Engineering Mathematics
for Semesters III and IV

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Preface

Engineering Mathematics for Semesters III and IV deals with the applications of applied Mathematics in the field of Engineering. This subject is generally taught in the 3rd and 4th semester of engineering. In the *Engineering Mathematics for Semesters I and II* we learnt about the basics of engineering mathematics as a branch of applied mathematics concerning mathematical models (mathematical methods and techniques) that are typically used in engineering and industry. This book on semesters III and IV will prepare students for their domain-specific study and applications in their respective branches.

This book will also introduce the students to the concepts of Fourier transform, *Z*-transform, complex variables, probability and numerical techniques.

Salient Features

- Engrossing problem sets based on **real life situations** like Modulation techniques and Heat Flow
- 360° coverage of subject matter: **Introduction–History–Pedagogy–Applications**
- Introduction to Fourier Transform, *Z*-transform, Complex Variable, Probability and Numerical Techniques with reference to **applications in the field of engineering**
- **582 Solved problems** with stepwise solutions
- **535 MCQs** for various competitive examinations
- Appendix includes Statistical Tables and List of Formulae
- Other pedagogical aids include:
    - Drill and Practice Problem: **1100**

Chapter Organization

The book is divided in fifteen chapters. In **Chapter 1**, we have discussed Fourier transform which includes Fourier transform of some basic functions and the properties of the Fourier Transform. **Chapter 2** deals with Z-Transform, inverse Z-transform, Cauchy's residue theorem, convolution theorem and properties of Z-transform. In **Chapter 3**, basic concepts of complex theory including basic concepts of complex numbers, Cauchy–Riemann equations, conjugate and conjugate harmonic equations, complex integrals, expansion of analytic functions as power series, zeros of analytic functions, calculus of residues, singularities, complex integrals, Cauchy's residue theorem, etc., are discussed. **Chapter 4** covers empirical laws and curve fitting along with scatter diagram and various methods of curve fitting. In **Chapter 5,** we present various statistical methods while in

Chapter 6 basic concepts of probability such as additive law, multiplicative and conditional probability, Baye's theorem, probability distribution (discrete and continuous in general) and some specific distributions such as binomial, Poisson, uniform, exponential and normal are discussed. **Chapter 7** deals with sampling, inference and testing of hypotheses which includes parameters and statistics, type I and II errors, confidence intervals and F, chi-square and Z statistic. **Chapter 8** deals with finite difference and interpolation while **Chapters 9** and **10** deal with numerical solution of differential equations and various formulas for numerical differentiation and integration, respectively. **Chapters 11 and 12** talk about numerical solutions of ordinary differential equations and partial differential equations, respectively. In **Chapter 13,** linear programming and various methods to solve linear programming including transportation and assignment problems, duality and dual simplex method, etc., have been discussed. **Chapters 14 and 15** cover the method of variational with fixed boundaries while **Chapter 15** deals with integral equations.

Online Learning Center

The Online Learning Center can be accessed at *http://www.mhhe.com/gupta/em3/4* and contains the *Instructor Elements*: Solutions Manual.

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1 Fourier Transforms

1.1 INTRODUCTION

Fourier transform is a mathematical tool which plays an important role in the field of science, engineering and medical science. The Fourier transform and inverse Fourier transform are defined in continuous and discrete domain. The discrete Fourier transform are very useful to solve the problem of image processing, image encoding, image enhancement and image restoration. It is also used to decompose time series signals into frequency components each having an amplitude and phase and using the inverse Fourier transform the time-series signal can be reconstructed from its frequency domain. It is one of the most important concept in digital signal processing. Besides, this Fourier transform are very useful to solve the integral equations, ordinary differential equations and partial differential equations.

Jean Baptiste Joseph Fourier was a French mathematician born on 21 March 1768 in Auxerre, Bourgogne, France. His father who was a tailor, who died when Fourier was 8 years old. In order to give the boyhim a proper education, his aunt and uncle put him in Ecole Royale Militaire where he proved to be a conscientious student showing high intellect particularly in mathematics. The teachers saw a his bright future in the field for him; however, Fourier seemed to have different plans. He then joined the Church for which he went to St. Benôit-sur-Loire to take his vows. Meanwhile, he taught mathematics to his fellow learners. Fourier was elected to the Acadêmie des Sciences in 1817. In 1822 Delambre, who was the Secretary to the mathematical section of the Academié des Sciences. Later in 1822, he succeeded Jean Baptiste Joseph Delambre as Permanent Secretary of the French Academy of Sciences. He died on 16 May 1830 in Paris, France.

1.1.1 Integral Transform

Integral transform theory is a mathematical tool which can be used to solve various initial and boundary value problem in engineering like, conduction of heat, transverse oscillation of an elastic beam, etc. The integral transform of function $f(x)$ is denoted by $f(s)$ and it is defined as

$$I\{F(x)\} = f(s) = \int_a^b K(s,x)\, f(x)\, dx \quad \text{or} \quad f(s) = \int_{-\infty}^{\infty} K(s,x)\, f(x)\, dx$$

where $K(s, x)$ is called kernel of the transform and s is a parameter independent of x.

The interpretation of $f(s)$ depends on the kernel $K(s, x)$

When $K(s, x) = 0$, then $f(s) = 0$

It means meaning less.

Now if $K(s, x) = x^s$ will give us the s^{th} moment of $f(x)$ whenever $f(x)$ is probability density function and if $s = 1$, this is just mean of distribution $f(x)$. The some of the well known transform are according to the kernel $K(s, x)$.

(i) If $K(s, x) = e^{-sx}$, then $L\{f(x)\} = f(s) = \int_0^\infty f(x)e^{-sx}dx$ called Laplace transform.

(ii) If $K(s, x) = x\,J_n(sx)$ (Bessel function), then

$$H\{f(x)\} = f(s) = \int_0^\infty f(x)\cdot x\,J_n(sx)\,dx \text{ is called Hankel transform.}$$

(iii) If $K(s, x) = x^{s-1}$, then

$$M\{f(x)\} = f(s) = \int_0^\infty f(x)x^{s-1}ds \text{ is called Mellin transform.}$$

(iv) If $K(s, x) = e^{-isx}$

$$F\{f(x)\} = \hat{f}(s) = \int_{-\infty}^{\infty} f(x)\,e^{-isx}dx \text{ is called Fourier transform, it is also defined as}$$

$$F\{f(x)\} = \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}dx \text{ when parameter } s \text{ replace by } \lambda.$$

***Note*:** We know that $e^{-i\lambda x} = \cos\lambda x - i\sin\lambda x$, if x is time, then $f(x)$ can appears as a waveform in time and $|\hat{f}(\lambda)|$ represents the strength of the frequency in the original signals.

1.2 FOURIER INTEGRAL THEOREM

If a function $f(x)$ is piecewise and periodic on $[-l, l]$ or $[0, l]$, then $f(x)$ can be represented by a Fourier series. However, we may be able to represent the function $f(x)$ is an integral form.

Consider $f(x)$ have the following properties:

(i) Function $f(x)$ is a piecewise continuous on $[-l, l]$.

(ii) Function $f(x)$ has left and right hand derivatives, at every x on the real axis.

(iii) Function $f(x)$ is absolutely integrable on the real axis, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|dx \text{ converges}$$

Then in the interval $[-l, l]$, the Fourier series representation of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right] \tag{1}$$

where $$a_0 = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$$

and $$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$$

Putting the values of a_0, a_n and b_n in Eq. (1), we get

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cdot\left[\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l} + \sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\right]dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\left[\cos\frac{n\pi(t-x)}{l}\right]dt \tag{2}$$

Now, consider the limit $l \to \infty$, then

$$\left|\frac{1}{2l}\int_{-l}^{l} f(t)\,dt\right| \le \frac{1}{2l}\int_{-\infty}^{\infty} |f(t)|\,dt$$

As $l \to \infty$, the first term of Eq. (2), tends to zero.

Putting $\frac{\pi}{c} = \Delta\lambda,$ then the second term of Eq. (2) becomes

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\Delta\lambda\int_{-l}^{l} f(t)\cos[n\Delta\lambda(t-x)]\,dt$$

This is of the form $\sum_{n=1}^{\infty} f(n\Delta\lambda)\,\Delta\lambda,$ whose limit as $\Delta\lambda \to 0$ is $\int_{0}^{\infty} f(\lambda)\,d\lambda$

$\therefore$ As $l \to \infty$, Eq. (2) becomes

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\,\cos\lambda(t-x)\,dt\;d\lambda \tag{3}$$

The integral on the right of Eq. (3) is known as the Fourier integral representation for $f(x)$.

At a point of discontinuity the value of the integral on the right is $\frac{1}{2}[f(x+0)+f(x-0)]$.

Expand cos $\lambda(t-x)$ and separating the integral in two parts in Eq. (3), can be written as:

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\cos\lambda x\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\,d\lambda + \frac{1}{\pi}\int_{0}^{\infty}\sin\lambda x\int_{-\infty}^{\infty} f(t)\sin\lambda t\,dt\,d\lambda \tag{4}$$

If $f(t)$ is an even function, $f(t)$ cosλt is also even, while $f(t)$ sinλt is odd function.

Therefore, the second term in Eq. (4) vanishes and we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\, dt\, d\lambda \tag{5}$$

Similarly, if $f(t)$ is an odd function, $f(t)$ cos λt is odd, while $f(t)$ sin λt is even.

$\therefore$ The first term in Eq. (4) vanishes and we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt\, d\lambda \tag{6}$$

The integrals in Eqs (5) and (6) are respectively known as *Fourier cosine* and *sine integrals*.

In case of half range series, a function $f(x)$ defined over the interval $(0, \infty)$, it may be expressed either as a Fourier sine integral or as a cosine integral. Equation (4), may also be expressed in the form

$$f(x) = \int_0^\infty [a(\lambda)\cos\lambda x + b(\lambda)\sin\lambda x]\, d\lambda \tag{7}$$

where $a(\lambda) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt$ and

$$b(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt$$

1.3 FOURIER INTEGRAL IN COMPLEX FORM

Since, the function $\cos(\lambda(t-x))$ is an even of λ, therefore Eq. (3) can be written in the form of

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\, dt\, d\lambda \tag{8}$$

and $\sin(\lambda(t-x))$ is an odd function of λ, then

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\lambda(x-t)\, dt\, d\lambda \tag{9}$$

Multiplying Eq. (9) by i and adding to Eq. (8), we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t) + i\sin\lambda(x-t)\right] dt\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(x-t)} dt\, d\lambda \qquad [\because\ e^{i\theta} = \cos\theta + i\sin\theta]$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\, dt\, d\lambda \tag{10}$$

Example 1 Express the function

$$f(x)=\begin{cases}1 & \text{if } |x|\le 1\\ 0 & \text{if } |x|>1\end{cases}$$ as a Fourier integral hence evaluate $\displaystyle\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$

[Andhra 2000, Kerala 1990, U.T.U. 2007]

Solution The Fourier integral for the function $f(x)$ is given by

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda \tag{11}$$

$$=\frac{1}{\pi}\int_0^\infty\left[\int_{-1}^1 1\cdot\cos\lambda(t-x)\,dt\right]d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{-1}^1 d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\frac{\sin\lambda(1-x)+\sin\lambda(1+x)}{\lambda}\,d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\frac{1}{\lambda}\left[\frac{2\sin\lambda(1-x+1+x)}{2}\cdot\cos\frac{\lambda(1-x-1-x)}{2}\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\sin\lambda\cdot\cos\lambda x}{\lambda}\,d\lambda \tag{12}$$

Equation (12) represent the required Fourier integral.

Therefore, $$\int_0^\infty\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda=\frac{\pi}{2}\cdot f(x)$$

$$=\begin{cases}\dfrac{\pi}{2} & \text{if } |x|<1\\ 0 & \text{if } |x|>1\end{cases}$$

For $|x| = 1$, which is a point of discontinuity of $f(x)$, the integral has the value

$$\frac{\pi}{2}\frac{(1+0)}{2}=\frac{\pi}{4}$$

Example 2 Express as a Fourier integral representation of the function $f(x)$, where

$$f(x)=\begin{cases}0; & x<0\\ 1; & 0\le x\le 1\\ 0; & x>1\end{cases}$$

Hence, show that $\displaystyle\int_0^\infty\frac{\sin\frac{x}{2}}{x}\,dx=\frac{\pi}{2}.$

Solution The Fourier integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda \tag{13}$$

$$\therefore \quad f(x) = \frac{1}{\pi}\int_0^\infty \int_0^1 1\cdot\cos\lambda(t-x)\,dt\,d\lambda$$

or
$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda(t-x)}{\lambda}\right]_0^1 d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda(1-x)+\sin\lambda x}{\lambda}\right] d\lambda$$

or
$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{1}{\lambda}\left[2\sin\frac{\lambda(1-x+x)}{2}\cdot\cos\frac{\lambda(1-x-x)}{2}\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\frac{\lambda}{2}\cdot\cos\frac{\lambda(1-2x)}{2}}{\lambda}\,d\lambda \tag{14}$$

Equation (14) is the Fourier integral representation of the given function.

Let $\quad x = \frac{1}{2}$, then $f\left(\frac{1}{2}\right) = 1$

Hence,
$$I = \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda}\sin\frac{\lambda}{2}\cdot\cos\frac{\lambda}{2}\left(1-2\cdot\frac{1}{2}\right) d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda}\sin\frac{\lambda}{2}\cdot\cos 0\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda}\sin\frac{\lambda}{2}\,d\lambda$$

or
$$\int_0^\infty \frac{1}{\lambda}\sin\frac{\lambda}{2}\,d\lambda = \frac{\pi}{2}$$

or
$$\int_0^\infty \frac{1}{x}\sin\frac{x}{2}\,dx = \frac{\pi}{2}$$

Hence, proved.

Example 3 Find the Fourier sine integral for $f(x) = e^{-kx}$, $x \ge 0$, where k is a positive constant.

Hence show that $\int_0^\infty \frac{\lambda\sin\lambda x}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-kx}$. **[Gulbarga 1996, Andhra 2000, G.E.U. 2009]**

Solution The Fourier sine integral for the function $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt\, \lambda t \tag{15}$$

Putting $f(x) = e^{-kx}$ in Eq. (15), we get

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty e^{-kt}\sin\lambda t\, dt\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \cdot \left[\frac{e^{-kt}}{k^2+\lambda^2}(-k\sin\lambda t - \lambda\cos\lambda t)\right]_0^\infty d\lambda$$

$$\left[\because \int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[0 + \frac{\lambda}{k^2+\lambda^2}\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{k^2+\lambda^2}\, d\lambda$$

or $\displaystyle\int_0^\infty \frac{\lambda\sin\lambda x}{k^2+\lambda^2}\, d\lambda = \frac{\pi}{2}e^{-kx}$ **Hence, proved.**

Example 4 Find the Fourier cosine integral representation of $f(x) = \begin{cases} \cos x; & 0 < x < \dfrac{\pi}{2} \\ 0; & x > \dfrac{\pi}{2} \end{cases}.$

Solution The Fourier cosine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\, dt\, d\lambda \tag{16}$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^{\pi/2}\cos t\,\cos\lambda t\, dt + \int_{\pi/2}^\infty 0\cos\lambda t\, dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^{\pi/2}\cos t\cos\lambda t\, dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left(\frac{\cos\lambda\frac{\pi}{2}}{1-\lambda^2}\right)d\lambda$$

or $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda\frac{\pi}{2}}{1-\lambda^2}\cos\lambda x\, d\lambda$$

Example 5 Obtain complex Fourier integral representation for the function

$$f(x) = \begin{cases} |x|, & \text{if } -\pi < x < \pi \\ 0, & \text{otherwise} \end{cases}$$

Solution The Fourier integral in complex form for the function $f(x)$ is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} d\lambda \int_{-\infty}^{\infty} f(t)e^{-i\lambda t} dt \tag{17}$$

$$\therefore \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} d\lambda \left[\int_{-\pi}^{\pi} |t| e^{-i\lambda t} dt\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} d\lambda \left[\int_{-\pi}^{0} (-t) e^{-i\lambda t} dt + \int_0^{\pi} t\, e^{-i\lambda t} dt\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} d\lambda \left[\left(\frac{-te^{-i\lambda t}}{-\lambda i} + \frac{e^{-\lambda it}}{(-\lambda i)^2}\right)_{-\pi}^{0} + \left(+\frac{te^{-i\lambda t}}{-\lambda i} - \frac{e^{-i\lambda t}}{(-\lambda i)^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} \left[\left(\frac{1}{\lambda^2} - \frac{\pi i}{\lambda}\right)e^{\lambda\pi i} - \frac{1}{\lambda^2} - \frac{1}{\lambda^2} + \left(\frac{\pi i}{\lambda} - \frac{1}{\lambda^2}\right)e^{-\lambda\pi i}\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} \left[\left(\frac{\pi i}{\lambda} + \frac{1}{\lambda^2}\right)e^{-\lambda\pi i} - \left(\frac{\pi i}{\lambda} - \frac{1}{\lambda^2}\right)e^{\lambda\pi i} - \frac{2}{\lambda^2}\right] d\lambda$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\lambda^2}[\lambda\pi\sin\lambda\pi + \cos\lambda\pi - 1]e^{i\lambda x}\, d\lambda$$

1.4 FOURIER TRANSFORM

Let the function $f(x)$ be defined on $(-\infty, \infty)$ and it is piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$, then the equation (10) of Section (1.3) as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} d\lambda \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\, dt$$

or $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\, dt\, d\lambda$$

if $$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x} dx \; (-\infty < \lambda < \infty) \tag{18}$$

then $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda \; (-\infty < \lambda < \infty) \tag{19}$$

The function $\hat{f}(\lambda)$ in Eq. (18) is called the Fourier transform of $f(x)$ or some times known the exponential Fourier transform and integral in Eq. (19) recovers $f(x)$ from $\hat{f}(\lambda)$ is called inverse Fourier transform. Another notation that is often useful involves representing the Fourier transform of function $f(x)$ by $F\{f(x)\}$ so that $F\{f(x)\} = \hat{f}(\lambda)$ and when this notation is used the inverse Fourier transform is written as $F^{-1}\{\hat{f}(\lambda)\} = f(x)$.

Where F is called the Fourier transform operator, while F^{-1} is called inverse Fourier transform operator.

***Note*:** Some authors also define the Fourier transform in the following manner:

(i) $$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{i\lambda x}\, dx$$

and $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \hat{f}(\lambda)\, d\lambda$$

(ii) $$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx$$

and $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \hat{f}(\lambda)\, d\lambda$$

1.5 FOURIER COSINE AND SINE TRANSFORMS

The Fourier cosine and sine transforms arise as special cases of the Fourier transform, according as $f(x)$ is even or odd. Now, we considering the Fourier cosine transform of function $f(x)$ that can be defined when $f(x)$ is an even function. The Fourier transform defined in (18) in Section (1.4) can be written as

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)[\cos \lambda x - i \sin \lambda x]\, dx \tag{20}$$

but if $f(x)$ is an even function, the product $f(x) \cos \lambda x$ is also an even function, so its integral over $(-\infty, \infty)$ does not vanish, while the product $f(x) \sin \lambda x$ is an odd function, so its integral over $(-\infty, \infty)$ vanishes, thus Eq. (20) becomes

$$\hat{f}_C(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \lambda x \, dx$$

If we use the result $f(-x) = f(x)$, then change the interval of integration to $(0, \infty)$ this last integral becomes

$$F_C\{f(x)\} = \hat{f}_C(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \lambda x \, dx \tag{21}$$

The integral of Eq. (21) is called the Fourier cosine transform of $f(x)$.

Now, the inverse Fourier cosine transform is denoted by $f(x) = F_C^{-1}\{\hat{f}_C(\lambda)\}$ and is defined as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_C(\lambda) \cos \lambda x \, d\lambda \tag{22}$$

In a similar manner if $f(x)$ is an odd function, then Eq. (20) gives

$$\hat{f}_S(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \sin \lambda x \, dx$$

If we use the result $f(-x) = -f(x)$, then change the interval of integration to $(0, \infty)$, then we have

$$F_S\{f(x)\} = \hat{f}_S(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx \tag{23}$$

The integral of Eq. (23) is called the Fourier sine transform of $f(x)$.
Now, the inverse Fourier sine transform is denoted and defined as

$$f(x) = F_S^{-1}\{\hat{f}_S(\lambda)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_S(\lambda) \sin \lambda x \, d\lambda \tag{24}$$

The Fourier cosine transform of $f(x)$ in Eq. (21) only involves $f(x)$ for $x \geq 0$, though it was derived from the Fourier transform on the assumption that $f(x)$ was an even function defined for all x. Similarly, the Fourier sine transform of $f(x)$ in Eq. (23) only involves $f(x)$ for $x \geq 0$, though it was derived on the assumption that the function $f(x)$ was an old function.

Because Eqs (22) and (23) have been derived from Eq. (18) in Section (1.4), it follows that if $f(x)$ is discontinuous, the expression on the left must be replaced by $\frac{1}{2}[f(x+0) + f(x-0)]$, because the Fourier sine and cosine transforms have the same convergence properties as the Fourier transform.

Following examples will illustrate the methods that are explained above:

Example 6 Find the Fourier transform of

$$f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$$

[Bangalore 1994, J.N.T.U. 2000, RTU 2001, U.T.U. 2006]

Solution The Fourier transform of $f(x)$ is given

$$F\{f(x)\} = \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 \cdot e^{i\lambda x} dx + \int_{-a}^{a} 1 \cdot e^{-i\lambda x} dx + \int_a^\infty 0 \cdot e^{-i\lambda x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-i\lambda x}}{-i\lambda}\right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}}\left[\frac{-e^{-i\lambda a}+e^{-i\lambda a}}{\lambda i}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i\lambda a}-e^{-i\lambda a}}{\lambda i}\right]$$

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\cdot\frac{2\sin\lambda a}{\lambda};\ \lambda \neq 0$$

$$= \frac{2a}{\sqrt{2\pi}};\quad \text{if } \lambda = 0$$

Example 7 Find the Fourier sine and cosine transform of e^{-x}.

Solution Here $f(x) = e^{-x}$

The Fourier cosine transform of $f(x)$ is

$$\hat{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos\lambda x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+\lambda^2}(-\cos\lambda x+\lambda\sin\lambda x)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\lambda^2}\right)$$

Similarly, the Fourier sine transform is

$$\hat{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\lambda x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\lambda x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+\lambda^2}(-\sin\lambda x-\lambda\cos\lambda x\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\cdot\left(\frac{\lambda}{1+\lambda^2}\right)$$

Example 8 Find Fourier sine transform of $\frac{e^{-ax}}{x}$. **[V.T.U. 2003]**

Solution The Fourier sine transform of $f(x)$ is

$$\hat{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin \lambda x \, dx$$

Let $$I = \int_0^\infty \frac{e^{-ax}}{x} \sin \lambda x \, dx$$

$$\frac{dI}{d\lambda} = \int_0^\infty \frac{e^{-ax}}{x} \cdot x \cos \lambda x \, dx = \int_0^\infty e^{-ax} \cos \lambda x \, dx$$

$$= \left[\frac{e^{-ax}}{a^2 + \lambda^2} (-a \cos \lambda x + \lambda \sin \lambda x) \right]_0^\infty$$

$$\frac{dI}{d\lambda} = \frac{a}{a^2 + \lambda^2}$$

Integrating with respect to λ, we get

$$I = a \int \frac{d\lambda}{a^2 + \lambda^2} + C$$

$$I = \tan^{-1} \lambda/a + C$$

Hence, $$\hat{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} (I) = \sqrt{\frac{2}{\pi}} \left[\tan^{-1} \frac{\lambda}{a} + C \right]$$

Example 9 Find the cosine transform of e^{-x^2}. **[Calicut 1994]**

Solution Here $f(x) = e^{-x^2}$

The Fourier cosine transform of $f(x)$ is

$$\hat{f}_C(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos \lambda x \, dx$$

Let $$I = \int_0^\infty e^{-x^2} \cos \lambda x \, dx \tag{25}$$

then $$\hat{f}_C(\lambda) = \sqrt{\frac{2}{\pi}} I \tag{26}$$

Differentiating Eq. (25) w.r.t λ, we get

$$\frac{dI}{d\lambda} = -\int_0^\infty \frac{x e^{-x^2} \sin \lambda x \, dx}{2}$$

$$= \left[\frac{e^{-x^2} \sin \lambda x}{2}\right]_0^\infty - \frac{\lambda}{2}\int_0^\infty e^{-x^2} \cos \lambda x \, dx$$

$$\frac{dI}{d\lambda} = 0 - \frac{\lambda}{2} I$$

or $$\frac{dI}{I} = -\frac{\lambda}{2} d\lambda \qquad (27)$$

Integrating Eq. (27), we obtain

$$\log I = -\frac{\lambda^2}{4} + \log A$$

or $$I = A\, e^{-\lambda^2/4}$$

Now, if $\lambda = 0$, $I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$\therefore$ $$\frac{\sqrt{\pi}}{2} = A \text{ so } I = \frac{\sqrt{\pi}}{2} e^{-\lambda^2/4}$$

Hence, Eq. (26) becomes

$$\hat{f}_C(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} e^{-\lambda^2/4} = \frac{1}{\sqrt{2}} e^{-\lambda^2/4}$$

Example 10 Find the Fourier cosine transform of $\dfrac{1}{1+x^2}$.

[Bhopal 1998, V.T.U. 2003, U.P.T.U. 2005]

Solution Let $f(x) = \dfrac{1}{1+x^2}$.

The Fourier cosine transform of $f(x)$ is

$$\hat{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos \lambda x \, dx$$

Let $$I = \int_0^\infty \frac{\cos \lambda x}{1+x^2} \qquad (28)$$

then $$\hat{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \cdot I \qquad (29)$$

Differentiating Eq. (28) w.r.t. to λ, we get

$$\frac{dI}{d\lambda} = -\int_0^\infty \frac{x \sin \lambda x}{1+x^2} dx$$

$$= -\int_0^\infty \frac{x^2 \sin \lambda x}{x(1+x^2)} dx$$

$$= -\int_0^\infty \frac{(1+x^2-1)\sin \lambda x}{x(1+x^2)} dx$$

$$= -\int_0^\infty \frac{\sin \lambda x}{x} dx + \int_0^\infty \frac{\sin \lambda x}{x(1+x^2)} dx$$

$$\frac{dI}{d\lambda} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin \lambda x}{x(1+x^2)} dx$$

Again differentiating w.r. to λ, we have

$$\frac{d^2 I}{d\lambda^2} = \int_0^\infty \frac{x \cos \lambda x}{x(1+x^2)} dx = \int_0^\infty \frac{\cos \lambda x}{1+x^2} dx$$

or $$\frac{d^2 I}{d\lambda^2} = I \qquad \text{(using Eq. (28))}$$

or $$\frac{d^2 I}{d\lambda^2} - I = 0 \tag{30}$$

The solution of Eq. (30) is

$$I = Ae^{\lambda} + Be^{-\lambda}$$

But, when $\lambda = 0$ so $I = \int_0^\infty \frac{dx}{1+x^2}$

$$= (\tan^{-1} x)_0^\infty = \frac{\pi}{2}$$

and $$\frac{dI}{d\lambda} = -\frac{\pi}{2}$$

Therefore $$A + B = \frac{\pi}{2} \tag{31}$$

and $$A + B = -\frac{\pi}{2} \tag{32}$$

Solving Eqs (31) and (32), we get

$$A = 0 \text{ and } B = \frac{\pi}{2}$$

$\therefore \qquad I = \frac{\pi}{2} e^{-\lambda}$

Hence, Eq. (29) becomes

$$\hat{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} e^{-\lambda} = \sqrt{\frac{\pi}{2}} e^{-\lambda}$$

Example 11 Find the Fourier sine transform of $\frac{x}{1+x^2}$.

Solution Do same as above example. Here we conclude the cosine transform of $\frac{1}{1+x^2}$ and sine transform of $\frac{x}{1+x^2}$ are same i.e.,

$$F_S\left\{\frac{x}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\, e^{-\lambda} = F_c\left\{\frac{1}{1+x^2}\right\}.$$

Example 12 Use the sine inversion formula to find $f(x)$ if $\hat{f}_S(\lambda) = \frac{\lambda}{1+\lambda^2}$.

Solution Using inverse Fourier sine transform, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\lambda}{1+\lambda^2} \sin \lambda x \, d\lambda$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(\lambda^2+1-1)}{\lambda(1+\lambda^2)} \sin \lambda x \, d\lambda$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \lambda x}{\lambda} d\lambda - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \lambda x}{\lambda(1+\lambda^2)} d\lambda$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \lambda x}{\lambda(1+\lambda^2)} d\lambda$$

$$f(x) = \sqrt{\frac{\pi}{2}} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \lambda x}{\lambda(1+\lambda^2)} d\lambda \tag{33}$$

Differentiating Eq. (33) w.r.t. x, we get

$$\frac{df}{dx} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\lambda \cos \lambda x}{\lambda(1+\lambda^2)} d\lambda = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda \tag{34}$$

Again differentiating Eq. (34) w.r.t. x, we get

$$\frac{d^2 f}{dx^2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2} d\lambda$$

or $$\frac{d^2 f}{dx^2} = f$$

or $$\frac{d^2 f}{dx^2} - f = 0 \tag{35}$$

Solution of Eq. (35) is

$$f(x) = Ae^{x} + Be^{-x} \tag{36}$$

$$\therefore \quad \frac{df}{dx} = Ae^{x} - Be^{-x} \tag{37}$$

Now when $x = 0,\ f = \sqrt{\frac{\pi}{2}}$ (using Eq. (33))

and $$\frac{df}{dx} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d\lambda}{1+\lambda^2} = -\sqrt{\frac{\pi}{2}}$$

From Eq. (36), $\sqrt{\frac{\pi}{2}} = A + B$ and from Eq. (37)

$$-\sqrt{\frac{\pi}{2}} = A - B$$

Solving both equations, we get $A = 0$ and $B = \sqrt{\frac{\pi}{2}}$

Hence, Eq. (36) becomes

$$f(x) = \sqrt{\frac{\pi}{2}}\, e^{-x}$$

Example 13 Find the inverse Fourier transform of $\hat{f}(\lambda) = e^{-|\lambda| y}$.

Solution We know that

$$|\lambda| = \begin{cases} -\lambda; & \lambda < 0 \\ \lambda; & \lambda > 0 \end{cases}$$

The inverse Fourier transform of $\hat{f}(\lambda)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\lambda| y} e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{\lambda y} e^{i\lambda x}\, d\lambda + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda y} e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\lambda(y+ix)} d\lambda + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda(y-ix)} d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{\lambda(y+ix)}}{y+ix} \right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-\lambda(y-ix)}}{-(y-ix)} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{y+ix} + \frac{1}{y-ix} \right]$$

$$= \frac{2y}{\sqrt{2\pi}\,(x^2+y^2)}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2}$$

Example 14 Find $f(x)$ if its cosine transform is

$$\hat{f}_c(\lambda) = \begin{cases} \frac{1}{\sqrt{2\pi}} \left(a - \lambda/2\right) & \text{if } \lambda < 2a \\ 0 & \text{if } \lambda \geq 2a \end{cases}.$$

Solution Using inverse Fourier cosine transform, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_c(\lambda) \cos \lambda x \, d\lambda$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{2a} \left[\frac{1}{\sqrt{2\pi}} \left(a - \frac{\lambda}{2} \right) \right] \cos \lambda x \, d\lambda + \sqrt{\frac{2}{\pi}} \int_{2a}^{\infty} 0 \cdot \cos \lambda x \, d\lambda$$

$$= \left[\frac{1}{\pi x} \left(a - \frac{\lambda}{2} \right) \sin \lambda x \right]_0^{2a} - \left[\frac{1}{2\pi x^2} \cos \lambda x \right]_0^{2a}$$

$$= \frac{1 - \cos 2ax}{2\pi x^2}$$

Hence, $f(x) = \dfrac{\sin^2 ax}{\pi x^2}$.

Example 15 Find the Fourier sine transform of $e^{-|x|}$ and hence evaluate $\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx.$

[Kerala 1990, Madurai 1987, Madras 2003, V.T.U. 2004, U.P.T.U. 2003, 2004]

Solution Since, x is +ve in $(0, \infty)$ so that

$$e^{-|x|} = e^{-x} = f(x)$$

$\therefore$ The Fourier sine transform of e^{-x} is

$$F_S\{e^{-x}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+\lambda^2} (-\sin \lambda x - \lambda \cos \lambda x) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{1+\lambda^2} \right)$$

Using inversion formula for Fourier sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S\{f(x)\} \sin \lambda x \, d\lambda$$

or
$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{1+\lambda^2} \cdot \sin \lambda x \, d\lambda$$

Putting $x = m$, we obtain

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin m\lambda}{1+\lambda^2} \, d\lambda$$

or
$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+x^2} \, dx$$

Hence,
$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$
Hence proved.

Example 16 Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 - x^2; & \text{for } |x| < 1 \\ 0; & \text{for } |x| > 1 \end{cases}$$

and hence evaluated $\displaystyle\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx.$

[Warangal 1996, Karnataka 1993, Anna 2003, V.T.U. 2003 Madras 1991, 1993, 1996, 1997, U.P.T.U. 2005, G.E.U. 2009, 2011]

Solution The Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{-i\lambda x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \, e^{-i\lambda x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)(\cos \lambda x - i \sin \lambda x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-x^2) \cos \lambda x\, dx - i \int_{-1}^{1} (1-x^2) \sin \lambda x\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \cos \lambda x\, dx$$ [Using a property of definite integral on II$^{\text{nd}}$ integration]

or $$\hat{f}(\lambda) = \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1-x^2) \cos \lambda\, x dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ (1-x^2) \frac{\sin \lambda x}{\lambda} \right\}_0^1 - \int_0^1 (-2x) \cdot \frac{\sin \lambda x}{\lambda} dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 2x \frac{\sin \lambda x}{\lambda} dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{2}{\lambda} \left[\left\{ x \left(-\frac{\cos \lambda x}{\lambda} \right) \right\}_0^1 - \int_0^1 \left(-\frac{\cos \lambda x}{\lambda} \right) dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{2}{\lambda} \left[-\frac{\cos \lambda}{\lambda} + \frac{1}{\lambda} \left(\frac{\sin \lambda x}{\lambda} \right)_0^1 \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{2}{\lambda} \left[-\frac{\cos \lambda}{\lambda} + \frac{\sin \lambda}{\lambda^2} \right]$$

$$= 2 \cdot \sqrt{\frac{2}{\pi}} \cdot \left[\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right]$$

Now using inversion formula for Fourier transform, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda)\, e^{i\lambda x} d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} 2 \cdot \sqrt{\frac{2}{\pi}} \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) \cdot (\cos \lambda x + i \sin \lambda x)\, d\lambda$$

$$= \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \cos \lambda x \cdot \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) d\lambda + i \int_{-\infty}^{\infty} \sin \lambda x \cdot \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) d\lambda \right]$$

$$= \frac{2}{\pi} \cdot 2 \int_0^{\infty} \cos \lambda x \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) d\lambda$$ [Using a property of definite integral]

or $$f(x) = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x \, d\lambda$$

or $$\int_0^\infty \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x \, d\lambda = \frac{\pi}{4} f(x)$$

$$= \frac{\pi}{4}\begin{cases}(1-x^2), & \text{for } |x|<1 \\ 0, & \text{for } |x|>1\end{cases}$$

Putting $x = \frac{1}{2}$ in above expression, we have

$$\int_0^\infty \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\frac{\lambda}{2} d\lambda = \frac{3\pi}{16}$$

or $$\int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\frac{\lambda}{2} d\lambda = -\frac{3\pi}{16}$$

Hence $$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2} dx = -\frac{3\pi}{16}.$$

Example 17 Find the Fourier transform of the function

$$f(x) = \begin{cases} 1+\dfrac{x}{a}; & \text{for } -a<x<0 \\ 1-\dfrac{x}{a}; & \text{for } 0<x<a \\ 0; & \text{otherwise}\end{cases}$$

[G.E.U. 2009, U.T.U. 2007, U.P.T.U. 2002]

Solution The Fourier transform for $f(x)$ is given by

$$F\{f(x)\} = \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{0}\left(1+\frac{x}{a}\right)e^{-i\lambda x}dx + \int_0^a\left(1-\frac{x}{a}\right)e^{-i\lambda x}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} e^{-i\lambda x}dx + \frac{1}{a}\int_{-a}^{0} x\, e^{-i\lambda x} - \frac{1}{a}\int_0^a x\, e^{-i\lambda x}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{-i\lambda x}}{-i\lambda}\right)_{-a}^{a} + \frac{1}{a}\left\{x\frac{e^{-i\lambda x}}{-i\lambda} - \frac{e^{-i\lambda x}}{(-i\lambda)^2}\right\}_{-a}^{0} - \frac{1}{a}\left\{\frac{x\, e^{-i\lambda x}}{-i\lambda} - \frac{e^{-i\lambda x}}{(-i\lambda)^2}\right\}_0^a\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{i\lambda a} - e^{-i\lambda a}}{i\lambda}\right) + \frac{1}{a}\left(\frac{1}{\lambda^2} - \frac{a\, e^{i\lambda a}}{i\lambda} + \frac{e^{i\lambda a}}{\lambda^2}\right) - \frac{1}{a}\left(\frac{a\, e^{-i\lambda a}}{-i\lambda} - \frac{e^{-i\lambda a}}{-\lambda^2}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a}\left[\frac{1}{\lambda^2}(1-e^{-i\lambda a}) - \frac{1}{\lambda^2}(e^{i\lambda a}-1)\right]$$

$$= \frac{2}{\lambda^2 a\sqrt{2\pi}} - \frac{1}{\lambda^2 a\sqrt{2\pi}}(e^{i\lambda a}+e^{-i\lambda a})$$

or $$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\left[\frac{2}{\lambda^2 a}(1-\cos a\lambda)\right]; \lambda \neq 0$$

If $\lambda = 0$, then

$$\hat{f}(\lambda) = \int_{-a}^{a} 1\,dx + \frac{1}{a}\int_{-a}^{0} x\,dx - \frac{1}{a}\int_{0}^{a} x\,dx$$

$$= 2a + \frac{1}{a}\left(\frac{x^2}{2}\right)_{-a}^{0} - \frac{1}{a}\left(\frac{x^2}{2}\right)_{0}^{a}$$

$$= 2a - \frac{a}{2} - \frac{a}{2}$$

$$= a$$

Hence, $$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\left[\frac{2}{\lambda^2 a}(1-\cos a\lambda)\right] \quad \text{if } \lambda \neq 0$$

$$= a \quad \text{if } \lambda = 0$$

Example 18 Solve the integral equation

$$\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases}(1-\lambda) & \text{if } 0 \le \lambda \le 1\\ 0 & \text{if } \lambda > 1\end{cases}$$

and hence show that $\int_0^\infty \frac{\sin^2 u}{u^2}\,du = \frac{\pi}{2}$. **[Coimbatore 1988, Punjab 1990, Madras 1993]**

Solution Let

$$f(\lambda) = \int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases}1-\lambda; & 0 \le \lambda \le 1\\ 0; & \lambda > 1\end{cases}$$

$$\therefore \quad \sqrt{\frac{2}{\pi}}f(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx$$

or $$\hat{f}_C(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx \tag{38}$$

Using inversion formula for Fourier cosine transform, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_C(\lambda)\cos\lambda\, x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty f(\lambda)\cos\lambda x\,dx$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\cos\lambda x\,d\lambda + \int_1^\infty 0.\cos\lambda x\,d\lambda\right]$$

$$= \frac{2}{\pi}\left[(1-\lambda)\frac{\sin\lambda x}{x} - (-1)\cdot\frac{-\cos\lambda x}{x^2}\right]_0^1$$

$$= \frac{2(1-\cos x)}{\pi x^2} \tag{39}$$

Now, we have

$$\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda; & 0\le\lambda\le 1 \\ 0; & \lambda>1 \end{cases}$$

$$\therefore \quad \frac{2}{\pi}\int_0^\infty \frac{(1-\cos x)}{x^2}\cdot\cos\lambda x\,dx = \begin{cases} (1-\lambda); & 0\le\lambda\le 1 \\ 0; & \lambda>1 \end{cases} \quad \text{[Using Eq. (39)]}$$

or $$\int_0^\infty \frac{1-\cos x}{x^2}\cdot\cos\lambda x dx = \frac{\pi}{2}\begin{cases} (1-\lambda); & 0\le\lambda\le 1 \\ 0; & \lambda>1 \end{cases}$$

Put $\lambda = 0$, we get

$$\int_0^\infty \frac{1-\cos x}{x^2}dx = \frac{\pi}{2}$$

or $$\int_0^\infty \frac{2\sin^2\frac{x}{2}}{x^2}dx = \frac{\pi}{2}$$

Again put $x = 2u$ so that $dx = 2\,du$

$$\therefore \quad \int_0^\infty \frac{2\sin^2 u}{4u^2}\cdot 2du = \frac{\pi}{2}$$

or $$\int_0^\infty \frac{\sin^2 u}{u^2}du = \frac{\pi}{2}$$

Hence proved.

Example 19 Solve the integral equation

$$\int_0^\infty f(x)\cdot\sin\lambda x\,dx = \begin{cases} 1; & 0\le\lambda\le 1 \\ 2; & 1\le\lambda<2\,. \\ 0; & \lambda\ge 2 \end{cases}$$

Solution Let

$$f(\lambda) = \int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1; & 0 \le \lambda < 1 \\ 2; & 1 \le \lambda < 2 \\ 0; & \lambda \ge 2 \end{cases}$$

$$\therefore \quad \sqrt{\frac{2}{\pi}} f(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\,\sin\lambda x\,dx$$

or
$$\hat{f}_S(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\,\sin\lambda x\,dx$$

Using inversion formula for Fourier sine transform, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_S(\lambda)\,\sin\lambda x\,d\lambda$$

or
$$f(x) = \frac{2}{\pi}\int_0^\infty f(\lambda)\sin\lambda x d\lambda$$

$$= \frac{2}{\pi}\left[\int_0^1 \sin\lambda x\,d\lambda + \int_1^2 2\sin\lambda x\,d\lambda\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{\cos\lambda x}{x}\right)_0^1 - 2\left(\frac{\cos\lambda x}{x}\right)_1^2\right]$$

$$f(x) = \frac{2}{\pi}[1 + \cos x - 2\cos 2x]$$

Example 20 Find the Fourier transform of the function $f(x) = e^{-4(x-2)^2}$, when $F\{e^{-x^2}\} = \sqrt{\pi}\,e^{-\lambda^2/4}$.

Solution Given
$$F\{e^{-x^2}\} = \sqrt{\pi}\,e^{-\lambda^2/4} \tag{40}$$

$$\therefore \quad F\{e^{-4x^2}\} = F\{e^{-(2x)^2}\}$$

$$= \frac{\sqrt{\pi}}{2}\cdot e^{-\frac{1}{4}(\lambda/2)^2}$$

$$= \frac{\sqrt{\pi}}{2}\,e^{-\lambda^2/16}$$

Hence
$$F\{e^{-4(x-2)^2}\} = \frac{\sqrt{\pi}}{2}\,e^{2i\lambda}\,e^{-\lambda^2/16}$$

$$= \frac{\sqrt{\pi}}{2}\,e^{[2i\lambda - \lambda^2/16]}$$

1.6 PROPERTIES OF FOURIER TRANSFORM

Some properties of Fourier transform are as follows:

Property 1

Linearity property If $\hat{f}_r(\lambda)$ is the Fourier transform of the function $f_r(x)$ for $r = 1, 2, 3, \ldots, n$, then.

$$F\left\{\sum_{r=1}^{n} a_r f_r(x)\right\} = \sum_{r=1}^{n} a_r \hat{f}_r(\lambda)$$

where a_r is constant for $r = 1, 2, 3, \ldots, n$.

Proof By the definition of Fourier transform; we get

$$\hat{f}_r(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_r(x) e^{-i\lambda x}\, dx$$

Now,

$$F\left\{\sum_{r=1}^{n} a_r f_r(x)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{r=1}^{n} a_r f_r(x)\right) e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{r=1}^{n} a_r f_r(x) e^{-i\lambda x}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{r=1}^{n} \left(\int_{-\infty}^{\infty} a_r f_r(x) e^{-i\lambda x}\, dx\right)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{r=1}^{n} \left(a_r \int_{-\infty}^{\infty} f_r(x) e^{-i\lambda x}\, dx\right)$$

$$= \sum_{r=1}^{n} a_r \hat{f}_r(\lambda)$$

Thus, we obtain,

$$F\left(\sum_{r=1}^{n} a_r f_r(x)\right) = \sum_{r=1}^{n} a_r \hat{f}_r(\lambda)$$

where a_r is constant for $r = 1, 2, 3, \ldots, n$.

***Note*:**

(i) $$F_c\left\{\sum_{r=1}^{n} a_r f_{rc}(x)\right\} = \sum_{r=1}^{n} a_r \hat{f}_{rc}(\lambda)$$

where a_r is a constant for $r = 1, 2, 3, \ldots n$ and $\hat{f}_{rc}(\lambda)$ is the Fourier cosine transform of $f_r(x)$.

(ii) $$F_S\left\{\sum_{r=1}^{n} a_r f_{rS}(x)\right\} = \sum_{r=1}^{n} a_r \hat{f}_{rS}(\lambda)$$

where a_r is a constant for $r = 1, 2, 3, \ldots, n$ and $\hat{f}_{rS}(\lambda)$ is the Fourier sine transform of $f_r(x)$.

Property 2

Scalar property If $\hat{f}(\lambda)$ is the Fourier transform of the function $f(x)$, then

$$F\{f(ax)\} = \frac{1}{a}\hat{f}\left(\frac{\lambda}{a}\right)$$

where a is a non-zero constant.

Proof By the definition of Fourier transform, we have

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx = F\{f(x)\}$$

Therefore,

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax) e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{a}\int_{-\infty}^{\infty} f(z) e^{-i\lambda z/a}\, dz \qquad [\text{put } z = ax]$$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\cdot e^{-i\frac{\lambda}{a}z}\, dz$$

$$F\{f(ax)\} = \frac{1}{a}\hat{f}\left(\frac{\lambda}{a}\right)$$

Property 3

Shifting property If $\hat{f}(\lambda)$ is the Fourier transform of the function $f(x)$, then

$F\{f(x-a)\} = e^{i\lambda a}\hat{f}(\lambda)$ where a is constant.

Proof By the definition of Fourier transform, we have

$$\hat{f}(\lambda) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-i\lambda x}\, dx$$

$$\therefore \qquad F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a) e^{-i\lambda x}\, dx$$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z) e^{-i\lambda(z+a)} dz \qquad [\text{put } x-a = z]$$

$$= e^{-i\lambda a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\cdot e^{-i\lambda z} dz$$

$$= e^{-\lambda a}\cdot\hat{f}(\lambda)$$

This property is also known as first shifting property.

Property 4

Second Shifting property If $\hat{f}(\lambda)$ is the Fourier transform of the function $f(x)$ and $g(x) = e^{iax} f(x)$. Then $F\{g(x)\} = \hat{f}(\lambda - a)$, where a is any constant.

Proof By the definition of Fourier transform, we have

$$\hat{f}(\lambda) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx$$

$$\therefore \quad \hat{f}(\lambda - a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i(\lambda - a)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{iax} \cdot e^{-i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f(x) \cdot e^{iax}) e^{-i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\lambda x} dx$$

$$= \hat{g}(\lambda) = F\{g(x)\}$$

Hence, proved.

Property 5

Modulation property If $\hat{f}(\lambda)$ is the Fourier transform of the function $f(x)$, then $F\{f(x) \cdot \cos ax\} = \frac{1}{2}\{\hat{f}(\lambda + a) + \hat{f}(\lambda - a)],$ where a is constant.

Proof We know that the Fourier transform of $f(x)$ is

$$F\{f(x)\} = \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$$

$$\therefore \quad \hat{f}(\lambda + a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(\lambda + a)x} dx$$

$$\text{and} \quad \hat{f}(\lambda - a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(\lambda - a)x} dx$$

Now,

$$\hat{f}(\lambda + a) + \hat{f}(\lambda - a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(\lambda + a)x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(\lambda - a)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) [e^{-i(\lambda + a)x} + e^{-i(\lambda - a)x}]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} (e^{-iax} + e^{iax})\, dx$$

$$\therefore \quad \frac{1}{2}[\hat{f}(\lambda + a) + \hat{f}(\lambda - a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} \left(\frac{e^{iax} + e^{-iax}}{2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f(x) \cos ax) e^{-i\lambda x} dx$$

$$= F\{f(x) \cos ax\}$$

Property 6

Fourier transform of the derivative Let $f(x)$ is a continuous function and $\lim\limits_{x \to 0} f(x) = 0$, also the derivative of $f(x)$ is absolutely integrable then

$$F\{f'(x)\} = i\lambda \hat{f}(\lambda), \text{ where } \hat{f}(\lambda) = F\{f(x)\}.$$

Proof We know that

$$F\{f'(x)\} = \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$$

$$\therefore \quad F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\lambda x} dx = \frac{1}{\sqrt{2\pi}} \left[\left\{ e^{-i\lambda x} f(x) \right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\lambda)\, e^{-i\lambda x} f(x)\, dx \right]$$

$$= \frac{i\lambda}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) dx$$

$$= i\lambda\, F\{f(x)\}$$

$$F\{f'(x)\} = i\lambda\, \hat{f}(\lambda)$$

In general

$$F\{f^{(n)}(x)\} = (i\lambda)^n \hat{f}(\lambda), \text{ where } n \text{ is any integer and } f^{(n)}(x) \text{ exists.}$$

Property 7

Convolution theorem Let $f(x)$ and $g(x)$ be two piece wise continuous, bounded and absolutely integrable functions in $-\infty < x < \infty$ and Fourier transform of $f(x)$ and $g(x)$ are given by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx \text{ and}$$

$$\hat{g}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{-i\lambda x} dx$$

respectively, then their convolution is denoted and defined as

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$$

Then

(i) $F\{f(x) * g(x)\} = F[f(x)] \cdot F[g(x)] = \hat{f}(\lambda) \cdot \hat{g}(\lambda)$

(ii) $f(x) * g(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\lambda)\, \hat{g}(\lambda)\, e^{i\lambda x} d\lambda$ (Inverse Fourier transform)

Proof We know that

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx$$

$$\therefore \quad F\{f(x) * g(x)\} = F\left\{\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cdot g(x-t) dt\right) e^{-i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{-i\lambda x} dt\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(z)\, e^{-i\lambda(x+z)} dz\, dt$$

[Put $x - t = z$
$x = t + z$
$dx = dt + dz$
or $dxdt = dzdt$]

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} f(t) \cdot e^{-i\lambda t} dt\right)\left(\int_{-\infty}^{\infty} g(z) \cdot e^{-i\lambda t} dz\right)$$

$$= F\{f(x)\} \cdot F\{g(x)\} = \hat{f}(\lambda) \cdot \hat{g}(\lambda)$$

(ii) Now $\displaystyle\int_{-\infty}^{\infty} \hat{f}(\lambda)\, \hat{g}(\lambda)\, e^{i\lambda x} d\lambda$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t} dt\right] \hat{g}(\lambda)\, e^{i\lambda x}\, d\lambda$$

$$= \int_{-\infty}^{\infty} f(t \left[\int_{-\infty}^{\infty} \hat{g}(\lambda) e^{i\lambda(x-t)} d\lambda\right] dt$$

$$= \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt \qquad \text{[Using I.F.T]}$$

$$= f(x) * g(x) \qquad \text{[Using definition of convolution]}$$

Hence, proved.

Property 8

If $$\hat{f}_S(\lambda) = \int_0^{\infty} f(x) \sin \lambda\, x\, dx \text{ and}$$

$$\hat{f}_C(\lambda) = \int_0^{\infty} f(x) \cos \lambda x\, dx, \text{ then}$$

(i) $$F_S\,\{f(x)\cdot \cos ax\} = \frac{1}{2}\left[\hat{f}_S\,(\lambda + a) + \hat{f}_S(\lambda - a)\right]$$

(ii) $$F_C\,\{f(x)\cdot \sin ax\} = \frac{1}{2}\left[\hat{f}_S\,(\lambda + a) - \hat{f}_S(\lambda - a)\right]$$

1.7 PARSEVAL'S THEOREM

(a) The Fourier transform of $f(x)$ and $g(x)$ are $\hat{f}(\lambda)$ and $\hat{g}(\lambda)$, respectively, then

(i) $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\lambda)|^2\, d\lambda$$

(ii) $$\int_{-\infty}^{\infty} f(x)\,\overline{g}(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda)\, \hat{\overline{g}}(\lambda)\, d\lambda$$

where $\overline{g}(x)$ is the complex conjugate of $g(x)$.

(b) The Fourier sine and cosine transform of the functions $f(x)$ and $g(x)$ are $\hat{f}_S(\lambda), \hat{g}_S(\lambda)$ and $\hat{f}_C(\lambda), \hat{g}_C(\lambda)$ respectively, then.

(i) $$\int_0^{\infty} f(x)\, g(x)\, dx = \frac{2}{\pi} \int_0^{\infty} \hat{f}_S(\lambda)\, \hat{g}_S(\lambda)\, d\lambda$$

(ii) $$\int_0^{\infty} f(x)\, g(x)\, dx = \frac{2}{\pi} \int_0^{\infty} \hat{f}_C(\lambda)\, \hat{g}_C(\lambda)\, d\lambda$$

(iii) $$\int_0^{\infty} |f(x)|^2\, dx = \frac{2}{\pi} \int_0^{\infty} [f_S(\lambda)]^2\, d\lambda$$

(iv) $$\int_0^{\infty} |f(x)|^2\, dx = \frac{2}{\pi} \int_0^{\infty} [f_C(\lambda)]^2\, d\lambda$$

Example 21 If Fourier transform of $f(x) = e^{-\frac{x^2}{2}}$ is $\sqrt{2\pi}\, e^{\frac{\lambda^2}{2}}$ then find $F\{f(\sqrt{2}x)\}$.

Solution Given

$$F\{f(x)\} = F\left\{e^{-\frac{x^2}{2}}\right\} = \sqrt{2\pi}\, e^{-\frac{\lambda^2}{2}} = \hat{f}(\lambda).$$

Now

$$F\{f(\sqrt{2}x)\} = \frac{1}{\sqrt{2}} F\left\{\frac{\lambda}{\sqrt{2}}\right\} \qquad \text{[Using property 2]}$$

$$= \frac{1}{\sqrt{2}} \cdot \sqrt{2\pi}\, e^{-\frac{1}{2}\left(\frac{\lambda}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\pi}\, e^{-\frac{\lambda^2}{4}}$$

Example 22 If $f(x) = \begin{cases} 1 & ; \quad |x| < a \\ 0 & ; \quad |x| > a \end{cases}$ and

$\hat{f}(\lambda) = \dfrac{2\sin \lambda a}{\lambda}\ (\lambda \neq 0)$, then prove that $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2}\, dx = \frac{\pi a}{2}$

Solution Using Parseval's theorem for Fourier transform, we have

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2\, d\lambda$$

or
$$\int_{-\infty}^{-a} 0 \cdot dx + \int_{-a}^{a} 1^2\, dx + \int_{a}^{\infty} 0 \cdot dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{2\sin \lambda a}{\lambda}\right|^2 d\lambda$$

or
$$\int_{-a}^{a} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4\sin^2 \lambda a}{\lambda^2}\, d\lambda$$

or
$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \lambda a}{\lambda}\, d\lambda = 2a$$

or
$$\int_{-\infty}^{\infty} \frac{\sin^2 \lambda a}{\lambda}\, d\lambda = \frac{\pi}{2} \cdot 2a$$

or
$$\int_{-\infty}^{\infty} \frac{\sin^2 \lambda a}{\lambda}\, d\lambda = \pi a$$

or $$2\int_0^\infty \frac{\sin^2 \lambda a}{\lambda}\, d\lambda = \pi a$$

or $$\int_0^\infty \frac{\sin^2 \lambda a}{\lambda}\, d\lambda = \frac{\pi a}{2}$$

$\therefore$ $$\int_0^\infty \frac{\sin^2 2x}{x}\, dx = \frac{\pi a}{2}$$ **Hence, proved.**

Example 23 Using modulation theorem, to find the Fourier transform of $f(x)$ cos bx, where $f(x)$ is defined by

$$f(x) = \begin{cases} 1; & \text{if } |x| < a \\ 0; & \text{if } |x| > a \end{cases}$$

Solution Fourier transform of the function $f(x)$ is

$$\hat{f}(\lambda) = \begin{cases} \dfrac{2\sin \lambda a}{\lambda} & \text{if } \lambda \neq 0 \\ 2a & \text{if } \lambda = 0 \end{cases}$$

Now

$$F\{f(x)\cdot \cos\ bx\} = \frac{1}{2}\left[\hat{f}\,(\lambda + b) + \hat{f}\,(\lambda - b)\right]$$ [Using modulation property]

or $$= \begin{cases} \dfrac{\sin(\lambda + b)a}{\lambda + b} + \dfrac{\sin(\lambda - b)a}{\lambda - b} \\ 2a \end{cases}$$

Example 24 Evaluate the energy spectrum of the function

$$f(x) = \begin{cases} e^{-ax}; & \text{for } x \geq 0 \\ 0; & \text{for } x < 0 \end{cases}$$

Solution The Fourier transform of the function $f(x)$ is

$$f(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} e^{-i\lambda x} dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(a+\lambda i)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(a+\lambda i)x}}{-(a+\lambda i)}\right]_0^\infty = \frac{1}{\sqrt{2\pi}}\ \frac{1}{a+\lambda i}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{a - \lambda i}{(a^2 + \lambda^2)}$$

Now, the energy spectrum is

$$= |f(\lambda)|^2$$

$$= f(\lambda)\,\bar{f}(\lambda) = \frac{(a - \lambda i)}{\sqrt{2\pi}\,(a^2 + \lambda^2)} \cdot \frac{a + \lambda i}{\sqrt{2\pi}\,(a^2 + \lambda^2)}$$

$$= \frac{1}{2\pi\,(a^2 + \lambda^2)}$$

Example 25 Show that the Fourier transform of 1 is $2\pi\,\delta(\lambda)$.

Solution We know that

$$F\{\delta(x)\} = 1$$

By inverse Fourier transform, we have

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} dx$$

or $$\delta(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} dx \quad \text{[Interchange } \lambda \text{ and } x\text{]}$$

Since delta function is even, we have

$$\delta(\lambda) = \delta(-\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} F\{1\}$$

$\therefore$ $$F\{1\} = 2\pi\,\delta(\lambda) \quad \text{Proved.}$$

1.8 FOURIER TRANSFORM OF SOME BASIC FUNCTIONS

In this section, we shall discuss the Fourier transform of the some basic functions.

(i) *The top-hat function* The top-hat function is denoted by π_a and is defined as

$$\pi_a(x) = \begin{cases} 0 & ; \; -\infty < x < -\dfrac{a}{2} \\ 1 & ; \; -\dfrac{a}{2} < x < \dfrac{a}{2} \\ 0 & ; \; \dfrac{a}{2} < x < \infty \end{cases}$$

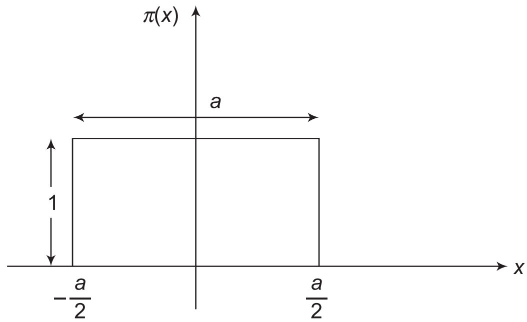


Fig. 1.1

Fourier transform is as follows:

$$F\{\pi_a(x)\} = \int_{-\infty}^{\infty} \pi_a(x)\, e^{-i\lambda x} dx$$

$$= \int_{-a/2}^{a/2} 1 \cdot\ e^{-i\lambda x} dx$$

$$= \left[\frac{e^{-i\lambda x}}{-i\lambda}\right]_{0a/2}^{a/2} = \left[\frac{-e^{-\frac{i\lambda a}{2}} + e^{\frac{i\lambda a}{2}}}{\lambda i}\right]$$

$$= \frac{\sin\left(\frac{\lambda a}{2}\right)}{\lambda / 2} = \sin c(\lambda a)$$

The sine function is defined as $\sin c(x) = \dfrac{\sin x}{x}$.

(ii) *The Dirac-Delta function* The Dirac-delta function is denoted by $\delta(t - a)$ and it is defined as

$$\delta(t-a) = \begin{cases} \infty & ; \quad t = a \\ 0 & ; \quad \text{otherwise} \end{cases}$$

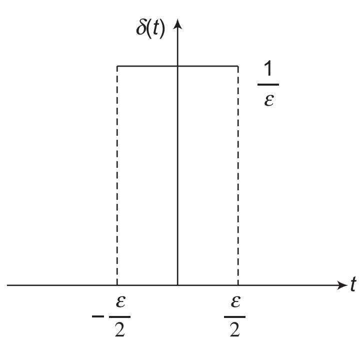


Fig. 1.2

Then, its Fourier transform is

$$F\{\delta(t-a)\} = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-i\lambda x}\, dt$$

$$= e^{-i\lambda a} \qquad \left[\because \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \right] \text{ and } \int_{-\infty}^{\infty} f(t)\delta(t-a)\, dt = f(a)$$

(iii) *The shah function:* The shah function is denoted by $\perp\!\perp\!\perp_T(t)$ and is defined as $\perp\!\perp\!\perp_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

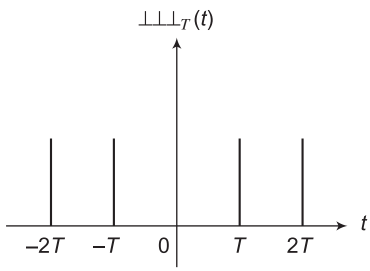


Fig. 1.3

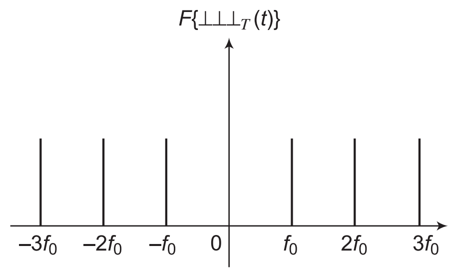


Fig. 1.4

its Fourier transform $F\{\perp\!\perp\!\perp_T(t)\} = f_0 \perp\!\perp\!\perp_{f_0}(f)$ where $f_0 = \dfrac{1}{T}$

(iv) *The triangle function* The triangle function is defined as

$$A(x) = \begin{cases} -\dfrac{A}{T_1}|x| + A & -T_1 < x < T_1 \\ 0 & \text{otherwise} \end{cases}$$

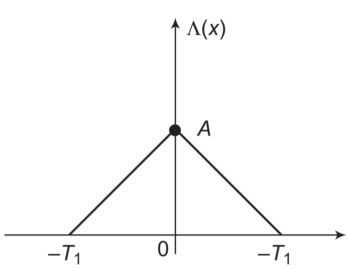


Fig. 1.5

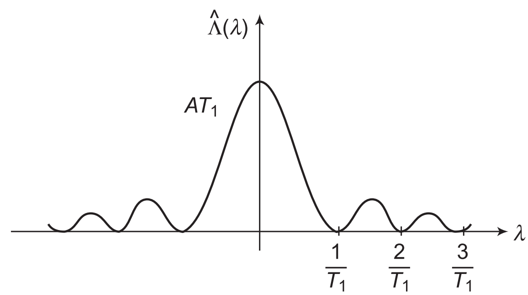


Fig. 1.6

Its Fourier transform

$$F\{\Lambda(x)\} = \hat{\Lambda}(\lambda) = AT_1 \sin c\,(\lambda T_1)$$

(v) *The Gate Function:* The gate function is defined as

$$f(t) = \begin{cases} 1; & \text{if} \quad |t| < a \\ 0; & \text{if} \quad |t| > a \end{cases}$$

Then its Fourier transform is

$$\hat{f}(\lambda) = F\{f(t)\} = 2\frac{\sin \lambda a}{\lambda}\ ;\ \lambda \neq 0$$

$$= 2a\ ;\ \lambda = 0$$

(vi) *Signum Function* The Signum function is denoted by sgn(t) and is defined as

$$Sgn(t) = \begin{cases} 1; & \text{if} \quad t > 0 \\ -1; & \text{if} \quad t < 0 \end{cases}$$

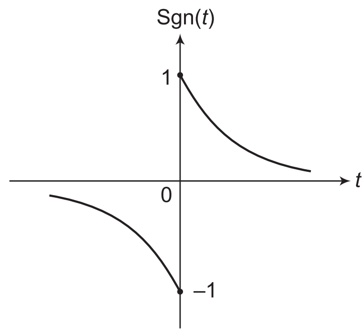


Fig. 1.7

Its Fourier transform is

$$F\{Sgn(t)\} = \frac{2}{\lambda i}$$

(vii) The Gaussian function is defined as

$$f(t) = e^{-at^2}$$

its Fourier transform is

$$\hat{f}\ (\lambda) = \sqrt{\frac{\pi}{a}}\ e^{-\lambda^2/4a}$$

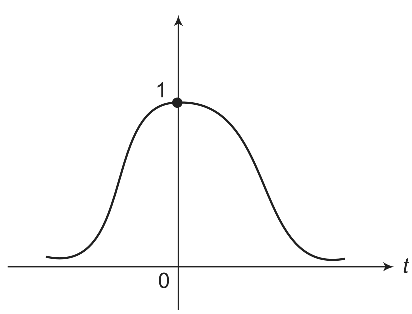


Fig. 1.8

1.9 DISCRETE FOURIER TRANSFORM

The discrete Fourier transform (DFT) is the transform that deals with a finite number of discrete time signal and a finite or discrete number of frequencies is of the form

$$W_r = \frac{2\pi}{N} r \;;\; r = 0, 1, 2, \cdots N-1$$

Let $f(r)$ be the continuous functions. Let N samples be denoted as $f(0), f(1), f(2), \ldots, f(N-1)$, then the discrete Fourier transform of $f(r)$ is

$$F(\lambda) = \frac{1}{N} \sum_{r=0}^{N-1} f(r)\, e^{-\frac{2\pi i \lambda r}{N}} \quad \text{for } \lambda = 0, 1, 2, \ldots N-1$$

and discrete inverse Fourier transform of $F(\lambda)$ is

$$f(r) = \sum_{\lambda=0}^{N-1} F(\lambda)\, e^{\frac{2\pi i \lambda r}{N}}$$

1.9.1 Discrete 2-D Fourier Transform

The discrete 2D Fourier transform of $f(r, s)$ is

$$F(\lambda, \mu) = \frac{1}{MN} \sum_{r=0}^{M-1} \sum_{s=0}^{N-1} f(r,s) e^{-2\pi i\left(\frac{\lambda r}{M} + \frac{\mu s}{N}\right)}$$

for $r = 0, 1, 2, \ldots, M-1$, $s = 0, 1, 2 \ldots, N-1$ and the discrete 2D inverse Fourier transform of $F(\lambda, \mu)$ is

$$f(r, s) = \sum_{\lambda=0}^{M-1} \sum_{\mu=0}^{N-1} F(\lambda, \mu)\, e^{2\pi i\left(\frac{\lambda r}{M} + \frac{\mu s}{N}\right)}$$

for $r = 0, 1, 2, \ldots, M-1$, $s = 0, 1, 2 \ldots, N-1$

1.9.2 Properties of Discrete Fourier Transform

(i) ***Linearity Property:*** Let $F(\lambda)$ and $G(\lambda)$ be the DFT of discrete time signals $f(r)$ and $g(r)$ respectively, then DFT of
$af(r) + bg(r) = aF(\lambda) + bG(\lambda)$, where a and b are any constants.

(ii) ***Time Reversal:*** Let $F(\lambda)$ be the DFT of periodic discrete time signal $f(r)$, then DFT of $f(-r)$ is $F(-\lambda)$.

(iii) ***Reciprocity:*** Let $F(\lambda)$ be the DFT of $f(r)$, then the DFT of $F(r)$ is $Nf(-\lambda)$.

1.9.3 Fourier Transform of Discrete Time Aperiodic Signals (DTFT)

Transformation is a mathematical tool, which transform the domain of the signals in order to find out the specific characteristic of it.

Mathematically it is defined as

$$X(w) = \sum_{n=-\infty}^{\infty} x(n)\, \mathrm{e}^{-jwn} \tag{41}$$

where $X(w)$ is Fourier Transform of $x(n)$ and $x(n)$ is discrete time aperiodic signals.

We can obtain $x(n)$ from $X(w)$ by taking inverse Fourier transform and it is given by

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(w)\,e^{jwn}\,dw \tag{42}$$

The frequency range of w is $-\pi$ to π or 0 to 2π. So $X(w)$ becomes periodic with period 2π.

The Fourier transform is convergent if

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

It means Fourier transform exists only if discrete time signal is absolutely summable. This is necessary and sufficient condition for the existence of Fourier transform.

Now, DTFT explain with the help of energy discrete time signal $x(n)$ is given by

$$E_X = \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{43}$$

or $$E_X = \sum_{n=-\infty}^{\infty} x(n)\cdot x^*(n) \qquad [\because \quad x^*(n) \text{ is complex conjugate of } x(n)] \tag{44}$$

where the inverse DTFT

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(w)e^{jwn}\,dw$$

$$\therefore \qquad x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(w)e^{-jwn}\,dw \tag{45}$$

Putting Eq. (45) in Eq. (44), we get

$$E_X = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(w)X(w)\,dw$$

But $X^*(w)\cdot X(w) = |X(w)|^2$

$$\therefore \qquad E_X = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(w)|^2\,dw \tag{46}$$

From Eqs (43) and (46), we have

$$E_X = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(w)|^2\,dw \tag{47}$$

Equation (47) is the Parseval's relation for discrete time periodic signal with finite energy.

Basically $X(w)$ is complex frequency function. Therefore, we can define it in terms of magnitude and angle as

$$X(w) = |X(\omega)|\,e^{j\theta(\omega)}$$

where $|X(\omega)|$ is magnitude spectrum and $\theta(\omega)$ is the phase spectrum.

Now, the energy density spectrum of $x(n)$ is denoted by $S_{XX}(\omega)$ and is given by

$$S_{XX} = |X(\omega)|^2.$$

We explain this theory with the help of the following example:

Example 26 Determine the Fourier transform of single sided exponential pulse and energy density spectrum for $a = 0.5$ and $a = -0.5$ where the energy, discrete time signal is

$$x(n) = a^n u(n); -1 < a < 1.$$

Solution The exponential signal $a^n\, u(n)$ exists from 0 to ∞. It is shown in the Fig. 1.9.

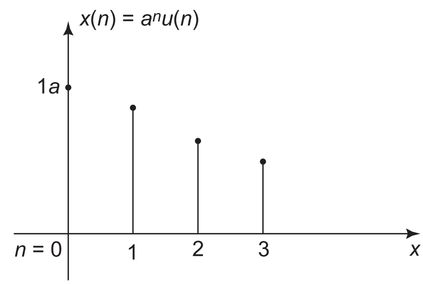


Fig. 1.9

Using Eq. (41), we get

$$X(\omega) = \sum_{n=0}^{\infty} a^n\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a\, e^{-j\omega})^n$$

or $$X(\omega) = \frac{1}{1 - ae^{-j\omega}};\ \left|a\, e^{-j\omega}\right| < 1$$

$$X(\omega) = \frac{1}{1 - a\, e^{j\omega}};\ |a| < 1 \qquad \left[\because\ \left|e^{-j\omega}\right| = \left|\cos\omega - j\sin\omega\right| = 1\right]$$

which is known as frequency spectrum or function.
Now, the energy density spectrum is given by

$$S_{XX}(\omega) = \left|X(\omega)\right|^2 = X(\omega)\cdot X^*(\omega)$$

where $X^*(\omega)$ is complex conjugate of $X(\omega)$

$\therefore$ $$X^*(\omega) = \frac{1}{1 - a^{ej\omega}}$$

Thus $$S_{XX} = \frac{1}{1 - a\, e^{-j\omega}} \cdot \frac{1}{1 - a}$$

$$= \frac{1}{1 - a(e^{j\omega} - e^{-j\omega}) + a^2} = \frac{1}{1 - 2a\cos\omega + a^2}$$

Case I

When $a = 0.5$

$\therefore$ $$x(n) = (0.5)^n\, u(n) \text{ and } S_{XX}(\omega) = \frac{1}{1 - \cos\omega + 0.25}$$

Case II

When $a = -0.5$

$\therefore$ $$x(n) = (-0.5)^n\, u(n) \text{ and } S_{XX}(\omega) = \frac{1}{1 + \cos\omega + 0.25}$$

Example 27 Determine the discrete time signal $x(n)$, whose Fourier Transform is

$$X^{(j\omega)} = \begin{cases} 1; & \text{if } -\omega_c \le \omega \le \omega_c \\ 0; & \text{if } -\omega_c \le |\omega| \le \pi \end{cases}$$

Solution We want to evaluate the discrete time signal $x(n)$ so we take the inverse Fourier transform of the given signal. (See the Fig. 1.10)

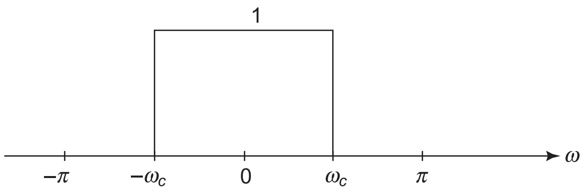


Fig. 1.10

Using Eq. (42), we get

$$x(n) = \frac{1}{2\pi}\int_{-w_c}^{w_c} 1\cdot e^{jwn}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-w_c}^{w_c}$$

$$= \frac{1}{2\pi jn}\left[e^{jw_c n} - e^{-jw_c n}\right] = \frac{1}{2\pi n}\cdot 2\sin w_c n$$

or $$x(n) = \frac{\sin w_c n}{\pi n}$$

Example 28 Calculate the 4-point discrete Fourier transform of $f(r)$ with period 4 and given by $f(-2) = 1, f(-1) = 0, f(0) = 2$ and $f(1) = 0$.

Solution The 4-point discrete Fourier transform of the given signal is

$$F(\lambda) = \sum_{r=-2}^{1} f(r)\, e^{-\frac{2\pi i r\lambda}{4}}$$

$$= f(-2)\, e^{\frac{4\pi i\lambda}{4}} + f(-1)\, e^{\frac{2\pi i\lambda}{4}} + f(0)\, e^{0} + f(1)\, e^{-\frac{2\pi i\lambda}{4}}$$

$$= 1\cdot e^{\lambda\pi i} + 0 + 2\times 1 + 0$$

$$= e^{\lambda\pi i} + 2$$

Now

$$F(-2) = e^{-2\pi i} + 2 = \cos 2\pi + i\sin 2\pi + 2 = 1 + 2 = 3$$

$$F(-1) = e^{\pi i} + 2 = \cos\pi - i\sin\pi + 2 = -1 + 2 = 1$$

$$F(0) = e^0 + 2 = 1 + 2 = 3$$

$$F(1) = e^{\pi i} + 2 = -1 + 2 = 1$$

Thus $\qquad F(\lambda) = \{3, 1, 3, 1\}$

The amplitude spectrum of $F(\lambda)$ is shown in the following Fig. (1.11).

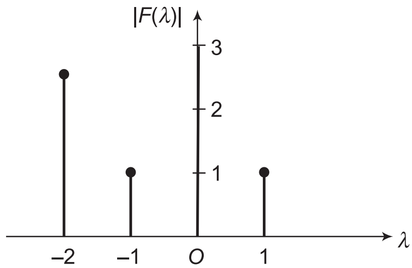


Fig. 1.11 *Amplitude Spectrum of F(λ)*

Example 29 Find 4-point discrete Fourier transform of $F(r) = \{1, 1, 0, 0\}$

Solution The 4-point discrete Fourier transform is

$$F(\lambda) = \sum_{r=0}^{3} f(r)\, e^{-\frac{2\pi i\lambda r}{4}}$$

$$= f(0) + f(1)\, e^{-\frac{2\pi i\lambda}{4}} + f(2) e^{-\frac{4\pi i\lambda}{4}} + f(3)\, e^{-\frac{6\pi i\lambda}{4}}$$

$$= 1 + 1\, e^{-\frac{\pi i\lambda}{2}} + 0 + 0 = 1 + e^{-\frac{\pi i\lambda}{2}}$$

Now $\qquad F(0) = 1 + 1 = 2,\ F(1) = 1 + e^{-\frac{\pi i}{2}} = 1 - i$

$$F(2) = 1 + e^{-\pi i} = 1 = 1 = 0,\ \ F(3) = 1 + e^{-\frac{3\pi i}{2}} = 1 - i$$

Thus $\qquad F(\lambda) = \{F(0), F(1), F(2), F(3)\} = \{2, 1 - i, 0, 1 + i\}$

The amplitude spectrum of $F(\lambda)$ is shown in the following Fig (1.12)

$$|\, F(\lambda)\, | = \left(2, \sqrt{1^2 + (-1)^2}, 0, \sqrt{1^2 + 1^2}\right) = (2, 1.4142, 0, 1.4142)$$

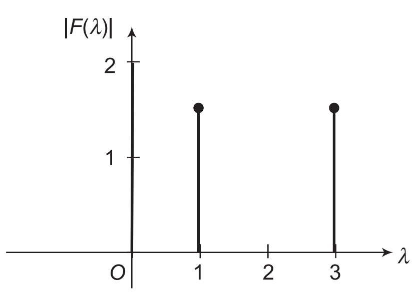


Fig. 1.12 *Amplitude spectrum of F(λ)*

Example 30 Calculate 4-point inverse DFT of the discrete signal $F(\lambda)$ with period 4 given by $F(0) = 1$, $F(1) = 0$, $F(2) = 0$, $F(3) = 1$.

Solution Given $F(\lambda) = \{1, 0, 0, 1\}$

Using discrete inverse FT, we have

$$f(r) = \sum_{\lambda=0}^{3} F(\lambda)\, e^{\frac{2\pi i \lambda r}{4}}$$

$$= F(0) + F(1)\, e^{\frac{2\pi ir}{4}} + F(2)\, e^{\frac{4\pi ir}{4}} + F(3)\, e^{\frac{6\pi ir}{4}}$$

$$= 1 + 0 + 0 + e^{\frac{3\pi ir}{2}}$$

$$f(r) = 1 + e^{\frac{3\pi ir}{2}}$$

Now

$$f(0) = 1 + 1 = 2$$

$$f(1) = 1 + e^{\frac{3\pi i}{2}} = (1 - i)$$

$$f(2) = 1 + e^{3\pi i} = 1 - 1 = 0$$

$$f(3) = 1 + e^{\frac{9\pi i}{2}} = (1 + i)$$

Hence $f(r) = \{2, (1 - i), 0, (1 + i)\}$

EXERCISE 1.1

(a) Fourier Integral

Obtain the Fourier integral representations of the following functions:

1. (i) $f(x) = \begin{cases} 0 & ; \quad x < 0 \\ \dfrac{1}{2} & ; \quad x = 0. \\ e^{-x} & ; \quad x > 0 \end{cases}$ (ii) $f(x) = \begin{cases} x & ; \quad |x| < 1 \\ 0 & ; \quad |x| > 1 \end{cases}.$

2. Find Fourier sine integral of the function $f(x) = e^{-\alpha x}$.

3. Find the Fourier sine and cosine integral of the function.

$$f(x) = \begin{cases} \sin x & \text{if} \quad 0 \le x \le \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$$

4. Find the complex form of the Fourier integral representations of

$$f(x) = \begin{cases} 0 & ; \quad -\infty < x < -1 \\ x & ; \quad -1 < x < 0 \\ 0 & ; \quad x > 0 \end{cases}$$

(b) Fourier Transform

5. Find the Fourier transform of

$$f(x) = \begin{cases} x \ ; & \text{if } |x| \le a \\ 0 \ ; & \text{if } |x| > a \end{cases}$$

6. Show that the Fourier transform of $f(x) = e^{-\frac{x^2}{2}}$ is $e^{-\frac{\lambda^2}{2}}$

7. Find the Fourier sine and cosine transform of the function.

$$f(x) = \begin{cases} x & ; \ \text{if } 0 < x < 1 \\ 2 - x & ; \ \text{if } 1 < x < 2 \\ 0 & ; \ \text{if } x > 2 \end{cases}$$

8. Find the Fourier cosine transform of

$$f(x) = \begin{cases} x & ; \quad \text{if } \ 0 < x < \frac{1}{2} \\ 1 - x & ; \quad \text{if } \ \frac{1}{2} < x < 1 \\ 0 & ; \quad \text{if } \ x > 1 \end{cases}$$

9. Find the Fourier cosine transform of $f(x) = e^{-2x} + 4e^{-3x}$

10. Evaluate the following using Parseval's theorem

(i) $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^2} dx$ (ii) $\displaystyle\int_0^\infty \left(\frac{1 - \cos x}{x}\right)^2 dx$

11. If $\hat{f}_C(\lambda) = \frac{1}{2}\tan^{-1}\left(\frac{2}{\lambda^2}\right)$, then find $f(x)$.

12. If $f(x) = \begin{cases} 1 \ ; & 0 < x < \pi \\ 0 \ ; & x > \pi \end{cases}$ then show that $\displaystyle\int_0^\infty \left(\frac{1 - \cos \lambda\pi}{\lambda}\right) \cdot \sin \lambda x \, d\lambda = \begin{cases} \frac{\pi}{2} & ; \ 0 < x < \pi \\ 0 & ; \ x > \pi \end{cases}$

13. Find the Fourier transform of the single gate function (rectangular pulse) shown in Fig. 1.13.

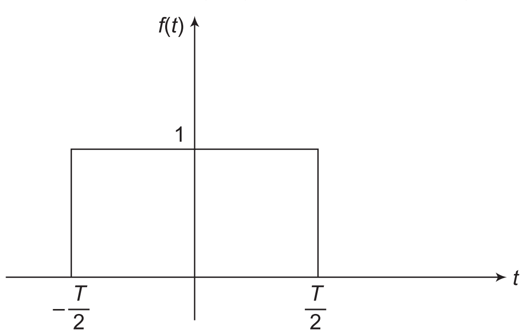


Fig. 1.13

14. Find the Fourier transform of the function shown in Fig. 1.14.

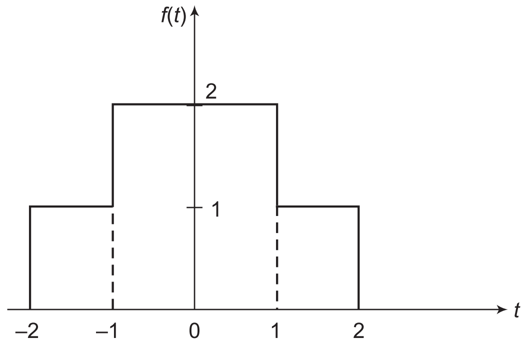


Fig. 1.14

15. Find Fourier sine and cosine transform of $f(x) = \cosh x - \sinh x$.

16. If $\hat{f}_C(\lambda) = \dfrac{\lambda}{1+\lambda^2}$, find $f(x)$

17. If $\hat{f}_C(\lambda) = \begin{cases} \sin\lambda & ; \quad 0 < \lambda < \pi \\ 0 & ; \quad \lambda \geq \pi \end{cases}$, then find $f(x)$

18. Find $F_S^{-1}\left\{\dfrac{e^{-a\lambda}}{\lambda}\right\}$ and hence evaluate $F_S^{-1}\left\{\dfrac{1}{\lambda}\right\}$.

