclined plane as m, and maximum mass of the m_0 as $(m_0)_{max}$

$$(m_0)_{max} = m[\mu \cos \theta + \sin \theta] = 100 \times [0.3 \cos 20° + \sin 20°] = 62.4 \text{ kg}$$

and

$$(m_0)_{min} = m[\sin \theta - \mu \cos \theta] = 100 \times [\sin 20° - 0.3 \cos 20°] = 6 \text{ kg}$$

Thus the range of values of m_0 becomes $6 \text{ kg} \le m_0 \le 62.4$ kg

4. (a) State Varignon's principle.

Answer

The algebraic sum of the moments of a system of coplanar forces about a moment centre in their plane is equal to the moment of their resultant force about the same moment centre.

Written analytically,

$$\sum_i \mathbf{M}_i = \mathbf{r} \times \sum_i \mathbf{F}_i$$

where $\sum_i \mathbf{M}_i$ = algebraic sum of the moments, $\mathbf{r}$ = position vector, $\sum_i \mathbf{F}_i$ = resultant force.

(b) A circular roller of 100-N weight and 10-cm radius hangs by a ties rod AB = 20 cm and rest against a smooth vertical wall at C as shown in the Fig. 3. Determine the force F in the rod.

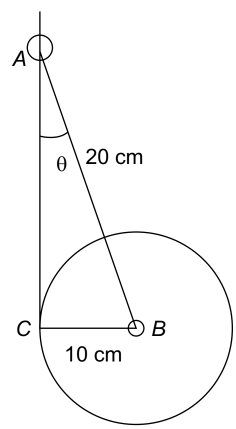


Figure 3

Answer

From the given geometry, θ is computed to be 30°.

Under condition of equilibrium (Fig. 3 (a)), considering $\sum Y = 0$;

$$F \cos \theta = W$$

or, $$F = W/\cos \theta = \frac{100}{\cos 30} = 115.5 \text{ N}$$

Thus, the tension in the rod F is 115.5 N.

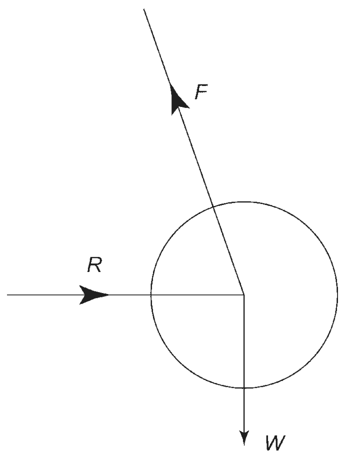


Figure 3 (a)

5. **Referring to Fig. 4, r = 12 cm, Q = 500 N and h = 6 cm. Find the magnitude of P required to start the roller over curb.**

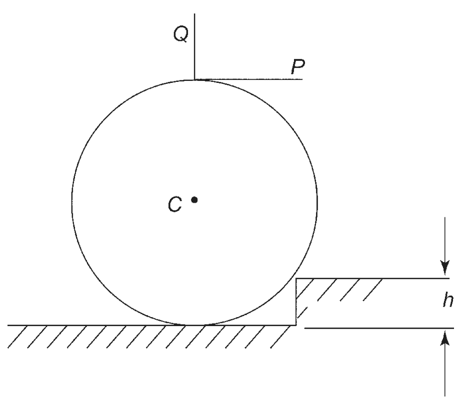


Figure 4

The FBD is shown in Fig. 4 (a). The reactions from the curb R_B, the applied force P and the weight of the roller Q is represented by a closed triangle (ΔBDE) under equilibrium.

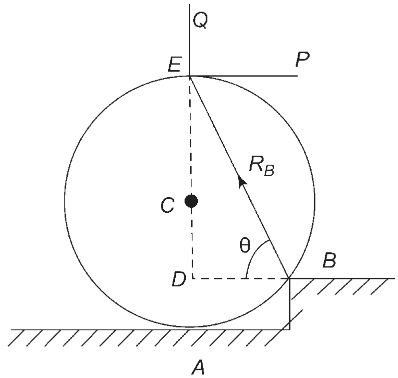


Figure 4 (a)

From the geometry;

$$BD = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2} = 10.4 \text{ cm}; \; DE = 2r - h = 18 \text{ cm}$$

Considering the equilibrium of the roller and taking moment about B,

$$\Sigma M_B = 0$$

$$Q \times BD = P \times DE$$

or
$$P = 500 \times \frac{10.4}{18} = 288.88 \text{ N}$$

Thus, P_{min} is 288.88 N to initiate movement of the roller.

6. Two smooth circular cylinders of Fig. 5 each of weight $W = 100$ N and radius $r = 6$ cm are connected by a string AB of length $l = 16$ cm and rest upon a horizontal plane, supporting a third cylinder of weight $Q = 200$ N and radius $r = 6$ cm above them. Find the tension S in the string AB and the pressure produced by the floor at points of contact D and E.

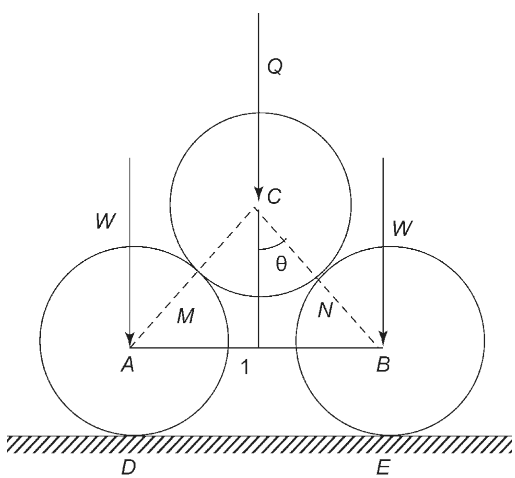


Figure 5

Answer

The FBD of the upper roller is shown in Fig. 5 (a).

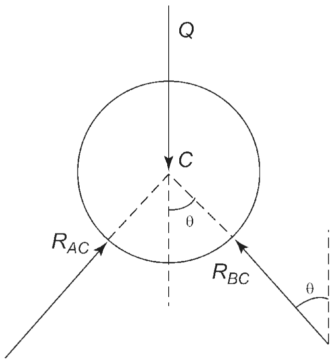


Figure 5 (a)

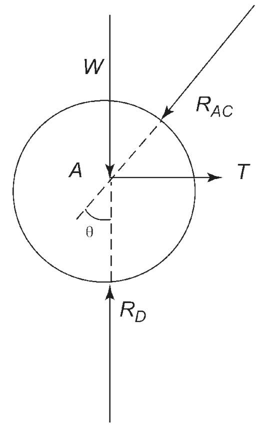


Figure 5 (b)

Under condition of equilibrium, considering $\sum X = 0$ yields that R_{AC} and R_{BC} are equal (say R). From the geometry,

$$\sin\theta = \frac{l/2}{2r}$$

$$\therefore\ \theta = 41.8°$$

From $\sum Y = 0$, we get

$$2R\cos\theta = Q$$

or,
$$R = \frac{Q}{2\cos\theta}$$

Since the lower two rollers are identical in all respect, R_D and R_E is same.
The FBD of the lower-left roller is shown in the Fig. 5 (b).

$$\sum X = 0;$$

$$R\sin\theta = T$$

or,
$$T = \frac{Q}{2}\tan\theta = \frac{200}{2}\tan 41.8° = 89.4 \text{ N}$$

$$\sum Y = 0;$$

or,
$$R_D = W + R\cos\theta = W + \frac{Q}{2} = 200 \text{ N} = R_E$$

Group – C
(Long-Answer Type Questions)

Answer any *three* questions $3 \times 15 = 45$

7. (a) A 150-kg man stands on the midpoint of a 50-kg ladder as shown in Fig. 6. Assuming that the floor and the wall are perfectly smooth, find the reactions at points *A* and *B*. 8

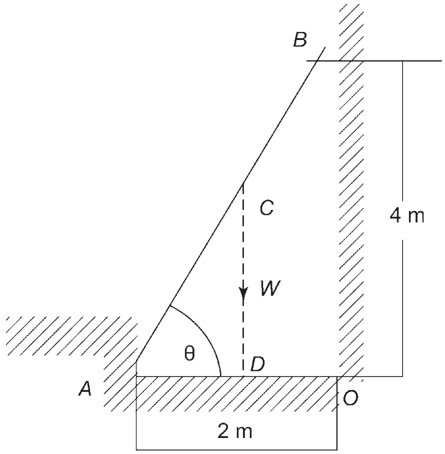


Figure 6

Answer

This problem is similar to the worked out Example 4.15.

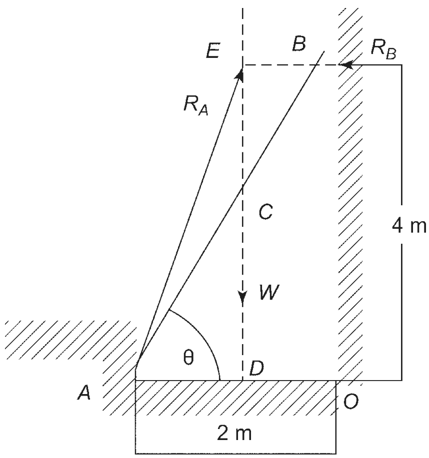


Figure 6 (a)

The forces acting on the ladder are shown in Fig. 6 (a).

The inclination of the ladder with the horizontal is θ (say)

$$\tan\ \theta = \frac{4}{2} = 2$$

or, $\theta = 63.4°$

Total vertical downward load on the ladder is $W = W_{man} + W_{ladder} = 150 + 50 = 200$ kg.

Considering the equilibrium of the ladder,

$$\sum M_A = 0$$

$$W \times \frac{l}{2} \cos \alpha = R_B \times l \sin \alpha$$

$$\frac{W}{2} = R_B \tan \alpha$$

$$R_B = \frac{W}{2 \tan \alpha} = \frac{200 \times 9.81}{2 \tan 63.4} = 491.25 \text{ N}$$

$$\therefore R_A = \sqrt{(R_B)^2 + (W)^2} = \sqrt{(491.25)^2 + (200 \times 9.81)^2} = 2022.5 \text{ N}$$

(b) Determine the moment of inertia for the T section (as shown in Fig. 7) with respect to a centroidal axis parallel to *x*-axis. All the dimensions are in mm. **7**

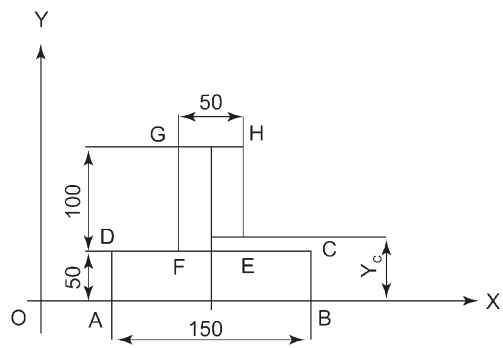


Figure 7

Answer

This problem is identical to the worked out Example 7.8.

For composites, it is imperative to calculate various parameters of the individual areas into which the entire composite is decomposed and present these in a tabular form to improve legibility and to reduce errors. Following the previous problem,

Identification	Area (A)	I_{xi}	y
ABCD	$A_1 = 150 \times 50 = 7500$	$I_{x1} = \dfrac{150 \times 50^3}{3} = 6250000$	$y_1 = 25$
EFGH	$A_2 = 100 \times 50 = 5000$	$I_{x2} = \dfrac{150 \times 100^3}{12} + 5000 \times 100^2 = 54170000$	$y_2 = 50 + 50 = 100$
Composite	$A_1 + A_2 = 12500$	$\sum_{i=1}^{2} I_{xi} = I_{x1} + I_{x2} = 60420000$	$y_c = 55$

Thus by using parallel axes theorem; $I_{xc} = \sum_{i=1}^{2} I_{xi} - d_x{}^2 \sum_{i=1}^{2} A_i$

$I_{xc} = 60420000 - 55^2 \times 12500 = 22607500 \text{ mm}^4$

8. (a) **Prove that the volumetric strain of a rectangular bar is the algebraic sum of strains of length, width and height.** **6**

Answer

This problem is identical to the worked out Example 8.12.

(b) **Show that elongation of a conical bar under its weight is independent of its base diameter but on length only.** **4**

Answer

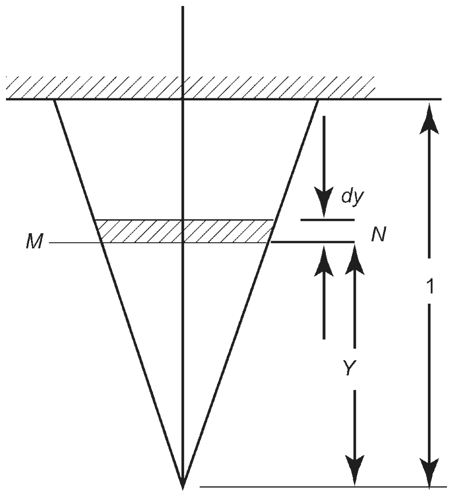


Figure 8

With reference to Fig. 8 of a solid cone, a section *M-N* is considered at a distance y from the vertex. The cross-section area at section *M-N* is say *A*.

The weight of the lower part of section *M-N* is $= \frac{1}{3} Ay\rho$, where ρ is the density of the cone material.

Thus stress at this section becomes $\sigma = \frac{1}{3} y\rho$

The elongation of the elemental length *dy* becomes $\frac{1}{3E}\; y\rho.dy$

The total elongation of the conical bar becomes $\int_0^l \frac{1}{3E}\; \rho y dy = \frac{\rho l^2}{6E}$

This expression shows that it is independent of base diameter but depends on length.

(c) **Determine the strain energy stored within a bar of length *l*, cross-sectional area *A*, density ρ and modulus of elasticity *E*, hanging vertically due to its height.** **5**

Answer

Refer to Article 8.7, the gravitational force acting on a small elemental length *dx* of the bar at a distance *x* from the support is $F_x = \rho Ax$.

Thus its elongation $d\delta = \frac{F_x dx}{AE}$

Therefore strain energy induced in this small element of the bar is

$$du = \frac{F_x.d\delta}{2} = \frac{(\rho Ax).(\rho Ax).dx}{2AE}$$

Thus total strain energy induced in the bar is $u = \int du = \frac{\rho^2 A}{2E}\int_0^l x^2 dx = \frac{\rho^2 A l^3}{6E}$

This is the required expression.

9. (a) Two spheres *P* and *Q* rests in the channel as shown in Fig. 9. The sphere *P* has a diameter 400 mm and weight of 200 N, whereas the sphere *Q* has diameter 500 mm and weight 500 N. If bottom width of the channel is 500 mm with one side vertical and other side inclined at 60°, determine the reaction induced in the contacts. **8**

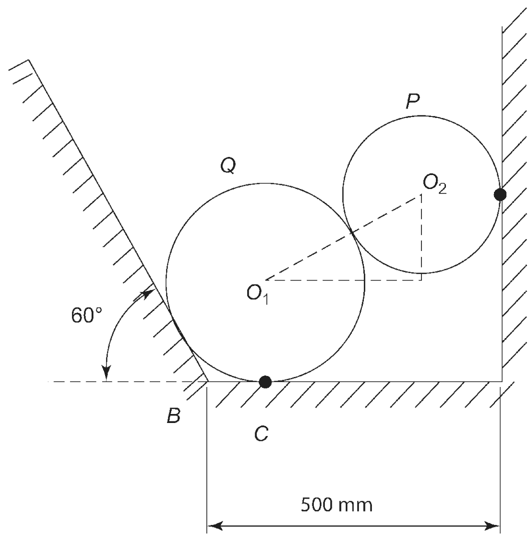


Figure 9

Answer

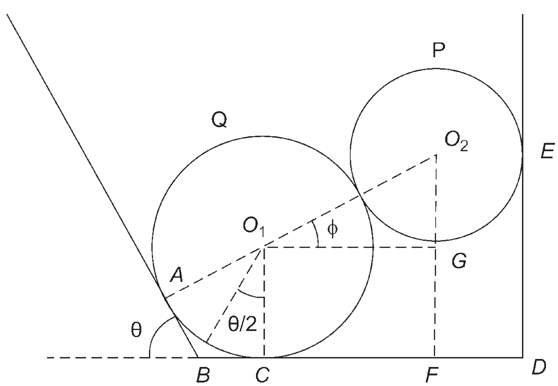


Figure 9 (a)

The geometry of the system is shown in Fig. 9 (a).
From the ΔO_1BC,

$$\tan\frac{\theta}{2} = \frac{BC}{O_1C},$$

or, $BC = 250 \tan 30°$

$$\therefore\ O_1G = CF = BD - DF - BC = 500 - 250 \tan 30° - 200 = 155.66 \text{ mm}$$

$$\cos\phi = \frac{O_1G}{O_1O_2} = \frac{155.66}{450};$$

or $\phi = 69.76°$

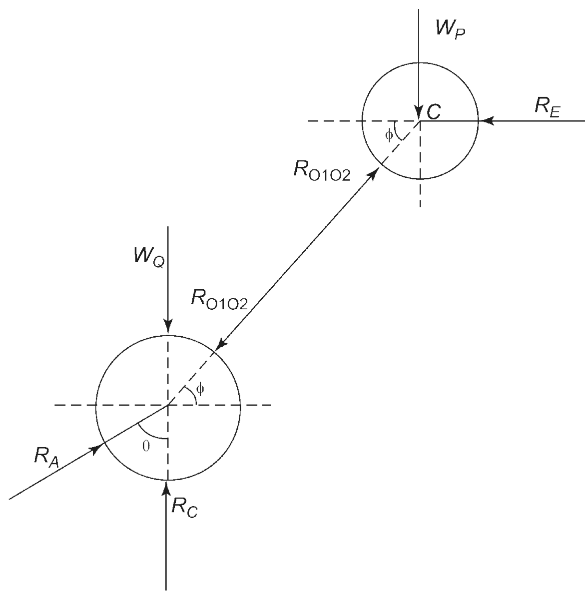


Figure 9 (b)

The FBD of the two rollers is shown in the Fig. 9 (b).