19. Find $f(x)$ if $\hat{f}_C(\lambda) = \dfrac{1}{1+\lambda^2}$.

20. Using Parseval's property, find the value of the integral

$$\int_0^\infty \frac{dx}{(a^2 + x^2)(b^2 + x^2)}.$$

21. Calculate the N-point DFT of a finite-time sequence $f(r)$ defined by

$$f(r) = \begin{cases} 1 \; ; & \text{for} \quad 0 \leq r \leq L-1 \\ 0 \; ; & \text{otherwise} \end{cases}$$

22. Find 4-point discrete Fourier transform of discrete-time signal $f(r) = (-1)^r$ for all r.

23. Find 5-point discrete Fourier transform of discrete-time signal $f(r)$ with period 5 and defined by

$$f(-2) = -1, f(-1) = -2, f(0) = 0, f(1) = 2 \text{ and } f(2) = 1$$

24. Find 4-point discrete Fourier transform of the discrete time signals $f(r)$ with period 4 and given by

$$f(0) = 3, f(1) = 4, f(2) = 5, f(3) = 5.$$

Answers

1. (i) $f(x) = \dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{\lambda \sin \lambda x + \cos \lambda x}{1+\lambda^2}$ (ii) $f(x) = \displaystyle\int_0^\infty \dfrac{2(\sin\lambda - \lambda\cos\lambda)}{\pi\lambda^2}\sin\lambda x\, d\lambda$

2. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{\lambda\sin\lambda x}{\alpha^2+\lambda^2}\,d\lambda$

3. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{\sin\cdot\lambda x}{1-\lambda^2}\sin\lambda x\, d\lambda;\ \dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{1+\cos\lambda\pi}{1-\lambda^2}\cos\lambda x\, d\lambda$

4. $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{i\lambda x}\left[\dfrac{1}{\lambda^2}+\left(\dfrac{1}{\lambda i}-\dfrac{1}{\lambda^2}\right)e^{-\lambda i}\right]d\lambda$

5. $\sqrt{\dfrac{2}{\pi}}\,\dfrac{i(a\lambda\cos\lambda a - \sin\lambda a)}{\lambda^2}$

7. $\dfrac{2}{\lambda^2}\sqrt{\dfrac{2}{\pi}}\sin\lambda(1-\cos\lambda);\ \dfrac{2}{\lambda^2}\sqrt{\dfrac{2}{\pi}}\cos\lambda\,(1-\cos\lambda)$

8. $\sqrt{\dfrac{2}{\pi}}\left[\dfrac{2\cos\lambda/2}{\lambda^2}-\dfrac{\cos\lambda}{\lambda^2}-\dfrac{1}{\lambda^2}\right]$

9. $\sqrt{\dfrac{2}{\pi}}\left[\dfrac{2}{\lambda^2+4}+\dfrac{12}{\lambda^2+9}\right]$

10. (i) $\dfrac{\pi}{2}$ (ii) $\dfrac{\pi}{2}$

11. $\dfrac{e^{-x}\sin x}{x}$

13. $T\sin c\left(\dfrac{\lambda T}{2}\right)$

14. $\dfrac{1}{\sqrt{2}}\left[\dfrac{2}{\lambda}\sin\lambda\,(1+2\cos\lambda)\right]$

15. $\dfrac{\lambda}{1+\lambda^2}, \dfrac{1}{1+\lambda^2}$

16. e^{-x}

17. $\dfrac{2}{\pi}\dfrac{\sin\pi x}{(1-x^2)}$

18. $\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{x}{a}\right); \sqrt{\dfrac{\pi}{2}}.$

19. e^{-x}

20. $\dfrac{\pi}{2ab\,(a+b)}$

21. $\dfrac{1-e^{-\frac{2\pi irL}{N}}}{1-e^{-\frac{2\pi ir}{N}}}$, , $r = 0, 1, 2, \ldots, N-1.$

22. $F(\lambda) = \{0,0,4,0\}$

23. $-2i\sin\dfrac{4\pi r}{5} - 4i\sin\dfrac{2\pi r}{5}$

24. $F(\lambda) = \left\{4.25, \dfrac{\sqrt{5}}{2}, \dfrac{1}{4}, \dfrac{\sqrt{5}}{4}\right\}$

1.10 FINITE FOURIER TRANSFORM

The finite Fourier sine transform of a function $f(x)$ which is sectionly continuous over some finite interval $(0, l)$ of the variable x it is defined as

$$\hat{f}_s(\lambda) = \int_0^l f(x) \sin \frac{\lambda \pi x}{l} dx; \ \lambda \in I$$

Similarly, the finite Fourier cosine transform of $f(x)$ in $(0, l)$ is defined as

$$\hat{f}_c(\lambda) = \int_0^l f(x) \cos \frac{\lambda \pi x}{l} dx; \ \lambda \in I$$

In the interval $(0, \pi)$ of the variable x,

$$\hat{f}_s(\lambda) = \int_0^\pi f(x) \sin \lambda x \, dx \text{ and}$$

$$\hat{f}_c(\lambda) = \int_0^\pi f(x) \cos \lambda x \, dx$$

1.11 INVERSE FINITE FOURIER TRANSFORM

(i) In the interval $(0, l)$

$f(x) = \frac{2}{l} \sum_{\lambda=1}^{\infty} \hat{f}_s(\lambda) \sin \frac{\lambda \pi x}{l}$ is called inverse Finite Fourier sine transform.

and $\quad f(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{\lambda=1}^{\infty} f_c(\lambda) \cos \frac{\lambda \pi x}{l}$ is

called inverse finite Fourier cosine transform where $f_c(0) = \int_0^l f(x) \, dx$

(ii) In the interval $(0, \pi)$
For sine transform

$$f(x) = \frac{2}{\pi} \sum_{\lambda=1}^{\infty} f_s(\lambda) \sin \lambda x$$

For cosine transform

$$f(x) = \frac{1}{\pi} f_c(0) + \frac{2}{\pi} \sum_{\lambda=1}^{\infty} f_c(\lambda) \cos \lambda x$$

where $\quad f_c(0) = \int_0^\pi f(x) \, dx$

Example 31 Find Finite Fourier sine and cosine transforms of $f(x) = 2x$; $0 < x < 4$.

Solution Given $f(x) = 2x$; $0 < x < 4$

(i) $$f_s(\lambda) = \int_0^4 2x \cdot \sin\frac{\lambda\pi x}{4}\,dx$$

$$= \left(2x \cdot \frac{-\cos\frac{\lambda\pi x}{4}}{\frac{\lambda\pi}{4}}\right)_0^4 - \int_0^4 2 \cdot \left(\frac{-\cos\frac{\lambda\pi x}{4}}{\frac{\lambda\pi}{4}}\right) dx$$

$$= \begin{cases} \dfrac{32}{\lambda\pi}(1-\cos\lambda\pi); & \lambda \neq 0 \\ 0; & \lambda = 0 \end{cases}$$

(ii) $$f_c(\lambda) = \int_0^4 2x \cdot \cos\frac{\lambda\pi x}{4}\,dx$$

$$= \left(2x \cdot \frac{\sin\frac{\lambda\pi x}{4}}{\frac{\lambda\pi}{4}}\right)_0^4 - \int_0^4 2 \cdot \left(\frac{\sin\frac{\lambda\pi x}{4}}{\frac{\lambda\pi}{4}}\right) dx$$

$$= \frac{32}{\lambda\pi}\sin\lambda\pi - \frac{8}{\lambda\pi}\left(\frac{-\cos\frac{\lambda\pi x}{4}}{\frac{\lambda\pi}{4}}\right)_0^4$$

$$= \frac{32}{\lambda^2\pi^2}(\cos\lambda\pi - 1);\ \lambda \neq 0$$

$$= 16;\ \lambda = 0$$

EXERCISE 1.2

1. Find the finite Fourier transforms (sine and cosine) of the function $f(x) = 1$; $0 < x < \pi$.
2. Find finite Fourier sine transform of the function $f(x) = \sin nx$, where n is an integer.
3. Find the finite Fourier cosine transform of $f(x) = \left(1 - \dfrac{x}{\pi}\right)^2$.
4. Find the finite Fourier sine and cosine transforms of $f(x) = x^2$ where $0 < x < \pi$.
5. Find $F_S^{-1}\{f_S(\lambda)\}$, if $f_S(\lambda) = \dfrac{2\pi}{\lambda^3}(-1)^{\lambda-1}$, $\lambda = 1, 2, \ldots$ where $0 < x < \pi$.
6. Find $F_C^{-1}\{f_C(\lambda)\}$, if $f_C(\lambda) = \dfrac{\sin\frac{\lambda\pi}{2}}{2\lambda}$ if $\lambda = 1, 2, 3, \ldots$ and $\dfrac{\pi}{4}$ if $\lambda = 0$, where $0 < x < 2\pi$.

7. Find $f(x)$ if its finite sine transform is given by $f_S(\lambda) = \dfrac{1-\cos\lambda\pi}{(\lambda\pi)^2}$ where $0 < x < \pi$.

8. Find the finite cosine transform of $f(x)$ if $f(x) = -\dfrac{\cos p(\pi - x)}{p \sin \lambda\pi}$

Answers

1. $f_S(\lambda) = \dfrac{1}{\lambda}\left[1-(-1)^{\lambda}\right],\ f_C(\lambda) = \begin{cases} 0 & \text{if } \lambda = 1, 2, 3\cdots \\ \pi & \text{if } \lambda = 0 \end{cases}$

2. $f_S(\lambda) = \begin{cases} 0 & ; \quad \text{if } \lambda \neq n \\ \dfrac{\pi}{2} & ; \quad \text{if } \lambda = n \end{cases}$

3. $f_C(\lambda) = \begin{cases} \dfrac{2}{\pi\lambda^2} & ; \quad \text{if } \lambda \neq 0 \\ \dfrac{\pi}{3} & ; \quad \text{if } \lambda = 0 \end{cases}$

4. $f_S(\lambda) = \begin{cases} \dfrac{2}{\lambda^3}(\cos\lambda\pi - 1) - \dfrac{\pi^2}{\lambda}\cos\lambda\pi & ; \quad \text{if } \lambda = 1, 2, 3, \ldots \\ \dfrac{\pi^3}{3} & ; \quad \text{if } \lambda = 0 \end{cases}$

$f_C(\lambda) = \dfrac{2\pi}{\lambda^2}(\cos\lambda\pi - 1)$

5. $\dfrac{2}{\pi}\sum_{\lambda=1}^{\infty}\dfrac{2\pi(-1)^{\lambda-1}}{\lambda^3}\sin\lambda x$

6. $\dfrac{1}{4} + \dfrac{1}{\pi}\sum_{\lambda=1}^{\infty}\dfrac{\sin\lambda\,\pi/2}{2\lambda}\cos\lambda x$

7. $\dfrac{2}{\pi^3}\sum_{\lambda=1}^{\infty}\left(\dfrac{1-\cos\lambda\pi}{\lambda^2}\right)\sin\lambda x$

8. $\dfrac{1}{\lambda^2 - K^2},\ K \neq 0,\ k = 1, 2, 3, \ldots$

1.11.1 Ordinary Differential Equation with Constant Coefficients

Suppose nth order linear differential equation with constant coefficients is

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = f(x) \tag{48}$$

Taking Fourier transform both sides on Eq. (48), we have

$$[a_0(i\lambda)^n + a_1(i\lambda)^{n-1} + a_2(i\lambda)^{n-2} + \cdots + a_n]\, F\{y(x)\} = F\{f(x)\}$$

If $F\{y(x)\}$ exists, then

$$F\{y(x)\} = \frac{F\{f(x)\}}{P(i\lambda)}$$

where $P(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ is the operator on the left hand side of the differential equation.

Let $\dfrac{1}{P(i\lambda)}$ has an inverse Fourier transform

$$g(x) = \frac{1}{2\pi} \int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{i\lambda x}}{P(i\lambda)}\, d\lambda$$

Then using convolution theorem, we get

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, g(x-t)\, dx$$

Example 32 Solve $y' - 2y = H(t)e^{-2t}$; $-\infty < t < \infty$. Using Fourier transform, where $H(t) = u_0(t)$ is the unit step function.

Solution Given

$$y' - 2y = H(t)e^{-2t} \tag{49}$$

Taking Fourier transform both sides on Eq. (48) we have

$$F\{y' - 2y\} = F\{H(t)e^{-2t}\}$$

$$(i\lambda - 2)\, F\{y(t)\} = \frac{1}{2 + i\lambda}$$

or

$$F\{y(t)\} = -\frac{1}{(2+\lambda i)(2-\lambda i)} = -\frac{1}{4+\lambda^2}$$

Taking inverse Fourier transform, we get

$$y(t) = F^{-1}\left\{\frac{-1}{4+\lambda^2}\right\} = -\frac{1}{4} e^{-2|t|}$$

$$= \begin{cases} -\dfrac{1}{4} e^{2} t; & t < 0 \\[2ex] -\dfrac{1}{4} e^{-2t}; & t > 0 \end{cases}$$

It is the required solution of the given equation.

Example 33 Using convolution theorem for Fourier transform, solve the differential equation $y'' - y = -H(1 - |x|)$; $-\infty < x < \infty$ with $y(x) \to 0$ and $y'(x) \to 0$ as $|x| \to \infty$.

Solution Given $y'' - y = -H(1 - |x|)$ (50)

where H is the Heaviside's unit step function and it is defined as

$$H(1 - |x|) = \begin{cases} 1; & \text{if } |x| < 1 \\ 0; & \text{if } |x| > 1 \end{cases}$$

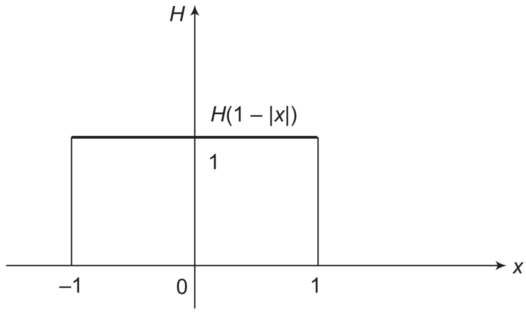


Fig. 1.15

Now, taking Fourier transform of both sides of the given Eq. (50), we obtain

$$[(i\lambda)^2 - 1]\, F\{y(x)\} = -F\{H(1 - |x|\}$$

$$= -\int_{-1}^{1} e^{-i\lambda x} dx$$

$$= \frac{-2\sin\lambda}{\lambda}$$

or $$(\lambda^2 + 1)\, F\{y(x)\} = \frac{2\sin\lambda}{\lambda}$$

or $$F\{y(x)\} = \frac{2\sin\lambda}{\lambda(\lambda^2 + 1)}$$

But $$F^{-1}\left\{\frac{2\sin\lambda}{\lambda}\right\} = H(1 - |x|) \quad \text{and}$$

$$F^{-1}\left\{\frac{1}{\lambda^2 + 1}\right\} = \frac{1}{2} e^{-|x|}$$

Using convolution theorem, we have

$$y(x) = F^{-1}\left\{\frac{2\sin\lambda}{\lambda} \cdot \frac{1}{\lambda^2 + 1}\right\}$$

$$= \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|} \cdot H(1 - |x - t|)\, dx$$

$$= \frac{1}{4\pi}\int_{x-1}^{x+1} e^{-|x|}dx$$

$$= \frac{1}{2\pi}\begin{cases} \sinh(1)e^{x}, & \text{for } -\infty < x < -1 \\ 1-\dfrac{\cosh x}{e}, & \text{for } -1 \le x \le 1 \\ \sinh(1)e^{-x}, & \text{for } 1 < x < \infty \end{cases}$$

1.11.2 Fourier Transform of the Derivative of Function

Let $\hat{u}(\lambda, t)$ be the Fourier transform of the function $u(x, t)$, then

$$\hat{u}(\lambda, t) = F\{u(x,t)\} = \int_{-\infty}^{\infty} u\, e^{-i\lambda x}\, dx$$

Now, the Fourier transform of $\dfrac{\partial^2 u}{\partial x^2}$ is

$$F\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-i\lambda x} dx$$

$$= \left[e^{-i\lambda x}\frac{\partial u}{\partial x} + e^{-i\lambda x}\cdot u \cdot (i\lambda)\right]_{-\infty}^{\infty} + (i\lambda)^2 \int_{-\infty}^{\infty} u\, e^{-i\lambda x}\, dx$$

$$F\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\lambda^2 F\{u\} \text{ As } x \to \pm\infty,\ u \to \infty \text{ and } \frac{\partial u}{\partial x} \to 0$$

Similarly, the Fourier sine and cosine transform, we have

$$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \lambda(u)_{x=0} - \lambda^2 F_s\{u\}$$

and $$F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - \lambda^2 F_c\{u\}$$

In general, the Fourier transform of the nth derivative of $u(x, t)$ is given by

$$F\left\{\frac{d^n u}{dx^n}\right\} = (-\lambda i)^n F\{u\}$$

1.11.3 Selection of Fourier Sine and Cosine Transform

Suppose we want to remove a term $\dfrac{\partial^2 u}{\partial x^2}$ in a differential equation, so we require the knowledge of the value.

(i) If $(u)_{x=0}$, is given, then we use Fourier sine transform and

(ii) If $\left(\dfrac{\partial u}{\partial x}\right)_{x=0}$, is given, then we use Fourier cosine transform.

1.11.4 Application of Initial Boundary Value Problem

In this section, we shall discuss the some of the Boundary Value Problem (BVP) using Fourier transforms.

Example 34 If u is the temperature at time t and K is the diffusivity of the material, find u from the partial differential equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2};\ x > 0, t > 0$$

With the boundary condition $u = u_0$ when $x = 0$, $t > 0$ and the initial condition $u = 0$ when $t = 0$, $x > 0$.

Solution Since $u = 0$ when $x = 0$ is given, so we apply the Fourier sine transform.

Taking Fourier sine transform on the given PDE, we get

$$\int_0^\infty \frac{\partial u}{\partial t}\sin\lambda x\,dx = K\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\lambda x\,dx$$

or $$\frac{d}{dt}\int_0^\infty u\sin\lambda x\,dx = K[\lambda(u)_{x=0} - \lambda^2 F_s\{u\}]$$

or $$\frac{d}{dt}(\hat{u}_s) = K\lambda\, u_0 - K\lambda^2\hat{u}_s$$

or $$\frac{d\hat{u}_s}{dt} + K\lambda^2\hat{u}_s = K\lambda u_0 \tag{51}$$

With the boundary condition $\hat{u}_s = 0$ when $t = 0$ (52)

Equation (51) is an ordinary linear differential equation. The solution of (51) is

$$\hat{u}_s\cdot e^{K\lambda^2 t} = u_0 K\lambda\int e^{K\lambda^2 t}dt + c$$

or $$\hat{u}_s e^{K\lambda^2 t} = \frac{u_0}{\lambda}e^{K\lambda^2 t} + c \tag{53}$$

Using Eq. (52), in Eq. (53), we get $c = -\dfrac{u_0}{\lambda}$

$\therefore$ $$\hat{u}_s = \frac{u_0}{\lambda}[1 - e^{-\lambda^2 Kt}] \tag{54}$$

Taking inverse Fourier sine transform, we get

$$u = \frac{2u_0}{\pi}\int_0^\infty [1 - e^{-\lambda^2 Kt}]\frac{\sin\lambda x}{\lambda}d\lambda \tag{55}$$

But $\int_0^\infty \frac{\sin \lambda x}{\lambda} d\lambda = \pi/2$, then (48) becomes

$$u = u_0 \left[1 - \frac{2}{\pi} \int_0^\infty e^{-\lambda^2 Kt} \frac{\sin \lambda x}{\lambda} d\lambda \right]$$

Example 35 Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$; $x > 0, t > 0$ under the conditions

(i) $u = 0$, when $x = 0, t > 0$

(ii) $u = \begin{cases} 1; & 0 < x < 1 \\ 0; & x \geq 1 \end{cases}$ when $t = 0$, and

(iii) $u(x, t)$ is bounded.

Solution Since $u = 0$ when $x = 0$ is given, so here we apply the Fourier sine transform.

Now, taking the Fourier sine transform both sides on the given PDE, we have

$$\int_0^\infty \frac{\partial u}{\partial t} \sin \lambda x \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \lambda x \, dx$$

or
$$\frac{d}{dt} \int_0^\infty u \sin \lambda x \, dx = \lambda (u)_{x=0} - \lambda^2 \hat{u}_s$$

or
$$\frac{d}{dt} \hat{u}_s = \lambda \cdot 0 - \lambda^2 \hat{u}_s$$

or
$$\frac{d\hat{u}_s}{dt} + \lambda^2 \hat{u}_s = 0 \tag{56}$$

With the boundary condition $\hat{u}_s = \begin{cases} 1; & 0 < x < 1 \\ 0; & x \geq 0 \end{cases}$ when $t = 0$ (57)

Equation (56) is an ordinary differential equation. The solution of (57) is

$\hat{u}_s = c_1 e^{-\lambda^2 t}$ where c_1 is a constant.

Now, using (57), we have

$$(\hat{u}_s)_{t=0} = \int_0^\infty u(x,0) \sin \lambda x dx$$

$$= \int_0^1 1 \cdot \sin \lambda x \, dx + \int_1^\infty 0 \cdot \sin \lambda x dx$$

$$= \left(-\frac{\cos \lambda x}{\lambda} \right)_0^1 = \frac{1 - \cos \lambda}{\lambda}$$

Now $(\hat{u}_s)_{t=0} = c_1 = \dfrac{1-\cos\lambda}{\lambda}$

$$\therefore \qquad \hat{u}_s = \left(\frac{1-\cos\lambda}{\lambda}\right)e^{-\lambda^2 t} \tag{58}$$

Taking inverse Fourier sine transform, on (58), we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos\lambda}{\lambda}\right)e^{-\lambda^2(t)}\sin\lambda x\, d\lambda$$

Example 36 If the flow of heat is linear so that the variation of θ (temperature) with z and y axis may be neglected and if it is assumed that no heat is generated in the medium, then solve the equation governing the conduction of heat in solids.

$$\frac{\partial\theta}{\partial t} = K\frac{\partial^2\theta}{\partial x^2}, \text{ where } -\infty < x < \infty.$$

with $\theta = f(x)$ when $t = 0$, $f(x)$ being a given function of x.

Solution Taking the Fourier transform both sides on the given equation, we have

$$\int_0^\infty \frac{\partial\theta}{\partial t}e^{-i\lambda x} = K\int_0^\infty \frac{\partial^2\theta}{\partial x^2}e^{-i\lambda x}dx$$

$$\text{or} \qquad \frac{d}{dt}\int_0^\infty \theta e^{-i\lambda x}dx = -K\lambda^2\hat{\theta}$$

$$\text{or} \qquad \frac{d\hat{\theta}}{dt} = -K\lambda^2\hat{\theta} \tag{59}$$

with the initial condition

$$\theta = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx = \hat{f}(\lambda) \text{ when } t = 0 \tag{60}$$

Solution of (59), we get

$$\hat{\theta} = c_1 e^{-K\lambda^2 t} \tag{61}$$

Using $(\hat{\theta})_{t=0} = \hat{f}(\lambda)$, then $c_1 = \hat{f}(\lambda)$

$$\therefore \qquad \hat{\theta} = \hat{f}(\lambda)\, e^{-K\lambda^2 t} \tag{62}$$

Taking inverse Fourier transform on (62), we get

$$\theta(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\lambda)\, e^{-K\lambda^2 t}d\lambda$$

where $\hat{f}(\lambda)$ is the F.T. of $f(x)$.

Example 37 Determine the distribution of temperature in the semi-infinite medium $x \geq 0$ when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.

Solution Let $u(x, t)$ be the temperature at point x and time t.

The heat flow equation in semi-infinite medium is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x > 0, t > 0 \tag{63}$$

With the boundary condition $u(0, t) = 0$ and the initial condition $u(x, 0) = f(x)$

Since $(u)_{x=0} = 0$ is given, so here we use the Fourier sine transform, we have

$$\int_0^\infty \frac{\partial u}{\partial t} \sin \lambda x dx = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \lambda x \, d\lambda$$

or $$\frac{d\hat{u}_s}{dt} = c^2[\lambda(u)_{x=0} - \lambda^2 \hat{u}_s] = -c^2\lambda^2\hat{u}_s$$

or $$\frac{d\hat{u}_s}{dt} + c^2\lambda^2\hat{u}_s = 0 \tag{64}$$

with the conditions $(\hat{u}_s)_{t=0} = f_s(\lambda)$ (65)

Solution of Eq. (64) is

$$\hat{u}_s = c_1 e^{-c^2\lambda^2 t} \tag{66}$$

using Eq. (65), $c_1 = f_s(\lambda)$

$$\therefore \qquad \hat{u}_s = f_s(\lambda) e^{-c^2\lambda^2 t} \tag{67}$$

Taking inverse Fourier sine transform, we get

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(\lambda) e^{-c^2\lambda^2 t} \cdot \sin \lambda x \, d\lambda$$

Example 38 Using Fourier cosine transform to show that the steady temperature u in the semi-infinite solid $y > 0$ when the temperature on the surface $y = 0$ is kept at unity over the strip $|x| < a$ and at zero outside the strip is

$$\frac{1}{\pi}\left[\tan^{-1}\left(\frac{x+a}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right)\right]$$

The result $\int_0^\infty e^{-\lambda x} \frac{\sin rx}{x} dx = \tan^{-1}\left(\frac{r}{\lambda}\right), r > 0, \lambda > 0$ may be assumed.

Solution Here the steady temperature $u(x, y)$ in the semi-infinite solid is governed by two-dimensional Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{68}$$

$0 < y < \infty, -\infty < x < \infty.$

Subject to the conditions

$u = 1, y = 0, -a < x < a$

and $u = 0, y = 0$ when $|x| < a$

Now taking Fourier cosine transform both sides on (68), we have

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos \lambda x \, dx + \int_0^\infty \frac{\partial^2 u}{\partial y^2} \cos \lambda x \, dx = 0$$

or $$-\left(\frac{\partial u}{\partial x}\right)_{x=0} - \lambda^2 \hat{u}_c + \frac{d^2 \hat{u}_c}{dy^2} = 0$$

or $$\frac{d^2 \hat{u}_c}{dy^2} - \lambda^2 \hat{u}_c = 0 \tag{69}$$

Its solution is $\hat{u}_c = c_1 e^{\lambda y} + c_2 e^{-\lambda y}$ (70)

Now as $y \to \infty, \hat{u}_c \to 0$

$\therefore$ $c_1 = 0$

Hence, $\hat{u}_c = c_2 e^{-\lambda y}$ (71)

But $$\hat{u}_c = \int_0^\infty u \cos \lambda \, x d\, x$$

$\therefore$ $$(\hat{u}_c)_{y=0} = \int_0^\infty (u)_{y=0} \cos \lambda x \, dx$$

$$= \int_0^a 1 \cos \lambda x \, dx = \frac{\sin \lambda a}{\lambda}$$

$\therefore$ Eq. (71) gives $c_2 = \dfrac{\sin \lambda a}{\lambda}$

Hence, $$\hat{u}_c = \frac{\sin \lambda a}{\lambda} e^{-\lambda y}.$$

Taking IFCT, we get

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda a}{\lambda} e^{-\lambda y} \cos \lambda x \, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \frac{e^{-\lambda y}}{\lambda} (2 \sin \lambda a \cos \lambda x) \, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \frac{e^{-\lambda y}}{\lambda} [\sin(a+x)\lambda + \sin(a-x)\lambda] \, d\lambda$$

$$= \frac{1}{\lambda}\left[\tan^{-1}\left(\frac{a+x}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right)\right]$$

Hence, proved.

EXERCISE 1.3

1. Use Fourier sine transform to solve the equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

under the conditions

(i) $u(0, t) = 0$ (ii) $u(x, 0) = e^{-x}$ (iii) $u(x, t)$ is bounded

2. The temperature u in the semi-infinite rod $0 \le x < \infty$ is determined by the equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$$

under the conditions

(i) $u(x, 0) = 0$

(ii) $\left(\frac{\partial u}{\partial x}\right)_{x=0} = -\mu$ (a constant), $t > 0$ using cosine transform, show that

$$u(x, t) = \frac{2\mu}{\pi}\int_0^\infty (1 - e^{-K\lambda^2 t})\frac{\cos \lambda x}{\lambda^2}\, d\lambda$$

3. Use the complex form of the Fourier transform to show that $u = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\lambda) e^{-\frac{(x-\lambda)^2}{4t}}\, d\lambda$ is the solution of the boundary value problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$ and $u = f(x)$ when $t = 0$.

4. Find the temperature $u(x, t)$ in a slab $0 < x < \pi$, initially at temperature $u = 1$, its faces being kept at temperature zero and K is the diffusivity, being taken to be 1.

5. Solve $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, t > 0, 0 \le x \le \pi$, under the conditions $u(x, 0) = 2x$ $u(0, t) = u(\pi, t) = 0$

6. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, t > 0$, subject to the condition $u(x, 0) = e^{-x^2}$

7. Solve $y'' + 3y' + 2y = H(t)\sin \lambda t$, for $t > 0$ satisfying $\lim_{t\to 0^+} y(t) = 0$ and $\lim_{y\to 0^-} y'(t) = 1$. .

8. Solve $y'' + 3y' + 2y = e^{-|t|}$, using Fourier transform.

Answers

1. $u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2} e^{-2\lambda^2 t} \sin \lambda x\, d\lambda$

4. $u(x, t) = \frac{2}{\pi}\sum_{\lambda=1}^{\infty}\left[-\frac{2\pi}{\lambda} e^{-\lambda^2 t} \cos \lambda\pi\right] \sin \lambda\pi$

5. $u(x, t) = -4\int_0^{\infty} \frac{\cos\lambda\pi}{\lambda} e^{-\lambda^2 \pi t} \sin\lambda x \, d\lambda$ 6. $u(x, t) = \frac{2}{\sqrt{(1+4t)}} e^{-x^2/1+4t}$

7. $y(t) = \frac{\lambda e^{-t}}{\lambda^2+1} - \frac{\lambda e^{-2t}}{\lambda^2+4} + \frac{ie^{i\lambda t}}{2(\lambda^2 - 3\lambda i - 2)} - \frac{ie^{-i\lambda t}}{2(\lambda^2 + 3\lambda i - 2)}$

8. $y(t) = \frac{2}{3}e^{-2t} + te^{-t} - \frac{1}{2}e^{-t}$

SUMMARY

Following topics have been discussed in this chapter:

1. Fourier Integral in Complex Form

Since the function cos ($\lambda(t - x)$) is an even of λ, therefore equation (3) can be written in the form of

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\,dt\,d\lambda$$

2. Fourier Transform

Let the function $f(x)$ be defined on $(-\infty, \infty)$ and it is piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$, then the Fourier transform is given by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x} dx (-\infty < \lambda < \infty)$$

then $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda x}\, \lambda x (-\infty < \lambda < \infty)$$

The function $\hat{f}(\lambda)$ in above equation called the Fourier transform of $f(x)$ or some times known the exponential Fourier transform and its integral recovers $f(x)$ from $\hat{f}(\lambda)$ is called inverse Fourier transform.

3. Fourier Cosine and Sine Transform

The Fourier cosine and sine transforms arise as special cases of the Fourier transform, according as $f(x)$ is even or odd. Now, we considering the Fourier cosine transform of function $f(x)$ that can be defined when $f(x)$ is an even function.

$$F_C\{f(x)\} = \hat{f}_C(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\lambda x\,dx$$

The integral given in the equation is called the Fourier cosine transform of $f(x)$.
Now, the inverse Fourier cosine transform is denoted by $f(x) = F_C^{-1}\{\hat{f}_C(\lambda)\}$ and is defined as

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}_C(\lambda)\cos\lambda x\,d\lambda$$

In a similar manner if $f(x)$ is an odd function, then

$$F_S\{f(x)\} = \hat{f}_S(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx$$

The integral in the given equation is called the Fourier sine transform of $f(x)$.
Now, the inverse Fourier sine transform is denoted and defined as

$$f(x) = F_S^{-1}\{\hat{f}_S(\lambda)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_S(\lambda) \sin \lambda x \, d\lambda$$

4. Properties of Fourier Transform

Some properties of Fourier transform as are as follows:

Property 1: *Linearity property* If $\hat{f}_r(\lambda)$ is the Fourier transform of the function $f_r(x)$ for $r = 1, 2, 3, \ldots, n$, then.

$$F\left\{\sum_{r=1}^{n} a_r f_r(x)\right\} = \sum_{r=1}^{n} a_r \hat{f}_r(\lambda)$$

where a_r is constant for $r = 1, 2, 3, \ldots, n$.

Property 2: *Scalar property* If $\hat{f}(\lambda)$ is the Fourier transform of the function $f(x)$, then

$$F\{f(ax)\} = \frac{1}{a} \hat{f}\left(\lambda/a\right)$$

where a is a non-zero constant.

Property 3: *Shifting property* If $\hat{f}(\lambda)$ is the Fourier transform of the function $f(x)$, then

$$F\{f(x-a)\} = e^{i\lambda a} \, \hat{f}(\lambda) \text{ where } a \text{ is constant.}$$

Property 4: *Second shifting property* If $\hat{f}(\lambda)$ is the Fourier transform of the function $f(x)$ and $g(x) = e^{iax} f(x)$. Then $F\{g(x)\} = \hat{f}(\lambda - a)$, where a is any constant.

Property 5: *Modulation property* If $\hat{f}(\lambda)$ is the Fourier transform of the function $f(x)$, then

$$F\{f(x) \cdot \cos ax\} = \frac{1}{2} \hat{f}(\lambda + a) + \hat{f}(\lambda - a)],$$

Property 6: *Fourier transform of the derivative* Let $f(x)$ is a continuous function and $\lim_{x \to 0} f(x) = 0$, also the derivative of $f(x)$ is absolutely integrable then

$$F\{(f')x\} = i\lambda \hat{f}(\lambda), \text{ where } \hat{f}(\lambda) = F\{f(x)\}$$

Property 7: *Convolution theorem* Let $f(x)$ and $g(x)$ be two piece wise continuous, bounded and absolutely integrable functions in $-\infty < x < \infty$ and Fourier transform of $f(x)$ and $g(x)$ are given by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} \, dx \text{ and } \hat{g}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\lambda x} \, dx$$

respectively, then their convolution is denoted and defined as

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Then

(i) $F\{f(x) * g(x)\} = F[f(x)] \cdot F[g(x)] = \hat{f}(\lambda) \cdot \hat{g}(\lambda)$

(ii) $f(x) * g(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\lambda)\hat{g}(\lambda) e^{i\lambda x} d\lambda$ (Inverse Fourier transform)

Property 8: $\hat{f}_s(\lambda) = \displaystyle\int_0^{\infty} f(x) \sin \lambda x\, dx$ and $\hat{f}_C(\lambda) = \displaystyle\int_0^{\infty} f(x) \cos \lambda x\, dx$, then

(i) $F_S\{f(x) \cdot \cos ax\} = \dfrac{1}{2}\left[\hat{f}_S(\lambda + a) + \hat{f}_S(\lambda - a)\right]$

(ii) $F_C\{f(x) \cdot \sin ax\} = \dfrac{1}{2}\left[\hat{f}_S(\lambda + a) - \hat{f}_S(\lambda - a)\right]$

5. Parseval's Theorem

(a) The Fourier transform of $f(x)$ and $g(x)$ are $\hat{f}(\lambda)$ and $\hat{g}(\lambda)$, respectively, then

(i) $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\lambda)|^2\, d\lambda$$

(ii) $$\int_{-\infty}^{\infty} f(x)\, \overline{g}(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda)\, \hat{\overline{g}}(\lambda)\, d\lambda$$

where $\overline{g}(x)$ is the complex conjugate of $g(x)$.

(b) The Fourier sine and cosine transform of the function $f(x)$ and $g(x)$ are $\hat{f}_S(\lambda), \hat{g}_S(\lambda)$ and $\hat{f}_C(\lambda), \hat{g}_C(\lambda)$ respectively, then

(i) $$\int_0^{\infty} f(x)\, g(x)\, dx = \frac{2}{\pi} \int_0^{\infty} \hat{f}_S(\lambda)\, \hat{g}_S(\lambda)\, d\lambda$$

(ii) $$\int_0^{\infty} f(x)\, g(x)\, dx = \frac{2}{\pi} \int_0^{\infty} \hat{f}_C(\lambda)\, \hat{g}_C(\lambda)\, d\lambda$$

(iii) $$\int_0^{\infty} |f(x)|^2\, dx = \frac{2}{\pi} \int_0^{\infty} [f_S(\lambda)]^2\, d\lambda$$

(iv) $$\int_0^{\infty} |f(x)|^2\, dx = \frac{2}{\pi} \int_0^{\infty} [f_C(\lambda)^2]\, d\lambda$$

6. Discrete Fourier Transform

The discrete Fourier transform (DFT) is the transform that deals with a finite number of discrete time signal and a finite or discrete number of frequencies is of form

$$W_r = \frac{2\pi}{N} r; \; r = 0, 1, 2, \ldots N-1$$

Let $f(r)$ be the continuous functions. Let N samples be denoted as $f(0), f(1), f(2), \ldots, f(N-1)$, then the discrete Fourier transform of $f(r)$ is

$$F(\lambda) = \frac{1}{N} \sum_{r=0}^{N-1} f(r) e^{-\frac{2\pi/\lambda r}{N}} \quad \text{for } \lambda = 0, 1, 2, \ldots, N-1$$

and discrete inverse Fourier transform of $F(\lambda)$ is

$$F(r) = \sum_{\lambda=0}^{N-1} F(\lambda) e^{\frac{2\pi/\lambda r}{N}}$$

Discrete 2-D Fourier Transform

The discrete $2D$ Fourier transform of $f(r, s)$ is

$$F(\lambda, \mu) = \frac{1}{MN} \sum_{r=0}^{M-1} \sum_{s=0}^{N-1} f(r, s) e^{-2\pi i\left(\frac{\lambda r}{M} + \frac{\mu s}{N}\right)}$$

for $r = 0, 1, 2, \ldots, M-1$, $s = 0, 1, 2, \ldots, N-1$ and the discrete $2D$ inverse Fourier transform of $F(\lambda, \mu)$ is

$$f(r, s) = \sum_{\lambda=0}^{M-1} \sum_{\mu=0}^{N-1} F(\lambda, \mu) e^{2\pi i\left(\frac{\lambda r}{M} + \frac{\mu s}{N}\right)}$$

for $r = 0, 1, 2, \ldots, M-1$, $s = 0, 1, 2, \ldots, N-1$

Properties of Discrete Fourier Transform

(i) *Linearity Property* Let $F(\lambda)$ and $G(\lambda)$ be the DFT of discrete time signals $f(r)$ and $g(r)$ respectively, then DFT of

$$af(r) + bg(r) = aF(\lambda) + bG(\lambda)$$

(ii) *Time Reversal* Let $F(\lambda)$ be the DFT of periodic discrete time signal $f(r)$, then DFT of $f(r)$ is $F(\lambda)$.

(iii) *Reciprocity* Let $f(\lambda)$ be the DFT of $f(r)$, then the DFT of $F(r)$ is $Nf(-\lambda)$.

7. Finite Fourier Transform

The finite Fourier sine transform of a function $f(x)$ which is sectionally continuous over some finite interval $(0, l)$ of the variable x it is defined as

$$\hat{f}_s(\lambda) = \int_0^t f(x) \sin \frac{\lambda \pi x}{l} dx; \; \lambda \in I$$

Similarly, the finite Fourier cosine transform $f(x)$ in $(0, l)$ is defined as

$$\hat{f}_c(\lambda) = \int_0^t f(x) \cos \frac{\lambda \pi x}{l} dx; \; \lambda \in I$$

In the interval $(0, \pi)$ of the variable x,

$$\hat{f}_s(\lambda) = \int_0^{\pi} f(x) \sin \lambda x \, dx \text{ and } \hat{f}_c(\lambda) = \int_0^{\pi} f(x) \cos \lambda x \, dx$$

8. Inverse Finite Fourier Transform

(i) In the interval $(0, l)$

$$f(x) = \frac{2}{l} \sum_{\lambda=1}^{\infty} \hat{f}_s(\lambda) \sin \frac{\lambda \pi x}{l}$$ is called inverse Finite Fourier sine transform.

and $$f(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{\lambda=1}^{\infty} f_c(\lambda) \cos \frac{\lambda \pi x}{l}$$ is called inverse finite Fourier cosine transform where

$$f_c(0) = \int_0^{l} f(x) \, dx$$

(ii) In the interval $(0, \pi)$
For sine transform

$$f(x) = \frac{2}{\pi} \sum_{\lambda=1}^{\infty} f_s(\lambda) \sin \lambda x$$

For cosine transform

$$f(x) = \frac{1}{\pi} f_c(0) + \frac{2}{\pi} \sum_{\lambda=1}^{\infty} f_c(\lambda) \cos \lambda x$$

where $$f_c(0) = \int_0^{\pi} f(x) \, dx$$

Ordinary Differential Equation with Constant Coefficients

Suppose nth order linear differential equation with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = f(x)$$

Taking Fourier transform both sides of above equation, we have

$$[a_0 (i\lambda)^n + a_1 (i\lambda)^{n-1} + a_2 (i\lambda)^{n-2} + \cdots + a_n] F\{y(x)\} = F\{f(x)\}$$

If $F\{y(x)\}$ exists, then

$$F\{y(x)\} = \frac{F\{f(x)\}}{P(i\lambda)}$$

where $P(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ is the operator on the left hand side of the differential equation.

Let $\dfrac{1}{P(i\lambda)}$ has an inverse Fourier transform

$$g(x) = \frac{1}{2\pi} \int_{-\infty + i\gamma}^{\infty + i\gamma} \frac{e^{i\lambda x}}{P(i\lambda)} \, d\lambda$$

Then using convolution theorem, we get

$$y(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,g(x-t)\,dx.$$

9. Selection of Fourier Sine and Cosine Transform

Suppose we want to remove a term $\frac{\partial^2 u}{\partial x^2}$ in a differential equation, so we require the knowledge of the value.

(i) $(u)_{x=0}$, if we use Fourier sine transform

(ii) $\left(\frac{\partial u}{\partial x}\right)$ if we use Fourier cosine transform

OBJECTIVE TYPE QUESTIONS

1. The function $f(t)$ has the Fourier transform $g(w)$. The Fourier transform

$$F\{f(t)\}=\int_{-\infty}^{\infty} g(t)\,e^{-j\omega t}\,dt \text{ is}$$

(a) $\frac{1}{2\pi}f(w)$ (b) $\frac{1}{2\pi}f(-w)$

(c) $2\pi f(-w)$ (d) None of these

[GATE (ECE) 1997]

2. If the Fourier transform of deterministic signal $g(t)$ is $G(f)$, then

(i) The Fourier transform of $g(t-2)$ is

(a) $G(f)\cdot e^{-j4\pi f}$

(ii) The Fourier transform of $g(t/2)$ is

(b) $G(2f)$

(c) $2G(2f)$

(d) $G(f(-2))$

[GATE (CE) 1997]

3. The Fourier transform of a function $X(t)$ is $X(f)$. The Fourier transform of $\frac{d}{df}X(f)$ will be

(a) $\frac{dX(f)}{df}$ (b) $j2\pi fX(f)$

(c) $jfX(f)$ (d) $\frac{X(f)}{jf}$

[GATE (ECE) 1998]

4. A signal $X(t)$ has a Fourier transform $X(w)$. If $X(t)$ is a real and odd function of t then $X(w)$ is

(a) A real and even function of w

(b) An imaginary and odd function of w

(c) An imaginary and even function of w

(d) A real and odd function of w

[GATE – (ECE) 1999]

5. The Fourier transform of the signal $X(t)=e^{-3t^2}$ is of the following form, where A and B are constants:

(a) $A\,e^{-B|f|}$ (b) $A\,e^{-Bf}$

(c) $A+B|f|^2$ (d) Ae^{-Bf^2}

[GATE(ECE) 2000]

6. The complex Fourier transform of dirac delta function $\delta\{x-a\}$ is

(a) 2 (b) –1

(c) 0 (d) e^{isa}

[GATE (ECE) 2002]

7. The Fourier transform of a conjugate symmetric function is always

(a) imaginary

(b) conjugate anti-symmetric

(c) real

(d) conjugate symmetric

[GATE (EC) 2004]

8. Let $X(n) = \left(\frac{1}{2}\right)^n u(n); Y(n) = X^2(n)$, and $Y(e^{jw})$ be the Fourier transform of $y(n)$. Then $Y(e^{j0})$ is

(a) 1/4 (b) 2
(c) 4 (d) 4/3

[GATE (EC) 2005]

9. Let $X(t) = \text{rect}\left(t - \frac{1}{2}\right)$, where rect (x) = 1 for $-\frac{1}{2} \le x \le \frac{1}{2}$ and = 0, otherwise

Then if $\text{Sinc}(x)\frac{\sin \pi x}{\pi x}$, the Fourier transform of $X(t) + X(-t)$ is

(a) $\text{Sinc}\left(\frac{\omega}{2\pi}\right)$

(b) $2\ \text{Sinc}\left(\frac{\omega}{2\pi}\right)$

(c) $2\ \text{Sinc}\left(\frac{\omega}{2\pi}\right) \cdot \cos\left(\frac{w}{2}\right)$

(d) $\text{Sinc}\left(\frac{\omega}{2\pi}\right) \cdot \sin\left(\frac{w}{2}\right)$

[GATE (EE) 2008]

10. The Fourier transform of $X(t) = e^{-at}u(-t)$, where $u(t)$ is the unit step function

(a) Exists for any real value of a
(b) Does not exist for any real value of a
(c) Exists if the real value of a is strictly –ve
(d) Exists if the real value of a is strictly +ve

[GATE (IN) 2008]

11. The 4-point discrete Fourier transform of a discrete time sequence {1, 0, 2, 3} is

(a) [0, –2 + 2j, 2, – 2 – 2j]
(b) [2, 2 + 2j, 6, 2 – 2j]
(c) [6, 1 – 3j, 2, 1 + 3j]
(d) [6, –1 + 3j, 0, –1 – 3j]

[GATE (EC) 2009]

12. Consider the signal $X(t) = \begin{cases} e^{-t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$. Let $X(w)$ denote the Fourier transform of this signal. The integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, d\omega$ is

(a) 0 (b) 1/2
(c) 1 (d) ∞

[GATE (IN) 2001]

13. The Fourier transform of a signal $h(t)$ is $H(jw)$ = (2 cos w) (sin 2 w)/w. The value of $h(0)$ is

(a) 1/4 (b) 1/2
(c) 1 (d) 2

[GATE (EE & IN) 2012]

14. Let $g(t) = e^{-\pi t^2}$ and $h(t)$ j is a filter matched to $g(t)$. If $g(t)$ is applied as input to $h(t)$ then the Fourier transform of the output is

(a) $e^{-\pi f^2}$ (b) $e^{-\pi f^2 c}$
(c) $e^{-\pi|f|}$ (d) $e^{-2\pi f^2}$

[GATE (EC) 2013]

15. Let $f(t)$ be a continuous time signal and let $F(w)$ be its Fourier transform; defined by $F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt$ and $g(t)$ defined by

$$g(t) = \int_{-\infty}^{\infty} F(u)\, e^{-jut} du$$

What is the relation ship between $f(t)$ and $g(t)$?

(a) $g(t)$ would always be proportional to $f(t)$
(b) $g(t)$ would be proportional to $f(t)$, if $f(t)$ is an even function
(c) $g(t)$ would be proportional to $f(t)$ only if $f(t)$ is a sinusoidal function
(d) $g(t)$ would never be proportional to $f(t)$

[GATE (EE) 2014]

16. A signal is represented by

$$X(t) = \begin{cases} 1 & \text{if } |t| > 1 \\ -1 & \text{if } |t| < 1 \end{cases}$$

The Fourier transform of the convolved signal $y(t) = X(2t) * X\left(\frac{t}{2}\right)$ is

(a) $\frac{4}{w^2}\sin\left(\frac{w}{2}\right) \cdot \sin(2w)$

(b) $\frac{4}{w^2}\sin\left(\frac{w}{2}\right)$

(c) $\frac{4}{w^2}\sin(2w)$

(d) $\dfrac{4}{w^2}\sin^2 w$

[GATE (EE) 2014]

17. If $X(K)$ is the discrete Fourier transform of a 6-point real sequence $X(n)$. If $X(0) = 9 + j0$, $X(2) = 2 + j2$, $X(3) = 3 - j0$, $X(5) = 1 - j1$, then $X(0)$ is

(a) 3 (b) 9
(c) 15 (d) 18

[GATE (IN) 2014]

18. If $F\{f(x)\} = F(\lambda)$, then $F\{f(x-a)\}$ is

(a) $e^{i\lambda a} F(\lambda)$ (b) $e^{-i\lambda a} F(\lambda)$
(c) $e^{\lambda a} F(\lambda)$ (d) zero

19. If $F_C\{f(ax)\} = K\, F_C\{\lambda/a\}$, then K is

(a) $1/a$ (b) $a/2$
(c) $-1/a$ (d) λ/a

20. Fourier sine transform of $1/x$ is

(a) $\lambda/2$ (b) $\lambda^2/2$
(c) $\lambda^2/3$ (d) $\lambda^3/2$

21. If $F_S\{f(x)\} = F_S(\lambda)$, then $f(x)$ is

(a) $\dfrac{2}{\pi}\displaystyle\int_0^\infty F_S(\lambda)\sin\lambda x\, d\lambda$

(b) $\dfrac{2}{\pi}\displaystyle\int_0^1 F_S(\lambda)\sin\lambda x\, d\lambda$

(c) $\dfrac{2}{\pi}\displaystyle\int_0^\infty F_S(\lambda)\cos\lambda x\, d\lambda$

(d) $\dfrac{2}{\pi}\displaystyle\int_{-\infty}^\infty F_S(\lambda)\sin\lambda x\, d\lambda$

22. If $F_C\{f(x)\} = F_C(\lambda)$, then $f(x)$ is

(a) $\dfrac{2}{\pi}\displaystyle\int_0^\infty F_c(\lambda)\cos\lambda x d\lambda$

(b) $\displaystyle\int_{-\infty}^\infty F_C(\lambda)\cos\lambda x\, d\lambda$

(c) $\dfrac{2}{\pi}\displaystyle\int_0^\infty F_C(\lambda)\cos\lambda x\, dx$

(d) $\dfrac{2}{\pi}\displaystyle\int_0^\infty f(x)\cos\lambda x\, dx$

23. The Fourier transform of $x^n f(x)$ is

(a) $\displaystyle\int_0^\infty t^n f(t)e^{i\lambda t}dt$ (b) $\displaystyle\int_{-\infty}^\infty t^n f(t)e^{i\lambda t}dt$

(c) $\displaystyle\int_{-\infty}^\infty f(t)e^{-i\lambda t}dt$ (d) none of these

24. The Fourier cosine transform of $f(x)$ is

(a) $\displaystyle\int_0^\infty f(x)\sin\lambda x\, dx$

(b) $\displaystyle\int_0^\infty f(x)\cos\lambda x\, dx$

(c) $\displaystyle\int_{-\infty}^\infty f(x)\cos\lambda x\, dx$

(d) $\displaystyle\int_0^\infty f(x)\cos x\, dx$

25. If $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$, then $F_S\{f(x)\}$ is

(a) $\dfrac{\{1+(-1)^\lambda\}}{\lambda}$ (b) $\dfrac{1}{\lambda}(1-\cos\lambda\pi)$

(c) $\dfrac{1}{\lambda}(1-\sin\lambda)$ (d) $\dfrac{1}{\lambda(1+\cos\lambda\pi)}$

ANSWERS

1. (c)	2. (i) a (ii) c		3. (c)	4. (d)	5. (d)	6. (d)	7. (c)	8. (a)	9. (b)
10. (a)	11. (d)	12. (d)	13. (c)	14. (d)	15. (b)	16. (a)	17. (a)	18. (a)	19. (a)
20. (b)	21. (a)	22. (a)	23. (b)	24. (b)	25. (b)				

2 Z-Transforms

2.1 INTRODUCTION

The Z-transform plays an important role in the field of science and engineering such as radar detection, signal processing circuit, coding theory, etc. The Z-transform is simply a power series representation of a discrete-time sequence and it is an essential mathematical tool which is used for the analysis and design of discrete-time control systems. Basically, the Z-transform is a discrete-time counter part of the Laplace transform. The properties of the Z-transform is similar to the properties of the Laplace transform except the difference between the discrete-time signals and continuous time functions. The application of Z-transform in discrete-time systems is similar to that of the Laplace transform is continuous time systems.

In this chapter, we shall discuss the fundamentals of Z-transform, properties and application to solve difference equations.

Pierre-Simon Laplace was born on 23 March 1749. He was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy and statistics. He formulated Laplace's equation, and pioneered the Laplace transform which appears in many branches of mathematical physics. The Laplacian differential operator, widely used in mathematics, is also named after him. He restated and developed the nebular hypothesis of the origin of the solar system and was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse and he died on 5 March 1827.

The basic idea of the Z-transform first given by Laplace, and reintroduced in 1947 by W. Hurewicz as a tractable way to solve linear, constant-coefficient difference equations. The modified or advanced Z-transform was later developed and popularized by EI Jury.

2.2 BASIC CONCEPT OF SEQUENCE

***Sequence*:** A function whose domain is a set of natural numbers N and range, a subset of real number R is called a sequence.

Let $f(n)$ be an element of a sequence S and it is defined as

$$S = \{f(n)\};\ -\infty < n < \infty, \text{ where } n \text{ is an integer.}$$

2.2.1 *Z*-Transform for Discrete Values of *t*

Let $f(t)$ is a function defined for discrete values of t, where $t = nT$, $n = 0, 1, 2, \ldots, T$ being the period of sampling, then Z-transform of $f(t)$ is defined as

$$Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)z^{-n} = F(z)$$

2.2.2 Some Basic Discrete Time Signals Sequences and Their Graphical Representation

1. *Unit Impulse Sequence*: The unit impulse sequence is defined by

$$\delta[n] = \begin{cases} 1; & \text{for } n = 0 \\ 0; & \text{for } n \neq 0 \end{cases}$$

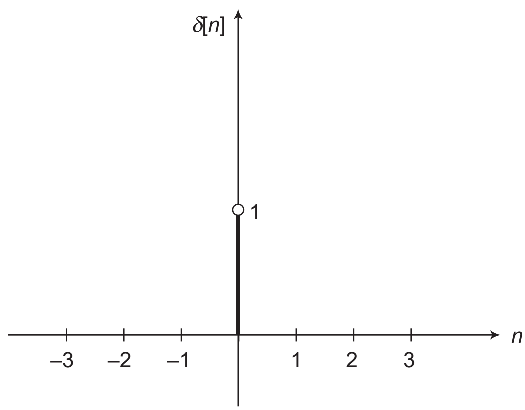


Fig. 2.1

2. *Unit-Step Sequence*: The unit-step sequence is defined by

$$u[n] = \begin{cases} 1; & \text{for } n \geq 0 \\ 0; & \text{for } n < 0 \end{cases}$$

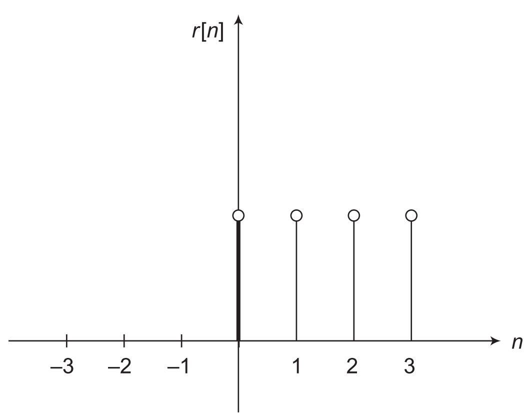


Fig. 2.2

3. *Sinusoidal Sequence*: It is defined as

$$S[n] = A \cos (w_0 n + \phi)$$

where A is the amplitude, w_0 is the angular frequency and ϕ is the phase of $S[n]$. The graphical representation in the Fig. 2.3.

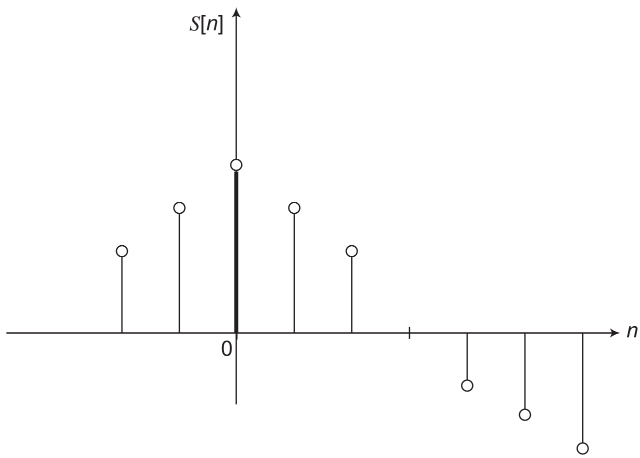


Fig. 2.3

4. *Unit-Ramp Sequence*: It is defined as

$$r[n] = \begin{cases} n; & \text{for } n \geq 0 \\ 0; & \text{for } n < 0 \end{cases}$$

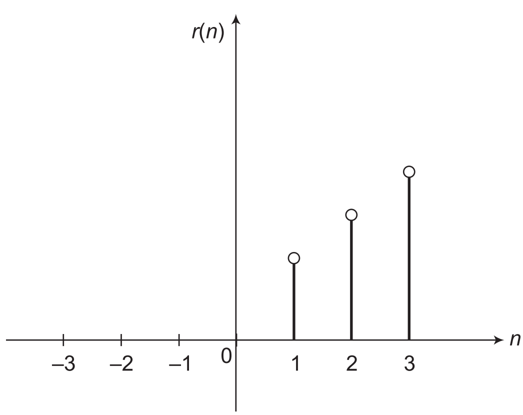


Fig. 2.4

2.3 *Z*-TRANSFORM

The *Z*-transform of a given discrete sequence $f[n]$ is denoted by $F(z)$ or $Z\{f(n)\}$ and is defined as

$$Z\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n)\cdot z^{-n}, \tag{1}$$

where z is a continuous complex variable and $Z\{\cdot\}$ is the *Z*-transform operator.

Equation (1) represents the bilateral *Z*-transform.

The unilateral *Z*-transform of a given sequence $f[n]$ is defined as

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)\cdot z^{-n} \tag{2}$$

2.3.1 Geometrical Representation

In the *Z*-transform, the complex variable z can be expressed as

$$z = \text{Re}(z) + \text{Im}\,(z),$$

where Re(z) is the real part of z and Im(z) is the imaginary part of z

$$\therefore \qquad z = re^{i\theta} = r\cos\theta + ir\sin\theta$$

Hence, $\quad \text{Re}(z) = r\cos\theta,\ \text{Im}(z) = r\sin\theta.$

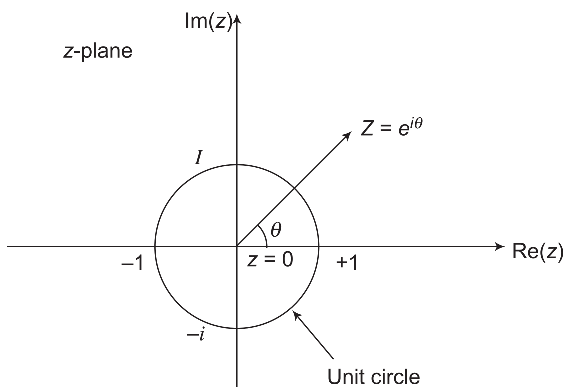


Fig. 2.5 Complex number *z* in complex plane

In Fig. 2.5 the position of the point i.e., z is represent in the complex *Z*-plane. If modulus of z, i.e., $|z| = 1$, then $r = 1$.

The *Z*-transform of $f[n]$ is

$Z\{f[n]\} = F(e^{i\theta})$ which is the discrete-time Fourier transform of $f[n]$.

Example 1 Find the *Z*-transform of a unit step sequence $u[n]$ is defined as

$$u[n] = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

Solution By the definition of Z-transform

$$F(z) = Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

$$\therefore \qquad Z\{u(n)\} = U(z) = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}; \text{ Provided } |z| > 1.$$

Example 2 Find the Z-transform of a polynomial function defined by

$$P[n] = \begin{cases} a^n; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

Solution The Z-transform of a function $P[n]$ is

$$Z\{P[n]\} = P(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= 1 + (a z^{-1}) + (a z^{-1})^2 + \cdots$$

$$= \frac{1}{1-(a z^{-1})}$$

$$= \frac{z}{z-a} \qquad \text{for } |z| > a$$

Example 3 Find the Z-transform of the exponential function. $E[n] = \begin{cases} e^{-anT}; & n \geq 0 \\ 0; & n < 0 \end{cases}$.

Solution The Z-transform of a function $E[n]$ is

$$Z\{E[n]\} = E(z) = \sum_{n=0}^{\infty} E[n] z^{-n} = \sum_{n=0}^{\infty} e^{-anT} z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{aT} z)^{-n}$$

or $$E(z) = \frac{1}{1-(e^{aT} z)^{-1}}$$

$$= \frac{z}{z-e^{-aT}} \qquad \text{for } |z| > e^{-aT}$$

Example 4 Find the Z-transform of sinusoidal function $S(n) = \begin{cases} \sin(\omega nT); & n \geq 0 \\ 0; & n < 0 \end{cases}$.

Solution The Z-transform of $S(n)$ is

$$Z\{S(n)\} = S(z) = \sum_{n=0}^{\infty} \sin(\omega nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{e^{i\omega nT} - e^{-i\omega nT}}{2i} \cdot z^{-n}$$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty} [e^{i\omega nT} - e^{-i\omega nT}\right] z^{-n}$$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty} [e^{i\omega nT} z^{-1})^n - \sum_{n=0}^{\infty} (e^{i\omega T} z)^{-n}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{1-e^{i\omega T} z^{-1}} - \frac{1}{1-e^{-i\omega T} z^{-1}}\right]$$

$$= \frac{1}{2i}\left[\frac{(e^{i\omega T} - e^{-i\omega T}) z^{-1}}{1-(e^{i\omega T} + e^{-i\omega T}) z^{-1} + z^{-2}}\right]$$

or
$$S(z) = \frac{z^{-1}\sin(\omega T)}{1-2z^{-1}\cos(\omega T) + z^{-2}}$$

$$= \frac{z\sin(\omega T)}{z^2 - 2z\cos(\omega T) + 1} \quad \text{for } |z| > 1.$$

Example 5 Find the Z-transform for the finite length sequence.

$$f(n) = \begin{cases} a^n; & \text{for } M \leq n \leq N-1 \\ 0; & \text{otherwise} \end{cases}$$

Solution The Z-transform of $f(n)$ is

$$Z\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}$$

$$= \sum_{n=M}^{N-1} a^n z^{-n}$$

$$= \sum_{n=M}^{N-1} (az^{-1})^n$$

$$= \frac{(az^{-1})^M \left[1-(az^{-1})^{N-M}\right]}{1-az^{-1}}$$

$$= \frac{a^M z^{-M} - a^N z^{-N}}{1 - az^{-1}}$$

2.3.2 Region of Convergence (ROC)

The Z-transform does not converge for all sequences or for all values of z. The set of values of z for which the Z-transform converges is called the region of convergence.

The Z-transform of $f[n]$ exists if the sum $\sum_{n=-\infty}^{\infty} |f(n)|$ converges. However, the Z-transform of $f(z)$, i.e., the discrete time Z-transform of the sequence $f(n)\, r^{-n}$ exists (or converges) if

$$\sum_{n=-\infty}^{\infty} |f(n) r^{-n}| < \infty, \quad \text{where } z = r\,e^{i\theta}$$

This $\Rightarrow$ $$\sum_{n=-\infty}^{\infty} |f(x)||z|^{-n} < \infty \text{ for the existence of the } Z\text{-transform.}$$

Example 6 Find the Z-transform and region of convergence for the right-sided exponential sequence.

Solution Consider $f[n] = a^n u[n]$, where $u[n]$ is unit step sequence.

The Z-transform of $f[n]$ is

$$F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= 1 + (az^{-1}) + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

$$= \frac{1}{1 - (az^{-1})}$$

$$F(z) = \frac{z}{z - a}$$

For the convergence

$$\sum_{n=0}^{\infty} |az^{-1}| < \infty$$

which is only the case if $|az^{-1}| < 1$ or $|z| > |a|$.

Hence, in the ROC, $F(z) = \dfrac{z}{z-a}; \quad |z| > |a|$

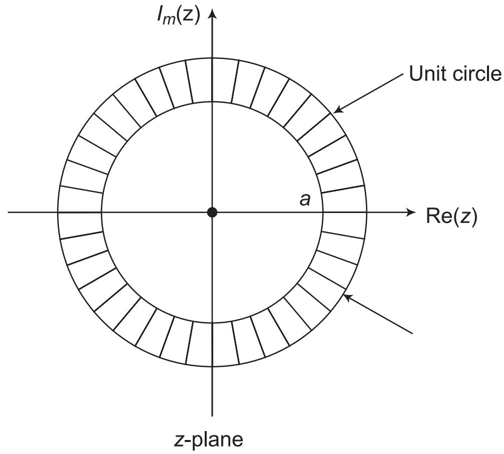


Fig. 2.6

***Note*:** The Z-transform of $f[n]$ only exists if the ROC includes the circle of unit radius, when $|a| < 1$ or $|a| > 1$, then the ROC does not include the unit circle, and the Z-transform does not exist.

Example 7 Find the Z-transform for the sum of two exponentials.

Solution The sum of two exponentials is

$$f[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n], \tag{3}$$

where unit step sequence $u[n] = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$

The Z-transform of sequence in Eq. (3) is

$$\begin{aligned} F(z) &= \sum_{n=-\infty}^{\infty} f[n]\, z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right) u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n} \end{aligned}$$

or $$F(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 - \left(-\frac{1}{3}\right) z^{-1}}.$$

From the above problem, the first term is converges for $|z| > \frac{1}{2}$ and the second term for $|z| > \frac{1}{3}$. $F(z)$ is converges in the intersection of these regions.

$$\therefore \qquad F(z) = \frac{2 - z^{-1}/6}{\left(1 - \frac{z^{-1}}{2}\right)\left(1 + \frac{z^{-1}}{3}\right)} \qquad \text{for } |z| > \frac{1}{3}$$

$$= \frac{2Z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)} \qquad \text{for } |z| > \frac{1}{3}$$

EXERCISE 2.1

1. Find the Z-transform of the following:

   (i) $\frac{1}{n}$ (ii) n^2 (iii) $\cos\frac{n\pi}{2}$

   (iv) $n(n-1)$ (v) $u(n-1)$ (vi) $a^n \cos n\pi$

2. Find $Z\{f(n)\}$, where the sequence $f(n)$ is defined as

   (i) $f(n) = \begin{cases} 2 & \text{if } n = 0, 2, 4, ..., 2k, ... \\ 1 & \text{if } n = 1, 3, 5, ..., 2k+1, ... \end{cases}$

   (ii) $f(n) = 1, n \geq 1$ and $f(0) = 0$

   (iii) $f(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 3\}; 0 \leq n \leq 3$

3. Find the Z-transform of

   (i) $e^n \cos(n\alpha), n = 0$

   (ii) $e^t \sin 2t$

Answers

1. (i) $\log\left(\frac{z}{z-1}\right), |z| > 1$ (ii) $\frac{z(z+1)}{(z-1)^3}$

   (iii) $\frac{z^2}{z^2+1}, \ |z| > 1$ (iv) $\frac{2z}{(z-1)^3}; |z| > 1$

   (v) $\frac{1}{z-1}, |z| > 1$ (vi) $\frac{z}{z+a}, |z| > |a|$

2. (i) $\frac{2z^2 + z}{z^2 - 1}$ (ii) $\frac{1}{z-1}$ (iii) $z^{-2} + 2z^{-3} + 3z^{-4}$

3. (i) $\frac{z(z - e\cos\alpha)}{z^2 - 2ez\cos\alpha + e^2}$ (ii) $\frac{ez\sin 2}{z^2 - 2ez\cos 2 + e^2}$

2.4 THE INVERSE Z-TRANSFORM

The process of inverse Z-transform is the reverse process of Z-transform. The inverse Z-transform is given by a complex integral.

$$f[n] = Z^{-1}\{F(z)\} = \frac{1}{2\pi i}\oint_C F(z)\cdot z^{n-1}dz, \tag{4}$$

where C is a simple closed curve enclosing the origin and lying outside the circle $|z| = r$.

In this section, we shall discuss some methods for finding the inverse Z-transform.

2.4.1 The Cauchy Residue Theorem

Example 8 Find the inverse Z-transform of the function $F(z) = \dfrac{z}{(z-1)(z-2)}$, using Residue theorem.

Solution Given $\left[F(z) = \dfrac{z}{(z-1)(z-2)}\right]$

The given function has the poles at $z = 1$ and $z = 2$. The inverse Z-transform of $F(z)$, using Eq. (4) is

$$f(n) = \frac{1}{2\pi i}\oint_C \frac{z}{(z-1)(z-2)} z^{n-1}dz$$

$$f(n) = \frac{1}{2\pi i}\cdot 2\pi i \text{ (sum of the residue of the integral)}$$

$$= \sum\left[(z-z_i)\, F(z)\, z^{n-1}\right]_{z=z_i}$$

$$= \sum\left[(z-z_i)\cdot\frac{z}{(z-1)(z-2)}\cdot z^{n-1}\right]_{z=z_i}$$

$$= \left[(z-z_1)\cdot\frac{z}{(z-1)(z-2)} z^{n-1}\right]_{z=z_1} + \left[\frac{(z-z_2)\cdot z}{(z-1)\,(z-2)} z^{n-1}\right]_{z=z_2}$$

$$= \left[\frac{z}{z-2} z^{n-1}\right]_{z=1} + \left[\frac{z}{(z-1)} z^{n-1}\right]_{z=2} \quad [\because\ z_1 = 1, z_2 = 2]$$

$$= -(1)^{n-1} + 2(2)^{n-1}$$

$$= (2^n - 1)$$

Example 9 Find the inverse Z-transform of the function $F(z) = \dfrac{z}{(z-2)\,(z-4)}$, using Residue theorem.

Solution Given $F(z) = \dfrac{z}{(z-2)\,(z-4)}$

The given function has the poles $z = 2$ and $z = 4$.

$\therefore$ The inverse Z-transform of $F(z)$ is

$$f(n) = \frac{1}{2\pi i}\oint_C F(z)\, z^{n-1}\, dz$$

$$= \frac{1}{2\pi i}\cdot(2\,\pi i\cdot \text{ sum of the residue of the integral})$$

$$= \sum\left[(z-z_i)\,F(z)\,z^{n-1}\right]_{z=z_i}$$

$$= \left[\frac{z}{z-2}\,z^{n-1}\right]_{z=4} + \left[\frac{z}{z-4}\cdot z^{n-1}\right]_{z=2}$$

$$= \frac{4}{2}(4)^{n-1} + \frac{2}{-2}(2)^{n-1}$$

$$= \frac{1}{2}(4)^n - \frac{1}{2}(2)^n$$

$$= \frac{1}{2}\left[(4)^n - (2)^n\right]$$

2.4.2 Using Partial Fraction Method

Example 10 Find the inverse Z-transform of

$$F(z) = \frac{3z^2 - z}{(z-1)\,(z-2)^2}, \text{ using partial fraction method.}$$

Solution We write $F(z)$ as partial fractions

$$F(z) = \frac{3z^2-2}{(z-1)(z-2)^2} = 2\cdot\frac{z}{z-1} - 2\cdot\frac{z}{z-2} + \frac{5}{2}\cdot\frac{2z}{(z-2)^2}$$

so that its inverse Z-transform.

$$f(n) = Z^{-1}[F(z)]$$

$$= Z^{-1}\left\{2\cdot\frac{z}{z-1} - 2\cdot\frac{z}{z-2} + \frac{5}{2}\frac{2z}{(z-2)^2}\right\}$$

$$= 2Z^{-1}\left\{\frac{z}{z-1}\right\} - 2Z^{-1}\left\{\frac{z}{z-1}\right\} + \frac{5}{2}Z^{-1}\left\{\frac{2z}{(z-2)^2}\right\}$$

$$= 2\cdot 1 - 2\cdot 2^n + \frac{5}{2}\cdot n\,2^n \qquad \left[\begin{array}{l} \because\ Z^{-1}\left\{\dfrac{z}{z-a}\right\} = a^n \\ Z^{-1}\left\{\dfrac{az}{(z-a)^2}\right\} = na^n \end{array}\right]$$

$$= 2 - 2^{n+1} + 5n2^{n-1}$$

2.4.3 Power Series Expansion

If the Z-transform of a sequence $f(n)$ is given as a power series in the form

$$F(z) = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}$$

$$= \cdots + f(-2)\,z^{2} + f(-1)\,z^{-1} + f(0) + f(1)\,z^{-1} + f(2)\,z^{-2} + \cdots$$

then any value in the sequence can be obtain in the coefficient of the appropriate power of Z^{-1}.

Example 11 Find inverse Z-transform of

$F(z) = \log(1 + az^{-1})$, $|z| > |a|$, by power series expansion.

Solution Here

$$F(z) = \log(1 + az^{-1})$$

$$\therefore \qquad F(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

[Using the power series expansion of $\log(1 + z)$ in $|z| < 1$.]

Therefore, the required sequence is

$$f(n) = \begin{cases} \dfrac{(-1)^{n+1} a^n}{n}; & n \geq 1 \\ 0; & n \leq 0 \end{cases}$$

Example 12 Find the inverse Z-transform of $F(z) = \dfrac{z}{(z+1)^2}, |z| < 1$ by power series method.

Solution Here,

$$F(z) = \frac{z}{(z+1)^2}$$

$$= \frac{z}{z^2(1+z^{-1})^2}$$

$$= z^{-1}(1+z^{-1})^{-2}$$

$$= z^{-1}(1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + 5z^{-4} - \cdots)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n\, z^{-n}$$

Therefore, the required sequence is

$$f(n) = (-1)^{n-1} n$$

Example 13 Find the inverse Z-transform of $F(z) = z(z - a)^{-1}$ by power series expansion.

Solution Here,

$$F(z) = z(z - a)^{-1}$$

$$= z z^{-1}(1 - a z^{-1})^{-1}$$

$$= (1 - a z^{-1})^{-1}$$

or $$F(z) = 1 + a z^{-1} + a^2 z^{-2} + \cdots$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

Therefore, the required sequence is

$$f(n) = a^n.$$

EXERCISE 2.2

1. Find the inverse Z-transform of $F(z) = \dfrac{z}{z^2 - 5z + 6}$.
2. Using convolution theorem, find
$$Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\}.$$
3. If $Z^{-1}\left\{\dfrac{z}{z+1}\right\} = (-1)^n$, then find $Z^{-1}\left\{\dfrac{1}{z+1}\right\}$.
4. Find $f(n)$ if $F(z) = \dfrac{3}{3z-1}$.
5. Find $Z^{-1}\left\{\dfrac{8\,z^2}{(2z-1)\,(4z-1)}\right\}$
6. Using residue method to find.

   (i) $z^{-1}\left\{\dfrac{z}{z^2 + 7z + 10}\right\}$ (ii) $z^{-1}\left\{\dfrac{z(z^2-1)}{(z^2+1)^2}\right\}$
7. If $F(z) = \dfrac{z}{z - e^{-T}}$, find $\lim\limits_{t\to\infty} f(t)$
8. Using power series method to find $Z^{-1}\left\{\log\left(\dfrac{z}{z+1}\right)\right\}$.

Answers

1. $3^n - 2^n$
2. $\dfrac{1}{2}(3^{n+1} - 1)$
3. $(-1)^{n-1}, n = 1, 2, 3, \ldots$
4. $\left(\dfrac{1}{3}\right)^{n-1}$ or $\left(\dfrac{1}{3}\right)^{n-1} u(n-1)$

5. $2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n, n = 0,1,2,..$

6. (i) $\frac{1}{3}\{(-2)^n - (-5)^n\}$ (ii) $\frac{n}{2}\{(i)^{n-1} + (-i)^{n-1}\}, n = 0,1,2,...$

7. 0

8. $f(n) = \begin{cases} 0, & \text{for } n = 0 \\ \frac{(-1)^n}{n}, & \text{otherwise} \end{cases}$

2.5 PROPERTIES OF THE Z-TRANSFORM

In this section, if $F(z)$ denotes the Z-transform of a sequence $f(n)$ and all the ROC of $F(z)$ is denoted by R_x, then this relationship is defined as

$$f(n) \xleftrightarrow{z} F(z), \text{ROC} = R_x$$

Property 1 (*Linearty*) If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$, then

$$Z\{af(n) + bg(n)\} = aZ\{f(n)\} + bZ\{g(n)\}$$
$$= aF(z) + bG(z)$$

Proof: $Z\{af(n) + bg(n)\} = \sum_{n=0}^{\infty} \{a\, f(n) + b\, g(n)\}\, z^{-n}$

$$= a \sum_{n=0}^{\infty} f(n)\, z^{-n} + b \sum_{n=0}^{\infty} g(n)\, z^{-n}$$
$$= a\, Z\{f(n)\} + bZ\{g(n)\}$$
$$= a\, F(z) + bG(z)$$

Hence proved.

Property 2 (*Shifting property*) Let the Z-transform of a casual sequence $f(n)$ is $F(z)$, i.e., $Z\{f(n)\} = F(z)$ and m is any positive integer then

$$Z\{f(n-m)\} = z^{-m} F(z);\ n \geq m \text{ (shifting to the right)}$$

***Proof*:** We have

$$Z\{f(n-m\} = \sum_{n=0}^{\infty} f(n-m)\, z^{-n}$$

putting $n - m = l$, we get

$$Z\{f(n-m)\} = \sum_{l=-m}^{\infty} f(l) \cdot z^{-m-l}$$
$$= \sum_{l=0}^{\infty} f(l)\, z^{-l} \cdot z^{-m}$$
$$= z^{-m}\, F(z)$$

Hence, proved.

Note: $Z\{f(n+m)\} = z^m \left[F(z) - \sum_{n=0}^{m-1} f(n)\, z^{-n} \right]$ (shifting to the left).

Property 3 (*Scaling in the Z-domain*) Let $F(z)$ be the Z-transform of $f(n)$. Then

$$Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$$

Proof: We know that

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n} = F(z)$$

$$\therefore \quad F\left(\frac{z}{a}\right) = \sum_{n=-\infty}^{\infty} f(n) \cdot \left(\frac{z}{a}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} f(n) \cdot \frac{z^{-n}}{a^{-n}}$$

$$= \sum_{n=-\infty}^{\infty} f(n) \cdot a^n z^{-n}$$

$$F\left(\frac{z}{a}\right) = Z\{a^n \cdot f(n)\}$$ **Hence, proved.**

Property 4 (*Time Reversal*) Let $F(z)$ be the Z-transform of $f(n)$. Then

$$Z\{f(-n)\} = F\left(\frac{1}{z}\right)$$

where $f(-n)$ represents mirror image of the signal $f(n)$.

Proof: We have

$$Z\{f(-n)\} = \sum_{n=-\infty}^{\infty} f(-n)\, z^{-n}$$

Putting $n = -k$, we get

$$Z\{f(-n) = \sum_{n=-\infty}^{\infty} f(k)\, z^k = \sum_{n=-\infty}^{\infty} f(k) \cdot \left(\frac{1}{z}\right)^{-k} = F\left(\frac{1}{z}\right)$$ **Hence, proved.**

Property 5 (*First Shifting Theorem*) If $Z\{f(t)\} = F(z)$, then $Z\{e^{-at} f(t)\} = F(ze^{aT})$.

Proof: We have

$$Z\{e^{-at} f(t)\} = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\, (z\, e^{aT})^{-n}$$

$$= F\,(ze^{aT})$$ **Hence, proved.**

Property 6 (*Second Shifting Theorem*) If $Z\{f(t)\} = F(z)$, then $Z\{f(t+T)\} = z[F(z) - f(0)]$.

Proof: We have

$$Z\{f(t+T)\} = \sum_{n=0}^{\infty} f(nT+T)\,z^{-n}$$

$$= z\sum_{n=0}^{\infty} f[(n+1)T]\cdot z^{-(n+1)}$$

$$= z\sum_{k=1}^{\infty} f(kT)\,z^{-k}$$

$$= z\left[\sum_{k=0}^{\infty} f(kT)z^{-k} - f(0)\right]$$

$$= z\big[F(z) - f(0)\big]$$ **Hence, proved.**

Property 7 (*Initial Value Theorem*) If $Z\{f(t)\} = F(z)$, then $\lim\limits_{z\to\infty} F(z) = f(0)$.

Proof: We have

$$F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\,z^{-n}$$

$$= f(0.T) + f(1.T)\,z^{-1} + f(2\cdot T)\,z^{-2} + \cdots\infty$$

$$= f(0) + \frac{1}{z}f(T) + \frac{1}{z^2}f(2T) + \cdots\infty$$

$\therefore$ $$\lim_{z\to\infty} F(z) = f(0)$$ **Hence, proved.**

Note: If $f(0) = 0$, then $f(1) = \lim\limits_{z\to\infty} z\,f(z)$.

Property 8 (*Final Value Theorem*) If $Z\{f(t)\} = F(z)$, then $\lim\limits_{t\to\infty} f(t) = \lim\limits_{z\to 1}(z-1)\,F(z)$.

Proof: We have

$$Z\{f(t+T) - f(t)\} = \sum_{n=0}^{\infty}\big[f(nT+T) - f(nT)\big]z^{-n}$$

$$Z\{f(t+T)\} - Z\{f(t)\} = \sum_{n=0}^{\infty}\big[f(nT+T) - f(nT)\big]z^{-n}$$

or $$z\,F(z) - z\,f(0) - F(z) = \sum_{n=0}^{\infty}\big[f(nT+T) - f(nT)\big]z^{-n}\quad \text{[By Property (6)]}$$

$$\lim_{Z\to 1}\big[(z-1)\,F(z) - z\,f(0)\big] = \lim_{z\to 1}\left[\sum_{n=0}^{\infty}\big[f(nT+T) - f(nT)\big]z^{-n}\right]$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right]-f(0)=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]$$

$$=\lim_{n\to\infty}\left[f(T)-f(0)+f(2T)-f(T)+\cdots+f\{(n+1)T\}-f(nT)\right]$$

$$=\lim_{n\to\infty} f\{(n+1)T-f(0)\}$$

$$=f(\infty)-f(0)$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right]-f(0)=\lim_{t\to\infty} f(t)-f(0)$$

or $$\lim_{z\to 1}\left[(z-1)\,F(z)\right]=\lim_{t\to\infty} f(t)$$

Property 9 (*Differentiation in the Z-domain*) Let *Z*-transform of a sequence $f(n)$

$$Z\{f(n)\}=F(z) \text{ exist in the region } |z|>\frac{1}{R}$$

where *R* is ROC. Then

$$Z\{nf(n)\}=-z\frac{d}{dz}\left[F(z)\right]$$

which is also convergent in the region $|z|>\frac{1}{R}$.

Note: $Z\left[\binom{n}{K}a^n u[n]\right]=\dfrac{a_z^K}{(z-a)^{K+1}}$; $K=0,1,2,...,$ and $|z|>|a|$

2.6 CONVOLUTION OF SEQUENCES

Let $\{f(n)\}$ and $\{g(n)\}$ be two sequences. Then the convolution of these sequences is defined as

$$\{f(n)\}*\{g(n)\}=\{f(n)*g(n)\}=\sum_{m=-\infty}^{\infty} f(m)\,g(n-m)$$

Note: If it is one sided (right) sequence, let

$f(m)=0=g(m)$ for $m<0$, then

$$\{f(n)*g(n)\}=\sum_{m=0}^{\infty} f(m)\cdot g(n-m)$$

2.6.1 Convolution Theorem

Let $\{f(n)\}$ and $\{g(n)\}$ be any two sequences.

Let the *Z*-transform of $\{f(n)\}$, $z(f(n)\}=F(z)$ exist in the region $|z|>\frac{1}{R_1}$ and $Z\{g(n)\}=G(z)$ exist in the region $|z|>\frac{1}{R_2}$.

Then $z\{f(n) * g(n)\} = F(z) \cdot G(z)$ exist in the region $|z| > \frac{1}{R}$, where $\frac{1}{R} = \text{Max}\left\{\frac{1}{R_1}, \frac{1}{R_2}\right\}$.

***Proof*:** We have

$$F(z)\, G(z) = Z\{f(n)\} \cdot Z\{g(n)\}$$

$$= \left[\sum_{n=0}^{\infty} f(n)\, z^{-n}\right] \cdot \left[\sum_{n=0}^{\infty} g(n)\, z^{-n}\right]$$

$$= \begin{matrix}\left[f(0) + f(1) z^{-1} + f(2) z^{-2} + \cdots + f(n) z^{-n} + \cdots\right] \cdot \\ \left[g(0) + g(1) z^{-1} + g(2) z^{-2} + \cdots + g(n) z^{-n} + \cdots\right]\end{matrix}$$

$$= f(0)\, g(0) + \left[f(0) g(1) + f(1) g(0)\right] z^{-1} + \left[f(0) g(2) + f(1) g(1) + f(2) g(0) z^{-2}\right] \cdot z^{-2}$$
$$+ \cdots + \left[f(0) g(n) + f(1) g(n-1) + \cdots + f(n) g(0)\right] z^{-n} + \cdots$$

$$= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} f(m)\, g(n-m)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty}\left[f(n) * g(n)\right] z^{-n}$$

$$= z\{f(n) * g(n)\}$$

Hence, proved.

***Note*:** $Z^{-1}\{f(z)\, G(z)\} = f(n) * g(n)$.

Example 14 Using convolution theorem to find the inverse Z-transform of

$$H(z) = \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}.$$

Solution Here, $H(z) = \dfrac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} = \dfrac{z}{\left(z - \frac{1}{2}\right)} \cdot \dfrac{z}{\left(z - \frac{1}{3}\right)}$

Let $F(z) = \dfrac{z}{\left(z - \frac{1}{2}\right)}$ and $G(z) = \dfrac{z}{\left(z - \frac{1}{3}\right)}$

So that

$$f(n) = Z^{-1}\{F(z)\} = Z^{-1}\left[\frac{z}{z - \frac{1}{2}}\right] = \left(\frac{1}{2}\right)^n$$

$$g(n) = Z^{-1}\{G(z)\} = Z^{-1}\left[\frac{z}{z - \frac{1}{3}}\right] = \left(\frac{1}{3}\right)^n$$

Thus, the convolution theorem gives

$$Z^{-1}\{F(z)\cdot G(z)\} = \sum_{m=0}^{n}\left(\frac{1}{2}\right)^m \cdot \left(\frac{1}{3}\right)^{n-m}$$

$$= \left(\frac{1}{3}\right)^n \sum_{m=0}^{n}\left(\frac{1}{2}\right)^m \cdot \left(\frac{1}{3}\right)^{-m} = 3^{-n}\sum_{m=0}^{n}\left(\frac{3}{2}\right)^m$$

$$= 3^{-n}\left[1+\frac{3}{2}+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\cdots\right]$$

$$= 3^{-n}\left[2\left(\frac{3^{n+1}}{2^{n+1}}-1\right)\right]$$

$$= (3.2^{-n} - 2.3^{-n})$$

Example 15 Using convolution theorem to show that $Z^{-1}\left\{\frac{z(z+1)}{(z-1)^3}\right\} = n^2$.

Solution We write

$$\frac{z(z+1)}{(z-1)^3} = \frac{z}{(z-1)^2}\cdot\left(\frac{z+1}{z-1}\right)$$

$$= \frac{z}{(z-1)^2}\left[\frac{z}{z-1}+\frac{1}{z-1}\right]$$

We take

$$F(z) = \frac{z}{(z-1)^2} \text{ and } G(z) = \left(\frac{z}{z-1}+\frac{1}{z-1}\right)$$

So that

$$f(n) = Z^{-1}\{F(z)\} = Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n$$

$$g(n) = Z^{-1}\{G(z)\} = Z^{-1}\left\{\frac{z}{z-1}+\frac{1}{z-1}\right\}$$

$$= H(n) + H(n-1)$$

Thus, the convolution theorem gives

$$Z^{-1}\left\{\frac{z(z+1)}{(z-1)^3}\right\} = Z^{-1}\{F(z)\cdot G(z)\}$$

$$= f(n) * g(n)$$

$$= \sum_{m=0}^{n} m\left[H(n-m)+H(n-m-1)\right]$$

$$= n^2$$

Hence, proved.

Example 16 Verify the initial value theorem for the function $F(z)$ in the above example.

Solution

Here $F(z)=\dfrac{z(z+1)}{(z-1)^3}$ and $f(n)=n^2$

We have $f(0)=0$

and $\lim_{z\to\infty} F(z)=\lim_{z\to\infty}\dfrac{z(z+1)}{(z-1)^3}=0$

$\therefore$ $f(0)=\lim_{z\to\infty} F(z)$. Hence, theorem is verified.

Example 17 Find the Z-transform of $\{f(n)\}$, where $f(n)=\begin{cases}3^n; & n<0\\ 5^n; & n\ge 0\end{cases}$.

Solution

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 3^n\cdot z^{-n} + \sum_{n=0}^{\infty} 5^n z^{-n}$$

$$= \left[\cdots+3^{-3}z^3+3^{-2}z^2+3^{-1}z^1\right]+\left[1+\frac{5}{z}+\frac{25}{z^2}+\frac{125}{z^3}+\cdots\right]$$

$$= \frac{3^{-1}z}{1-3^{-1}z}+\frac{1}{1-\dfrac{5}{z^{-1}}}$$

$$= \frac{z}{3-z}+\frac{z}{z-5}$$

$$= \frac{z\left[(z-5)+(3-z)\right]}{(z-5)(3-z)}$$

$$= \frac{z[-2]}{-(z-5)(z-3)}$$

$$= \frac{2z}{(z-3)\,(z-5)};\ \left|\frac{z}{3}\right|<1, \left|\frac{5}{z}\right|<1.$$

Example 18 Find the Z-transform of $\left\{\left(\frac{1}{3}\right)^{|n|}\right\}$.

Solution

$$Z\left\{\left(\frac{1}{3}\right)^{|n|}\right\} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{|n|}\cdot z^{-n}$$

$$= \sum_{n=-\infty}^{-1}\left(\frac{1}{3}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n} z^{-n}$$

$$= \sum_{n=-\infty}^{-1}\frac{z^{-n}}{3^{-n}} + \sum_{n=0}^{\infty}\frac{z^{-n}}{3^{n}}$$

$$= \sum_{n=-\infty}^{-1}\left(\frac{3}{z}\right)^{n} + \sum_{n=0}^{\infty}\frac{1}{(3z)^{n}}$$

$$= \left[\cdots+\left(\frac{3}{z}\right)^{-3}+\left(\frac{3}{z}\right)^{-2}+\left(\frac{3}{z}\right)^{-1}\right]+\left[1+\frac{1}{3z}+\frac{1}{(3z)^{2}}+\frac{1}{(3z)^{3}}+\cdots\right]$$

$$= \left[\cdots+\left(\frac{z}{3}\right)^{3}+\left(\frac{z}{3}\right)^{2}+\left(\frac{z}{3}\right)\right]+\left[1+\frac{1}{3z}+\frac{1}{(3z)^{2}}+\frac{1}{(3z)^{3}}+\cdots\right]$$

$$= \frac{z}{3-z}+\frac{3z}{3z-1};\ \frac{1}{3}<|z|<3 \qquad \left[\text{Sum of G.P} = \frac{a}{1-r}\right]$$

$$= \frac{8z}{(3z-1)(3-z)}.$$

Example 19 Find the Z-transform of the sequence $\{f(n)\} = \sum_{n=0}^{\infty} 2^{n}\cdot\sum_{n=0}^{\infty} 3^{n}$.

Solution We know that

$$Z\{2^{n}\} = \sum_{n=0}^{\infty} 2^{n} z^{-n} = 1+2z^{-1}+2^{2}z^{-2}+2^{3}z^{-3}+\cdots$$

$$= \frac{1}{1-2z^{-1}}$$

Similarly $\quad Z\{3^{n}\} = \dfrac{1}{1-3z^{-1}}$

$$\therefore \quad Z\{f(n)\} = Z\{2^{n}\}\ \{3^{n}\}$$

$$= \frac{1}{1-2z^{-1}}\cdot\frac{1}{1-3z^{-1}}$$

$$= \frac{z^2}{(z-2)} \cdot \frac{z}{(z-3)}$$

$$= \frac{z^2}{(z-2)(z-3)}.$$

Example 20 Let $f(n) = \dfrac{(n+1)\,a^n}{n!}$. Determine Z-transform of $f(n)$.

Solution We know that

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

Given $$f(n) = \frac{(n+1)\,a^n}{n!}$$

$$f(n) = \frac{n}{n!}\,a^n + \frac{1}{n!}\,a^n$$

$\therefore$ $$Z\{f(n)\} = Z\left\{n \cdot \frac{a^n}{n!}\right\} + Z\left\{\frac{a^n}{n!}\right\}$$

$$= -z\frac{d}{dz}\left[Z\left\{\frac{a^n}{n!}\right\}\right] + Z\left\{\frac{a^n}{n!}\right\} \tag{5}$$

Now $$Z\left\{\frac{a^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \cdots$$

$$= e^{\frac{a}{z}}$$

Hence, Eq. (5) gives

$$Z\{f(n)\} = -z\frac{d}{dz}\{e^{\frac{a}{z}}\} + e^{\frac{a}{z}}$$

$$= -z \cdot e^{\frac{a}{z}}\left(-\frac{a}{z^2}\right) + e^{\frac{a}{z}}$$

$$= \frac{a}{z}\,e^{\frac{a}{z}} + e^{\frac{a}{z}}$$

$$= \left(1 + \frac{a}{z}\right) e^{\frac{a}{z}}.$$

Example 21 Using the convolution of sequences, to show that

$$\left\{\frac{1}{n!}\right\}*\left\{\frac{1}{n!}\right\}=\left\{\frac{2^n}{n!}\right\}.$$

Solution We know that

$$\left\{\frac{1}{n!}\right\}*\left\{\frac{1}{n!}\right\}=\sum_{m=0}^{n}\left(\frac{1}{m!}\right)\cdot\left(\frac{1}{(n-m)!}\right)$$

$$=\frac{1}{n!}+\frac{1}{1!(n-1)}+\frac{1}{2!(n-2)!}+\cdots+\frac{1}{n!}$$

$$=\frac{1}{n!}\left[1+n+\frac{n(n-1)}{2!}+\cdots+\right]$$

$$=\frac{1}{n!}\left[{}^nC_0+{}^nC_1+{}^nC_2+\cdots+{}^nC_n\right]$$

$$=\frac{2^n}{n!}.$$

Hence, proved.

Example 22 Using convolution theorem, find $Z^{-1}\left\{\frac{z^2}{(z-3)(z-5)}\right\}$.

Solution Let

$$F(z)=\frac{Z}{z-3}\text{ and }G(z)=\frac{Z}{z-5}$$

$$\therefore \qquad f(n)=Z^{-1}\left\{\frac{Z}{z-3}\right\}=3^n\text{ and }g(n)=Z^{-1}\left\{\frac{z}{z-5}\right\}=5^n$$

Using convolution theorem, we get

$$Z^{-1}\{F(z)\cdot G(z)\}=f(n)*g(n)$$

$$=\{3^n*5^n\}$$

$$=\sum_{m=0}^{n}(3)^m\,(5)^{n-m}$$

$$=5^n+3^1(5)^{n-1}+3^2(5)^{n-2}+\cdots+3^n$$

$$=5^n+\frac{3}{5}5^n+\frac{9}{25}5^n+\cdots+3^n.$$

Example 23 Find $Z^{-1}\left\{\frac{2z^2+3z}{(z+2)(z-4)}\right\}$. **[J.N.T.U. 2002]**

Solution Let

$$F(z)=\frac{2z^2+3z}{(z+2)(z-4)}$$

or $$\frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)}$$

$$\frac{F(z)}{z} = \frac{A}{z+2} + \frac{B}{z-4} \quad \text{[By partial fraction]}$$

or $$\frac{F(z)}{z} = \frac{1}{6(z+2)} + \frac{11}{6} \cdot \frac{1}{z-4}$$

or $$F(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

$\therefore$ $$Z^{-1}\{F(z)\} = \frac{1}{6} Z^{-1}\left\{\frac{z}{z+2}\right\} + \frac{11}{6} Z^{-1}\left\{\frac{z}{z-4}\right\}$$

or $$f(n) = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n.$$

Example 24 Find $Z^{-1}\left\{\dfrac{z^2}{z^2+1}\right\}$.

Solution Let

$$F(z) = \frac{z^2}{z^2+1}$$

Then $$\frac{F(z)}{z} = \frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)}$$

$$= \frac{1}{2}\left[\frac{1}{z+i} + \frac{1}{z-i}\right] \quad \text{[Using partial fraction]}$$

$$= \frac{1}{2}\left[\frac{1}{z - e^{-\frac{\pi i}{2}}} + \frac{1}{z - e^{\frac{\pi i}{2}}}\right]$$

or $$F(z) = \frac{1}{2}\left[\frac{z}{z - e^{-\frac{\pi i}{2}}} + \frac{z}{z - e^{\frac{\pi i}{2}}}\right]$$

$\therefore$ $$Z^{-1}\{F(z)\} = \frac{1}{2}\left[Z^{-1}\left\{\frac{z}{z - e^{-\frac{\pi i}{2}}}\right\} + Z^{-1}\left\{\frac{z}{z - e^{\frac{\pi i}{2}}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{-\frac{n\pi i}{2}} + e^{\frac{n\pi i}{2}}\right]$$

$$Z^{-1}\{F(z)\} = \cos\left(\frac{n\pi}{2}\right).$$

Example 25 If $F(z)=\dfrac{2z^2+5z+14}{(z-1)^4}$. Find f_2 and f_3.

Solution We can write the given function as

$$F(z)=\frac{1}{z^2}\cdot\frac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4}$$

Using initial value theorem, we obtain

$$f_0=\lim_{z\to\infty} F(z)=0$$

$$f_1=\lim_{z\to\infty}\left[\{F(z)-f_0\}\right]$$

$$=\lim_{z\to\infty}\left[z\left(\frac{2+5z^{-1}+14z^{-2}}{z^2(1-z^{-1})^4}-0\right)\right]=0$$

Now $$f_2=\lim_{z\to\infty}\left[z^2\{F(z)-f_0-f_1z^{-1}\}\right]$$

$$=\lim_{z\to\infty}\left[z^2\left\{\frac{2+5z^{-1}+14z^{-2}}{z^2(1-z^{-1})^4}-0-0\right\}\right]$$

$$=2$$

and $$f_3=\lim_{z\to\infty}\left[z^3\{F(z)-f_0-f_1z^{-1}-f_2z^{-2}\}\right]$$

$$=\lim_{z\to\infty}\left[z^3\cdot\left\{\frac{2+5z^{-1}+14z^{-2}}{z^2(1-z^{-1})^4}-0-0-\frac{2}{z^2}\right\}\right]$$

$$=\lim_{z\to\infty}\left[z^3\cdot\left\{\frac{2z^2+5z+14z}{(z-1)^4}-\frac{2}{z^2}\right\}\right]$$

$$=\lim_{z\to\infty} z^3\left[\frac{13z^3+2z^2+8z-2}{z^2(z-1)^4}\right]$$

$$=13.$$

Example 26 If $Z\{f(n)\}=F(z)=\dfrac{3z^3+5z^2-7z+1}{(z-1)(z+2)^2}$, then find $\lim\limits_{n\to\infty} f(n)$.

Solution Using final value theorem.

$$\lim_{n\to\infty}\cdot f(n)=\lim_{z\to 1}\left[(z-1)\cdot F(z)\right]$$

or $$\lim_{n\to\infty} f(n)=\lim_{z\to 1}\left[(z-1)\cdot\frac{3z^3+5z^2-7z+1}{(z-1)(z+2)^2}\right]$$

$$= \lim_{z \to 1} \left[\frac{3z^3 + 5z^2 - 7z + 1}{(z+2)^2} \right]$$

$$= \frac{3 + 5 - 7 + 1}{9}$$

$$= \frac{2}{9}.$$

2.7 TABLE OF Z-TRANSFORMS

Table of Z-transforms is as follows:

Sequence	Z-transform	ROC
$\delta[n]$	1	all z
$u[n]$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$u[-n]$	$\frac{1}{1-z}$	$\lvert z\rvert > 1$
$u[n-1]$	$\frac{1}{z-1}$	$\lvert z\rvert > 1$
$u[n+1]$	$\frac{z^2}{z-1}$	$\lvert z\rvert > 1$
$a^n u[n], a \neq 0$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-u[-n-1]$	$\frac{z}{z-1}$	$\lvert z\rvert < 1$
$-a^n u[-n-1]$	$\frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az}{(z-a)^2}$ or $\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$n\, u\, [n]$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$

(*Contd.*)

Sequence	Z-transform	ROC
$(n-1)\, u\,[n-1]$	$\dfrac{1}{(z-1)^2}$	$\lvert z\rvert > 1$
$\dfrac{1}{n!}$	$e^{\frac{1}{z}}$	all z
$\cos(\omega_0 n)\cdot u[n]$	$\dfrac{z^2 - z\cos\omega_0}{z^2 - 2z\cos\omega_0 + 1}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)\cdot u[n]$	$\dfrac{z\sin\omega_0}{z^2 - 2z\cos\omega_0 + 1}$	$\lvert z\rvert > 1$
$\delta(n-k)$	z^{-k}	all z
$\dfrac{1}{n+1}$	$z\log\left(\dfrac{z}{z-1}\right)$	$\lvert z\rvert > 1$
$u(n-k)$	$z^{-k}\left(\dfrac{z}{z-1}\right)$	$\lvert z\rvert > 1$
$\cos\dfrac{n\pi}{2}\cdot u(n)$	$\dfrac{z^2}{z^2+1}$	$\lvert z\rvert > 1$
$\sin\dfrac{n\pi}{2}\cdot u(n)$	$\dfrac{z}{z^2+1}$	$\lvert z\rvert > 1$
n^k	$\dfrac{k!z}{(z-1)^{k+1}}$	$\lvert z\rvert > 1$
$a^n f(n)$	$F\left(\dfrac{z}{a}\right)$	all z
$n\cdot f(n)$	$-z\dfrac{d}{dz}F(z)$	all z
n^{m+1}	$(-z)^m \dfrac{d^m}{dZ^m} z(n)$	all z
$\sinh(n\theta)$	$\dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$	$\lvert z\rvert > 1$
$\cosh(n\theta)$	$\dfrac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$	$\lvert z\rvert > 1$

(*Contd.*)

Sequence	Z-transform	ROC
$a^n \sin n\theta$	$\dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$	$\|z\| > 1$
$a^n \cos n\theta$	$\dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$	$\|z\| > 1$

2.8 SOME USEFUL INVERSE Z-TRANSFORM

Some useful inverse *Z*-transforms have been given in the table as following:

$F(z)$	Inverse of $F(z)$ i.e., $Z^{-1}\{F(z)\}$
$\dfrac{1}{z-a}$	$a^{n-1}u[n-1]$
$\dfrac{z}{z-a}$	$a^n u[n]$
$\dfrac{z^2}{(z-a)^2}$	$(n+1)\, a^n\, u[n]$
$\dfrac{z^3}{(z-a)^3}$	$\dfrac{1}{2!}(n+1)(n+2)a^n u[n]$
$\dfrac{1}{(z-a)^2}$	$(n-1)\, a^{n-2}\, u[n-2]$
$\dfrac{1}{(z-a)^3}$	$\dfrac{1}{2}(n-1)\,(n-2)\, a^{n-3}\, u\,[n-3]$

2.9 SOLUTION OF DIFFERENCE EQUATIONS USING Z-TRANSFORMS

Consider a relation is of the form

$$y_{n+k} + a_1 y_{n+k-1} + a_2 y_{n+k-2} + \cdots + a_k y_n = f(n) \tag{6}$$

where $a_1, a_2, \ldots, a_k$ are all constants, is called a linear difference equation with constant coefficient of order k.

The order of a difference equation is the difference between the largest and the smallest arguments in the difference equation.

Equation (6) is called homogenous if $f(n) = 0$, and non-homogeneous if $f(n) \neq 0$.

Equation (6) can be solved with the help of *Z*-transform.

Following steps can be use to solve:

1. Take the *Z*-transform of both sides of the given difference equation.
2. Using the given conditions (Initial or boundary), transpose all the terms without $Y(z)$ to the right, where $Y(z) = Z\{y_n\}$. Here $y_n = y(n)$.
3. Simplify and find $Y(z)$.

4. Take the inverse Z-transform of $Y(z)$ and compute y_n, which is the required solution to the given difference equation.

***Note*:**

1. $Z\{y_n\} = Y(z)$

2. $Z\{y_{n+k}\} = z^k\left[Y(z) - y_0 - \dfrac{y_1}{z} - \dfrac{y_2}{z^2} - \cdots - \dfrac{y_{k-1}}{z^{k-1}}\right]$

   In particular

   (i) $Z\{y_{n+1}\} = z\{Y(z) - y_0\}$

   (ii) $Z\{y_{n+2}\} = z^2\left\{Y(z) - y_0 - \dfrac{y_1}{z}\right\}$

   (iii) $Z\{y_{n+3}\} = z^3\left\{Y(z) - y_0 - \dfrac{y_1}{z} - \dfrac{y_2}{z^2}\right\}$ and so on.

3. $Z\{y_{n-k}\} = z^{-k}Y(z)$

Example 27 Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$; given $y_0 = 0 = y_1$.

Solution Given

$$y_{n+2} + 6y_{n+1} + 9y_n = 2^n \tag{7}$$

Taking Z-transform of both sides on Eq. (7),

$$Z\{y_{n+2} + 6y_{n+1} + 9y_n\} = Z\{2^n\}$$

or
$$Z\{y_{n+2}\} + 6Z\{y_{n+1}\} + 9Z\{y_n\} = Z\{2^n\}$$

or
$$Z^2\left[Y(z) - y_0 - \frac{y_1}{z}\right] + 6Z\left[Y(z) - y_0\right] + 9Y(z) = \frac{z}{z-2}$$

Since $y_0 = 0$ and $y_1 = 0$, we have

$$Z^2\{Y(z)\} + 6ZY(z) + 9Y(z) = \frac{z}{z-2}$$

or
$$(z^2 + 6z + 9)\,Y(z) = \frac{z}{z-2}$$

or
$$Y(z) = \frac{z}{(z-2)\,(z^2 + 6z + 9)}$$

$$= \frac{z}{(z-2)\,(z+3)^2}$$

or
$$\frac{Y(z)}{z} = \frac{1}{(z-2)\,(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2} \tag{8}$$

Equation (8) gives, $A = \frac{1}{25}$, $B = -\frac{1}{25}$ and $C = -\frac{1}{5}$

Therefore Eq. (8), becomes

$$\frac{Y(z)}{z} = \frac{1}{25}\cdot\frac{1}{(z-2)} - \frac{1}{25}\frac{1}{(2+3)} - \frac{1}{5}\frac{1}{(z+3)^2}$$

or $$Y(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

Taking inverse Z-transform on both sides, we get

$$y_n = Z^{-1}\{Y(z)\} = \frac{1}{25}Z^{-1}\left(\frac{z}{z-2}\right) - \frac{1}{25}Z^{-1}\left(\frac{z}{z+3}\right) - \frac{1}{5}Z^{-1}\left\{\frac{z}{(z+3)^2}\right\}$$

$$y_n = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^{n-1}$$

or $$y_n = \frac{1}{25}\left[(2)^n - (-3)^n + \frac{5}{3}n(-3)^n\right]$$

Example 28 Solve $y_{n+2} - 3y_{n+1} + 2y_n = 0$, given $y_0 = 0$ and $y_1 = 1$.

Solution Given

$$y_{n+2} - 3y_{n+1} + 2y_n = 0 \tag{9}$$

Taking Z-transform both sides on Eq. (9), we have

$$Z\{y_{n+2}\} - 3Z\{y_{n+1}\} + 2Z\{y_n\} = 0$$

$$z^2\left\{y(z) - y_0 - \frac{y_1}{z}\right\} - 3z\{Y(z) - y_0\} + 2Y(z) = 0$$

Since $y_0 = 0$ and $y_1 = 1$, we have

$$z^2\left\{Y(z) - \frac{1}{2}\right\} - 3z\{Y(z)\} + 2Y(z) = 0$$

or $$(z^2 - 3z + 2)\,Y(z) - z = 0$$

or $$Y(z) = \frac{z}{z^2 - 3z + 2}$$

$$= \frac{z}{(z-1)(z-2)}$$

or $$\frac{Y(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

or $$Y(z) = \frac{z}{z-2} - \frac{z}{z-1} \tag{10}$$

Taking inverse Z-transform both sides on Eq. (10), we get

$$y_n = Z^{-1}\{Y(z)\} = Z^{-1}\left\{\frac{z}{z-2}\right\} - Z^{-1}\left\{\frac{z}{z-1}\right\}$$

or $$y_n = (2^n - 1);\ n = 0,1,2,3,...$$

Example 29 Solve $y_{n+2} - 2y_{n+1} + y_n = 3n + 5$, subject to the condition $y_0 = 0 = y_1$.

Solution Given $y_{n+2} - 2y_{n+1} + y_n = 3n + 5$ (11)

Taking Z-transform, both sides on Eq. (11), we have.

$$Z\{y_{n+2}\} - 2Z\{y_{n+1}\} + Z\{y_n\} = Z\{3n+5\}$$

$$z^2\left\{Y(z) - y_0 - \frac{y_1}{z}\right\} - 2z\{Y(z) - y_0\} + Y(z) = 3\cdot\frac{z}{(z-1)^2} + 5\frac{z}{z-1}$$

Since, $y_0 = 0$ and $y_1 = 0$, we have

$$z^2Y(z) - 2zY(z) + Y(z) = 3\frac{z}{(z-1)^2} + 5\frac{z}{(z-1)}$$

or $$(z^2 - 2z + 1)\,Y(z) = 3\cdot\frac{z}{(z-1)^2} + 5\frac{z}{z-1}$$

or $$Y(z) = \frac{3z}{(z-1)(z^2-2z+1)} + \frac{5z}{(z-1)(z^2-2z+1)}$$

$$Y(z) = \frac{3z}{(z-1)^4} + \frac{5z}{(z-1)^3} \tag{12}$$

Taking inverse Z-transform both sides on Eq. (12), we get

$$y_n = Z^{-1}\{Y(z)\} = 3Z^{-1}\left\{\frac{z}{(z-1)^4}\right\} + 5\,Z^{-1}\left\{\frac{z}{(z-1)^3}\right\}$$

$$= 3\binom{n}{3}u[n] + 5\binom{n}{2}u[n] \quad \left[\because \quad Z^{-1}\left\{\frac{a^K z}{(z-a)^{K+1}} = \binom{n}{K}a^n u[n]\right\}\right]$$

$$= \left[\frac{1}{2}n(n-1)(n-2) + \frac{5}{2}n(n-1)\right]u[n]$$

$$= \frac{1}{2}n(n-1)\,[n-2+5]\,u[n]$$

$$y_n = \frac{1}{2}n(n-1)\,(n+3)\,u[n]$$

or $$y_n = \frac{1}{2}n(n-1)\,(n+3) \qquad [\because \quad u[n] = 1]$$

Example 30 Solve $x_{n+1} = 7\,x_n + 10\,y_n$

$$y_{n+1} = x_n + 4\,y_n$$

given $x_0 = 3$ and $y_0 = 2$.

Solution Taking the Z-transform, to $x_{n+1} = 7\,x_n + 10\,y_n$, we get

$$Z\{x_{n+1}\} = 7Z\{x_n\} + 10\,Z\{y_n\}$$

or $$z\{X(z) - x_0\} = 7X(z) + 10Y(z)$$

Since $x_0 = 3$, we have

$$z[X(z) - 3] = 7X(z) + 10Y(z)$$

or $$(7 - z)\,X(z) + 10Y(z) = -3z \tag{13}$$

Again, taking Z-transform, to $y_{n+1} = x_n + 4y_n$, we get

$$Z\{y_{n+1}\} = Z\{x_n\} + 4Z\{y_n\}$$

$$Z\{Y(z) - y_0\} = X(z) + 4Y(z)$$

or $$X(z) - (z-4)Y(z) = -2z \qquad [\because \quad y_0 = 2] \tag{14}$$

Eliminate $X(z)$ from Eqs (13) and (14), we get

$$(z^2 - 11z + 18)\,Y(z) = 2z^2 - 11z$$

or $$Y(z) = \frac{2z^2 - 11z}{(z-2)\,(z-9)} = \frac{z}{z-9} + \frac{z}{z-2} \qquad \text{[By partial fraction]}$$

Taking inverse Z-transform, we have

$$y_n = Z^{-1}\{Y(z)\} = Z^{-1}\left\{\frac{z}{z-9}\right\} + Z^{-1}\left\{\frac{z}{z-2}\right\}$$

$\therefore$ $$y_n = 9^n + 2^n$$

From the given Eq (14), we get

$$x_n = y_{n+1} - 4y_n$$

$$= 9^{n+1} + 2^{n+1} - 4(9^n + 2^n)$$

$$= 9.9^n + 2.2^n - 4(9^n + 2^n)$$

$$x_n = 5.9^n - 2.2^n$$

Hence, $$x_n = 5.9^n - 2.2^n \text{ and } y_n = 9^n + 2^n.$$

EXERCISE 2.3

Solve the following difference equation by Z-transform:

1. Solve $y_{n+1} + y_n = 1$; given $y_0 = 0$.
2. Solve $y_{n+2} + y_n = 5.2^n$; given $y_0 = 1,\ y_1 = 0$.
3. Solve $y_{n+2} - 5y_{n+1} + 6y_n = 6n$; if $y_0 = 0 = y_1$.
4. Solve $y_n - y_{n-1} = u[n] + u[n-1]$.
5. Solve $y_{n+1} - x_n = 0;\ x_{n+1} - y_n = 1;\ x_0 = 0,\ y_0 = -1$.
6. Solve $y_{n+3} - 3y_{n+1} - 2y_n = 3^n;\ y_0 = 2,\ y_1 = 1,\ y_2 = 6$.
7. Solve $y_{n+2} + 10\,y_{n+1} + 25\,y_n = n;\ y_0 = 1,\ y_1 = -5$.
8. Solve $y_{n+2} + 4y_n = (n+1);\ y_0 = -1$.
9. Find the Z-transform of the following sequence

   (i) a^n (ii) a^{-n} (iii) e^{ax} (iv) $\dfrac{e^n}{n!}$

10. Using convolution theorem, to find the inverse Z-transform of the

    (i) $F(z) = \dfrac{2z}{(z-1)\,(z^2+1)}$ (ii) $\dfrac{10\,z}{(z-1)\,(z-2)}$

Answers

1. $y_n = \dfrac{1}{2}\{1-(-1)^n\}$
2. $y_n = 2^n - 2\sin\dfrac{n\pi}{2}$
3. $y_n = \dfrac{1}{12}6^n - \dfrac{1}{3}3^n + \dfrac{1}{4}2^n$
4. $y_n = 1 + 2n$
5. $x_n = \dfrac{1}{2}\left[1 - \cos\dfrac{n\pi}{2} - \sin\dfrac{n\pi}{2}\right]$

   $y_n = \dfrac{1}{2}\left[\sin\dfrac{n\pi}{2} - \cos\dfrac{n\pi}{2} - 1\right]$
6. $y_n = \dfrac{1}{16}\left[16(2)^n + (3)^n + 15 + 4n(-1)^n\right]$
7. $y_n = \dfrac{1}{540}\left[(545-3n)(-5)^n + 5(3n-1)\right]$
8. $y_n = \dfrac{1}{25}\left[-29(-4)^n + 4 + 5n\right]$
9. (i) $\dfrac{z}{z-a}$ (ii) $\dfrac{az}{az-1}$ (iii) $\dfrac{z}{z-e^{-a}}$ (iv) $e^{a/z}$
10. (i) $f(n) = 1 - \dfrac{(i)^n}{1+i} - \dfrac{(-i)^n}{1-i}$ (ii) $f(n) = 10(2^n - 1);\ n = 0,1,2,3,\ldots,$

EXERCISE (MIXED PROBLEMS)

1. Show that $Z\left\{\frac{1}{n}\right\} = z\log\left(\frac{z}{z-1}\right)$ **[Madras 2003]**

2. Find Z-transform of $\frac{1}{n(n+1)}$. **[Mangalore 1999]**

3. If $F(z) = \frac{2z^2+3z+12}{(z-1)^4}$, then find the values of f_2 and f_3.

4. Find $\lim\limits_{n\to\infty} f(n)$, where $F(z) = z\{f(n)\} = \frac{z^2-3z+5}{(z-1)(z+2)}$.

5. Find the inverse Z-transform of $F(z) = \frac{10\,z}{(z-1)\,(z-2)}$. **[Madras 2003]**

6. Find the inverse Z-transform $\frac{5z}{(3z-1)(2-z)}$. **[Madras 1999]**

7. Find $Z^{-1}\left[\frac{z+z^2}{(z-1)(z^2+1)}\right]$ **[Madras 2003]**

8. If $Z\{f(n)\} = \frac{z^3-3z^2+2z+1}{(z-1)(z+3)^2}$, using final value theorem to evaluate $\lim\limits_{n\to\infty} f(n)$.

9. Determine f_0, f_1 and f_2 when $Z\{f(n)\} = F(z)$ is as given below:

   (i) $\frac{z^2}{z^2+1}$ (ii) $\frac{(z-1)^2\,(z+2)}{(z+3)(z+5)^2}$

10. Determine $Z\{nf(n)\}$ and $Z\{n^2f(n)\}$, where $f(n) = a^n$.

11. Find the inverse Z-transform of $F(z) = \left(\frac{z}{z-a}\right)^2$.

12. Find $Z^{-1}\left\{\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right\}$ when

    (i) $\frac{1}{3} < |z| < \frac{1}{2}$ (ii) $|z| > \frac{1}{2}$.

13. Using residues method to evaluate the inverse Z-transform of $\frac{9z^3}{(3z-1)^2(z-2)}$.

14. Using Z-transform to solve the difference equation $y_{n+1}+\frac{1}{4}y_n=\left(\frac{1}{4}\right)^n$ $(n\geq 0)$ with $y(0)=0$

15. Using Z-transform to solve the equation $y_n+\frac{1}{4}y_{n-1}=u(n)+\frac{1}{3}u(n-1)$.

16. Solve $y_n+\frac{1}{25}y_{n-2}=\left(\frac{1}{5}\right)^n\cos\left(\frac{n\pi}{2}\right)$ $(n\geq 0)$.

Answers

3. $f_2=2, f_3=11$

4. 1

5. $f(n)=10(2^n-1)$, $n=0, 1, 2, 3, \ldots$

6. $f(n)=\left(\frac{1}{3}\right)-2^n$

7. $f(n)=1+\frac{1}{2}\left[(i)^{n-2}+(-i)^{n-2}\right]$

8. $\frac{1}{16}$

9. (i) $f_0=1, f_1=0, f_2=-1$ (ii) $f_0=1, f_1=-13, f_2=111$

10. $\frac{az}{(z-a)^2}, \frac{az(z+a)}{(z-a)^3}$

11. $(n+1)a^n$.

12. (i) $f(n)=\begin{cases}-\frac{6}{3^{n-1}} & \text{if } n>0\\ -12.2^{-n} & \text{if } n<0\end{cases}$ (ii) $f(n)=6\left[\frac{1}{2^{n-1}}-\frac{1}{3^{n-1}}\right], n\geq 1$

13. $f(n)=\frac{9}{25}2^{n+2}-\frac{1}{5}\frac{(n+2)}{3^n}$

14. $y(n)=-2\left(-\frac{1}{4}\right)^n+2\left(\frac{1}{4}\right)^n$

15. $y(n)=\frac{1}{12}\left(-\frac{1}{4}\right)^{n-1}$

16. $y(n)=\frac{1}{2}(n+2)\cdot\frac{1}{5^n}\cos\left(\frac{n\pi}{2}\right)$

SUMMARY

Following topics have been discussed in this chapter:

1. Sequence

A function whose domain is the set of natural numbers N and range, a subset of real number R is called a sequence.

Let $f(n)$ be an element of a sequence S and it is defined as

$$S=\{f(n)\}; -\infty<n<\infty, \text{ where } n \text{ is an integer.}$$

2. Z-Transform for Discrete Values of *t*

Let $f(t)$ is a function defined for discrete values of t, where $t = nT$, $n = 0, 1, 2, \ldots$, T being the period of sampling, then z-transform of $f(t)$ is defined as

$$Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)z^{-n} = F(z)$$

3. *Z*-Transform

The z-transform of a given discrete sequence $f[n]$ is denoted by $F(z)$ or $Z\{f(n)\}$ and is defined as

$$Z\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n)\cdot z^{-n}$$

where Z is a continuous complex variable and $Z\{\cdot\}$ is the Z-transform operator.
Given equation represent the bilateral Z-transform.
The unilateral Z-transform of a given sequence $f[n]$ is defined as

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)\cdot z^{-n}$$

Region of Convergence (ROC)

The Z-transform does not converge for all sequences or for all values of Z. The set of values of Z for which the Z-transform converges is called the region of convergence.

The Z-transform of $f[n]$ exists if the sum $\sum_{n=-\infty}^{\infty} |f(n)|$ converges. However, the Z-transform $F(z)$ of $f(n)$, i.e., the discrete time Z-transform of the sequence $f(n)\ r^{-n}$ exists (or converges) if $\sum_{n=-\infty}^{\infty} |f(n)r^{-n}| < \infty$, where $Z = r\,e^{i\theta}$

$$\Rightarrow \qquad \sum_{n=-\infty}^{\infty} |f(x)||z|^{-n} < \infty$$

for the existence of the Z-transform.

4. The Inverse *Z*-Transform

The process of inverse Z-transform is the reverse process of Z-transform. The inverse Z-transform is given by the complex integral.

$$f[n] = Z^{-1}\{F(z)\} = \frac{1}{2\pi i}\oint_C F(z)\cdot z^{n-1}dz$$

where C is a simple closed curve enclosing the origin and lying outside the circle $|z| = R$.
In this section, we shall discuss some methods for finding the inverse Z-transform.

5. Properties of the *Z*-Transform

Property 1 (*Linearty*) If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$, then

$$Z\{af(n) + bg(n)\} = aZ\{f(n)\} + bZ\{g(n)\} = aF(z) + bG(z)$$

Property 2 (*Shifting property*) Let the *Z*-transform of a casual sequence $f(n)$ is $F(z)$, i.e.,

$Z\{f(n)\} =$ $F(z)$ and m is any positive integer then

$$Z\{f(n-m)\} = z^{-m}F(z);\ n \geq m \quad \text{(shifting to the right)}$$

Property 3 (*Scaling in the Z-domain*) Let $F(z)$ be the *Z*-transform of $f(n)$. Then

$$Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$$

Property 4 (*Time Reversal*) Let $F(z)$ be the *Z*-transform of $f(n)$. Then

$$Z\{f(-n)\} = F\left(\frac{1}{2}\right)$$

where $f(-n)$ represents mirror image of the signal $f(n)$.

Property 5 (First Shifting Theorem) If $Z\{f(t)\} = F(z)$, then $Z\{e^{-at} f(t)\} = F(ze^{aT})$.

Property 6 (*Second Shifting Theorem*) If $Z\{f(t)\} = F(z)$, then $Z\{f(t+T)\} = z[F(z) - f(0)]$.

Property 7 (*Initial Value Theorem*) If $Z\{f(t)\} = F(z)$, then $\lim_{z\to\infty} F(z) = f(0)$.

***Note*:** If $f(0) = 0$, then $f(1) = \lim_{z\to\infty} zf(z)$.

Property 8 (Final Value Theorem) If $Z\{f(t)\} = F(z)$, then $\lim_{t\to\infty} f(t) = \lim_{z\to 1}(z-1)\, F(z)$.

Property 9 (*Differentiation in the Z-domain*): Let Z-transform of a sequence $f(n)$

$$Z\{f(n)\} = F(z) \text{ exist in the region } |z| > \frac{1}{R}$$

where R is Radius of Convergence, then

$$Z\{nf(n)\} = -z\frac{d}{dz}\left[F(z)\right]$$

which is also convergent in the region $|z| > \dfrac{1}{R}$.

6. Convolution of Sequences

Let $[f(n)\}$ and $\{g(n)\}$ be two sequences. Then the convolution of these sequences is defined as

$$\{f(n)\} * \{g(n)\} = \{f(n) * g(n)\} = \sum_{m=-\infty}^{\infty} f(m)\, g(n-m)$$

***Note*:** If it is one sided (right) sequence, let

$$f(m) = 0 = g(m) \text{ for } m < 0, \text{ then}$$

$$\{f(n) \times g(n)\} = \sum_{m=0}^{\infty} f(m) \cdot g(n-m)$$

Convolution Theorem

Let $\{f(n)\}$ and $\{g(n)\}$ be any two sequences.

Let the Z-transform of $\{f(n)\}$, $Z(f(n)\} = F(z)$ exist in the region $|z| > \frac{1}{R_1}$ and $Z\{g(n)\} = G(z)$ exist in the region $|z| > \frac{1}{R_2}$.

Then $Z\{f(n) * g(n)\} = F(z) \cdot G(z)$ exist in the region $|z| > \frac{1}{R}$, where $\frac{1}{R} = \text{Max}\left\{\frac{1}{R_1}, \frac{1}{R_2}\right\}$.

***Note*:** $Z^{-1}\{f(z)\, G(z)\} = f(n) * g(n)$.

7. Solution of Difference Equations Using *Z*-Transforms

Consider a relation is of the form.

$$y_{n+k} + a_1 y_{n+k-1} + a_2 y_{n+k-2} + \cdots + a_k y_n = f(n)$$

where $a_1, a_2, \ldots, a_k$ are all constants, is called a linear difference equation with constant coefficient of order k.

The order of a difference equation is the difference between the largest and the smallest arguments in the difference equation.

Given equation is called homogenous if $f(n) = 0$, and non-homogeneous if $f(n) \neq 0$.

Given equation can be solved by Z-transform if we follow the following steps:

1. Take the Z-transform of both sides of the given difference equation.
2. Using the given conditions (Initial or boundary), transpose all the terms without $Y(z)$ to the right, where $Y(z) = Z\{y_n\}$, Here $y_n = y(n)$.
3. Simplify and find $Y(z)$.
4. Take the inverse Z-transform of $Y(z)$ and compute y_n, which is the required solution to the given difference equation.

***Note*:**

1. $Z\{y_n\} = Y(z)$
2. $$Z\{y_{n+k}\} = z^k\left[Y(z) - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2} - \cdots - \frac{y_{k-1}}{z^{k-1}}\right]$$

   In particular

   (i) $Z\{y_{n+1}\} = z\{Y(z) - y_0\}$

   (ii) $Z(y_{n+2}) = z^2\left\{Y(z) - y_0 - \frac{y_1}{z}\right\}$

   (iii) $Z\{y_{n+3}\} = z^3\left\{Y(z) - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2}\right\}$ and so on.
3. $Z\{y_{n-k}\} = z^{-k} Y(z)$

OBJECTIVE TYPE QUESTIONS

1. The Z-transform $F(z)$ of the function $f(nT) = a^{nT}$ is

(a) $\frac{z}{z-a^T}$ (b) $\frac{z}{z+a^T}$

(c) $\frac{z}{z-a^{-T}}$ (d) $\frac{z}{z+a^{-T}}$

[GATE (EC) 1999]

2. The region of convergence of the z-transform of a unit step function is

(a) $|z| > 1$ (b) $|z| < 1$

(c) $\text{Re}(z) > 0$ (d) $\text{Re}(z) < 0$

[GATE (EC) 2001]

3. The z-transform of a system is $H(z) = \frac{z}{z-0.2}$.

If the ROC is $|z| < 0.2$, then the impulse response of the system is

(a) $(0.2)^n u(n)$ (b) $(0.2)^n u(-n-1)$

(c) $-(0.2)^n u(n)$ (d) $-(0.2)^n u(-n-1)$

[GATE (EC) 2004]

4. The region of convergence (ROC) of z–transform of the sequence.

$\left(\frac{5}{6}\right)^n u(n) - \left(\frac{6}{5}\right)^n u(-n-1)$ must be

(a) $|z| < \frac{5}{6}$ (b) $|z| > \frac{6}{5}$

(c) $\frac{5}{6} < |z| < \frac{6}{5}$ (d) $\frac{6}{5} < |z| < \infty$.

[GATE (EC) 2005]

5. If $u(t)$ is the unit step function and $\delta(t)$ is the unit impulse function, then the inverse z-transform of $F(z) = \frac{1}{z+1}$ for $K \geq 0$ is

(a) $(-1)^K \delta(K)$ (b) $\delta(K) - (-1)^K$

(c) $(-1)^K u(K)$ (d) $u(K) - (-1)^K$

[GATE (EE) 2005]

6. The region of convergence (ROC) of $X_n(n) + X_2(n)$ is $\frac{1}{3} < |z| < \frac{2}{3}$, then the ROC of $X_n(n) - X_2(n)$ includes

(a) $\frac{1}{3} < |z| < 3$ (b) $\frac{2}{3} < |z| < 3$

(c) $\frac{3}{2} < |z| < 3$ (d) $\frac{1}{3} < |z| < \frac{2}{3}$

[GATE (EC) 2006]

7. The region of convergence of the z-transform of the discrete-time signal $X(n) = 2^n u[n]$ will be

(a) $|z| > 2$ (b) $|z| < 2$

(c) $|z| > \frac{1}{2}$ (d) $|z| < \frac{1}{2}$

[GATE (IN) 2008]

8. For input $X(t)$, an ideal impulse sampling system produces the output

$$y(t) = \sum_{K=-\infty}^{\infty} X(KT)\delta(t-KT),$$

where $\delta(t)$ is the Dirac-delta function. The system is

(a) Non-linear and time invariant
(b) Non-linear and time variant
(c) Linear and time invariant
(d) Linear and time variant

[GATE (IN) 2009]

9. The region of convergence of the discrete–time sequence

$$X(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1) \text{ is}$$

(a) $\frac{1}{3} < |z| < \frac{1}{2}$ (b) $|z| > \frac{1}{2}$

(c) $|z| < \frac{1}{3}$ (d) $2 < |z| < 3$

[GATE (EC) 2009]

10. The z-transform of a signal. $X(n)$ is given by $4z^{-3} + 3z^{-1} + 2 - 6z^2 - 2z^3$. It is applied

to a system, with a transfer function $H(z) = 3z^{-1} - 2$. Let the output by $y(n)$. Which of the following is true?
(a) $y(n)$ is non-casual with finite support
(b) $y(n)$ is casual with infinite support
(c) $y(n) = 0$; $|n| > 3$
(d) $\text{Re}\{Y(z)\}_{z=e^{j\theta}} = -\text{Re}\{Y(z)\}_{z=e^{-j\theta}}$;

$\text{Im}\{Y(z)\}_{z=e^{j\theta}} = \text{Im}\{Y(z)\}_{z=e^{-j\theta}}; -\pi \le \theta < \pi$

[GATE (EE) 2009]

11. Consider the z-transform $X(z) = 5z^2 + 4z^{-1} + 3$; $0 < |z| < \infty$. The inverse z-transform $X(n)$ is
(a) $5\delta(n+2) + 3\delta(n) + 4\delta(n-1)$
(b) $5\delta(n-2) + 3\delta(n) + 4\delta(n+1)$
(c) $5\,u(n+2) + 3u(n) + 4u(n-1)$
(d) $5u(n-2) + 3u(n) + 4u(n+1)$

[GATE (EC) 2010]

12. A system is defined by its impulse response $H(n) = 2^n u(n-2)$. The system is
(a) Stable and casual
(b) Casual but not stable
(c) Stable not casual
(d) Unstable and non-casual

[GATE (EC) 2011]

13. Consider the difference equation $y(n) - \frac{1}{3}y(n-1) = X(n)$ and suppose that $X(n) = \left(\frac{1}{2}\right)^n u(n)$. Assuming the condition of initial rest, the solution for $y(n)$, $n \ge 0$ is

(a) $3\left(\frac{1}{3}\right)^n - 2\left(\frac{1}{2}\right)^n$

(b) $-2\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{2}\right)^n$

(c) $\frac{2}{3}\left(\frac{1}{3}\right)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n$

(d) $\frac{1}{3}\left(\frac{1}{3}\right)^n + \frac{2}{3}\left(\frac{1}{2}\right)^n$

[GATE (IN) 2011]

14. If $X(n) = \left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{2}\right)^n u(n)$, then the region of convergence of its z-transform in the z-plane will be

(a) $\frac{1}{3} < |z| < 3$
(b) $\frac{1}{3} < |z| < \frac{1}{2}$
(c) $\frac{1}{2} < |z| < 3$
(d) $\frac{1}{3} < |z|$

[GATE (IN & EE) 2012]

15. Let $y(n)$ denote the convolution of $h(n)$ and $g(n)$, where $h(n) = \left(\frac{1}{2}\right)^n u(n)$ and $g(n)$ is a casual sequence. If $y(0) = 1$ and $y(1) = \frac{1}{2}$, then $g(1)$ equals

(a) 0
(b) $\frac{1}{2}$
(c) 1
(d) $\frac{3}{2}$

[GATE (EE & IN) 2012]

16. Let $X(n) = \left(-\frac{1}{9}\right)^n u(n) - \left(-\frac{1}{3}\right)^n u(-n-1)$.

The region of convergence (ROC) of the z-transform of $X(n)$ is

(a) $|z| > \frac{1}{9}$
(b) $|z| < \frac{1}{3}$
(c) $\frac{1}{9} < |z| < \frac{1}{3}$
(d) does not exist

[GATE (EC) 2014]

17. Consider a discrete time signal $X(n) = \begin{cases} n, & 0 \le n \le 10 \\ 0, & \text{otherwise} \end{cases}$. If $y(n)$ is the convolution of $X(n)$ with itself, the value of $y(4)$ is$^{(10)}$ **[GATE (EC) 2014]**

18. Let $X(z) = \dfrac{1}{1 - z^{-3}}$ be the z-transform of a casual signal $x(n)$. Then the value of $X(2)$ and $X(3)$ are
(a) 0 and 0
(b) 0 and 1
(c) 1 and 0
(d) 1 and 1

19. Two casual discrete-time signals $X(n)$ and $Y(n)$ are related as $Y(n) = \sum_{n=0}^{n} X(m)$. If the z-transform of $y(n)$ is $\frac{2}{z(z-1)^2}$, the value of $X(2)$ is______(0).

[GATE (EC) 2015]

20. Consider two real sequences with time origin marked by the value,

$x_1(n) = \{1, 2, 3, 0\}$, $x_2(n) = \{1, 3, 2, 1\}$.

Let $X_1(K)$ and $X_2(K)$ be 4–point discrete Fourier transform of $x_1(n)$ and $x_2(n)$ respectively. Another sequence $x_3(n)$ is derived by taking 4-point inverse DFT of $x_3(n) = X_1(K)\cdot X_2(K)$. The vale of $x_3(n)$ is (11)

[GATE (EC) 2015]

21. Consider a discrete time signal given by

$x(n) = (-0.25)^n u(n) + (0.5)^n u(-n-1)$

The region of convergence of its z-transform would be

(a) The region inside the circle of radius 0.5 and centered at origin
(b) The region outside the circle of radius 0.25 and centered at origin
(c) The annular region between the two circles, both centered at origin and having radii 0.25 and 0.5
(d) The entire z-plane

[GATE (EE) 2015]

22. If $X(n) = \alpha^{|n|}$; $0 < |\alpha| < 1$, then the region of convergence of $X(n)$ is

(a) $\alpha < |z| < \frac{1}{\alpha}$ (b) $|z| > \alpha$

(c) $|z| < \frac{1}{\alpha}$ (d) $|z| > 1$

[GATE (IN) 2015]

23. The Z-transform of na^n is

(a) $\frac{Z}{(Z-a)}$ (b) $\frac{az}{(z-a)^2}$

(c) $\frac{az}{(z+a)^2}$ (d) $\frac{az}{(z-a)^3}$

24. The Z-transform of $\frac{1}{n!}$ is

(a) $e^{-\frac{1}{z}}$ (b) $e^{\frac{1}{z}}$
(c) e^{z} (d) e^{-z}

25. $Z^{-1}\left\{\frac{1}{Z-2}\right\}$ is

(a) 2^{n+1} (b) 2^{n-1}
(c) 2^{n} (d) 2^{-n}

ANSWERS

1. (a)	2. (a)	3. (d)	4. (c)	5. (c)	6. (d)	7. (a)
8. (d)	9. (a)	10. (b)	11. (a)	12. (b)	13. (b)	14. (c)
15. (a)	16. (c)	17. (10)	18. (b)	19. (0)	20. (11)	21. (c)
22. (a)	23. (b)	24. (b)	25. (b)			

3 Complex Variables and Calculus

3.1 COMPLEX NUMBER

An ordered pair of real numbers x and y to be written as $z = (x, y)$ is called a complex number. Also, we may write $z = (x, y) = x + iy$, where imaginary unit i(iota) is defined as $i = (0, 1)$. Here x is called real part of z and y is called imaginary part of z. The real part of z is denoted by Re(z) and the imaginary part of z by Im(z).

Set of Complex Numbers

The set of all complex numbers is denoted by C, i.e.,

$$C = \{x + iy \mid x, y \in R\}.$$

Since a real number 'x' can be written as $x + oi$,

$\therefore$ Every real number is a complex number. Hence $R \subset C$, where R is the set of all real numbers.

3.2 EQUALITY OF COMPLEX NUMBERS

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if $x_1 = x_2$ and $y_1 = y_2$, i.e. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

3.3 FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then

(i) Addition of Complex Numbers

Addition of two complex numbers z_1 and z_2 is denoted by $z_1 + z_2$ and is defined as

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

i.e., $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$

and $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$.

Properties of Addition of Complex Numbers

(a) Addition is commutative for any two complex numbers z_1 and z_2, we have

$$z_1 + z_2 = z_2 + z_1.$$

(b) *Addition is associative* For any three complex numbers z_1, z_2, z_3, we have

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

(c) *Existence of additive identity* The complex number $0 = 0 + 0i$ is an identity element for addition, i.e. $z + 0 = z = 0 + z$ for al $z \in C$.

(d) *Existence of additive inverse* For every non-zero complex number $z = x + iy$ and there exists an additive inverse $z = -(x + iy)$, then

$$\begin{aligned} z + (-z) &= (x + iy) - (x + iy) \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

(ii) Subtraction of Complex Numbers

The subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and z_2. Thus, $z_1 + (-z_2) = z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$$= (x_1 - x_2) + i(y_1 - y_2).$$

(iii) Multiplication of Complex Numbers

The multiplication of z_1 with z_2 is denoted by z_1z_2 and is defined as the complex number

$$(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Thus,

$$\begin{aligned} z_1z_2 &= (x_1 + i\, y_1) \cdot (x_2 + \mathrm{i}y_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

$\Rightarrow$

$$z_1z_2 = [\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\,\mathrm{Im}(z_2)] + i\,[\mathrm{Re}(z_1)\,\mathrm{Im}(z_2) + \mathrm{Re}(z_2)\,\mathrm{Im}(z_1)]$$

Properties of Multiplication

(a) *Multiplication is commutative* For any two complex numbers z_1 and z_2, we have

$$z_1z_2 = z_2z_1.$$

(b) *Multiplication is associative* For any three complex numbers z_1, z_2, z_3, we have

$$(z_1\, z_2)\, z_3 = z_1(z_2\, z_3).$$

(c) *Existence of identity element for multiplication* The complex number $1 = 1 + oi$ is the identity element for multiplication, i.e. for every complex number z, we have
$z \cdot 1 = z = 1 \cdot z$.

(d) *Existence of multiplicative inverse* For every non-zero complex number $z = x + iy$, there exists a complex number $z_1 = x_1 + iy_1$. Such that

$$z\, z_1 = 1 = z_1\, z.$$

(e) *Multiplication of complex numbers is distributive over addition of complex numbers* For any three complex numbers z_1, z_2, z_3, we have
(i) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (Left distributivity)
(ii) $(z_2 + z_3)z_1 = z_2z_1 + z_3z_1$ (Right distributivity)

3.4 DIVISION OF COMPLEX NUMBERS

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\dfrac{z_1}{z_2} = z_1 \cdot (z_2)^{-1} = z_1 \cdot \left(\dfrac{1}{z_2}\right)$.

3.5 MODULUS OF A COMPLEX NUMBER

The modulus of a complex number $z = x + iy$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2} = \sqrt{(x^2 + y^2)}$$

Clearly, $|z| \geq 0 \ \forall z \in C$.

Properties of Modulus

If z, z_1 and z_2 be the three complex numbers, then

(i) $|z| = 0 \Leftrightarrow z = 0$, i.e. $\text{Re}(z) = \text{Im}(z) = 0$

(ii) $|z| = |\overline{z}| = |-z|$

(iii) $|z|^2 = z\overline{z}$

(iv) $|z_1 z_2| = |z_1||z_2|$

(v) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $z_2 \neq 0$

(iv) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1\,\overline{z}_2)$

(vii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\,(z_1\,\overline{z}_2)$

(viii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$

3.6 GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

Every complex number $z = x + iy$ can be represented geometrically as a point in the XY-plane. The complex number z can be represented by a point $P = (x, y)$, whose co-ordinates are x and y relative to rectangular axes X and Y. To every complex number there corresponds one and only one point in the XY-plane, conversely to every point in the plane there exists one and only one complex number. This plane is known complex plane or Argand plane. The representation of z is called Argand diagram. X and Y axes are called real and imaginary axes respectively.

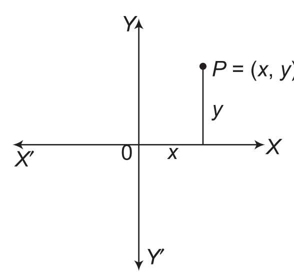


Fig. 3.1

3.7 POLAR FORM OF A COMPLEX NUMBERS

Let $P = (x, y)$ be any point in the complex plane corresponding to a complex number $z = (x, y)$. The polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, Then,

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{1}$$

$$\therefore \quad |z| = |x + iy| = \sqrt{(x^2 + y^2)} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right).$

Equation (1) is called the polar form of the complex number z. r and θ are called polar coordinates of z. r is called the absolute value of z and angle θ is called argument or amplitude of complex number z and it is denoted as $\theta = \text{amp}(z)$ or $\theta = \arg(z)$. If $-\pi \le \theta \le \pi$ and satisfies Eq. (1) then value of θ is said principal value of the amplitude.

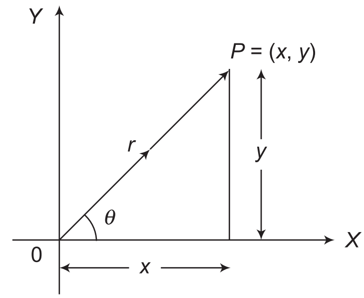


Fig. 3.2

3.8 CONJUGATE COMPLEX NUMBER

If $z = x + iy$ is any complex number, then its conjugate denoted by $\bar{z}$ is $\bar{z} = x - iy$ or $\bar{z} = (x, -y)$.

Thus, $\bar{z}$ is the mirror image of the complex point z in to real axis. This shows that $\bar{z} = z \Leftrightarrow z$ is purely a real number.

Following remarks are easy consequences of the above definition:

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

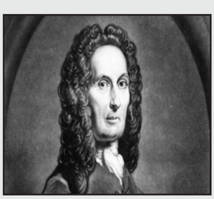
(ii) $z + \bar{z} = 2\,\text{Re}(z)$

(iii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iv) $z\bar{z}$ is real and positive unless $z = 0$

(iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

3.9 DE MOIVRE'S THEOREM



Abraham De Moivre was a French mathematician born on 26 May 1667 in Champagne, France. His father's belief in education prompted Moivre to gain a good education but he never gained a proper degree in mathematics. Due to the beginning of religious persecution in France, De Moivre's family moved to London. During his stay in London, he started excelling in mathematics and soon became a proficient mathematician with knowledge of the standard texts. He published his first paper about fluxions in Principia. After the paper was published, De Moivre also generalized Newton's renowned 'Binomial Theorem' to 'Multinomial Theorem'. He put forward many theories such as 'the centripetal force of any planet is directly related to its distance from the center of the forces and reciprocally related to the product of the diameter of the evolute and the cube of the perpendicular on the tangent. His book '*Doctrine of Chances*' published in 1718 was on the subject of probability containing many innovations such as method of approximating to functions of large numbers. He is also known for his 'De Moivre's Formula' about complex numbers. He predicted his death date based upon his minutes of sleep added up to 24 hours, which turned out to be the correct date. De Moivre died on 27 November 1754 in London.

Statement

(i) If $n \in z$ (the set of integers), then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(ii) If $n \in Q$ (the set of rational numbers), then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Remarks

(a) Notation of $\cos\theta + i\sin\theta = \text{cis}\,\theta$.

Let $z = \cos\theta + i\sin\theta$, then

$$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$$
$$= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = \text{cis}\,(-\theta).$$

(b) $(\text{cis}\,\theta)^n = \text{cis}\,n\theta$

(c) $(\text{cis}\,\theta)^{-n} = \dfrac{1}{(\text{cis}\,\theta)^n} = \cos n\theta - i\sin n\theta$

(d) $(\text{cis}\,\theta)(\text{cis}\,\alpha) = \text{cis}(\theta + \alpha)$

(e) $\dfrac{\text{cis}\,\theta}{\text{cis}\,\alpha} = \text{cis}\,(\theta - \alpha)$.

Example 1 Prove that $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\dfrac{n\pi}{2}-n\theta\right)+i\sin\left(\dfrac{n\pi}{2}-n\theta\right)$.

Solution We know that

$$(\sin\theta+i\cos\theta)(\sin\theta-i\cos\theta) = \sin^2\theta - i^2\cos^2\theta$$
$$= \sin^2\theta + \cos^2\theta \qquad [\because \; i^2 = -1]$$
$$= 1.$$

Therefore,

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \left[\frac{(\sin\theta+i\cos\theta)(\sin\theta-i\cos\theta)+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right]^n$$

$$= \left[\frac{(\sin\theta+i\cos\theta)\,(1+\sin\theta-i\cos\theta)}{(1+\sin\theta-i\cos\theta)}\right]^n = [\sin\theta + i\cos\theta]^n$$

$$= \left[\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n = \cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

Hence, proved.

Example 2 If $\cos\alpha+\cos\beta+\cos\gamma = \sin\alpha+\sin\beta+\sin\gamma = 0$, then prove that $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$.

Solution

Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$, then

$$a + b + c = (\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma)$$

$$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$
$$= 0 + i0 = 0.$$

$\therefore \quad a^3 + b^3 + c^3 = 3abc \qquad [\because \; a + b + c = 0]$

$$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta)\cdot(\cos\gamma + i\sin\gamma)$$

or $\quad (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma) = 3\left[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\right]$

or $\quad (\cos 3\alpha + \sin 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3\left[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\right]$

Equating real and imaginary parts, we get

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma) \text{ and}$$
$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$$

Hence, proved.

Example 3 Prove that $(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}} = 2(a^2+b^2)^{\frac{m}{2n}}\cdot\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$.

Solution Put

$$a = r\cos\theta \text{ and } b = r\sin\theta, \text{ we have}$$
$$r^2 = a^2 + b^2 \quad \text{or} \quad r = \sqrt{a^2+b^2}$$

and $\quad \tan\theta = \frac{b}{a} \quad$ or $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore \quad (a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}} = \left[r(\cos\theta + i\sin\theta)\right]^{\frac{m}{n}} + \left[r(\cos\theta - i\sin\theta)\right]^{\frac{m}{n}}$$
$$= r^{\frac{m}{n}}(\cos\theta + i\sin\theta)^{\frac{m}{n}} + r^{\frac{m}{n}}(\cos\theta - i\sin\theta)^{\frac{m}{n}}$$
$$= r^{\frac{m}{n}}\left[\left(\cos\frac{m}{n}\theta + i\sin\frac{m}{n}\theta\right) + \left(\cos\frac{m}{n}\theta - i\sin\frac{m}{n}\theta\right)\right]$$
$$= r^{\frac{m}{n}}\left[2\cos\frac{m}{n}\theta\right]$$
$$= 2\cdot(a^2+b^2)^{\frac{m}{2n}}\cdot\cos\left(\frac{m}{n}\cdot\tan^{-1}\frac{b}{a}\right)$$

Hence, proved.

Example 4 If $(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, show that

(i) $a_0 - a_2 + a_4 \cdots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$ (ii) $a_0 - a_3 + a_5 \cdots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$.

Solution Given $(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \qquad (2)$

Put $x = i$ both sides in Eq. (2), we get

$$(1+i)^n = a_0 + a_1 i + a_2 i^2 + a_3 i^3 + a_4 i^4 + a_5 i^5 + \cdots$$

$$= a_0 + ia_1 - a_2 - ia_3 + a_4 + ia_5 - \cdots \qquad \left[\because \quad i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i\right]$$

$$(1+i)^n = (a_0 - a_2 + a_4 - \cdots) + i(a_1 - a_3 + a_5 - \cdots) \tag{3}$$

Let $\quad 1 + i = r(\cos\theta + i\sin\theta) \qquad (4)$

Equating real and imaginary parts, both sides on Eq. (3), we get

$$r\cos\theta = 1,\ r\sin\theta = 1$$

$\therefore \quad r^2 = 1 + 1 = 2$ or $r = \sqrt{2}$

and $\quad \tan\theta = 1$ or $\theta = \dfrac{\pi}{4}$.

Thus, $\quad 1 + i = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

$$\therefore \quad (1+i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$$

$$= 2^{\frac{n}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$

or $$(1+i)^n = 2^{\frac{n}{2}}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right] \tag{5}$$

Using Eq. (5) in (3), we have

$$2^{\frac{n}{2}}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right] = (a_0 - a_2 + a_4 - \cdots) + i(a_1 - a_3 + a_5 - \cdots)$$

Equating real and imaginary parts, we get

$$a_0 - a_2 + a_4 = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

and $$a_1 - a_3 + a_5 = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$$ **Hence, proved.**

3.10 ROOTS OF A COMPLEX NUMBER

We can find all n-roots of a complex number by De-Moivre's theorem.

Since $\quad \sin\theta = \sin(2n\pi + \theta)$

and $\quad \cos\theta = \cos(2n\pi + \theta)$

or $\quad \text{cis}\,\theta = \text{cis}\,(2n\pi + \theta)$, where n is an integer.

$\therefore \quad (\text{cis}\,\theta)^{1/k} = [\text{cis}\,(2n\pi + \theta]^{1/k}$

$$= \text{cis}\left(\frac{2n\pi + \theta}{k}\right); \quad n = 0, 1, 2, ..., n-1$$

Thus, $(\text{cis } \theta)^{1/k}$ has k distinct roots.

3.11 EULER'S FORMULA

For any value of x, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Using above series, we get

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

or
$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$e^{ix} = \cos x + i \sin x \tag{6}$$

Similarly,
$$e^{-ix} = \cos x - i \sin x \tag{7}$$

Equations (6) and (7) are called Euler's formulae.

3.12 EXPONENTIAL (OR EULERIAN) FORM OF A COMPLEX NUMBER

We know that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Let z be any complex number, then in polar form z can be written as

$$z = r(\cos \theta + i \sin \theta)$$

or
$$z = r\,e^{i\theta}. \qquad \text{(Using Euler's notation)}$$

This form of z is known as exponential or Eulerian form.

3.13 CIRCULAR FUNCTIONS

For any real or complex number 'x', we have the Euler's formula

$$e^{ix} = \cos x + i \sin x \tag{8}$$

and
$$e^{-ix} = \cos x - i \sin x \tag{9}$$

Adding Eqs (8) and (9), we get

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Subtracting Eq. (9) from Eq. (8), we get

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Thus, $$\tan x = \frac{1}{i}\left[\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\right]$$

$$\cot x = i\left[\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}\right]$$

$$\operatorname{cosec} x = \frac{2i}{e^{ix} - e^{-ix}}$$

$$\sec x = \frac{2}{e^{ix} + e^{-ix}}.$$

3.13.1 Inverse Circular Function of a Complex Number

If two complex numbers $(x + iy)$ and $(u + iv)$ are connected by the relation

$\cos (x + iy) = u + iv$, then $(x + iy)$ is called cosine inverse of $(u + iv)$ and it is written as

$$(x + iy) = \cos^{-1}(u + iv).$$

Similarly, if $\sin (x + iy) = u + iv$, then $(x + iy)$ is called sine inverse of $u + iv$ and it is written as

$$(x + iy) = \sin^{-1}(u + iv).$$

Similarly, $$\tan^{-1}(u + iv) = x + iy$$

$$\cot^{-1}(u + iv) = x + iy, \text{ etc.}$$

3.13.2 The Principal Value and General Value of a Inverse Circular Function

We know that

$$u + iv = \cos (x + iy)$$

$$u + iv = \cos[2n\pi \pm (x+iy)] \qquad [\because \quad \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha]$$

$\therefore$ By the above definition the general value of inverse cosine of $u + iv$ is $2n\pi \pm (x + iy)$, and is denoted by $\cos^{-1}(u + iv)$ i.e., the first letter 'C' as capital.

The inverse cosine of $u + iv$ is a many-valued function. Its principal value is that value of $2n\pi \pm (x + iy)$ in which the real part lies between 0 and π, and it is denoted by $\cos^{-1}(u + iv)$.

Thus $$\text{Cos}^{-1} (u + iv) = 2n\pi \pm (x + iy)$$

$$\text{Cos}^{-1}(u + iv) = 2n\pi \pm \cos^{-1}(u + iv).$$

In a similar manner if $\sin (x + iy) = u + iv$, then its general value is $n\pi + (-1)^n (x + iy)$ and is denoted by $\sin^{-1}(u + iv)$.

Its principal value is that value of $n\pi + (-1)^n (x + iy)$ for which the real part lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

Thus, $\text{Sin}^{-1}(u + iv) = n\pi + (-1)^n \sin^{-1}(u + iv)$.

In the same way the other inverse circular functions defined as:

$$\text{Tan}^{-1}(u + iv) = n\pi \pm \tan^{-1}(u + iv)$$

$$\text{Sec}^{-1}(u + iv) = 2n\pi \pm \sec^{-1}(u + iv)$$

$$\text{Cot}^{-1}(u + iv) = n\pi + \cot^{-1}(u + iv)$$

$$\text{Cosec}^{-1}(u + iv) = n\pi + (-1)^n \text{cosec}^{-1}(u + iv).$$

It should be noted that the principal value for the case of cos and sec is that value for which the real part lies between 0 and π while for the case of sin, cosec, tan and cot is that value for which its real part lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

3.14 HYPERBOLIC FUNCTIONS

For any real or complex 'x', the hyperbolic sine and cosine of x is defined as:

$$\sin hx = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Other hyperbolic functions are defined as

$$\tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cot hx = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{cosec } hx = \frac{2}{e^x - e^{-x}}$$

$$\sec hx = \frac{2}{e^x + e^{-x}}.$$

3.14.1 Inverse Hyperbolic Functions

If $\cos hy = x$, then y is called inverse hyperbolic cosine of x and is defined as

$$y = \cos h^{-1} x.$$

Similarly, we can define $\sin h^{-1}x$, $\text{cosec } h^{-1}x$, $\sec h^{-1} x$, $\tan h^{-1}x$ and $\cot h^{-1}x$.

Let z be any real number, then

(i) $\cos h^{-1} z = \log\left[z + \sqrt{(z^2 - 1)}\right]$

(ii) $\sin h^{-1} z = \log\left[z + \sqrt{(z^2 + 1)}\right]$

(iii) $\tan h^{-1} z = \dfrac{1}{2} \log\left(\dfrac{1+z}{1-z}\right)$

3.14.2 Relation between Circular and Hyperbolic Function

(i) $\sin h(ix) = i \sin x$, $\cos h(ix) = \cos x$, $\tan h(ix) = i \tan x$

(ii) $\sec h(ix) = \sec x$, $\operatorname{cosec} h(ix) = -i \operatorname{cosec} x$, $\cot h(ix) = -i \cot x$

(iii) $\sin (ix) = i \sin h\, x$, $\tan (ix) = i \tan hx$, $\cos (ix) = \cos hx$

3.14.3 Real and Imaginary Parts of Hyperbolic Functions

(i) $\sin h(x + iy) = \sin hx \cos y + i \cos hx \sin y$

(ii) $\cos h(x + iy) = \cos hx \cos y - i \sin hx \sin y$

(iii) $\tan h(x + iy) = \dfrac{\sin h\, 2x}{\cos h\, 2x + \cos 2y} + i \dfrac{\sin 2y}{\cos h\, 2x + \cos 2y}$

3.15 REAL AND IMAGINARY PARTS OF CIRCULAR FUNCTION

(i) $\sin (x + iy) = \sin x \cos (iy) + \cos x \sin (iy)$

$= \sin x \cos hy + i \cos x \sin hy$

(ii) $\cos (x + iy) = \cos x \cos (iy) - \sin x \sin (iy)$

$= \cos x \cos hy - i \sin x \sin hy$

(iii) $\tan (x + iy) = \dfrac{\sin (x+iy)}{\cos (x+iy)} = \dfrac{2 \sin (x+iy) \cos (x-iy)}{2 \cos (x+iy) \cos (x-iy)}$

$= \dfrac{\sin 2x + \sin (2iy)}{\cos 2x + \cos (2iy)} = \dfrac{\sin 2x + i \sin h2y}{\cos 2x + \cos h2y}$

$= \dfrac{\sin 2x}{\cos 2x + \cos h\, 2y} + i \dfrac{\sin h2y}{\cos 2x + \cos h\, 2y}$

3.16 LOGARITHM OF A COMPLEX NUMBER

Let $(x + iy)$ and $(a + ib)$ be two complex numbers such that $(a + ib) = e^{(x+iy)}$, then $(x + iy)$ is said to the logarithm of $(a + ib)$ to the base e.

Thus $(x + iy) = \log_e(a + ib)$.

Since $e^{i2n\pi} = \cos 2n\pi + i \sin 2n\pi = 1$

$\therefore$ $e^{(x+iy)} = e^{(x+iy)} \cdot e^{i2n\pi} = e^{x+i(y+2n\pi)}$ for all $n \in Z$.

If $(x + iy)$ is the logarithm of $(a + ib)$, then $x + i(2n\pi + y)$ is also logarithm of $(a + ib)$ for all $n \in z$. The value $[x + i(2n\pi + y)]$ is called the general value of $\log (a + ib)$ and it is denoted by $\text{Log}\,(a + ib)$ and is defined as:

$$\text{Log}(a + ib) = 2n\pi i + \log (a + ib).$$

Let $a = r \cos \theta$, $b = r \sin \theta$, then $(a + ib) = r(\cos \theta + i \sin \theta) = r\, e^{i\theta}$, where $\quad r = |a+ib| = \sqrt{(a^2+b^2)}$ and $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$.

Then $\log(a+ib) = \log(re^{i\theta}) = \log r + i\theta$

or $\quad \log(a+ib) = \log(\sqrt{a^2+b^2}) + i\tan^{-1}\left(\dfrac{b}{a}\right)$

is called the principal value of the logarithm of $(a + ib)$.

Example 5 To show that $\left[i^{19} + \left(\dfrac{1}{i}\right)^{25}\right]^2 = -4$

Solution We have

$$\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = \left[i^{16}\cdot i^3 + \frac{1}{i^{24}\cdot i}\right]^2$$

$$= \left[i^3 + \frac{1}{i}\right]^2 = \left[-i + \frac{i^3}{i^4}\right]^2$$

$$= (-i + i^3)^2 = (-i - i)^2 = (-2i)^2$$

$$= 4i^2 = -4$$

Hence, proved.

Example 6 Find the least positive value of n, for which $\left(\dfrac{1+i}{1-i}\right)^n$ is real.

Solution We know that

$$\frac{1+i}{1-i} = \frac{1+i}{1-i}\times\frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2}$$

$$= \frac{1+i^2+2i}{1+1} = \frac{1-1+2i}{2} = i$$

Therefore, $\left(\dfrac{1+i}{1-i}\right)^n$ is real

$\Rightarrow$ i^n is real

$\Rightarrow$ n is a multiple of 2.

Hence, the smallest positive value of n is 2.

Example 7 Express the complex number $-1-\sqrt{3}\,i$ in Eulerian form.

Solution Let $\quad z = -1-\sqrt{3}\,i$

Modulus of z is $|z| = |-1-\sqrt{3}\,i|$

$$= \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

and argument of z is

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3} \Rightarrow \theta = \tan^{-1}\sqrt{3} = \frac{4\pi}{3}.$$

Hence, the required Eulerian form of z is $2\,e^{\frac{4\pi i}{3}}$.

Example 8 Prove that $(1+i)^n + (1-i)^n = 2^{\left(\frac{n}{2}+1\right)}\cos\left(\frac{n\pi}{4}\right)$.

Solution Let $1 + i = r(\cos\theta + i\sin\theta)$ (10)

Now $r = |1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

and $\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore$ $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (from Eq. (10))

$$(1+i)^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$

$$(1+i)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \tag{11}$$

Similarly, $(1-i)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$ (12)

Adding Eqs (10) and (11), we get

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]$$

$$= 2^{\frac{n}{2}}\cdot 2^1\cos\frac{n\pi}{4}$$

$$(1+i)^n + (1-i)^n = 2^{\left(\frac{n}{2}+1\right)}\cos\frac{n\pi}{4}$$ **Hence, proved.**

Example 9 Find $\log(1+i)$.

Solution $\log(1+i) = \log|1+i| + i\,\mathrm{amp}\,(1+i)$ $[\because \log z = \log|z| + i\,\mathrm{amp}\,(z)]$

$$= \log(\sqrt{2}) + i\cdot\frac{\pi}{4}$$

$$\log(1+i) = \frac{1}{2}\log 2 + \frac{\pi i}{4}$$

Example 10 If $\left|z - \frac{4}{z}\right| = 2$, find the maximum value of $|z|$.

Solution We have

$$|z| = \left|z - \frac{4}{z} + \frac{4}{z}\right|$$

$$\leq \left|z - \frac{4}{z}\right| + \left|\frac{4}{z}\right| \qquad (\because |z_1 + z_2| \leq |z_1| + |z_2|)$$

or $$|z| \leq 2 + \frac{4}{|z|} \qquad \left(\because \left|z - \frac{4}{z}\right| = 2\right)$$

$\Rightarrow$ $$|z|^2 - 2|z| - 4 \leq 0$$

$\Rightarrow$ $$(|z| - 1 + \sqrt{5})(|z| - 1 - \sqrt{5}) \leq 0$$

$\Rightarrow$ $$(1 - \sqrt{5}) \leq |z| \leq (1 + \sqrt{5})$$

Hence, the maximum value of $|z|$ is $(1 + \sqrt{5})$.

Example 11 Find log (log i).

Solution

$$\begin{aligned}\log(\log i) &= \log[\log(0 + i)]\\ &= \log[\log|0 + i| + i \arg(0 + i)]\\ &= \log[\log 1 + i \tan^{-1}(1/0)]\\ &= \log\left[0 + i\frac{\pi}{2}\right] = \log\left|0 + \frac{\pi i}{2}\right| + i \arg\left(0 + \frac{\pi i}{2}\right)\\ &= \log\sqrt{\frac{\pi^2}{4}} + i\tan^{-1}\left(\frac{\pi/2}{0}\right) = \log\frac{\pi}{2} + i\frac{\pi}{2}.\end{aligned}$$

Example 12 If z is a complex number having least absolute value and $|z - 2 + 2i| = 1$, then find z.

Solution We have

$$|z - 2 + 2i| = 1$$

$\Rightarrow$ $|z - (2 - 2i)| = 1$

$\Rightarrow$ z lies on a circle having centre at (2, –2) and radius 1.

In the Fig. (3.3) the complex number z is given by the point P. OP makes an angle 45° with X-axis and

$$OP = OC - CP$$

$$= \sqrt{2^2 + 2^2} - 1 = 2\sqrt{2} - 1.$$

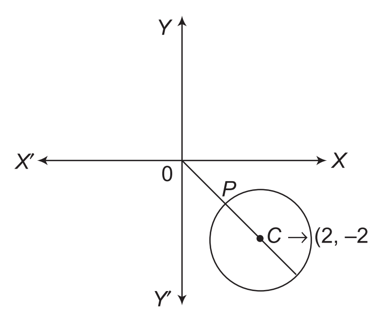


Fig. 3.3

$\therefore$ The coordinates of P are

$$\left[(2\sqrt{2}-1)\cos\frac{\pi}{4},\ -(2\sqrt{2}-1)\sin\frac{\pi}{4}\right] \text{ i.e., } \left[\left(2-\frac{1}{\sqrt{2}}\right),-\left(2-\frac{1}{\sqrt{2}}\right)\right]$$

Hence, $$z=\left(2-\frac{1}{\sqrt{2}}\right)+\left[-\left(2-\frac{1}{\sqrt{2}}\right)\right]i=\left(2-\frac{1}{\sqrt{2}}\right)(1-i).$$

Example 13 Find the smallest positive integer n for $(1+i)^{2n}=(1-i)^{2n}$.

Solution We have

$$(1+i)^{2n}=(1-i)^{2n}$$

or $$\left(\frac{1+i}{1-i}\right)^{2n}=1$$

or $$\left[\frac{(1+i)^2}{(1+i)(1-i)}\right]^{2n}=1$$

or $$i^{2n}=1$$

$\Rightarrow$ $2n$ is a multiple of 4.

Hence, the smallest positive value of n is 2.

EXERCISE 3.1

1. Find the maximum value of $|z|$, when z satisfies the condition $\left|z+\frac{2}{z}\right|=2$.
2. Find the value of i^i.
3. If $z=x+iy$ and $w=\frac{1-iz}{z-i}$, show that $|w|=1 \Rightarrow z$ is purely real.
4. If n is a positive integer, prove that $(\sqrt{3}+i)^n+(\sqrt{3}-i)^n=2^{n+1}\cos\frac{n\pi}{6}$.
5. Find the value of $\sum_{n=1}^{13}(i^n+i^{n+1})$, where $i=\sqrt{-1}$.
6. Find the argument of $\frac{1+\sqrt{3}\,i}{1+\sqrt{3}}$.
7. If $iz^2-\bar{z}=0$ and $\bar{z}$ is the complex conjugate of z, where $i=\sqrt{-1}$ find values of $|z|$.
8. Show that $\arg(z)+\arg(\bar{z})=2n\pi$.
9. The sum and the product of two complex numbers are both real, show that the two numbers are either both real or complex conjugates.

Answers

1. $(\sqrt{3}+1)$ 2. $e^{-\frac{\pi}{2}}$

5. $i-1$ 6. $\frac{\pi}{3}$

7. $|z| = 0, 1$

3.17 SUMMATION OF TRIGONOMETRIC SERIES – (C + iS) METHOD

This method can be applied to find the sum of the series of the form:

$$a_0 \sin\alpha + a_1 \sin(\alpha+\beta) + a_2 \sin(\alpha+2\beta) + a_3 \sin(\alpha+3\beta) + \cdots$$

or
$$a_0 \cos\alpha + a_1 \cos(\alpha+\beta) + a_2 \cos(\alpha+2\beta) + a_3 \cos(\alpha+3\beta) + \cdots$$

Method Putting the given series is equal to S(or C) according as it is a series of sines (or cosines).

Let
$$S = a_0 \sin\alpha + a_1 \sin(\alpha+\beta) + a_2 \sin(\alpha+2\beta) + a_3 \sin(\alpha+3\beta) + \cdots$$

and
$$C = a_0 \cos\alpha + a_1 \cos(\alpha+\beta) + a_2 \cos(\alpha+2\beta) + a_3 \cos(\alpha+3\beta) + \cdots$$

Multiplying the series of sines by i and adding to the sum of cosines, then we get the series of complex numbers as

$$C + iS = a_0(\cos\alpha + i\sin\alpha) + a_1\left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right] + a_2\left[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)\right] + a_3\left[\cos(\alpha+3\beta) + i\sin(\alpha+3\beta)\right] + \cdots$$

$$= a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha+2\beta)} + a_3 e^{i(\alpha+3\beta)} + \cdots \qquad \left[\because \cos\theta + i\sin\theta = e^{i\theta}\right]$$

or
$$C + iS = e^{i\alpha}\left[a_0 + a_1 e^{i\beta} + a_2 e^{2i\beta} + a_3 e^{3i\beta} + \cdots\right]$$

$$= e^{i\alpha}\cdot\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right], \text{ where } x = e^{i\beta}$$

or
$$C + iS = e^{i\alpha}\cdot f(x).$$

The series represented by $f(x)$ can be sum up, if it is in any one of the following:

(i) Gregory's series
(ii) Exponential series
(iii) Logarithmic series
(iv) Geometric series
(v) sine, cosine, sinh or cosh series
(vi) Binomial series

The following standard series will be used:

(i) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\infty$

$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\infty = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

(ii) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \infty$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \infty$$

(iii) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots \infty$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \infty \right)$$

(iv) $a + ar + ar^2 + \cdots \text{to } n \text{ terms} = \dfrac{a(1-r^n)}{1-r}$

$$a + ar + ar^2 + \cdots \infty = \frac{a}{1-r} \text{ if } |r| < 1$$

(v) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots \infty$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \infty$$

(vi) $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!} x^2 + \dfrac{n(n-1)(n-2)}{3!} x^3 + \cdots \infty$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \cdots \infty$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \cdots \infty$$

Example 14 If $C = \cos^2\theta - \dfrac{1}{3}\cos^3\theta \cos 3\theta + \dfrac{1}{5}\cos^5\theta \cos 5\theta - \cdots \infty$, then show that $\tan 2C = 2\cot^2\theta$.