Considering equilibrium of the roller P and having $\sum X = 0$, we have

$$R_E = R_{O1O2} \cos\ \phi$$

From $\sum Y = 0$, we get

$$W_P = R_{O1O2} \sin\ \phi$$

Solving the above two equations, we have

$$R_E = 73.75 \text{ N and } R_{O1O2} = 213.2 \text{ N}$$

Similar considerations of the roller Q yields $\sum X = 0$;

$$R_A \sin\ \theta = R_{O1O2} \cos\ \phi$$

and

$$\sum Y = 0;$$

$$R_C + R_A \cos\ \theta = W_Q + R_{O1O2} = \sin\ \phi$$

Solving these two equations, we have

$$R_A = 85.2 \text{ N and } R_C = 657.5 \text{ N}$$

(b) In Fig. 10, find the minimum value of horizontal force *P* applied to the lower block that will keep the system in equilibrium. Given, coefficients of friction between lower block and floor = 0.25, between the upper block and the vertical wall = 0.30, between the two blocks = 0.20. **7**

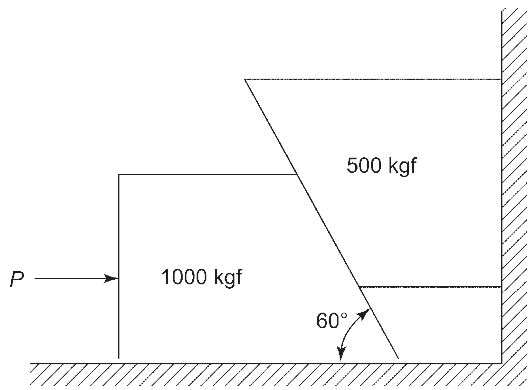


Figure 10

Answer

This problem is similar to the worked out Example 6.10.

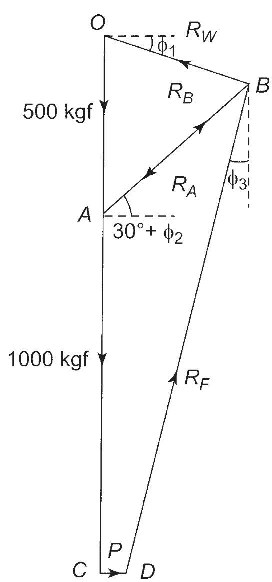


Figure 10 (a)

Introducing suffice W for wall; F for floor and B for common contact surface between blocks $F_w = \mu_1.N_w$; $F_B = \mu_2.N_B$ and $F_F = \mu_3.N_F$

Given $\mu_1 = 0.30 \Rightarrow \phi_1 = 16.7°$.

Similarly, $\mu_2 = 0.20$

or, $\phi_2 = 11.3°$ and

$\mu_3 = 0.25$

or, $\phi_2 = 14°$

R_W, R_B and R_F are the resultant of normal reaction and frictional force at the wall, between blocks and the floor respectively.

Selecting a particular scale and drawing the vector diagram for the two blocks [refer Fig. 10 (a)], P becomes 82 N.

10. (a) State the principle of virtual work. **3**

Answer

The work done is expressed by $U = \int F_x dx + F_y dy + F_z dz$. In view of equilibrium condition, the algebraic sum of all forces along the three mutually perpendicular directions is zero i.e. $\sum F_x = 0; \sum F_y = 0; \sum F_z = 0.$

This leads to the situation that U is also zero.

Another condition of equilibrium $\sum M = 0$ also converges to the same conclusion that work done by the moment of a couple is also zero.

These two situations can be extended to conclude that "***if a rigid body is in equilibrium, the algebraic sum of all the work done by external forces for any virtual displacements consistent with the geometrical configurations of the body is zero.***" This is popularly known as **Principle of Virtual Work.**

(b) Two blocks weighing W_1 and W_2 resting on smooth inclined planes are connected by an inextensible string passing over a smooth pulley as shown in Fig. 11. Find the value of W_2, when W_1 = 500 N and $\alpha = 30°$, $\beta = 60°$. **7**

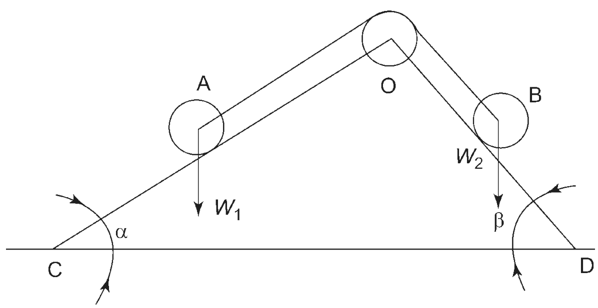


Figure 11

Answer

Since the two rollers are connected by a common string, the tension is same.

Considering the roller A, $T = W_1 \sin \alpha$ and from the roller B, $T = W_2 \sin \beta$

Comparing the above two equations,

$$W_2 \sin \beta = W_1 \sin \alpha.$$

or,
$$W_2 = \frac{500 \sin 30}{\sin 60} = 288.67\,\text{N}$$

(c) Determine velocity V of the falling weight W of the system as shown in Fig. 12 as a function of its displacement from the initial position of rest. Assume weight of the cylinder as 2 W. **5**

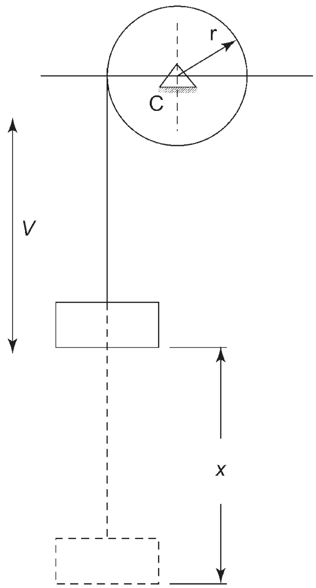


Figure 12

Answer

The total K.E. of the system = (K.E.)$_{\text{Cylinder}}$ + (K.E.)$_{Block}$

$$= \frac{1}{2} I_m \omega^2 + \frac{1}{2}\frac{W}{g} V^2$$

$$= \frac{1}{2}\left(\frac{1}{2} \times \frac{2W}{g} \times r^2\right)\omega^2 + \frac{1}{2}\frac{W}{g} V^2$$

$$= \frac{1}{2}\left(\frac{W}{g} \times \omega^2 r^2\right) + \frac{1}{2}\frac{W}{g} V^2$$

$$= \frac{1}{2}\left(\frac{W}{g} \times V^2\right) + \frac{1}{2}\frac{W}{g} V^2$$

$$= \frac{W}{g} V^2$$

The work done = W_x

Considering conservation of energy, we get

$$= \frac{W}{g} V^2 W.x$$

or, $$V = \sqrt{gx}$$

This is the requisite relation between V and x.

11. (a) From top of a 60-m high, tower, a bullet is fired at an angle of 20° up the horizontal with a velocity of l20 m/s. Determine

(i) time of flight

(ii) horizontal range of ground

(iii) maximum height of the bullet from ground

(iv) velocity of the bullet after 8 seconds

Assume horizontal ground at the foot of the tower **10**

Answer

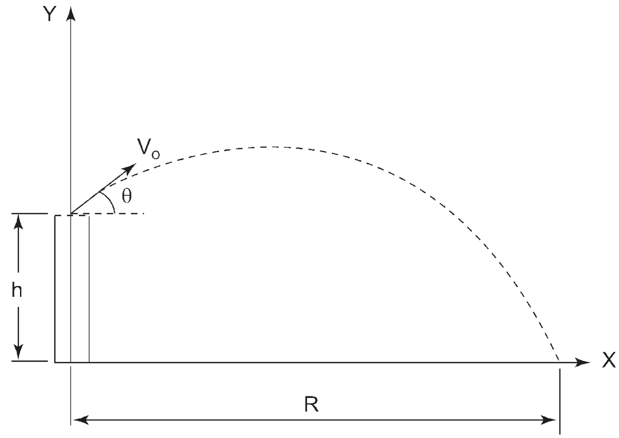


Figure 13

The initial velocity v_0 has got two components — the horizontal component being $v_0 \cos \theta$, the vertical component is $v_0 \sin \theta$.

Let the total time of flight is T seconds. Also consider that the projectile will cover h' meter height above the tower to reach its peak and its corresponding time is t' and during fall, it covers h meter in t seconds.

Thus $T = 2t' + t$

$$0 = v_0 \sin \theta - gt'$$

$$2t' = \frac{2v_0 \sin\theta}{g} = \frac{2 \times 120 \sin 20}{9.81} = 8.367\text{s}$$

Further $$h = v_0 \sin\theta . t + \frac{1}{2} g t^2;$$

$$60 = 120 \sin 20 t + \frac{1}{2} \times 9.81 \times t^2 \Rightarrow t = 1.27\text{s}$$

Thus total time of flight becomes $T = 2t' + t = 8.367 + 1.27 = 9.637\text{s}$

$$0 = (v_0 \sin \theta)^2 - 2gh'$$

or, $$h' = \frac{(120 \sin 20)^2}{2 \times 9.81} = 85.85 \text{ m}$$

$\therefore$ Total height above the ground becomes $H = h' + h = 85.85 + 60 = 145.85$ m

The range $R = (v_0 \cos \theta).T = 120 \cos 20° \times 9.637$ m = 1086.7 m.
Since $t' = 4.183$ s, after 8 s the projectile on its downward movement from the peak height. Thus the downward component of the velocity after 8 seconds if denoted by v_{v8}, then

$$v_{v8} = g \times 3.817 = 37.45 \text{ m/s}$$

The horizontal component being unchanged at $v_0 \cos \theta$ implying

$$v_{h8} = 120 \cos 20 = 112.76 \text{ m/s}$$

Thus resultant velocity after 8 s is

$$v_8 = \sqrt{(v_{h8})^2 + (v_{v8})^2} = \sqrt{112.76^2 + 37.45^2} = 118.82 \text{ m/s}$$

(b) Determine the tension in the strings and accelerations of two blocks of masses 150 kg and 50 kg connected by a string and a frictionless, weightless pulley as shown in Fig. 12. **5**

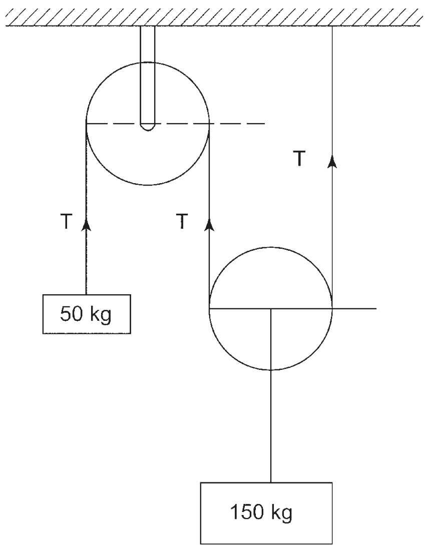


Figure 14

Answer
This problem is similar to the worked out Example 11.3.

For the given system, the 50-kg mass (m_1) will go up and the 150-kg mass (m_2) will come down. Let the tension in the string is T.

If the mass m_1 moves up with accelerations a; the mass m_2 moves down with acceleration

$$a' = -\frac{a}{2}$$

Considering the equilibrium of the mass m_1 and applying Newton's second law of motion

$$T - m_1 g = m_1 a$$

or, $$T = m_1(g + a)$$

Similar consideration of the mass m_2 yields

$$m_2 g - 2T = m_2 a' = m_2 \frac{a}{2} \Rightarrow 2T = m_2\left(g - \frac{a}{2}\right)$$

Comparing above two equations

$$m_2(g - \frac{a}{2}) = 2m_1(g + a)$$

Solving $$a = \frac{2g}{7}$$

Thus the acceleration of the block of mass 50 kg is a $= \frac{2g}{7} \uparrow$ and the block of mass 150 kg is $a' = \frac{g}{7} \downarrow$

Tension in the string is $T = m_1(g + a) = 50 \times 9 \times 9.81/7$ N = 630.65 N

CS/B.TECH/SEM–1/ME-101/2010-11

ENGINEERING MECHANICS

SEMESTER – 1

Group – A
(Multiple Choice Questions)

1. Choose the correct alternatives for the following: **10×1**

(i) Mass moment of inertia of a body is
(a) moment of its inertia
(b) rotational analogue of mass
(c) inertial moment about the centroidal axis
(d) none of these
Answer (b)

(ii) The centre of percussion of a rigid body is a point
(a) through which the resultant of all forces acts
(b) where minimum external forces act
(c) where impact is made
(d) all of these
Answer (a)

(iii) Two coplanar couples having equal and opposite moments
(a) balance each other
(b) produce a couple and an unbalance force
(c) an equivalent
(d) all of these
Answer (a)

(iv) The moment of inertia of a rectangle having base b and height h with respect to its base is
(a) $\frac{1}{3}h^2b^2$
(b) $\frac{1}{3}b^3h$
(c) $\frac{1}{3}bh^3$
(d) $\frac{1}{3}b^2h^2$
Answer (c)

(v) D'Alembert's principle
 (a) is based upon the presence of inertia force
 (b) provides advantage over Newton's law
 (c) is purely a hypothetical principle
 (d) allows a dynamic problems to be treated as a static ones
 Answer (d)

(vi) Centroid of a line segment
 (a) must lie on the line
 (b) must not lie on the line
 (c) must be same as the centre of gravity
 (d) none of these
 Answer (b)

(vii) Volumetric strain of a rectangular body subjected to an axial force, in terms of linear strain ε and Poisson's ratio μ, is given by
 (a) $\varepsilon(1+2\mu)$
 (b) $\varepsilon(1-2\mu)$
 (c) $\varepsilon(1+\mu)$
 (d) $\varepsilon(1-\mu)$
 Answer (b)

(viii) Poisson's ratio is defined as
 (a) lateral stress and lateral strain
 (b) longitudinal stress and longitudinal strain
 (c) lateral stress and longitudinal stress
 (d) none of these
 Answer (d)

(ix) Relative velocity of $\vec{A}$ with respect to $\vec{B}$ is defined as
 (a) $\vec{V}_{A/B} = \vec{V}_B - \vec{V}_A$
 (b) $\vec{V}_{A/B} = \vec{V}_A - \vec{V}_B$
 (c) $\vec{V}_{A/B} = \vec{V}_B + \vec{V}_A$
 (d) none of these
 Answer (a)

(x) When a change in length takes place, the strain is known as
 (a) linear strain
 (b) lateral strain
 (c) shear strain
 (d) volumetric strain
 Answer (a)

Group – B
(Short-Answer Questions)

Answer any *three* questions 3 × 5 = 15

2. The position vector of a particle moving in the *x-y* plane at time *t* = 3.60 s is 2.76*j* m. At *t* = 3.62 s, its position vector has become 2.79*i* – 3.33*j* m. Determine the magnitude *v* of its average velocity during this interval and the angle *θ* made by *v* with the *x*-axis.