Solution Given

$$C = \cos\theta \cdot \cos\theta - \frac{1}{3}\cos^3\theta \cdot \cos 3\theta + \frac{1}{5}\cos^5\theta \cos 5\theta - \cdots \infty$$

Let $S = \cos\theta \cdot \sin\theta - \dfrac{1}{3}\cos^3\theta \cdot \sin 3\theta + \dfrac{1}{5}\cos^5\theta \cdot \sin 5\theta - \cdots \infty$

$$\therefore \quad C + iS = \cos\theta(\cos\theta + i\sin\theta) - \frac{1}{3}\cos^3\theta(\cos 3\theta + i\sin 3\theta) + \frac{1}{5}\cos^5\theta(\cos 5\theta + i\sin 5\theta) - \cdots$$

$$= \cos\theta \cdot e^{i\theta} - \frac{1}{3}\cos^3\theta \cdot e^{3i\theta} + \frac{1}{5}\cos^5\theta e^{5i\theta} - \cdots\infty$$

$$= \cos\theta\, e^{i\theta} - \frac{1}{3}(\cos\theta \cdot e^{i\theta})^3 + \frac{1}{5}(\cos\theta\, e^{i\theta})^5 - \cdots$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots \qquad \left[\because \quad x = \cos\theta \cdot e^{i\theta}\right]$$

$$= \tan^{-1}x$$

$$C + iS = \tan^{-1}(\cos\theta \cdot e^{i\theta})$$

$$C + iS = \tan^{-1}[\cos\theta(\cos\theta + i\sin\theta)] \tag{13}$$

Replace $-i$ by i in Eq. (13) we get,

$$C - iS = \tan^{-1}[\cos\theta\,(\cos\theta - i\sin\theta] \tag{14}$$

Adding Eqs (13) and (14), we get

$$2C = \tan^{-1}[\cos\theta(\cos\theta + i\sin\theta)] + \tan^{-1}[\cos\theta(\cos\theta - i\sin\theta)]$$

or $$2C = \tan^{-1}\left[\frac{\cos\theta(\cos\theta + i\sin\theta) + \cos\theta(\cos\theta - i\sin\theta)}{1 - \cos\theta(\cos\theta + i\sin\theta)\cdot\cos\theta(\cos\theta - i\sin\theta)}\right]$$

$$= \tan^{-1}\left[\frac{2\cos^2\theta}{1 - \cos^2\theta\,(\cos^2\theta + \sin^2\theta)}\right]$$

$$= \tan^{-1}\left[\frac{2\cos^2\theta}{1 - \cos^2\theta}\right] = \tan^{-1}\left[\frac{2\cos^2\theta}{\sin^2\theta}\right]$$

$$2C = \tan^{-1}\left[2\cot^2\theta\right]$$

or $\boxed{\tan 2C = 2\cot^2\theta}$ **Hence, proved.**

Example 15 Find the sum to infinite terms of the series

$$1 + \frac{x^2\cos 2\theta}{2!} + \frac{x^4\cos 4\theta}{4!} + \cdots$$

Solution Let

$$C = 1 + \frac{x^2\cos 2\theta}{2!} + \frac{x^4\cos 4\theta}{4!} + \cdots$$

$$S = \frac{x^2\sin 2\theta}{2!} + \frac{x^4\sin 4\theta}{4!} + \cdots$$

$$\therefore \qquad C + iS = 1+\frac{x^2}{2!}[\cos 2\theta + i\sin 2\theta] + \frac{x^4}{4!}[\cos 4\theta + i\sin 4\theta] + \cdots$$

$$= 1+\frac{x^2}{2!}e^{2i\theta}+\frac{x^4}{4!}e^{4i\theta}+\cdots$$

$$= 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots, \text{ where } z = xe^{i\theta}$$

$$= \cosh z.$$

or

$$C + iS = \cosh (x\, e^{i\theta}) = \cosh [x(\cos\theta + i\sin\theta)]$$

$$= \cos i[x(\cos\theta + i\sin\theta)] = \cos[ix\cos\theta - x\sin\theta]$$

$$= \cos(ix\cos\theta)\cos(x\sin\theta) + \sin(ix\cos\theta)\cdot\sin(x\sin\theta)$$

$$C + iS = \cosh (x\cos\theta)\cdot\cos(x\sin\theta) + i\sinh (x\cos\theta)\cdot\sin(x\sin\theta).$$

Equating the real parts, we have

$$C = \cosh (x\cos\theta)\cdot\cos(x\sin\theta)$$

3.18 INTRODUCTION TO THEORY OF COMPLEX VARIABLES

When we study the real number system, we have seen that there does not exist any real number whose square is a negative real number. Thus, the concept of $\sqrt{-1}$ is not valid. Euler (1707–1783) was the first mathematician who introduced the symbol i with the proper $i = \sqrt{-1}$. It was Gauss (1777–1855) who first studied that an algebraic equation with real coefficients has complex roots of the form $x + iy$, where x and y are real numbers. It is a powerful method which is useful in the study of fluid dynamics, electrostatics and heat flow.

3.19 BASIC CONCEPTS OF THE COMPLEX VARIABLE

1. *Point set:* Any collection of points in the complex plane is called a point set and each point of the set is called a member or element of the set.
2. *Neighbourhoods:* The neighbourhood of a point z_0 is set of all points z, such that.

   $$|z - z_0) < \delta, \text{ where } \delta \text{(delta) is any given positive number.}$$

   A deleted δ-neighbourhood of z_0 is a neighbourhood of z_0 in which the point z_0 is omitted, i.e. $0 < |z - z_0| < \delta$.
3. *Limit point:* A point z_0 is called a limit point of a point set S if every deleted δ-neighbourhood of z_0 contains points of S. The limit of a set S may or may not belong to the set S, and it is also known as limiting point, cluster point or point of accumulation.
4. *Closed set:* A point set S is said to be closed if every limit point of S belongs to S, i.e.,
   If set S contains all its limit points.
   For example, the set of all points z such that

   $$|z| \le 1 \text{ is a closed set.}$$
5. *Bounded set:* A set S is called bounded if we can find a constant M, such that $|z| < M$ for every point $z \in S$. An unbounded set is one which is not bounded.
   A set which both bounded and closed is called compact.

6. *Interior, exterior and boundary points:* A point z_0 is said to be an interior point of a set S if their exists a δ-neighbourhood of z_0 contains points belong to S. If every δ-neighbourhood of z_0 contains points belonging to set S and also points not belonging to S, then z_0 is called a boundary point.
   If a point is not an interior or boundary point of a set S, then it is an exterior point of S.
7. *Open set:* An open set is a set which consists only of interior points.
   For example, the set of points z such that $|z| < 1$ is an open set.
8. *Connected set:* An open set S is said to be connected if any two points of the set S are joined by a polygonal path, all the points of which lie in set S.
9. *Domain:* A set S is said to be domain if every point of set S is an interior point and connected; a domain is denoted by D.
10. *Region:* A region is a domain together with all, some or none of its boundary points. Thus, a domain is always a region but a region may or may not be a domain.
    For example, an open disk is both a domain and a region but a closed disk is a region and not a domain. A region is denoted by R.
11. *Circle:* $|z - z_0| < \rho$, represents a circle with centre z_0 and radius ρ.
12. *Complex variable:* If a symbol z which can stand for any one of a set of complex numbers is called a complex variable.
13. *Function of a complex variable:* We define the function of a variable in a similar way as the function of a real variable. Let S_1 and S_2 be two non-empty sets of complex numbers. If there is a rule f, which assigns a complex number w in S_2 for each z in S_1, then f is said to be a complex valued function of a complex variable z and is denoted by

$$w = f(z).$$

The set S_1 is called the domain of definition of f and the set S_2 is called the range of f.

$$w = f(z) = u(x, y) + iv(x, y)$$

Here $u(x, y)$ and $v(x, y)$ real valued functions of x and y, are known as the real and imaginary parts of the functions w.

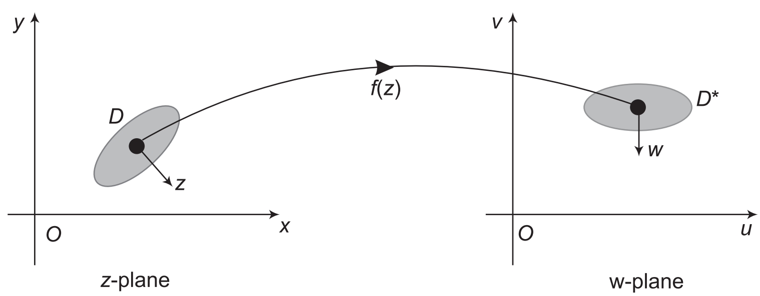


Fig. 3.4

14. *Single valued and multiple valued functions:* w is said to be single-valued or multiple valued function of z according as for a given value of z corresponds, one or more than one values of w.
15. *Limit:* A number l is said to be the limit of $f(z)$ as $z \to z_0$ and is denoted by

$$\lim_{z \to z_0} f(z) = l.$$

If for every $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that $|f(z) - l| < \varepsilon$ whenever $|z - z_0| < \delta$.

16. *Continuity:* A function $f(z)$ is said to be continuous at $z = z_0$ if an arbitrary $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\left|f(z) - f(z_0)\right| < \varepsilon, \text{ whenever } \left|z - z_0\right| < \delta.$$

If follows from the above definition that $f(z)$ will be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

If a function $f(z)$ is said to be continuous in a domain D if it is continuous at every point of D.

A function $f(z)$ is not continuous at $z = z_0$ if $\lim\limits_{z \to z_0} f(z)$ does not exist or $\lim\limits_{z \to z_0} f(z) \neq f(z_0)$ is called discontinuous at $z = z_0$.

***Remarks*:**

(1) If $f(z)$ and $g(z)$ are two continuous functions in a domain D, then their sum $f + g$, difference $f - g$, product $f \cdot g$ and quotient f/g are all continuous in D.

(2) If $f(z) = u(x, y) + iv(x, y)$ is continuous if both u and v are continuous.

17. *Differentiability:* Let $f(z)$ be a single valued function defined in a domain D. The function $f(z)$ is said to be differentiable at a point $z = z_0$, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

This limit is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$

$$\therefore \qquad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

***Note*:** If function $f(z)$ is differentiable at $z = z_0$, then it must be continuous at $z = z_0$.

18. *Analytic function:* A function $f(z)$ is said to be analytic at a point $z = z_0$, if it is differentiable at the point z_0 and also at each point in some neighbourhood of the point z_0. Thus, the analyticity at z_0 means $f(z)$ is differentiable is some open disk about z_0.

A function $f(z)$ is said to be analytic in a domain D, if it is analytic at every point in D. Analytic function is also known as regular or holomorphic functions.

19. Entire function: A function $f(z)$ which is analytic everywhere is said to be an entire function. For example, a polynomial of any degree is an entire function.

***Note*:** Entire $\Rightarrow$ Analytic $\Rightarrow$ Differentiable $\Rightarrow$ Continuous, but not vice-versa.

3.20 CAUCHY–REIMANN EQUATIONS

Cauchy–Reimann (C.R.) equations are used to determine whether a given function $f(z)$ is analytic or not.

Necessary conditions for a function to be analytic: According to the following theorems.

Theorem Suppose that the function $f(z) = u(x, y) + iv(x, y)$ is continuous in some neighbourhood of the point $z = x + iy$ and is differentiable at z. Then, the first order partial derivatives of $u(x, y)$ and $v(x, y)$ exist and satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at the point } z.$$

Proof: Since the given function $f(z)$ is differentiable at z, we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\right] - \left[u(x, y) + iv(x, y)\right]}{(\Delta x + i\Delta y)} \tag{15}$$

Since the limit exists, it must have the same value independent of the path along which $\Delta z \to 0$. We consider the following two parts.

(i) Let $\Delta y \to 0$ and then $\Delta x \to 0$. The limit in Eq. (15) becomes, $(\Delta z = \Delta x)$

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{\left[u(x+\Delta x, y) - u(x, y)\right]}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right]$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial V}{\partial x} \tag{16}$$

(ii) Let $\Delta x \to 0$ first and then $\Delta y \to 0$. The limit of Eq. (15) becomes $(\Delta z = i\Delta y)$

$$f'(z) = \lim_{\Delta y \to 0} \left[\frac{\left[u(x, y+\Delta y)\right]}{i\Delta y} + i \frac{v(x, y+\Delta y) - v(x, y)}{i\Delta y} \right] = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

or $$f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \tag{17}$$

Since, $f(z)$ is differentiable at z, the two limits given in Eqs. (16) and (17) must be equal. Therefore,

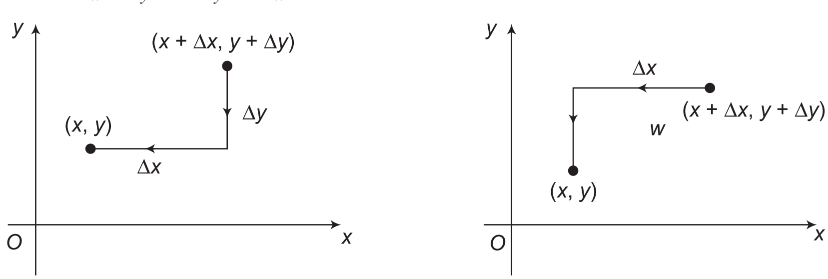
$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Comparing the real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{18}$$

Or in short notation

$$u_x = v_y \text{ and } u_y = -v_x.$$



(a) Along the path $\Delta y \to 0$, then $\Delta x \to 0$ (b) Along the path $\Delta x \to 0$, then $\Delta y \to 0$

Fig. 3.5

The Eq. (18) is called the Couchy–Riemann equations and are the necessary conditions for differentiability and analyticity of the function $f(z)$ at a given point. Thus if the function $f(z)$ does not satisfy the Couchy–Reimann equations at a point, it is not differentiable and hence not analytic at that point.

Sufficient conditions for a function f(z) to be analytic:

Theorem Let $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of the function $f(z) = u(x, y) + iv(x, y)$ and have the continuous first order partial derivatives in a domain D. If $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Reimann equations at all points in D, then the function $f(z)$ is analytic in D and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Proof Since the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous, we can write

$$\Delta u = u(x+\Delta x,\ y+\Delta y) - u(x,\ y) = u_x \Delta x + u_y\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\Delta v = v(x+\Delta x,\ y+\Delta y) - v(x.\ y) = v_x \Delta x + v_y\ \Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Now $\quad \Delta w = f(z+\Delta z) - f(z) = \Delta u + i\ \Delta v$

$$\Delta w = (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y.$$

Using the Cauchy–Reimann equations, we get

$$\Delta w = (u_x + iv_x)\Delta x + (-v_x + iu_x)\Delta y + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y$$

Now, $$\left|\frac{f(z+\Delta z) - f(z)}{\Delta z} - (u_x + iv_x)\right| \le |\varepsilon_1 + i\varepsilon_3| \cdot \left|\frac{\Delta x}{\Delta z}\right| + |\varepsilon_2 + i\varepsilon_4| \cdot \left|\frac{\Delta y}{\Delta z}\right|.$$

Since $\left|\frac{\Delta x}{\Delta z}\right| \le 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \le 1$, we obtain $\lim\limits_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z) = u_x + iv_x = v_y - iu_y.$

Therefore, $f(z)$ is differentiable at an arbitrary point z in D, i.e., the function $f(z)$ is analytic in D.

3.20.1 Cauchy–Reimann Equation of the Function *f*(*z*) = *u*(*x*, *y*) + *iv*(*x*, *y*) at Origin

$$f(z) = u(x, y) + iv(x, y) \tag{19}$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \cdot \frac{u(x, 0) - u(0, 0)}{x}$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \cdot \frac{u(0, y) - u(0, 0)}{y}$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \cdot \frac{v(x, 0) - v(0, 0)}{x}$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \cdot \frac{v(0, y) - v(0, 0)}{y}$$

If at $z = 0$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Thus the C–R equations are satisfied at $z = 0$.

3.20.2 Polar Form of the Cauchy–Reimann Equations

Let $f(z) = u(r, \theta) + iv(r, \theta)$, where $z = re^{i\theta}$

We have $x = r\cos\theta,\ y = r\sin\theta$

$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

and $$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Using chain rule of differentiation, we get

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial u}{\partial x} + (\sin\theta)\frac{\partial u}{\partial y} \tag{20}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial u}{\partial x} + (r\cos\theta)\frac{\partial u}{\partial y} \tag{21}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial r}{\partial x} + (\sin\theta)\frac{\partial v}{\partial y} \tag{22}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial v}{\partial x} + (r\cos\theta)\frac{\partial v}{\partial y} \tag{23}$$

Using the Cauchy–Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we can write Eqs (22) and (23) as

$$\frac{\partial v}{\partial r} = (-\cos\theta)\frac{\partial u}{\partial y} + (\sin\theta)\frac{\partial u}{\partial x}$$

$$= -\frac{1}{r}\left[(-r\sin\theta)\frac{\partial u}{\partial x} + (r\cos\theta)\frac{\partial u}{\partial y}\right] = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{(Using Eq. 21)}$$

and $$\frac{\partial v}{\partial \theta} = (-r\sin\theta)\left(-\frac{\partial y}{\partial y}\right) + (r\cos\theta)\frac{\partial u}{\partial x}$$

$$= r\left[(\cos\theta)\frac{\partial u}{\partial x} + (\sin\theta)\frac{\partial u}{\partial y}\right] = r\frac{\partial u}{\partial r}. \quad \text{(Using Eq. 20)}$$

Therefore, the Cauchy–Reimann equations in polar form are as follows:

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ and } \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}.$$

George Friedrich Bernhard Riemann was born on 17 September 1826, in the village of Breselenz near Dannenberg, Germany. His father Friedrich Bernhard Riemann was a pastor and his mother was Charlotte Ebell who died when he was just a child. Riemann was under confident as a child with a fear of public speaking and had many nervous breakdown attacks. On the other hand, he was a gifted mathematics genius with exceptional calculation aptitude. He used to solve extremely complex math problems surprising even his teachers. His teacher was Carl Friedrich Gauss, who encouraged him to talk to his parents and switch to a degree in Mathematics rather than theology. Once getting their approval, Riemann transferred to the University of Berlin in 1847 and remained there for the next two years. He had an extraordinary command over complex analysis which he interconnected with topology and number theory. Other revolutionary contributions include the tensor analysis, theory of functions, differential geometry and the most notable being the theory of manifolds. His work in geometry defined new probabilities by generalizing the notions of distance and curvature. Many theorems are named after him; for example, the Reimann-Roch theorem. Seeing the brilliance of Reimann, efforts were made in order to promote him to a position of an extraordinary Professor. This, however, could not be done and he was paid like any other professor in the University of Göttingen. He was later made the Head of Mathematics Department. He spent the final days of his life in Italy in the village of Selasca with his wife and daughter. Riemann died on 20 July 1866.

Example 16 Prove that the function $f(z) = |z|^2$ is continuous everywhere but nowhere differentiable except at the origin.

Solution The given function $f(z) = |z|^2 = x^2 + y^2$ is continuous everywhere.

Now $$f'(z_0) = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)\cdot(\bar{z}_0 + \Delta \bar{z}) - z_0\,\bar{z}_0}{\Delta z} = \lim_{\Delta z \to 0}\left[\bar{z}_0 + \Delta \bar{z} + z_0 \cdot \frac{\Delta \bar{z}}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0}\left[\bar{z}_0 + z_0 \cdot \frac{\Delta \bar{z}}{\Delta z}\right] \qquad (\because \quad \Delta \bar{z} \to 0, \text{ as } \Delta z \to 0)$$

Therefore at $z_0 = 0$, this limit is zero so that $f'(0) = 0$.

When $z_0 \neq 0$, let $\Delta z = r(\cos\theta + i\sin\theta)$, then $\Delta \bar{z} = r(\cos\theta - i\sin\theta)$

$$\frac{\Delta \bar{z}}{\Delta z} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} = \cos 2\theta - i\sin 2\theta.$$

This limit depends upon the arg Δz, so that which does not tend to unique unit.

Hence, the function $f(z)$ is not differentiable for any non-zero value of z.

Example 17 If $\omega = f(z) = u + iv$ be an analytic function of $z = x + iy$, show that the curves u = constant, v = constant represented on the z-plane intersect at right angles.

Solution Since $f(z) = u + iv$ be an analytic function of z, the functions u and v will satisfy C–R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Multiplying these, we obtain.

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}=0$$

which is the condition that the curves u = constant and v = constant intersect at right angles.

Hence, if $f(z)$ is a regular function of z, then the curves

$u = R[f(z)]$ = constant and $v = I[f(z)]$ = constant form an orthogonal system, i.e. they intersect at right angles.

Example 18 Determine whether the following function $f(z)=\frac{x}{x^2+y^2}+i\cdot\frac{y}{x^2+y^2}$ of the complex variable $z = x + iy$ is an analytic function.

Solution Here

$$u(x, y)=\frac{x}{x^2+y^2} \text{ and } v(x, y)=\frac{y}{x^2+y^2}.$$

Therefore,
$$\frac{\partial u}{\partial x}=\frac{(x^2+y^2)\cdot 1-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

and
$$\frac{\partial u}{\partial y}=x\cdot(-1)(x^2+y^2)^{-2}\cdot(2y)=-\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x}=y(-1)(x^2+y^2)^{-2}(2y)=-\frac{2xy}{(x^2+y^2)^2}$$

and
$$\frac{\partial v}{\partial y}=\frac{(x^2+y^2)\cdot 1-y\cdot 2y}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

Since, $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$

∴ Cauchy–Riemann conditions are not satisfied. Hence, the given function is not analytic.

Example 19 Show that the function $f(z) = xy + iy$ is everywhere continuous but not analytic.

[Bundelkhand 2003]

Solution Given $f(z) = u + iv = xy + iy$

∴ $u(x, y) = xy$ and $v(x, y) = y$.

Since u and v are polynomials of x and y. Thus both u and v are continuous everywhere. Hence, $f(z)$ is continuous everywhere.

Now,
$$\frac{\partial u}{\partial x}=y,\ \frac{\partial u}{\partial y}=x,\ \frac{\partial v}{\partial x}=0 \text{ and } \frac{\partial v}{\partial y}=1.$$

Thus, we have $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}$.

∴ Cauchy–Riemann equations/conditions are not satisfied anywhere.

Hence, $f(z)$ is not analytic at any point. **Hence, proved.**

Example 20 Prove that $f(z) = \sinh z$ is an analytic function of the complex variable $z = x + iy$.

Solution Given $f(z) = u + iv = \sinh z$

$$= \frac{e^z - e^{-z}}{2}$$

or $$u + iv = \frac{e^{x+iy} - e^{-(x+iy)}}{2} = \frac{e^x \cdot e^{iy} - e^{-x} \cdot e^{-iy}}{2}$$

$$= \frac{1}{2}\left[e^x(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)\right]$$

or $$u + iv = \frac{1}{2}\left[(e^x - e^{-x})\cos y + i(e^x + e^{-x})\cdot \sin y\right]$$

$$= \left(\frac{e^x - e^{-x}}{2}\right)\cos y + i\left(\frac{e^x + e^{-x}}{2}\right)\sin y$$

$$u + iv = \sinh x \cdot \cos y + i \cosh x \sin y.$$

$\therefore$ $$u(x, y) = \sinh x \cos y \quad \text{and} \quad v(x, y) = \cosh x \sin y$$

Now, $$\frac{\partial u}{\partial x} = \cosh x \cos y, \quad \frac{\partial u}{\partial y} = -\sinh x \sin y$$

and $$\frac{\partial v}{\partial x} = \sinh x \sin y, \quad \frac{\partial v}{\partial y} = \cosh x \cos y.$$

Thus, we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$\therefore$ Cauchy–Reimann equations are satisfied. Hence, the given function is analytic.

3.21 HARMONIC AND CONJUGATE HARMONIC FUNCTIONS

Harmonic function: Any function of x, y which has first and second order continuous partial derivatives and satisfies the Laplace equation is called harmonic function.

3.21.1 Conjugate Harmonic Function

If $f(z) = u(x, y) + iv(x, y)$ be an analytic function, then the real part $u(x, y)$ of $f(z)$ is known as the conjugate harmonic function of $v(x, y)$ and vice-versa.

Theorem The real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ are harmonic.

Proof: $f(z) = u(x, y) + iv(x, y)$ be an analytic function.

Then Cauchy–Reimann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{are satisfied}$$

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{24}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{25}$$

Differentiating Eqs (24) and (25) partially w.r.t. x and y respectively, we get

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial x\,\partial y} \\ \frac{\partial^2 u}{\partial y^2} &= -\frac{\partial^2 v}{\partial y\,\partial x} \end{aligned}\right\} \tag{26}$$

Adding Eqs (26), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} = 0; \quad \text{Since } \frac{\partial^2 v}{\partial y\,\partial x} = \frac{\partial^2 v}{\partial x\,\partial y}$$

or $$\nabla^2 u = 0.$$

Thus, u is a solution of Laplace equation. Hence, u is a harmonic function.

Similarly, differentiating Eqs (24) and (25) partially w.r.t. y and x, we get

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 v}{\partial y^2} \tag{27}$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{\partial^2 v}{\partial x^2}. \tag{28}$$

Adding Eqs (27) and (28), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial^2 u}{\partial y\,\partial x} = 0$$

or $$\nabla^2 v = 0.$$

Thus v is a solution of Laplace equation. Hence, v is a harmonic function.

3.22 METHOD OF CONSTRUCTING CONJUGATE FUNCTION

If $f(z) = u(x, y) + iv(x, y)$ be an analytic function, where $u(x, y)$ is known then the conjugate function $v(x, y)$ is determined as follows:

We know that $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy. \tag{29}$$

Since function $f(z)$ is analytic, so by C–R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \text{ then Eq. (29) becomes}$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy. \tag{30}$$

Equation (30) is of the form $dv = M(x, y)\,dx + N(x, y)\,dy$, where

$$M = -\frac{\partial u}{\partial y} \text{ and } N = \frac{\partial u}{\partial x}$$

$$\therefore \qquad \frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}.$$

Function $u(x, y)$ is harmonic if $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

or $$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \text{ so that } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

∴ Equation (30) satisfy the condition of an exact differential equation. Then $v(x, y)$ can be determined by integrating of Eq. (30).

Thus, we have

$$\begin{aligned} v(x, y) &= -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy \\ &= -\int M\, dx + \int (\text{the terms of } N \text{ which is independent of } x)\, dy + c. \end{aligned}$$

If $u(u, y)$ is given, $v(x, y)$ can be determined. If $v(x, y)$ is known, then $u(x, y)$ can be determined by using $u(x, y) = -\int M_1 dx + \int (\text{the terms of } N_1 \text{ independent of } x)\, dy + d$ (constant)

When $$M_1 = \frac{\partial v}{\partial y}$$

$$N_1 = \frac{\partial v}{\partial x}.$$

3.23 METHOD OF CONSTRUCTING AN ANALYTIC FUNCTION OR A REGULAR FUNCTION

Milne–Thomson's Method

Using Milne–Thomson's method, the analytic function $f(z) = u(x, y) + iv(x, y)$ is directly constructed, without finding $v(x, y)$, if $u(x, y)$ is given and vice-versa.

Since $z = x + iy$ and $\overline{z} = x - iy$

Therefore, $x = \dfrac{z + \overline{z}}{2}$, and $y = \dfrac{z - \overline{z}}{2i}$, then

$f(z) = u(x, y) + iv(x, y)$ can be written as

$$f(z) = u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + iv\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) \tag{31}$$

Equation (31) can be regarded as a formal identity in two independent variables z and $\overline{z}$

On putting $\overline{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0) \tag{32}$$

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

or $$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(By C–R equations)}$$

If we write $\frac{\partial u}{\partial x} = \phi_1(x, y)$ and $\frac{\partial u}{\partial y} = \phi_2(x, y)$ then

$$f'(z) = \phi_1(x, y) - i\,\phi_2(x, y)$$

$$f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0). \qquad [\text{Replacing } x \text{ by } z \text{ and } y \text{ by } 0]$$

Integrating both sides, we get

$$f(z) = \int \phi_1(z, 0)\,dz - i\int \phi_2(z, 0)\,dz + C,$$

where C is an arbitrary constant.

Similarly if $v(x, y)$ is given, we have

$$f(z) = \int \psi_1(z, 0)\,dz - i\int \psi_2(z, 0)\,dz + D,$$

where $\frac{\partial v}{\partial y} = \psi_1(x, y), \frac{\partial v}{\partial x} = \psi_2(x, y)$ and D is an arbitrary constant.

Example 21 Find the analytic function $f(z) = u(x, y) + iv(x, y)$ of which the real part $u(x, y) = e^x(x \cos y - y \sin y)$.

Solution Given $u(x, y) = e^x(x \cos y - y \sin y)$

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x \cos y$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$$

$$\therefore \quad \phi_1(x, y) = \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y).$$

By Milne's Thomson method;

$$f(z) = \int\left[\phi_1(z, 0) - i\,\phi_2(z, 0)\right]dz + C$$

$$= \int\left[e^z(z+1) - i\,0\right]dz + C = \int(ze^z + e^z)\,dz + C$$

$$= (z - 1)\,e^z + e^z + C = z\,e^z + C$$

Another method

We have $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \text{[By } C\text{–}R \text{ equations]}$$

$$dv = -e^x(-x\sin y - y\cos y - \sin y)\,dx + e^x(x\cos y - y\sin y + \cos y)\,dy.$$

Integrating, both sides, we get

$$v = \int e^x(x\sin y + y\cos y + \sin y)\,dx + \int(\text{those terms which do not contain } x)\,dy + C.$$

$$v = \sin y\int xe^x\,dx + (y\cos y + \sin y)\int e^x dx + \int 0\cdot dy + C$$

$$v = \left[(x-1)\sin y + y\cos y + \sin y\right]e^x + C$$

$$v(x, y) = (x\sin y + y\cos y)\,e^x + C.$$

$$\therefore \qquad f(z) = u(x, y) + iv(x, y)$$

$$= e^x(x\cos y - y\sin y) + i\left[e^x(x\sin y + y\cos y) + C\right]$$

$$= xe^x(\cos y + i\sin y) + iy\,e^x(\cos y + i\sin y) + iC.$$

$$= (x + iy)\,e^x\cdot e^{iy} + D = ze^z + D.$$

Example 22 Find the analytic function of which the real part is $e^{-x}[(x^2 - y^2)\cos y + 2xy\sin y]$.

Solution Given $u(x, y) = e^{-x}[(x^2 - y^2)\cos y + 2xy\sin y]$, then

$$\frac{\partial u}{\partial x} = e^{-x}[2x\cos y + 2y\sin y] - e^{-x}\left[(x^2 - y^2)\cos y + 2xy\sin y\right]$$

$$\frac{\partial u}{\partial y} = e^{-x}\left[-2y\cos y - (x^2 - y^2)\sin y + 2x\sin y + 2xy\cos y\right]$$

$$\therefore \qquad \phi_1(x, y) = \frac{\partial u}{\partial x} = e^{-x}\left[2x\cos y + 2y\sin y\right] - e^{-x}\left[(x^2 - y^2)\cos y + 2xy\sin y\right]$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = e^{-x}\left[-2y\cos y - (x^2 - y^2)\sin y + 2x\sin y + 2xy\cos y\right].$$

By Milne's Thomson method

$$f(z) = \int\left[\phi_1(z, 0) - i\,\phi_2(z, 0)\right]dz + C$$

$$= \int\left[e^{-z}(2z - z^2)\right]dz + C = -(2z - z^2)\,e^{-z} + \int(2 - 2z)\,e^{-z}dz + C$$

$$= -(2z - z^2)\,e^{-z} + (2 - 2z)(-e^{-z}) - \int 2e^{-z}dz + C$$

$$f(z) = z^2\,e^{-z} + C$$

Example 23 If $w = u + iv = \log z$, find $\dfrac{dw}{dz}$ and determine, whether w is non-analytic.

Solution Given $\quad w = u + iv = \log z = \log(x + iy)$

$$w = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore$ $$u(x, y) = \frac{1}{2}\log(x^2 + y^2) \text{ and } v(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}, \frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2} \text{ and } \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

Here $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Since the *C–R*. Equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence, w is analytic everywhere except at $z = 0$.

Now, $$\frac{dw}{dz} = \frac{\partial y}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{dw}{dz} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{(x + iy)} = \frac{1}{z}$$

$$\frac{dw}{dz} = \frac{1}{z}; z \neq 0.$$

Example 24 Determine whether $\frac{1}{z}$ is analytic or not?

Solution Given $f(z) = u + iv = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$

$\therefore$ $$u(x, y) = \frac{x}{x^2 + y^2} \text{ and } v(x, y) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2 + y^2)\cdot 1 - x \cdot 2x}{(x^2 + y^2)} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2} \text{ and } \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Since $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus the *C–R* equations are satisfied. Also the partial derivatives are continuous except (0, 0).

Therefore, $f(z) = \frac{1}{z}$ is analytic except $z = 0$.

Also $$f'(z) = -\frac{1}{z^2}$$

$\Rightarrow$ $f'(z)$ exists everywhere except at $z = 0$.

Hence $\frac{1}{z}$ is analytic except at $z = 0$.

Example 25 Find the values of C_1 and C_2 such that the function $f(z) = x^2 + C_1 y^2 - 2xy + i(C_2 x^2 - y^2 + 2xy)$ is analytic. Also find $f'(z)$.

Solution Given $f(z) = u + iv = (x^2 + C_1 y^2 - 2xy + i(C_2 x^2 - y^2 + 2xy)$.

$$\therefore \qquad \frac{\partial u}{\partial x} = 2x - 2y, \frac{\partial v}{\partial x} = 2C_2 x + 2y$$

$$\frac{\partial u}{\partial y} = 2C_1 y - 2x \text{ and } \frac{\partial v}{\partial y} = 2x - 2y$$

Since function $f(z)$ is analytic, so C–R equations are satisfies.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 2y \tag{33}$$

and
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2C_1 y - 2x = -2C_2 x - 2y.$$

By equating the coefficients of x and y, we get

$$2C_1 = -2 \Rightarrow C_1 = -1 \text{ and } -2C_2 = -2 \Rightarrow C_2 = 1.$$

Putting $C_2 = 1$ in $\frac{\partial v}{\partial x}$, we get $\frac{\partial v}{\partial x} = 2x + 2y$

Now
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x - 2y) + i(2x + 2y)$$

$$= 2(1 + i)\,x + 2(-1 + i)\,y$$

$$= 2(1 + i)\,x + 2i(1 + i)\,y = 2(1 + i)\,(x + iy)$$

$$\therefore \qquad f'(z) = 2(1 + i)\,z.$$

Example 26 Show that the function $f(z) = z|z|$ is not analytic anywhere.

Solution Given $f(z) = z|z|$

$$= (x + iy)\,|x + iy|$$

$$= (x + iy)\sqrt{(x^2 + y^2)}$$

Also
$$f(z) = u + iv = x\sqrt{x^2 + y^2} + iy\sqrt{x^2 + y^2}$$

$$\therefore \qquad u(x, y) = x\sqrt{x^2 + y^2} \text{ and } v(x, y) = y\sqrt{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2 + y^2}} = \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}}, \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}} \text{ and } \frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

Since $\quad \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$.

Hence, the *C–R*, equations are not satisfied at any point, the function $z|z|$ is not analytic anywhere.

Example 27 Show that the function $f(z) = |z|^2$ is analytic at $z = 0$, although the Cauchy–Reimann equations are satisfied at that point.

Solution Given $f(z) = |z|^2 = z\,\bar{z} = (x + iy)(x - iy) = x^2 + y^2$

$\therefore \quad f(z) = u + iv = x^2 + y^2$

$\Rightarrow \quad u = x^2 + y^2$ and $v = 0$.

At $z = 0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x^2-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{y^2-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Hence, $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Thus, *C–R* equations are satisfied at the origin.

Also, $$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{(x^2+y^2)-0}{x+iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x\to 0} \frac{x^2+m^2x^2}{x+imx} = \lim_{x\to 0} \frac{(1+m^2)x}{(1+im)} = 0.$$

Therefore, $f'(0)$ is unique. Hence, the function $f(z) = |z|^2$ is analytic at $z = 0$.

Example 28 Show that the function $f(z) = |xy|^{1/2}$ is not analytic at the origin, although the Cauchy–Riemann equations are satisfied at that point.

Solution Given $f(z) = u + iv = |xy|^{1/2}$

$\therefore \quad u(x, y) = |xy|^{1/2}, v(x, y) = 0$

At $z = 0$, $$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{(0-0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{(0-0)}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{(0-0)}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = 0.$$

Thus $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, the C–R. equations are satisfied at $z = 0$

Now, $$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{|xy|^{1/2}-0}{x+iy}.$$

If $z \to 0$ along the path $y = mx$, then we have

$$f'(0) = \lim_{x\to 0} \frac{\left|mx^2\right|^{1/2}}{(x+mix)} = \lim_{x\to 0} \frac{|m|^{1/2}}{(1+im)}.$$

Limits depends on m. So $f'(z)$ is not unique at origin. Thus $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin.

Example 29 Show that the function $f(z) = u + iv$,

where $$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}; \; z \neq 0$$

$$f(0) = 0$$

is continuous and that the Cauchy–Riemann equations are satisfied at the origin yet $f'(z)$ does not exist at $z = 0$.

Solution Given

$$f(z) = u + iv = \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}; \; z \neq 0$$

$$\therefore \quad u(x,y) = \frac{x^3-y^3}{x^2+y^2} \text{ and } v(x,y) = \frac{x^3+y^3}{x^2+y^2}; \; x, y \neq 0.$$

(i) To prove that $f(z)$ is continuous everywhere. When $z \neq 0$, u and v both are rational functions of x and y with non-zero denominators. It follows that u, v and with non-zero denominators. It follows that u, v and therefore also $f(z)$ are continuous functions everywhere except at $z = 0$. To test the continuity of u and v at the origin, we change u, v to polar co-ordinates.

$$u = r(\cos^3\theta - \sin^3\theta), \; v = r(\cos^3\theta + \sin^3\theta).$$

As $z \to 0$, $r \to 0$

Now, $$\lim_{r\to 0} u = 0 = \lim_{r\to 0} v$$

$$\Rightarrow \qquad \lim_{z\to 0} f(z) = 0$$

$$\Rightarrow \qquad \lim_{z\to 0} f(z) = 0 = f(0)$$

$\Rightarrow$ $f(z)$ is continuous at $z = 0$.

Hence $f(z)$ is continuous everywhere.

(ii) To show that C–R equations are satisfied at $z = 0$.

$$f(0) = 0 \Rightarrow u(0, 0) + iv(0, 0) = 0$$

$$\Rightarrow \qquad u(0, 0) = 0 = v(0, 0)$$

Now, $$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y\to 0} \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y\to 0} \frac{y - 0}{y} = 1$$

Thus, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at } z = 0.$$

Thus, the C–R equations are satisfied.

(iii) To prove that $f'(0)$ does not exist.

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{f(z) - 0}{z} = \lim_{z\to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}.$$

Let $z \to 0$ along the path $y = x$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3 - x^3 + i(x^3 + x^3)}{(x^2 + x^2)(x + ix)} = \frac{i}{(1 + i)}.$$

Let $z \to 0$ along x-axis, then

$$f'(0) = \lim_{x\to 0} \frac{x^3 - 0 + i(x^3 + 0)}{(x^2 + 0)(x + i0)} = 1 + i$$

$$f'(0) = \begin{cases} \dfrac{i}{1+i} & \text{along the path } y = x \\ 1 + i & \text{along the path } y = 0 \end{cases}$$

Since the values of $f'(0)$ are not unique along different paths. Hence $f'(0)$ does not exist.

Example 30 If $f(z) = u + iv$ is analytic function and $u - v = e^x (\cos y - \sin y)$, find $f(z)$ in terms of z.

Solution Given

$$f(z) = u + iv \tag{34}$$

$$\therefore \quad i\,f(z) = i(u + iv) \tag{35}$$

$$i\,f(z) = iu - v$$

Adding Eqs (34) and (35), we get

$$(1 + i)\,f(z) = (u - v) + i(u + v)$$

Taking $u - v = U$, $u + v = V$ and $(1 + i)f(z) = F(z)$, we obtain

$$F(z) = u + iv. \tag{36}$$

Since $\quad f(z) = u + iv$ is analytic $\Rightarrow F(z) = U + iV$ is analytic.

Now, $\quad u - v = U = e^x(\cos y - \sin y)$

$$\therefore \quad \frac{\partial U}{\partial x} = \phi_1(x, y) = e^x(\cos y - \sin y)$$

$$\frac{\partial U}{\partial y} = \phi_2(x, y) = e^x(-\sin y - \cos y)$$

$$\Rightarrow \quad \phi_1(z, 0) = e^z(\cos 0 - \sin 0) = e^z$$

$$\phi_2(z, 0) = e^z(-\sin 0 - \cos 0) = -e^z.$$

By Milne's Thomson method,

$$F(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + C$$

$$F(z) = \int e^z\,(1 + i)\,dz + C$$

$$F(z) = (1 + i)\,e^z + C$$

or $\quad (1 + i)\,f(z) = (1 + i)\,e^z + C$

or $$f(z) = e^z + \left(\frac{1}{1+i}\right)C$$

or $$f(z) = e^z + C_1 \qquad \left(\because \quad C_1 = \frac{1}{1+i}C\right)$$

Example 31 If $f(z)$ is a regular function of z, then prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Solution Let $f(z) = u + iv$, so that

$$|f(z)|^2 = u^2 + v^2 = \phi(x, y) \text{ say.}$$

$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2} = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2v\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2$$

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]. \tag{37}$$

Similarly,

$$\frac{\partial^2\phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]. \tag{38}$$

Adding Eqs (37) and (38), we get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]. \tag{39}$$

Since $f(z)$ is a regular function, then u and v have to satisfy C–R equations and the Laplace equation. So we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Equation (39), becomes.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 2\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2\right] = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4|f'(z)|^2$$

$$\Rightarrow \qquad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}|f(z)|^2 = 4|f'(z)|^2$$

Hence, proved.

3.24 DETERMINATION OF VELOCITY POTENTIAL AND STREAM FUNCTION

Consider the irrotational motion of an incompressible fluid in two dimensions. We assume the flow to be in planes parallel to the xy-plane, the velocity $\vec{v}$ of a fluid particle as follows:

$$\vec{v} = v_x\,\hat{i} + v_y\,\hat{j}. \tag{40}$$

Since the motion is irrotational, therefore, there exists a scalar function $\phi(x, y)$ such that

$$\vec{v} = \vec{\nabla}\,\phi\,(x, y) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)\phi\,(x, y)$$

$$\vec{v} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}. \tag{41}$$

The function $\phi(x, y)$ is called the velocity potential and the family of curve $\phi(x, y) = C$ are known as equipotential lines. From Eqs (40) and (41), we have

$$v_x = \frac{\partial\phi}{\partial x} \text{ and } v_y = \frac{\partial\phi}{\partial y}. \tag{42}$$

Since, the fluid is incompressible, so that

$$\text{Div } \vec{v} = 0 \Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{43}$$

Using Eq. (42) in Eq. (43), we get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{44}$$

Equation (44) shows that the velocity potential ϕ is harmonic. It follows that there must exist a conjugate harmonic function $\psi(x, y)$, such that

$$F(z) = \phi(x, y) + i\psi(x, y) \tag{45}$$

Now, the slope at any point of the curve $\psi(x, y) = C_1$ is given by

$$\frac{dy}{dx} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y}$$

$$\frac{dy}{dx} = -\frac{\partial\phi/\partial y}{\partial\phi/\partial x} \qquad \text{(by C–R. equations)}$$

or $$\frac{dy}{dx} = \frac{v_y}{v_x}. \qquad \text{(by Eq. (42))} \tag{46}$$

Equation (46) shows that the velocity potential of the fluid particle is along the tangent to the curve $\psi(x, y) = C_1$, i.e. the particle moves along this curve. Such curves are known as stream lines and $\psi(x, y)$ is called the stream function. Also, the equipotential lines $\phi(x, y) = C$ and the stream lines $\psi(x, y) = C_1$ are orthogonal.

By Eq. (45), we have

$$\frac{dF}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} \qquad \text{(by C–R equations)}$$

$$\frac{dF}{dz} = v_x - iv_y \qquad \text{(by Eq. (47))}$$

Thus, the flow pattern is fully expressed by the function $F(z)$ which is known as the complex potential. Similarly, the complex potential $F(z)$ can be taken to represent any other type of 2-dimensional steady and heat lines.

If $\phi(x, y)$ is given, then we can find $\psi(x, y)$ and vice-versa.

Example 32 In a two-dimensional fluid flow, the stream function is $\psi(x, y) = \tan^{-1}(y/x)$. Find the velocity potential.

Solution Given

$$\psi(x, y) = \tan^{-1}(y/x) \tag{47}$$

$$\therefore \quad \frac{d\psi}{dx} = -\frac{y}{x^2 + y^2} \text{ and } \frac{d\psi}{dy} = \frac{x}{x^2 + y^2}$$

$$d\phi = \frac{\partial\phi}{dx}dx + \frac{\partial\phi}{\partial y}dy$$

$$d\phi = \frac{\partial\psi}{dy}dx - \frac{\partial\psi}{\partial x}dy \qquad \text{(by C–R equations).}$$

or
$$d\phi = \frac{x}{x^2 + y^2}dx + \frac{y}{x^2 + y^2}dy \tag{48}$$

which is an exact differential equation.

Integrating Eq. (48), both sides, we get

$$\phi = \int \frac{x}{x^2 + y^2}dx + 0 + C$$

$$\phi = \frac{1}{2}\log(x^2 + y^2) + C$$

which is the required velocity potential.

EXERCISE 3.2

1. Obtain Cauchy–Riemann equations for an analytic function.
2. Find an analytic function whose real part is $e^x \cos y$.
3. Find an analytic function whose real part is:
   (i) $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$
   (ii) $u(x, y) = e^x \cos y + y$
   (iii) $u(x, y) = x^3 - 3xy^3$
4. Show that the function $f(z) = e^{-z^{-4}}$; $z \neq 0$ and $f(0) = 0$ is not analytic at $z = 0$ although the C–R equations are satisfied at that point.
5. Prove that the function $f(z) = e^{2z}$ is analytic and find $f'(z)$.
6. Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is continuous as well as analytic everywhere.
7. Show that the function $f(z) = \dfrac{xy^3}{x^2 + y^6}$; $z \neq 0$, $f(0) = 0$ is not continuous at the origin.
8. Show that the function
   (i) $u(x, y) = \dfrac{1}{2}\log(x^2 + y^2)$
   (ii) $u(x, y) = \cos x \cosh y$ are harmonic, and find their harmonic conjugates.

9. Find the analytic function whose imaginary part is
   (i) $v(x, y) = e^{-x}(x \cos y + y \sin y)$
   (ii) $v(x, y) = e^{x} \cdot \sin y$.
10. Prove that $u(x, y) = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function $v(x, y)$ such that $f(z) = u + iv$ is analytic.
11. If $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$; $z \neq 0$, $f(0) = 0$, prove that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$, along any radius vector, but not as $z \to 0$ in any manner.
12. An electrostatic field in xy-plane is given by the potential function $\phi(x, y) = x^2 - y^2$, find a stream function. **[Grad. ITE 1979]**
13. If the potential function of an electrostatic field is $\phi(x, y) = 3x^2y - y^3$, find the stream function. **[Grad. ITE 1978]**
14. Find the analytic function whose imaginary part is $e^x(x \sin y + y \cos y)$. **[Madurai 1973]**
15. Find an analytic function whose real part is $\sin 2x/(\cosh 2x - \cos 2x)$. **[K.U.K. 1983, Gulbarga 1984, Coimbatore 1984]**
16. Show that an analytic function cannot have a constant modulus without reducing to a constant.
    or
    Show that an analytic function with constant modulus is constant. **[Meerut 1992, 94, 95, 96, 97, 98]**
17. Examine the nature of the function $f(z) = \dfrac{x^2y^5(x+iy)}{x^4+y^{10}}$; $z \neq 0$, $f(0) = 0$ in a region including origin. **[Gorakhpur 2009]**
18. Find an analytic function whose real part is $\log\sqrt{(x^2+y^2)}$, also find its imaginary part. **[Ranchi 1980]**
19. Show that the function $f(z) = |z|^{2013}$ satisfies the Cauchy–Riemann equations only at the origin.
20. Determine the analytic function $f(z) = u + iv$, if the imaginary part is $v(x, y) = \log(x^2 + y^2) + x - 2y$.
21. If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of z, then find $f(z)$ in terms of z. **[Barilly 2009, Gorakhpur 2010]**
22. If $f(z) = u + iv$ is an analytic function of z and $u = 4xy + x + 1$. Prove that $f(z) = z - 2i z^2 + 3i + 1$, when $f(1) = 2i$.
23. Show that the function $f(z) = z^3$ is analytic everywhere.
24. Show that the function $f(z) = \dfrac{z}{z+2}$ is analytic at $z = \infty$.

Answers

2. $f(z) = e^z$
3. (i) $f(z) = z^3 + 3z^2 + 1 + iC$ (ii) $f(z) = c^z - i^z + C$ (iii) $f(z) = z^3 + iC$
4. (i) $ize^{-z} + C$ (ii) $e^z + C$

12. $\psi(x, y) = 4xy$

13. $\psi(x, y) = 6xy^2 - x^3$

14. $\overline{f}(z) = z\,e^z + C$

15. $f(z) = \cot z + iC$

17. Not analytic at origin.

18. I.P. is $\tan^{-1}\left(\dfrac{y}{x}\right)$.

20. $f(z) = (i - 2)z + 2i \log z + C.$

21. $f(z) = -i\,z^3 + c_1$, where $c_1 = \dfrac{c}{1+i}$.

3.25 INTRODUCTION TO COMPLEX INTEGRATION

In this section we define definite integrals of complex variable are known as line integrals. As in the case of real variable, an indefinite integral of a complex variable is a function whose derivative equals a given analytic function.

The theory of line integrals, along with the theory of residues and power series forms play a very important role in the theory of functions of a complex variable. These theories contain some of the most powerful theorems which have the application in pure and applied mathematics as well as in engineering.

3.26 LINE INTEGRAL IN COMPLEX PLANE

(i) *Continuous Arc:* A set of points (x, y) defined by $x = \phi(t)$ and $y = \psi(t)$ with parameter 't' in the interval (a, b), defined a continuous arc provided ϕ and ψ are continuous functions.

(ii) *Domain (Region):* A set S of points in the Argand plane is said to be connected set if any two of its points can be joined by a continuous curve, all of whose points belong to S.
An open connected set is said to be an open domain. If the boundary points of S are also added to an open domain, then it is called a closed domain.

(iii) *Contours:* A contour is a continuous chain of a finite number of regular arcs.
If the contour is closed and does not intersect itself, then it is said closed contour. For example, the boundaries of triangles and rectangles.

(iv) *Simply and multiply connected regions:* A region in which every closed curve can be shrunk to a point without passing out of the region is called a simply connected region. Otherwise it is said to be multiply connected region.
Example:

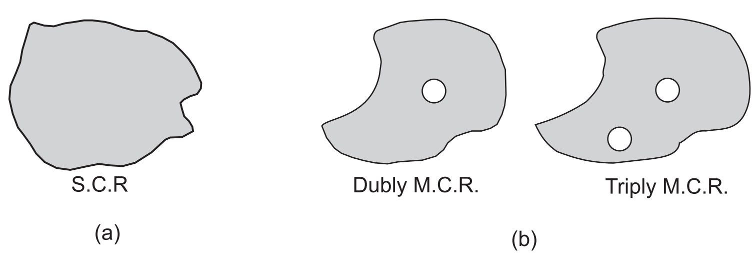


Fig. 3.6

3.27 COMPLEX FUNCTION INTEGRALS

Let $f(z)$ is continuous at every point of a closed curve C having a finite length, i.e. C is rectifiable curve.

Subdivided C into n parts by means of points $z_0, z_1, z_2, \ldots, z_n$, let $a = z_0$, $b = z_n$.

We choose a point n_k on each arc joining z_{k-1} to z_{k-2} form the sum.

$$S_n = \sum_{r=1}^{n} f(n_r)(z_r - z_{r-1}).$$

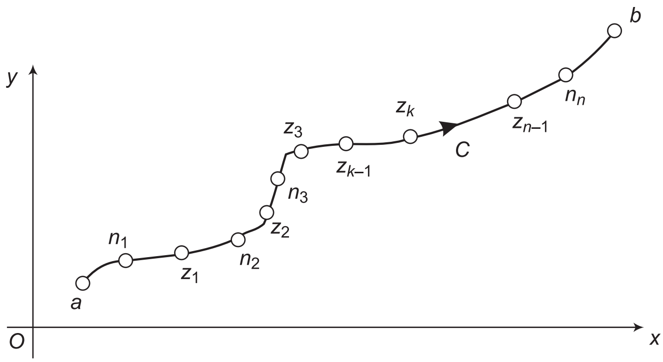


Fig. 3.7

Suppose maximum value of $(z_n - z_{n-1}) \to 0$ as $n \to \infty$. Then the sum S_n tends to a fixed limit which does not depend upon the mode of subdivision and denoted this limit by

$$\int_a^b f(z)dz \text{ or } \int_C f(z)dz$$

which is called the complex line integral or line integral of $f(z)$ along the curve C.

3.28 PROPERTIES OF COMPLEX INTEGRALS

Property 1 $\displaystyle\int_C [f(z) + g(z)]\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz.$

It can be easily generalized for a finite number of functions.

Property 2 $\displaystyle\int_C f(z)\, dz = -\int_{-C} f(z)\, dz,$

where $-C$ is the curve traversed in the opposite direction.

Property 3 $\displaystyle\int_{C_1 + C_2} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)dz$

Property 4 $\displaystyle\int_C K\, f(z)\, dz = K \int_C f(z)\, dz,$ where K is any complex constant.

Property 5 $\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\,|dz|.$

Example 33 Evaluate $\int_C \frac{1}{z-a}\,dz$, where C is a circle $|z - a| = r$.

Solution

$$C: |z - a| = r$$

or $\quad z - a = r\,e^{i\theta}; \qquad 0 \le \theta \le 2\pi$

or $\quad z = a + r\,e^{i\theta}$

or $\quad \frac{dz}{d\theta} = r\,i\,e^{i\theta}$

or $\quad dz = r\,i\,e^{i\theta}\,d\theta; \qquad 0 \le \theta \le 2\pi.$

Now $\quad \int_C \frac{1}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{1}{r\,e^{i\theta}} \cdot r\,i\,e^{i\theta}\,d\theta$

$$\theta = 0$$

$$= \int_0^{2\pi} i\,d\theta = i\int_0^{2\pi} d\theta = i\,[\theta]_0^{2\pi}$$

$$\int_C \frac{1}{z-a}\,dz = 2\pi i$$

Example 34 Integrate z^2 along the straight line OA and also along the path OAB consisting of two straight line segments OB and OA, where O is the origin, B is the point $z = 3$ and A is the point $z = 3 + i$. Hence show that the integral of z^2 along the closed path $OBAO$ is zero.

Solution

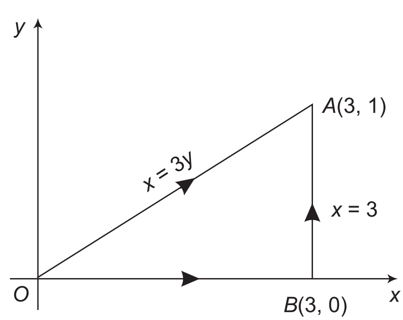


Fig. 3.8

Given $f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2\,ixy$

On the line OA, $x = 3y$ so that $dx = 3dy$ and y varies from 0 to 1 as z moves on OA from O to A.

$$\therefore \qquad I_1 = \int_{OA} f(z)\, dz = \int_{OA} z^2\, dz$$

$$= \int_0^1 (x^2 - y^2 + 2\, i\, xy)\,(dx + i\, dy) = \int_0^1 (9y^2 - y^2 + 6\, iy^2)\cdot(3dy + i\, dy)$$

$$= \int_0^1 (8y^2 + 6\, i\, y^2)\,(3 + i)\, dy = \int_0^1 (8 + 6\, i)\,(3 + i)\, y^2 dy$$

$$= (8 + 6i)\,(3 + i)\left(\frac{y^3}{3}\right)_0^1 = \frac{1}{3}(8 + 6i)\,(3 + i)$$

$$I_1 = 6 + \frac{26}{3}\, i. \tag{49}$$

Now, the integral of z^2 along the path OBA. On the line OB, $y = 0$ so that $dy = 0$ and on the line BA, $x = 3$ so that $dx = 0$.

Also varies from 0 to 3 as z moves along OB and y varies from 0 to 1 as z moves along BA.

Hence,

$$I_2 = \int_{OBA} z^2\, dz = \int_{OB} z^2\, dz + \int_{BA} z^2\, dz$$

$$= \int_{OB} (x^2 - y^2 + 2ixy)\, dz + \int_{BA} (x^2 - y^2 + 2ixy)\, dz$$

$$= \int_{OB} (x^2 - y^2 + 2ixy)\,(dx + idy) + \int_{BA} (x^2 - y^2 + 2ixy)\,(dx + idy)$$

$$= \int_0^3 x^2\, dx + \int_0^1 (9 - y^2 + 6iy)\cdot i\, dy = \left(\frac{x^3}{3}\right)_0^3 + i\left(9y - \frac{y^3}{3} + 6i\,\frac{y^2}{2}\right)_0^1$$

$$= 9 + i\left(9 - \frac{1}{3} + 3i\right) = 9 + \frac{26}{3}i - 3$$

$$I_2 = 6 + \frac{26}{3}i. \tag{50}$$

Also, $$\int_{OA} z^2\, dz = 6 + \frac{26}{3}i \Rightarrow \int_{AO} z^2\, dz = -\left(6 + \frac{26}{3}i\right). \tag{51}$$

Now, the integral of z^2 along the path $OBAO$ is

$$\int_{OBAO} z^2\, dz = \int_{OBA} z^2 dz + \int_{AO} z^2\, dz = \left(6 + \frac{26}{3}\, i\right) - \left(6 + \frac{26}{3}\, i\right) = 0.$$

3.28.1 Relation between the Real and Complex Line Integrals

If $f(z) = u + iv$; where $z = x + iy$, then the complex line integral $\int_C f(z)\,dz$ can be expressed in terms of the real line integrals as

$$\int_C f(z)\,dz = \int_C (u+iv)\,(dx+idy) \qquad \begin{bmatrix} \because & z = x+iy \\ \therefore & dz = dx+idy \end{bmatrix}$$

$$= \int_C (udx - v\,dy) + i\int_C (v\,dx + udy)$$

Thus, the complex line integral can be expressed in terms of real line integral.

***Note*:** For the integral $\int_C [Mdx + Ndy]$

(i) If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then the line integral does not depend upon the path of integration.

(ii) If $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, then the line integral depends upon the path of integration.

(iii) If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then the value of the integral round a closed path is zero, i.e.,

$$\oint_C Mdx + Ndy = 0.$$

Example 35 Evaluate $\displaystyle\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + 3y^2)\,dx + (x^2 + 3xy + 4y^2)\,dy$

(i) along $y = x^2$ and (ii) along $y = x$. Does the value of the integral depend upon the path?

Solution Let

$$I = \int_{(0,0)}^{(1,1)} (3x^2 + 4xy + 3y^2)\,dx + (x^2 + 3xy + 4y^2)\,dy \tag{52}$$

(i) Along the curve $y = x^2$ so that $dy = 2xdx$ and x varies from 0 to 1.
Thus, Eq. (52) gives

$$I = \int_{x=0}^{1} \left[3x^2 + 4x\cdot x^2 + 3\cdot(x^2)^2\right]dx + \left[x^2 + 3x\cdot x^2 + 4(x^2)^2\right]\cdot 2xdx$$

$$= \int_{x=0}^{1} \left[3x^2 + 4x^3 + 3x^4 + 2x^3 + 6x^4 + 8x^5\right]dx$$

$$= \left[\frac{3x^3}{3} + \frac{4x^4}{4} + \frac{3x^5}{5} + \frac{2x^4}{4} + \frac{6x^5}{5} + \frac{8x^6}{6}\right]_0^1$$

$$= \left[x^3 + x^4 + \frac{3}{5}x^5 + \frac{1}{2}x^4 + \frac{6}{5}x^5 + \frac{4}{3}x^6 \right]_0^1$$

$$= \left[\frac{4}{3}x^6 + \frac{9}{5}x^5 + \frac{3}{2}x^4 + x^3 \right]_0^1$$

$$= \left(\frac{4}{3} + \frac{9}{5} + \frac{3}{2} + 1 \right) - 0 = \frac{169}{30}$$

(ii) Along the line $y = x$ so that $dy = dx$ and x varies from 0 to 1.
Thus, Eq. (52) gives

$$I = \int_0^1 \left[3x^2 + 4x \cdot x + 3x^2 \right] dx + \left[x^2 + 3x \cdot x + 4x^2 \right] dx$$

$$= \int_0^1 \left[3x^2 + 4x^2 + 3x^2 + x^2 + 3x^2 + 4x^2 \right] dx$$

$$= \int_0^1 18x^2\, dx = 18 \left(\frac{x^3}{3} \right)_0^1 = 18 \cdot \left(\frac{1}{3} - 0 \right) = 6$$

Since, both the values of integration along the path $y = x^2$ and $y = x$ are not same.
Hence, the value of integration depend upon the path of integration.

Example 36 Evaluate $\int_{1-i}^{2+3i} (z^2 + z)\, dz$ along the line joining the points (1, –1) and (2, 3).

Solution Equation of the line joining the points (1, –1) and (2, 3) is

$$y - (-1) = \frac{3 - (-1)}{2 - 1}(x - 1)$$

or $$y + 1 = \frac{3 + 1}{1}(x - 1)$$

or $$y = 4x - 5$$

The complex variable $z = x + iy$

or $$z = x + i(4x - 5)$$

or $$z = (1 + 4i)x - 5i$$

$\therefore$ $$dz = (1 + 4i)dx$$

Now $$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = x^2 - (4x - 5)^2 + 2ix(4x - 5)$$
$$= (-15x^2 + 40x - 25) + i(8x^2 - 10x)$$

Now $$\int_{1-i}^{2+3i} (z^2 + z)\, dz = \int_{x=1}^{2} \left[(-15x^2 + 40x - 25) + i(8x^2 - 10x) + x + i(4x - 5) \right] (1 + 4i)\, dx$$

$$= (1 + 4i) \int_1^2 \left[(-15x^2 + 41x - 25) + i(8x^2 - 6x - 5) \right] dx$$

$$= (1+4i)\left[(-15+8i)\frac{x^3}{3}+(4i-6x)\frac{x^2}{2}-(25+5i)\,x\right]_1^2$$

$$= (1+4i)\left[\frac{3}{2}+\frac{14i}{3}\right] = \frac{1}{6}(64i-103)$$

Example 37 Evaluate $\int_0^{1+i} (x-y+ix^2)\,dz$.

(i) Along the straight line from $z = 0$ to $z = 1 + i$.

(ii) Along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = i$ to $z = 1 + i$.

(iii) Along the imaginary axis from $z = 0$ to $z = i$ and then along a line parallel to real axis from $z = i$ to $z = 1 + i$.

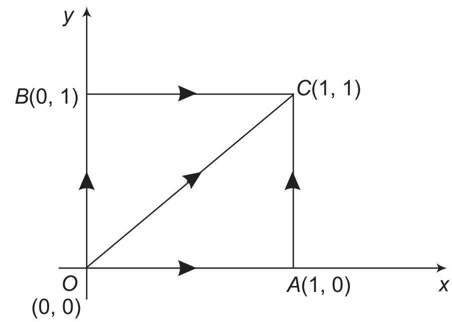


Fig. 3.9

Solution

(i) Along the straight line OC joining the points $O(z = 0)$ and $C(z = 1 + i)$, i.e., (0, 0) to (1, 1) Equation of line is

$$y = x \text{ so that } dy = dx$$

and x varies from $x = 0$ to 1.
Therefore,

$$\int_0^{1+i} (x-y+ix^2)\,dz = \int_0^{1+i} (x-y+ix^2)\,(dx+idy) \qquad \left[\begin{array}{ll}\because & z = x+iy \\ \therefore & dz = dx+idy\end{array}\right]$$

$$= \int_0^1 (x-x+ix^2)\,(1+i)\,dx = \int_0^1 i\,x^2(1+i)\,dx$$

$$= (i-1)\left(\frac{x^3}{3}\right)_0^1 = -\frac{1}{3}+\frac{i}{3}.$$

(ii) Along the path OAC

$$\int_0^{1+i} (x-y+i\,x^2)\,dz = \int_{OA} (x-y+ix^2)\,dz + \int_{AC} (x-y+ix^2)\,dz \tag{53}$$

Now along OA: $y = 0$, $dz = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_{OA} (x-y+ix^2)\,dz = \int_0^1 (x+ix^2)\,dx = \left[\frac{x^2}{2}+\frac{ix^3}{3}\right]_0^1 = \frac{1}{2}+\frac{i}{3}.$$

Also, along AC: $x = 1$ so that $dz = idy$ and y varies from 0 to 1.

$$\therefore \quad \int_{AC} (x-y+ix^2)\,dz = \int_0^1 (1-y+i)\,idy = \int_0^1 (i-iy-1)\,dy$$

$$= \left[iy - i\frac{y^2}{2} - y\right]_0^1 = i - \frac{1}{2}i - 1 = \frac{i}{2} - 1.$$

Hence, from Eq. (53), we get

$$\int_0^{1+i} (x - y + ix^2)\,dz = \frac{1}{2} + \frac{i}{3} + \frac{i}{2} - 1 = -\frac{1}{2} + \frac{5}{6}i.$$

(iii) Along the path OBC

$$\int_0^{1+i} (x - y + ix^2)\,dz = \int_{OB} (x - y + ix^2)\,dz + \int_{BC} (x - y + ix^2)\,dz \tag{54}$$

Now, along the path OB: $x = 0$ so that $dz = idy$ and y varies from 0 to 1.

$$\therefore \quad \int_{OB} (x - y + ix^2)\,dz = \int_0^1 (-y)\,idy = -i\left(\frac{y^2}{2}\right)_0^1 = -\frac{i}{2}.$$

Also, along the path BC: $y = 1$ so that $dz = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_{BC} (x - y + ix^2)\,dz = \int_0^1 (x - 1 + ix^2)\,dx = \left[\frac{x^2}{2} - x + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - 1 + \frac{i}{3} = -\frac{1}{2} + \frac{i}{3}.$$

Hence, from Eq. (54), we get

$$\int_0^{1+i} (x - y + ix^2)\,dz = -\frac{i}{2} - \frac{1}{2} + \frac{i}{3} = -\frac{1}{2} - \frac{i}{6}.$$

Example 38 Evaluate $\int_{(1,1)}^{(2,4)} (x^2 + ixy)\,dz$ along the curve $x = t$ and $y = t^2$.

Solution Equations of the path of integration are $x = t$, $y = t^2$.

So that $dx = dt$ and $dy = 2t\,dt$,

At (1, 1), $t = 1$ and (2, 4), $t = 2$

$$\therefore \quad \int_{(1,1)}^{(2,4)} (x^2 + ixy)\,dz = \int_{(1,1)}^{(2,4)} (x^2 + ixy)(dx + idy) = \int_{t=1}^{2} (t^2 + it^3)(dt + 2it\,dt)$$

$$= \int_1^2 \left[(t^2 - 2t^4) + i(3t^3)\right] dt = \left[\left(\frac{t^3}{3} - \frac{2t^5}{5}\right) + i\left(\frac{3t^4}{4}\right)\right]_1^2$$

$$= \left[\left(\frac{8}{3} - \frac{2}{5}\times 32\right) + \frac{3i}{4}\times 16\right] - \left[\left(\frac{1}{3} - \frac{2}{5}\right) + i\frac{3}{4}\right]$$

$$= \left(-\frac{151}{15} + \frac{45}{4}i\right)$$

EXERCISE 3.3

1. Evaluate the integral $\int_C (x^2 + y^2)\,dz$ from $z = 0$ to $z = 2 + 4i$ along the line segment joining the points (0, 0) and (2, 4).

2. Evaluate $\int_0^{4+2i} (x - iy)\,dz$, along the curve given by $z = t^2 + it$.

3. Evaluate $\int_C \frac{1}{z}\,dz$, where C is the semi-circular arc $|z| = 1$ from $z = -1$ to $z = 1$ above or below the real axis.

4. Show that $\oint_C (z+1)\,dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$.

5. Evaluate $\int_C \frac{2z+3}{z}\,dz$, where c is
   (i) upper half of the circle $|z| = 2$ in clockwise direction
   (ii) lower half of the circle $|z| = 2$ in anticlockwise direction
   (iii) the circle $|z| = 2$ in anticlockwise direction.

6. Evaluate $\int_C (z - z^2)\,dz$, where c is the upper half of the circle $|z| = 1$. **[Mysore 1980]**

7. Show that for every path between the limits.
$$\int_{-2}^{2+i} (2+z)^2\,dz = -\frac{i}{3}.$$
[Andhra 1977]

8. Prove that $\int_C \frac{dz}{z - z_0} = 2\pi i$, where C is the circle $|z - z_0| = r$. **[Punjab 1983]**

9. Evaluate the integral $\int z^2\,dz$, along the rectilinear path joining the points $z = 0$ to $2 + i$.

10. Find the value of the integral $\int_C |z|\,dz$, where c is the contour, left half of the circle $|z| = 1$ from $z = -1$ to i.

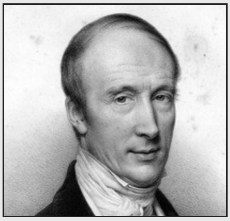
Answer

1. $-8(1 + 2i)$
2. $\left(10 - \frac{8i}{3}\right)$
3. $-\pi i$ or πi
5. (i) $8 - 3\pi i$ (ii) $8 + 3\pi i$ (iii) $6\pi i$
6. $\frac{2}{3}$
9. $(3i - 1)$
10. $(4 + 8i)$

3.29 CAUCHY FUNDAMENTAL THEOREM

Let $f(z)$ be an analytic function in simply connected domain D, and C be any closed continuous curve in D, then

$$\int_C f(z)\, dz = 0.$$



Augustin-Louis Cauchy was born on 21 August 1789 in Paris, France. He was a French mathematician. He laid the foundation for modern day analysis and did many significant works in the field of mathematics. After his graduation in 1810, Cauchy got a job in Cherbourg as 'junior engineer'. Even while at this extremely time taking job, he managed to write three mathematical manuscripts which he submitted to 'Institute de France'. He was also appointed as a member of the 'Académie des Sciences' in 1816. He got the position of associate professor of mathematics at Polytechnique in 1815 and by the next year he was promoted to the position of a full professor. He left for Prague to tutor Duke of Bordeaux till 1838. Their relationship as teacher and student was not great but Cauchy still tried his best to teach his pupil the most he could. Cauchy did not gain his academic positions again, however, tried to remain in touch with his roots till his death in 1857.

3.30 CAUCHY'S THEOREM

Let D be a simply connected region and let $f(z)$ be single valued continuously differentiable function of D, i.e. $f'(z)$ exists and is continuous at each point of D. Then

$$\int_C f(z)\, dz = 0; \text{ where } C \text{ is any closed contour in } D.$$

Proof: Let $f(z) = u + iv$, where $z = x + iy \quad \Rightarrow \quad dz = dx + idy$

$$\therefore \quad \int_C f(z)\, dz = \int_C (u + iv)(dx + idy)$$

$$\int_C f(z)\, dz = \int_C (udx - v\,dy) + i\int_C (v\,dx + u\,dy). \tag{55}$$

Now $\quad f(z) = u + iv$

$$\Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \qquad \text{(by C–R equation)} \tag{56}$$

Since $f'(z)$ is continuously differentiable, it follows from Eq. (56) that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist and are continuous in D. Then, by Green's theorem R.H.S. Eq. (55) becomes

$$\int_C f(z)\, dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) dx\, dy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy \quad \text{(By C–R equations).}$$

$$\int_C f(z)\, dz = 0.$$

***Note* 1:** Cauchy's theorem is also known as Cauchy's integral theorem.

2: $f'(z)$ is not continuous, then Cauchy's theorem is known as Cauchy's Goursal theorem.

3.31 CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is an analytic function with in and on a closed contour C, and if z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)}.$$

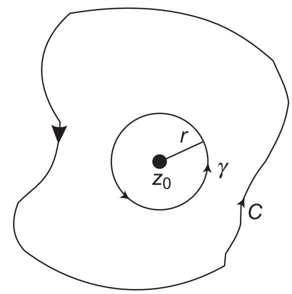


Fig. 3.10

Proof: Suppose $f(z)$ is an analytic function within and on a closed contour C and z_0 is an interior point of C.

To prove that $f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)\, dz}{z - z_0}$. To describe a circle γ about the centre $z = z_0$ of small radius r; such that the circle γ: $|z - z_0| = r$ does not intersect the curve C. Then the function $\phi(z) = \dfrac{f(z)}{z - z_0}$ is analytic in the double connected region C and γ.

Since C be a closed contour contain another closed counter γ and $f(z)$ be analytic at every point in C and γ, then

$$\int_C \frac{f(z)\, dz}{z - z_0} = \int_\gamma \frac{f(z)\, dz}{z - z_0} \tag{57}$$

or
$$\int_C \frac{f(z)\, dz}{z - z_0} = \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz + \int_\gamma \frac{f(z_0)\, dz}{z - z_0}. \tag{58}$$

Since $f(z)$ is analytic within C and so it is continuous at $z = z_0$ so that given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ for $|z - z_0| < \delta$. Now for any point z on γ, $|z - z_0| = r$ or $(z - z_0) = re^{i\theta}$ and $0 \le \theta \le 2\pi$

$$\therefore \quad \int_\gamma \frac{f(z_0)}{z - z_0}\, dz = \int_O^{2\pi} \frac{f(z_0) \cdot r\, e^{i d\theta}}{r\, e^{i\theta}}\, i d\theta = 2\pi i\, f(z_0).$$

Using Eq. (58),

$$\left| \int_C \frac{f(z)\, dz}{z - z_0} - 2\pi i\, f(z_0) \right| = \left| \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \le \int_\gamma \frac{|f(z) - f(z_0)|}{|z - z_0|} |dz|$$

$$< \frac{\varepsilon}{r} \int_\gamma |dz| = \frac{\varepsilon}{r} \cdot 2\pi r = 2\pi\varepsilon$$

or $$\left|\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i\, f(z_0)\right| < 2\pi\varepsilon.$$

Since ε is arbitrary and so making $\varepsilon \to 0$, we get

$$\int \frac{f(z)\,dz}{z-z_0} - 2\pi i\, f(z_0) = 0$$

or $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}$$ **Hence, proved.**

3.32 CAUCHY INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

Let $f(z)$ be an analytic function within and on a closed contour C and z_0 is any point lying in it, then

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2}.$$

Proof: Let $(z_0 + h)$ be a neighbouring point of a point z_0, then by Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^2}.$$

and $$f(z_0 + h) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-(z_0+h)}.$$

Now $$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{h}\left[\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)h}\left[\left(1-\frac{h}{z-z_0}\right)^{-1} - 1\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)h}\left[\frac{h}{z-z_0} + \left(\frac{h}{z-z_0}\right)^2 + \left(\frac{h}{z-z_0}\right)^3 + \cdots\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)}\left[\frac{1}{z-z_0} + \frac{h}{(z-z_0)^2} + \frac{h^2}{(z-z_0)^3} + \cdots\right]dz.$$

Taking limit as $h \to 0$, we get

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)}\left[\frac{1}{(z-z_0)} + 0 + 0 + 0 + \cdots\right]dz$$

or $$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2}.$$

3.33 CAUCHY INTEGRAL FORMULA FOR HIGHER ORDER DERIVATIVES

Let $f(z)$ be an analytic function in a simply connected region D, and C be a closed contour in D and z_0 is any point in C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

Proof: We prove this theorem by using mathematical induction.

Axiom-I Statement must be true for $n = 1$

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^2}$$

which is true, this is Cauchy integral formula for derivative.

Axiom-II Let the statement is true for $n = k$

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{k+1}} \tag{59}$$

Axiom-III The statement is true for $n = k$, then we want to prove that the statement is true for $n = k + 1$. Now, differentiate Eq. (59) both sides w.r.to z_0, we get,

$$f^{(k+1)}(z_0) = \frac{k!}{2\pi i} \int_C (-k-1)(z - z_0)^{-k-2}(-1)\, f(z)\, dz$$

$$= \frac{k!}{1\pi i} \int_C \frac{(k+1)\, f(z)}{(z - z_0)^{k+2}}\, dz = \frac{(k+1)k!}{2\pi i} \int \frac{f(z)}{(z - z_0)^{k+2}}\, dz$$

$$f^{(k+1)}(z_0) = \frac{(k+1)!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{(k+1)+1}}\, dz$$

The above statement is true for $n = k + 1$.

Hence, this is true in general

$$\therefore \qquad f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}.$$

3.34 POISSON'S INTEGRAL FORMULA

If $f(z)$ is an analytic function within and on a circle C defined by $|z| = R$ and if z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{(R^2 - z_0\,\overline{z}_0)\, f(z)}{(z - z_0)(R^2 - z\,\overline{z}_0)}\, dz.$$

3.35 MORERA'S THEOREM (CONVERSE OF CAUCHY'S THEOREM)

If $f(z)$ is a continuous function in a domain D and if for every closed contour C in the domain D,

$$\int_C f(z)\, dz = 0$$

Then $f(z)$ is analytic within D.

Proof: Let z_0 be a fixed point and $f(z)$ be a continuous function in D, then by the definition of continuity for any arbitrary $\varepsilon > 0$ there exist a positive number δ, such that

Consider an auxiliary function.

$$F(z) = \int_{z_0}^{z} f(t)\, dt$$

$$F(z+h) = \int_{z_0}^{z+h} f(t)\, dt$$

$$\therefore \quad F(z+h) - F(z) = \int_{z_0}^{z+h} f(t)\, dt - \int_{z_0}^{z} f(t)\, dt = \int_{z}^{z+h} f(t)\, dt$$

Now $$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{z}^{z+h} f(t)\, dt - f(z)\right|$$

$$= \frac{1}{|h|}\left|\int_{z}^{z+h} \left[f(t) - f(z)\, dt\right]\right|$$

$$\leq \frac{1}{|h|}\int_{z}^{z+h} \left|\left[f(t) - f(z)\right]\right| |dt| \tag{60}$$

Since $f(z)$ is continuous at each point in D, i.e. $|f(t) - f(z)| < \varepsilon$, when ever $|t - z| < \delta$

$$\therefore \qquad \leq \frac{1}{h}\int_{z}^{z+h} \varepsilon \cdot dt \leq \frac{1}{h}\cdot \varepsilon \cdot \int_{z}^{z+h} 1 \cdot dt \leq \frac{\varepsilon}{h}\cdot h \leq \varepsilon$$

As $\varepsilon \to 0$

$$\left|\frac{F(z+h)-F(z)}{t} - f(z)\right| \to 0$$

$$\Rightarrow \qquad \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$$

$$\Rightarrow \qquad F'(z) = f(z)$$

Hence, $f(z)$ is analytic in D.

3.36 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let $f(z)$ be single valued analytic function in a simple connected domain D. If $a, b \in D$, then

$$\int_{a}^{b} f(z)dz = F(b) - F(a),$$

where $f(z)$ is an indefinite integral of $f(z)$.

Proof: We prove these theorem with the help of indefinite integral,

$$F(z) = \int_{z_0}^{z} f(t)\,dt$$

$$F(b) - F(a) = \int_{z_0}^{b} f(t)\,dt - \int_{z_0}^{a} f(t)\,dt = \int_{z_0}^{b} f(t)\,dt + \int_{a}^{z_0} f(t)\,dt = \int_{a}^{b} f(t)\,dt$$

or $$\int_{a}^{b} f(z)\,dz = F(b) - F(a)$$ **Hence, proved.**

3.37 CAUCHY'S INEQUALITY THEOREM

Let $f(z)$ be an analytic function in a domain D and let D contain the interior and the boundary of the circle γ defined by $|z - z_0| = \rho$. If $|f(z)| \leq M$ on γ, then

$$\left|f^n(z_0)\right| \leq n! \cdot \frac{M}{\rho^n}$$

Entire function: A function $f(z)$ which is analytic in every finite region of the z-plane is called an entire function or an integral function.

3.38 LIOUVILLE'S THEOREM

Let $f(z)$ be an integral function satisfying the inequality $|f(z)| \leq M$ for all values of z, where M is a positive constant. Then $f(z)$ is constant.

Proof: Consider z_1, z_2 be any two points in the z-plane. Let Γ be a circle with centre z_1 and radius R such that the point z_2 is interior to C. Then by Cauchy's integral formula, we have

$$f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_1}\,dz \quad \text{and} \quad f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_2}\,dz$$

$$\therefore \quad f(z_2) - f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_1} - \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_1}$$

$$= \frac{1}{2\pi i}\int_C \frac{(z_2 - z_1)\,f(z)\,dz}{(z - z_1)(z - z_2)}$$

$$= \frac{(z_2 - z_1)}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_1)(z - z_2)}. \qquad (61)$$

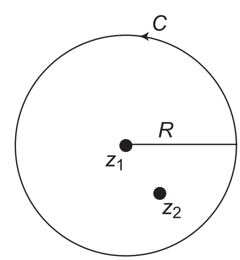


Fig. 3.11

We choose R is larger so that $|z_2 - z_1| < R/2$.

Then since $[z - z_1] = R$, we have

$$|z - z_2| = |z - z_1 + z_1 - z_2| = |(z - z_1) - (z_2 - z_1)|$$

$$\geq |z - z_1| - |z_2 - z_1| \geq R - \frac{R}{2} = \frac{R}{2}.$$

Also $|f(z)| \leq M$.

Hence, from Eq. (61)

$$|f(z_2) = f(z_1)| = \left|\frac{z_2 - z_1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_1)(z-x_2)}\right| \le \frac{|z_2 - z_1|}{|2\pi i|}\int_C \frac{|f(z)||(dz)|}{|z-z_1|\cdot|z-z_2|}$$

$$\le \frac{|z_2 - z_1|}{2\pi}\int_C \frac{M}{R\cdot\dfrac{R}{2}}|dz|$$

$$= \frac{M|z_2 - z_1|}{2\pi}\cdot\frac{2}{R^2}\cdot\int_C dz = \frac{M|z_2 - z_1|}{2\pi}\cdot\frac{2}{R^2}\cdot 2\pi R$$

$$= \frac{2M|z_2 - z_1|}{R} \tag{62}$$

$f(z_2) - f(z_1) = 0$ as $R \to \infty$

or $f(z_2) = f(z_1)$.

This shows that the function $f(z)$ is constant.

3.39 EXPANSION OF ANALYTIC FUNCTIONS AS POWER SERIES

3.39.1 Taylor's Theorem

If a function $f(z)$ is analytic within a circular C_1 with its centre z_0 and radius R, then at every point z in side C_2,

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)\frac{(z-z_0)^n}{n!}$$

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots$$

Proof: Let z be any point inside the circle C_1 with centre z_0 and radius R. Let $|z - z_0| = r$ and let C_2 be the circle with centre z_0 and radius ρ, such that $r < \rho < R$. So that the point z lies inside C_2 (see the figure). Then by Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-z)}\,dt \tag{63}$$

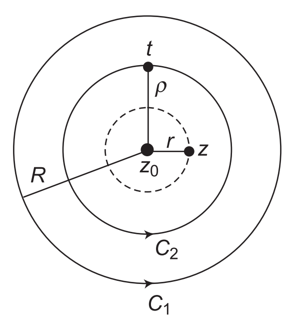


Fig. 3.12

To obtain the desired result, we consider

$$\frac{1}{t-z} = \frac{1}{(t-z_0)-(z-z_0)} = \frac{1}{(t-z_0)}\left[\frac{1}{1-\left(\dfrac{z-z_0}{t-z_0}\right)}\right]$$

$$= \frac{1}{(t-z_0)}\left[1-\left(\frac{z-z_0}{t-z_0}\right)\right]^{-1}$$

$$= \frac{1}{(t-z_0)}\left[1+\left(\frac{z-z_0}{t-z_0}\right)+\left(\frac{z-z_0}{t-z_0}\right)^2+\cdots\left(\frac{z-z_0}{t-z_0}\right)^{n-1}+\left(\frac{z-z_0}{t-z_0}\right)^n\cdot\frac{1}{1-\dfrac{z-z_0}{t-z_0}}\right]$$

$$= \left[\frac{1}{t-z_0}+\frac{z-z_0}{(t-z_0)^2}+\frac{(z-z_0)^2}{(t-z_0)^3}+\cdots+\frac{(z-z_0)^{n-1}}{(t-z_0)^n}+\frac{(z-z_0)^n}{(t-z_0)^n}\cdot\frac{1}{(t-z)}\right]. \tag{64}$$

Using Eq. (64) in (63), we get

$$f(z) = \frac{1}{2\pi i}\int_C\left[\frac{1}{t-z_0}+\frac{z-z_0}{(t-z_0)^2}+\frac{(z-z_0)^2}{(t-z_0)^3}+\cdots+\frac{(z-z_0)^{n-1}}{(t-z_0)^n}+\frac{(z-z_0)^n}{(t-z_0)^n}\cdot\frac{1}{(t-z)}\right]f(t)\,dt. \tag{65}$$

Using Cauchy's integral formula for n^{th} derivative

$\dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(t)\,dt}{(t-z_0)^{n+1}}$ in Eq. (64), we get

$$f(z) = f(z_0)+(z-z_0)\,f'(z_0)+\frac{(z-z_0)^2}{2!}f''(z_0)+\cdots+\frac{(z-z_0)^{n-1}}{(n-1)!}f^{(n-1)}(z_0)+(z-z_0)^n R_n, \tag{66}$$

where
$$R_n = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-z_0)^n(t-z)} \tag{67}$$

Theorem will be proved if we show that $R_n \to 0$ as $n \to \infty$. To prove this, we know that $|z - z_0| = r$, $|(t-z_0)| = \rho$ and therefore $|t-z| = |(t-z_0)-(z-z_0)| \geq |t-z_0| - |z-z_0| = \rho - r$.

Hence, when M denotes the maximum value of $f(t)$ on C_2, we get Eq. (67)

$$|R_n| = \left|\frac{1}{(2\pi i)}\int_C \frac{f(t)\,dt}{(t-z_0)^n\,(t-z)}\right| \leq \frac{1}{|2\pi i|}\int_C \frac{|f(t)|\,|dt|}{|t-z_0|^n\,|t-z|}$$

$$|R_n| \leq \frac{1}{2\pi}\int_C \frac{M\,|dt|}{(\rho-r)\cdot\rho^n} \leq \frac{M}{2\pi(\rho-r)\rho^n}\int_C |dt|$$

$$= \frac{M}{2\pi\,(\rho-r)\,\rho^n}\cdot 2\pi\rho$$

$$= \frac{M\rho}{(\rho-r)\,\rho^n}. \tag{68}$$

Since $r < \rho$, the RHS of (68) tends to zero as $n \to \infty$ and consequently $R_n \to 0$ as $n \to \infty$. Thus as $n \to \infty$, the limit of the sum of the first n terms on the RHS of Eq. (66) is $f(z)$.

$\therefore$ $f(z)$ is represented by the infinite series.

$$f(z) = f(z_0)+\sum_{n=1}^{\infty}\frac{(z-z_0)^n}{n!}f^{(n)}(z_0) \tag{69}$$

Equation (69) known as Taylor's series/Theorem.

When $z_0 = 0$ then Eq. (69) reduces to

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} f^{(n)}(0)$$

which is known as Maclaurin's series.

3.39.2 Laurent's Theorem

Let $f(z)$ be analytic in the ring shaped region D bounded by two concentric circles C_1 and C_2 with centre z_0 and radii ρ_1 and ρ_2 $(\rho_1 > \rho_2)$ and let z be any point of D.

Then $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

where $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t - z_0)^{n+1}} dt, \quad b_n = \frac{1}{2\pi i} \oint_{C_2} (t - z_0)^{n-1} f(t)\, dt.$$

Proof: Let D be an annulous region bounded by two concentric circle C_1: $|t - z_0| = \rho_1$ and C_2: $|t - z_0| = \rho_2$ $(\rho_1 > \rho_2)$.

Obviously region is multiply connected region, so we convert it in to simply connected region by making a cut $ABCD$.

Then by Cauchy integral formula for any fixed z in D,

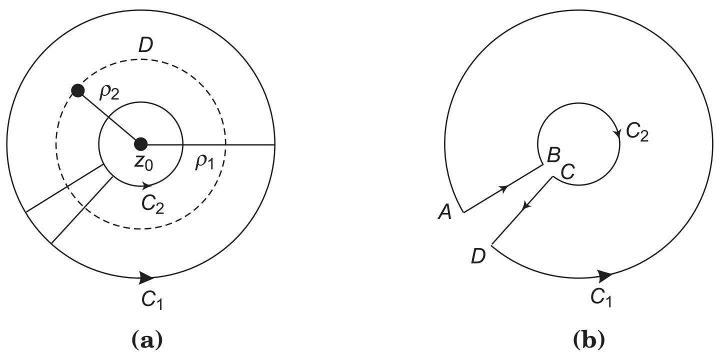


Fig. 3.13

$$f(z) = \frac{1}{2\pi i} \oint_D \frac{f(t)}{(t - z)} dt = \frac{1}{2\pi i} \oint_{C_1 + AB - C_2 + CD} \frac{f(t)\, dt}{(t - z)}$$

$$= \frac{1}{2\pi i} \oint_{C_1 - C_2} \frac{f(t)\, dt}{(t - z)} \qquad (\because AB = CD, \text{ but opposite in direction}).$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)\, dt}{(t - z)} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)\, dt}{(t - z)}$$

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)\, dt}{(t - z)} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)\, dt}{(z - t)}$$

$$= I_1 + I_2. \tag{70}$$

Now $$I_1 = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-z)}\, dt.$$

Consider

$$\frac{1}{t-z} = \frac{1}{(t-z_0)+z_0-z} = \frac{1}{(t-z_0)\left[1-\dfrac{z-z_0}{t-z_0}\right]}$$

$$= \frac{1}{(t-z_0)}\left[1-\frac{z-z_0}{t-z_0}\right]^{-1}$$

$$= \frac{1}{(t-z_0)}\left[1+\frac{z-z_0}{t-z_0}+\left(\frac{z-z_0}{t-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{t-z_0}\right)^n+\left(\frac{z-z_0}{t-z_0}\right)^{n+1}\cdot\frac{1}{\left(1-\dfrac{z-z_0}{t-z_0}\right)}\right]$$

$$= \frac{1}{(t-z_0)}+\frac{(z-z_0)}{(t-z_0)^2}+\frac{(z-z_0)^2}{(t-z_0)^3}+\cdots+\frac{(z-z_0)^n}{(t-z_0)^{n+1}}+\frac{(z-z_0)^{n+1}}{(t-z_0)^{n+1}}\cdot\frac{1}{(t-z)}.$$

Substitute the value of $\dfrac{1}{t-z}$ in I_1, we get

$$I_1 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{(t-z_0)}+\frac{z-z_0}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{(t-z_0)^2}+\cdots$$

$$+\frac{(z-z_0)^n}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{(t-z_0)^{n+1}}+\frac{(z-z_0)^{n+1}}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-z_0)^{n+1}}\cdot\frac{1}{(t-z)}\,dt.$$

Putting $$a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{(t-z_0)^{n+1}}$$

$\therefore$ $$I_1 = a_0 + (z-z_0)\,a_1 + (z-z_0)^2\,a_2 + \ldots + (z-z_0)^n\,a_n + R_{n+1}$$

$$= \sum_{n=0}^{\infty} a_n\,(z-z_0)^n + R_{n+1},$$

where $$R_{n+1} = \frac{(z-z_0)^{n+1}}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{(t-z_0)^{n+1}}\cdot\frac{1}{(t-z)}.$$

Now our aim is to show that $R_{n+1} \to 0$ as $n \to \infty$.

$$|R_{n+1}| = \left|\frac{(z-z_0)^{n+1}}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{(t-z_0)^{n+1}}\cdot\frac{1}{(t-z)}\right| \le \frac{|z-z_0|^{n+1}}{2\pi}\oint_{C_1}\frac{|f(t)|\cdot|dt|}{|t-z_0|^{n+1}\cdot|t-z|}.$$

Since $\quad |z - z_0| = r,\ |t - z_0| = \rho_1$

$$\begin{aligned} |t - z| &= |t - z_0 + z_0 - z| = |(t - z_0) - (z - z_0)| \\ &\geq |t - z_0| - |z - z_0| \\ &\geq \rho_1 - r \end{aligned}$$

and $\quad |f(t)| \leq M.$

$$\therefore \quad |R_{n+1}| \leq \frac{r^{n+1}}{2\pi} \oint_{C_1} \frac{M \cdot |dt|}{\rho_1^{n+1} \cdot (\rho_1 - r)} \leq \frac{r^{n+1}}{2\pi} \cdot \frac{M}{\rho_1^{n+1} \cdot (\rho_1 - r)} \cdot 2\pi\, \rho_1$$

$$|R_{n+1}| \leq \frac{M \cdot r^{n+1}\, \rho_1}{\rho_1^{n+1} \cdot (\rho_1 - r)}.$$

As $n \to \infty$, $R_{n+1} \to 0$

$$\therefore \quad I_1 = \sum_{n=0}^{\infty} a_n\, (z - z_0)^n.$$

Now, $\quad I_2 = \dfrac{1}{2\pi i} \displaystyle\oint_{C_2} \frac{f(t)\, dt}{(z - t)}.$

Consider

$$\frac{1}{z - t} = \frac{1}{z - z_0 + z_0 - t} = \frac{1}{(z - z_0) - (t - z_0)} = \frac{1}{(z - z_0)} \left[1 - \frac{t - z_0}{z - z_0}\right]^{-1}$$

$$= \frac{1}{(z - z_0)} \left[1 + \left(\frac{t - z_0}{z - z_0}\right) + \left(\frac{t - z_0}{z - z_0}\right)^2 + \cdots + \left(\frac{t - z_0}{z - z_0}\right)^n + \left(\frac{t - z_0}{z - z_0}\right)^{n+1} \cdot \frac{1}{\left(1 - \dfrac{t - z_0}{z - z_0}\right)}\right]$$

$$= \frac{1}{z - z_0} + \frac{t - z_0}{(z - z_0)^2} + \frac{(t - z_0)^2}{(z - z_0)^3} + \cdots + \frac{(t - z_0)^n}{(z - z_0)^{n+1}} + \frac{(t - z_0)^{n+1}}{(z - z_0)^{n+1}} \cdot \frac{1}{(z - t)}.$$

Substitute the value of $\dfrac{1}{z - t}$ in I_2, we get

$$I_2 = \frac{1}{(z - z_0)} \cdot \frac{1}{2\pi i} \oint_{C_2} f(t)\, dt + \frac{1}{(z - z_0)^2} \frac{1}{2\pi i} \oint_{C_2} f(t) \cdot (t - z_0)\, dt$$
$$+ \cdots + \frac{1}{(z - z_0)^{n+1}} \cdot \frac{1}{2\pi i} \oint_{C_2} (t - z_0)^n\, f(t)\, dt + S_{n+1}$$

Put $\quad b_n = \dfrac{1}{2\pi i} \displaystyle\oint_{C_2} (t - z_0)^{n-1}\, f(t).$

Then

$$I_2 = \frac{1}{z - z_0} b_1 + \frac{1}{(z - z_0)^2} b_2 + \cdots + \frac{1}{(z - z_0)^{n+1}} \cdot b_{n+1} + S_{n+1}.$$

$$= \sum_{n+1}^{\infty} b_n (z - z_0)^{-n} + S_{n+1},$$

where $$S_{n+1} = \frac{1}{2\pi i} \oint_{C_2} \frac{(t - z_0)^{n+1} f(t)}{(z - z_0)^{n+1} \cdot (z - t)} dt.$$

Now our aim is to show that $S_{n+1} \to 0$ as $n \to \infty$

$$|S_{n+1}| = \left| \frac{1}{2\pi i} \oint_{C_2} \frac{(t - z_0)^{n+1}}{(z - z_0)^{n+1}} \cdot \frac{f(t)}{(z - t)} dt \right| \le \frac{1}{2\pi} \oint_{C_2} \frac{|t - z_0|^{n+1} \cdot |f(t)| |dt|}{|z - z_0|^{n+1} \cdot |z - t|}.$$

Since $|t - z_0| = \rho_2$, $|z - z_0| = r$ and $|t - z| = \rho_2 - r$, $|(f(t)| \le M$

$$\therefore \quad |S_{n+1}| \le \frac{\rho_2^{n+1} \cdot M \cdot \rho_2}{r^{n+1} \cdot (\rho_2 - r)}$$

$S_{n+1} \to 0$ as $n \to \infty$

$$\therefore \quad I_2 = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}.$$

Substitute the values of I_1 and I_2 in Eq. (71), we get

$$I = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

where $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)\, dt}{(t - z_0)^{n+1}}, \quad b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)\, dt}{(t - z_0)^{-n+1}}.$$

Obviously $b_n = a_{-n}$

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Hence, proved.

3.39.3 Problems on Cauchy's Integral Formula

Example 39 Use Cauchy's integral formula, evaluate

$$\int_C \frac{1}{z(z + \pi i)} dz, \text{ where } C : |z + 3i| = 1$$

Solution By Cauchy's integral formula, $f(z_0) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)\,dz}{z - z_0}$,

where $z = z_0$ is a point inside the contour C.

Here $$f(z) = \frac{1}{z}, \text{ then } I = \oint_C \frac{f(z)}{z-(-\pi i)}\,dz$$

$$= 2\pi i\, f(-\pi i)$$

$$= 2\pi i \cdot \left(\frac{1}{-\pi i}\right) = -2.$$

Example 40 Evaluate $\oint_C \dfrac{\sin z}{\left(z - \frac{\pi}{4}\right)^3}\,dz$, where $C: \left|z - \dfrac{\pi}{4}\right| = \dfrac{1}{2}$.

Solution Let $I = \oint \dfrac{\sin z}{\left(z - \frac{\pi}{4}\right)^3}\,dz\,;\, C: \left|z - \dfrac{\pi}{4}\right| = \dfrac{1}{2}$

Here $f(z) = \sin z$, and $f'(z) = \cos z$, $f''(z) = -\sin z$.

By Cauchy integral formula

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\, f^{(n)}(z_0)$$

$$\therefore \quad \oint_C \frac{\sin z}{\left(z - \frac{\pi}{4}\right)^3}\,dz = \frac{2\pi i}{2!}\, f''\left(\frac{\pi}{4}\right) = \pi i\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi i}{\sqrt{2}}$$

Example 41 Evaluate $\oint_C \dfrac{z\,dz}{(9-z^2)(z+i)}$, where C: $|z| = 2$ described in positive sense.

Solution Let

$$I = \oint_C \frac{z\,dz}{(9-z^2)(z+i)}$$

Here $$f(z) = \frac{z}{9-z^2}, \text{ then.}$$

$$I = \oint_C \frac{f(z)\,dz}{z+i}, \quad z = -i \text{ lies in side 'C'},$$

$$= 2\pi i\, f(-i)$$

$$= 2\pi i\left[\frac{-i}{9-(-i)^2}\right] = 2\pi i \cdot \frac{-i}{9+1} = \frac{\pi}{5}$$

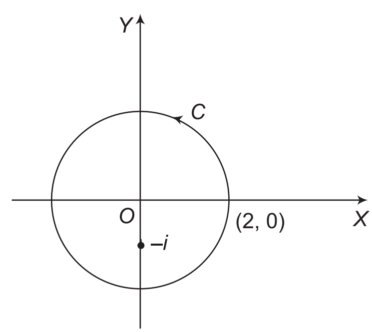


Fig. 3.14

Example 42 Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}\, dz$, where $C: |z| = 3$

Solution By Cauchy's integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\, dz}{(z - z_0)^{n+1}}$$

Here $f(z) = e^{2z},\ n = 3,\ z_0 = -1$

$$f^{(3)}(z) = 8\, e^{2z} \Rightarrow f^{(3)}(-1) = \frac{8}{e^2}$$

$$\therefore \quad \int \frac{e^{2z}}{(z+1)^4}\, dz = \frac{2\pi i}{3!} \cdot f^{(3)}(-1) = \frac{2\pi i}{6} \cdot \frac{8}{e^2} = \frac{8\pi i}{3e^2}$$

Example 43 Use Cauchy's integral formula, evaluate

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz, \text{ where } C: |z| = 3$$

Solution Here $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic in side $|z| = 3$.

$z = 1, 2$ lie inside C, where $\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}$ is not analytic, then

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = \oint_{C_1} \frac{(\sin \pi z^2 + \cos \pi z^2)/(z-2)}{(z-1)}\, dz + \oint_{C_2} \frac{(\sin \pi z^2 + \cos \pi z^2)/(z-1)}{(z-2)}\, dz$$

$$= 2\pi i \left[\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}\right]_{z=1} + 2\pi i \left[\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}\right]_{z=2} \quad \text{(By Cauchy integral formula)}$$

$$= 2\pi i \cdot \left(\frac{0-1}{-1}\right) + 2\pi i \left(\frac{0+1}{1}\right)$$

$$= 2\pi i + 2\pi i = 4\pi i$$

Example 44 Evaluate $\oint_C \frac{e^{3iz}}{(z+\pi)^3}\, dz$, where $C: |z - \pi| = \frac{16}{5}$.

Solution Here $f(z) = e^{3iz}$ and $z = -\pi$ (triple pole) lies outside C, then by Cauchy integral formula.

$$\oint_C \frac{e^{3iz}}{(z+\pi)^3}\, dz = 0$$

Example 45 Evaluate $\oint_C \frac{z^2+1}{z^2-1}\,dz$, where $C : |z| = \frac{1}{2}$

Solution Here $f(z) = z^2 + 1$, and the poles $z = 1, -1$ lies outside C, i.e. $C : |z| = \frac{1}{2}$, then by Cauchy integral formula

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 0.$$

Example 46 Evaluate $\int_C \frac{z^2+28}{(z-2)(z-3)}\,dz$, if c is the circle $|z - 4i| = 3$.

Solution Here $f(z) = \frac{z^2+28}{(z-2)(z-3)}$

Putting the denominator equal to zero, i.e.,

$(z - 2)(z - 3) = 0$ or $z = 2, 3$.

Thus, the circle c: $|z - 4i| = 3$ with centre $4i$ and radius 3 does not enclose $z = 2$ and $z = 3$.

Hence, by Cauchy's integral theorem

$$\int_C \frac{z^2+28}{(z-2)(z-3)}\,dz = 0$$

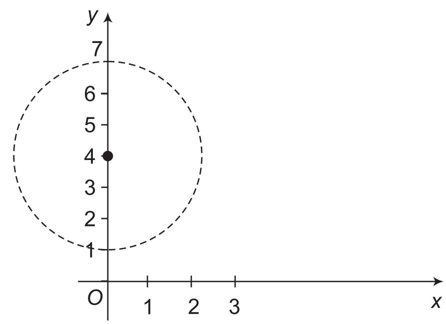


Fig. 3.15

Example 47 Evaluate $\int_C \frac{2z^2+z}{z^2-1}\,dz$, where c is the circle of unit radius with centre at $z = 1$. **[U.P.T.U. 2004]**

Solution Here $F(z) = \frac{2z^2+z}{z^2-1}$

Putting $z^2 - 1 = 0$ or $z = 1, -1$.

The given circle c: $|z - 1| = 1$.
which includes the point $z = 1$ only.

Let $$f(z) = \frac{2z^2+z}{z+1}$$

Clearly, $f(z)$ is analytic in the circle $|z - 1| = 1$.

Hence, by Cauchy's integral theorem, we have

$$\int_C \frac{2z^2+z}{z^2-1}\,dz = \int_C \frac{f(z)}{(z-1)}\,dz = 2\pi i\, f(1)$$

$$= 2\pi i \left[\frac{2z^2+z}{z+1}\right]_{z=1} = 2\pi i \cdot \frac{3}{2} = 3\pi i.$$

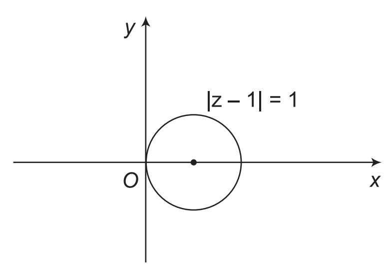


Fig. 3.16

Example 48 Evaluate $\int_C \frac{\sinh(z^{2013})}{z^3} dz$, where c is the circle $|z| = 1$.

Solution Put $z^3 = 0$ or $z = 0$ is a singularity of order 3.

Clearly $z = 0$ lies inside the circle $|z| = 1$.

Let $f(z) = \sinh(z^{2013})$, clearly $f(z)$ is analytic within and on the circle $|z| = 1$.

Hence, by Cauchy's integral formula for derivative, we have

$$\int_c \frac{\sinh(z^{2013})}{z^3} dz = \int_c \frac{f(z)}{z^3} dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2} f(z)\right]_{z=0}$$

$$= \pi i \left[\frac{d}{dz}\left\{\frac{d}{dz}\sinh(z^{2013})\right\}\right]_{z=0}$$

$$= \pi i \left[\frac{d}{dz}\left\{2013\, z^{2012} \cosh(z^{2013})\right\}\right]_{z=0}$$

$$= 2013\,\pi i \left[\left\{2012\, z^{2011} \cosh(z^{2013})\right\} + z^{2012} \cdot 2013 \cdot z^{2012} \sinh(z^{2013})\right]_{z=0}$$

$$= 0.$$

Example 49 Evaluate $\int_c \frac{e^{ez}}{z - ei} dz$, where c is the ellipse $|z - 2| + |z + 2| = 6$.

Solution Let

$$I = \int_c \frac{e^{ez}}{z - ei} dz \tag{71}$$

Since c is an ellipse

$$|z - 2| + |z + 2| = 6$$

or $$|(x-2)+iy| + |(x+2)+iy| = 6$$

or $$\left[(x-2)^2 + y^2\right]^{\frac{1}{2}} = 6 - \left[(x+2)^2 + y^2\right]^{\frac{1}{2}}$$

Squaring both sides, we get

$$x^2 + y^2 - 4x + 4 = 36 + (x^2 + y^2 + 4x + 4) - 12\left[(x+2)^2 + y^2\right]^{\frac{1}{2}}$$

or $$12(x^2 + y^2 + 4x)^{\frac{1}{2}} = 36 + 8x$$

or $$3(x^2 + y^2 + 4x)^{\frac{1}{2}} = 9 + 2x$$

Again squaring, we get

$$9(x^2 + y^2 + 4x) = (9 + 2x)^2$$

or $$9(x^2 + y^2 + 4x) = 81 + 4x^2 + 36x.$$

or $$5x^2 + 9y^2 = 45$$

or $$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Compare $\frac{x^2}{9} + \frac{y^2}{5} = 1$ with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$a^2 = 9, b^2 = 5$$

or $$a = 3,\ b = \sqrt{5} = 2.2 \text{ (approx.)}$$

Clearly, $z = ei = 2.71\ i$, lies outside the c.
Hence, by Cauchy's integral theorem.

$$\int_c \frac{e^{ez}}{z - ei} dz = 0.$$

Example 50 Using Cauchy–integral formula to evaluate $\int_c \frac{z\,dz}{z^2 - 3z + 2}$, where c is the circle $|z - 2| = \frac{1}{2}$.

[U.P.T.U. 2009]

Solution Poles of the integrand function is

Put $\quad z^2 - 3z + 2 = 0$

or $\quad (z - 1)(z - 2) = 0$ or $z = 1, 2$

The given circle $|z - 2| = \frac{1}{2}$ has centre 2 and radius $\frac{1}{2}$.

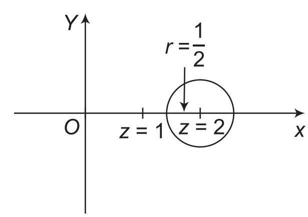


Fig. 3.17

Clearly $z = 2$ lies inside the given circle [See Fig. 3.17]
Hence, by Cauchy integral formula.

$$\int_c \frac{z\,dz}{z^2 - 3z + 2} = \int_c \frac{\left(\frac{z}{z-1}\right)}{z - 2} dz$$

or $$= \int_c \frac{f(z)}{z - 2} dz, \quad \text{where } f(z) = \frac{z}{z - 1}$$

$$= 2\pi i \left[\frac{z}{z - 1}\right]_{z=2} = 2\pi i \left[\frac{2}{1}\right] = 4\pi i$$

Example 51 Using Cauchy's integral formula to evaluate $\int_c \frac{e^{zt}}{z^2 + 1} dz$, where c is the circle $|z| = 3$.

[U.P.T.U. 2009]

Solution Poles of the integrand are given by

$$z^2 + 1 = 0 \text{ or } z = \pm i.$$

The circle $|z| = 3$ has centre at $z = 0$ and radius 3.
Clearly, both the poles $z = i$ and $z = -i$ lies inside the given circle.
Hence, by Cauchy's integral formula

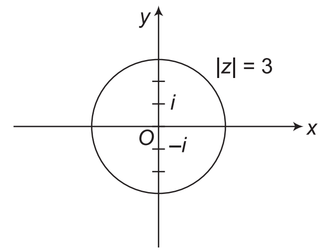


Fig. 3.18

$$\int \frac{e^{zt}}{z^2+1}dz = \int_{c_1} \frac{e^{zt}/(z+i)}{z-i}\,dz + \int_{c_2} \frac{e^{zt}/(z-i)}{z+i}\,dz$$

$$= 2\pi i\left(\frac{e^{zt}}{z+i}\right)_{z=i} + 2\pi i\left(\frac{e^{zt}}{z-i}\right)_{z=-i}$$

$$= 2\pi i\left[\frac{e^{it}}{2i} + \frac{e^{-it}}{-2i}\right] = 2\pi i\left[\frac{e^{it}-e^{it}}{2i}\right] = 2\pi i \sin t.$$

Example 52 Using Cauchy's integral theorem, to evaluate $\int_c \frac{e^z}{(z^2+\pi^2)^2}dz,$ where c is the circle $|z| = 4$.

[U.P.T.U. 2008]

Solution Put $(z^2 + \pi^2)^2 = 0$

or $\quad z = \pm\pi i$ are the poles of order 2.

The given circle $|z| = 4$ has centre zero and radius 4.
Clearly, the given circle encloses both the poles. Hence, by Cauchy's theorem, we have

$$\int \frac{e^z}{(z^2+\pi^2)^2}dz = \int_{c_1} \frac{e^z/(z+\pi i)^2}{(z-\pi i)^2}dz + \int_{c_2} \frac{e^z/(z-\pi i)^2}{(z+\pi i)^2}dz$$

$$= 2\pi i\left[\frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\}\right]_{z=\pi i} + 2\pi i\left[\frac{d}{dz}\left\{\frac{e^z}{(z-\pi i)^2}\right\}\right]_{z=-\pi i}$$

$$= 2\pi i\left[\frac{e^z(z+\pi i-2)}{(z+\pi i)^3}\right]_{z=\pi i} + 2\pi i\left[\frac{e^z(z-\pi i-2)}{(z-\pi i)^3}\right]_{z=-\pi i}$$

$$= \left(\frac{\pi i-1}{2\pi^2}\right) + \left(\frac{\pi i+1}{2\pi^2}\right) = \frac{i}{\pi}. \qquad [\because e^{\pi i} = \cos\pi + i\sin\pi = 1]$$

Example 53 Evaluate $f(z)$, if $f(z) = a + bz + cz^2$ and $\oint_c \frac{f(z)}{z}dz = \oint_c \frac{f(z)}{z^2}dz = \oint_c \frac{f(z)}{z^3}dz = 2\pi i$ where c is the circle $|z| = 1$.

Solution Given

$$\oint_c \frac{f(z)}{z}dz = 2\pi i \tag{72}$$

Here $z = 0$ is a simple pole which lies inside the circle $|z| = 1$.
Hence, by Cauchy's integral formula, Eq. (72) gives

$$2\pi i[f(z)]_{z=0} = 2\pi i$$

or $$2\pi i\,[a + bz + cz^2]_{z=0} = 2\pi i$$

or $$2\pi i\,(a) = 2\pi i$$

or $$\boxed{a = 1}$$

Now, $$\oint_c \frac{f(z)}{z^2}\,dz = 2\pi i \tag{73}$$

Here, $z = 0$ is a pole of order 2, which lies inside the circle $|z| = 1$.
From Eq. (73)

$$\frac{2\pi i}{1!}\left[\frac{d}{dz} f(z)\right]_{z=0} = 2\pi i \qquad \text{[By C–I formula for derivative]}$$

or $$2\pi i\left[\frac{d}{dz}(a + bz + cz^2)\right]_{z=0} = 2\pi i$$

or $$2\pi i\,[b + 2cz]_{z=0} = 2\pi i$$

or $$2\pi i(b) = 2\pi i$$

or $$\boxed{b = 1}$$

Also given

$$\oint_c \frac{f(z)}{z^3}\,dz = 2\pi i \tag{74}$$

Here $z = 0$ is a pole of order 3, which lies inside the circle $|z| = 1$.
From Eq. (74)

$$\frac{2\pi i}{2!}\left[\frac{d^2}{dz^2} f(z)\right]_{z=0} = 2\pi i \qquad \text{[By C–I formula for derivative]}$$

or $$\frac{2\pi i}{2}\left[\frac{d^2}{dz^2}(a + bz + cz^2)\right]_{z=0} = 2\pi i$$

or $$\pi i\,[2c]_{z=0} = 2\pi i$$

or $$\boxed{c = 1}$$

Hence, $$f(z) = (1 + z + z^2)$$

3.39.4 Problems on Taylor's and Laurent's Theorem/Series

Example 54 Expand the following functions in a Taylor's series about $z = 0$ and determine the region of convergence in each case.
(i) $\sin z$ (ii) e^z (iii) $\cos z$.

Solution

(i) Let $f(z) = \sin z$, then

$$f'(z) = \cos z,\ f''(z) = -\sin z,\ f'''(z) = -\cos z,\ f^{iv}(z) = \sin z \text{ etc.}$$

$\therefore \quad f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{\text{iv}}(0) = 0$

In general $f^{(2n-1)}(0) = (-1)^{n+1}$.

Since $f(z) = \sin z$ is analytic for all values of z, we have

$$f(z) = \sin z = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots + \frac{z^{2n-1}}{(2n-1)} f^{(2n-1)}(0) + \cdots$$

$$= 0 + z + 0 - \frac{z^3}{3!} + 0 + \cdots + \frac{z^{2n-1}}{(2n-1)!}(-1)^{n+1} + \cdots$$

$$= z - \frac{z^3}{3!} + \cdots + \frac{z^{2n-1}}{(2n-1)!}(-1)^{n+1} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{z^{2n-1}}{(2n-1)!}, \text{ when } |z| < \infty.$$

(ii) $f(z) = e^z$, then $f'(z) = e^z, f''(z) = e^z, f'''(z) = e^z, f^{(\text{iv})}(z) = e^z, \ldots, f^{(n)}(z) = e^z$

$\therefore \quad f(0) = 1, f'(0) = 1, f''(0) = 1, \ldots, f^{(n)}(0) = 1.$

Since $f(z) = e^z$ is analytic for every value of z, we have

$$f(z) = e^z = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \cdots + \frac{z^n}{n!} f^{(n)}(0) + \cdots$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ when } |z| < \infty.$$

(iii) Proceeding as in (i), we can obtain

$$\cos z = 1 + \sum_{n=1}^{\infty} (-1)^2 \frac{z^{2n}}{(2n)!}, \text{ when } |z| < \infty.$$

Example 55 Find the Taylor's and Laurent's series which represents the function $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$ in the regions

(i) $|z| < 2$, (ii) $2 < |z| < 3$, (iii) $|z| > 3$.

Solution

$$f(z) = \frac{z^2 - 1}{(z+2)(2+3)} = 1 - \frac{5z+7}{(z+2)(z+3)}$$

$$F(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} \tag{75}$$

(i) When $|z| < 2$, then $\dfrac{|z|}{2} < 1$

$$\therefore \quad f(z) = 1 + \frac{3}{2}\cdot\frac{1}{\left(1-\frac{z}{2}\right)} - \frac{8}{3}\cdot\frac{1}{\left(1+\frac{z}{3}\right)} = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{2}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n \cdot \frac{z^n}{2^n} - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{3^n}$$

$$f(z) = 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n.$$

This is the Taylor's series valid for $|z| < 2$.

(ii) When $2 < |z| < 3$, then $|z| > 2$ and $|z| < 3$

$$\therefore \quad \frac{2}{|z|} < 1 \text{ and } \frac{|z|}{3} < 1$$

$$\therefore \quad f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3.2^n}{z^{n+1}} - \frac{8\,z^n}{3^{n+1}}\right]\cdot z^n$$

(iii) When $|z| > 3$, then $\dfrac{3}{|z|} < 1, \dfrac{2}{|z|} < \dfrac{2}{3} < 1$

$$\therefore \quad f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)\left(\frac{3}{z}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\,[3.2^n - 3^n \cdot 8]\cdot\frac{1}{z^{n+1}}$$

Example 56 Find the Laurent series expansion of the function $f(z) = \dfrac{1}{z^2(1-z)}$ about $z = 0$.

Solution

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}(1-z)^{-1} = \frac{1}{z^2}(1+z+z^2+z^3+\cdots)$$

$$f(z) = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n.$$

or
$$f(z) = \frac{1}{z^2} + \frac{1}{z} + 1 + \sum_{n=1}^{\infty} z^n$$

Example 57 Show that

$$e^{\frac{c}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n, \text{ where } a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos[n\theta - c\sin\theta]\, d\theta$$

Solution The function $f(z) = e^{c/2\left(z-\frac{1}{z}\right)}$ is analytic except at $z = 0$ and $z = \infty$. Hence $f(z)$ is analytic in the annulus region $r_1 \le |z| \le r_2$ where r_1 is small and r_2 is large.

$\therefore$ $f(z)$ can be expanded in Laurent's series in the form.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n},$$

where
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z^{n+1}},\ b_n = a_{-n}. \qquad C: |z| = 1 \text{ or } z = re^{i\theta},\ 0 \le \theta \le 2\pi$$

$\therefore$
$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{ic\sin\theta}\cdot\frac{e^{i\theta}\, i\, d\theta}{e^{i(n+1)\theta}} = \frac{1}{2\pi}\int_0^{2\pi} e^{i[c\sin\theta - n\theta]}\, d\theta$$

or
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(c\sin\theta - n\theta)\, d\theta - \frac{i}{2\pi}\int_0^{2\pi} \sin(c\sin\theta - n\theta)\, d\theta. \tag{76}$$

II$^{\text{nd}}$ integral of Eq. (76) is zero, because, by the property of definite integral $\int_0^{2a} f(x)\,dx = 0)$ if $f(2a - x) = -f(x)$.

$\therefore$
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos[c\sin\theta - n\theta]\, d\theta. \tag{77}$$

Equation (76) is also expressible as

$$f(x) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-1}^{-\infty} a_n z^n$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \, z^n. \tag{78}$$

Equation (77) and (78) shows the required result. **Hence, proved.**

EXERCISE 3.4

1. Prove that the value of the integral of $\frac{1}{z}$ along a semi-circular are $|z| = 1$ from -1 to $+1$ is $-\pi i$ or πi according as the are lies above or below the real axis.
2. Evaluate $\int_C \bar{z}\, dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C defined by
   (i) $z = t^2 + \text{it}$
   (ii) the line from $z = 0$ to $z = 2i$ and the line from $z = 2i$ to $z = 4 + 2i$.

Brook Taylor was born on 18 August 1685 in Edmonton, England. Taylor was home tutored before starting his studies in St. John's College, Cambridge. He acquired the degrees of LLB in 1709 and LLD in 1714. He was interested in Art and Music, but his first love was mathematics. He portrayed exceptional abilities in mathematics by writing a very important paper even before his graduation. It gave the explanation of the oscillation of a body. Noticing Taylor's extraordinary expertise in the subject, he was elected as a member of the Royal Society by Machin and Keill. In 1714, Brook Taylor became the secretary of the Royal Society. He died on 29 December 1731.

Pierre Alphonse Laurent born in 18 July 1813 Paris, France. He was a French mathematician best known as the discoverer of the Laurent series, an expansion of a function into an infinite power series, generalizing the Taylor series expansion. Laurent graduated from the École Polytechnique in 1832, and entered the engineering corps as second lieutenant. He then attended the École d'Application at Metz until he was sent to Algeria. His result was contained in a memoir submitted for the Grand Prize of the Académie des Sciences in 1843, Laurent died at age 41 in 2 September 1854 in Paris, France.

3.40 ZEROS OF AN ANALYTIC FUNCTION

Let $f(z)$ be an analytic function in a domain D. Then it can be expanded in Taylor's theorem about the point $z = z_0$ in D is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ where, } a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

If $a_0 = a_1 = a_2 = \dots = a_{m-1} = 0$ but $a_m \neq 0$, then $z = z_0$ is a zero of $f(z)$ of n^{th} order.

If $f(z)$ satisfies the conditions of the Laurants theorem, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

where $$a_n = \frac{1}{2\pi i} \oint \frac{f(t)\, dt}{(t - z_0)^{n+1}}, \quad b_n = \frac{1}{2\pi i} \oint \frac{f(t)\, dt}{(t - z_0)^{-n+1}}$$

the term $\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ is called the principal part of the function $f(z)$ at $z = z_0$.

3.41 SINGULARITIES

Consider a function $f(z)$ which is analytic at all points of a circular bounded region D except at a finite number of points, these exceptional points are known as singular point or singularities.

There are two types of the singularities:

(i) Isolated Singularities

A point $z = z_0$ is said to be an isolated singularity of $f(z)$, if $f(z)$ is analytic at each point in the neighbourhood of z_0. For example

$$f(z) = \frac{z^2 + 5}{z(z-1)}$$

At $z = 0$ or 1, function $f(z)$ does not define.

$\therefore \quad z = 0$ or $z = 1$ are singularities but $f(z)$ is analytic in the neighbourhood of $z = 0$ or $z = 1$.

Hence, $z = 0$ and 1 are isolated singularity.

(ii) Non-isolated Singularities

A point $z = z_0$ is said to be a non-isolated singularity of $f(z)$, if $f(z)$ is not analytic at $z = z_0$ and in the neighbourhood of z_0.

***Note*:** Isolated singularities are further specify by Pole's, essential singularity and removable singularity.

Let a function $f(z)$ is analytic in a domain except at $z = z_0$. Then $f(z)$ can be expanded by using Laurent series expansion in to the form of

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{I}} + \underbrace{\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}}_{\text{II}}$$

The second term of the expansion in the RHS is known as principal part of $f(z)$ at $z = z_0$.

Now there are the following three possibilities:

(a) *Pole:* If the principal part of $f(z)$ consist only a finite number of terms say m. Then $z = z_0$ is said to be a pole of order m. If $m = 1$, then z_0 is called a simple pole.

(b) *Essential Singularity:* If the principal part in Laurent expansion of $f(z)$ consists an infinite number of terms, then $z = z_0$ is called an isolated essential singularity.

(c) *Removable Singularity:* If the principal part in Laurent expansion of $f(z)$ does not contain any terms, that is, all b_n are zero, then $z = z_0$ is called a removable singularity.

Example 58 Find Laurent expansion of $\dfrac{1}{z^2\,(z+2)}$ about $z = 0$ and indicate the character of the singularity.

Solution We have

$$f(z) = \frac{1}{z^2\,(z+2)} = \frac{1}{2z^2}\cdot\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2z^2}\cdot\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\cdots\right] = \frac{1}{2}\left[\frac{1}{z^2}-\frac{1}{2z}+\frac{1}{4}-\frac{z}{8}+\cdots\right]$$

Thus, the Laurent expansion about $z = 0$ has only two terms in the principal part.
Hence, $z = 0$ is a pole of 2^{nd} order.

Example 59 Find the Laurent expansion of $e^{1/z}$ about $z = 0$ and indicate the character of the singularity.

Solution We have

$$f(z) = e^{\frac{1}{z}} = 1+\frac{1}{z}+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^3+\cdots = 1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\cdots$$

Thus, the Laurent expansion about $z = 0$ contain an infinite number of terms in the principal part of $f(z)$.
Hence, $z = 0$ is an essential singularity.

Example 60 Find the Laurent expansion of $\dfrac{\sin z}{z}$ about z = 0 and name of the singularity.

Solution We have

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}(\sin z)$$

$$= \frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right) = 1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^7}{7!}+\cdots$$

Thus, the Laurent expansion about $z = 0$ does not contain no terms in the principal part of $f(z)$.
Hence, $z = 0$ is a removable singularity.

Example 61 Find the singularities of $f(z) = \dfrac{z}{(z^2+4)^2}$ and indicate the character of the singularities.

Solution We have

$$f(z) = \frac{z}{(z^2+4)^2} = \frac{z}{[(z+2i)(z-2i)]^2} = \frac{z}{\left[(z+2i)^2\,(z-2i)^2\right]}.$$

Since $\lim\limits_{z \to -2i} (z + 2i)^2 \cdot f(z) = \lim\limits_{z \to -2i} \dfrac{z}{(z - 2i)^2} = -\dfrac{i}{8} \neq 0,$ it follows that $z = -2i$ is a pole of order 2.

Similarly, $z = +2i$ is a pole of order 2. Further, we can find such that no other singularity other than $z = 2i$ lies inside the circle $|z - 2i| = \delta$. For example if $\delta = 1$. Hence $z = 2i$ is an isolated singularity. Similarly, $z = -2i$ is also an isolated singularity.

Example 62 Specify the nature of singularity at $z = -2$ of

$$f(z) = (z-3)\cdot\sin\left(\frac{1}{z+2}\right)$$

Solution Zero's of $f(z)$ are given by

$$f(z) = 0 \Rightarrow (z-3)\sin\left(\frac{1}{z+2}\right) = 0$$

$$\Rightarrow \qquad z = 3 \text{ and } \sin\left(\frac{1}{z+2}\right) = 0 = \sin 0 \Rightarrow \frac{1}{z+2} = n\pi + (-1)^n(0) = n\pi$$

$$\Rightarrow \qquad z = \frac{1}{n\pi} - 2 \text{ for } n = 0, 1, 2, 3, \ldots$$

$\therefore$ $z = -2$ is an isolated singularity.

EXERCISE 3.5

1. Using Cauchy's integral formula to evaluate $\int_c \dfrac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz,$ where C is the circle $|z| = 4$. **[U.P.T.U. 2008]**
2. Evaluate $\int_c \dfrac{3z^2 + z}{z^2 - 1} dz,$ where c is the circle $|z - 1|$.
3. Using Cauchy's integral formula to evaluate $\int_c \dfrac{z-1}{(z-2)(z+1)^2} dz,$ where c is the circle $|z - i| = 2$.
4. Using Cauchy's integral formula to evaluate $\int_c \dfrac{1}{(z^3 - 1)} dz,$ where c is the circle $|z - 1| = 1$.
5. Using Cauchy's integral theorem to evaluate $\int_c \dfrac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz,$ where c is the circle $|z| = 3$. **[U.P.T.U. 2005]**
6. Evaluate $\oint_c \dfrac{e^z + \sin \pi z}{(z-3)^2 (z+4)(z-1)} dz,$ where c is the circle $|z| = 2$.
7. Evaluate $\oint_c \dfrac{e^z}{(z+1)^3 . z^2} dz,$ where c is the circle $|z| = 2$.
8. Expand the function $f(z) = \dfrac{1}{z^2 - 3z + 2}$ in the region (i) $1 < |z| < 2$ (ii) $0 < |z - 1| < 1$. **[U.P.T.U. 2006, 2008, 2010]**

9. Using Laurent's series, expand the function $f(z)=\dfrac{1-\cos z}{z^3}$ about the point $z = 0$. **[U.P.T.U. 2002]**

10. Using Taylor's theorem; show that $\log z=(z-1)-\dfrac{(z-1)^2}{2}+\dfrac{(z-1)^3}{3}-\cdots$ where $|z-1|<1$. **[U.P.T.U. 2004]**

11. Find the Laurent's series expansion for the function $f(z)=\dfrac{7z-2}{z^3-z^2-2z}$ in the regions given by
(i) $0<|z+1|<1$ (ii) $1<|z+1|<3$ (iii) $|z+1|>3$. **[U.P.T.U. 2003, 2005]**

12. Find the Laurent series that represents the function $f(z)=z^2\sin\left(\dfrac{1}{z^2}\right)$ in the domain $0<|z|<\infty$.

13. Find the Taylor's series expansion of the following functions:
(i) $\sin^{-1}z$ in powers of z
(ii) $\dfrac{4z-1}{z^4-1}$ about the point $z = 0$ **[U.P.T.U. 2007]**

14. Define the Laurent series expansion of a function. Expand the function $f(z)=e^{z/(z-2)}$ in a Laurent series about the point $z = 2$. **[U.P.T.U. 2009]**

15. The series expansions of the functions $\dfrac{1}{1-z}$ and $\dfrac{1}{z-1}$ are $\dfrac{1}{1-z}=1+z+z^2+z^3+\cdots$ and $\dfrac{1}{z-1}=\dfrac{1}{z}\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\dfrac{1}{z^3}+\cdots\right)$ on adding, we get $(1+z+z^2)+\dfrac{1}{z}\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\cdots\right)=0$. Is this result true? If not, give the region.

16. Discuss the singularity of the function $f(z)=\sin\left(\dfrac{1}{1-z}\right)$ at $z=1$.

17. Discuss the nature of singularity of the function $f(z)=\dfrac{z-\sin z}{z^3}$ at $z=0$.

18. Discuss the nature of the singularity, of $f(z)=\left(\dfrac{z-2}{z^2}\right)\sin\left(\dfrac{1}{z-1}\right)$ at $z=1$.

19. Discuss the nature of the function $f(z)=\dfrac{1-e^z}{1+e^z}$ at $z=\infty$.

20. Show that the function $f(z)=e^{-\frac{1}{z^2}}$ has no singularities.

21. Discuss the singularity of $f(z)=\dfrac{\cot \pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$.

22. Prove that the singularity of $f(z) = \cot z$ at $z = \infty$ is a non-isolated essential singularity.

23. Show that $\operatorname{cosec} z=\dfrac{1}{z}+\dfrac{1}{3!}z+\dfrac{7}{360}z^3+\cdots;\ 0<|z|<\pi$.

24. Prove that $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$ when $|z| < 1$.

25. Expand $\sin z$ in a Taylor's series about $z = \frac{\pi}{4}$.

Answer

1. $4\pi i$
2. $4\pi i$
3. $-\frac{2\pi i}{9}$
4. $-\frac{4\pi i}{9}$
5. $-4\pi i$
6. $\frac{\pi i e}{10}$
7. $\left(\frac{11}{e} - 4\right)\pi i$
8. (i) $-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$ (ii) $-\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n$
9. $\frac{1}{2!z} - \frac{1}{4!}z + \frac{1}{6!}z^3 - \cdots$
11. (i) $-\frac{3}{z+1} - \sum_{n=0}^{\infty}(z+1)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$ (ii) $\frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{z+1} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$

    (iii) $\frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{z+1} - \frac{2}{z+1}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n$
12. $1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}\cdot\frac{1}{z^{4n}}$.
13. (i) $z + \frac{z^3}{6} + \frac{3z^5}{40} + \cdots$ (ii) $-\frac{3}{4}\sum_{n=0}^{\infty}z^n + \frac{5}{4}\sum_{n=0}^{\infty}(-1)^n z^n + \left(\frac{1}{2} - 2z\right)\sum_{n=0}^{\infty}(-1)^n z^{2n}$.
14. $e\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2}{z-2}\right)^n$.
15. No, the given first series is valid only for $|z| < 1$ and the 2nd series is valid only for $|z| > 1$. There is no common point where both the series are valid.
16. $z = 1$ is an isolated essential singularity.
17. $z = 0$ is a removable singularity.
18. $z = 1$ is an isolated essential singularity.
19. Non-isolated essential singularity.
21. $z = a$ is a double pole and $z = \infty$ is a non-isolated essential singularity.

3.42 THE CALCULUS OF RESIDUES

3.42.1 Residue at a Pole

Suppose a single valued function $f(z)$ has a pole of order m at $z = z_0$, then the principal part of Laurent expansion of (z) consist only m terms so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \tag{79}$$

where $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz, b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

C being a circle $|z - z_0| = r$.

The coefficient b_1, in the principal part of the expansion, given by

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz \tag{80}$$

The coefficient b_1 is called the residue of $f(z)$ at the pole $z = z_0$ and is denoted by the symbol $\text{Res}(z = z_0) = b_1$. Evidently the value of b_1, given by Eq. (80), does not depend upon the order of the pole and hence it represents a general definition of the residue at a pole then Eq. (80) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0},$$

Thus, the residue at $z = z_0$ is

$$\begin{aligned} \text{Res}(z = z_0) &= \lim_{z \to z_0} (z - z_0) \cdot f(z) = b_1 \\ &= \frac{1}{2\pi i} \int_C f(z)\, dz. \qquad \text{(using Eq. (80))} \end{aligned}$$

3.42.2 Residue at Infinity

Residue of $f(z)$ at $z = \infty$ is defined as $-\frac{1}{2\pi i} \int_C f(z)\, dz$, where the integration is taken round C in anti-clockwise direction.

3.42.3 Method of Finding the Residues

1. *If $f(z)$ has a pole of order one at $z = z_0$:*

   Since $z = z_0$ is a pole of order 1, the Laurent series expansion becomes

   $$f(z) = a_0 + a_1(z - z_0) + a^2(z - z_0)^2 + \ldots + b_1(z - z_0)^{-1}.$$

   Multiplying both sides by $(z - z_0)$, we get

   $$(z - z_0) f(z) = a_0 (z - z_0) + a_1 (z - z_0)^2 + \ldots + b_1$$

   $$\therefore \qquad \lim_{z \to z_0} (z - z_0) f(z) = b_1 = \text{Res}\,(z = z_0)$$

2. *If $f(z)$ has a pole of order 'm' at $z = z_0$:*
   Since $z = z_0$ is a pole of order 'm', the Laurent expansion becomes
   $$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots + b_1(z - z_0)^{-1} + \dots + b_m(z - z_0)^{-m}.$$
   Multiplying both sides by $(z - z_0)^m$, we get
   $(z - z_0)^m f(z) = a_0 (z - z_0)^m + a_1(z - z_0)^{m+1} + \dots + b_1(z - z_0)^{m-1} + \dots + b_m.$
   Differentiating both sides $(m - 1)$ times w.r.to z and taking the limit as $z \to z_0$, we get
   $$\lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m \; f(z) \right] \right] = b_1 \; (m-1)!$$
   or $\qquad \text{Res}(z = z_0) = b_1 = \dfrac{1}{(m-1)!} \lim\limits_{z \to z_0} \left[\dfrac{d^{m-1}}{dz^{m-1}} \left\{ (z - z_0)^m \cdot f(z) \right\} \right].$
3. If $f(z)$ in the form given by
   $$f(z) = \frac{\phi(z)}{\psi(z)}; \psi(z_0) = 0, \phi(z_0) \neq 0,$$
   where $z = z_0$ is a pole of order 1, then
   $$\text{Res } (z = z_0) = \frac{\phi(z_0)}{\psi'(z_0)}.$$
4. Residue of $f(z)$ at a simple pole $(z = z_0)$ = coefficient of $\dfrac{1}{t}$ in $f(z_0 + t)$ expanded in powers of t, where t is sufficiently small.
5. Residue of $f(z)$ at $(z = \infty) = \lim\limits_{z \to \infty} [-z \cdot f(z)]$
   or $\;$ = [Coefficient of $1/z$ in the expansion of $f(z)$ for values of z in the neighbourhood of $z = \infty$].

3.43 CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic within and on a closed contour C, except at a finite number of poles $z_1, z_2, z_3, \dots, z_n$ within C, then

$$\int_C f(z)\, dz = 2\pi i \sum_{r=1}^{n} \text{Res } (z = z_r),$$

where RHS denotes sum of residues of $f(z)$ at its poles lying within 'C'.

Proof: Suppose $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n$ are the circles with centres at $z_1, z_2, \dots, z_n$, respectively and radii so small that they lie within closed curve C and do not overlap. Then $f(z)$ is analytic within the region enclosed by the curve C and these circles. Hence by Cauchy's theorem for multi-connected regions, we have

$$\int_C f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz + \cdots + \int_{\gamma_n} f(z)\, dz \tag{81}$$

But $$\frac{1}{2\pi i}\int_{\gamma_1} f(z)\,dz = \text{residue of } f(z) \text{ at } (z = z_1) = \text{Res}\,(z = z_1)$$

$\therefore$ $$\int_{\gamma_1} f(z)\,dz = 2\pi i \,\text{Res}\,(z = z_1)$$

Using this in Eq. (81), we get

$$\int_C f(z)\,dz = 2\pi i \,\text{Res}\,(z = z_1) + 2\pi i\,\text{Res}\,(z = z_2) + \cdots + 2\pi i\,\text{Res}\,(z = z_n)$$

$$\int_C f(z)\,dz = 2\pi i \sum_{r=1}^{n} \text{Res}\; z = z_r)$$

Example 63 Evaluate the residues of $\dfrac{z^2}{(z-1)\,(z-2)(z-3)}$ at z = 1, 2, 3 and infinity and show that their sum is zero.

Solution Suppose $f(z) = \dfrac{z^2}{(z-1)\,(z-2)(z-3)}$

$$\text{Res}\,(z=1) = \lim_{z\to 1}\left[(z-1)\,f(z)\right] = \lim_{z\to 1}\left[(z-1)\cdot\frac{z^2}{(z-1)(z-2)(z-3)}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^2}{(z-2)(z-3)}\right] = \frac{1}{(1-2)(1-3)} = \frac{1}{2}$$

$$\text{Res}\,(z=2) = \lim_{z\to 2}\left[(z-2)\cdot f(z)\right] = \lim_{z\to 2}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{4}{(2-1)(2-3)} = -4$$

$$\text{Res}\,(z=3) = \lim_{z\to 3}\left[(z-3)\cdot f(z)\right] = \lim_{z\to 3}\left[\frac{z^2}{(z-1)(z-2)}\right] = \frac{9}{(3-1)\,(3-2)} = \frac{9}{2}$$

Now,

$$\text{Res}(z=+\infty) = \lim_{z\to\infty}\left[-z\,f(z)\right]$$

$$\lim_{z\to\infty}\frac{-z^3}{(z-1)(z-2)\,(z-3)} = -1$$

Sum of the residues

$$R_+ = \text{Res}(z=1) + \text{Res}\,(z=2) + \text{Res}\,(z=3) + \text{Res}\,(z=\infty)$$

$$= \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0$$

Hence, proved.

Example 64 Using residue theorem, evaluate $\int_C \frac{e^z}{z(z-1)^2}\,dz$ where C is circle $|z| = 2$.

Solution Here $f(z) = \frac{e^z}{z(z-1)^2}$

For pole, put denominator of $f(z) = 0$

i.e. $z(z-1)^2 = 0$ or z = 0, $z = 1$. (Twice)

$z = 0$ and $z = 1$ lying within $|z| = 2$.

$\therefore$ $z = 0$ is a simple pole and $z = 1$ is a pole of order two.

Now

$$\text{Res}\,(z=0) = \lim_{z\to 0}\left[(z-0)\cdot f(z)\right] = \lim_{z\to 0}\left[\frac{e^z}{(z-1)^2}\right] = 1$$

$$\text{Res}\,(z=1) = \lim_{z\to 1}\left[\frac{1}{1!}\frac{d}{dz}\left\{(z-1)^2\cdot f(z)\right\}\right]$$

$$\text{Res}\,(z=1) = \lim_{z\to 1}\left[\frac{d}{dz}\cdot\left\{\frac{e^z}{z}\right\}\right] = \lim_{z\to 1}\left[\frac{e^z\cdot z - 1\cdot e^z}{z^2}\right] = \lim_{z\to 1}\left[\frac{(z-1)\,e^z}{z^2}\right] = 0.$$

$\therefore$ By Cauchy residue theorem

$$\int \frac{e^z}{z(z-1)^2}\,dz = 2\pi i\,[\text{Res}\,(z=0) + \text{Res}\,(z=1)]$$

$$= 2\pi i\,[1+0] = 2\pi i.$$

Example 65 Find the residue of $\frac{z^3}{z^2-1}$ at $z = \infty$.

Solution Here

$$f(z) = \frac{z^3}{z^2-1} = \frac{z^3}{z^2}\left(1 - \frac{1}{z^2}\right)^{-1}$$

$$= z\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots\right) = z + \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \cdots$$

$$\therefore \quad \text{Res}\,(z=\infty) = -\left(\text{coefficient of } \frac{1}{z}\right) = -(1) = -1.$$

Example 66 Find the poles and residue of the function $\frac{1-e^{2z}}{z^4}$.

Solution Here

$$f(z) = \frac{1-e^{2z}}{z^4}$$

The pole of $f(z)$ is evidently $z = 0$ (not of the 4^{th} order).
Since

$$\frac{1-e^{2z}}{z^4} = \frac{1}{z^4}\left[1-\left(1+2z+\frac{4z^2}{2!}+\frac{2}{3}z^3\right)+\cdots\right] = \frac{2+2z+\frac{4}{3}z^2+\frac{2}{3}z^3+\cdots}{z^3}$$

$\therefore$ The pole is of order 3.

Now, $$\text{Res}(z=0) = \lim_{z\to 0}\left[\frac{1}{2!}\frac{d^2}{dz^2}\left\{(z-0)^3\cdot\frac{1-e^{2z}}{z^4}\right\}\right]$$

$$= \lim_{z\to 0}\left[\frac{1}{2}\cdot\frac{d^2}{dz^2}\cdot\left(\frac{1-1-2z-\frac{4z^2}{2}-\frac{8z^3}{6}-\cdots}{z}\right)\right]$$

$$= \lim_{z\to 0}\left[\frac{1}{2}\frac{d^2}{dz^2}\cdot\left(\frac{-2z-\frac{4}{3}z^2-\cdots}{z}\right)\right] = \frac{1}{2}\lim_{z\to 0}\left[-\frac{8}{3}-\frac{2}{3}\cdot 6z-\cdots\right] = -\frac{4}{3}$$

Example 67 Evaluate $\oint_C \frac{2z^2+3}{(z+2)^2\,(z^2+4)}\,dz$, where C is the square with the vertices at $1+i$, $2+i$, $2+2i$ and $1+2i$.

Solution Here

$$f(z) = \frac{2z^2+3}{(z+2)^2\,(z^2+4)}$$

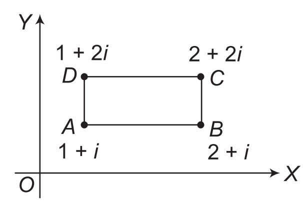


Fig. 3.19

The poles of $f(z)$ are

$z = -2$ (order two)

$z = \pm 2i$ (simple pole).

Since the poles $z = -2$ and $2i$ does not lie in side the C with vertices $1+i$, $2+i$, $2+2i$ and $1+2i$, Hence by Cauchy integral theorem.

$$\oint_C \frac{2z^2+3}{(z+2)^2\,(z^2+4)}\,dz = 0.$$

Example 68 Find the residue at the poles of the function $f(z) = \frac{\cot \pi z}{(z-a)^2}$.

Solution We have

$$f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{(z-a)^2 \sin \pi z}$$

The poles of the given function $f(z)$ are given by $(z-a)^2 \sin \pi z = 0$

$(z-a)^2 = 0$ and $\sin \pi z = 0$

or $z = a$ (pole of order 2)

and $\quad z = n;\ n \in I$ (simple pole if n is finite)

Now, residues at $z = a$ is

$$\text{Res}(z=a) = \lim_{z\to a} \frac{1}{1!}\left[\frac{d}{dz}\cdot \cot \pi z\right] = \lim_{z\to a}\left[-\pi \operatorname{cosec}^2 \pi z\right] = -\pi \operatorname{cosec}^2 \pi a$$

and Residues at $z = n$ is

$$\text{Res }(z=n) = \lim_{z\to n}\left[\frac{\cos \pi z \big/ (z-a)^z}{\pi \cos \pi z}\right] \qquad \left[\because \quad \text{Res }(z=a) = \lim_{z\to a}\left\{\frac{\phi(z)}{\psi'(z)}\right\}\right]$$

$$= \left[\frac{\cos \pi n \big/ (n-a)^2}{\pi \cos \pi n}\right] = \frac{1}{\pi (n-a)^2}.$$

Example 69 Find the sum of the residues of the function $f(z) = \dfrac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.

Solution We have $f(z) = \dfrac{\sin z}{z\cos z}$. The poles of $f(z)$ are given by

$$z \cos z = 0 \Rightarrow z = 0 \text{ and } \cos z = 0$$

$\therefore \quad z = 0$ is a simple pole and

$$z = (2n+1)\frac{\pi}{2};\ n \in I \quad \text{(simple poles if } n \text{ is finite)}$$

Out of these poles, $z = 0,\ z = \pm\dfrac{\pi}{2}$ lie inside the circle $|z| = 2$.

Residues at the pole $z = 0$ is

$$\text{Res }(z=0) = \lim_{z\to 0}\left[(z-0)\cdot\frac{\sin z}{z\cos z}\right] = \lim_{z\to 0}\left[\frac{\sin z}{\cos z}\right] = 0$$

Residues at the pole $z = \dfrac{\pi}{2}$ and $-\dfrac{\pi}{2}$ are

$$\text{Res}\left(z = \frac{\pi}{2}\right) = \lim_{z\to \pi/2}\left[\left(z - \frac{\pi}{2}\right)\cdot\frac{\sin z}{z\cos z}\right] \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{z\to\pi/2}\left[\frac{\left(z-\frac{\pi}{2}\right)\cdot\cos z + \sin z}{-z\sin z + \cos z}\right] = -\frac{2}{\pi}$$

Similarly, $\text{Res}\left(z = -\dfrac{\pi}{2}\right) = \dfrac{2}{\pi}$

Hence, sum of residues $= 0 - \dfrac{2}{\pi} + \dfrac{2}{\pi} = 0.$

Example 70 Using residue theorem to evaluate the integral $\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2(z^2+2z+2)}dz$ at its poles inside the circle $|z| = 3$.

Solution Here

$$f(z)=\frac{e^{zt}}{z^2(z^2+2z+2)}$$

The poles of $f(z)$ are given by

$$z^2(z^2 + 2z + 2) = 0$$

or $\qquad z = 0$ (double pole) and

$$z^2+2z+2=0 \Rightarrow z=-1\pm i \text{ (simple poles)}$$

Residue at $z = 0$ is

$$\text{Res}(z=0)=\lim_{z\to 0}\left[\frac{1}{1!}\frac{d}{dz}(z-0)^2\cdot\frac{e^{zt}}{z^2\cdot(z^2+2z+2)}\right]$$

$$=\lim_{z\to 0}\left[\frac{d}{dz}\left(\frac{e^{zt}}{z^2+2z+2}\right)\right]=\lim_{z\to 0}\left[\frac{(z^2+2z+2)(t\,e^{zt})-(e^{zt})(2z+2)}{(z^2+2z+2)^2}\right]$$

$$=\frac{t-1}{2}$$

Residue at $z = (-1 + i)$ is

$$\text{Res}\,(z=-1+i)=\lim_{z\to(-1+i)}\left[\{z-(-1+i)\}\frac{e^{zt}}{z^2(z^2+2z+2)}\right]$$

$$=\lim_{z\to(-1+i)}\left(\frac{e^{zt}}{z^2}\right)\cdot\lim_{z\to(-1+i)}\frac{(z+1-i)}{(z^2+2z+2)}$$

$$=\frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i}=\frac{e^{(-1+i)t}}{4}$$

Similarly,

$$\text{Res}\,(z=-1-i)=\lim_{z\to(-1-i)}\left[[z-(1-i)]\cdot\frac{e^{zt}}{z^2(z^2+2z+2)}\right]=\frac{e^{(-1-i)t}}{4}$$

Hence, by residue theorem

$$\oint_c \frac{e^{zt}}{z^2(z^2+2z+2)}dz=2\pi i\text{ (sum of residues)}$$

$$=2\pi i\left[\frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}\right]=2\pi i\left[\frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t\right]$$

or $$\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \left[\left(\frac{t-1}{2}\right)+\frac{1}{2}e^{-t}\cos t\right].$$

Example 71 Find the residue of the function $f(x)=\dfrac{\cot z\cdot\coth z}{z^3}$ at $z=0$.

Solution We have

$$f(z) = \frac{\cot z\cdot\coth z}{z^3}$$

or $$f(z) = \frac{\cos z\cosh z}{z^3\sin z\cdot\sinh z} = \frac{1}{z^3}\left[\frac{\left(1-\frac{z^2}{z!}+\frac{z^4}{4!}-\cdots\right)\left(1+\frac{z^2}{z!}+\frac{z^4}{4!}+\cdots\right)}{\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right)}\right]$$

$$= \frac{\left(1-\frac{z^6}{6}+\cdots\right)}{z^5\left(1-\frac{z^4}{90}+\cdots\right)} = \frac{1}{z^5}\left(1-\frac{z^6}{6}+\cdots\right)\left(1-\frac{z^4}{90}+\cdots\right)^{-1} = \frac{1}{z^5}\left(1-\frac{7}{45}z^4+\cdots\right)$$

Hence, the residue at $z=0$ is

$$= \text{coefficient of } \frac{1}{z} \text{ in the expansion of } f(z)$$

$$= -\frac{7}{45}.$$

Example 72 Find the residue of $f(z)=z\cdot\cos\dfrac{1}{z}$ at $z=0$.

Solution Let

$$f(z) = z\cos\frac{1}{z}$$

or $$f(z) = z\cdot\left[1-\frac{1}{2!z^2}+\frac{1}{4!}\cdot\frac{1}{z^4}-\cdots\right] = z-\frac{1}{2z}+\frac{1}{24z^3}-\cdots$$

$\therefore$ Residue at $z=0$ = coefficient of $\dfrac{1}{z}$ in the expansion of $f(z)$

$$= -\frac{1}{2}.$$

Example 73 Find the residue of $f(z)=\dfrac{z}{\sin z}$.

Solution The poles of $f(z)$ are given by

$$\sin z = 0 \Rightarrow z = n\pi;\ n\in I$$

$z = 0$ is not a pole; since $\dfrac{\sin z}{z} \to 1$ as $z \to 0$.

$$\therefore \quad \operatorname{Re} s\,(z = n\pi) = \lim_{z\to n\pi}\left[(z-n\pi)\cdot\frac{z}{\sin z}\right] \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{z\to n\pi}\left[\frac{(2z-n\pi)}{\cos z}\right] = \frac{n\pi}{(-1)^n} = (-1)^{-n}\cdot n\pi.$$

EXERCISE 3.6

Evaluate the following integrals using Cauchy's residue theorem.

1. $\displaystyle\oint_C \frac{z}{(z-1)(z-2)^2}\,dz,$ where C is the circle $|z-2| = \dfrac{1}{2}$.

2. $\displaystyle\oint_C \frac{1}{(z-1)(z+1)}\,dz,$ where C is the circle $|z| = 3$.

3. $\displaystyle\oint_C \frac{(z+1)}{z^2(z-3)}\,dz,$ where C circle $|z| = 4$.

4. $\displaystyle\oint_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z-2)}\,dz,$ where C is circle $|z| = 3$.

5. $\displaystyle\oint_C \frac{z\cdot\cos z}{\left(z-\frac{\pi}{2}\right)^3}\,dz,$ where C is circle $|z-1| = 1$.

6. $\displaystyle\oint_C \frac{(3z^2+z+1)}{(z^2-1)(z+3)}\,dz,$ where C is circle $|z| = 2$.

7. $\displaystyle\oint_C \frac{z^2}{(z+1)^2(z-2)}\,dz,$ where C is the circle $|z| = 5/2$.

8. $\displaystyle\oint_C \frac{1}{(z^2+1)(z^2-4)}\,dz,$ where C is the circle $|z| = 3/2$.

9. Show that the residue of the function $f(z) = \dfrac{1}{z^3}(\operatorname{cosec} z \operatorname{cosec} h\, z)$ at $z = 0$ is $-\dfrac{1}{60}$.

10. Let C be a circle with equation $|z| = 4$. Determine the value of the integral $\displaystyle\oint_C z^2\cdot\operatorname{cosec}\frac{1}{z}\,dz$ if it exists.

11. Determine the residues at the poles of the function $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z}$.

12. Prove that $\oint_C \frac{\cosh z}{z^3} dz = \pi i$, where C is the square with vertices at $\pm 2 \pm 2i$.

13. Evaluate $\oint_C \frac{2z^2+5}{z^2(z+2)^2(z^2+4)} dz$, where C is a circle $|z-2i| = 6$.

14. Find the residue of $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ at its poles and hence evaluate $\oint_C f(z)\,dz$, where C is the circle $|z| = \frac{5}{2}$. **[U.P.T.U. 2003]**

15. Find the residue of the following functions:

(i) $z^2 \sin\frac{1}{z}$ at $z = 0$

(ii) $\frac{z^3}{z-1}$ at $z = \infty$

(iii) $\frac{1}{(z^2+1)^3}$ at $z = i$

(iv) $\frac{z^2}{(z-1)^2(z+2)}$ at $z = 1$

Answers

1. $-2\pi i$
2. 0
3. 0
4. $-4\pi i$
5. $-2\pi i$
6. $-\frac{\pi i}{4}$
7. $2\pi i$
8. 0

11. Poles $z = 0, -1 \pm i$, Res$(z = 0) = 2$, Res $(z = -1 + i) = -\frac{(1-3i)}{2}$, Res $(z = -1 - i) = -\frac{1}{2}(1+3i)$.

14. $-\frac{27\pi i}{8}$

15. (i) $-\frac{1}{6}$ (ii) -1 (iii) $-\frac{3i}{16}$ (iv) $\frac{5}{9}$.

3.44 EVALUATION OF REAL DEFINITE INTEGRALS

In this section we discuss the applications of Cauchy's residue theorem to evaluate real definite integrals.

3.44.1 Integration Around the Unit Circle

Suppose the integral of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$, (82)

where the integrand function is a rational function of $\sin\theta$ and $\cos\theta$.

Let C: $|z| = 1$, i.e., C is a circle with centre zero and radius 1.

Putting $\qquad z = e^{i\theta}$

Then $\qquad dz = i\, e^{i\theta}\, d\theta = iz\, d\theta$ and

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$= \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

then Eq. (82) converts into the integral.

$$\int_C f(z)\, dz \tag{83}$$

where $f(z)$ is a rational function of z and C is the unit circle $|z| = 1$.

Thus, integral (83) can be solved by using Cauchy's residue's theorem.

Example 74 Evaluate the integral $I = \int_0^{2\pi} \frac{d\theta}{2 + \sin\theta}$.

Solution Substitute $z = e^{i\theta}$, we get $dz = e^{i\theta} \cdot i\, d\theta$

or $\qquad d\theta = \dfrac{dz}{iz}$

$$\therefore \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2 - 1}{2\, iz}$$

$$2 + \sin\theta = 2 + \frac{z^2 - 1}{2\, iz} = \frac{z^2 + 4\, iz - 1}{2\, iz}.$$

Now,
$$I = \int_0^{2\pi} \frac{d\theta}{2 + \sin\theta} = \oint_C \frac{2\, iz}{z^2 + 4\, iz - 1}\left(\frac{dz}{iz}\right) = \oint_C \frac{2dz}{z^2 + 4\, iz - 1} = \oint_C f(z)\, dz$$

where $\qquad f(z) = \dfrac{2}{z^2 + 4\, iz - 1}$ and C: $|z| = 1$.

Putting $z^2 + 4\, iz - 1 = 0 \Rightarrow z = (-2 \pm \sqrt{3})i$

Let $\qquad z_1 = (-2 + \sqrt{3})i$ and $z_2 = -(2 + \sqrt{3})i$.

The pole $z = z_1$ lies inside the contour C: $|z| = 1$.

$$\text{Res}\,(z = z_1) = \lim_{z \to z_1} [(z - z_1)\, f(z)] = 2 \lim_{z \to z_1} \left[\frac{1}{z - z_2}\right] = \frac{z}{z_1 - z_2} = \frac{1}{\sqrt{3}\, i}$$

Hence,
$$I = \oint_C f(z)\, dz = 2\,\pi i\, [\text{Res}\,(z = z_1)] = \frac{2\pi}{\sqrt{3}}$$

Example 75 Evaluate $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$, where $a > |b|$. Hence, or otherwise evaluate

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta}.$$

Solution We have

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$$

or $$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} \tag{84}$$

Substitute $z = e^{i\theta}$ so that $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$ and $d\theta = \frac{dz}{iz}$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \oint_C \frac{1}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz} = \oint_C \frac{2\,dz}{i\,(bz^2+2az+b)} = \frac{2}{i}\oint_C f(z)\,dz,$$

where $f(z) = \frac{1}{(bz^2+2az+b)}$ and C is the circle $|z| = 1$.

Putting $bz^2 + 2\,az + b = 0 \Rightarrow z = \frac{-a\pm\sqrt{a^2-b^2}}{b}$

Let $$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b},\ z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}$$

If $a > |b|$, then $z = z_1$ lies inside the C: $|z| = 1$.

$$\text{Res}\,(z=z_1) = \lim_{z\to z_1} [(z-z_1)\cdot f(z)]$$

$$= \lim_{z\to z_1}\left[(z-z_1)\cdot\frac{1}{b(z-z_1)(z-z_2)}\right] = \frac{1}{b(z_1-z_2)}$$

$$\text{Res}\,(z=z_1) = \frac{b}{b\cdot 2\sqrt{a^2-b^2}} = \frac{1}{2\sqrt{a^2-b^2}}$$

$\therefore$ By Cauchy residue theorem

$$I = \frac{2}{i}\oint_C \frac{dz}{bz^2+2\,az+b} = 2\,\pi i\cdot\frac{2}{i}\cdot\frac{1}{2\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Hence by Eq. (85)

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

Hence, $$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = \frac{\pi}{\sqrt{2-1}} = \pi \qquad [\because a = \sqrt{2}, b = -1]$$

Example 76 Apply residue theorem to prove that $\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{2\pi}{b^2}\left[a - \sqrt{a^2 - b^2}\right]$; where $0 < b < a$.

Solution Suppose

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \int_0^{2\pi} \frac{1 - \cos 2\theta}{2(a + b\cos\theta)}\, d\theta$$

$$= \text{Real part of } \int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + 2b\cos\theta} \tag{85}$$

Substitute $z = e^{i\theta}$ so that $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$.

Then

$$\int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + 2b\cos\theta} = \oint_C \frac{1 - z^2}{2a + b\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= \oint_C \frac{1 - z^2}{i\,(bz^2 + 2az + b)}\, dz,$$

where C is circle $|z| = 1$.

Poles of the integrand are the roots of $bz^2 + 2\,az + b = 0$

i.e., $$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

let $$z_1 = \frac{-a + \sqrt{a^2 - b^2}}{b} \text{ and } z_2 = \frac{-a - \sqrt{a^2 - b^2}}{b}.$$

Here $|z_2| > 1$ so that $z = z_1$ is lies inside C.

Also $bz^2 + 2\,az + b = b(z - z_1)\,(z - z_2)$, then.

$$\text{Res}\,(z = z_1) = \lim_{z \to z_1}\, (z - z_1) \cdot \frac{1 - z^2}{i\,b(z - z_1)\,(z - z_2)}$$

$$\text{Res}\,(z = z_1) = \lim_{z \to z_1} \frac{1 - z^2}{i\,b(z - z_2)} = \frac{1 - z_1^2}{i\,b(z_1 - z_2)}$$

$$= \frac{z_1\left(\dfrac{1}{z_1} - z_1\right)}{i\,b(z_1 - z_2)} = \frac{z_1\,(z_2 - z_1)}{i\,b(z_1 - z_2)} \qquad [\because\ z_1 z_2 = 1]$$

$$= -\frac{z_1}{ib} = \frac{a - \sqrt{a^2 - b^2}}{ib^2}$$

Hence by Cauchy's residue theorem

$$\oint_C \frac{1 - z^2}{i\,(bz^2 + 2az + b)}\,dz = 2\pi\,i[\text{Res}\,(z = z_1)]$$

$$= 2\pi i \frac{a - \sqrt{a^2 - b^2}}{i\,b^2} = \frac{2\pi}{b^2}(a - \sqrt{a^2 - b^2})$$

$$\therefore \qquad I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\,d\theta = \frac{2\pi}{b^2}(a - \sqrt{a^2 - b^2}).$$

Hence, proved.

Example 77 Evaluate the integral $\int_0^{\pi} \sin^4\theta\,d\theta$.

Solution We have

$$I = \int_0^{\pi} \sin^4\theta\,d\theta = \int_0^{\pi} \sin^4\theta\,d\theta + \int_{\pi}^{2\pi} \sin^4\theta\,d\theta$$

Putting $\theta = 2\pi - \phi$ in the 2[nd] integral and simplifying, we get

$$\int_0^{2\pi} \sin^4\theta\,d\theta = \int_0^{\pi} \sin^4\theta\,d\theta + \int_0^{\pi} \sin^4\phi\,d\phi$$

$$= 2\int_0^{\pi} \sin^4\theta\,d\theta \qquad \left[\because\ \int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(y)\,dy\right].$$

$$\therefore \qquad I = \frac{1}{2}\int_0^{2\pi} \sin^4\theta\,d\theta.$$

Substitute $z = e^{i\theta}$, so that $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right) = \dfrac{z^2 - 1}{2\,iz}$ and $d\theta = \dfrac{dz}{iz}$.

$$\therefore \qquad I = \frac{1}{2}\oint_C \left(\frac{z^2 - 1}{2\,iz}\right)^4 \cdot \frac{dz}{iz} = \oint_C f(z)dz,$$

where $f(z) = \dfrac{(z^2-1)^4}{32\,z^5\,i}$ and C: $|z| = 1$.

The integrand has the pole of order 5 at $z = 0$.
Now,

$$\text{Res}\,(z=0) = \frac{1}{(5-1)!}\lim_{z\to 0}\left[\frac{d^4}{dz^4}(z-0)^5\cdot f(z)\right]$$

$$= \frac{1}{4!}\lim_{z\to 0}\left[\frac{d^4}{dz^4}\cdot z^5\cdot\frac{(z^2-1)^4}{32\,i\,z^5}\right] = \frac{1}{4!}\lim_{z\to 0}\left[\frac{d^4}{dz^4}\cdot\frac{(z^2-1)^4}{32\,i}\right]$$

$$= \frac{1}{786i}\lim_{z\to 0}\left[\frac{d^4}{dz^4}\cdot\{(z^8-4z^6)+6z^4-4z^2+1)\}\right]$$

$$= \frac{144}{786\,i} \qquad [\because \quad D^n z^n = n!]$$

$$= \frac{3}{16\,i}$$

$\therefore$ By Cauchy residue theorem

$$I = 2\,\pi i\cdot\frac{3}{16\,i} = \frac{3\pi}{8}.$$

Example 78 Evaluate $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cdot\cos(\sin\theta - n\theta)\,d\theta$ by Cauchy residue theorem.

Solution Let

$$I = \int_0^{2\pi} e^{\cos\theta}\cdot\cos(\sin\theta - n\theta)\,d\theta$$

$$= \text{Real part of} \int_0^{2\pi} e^{\cos\theta}\cdot e^{i(\sin\theta - n\theta)}\,d\theta$$

$$= \text{Real part of} \int_0^{2\pi} e^{[\cos\theta + i(\sin\theta - n\theta)]}\,d\theta$$

$$= \text{Real part of} \int_0^{2\pi} e^{(\cos\theta + i\sin\theta)}\cdot e^{-in\theta}\,d\theta$$

$$I = \text{Real part of} \int_0^{2\pi} e^{e^{i\theta}}\cdot e^{-in\theta}\,d\theta$$

Substitute $z = e^{i\theta}$ so that $d\theta = \dfrac{dz}{iz}$

$$\therefore \qquad I = \text{Real part of} \oint_C e^z \cdot z^{-n} \cdot \frac{dz}{iz} = \text{R.P.} \oint_C \frac{e^z}{i\, z^{n+1}}\, dz.$$

$$= \frac{1}{i} \oint_C f(z)\, dz$$

where $\qquad f(z) = \dfrac{e^z}{z^{n+1}}$

The integrand has the pole of order $(n + 1)$ at $z = 0$

Now,

$$\text{Res}\,(z = 0) = \frac{1}{n!} \lim_{z \to 0} \left[\frac{d^n}{dz^n} (z^{n+1} \cdot f(z)) \right]$$

$$= \frac{1}{n!} \lim_{z \to 0} \left[\frac{d^n}{dz^n} \cdot e^z \right] = \frac{1}{n!} \lim_{z \to 0} e^z = \frac{1}{n!}$$

Hence, by Cauchy's residue theorem

$$I = \frac{1}{i} \oint_C f(z)\, dz = R.P \left[2 \frac{\pi i}{i} \cdot \frac{1}{n!} \right] = \frac{2\pi}{n!}.$$

3.45 IMPROPER REAL INTEGRALS OF THE FORM $\int_{-\infty}^{\infty} f(z)\, dz$

Any integral of the form $\int_a^b f(x)dx$ is called an improper integral if either (i) one or both of the limits of integration are not finite or (ii) the integrand function has an infinite discontinuity at a or at $b(a, b$ finite) or at some point c, $a < c < b$.

The integral $\int_{-\infty}^{\infty} f(z)dz,$ where the function $f(z)$ is such that no pole of $f(z)$ lies on the real line.

But the poles lie in the upper half of z-plane. Now, evaluate the above integral along the closed. Contour 'C' consisting of

(i) semi circle Γ: $|z| = R$ in the upper half plane.

(ii) real axis from $-R$ to R.

Then, we want to show that integral along Γ vanishes as $|z| \to \infty$.

$$\therefore \qquad \int_C f(z)dz = \int_\Gamma f(z)dz + \int_{-R}^{R} f(z)dz.$$

Taking limit as $R \to \infty$, then.

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(z)dz$$

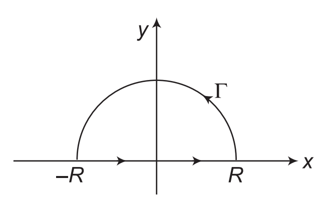


Fig. 3.20

By Cauchy's residues theorem

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \text{ (sum of residues within } C\text{)}.$$

Example 79 Using contour integration, show that $\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$.

Solution Let

$$I = \int_C f(z)\,dz, \text{ where } f(z) = \frac{1}{1+z^2}$$

$$\therefore \qquad \int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx \tag{86}$$

(Refer Fig. 3.20)

Here C is the closed contour consisting of Γ, the upper half of the semi circle C: $|z| = R$ and the real axis from $-R$ to R.

The poles of the function $f(z)$ are

$$z = \pm i$$

Only the pole $z = i$ lie inside C, then

$$\text{Res}\,(z = i) = \lim_{z \to i} (z - i)\, f(z) = \lim_{z \to i} (z - i)\cdot\frac{1}{(z+i)(z-i)} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}.$$

By Cauchy's residue, theorem

$$\int_C f(z)\,dz = 2\,\pi i\cdot\text{Res}\,(z = i) = 2\pi i\cdot\frac{1}{2i} = \pi$$

from Eq. (86), we get

$$\int_\Gamma f(z)\,dz + \int_{-R}^{R} \frac{dx}{1+x^2} = \pi \tag{87}$$

Taking $R \to \infty$ in Eq. (87), we get

$$\lim_{R\to\infty} \int_\Gamma f(z)\,dz + \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

or $0 + \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$ [By the definition of Gauss Jordan Lemma $\lim_{R\to\infty} \int_T f(z) = 0$]

or $$2\int_0^{\infty} \frac{dx}{1+x^2} = \pi$$

or $$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$ **Hence, proved.**

Example 80 Evaluate $\int_0^\infty \frac{\cos ax}{1+x^2}\,dx$.

Solution Let the integral

$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)dx, \tag{88}$$

where $f(z) = \frac{e^{iaz}}{z^2+1}$, here C is the closed contour. Consisting of Γ, the upper half of the circle $|z| = R$ and the real axis from $-R$ to R.

The poles of $f(z)$ are $z = \pm i$

Since the pole $z = i$ lies in side the circle C, then the residue of $f(z)$ at $z = i$ is

$$\text{Res}\,(z = i) = \lim_{z\to i}\left[(z-i)\cdot\frac{e^{iaz}}{(z+i)(z-i)}\right] = \frac{e^{-a}}{2i}$$

$$\therefore \quad \int_C f(z)dz = \int_\Gamma f(z)dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\,[\text{Res}\,(z=i)] = 2\pi i\cdot\frac{e^{-a}}{2i} = \pi e^{-a}$$

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)\,dx = \pi e^{-a}$$

or $$\int_\Gamma f(z)dz + \int_{-R}^{R} \frac{e^{iax}}{x^2+1}\,dx = \pi\, e^{-a}$$

Taking $R \to \infty$, we get

$$\lim_{R\to\infty}\int_\Gamma f(z)dz + \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$$

By Gauss Jordan Lemma, $\lim_{R\to\infty}\int_\Gamma f(z)dz = 0$,

$$\therefore \quad \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi\, e^{-a}$$

Equating real part both side, we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi\, e^{-a}$$

or $$\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}\, e^{-a}.$$

3.46 IMPROPER INTEGRALS WITH POLES ON THE REAL AXIS

In the previous section we have supposed that the function $f(z)$ has no pole on the real axis. In the present section the function $f(z)$ has poles within the semicircle Γ as well as on the real axis. We exclude the poles on the real axis by enclosing them with semi circles of small radii. This procedure is said identing the semi circle contour.

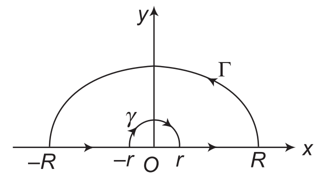


Fig. 3.21

Example 81 Evaluate the integral $\int_0^{\infty} \frac{\sin x}{x}\, dx$.

Solution Let $f(z) = \frac{e^{iz}}{z}$

Suppose the integral $\int_C f(z)dz$, where C is the closed contour consisting of Γ, upper half circle $|z| = R$ and real axis from $-R$ to R idented at $z = 0$. Consider r be the radius of identation. Since the function $f(z)$ has no singularity within C and by Cauchy residue theorem.

$$\int_C f(z)dz = 2\pi i \text{ (sum of residues within } C)$$

$$= 2\,\pi i \cdot (0) = 0$$

or $$\int_{-R}^{-r} f(x)dx + \int_{\gamma} f(z)dz + \int_{r}^{R} f(x)dx + \int_{\Gamma} f(z)dz = 0$$

By Gauss's Jordan's Lemma, we have

$$\lim_{R\to\infty} \int_{\Gamma} f(z)dz = 0.$$

Further, Since $\lim_{z\to 0} z\, f(z) = \lim_{z\to 0} z\cdot\frac{e^{iz}}{z} = 1$

We have

$$\lim_{r\to 0} \int_{\gamma} f(z)dz = i(0-\pi)\cdot 1 = -\pi i$$

$\Bigg[\because$ If AB is the arc $\theta_1 < \theta < \theta_2$ of the circle $|z - z_0| = r$ and if $\lim_{z\to z_0} (z - z_0)\, f(z) = K(\text{constant})$ then $\lim_{r\to\infty} \int_{AB} f(z)dz = i(\theta_2 - \theta_1)\cdot K\Bigg]$

Hence, as $r \to 0$ and $R \to \infty$, we get

$$\int_0^{\infty} f(x)dx + \int_{-\infty}^{0} f(x)dx - \pi\, i = 0$$

or $$\int_{-\infty}^{\infty} f(x)dx = \pi i$$

or $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = \pi i$ or $\int_{0}^{\infty} \frac{e^{ix}}{x}\, dx = \frac{\pi}{2} i$

Equating real and Imaginary parts, we get

$$\int_{0}^{\infty} \frac{\cos x}{x}\, dx = 0 \text{ and } \boxed{\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}}$$

EXERCISE 3.7

1. Show that $\int_{0}^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$.

2. Use calculus of residues to prove that
$$\int_{0}^{2\pi} \frac{\cos 2\theta\, d\theta}{5 + 4\cos\theta} = \frac{\pi}{6}.$$

3. Use calculus of residues to prove that
$$\int_{0}^{\pi} \frac{a\, d\theta}{a^2 + \cos^2\theta} = \frac{\pi}{\sqrt{(1 + a^2)}}.$$

4. Prove that $\int_{0}^{2\pi} e^{-\cos\theta} \cdot \cos(n\theta + \sin\theta)\, d\theta = \frac{2\pi(-1)^n}{n!}$ where n is a positive integer.