Answer

The position vector at t = 3.60 s is $\vec{r}_1 = 2.76j$ and the same at t = 3.62 s is $\vec{r}_2 = 2.79i - 3.33j$
Thus,

$$\Delta r = \vec{r}_2 - \vec{r}_1 = 2.79i - 3.33j - 2.76j = 2.79i - 6.09j$$

$$\therefore v_{av} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}i + \frac{\Delta y}{\Delta t}j = \frac{2.79}{0.02}i - \frac{6.09}{0.02} = 139.5i - 304.5j$$

Therefore, the magnitude v of its average velocity during this interval becomes

$$\sqrt{(139.5)^2 + (-304.5)^2} \text{ m/s} = 334.9 \text{ m/s}$$

The angle θ made by v with x-axis is $\tan^{-1}\dfrac{-304.5}{139.5} = -65.38°$

3. What do you mean by a free-body diagram? Draw the FBD for the Fig.1

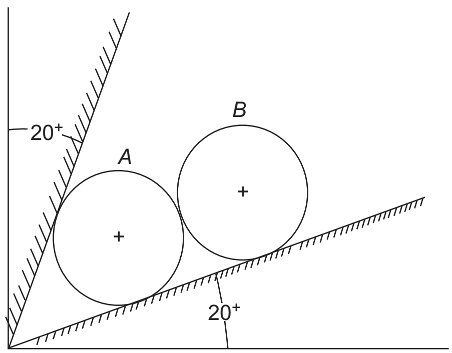


Figure 1

Answer

See Section 4.4, Page 4.4.

The FBD is shown in Fig. 1.(a).

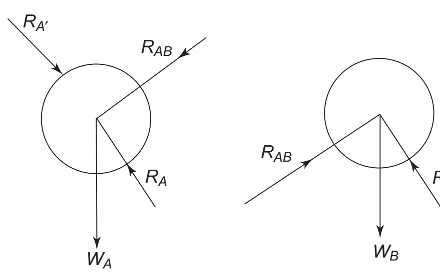


Figure 1(a)

4. If two equal tensions T in a pulley cable are 400 N, as shown in Fig. 2, express in vector notation the force R exerted on the pulley by the two tensions. Determine the magnitude of R.

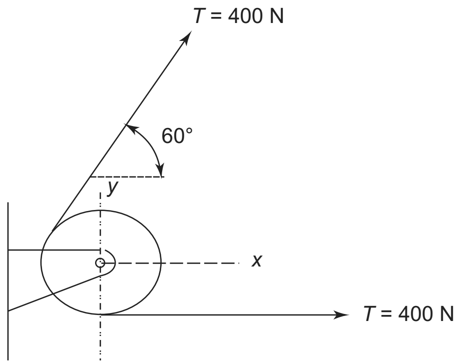


Figure 2

Answer

This problem is identical to the worked out Example 3.5.

5. (i) State and prove Lami's theorem.

Answer

If a body is in equilibrium under the action of three concurrent forces, each force will be proportional to the sine of the angle between the other two forces.

Hence for the system of forces shown above,

$$\frac{P_1}{\sin\alpha} = \frac{P_2}{\sin\beta} = \frac{P_3}{\sin\gamma}$$

Proof:

Forming a triangle of forces as shown in Fig. 2 (a), we have

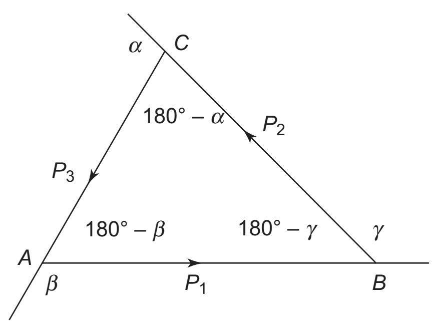


Figure 2 (a)

where $AB = P_1$, $BC = P_2$, $CA = P_3$

Now applying sine rule for the triangle ABC,

$$\frac{AB}{\sin(180° - \alpha)} = \frac{BC}{\sin(180° - \beta)} = \frac{CA}{\sin(180° - \gamma)}$$

or,
$$\frac{P_1}{\sin\alpha} = \frac{P_2}{\sin\beta} = \frac{P_3}{\sin\gamma}$$

ii) Define free-body diagram.
Answer
See Section 4.4, Page 4.4.

6. A bullet of mass *m*, moving with a horizontal velocity *v*, hits a stationary block of mass *M*, suspended by a massless string of length *L*. The bullet gets embedded in the block after impact and the two together swing up. Show that the maximum angle of swing (i.e., angle made by the string with the vertical is

$$\theta = \cos^{-1}\left(1 - \frac{m^2v^2}{2gL(M+m)^2}\right)$$

Answer

The arrangement is shown in Fig. 2.(b).

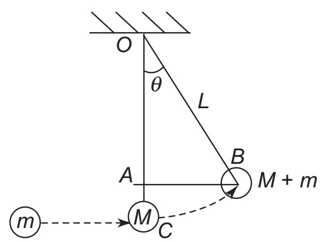


Figure 2 (b)

Let after collision, the velocity of the mass (M + m) becomes v'.

$\therefore M \times 0 + mv = (m+m)v'$

or, $$v' = \frac{mv}{M+m}$$

From Fig. 2 (b), we get

$$AC = OC - OA = L - L\cos\theta$$

Now energy of the system at points C and B will remain same.

$\therefore$ energy at the point C = Energy at the point B

$$\frac{1}{2}(M+m)v'^2 = (M+m)g \times AC$$

or, $$(M+m)g \times AC = \frac{1}{2}(M+m)v'^2$$

or, $$(M+m)g \times (L - L\cos\theta) = \frac{1}{2}(M+m)\frac{m^2v^2}{(M+m)^2}$$

or, $$(1-\cos\theta) = \frac{1}{2gL}\frac{m^2v^2}{(M+m)^2}$$

or, $$\cos\theta = 1 - \frac{1}{2gL}\frac{m^2v^2}{(M+m)^2}$$

or, $$\theta = \cos^{-1}\left(1 - \frac{1}{2gL}\frac{m^2v^2}{(M+m)^2}\right)$$

Group – C
(Long-Answer Questions)

Answer any *three* questions $3 \times 15 = 45$

7. (a) A 50 N block is released from rest on an inclined plane which is making an angle of 35° to the horizontal (Fig. 3). The block starts from '*A*', slides down a distance of 1.2 m and strikes a spring with a stiffness of 8 kN/m. The coefficient of friction between the inclined plane and the block is 0.25. Determine (i) the amount the spring gets compressed, and (ii) distance the block will rebound up the plane from the compressed position.

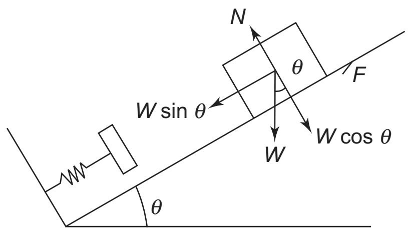


Figure 3

Answer

Normal reaction on the block = $N = W\cos\theta = 50\cos 35° = 40.96\text{N}$

Friction force on the block $F = \mu N = 0.25 \times 40.96 = 10.24\text{N}$

The net downward force acting on the block is $F_{\text{net}} = W\sin\theta - \mu N = 18.44\text{N}$

Let x be the displacement (deformation) of the spring when the block strikes.

Thus total displacement of the block becomes $(1.2 + x)$ m

Thus work done by the block is $U = F_{\text{net}}.x = 18.44(1.2 + x)\text{N-m}$

Work done by the spring is $-\frac{1}{2}(kx)\cdot x = -\frac{1}{2} \times 8000 \times x^2 \text{N-m}$

The net work done of the system is $U_{\text{net}} = 18.44(1.2 + x) - 4000x^2 \text{N-m}$

Since the block starts from rest and after striking, it also comes to rest, the net change in kinetic energy of the system is zero.

Following work-energy principle, $18.44(1.2 + x) - 4000x^2 = 0$

Solving, we get

$$x = 0.0766 \text{ m}$$

(a) Thus the compression of the spring is 0.0766 m.

During rebounds, the net force on the block is $F^1_{\text{net}} = W\sin\theta + \mu N = 38.92\text{N}$

If s is the distance travelled by the block along the plane (upward) then work done is

$$F^1_{\text{net}}.s = 38.92 \times s.\text{N-m}$$

$$\therefore \quad \frac{1}{2}kx^2 = 38.92 \times s \text{ N-m}$$

$$\text{or,} \quad \frac{1}{2} \times 8000 \times (0.0766)^2 = 38.92 \times s.$$

or, $s = 0.603$ m

(b) Thus the block rebounds by 0.603 m up along the plane.

(b) A reinforced concrete column having a cross-section of 300 mm X 300 mm is provided with 9 bars of 20 mm diameter (Fig. 4). The column carries a load of 300 KM. Find the stress developed in the steel bars and concrete. Take $E_s = 2.1 \times 10^5$ N/mm² and $E_c = 2.1 \times 10^5$ N/mm².

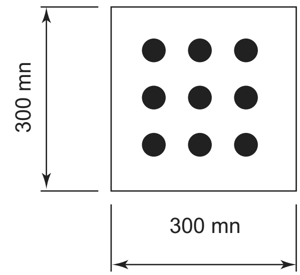


Figure 4

Answer

Applied load = P = 300 kN = 300000 N.

Total area of the column = 300 × 300 mm².

Area of steel bars = $A_{\text{steel}} = 9 \times \frac{\pi}{4} \times (20)^2 \text{mm}^2 = 32827.5 \text{ mm}^2$

Area of concrete therefore becomes A_{con} = (90000 – 2827.5) mm² = 87172.5 mm²

Let the stresses induced in the concrete and steel be σ_{con} and σ_{steel} respectively.

Due to loading, the deformation of both the materials would be same.

$$\therefore \quad \frac{\sigma_{\text{steel}}}{E_{\text{steel}}} = \frac{\sigma_{\text{con}}}{E_{\text{con}}}. \tag{1}$$

Further total load = Load shared by steel + Load shared by concrete

$$\therefore \quad \sigma_{\text{steel}} \times A_{\text{steel}} + \sigma_{\text{con}} \times A_{\text{con}} = P \tag{2}$$

Solving Eqs (1) and (2), we have

σ_{con} = 2.315 N/mm² and σ_{steel} = 34.726 N/mm²

8. A rigid bar *AB* is hinged to a vertical wall and supported horizontally by a tie bar *CD* as shown in Fig 5. The cross-sectional area of *CD* is A = 0.5 sq cm and its allowable stress in tension is 1500 kg/cm². Find the safe value of *P* and the corresponding vertical deflection Δ_B of *B*. The modulus of elasticity of the tie-rod $E = 2 \times 10^6$ kg/cm².

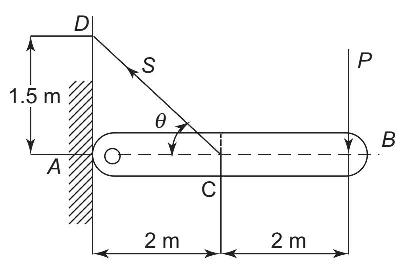


Figure 5

Answer

This problem is identical to the worked out Example 8.7.

9. (a) A block of weight W_1 = 200 N rests on the horizontal surface and supports on top of it, another block of weight W_2 = 50 N. The block W_2 is attached to a vertical wall by the inclined string *AB*. Find the magnitude of the horizontal force *P* applied to the lower block, as shown in Fig. 6, that will be necessary to cause slipping to impend. Take coefficient of static friction for all continuous surfaces is μ =0.3.

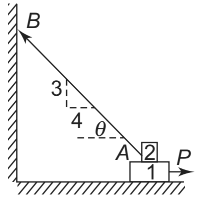


Figure 6

Answer

The coefficient of friction between the supporting surface and block μ_1 = 0.4

And the coefficient of friction between the blocks μ_2 = 0.25

The angle θ is given by

$$\theta = \tan^{-1}\frac{3}{4} = 36.87°$$

Let us draw the FBD of the two blocks [Fig. 6 (a)].

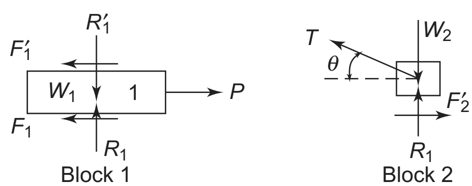


Figure 6 (a)

For Block-1,

$$\sum F_x = 0$$

or, $$P = F_1 + F_1' \tag{3}$$

$$\sum F_y = 0$$

or, $$R_1 = W_1 + R_1' \tag{4}$$

For limiting equilibrium,

$$F_1 = \mu_1 R_1 \tag{5}$$

$$F_1' = \mu R_1' \tag{6}$$

$$R_1 = W_1 + R_1'$$

$$\frac{F_1}{\mu_1} = W_1 + \frac{F_1'}{\mu} \tag{7}$$

For Block-2,

$$\sum F_x = 0$$

or, $$F_1' = T\cos\theta$$

or, $$T = \frac{F_1'}{\cos\theta} \tag{8}$$

$$\sum F_y = 0$$

or, $$R_1' + T\sin\theta = W_2 \tag{9}$$

$$\frac{F_1'}{\mu} + \frac{F_1'}{\cos\theta} \times \sin\theta = W_2$$

$$F_1' = \frac{W_2}{\frac{1}{\mu} + \tan\theta} = \frac{570}{\frac{1}{0.25} + \frac{3}{4}} = 120\,\text{N}$$

From Eq.(7), we have

$$F_1 = \mu_1\left[1290 + \frac{120}{0.25}\right]$$

$= 0.4 \times 1770 = 708$ N

From Eq.(3), we find

$$P = F_1 + F_1' = 120 + 708$$

$= 828$ N

(b) Two smooth circular cylinders of Figure 5, each of weight W = 100 N and radius r = 6 cm are connected by a string AB of length l = 16 cm. They rest upon a horizontal plane, supporting a third cylinder of weight Q = 200 N and radius r = 6 cm above them. Find the tension S in the string AB and the pressure produced by the floor at points of contact D and E.

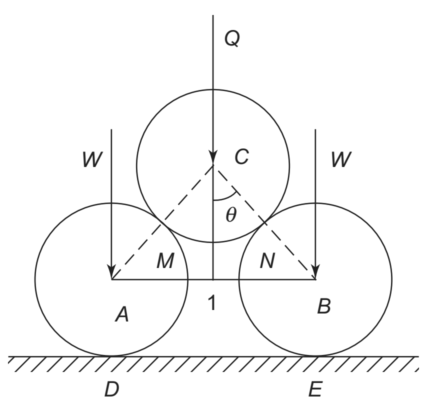


Figure 7

Answer

The FBD of the upper roller is shown in Fig. 7(a).

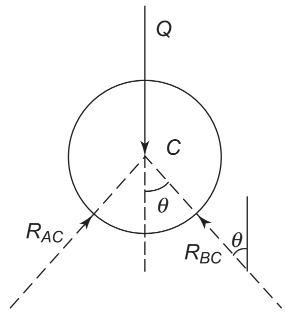


Figure 7 (a)

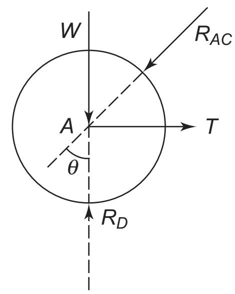


Figure 7 (b)

Under condition of equilibrium, considering $\sum X = 0$ yields that R_{AC} and R_{BC} are equal (say R). From the geometry,

$$\sin\theta = \frac{l/2}{2r}$$

$\therefore \quad \theta = 41.8°$

From $\sum Y = 0$, we get

$$2R\cos\theta = Q$$

or, $$R = \frac{Q}{2\cos\theta}$$

Since the lower two rollers are identical in all respect, R_D and R_E are same.

The FBD of the lower-left roller is shown in Fig. 7(b).

$\sum X = 0;$

$$R\sin\theta = T$$

or, $$T = \frac{Q}{2}\tan\theta = \frac{200}{2}\tan 41.8° = 89.4 \text{ N}$$

$\sum Y = 0;$

or, $$R_D = W + R\cos\theta = W + \frac{Q}{2} = 200N = R_E$$

10. (a) Two rollers of diameters 60 mm and 30 mm weigh 200 N and 150 N respectively. They are supported by an inclined plane and vertical walls as shown in Fig. 8. Assuming smooth surfaces, determine the reactions at the contact surfaces.

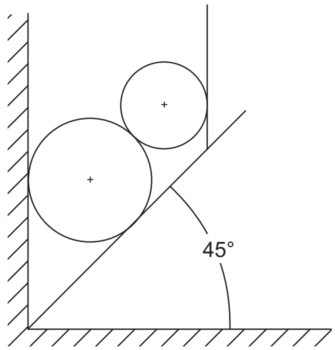


Figure 8

Answer

This problem is identical to the worked out Example 4.5.

(b) Find out the moment of inertia about centroidal axes of an area as shown in Fig. 9.

Answer

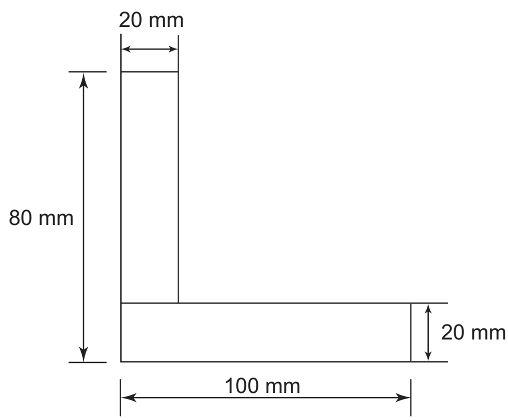


Figure 9

This problem is identical to the worked out Example 8.8.

11. (a) Calculate the increase in stress for each segment of the compound bar shown in Fig. 10. If the temperature increases by 100°F, assume that the supports are unyielding and that the bar is suitably braced against buckling.

$E_{Al} = 10 \times 10^6$ **psi,** $A_{Al} = 2.0$ **in**^{2}**,** $\alpha_{Al} = 12.8 \times 10^{-6}$**/°F and**

$E_{St} = 29 \times 10^6$ **psi,** $A_{St} = 1.5$ **in**^{2}**,** $\alpha_{St} = 6.5 \times 10^{-6}$**/°F**

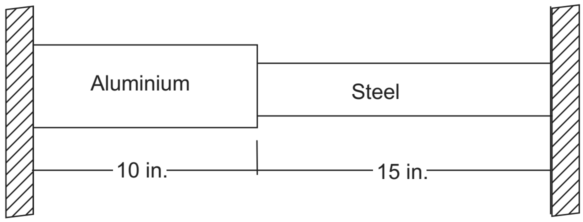


Figure 10

Answer

Change (rise) in temperature is $\Delta t = 100°F$

Both the bars will be under compression.

Force balance of the two bars yields

$$\sigma_{St} \times A_{St} = \sigma_{Al} \times A_{Al}$$

or, $\sigma_{St} \times 1.5 = \sigma_{Al} \times 2$

or, $\sigma_{St} = 1.33\ \sigma_{Al}$ (10)

Now,

$$\sigma = E\varepsilon = E \times \frac{\delta}{l}$$

$$\therefore \quad \delta_{St} = \frac{\sigma_{St} \times l_{St}}{E_{St}} = \frac{15\sigma_{St}}{29 \times 10^6} = 0.517 \times 10^{-6} \times \sigma_{St}$$

Similarly,

$$\delta_{Al} = \frac{\sigma_{Al} \times l_{Al}}{E_{Al}} = \frac{10\sigma_{Al}}{10 \times 10^6} = 10^{-6} \times \sigma_{Al}$$

Free contraction = $\delta_{St} + \delta_{Al}$

Further, free contraction = $l_{St}\alpha_{St}\Delta t + l_{Al}\alpha_{Al}\Delta t$

$$= 100\left[15 \times 6.5 \times 10^{-6} + 10 \times 12.8 \times 10^{-6}\right]$$

$= 0.2255$ inch

Therefore,

$$0.2255 = 0.517 \times 10^{-6} \times \sigma_{St} + 10^{-6} \times \sigma_{Al} \tag{11}$$

Solving Eqs (10) and (11), we get

σ_{St} = 17,690 psi and σ_{Al} = 13,352 psi

(b) In Fig. 11, a lever is attached to a spindle that is 2.5 cm in diameter by means of a square key of dimensions 6 mm × 6 mm. If the average shear stress in the key is not to exceed 700 N/cm², what is the safe value of the load *P* applied at the free end of the lever?

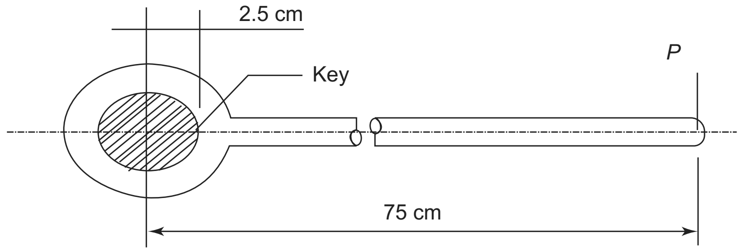


Figure 11

Answer

This problem is identical to the worked out Example 8.4.

ENGINEERING MECHANICS

SEMESTER – 1

Time: 3 Hours **Full Marks: 70**

Group – A
(Multiple Choice Questions)

1. Choose the correct alternatives for any *ten* of the following: **$10 \times 1 = 10$**

(i) Coulomb friction is between
(a) solids and liquids
(b) dry surfaces
(c) between bodies having relative motion
(d) none of these
Answer (b)

(ii) The velocity of a simple wheel and axle, with D and d as the diameters of effort respectively is
(a) $(D + d)$
(b) $(D - d)$
(c) d/D
(d) D/d
Answer (d)

(iii) For stable equilibrium the potential energy will be
(a) maximum
(b) minimum
(c) zero
(d) equal to kinetic energy
Answer (b)

(iv) The centroid of a semicircular area of radius r from the base is
(a) $\frac{4r}{3\pi}$
(b) $\frac{2r}{3\pi}$
(c) $\frac{3r}{2\pi}$
(d) r
Answer (a)

(v) Materials having same elastic properties in all directions are called
(a) isotropic
(b) orthotropic
(c) composite
(d) elastic
Answer (a)

(vi) The work done against any conservative force is stored in the body in the form of
(a) energy
(b) potential energy
(c) elastic energy
(d) strain energy
Answer (b)

(vii) A pair of a force and a couple in the same plane upon a rigid body
(a) balance each other
(b) cannot modify each other
(c) produce a moment
(d) none of these
Answer (c)

(viii) A particle inside a hollow sphere of radius r having coefficeient of friction $\frac{1}{\sqrt{3}}$, can be in rest up to a height of
(a) $r/2$
(b) $r/4$
(c) $3r/8$
(d) none of these
Answer (d)

(ix) Hooke's law is valid up to
(a) yield point
(b) elastic limit
(c) proportional limit
(d) ultimate stress
Answer (c)

(x) A jet engine works on the principle of conservation of yield point
(a) energy
(b) angular momentum
(c) linear momentum
(d) none of these
Answer (c)

(xi) Moment of inertia of a triangle of base and height about the centroidal axis parallel to base is
(a) $\frac{bh^3}{36}$
(b) $\frac{bh^3}{12}$
(c) $\frac{bh^3}{3}$

(d) none of these

Answer (a)

(xii) Couple is a

(a) bound vector

(b) free vector

(c) sliding vector

(d) none of these

Answer (b)

xiii) Angle between the vectors ($i + j$) and ($i - j$) is

(a) 90°

(b) 45°

(c) 0

(d) none of these

Answer (b)

Group – B
(Short-Answer Questions)

Answer any *three* questions $3 \times 5 = 15$

2. (a) Define moment.

Answer

See Section 3.7

(b) In the given Figure 1, the weight of the block is 1600 N and μ = 0.2. Find the value of P for impending motion.

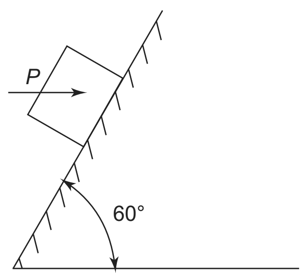


Figure 1

Solution

Considering equilibrium condition of the block when motion impends,

$\sum X = 0$

or, $P \cos 60° + F = W \sin 60°$ (1)

$\sum Y = 0$

or, $N = W \cos 60° + P \sin 60°$ (2)

Further, under limiting condition

$F = \mu N$ (3)

Combing Eqs (1), (2) and (3), we have

$$P = \frac{W(\sin 60° - \mu \cos 60°)}{(\cos 60° + \mu \sin 60°)} = \frac{1600(\sin 60° - 0.2\cos 60°)}{(\cos 60° + 0.2\sin 60°)} = 1820.54\text{N}$$

3. The position coordinate of particle which is confined to move in a straight line is given by $S = 2t^3 - 24t + 6$, where S is in m and t is in seconds.

Determine,

(a) the time required for the particle to reach a velocity of 72 m/s from its initial condition at $t = 0$.
(b) the acceleration of the particle when $v = 30$ m/s.
(c) the net displacement of the particle during the interval from $t = 1$ second to $t = 4$ seconds.

Solution

It is given that $S = 2t^3 - 24t + 6$

Now, $v = \dfrac{dS}{dt} = 6t^2 - 24$

(a) When $v = 72$ m/s, we have

$$72 = 6t^2 - 24$$

or, $t = 4$ s

(b) When $v = 30$ m/s, we get

$$30 = 6t^2 - 24$$

or, $t = 3$ s

The acceleration of the particle is given by

$$a = \frac{dv}{dt} = 12t$$

$\therefore$ $a_{t=3\text{ s}} = 12 \times 3 = 36$ m/s^2

(c) The net displacement of the particle during the interval from $t = 1$ second to $t = 4$ second is

$$\Delta S = 2(4^3 - 1^3) - 24(4 - 1) = 54 \text{ m}$$

4. Define (i) Malleability, (ii) Resilience, (iii) Toughness, (iv) Ductility, and (v) Proof Resilience.

Answer

See sections 9.3.1, 9.3.4, 9.3.5, and 9.4.1.

5. A force $F = 3i - 4j + 12k$ acts at a point A whose coordinates are (1, –2, 3). Compute,

(a) moment of force about origin,
(b) moment of force about point (2, 1, 2).

Solution

The position vector $\vec{r}_A = xi + yj + zk. = i - 2j + 3k$

(a) The moment of force about origin becomes

$$M_o = \vec{r}_A \times \vec{F} = (i - 2j + 3k) \times (3i - 4j + 12k)$$
$$= -12i - j + 2k$$

(b) To compute moment of force about point $B(2, 1, 2)$, it is essential to find out position vector $\vec{r}_{BA}$ as

$$\vec{r}_{BA} = \vec{r}_A - \vec{r}_B = (i - 2j + 3k) - (2i + j + 2k)$$
$$= -i - 3j + k$$

The moment of force about the point (2, 1, 2) becomes

$$M_B = \vec{r}_{BA} \times \vec{F} = (-i-3j+k)\times(3i-4j+12k)$$
$$= -32i+15j+13k$$

6. (a) State and Prove Lami's theorem.

Answer

See answer to Question 5 (i) of 2010 Solved Question Paper.

(b) Two equal loads of 2500 N are supported by a flexible string *ABCD* at points *B* and *D* as shown in Figure 2. Find the tensions in the portions *AB*, *BC* and *CD* of the string.

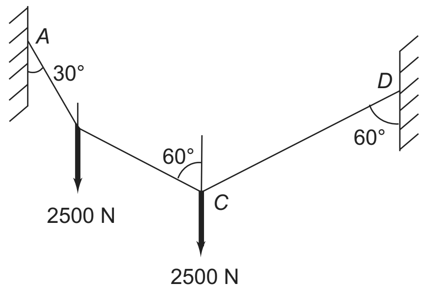


Figure 2

Solution

Considering equilibrium condition of the point B

$\sum X = 0$

or, $T_{AB} \sin 30° = T_{BC} \cos 30°$ (4)

$\sum Y = 0$

or, $T_{AB} \cos 30° = T_{BC} \sin 30° + 2500$ (5)

Dividing Eq.(4) by Eq.(5), we get

$$\tan 30° = \frac{T_{BC}\cos 30°}{T_{BC}\sin 30° + 2500}$$

or, $T_{BC} = 2500$ N

From Eq. (4), we have

$T_{AB} = T_{BC} \cot 30° = 4330.13$ N

Equilibrium of the point C yields $T_{CD} = T_{BC} = 2500$ N [From symmetry]

Group – C
(Long-Answer Questions)

Answer any *three* questions $3 \times 15 = 45$

7. (a) A block of weight $W_1 = 200$ kgf rests on the horizontal surface and supports on top of it, another block of weight $W_2 = 50$ kgf. The block W_2 is attached to a vertical wall by the inclined string *AB*. Find the magnitude of the horizontal force *P* applied to the lower block as shown, that will be necessary to cause slipping to impend. Take coefficient of static friction for all continuous surfaces is $\mu = 0.3$.

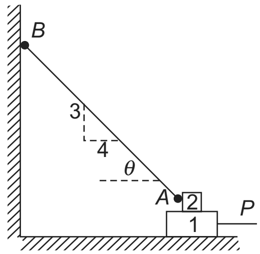


Figure 3

Solution

This problem is identical to solution to Question 9 (a) of 2010 Solved Question Paper.

(b) **A shot is fired with a bullet with an initial velocity of 40 m/s from a point 20 m in front of a 10 m high vertical wall. Find the angle of projection with the horizontal to enable the shot to just clear the wall.**

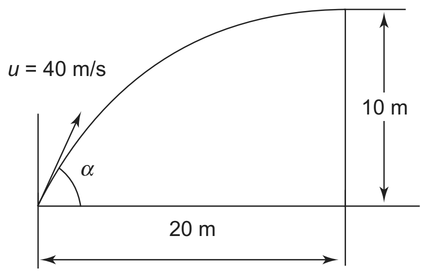


Figure 4

Solution

The trajectory of the bullet can be expressed by

$$y = \tan\theta x - \left(\frac{g}{2v_0^2\cos^2\theta}\right)x^2$$

From the given situation, the top of the wall will lie on the trajectory. Thus the coordinate of the top of the wall (20 m, 10 m) satisfies the equation.

Thus $$10 = \tan\theta \times 20 - \left(\frac{g}{2\times 40^2\cos^2\theta}\right)\times 20^2$$

Solving the above equation, we get

$$\theta = 86.36° \quad \text{or} \quad 30.27°$$

8. (a) **The bar shown in Figure 5 is subjected to a tensile load of 152 kN. Find the diameter of the middle portion if the stress there is to be limited to 140 N/mm². Find also the length of the middle portion if the total elongation of the bar is to be 0.16 mm. Take $E = 2 \times 10^5$ N/mm².**

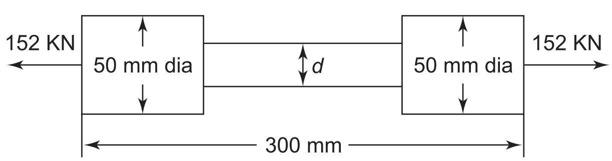


Figure 5

Solution

Let the diameter of the middle portion of the bar is d mm.

$$\therefore \qquad \sigma = \frac{P}{A} = \frac{152 \times 1000}{\frac{\pi}{4} d^2}$$

$$\text{or,} \qquad 140 \times 10^6 = \frac{152 \times 1000}{\frac{\pi}{4} d^2}$$

or, $\quad d = 0.3718$ m $= 37.18$ mm

Let the length of the two ends (having 50 mm diameters) of the bar is l_1 mm and that of middle portion is l_2 mm respectively and the corresponding areas are A_1 mm^2 and A_2 mm^2.

$$\therefore \qquad \delta = 2\delta_1 + \delta_2 = \frac{P}{E}\left[2\frac{l_1}{A_1} + \frac{l_2}{A_2}\right]$$

$$0.16 \times 10^{-3} = \frac{152 \times 10^3}{2 \times 10^5 \times 10^6}\left[2\frac{l_1}{\frac{\pi}{4}(0.05)^2} + \frac{l_2}{\frac{\pi}{4}(0.03718)^2}\right]$$

or,

or, $\quad 800 l_1 + 723.4 l_2 = 165.35 \qquad (6)$

Again, $\quad 2l_1 + l_2 = 0.3 \qquad (7)$

Solving Eqs.(6) and (7), we get

$$l_2 = 0.140 \text{ m} = 140 \text{ mm}$$

(b) Determine the coordinate of the centroid with respect to the given axis of the shaded area as shown in Figure 6.