5. Show that $\int_{0}^{\pi} \tan(\theta + ia)\, d\theta = \pi i$, where $R(a) > 0$.

6. Use calculus of residues to prove that
$$\int_{0}^{\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{\pi a^2}{1 - a^2}, \quad \text{where } -1 < a < 1.$$

7. Prove that $\int_{0}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}$, where $a > 0$.

8. Use method of contour integration, to prove that
$$\int_{0}^{\infty} \frac{dx}{(1 + x^2)^2} = \frac{\pi}{4}.$$

9. Prove that $\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2 + 1)(x^2 + 4)} = \frac{\pi}{3}$.

10. Prove that $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right], a > b > 0.$

11. Prove that $\int_{0}^{\infty} \frac{1-\cos x}{x^2} dx = \frac{\pi}{2}.$

12. Use calculus of residues to prove that

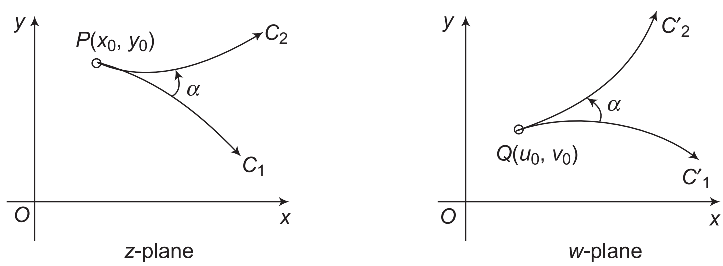
$$\int_{0}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}(b-a), a > b > 0.$$

3.47 CONFORMAL MAPPING

A function $w = u(x, y) + iv(x, y)$ defined in a domain D of the z-plane. A transformation or mapping which maps the domain D of the z-plane in to a domain D' of the w-plane.

Thus the point $P(x_0, y_0)$ of the z-plane is mapped in to the point $Q(u_0, v_0)$ of the w-plane.

Let the curves c_1 and c_2 intersecting at (x_0, y_0) be mapped respectively into the curves c_1' and c_2' intersection at (u_0, v_0). Then if the transformation is such that the angle between c_1 and c_2 at (x_0, y_0) is equal both in magnitude and sense to the angle between c_1' and c_2' at (u_0, v_0), it is said to be conformal at (x_0, y_0).



(a) **(b)**

Fig. 3.22

Before, we derive the conditions for testing whether a mapping $w = f(z)$ is conformal in a domain, we define a few important transformations.

3.47.1 Linear Transformation

A transformation or mapping of the form

$$w = az + b \tag{89}$$

where a, b are real or complex constants, is said a linear mapping or a linear function or a linear transformation.

3.47.2 Translation Mapping

The mapping $w = z + c$, where c is a complex constant.

Let $\quad z = x + iy, c = c_1 + i\, c_2$ and

$$w = (x + iy) + c_1 + i\,c_2 = (x + c_1) + i(y + c_2)$$
$$w = u + i\,v\text{: where } u = x + c_1,\ v = y + c_2.$$

Thus, the image in the w-plane of any region of the z-plane is the translation of that region in the direction of the $c_1 + i\,c_2$.

Hence, the image of w plane is the same as image in z-plane with different origin.

3.47.3 Rotation Mapping

The mapping $w = z\,e^{i\phi}$.

Let $\qquad z = r\,e^{i\theta}$, then.

$$w = r\,e^{i(\theta + \phi)},\ |w| = r \text{ and } \arg(w) = \theta + \phi.$$

Under this transformation, the point $z(r, \theta)$ in z-plane is mapped as a point $w(r, \theta + \phi)$ in w-plane.

$\therefore$ The mapping rotates a region in the z-plane through an angle ϕ in w-plane.

Hence, if $\phi > 0$, the rotation is anti-clockwise and if $\phi < 0$, the rotation is clock-wise.

3.47.4 Magnification

The mapping $w = cz$, where c is a real and $c > 0$.

Let $\qquad z = r\,e^{i\theta}$, then $w = cr\,e^{i\theta}$, $c > 0$.

$\therefore \qquad |w| = cr$ and $\arg(w) = \theta$.

A point in z-plane is moved radilly to a new position at a distance cr from the origin.

Hence, if $c > 1$, then vector in the z-plane is magnified, if $c < 1$, then vector is contracted and $w = z$ is an identity mapping, if $c > 1$.

3.47.5 Inverse Mapping

The mapping $w = \dfrac{1}{z}$, let $z = r\,e^{i\theta}$, then

$$w = \frac{1}{r}\,e^{-i\theta} = R\,e^{i\phi},\ \text{ where, } R = \frac{1}{r} \text{ and } \phi = -\theta$$

$\therefore$ Under this transformation $w = \dfrac{1}{z}$, a point $z(r, \theta)$ in z-plane is mapped in to the point $w(R, \phi)$ in w-plane.

Thus, the image in z-plane is mapped upon the reciprocal image in w-plane.

Example 82 Determine the region in the w-plane, corresponding to the triangular region bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$ in the z-plane under the transformation.

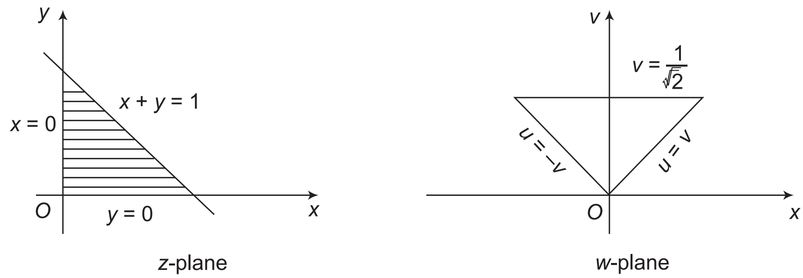
$$w = z\,e^{\frac{\pi i}{4}}$$

Solution Given $\qquad w = z\,e^{\frac{\pi i}{4}}$

$$u + i\,v = (x + i\,y)\,e^{\frac{\pi i}{4}} = (x + i\,y)\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$= \frac{1}{\sqrt{2}}(x + i\,y)\,(1 + i) \;=\; \frac{1}{\sqrt{2}}(x - y) + i\,(x + y)$$

$$\Rightarrow \qquad u = \frac{1}{\sqrt{2}}(x - y),\; \mathrm{v} = \frac{1}{\sqrt{2}}(x + y)$$



(a) **(b)**

Fig. 3.23

The line $x = 0$ maps into $u = -\frac{y}{\sqrt{2}},\, v = \frac{y}{\sqrt{2}}$ or into $u = -v$

The line $y = 0$ maps in to $u = \frac{x}{\sqrt{2}},\, v = \frac{x}{\sqrt{2}}$, i.e. into $u = v$

The lines $x + y = 1$ maps into $v = \frac{1}{\sqrt{2}}$

Thus, the given triangular region in the z-plane is transformed into the triangular region in the w-plane bounded by the lines $u = v$, $u = -v$ and $v = \frac{1}{\sqrt{2}}$. The regions are shown in the Fig. 3.23.

Example 83 Consider the rectangular region in the z-plane be bounded by $x = 0, y = 0, x = 2$ and $y = 3$. Find the region of the w-plane in to mapped in the z-plane under the transformation $w = \sqrt{2}e^{i\frac{\pi}{4}}.z$.

Solution Given $\quad w = \sqrt{2}e^{i\frac{\pi}{4}}.z$

$$u + iv = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right].(x + iy)$$

$$= \frac{\sqrt{2}}{\sqrt{2}}\left[(1 + i)(x + iy)\right] = (1 + i)\,(x + iy)$$

$$\Rightarrow \qquad u = x - y,\; v = x + y$$

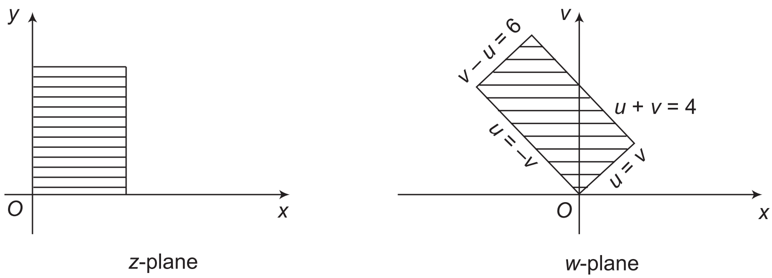
Now, the line $x = 0$ is maps into $u = -y,\, v = y$, i.e. $u = -v$

The line $y = 0$ is maps into $u = x,\, v = x$, i.e. $u = v$

The lines $x = 2$ is maps into $u = 2 - y,\, v = 2 + y$ or in to $u + v = 4$

and the line $y = 3$ is maps in to $u = x - 3$, $v = x + 3$

or in to $\quad v - u = 6$



(a) **(b)**

Fig. 3.24

Example 84 Find the image of the region bonded by the lines $x + y > 2$ and $x - y < 2$ under the mapping $w = \dfrac{1}{z}$.

Solution Putting $z = x + iy$ and $w = u + iv$ in $w = \dfrac{1}{z}$,

i.e. $$u + iv = \frac{1}{x + iy}$$

or $$x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

Compare both sides real and imaginary parts, we get $x = \dfrac{u}{u^2 + v^2}$ and $y = -\dfrac{v}{u^2 + v^2}$.

The region $x + y > 2$ is transformed as $\dfrac{u - v}{u^2 + v^2} > 2$

or $$u^2 + v^2 - \frac{1}{2}(u - v) < 0 \text{ or } \left(u - \frac{1}{4}\right)^2 + \left(v + \frac{1}{4}\right)^2 < \frac{1}{8}.$$

The boundary of this region is a circle

$\left|w - \dfrac{1}{4}(1 - i)\right| = \dfrac{1}{2\sqrt{2}}$ with center at $\left(\dfrac{1}{4}, -\dfrac{1}{4}\right)$ and radius $\left(\dfrac{1}{2\sqrt{2}}\right)$.

Similarly, the region $x - y < 2$ is transformed as $\dfrac{u + v}{u^2 + v^2} < 2$ or

$$u^2 + v^2 - \frac{1}{2}(u + v) > 0 \quad \text{or} \quad \left(u - \frac{1}{4}\right)^2 + \left(v - \frac{1}{4}\right)^2 > \frac{1}{8}$$

The boundary of this region is a circle

$$\left| w - \frac{1}{4}(1+i) \right| = \frac{1}{2\sqrt{2}}$$

with center at $\left(\frac{1}{4}, \frac{1}{4}\right)$ and radius $\frac{1}{2\sqrt{2}}$.

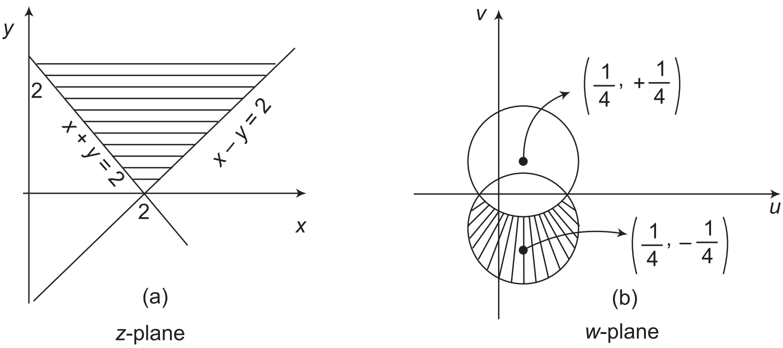


Fig. 3.25

Example 85 Determine the image in to w-plane of the circle $|z - 3| = 2$ in the z-plane under the inverse transformation $w = \frac{1}{z}$.

Solution Substitute $z = x + iy$ and $w = u + iv$ in $w = \frac{1}{z}$, i.e.,

$$u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \quad \text{or,} \quad z = x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow \qquad u = \frac{x}{x^2+y^2} \text{ and } v = -\frac{y}{x^2+y^2}$$

Now, the circle $|z - 3| = 2$ is transformed as $|x + iy - 3| = 2$ or $\left|\frac{u-iv}{u^2+v^2} - 3\right| = 2$

$$\text{or} \qquad \left|\frac{u-iv}{u^2+v^2} - 3\right|^2 = 4$$

$$\text{or} \qquad \left[\left(\frac{u}{u^2+v^2} - 3\right) - i\frac{v}{u^2+v^2}\right] \cdot \left[\left(\frac{u}{u^2+v^2} - 3\right) + i\frac{v}{u^2+v^2}\right] = 4$$

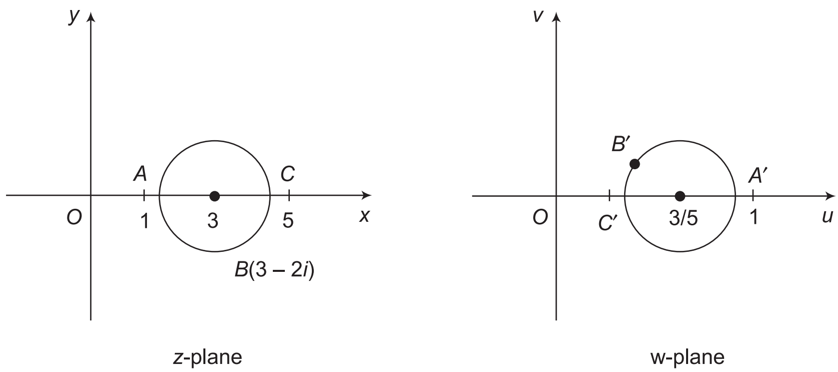
$$\text{or} \qquad \left(\frac{u}{u^2+v^2} - 3\right)^2 + \frac{v^2}{(u^2+v^2)^2} = 4 \quad \text{or} \quad \frac{u^2+v^2}{(u^2+v^2)^2} - \frac{6u}{u^2+v^2} + 5 = 0$$

$$\text{or} \qquad \frac{1-6u}{u^2+v^2} + 5 = 0$$

or $$1 - 6u + 5(u^2 + v^2) = 0$$

or $$\left(u - \frac{3}{5}\right)^2 + v^2 = \left(\frac{2}{5}\right)^2.$$

The image of $|z - 3| = 2$ is a circle with center $\left(\frac{3}{5}, 0\right)$ and radius $\frac{2}{5}$.



(a) **(b)**

Fig. 3.26

The center (3, 0) of the circle in z-plane is mapped in to $(u, v) = \left(\frac{1}{3}, 0\right)$ in the w-plane, which is inside the mapped circle.

$\therefore$ Under the transformation $w = \frac{1}{z}$, the region under the circle $|z - 3| = 2$ is mapped onto the region inside the circle in the w-plane.

3.48 BILINEAR (OR MOBIUS OR FRACTIONAL) TRANSFORMATION

The mapping as transformation

$$w = \frac{az + b}{cz + d}, \tag{90}$$

where a, b, c, d, are complex constant and $ad - bc \neq 0$ is called bilinear or mobius transformation.

The transformation Eq. (90) is said to be normalized if $ad - bc = 1$

The transformation Eq. (90) is expressed as

$$cwz + dw - az - b = 0 \tag{91}$$

Evidently it is linear both in w and z, it is called a bilinear transformation.

A transformation $w = f(z)$ is said to be univalent if $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$

Some Important Points

(i) If $c = 0$, $d = 1$, then the transformation Eq. (90) becomes

$$w = az + b \tag{92}$$

The transformation Eq. (92) will always transform a circle in z-plane into a circle in w-plane.

(ii) If $a = d = 0$, $b = c$, than Eq. (90) becomes

$$w = \frac{1}{z} \tag{93}$$

The transformation Eq. (93) is called inversion, will always transforms a circle into a circle.

(iii) Every bilinear transformation of the form given in Eq. (90) maps the circle and straight lines in the z-plane on to circles and straight lines in the w-plane.

(iv) The product of two bilinear transformations is a bilinear transformation.

(v) A linear fractional transformation with one fixed point z_0 is called parabolic and is expressible as

$$\frac{1}{w - z_0} = \frac{1}{z - z_0} + h \text{ if } z_0 \neq \infty$$

or $$w = z + h \text{ if } z_0 = \infty.$$

(vi) A linear fractional transformation with two different fixed points z_1 and z_2 is expressible as

$$\frac{w - z_1}{w - z_2} = k\left(\frac{z - z_1}{z - z_2}\right) \quad \text{if} \quad z_1, z_2 \neq \infty$$

If $z_2 = \infty$, then it becomes $w - z = k(z - z_1)$.

A transformation with two different fixed points is called hyperbolic if $k > 0$, and elliptic if $k = e^{i\alpha}$ and $\alpha \neq 0$ and loxodromic if $k = a\,e^{i\alpha}$, where $a \neq 1$, $\alpha \neq 0$, α and a both are real numbers and $a > 0$.

3.49 CROSS RATIO

If z_1, z_2, z_3 and z_4 are distinct points, then the ratio $\dfrac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$ is called the cross ratio of z_1, z_2, z_3, z_4 and is denoted as (z_1, z_2, z_3, z_4).

Example 86 Determine the bilinear transformation which maps the points $z_1 = 2, z_2 = i, z_3 = -2$ into the points $w_1 = 1, w_2 = i, w_3 = -1$.

Solution Let the required bilinear transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

Putting the values, we get

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

or $$-\frac{(w-1)(i+1)^2}{(w+1)(1-i)(1+i)} = -\frac{(z-2)(2+i)^2}{(z+2)(2-i)(2+i)}$$

or $$-\frac{(w-1)(1+i)^2}{(w+1).2} = -\frac{(z-2)(2+i)^2}{(z+2)(4+1)}$$

or $$-\frac{(w-1).2i}{(w+1).2} = -\frac{(z-2).(3+4i)}{(z+2).5}$$

or $$\frac{(w-1)}{(w+1)} = \frac{(z-2)(3+4i)}{5i(z+2)} = \frac{(z-2)(4-3i)}{5(z+2)}$$

or $$\frac{(w-1)+(w+1)}{(w+1)-(w-1)} = \frac{(z+2)(4-3i)+5(z+2)}{5(z+2)-(z+2)(4-3i)}$$

or $$\frac{2w}{2} = \frac{(z-2)(4-3i)+(5z+10)}{5z+10-(z+2)(4-3i)}$$

$$w = \frac{4z-8-3iz+6i+5z+10}{5z+10-4z+3iz-8+6i}$$

$$= \frac{3(3-i)z+2(1+3i)}{(1+3i)z+6(3-i)} = \frac{3z+2(1+3i)/(3-i)}{6+(1+3i)z/(3-i)}$$

or $$w = \frac{3z+2i}{6+zi} \quad \left[\because \frac{1+3i}{3-i} = i\right] \quad \text{By simplification}$$

which is the required bilinear transformation.

Example 87 Find the bilinear transformation which maps outside. $|z| = 1$, on the half plane $R(w) \geq 0$ so that the points $z = 1, -i, -1$ correspond to $w = i, 0, -i$ respectively.

Solution Let $z_1 = 1$, $z_2 = -i$, $z_3 = -1$, $w_1 = i$, $w_2 = 0$, $w_3 = -i$

Suppose the required bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \tag{94}$$

Putting the values in (94), we get

$$\frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(-i+1)}{(1+i)(-1-z)}$$

or $$\frac{w-i}{w+i} = -i\left(\frac{z-1}{z+1}\right) = \frac{-iz+i}{z+1}$$

or $$\frac{(w-i)+(w+i)}{(w-i)-(w+i)} = \frac{(-iz+i)+(z+1)}{(-iz+i)-(z+1)}$$

$$\frac{2w}{-2i} = \frac{z(1-i)+(1+i)}{-z(1+i)-(1-i)}$$

or $$w = i\left(\frac{1-i}{1+i}\right)\left[\frac{z+\left(\frac{1+i}{1-i}\right)}{z+\left(\frac{1-i}{1+i}\right)}\right]$$

$$= i(-i)\left[\frac{z+i}{z-i}\right] \qquad \left[\because \frac{1-i}{1+i} = i\right]$$

$$w = \frac{z+i}{z-i} \tag{95}$$

which is the required transformation.

From Eq. (95), $w(z-i) = z + i \Rightarrow z = i.\left(\frac{w+1}{w-1}\right)$ (96)

$|z| \geq 1$ is transformed in to $\left|\frac{w+1}{w-1}\right| \cdot |i| \geq 1$

or $(w+1)^2 \geq (w-1)^2$

or $(u+1)^2 + v^2 \geq (u-1)^2 + v^2$

or $R(w) = 4 \geq 0$

Hence, the exterior of the circle $|z| = 1$ is transformed in to half plane $R(w) \geq 0$.

Example 88 Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ on to the straight line $4u + 3 = 0$ and explain why the curve obtained is not a circle.

Solution The inverse transformation is $z = \frac{4w+3}{w-2}$ (97)

Now, the equation $x^2 + y^2 - 4x = 0$ can be written as $z\bar{z} + 2(z + \bar{z}) = 0$

Putting for z and $\bar{z}$ from Eq. (97), we get

$$\frac{4w+3}{w-2} \cdot \frac{4\bar{w}+3}{\bar{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

or $16w\bar{w} + 12w + 12\bar{w} + 9 - 2(4w\bar{w} + 3\bar{w} - 8w - 6 + 4w\bar{w} + 3w - 8\bar{w} - 6) = 0$

or $22(w + \bar{w}) + 33 = 0$ $[\because w = u + iv]$

or $44u + 33 = 0$

or $4u + 3 = 0$ as required.

Thus, the circle is transformed in to a straight line which is possible under a bilinear transformation. Since we regard a straight line as a particular case of a circle .

Example 89 Find a mobius transformation of the upper half plane $I(z) > 0$ on to the interior $|w| < 1$ of the unit circle $|w| = 1$.

Solution Choose $z_1 = -1, z_2 = 0, z_3 = 1$ on the real axis so that the upper half-plane is on the left of an observer moving along the real axis in the direction from z_1 to z_3 through z_2.

Now, we choose three points w_1, w_2, w_3 on the circle.

$|w| = 1$ such that its interior is on the left of an observer moving along the circle in the direction from w_1 to w_3 through w_2.

Suppose $w_1 = 1$, $w_2 = i$, and $w_3 = -1$, then the required mobius Transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

or $$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}.$$

or $$\frac{w-1}{w+1} = \frac{(z+1)(1-i)}{(z-1)(1-i)} = \frac{(z+1)(1-i)^2}{(z-1)(1-i^2)} = \frac{-i(z+1)}{(z-1)}$$

or $$w = \frac{(i-1)z+(i+1)}{-(i+1)z+1-i} = -\frac{(i-1)}{(i+1)}\cdot\frac{z+\left(\frac{i+1}{i-1}\right)}{z+\left(\frac{i-1}{i+1}\right)}$$

$$w = -i\,\frac{z-i}{z+i}$$

which is the required mobius transformation.

Example 90 Find the bilinear transformation that maps the points $z_1 = \infty$, $z_2 = i$ and $z_3 = 0$ into the points $w_1 = 0$, $w_2 = i$ and $w_3 = \infty$. **[Meerut 1994, 96, 97, 98, 99, GEU 2010, 13]**

Solution The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

or $$\frac{(w-0)(i-\infty)}{(0-i)(\infty-w)} = \frac{(z-\infty)(i-0)}{(\infty-i)(0-z)}$$

or $$\frac{w}{i} = \frac{i}{z}$$

or $$w = -\frac{1}{z}$$

which is the required transformation.

Example 91 Find a bilinear transformation which maps the upper half of the z-plane into the unit circle in the w-plane in such a way that $z = i$ is mapped into $w = 0$ while the point at infinity is mapped into $w = -1$.

Solution We have

$w = 0$ corresponding to $z = i$

and $w = -1$ corresponding to $z = \infty$.

Then, the bilinear transformation is given by

$$w = \left(\frac{z-z_0}{z-\overline{z}_0}\right)e^{i\alpha}$$

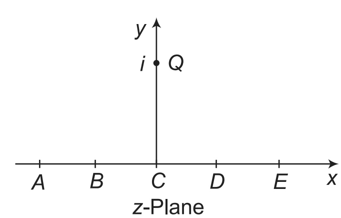


Fig. 3.27(a)

Corresponding to $z = i$

$$\therefore \qquad 0 = \frac{i - z_0}{i - \bar{z}_0} \text{ so that } z_0 = i.$$

Now, corresponding to $z = \infty$, we have

$$-1 = \left(\frac{\infty - z_0}{\infty - \bar{z}_0}\right) e^{i\alpha} \Rightarrow e^{i\alpha} = -1$$

Hence, the required bilinear transformation is

$$w = \left(\frac{z-i}{z+1}\right)(-1) \quad \left[\because \quad z_0 = i \text{ so that } \bar{z}_0 = -i\right]$$

$$w = \frac{i-z}{i+z}$$

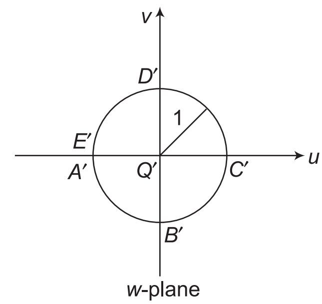


Fig. 3.27(b)

Example 92 Prove that the bilinear transformation can be considered as a combination of the transformations of translation, rotation, stretching and inversion.

Solution We know that

$$w = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{98}$$

we can write Eq. (98) as

$$w = \frac{\alpha}{\gamma} + \frac{\beta\gamma - \alpha\delta}{\gamma(\gamma z + \delta)} = \frac{\alpha}{\gamma} + \frac{(\beta\gamma - \alpha\delta)}{\gamma^2\left(z + \delta/\gamma\right)}$$

$$w = r + \frac{s}{z+t} \tag{99}$$

where $r = \dfrac{\alpha}{\gamma}$, $s = \dfrac{(\beta\gamma - \alpha\delta)}{\gamma^2}$ and $t = \dfrac{\delta}{\gamma}$ are constants.

Then the transformation is equivalent to

$$\mu = z + t, \;\; \tau = \frac{1}{\mu} \text{ and } w = r + \tau s$$

which are the combinations of the transformations of translation, rotation, stretching and inversion. **Hence, proved.**

3.50 APPLICATIONS OF COMPLEX VARIABLES

Complex variables play a very important role in the field of engineering and science. In this section we shall discuss the applications to heat flow, electrostatics and fluid flow problems.

3.50.1 Applications to Heat Flow

Let a solid having a temperature distribution and the quantity of heat conducted per unit area per unit time across a surface of the solid is called the heat flux across the surface, which is given by

$$q = -K \text{ grad } \phi \tag{100}$$

where ϕ is the temperature and K is the thermal conductivity which depends on the material of solid is made.

Now Eq. (100) can be written as

$$q = -K\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right)$$

$$= Q_x + iQ_y, \text{ where } Q_x = -K\frac{\partial\phi}{\partial x}, Q_y = -K\frac{\partial\phi}{\partial y}$$

Let C be any simple closed curve in z-plane, which represents the cross-section of a cylinder. If Q_n and Q_t denotes the normal and tangential components of the heat flux.

If it is in steady state condition, then the total heat flow is zero inside the curve C.

$$\therefore \quad \oint_C Q_n ds = 0 \Rightarrow \oint_C Q_x dy - Q_y dx = 0 \tag{101}$$

and

$$\oint_C Q_t ds = 0 \Rightarrow \oint_C Q_x dx + Q_y dy = 0 \tag{102}$$

We assume, no sources or sinks inside the curve C. Then the Eq. (101) yields.

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \tag{103}$$

or

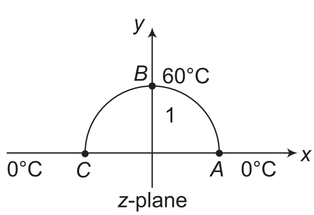
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \left[\because Q_x = -K\frac{\partial\phi}{\partial x} \text{ and } Q_y = -K\frac{\partial\phi}{\partial y}\right] \tag{106}$$

Equation (104) represents the condition of harmonic function so that the function ϕ is harmonic.

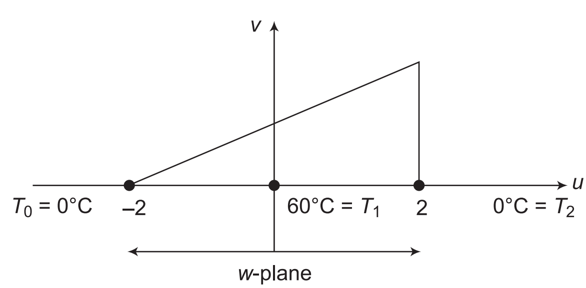
If ψ is the harmonic conjugate function, then

$\Omega(z) = \phi(x, y) + i\psi(x, y)$ is analytic function and it is called the complex temperature. The families of curves $\phi(x, y) = c_1$ and $\psi(x, y) = c_2$ are called isothermal and flux lines respectively.

Example 93 Determine the steady-state temperature at any point of the region shown in the following figure if the temperature are maintained as indicated.



(a)



(b)

Fig. 3.28

Solution The region of z-plane is mapped to the upper half of the w-plane by the mapping $w = z + \dfrac{1}{z}$.

$$\therefore \qquad u + iv = (x+iy) + \frac{1}{(x+iy)} = (x+iy) + \frac{(x-iy)}{(x+iy)(x-iy)}$$

$$= (x+iy) + \frac{(x-iy)}{x^2+y^2} = x+iy+\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

or $$u + iv = \left(x+\frac{x}{x^2+y^2}\right) + i\left(y-\frac{y}{x^2+y^2}\right)$$

Now, the solution to the given problem in w-plane is given by

$$\phi = \left(\frac{T_0-T_1}{\pi}\right)\tan^{-1}\left(\frac{v}{u-(-2)}\right) + \left(\frac{T_1-T_2}{\pi}\right)\tan^{-1}\left(\frac{v}{u-2}\right) + T_2 \tag{105}$$

or $$= \left(\frac{0-60}{\pi}\right)\tan^{-1}\left(\frac{v}{u+2}\right) + \left(\frac{60-0}{\pi}\right)\tan^{-1}\left(\frac{v}{u-2}\right) + 0$$

or $$= -\frac{60}{\pi}\tan^{-1}\left(\frac{v}{u+2}\right) + \frac{60}{\pi}\tan^{-1}\left(\frac{v}{u-2}\right)$$

or $$\phi = \frac{60}{\pi}\left[\tan^{-1}\left(\frac{v}{u-2}\right) - \tan^{-1}\left(\frac{v}{u=2}\right)\right]$$

where $$u = x+\frac{x}{x^2+y^2} \quad \text{and} \quad v = y-\frac{y}{x^2+y^2}.$$

***Note*:** Equation (106) is the solution of the equation

$$\phi_{xx} + \phi_{yy} = 0,\ y > 0 \ \text{ and } \ \phi(x, y) = \begin{cases} T_0 & \text{if } x < -a \\ T_1 & \text{if } -a < x < a \\ T_2 & \text{if } x > a \end{cases}$$

where T_0, T_1 and T_2 are constants.

Example 94 Find the steady-state temperature at any point of the region shown in the Figures 3.29 and 3.30. Also find the isothermal and flux lines.

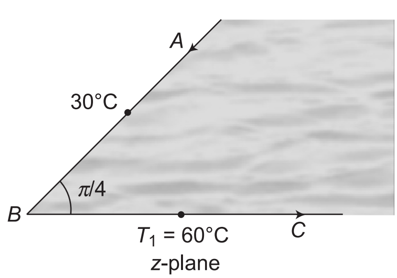


Fig. 3.29

Solution The shaded region of the z-plane is mapped on to the upper half of the w-plane by the mapping function $w = z$; $\left(w = z^m; m \geq \frac{1}{2}\right)$.

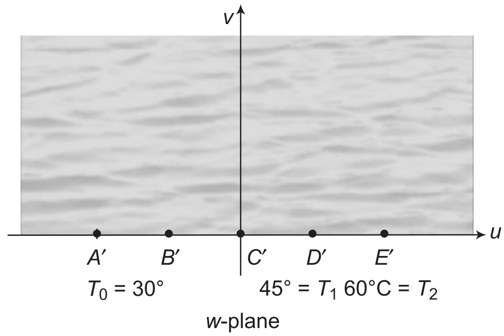


Fig. 3.30

$\therefore \quad u + iv = x + iy$

$\Rightarrow \quad u = x$ and $v = y$.

Now, the solution of the given problem in w-plane is given by

$$\phi = \left(\frac{T_0 - T_1}{\pi}\right)\tan^{-1}\left(\frac{v}{u}\right) + \left(\frac{T_1 - T_2}{\pi}\right)\tan^{-1}\left(\frac{v}{u}\right) + T_2$$

or
$$= \left(\frac{30 - 45}{\pi}\right)\tan^{-1}\left(\frac{v}{u}\right) + \left(\frac{45 - 60}{\pi}\right)\tan^{-1}\left(\frac{v}{u}\right) + 60$$

or
$$= 60 - \frac{15}{\pi}\tan^{-1}\left(\frac{v}{u}\right) - \frac{15}{\pi}\tan^{-1}\left(\frac{v}{u}\right)$$

or
$$\phi = 60 - \frac{30}{\pi}\tan^{-1}\left(\frac{v}{u}\right), \tag{106}$$

where $u = x$ and $v = y$

Equation (106) represents the steady-state temperature and if $\phi(x, y)$ = constant, then it is known as isothermal lines and $\psi(x, y)$ = constant is the flux lines determine, using analytic function.

3.50.2 Applications to Fluid Flow

The solution of the various fluid flow problems obtained by the method of complex variable techniques under the following:

(a) *The fluid flow is stationary*: When the velocity of fluid at any point depends only on the position (x, y) and on the time.

(b) *Velocity potential*: Let V_x and V_y be the velocity components of the fluid at (x, y) in the positive directions of x and y, then there exists a velocity potential ϕ, such that

$$V_x = \frac{\partial \phi}{\partial x} \text{ and } V_y = \frac{\partial \phi}{\partial y} \tag{107}$$

Let c be any simple closed curve in the z-plane, and velocity tangential component on c is V_t, then

$$\oint_c V_t ds = \oint_c V_x dx + V_y dy = 0 \tag{108}$$

Equation (108) represents the circulation of the fluid along the curve c. If the circulation is zero, then the motion of flow is irrotational.

(c) *Incompressible fluid*: If the mass or density per unit volume of the fluid is constant, then the fluid is incompressible. Let the normal component of velocity is V_n on the curve c, then

$$\oint_c V_n dS = \oint_c V_x dy - V_y dx = 0$$

or
$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \tag{109}$$

Equation (109) represents the equation of continuity.

(d) *Complex potential*: Using Eqs (107) and (109) become

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

which is represents the velocity potential ϕ is harmonic.

If $\psi(x, y)$ is conjugate harmonic function, such that $\Omega(z) = \phi(x, y) + i\,\psi(x, y)$ is analytic and $\psi(x, y)$ is called stream function.

Thus, the function $\Omega(z)$ is called complex potential.

Also, the families of curves $\phi(x, y) = c_1$ and $\psi(x, y) = c_2$, where c_1 and c_2 are constants are orthogonal called the equipotential lines and streamlines of the flow.

Example 95 Determine the complex potential for a fluid moving with constant speed V_0 in the direction making an angle θ with the positive x-axis. Also find the velocity potential and stream function.

Solution

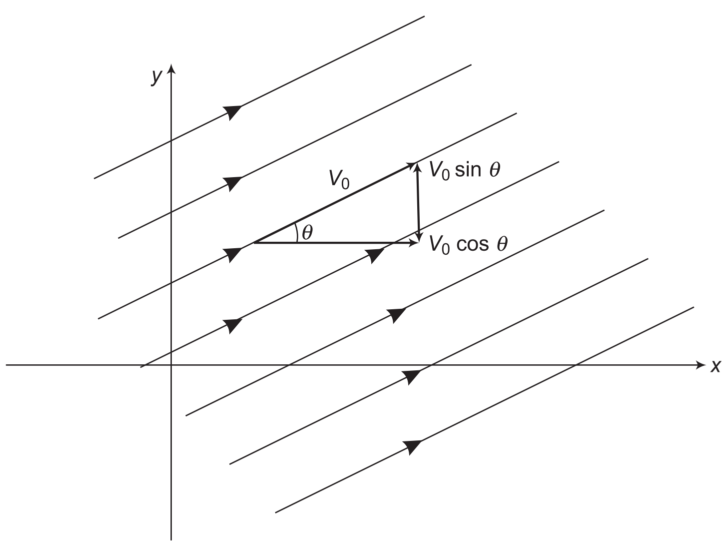


Fig. 3.31

The velocity components along x and y direction are

$$V_x = V_0 \cos\theta \text{ and}$$
$$V_y = V_0 \sin\theta$$

The complex velocity $(V) = V_x + iVy$

$$= V_0 \cos\theta + i\, V_0 \sin\theta$$
$$V = V_0\, e^{i\theta}$$

The complex potential $\Omega(z) = \dfrac{d\Omega}{dz} = \overline{V} = V_0\, e^{-i\theta}$.

on integrating, we get

$$\Omega(z) = V_0\, e^{-i\theta} z.$$

Now $\quad \Omega(z) = \phi(x, y) + i\psi(x, y) = V_0 e^{-i\theta} \cdot z$

$$= V_0(\cos\theta - i\sin\theta)\,(x+iy) = V_0(x\cos\theta + y\sin\theta) + iV_0(y\cos\theta - x\sin\theta)$$

$\therefore \quad \phi(x, y) = V_0(x\cos\theta + y\sin\theta)$

and $\quad \psi(x, y) = V_0(y\cos\theta - x\sin\theta)$ are the velocity potential and stream function.

3.50.3 Applications to Electrostatics

We know that the function

$\Omega(z) = \phi(x, y) + i\ \psi(x, y)$ is analytic in the region which is not a occupied charge and the function $\Omega(z)$ is called the complex electrostatic potential or complex potential.

Then the electrostatic potential

$$\in\ = -\text{grad}\ \phi = -\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}$$

$$= -\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial y} \quad [\text{Using C–R equations}]$$

$$= -\overline{\frac{d\Omega}{dz}} = -\overline{\Omega'(z)}.$$

Its magnitude is $E = |\in| = \left|-\overline{\Omega'(z)}\right| = |\Omega'(z)|$.

The families of curves $\phi(x, y) = c_1$ and $\psi(x, y) = c_2$ are known as equipotential lines and flux lines respectively.

Thus, the electrostatic potential due to a line charge (q) per unit length at z_0 is given by

$$\Omega(z) = -2q\log(z - z_0)$$

and represents a source or sink according as $q < 0$ or $q > 0$.

***Note*:** When the medium is not a vacuum then the charge $q = \dfrac{q}{K}$, where K is any constant.

Example 96 Determine the complex electrostatics potential due to a line of charge q per unit length perpendicular to the z-plane at $z = 0$. Also discuss the similarity with the complex potential for a line sink or source in fluid flow.

Solution The normal component of electric vector is V_n (which is a constant) while the tangential component of electric V_t is zero.

Let c be any cylinder of radius r with axis at $z = 0$, then by Gauss's divergence theorem;

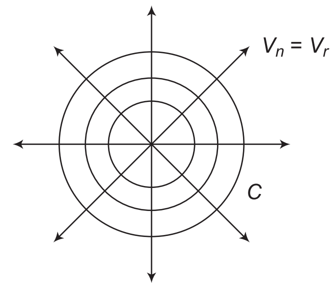


Fig. 3.32

$$\oint_c V_n ds = V_r \oint_c ds, \quad \text{(where } V_r \text{ is the radial velocity)}$$

$$= V_r \cdot 2\pi r$$

$$= 4\pi q, \qquad \text{where } q = \frac{r \cdot V_r}{2}.$$

Since $\quad V_r = -\dfrac{\partial \phi}{\partial r} \Rightarrow \phi = -2q \log r$ (omitting the constant of integration)

$\therefore$ The real part of the required complex electrostatics potential is

$$\Omega(z) = -2q \log z$$

Now, the complex potential has the same form of a line sink of fluid if $K = +2q$ because $V_r = \dfrac{K}{r}$. If q is a negative charge then it is a line source.

EXERCISE 3.8

1. Find the mobius transformation which maps the circle $|w| = 1$ in to the circle $|z - 1| = 1$ and maps $w = 0$, $w = 1$ in to $z = \dfrac{1}{2}, z = 0$ respectively.
2. Find the bilinear transformation which maps the points $z = -2, 0, 2$ in to the points $w = 0, i, -i$ respectively.
3. Find the mobius transformation which maps $1, -1, \infty$ on to $1 + i, 1 - i, 1$ respectively.
4. Find the image of the region $x \geq 2$ in the z-plane under the mapping $w = \dfrac{(4z+1)}{(z-2-i)}$.
5. Find the image of the annulus region $1 < |z| < 2$ under the mapping $w = \dfrac{z}{z-1}$.
6. Find the condition that the transformation $w = \dfrac{az+b}{cz+d}$ transforms a straight line of z-plane in to the unit circle in w-plane.
7. Show that the transforms $w = i\left(\dfrac{1-z}{1+z}\right)$ transforms the circle $|z| = 1$ on to the real axis of w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane.
8. Show that the relation $w = \dfrac{i}{4} \cdot \dfrac{z+2}{z+1}$ transforms the real axis in z-plane to a circle in w-plane. Find the centre and radius of circle and the point in z-plane which is mapped onto the centre of the circle.
9. Show that the transformation $w = 2z - 3i\bar{z} + 5 - 4i$ is equivalent to $u = 2x + 3y + 5$ and $v = 2y - 3z - 4$.

10. The straight lines $y = 2x$, $x + y = 6$ in the xy-plane are mapped on to the w-plane by the mapping $w = z^2$.
11. If $a \neq b$ are two fixed points of the bilinear transformation, show that it can be written in the form

$$\lambda\left(\frac{z-a}{z-b}\right) = \frac{w-a}{w-b}, \text{ where } \lambda \text{, is a constant.}$$

12. Fluid emanates at a constant rate from an infinite line source perpendicular to the z-plane at $z = 0$, determine the speed of the fluid at a distance r from the source. Also find the complex potential.
13. Determine the potential at any point of the region shown in the figure given below, if the potential on the x-axis is given as

$$\phi(x, y) = \begin{cases} V_0 & \text{if } x > 0 \\ -V_0 & \text{if } x < 0 \end{cases}$$

Hence, find the equipotential and flux lines.

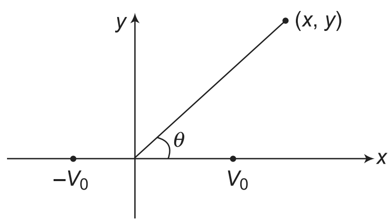


Fig. 3.33

14. Show that the complex potential due to a source of strength $K > 0$ in a fluid moving with speed V_0 is $\Omega(z) = V_0(z) + K \log z$.

Answer

1. $z = \dfrac{1-w}{2+w}$ 2. $w = i\left(\dfrac{2+z}{2-3z}\right)$ 3. $w = \dfrac{z+i}{z}$

4. $9u + 4v - 36 \geq 0$ 5. $|w| = 1$ if $|a| = |c|$.

12. Speed $(V) = \dfrac{K}{r}$, K is a constant

$\Omega(z) = K \log z$.

13. The required potential is $V_0\left(1 - \dfrac{2\theta}{\pi}\right)$, where $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.

The equipotential and fluxlines are

$$V_0\left(1 - \frac{2}{\pi}\tan^{-1}\frac{y}{x}\right) = c_1 \text{ and } x^2 + y^2 = c_2.$$

SUMMARY

Following topics have been discussed in this chapter:

1. Complex Number

An ordered pair of real numbers x and y to be written as $z = (x, y)$ is called a complex number. Also, we may write $z = (x, y) = x + iy$, where imaginary unit i(iota) is defined as $i = (0, 1)$. Here x is called real part of z and y is called imaginary part of z. The real part of z is denoted by Re(z) and the imaginary part of z by Im(z).

(i) Set of Complex Numbers

The set of all complex numbers is denoted by C, i.e.

$$C = \{x + iy \mid x, y \in R\}.$$

Since a real number 'x' can be written as $x + oi$

$\therefore$ Every real number is a complex number. Hence, $R \subset C$, where R is the set of all real numbers.

2. Conjugate Complex Number

If $z = x + iy$ is any complex number, then its conjugate denoted by $\overline{z} = x - iy$ or $\overline{z} = (x, -y)$.

Thus, $\overline{z}$ is the mirror image of the complex point z in to real axis. This shows that $\overline{z} = z \Leftrightarrow z$ is purely a real number.

3. De-Moivers's Theorem

Statement:

(i) If $n \in z$ (the set of integers), then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(ii) If $n \in \theta$ (the set of rational numbers), then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

4. Exponential (or Eulerian) Form of a Complex Number

We know that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Let z be any complex number, then in polar form z can be written as

$$z = r(\cos\theta + i\sin\theta)$$

or $$z = r\,e^{i\theta} \qquad \text{(Using Euler's notation)}$$

This form of z is known as exponential or Eulerian form.

5. Hyperbolic Functions

For any real or complex 'x', the hyperbolic sine and cosine of x is defined as:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Other hyperbolic functions are defined as

$$\tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \cot hx = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \operatorname{cosec} hx = \frac{2}{e^x - e^{-x}}, \quad \sec hx = \frac{2}{e^x + e^{-x}}$$

6. Logarithm of a Complex Number

$$\log (a + ib) = \log \left(\sqrt{a^2 + b^2}\right) + i \tan^{-1} \left(\frac{b}{a}\right)$$

is called the principal value of the logarithm of $(a + ib)$.

7. Cauchy–Reimann Equations

Cauchy–Reimann (C.R.) equations are used to determine whether a given function $f(z)$ is analytic or not.

Necessary Conditions for a Function to be Analytic

Theorem: Suppose that the function $f(z) = u(x, y) + iv(x, y)$ is continuous in some neighborhood of the point $z = x + iy$ and is differentiable at z. Then, the first order partial derivatives of $u(x, y)$ and $v(x, y)$ exist and satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at the point } z.$$

Sufficient Conditions for a Function $f(z)$ to be Analytic

Theorem: Let $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of the function $f(z) = u(x, y) + iv(x, y)$ and have the continuous first order partial derivatives in a domain D. If $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Reimann equations at all points in D, then the function $f(z)$ is analytic in D and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

8. Polar form of the Cauchy–Riemann Equations

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ and } \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

9. Method of Constructing an Analytic Function or a Regular Function

Milne–Thomson's Method Using Milne–Thomson's method, the analytic function $f(z) = (u(x, y) + iv(x, y))$ is directly constructed without finding $v(x, y)$, if $u(x, y)$ is given and vice-versa.

$$f(z) = \int \phi_1 (z, 0)\, dz - i \int \phi_2 (z, 0)\, dz + C, \text{ where } C \text{ is an arbitrary constant.}$$

Similarly, if $v(x, y)$ is given, we have

$$f(z) = \int \psi_1 (z, 0)\, dz - i \int \psi_2 (z, 0)\, dz + D$$

where $\frac{\partial v}{\partial y} = \psi_1 (x, y), \frac{\partial v}{\partial x} = \psi_2 (x, y)$ and D is an arbitrary constant.

10. Cauchy's Theorem

Let D be a simply connected region and let $f(z)$ be single valued continuously differentiable function of D, i.e. $f'(z)$ exists and is continuous at each point of D.

Then $\int_C f(z)\,dz = 0$; where C is any closed contour in D.

11. Cauchy's Integral Formula

If $f(z)$ is an analytic function within and on a closed contour C, and if z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)}$$

12. Cauchy Integral Formula for the Derivative of an Analytic Function

Let $f(z)$ be an analytic function within and on a closed contour C and z_0 is any point lying in it, then

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)^2}$$

13. Cauchy Integral Formula for Higher Order Derivatives

Let $f(z)$ be an analytic function in a simply connected region D, and C be a closed contour in D and z_0 is any point in C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

14. Poisson's Integral Formula

If $f(z)$ is an analytic function within and on a circle C defined by $|z| = R$ and if z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{(R^2 - z_0\,\overline{z}_0)\,f(z)}{(z - z_0)(R^2 - z\,\overline{z}_0)}\,dz$$

15. Liouville's Theorem

Let $f(z)$ be an integral function satisfying the inequality $|f(z)| \le M$ for all values of z, where M is a positive constant, then $f(z)$ is constant.

16. Expansion of Analytic Functions as Power Series

(i) Taylor's Theorem If a function $f(z)$ is analytic within a circular C_1 with its centre z_0 and radius R, then at every point z in side C_2,

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)\frac{(z - z_0)}{n!} = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!}f''(z_0) + \cdots$$

(ii) Laurent's Theorem Let $f(z)$ be an analytic function in the ring shaped region D bounded by two concentric circles C_1 and C_2 with centre z_0 and radii ρ_1 and ρ_2 $(\rho_1 > \rho_2)$ and let z be any point of D.

Then $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

where $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t - z_0)^{n+1}} dt, b_n = \frac{1}{2\pi i} \oint_{C_2} (t - z_0)^{n-1} f(t)\, dt$$

17. Singularities

Consider a function $f(z)$ which is analytic at all points of a circular bounded region D except at a finite number of points, these exceptional points are known as singular point or singularities.

There are two types of the singularities, which are as follows:

(i) Isolated Singularities A point $z = z_0$ is said to be an isolated singularity of $f(z)$, if $f(z)$ is analytic at each point in the neighbourhood of z_0.

(ii) Non-isolated Singularities A point $z = z_0$ is said to be a non-isolated singularity of $f(z)$, if $f(z)$ is not analytic at $z = z_0$ and is the neighbourhood of z_0.

18. Cauchy's Residue Theorem

If $f(z)$ is analytic within and on a closed contour C, except at a finite number of poles $z_1, z_2, z_3, \ldots z_n$ within C, then

$$\int_C f(z)\, dz = 2\pi i \sum_{r=1}^{n} \text{Res}\,(z = z_r)$$

where RHS denotes sum of residues of $f(z)$ at its poles lying within 'C'

19. Conformal Mapping

A function $w = u(x, y) + i\, v(x, y)$ defined in a domain D of the z-plane defines a transformation or mapping which maps the domain D of the z-plane in to a domain D of the w-plane. We have define a few important transforms.

(i) Linear Transformation A transformation or mapping of the form

$$w = az + b$$

where a, b are real or complex constants, is said a linear mapping or a linear function or a linear transformation.

(ii) Translation Mapping The mapping $w = z + c$, where c is a complex constant.

Let $z = x + iy$, $c = c_1 + i\, c_2$ and

$$w = (x + iy) + c_1 + i\, c_2 = (x + c_1) + i(y + c_2)$$

$$w = u + i\, v: \text{ where } u = x + c_1,\ v = y + c_2.$$

Thus, the image in the w-plane of any region of the z-plane is the translation of that region in the direction of the $c_1 + i\, c_2$.

Hence, the image of w plane is the same as image in z-plane, with different origin.

(iii) **Rotation mapping** The mapping $w = z\,e^{i\theta}$.

Let $z = r\,e^{i\theta}$, then.

$$w = re^{i(\phi+\theta)},\ |w| = r \text{ and } \arg(w) = \theta + \phi.$$

Under this transformation, the point $z(r, \theta)$ in z-plane is mapped as a point $w(r, \theta + \phi)$ in w-plane.

Therefore, the mapping rotates a region in the z-plane through an angle ϕ in w-plane.

Hence, if $\phi > 0$, the rotation is anti-clockwise and if $\phi < 0$, the rotation is clock-wise.

(iv) **Magnification** The mapping $w = cz$, where c is a real $c > 0$.

Let $z = r\,e^{i\theta}$, then $w = cr\,e^{i\theta}$, $c > 0$.

Therefore, $|w| = cr$ and $\arg(w) = \theta$.

A point in z-plane is moved radilly to a new position at a distance cr from the origin.

Hence, if $c > 1$, then vector in the z-plane is magnified, if $c < 1$, then vector is contracted and $w = z$ is an identity mapping, if $c > 1$.

(v) Inverse Mapping The mapping $w = \dfrac{1}{z}$ let $z = r\,e^{i\theta}$, then

$$w = \frac{1}{r}e^{-i\theta} = R\,e^{i\phi},\ \text{where } R = \frac{1}{r} \text{ and } \phi = -\theta$$

Under this transformation $w = \dfrac{1}{z}$, a point $z(r, \theta)$ in z-plane is mapped in to the point $w(R, \phi)$ in w-plane.

Thus, the image in z-plane is mapped upon the reciprocal image in w-plane.

20. Bilinear (or Mobius or Fractional) Transformation

The mapping as transformation

$$w = \frac{az+b}{cz+d}$$

where a, b, c, d are complex constant and $ad - bc \neq 0$ is called Bilinear or Mobius transformation.

The transformation of $w = \dfrac{az+b}{cz+d}$ is said to be normalized if $ad - bc = 1$.

The transformation $w = \dfrac{az+b}{cz+d}$ is expressible as $cwz + dw - az - b = 0$

Evidently it is linear both in w and z, it is called a bilinear transformation.

A transformation $w = f(z)$ is said to be univalent if $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$.

21. Cross Ratio

If z_1, z_2, z_3 and z_4 are distinct points, then the ratio $\dfrac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$ is called the cross ratio of z_1, z_2, z_3 and z_4 and is denoted as (z_1, z_2, z_3 and z_4).

OBJECTIVE-TYPE QUESTIONS

1. The modulus of the complex number $\dfrac{3+4i}{1-2i}$ is

(a) 5 (b) $\sqrt{5}$

(c) $\dfrac{1}{\sqrt{5}}$ (d) $\dfrac{1}{5}$

2. The function $u(x, y) = \dfrac{1}{2}\log(x^2 + y^2)$ is

(a) Not a harmonic function
(b) The harmonic conjugate of u is $\tan^{-1}\left(\dfrac{y}{x}\right) + c$
(c) Satisfies the Laplace equation
(d) None of the above

3. The function $f(z) = |z|^2$ is
(a) Continuous and differentiable every where
(b) Continuous at $z = 0$ but not differentiable at $z = 0$
(c) Continuous every where but nowhere differentiable
(d) None of the above

4. A function is said to be harmonic if
(a) $u_{xx} + u_{yy} = 0$ (b) $u_{xx} + u_{xy} = 0$
(c) $u_y - iv_x = 0$ (d) None of the above

5. The function $f(z) = z\bar{z}$ is
(a) Analytic at $z = 0$
(b) Not analytic at $z = 0$
(c) No where analytic
(d) None of these

6. The Cauchey–Riemann equations for $f(z) = u + iv$ are
(a) $u_x = v_x$, $u_y = v_y$ (b) $u_x = v_y$, $u_y = -v_x$
(c) $u_x = v_y$, $u_y = v_x$ (d) $u_x = -v_y$, $u_y = v_x$

7. The polar form of Cauchey–Riemann equations are

(a) $u_r = \dfrac{1}{r} v_\theta, u_\theta = -r\, v_r$

(b) $u_r = v_\theta, u_\theta = v_r$

(c) $u_r = r\, v_\theta, u_\theta = -r\, v_r$

(d) None of the above

8. The value of $\dfrac{1}{2\pi i}\oint\limits_{|z|=3} \dfrac{z^2 + 3z + 4}{(z-1)^3}\,dz$ is

(a) 2 (b) πi
(c) 0 (d) $-\pi i$

9. If C is a circle $|z| = 1$, then $\oint\limits_C \bar{z}\,dz$ is

(a) πi (b) $2\pi i$
(c) 0 (d) $-\pi i$

10. The value of the integral $\int_c \dfrac{\cos 2\pi z}{(2z-1)(z-3)}\,dz$, where c is a closed curve $|z| = 1$ is

(a) $\dfrac{\pi i}{5}$ (b) $\dfrac{2\pi i}{5}$

(c) $-\pi i$ (d) πi

11. The value of $\oint_c \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}\,dz$, where c is the circle $|z| = 3$ is
(a) $4\pi(\pi + 1)i$ (b) $-4\pi(\pi - 1)i$
(c) 0 (d) $2\pi i$

12. The function $f(z) = \dfrac{1}{z^n}$ is

(a) Analytic for all z
(b) Singularities at $z = 0$
(c) Singularities at $n = 0$
(d) None of these

13. For the function $f(z) = \dfrac{e^{2z}}{(z-1)^4}$, $z = 1$ is a

(a) Pole
(b) Removable singularity
(c) Essential singularity
(d) None of these

14. For the function $f(z) = z\,e^{\frac{1}{z^2}}$, $z = 0$ is
(a) Pole
(b) Removable singularity
(c) Essential singularity
(d) None of these

15. Using Cauchy's integral theorem, the value of the integral (integration being taken in counter clockwise direction)

$\oint_{|z|=1} \frac{z^3 - 6}{3z - i} dz$ is

(a) $\frac{2\pi}{81} - 4\pi i$ (b) $\frac{\pi}{8} - 6\pi i$

(c) $\frac{4\pi}{81} - 6\pi i$ (d) 1

[GATE (CE) 1996]

16. Consider the circle $|z - 5 - 5i| = 2$ in the complex plane (x, y) with $z = x + iy$. The minimum distance from the origin to the circle is

(a) $5\sqrt{2} - 2$ (b) $\sqrt{54}$

(c) $\sqrt{34}$ (d) $5\sqrt{2}$

[GATE (AEI) 2005]

17. Consider likely applicability of Cauchy's integral theorem of evaluate the following integral, contour around the unit circle c, $I = \oint_c \sec z\, dz,$ then the value of I is

(a) $I = 0$; singularity set = $\{\phi\}$

(b) $I = 0$; singularity set $= \left\{\pm\frac{2n+1}{2}\pi, n = 0, 1, 2, ...\right\}$

(c) $I = \frac{\pi}{2}$; singularity set = $\{\pm n\pi, n = 0, 1, 2, \ldots\}$

(d) None of the above

[GATE (CE) 2005]

18. Let $z^3 = \bar{z},$ where z is a complex number not equal to zero, then z is a solution of

(a) $z^2 = 1$ (b) $z^3 = 1$

(c) $z^4 = 1$ (d) $z^9 = 1$

[GATE (AIE) 2005]

19. For the function $\frac{\sin z}{z^3}$ of a complex variable z, the point $z = 0$ is a

(a) Pole of order 3 (b) Pole of order 2

(c) Pole of order 1 (d) Not a singularity

[GATE (AIE) 2006]

20. The value of the integral of the complex function $f(z) = \frac{3z+4}{(z+1)(z+2)}$ along the path $|z| = 3$ is

(a) $2\pi i$ (b) $4\pi i$

(c) $6\pi i$ (d) $8\pi i$

[GATE (AEI) 2006]

21. Assuming $i = \sqrt{-1}$ and t is a real number, the value of integral $\int_0^{\pi/6} e^{it}\, dt$ is

(a) $\frac{\sqrt{3}}{2} + \frac{i}{2}$ (b) $\frac{\sqrt{3}}{2} - \frac{i}{2}$

(c) $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ (d) $\frac{1}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) i$

[GATE (ME) 2006]

22. The value of the contour integral $\oint_{|z-i|=2} \left(\frac{1}{z^2+4}\right) dz$ in positive sense is

(a) $\frac{\pi i}{2}$ (b) $-\frac{\pi}{2}$

(c) $-\frac{\pi i}{2}$ (d) $\frac{\pi}{2}$

[GATE (ECE) 2006]

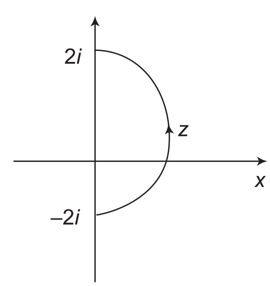
23. Let $i = \sqrt{-1},$ the value of $(i)^i$ is

(a) $\sqrt{i}$ (b) -1

(c) $\frac{\pi}{2}$ (d) $e^{-\pi/2}$

[GATE (AIE) 2007]

24. The semi-circular contour D of radius 2 is shown in the figure



then the value of the integral $\oint_D \frac{1}{z^2 - 1} dz$ is

(a) πi (b) $-\pi i$

(c) π (d) $-\pi$

[GATE (EC) 2007]

25. The value of the integral $\oint_c \frac{1}{1+z^2} dz$, where c is the contour $|z - i/2| = 1$ is

(a) $2\pi i$ (b) π

(c) $\tan^{-1}(z)$ (d) $\pi i \tan^{-1}(z)$

[GATE (ECE) 2007]

26. The integral $\oint f(z)\, dz$ evaluated round the unit circle on the complex plane for $f(z) = \frac{\cos z}{z}$ is

(a) $2\pi i$ (b) $4\pi i$

(c) $-2\pi i$ (d) 0

[GATE (ME) 2008]

27. The equation $\sin z = 10$ has

(a) No real or complex solutions

(b) Exactly two distinct complex solutions

(c) A unique solution

(d) An infinite number of complex soluting

[GATE (ECE) 2008]

28. The residue of the function

$$f(z) = \frac{1}{(z+2)^2 (z-2)^2} \text{ at } z = 2 \text{ is}$$

(a) $-\frac{1}{32}$ (b) $-\frac{1}{16}$

(c) $\frac{1}{16}$ (d) $\frac{1}{32}$

[GATE (ECE) 2008]

29. Given $X(z) = \frac{z}{(z-a)^2}$ with $|z| > a$, the residue of $X(z)z^{n-1}$ at $z = a$ for $n \geq 0$ will be

(a) a^{n-1} (b) a^n

(c) $n\, a^n$ (d) $n\, a^{n-1}$

[GATE (EE) 2008]

30. The value of the integral $\oint_c \frac{\cos 2\pi z}{(2z-1)(z-3)} dz$, where c is a closed curve given by $|z| = 1$ is

(a) $-\pi i$ (b) $\frac{\pi i}{5}$

(c) $\frac{2\pi i}{5}$ (d) πi

[GATE (CE) 2009]

31. If $z = x + iy$, where x and y are real, the value of $|e^{iz}|$ is

(a) 1 (b) e

(c) e^y (d) e^{-y}

[GATE (AIE) 2009]

32. The value of $\oint \frac{\sin z}{z} dz$, where the contour of integration is a simple closed curve around the origin is

(a) 0 (b) $2\pi i$

(c) ∞ (d) $\frac{1}{2\pi i}$

[GATE (AIE) 2009]

33. If $f(z) = c_0 + c_1 z^{-1}$, then $\oint_{|z|=1} \frac{1+f(z)}{z} dz$ is given by

(a) $2\pi c_1$ (b) $2\pi(1 + c_0)$

(c) $2\pi i c_1$ (d) $2\pi i(1 + c_0)$

[GATE (ECE) 2009]

34. The analytic function $f(z) = \frac{z-1}{z^2+1}$ has singularities at

(a) 1 and –1 (b) 1 and i

(c) 1 and $-i$ (d) i and $-i$

[GATE (CE) 2009]

35. An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + iv(x, y)$, where $i = \sqrt{-1}$. If $u = xy$, the expression for v should be

(a) $\frac{(x+y)^2}{2} + K$ (b) $\frac{x^2 - y^2}{2} + K$

(c) $\frac{-x^2 + y^2}{2} + K$ (d) $\frac{(x-y)^2}{2} + K$

[GATE (ME) 2009]

36. The residues of a complex function $X(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles are

(a) $\frac{1}{2}, -\frac{1}{2}$ and 1 (b) $\frac{1}{2}, -\frac{1}{2}$ and -1

(c) $\frac{1}{2}, -1$ and $-\frac{3}{2}$ (d) $\frac{1}{2}, -1, \frac{3}{2}$

[GATE (ECE) 2010]

37. The value of the integral $\oint_c \frac{4-3z}{(z^2+4z+5)}dz$, where c is the circle $|z| = 1$ is given by

(a) 0 (b) $\frac{1}{10}$
(c) $\frac{4}{5}$ (d) 1

[GATE (ECE) 2011]

38. Given $f(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If c is a counter clockwise path in the z-plane such that $|z + 1| = 1$, the value of $\frac{1}{2\pi i}\oint_c f(z)\,dz$ is

(a) –2 (b) –1
(c) 1 (d) 2

[GATE (ECE) 2012]

39. Integral $\oint_c \frac{z^2-4}{z^2+4}dz$ evaluated anticlockwise around the circle $|z - i| = 2$, where $i = \sqrt{-1}$, is

(a) -4π (b) 0
(c) $2 + \pi$ (d) $2 + 2i$

[GATE (EE) 2013]

40. For a 2–D flow field, the stream function ψ is given as $\psi = \frac{3}{2}(y^2 - x^2)$. The magnitude of discharge occurring between the stream line passing through points (0, 3) and (3, 4) is

(a) 6 units (b) 3 units
(c) 1.5 units (d) 2 units

[GATE (CE) 2013]

41. For the function $f(z) = \frac{1}{(2-z)(z+2)}$, the residue at $z = 2$ is ______. **[GATE (CH) 2013]**

42. Square root of $-i$, where $i = \sqrt{-1}$ are

(a) $i, -i$

(b) $\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

(c) $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

(d) $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right), \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

[GATE (EE) 2013]

43. The real part of an analytic function $f(z)$, where $z = x + iy$ is given by $e^{-y}\cos x$. The imaginary part of $f(z)$ is

(a) $e^y \cos x$ (b) $e^{-y}\sin x$
(c) $-e^y \sin x$ (d) $-e^{-y}\sin x$

[GATE (ECE) 2014]

44. If $z = (xy)\log(xy)$, then

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$ (b) $y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$

(c) $x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$ (d) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

[GATE (ECE) 2014]

45. Integration of the complex function $f(z) = \frac{z^2}{z-1}$, in the counter clockwise direction, around $|z - 1| = 1$ is

(a) $-\pi i$ (b) 0
(c) πi (d) $2\pi i$

[GATE (EE) 2014]

46. An analytic function of a complex variable. $z = x + iy$ is expressed as $f(z) = u(x, y) + iv(x, y)$, where $i = \sqrt{-1}$. If $u(x, y) = 2xy$, then $v(x, y)$ must be

(a) $x^2 + y^2 + K$ (b) $x^2 - y^2 + K$
(c) $-x^2 + y^2 + K$ (d) $-x^2 - y^2 + K$

[GATE (ME) 2014]

47. If c is a closed path in the z-plane given by $|z| = 3$. The value of the integral

$$\oint_c \left(\frac{z^2 - z + 4j}{z + 2j}\right)dz \text{ is}$$

(a) $-4\pi(1 + 2j)$ (b) $4\pi(3 - 2j)$
(c) $-4\pi(3 + 2j)$ (d) $4\pi(1 - 2j)$

[GATE (ECE) 2014]

48. Given $i = \sqrt{-1}$, the value of the definite integral, $I = \int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}dx$ is

(a) 1 (b) –1
(c) i (d) $-i$

[GATE (CE) 2015]

49. Consider the following complex function

$$f(z) = \frac{9}{(z-1)(z+2)^2}$$

which of the following is one of the residues of the above function?

(a) -1 (b) $\frac{9}{16}$

(c) 2 (d) 9

[GATE (CE) 2015]

50. Let $f(z) = \frac{az+b}{cz+d}$. If $f(z_1) = f(z_2)$ for all $z_1 \neq z_2$, $a = 2$, $b = 4$ and $c = 5$, then d should be equal to ______.

[GATE (ECE) 2015]

51. If c denotes the counterclockwise unit circle, the value of the contour integral $\frac{1}{2\pi j}\oint_c \text{Re}(z)\,dz$ is

[GATE (ECE) 2015]

52. If c is a circle of radius r with centre z_0, in the complex z-plane and if n is a non-zero integer, then $\oint_c \frac{1}{(z-z_0)^{n+1}}dz$ equals

(a) $2\pi n j$ (b) 0

(c) $\frac{\pi j}{2n}$ (d) $2\pi n$

[GATE (ECE) 2015]

53. Let $z = x + iy$ be a complex variable. Consider that contour integration is performed along the unit circle in anticlockwise direction. Which one of the following statement is *not true*?

(a) The residue of $\frac{z}{z^2-1}$ at $z = 1$ is $\frac{1}{2}$

(b) $\oint_c z^2\,dz = 0$

(c) $\frac{1}{2\pi i}\oint_c \frac{1}{z}dz = 1$

(d) $\bar{z}$ (complex conjugate of z) is analytical function

[GATE (ECE) 2015]

54. Evaluate the integral $\oint \frac{1}{z}dz$, where the contain is unit circle taken in clockwise direction is

(a) $2\pi i$ (b) 0

(c) $-2\pi i$ (d) $4\pi i$

[GATE (IN) 2015]

55. Given two complex numbers $z_1 = 5 + (5\sqrt{3})i$ and $z_2 = \frac{2}{\sqrt{3}} + 2i$, the argument of $\frac{z_1}{z_2}$ in degree is

(a) 0 (b) 30

(c) 60 (d) 90

[GATE (ME) 2015]

56. The bilinear transformation which maps the points $z = 1$, $z = 0$, $z - 1$ of z-plane in to $w = i$, $w = 0$, $w = -i$ of w-plane respectively is

(a) $w = z$ (b) $w = i(z + 1)$

(c) $z = iz$ (d) None of these

57. Under the transformation $w = (1 + i)\,z + 2 - i$, the line $x = 0$ is mapped into the line

(a) $3u + 2v = 1$ (b) $u + v = 1$

(c) $2v - u = 1$ (d) None of these

58. The bilinear transformation $w = \frac{i(1-z)}{1+z}$ maps

(a) i, 1, -1 on to 1, 0, ∞

(b) -1, 0, 1 onto 0, i, $3i$

(c) both (a) and (b)

(d) Neither (a) nor (b)

59. The invariant points of the transformation $w = \frac{z-1}{z+1}$ are

(a) $z = \frac{2i}{5}$ (b) $z = i$

(c) $z = \pm i$ (d) $z = 3i - 2$

60. The condition of a conformal mapping is

(a) $J\left(\frac{u, v}{x, y}\right) = 0$ (b) $J\left(\frac{x, y}{u, v}\right) = 0$

(c) $J\left(\frac{u, v}{x, y}\right) \neq 0$ (d) $J\left(\frac{x, y}{u, v}\right) \neq 0$

61. The bilinear transformation which maps the half plane $\text{Im}(z) \geq 0$ on to the circular disc $|w| \leq 1$, is

(a) $w = e^{i\lambda}\left(\frac{z-\alpha}{z+\bar{\alpha}}\right)$

(b) $w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$

(c) $w = e^{-i\lambda}\left(\dfrac{z-\alpha}{\bar{\alpha}z-1}\right)$

(d) None of these

62. A transformation $w = \dfrac{5-4z}{4z-2}$ transform the circle $|z| = 1$ into a circle in w-plane whose centre is

(a) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(-\frac{1}{2}, 0\right)$

(c) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, 0\right)$

63. The fixed points of the bilinear transformation $w = \dfrac{z}{2-z}$ are

(a) 0, 1 (b) 0, ∞

(c) 1, 2 (d) 3, 5

64. The cross ratio z_1, z_2, z_3, z_4 is real if and only if the four points z_1, z_2, z_3, z_4 lies on a

(a) Circle (b) Square

(c) Straight line (d) None of these

65. Laurent's series of $f(z) = \dfrac{1}{(z+1)(z+3)}$ in the region $1 < |z| < 3$ is

(a) $\dfrac{1}{2z} + \dfrac{1}{2z^2} + \dfrac{1}{2z^3} + \cdots$

(b) $\dfrac{1}{2z} - \dfrac{1}{2z^2} + \dfrac{1}{2z^3} - \cdots$

(c) $\dfrac{1}{z} - \dfrac{1}{z^2} + \dfrac{1}{z^3} - \cdots$

(d) None of these

66. Expansion of the function $f(z) = \dfrac{\sin z}{(z-\pi)}$ about $z = \pi$ is

(a) $-1 + \dfrac{(z-\pi)^2}{3!} - \dfrac{(z-\pi)^4}{5!} + \cdots$

(b) $-1 + \dfrac{(z-\pi)^2}{2!} - \dfrac{(z-\pi)^4}{4!} + \cdots$

(c) $-1 + \dfrac{(z-\pi)^2}{3} - \dfrac{(z-\pi)^4}{5} + \cdots$

(d) $1 - \dfrac{(z-\pi)^2}{2!} + \dfrac{(z-\pi)^4}{5!} - \cdots$

67. The coefficient of $\dfrac{1}{z}$ in the Laurent's series of $\dfrac{\sin 2z}{z^2}$ is

(a) -1 (b) -2

(c) 2 (d) 0

68. Taylor's series of $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ for $|z| < 2$ is

(a) $1 - \dfrac{3}{2}\sum \dfrac{(-1)^n}{2} + 9\sum (-1)^{n+1}\left(\dfrac{z}{3}\right)^3$

(b) $1 + \dfrac{3}{2}\sum (-1)^n \left(\dfrac{z}{2}\right)^n - \dfrac{8}{3}\sum (-1)^n \left(\dfrac{z}{3}\right)^n$

(c) $1 - \dfrac{3}{2}\sum (-1)^n \left(\dfrac{z}{2}\right)^n + \dfrac{8}{3}\sum (-1)^n \left(\dfrac{z}{3}\right)^n$

(d) None of these

69. The analytic part of Laurent's series is

(a) $\sum_{n=0}^{\infty} a_n (z-a)^n$ (b) $\sum_{n=0}^{\infty} a_n (z-a)^{-n}$

(c) $\sum_{n=1}^{\infty} a_n (z-a)^n$ (d) None of these

70. The coefficient of $\dfrac{1}{z}$ in the expansion of $\log\left(\dfrac{z}{z-1}\right)$ valid in $|z| > 1$ is

(a) 1 (b) –2

(c) $\dfrac{1}{2}$ (d) –1

71. The residue of $\dfrac{1}{\sin z - \cos z}$ at $z = \dfrac{\pi}{4}$ is

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{\sqrt{2}}$

(c) –1 (d) 0

72. The residue of $f(z) = \dfrac{z^3}{(z-1)(z-2)(z-3)}$ at $z = \infty$ is

(a) –6 (b) 9

(c) 6 (d) –9

73. The residue of $\dfrac{\log z}{(1+z^2)^2}$ at $z = i$ is

(a) $\frac{\pi}{2} - i$ (b) $\left(\frac{\pi}{2} + i\right)$

(c) $\frac{1}{4}\left(\frac{\pi}{2} + i\right)$ (d) $\frac{\pi}{2}$

74. The residue of $\frac{1}{z^3(1-z^2)}$ at $z = \infty$ is

(a) 0 (b) 1
(c) –1 (d) 4

75. If $z = 0$ is a simple pole of the function $f(z)$, then the residue at $z = 0$ is

(a) $\lim_{z\to 0} z \cdot f(z)$ (b) $\lim_{z\to 0} \frac{f(z)}{z}$

(c) $\lim_{z\to \infty} z \cdot f(z)$ (d) $\lim_{z\to \infty} z/f(z)$

76. The residue of $z^3 \cdot \cos\left(\frac{1}{z-2}\right)$ at $z = 2$ is

(a) $-\frac{143}{124}$ (b) $\frac{143}{134}$

(c) $\frac{143}{142}$ (d) $\frac{134}{421}$

77. The residue of the function $f(z)$ at $z = \infty$ is

(a) $\lim_{z\to\infty}\left[-z f\left(\frac{1}{z}\right)\right]$ (b) $\lim_{z\to\infty}\left[-z\, f(z)\right]$

(c) $\lim_{z\to\infty}\left[-f(z)\right]$ (d) $\lim_{z\to\infty}\left[(z-a)\, f(z)\right]$

78. Residue of $f(z) = \frac{z^2}{(z-a)(z-b)(z-c)}$ at $z = \infty$ is

(a) 1 (b) –1

(c) 0 (d) $-\frac{1}{abc}$

79. The value of $\oint_c \frac{dz}{z-a}$, where c is a circle $|z - a| = r$ is

(a) πi (b) $2\pi i$
(c) $4\pi i$ (d) $-\pi i$

80. What is the form of a bilinear transformation if ∞ is the only fixed point

(a) $w = z + \frac{b}{d}$ (b) $w = z - \frac{b}{d}$

(c) $w = z - \frac{d}{b}$ (d) $w = az + \frac{b}{d}$

81. What is the form of a bilinear transformation if it is maps the half plane $I(z) \geq 0$ on to the circular disc $|w| \leq 1$

(a) $e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}}$ (b) $e^{-i\lambda}\frac{z+\alpha}{z+\bar{\alpha}}$

(c) $e^{i\lambda}\frac{z+\alpha}{z+\bar{\alpha}}$ (d) $\frac{z-\alpha}{z+\bar{\alpha}}$

82. The transformation of $f(z) = \frac{1}{z}$ maps

(a) $|z| < 1$ on to $|f(z)| < 1$
(b) $|z| < 1$ on to $|f(z)| > 1$
(c) $|z| > 1$ on to $|f(z)| > 1$
(d) $|z| < 1$ on to $|f(z)| > 1$

83. The bilinear transformation that maps the points $z_1 = \infty$, $z_2 = i$ and $z_3 = 0$ into the points $w_1 = 0$, $w_z = i$ and $w_3 = \infty$ is

(a) $\frac{1}{z}$ (b) $\frac{1}{2z}$

(c) $-\frac{1}{z}$ (d) $\frac{2}{z}$

84. The bilinear transformation $w = \frac{2z}{z-2}$ maps $\{z: |z - 1| < 1\}$ on to

(a) $\{w: \text{Re}(w) < 0\}$ (b) $\{w: \text{Im}(w) > 0\}$
(c) $\{w: \text{Re}(w) > 0\}$ (d) $\{w: |w + 2| < 1\}$

85. The transformation $w = e^{i\theta}\left[\frac{z-\alpha}{\bar{\alpha}z-1}\right]$, where α is a constant, maps, $|z| < 1$ on to

(a) $|w| > 1$ if $|\alpha| > 1$ (b) $|w| < 1$ if $|\alpha| < 1$
(c) $|w| = 1$ if $|\alpha| = 1$ (d) $|w| = 3$ if $|\alpha| = 0$

86. The magnification factor of the conformal mapping $w = \sqrt{2}\, z e^{\frac{\pi i}{4}} + (1-2i)$ is

(a) $\sqrt{2}$ (b) 3
(c) 2 (d) 1

87. If $z = z_0$ is an essential singularity of $f(z)$ and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is its Laurent's expansion in annulus (z_0, 0, R). Then $z = z_0$ is an essential singularity if

(a) $a_n \neq 0$ for many positive n
(b) $a_n \neq 0$ for infinitely many negative n
(c) $a_n \neq 0$ for all integers n
(d) $a_n = 0$ for all integers n

88. If a function $f(z)$ is analytic at all points of a bounded region except at infinitely many points, then these exceptional points are known

(a) Singularity (b) Poles
(c) Zeros (d) Simple points

89. For the function $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$; $z = 0$ is a

(a) Pole
(b) Essential singularity
(c) Removable singularity
(d) Non-essential singularity

90. The least positive integer n for which $\left(\dfrac{1+i}{1-i}\right)^n$ is real, is

(a) 2 (b) 8
(c) 4 (d) 0

91. If $\sqrt{-1} = i$, then

$$4+5\left[-\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right]^{334}+3\left[-\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right]^{365}$$

is equal to

(a) $-1+\sqrt{3}\,i$ (b) $1-\sqrt{3}\,i$
(c) $\sqrt{3}\,i$ (d) $-\sqrt{3}\,i$

92. If $i = \sqrt{-1}$ and n is a positive integer, then $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is equal to

(a) 1 (b) i
(c) 0 (d) i^n

93. The smallest positive integer n for which $(1 + i)^{2n} = (1 - i)^{2n}$ is

(a) 4 (b) 8
(c) 10 (d) 2

94. $\dfrac{\left(\sin\dfrac{\pi}{8}+i\cos\dfrac{\pi}{8}\right)^8}{\left(\sin\dfrac{\pi}{8}-i\cos\dfrac{\pi}{8}\right)^8}$ is equal to

(a) 1 (b) -1
(c) 0 (d) i

95. The value of $\left(\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^6+\left(\dfrac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)^6$ is

(a) 1 (b) 2
(c) -2 (d) -1

96. The value of i^i is

(a) $-w^2$ (b) w
(c) $\dfrac{\pi}{2}$ (d) $e^{-\pi/2}$

97. If $x = \cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}$, then the value of $\left(\dfrac{1+x}{2}\right)^{3x}$ is

(a) $(-1)^n$ (b) $(-1)^{2n}$
(c) $\dfrac{(-1)^n}{2^{3n}}$ (d) 1

98. The value of log (log i) is

(a) $\log\left(\dfrac{\pi}{2}-i\dfrac{\pi}{2}\right)$ (b) $\log\dfrac{\pi}{2}+\dfrac{\pi i}{2}$
(c) $\log\dfrac{\pi}{2}-\dfrac{\pi i}{2}$ (d) None of these

99. If $(1 - i)^n = 2^n$, then n is equal to

(a) 0 (b) 1
(c) 2 (d) -1

100. If $x + iy = (1-i\sqrt{3})^{100}$, then the value of (x, y) is

(a) $(2^{99}, 2^{99}\sqrt{3})$ (b) $(-2^{99}, 2^{99}\sqrt{3})$
(c) $(2^{99}, -2^{99}\sqrt{3})$ (d) $(0, 0)$

ANSWERS

1.(b)	2.(b)	3.(c)	4.(a)	5.(c)	6.(b)	7.(a)	8.(a)	9.(b)	10.(b)
11.(a)	12.(b)	13.(a)	14.(c)	15.(a)	16.(a)	17.(b)	18.(c)	19.(b)	20.(c)
21.(d)	22.(d)	23.(d)	24.(a)	25.(b)	26.(a)	27.(b)	28.(a)	29.(d)	30.(c)
31.(d)	32.(a)	33.(b)	34.(a)	35.(c)	36.(c)	37.(a)	38.(c)	39.(a)	40.(b)
41. $\left(-\frac{1}{4}\right)$	42.(b)	43.(b)	44.(c)	45.(c)	46.(c)	47.(c)	48.(c)	49.(a)	50.(10)
51.(0)	52.(b)	53.(d)	54.(b)	55.(a)	56.(c)	57.(b)	58.(a)	59.(c)	60.(c)
61.(b)	62.(b)	63.(a)	64.(a)	65.(b)	66.(a)	67.(c)	68.(b)	69.(a)	70.(a)
71.(b)	72.(a)	73.(c)	74.(a)	75.(a)	76.(a)	77.(b)	78.(b)	79.(b)	80.(a)
81.(a)	82.(b)	83.(c)	84.(a)	85.(b)	86.(c)	87.(b)	88.(a)	89.(c)	90.(a)
91.(c)	92.(c)	93.(d)	94.(a)	95.(b)	96.(d)	97.(c)	98.(b)	99.(a)	100.(b)

4 Empirical Laws and Curve Fitting

4.1 INTRODUCTION

Quite often we come across situations where we need to express the given data, obtained from observations, in the form of law, which involves two or more than two variables. This law is generally called empirical relation law. For example, we are interested to study the law between the height of father and elder son, height and weight of an individual, etc. This relation is used for future studies.

4.2 SCATTER DIAGRAM

It is the simplest way of diagrammatically depicting the data which involves two variables known as bivariate data. Thus, for a bivariate distribution (x_i, y_i), $i = 1, 2, \ldots, n$, if the values of variable X plotted along the x-axis and value of variable Y is plotted along y-axis, respectively, in the xy-plane, the diagram of dots so obtained is known as scatter diagram.

From the scatter diagram, we can form a fairly good, though vague idea whether the variables are correlated or not. But if number of observations are fairly large, this method, however, is not suitable.

4.3 CURVE FITTING

To express the given data approximately, a number by equations of various types can be obtained. But question arises how to find an equation of the curve which fits the data best and can be used to predict the unknown values. The method of finding such an equation of 'best fit' is called curve fitting.

Let $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ be n pairs of observed data then the given data that can be fitted in an equation which will contain n arbitrary constants and to find these n arbitrary constants, n simultaneous equations for n unknown values can be solved. But if we want equations to have less than n unknown values then the following four methods are used for this purpose:

(i) Graphical Method
(ii) Least Square Method
(iii) Group Averages Method
(iv) Moments Method

It is difficult to find the values of unknown uniquely by Graphical methods amongst help of the remaining three methods, the least square method is considered of best to fit to a curve for a given data. This method has wide applications and can be easily implemented in computer. Now, in the coming sections we shall discuss all the four methods in details.

4.4 GRAPHICAL METHOD

Let the linear law $y = ax + b$ be the curve representing the given data, then we proceed as per the following steps:

(i) Plot the given data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of n pairs on the graph paper on xy plane by using suitable scale.

(ii) Draw the straight line which fits the data best.

(iii) Calculate slop of line a and intercept on y-axis b by taking any two points of n pairs of given prints.