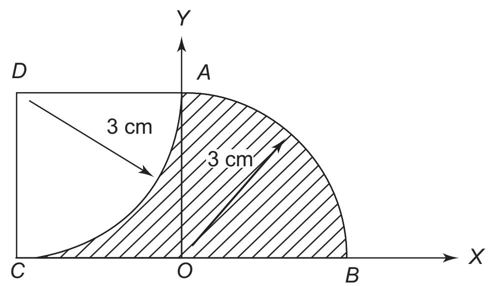


Figure 6

Solution

The area of interest can be considered as

Area of the quarter circle $OAB(A_1)$ + Area of the rectangle $OADC(A_2)$ – Area of the quarter circle $ACD(A_3)$

$A_1 = \frac{1}{4} \times \pi r^2$; $\qquad x_1 = \frac{4r}{3\pi}$; $\qquad y_1 = \frac{4r}{3\pi}$

$A_2 = r^2$; $\qquad x_2 = \frac{-r}{2}$; $\qquad y_2 = \frac{r}{2}$

$$A_3 = \frac{1}{4}\times\pi r^2 ; \qquad x_3 = -r\left(1-\frac{4}{3\pi}\right); \; y_3 = r\left(1-\frac{4}{3\pi}\right)$$

$$\therefore \qquad x_c = \frac{A_1x_1 + A_2x_2 - A_3x_3}{A_1 + A_2 - A_3}$$

$$= \frac{\frac{\pi r^2}{4}\times\frac{4r}{3\pi} + r^2\times\left(-\frac{r}{2}\right) - \frac{\pi r^2}{4}r\left(\frac{4}{3\pi}-1\right)}{\frac{\pi r^2}{4} + r^2 - \frac{\pi r^2}{4}} = \frac{\pi r}{4} - \frac{r}{2} = 0.856 \text{ mm}$$

$$y_c = \frac{A_1y_1 + A_2y_2 - A_3y_3}{A_1 + A_2 - A_3}$$

$$= \frac{\frac{\pi r^2}{4}\times\frac{4r}{3\pi} + r^2\times\frac{r}{2} - \frac{\pi r^2}{4}r\left(1-\frac{4}{3\pi}\right)}{\frac{\pi r^2}{4} + r^2 - \frac{\pi r^2}{4}} = \frac{2r}{3} - \frac{\pi r}{4} + \frac{r}{2} = 1.144 \text{ mm}$$

9. (a) State principle of transmissibility.

Answer

See Section 3.2.2

(b) Given a force *F* = 10*i* + 5*j* + *Ak*. If this force is to have a rectangular component of 8 N along a line having unit vector *r* = 0.6*i* + 0.8*k*, what should be the value of *A*? What is the angle between *F* and *r*?

Solution

Work done by this force would be

$$\vec{F}\cdot\vec{r} = (10i + 5j + Ak)\cdot(0.6i + 0.8k)$$
$$= 10\times0.6 + 5\times0 + A\times0.8 = 6 + 0.8A$$

$\therefore$ $\quad 6 + 0.8\,A = 8$

or, $\quad A = 2.5$

$\therefore$ $\quad |F| = \sqrt{10^2 + 5^2 + 2.5^2} = 11.456$

And $\quad |r| = \sqrt{0.6^2 + 0.8^2} = 1$

Further, $\quad \vec{F}\cdot\vec{r} = |F||r|\cos\theta$

or, $\quad \cos\theta = \frac{\vec{F}\cdot\vec{r}}{|F||r|} = \frac{2.5}{11.456\times1} \Rightarrow \theta = 45.7°$

or, $\quad \theta = 45.7°$

(c) Two identical blocks *A* and *B* each having weight *W* are connected by rigid link and supported by a vertical wall and a horizontal plane having same co-efficient of friction (μ) as shown in Figure 7. If sliding impends for θ = 45°, calculate μ.

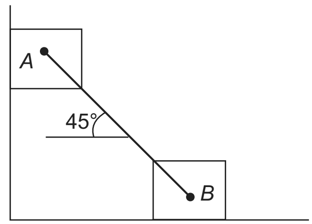


Figure 7

Solution

This problem is identical to the worked out Example 6.2.

10. (a) If the string *AB* is horizontal, find the angle that the string *AC* makes with the horizontal when the ball is in a position of equilibrium. Also find the pressure *R* between the ball and the plane.

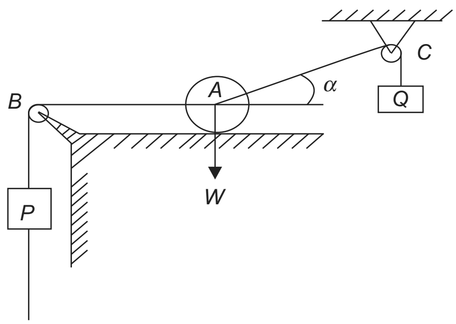


Figure 8

Solution

This problem is identical to solution to Question 5 of 2008 Solved Question Paper.

(b) A roller of radius r = 12 cm and weight Q = 500 kgf is to be rolled over a curb of height h = 6 cm by a horizontal force P applied to the end of a string wound around the circumference of the roller. Find the magnitude of P required to start the roller over the curb. There is sufficient friction between the roller surface and the edge of the curb to prevent slip at A.

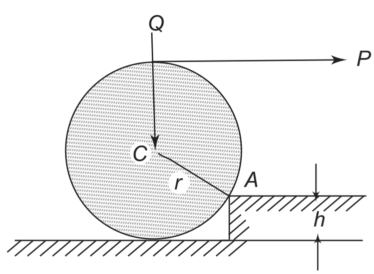


Figure 9

Solution
This problem is identical to solution to Question 5 of 2009 Solved Question Paper.

11. (a) State parallel axis and perpendicular axis theorem for moment of inertia.
Answer
See Sections 7.3 and 7.2.1

(b) Define radius of gyration. How is it related to mass moment of inertia?
Answer
See Sections 7.2.2 and 7.6

(c) Determine the center of a quarter circular arc of radius
Answer
This problem is identical to worked out Example 5.2.

ENGINEERING MECHANICS

Group – A
(Multiple Choice Questions)

1. Choose the correct alternatives for any *ten* of the followings: **$10 \times 1 = 10$**

(i) Two non-collinear parallel equal forces acting in opposite directions
- (a) balance each other
- (b) constitute a moment
- (c) constitute a couple
- (d) constitute a moment of a couple

Answer (c)

(ii) The centre of gravity of a uniform lamina lies at the
- (a) centre of the heavy portion
- (b) bottom surface
- (c) midpoint of its axis
- (d) all of these

Answer (c)

(iii) Materials having the same elastic properties in all directions are called
- (a) ideal materials
- (b) isotropic materials
- (c) elastic materials
- (d) uniform materials

Answer (b)

(iv) Given $\vec{F}_1 = 5\hat{j} + 4\hat{k}$ and $\vec{F}_2 = 3\hat{i} + 6\hat{k}$. The magnitude of the scalar product of these vectors is

(a) 15 (b) 30 (c) 24 (d) 12

Answer (c)

(v) The moment of inertia of a semicircle of radius R about its centroidal axis x-x is
- (a) $0.22R^4$
- (b) $0.055R^4$
- (c) $0.11R^4$
- (d) none of these

Answer (c)

(vi) The first moment of an area about the centroidal axis of that area is

(a) maximum (b) minimum

(c) zero (d) cannot be defined

Answer (c)

(vii) A projectile is fired at an angle θ to the vertical. Its horizontal range will be maximum when θ is
(a) 0 (b) 30° (c) 45° (d) 60°
Answer (c)

(viii) When a body slides down an inclined surface of inclination θ, the acceleration of the body is given by
(a) $f = g$ (b) $f = g \sin \theta$
(c) $f = g \cos \theta$ (d) $f = g/\sin \theta$
Answer (b)

(ix) A body is testing on a plane inclined at an angle of 30° to the horizontal. What forces would be required to slide down, if the coefficient of friction between body and plane is 0.3?
(a) zero (b) 1 kg
(c) 5 kg (d) none of these
Answer (a)

(x) Poisson's ratio is defined as
(a) longitudinal stress and longitudinal strain
(b) longitudinal strain and lateral strain
(c) lateral stress and longitudinal stress
(d) lateral strain and longitudinal strain
Answer (c)

(xi) The maximum strain energy that can be stored under elastic limit in a body is known as
(a) impact energy (b) resilience
(c) proof resilience (d) toughness
Answer (c)

(xii) Coulomb friction is
(a) the friction between solids and liquids
(b) the friction between dry surfaces
(c) the friction between bodies having reactive motion
(d) none of these
Answer (b)

(xiii) The deformation of a bar per unit length in the direction of force is known as
(a) linear strain
(b) lateral strain
(c) shear strain
(d) volumetric strain
Answer (a)

Group – B
(Short-Answer Questions)

Answer any *three* questions

2. (a) State D'Alembert principle.
Answer See Section 11.3, page 11.7

(b) A smooth circular cylinder of radius 1.5 cm is lying in a rectangular groove is shown in Figure 1. Find the reactions at the surfaces of contact, if there is no friction and cylinder weighs 1000 N.

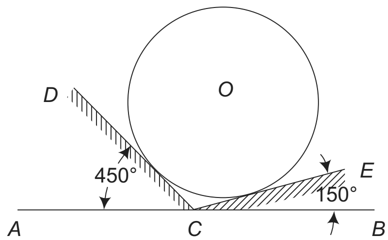


Figure 1

Solution

This problem is identical to solution to Question 9 (a) of 2009 Solved Question Paper.

3. A horizontal bar *AB* is hinged to a vertical wall at *A* and supported at its mid point *C* by a cable *CD* as shown in Figure 2. The bar is subjected to a vertical load *P* applied at the free end *B*. The bar maintains horizontal position. Find the tension *T* in the cable and the reaction at *A*. Neglect the weight of the bar.

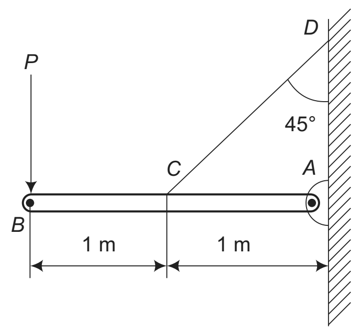


Figure 2

Solution

This problem is identical to the worked out Example 4.13.

4. (a) State the parallel axes theorem of moment of inertia of lamina.

Answer See Section 7.3, page 7.3

(b) Calculate the location of the centroid of the *L* section shown in Figure 3.

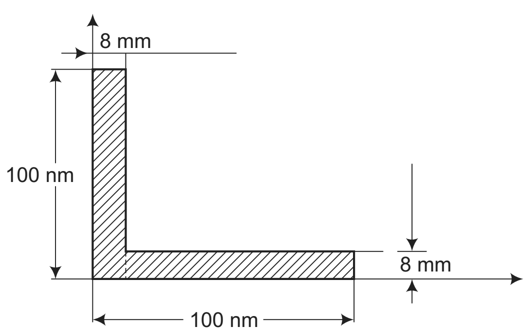


Figure 3

Solution
This problem is identical to the worked out Example 7.7.

5. **A bar of variable cross-sectional areas as shown in Figure 4 is subjected to different forces. Find the total elongation of the bar. Take $E = 2 \times 10^5$ N/mm^2**

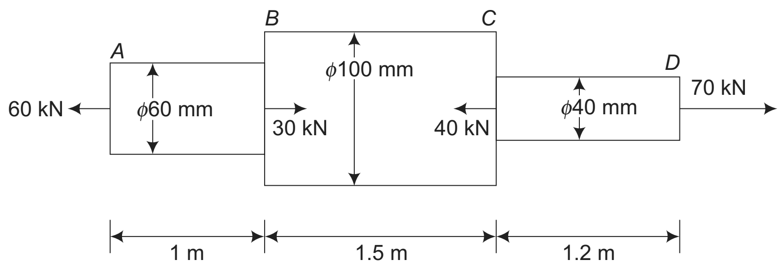


Figure 4

Solution
This problem is identical to solution to Question 8 (a) of 2011 Solved Question Paper.

6. **The motion of a particle is expressed as $x = x_0 + v_0 t + \frac{1}{2}at^2$. Calculate the displacement and velocity at time $t = 5$ second. $x_0 = 12$ m, $v_0 = 5$ m/s, $a = 20$ m/s^2.**

Solution
This problem is identical to the worked out Example 10.1.

Group – C
(Long-Answer Questions)
Answer any *three* questions

7. (a) **A cart of mass M rolls down a track inclined at an angle θ. The cart starts from rest a distance l up the track from a spring, and rolls down to collide with the spring as shown in Figure 5.**

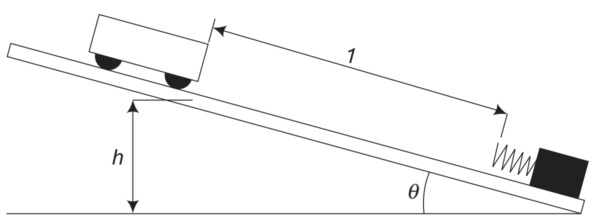


Figure 5

(a) **Assuming no non-conservative work is done, what is the speed of the cart when it first contacts the spring? (Express your answer in terms of the given variables and the gravitational acceleration g).**
(b) **Suppose the spring has a force constant k. What is the peak force compressing the spring during the collision?**

Solution

(a) Following conservation of energy, loss in PE = gain in KE

$$\therefore \qquad Mg \cdot l \sin\theta = \frac{1}{2}Mv^2 \Rightarrow v = \sqrt{2gl\sin\theta}$$

(b) Let the maximum compression of the spring is x

Since the cart will come to rest after collision, final velocity would be zero

According to conservation of energy, potential energy loss by the cart = Mechanical energy gain by the spring.

$$\therefore \qquad Mg(l + x)\sin\theta = \frac{1}{2}(kx)x$$

or $\quad kx^2 - (2Mg\sin\theta)x - 2Mgl\sin\theta = 0$

or
$$x = \frac{2Mg\sin\theta \pm \sqrt{(2Mg\sin\theta)^2 + 4k \times 2Mgl\sin\theta}}{2k}$$

$$= \frac{Mg\sin\theta}{k}\left[1 \pm \sqrt{1 + \frac{2kl}{Mg\sin\theta}}\right]$$

Neglecting the minus sign, $x_{\max} = \dfrac{Mg\sin\theta}{k}\left[1 + \sqrt{1 + \dfrac{2kl}{Mg\sin\theta}}\right]$

The maximum compressive force therefore becomes

$$F_{\max} = kx_{\max} = Mg\sin\theta\left[1 + \sqrt{1 + \frac{2kl}{Mg\sin\theta}}\right]$$

8. A block of weight W_1 = 200 kgf rests on the horizontal surface and supports on top of it, another block of weight W_2 = 50 kgf. The block W_2 is attached to a vertical wall by the inclined string *AB*. Find the magnitude of the horizontal force *P* applied to the lower block as shown in Figure 6, that will be necessary to cause slipping to impend. Take coefficient of static friction for all contiguous surfaces is μ = 0.3.

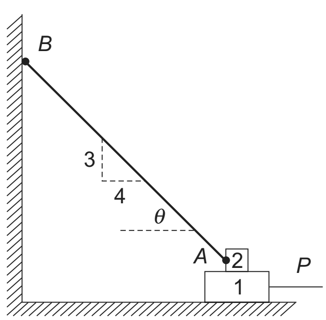


Figure 6

Solution This problem is identical to the worked out Example 6.16.

9. (a) **Determine the moment of inertia of the shaded area with respect to the given axis as shown in Figure 7.**

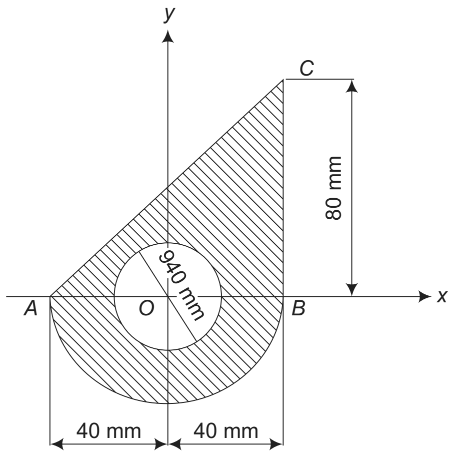


Figure 7

Solution
This problem is identical to the worked out Example 7.6.

(b) **Explain D'Alembert principle.**
Answer See Section 11.3, page 11.7

(c) **Two shots are fired from a rifle with an initial velocity of 800 m/s from a point 5 km in front of a vertical wall of 1.5 km high. Find the two angles of projection with horizontal to enable the short to just clear the wall (g = 9.81 m/s^2).**

Solution
This problem is identical to the worked out Example 13.9.

10. (a) **In the following Figure 8, F = 1000 N while O(0, 0, 0), A(0, 10, 0) and B(5, 0, 4). Calculate the moment of force about O.**

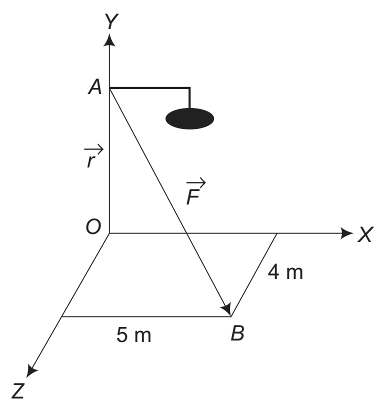


Figure 8

Solution

The force F can be expressed as

$$\mathbf{F} = F\mathbf{n} = F\frac{(x_2-x_1)i+(y_2-y_1)j+(z_2-z_1)k}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}} = 1000\frac{(5-0)i+(0-10)j+(4-0)k}{\sqrt{5^2+(-10)^2+4^2}}$$

$$1000\frac{5i-10j+4k}{\sqrt{5^2+(-10)^2+4^2}} = 421i - 842j + 337k$$

$$\mathbf{r} = xi + yj + zk = 0i + 10j + 0k = 10j$$

Therefore the moment of force **F** about O is $\mathbf{M_o} = \mathbf{r} \times \mathbf{F} = (337 \times 10)i - (421 \times 10)j = 3370i - 4210j$

(b) Find the perpendicular distance from the point $A(1, 2, 3)$ to the line joining the origin O and the point $B(2, 10, 5)$

Solution

Equation of the line joining the origin O and the point $B(2, 10, 5)$ is given by

$$\frac{x-0}{2-0} = \frac{y-0}{10-0} = \frac{z-0}{5-0} = t(say)$$

or
$$x = 2t,\ y = 10t,\ z = 5t$$

If d be the distance from the point $A(1, 2, 3)$ to the line joining the origin O and the point $B(2, 10, 5)$, then one can write

$$d^2 = (2t-1)^2 + (10t-2)^2 + (5t-3)^2$$

For minimization of d, we have

$$\frac{d(d^2)}{dt} = 2(2t-1)2 + 2(10t-2)10 + 2(5t-3)5 = 0$$

or
$$t = \frac{37}{129}$$

The perpendicular distance from the point $A(1, 2, 3)$ to the line joining the origin O and the point $B(2,10, 5)$ is

$$d = \sqrt{\left(2\times\frac{37}{129}-1\right)^2+\left(10\times\frac{37}{129}-2\right)^2+\left(5\times\frac{37}{129}-3\right)^2}$$

$$= 1.939 \text{ unit}$$

11. (a) Determine velocity *V* of the falling weight *W* of the system as shown in Figure 9 as a function of its displacement from the initial position of rest. Assume weight of the cylinder as 2*W*.

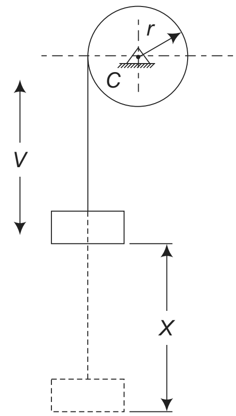


Figure 9

Solution

This problem is identical to solution to Question 10(c) of 2009 Solved Question Paper.

(b) From the top of a tower, 60 m high a bullet is fired at an angle of 20° up the horizontal with velocity 120 m/s. Determine:

(i) Time of flight
(ii) Horizontal range of ground
(iii) Maximum height of bullet from ground
(iv) Velocity of bullet after 8 sec.

Assume horizontal ground at the foot of the tower.

Solution

This problem is identical to the worked out Example 13.10.

CS/B.TECH (NEW)/SEM-1/ME 101/2013-14
ENGINEERING MECHANICS

Group–A
(Multiple-Choice-Type Questions)

1. Choose the correct alternatives for any *ten* of the followings: **10 × 1 = 10**

*(i) Lami's theorem is applicable to
(a) equilibrium of two coplanar, concurrent forces
(b) equilibrium of three coplanar, concurrent forces
(c) equilibrium of three coplanar, non-concurrent forces
(d) none of these
Answer (b)

*(ii) The angle between the vectors $(i + j)$ and $(i - j)$ is
(a) 90° (b) 45° (c) 0° (d) none of these
Answer (a)

**(iii) When a body slides down an inclined surface of inclination θ with the horizontal, the acceleration θ of the body is given by
(a) $a = g$ (b) $a = g \cos \theta$ (c) $a = g \sin \theta$ (d) $a = \dfrac{g}{\cos\theta}$
Answer (c)

*(iv) The values of $\hat{i} \cdot \hat{i}$ and $\hat{i} \times \hat{i}$ are
(a) 1 and 0 (b) 1 and 1 (c) 0 and 0 (d) 0 and 1
Answer (a)

*(v) The moment of inertia of a circle with its centroidal x-axis is
(a) $\pi d^4/32$ (b) $\pi d^4/256$ (c) $\pi d^4/64$ (d) $\pi d^4/128$
Answer (c)

**(vi) A particle moves along the horizontal direction and its position at any instance is prescribed by the relation $x = 3t^3 - 5t^2$ where x is in metres and t is in seconds. What distance will be covered by the particle during $t = 2$ seconds to 5 seconds?
(a) 246 m (b) 146 m (c) 200 m (d) 216 m
Answer (a)

*(vii) When a rectangular bar of length l, breadth b and thickness t is subjected to an axial pull of P, the linear strain is given by
(a) $bt\ E/P$ (b) $P/bt\ E$ (c) bt/PE (d) PE/bt
Answer (b)

**(viii) Given $\vec{F}_1 = 5\hat{j} + 4\hat{k}$ and $\vec{F}_2 = 3\hat{i} + 6\hat{k}$. The magnitude of the scalar product of these vectors is
(a) 15 (b) 30 (c) 24 (d) 12
Answer (c)

Note: * Indicates Level 1 difficulty
** Indicates Level 2 difficulty
*** Indicates Level 3 difficulty

*(ix) The equation of motion of a particle is $s = 2t^3 - t^2 - 2$, where s is the displacement in metres and t is time in seconds. The acceleration of the particle after 1 second will be

(a) 8 m/s^2 (b) 9 m/s^2 (c) 10 m/s^2 (d) 5 m/s^2

Answer (c)

*(x) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ then $\vec{A}\cdot\vec{B}$ is given by

(a) $A_xB_y + B_xA_y + A_zB_z$ (b) $A_xA_y + A_zB_y + B_zA_x$

(c) $A_xA_y + B_xB_y + B_zA_x$ (d) $A_xB_x + A_yB_y + A_zB_z$

Answer (d)

*(xi) D' Alembert's principle

(a) is based upon the presence of inertia force
b) provides an advantage over Newton's law
(c) is purely a hypothetical law
(d) allows a dynamic problem to be treated as a static one

Answer (d)

***(xii) A single force and couple action in the same plane upon a rigid body

(a) balance each other (b) cannot balance each other

(c) produce moment of a couple (d) are equivalent

Answer (b)

Group–B
(Short-Answer-Type Questions)

Answer any *three* of the following.

3 × 5 = 15

****2. A member is shown in Fig. 1. Replace the force (100 N) acting at the point *D*, into an equivalent force-couple system at the point *C*. Find the reaction forces at points *A* and *B*.**

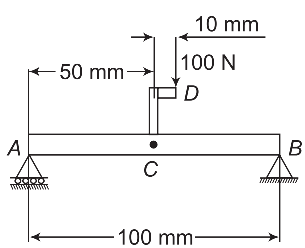


Figure 1

Solution

At the point *C*, a pair of 100 N force, one vertically upward and another vertically downward, is added as shown in Fig. 2(a).

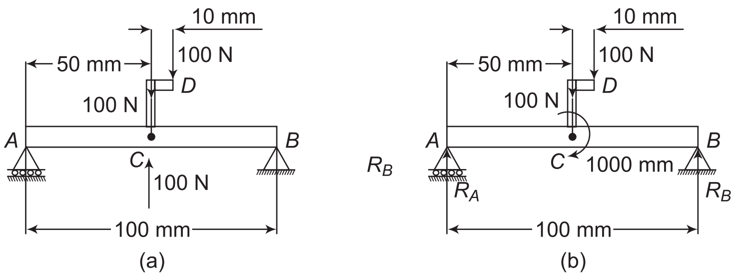


Figure 2

Thus, the 100 N force applied vertically downward at D can be replaced by an equivalent clockwise couple of moment 1000 N-mm at C along with a 100 N force applied vertically downward at C as shown in Fig. 2(b).

Let the reactions at A and B be denoted by R_A and R_B respectively. Taking moment with respect to the point A, we obtain

$$\sum M_A = 0$$

or $\quad R_B \times 100 = 1000 + 100 \times 50$

or $\quad R_B = 60$ N and

From the condition of equilibrium, we get

$$R_A = 100 - 60 = 40 \text{ N}$$

***3. A load P = 50 kg is suspended from A. The spring AB is deformed by an amount of 0.025 m and is horizontal in the equilibrium position according to Fig. 3. Determine the stiffness of the spring.**

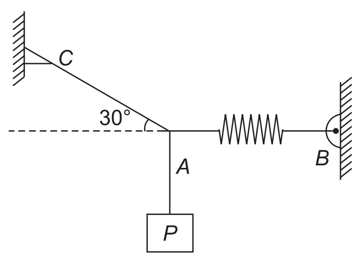


Figure 3

Solution From the conditions of equilibrium at the point A, we get

$$\sum Y = 0$$

or $\quad T \sin 30° = P$

or $\quad T = \dfrac{50}{\sin 30°} = 100 \text{ kg}$

$$\sum X = 0$$

or $\quad T \cos 30° = F$

or $\quad F = 100 \cos 30° = 86.6$ kg

It is given that the deformation amount δ = 0.025 m. Let the stiffness of the spring is k.

For a spring, $F = k\delta$

Thus, $k = F/\delta = 86.6/0.025 = 3464$ kg/m

****4. A bar of uniform cross section A and length L is vertically hung subjected to its own weight. Prove that strain energy within the bar U = $A\omega^2L^3/6E$, where ω = sp. weight, E = modulus of elasticity.**

Solution This problem is identical to Example 9.7.

5. *(a) State the parallel axes theorem of moment of inertia of lamina.

Solution Refer Section 7.3.

***(b) Calculate the location of the centroid of the L-section as shown in Fig. 4.**

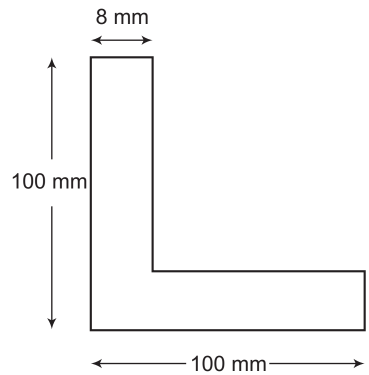


Figure 4

Solution This problem is identical to Example 5.12.

***6. A force $\vec{F} = (3i - 4j + 2k)$ N acts at a point A whose coordinates are (1, –2, 3) m. Compute (a) moment of force about the origin, and (b) moment of force about the point (2, 1, 2) m.**

Solution It is given that $\vec{F} = 3i - 4j + 2k$ and $\vec{r} = i - 2j + 3k$.

(a) Moment of the force about the origin is

$$\vec{M}_O = \vec{r} \times \vec{F} = (i - 2j + 3k) \times (3i - 4j + 2k) = 8i + 7j + 2k$$

(b) Now, $\vec{r}_{AB} = -i - 3j + k$.

Moment of the force about the point (2,1,2) m is

$$\therefore\ \vec{M}_B = \vec{r}_{AB} \times \vec{F} = (-i - 3j + k) \times (3i - 4j + 2k) = -2i + 5j + 13k$$

Group–C
(Long-Answer-Type Questions)

Answer any *three* of the following.

3 × 15 = 45

7. *(a) A ball of weight W is resting upon a smooth plane and is attached at its centre to the strings which pass over a smooth pulley and carry loads P and Q as shown in Fig. 5. Find the angle θ and pressure between the ball and the plane.

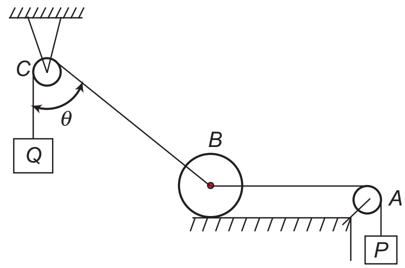


Figure 5

Solution This problem is identical to solution to Question 5 of 2008 Solved Question Paper.

****(b) A uniform wheel of 60 cm diameter rests against a rigid rectangular block of 15 cm height as shown in Fig. 6. Find the magnitude and direction of the least pull through the centre of the wheel that will just turn the wheel over the corner of the block. All surfaces are smooth. Determine the reaction of the block at the point *C*. Weight of the wheel is 5 kN.**

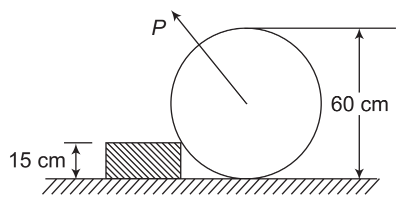


Figure 6

Solution This problem is identical Example 4.17.

8.*(a) State D' Álembert's principle.
Answer Refer Section 11.3.

*****(b) Two blocks weighing 600 N and 1200 N are placed on 30° and 60° planes respectively as shown in Fig. 7. The blocks are connected by an extensible string passing over a friction pulley. If μ = 0.25 for both the planes, find the tension in the string and the acceleration of blocks.**

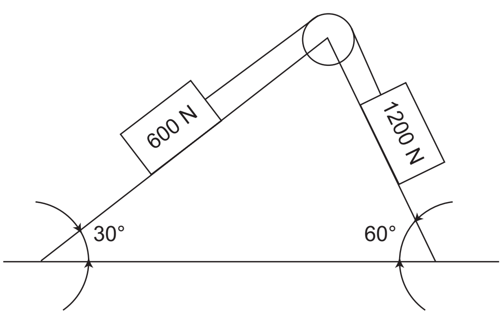


Figure 7

Solution Let the tension in the string be T and the acceleration of the blocks is a

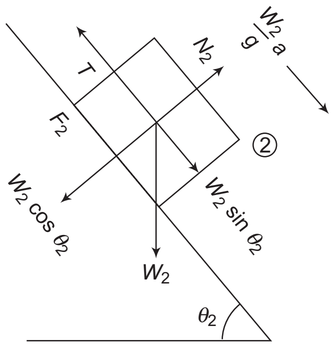


Figure 8 (a)

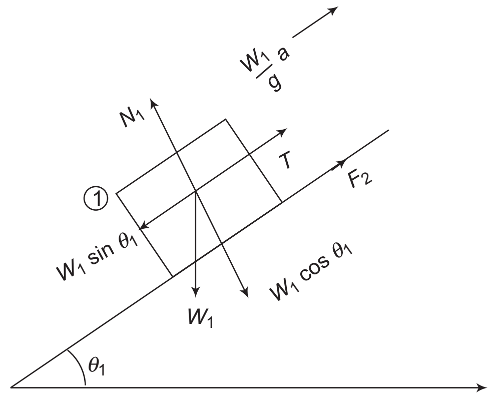


Figure 8 (b)

Considering the free-body diagram of the block 1 [Refer Fig. 8 (a)] and using dynamic equilibrium,

$$T - F_1 - W_1 \sin\theta_1 = \frac{W_1}{g}a$$

or $$T - \mu W_1 \cos\theta_1 - W_1 \sin\theta_1 = \frac{W_1}{g}a \qquad (1)$$

Similar condition of the block 2 [Refer Fig. 8(b)] yields

$$W_2 \sin\theta_2 - T - F_2 = \frac{W_2}{g}a$$

or $$W_2 \sin\theta_2 - T - \mu W_2 \cos\theta_2 = \frac{W_2}{g}a \qquad (2)$$

Eliminating T from the above two equations;

$$W_2[\sin\theta_2 - \mu\cos\theta_2] - W_1[\sin\theta_1 + \mu\cos\theta_1] = \frac{a}{g}(W_1 + W_2)$$

or $$1200(\sin 60° - 0.25\cos 60°) - 600(\sin 30° + 0.25\cos 30°) = \frac{a}{g} \times 1800$$

or $a = 2.5$ m/s^2

From Eq. (1), we get

$$T = W_1\left(\frac{a}{g} + \mu\cos\theta_1 + \sin\theta_1\right)$$

or $$= 600\left(\frac{2.5}{9.81} + 0.25\cos 30° + \sin 30°\right) = 583\text{N}$$

9. **(a) A gun is fired, so that the initial velocity of its bullet is 200 m/s and can hit the target located 500 m above the level of gunpoint and at horizontal distance of 3000 m. Neglecting the air resistance, determine the firing angle.