If we do not get an approximate straight line with these plotted n pairs of given data, then we draw a smooth curve through them and from the shape of graph, we try to find a law of the curve by which we can reduce this to the curve of the form.

$$y = ax + b$$

4.4.1 Linear Laws

Following are some of the laws which are commonly used and indicating the way how these can be reduced to linear form by suitable substitutions:

(i) Fitting of the curve $y = ax^m + b$. Let $y = ax^m + b$ (1)

Taking $x^m = X$ and $Y = y$, then Eq. (1) becomes

$$Y = aX + b \tag{2}$$

(ii) Fitting of the curve $y = ax^b$. When $y = ax^b$ (3)

Taking log of both sides, we get

$$\log_{10} y = \log_{10} a + b \log_{10} x \tag{4}$$

Let $\log_{10} y = Y$, $\quad \log_{10} a = A$, $\quad \log_{10} x = X$

then Eq. (3) reduces to

$$Y = A + bX \tag{5}$$

(iii) Fitting of the curve $y = ax^m + b \log x$. Let $y = ax^m + b \log x$ (6)

Dividing both sides of Eq. (6) by $\log x$, we get

$$\frac{y}{\log x} = \frac{a}{\log x} x^m + b \tag{7}$$

Put $\dfrac{y}{\log x} = Y$, $\quad \dfrac{x^m}{\log x} = X$

then the given law becomes

$$Y = ax + b \tag{8}$$

(iv) Hitting of an exponential curve $y = a\, e^{bx}$. Suppose $y = a\, e^{bx}$ (9)

Taking log of both sides

$$\log_{10} y = \log_{10} a + bx \tag{10}$$

Let $\log_{10} y = Y$ and $\log_{10} a = A$

then law reduces to

$$Y = A + bx \tag{11}$$

Example 1 For the following data, find a law of the type $y = ax^2 + b$

x	1	2	3	4	5	6	7	8
y	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution Given law

$$y = ax^2 + b$$

Let $x^2 = X$

then $y = aX + b$

Table for X and Y is given as follows:

$X = x^2$	1	4	9	16	25	36	49	64
Y	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Plot these points and draw the straight line of best fit through these points as shown in Fig. 4.1.

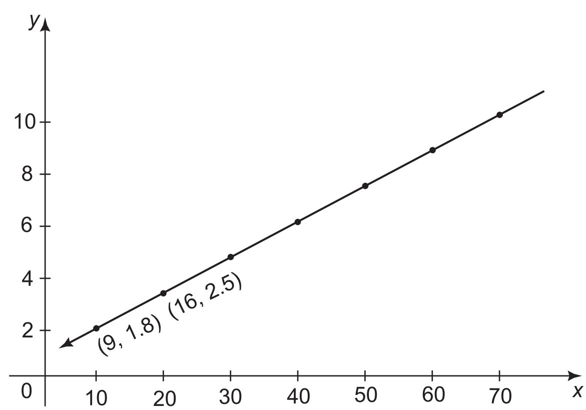


Fig. 4.1

Slope of the line is as follows:

$$\frac{2.5-1.8}{16-9} = \frac{0.7}{7} = 0.1$$

and using $y = aX + b$

$$2.5 = a \times 16 + 0.1$$

$$16\,a = 2.4$$

$$a = \frac{2.4}{16} = 0.15$$

Example 2 The following values of x and y follow the law $y = ax^2 + b\log_{10}x$. Find graphically the suitable values of a and y.

x	2	3	4	5
y	2	4	6	10

Solution

Given $$y = ax^2 + b\log_{10} x \tag{12}$$

$$\Rightarrow \quad \frac{y}{\log_{10} x} = a\frac{x^2}{\log_{10} x} + b \tag{13}$$

Let $$\frac{y}{\log_{10} x} = Y \text{ and } \frac{x^2}{\log_{10} x} = X$$

Then Eq. (13) becomes

$$Y = ax + b$$

The table for Y and X is given as follows:

$X = \dfrac{x^2}{\log_{10} x}$	13.29	18.86	26.58	35.77
$Y = \dfrac{y}{\log_{10} y}$	6.64	8.38	9.97	14.31

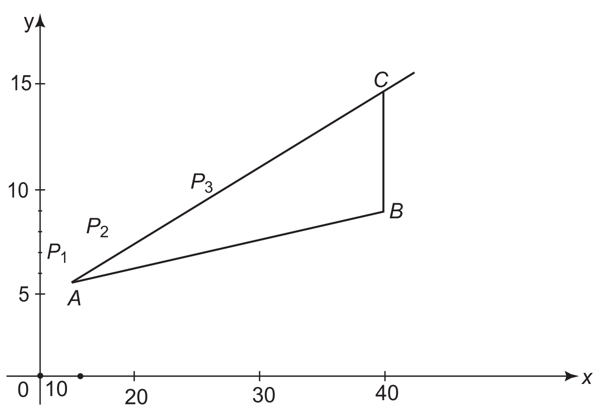


Fig. 4.2

Slope can be calculated using Fig. 4.2.

Slope $$b = \frac{BC}{AB} = \frac{14.31 - 6.64}{35.77 - 13.29} = \frac{7.67}{22.48} = 0.34$$

$\therefore$ $a = P_1$ lies on straight line

$\therefore$ $6.64 = a(13.29) + .34$

$a = 0.47$

$\therefore$ $y = 0.47x^2 + (0.34)\log_{10} x$

Example 3 Fit a curve $y = ae^{bx}$ for the following data:

x	1	2	3	4	5	6	7	8
y	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution

Given curve is $y = ae^{bx}$ (14)

Taking log of both sides we get

$$\log_{10}y = \log_{10}a + bx \tag{15}$$

Let $\log_{10}y = Y$

$\log_{10}a = A$

Therefore, Eq. (15) becomes

$$Y = A + bx$$

Table for x and y is given as follows:

x	1	2	3	4	5	6	7	8
$Y = \log_{10}y$	0.00 P_1	0.08 P_2	0.26 P_3	0.40 P_4	0.56 P_5	0.67 P_6	0.82 P_7	0.96 P_8

The points on x-y plane are depicted in Fig. 4.3. Draw the straight line of best and find the values of A and b.

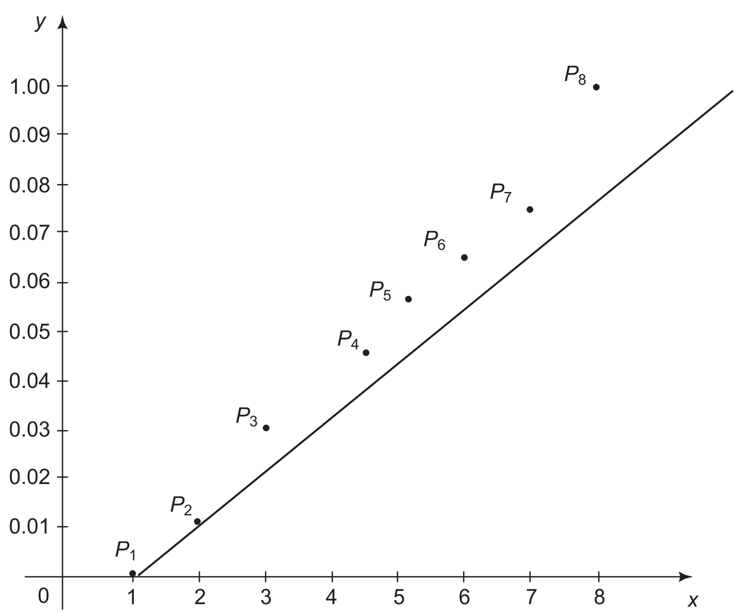


Fig. 4.3

EXERCISE 4.1

1. Find a law of the type $y = a + bx^2$ for the following data:

x	10	20	30	40	50
y	8	10	15	21	30

2. For the following data, fit a linear curve $y = ax + b$ and compute y when $x = 100$.

x	50	70	100	120
y	12	15	21	25

3. Fit the curve $y = ae^{bx}$ to the following data:

x	0	2	4
y	5.1	10	31.1

(Coimbatore 1997)

4. For the following data, fit a linear curve $y = a + bx$.

x	1	1	2	2	3	3	4	5	6	7
y	2	7	7	10	8	12	10	14	11	14

5. Find the values of c and d for the following data which follows the law $v = cu^2 + d\log_{10}v$.

u	2.85	3.88	4.66	5.69	6.65	7.77	8.67
v	16.7	26.4	35.1	47.5	60.6	77.5	93.4

6. Find the best values of b and m for the following data which follows the law $y = bx^m$.

x	25	56.2	100	1.56
Y	1.0	1.5	2.0	2.5

7. Following data follow the law $N = c + d\sqrt{M}$, find the values of c and d graphically which fits the data best.

M	500	1000	2000	4000	6000
N	0.20	0.33	0.38	0.45	0.51

8. Fit the curve $y = a + bx$ to the following data:

y	1.8	3.0	4.8	5.0	6.5	7.0	9.0	9.1
x	20.0	30.5	40.0	55.1	60.3	74.9	88.4	95.2

Answer

1. $a \cong 7.35$ and $b \cong 0.0085$; $y = 7.35 + 0.0085\,x^2$
2. $a = 0.1879$, $b = 2.2759$; $y = 21.07$
3. $a = 4.1$; $b = 0.43$; $y = 4.1\,e^{0.43\,x}$
4. $a = 4.71$, $b = 1.41$; $y = 4.71 + 1.41\,x$
5. $c = 0.99$, $d = 20.2$; $v = (0.99)u^2 + (20.20)\log_{10}u$
6. $b = 0.5012$, $m = 0.5$
7. $c = 0.20$, $d = 0.0044$
8. $y = 0.2177 + 0.0957\,x$

4.5 LEAST SQUARE METHOD

The principle of least squares is the most popular and widely used method of fitting a mathematical function best to a given set of data. This method gives very good results. The principle of least square method is that the sum of the difference between observed values of yield and the expected values of y_i $(i = 1, 2, \ldots, n)$ should be minimum. The various curves that may be used to describe the given data are as follows:

If y be the value of a variable corresponding to independent variable x, then

(i) A straight line: $y = a + bx$

(ii) A second degree parabola: $y = a + bx + cx^2$

(iii) k^{th} degree polynomial $= y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

(iv) Exponential curve: $y = ab^x$

$$\Rightarrow \quad \log_{10}y = \log_{10}a + x\log_{10} b$$

Let $\log_{10}y = Y$, $\log_{10}a = A$ and $\log_{10}b = B$

$$\Rightarrow \quad Y = A + Bx$$

4.5.1 Fitting of a Straight Line

Let the following equation be required to fit

$$y = a + bx \tag{16}$$

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n observations of an experiment then for a value x_i, the observed value y_i and the expected value is $\hat{y}_i = a + bx_i$, so that the error is $E_i = y_i - \hat{y}_i = y_i - a - bx_i$. Therefore, the sum of squares of these errors for the data is as follows:

$$\sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n}(y_i - a - bx_i)^2 = \left[(y_1 - a - bx_1)^2 + (y_2 - a - bx_2)^2 + \cdots + (y_n - a - bx_n)^2\right]$$

E will be minimum if $\dfrac{\partial E}{\partial a} = 0$ and $\dfrac{\partial E}{\partial b} = 0$

$$\therefore \quad 2((y_1 - a - bx_1)(-1) - 2 \in y_2 - a - bx_2)\ldots - 2(y_n - a - bx_2) = 0 \tag{17}$$

and
$$\frac{\partial E}{\partial b} = 0 - 2x_1(y_1 - a - bx_1) - 2x_2(y_2 - a - bx_2)\ldots 2x_n(y_n - a - bx_n) = 0 \tag{18}$$

by Eq. (17) we get

$$y_1 + y_2 + \cdots + y_n = na + b(x_1 + x_2 + \cdots + x_n)$$

i.e.,
$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \tag{19}$$

and by Eq. (18) we get

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n = ax_1 + ax_2 + \cdots + ax_n + bx_1^2 + bx_2^2 + \cdots + bx_n^2$$

i.e.,
$$\sum x_iy_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \tag{20}$$

The Eqs (19) and (20) are called normal equations and can be solved simultaneously to find the values of a and b. By putting the values of a and b in Eq. (1), we get the desired curve of best fit.

4.5.2 Fitting of a Second Degree Parabola

Suppose we want to fit a parabola

$$y = a + bx + cx^2$$

Write the normal equations based on the given data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ as follows:

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2 \tag{21}$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3 \tag{22}$$

$$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4 \tag{23}$$

By solving the Eqs (21), (22) and (23) we can find the values of constants a, b and c and putting these values of a, b, c in the equation $y = a + bx + cx^2$ will give the best fit of second degree parabola to the given data.

4.5.3 Fitting of a k^{th} Degree Polynomial

Suppose we want to fit a polynomial of k^{th} degree $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

Write the k+ normal equations based on the given data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$\sum_{i=1}^{n} y_i = na_0 + a_1\sum_{i=1}^{n} x + a_2\sum_{i=1}^{n} x_i^2 + \cdots + a_n\sum_{i=1}^{n} x^n$$

$$\sum_{i=1}^{n} x_i y_i = a_0\sum_{i=1}^{n} x_i + a_1\sum_{i=1}^{n} x_i^2 + a_2\sum_{i=1}^{n} x_i^3 + \cdots + a_n\sum_{i=1}^{n} x_i^{n+1}$$

$$\sum_{i=1}^{n} x_i y_i = a_0\sum_{i=1}^{n} x_i^2 + a_1\sum_{i=1}^{n} x_i^3 + a_2\sum_{i=1}^{n} x_i^4 + \cdots + a_n\sum_{i=1}^{n} x^{n+2}$$

$$\vdots$$

$$\sum_{i=1}^{n} x_i^{\,k} y_i = a_0\sum_{i=1}^{n} x_i^k + a_1\sum_{i=1}^{n} x_i^{k+1} + a_2\sum_{i=1}^{n} x_i^{4+2} + \cdots + a_n\sum_{i=1}^{n} x^{k+n}$$

Solve the above $k + 1$ normal equations and find the values of $a_0, a_1, \ldots, a_n$. Put the values of $a_0, a_1, \ldots, a_n$ the polynomial which will give the best fit of k^{th} degree polynomial to the given data.

4.5.4 Fitting of Exponential Curve

Suppose we want to fit an exponential curve $y = ab^x$ (24)

$\Rightarrow \quad \log_{10}y = \log_{10}a + x\log_{10}b$

Let $\quad \log_{10}y = Y$, $\log_{10}a = A$ and $\log_{10}b = B$

then the given curve becomes $Y = A + Bx$; which is now the equation of straight line. Now find the values of A and B in the similar manner as in fitting of a straight line. After finding the values of A and B, compute

$$a = \text{antilog } A$$

and $$b = \text{antilog } B$$

Put the values of a and b in given equation of exponential curve which will give you the best fit curve of given data.

Example 4 For the following data, fit a straight line $y = a + bx$

x	35.3	29.7	30.8	58.8	61.4	71.3	74.4	76.7	70.7	57.5	46.4	28.9	28.1	39.1	46.8
y	11.0	11.1	12.5	8.4	9.3	8.7	6.4	8.5	7.8	9.1	8.2	12.2	11.9	9.6	10.9
x	48.5	59.3	70.0	70.0	74.4	72.1	58.1	44.6	33.4	28.6					
y	9.6	10.1	8.1	6.8	8.9	7.7	8.5	8.9	10.4	11.1	Compute the value of y when x = 50				

Solution

Here $$n = 25$$

$$\sum_{i=1}^{25} x_i = 1314.90, \sum_{i=1}^{25} x_i^2 = 76\,308.53, \sum_{i=1}^{25} y_i = 235.70$$

$$\sum_{i=1}^{25} y_i^2 = 2286.07, \sum_{i=1}^{25} x_i y_i = 11\,824.44$$

Normal equations are as follows:

$$235.70 = 25a + 1314.90b$$

$$11824.40 = 1314.90\,a + 76308.53\,b$$

Solving the above two normal equations, we get

$$a = 13.64 \text{ and } b = -0.08\,x$$

∴ Curve of best fit $y = 13.64 - 0.8\,x$

when $$x = 50$$

$$y = 13.06 - (0.08)\,50 = 9.64$$

Example 5 For the following data fit a straight line $y = a + bx$

x	5	15	25	35	45	50
y	10	18	20	25	32	45

Solution

To find a straight line, normal equations are as follows:

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

Here $n = 6, \sum x = 175, \sum x^2 = 6625, \sum y = 150$

$$\sum xy = 5385$$

$\therefore$ $$150 = 6a + 175b$$

$$5385 = 175a + 6625b$$

Solving above two equations, we get

$$a = 5.75 \text{ and } b = 0.66$$

$\therefore$ Line of best fit $y = 5.75 + 0.66\,x$

Example 6 Following are measurements of air velocity (x) and revaporation coefficient (y) of burning fuel droplets in an impulse engine:

x	20	60	100	140	180	220	260	300	340	380
y	0.18	0.37	0.35	0.78	0.56	0.75	1.18	1.36	1.17	1.65

Fit a straight line $y = a + bx$ to the given data:

Solution

Here $n = 10, \Sigma x = 2000, \Sigma xy = 2175.40$

$$\Sigma y = 8.35, \Sigma x^2 = 532{,}000$$

$\therefore$ normal equations are as follows:

$$8.35 = 10\,a + 2000\,b$$

$$2175.40 = 2000\,a + 532000\,b$$

Solving the above equations, we get

$$a = 0.069 \text{ and } b = 0.00383$$

$\therefore$ Line of best fit is

$$y = 0.069 + 0.00383\,x$$

Example 7 Following are the data on the drying time of a certain varnish and the amount of an additive that is intended to reduce the drying time:

Amount of Varnish additive groups (x)	0	1	2	3	4	5	6	7	8
Drying time (hours) y	12.0	10.5	10.0	8.0	7.0	8.0	7.5	8.5	9.5

(a) Fit a second degree parabola $y = a + bx + cx^2$

(b) Use the result of part (a) to predict the drying time of the varnish when 6.5 g of additive is being used.

Solution

(a) The normal equations to fit the parabola $y = a + bx + cx^2$ are as follows:

$$\sum y = xa + b\Sigma x + c\Sigma x^2$$

$$\sum xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\sum x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Here $n = 9$, $\Sigma y = 80.5$, $\Sigma x = 36$, $\Sigma x^2 = 204$

$$\Sigma x^3 = 1296, \Sigma x^4 = 8772, \Sigma xy = 299, \Sigma x^2 y = 1697$$

$$\therefore \quad 80.5 = 9a + 36b + 204c$$

$$299 = 36a + 204b + 1296c$$

$$1697 = 204a + 1296b + 8772c$$

Solving the above three equations, we get

$$a = 12.2, b = -1.85 \text{ and } c = 0.183$$

$$\therefore \quad y = 12.2 - 1.85\,x + 0.183\,x^2$$

(b) When $x = 6.5$

$$y = 12.2 - (1.85)\,6.5 + (0.183)\,(6.5)^2$$

$$y = 7.9$$

Example 8 Find an exponential curve $y = ab^x$ to the following data:

x	1	2	3	4	5	6	7	8
y	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution

To fit the curve $y = ab^x$

Taking log of both the sides

$$\log_{10} y = \log_{10} a + x \log_{10} b \Rightarrow Y = A + Bx$$

where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = \log_{10} b$

x	1	2	3	4	5	6	7	8
y	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1
$Y = \log 10 y$	.0000	.0792	.2553	.3979	.5563	.6721	.8195	.9590
x^2	1	4	9	16	25	36	49	64
ΣxY	.0000	.1584	.7659	1.5916	2.7815	4.0326	5.7365	7.6726

Normal equations are as follows:

$$\Sigma Y = nA + B\Sigma x$$

$$\Sigma xY = A\Sigma x + B\Sigma x^2$$

Here $n = 8$, $\Sigma x = 36$, $\Sigma xY = 22.7385$, $\Sigma Y = 3.7393$, $\Sigma x^2 = 204$

$$\therefore \quad 3.7393 = 8\,A + 36\,B$$

$$22.7385 = 36\,A + 204\,B$$

Solving above two equations we get

$$A = -0.1662 \Rightarrow a = \text{antilog} A = 0.6821$$
$$B = 0.1408 \Rightarrow b = \text{antilog } B = 1.383$$
$$\boxed{\therefore \quad y = 0.6821(1.383)^x}$$

EXERCISE 4.2

1. If P is the pull required to lift a load w by means of a pulley block, find a linear law of the form $P = mw + c$ connecting P and w, using the following data:

P	12	15	21	25
w	50	70	100	120

where P and w are taken in kg wt. Compute P when; $W = 150$ kg wt.

(U.P.T.U. 2007, V.T.U. 2002)

2. By the method of least squares, fit the straight line that best fits the following data:

x	1	2	3	4	5
y	14	27	40	55	68

(U.P.T.U. 2008)

3. Fit a straight line to the following data:

Year (x)	1961	1971	1981	1991	2001
Production (y)	8	10	12	10	16

and find the expected production in 2006 (in thousand tonnes).

4. Find the best possible curve of the form $y = a + bx$, using method of least squares to the data:

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

(V.T.U. 2011)

5. Fit a straight line to the following data:

x	1	2	3	4	5	6	7	8	9
y	9	8	10	12	11	13	14	16	5

(Bhopal 2008)

6. Fit a straight line to the following data:

x	6	7	7	8	8	8	9	9	10
y	5	5	4	5	4	3	4	3	3

(J.N.T.U. 2008)

7. Fit a second degree parabola to the following data:

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

(P.T.U. 2006)

8. Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the observations:

x	–3	–2	–1	0	1	2	3
y	4.63	2.11	0.67	0.09	0.63	2.15	4.58

(V.T.U. 2006, J.N.T.U. 2005)

9. Fit a parabola $y = a + bx + cx^2$ to the following data:

x	2	4	6	8	10
y	3.07	12.85	31.47	57.38	91.29

(V.T.U. 2003S)

10. Fit a second degree parabola to the following data:

x	1	2	3	4	5	6	7	8	9	10
y	124	129	140	159	228	289	315	302	263	210

(U.P.T.U. 2009)

11. Following data gives the results of the measurements of train resistance, V is the velocity in miles/hour. R is the resistance in pounds per ton.

V	20	40	60	80	100	120
R	5.5	9.1	14.9	22.8	33.3	46.0

If R is related to V by the relation $R = a + bV + cV^2$, find a, b and c.

(U.P.T.U. 2002)

12. The velocity V of a liquid is known to vary with temperature according to a quadratic law $V = a + bT + cT^2$, find the best values of a, b and c for the following data:

T	1	2	3	4	5	6	7
V	2.31	2.01	3.80	1.66	1.55	1.47	1.41

(U.P.T.U. M.C.A. 2010)

13. For the normal equations

$$\Sigma y = na + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

Prove that $a = \overline{y} - b\overline{x}$ and $b = \dfrac{n\Sigma xy - \Sigma x\Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$

14. Verify that $\sum_{i=1}^{n} E_i = 0$ where $E_i = y_i - (a + bx_i), i = 1, 2, ..., n$

15. Fit a second degree parabola to the following data:

x	1989	1990	1991	1992	1993	1994	1995	1996	1997
y	352	356	357	358	360	361	361	360	359

(U.P.T.U. 2009)

16. Fit an exponential curve $y = ab^x$ to the following data:

x	3	6	9	12	15	18
y	115,000	147,000	239,000	356,000	579,000	864,000

Answers

1. $m = 0.1879$, $c = 2.2785$; $P = 0.1879\,w + 2.2785$
2. $y = 13.6\,x$ as $a = 0$, $b = 13.6$
3. 15.2 thousand tonnes.
4. $a = 0.545$, $b = 0.636$; $y = 0.545 + (0.636)x$
5. $a = 4.193$, $b = 1.117$; $y = 4.193 + (1.117)x$
6. $a = 8.0$, $b = -0.5$; $y = 8 -(0.5)x$
7. $a = 1.42$, $b = -1.07$, $c = 0.55$; $y = 1.42 - (1.07)x + (0.55)x^2$
8. $a = 1.04$, $b = -0.198$, $c = 6.244$; $y = 1.04 - (0.198)x + (.244)x^2$
9. $a = 0.34$, $b = -0.78$, $c = 0.99$; $y = 0.34 - (0.78)x + (0.99)x^2$
10. $a = 18.866$, $b = 66.158$, $c = -4.333$;
    $y = 18.866 + (66.158)x - (4.333)x^2$
11. $a = 3.48$, $b = -0.002$, $c = 0.0029$;
    $R = 3.48 - 0.002\,V + (0.0029)V^2$
12. $a = 2.593$, $b = -0.326$, $c = 0.023$;
    $V = 2.593 - (0.326)T + (0.023)T^2$
15. $a = -10000106$, $b = 1034.29$, $c = -0.267$;
    $y = -10000106 + 1034.29x - (0.267)x^2$
16. $a = 69502.4$; $b = 1.149$; $y = 69502.4(1.149)^x$

4.6 FITTING OF OTHER CURVES

4.6.1

Suppose we want to fit a curve

$$y = ax^b \tag{25}$$

Taking log of both the sides of Eq. (25)

$$\log_{10} y = \log_{10} a + b \log_{10} x \tag{26}$$

Let $\log_{10} y = Y$, $\log_{10} a = A$, $\log_{10} x = X$ then (26) becomes

$$Y = A + bX \tag{27}$$

which is nothing but a linear curve. So normal equations of (27) will be

$$\Sigma Y = nA + b\Sigma X$$

$$\Sigma XY = A\Sigma X + bZX^2$$

Solving above two equations, we find the values of A and b. Then form the value of A, we find a = antilog A, Put the values of a and b in Eq. (25), this will give the best fit curve $y = ax^b$ to the given data.

4.6.2

Suppose $y = ae^{bx}$ (28)

Taking log of both the sides of Eq. (28)

$$\log_{10} y = \log_{10} a + bx \log_{10} e \tag{29}$$

Let $\log_{10} y = Y, \ \log_{10} a = A$

then (29) becomes

$$Y = A + bx \qquad (\because \log_{10} e = 1) \tag{30}$$

normal equations for (30) are

$$\Sigma Y = nA + b\Sigma x$$

$$\Sigma xY = A\Sigma x + b\Sigma x^2$$

Solving above equations, we find the values of a and b and then put these values in (28), which gives the best fit $y = ae^{bx}$ to the given data.

4.6.3

Suppose $xy^a = b$ (31)

Taking log of both the sides of Eq. (31)

$$\log_{10} x + a \log_{10} y = \log_{10} b$$

or $$\log_{10} y = \frac{1}{a}\log_{10} b - \frac{1}{a}\log_{10} x$$

Let $$\log_{10} y = Y, \frac{1}{a}\log_{10} b = A \text{ and } -\frac{1}{a} = B$$

then the above equation becomes

$$Y = A + Bx \tag{32}$$

and normal equations for the above equations are as follows:

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

By solving above equations, we find the values of a and b, and putting these values of a and b in Eq. (31), we get the best fit curve $x\,y^a = b$ to the given data.

Example 9 Find the least square fit of the form $y = a + bx^2$ to the following data:

X	0	1	2	3
y	1	4	2	0

Solution Let $x^2 = X$, then given curve becomes $y = a + bX$

Normal equations for the above curve are as follows:

$$\Sigma y = na + b\Sigma X$$

$\Sigma Xy = a\Sigma X + b\Sigma X^2$, the values of $\Sigma X, \Sigma X^2, \Sigma Xy$, etc., are given in the following table:

					Total
x	0	1	2	3	6
y	1	4	2	0	7
$X = x^2$	0	1	4	9	14
X^2	0	1	16	81	98
Xy	0	4	8	0	12

Here $n = 4, \Sigma y = 7, \Sigma X = 14, \Sigma Xy = 12, \Sigma X^2 = 98.$

$$\therefore \quad \begin{aligned} 7 &= 4a + 14b \\ 12 &= 14a + 98b \end{aligned} \quad \Rightarrow a = \frac{37}{14} \text{ and } b = 50/14$$

$\therefore$ The best fit curve is $y = \dfrac{37}{14} + \dfrac{50}{14}x^2$

Example 10 Fit the curve $y = ax^b$ to the following data:

x	100	200	300	400
y	50	30	10	25

Solution Given $y = ax^b$

Taking log of both the sides, we get

$$\log_{10} y = \log_{10} a + b \log_{10} x$$

Let $\log_{10} y = Y, \log_{10} a = A$ and $\log_{10} x = X$

then above equation becomes

$$Y = A + bx$$

Normal equations are as follows:

$$\Sigma Y = nA + b\Sigma X$$

$$\Sigma Xy = A\Sigma X + b\Sigma X^2$$

The values of $\Sigma X, \Sigma Y, \Sigma X^2$ and ΣXY are given in the following table:

x	y	$X = \log_{10} x$	$Y = \log_{10} Y$	X^2	XY
100	50	2.00	1.70	4.00	3.40
200	30	2.30	1.48	5.29	3.40
300	10	2.48	1.00	6.15	2.48
400	25	2.60	1.40	6.78	3.64

Here $n = 4$, $\Sigma X = 9.38, \Sigma Y = 5.58, \Sigma X^2 = 22.22, \Sigma XY = 12.92$

$\therefore \quad 5.58 = 4\,A + 9.38\,b$

$\quad 12.92 = 9.38A + 22.22\,b$

Solving above equations, we get $A = 3.11$ and $b = -0.73$

then a = Antilog A = Antilog 3.11 = 1288.2496 ≅ 1288.25

$\therefore \qquad y = 1288.25x^{(-0.73)}$

Example 11 The pressure and volume of a gas are related by the equation $pv^Y = k$, Y and k being constants. Fit this equation to the following set of observations:

p(kg/cm^2)	0.5	1.0	1.5	2.0	2.5	8.0
v(litres)	1.62	1.00	0.75	0.62	0.52	0.46

(V.T.U. 2011)

Solution The curve to be fitted is

$$pv^{\gamma} = k \tag{33}$$

Taking log of both the sides

$$\log_{10} p + \gamma \log_{10} v = \log_{10} k \Rightarrow \log_{10} v = -\frac{1}{\gamma}\log_{10} p + \frac{1}{\gamma}\log_{10} k \tag{34}$$

Let $\quad \log_{10} p = X, -\dfrac{1}{\gamma} = B$ and $\dfrac{1}{\gamma}\log_{10} k = A, \log_{10} v = Y$

then (34) becomes

$$Y = A + BX$$

Normal equation are as follows:

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

Using the above data, we get

$$\Sigma X = \Sigma \log_{10} p = 1.0511$$

$$\Sigma Y = \Sigma \log_{10} v = -0.7442$$

$$\Sigma XY = -0.4214$$

$$\Sigma X^2 = 0.5982$$

$$n = 6$$

$\therefore \qquad -0.7442 = 6A + 1.0511\,B$

$\qquad\quad -0.4214 = 1.0511\,A + 0.5982B$

Solving above equations, we get the values of A and B respectively, which are $B = -0.7836$ and $A = 0.0132$

$$\gamma = -\frac{1}{B} = -\frac{1}{(-7836)} = 1.2762$$

and $k = \text{Antilog}(A\gamma) = \text{Antilog}(0.0168) = 1.039$

$\therefore$ The best fit curve $pv^{\gamma} = k$ is

$$pv^{1.2762} = 1.039$$

EXERCISE 4.3

1. Using method of least squares, fit a relation of the form $y = ab^x$ to the following data:

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

(Tiruchirapalli 2001)

2. Fit the curve of the form $y = ae^{bx}$ to the following data:

x	77	100	185	239	285
y	2.4	3.4	7.0	11.1	19.6

(V.T.U. 2015, J.N.T.U. 2006)

3. Obtain the least squares fit of the form $f(t) = ae^{-3t} + be^{-2t}$ for the following data:

t	0.1	0.2	0.3	0.4
$f(t)$	0.76	0.58	0.44	0.35

(U.P.T.U. 2008)

4. Predict y at $x = 3.75$, by fitting a power curve $y = ax^b$ to the following data:

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.10	6.80	7.50

(J.N.T.U. 2003)

5. Find the least square curve $y = ax + \dfrac{b}{x}$ for the following data:

x	1	2	3	4
y	–1.5	0.98	3.88	7.66

(Madras 2003)

6. If V (km/h) and R(kg/ton) are related by a relation of the type $R = a + bV^2$, find by the method of least squares a and b with the help of the following data:

V	10	20	30	40	50
R	8	10	15	21	30

(Indore 2008)

7. Fit the curve $y = ax + \dfrac{b}{x}$ to the following data:

x	1	2	3	4	5	6	7	8
y	5.4	6.3	8.2	10.3	12.6	14.9	17.3	19.5

(U.P.T.U. 2010)

8. Estimate y at $x = 2.25$ by fitting the indifference curve of the form $xy = Ax + B$ to the following data:

x	1	2	3	4
y	3.0	1.5	6.0	7.5

(J.N.T.U. 2003)

9. Predict the mean radiation dose at an altitude of zero feet by fitting an curve $y = ab^x$ to the following data:

Altitude (x)	50	480	780	1200	4400	4800	5300
Dose of radiation (y)	28	30	32	36	51	58	69

(S.V.T.U. 2007, J.N.T.U. 2003)

10. Find the least squares fit of the form $y = a_0 + a_1 x^2$ to the following data:

x	–1	0	1	2
y	2	5	3	0

(U.P.T.U. 2008)

11. Find y at $x = 3000$, using curve $y = ae^{-bx}$ for the following data:

x	50	450	780	1200	4400	4800	5300
y	28	30	32	36	51	58	69

12. The number of inches which a newly built structure is setting into the ground is given by $y = 3 - 3e^{-ax}$, where x is its age in months.

x	2	4	6	12	18	24
y	1.07	1.88	2.26	2.78	2.97	2.99

Use the method of lead square to estimate the value of a.
(*Note*: Relation between log $(3 - y)$ and x is linear.)

Answers

1. $a = 99.86$, $b = 1.2$; $y = 99.86(1.2)^x$
2. $a = 0.1839$, $b = 0.0221$; $y = 0.1839\, e^{0.221\, x}$
3. $f(t) = 0.678\, e^{-3t} + 0.312\, e^{-2t}$
4. $y = 2.978\, x^{0.5143}$; at $x = 3.75$, $y = 5.8769$
5. $a = 0.988, b = 3.275;\ y = 0.988x + \dfrac{3.275}{x}$
6. $a = 6.32$, $b = 0.0095$; $R = 6.32 + 0.0095\ \mathrm{V}^2$
7. $a = 3, b = 2;\ y = 3x + \dfrac{2}{x}$
8. $A = 7.187$, $B = -5.16$; $xy = 7.187\, x - 5.16$ at $x = 2.25$; $y = 4.894$
9. $\log_{10}(y) = Y = 1.4521 + 0.000067\, x$; y at $x = 3000 = 44.9$ (Approx.)
10. $a_0 = 4.169$, $a_1 = -1.111$; $y = 4.167 - 1.111\, x^2$

11. 44.86
12. $a = 0.240$

4.7 GROUP AVERAGES METHOD

Suppose we want to fit a straight line $y = a + bx$ best on the set of n pairs of observations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

When x takes the value of x_1, then observed value of y is y_1 and expected value of

$$y = a + bx_1 \tag{35}$$

$\therefore$ The residuals or error between observed and expected value of y for $x = x_1$ is given by

$$E_1 = y_1 - (a + bx_1)$$

Similarly,

$$E_2 = y_2 - (a + bx_2) \text{ for } x = x_2$$
$$E_3 = y_3 - (a + bx_3) \text{ for } x = x_3$$
$$\vdots$$
$$E_n = y_n - (a + bx_n) \text{ for } x = x_3$$

These E_i's could be positive or negative or 0.

The method of group averages is based on the principle that $\sum_{i=1}^{n} E_i = 0$, i.e., the sum of errors is equal to zero. To find the values of a and b, we need two equations. Therefore, we divide the entire data into two group of sizes m and $n - m$ respectively.

i.e., $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ constitute first group and
$(x_{m+1}, y_{m+1}), (x_{m+2}, y_{m+2}), \ldots, (x_n, y_n)$ constitute second group.

Assuming that for both the groups sum of errors is zero, we get

$$\left[y_1 - (a + bx_1)\right] + \left[y_2 - (a + bx_2)\right] + \cdots + \left[y_m - (a + bx_m)\right] = 0$$
$$\left[y_{m+1} - (a + bx_{m+1})\right] + \left[y_{m+2} - (a + bx_{m+2})\right] + \cdots + \left[y_n - (a + bx_n)\right] = 0$$

$$\Rightarrow \quad a + b\frac{(x_1 + x_2 + \cdots + x_m)}{m} = \frac{y_1 + y_2 + \cdots + y_n}{m} \tag{36}$$

$$\text{and} \quad a + b\frac{(x_{m+1} + x_{m+2} + \cdots + x_n)}{n - m} = \frac{y_{m+1} + y_{m+2} + \cdots + y_n}{n - m} \tag{37}$$

$$\text{or} \quad a + b\sum_{i=1}^{m}\frac{x_i}{m} = \sum_{i=1}^{m}\frac{y_i}{m} \Rightarrow a + b\,\overline{x} = \overline{y} \tag{38}$$

Where $\overline{x}$ and $\overline{y}$ are the means of first group. Similarly

$$a + b\sum_{i=m+1}^{n} x_i \Big| n \cdot m = \sum_{i=m+1}^{n}\frac{y_i}{n - m}$$

$$\Rightarrow \quad a + b\overline{x} = \overline{y}, \text{ here } \overline{x} \text{ and } \overline{y} \text{ are means of second group.} \tag{39}$$

We solve Eq. (38) and (39) to find the values of a and b respectively and then putting these values in (35), we obtain the required equation.

4.8 FITTING OF A PARABOLA

Suppose we want to fit a second degree parabola

$$y = a + bx + cx^2 \tag{40a}$$

using method of averages based on the n pairs of observations $(x_1, y_1), (x_2, y_2),\ldots, (x_n, y_n)$. To fit this, we know $y = y_1$ for $x = x_1$

$$\therefore \qquad y_1 = a + bx_1 + cx_1^2 \tag{40b}$$

Subtracting Eq. (40b) from (40a), we get

$$\begin{aligned} y - y_1 &= b(x - x_1) + c(x^2 - x_1^2) \\ &= b(x - x_1) + c(x - x_1)(x + x_1) = (x - x_1)\left[b + c(x + x_1)\right] \end{aligned}$$

or $$\frac{y - y_1}{x - x_1} = b + c(x + x_1) \tag{41}$$

Let $\dfrac{y - y_1}{x - x_1} = Y$ and $x + x_1 = X$ then (41) becomes

$$Y = b + cX$$

Now we can find the values of b and c as before and then we can find the value of a. By putting the values of a, b and c in Eq. (40b), we shall obtain the equation of a parabola which fits the data best.

Remarks: The main draw-back of this method is that different sub groups of the given observations will yield different values of $\bar{x}$ and $\bar{y}$ and hence different values of a and b, therefore to avoid this we should divide the entire data in such a way that each subgroup contains almost equal number of observations.

Example 12 For the following data fit a straight line using method of averages:

x	0	5	10	15
y	10	15	20	25

Solution Here $n = 4$, let $m = 2$, then $n - m = 2$

$$\bar{y} = a + b\bar{x}$$

For first group $\bar{y} = y_1 + y_2 = \dfrac{10+15}{2} = 12.5$

For first group $\bar{x} = \dfrac{0+5}{2} = 2.5$

For second group $\bar{y} = \dfrac{20+25}{2} = 22.5$

For second group $\bar{x} = \dfrac{10+15}{2} = 12.5$

$\therefore$ For first group $12.5 = a + b\,(2.5)$

For second group $22.5 = a + b\,(12.5)$

Solving the two equations we get $a = 10$, $b = 1$

$\therefore$ Straight line $y = 10 + x$.

Example 13 For the following data, fit a straight line $y = a + bx$ using method of averages.

x	0	2	4	6	8	10
y	4	6	8	10	12	14

Solution

$$n = 6, \text{ let } m = 3 \Rightarrow n - m = 3$$

For first group $\bar{x} = \dfrac{0+2+4}{3} = 2,\ \bar{y} = \dfrac{4+6+8}{3} = 6$

For second group $\bar{x} = \dfrac{6+8+10}{3} = 8,\ \bar{y} = \dfrac{10+12+14}{3} = 12$

Putting the values of both the groups in equation

$$\bar{y} = a + b\bar{x}, \text{ we get}$$

For first group $6 = a + 2b$ and

For second group $12 = a + 8\,b$

Solving the equations, we get $a = 4$ $b = 1$

$\therefore$ $y = 4 + x$ is the straight line.

Example 14 Fit a straight line to the following data by method of averages

x	1	3	5	7	9	11	13	15
y	3	5	7	9	11	13	15	17

Solution Here

$$n = 8, \text{ Let } m = 4 = n - m$$

For first group: $\bar{x} = \dfrac{1+3+5+7}{4} = 4,\ \bar{y} = \dfrac{3+5+7+9}{4} = 6$

For second group: $\bar{x} = \dfrac{9+11+13+15}{4} = 12,\ \bar{y} = \dfrac{11+13+15+17}{4} = 14$

Putting these values of $\bar{x}$ and $\bar{y}$ in equation

$$\bar{y} = a + b\bar{x}, \text{ we get}$$

$$6 = a + 4b$$

and $14 = a + 12b$

Solving the equations, we get $a = 2$, $b = 1$

$\therefore$ $y = 2 + x$ is the equation of straight line.

Example 15 Fit a straight line to the following data:

x	0	4	8	12	16	20
y	6	12	18	24	30	36

Solution Here $n = 6$, let $m = 3 = n - m$

$\bar{x}$ and $\bar{y}$ for first group are: $\bar{x} = 4, \bar{y} = 12$

$\bar{x}$ and $\bar{y}$ for second group are: $\bar{x} = 16, \bar{y} = 30$

$\therefore$ For first group $12 = a + b \cdot 4$

and for second group $30 = a + 16 \cdot b$

by solving the equations, we get $a = 6$, $b = 1.5$

$\therefore$ $y = 6 + (1.5)\,x$ is the equation by straight line.

Example 16 For the following data, fit a parabola $y = a + bx + cx^2$

x	4	8	12	16	20	24
y	10	20	30	40	50	60

Solution To fit a parabola $y = a + bx + cx^2$, we need the values of a, b and c. For the purpose, let us take a value of $x = 12$, $y = 30$ a particular point on $y = a + bx + cx^2$, we get

$$30 = a + 12\,b + (12)^2\,c \tag{42}$$

$$\therefore \quad y \cdot 30 = b(x - 12) + c(x^2 - 12^2)$$

$$\Rightarrow \quad \frac{y-30}{x-12} = b + (x+12)\,c \tag{43}$$

Let $\dfrac{y-30}{x-12} = Y,\ x + 12 = X$

then (43) becomes

$$Y = b + cX$$

x	y	$X = x + 12$	$Y = \dfrac{(y-30)}{x-12}$
4	10	16	2.5
8	20	20	2.5
12	30	24	0
		$\Sigma X = 6.0$	$\Sigma Y = 5.0$
16	40	28	2.5
20	50	32	2.5
24	60	36	2.5
		$\Sigma X = 96$	$\Sigma Y = 7.5$

To find the values of b and c, we use the relation

$$Y = b + cX$$

i.e., $$\bar{Y} = b + c\bar{X}$$

or $$\frac{5}{3} = b + \frac{60c}{3} = b + 20c$$

$$\frac{7.5}{3} = b + \frac{96c}{3} = b + 32c$$

$$-\frac{2.5}{3} = -12c \Rightarrow c = 0.069$$

$$\therefore \quad b = \frac{5}{3} - 20(0.69) = 0.287$$

by (42)

$$30 = a + 12\,b + 144\,c$$

$$= a + 12\,(0.287) + 144\,(0.069)$$

$$\Rightarrow \quad a = 16.62$$

$\therefore$ $y = 16.62 + 0.287\,x + 0.069x^2$ is the equation of the parabola which fits the data.

Example 17 Fit a parabola to the following data:

x	0	2	4	6	8	10	12	14
y	4	6	8	10	12	14	16	18

Solution

$$y = a + bx + cx^2 \tag{44}$$

Equation (44) gives the parabola to be fitted on the basis of given data.

Let $x = 6$, $y = 10$

$$\therefore \quad 10 = a + 6b + 36c \tag{45}$$

$$\Rightarrow \quad (y - 10) = b(x - 6) + c(x^2 - 36)$$

$$\Rightarrow \quad \frac{y-10}{x-6} = b + (x+6)c$$

Let $$X = x + 6 \text{ and } Y = \frac{y-10}{x-6}$$

Then $$Y = b + cX$$

x	y	$X = x + 6$	$Y = \dfrac{y-10}{x-6}$
0	4	6	1
2	6	8	1
4	8	10	1
6	10	12	0
		$\Sigma X = 36$	$\Sigma Y = 3$
8	12	14	1
10	14	16	1
12	16	18	1
14	18	20	1
		$\Sigma X = 69$	$\Sigma Y = 4$

To find the values of b and c, use equation

$$Y = b + cX \Rightarrow \bar{Y} = b + c\bar{X}$$

by I subgroup: $$\frac{3}{4} = b + c\left(\frac{36}{4}\right) = b + 9c$$

$$\frac{4}{4} = b + c\left(\frac{68}{4}\right) = b + 17c$$

by solving above equations

$$0.25 = 8\,c \Rightarrow c = 0.031$$

$$\therefore \qquad b = 0.473$$

Putting the values of b and c in Eq. (45), we get

$$10 = a + 6(.473) + 36(0.031)$$

$$a = 6.046$$

$\therefore$ $y = 6.046 + 0.473x + 0.031x^2$ is the equation of the second degree parabola which fits the given data.

EXERCISE 4.4

1. Fit a parabola $y = a + bx + cx^2$ corresponding to the following data:

x	36.9	46.7	63.7	77.8	84.0	87.5
y	181	197	235	270	283	292

(Hint: To solve take $x = 84$ which lies on the parabola).

2. Fit a straightline of the form $y = a + bx$ to the following data by the method of averages:

x	0	5	10	15	20	25
y	12	15	17	22	24	30

(Tiruchirapalli 2001)

3. The latent heat of vaporization of steam r is given in the following table at different temperature t.

t	40	50	60	70	80	90	100	110
r	1069.1	1063.6	1058.2	1052.7	1049.3	1041.8	1036.3	1030.8

For the range of temperature, a relation of the form $r = a + bt$ is known to fit the data. Find the values of a and b by the method of group of averages.

(Madras 2003)

4. By the method of averages, fit a curve of the form $y = ae^{bx}$ to the following data:

x	5	15	20	30	35	40
y	10	14	25	40	50	62

(Madras 2002)

5. Fit a straight line of best fit $y = a + bx$ to the following data:

x	10	9	8	7	6	5	4	3	2	1
y	108.4	102.2	95.5	87.2	81.1	75.4	70.2	65.0	58.7	52.5

6. Fit a parabola $y = a_0 x^2 + a_1 x + a_2$ to the following data:

x	120	100	80	60	40	20
y	46.0	33.3	22.8	14.9	9.1	5.5

Answers

1. $y = 97 + 2.1x + 0.0014\,x^2$
2. $a = 11.1$, $b = 0.71$; $y = 11.1 + 0.71\,x$
3. $a = 1090.26$, $b = -0.534$; $r = 1090.26 - 0.534\,t$
4. $a = 1.459$, $b = 0.062$; $y = 1.459\,e^{0.062\,x}$
5. $a = 1.52$, $b = 0.49$; $y = 1.52 + 0.49\,x$
6. $a_0 = 3$, $a_1 = 2$, $a_2 = 0$.

4.9 MOMENTS METHOD

To fit a curve to the given data of n pairs of observations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ let $x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = k$.

First we define the moments of observed values of y as following:

First moment $m_1^1 = k\Sigma y$

Second moment $m_2^1 = k\Sigma xy$

Third moment $m_3^1 = k\Sigma x^2 y$

Fourth moment $m_4^1 = k\Sigma x^3 y$ and so on.

Now we define the moments of the calculated values of y of curve which fits the data as follows:

First moment $m_1 = \int_{x_1-\frac{k}{2}}^{x_n+\frac{k}{2}} y dy$

Second moment $m_2 = \int_{x_1-\frac{k}{2}}^{x_n+\frac{k}{2}} xy\, dy$

Third moment $m_3 = \int_{x_1-\frac{k}{2}}^{x_n+\frac{k}{2}} x^2 y\, dy$

Fourth moment $m_4 = \int_{x_1-\frac{k}{2}}^{x_1+\frac{k}{2}} x^3 y\, dy$

Then by equating $m_i^1\ (i = 1, 2, 3, ...)$ and $m_i\,(i = 1, 2, ...)$, we get the values of unknown constants a, b, c, …, etc. For example, suppose we want to fit $y = a + bx$ then calculate m_1^1, m_2^1, m_1 and m_2 and equate $m_1^1 = m_2$ and $m_1^2 = m_2$, we will find the values of a and b and for other curve as well.

Example 18 For the following data, fit a straight line $y = a + bx$ by the method of moments:

X	0	1	2
Y	1	5	10

Solution To fit the straight line $y = a + bx$

Here $k = 1$

We calculate

$$m_1^1 = k\Sigma y = 1(16) = 16$$

and $$m_2^1 = k\Sigma xy = 1(0+5+20) = 25$$

and $$m_1 = \int_{0-\frac{1}{2}}^{2+\frac{1}{2}} (a+bx)\, dx = \left(ax + \frac{bx^2}{2}\right)_{-0.5}^{2.5}$$

$$= 2.5\,a + \frac{6.25}{2}b + .5a - \frac{.25\,b}{2}$$

$$m_1 = 3a + 3b$$

$$m_2 = \int_{0-\frac{1}{2}}^{2+\frac{1}{2}} x(a+bx)\,\mathrm{d}x = \int_{-.5}^{.25} \frac{a}{2}x^2 + \frac{bx^3}{3}$$

$$= \frac{6.25\,a}{2} + \frac{15.625\,b}{3} - \frac{.25\,a}{2} + \frac{.125\,b}{3}$$

$$m_2 = 3a + 5.25\,b$$

To find values of a and b we put $m_1^1 = m_1$ and $m_2^1 = m_2$

$\therefore$ $3a + 3b = 16$

$3a + 5.25\ b = 25$

Solving these equations

we get $a = 4/3,\ b = 4$

$\therefore$ $y = \frac{4}{3} + 4x$ is the equation of straight line which fits the data.

Example 19 Use the method of moments to fit the straight line $y = a + bx$ to the following data:

X	1.5	2.5	3.5	4.5
y	.10	.20	.30	.40

Solution

$$m_1^1 = k\Sigma y = k.1 = 1,\ m_2^1 = k\Sigma xy = 1.(3.5) = 3.5$$

$$m_1 = \int_{x_1-\frac{1}{2}}^{x_n+\frac{1}{2}} ky\,dx = \int_{1.5-\frac{1}{2}}^{4.5+\frac{1}{2}} 1(a+bx)\,\mathrm{d}x = \int_1^5 (a+bx)\,\mathrm{d}x = \left(ax + \frac{bx^2}{2}\right)_1^5 \Rightarrow m_1 = 4a + 12b$$

and
$$m_2 = \int_1^5 k(xy)\,\mathrm{d}x = \int_1^5 (ax + bx^2)\mathrm{d}x = \left(\frac{ax^2}{2} + \frac{bx^3}{3}\right)_1^5$$

$$= \frac{25a}{2} + \frac{125}{3}b - \frac{9}{2} - \frac{1}{3}b = 12a + 48\,b$$

To find equate

$$m_1^1 = m_1 \text{ and } m_2^1 = m_2$$

$\Rightarrow$ $4a + 12\ b = 1,\ 12\ a + 48\ b = 3.5$

$\Rightarrow$ $a = 0.125,\ b = 0.417$

$\therefore$ $y = 0.125, + 0.0417\ x$ is the required equation.

Example 20 Using the method of moments, fit a parabola to the following data:

x	4	3	2	1	0
y	38	22	10	5	1

Solution We have to fit the parabola $y = a + bx + cx^2$ for which calculate m_1^1, m_2^1, m_3^1 and m_1, m_2, m_3

$$\therefore \quad m_1^1 = k\Sigma y = 1(76) = 76$$

$$m_2^1 = k\Sigma y = 1(243) = 243$$

$$m_3^1 = k(x^2 y) = 1(851) = 851$$

$$m_1 = k\int_{0-\frac{1}{2}}^{4+\frac{1}{2}} (a + bx + cx^2)\,\mathrm{d}x = 5a + 10b + 30.4c$$

$$m_2 = k\int_{-0.5}^{4.5} x(a + bx + cx^2)\,\mathrm{d}x = 10a + 30.4b + 102.5c$$

$$m_3 = k\int_{-0.5}^{4.5} x^2(a + bx + cx^2)\,\mathrm{d}x = 30.4a + 102.5b + 369.1c$$

By putting $m_1^1 = m_1, m_2^1 = m_2, m_3^1 = m_3,$ we get

$a = 0.4$, $b = 3.15$ and $c = 1.4$

$\therefore$ $y = 0.4 + 3.15\,x + 1.4\,x^2$ is the best fit parabola to the given data.

EXERCISE 4.5

1. Fit a straight line $y = a + bx$ to the following data by the method of moments:

x	1	2	3	4
y	16	19	23	26

(Madras 2015)

2. Fit a straight line to the following data, using the method of moments:

x	1	3	5	7	9
y	1.5	2.8	4.0	4.7	6.0

(Madras 2001)

3. By using the method of moments, fit a parabola to the following data:

x	1	2	3	4
y	0.30	0.64	1.32	5.40

(Madras 2000S)

4. By using the method of moments, fit a straight line $y = a + bx$ to the following data:

x	0	1	2	3
y	10	20	30	40

5. Fit a straight line $y = a + bx$ to the question 3 of the Exercise 4.5.
6. Fit a straight line $y = a + bx$ to the solved Example 20.

Answers

1. $a = 13.02$, $b = 3.19$; $y = 13.02 + 3.19x$
2. $a = 1.184$, $b = 0.523$; $y = 1.184 + 0.523x$
3. $a = 0.485$, $b = 0.397x$, $c = 0.124$; $y = 0.485 + 0.397x + 0.124x^2$
4. $a = \dfrac{100}{3}$, $b = 0$; $y = \dfrac{100}{3}$
5. $a = 7$, $b = 12$; $y = 7 + 12x$
6. $a = -2.3$, $b = 8.75$; $y = -2.3 + 8.75x$

SUMMARY

Following topics have been discussed in this chapter:

1. Empirical law

2. Scatter diagram

3. Curve fitting by following four methods:

(i) Graphical method

(ii) Least square method;

(iii) Group average method, and

(iv) Moments method

In least square for fitting of a kth degree polynomial,

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$$

The normal equations are

$$\Sigma y = na_0 + a_1 \Sigma x + a_2 \Sigma x^2 + \cdots + a_k \Sigma x^k$$

$$\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2 + a_2 \Sigma x^3 + \cdots + a_k \Sigma x^{k+1}$$

$$\vdots$$

$$\Sigma x^k y = a_0 \Sigma x^k + a_1 \Sigma x^{k+2} + a_2 \Sigma x^{k+3} + \cdots + a_k \Sigma x^{2k}$$

To find $a_0, a_1, \dots, a_n$ we solve the above equations.

OBJECTIVE TYPE QUESTIONS

1. The curve $y = a_0 x^2 + a_1 \log_{10} x$ reduces to linear of the form $Y = a_0 X + a_1$, if

(a) $Y = \dfrac{y}{\log_{10} x}, X = \dfrac{x^2}{\log_{10} x}$

(b) $Y = y$

(c) $X = \dfrac{x}{\log_{10} x}$

(d) None of these

2. The curve $y = 2x^2 + 3 \log_{10} x$ reduces to linear of the form $Y = 2X + 3$ if

(a) $Y = y$

(b) $X = \dfrac{x^2}{\log_{10} x}$

(c) $Y = \dfrac{y}{\log_{10} x}, X = \dfrac{x^2}{\log_{10} x}$

(d) None of these

3. The curve $y = a_0 x^2 + a_1 x$ converted to linear of the form $Y = a_0 X + a_1$ if

(a) $Y = \dfrac{y}{x}, X = x$ (b) $Y = y, X = x$

(c) $Y = \dfrac{y}{x}, X = x^2$ (d) None of these

4. The curve $xy^z = h$ can be converted to the linear of the form $Y = A + B X$, if

(a) $A = \dfrac{1}{z} \log_{10} p$

(b) $A = \dfrac{1}{z} \log_{10}, B = -\dfrac{1}{z};$

$X = \log_{10} p$ and $Y = \log_{10} y$

(c) $B = -\dfrac{1}{z}$

(d) None of these

5. In least of square method $\sum_{i=1}^{n} E_i^2$ (sum of squares all errors) is

(a) Minimum

(b) Maximum

(c) Anywhere between 1 and 2

(d) None of these

6. $y = ax^b + c$ can be reduced to linear form of type $y = aX + c$ if

(a) $X = x^b$ (b) $X = x$

(c) $X = \dfrac{1}{x^b}$ (d) $X \cdot x^b = 2$

7. $y = ax^b + 3$ can be reduced to linear form of type $y = ax + 3$ if

(a) $X = x^b$ (b) $X = \dfrac{1}{x^b}$

(c) $X = c$ (d) $X \cdot x^b = 3$

8. To fit a straight line $y = a + bx$, the normal equations are

(a) $\Sigma y = a + b\Sigma x$

$\Sigma xy = a\Sigma x + b\Sigma x^2$

(b) $\Sigma y = na + b\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x$

(c) $\Sigma y = na + b\Sigma x$

$\Sigma xy = a\Sigma x + b\Sigma x^2$

(d) None of these

9. To fit a straight line $a + bx = y$ by least square method the number of normal equations are

(a) 2 (b) 3

(c) 4 (d) 1

10. To fit a parabola $y = a + bx + cx^2$ by least square method, the number of normal equations are

(a) 2 (b) 3

(c) 4 (d) 5

11. To fit a polynomial of m degree by least square method, the number of normal equations are

(a) m (b) $m + 1$

(c) $m - 1$ (d) $m + 2$

12. To fit $y = ab^x$ by least square, the number of normal equations are

(a) 1 (b) 2

(c) 3 (d) 4

13. In the method of group averages, the sum of residuals is

(a) 0 (b) 1

(c) 2 (d) 3

14. The method in which moments of observed values of y are respectively equal to the moments of the calculated values of y is known as

(a) Graphical method
(b) Least square method
(c) Group averages method
(d) Moments method

15. If $y = ax^b$ and let $\Sigma x = 50$, $\Sigma y = 80$, $\Sigma xy = 1030$, $\Sigma x^2 = 750$ and $n = 10$, then a and b are respectively equal to

(a) $a = 1.5$, $b = 1.26$
(b) $a = 1.7$, $b = 1.26$
(c) $a = 1.7$, $b = 1.16$
(d) None of these

16. The curve $y = \dfrac{x}{ax+b}$ can be converted to the linear form $Y = a + bX$ if

(a) $X = x, Y = \dfrac{1}{y}$ (b) $X = \dfrac{1}{x}, Y = y$
(c) $X = \dfrac{1}{x}, Y = \dfrac{1}{y}$ (d) none of these

17. The first normal equation to the curve $y = a + bx$ to the following data is as follows:

x	0	1	2	3
y	2	3	4	6

(a) $15 = 4a + 6b$ (b) $6 = 4a + 6b$
(c) $15 = 6a + 4b$ (d) None of these

18. The first normal equation to the curve $y = a + bx + cx^2$ to the following data is

(a) $15 + 4a + 6b + 14c$
(b) $14 = 4a + 6b + 15c$
(c) $27 = 49 + 6b + 14c$
(d) None of these

19. If $y = 2x + 10$ is the best fit for 10 pairs of the values (x_i, y_i) by the method of least squares and $\Sigma y = 200$, then Σx is equal to

(a) 40 (b) 50
(c) 60 (d) 100

20. Given

x	0	1	2
y	0	2	4

then to fit the straight line of best fit $a + bx$ by least square method, the values of a and b are

(a) $a = 0$, $b = 2$ (b) $a = 2$, $b = 0$
(c) $a = 0$, $b = 0$ (d) $a = 2$, $b = 2$

ANSWERS

1. (a)	2. (c)	3. (a)	4. (b)	5. (a)	6. (a)	7. (a)	8. (c)	9. (a)	10. (b)
11. (b)	12. (b)	13. (a)	14. (d)	15. (b)	16. (c)	17. (a)	18. (a)	19. (b)	20. (a)

5 Statistical Methods

5.1 INTRODUCTION

According to Croxton and Cowden, statistics may be defined as the science of collection, presentation, analysis and interpretation of numerical data or in other words, we can say that statistics is a branch of applied mathematics which specializes in data. It has wide applications in diversified spheres of life—social as well as physical—such as biology, psychology, sociology, education, economics, management, engineering, etc. It is hardly possible to enumerate even a single department of human activity where statistics does not creep in. It has rather become indispensable in all phases of human endeavour.

Statistics, with its wide applications in almost every sphere of human activity, is not without limitations. Following are some of its important limitations:

(i) Statistics is not suited to study of qualitative phenomenon.
(ii) Statistics does not study individuals.
(iii) Statistics laws are not exact.
(iv) Statistics is liable to be misused.

According to kings, "Statistics are like clay of which one can make a God or Devil as one pleases". Therefore, the requirement of experience and skill for judicious use of statistical methods restricts their use to experts only and limits the chances of the mass popularity of this useful and important science.

5.2 STEPS OF STATISTICAL METHODS

The most important steps which are taken in statistical methods are collection and classification of data, which will be discussed in the following sections:

5.2.1 Collection of Data

For any statistical investigation, the collection of data is the starting point. Data may be collected for each and every unit of universe or lot under study which is called population or a part of items, or some of the whole items which is called a sample. The study based on each unit of population has more accuracy; but when number of units are more then it is not only difficult, but expensive and time consuming also. Therefore a sample is drawn from the population and conclusions are drawn based on the sample to the population, the most important point which is to be kept in mind while drawing a sample from the population is that it should be unbiased and representative of the entire population.

5.2.2 Classification of Data

Once the data is collected for a single characteristic of a large number of individuals, often it becomes necessary to condense the data as far as possible without losing any information of interest.

Example 1 Let us consider the marks in engineering mathematics obtained by 50 students selected at random from among those appearing in an university examination.

32	47	41	30	20	25	40	45	44	33
26	24	30	20	19	15	49	49	26	36
46	45	49	20	12	16	44	43	30	29
40	41	39	38	34	33	32	20	22	25
30	32	34	36	40	44	45	30	32	39

The above data does not provide any useful information and rather confusing in mind. A better way may be to express the figures in an ascending or descending order of magnitude but it will not reduce the bulk of data. It is represented in the following table.

A bar (|) called tally mark is put against the number when it appears. Having appeared 4 times, the 5th appearance is represented by putting a cross tally (1) on the first four tallies. In the end count the tally mark.

Marks	No. of Students Tally marks	Total	Marks	No. of Students Tally marks	Total
12	|	1	31		
13			32	||||	4
14			33	||	2
15	|	1	34	||	2
16	|	1	35		
17			36	||	2
18			37		
19	|	1	38	|	1
20	||||	4	39	||	2
21			40	|||	3
22	|	1	41	||	2
23			42		
24	|	1	43	|	1
25	||	2	44	|||	3
26	||	2	45	|||	3
27			46	|	1
28			47	|	1
29	|	1	48		
30	$\cancel{\vert\vert\vert\vert}$	5	49	|||	3
			50		
			Total		50

This kind of representation is called frequency distribution. Marks are known as variable x and the number of candidates against the marks is called frequency (f) of the variable. The frequency means how frequently a variable occurs. In the above case the frequency of 32 is 4, i.e., 4 students got 32 marks.

This data is better but still confusing. Another way of representing data is to divide the observed range of variables into a suitable number of class-intervals and write the number of observations in each class. Such a table showing the distribution of the frequencies in the different classes is called frequency table and the way in which the frequencies are distributed over the class intervals is called the grouped frequency distribution. The above data can be represented in the given table.

Marks (x)	No. of Students
10–14	01
15–19	03
20–24	06
25–29	05
30–34	13
35–39	05
40–44	09
45–49	08
Total	50

Although it has great importance in the analysis of statistical data, but there is no hard and fast rule to construct a grouped frequency distribution but still we should keep the following points in our mind:

1. The classes should be clearly defined.
2. The classes should be mutually exclusive and non-overlapping.
3. The classes should be exhaustive, i.e. each observation must fall in one or the other class.
4. The classes should be of equal width.
5. Open ended classes, i.e. less than x or greater than y should be avoided.
6. The number of classes should not be too large or too small. Preferably, it should not be greater than 20 or less than 15.

Magnitude of the Class Interval

First fix the number of classes. After that divide the range (difference between the greatest and smallest observation) by the number of classes and the nearest integer to this value will be the magnitude of the class interval.

Continuous Frequency Distribution: If we deal with continuous variable, it is not possible to arrange the data in the class interval of above type. Let us consider the distribution of age in years. If intervals are 10–14, 15–19, 20–24, then the persons of age between 14 and 15, 19 and 20 are not taken into consideration. Then the class intervals are formed in the following ways:

Age in years
Below 5 years
5 or more but less than 10 years
10 or more but less than 15 years

15 or more but less than 20 years
20 or more but less than 25 years
and so on.
But for all practical purposes, it is written as following:
0–5, 5–10, 10–15, 15–20, 20–25 and so on.
This kind of frequency distribution is called continuous frequency distribution.

5.3 GRAPHICAL REPRESENTATION OF FREQUENCY DISTRIBUTION

It is generally useful to represent a frequency distribution diagrammatically by which data becomes intelligent and gives the general behavior of the data. By the help of diagrammatic representation, we can compare two or more frequency distributions.

There are three main graphs to represent a frequency distribution which are as follows:

(i) Histogram
(ii) Frequency Polygon and
(iii) Cumulative Frequency curve or ogive

We shall discuss these three in details in the upcoming sections.

5.3.1 Histogram

While making a histogram for a given continuous frequency distribution, first of all mark all the intervals on a suitable scale, along the x-axis. On each interval, construct rectangulars with heights proportional to the frequency of corresponding class interval so that the area of the rectangle is proportional to the frequency of the interval. When the class intervals are not of equal width then the heights of rectangles will be proportional to the ratio of the frequencies to the width of the class.

***Note*:** If frequency distribution is not continuous, first make it continuous. Consider the example of marks of 50 students in engineering mathematics, since the frequency distribution is not continuous, to make it continuous we subtract $\frac{1}{2}$ from left and added $\frac{1}{2}$ to the right end of the class interval. By doing this we get the following continuous frequency distribution.

Marks	No. of Students
9.5–14.5	01
14.5–19.5	03
19.5–24.5	06
24.5–29.5	05
29.5–34.5	13
34.5–39.5	05
39.5–44.5	09
44.5–49.5	08
Total	50

Histogram of the above distribution is shown in Fig. 5.1.

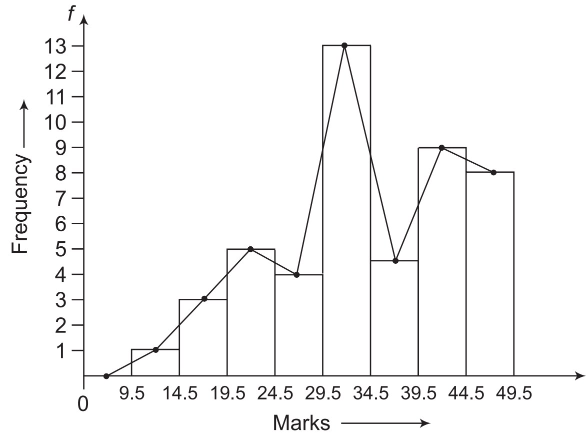


Fig. 5.1

5.3.2 Frequency Polygon

For an ungrouped distribution, frequency polygon is obtained by plotting points of x-axis as the variable and along y-axis the corresponding frequencies and these points are joined by straight lines. For a grouped distribution, we mark mid-value of the class intervals on x-axis and corresponding class frequencies along y-axis and then join them with dotted lines. If class intervals are of equal length then the mid points of the upper side of the rectangles of the histogram are joined by straight lines (Graph is shown along with Histogram in Fig. 5.1).

5.3.3 Cumulative Frequency Curve or Ogive Curve

Sometimes we are interested to know the frequency of the variable which takes more than or less than a given value. Such kind of curves are called ogive curves.

Consider the following example in which x takes the following values of frequencies:

(a)		(b)	
x	**frequency (f)**	x	f
Less than 10	7	Greater than or equal to 5	40
Less than 15	13	Greater than or equal to 10	33
Less than 20	18	Greater than or equal to 15	27
Less than 25	27	Greater than or equal to 20	22
Less than 30	31	Greater than or equal to 25	13
Less than 35	35	Greater than or equal to 30	09
Less than 40	38	Greater than or equal to 35	05
Less than 45	40	Greater than or equal to 40	02
		Greater than or equal to 45	00

The ogive curves of the above data are given in Fig. 5.2.

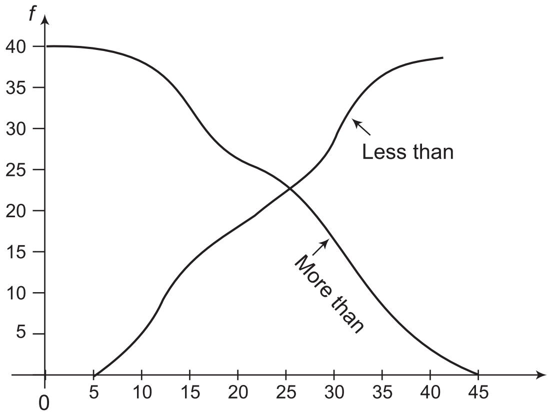


Fig. 5.2

Example 2 Draw the histogram, frequency polygon and the ogive less than and more than from the following distribution.

Class	Frequency	Cumulative Frequency	
		Less than	More than
0–5	5	5	75
5–10	10	15	70
10–15	15	30	60
15–20	20	50	45
20–25	25	75	25

Solution Histogram, polygon are shown in Fig. 5.3, while ogive curve (less than) and more than one shown in Fig. 5.4.

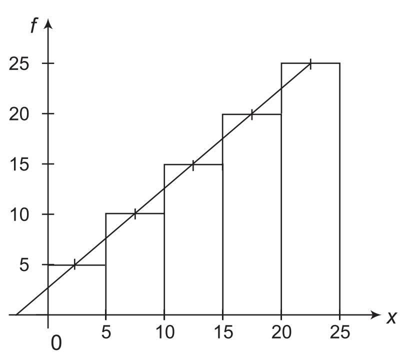


Fig. 5.3 *Histogram and Polygon*

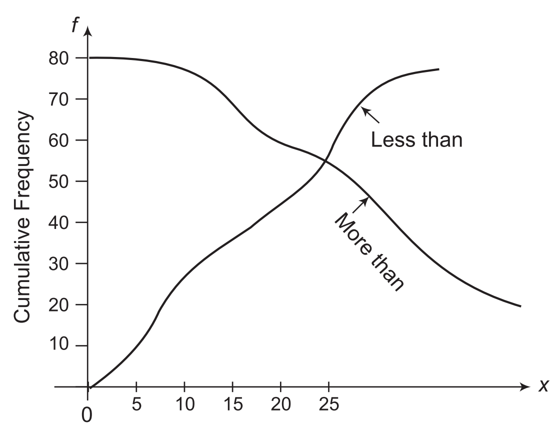


Fig. 5.4

5.4 COMPARISON OF FREQUENCY DISTRIBUTION

The conversion of given raw data into frequency distribution gives vital information which are useful in analyzing and interpreting the data. These informations are based on certain constants, calculated from the frequency distribution and after calculating these constants we can compare various frequency distributions. The main constants which tell about the fundamental characteristics of frequency distribution can be classified as following:

(i) Measures of Central Tendency
(ii) Measures of Dispersion and
(iii) Measures of Skewness

We shall now discuss all these in details in upcoming sections.

5.5 MEASURES OF CENTRAL TENDENCY

Measures of central tendency are also known as averages or measures of location. These are statistical constants which tell us about the significance of the whole data in a single effort. Averages provides the concentration of the whole data in the central part of the distribution.

We have the following five measures of central tendency:

1. Arithmetic mean or simply mean
2. Median
3. Mode
4. Geometric mean and
5. Harmonic mean

Before we discuss the above five averages, first we shall discuss about the requirements of an ideal measure of central tendency which are as follows:

(i) It should be rigidly defined,
(ii) It should be based on all observations,
(iii) It should be easy to calculate and readily comprehensible,
(iv) It should not be much affected by fluctuation of sampling and
(v) It should be suitable for further mathematical treatment.

5.5.1 Arithmetic Mean Or Simply Mean

Let $x_1, x_2, \ldots, x_n$ be n observations, then arithmetic mean of these observations are given by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \sum_{i=1}^{n} \frac{x_i}{n} \tag{1}$$

If $x_1, x_2, \ldots, x_n$ have the frequencies $f_1, f_2, \ldots, f_n$ respectively, then

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} f_i x_i \tag{2}$$

where $\sum_{i=1}^{n} f_i = N$

In case of grouped or continuous frequency distribution $x_i (i = 1, 2, n)$ is taken as mid-point of the class interval sometimes the values of x or (and) f are large, then to calculate $\bar{x}$ by Eq. (2) becomes quite tedious and time consuming, in such case we subtract a number A from each x_i and find the $\overline{x}$ as follows:

Let $\quad x_i - A = d_i$, then $f_i d_i = f_i(x_i - A) = f_i x_i - Af_i$

$$\Rightarrow \quad \sum_{i=1}^{n} f_i d_i = \sum_{i=1}^{n}\left[f_i x_i - A f_i\right] = \sum_{i=1}^{n} f_i x_i - A\,\Sigma f_i$$

$$\Rightarrow \quad \frac{1}{N}\sum_{i=1}^{n} f_i d_i = \frac{1}{N}\sum_{i=1}^{n} f_i x_i - A \qquad (\Sigma f_i = N)$$

$$\Rightarrow \quad \overline{x} = A + \frac{1}{N}\sum_{i=1}^{n} f_i d_i \tag{3}$$

In case of grouped or continuous frequency distribution, if h is the width of class interval, then $\overline{x}$ can be calculated as following:

Let $\quad d_i = \dfrac{x_i - A}{h} \Rightarrow hd_i = x_i - A$

and proceeding, we get

$$\overline{x} = A + \frac{h}{N}\sum_{i=1}^{n} f_i d_i \qquad \left(\because \; N = \sum f_i\right) \tag{4}$$

***Remark*:**

1. Let $x_1, x_2, \ldots, x_n$ be n values with $f_1, f_2, \ldots, f_n$ frequencies then $\sum_{i=1}^{n} f_i(x_i - \overline{x}) = 0$.

   ***Proof*:**

$$\sum_{i=1}^{n} f_i(x_i - \overline{x}) = \sum_{i=1}^{n} f_i x_i - \overline{x}\sum f_i = \sum f_i x_i - \frac{\sum f_i x_i}{\sum f_i}\cdot\sum f_i = 0 \qquad \left(\because \; \overline{x} = \frac{\sum f_i x_i}{\sum f_i}\right)$$

2. Let $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_k$ are the means of k-component series of sizes $n_1, n_2, \ldots, n_k$ respectively, then the mean $\overline{x}$ of the compute series is given by

$$\overline{x} = \frac{n_1\overline{x}_1 + n_2\overline{x}_2 + \cdots + n_k\hat{x}_k}{n_1 + n_2 + \cdots + n_k} = \frac{\sum_{i=1}^{n} n_i\overline{x}_i}{\sum_{i=1}^{n} n_i}$$

Example 3

(a) Find the arithmetic mean of the following data

x	0	2	3	4	5	6	7	8
f	4	5	10	15	16	10	10	5

(b) Calculate the arithmetic mean by the following distribution using Eqs (2), (3) and (4).

Marks	0–10	10–20	20–30	30–40	40–50	50–60
No. of students	12	18	27	20	17	6

Solution

(a) $\overline{x} = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{350}{75} = 4.67$

(b)

Marks	No. of Students (f)	Mid value (x)	fx	$d_i = x_i - 35$	$f_i d_i$
0–10	12	5	60	–30	–360
10–20	18	15	270	–20	–360
20–30	27	25	675	–10	–270
30–40	20	35	700	0	0
40–50	17	45	765	10	170
50–60	06	55	330	20	120
Total	100 = N		2800		–700

By using Eq. (1)

$$\overline{x} = \frac{1}{N}\sum fx = \frac{2800}{100} = 28$$

By using Eq. (2)

$$\overline{x} = A + \frac{1}{N}\sum f_i d_i = 35 + \frac{1}{100}(-700) = 28$$

By using Eq. (3)

$$\overline{x} = A + \frac{h}{N}\sum f_i d_i, \qquad \text{here } h = 10h,\ d_i = \frac{x_i - A}{h}$$

x	$d_i = \dfrac{x_i - A}{h}$	f	$f_i d_i$
5	–3	12	–36
15	–2	18	–36
25	–1	27	–27
35	0	20	0
45	1	17	17
55	2	06	12
Total	–3	100	–70

$$\therefore \qquad \overline{x} = 35 + \frac{10}{100}(-70) = 28$$

It is verified that using any of the three formulas, we get the same answer.

5.5.2 Median

The value of the variable which divides the distribution into two equal parts is called the median. It is the value which has equal number of observations above and below it.

(i) In case of ungrouped data the median is calculated as follows:
Let there are n observations, then arrange these in ascending or descending order. Then median

$$M_d = \text{the value of } x \text{ corresponding } \frac{n+1}{2} \text{obs. if } n \text{ is odd}$$

$$= \text{the average of } \frac{n}{2} \text{ and } \frac{n}{2}+1^{\text{th}} \text{ value if } n \text{ is even.}$$

(ii) In case of discrete distribution: The median is calculated by cumulative frequency. Let $N = \Sigma f_i$, total frequency. Find $\frac{N}{2}$, see the cumulative frequency just greater than $\frac{N}{2}$, the corresponding value of x is the value of median.

(iii) In the case of continuous frequency distribution. The class corresponding to cumulative frequency just greater than the $\frac{N}{2}$ is called median class and

$$M_d = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where l = lower limit of the median class
f = frequency of the median class
h = magnitude or width of the class
c = cumulative frequency of class preceding the median class.

Example 4 Find the median of the following distributions:

(a) 7, 12, 19, 8, 25, 2, 11

(b) 80, 70, 60, 90, 200, 100

(c)

x	1	2	3	4	5	6	7	8
f	5	4	8	2	12	20	21	8

(d) Find the median salary of the following distribution:

Salary in (\$)	20–30	30–40	40–50	50–60	60–70
Number of workers	3	5	20	10	5

Solution

(a) Arranging the data in ascending order

2, 7, 8, 11, 12, 19, 15, $n = 7 \text{ (odd)} \Rightarrow \frac{n+1}{2} = 4$

$\therefore$ Median = Fourth value in ascending order = 11.

(b) Arrange the data in ascending order

60, 70, 80, 90, 100, 200, $n = 6$, $\frac{n}{2} = 3$

the M_d = Average of the 3rd and 4th obs.

$$\therefore \qquad M_d = \frac{80+90}{2} = 85$$

(c)

x	1	2	3	4	5	6	7	8
f	5	4	8	2	12	20	21	8
cf (cumulative frequency)	5	9	17	19	31	51 ↑ Median class	72	80

$$N = \sum f_i = 80 \Rightarrow \frac{N}{2} = 40$$

$\therefore \quad M_d = 6$

(d)

Salary (in $)	No. of Workers (f)	Cumulative frequency
20–30	3	3
30–40	5	8
40–50	20	28 ← Median class
50–60	10	38
60–70	5	43

$$N = 43, \frac{N}{2} = 21.5, l = 40, h = 10, f = 20, c = 8$$

$$\therefore \quad M_d = l + \frac{h}{f}\left(\frac{N}{2} - c\right) = 40 + \frac{10}{20}(21.5 - 8) = 46.25$$

$\therefore$ Median salary = $ 46.25.

5.5.3 Mode

Mode is the value which occurs most frequently and around which the other items of the set cluster density. In the case of discrete frequency distribution, the mode of the value of x corresponding to the maximum frequency. In case of continuous frequency distribution, mode is given by

$$\text{Mode} = M_0 = l + \frac{h(f_1 - f_0)}{(f_1 - f_0) - (f_2 - f_1)} = l + \frac{h(f_1 - f_0)}{2f_1 + f_2 - f_0}$$

where l = lower limit

h = magnitude and f_1 is the frequency of model class

f_0 = frequency of the preceding the model class

f_2 = frequency of the succeeding the model class

Example 5 Find the mode of the following distributions:

(a)

x	1	2	3	4	5	6	7	8
f	2	8	10	15	25	17	3	2

(b)

Class interval	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Frequency	4	8	12	30	20	16	10

Solution

(a) The given distribution is discrete, the maximum frequency is 25 ⇒ Mode $M_0 = 5$

(b) Given distribution is continuous. Maximum frequency

$$= 30 \Rightarrow \text{model class} = 40\text{–}50$$

$$\therefore \quad l = 40, h = 10, f_1 = 30, f_0 = 12, f_2 = 20 \Rightarrow M_0 = l + \frac{h(f_1 - f_0)}{2f_1 - f_0 - f_2}$$

$$\therefore \text{ Mode} = M_0 = 40 + \frac{10(30-20)}{2(30) - 12 - 20} = 46.43$$

Remark If we know the values of two among three mean, median and mode, then the third can be calculated by using the relation.

$$\text{Mode} = 3 \text{ Median} - 2 \text{ Mean}$$

$$\Rightarrow \quad M_0 = 3M_d - 2\overline{x}$$

5.5.4 Geometric Mean

Let $x_1, x_2, \ldots, x_n$ be the n observations then their geometric mean G is the nth not of their product.

i.e.,
$$G = (x_1, x_2, \ldots, x_n)^{\frac{1}{n}}$$

$$\log G = \frac{1}{n}(\log x_1 + \log x_2 + \cdots + \log x_n) = \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

$$G = \text{Anti log}\left[\frac{1}{n}\sum_{i=1}^{n} \log x_i\right]$$

If $x_1, x_2, \ldots, x_n$ have the frequencies $f_1, f_2, \ldots, f_n$ respectively, then geometric mean,

$$G = \left[(x_1)^{f_1}(x_2)^{f_2}\cdots(x_n)^{f_n}\right]^{\frac{1}{N}}, \quad \text{where } N = \sum_{i=1}^{n} f_i$$

$$\Rightarrow \quad \log G = \frac{1}{N}[f_1 \log x_1 + f_2 \log x_2 + \cdots + f_n \log x_n] = \frac{1}{N}\sum_{i=1}^{n} f_i \log x_i$$

$$\Rightarrow \quad G = \text{Anti log}\left[\frac{1}{N}\sum_{i=1}^{n} f_i \log x_i\right]$$

5.5.5 Harmonic Mean

Let $x_1, x_2, \ldots, x_n$ be the n observations, then harmonic mean H is defined as

$$H = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} = \frac{1}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

i.e., Harmonic mean is the reciprocal of arithmetic mean of the reciprocal of the values.

In case of discrete frequency distribution where $x_1, x_2, \ldots, x_n$ take frequencies $f_1, f_2, \ldots, f_n$ respectively, here

$$H = \frac{1}{\sum_{i=1}^{n} \frac{f_i}{x_i}}.$$

In case of grouped frequency distribution or continuous frequency distribution $x_i (i = 1, 2, \ldots, n)$ is the mid value of the ith class.

Example 6 Ram goes to his college by scooter from his home at the speed of 40 km per hour and come back at the speed of 60 km per hour. Find the average speed.

Solution

Let the distance between Ram's home and college be x km. The time taken by Ram to go to college and coming back to home is $\frac{x}{40}$ and $\frac{x}{60}$ hours respectively.

$\therefore$ Total distance covered by Ram $= 2x$ km in $\left(\frac{x}{40} + \frac{x}{60}\right)$ hours

$$\therefore \quad \text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}}$$

$$= \frac{2x}{\frac{x}{40} + \frac{x}{60}} = 48 \text{ km/hour}$$

Remark From the above discussion from sections 5.5.1 to 5.5.5, we come to a conclusion that there is no single average or measure of central tendency which is suitable to all partical purposes. Therefore, the use of a particular average depends on the nature of data and purpose, however, arithmetic mean is an ideal measure of central tendency which has wide applications in statistical theory.

5.5.6 Partition Values

In addition the averages discussed above, there are certain values which divide a series in equal parts.

(i) *Quartiles:* Three points which divides the series into four equal parts are known as quartiles. Three quartiles Q_1, Q_2 and Q_3 are known as first (lower), second (median) and third (upper) quartiles. Q_1 is the first quartile which exceeds 25% data and exceeded by 75%, Q_2 is nothing but median and Q_3 exceeds 75% data and exceeded by 25% of data.

(ii) *Deciles:* Nine points $D_1, D_2, \ldots, D_9$ called first, second, ninth decile which divide the series into 10 equal parts. For example D_1 is the decides which exceeds 10% data and exceeded by 90% data, similarly D_9 is the value which exceeds 90% data and exceeded by 10% data.

(iii) *Percentiles:* 99 points $P_1, P_2, \ldots, P_{99}$ which divides the data into 100 equal points.

Remark For continuous distributions

1. Calculation of Quartiles: $Q_1 = l + \frac{h}{f}\left(\frac{N}{4} - c\right)$

$$Q_3 = l + \frac{h}{f}\left(\frac{3N}{4} - c\right)$$

2. Calculation of Decils: $D_i = l + \frac{h}{f}\left(i \cdot \frac{N}{10} - c\right), i = 1 \text{ to } 9$

3. Calculation of Percentiles: $P_i = l + \frac{h}{f}\left(i \cdot \frac{N}{100} - c\right), i = 1 \text{ to } 99$

   Where l, h, f, N and c have usual meanings as defined in median.
   Similarly, they can be calculated for ungrouped data.