Solution This problem is identical to Example 13.9.

****(b) Determine the centroid of the shaded area as shown in Fig. 9.**

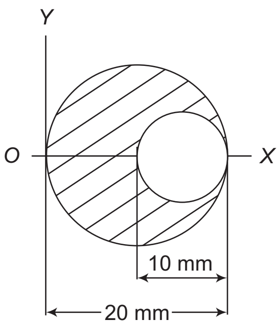


Figure 9

Solution Let the larger circle and the smaller circle be denoted by 1 and 2 respectively.

$$A_1 = \frac{\pi}{4}(20)^2 = 100\pi \text{ mm}^2; \quad A_2 = \frac{\pi}{4}(10)^2 = 25\pi \text{ mm}^2;$$

$x_1 = 10$ mm $x_2 = 15$ mm

$$\therefore \qquad x_c = \frac{A_1x_1 - A_2x_2}{A_1 - A_2} = \frac{100\pi \times 10 - 25\pi \times 15}{100\pi - 25\pi} = 8.33 \text{ mm}$$

Since the centroid lies on the x-axis, $y_c = 0$

*(c) Draw the stress-strain curve for ductile material and show various regions and points on it.

Answer Refer Section 9.4.

10. *(a) A force $F = 50i + 75j + 100k$ acts through E as shown in Fig. 10. Determine the moment of the force about x, y, and z axes respectively.**

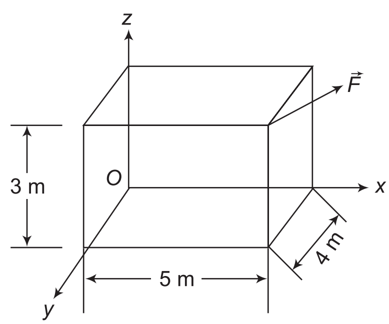


Figure 10

Solution

$$\vec{F} = 50i + 75j + 100k; \quad \vec{r} = 4i + 5j + 3k.$$

$$\vec{M}_O = \vec{r} \times \vec{F} = (50i + 75j + 100k) \times (4i + 5j + 3k) = 275i - 250j + 50k$$

$$\therefore \qquad M_x = 275 \text{ N-m}; \quad M_y = 250 \text{ N-m}; \quad M_x = 50 \text{ N-m}$$

*****(b) Two steel cylinders are supported in a right-angled wedge support as shown in Fig. 11. The side OL makes an angle of 30° with the horizontal. The diameters of the cylinders A and B are 250 mm and 500 mm, and their weights being 100 N and 400 N, respectively. Determine the reactions R between all contact points.**

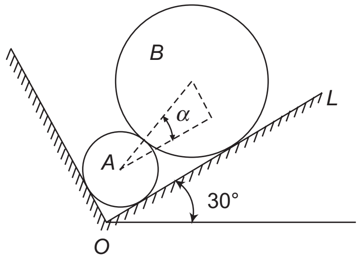


Figure 11

Solution From the geometry,

$$\sin\alpha = \frac{r_2 - r_1}{r_2 + r_1} = \frac{125}{375} = 0.33 \;\Rightarrow\; \alpha = 19.5°$$

The free-body diagrams of the two cylinders are shown in Fig. 12(a) and Fig. 12(b).

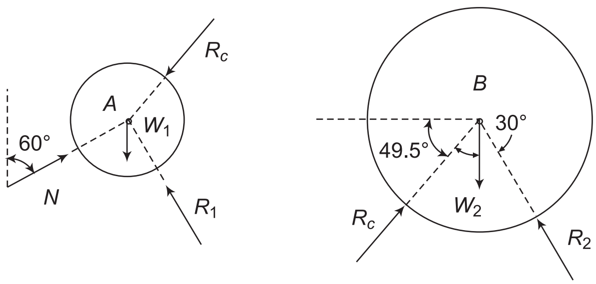


Figure 12(a) **Figure 12(b)**

Considering the equilibrium of the larger cylinder,

$$\sum X = 0 \;\Rightarrow\; R_C \cos 49.5 = R_2 \sin 30$$

$$\sum Y = 0 \;\Rightarrow\; R_2 \cos 30 + R_c \sin 49.5 = W_2 = 400$$

Solving the above two equations, R_C = 212.17 N and R_2 = 275.6 N

To calculate N and R_1 for the smaller cylinder, the force polygon is drawn and shown in Fig. 12 (c).

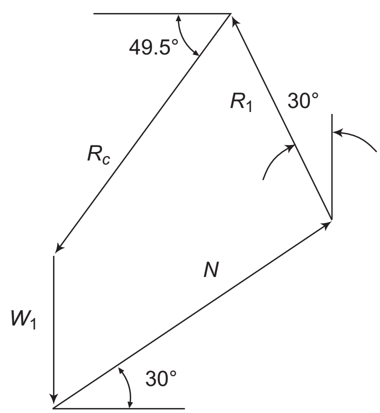


Figure 12(c)

From Fig. 12 (c), N = 250 N and R_1 = 157 N

11. **(a) Find the decrease in length of the steel bar loaded as shown in Fig. 13. Take $E = 2 \times 10^5$ N/mm²

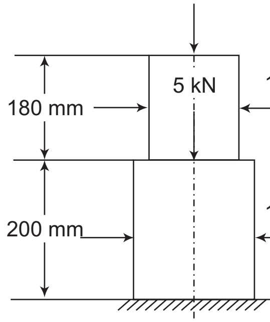


Figure 13

Solution

$$F_1 = 2 \text{ kN};\ F_2 = 5 \text{ kN}\ \ A_1 = \frac{\pi}{4}10^2 = 25\pi \text{ mm}^2;\ A_2 = \frac{\pi}{4}15^2 = 56.25\pi \text{ mm}^2;$$

$$\sigma_1 = \frac{F_1}{A_1} = \frac{2500}{25\pi} = 25.46 \text{ N/mm}^2; \qquad \sigma_2 = \frac{F_1}{A_1} = \frac{2000+5000}{56.25\pi} = 39.61 \text{ N/mm}^2$$

$$\therefore \qquad \delta = \delta_1 + \delta_2 = \frac{1}{E}(\sigma_1 l_1 + \sigma_2 l_2) = \frac{1}{2\times 10^5}(25.46\times 180 + 39.61 \times 200) = 0.0625 \text{ mm}$$

*****(b) t_1 is the time in which a projectile reaches a point P_1 along its path and t_2 is the time taken by the projectile from P_1 till it hits the horizontal plane passing through point P_2 as shown in Fig. 14. Show that the height of the point P_1 above the plane is 1/2 gt_1t_2.**

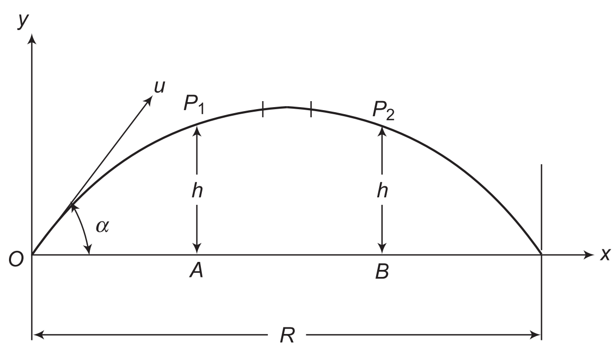


Figure 14

Solution The expression $y = v_0 \sin\theta t - \frac{1}{2}gt^2$ can be written in the following form in the context of the present problem.

$$h = v_0 \sin\alpha t - \frac{1}{2}gt^2$$

In the given figure (Fig. 14), P_1 and P_2 both are located at a height h from the horizontal plane.

It, therefore, implies that the time required by the projectile to reach from P_1 to the ground = time required by the projectile to reach P_2 from O (following the same path) = t_2

The above equation being a quadratic one, its two roots are t_1 and t_2 which corresponds to the time required to reach P_1 and P_2 from O respectively.

Rearranging the above equation, we have

$$t^2 - \frac{2v_0 \sin\alpha}{g} t + \frac{2h}{g} = 0$$

or $$t_1 t_2 = \frac{2h}{g}$$

or $$h = \frac{1}{2} g t_1 t_2$$

CS/B.TECH (NEW)/SEM-1/ME 101/2014-15
ENGINEERING MECHANICS

Group–A
(Multiple-Choice-Type Questions)

1. Answer any *ten* questions. **10 × 1 = 10**

*(i) The work done against any conservative forces is stored in the body in the form of
(a) energy (b) potential energy (c) elastic energy (d) strain energy
Answer (b)

**(ii) The magnitude of two forces, acting at right angles, produces a resultant force of $\sqrt{10}$ kg and when acting at 60°, produces a resultant of $\sqrt{13}$ kg. These forces are at
(a) 90° (b) 45° (c) 0° (d) none of these
Answer (b)

*(iii) If three forces acting in one plane upon a rigid body keep it in equilibrium then they must either
(a) meet in a point (b) be all parallel
(c) at least two of them must meet (d) all of the above are correct
Answer (a)

*(iv) A projectile is fired at an angle θ to the vertical. Its horizontal range will be maximum when θ is
(a) 0° (b) 30° (c) 45° (d) 60°
Answer (c)

*(v) Varignon's theorem is related to
(a) moment of forces(s) (b) friction
(c) deformation characteristics of rigid bodies (d) none of the above
Answer (a)

*(vi) Strain energy is the
(a) maximum energy which can be stored in a body
(b) energy stored in a body when stressed to the elastic limit
(c) energy stored in a body when stressed to the breaking point
(d) none of the above
Answer (b)

*(vii) The CG of a solid hemisphere lies on the central radius
(a) at distance $3r/2$ from the plane base (b) at distance $3r/4$ from the plane base
(c) at distance $3r/5$ from the plane base (d) at distance $3r/8$ from the plane base
Answer (d)

*(viii) If **i** and **j** are two Cartesian unit vectors then
(a) $\mathbf{i} \cdot \mathbf{j} = 1$ (b) $\mathbf{i} \cdot \mathbf{j} = 0$
(c) $\mathbf{i} \cdot \mathbf{j} = 2$ (d) none of these
Answer (b)

Note: * Indicates Level 1 difficulty
** Indicates Level 2 difficulty
*** Indicates Level 3 difficulty

**(ix) An elevator weighing 980 N attains an upward velocity of 3 m/s in 3 s following a uniform acceleration. The tension in the cable that supports the elevator is

(a) 1000 N (b) 1080 N (c) 880 N (d) 1150 N

Answer (b)

*(x) If momentum of a body is doubled, its kinetic energy

(a) gets doubled (b) gets halved (c) remain same (d) gets quadrupled

Answer (a)

*(xi) The condition of equilibrium of co-planar non-concurrent forces are

(a) $\sum F_x = 0; \sum F_y = 0$ (b) $\sum F_x = 0; \sum F_y = 0; \sum M = 0$

(c) $\sum F_y = 0; \sum M = 0$ (d) $\sum F_x = 0 ; \sum M = 0$

Answer (b)

*(xii) The equation of motion of a particle is $S = 2t^3 - t^2 - 2$, where S is the displacement in metres and t is time in seconds. The acceleration of the particle after 1 second will be

(a) 8 m/s^2 (b) 9 m/s^2 (c) 10 m/s^2 (d) 5 m/s^2

Answer (c)

Group–B
(Short-Answer-Type Questions)

Answer any *three* of the following.

3 × 5 = 15

****2. A string is connected at the point *C* of a structure *AB*, passing through a frictionless pulley and at the free end of the string, a weight is attached as shown in Fig. 1. Determine the reaction forces developed at points *A* and *B*. Neglect the mass of the structure *AB*.**

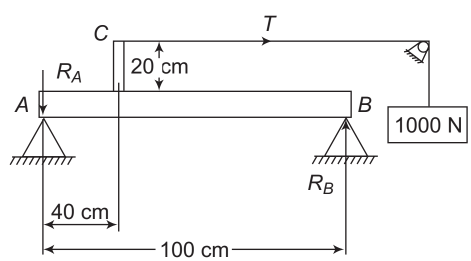


Figure 1

Solution Taking moment of all the forces with respect to *A*, we get

$$\sum M_A = 0$$

or $R_b \times 100 = T \times 20 = 1000 \times 20$

or $R_b = 200$ N

Considering the beam *AB*, we have

$$\sum Y = 0$$

or $R_A + R_B = 0$

or $R_A = -R_B = -200$N

***3. What is meant by toughness? What is meant by resilience? Draw a stress-strain diagram of a mild steel specimen and show the region of modulus of toughness and modulus of resilience.**
Solution Refer Section 9.4.

***4. By integration, determine the co-ordinate of the centroid of the plane area under the curve $y = kx^2$ and x-axis, between (0, 0) and (a, b) of Fig. 2.**

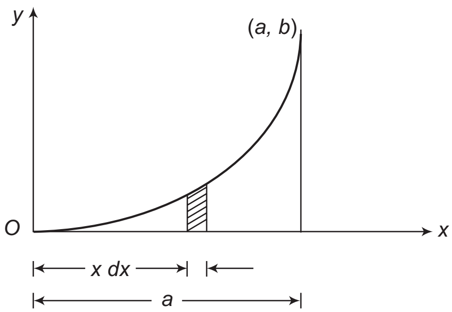


Figure 2

Solution This problem is identical to Example 5.5.

****5. By integration, determine the co-ordinate of the centroid of the plane area under 5, given a force F = 10i + 5j + Ak N. If this force is to have a rectangular component of 8 N along a line having unit vector r = 0.6i + 0.8k, what should be the value of A? What is the angle between F and r?**
Answer

$\mathbf{F} = 10i + 5j + Ak$ N and $\mathbf{r} = 0.6i + 0.8k$ N

According to the given statement, **F. r = 8 N**

Thus, $\mathbf{F}\cdot\mathbf{r} = (10i + 5j + Ak)\cdot(0.6i + 0.8k) = 6 + 0.8A$

or $6 + 0.8A = 8$

or $A = 2.5$

Therefore, $\mathbf{F} = 10i + 5j + 2.5k$

The magnitude of **F** is $\sqrt{10^2 + 5^2 + 2.5^2} = 11.456\text{ N}$ and the magnitude of **r** is $\sqrt{0.6^2 + 0.8^2} = 1$

Further, $\mathbf{F}.\mathbf{r} = |F||r|\cos\theta$

or $$\cos\theta = \frac{\mathbf{F}.\mathbf{r}}{|F||r|} = \frac{8}{11.456\times 1}$$

or $\theta = 45.7°$

Therefore, the angle between **F** and **r** becomes 45.7°.

***6.(a) State Lami's theorem.**
Solution Refer Section 4.2.

***(b) Two equal loads of 2500 N are supported by a flexible string $ABCD$ at points B and D as shown in Fig. 3. Find the tensions in the portions AB, BC, CD of the string.**

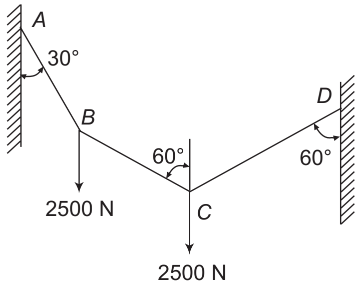


Figure 3

Solution This problem is identical to the solution to Question 6(a) of 2011 Solved Question Paper.

Group–C
(Long-Answer-Type Questions)

Answer any *three* of the following.

3 × 15 = 45

7. **(a) A force of 200 N is directed along the drawn from the point *P* (5, 2, 4) to the point *Q*(3, –5, 6). Determine the moment of this force about a point *A* (4, 3, 2). The distances are in metres.

Solution

$$\mathbf{F} = \mathbf{F}n = F\frac{(x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$= 200\frac{(3-5)i + (-5-2)j + (6-4)k}{\sqrt{(3-5)^2 + (-5-2)^2 + (6-4)^2}}$$

$$= -53i - 185.5j + 53k$$

The moment of force **F** about the point *A* (4, 3, 2), therefore, would be

$$\mathbf{M_A} = \mathbf{r_A} \times \mathbf{F}$$
$$= (4i + 3j + 2k) \times (-53i - 185.5j + 53k)$$
$$= 530i - 318j - 583k$$

***(b) Referring to Fig. 4, with what minimum horizontal velocity *u* can a boy throw a rock at *A* and have it just clear the obstruction at *B*?**

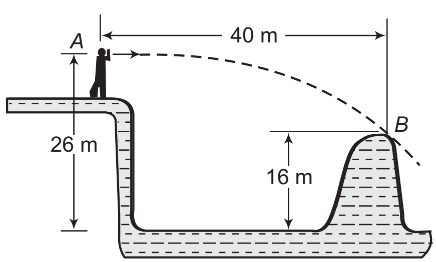


Figure 4

Solution The equation of the projectile is given by $y = \tan\theta x - \left(\dfrac{g}{2v_0^2 \cos^2\theta}\right)x^2$

The coordinate of the point B is (40, –10) and $\theta = 0$
The above equation, therefore, becomes

$$-10 = -\left(\frac{g}{2v_0^2 \cos^2 0}\right)40^2 \Rightarrow v_0 = 28 \text{ m/s}$$

or $v_0 = 28$ m/s

8. **(a) A block of weight W_1 = 200 kgf rests on a horizontal surface and supports on top of it another block of weight W_2 = 50 kgf.

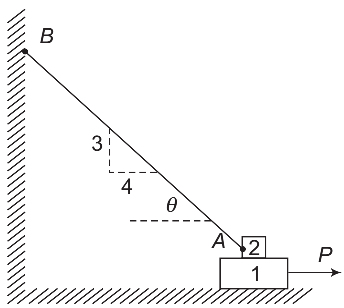


Figure 5

The block W_2 is attached to a vertical wall by the inclined string AB. Find the magnitude of the horizontal force P applied to the lower block as shown in Fig. 5, which will be necessary to cause slipping to impend. The coefficient of static friction for all contiguous surfaces is μ = 0.3.

Solution This problem is identical to Example 6.16.

****(b) A steel rod $ABCD$ of stepped section is loaded as shown in Fig. 6. The loads are assumed to act along the centreline of the load. Estimate the displacement of D relative to A. Assume $E = 2 \times 10^5$ N/mm^2.**

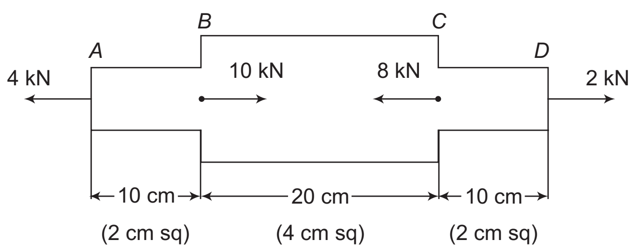


Figure 6

Solution The bar $ABCD$ can be divided into three segments AB, BC, and CD and the forces acting on them are shown in Fig. 7.

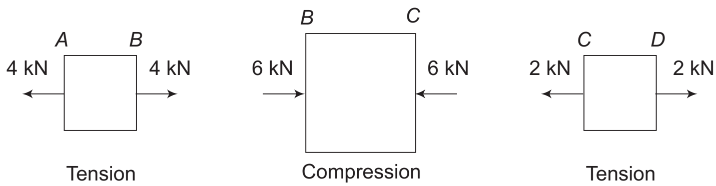


Figure 7

Total elongation is found to be

$$\delta_{AD} = \delta_{AB} - \delta_{BC} + \delta_{CD} = \frac{1}{E}\left[\frac{P_1 l_1}{A_1} - \frac{P_2 l_2}{A_2} + \frac{P_3 l_3}{A_3}\right]$$

or
$$\delta_{AD} = \frac{1}{200}\left[\frac{4\times100}{2\times100} - \frac{6\times200}{4\times100} + \frac{2\times100}{2\times100}\right] = 0.01 - 0.015 + 0.005 = 0$$

9. **(a) Determine velocity *V* of the falling weight *W* of the system as shown in Fig. 8, as a function of its displacement from the initial position of rest. Assume weight of the cylinder as 2*W*.

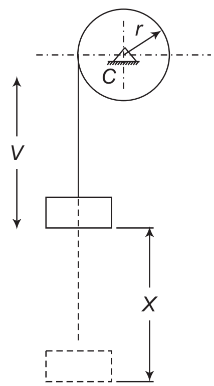


Figure 8

Solution This problem is identical to solution to Question 10(c) of 2009 Solved Question Paper.

***(b) Prove that the volumetric strain of a rectangular bar is the algebraic sum of strains of length, width, and height.**

Solution Refer Example 8.12.

***(c) A small block of weight *W* rests on a horizontal turntable at a distance *r* from the axis of rotation as shown in Fig. 9. If the coefficient of friction between the block and the surface of the turntable is μ, find the maximum uniform speed that the block can have due to rotation of the turntable without slipping off.**

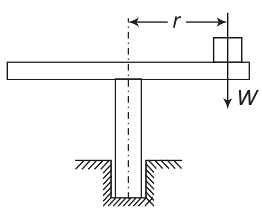


Figure 9

Answer This problem is identical to Example 12.11.

10. ***(a) **For the rectangle shown in Fig. 10, compute I_u, I_v and I_{uv} with respect to *u*-*v* axes inclined to *x*-*y* axes by 30°. Determine the principal axes and second moment of area about the principal axes.**

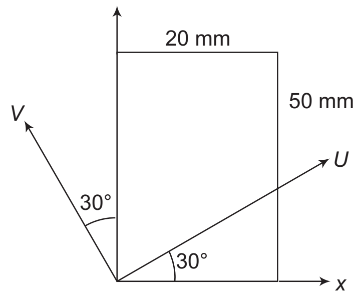


Figure 10

Solution We know that

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{uv} = \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta$$

For the given figure, we have

$$I_x = \frac{bh^3}{3} = \frac{20 \times 50^3}{3} = 8.33 \times 10^5 \text{ mm}^4$$

$$I_y = \frac{20^3 \times 50}{3} = 1.33 \times 10^5 \text{ mm}^4$$

$$I_{xy} = \frac{b^2h^2}{4} = \frac{20^2 \times 50^2}{4} = 2.5 \times 10^5 \text{mm}^4 \text{ and } \theta = 30°$$

Therefore, we get

$$I_u = 4.83 \times 10^5 + 3.5 \times 10^5 \cos 60 - 2.5 \times 10^5 \sin 60 = 441493.65 \text{ mm}^4$$

$$I_v = 4.83 \times 10^5 - 3.5 \times 10^5 \cos 60 + 2.5 \times 10^5 \sin 60 = 524506.35 \text{ mm}^4$$

$$I_{uv} = \frac{1}{2}(8.33 \times 10^5 - 1.33 \times 10^5)\sin 60 + 2.5 \times 10^5 \cos 60 = 428108.89 \text{ mm}^4$$

Principal moments of inertia are found to be

$$I_{\max} = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}$$

$$= \frac{8.33 \times 10^5 + 1.33 \times 10^5}{2} + \sqrt{\left(\frac{8.33 \times 10^5 - 1.33 \times 10^5}{2}\right)^2 + (2.5 \times 10^5)^2} = 913116.26 \text{ mm}^4$$

$$I_{\min} = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}$$

$$= \frac{8.33\times10^5 + 1.33\times10^5}{2} - \sqrt{\left(\frac{8.33\times10^5 - 1.33\times10^5}{2}\right)^2 + (2.5\times10^5)^2} = 52883.74 \text{ mm}^4$$

Principal axes are obtained as follows:

$$\tan 2\phi = \frac{-2I_{xy}}{I_y - I_x} = \frac{-2\times2.5\times10^5}{1.33\times10^5 - 8.33\times10^5}$$

or $2\phi = 35.54°$

or $\phi = 17.77°$ or $162.23°$

***(b) **Three forces F_1, F_2 and F_3 act on the box as shown in Fig. 11. The magnitudes of the given forces are 19 N, 23 N and 46 N respectively. Determine the resultant of 2.5×10^5 the forces and its magnitude.**

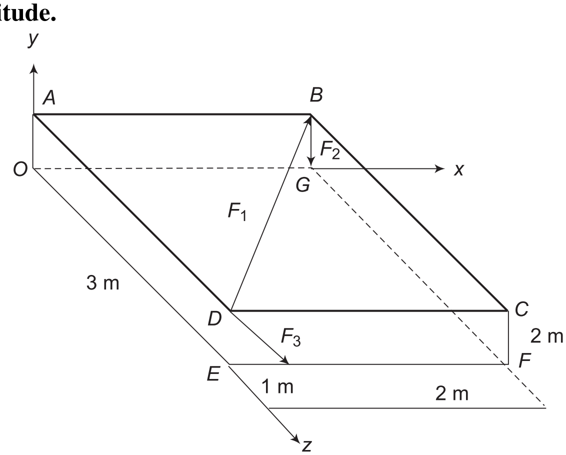


Figure 11

Solution

$$\mathbf{F_1} = 19\frac{(1-0)i + (0-2)j + (3-3)k}{\sqrt{(1-0)^2 + (0-2)^2 + (3-3)^2}} = 19\times\frac{i-2j}{\sqrt{5}} = 8.5(i-2j)$$

$$\mathbf{F_2} = 23\frac{(3-0)i + (2-0)j + (0-3)k}{\sqrt{(3-0)^2 + (2-0)^2 + (0-3)^2}} = 23\times\frac{3i+2j-3k}{\sqrt{22}} = 4.9(3i+2j-3k)$$

$$\mathbf{F_3} = 46\frac{(3-3)i + (0-2)j + (0-0)k}{\sqrt{(3-3)^2 + (0-2)^2 + (0-0)^2}} = 46(-j) = -46j$$

Therefore, $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} = 8.5(i-2j) + 4.9(3i+2j-3k) - 46j = 23.2i - 53.2j - 14.7k$

11. **(a) **A solid right circular cone of altitude h = 12 cm and radius r = 3 cm has its CG C on its geometric axis at a distance h/4 above the base. This cone rests on the inclined plane AB which makes an angle of 30° with the horizontal and for which the angle of friction is 0.5. A horizontal force P is applied to the vertex O of the cone and acts in the vertical plane of the figure. Find the maximum and minimum values of P consistent with equilibrium of the cone of weight W = 10 kgf.**

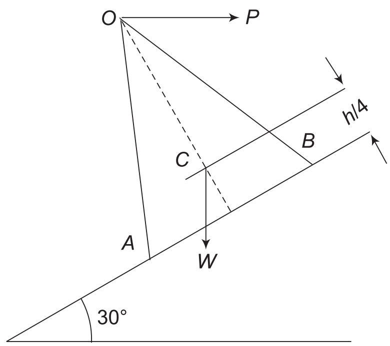


Figure 12

Solution The free-body diagram of the cone is shown in Fig. 13.

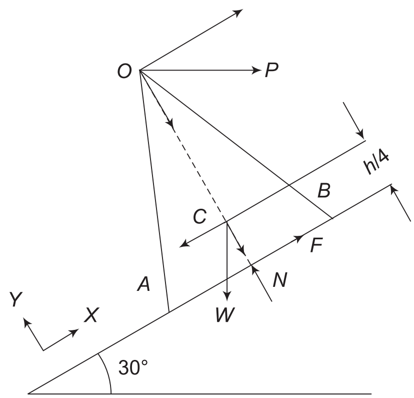


Figure 13

Two different situations may arise regarding the motion of the cone: (i) the cone may slide down along the inclined plane, and (ii) the cone may topple against the point B of the base.

Under the limiting condition, the force P required to maintain equilibrium in case (i) would be less than that required in case (ii).

Case (i) the cone may slide down along the inclined plane.

Considering the equilibrium of the cone, we have

$$\sum X = 0$$

or $\quad P \cos 30 + \mu N = W \sin 30$

or $$P \times \frac{\sqrt{3}}{2} + 0.5 \times N = 10 \times \frac{1}{2}$$

$$\sum Y = 0$$

or $\quad P \sin 30 + W \cos 30 = N$

or $$P \times \frac{1}{2} + 10 \times \frac{\sqrt{3}}{2} = N$$

Solving the above two equations, we get

$$P = P_{min} = 0.6 \text{ kgf}$$

Case (ii) The cone may topple against the point B of the base of the cone.

$$\sum M_B = 0$$

or $$W\sin 30 \times \frac{h}{4} + (W\cos 30 + P\sin 30)\times r = P\cos 30 \times h$$

or $$P = \frac{W\left[\frac{h}{4}\sin 30 + r\cos 30\right]}{[h\cos 30 - r\sin 30]} = 10\frac{[3\sin 30 + 3\cos 30]}{[12\cos 30 - 3\sin 30]} = 4.608$$

or $$P = P_{\max} = 4.608 \text{ kgf}$$

****(b) A block *A* weighing 1000 N rests on a rough inclined plane whose inclination to the horizontal is 45°. The block is connected to another block *B* weighing 3000 N resting on a rough horizontal plane, by a weightless rigid bar inclined at an angle of 30° to the horizontal as shown in Fig. 14. Find the horizontal force that has to be applied on the block *B* to just move the block *A* up the slope. Assume that the coefficient of friction for all contact surfaces is 0.26.**

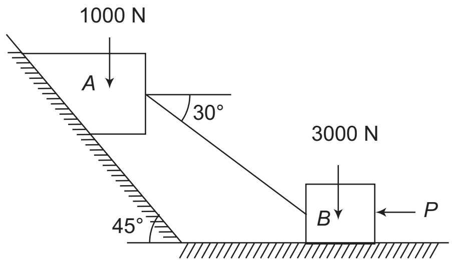


Figure 14

Solution Let the axial force in the bar that connects the two blocks be T. The free-body diagrams of the blocks A and B are shown in Fig. 15. Note that R_A is the resultant of N_A and F_A.

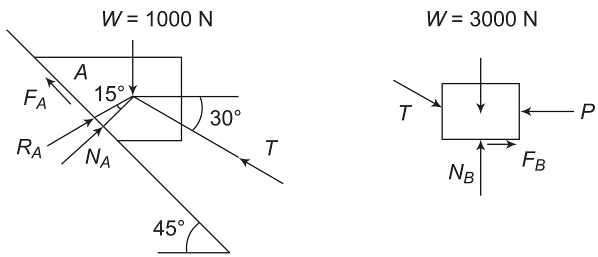


Figure 15

Considering the equilibrium of the block A and using Lami's theorem,

$$\frac{T}{\sin(180-45-15)} = \frac{1000}{\sin 120}$$

or $T = 1000$ N

Considering the equilibrium of the block B,

$$\sum Y = 0$$

or $T \sin 30 + 3000 = N$

or $N = 3500$ N

$$\sum X = 0$$

or $$P = T\cos 30 + \mu N = 1000\cos 30 + 0.26 \times 3500 = 1776.02\text{N}$$

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