Example 7 For the following data, calculate median, lower (Q_1) and upper (Q_3) quartiled, 6th decile and 30th percentile.

x	0	1	2	3	4	5	6	7	8
f	1	9	26	59	72	52	29	7	1

Solution

x	0	1	2	3	4	5	6	7	8
f	1	9	26	59	72	52	29	7	1
Cumulative frequency cf	1	10	36	95	167	219	248	255	256

For median $\quad M_d = \frac{N}{2} = \frac{256}{2} = 128 \Rightarrow M_d = 4$

For Q_1, $\quad \frac{N}{4} = \frac{256}{4} = 64, Q_1 = 3$

Q_3, $\quad \frac{3N}{4} = \frac{3 \times 256}{4} = 192, Q_3 = 5$

For D_6, $\quad \frac{6N}{10} = \frac{6 \times 256}{10} = 153.6 \Rightarrow D_6 = 4$

and for P_{30}, $\quad \frac{30\,N}{100} = \frac{30 \times 256}{100} = 76.8 \Rightarrow P_{30} = 3.$

Example 8 Find the missing frequencies of the following distribution, whose median is 46.

Variable	Frequency	Variable	Frequency
10–20	12	50–60	?
20–30	30	60–70	25
30–40	?	70–80	18
40–50	65	Total	229

Solution

Let f_1 and f_2 be the frequencies of the class 30–40 and 50–60 respectively. Given $M_d = 46 \Rightarrow$ 40–50 is the median class.

$$\therefore \quad M_d = l + \frac{h}{f}\left(\frac{N}{2} - C\right) = 40\left(\frac{10}{65}\right)\left[\frac{229}{2} - (42 + f_1)\right]$$

$\Rightarrow$ $f_1 = 33.5 \simeq 34$ (as frequency is over in fraction)

Now $229 = 12 + 30 + f_1 + 65 + f_2 + 25 + 18$

$= f_1 + f_2 + 150 \Rightarrow f_1 + f_2 = 79$

Now $f_1 = 34$

$\therefore$ $f_2 = 79.34 = 45$

EXERCISE 5.1

1. Represent the following distribution by (i) a histogram (ii) a frequency polygon and (iii) an ogive.

Scores	Frequency	Scores	Frequency
20–30	01	60–70	20
30–40	01	70–80	22
40–50	03	80–90	12
50–60	14	90–100	02

2. For the question 1, find the mean score of the distribution.
3. Following are the weekly salaries in rupees of 30 employees in a firm:
1400, 1390, 1260, 1140, 1000, 880, 620, 770, 990, 1030, 1080, 1290, 1440, 1480, 1340, 630, 690, 1480, 1320, 1180, 1420, 1160, 1230, 1040, 950, 800, 850, 1060, 1230 and 1330. The firm gave bonus of ₹ 100, 150, 200, 250, 300 and 350 for individuals in the respective salary group: exceeding 600 but not exceeding 750, exceeding 750 but not exceeding 900, exceeding 900 but not exceeding 1050, exceeding 1050 but not exceeding 1200, exceeding 1250 but not exceeding 1350, exceeding 1350 but not exceeding 1500. Find the average bonus paid.
4. Find the mean, median and mode for the following:

Mid value	15	20	25	30	35	40	45	50	55
Frequency	02	22	19	14	03	04	06	01	01

(Kerala 1990)

5. The table below gives the distribution of a sample of 50 people according to weight. Calculate the mean, median and mode.

Weight (kg)	45–50	50–55	55–60	60–65	65–70	70–75	75–80	80–85	85–90	90–95
Frequency	02	03	05	07	09	11	07	02	03	01

6. The mean of marks obtained by a group of 100 in an examination is found to be 49.96. The mean of the marks obtained in the same examination by another group of 200 students is 52.32. What is the mean of two combined group.
7. The mean marks got by 300 students in statistics are 45. The mean of top 100 of them was found to be 70 and the mean by the last 100 was 20. Find the mean of remaining 100 students.

8. Consider the following distribution:

Weekly earning (in \$)	No. of Persons
25–26	25
26–27	70
27–28	210
28–29	275
29–30	430
30–31	550
31–32	340
32–33	130
33–34	90
34–35	55
35–36	25

For the above data:
(a) What is the mean earning?
(b) What is median of earning?
(c) What is most usual (mode) earning?
(d) What are the wage limits for the central 50% of wage earners?
(e) What percentage earned less than 27.5 dollars?
(f) What percentage earned more than 31.5 dollars?
(g) What percentage of persons earned between 28.5 dollars to 30.5 dollars?

9. Following table shows the distribution of 100 families according to their expenditure per week. Number of families corresponds to expenditure groups Rs. (1000–2000) and Rs. (3000–4000) are missing from the table. The median and mode are given to be Rs. 2500 and Rs. 2400. Calculate the missing frequencies.

Expenditure	0–1000	1000–2000	2000–3000	3000–4000	4000–5000
Frequency	14	?	27	?	15

10. Following table gives the frequency distribution of marks in a class of 65 students.

Marks	No. of Students	Marks	No. of Students
0–4	10	16–20	05
4–8	12	20–24	03
8–12	18	24–28	04
12–16	07	28 and above	06
Total			65

Calculate Q_1 and Q_3.

11. The average weight for group of 25 adult was calculated 78.4 kg. It was later found that one weight was 60 kg instead of 96 kg. Calculate the correct weight.

Answers

2. 68.33
3. Average bonus Rs. 245
4. Mean $\overline{x} = 27.9$, Median $M_d = 25.66$, Mode $M_0 = 21.85$
5. Mean $\overline{x} = 68.9$, Median $M_d = 69.44$, Mode $M_0 = 71.6$
6. 51.53
7. 88
8. (a) $\overline{x} = \$ 30.14$, (b) $M_d = \$ 30.16$, (c) $M_0 = \$30.36$,
   (d) 28.89 to 31.26, (e) 9%, (f) 20%, (g) 40%
9. 25 and 24 respectively.
10. $Q_1 = 10.08$, $Q_2 = 17.40$
11. 79.48 kg.

5.6 MEASURES OF DISPERSION

So far we were discussing the averages of measures of central tendency, which tell us about the concentration of data in the central part of the distribution. But if we know the average, it does not give the complete information about the distribution. For example, consider the three series:

(i) 6, 12, 18, 24, 30, (ii) 14, 16, 18, 20, 22 and (iii) 16, 17, 18, 19, 20.

In all the three series, number of observations are 5 and mean $\overline{x} = 18$, therefore it is difficult to find which of the series we have considered. So we see that average is not able to give complete idea of the distribution. There must be some other measure, one of such measure is the measure of dispersion or measure of variation.

By studying the dispersion or variation, we have an idea the line homogeneity of heterogeneity of the distribution. By seeing we can say that series (iii) is more homogeneous (less scattered) than (i) and (ii) series is more homogeneous than (i).

Following are the measures of dispersion (variation):

(i) Range
(ii) Quartile deviation or Semi-Interquartile range
(iii) Mean deviation
(iv) Standard deviation

5.6.1 Range

Let A and B be the greatest and smallest observation in the distribution, then range R is the difference of A and B i.e.,

$$R = A - B$$

For example, for the following distribution 20, 19, 25, 28, 30, the range $R = 30 - 19 = 11$.

5.6.2 Quartile Deviation or Semi-Interquartile Range

Let Q_1 be the first (lower) and Q_3 be the third (upper) quartiles of the distribution, then quartile deviation Q is given by

$$Q = \frac{1}{2}(Q_3 - Q_1).$$

5.6.3 Mean Deviation

Let $x_1, x_2, \ldots, x_n$ be the observations of the distribution with frequencies $f_1, f_2, \ldots, f_n$ respectively then mean deviation about average A(A can be mean, median or mode) is given by

$$\text{Mean deviation} = \frac{1}{N}\sum_{i=1}^{n} f_i \,|\, x_i - A \,|, \quad \text{when} \quad N = \sum_{i=1}^{n} f_i.$$

When $|x_i - A|$ represents the absolute value of the deviation $(x_i - A)$, then the negative sign is ignored.

Remark Mean deviation is least when measured about median.

5.6.4 Standard Deviation

Standard deviation (S.D) denoted by sigma (σ) is calculated as follows:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \overline{x})^2}$$

i.e., if $x_1, x_2, \ldots, x_n$ observations of a frequency distribution has frequencies $f_1, f_2, \ldots, f_n$ respectively and its mean is equal to $\overline{x}$ and $N = \sum_{i=1}^{n} f_i$, then standard deviation of the distribution is defined as above.

Remark If deviations are measured from any other value A instead of $\overline{x}$, it is called the root mean square deviation.

5.6.5 Variance

The square of SD is called variance of the distributed and denoted by σ^2.

$$\therefore \qquad \sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \overline{x})^2$$

5.6.6 Different Formulas to Calculate Variance

(i) We know $\sigma^2 = \frac{1}{N}\sum f_i (x_i - \overline{x})^2$

$$= \frac{1}{N}\sum f_i (x_i^2 - 2x_i\overline{x} + \overline{x}^2)$$

$$= \frac{1}{N}\left[\sum f_i x_i^2 - 2\overline{x}\sum f_i x_i + \overline{x}^2 \sum f_i\right]$$

$$= \frac{1}{N}\left[\sum f_i x_i^2 - 2\overline{x}(N\overline{x}) + N\overline{x}^2\right] \qquad \because \ \overline{x} = \frac{1}{N}\sum f_i x_i$$

$$\sigma^2 = \frac{1}{N}\sum f_i x_i^2 - \overline{x}^2$$

(ii) If the values of x and f are large, then it is difficult to calculate fx and fx^2. In such case we take deviation from any arbitrary point A.

$$\therefore \qquad \sigma^2 = \frac{1}{N}\left[\sum f_i (x_i - \overline{x})^2\right]$$

$$\Rightarrow \qquad \sigma^2 = \frac{1}{N}\left[\sum f_i(x_i - A + A - \bar{x})^2\right]$$

Let $\qquad d_i = x_i - A$

$$= \frac{1}{N}\left[\sum f_i(d_i + A - \bar{x})^2\right]$$

On solving, we get

$$\sigma^2 = \frac{1}{N}\sum f_i d_i^2 - \left(\frac{1}{N}\sum f_i d_i\right)^2$$

(iii) If we take $d_i = \dfrac{x_i - A}{h}$

then $\qquad \sigma^2 = h^2\left[\frac{1}{N}\sum f_i d_i^2 - \frac{1}{N}\sum (f_i d_i)^2\right]$

Remark

1. The positive square root of variance is known as standard deviation.
2. SD has unit but variance does not.

5.7 COEFFICIENT OF VARIATION

Coefficient of variation (C.V.) is given by

$$\text{C.V.} = 100 \times \frac{\sigma}{\bar{x}}$$

i.e., C.V. is the percentage variation in the mean, standard deviation being considered as the total variation in the mean.

The coefficient of variation is calculated to compare the variability of two series. The series whose C.V. is more is said to be more variable than the other and the series whose C.V. is less is lesser variable than the other.

5.8 VARIANCE OF THE COMBINED SERIES

Let $\bar{x}_1, \bar{x}_2$ be the means σ_1 and σ_2 be the standard deviations of two series of sizes n_1 and n_2 respectively, then the variance of the combined series is given by:

$$\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)\right],$$

where $\qquad d_1 = \bar{x}_1 - \bar{x}$ and $d_2 = \bar{x}_2 - \bar{x}$,

$\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$ is the mean of combined series and σ = positive square root of σ^2 is known standard deviation of combined series.

Remark

1. Quartile deviation = $\dfrac{2}{3}$ SD
2. Mean deviation = $\dfrac{4}{5}$ SD

Example 9 Calculate the mean and SD for the following table giving the weight (in kg) distribution of 542 members.

Weight (in kg)	20–30	30–40	40–50	50–60	60–70	70–80	80–90
No. of members	03	61	132	153	140	51	2

Solution

Here $h = 10$, let $A = 55$ then $d_i = \dfrac{x_i - 55}{10}$

Age group	Mid value (x)	Frequency (f)	d_i	$f_i d_i$	$f_i d_i^2$
20–30	25	03	–3	–9	27
30–40	35	61	–2	–122	244
40–50	45	132	–1	–132	132
50–60	55	153	0	0	0
60–70	65	140	1	140	140
70–80	75	51	2	102	204
80–90	85	02	3	06	18
		$N = \sum f_i = 542$		$\sum f_i d_i = -15$	$\sum f_i d_i^2 = 765$

$$\therefore \qquad \sigma^2 = h^2\left[\frac{1}{N}\sum f_i d_i^2 - \left(\frac{1}{N}\sum f_i d_i\right)^2\right]$$

and $$\text{mean } \bar{x} = A + h\frac{\sum f_i d_i}{N} \Rightarrow 55 + 10\left(-\frac{15}{542}\right) = 54.72$$

and $$\sigma^2 = 10^2\left[\frac{1}{542}(765) - \left(\frac{-15}{542}\right)^2\right] = 141.07$$

$$\therefore \qquad \text{SD } \sigma = \sqrt{141.07} \simeq 11.9 \text{ kg.}$$

Example 10 A student obtained the mean and standing deviation of 100 observation as 40 and 5.1 respectively. It was later found that he had wrongly copied down an observation as 50 instead of 40. Find the correct mean and standard deviation.

Solution

$$\therefore \qquad \bar{x} = 40 = \frac{\sum x_i}{n} = \frac{\sum x_i}{100} \Rightarrow \sum x_i = 4000$$

$$\sigma^2 = \frac{1}{n}\sum x_i^2 - (\bar{x})^2$$

$$\sum x_i^2 = n(\sigma^2) + (\bar{x}^2)$$

$$= 100\ [(5.1)^2 + (40)^2] = 162601$$

Let x be the given variable corrected $\Sigma x_i = 4000 - 50 + 40 = 3990$

and corrected $\sum x_i^2 = 162601 - (50)^2 + (40)^2 = 161701$

$\therefore \quad$ Corrected mean $= \dfrac{3990}{100} = 39.9$

and corrected variance $\sigma^2 = \dfrac{161701}{100} - (39.9)^2 = 25$

and corrected standard deviation $\sigma = 5$

Example 11 The first of two samples has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and standard deviation $\sqrt{13.44}$. Find the standard deviation of the second group.

Solution

Given $\quad n_1 = 100,\ \bar{x}_1 = 15$ and $\sigma_1 = 3$

$$n = n_1 + n_2 = 250 \Rightarrow n_2 = 150, \bar{x} = 15.6 \text{ and } \sigma = \sqrt{13.44}$$

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} \Rightarrow 15.6 = \frac{100 \times 15 + 150 \times \bar{x}_2}{250} \Rightarrow \bar{x}_2 = 16$$

$$\therefore \quad d_1 = \bar{x}_1 - \bar{x} = -0.6$$

$$d_2 = \bar{x}_2 - \bar{x} = +0.4$$

$$\therefore \quad \sigma^2 = \frac{1}{n_1 + n_2}\left[n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)\right] = \frac{1}{250}\left[100(3^2 + (-0.6)^2) + 150(\sigma_2^2 + (0.4)^2)\right]$$

$$\therefore \quad \sigma_2^2 = 16 \text{ and } \sigma_2 = 4$$

Example 12 An analysis of monthly salary paid to the employees in two companies X and Y belonging to the same group gives the following result:

	Company X	Company Y
No. of Employees	500	600
Average monthly salary (in \$)	186.00	175.00
Variance of distribution of wages	81	100

(i) Which company X or Y has a large bill?

(ii) In which company X or Y has greater variability in individual's salary?

(iii) Calculate (a) the average monthly salary, (b) the variance of the distribution of salary of all the employees in company X and Y together.

Solution

(i) Company X: $n_1 = 500,\ \bar{x}_1 = \$\,186.00$

Average monthly salary $= \dfrac{\text{Total salary}}{\text{No. of employees}}$

$\therefore \quad 186 = \dfrac{\text{Total salary}}{500} \Rightarrow$ Total salary $= 500 \times 186 = \$93{,}000$

Company Y: $n_2 = 600,\ \bar{x}_2 = \$\,175$

$\therefore$ Total salary $= 600 \times 175 = \$105{,}000$

Therefore company Y has larger salary bill.

(ii) C.V. for company $X = 100\dfrac{\sigma_1}{\bar{x}_1} = \dfrac{100\times\sqrt{81}}{186} = 4.84$

C.V. for company $Y = 100\dfrac{\sigma_2}{\bar{x}_2} = \dfrac{100\times\sqrt{100}}{175} = 5.71$

$\Rightarrow$ Company Y has greater variability in comparison to company X as C.V. of $Y >$ C.V. of X.

(iii) (a) Let $\bar{x}$ be the average monthly salary of all the employees of company X and Y together and σ^2 is the variance.

Then $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2} = \dfrac{500\times186+600\times175}{1100} = \$\,180.00$

(b) $\sigma^2 = \dfrac{1}{n_1+n_2}[n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)],$

where $d_1 = \bar{x}_1 - \bar{x} = 186-180 = 6$

$d_2 = \bar{x}_2 - \bar{x} = 175-180 = -5$

$\therefore \quad \sigma^2 = \dfrac{1}{500+600}\left[500(81+(6)^2)+600(175+(-5)^2)\right] = 121.36$

Example 13 Calculate the mean and standard deviation for the following:

Size of item	6	7	8	9	10	11	12
Frequency	3	6	9	13	8	5	4

(V.T.U. 2001)

Solution

Size of item	Frequency (f)	$d_i = x_i - 9$	f_id_i	$f_id_i^2$
6	3	–3	–9	27
7	6	–2	–12	24
8	9	–1	–9	9
9	13	0	0	0
10	8	1	8	8
11	5	2	10	20
12	4	3	12	35
	$N=\sum f_i = 48$		$\sum f_id_i = 0$	$\sum f_id_i^2 = 124$

$\therefore$ Mean $\bar{x} = A + \dfrac{\sum f_id_i}{\sum f_i} = 9 + \dfrac{0}{48} = 9$

$$\text{SD} = \sqrt{\frac{\sum f_id_i^2}{\sum f_i} - \left(\frac{\sum f_id_i}{\sum f_i}\right)^2} = \sqrt{\frac{124}{48} - \left(\frac{0}{48}\right)^2} = 1.607$$

EXERCISE 5.2

1. Calculate the mean and standard deviation of the following distribution:

Group	Frequency	Group	Frequency
2.5–7.5	12	32.5–37.5	176
7.5–12.5	28	37.5–42.5	120
12.5–17.5	65	42.5–47.5	66
17.5–22.5	121	47.5–52.5	27
22.5–27.5	175	52.5–57.5	9
27.5–32.5	198	57.5–62.56	3

2. Show that the variance of first n positive integers is $\frac{1}{12}(n^2-1)$ **(V.T.U. 2013)**

3. The mean of five items of an observation is 4 and the variance is 5.2. If three of the items are 1, 2 and 6, then find the other two. **(V.T.U. 2002)**

4. Calculate (i) mean deviation about the mean, (ii) mean deviation about the median for the following distribution:

Class	3–4.9	5–6.9	7–8.9	9–10.9	11–12.9	13–14.9	15–16.9
f	5	8	30	82	45	24	6

(Madras, 2002)

5. The following table shows the marbles obtained by 100 candidates in an examination. Calculate the mean, median and standard deviation.

Marks obtained	1–10	11–20	21–30	31–40	41–50	51–60
No. of Candidate	3	16	26	31	16	8

(Osmania 2003S, V.T.U. 2003S)

6. Calculate the mean deviation from median age, for the following age distribution of 100 life insurance policy holders.

Age as on nearest birthday	Number
17–19.5	9
20–25.5	16
26–35.5	12
36–40.5	26
41–50.5	14
51–55.5	12
50–60.5	6
61–70.5	5

7. For a frequency distribution of marks in statistics of 200 students, the mean and standard deviation were found to be 40 and 15 respectively. Later it was discovered that the score 43 was misread 53 in obtaining the frequency distribution. Find the corrected mean and standard deviation corresponding to the corrected distribution.

8. Scores of two golfers for 24 rounds were as follows:
Golfer A: 74, 75, 78, 78, 72, 77, 79, 78, 81, 76, 72, 72, 77, 74, 70, 78, 79, 80, 81, 74, 80, 75, 71, 73
Golfer B: 86, 84, 80, 88, 89, 85, 86, 82, 82, 79, 86, 80, 82, 76, 86, 89, 87, 83, 80, 88, 86, 81, 84, 87
Find which golfer may be considered to be more consistent player.
9. Prove that the mean of first n natural numbers is $\frac{n+1}{2}$.
10. Lives of two models of refrigerators turned in for new models in a recent survey are:

Life (No. of years)	0–2	2–4	4–6	6–8	8–10	10–12
No. of refrigerators Model I	5	16	13	7	5	4
No. of refrigerators Model II	2	7	12	19	9	1

What is the average life of each model of these refrigerators? Which has more uniformity?
11. Particulars relating to the wage distribution of two manufacturing firms are as follows:

	Firm A	Firm B
Mean wage	Rs. 175	Rs. 180
Median wage	Rs. 172	Rs. 170
Model wage	Rs. 167	Rs. 162
Quartiles	$Q_1 = 162, Q_3 = 178$	$Q_1 = 165, Q_3 = 185$
S.D.	13	19

Calculate coefficient of variation for firm A and B.
12. Find the coefficient of variation of a frequency distribution, whose mean is 120 and variance is 25.
13. If the coefficient of variation and standard deviation of a frequency distribution are 5 and 2 respectively, find the mean of the distribution.
14. Obtain the range and semi-interquartile range for the data given as follows:
(i) Greatest and smallest observation are 31.8 and 6.2 respectively
(ii) $Q_4 = 14.95$, $Q_3 = 22.95$
15. Find the mean and standard deviation for the following distribution:

Group	0.5–5.5	5.5–10.5	10.5–15.5	15.5–20.5	20.5–25.5	25.5–30.5	30.5–35.5
Frequency	3	4	68	30	10	6	2

Answers

1. Mean = 30.005; SD = 0.01
3. 4 and 7
4. (i) 1.845; (ii) 1.8175
5. Mean 32, Median 32.6, SD = 12.4
6. Median = 38.25; Mean deviation = 10.605
7. Mean = 39.95; SD = 14.974
8. Golfer B is more consistent.

10. C.V. (Model I) = 54.9%
    C.V. (Model II) = 36.2%
    Model II has more uniformity.
11. C.V. for A = 7.43
    C.V. for B = 10.56
12. 4.17
13. Mean = 40
14. Range = 25.6
    Semi-Interquartile range = 4
15. Mean = 15.68, S.D. = 6.57

5.9 SKEWNESS

Before we discuss skewness of a distribution, we shall discuss moments about origin and moments about mean which have vital applications in the study of statistics and its applications in the various fields.

5.9.1 Moments

k^{th} moment about any point A of a variable x is defined by

$$\mu'_k = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^k; \quad \text{where } N = \sum f_i,\ \mu'_0 = 1$$

$$\therefore \qquad \mu'_1 = \frac{1}{N}\sum f_i(x_i - A)$$

if $A = 0$, then $\mu'_1 = \frac{1}{N}\sum f_i x_i = \overline{x}$ called first moment about origin, which is $\overline{x}$. mean of the distribution.

k^{th} moment about mean of a variable x about mean $\overline{x}$ is defined as:

$$\mu_k = \frac{1}{N}\sum_{i=1}^{k} f_i(x_i - \overline{x})^k, \quad N = \sum f_i$$

In particular,

$$\mu_1 = 0$$

$$\mu_2 = \frac{1}{N}\sum f_i(x_i - \overline{x})^2 = \sigma^2$$

Second moment about mean is called variance of the distribution.

$$\therefore \qquad \mu_2 = \frac{1}{N}\sum f_i\left[x_i^2 - 2x_i\overline{x} + \overline{x}^2\right]$$

$$= \frac{1}{N}\sum f_i x_i^2 - 2\frac{1}{N}\overline{x}\sum f_i x_i + \frac{1}{N}\sum f_i \overline{x}^2$$

$$= \frac{1}{N}\sum f_i x_i^2 - 2\overline{x}\sum \frac{f_i x_i}{N} + \overline{x}^2\frac{1}{N}\sum f_i \qquad \left(\text{but } \sum f_i = N\right)$$

$$= \frac{1}{N}\sum f_i x_i^2 - 2\overline{x}\cdot\overline{x} + \overline{x}^2$$

$$= \frac{1}{N}\sum f_i x_i^2 - 2\bar{x}^2 + \bar{x}^2$$

$$= \frac{1}{N}\sum f_i x_i^2 - \left(\sum \frac{f_i x_i}{N}\right)^2$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

Similarly,

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'\,\mu_1'^2 - 3\mu_1'^4$$

and so on.

5.9.2 Skewness

Skewness mean lack of symmetry. Skewness is studied to have an idea about the shape of the curve which is drawn with the help of the data. A distribution is said to be skewed if

1. Mean ≠ Median ≠ Mode i.e., Mean, Median and Mode fall at different points.
2. Quartiles are not equidistant from median i.e.,

$$M_d - Q_1 \neq Q_3 - M_d$$

3. The curve drawn with the help of given data is not symmetrical but sketched more to one side than to the other.

5.9.3 Measures of Skewness

Following are the measures of skewness (s_k):

(i) $s_k = \bar{x} - M_d$

(ii) $s_k = \bar{x} - M_0$

where $\bar{x}$ = Mean, M_d = Median and M_0 = Mode

(iii) $s_k = (Q_3 - M_d) - (M_d - Q_1)$

(i), (ii) and (iii) are three absolute measures of skewness. For comparing two series we do not use absolute measures of skewness, but we use relative measures which are called coefficients of skewness which are pure numbers and independent of units of measurements.

We have following coefficient of skewness

(a) $s_k = \dfrac{\bar{x} - M_0}{\sigma}$ where $\bar{x}$ = Mean, M_0 = Mode and σ = S.D.

Taking $M_0 = 3M_d - 2\bar{x}$, we get

$$s_k = 3\frac{(\bar{x} - M_d)}{\sigma} \tag{5}$$

Equation (5) is called Karl Pearson coefficient of skewness.

$\dfrac{\bar{x}-M_d}{\sigma}$ always lies between –1 and 1, therefore, s_k always lies between –3 and 3.

(b) Quartile coefficient of skewness: $s_k = \dfrac{Q_3+Q_1-2Q_2}{Q_3-Q_1}$

where Q_1 = first quartile

Q_3 = third quartile

Q_2 = Median

Here $-1 \le s_k \le 1$

(c) Coefficient of skewness based on third moment: $Y_1 = \sqrt{\beta_1}$,

where $\beta_1 = \mu_3^2 \mid \mu_2^2$

$\therefore$ $Y_1 = \sqrt{\beta_1}$ gives the simplest measure of skewness.

Remark If $\bar{x} > M_0$ or $\bar{x} > M_d$. Skewness is positive and if $\bar{x} < M_0$ or $\bar{x} < M_d$, skewness is negative.

5.10 KURTOSIS

By measures of Kurtosis, we check the peakness of a distribution and its measure is given by

$$\beta_2 = \mu_4 \mid \mu_2^2$$

$Y_2 = \beta_2 - 3$ defines the excess of kurtosis. If $\beta_2 = 3$, then curve is called normal curve or mesokurtic curve given in Fig. 5.5 (a), the curve with $\beta_2 > 3$ is called leptokurtic shown in Fig. 5.5 (b) and the curve $\beta_2 < 3$ is called platykurtic and shown in Fig. 5.5(c).

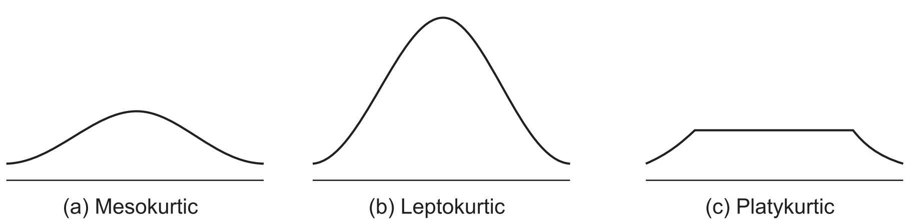


Fig. 5.5

Example 14 Calculate the Karl Pearson coefficient for the following distribution. Also, calculate quartile coefficient of skewness.

Group	1–5	6–10	11–15	16–20	21–25	26–30	31–35
Frequency	3	4	68	30	10	6	2

Solution

Group	Mid value x	f	$f{\cdot}x$	$f{\cdot}x^2$	Cumulative Frequency (C.F.)
0.5–5.5	3	3	9	27	3
5.5–10.5	8	4	32	256	7
10.5–15.5	13	68	884	11492	75
15.5–20.5	18	30	540	9720	105
20.5–25.5	23	10	230	5290	115
25.5–30.5	28	6	168	4704	121
30.5–35.5	33	2	66	4059	123
		$\sum f = N = 123$	$\sum fx = 1929$	$\sum fx^2 = 35548$	

$\therefore$ Karl Pearson coefficient of skewness

$$s_k = \frac{\bar{x} - M_0}{\sigma}$$

$$\therefore \quad \bar{x} = \frac{\sum fx}{N} = \frac{1929}{123} = 15.68$$

$$\sigma^2 = \frac{1}{N}\sum fx^2 - \bar{x}^2 = \frac{1}{23}(35548) - (15.68)^2 = 43.15$$

$$\therefore \quad \sigma = 6.57$$

To calculate M_0:
Model class = 10.5 – 15.5

$$\therefore \quad M_0 = l + \frac{h(f_1 - f_0)}{2f_i - f_0 - f_2} = 10.5 + \frac{5(68.4)}{2(68) - 4 - 30} = 13.64$$

$$\therefore \quad s_k = \frac{15.68 - 13.64}{6.57} = 0.31$$

To calculate quartile coefficient of skewness, we have to calculate Q_1, Q_3 and Q_2 (Median)

$$Q_1 = l + \left(\frac{N}{4} - c\right)\frac{h}{f}, \frac{N}{4} = \frac{123}{4} = 30.75$$

$$= 10.5 + (30.75 - 7)\cdot\frac{5}{68} = 10.5 + 2.44 = 12.94$$

$$= l + \left(\frac{3N}{4} - c\right)\frac{h}{f}, \frac{3N}{4} = 92.25$$

$$= 15.5 + (92.25 - 75)\cdot\frac{5}{30} = 18.38$$

$$Q_2 = M_d = l + \left(\frac{N}{2} - c\right)\cdot\frac{h}{f}, \frac{N}{2} = 61.5$$

$$= 10.5 + (61.5 - 7) \cdot \frac{5}{68} = 14.51$$

$\therefore$ Quartile coefficient of skewness

$$= \frac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}$$

$$= \frac{12.94 + 18.38 - 2(14.51)}{18.38 - 12.94} = \frac{31.32 - 29.02}{5.54} \simeq 0.42.$$

Example 15 The first moments about the working mean 28.5 of a distribution are 0.294, 7.144, 42.409 and 454.98. Calculate the moments about the mean. Also calculate β_1, β_2 and comment upon the skewness and kurtosis of the distribution. **(V.T.U. 2003S)**

Solution The first moments about the arbitrary point 28.5 are given as $\mu_1' = 0.294$, $\mu_2' = 7.144$, $m_3' = 42.409$ and $\mu_4' = 454.98$.

By definition

$$\mu_1' = \frac{1}{N} \sum f_i (x_i - 28.5) = \frac{1}{N} \sum f_i x_i - 28.5$$

$$\Rightarrow \qquad 0.294 = \bar{x} - 28.5 \Rightarrow \bar{x} = 28.794$$

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 7.144 - (0.294)^2 = 7.058$$

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^3$$

$$= 42.409 - 3(7.144)\,(0.294) + 2(0.294)^3$$

$$= 36.151$$

Similarly,

$$\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' \mu_1'^2 - 3\mu_1'^4 = 408.738$$

Therefore,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(36.151)^2}{(7.058)^3} = 3.717$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{408.738}{(7.058)^2} = 8.205$$

and $\gamma_1 = \sqrt{\beta_1} = \sqrt{3.717} = 1.928 \Rightarrow$ considerable skewness of the distribution

$\gamma_2 = \beta_2 - 3 = 8.205.3 = 5.205 > 3 \Rightarrow$ The given distribution is leptokurtic.

EXERCISE 5.3

1. Calculate the first four moments of the following distribution about the mean.

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

Also evaluate β_1 and β_2. **(V.T.U. 2004, Madras 2003)**

2. Find Karl Pearson's coefficient for the following data:

Class	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Frequency	5	9	14	20	25	15	8	4

(V.T.U. 2005)

3. The following table gives the monthly wages of 72 workers in a factory. Compute the standard deviation, quartile deviation, coefficient of variation and skewness. **(V.T.U. 2001)**

Monthly wages (in \$)	No. of workers	Monthly wages (in \$)	No. of workers
12.5–17.5	2	37.5–42.5	4
17.5–22.5	22	42.5–47.5	6
22.5–27.5	19	47.5–52.5	1
27.5–32.5	14	52.5–57.5	1
32.5–37.5	3		

4. Compute the quartile coefficient of skewness for the following distribution.

x	3–7	8–12	13–17	18–22	23–27	28–32	33–37	38–42
f	2	108	580	175	80	32	18	5

(Madras 2002, V.T.U. 2000)

5. Compute skewness and kurtosis, if the first four moments of a frequency distribution $f(x)$ about the value $x = 4$ are respectively 1, 4, 10 and 45. **(Coimbatore 1999)**
6. The first three moments of a distribution about the value 2 of the variable are 1, 16 and –40. Show that the mean = 3, the variance = 15 and $\mu_3 = -83$. **(V.T.U. 2003S)**
7. A frequency distribution gives the following results:
   (i) coefficient of variation = 5,
   (ii) Karl Pearson's coefficient of skewness = 0.5, and
   (iii) $\sigma = 2$. Find the mean and mode of the distribution.
8. Consider the question no. 11 of Exercise 5.2. Calculate Karl Pearson's coefficients of skewness for firm A and B.

Answers

1. $\mu_1 = \mu_3 = 0, \mu_2 = 2, \mu_4 = 11; \beta_1 = 0, \beta_2 = 2.75$
2. $s_k = -0.2064$
3. SD = 8.85, Quartile deviation = 5.25, C.V. = 0.32; skewness = 1.09
4. Quartile coefficient = 0.22; skewness = 1.157

5. Skewness = 0; Kurtosis = 2.9
6. $\beta_1 = 0.493$, $\beta_2 = 0.655$; platykurtic
7. Mean = 40, Mode = 39
8. Coefficient of skewness for firm $A = 0.615$
   Coefficient of skewness for firm $B = 0.947$

5.11 CORRELATION

So far we have discussed the analysis and interpretation of a single variate. But this is not true always, sometimes we come across the situation where two variables are under study. For example, we are interested in the study of height and weights of a group of persons, here a variable is related to height and another variable is related to weight, such kind of distribution is called a bivariate distribution.

In bivariate distribution we may be interested whether two variables have any relationship or not between them or they are correlated or not. If the change in one variable affects a change in the other variable, then the variables are correlated and if the change in one variable does not affect the change in other variable, they are not correlated or unrelated. If they deviate in the same direction i.e., if increase (decrease) in one variable results in a corresponding increase (decrease) in the other variable, then correlation is positive or direct and if the increase (decrease) in one variable results in a corresponding decrease (increase) in the other variable, then the correlation is negative or indirect. If change in one variable does not change in other variable then we say that both the variables are unrelated or they do not have any correlations.

For example: The correlation between

(i) the income and expenditure, and
(ii) the height and weight of a person is positive, while the correlation between
(iii) the volume and pressure of a gas
(iv) price and demand of a commodity is negative.

5.11.1 Karl Pearson Coefficient of Correlation

As a measure of linear relationship between two variables, Karl Pearson gave a formula, known as correlation coefficient.

Let X and Y be two variables, then correlation coefficient between them is denoted by r_{xy} or $r(X, Y)$ is a numerical measure of linear relationship between them and r_{xy} is defined by

$$r_{xy} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

Where Cov (X, Y) is covariance of (X, Y), σ_X and σ_Y are SD of X and Y respectively.

Then $$\text{Cov}(X,Y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[x_i y_i - x_i\bar{y} - \bar{x}y_i + \bar{x}\,\bar{y}\right]$$

$$= \frac{1}{n}\left[\sum_i x_i y_i - \bar{y}\sum_i y_i - \bar{x}\sum_i y_i + \bar{x}\,\bar{y}\sum_i\right]$$

$$= \frac{1}{n}\left[\sum_i x_i y_i - n\overline{x}\,\overline{y} - n\,\overline{x}\,\overline{y} + n\,\overline{x}\,\overline{y}\right] \qquad \because \quad \sum_i x_i = n\overline{x}$$

$$\text{and} \sum_i y_i = x\overline{y}$$

$$= \frac{1}{n}\left[\sum_i x_i y_i - \overline{x}\,\overline{y}\right.$$

Similarly, $$\sigma_X^2 = \frac{1}{n}\left[\sum_i x_i^2 - \overline{x}^2\right]$$

and $$\sigma_Y^2 = \frac{1}{n}\left[\sum_i y_i^2 - \overline{y}^2\right]$$

$\therefore$ $$r_{xy} = \frac{\frac{1}{n}\sum_i (x_i y_i) - \overline{x}\,\overline{y}}{\sqrt{\left(\frac{1}{n}\sum x_i^2 - \overline{x}^2\right)}\sqrt{\frac{1}{n}\sum y_i^2 - \overline{y}^2}}$$

r_{xy} also calculated by the formula

$$r_{xy} = \frac{s_{xy}}{s_{xx}s_{yy}},$$

where $$s_{xy} = \sum_i x_i y_i - \frac{\sum_i x_i \sum_i y_i}{n}$$

$$s_{xx} = \sum_i x_i^2 - \frac{\left(\sum_i x_i\right)^2}{n}$$

$$s_{yy} = \sum_i y_i^2 - \frac{\left(\sum_i y_i\right)^2}{n}$$

$\therefore$ $$r_{xy} = \frac{n\sum_i x_i y_i - \sum_i x_i \sum_i y_i}{\sqrt{n\sum_i x_i^2 - \left(\sum_i x_i\right)^2}\sqrt{n\sum_i y_i^2 - \left(\sum_i y_i\right)^2}}.$$

Figures 5.6(a), 5.6(b), 5.6(c), 5.6(d) and 5.6(e) present scattered data showing $r > 0$, $r < 0$, $r = 0$, $r = 1$, and $r = -1$.

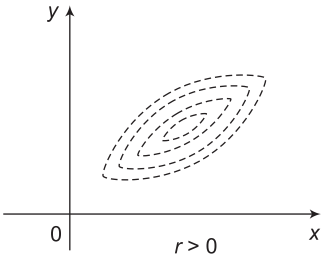


Fig. 5.6 (a)

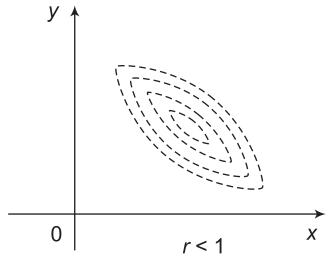


Fig. 5.6 (b)

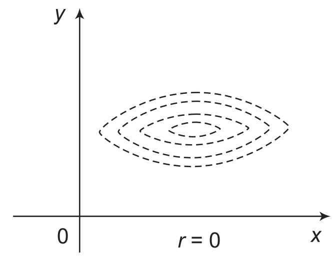


Fig. 5.6 (c)

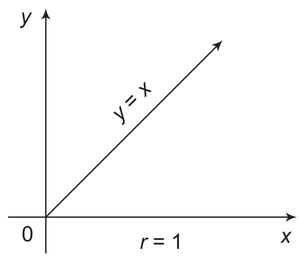


Fig. 5.6 (d)

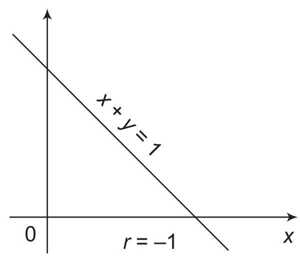


Fig. 5.6 (e)

Limits of correlation coefficient: Correlation coefficient lies between –1 and 1, i.e. $-1 \le r_{xy} \le 1$.

If $r = +1$, then correlation is perfect and positive, and

if $r = -1$, then correlation is perfect and negative. If $r = 0$, then there is no correlation between two variables and X, Y are independent or unrelated.

Example 16 Calculate the correlation coefficient for the following heights in inches of the fathers (x) and their sons (y).

x	65	66	67	67	68	69	70	72
y	67	68	65	68	72	72	69	71

Solution

$n = 8$									Total
x	65	66	67	67	68	69	70	72	544
y	67	68	65	68	72	72	69	71	552
x^2	4225	4356	4489	4489	4624	4761	4900	5184	37028
y^2	4489	4624	4225	4624	5184	5184	4761	5041	38132
xy	4355	4484	5355	4556	4896	4968	4830	5112	37560

$$\bar{x} = \frac{1}{n}\sum_i x_i = \frac{1}{8}(544) = 68, \ \bar{y} = \frac{1}{n}\sum y_i = \frac{1}{8}(552) = 69$$

$$\therefore \qquad r_{xy} = \frac{\frac{1}{n}\sum_i x_i y_i - \bar{x}\,\bar{y}}{\sqrt{\frac{1}{n}\sum x_i^2 - (\bar{x})^2}\ \sqrt{\frac{1}{n}\sum y_i^2 - (\bar{y})^2}}$$

$$= \frac{\frac{37560}{8} - (68)(69)}{\sqrt{\frac{1}{8}(37028) - (68)^2}\sqrt{\frac{1}{8}(38132) - (69)^2}} = 0.603$$

$\Rightarrow \qquad r_{xy} = 0.603$

Example 17 Using formula $r_{xy} = \frac{S_{xy}}{\sqrt{S_{xx}}\sqrt{S_{yy}}}$, find the correlation coefficient of the following data:

Air velocity (x)	20	60	100	140	180	220	260	300	340	380
Evaporation coefficient (y)	0.18	0.37	0.35	0.78	0.56	0.75	1.18	1.30	1.17	1.65

Solution

For the above data: $n = 10$

$$\sum_{i=1}^{10} x_i = 2000, \sum_{i=1}^{10} y_i = 8.35, \sum_{i=1}^{10} x_i y_i = 2175.4$$

$$\sum_{i=1}^{10} x_i^2 = 532000, \sum_{i=1}^{n} y_i^2 = 9.1097$$

$$\therefore \qquad s_{xx} = \sum_i x_i^2 - \frac{1}{n}\left(\sum_i x_i\right)^2 = 532000 - \frac{1}{10}(2000)^2 = 132000$$

$$s_{yy} = \sum_i y_i^2 - \frac{1}{n}\left(\sum_i y_i\right)^2 = 9.1097 - \frac{1}{10}(8.35)^2 = 2.13745$$

$$s_{xy} = \sum_i x_i y_i - \frac{1}{n}\left[\left(\sum x_i\right)\sum y_i\right] = 2175.4 - \frac{1}{10}(2000)(8.35) = 505.4$$

$$\therefore \qquad r_{xy} = \frac{s_{xy}}{\sqrt{s_{xx}}\sqrt{s_{yy}}} = \frac{505.4}{\sqrt{132000}\sqrt{2.13745}} = 0.9515$$

5.12 RANK CORRELATION

Let a group of n individuals is arranged in order of merit of proficiency having two characteristics A and B. The ranks of two characteristics A and B are generally different. For example, if we are interested to find relation between intelligence and beauty, it is not always true that an intelligent person is always beautiful or vice-versa.

Let $(x_i\ y_i)$, $i = 1, 2_i, \ldots, n$ be the ranks of i^{th} individual in two characteristics A and B respectively. The correlation coefficient between the ranks of $x_1's$ and $y_i's$ is called the rank correlation coefficient between characteristics A and B.

Case I When no two individuals are bracketed i.e., no two ranks are same, i.e. each variable takes values $1, 2_i, \ldots, n$ then

$$\bar{x} = \frac{1}{n}(1 + 2 + \cdots + n) = \bar{y} = \frac{n+1}{2}$$

Let σ_x^2 and σ_y^2 are the variances of x and y respectively, then

$$\sigma_x^2 = \frac{1}{n}\sum_i x_i^2 - (\overline{x})^2$$

$$= \frac{1}{n}[1^2 + 2^2 + \cdots + n^2] - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12} = \sigma_y^2$$

Let $d_i = x_i - y_i \Rightarrow (x_i - \overline{x}) - (y_i - \overline{y}),$ squaring and summing over i from 1 to n and dividing by n we get

$$\frac{1}{n}\sum_i d_i^2 = \frac{1}{n}\sum_i \left[(x_i - \overline{x})^2 + (y_i - \overline{y})^2 - 2(x_i - \overline{x})\,(y_i - \overline{y})\right]$$

$$= \frac{1}{n}\sum_i (x_i - \overline{x})^2 + \frac{1}{n}\sum_i (y_i - \overline{y})^2 - 2\frac{1}{n}\sum_i (x_i - \overline{x})\,(y_i - \overline{y})$$

$$\frac{1}{n}\sum_i d_i^2 = \sigma_x^2 + \sigma_y^2 - 2\,\text{cov}\,(x_1 y)$$

$$\left[\begin{array}{l} \because \quad \dfrac{1}{n}\sum_i (x_i - \overline{x})^2 = \sigma_x^2, \dfrac{1}{n}\sum_i (y_i - \overline{y})^2 = \sigma_0^2 \\ \text{and } \dfrac{1}{n}\sum_i (x_i - \overline{x})\,(y_i - \overline{y}) = \text{cov}\,(x, y) \end{array}\right].$$

We also know that

$$r = \frac{\text{cov}\,(x, y)}{\sigma_x \sigma_y} \Rightarrow \text{cov}\,(x, y) = r\,\sigma_x\,\sigma_y$$

$\therefore \quad \frac{1}{n}\sum_i d_i^2 = \sigma_x^2 + \sigma_y^2 - 2r\,\sigma_x\,\sigma_y$ where r is the rank correlation coefficient between A and B.

$$\therefore \quad \frac{1}{n}\sum d_i^2 = 2\sigma_x^2 - 2r\,\sigma_x^2 \qquad \left[\because \quad \sigma_x^2 = \sigma_y^2\right]$$

$$\Rightarrow \quad r = 1 - 6\sum_{i=1}^{n} \frac{d_i^2}{2n\sigma_x^2}$$

$$r = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} \tag{6}$$

Equation (6) is called Spearman's formula for the rank correlation coefficient. Rank correlation always lies between –1 and 1 i.e., $-1 \le r \le 1$.

Note: $\sum_{i=1}^{n} d_i$ is always zero.

Case (ii) Repeated Ranks: If two or more individuals are bracketed equally in any classification with respect to characteristic A and B i.e., ranks are repeated. In such case we cannot use Spearman's rank correlation coefficient because $\bar{x} \neq \bar{y}$.

In this case common ranks are given to the repeated item. This common rank is the average of the normal ranks of these items would have taken if they were different, and we add $\frac{m(m^2-1)}{12}$ to $\sum d^2$ in the formula.

$$\therefore \qquad r = 1 - \frac{6\left[\sum d^2 + \frac{m(m^2-1)}{12}\right]}{n(n^2-1)}$$

where m is the number of times and item is repeated. $\frac{m(m^2-1)}{12}$ is called correction factor and is to be added for each repeated item.

Example 18 The ranks of same students in statistics and physics are as follows: The numbers within brackets denote the ranks of the students in statistics and physics.

(1, 1), (2, 10), (3, 3), (4, 4), (5, 5) (6, 7), (7, 12), (8, 6), (9, 8), (10, 11), (11, 15), (12, 9), (13, 14), (14, 12), (15, 16), (16, 15).

Calculate the Spearman's rank correlation coefficient for proficiencies of this group in statistics and physics.

Rank in statistics marks (x)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Ranks in Physics marks (y)	1	10	3	4	5	7	12	6	8	11	15	9	14	12	16	15
$d_i = x_i - y_i$	0	–8	0	0	0	–1	–5	–2	1	–1	–4	3	–1	2	–1	1
d_i^2	0	64	0	0	0	1	25	4	1	1	16	9	1	4	1	1

Solution

$$\sum d_i^2 = 136, n = 16$$

$\therefore$ Spearman's rank correlation coefficient $r = 1 - \frac{6\sum d_i^2}{n(n^2-1)} = 1 - \frac{6(136)}{16(16^2-1)} = 0.80$

Example 19 Obtain the rank correlation coefficient for the following data:

X	68	64	75	50	64	80	75	40	55	64
Y	62	58	68	45	81	60	68	48	50	70

Solution

X	Y	Rank of X (x)	Rank of Y (y)	$d_i = x_i - y_i$	d_i^2
68	62	4	5	–1	1
64	58	6	7	–1	1
75	68	2.5	3.5	–1	1
50	45	9	10	–1	1

64	81	6	1	5	25
80	60	1	6	–5	25
75	68	2.5	3.5	–1	1
40	48	10	9	1	1
55	50	8	8	0	0
64	70	6	2	4	16
			Total	$\sum d_i = 0$	$\sum d_i^2 = 72$

In X series 75 is repeated twice, 64 repeated thrice and in Y-series 68 also repeated twice so we add $\frac{m(m^2-1)}{12}$ to Σd^2 each times.

For *X*-series: that correction factor is

$$= \sum \frac{m(m^2-1)}{12} = \frac{2(2^2-1)}{12} + \frac{3(3^2-1)}{12} = \frac{5}{2}$$

For *Y*-series: Correction factor is

$$\frac{m(m^2-1)}{n} = \frac{2(2^2-1)}{12} = \frac{1}{2}$$

Total correction factor to be added to $\sum d^2 = \frac{5}{2} + \frac{1}{2} = 3$

∴ Rank correlation coefficient

$$r = 1 - 6\frac{\left[\sum d^2 + 5/2 + 1/2\right]}{n(n^2-1)}$$

$$= 1 - 6\left[\frac{72+3}{10(10^2-1)}\right] = 0.545$$

5.13 REGRESSION

The term regression means stepping back towards the average. But now this term in statistics is only a convenient term without having reference to biometry.

5.13.1 Lines of Regression

If the variables in a bi-variate distribution are related, then the points in scatter diagram cluster around some curve and this curve is called curve of regression. If the curve is a straight line, it is called the line of regression and we say that two variables have linear regression, otherwise it is curve linear regression between them.

The line of regression is the line which gives the best estimate to the value of one of variable for any specific value of the other variable. Thus, the line of regression is line of 'best fit' and obtained by 'principle of least squares'.

Let (x_i, y_i), $i = 1, 2_i, ..., n$ represents a bi-variate distribution in which x is independent and y is dependent variable.

Let the line of y on x is $y = a + bx$ where a and b are constants. To find the value of a and b, we use the normal equation as follows:

$$\sum_i y_i = \sum_i a + b \sum_i x_i$$

$$\Rightarrow \qquad \sum_i y_i = na + b \sum_i x_i \tag{7}$$

$$\sum_i x_i y_i = a \sum_i x_i + b \sum_i x_i^2 \tag{8}$$

dividing Eq. (7) by n, we get

$$\overline{y} = a + b\overline{x} \tag{9}$$

Thus the line of regression of y on x passing through the point $(\overline{x}, \overline{y})$ is given by

$$y - \overline{y} = b(x - \overline{x}) \tag{10}$$

The value of b can be found in the following way:

Let $\qquad \operatorname{cov}(x, y) = \mu_{11} = \dfrac{1}{n} \sum_i x_i y_i - \overline{x}\,\overline{y}$

$$\frac{1}{n} \sum_i x_i y_i = \mu_{11} + \overline{x}\,\overline{y} \tag{11}$$

and variance of $x, \sigma_x^2 = \dfrac{1}{n} \sum_i x_i^2 - (\overline{x})^2 \Rightarrow \dfrac{1}{n} \sum x_i^2 = \sigma_x^2 + \overline{x}^2$ (12)

Dividing Eq. (8) by n we get

$$\sum_i \frac{x_i y_i}{n} = a_i \frac{\sum x_i}{n} + b \frac{\sum x_i^2}{n}$$

$$\mu_{11} + \overline{x}\,\overline{y} = a\overline{x} + b(\sigma_x^2 + \overline{x}^2) \tag{13}$$

Multiply Eq. (9) by $\overline{x}$, we get

$$\overline{x}\,\overline{y} = a\overline{x} + b\overline{x}^2 \tag{14}$$

Substituting Eq. (14) from Eq. (13) we get,

$$\mu_{11} = b\sigma_x^2 \Rightarrow b = \frac{\mu_{11}}{\sigma_x^2} \tag{15}$$

'b' is called the slope of the regression line of y on x

$\therefore$ Regression line of y on x is

$$y - \overline{y} = \frac{\mu_{11}}{\sigma_x^2}(x - \overline{x}) \tag{16}$$

If r be the correlation coefficient between x and y, then

$$r = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y} = \frac{\mu_{11}}{\sigma_x \sigma_y} \Rightarrow \mu_{11} = r\sigma_x \sigma_y$$

then Eq. (16) becomes

$$y-\overline{y}=\frac{r\sigma_x\sigma_y}{\sigma_x^2}(x-\overline{x})$$

or
$$y-\overline{y}=\frac{r\sigma_y}{\sigma_x}(x-\overline{x}) \tag{17}$$

Equation (17) is called regression line of y on x.

Similarly, the regression line of x on y is

$$x-\overline{x}=r\frac{\sigma_x}{\sigma_y}(y-\overline{y}) \tag{18}$$

How to find values of a, b: Let $y = a + bx$ be the line of regression of y on x, then

$$a=\frac{\sum_i y_i - b\sum_i x_i}{n}$$

and
$$b=\frac{n(\sum_i x_i y_i)-\left(\sum_i x_i\right)\left(\sum_i y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right)-\left(\sum_{i=1}^{n} x_i\right)^2}$$

These values are obtained by principle of least squares which gives normal equations (7) and (8) and by solving Eqs (7) and (8), we get the values of a and b respectively.

$\therefore$ Line of best fit $y = a + bx$

***Note*:** If $r=\pm 1$, then

$$y-\overline{y}=\pm(x-\overline{x})$$

$\Rightarrow$ that two lines of regression coincide and we get only one line of regression.

Regression Coefficient: 'b' the slope of the line of regression of y on x is called the coefficient of regression of y on x. It can be written as

$$b_{yx} = \text{regression coefficient of } y \text{ on } x = \frac{\operatorname{cov}(x,y)}{\sigma_x^2}=\frac{r\sigma_y}{\sigma_x}$$

Similarly, b_{xy} = regression coefficient of x on $y = \dfrac{\operatorname{cov}(x,y)}{\sigma_y^2}=r\dfrac{\sigma_x}{\sigma_y}$

Example 20 Show that correlation coefficient is geometric mean of regression coefficients.

Solution

Proof Let $b_{yx}=r\dfrac{\sigma_y}{\sigma_x}$ and $b_{xy}=r\dfrac{\sigma_x}{\sigma_y}$ are the regression coefficients of y on x and x on y respectively.

Then geometric mean of b_{yx} and $b_{xy}=\sqrt{b_{yx}\cdot b_{xy}}$

$$= \sqrt{\frac{r\sigma_y}{\sigma_x} \cdot r\frac{\sigma_x}{\sigma_y}} = \sqrt{r^2}$$

$$\Rightarrow \qquad r = \sqrt{b_{yx} \cdot b_{xy}}$$

Example 21 Establish the relation

$$\frac{1}{2}(b_{yx} + b_{xy}) \geq r$$

i.e., arithmetic mean of regression coefficients is greater than or equal to correlation coefficient.

Solution

Let b_{yx} and b_{xy} are the coefficient of regression of y on x and x on y respectively.

$$\therefore \quad \text{A.M of } b_{yx} \text{ and } b_{xy} = \frac{1}{2}(b_{yx} + b_{xy})$$

$$= \frac{1}{2}\left(r\frac{\sigma_y}{\sigma_x} + r\frac{\sigma_x}{\sigma_y}\right)$$

To prove $\frac{1}{2}(b_{yx} + b_{xy}) \geq r$

$$\Rightarrow \qquad \frac{1}{2}\left(\frac{r\sigma_y}{\sigma_x} + \frac{r\sigma_x}{\sigma_x}\right) \geq r$$

$$\frac{\sigma_y}{\sigma_x} + \frac{\sigma_x}{\sigma_y} \geq 2$$

$$= \sigma_y^2 + \sigma_x^2 - 2\sigma_x\sigma_y \geq 0$$

$$= (\sigma_y - \sigma_x)^2 \geq 0$$

which is always true because $(\sigma_y - \sigma_x)^2$ is the square of real quantity

$$\Rightarrow \qquad \frac{1}{2}(b_{yx} + b_{xy}) \geq r$$

Example 22 If θ be the angle between lines of regression, then show that

$$\tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x + \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

(U.P.T.U. 2007, V.T.U. 2007)

Explain the significance when $r = 0$ and $r = \pm 1$.

Solution Let

$y - \overline{y} = r\frac{\sigma_y}{\sigma_x}(x - \overline{x})$ and $x - \overline{x} = r\frac{\sigma_x}{\sigma_y}(y - \overline{y})$ be the lines of regression of y on x and x on y respectively.

Slopes of these two lines one given by

$$r\frac{\sigma_y}{\sigma_x} \text{ and } r\frac{\sigma_x}{\sigma_y}$$

If θ is the angle between two lines then

$$\tan\theta = \frac{\frac{r\sigma_y}{\sigma_x} - \frac{\sigma_y}{r\sigma_x}}{1 + \frac{r\sigma_y}{\sigma_x}\cdot\frac{\sigma_y}{r\sigma_x}} = \frac{\frac{r\sigma_y}{\sigma_x} - \frac{\sigma_y}{r\sigma_x}}{1 + \frac{\sigma_y^2}{\sigma_x^2}}$$

$$\tan\theta = \frac{(r^2-1)(\sigma_x\sigma_y)}{r(\sigma_x^2+\sigma_y^2)} = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right) \qquad (\because \;\; r^2 \le 1)$$

(i) If $r = 0$

$$\tan\theta = \infty, \Rightarrow \tan^{-1}(\infty) \Rightarrow \theta = \pi/2$$

$\Rightarrow$ If two variables are unrelated then lines regression are perpendicular to each other.

$\Rightarrow$ If $r = \pm 1$

then tan $\theta = 0 \Rightarrow \theta = 0$ or π

If two variables are perfectly correlated then lines of regression either coincide or parallel to each other as both lines pass through $(\overline{x}, \overline{y})$, so they can not be parallel to each other. $\Rightarrow$ They coincide with each other $\Rightarrow$ we get only one line.

Example 23 Various dozes of a poisonous substances were given to groups of 25 mice and the following results were observed:

Dose (mg) (x)	4	6	8	10	12	14	16
No. of deaths (y)	1	3	6	8	14	16	20

(a) Find the equation of regression line of y on x, which fits the date best.

(b) Estimate the number of deaths in a group of 25 mice who receive a doze of 7 mg of this poison.

Solution

(a) $\sum x_i = 70, \sum y_i = 68, \sum x_i^2 = 812, \sum x_i y_i = 862, n = 7$

$\therefore$ line of regression of best fit of y on x is

$$y = a + bx$$

where $$a = \frac{\sum y_i - b\sum x_i}{n}$$

and $$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{7(862) - (70)\,(68)}{7(812) - (70)^2} = 1.625$$

$$a = \frac{68 - (1.625)(70)}{7} = -6.536$$

$\therefore$ $y = -6.536 + 1.625\,x$

(b) when $x = 7$

$$y = -6.536 + (1.625) \times 7 = 4.839 \simeq 5 \text{ mice.}$$

Example 24 Find the most likely price in city C_2 corresponding to the price of Rs. 70 at city C_1 from the following data:

Average price	city C_1	city C_2
	65	67
Standard deviation	2.5	3.5

Correlation coefficient between the prices of commodities in two cities is 0.8.

Solution Let the prices in city C_1 and city C_2 be denoted by x and y respectively. Then, given

$$\bar{x} = 65, \bar{y} = 67, \sigma_x = 2.5, \sigma_y = 3.5, r = r_{xy} = 0.8$$

We want to calculate value of y when $x = 70$

Line of regression of y on x is

$$y - \bar{y} = r\frac{\sigma_x}{\sigma_y}(x - \bar{x})$$

$$\Rightarrow \quad y - 67 = 0.8\left(\frac{3.5}{2.5}\right)(x - 65)$$

$$\Rightarrow \quad y = 67 + 1.12\,(x - 65)$$

$$\Rightarrow \quad y = 1.12\,x - 5.8$$

when $x = 70$

$$y = 1.12\,(70) - 5.8 = 72.6$$

Example 25 While calculating correlation coefficient between two variable x and y from 25 pairs of observations, the following results were obtained:

$$n = 25, \Sigma x = 125, \Sigma x^2 = 650, \Sigma y = 100, \Sigma y^2 = 460, \Sigma xy = 508,$$

It was however, later discovered at the time of checking that the hard disk copied down two pairs as

$\begin{array}{c|c} x & y \\ \hline 6 & 14 \\ \hline 8 & 6 \end{array}$ while the correct values were $\begin{array}{c|c} x & y \\ \hline 8 & 12 \\ \hline 6 & 8 \end{array}$. Find the correct value of correlation coefficient.

Solution

$$\text{Corrected } \Sigma x = 125 - 6 - 8 + 5 + 6 = 125$$

$$\text{Corrected } \Sigma y = 100 - 14 - 6 + 12 + 8 = 100$$

$$\text{Corrected } \Sigma x^2 = 650 - 6^2 - 8^2 + 8^2 + 6^2 = 650$$

Corrected $\Sigma y^2 = 100 - 14^2 - 6^2 + 12^2 + 8^2 = 436$

Corrected $\Sigma xy = 508 - (6)(14) - (8)(6) + (8)(12) + (6)(8) = 520$

$$\therefore \quad \text{Corrected } r_{xy} = \frac{n\sum xy - \sum x \sum y}{\sqrt{n\sum x^2 - \left(\sum x\right)^2}\ \sqrt{n\sum y^2 - \left(\sum y\right)^2}}$$

$$= \frac{25(520) - 125 \times 100}{\sqrt{25(650) - (100)^2}\ \sqrt{25(436) - (100)^2}} = \frac{2}{3} = 0.67$$

Line of regression of y on x is

$$(y - \bar{y}) = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$\therefore \qquad \bar{y} = \frac{\sum y}{x} = \frac{100}{25} = 4,\ \bar{x} = \frac{125}{25} = 5$$

$$\sigma_x = \sqrt{\frac{1}{n}\sum x^2 - (\bar{x})^2} = \sqrt{\frac{1}{25}(650) - (25)^2} = \sqrt{26 - 25} = 1$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum y^2 - \left(\sum \bar{y}\right)^2} = \sqrt{\frac{1}{25}(436) - (4)^2} = \sqrt{1.44} = 1.2.$$

$\therefore$ Line of regression of y on x is

$$y - 4 = \frac{2}{3}\left(\frac{1.2}{1}\right)(\bar{x} - 5)$$

$$\Rightarrow \qquad y - 4 = 0.8\,(x - 5)$$

$$\Rightarrow \qquad y = 4 + 0.8x - 4 \Rightarrow \boxed{y = 0.8x}.$$

5.13.2 Multiple Regression

In 5.13.1 we have studied the simple linear regression model. This model gives the idea, how one variable gives the best estimate for any specific value of other variable. This idea is extended in this section by considering two different models: the polynomial model, in which the single independent variable can appear to a power greater than one, and the multiple linear model, in which more than one distinct independent variables can be used. The techniques employed in each case are similar and conceptually easy.

5.13.3 Least-Square Procedures for Model Fitting

(i) Let $X_1, X_2, \ldots, X_p$ be random variables and $x_1, x_2, \ldots, x_p$ be their numerical values and $y/x_1, x_2, \ldots, x_p$ be the numerical value of dependent variable Y corresponding to the independent variable's numerical values $x_1, x_2, \ldots, x_p$. Then the general linear model is defined as:

$$y = a_0 + a_1x_1 + a_2x_2 + \cdots + a_px_p \tag{19}$$

Equation (19) is a straight forward generalization of simple linear regression.

(ii) The polynomial model of regression is defined as:

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p \tag{20}$$

where Eq. (20) is a polynomial regression model of degree p for the variables x and y.

Normal equations to solve polynomial model of regression are as follows:

Let (x_i, y_i), $i = 1, 2, \ldots, n$ represents a bivariate distribution in which x is independent and y is dependent variable. Multiple regression linear model between x and y is defined as Eq. (20), then to find the values of $a_0, a_1, a_2, \ldots, a_p$, we use the following normal equations:

$$\Rightarrow \left.\begin{aligned}
\sum_{i=1}^{n} y_i &= \sum_{i=1}^{n} a_0 + a_1 \sum_{i} x_i + a_2 \sum_{i=1}^{n} x_i^2 + \cdots + a_p \sum x_i^p \\
\sum_{i=1}^{n} y_i &= n a_0 + a_1 \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_i^2 + \cdots + a_p \sum_{i=1}^{n} x_i^p \\
\sum_{i=1}^{n} x_i y_i &= a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + a_2 \sum_{i=1}^{n} x_i^3 + \cdots + a_p \sum_{i=1}^{n} x_i^{p+1} \\
&\vdots \\
\sum_{i=1}^{n} x_i^p y_i &= a_0 \sum_{i=1}^{n} x_i^p + a_1 \sum_{i=1}^{n} x_i^{p+1} + a_2 \sum_{i=1}^{n} x_i^{p+2} + \cdots + a_p \sum_{i=1}^{n} x_i^{2p}
\end{aligned}\right\} \tag{21}$$

These Eqs (19), (20) and (21) are solved simultaneously for the $p + 1$ unknown $a_0, a_1, \ldots, a_p$.

Special Model:

(a) Let $p = 2$, then

$$y = a_0 + a_1 x + a_2 x^2$$

which is quadratic in nature.

(b) Let $p = 3$, then

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

which is cubic model, and so on.

(c) In general model if $x_1 = x$, $x_2 = x^2$, $\ldots$, $x_p = x^p$ then (19) becomes:

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_p x^p$$

and the values of $a_0, a_1, \ldots, a_p$ can be estimated using normal equations by Eq. (21).

But, if $y = a_0 + a_1 x_1 + a_2 x_2 + \ldots + a_p x^p$

then to find the values of $a_0, a_1, a_2, \ldots, a_p$ we use the following normal equations:

$$\left.\begin{aligned}
\sum_{i=1}^{n} y_i &= n a_0 + a_1 \sum_{i=1}^{n} x_{1i} + a_2 \sum_{i=1}^{n} x_{2i} + \cdots + a_p \sum_{i=1}^{n} x_{pi} \\
\sum_{i=1}^{n} x_i y_i &= a_0 \sum_{i=1}^{n} x_{1i} + a_1 \sum_{i=1}^{n} x_{1i}^2 + a_2 \sum_{i=1}^{n} a_{1i} x_{2i} + \cdots + a_p \sum_{i=1}^{n} x_{1i}\, x_{pi} \\
&\vdots \\
\sum_{i=1}^{n} x_{pi} y_i &= a_0 \sum_{i=1}^{n} x_{pi} + a_1 \sum_{i=1}^{n} x_{pi} x_{1i} + a_2 \sum_{i=1}^{n} p_{1i} x_{2i} + \cdots + a_p \sum_{i=1}^{n} x_{pi}^2
\end{aligned}\right\} \tag{22}$$

where $(x_{1i}, x_{2i}, \ldots, x_{ki})$ are the n values corresponding to k-independent variables $x_1, x_2, \ldots, x_k$.

Example 26 A study is conducted to develop an equation by which the unit cost of producing a new drug (Y) can be predicted based on the number of units produced (X). Estimate the value of Y corresponding to the value of $X = 30$, using a polynomial of degree 2.

No. of units produced (x)	5	5	10	10	15	15	20	20	25	25
Cost in hundreds of dollars (y)	14.0	12.5	7.0	5.0	2.1	1.8	6.2	4.9	13.2	14.6

Solution

The proposed model is a polynomial of degree 2.

i.e., $y = a_0 + a_1x + a_2x^2$

$\therefore$ normal equations are

$$\sum_{i=1}^{n} y_i = na_0 + a_1\sum_{i=1}^{n} x_i + a_2\sum_{i=1}^{n} x_i^2$$

$$\sum_{i=1}^{n} x_i y_i = a_0\sum_{i=1}^{n} x_i + a_1\sum_{i=1}^{n} x_i^2 + a_2\sum_{i=1}^{n} x_i^3$$

$$\sum_{i=1}^{n} x_i^2 y_i = a_0\sum_{i=1}^{n} x_i^2 + a_1\sum_{i=1}^{n} x_i^3 + a_2\sum_{i=1}^{n} x_i^4$$

From these data

$$n = 10, \sum_{i=1}^{10} x_i = 150, \sum_{i=1}^{10} x_i^2 = 2750, \sum_{i=1}^{10} x_i^3 = 56,250, \sum_{i=1}^{10} x_i^4 = 1223,750$$

$$\sum_{i=1}^{10} y_i = 81.3, \sum_{i=1}^{10} x_i y_i = 1228, \sum_{i=1}^{10} x_i^2 y_i = 24,555$$

Substituting these values in normal equations, we get

$$81.3 = 10\,a_0 + 150\,a_1 + 2750\,a_2$$

$$1228 = 150\,a_0 + 2750\,a_1 + 56250\,a_2$$

$$24555 = 2750\,a_0 + 56250\,a_1 + 1223750\,a_2$$

Solving the above three equations in a_0, a_1 and a_2, we get

$$a_0 = 27.3,\ a_1 = -3.313 \text{ and } a_2 = 0.111$$

$\therefore$ The regression model of degree 2 becomes

$$y = 27.3 - 3.313\,x + 0.111\,x^2$$

If $x = 30$ then estimated value of y is

$$y = 27.3 - (3.313)\,30 + 0.111(30)^2$$

$$y = 27.3 - 99.39 + 99.90 = 27.81 \text{ dollars}$$

Example 27 The following are data for the gasoline mileage (y) of an automobile based on its weight (x_1) and temperature (x_2) at the time of operation. Estimate y for $x_1 = 1.2$ and $x_2 = 35$.

Car Number	1	2	3	4	5	6	7	8	9	10
Miles per gallon (y)	17.9	16.5	16.4	16.8	18.8	15.5	17.5	16.4	15.9	18.3
Weight in tons (x_1)	1.35	1.90	1.70	1.80	1.30	2.05	1.60	1.80	1.85	1.40
Temperature in $F(x_2)$	90	30	80	40	35	45	50	60	65	30

Solution

Here regression model is

$$y = a_0 + a_1 x_1 + a_2 x_2$$

where x_1 takes values $x_1 i$, ($i = 1, 2, \ldots, 10$) and x_2 takes values x_{2i}($i = 1, 2, \ldots, 10$)

The normal equations are

$$\sum_{i=1}^{10} y_i = na_0 + a_1 \sum_{i=1}^{n} x_{1i} + a_2 \sum_{i=1}^{n} x_{2i}$$

$$\sum_{i=1}^{10} x_{1i}\, y_i = a_0 \sum_{i=1}^{n} x_{1i} + a_1 \sum_{i=1}^{n} x_{1i}^2 + a_2 \sum_{i=1}^{n} a_{2i}\, a_{ix}$$

$$\sum_{i=1}^{n} x_{2i}\, y_i = a_0 \sum_{i=1}^{n} x_{2i} + a_1 \sum_{i=1}^{n} x_{2i} x_{1i} + a_2 \sum_{i=1}^{n} x_{2i}{}^2$$

Here $n = 10$, $\displaystyle\sum_{i=1}^{10} x_{1i} = 16.75, \sum_{i=1}^{10} x_{2i} = 525, \sum_{i=1}^{10} x_{1i}^2 = 28.6375$

$$\sum_{i=1}^{10} x_{2i}^2 = 31475, \sum_{i=1}^{10} x_{1i}\, x_{2i} = 874.5, \sum_{i=1}^{10} y_i = 170$$

$$\sum_{i=1}^{10} x_{1i}\, y_i = 282.405, \sum_{i=1}^{10} x_{2i} y_i = 8887.0$$

Substituting these values in the above normal equations we get

$$170 = 10\, a_0 + 16.75\, a_1 + 525\, a_2$$

$$282.405 = 16.75\, a_0 + 28.6375\, a_1 + 874.5\, a_2$$

$$8887 = 525\, a_0 + 874.5\, a_1 + 31475\, a_2$$

Solving the above equations, we get

$$a_0 = 24.75,\ a_1 = -4.16 \text{ and } a_2 = -0.014897$$

$\therefore$ Model of regression is

$$y = 24.75 - 4.16\, x_1 - 0.014897\, x_2$$

If $x_1 = 1.2$ and $x_2 = 35$

then $y = 24.75 - 4.16 \times 1.2 - 0.14897\, x_2$

If $x_1 = 1.2$ and $x_2 = 35$

then

$$y = 24.75 - 4.16 \times 1.2 - .014897 \times 35$$

$$y = 24.75 - 4.992 - 0.521395 = 19.2366$$

EXERCISE 5.4

1. Psychological tests of intelligence of engineering ability were applied to 10 students. Here is a record of ungrouped data showing intelligence ratio (IR) and engineering ratio (ER). Calculate the coefficient of correlation.

Student	A	B	C	D	E	F	G	H	I	J
I.R.	105	104	102	101	100	99	98	96	93	92
E.R.	101	103	100	98	95	96	104	92	97	94

(Andhra 2000)

2. The given correlation table shows that the ages of husbands and wives of 53 married couples living together on the census night of 1991. Calculate the coefficient of correlation between the age of the husband and that of two wives.

(J.N.T.U. 2003)

Age of husband	Age of wife						Total
	15–25	25–35	35–45	45–55	55–65	65–75	
15–25	1	1	-	-	-	-	2
25–35	2	12	1	-	-	-	15
35–45	-	4	10	1	-	-	15
45–55	-	-	3	6	1	-	10
55–65	-	-	-	2	4	2	8
65–75	-	-	-	-	1	2	3
Total	3	17	14	9	6	4	53

3. Calculate the correlation coefficient for the following data:

x	10	7	12	12	9	16	12	18	8	12	14	16
y	6	4	7	8	10	7	10	15	5	6	11	13

4. Find the correlation coefficient and the regression lines of y on x and x on y for the following data:

x	1	2	3	4	5
y	2	5	3	8	7

(V.T.U. 2010)

5. Find the correlation coefficient between x and y from the given data:

x	78	89	97	69	59	79	68	57
y	125	137	156	112	107	138	123	108

(J.N.T.U. 2005)

6. Find the coefficient of correlation between industrial production and export using the following data and comment on the result.

Production (in crore tons)	55	56	58	59	60	60	62
Exports (in crore tons)	35	38	38	39	44	43	45

(Madras 2000)

7. Find the rank correlation for the following data:

x	56	42	72	36	63	47	55	49	38	42	68	60
y	147	125	160	118	149	128	150	145	115	140	152	155

(S.V.T.U. 2009, J.N.T.U. 2003)

8. Obtain the rank correlation coefficient for the following data which give the I.Q. of a group of 6 persons who sat in an examination.

I.Q.(x)	110	100	140	120	80	90
Marks(y)	70	60	80	60	10	20

9. A sample of 12 fathers and their eldest sons gave the following data about their heights in inches:

Father	62	63	67	64	68	62	70	66	68	67	69	71
Sons	68	66	68	65	69	66	68	65	71	67	68	70

Find the coefficient of rank correlation.

10. The ranking of 10 students in two subjects A and B are given as follows, find coefficient of rank correlation.

A	3	5	8	4	7	10	2	1	6	9
B	6	4	9	8	1	2	3	10	5	7

11. The following are the numbers of hours which 10 students studied for an examination and the score they obtained:

No. of hours studied (x)	8	5	11	13	10	5	18	15	2	8
Scores (y)	56	44	79	72	70	54	94	85	33	65

Calculate rank correlation coefficient.

12. Find two times of regression and coefficient of correlation for the data given below:

$$n = 18, \Sigma x = 12, \Sigma y = 18, \Sigma x^2 = 60, \Sigma y^2 = 96, \Sigma xy = 48$$

(U.P.T.U., M.C.A. 2009)

13. If the coefficient of correlation between two variables x and y in 0.5 and the acute angle between their lines of regression is $\tan^{-1}(3/8)$, show that $\sigma_x = \frac{1}{2}\sigma_y$.

(V.T.U. 2004)

14. For two random variables x and y with the same mean, the two regression lines are $y = a\,x + b$ and $x = \alpha y + \beta$. Show that $\frac{b}{\beta} = \frac{1-a}{1-\alpha}$. Find also the common mean.

(U.P.T.U. 2010)

15. Two random variables have the regression lines with equation $3x + 2y = 26$ and $6x + y = 31$. Find the mean values and the correlation coefficient between x and y.

(Madras 2002)

16. The regression equations of two variables x and y are $x = 0.7\,y + 5.2$, $y = 0.3x + 2.8$. Find the means of the variables and the coefficient of correlation between them.

(Osmania 2002)

17. Consider the model $y = a_0 + a_1x_1 + a_2x_2$. The following data are available:

y	9	8	7
x_1	0	2	4
x_2	8	9	8

(a) Find $\sum_{i=1}^{3} x_{1i}, \sum_{i=1}^{3} x_{2i}, \sum_{i=1}^{3} x_{1i}\,x_{2i}, \sum_{i=1}^{3} x_{1i}y_i, \sum_{i=1}^{3} x_{1i}^2, \sum_{i=1}^{3} x_{2i}^2, \sum_{i=1}^{3} y_i, \sum_{i=1}^{3} x_{1i}y_i$

(b) Find the normal equations

(c) Show that $a_0 = 9$, $a_1 = -0.5$ and $a_2 = 0$ are the solutions of the normal equations

(d) Find the value of y when $x_1 = 3$ and $x_2 = 10$.

18. The following data represent carbon dioxide (CO_2) emission from coal-fluid boilers (in units of 1000 tons) over a period of years 1965 and 1977. The independent variable (year) has been standardized yield the following table:

Year (x)	0	5	8	9	10	11	12
CO_2 emission (y)	910	680	520	450	370	380	340

Write the normal equation for regression model of degree 2, i.e. $y = a_0 + a_1x + a_2x^2$

Answers

1. $r_{xy} = 0.59$
2. $r_{xy} = 0.91$
3. $r_{xy} = 0.749$
4. $r_{xy} = 0.81$; line of regression of y on x: $y = 1.3x + 1.1$; line of regression of x on y; $x = 0.5y + 0.5$
5. $r_{xy} = 0.96$
6. $r_{xy} = 0.92$
7. 0.932
8. 0.882
9. 0.722
10. –0.30
11. 0.98
12. $r_{xy} = 0.632$; regression line of y on x; $y = 0.467 + 0.8x$ and regression line of x on y; $x = 0.167 + 0.5y$
14. Common mean $= \dfrac{\beta - b}{a - \alpha}$
15. $\overline{x} = 4, \overline{y} = 7, r_{xy} = -0.5$
16. $\overline{x} = 9.06, \overline{y} = 5.52; r_{xy} = 0.46$
17. (a) $\sum x_{1i} = 6, \sum x_{2i} = 25, \sum x_{1i}\,x_{2i} = 50, \sum x_{1i}y_i = 44, \sum x_{1i}^2 = 20, \sum x_{2i}^2 = 209,$
$\sum y_i = 24, \sum x_{2i}\,y_i = 200$

(b) $24 = 3a_0 + 6a_1 + 25a_2$
$44 = 6a_0 + 20a_1 + 50\,a_2$

$$200 = 25a_0 + 50a_2 + 209\,a_2$$

(d) $y = 7.5$

18. Normal equations are:

$$3650 = 7a_0 + 55a_1 + 535a_2$$
$$23570 = 55\,a_0 + 535a_1 + 5169\,a_2$$
$$218670 = 535\,a_0 + 5169\,a_1 + 56659\,a_2$$

(Find the values of a_0, a_1 and a_2 to fit a regression model of degree 2.)

SUMMARY

Following topics have been discussed in this chapter:

1. Classification of Data

(i) Frequency distribution grouped and ungrouped.

2. Graphical Representation

(i) Histogram (ii) Frequency polygon and (iii) ogive

3. Measures of Central Tendency

$$\text{Mean} = \frac{1}{n}\sum f_i x_i$$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$\text{Mode} = l + \frac{h(f_1 - f_0)}{2f_1 - f_2 - f_0}$$

$$\text{Geometric Mean} = \text{Antilog}\left[\frac{1}{N}\sum_{i=1}^{n} f_i \log x_i\right]$$

$$\text{Harmonic Mean} = \frac{1}{\sum \frac{f_i}{x_i}}$$

4. Measure of Dispersion

Range: A–B

Quartile deviation: $\frac{1}{2}(Q_3 - Q_1)$

Standard deviation $\sigma = \sqrt{\frac{1}{N}\sum_i f_i (x_i - \overline{x})^2}$

Variance $\sigma^2 = \frac{1}{N}\sum_i f_i (x_i - \overline{x})^2$

Coefficient of variation = $100 \times \dfrac{\sigma}{x}$

Quartile deviation = $\dfrac{2}{3}\sigma$

Mean deviation = $\dfrac{4}{5}\sigma$

k^{th} moment about any point A

$$\mu_k' = \frac{1}{N}\sum f_i\,(x_i - A)^k$$

k^{th} moment about mean $\overline{x}$

$$\mu_k = \frac{1}{N}\sum f_i\,(x_i - \overline{x})^k$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\,\mu_3'\,\mu_1' + 6\mu_2'\,\mu_1'^2 - 3\,\mu_1'^4$$

5. Measures of Skewness

$$s_k = \overline{x} - M_d$$

$$s_k = \overline{x} - M_0$$

$$s_k = (Q_3 - M_d) - (M_d - Q_1)$$

Karl Pearson coefficient of skewness

$$s_k = \frac{3(\overline{x} - M_d)}{\sigma}$$

Quartile coefficient of skewness:

$$s_k = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$$

$$\gamma_1 = \sqrt{\beta_1}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

Measures of Kurtosis:

$$\gamma_2 = \beta_2 - 3$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

6. Correlation

Correlation coefficient $r_{xy} = \dfrac{\frac{1}{n}\sum x_i y_i - \overline{x}\,\overline{y}}{\sqrt{\frac{1}{n}\sum x_i^2 - \overline{x}^2}\sqrt{\frac{1}{n}\sum y_i^2 - \overline{y}^2}}$

Rank correlation coefficient:

$$r = 1 - 6\frac{\sum d_i^2}{n(n^2-1)}$$

Line of regression of y on x:

$$y - \overline{y} = b_{yx}(\overline{x} - \overline{x}) \Rightarrow y - \overline{y} = r\frac{\sigma_y}{\sigma_y}(x - \overline{x})$$

and line of regression of x on y:

$$x - \overline{x} = b_{xy}(y - \overline{y}) \Rightarrow x - \overline{x} = r\frac{\sigma_x}{\sigma_y}(y - \overline{y})$$

$$r = \sqrt{b_{yx} \times b_{xy}}$$

If θ is angle between regression lines of y on x and x on y then $\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)$

7. Multiple regression of single variable of degree p is defined as:

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_px^p$$

8. Multiple regression of k variables is defined as:

$$y = a_0 + a_1x_1 + a_2x_2 + \cdots + a_kx^k$$

OBJECTIVE TYPE QUESTIONS

1. The mean of 9, 10, 11, 12 and 13 is
(a) 11 (b) 12
(c) 13 (d) none of these

2. The median of 13, 12, 10, 9 and 11 is
(a) 10 (b) 11
(c) 12 (d) 13

3. The median of 4, 5, 6, 7, 8, 9 is
(a) 6 (b) 6.5
(c) 7 (d) 7.5

4. The mode of 5, 5, 5, 7, 8, 9, 9, 9, 10 is
(a) 5 and 9 (b) 5 only
(c) 9 only (d) none of these

5. Variance is defined by
(a) $\dfrac{\sum f(x-\overline{x})}{\sum f}$ (b) $\dfrac{\sum f(x-\overline{x})^2}{\sum f}$
(c) $\sum f(x-\overline{x})^2$ (d) $\sqrt{\dfrac{\sum f(x-\overline{x})^2}{\sum f}}$

6. $\sum_{i=1}^{n} f_i(x_i - \overline{x})$ is always equal to
(a) 0 (b) 1
(c) 2 (d) 3

7. Median is the measure of
(a) Central tendency (b) Dispersion
(c) Kurtosis (d) Skewness

8. In the relation, Mode = x(Median) -2 mean, the missing value x is equal to
(a) 1 (b) 2
(c) 3 (d) 4

9. Median of the distribution divides the entire data in parts equal to
(a) 1 (b) 2
(c) 3 (d) 4

10. If $Q_1 = 3$, $Q_3 = 5$, then semi-interquartile is equal to
(a) 1 (b) 4
(c) 8 (d) none of these

11. The range for the following distribution 9, 15, 25 and 40 is
(a) 9 (b) 15
(c) 25 (d) 31

12. If $n = 5$, $\Sigma x_i^2 = 125$, $\overline{x} = 5$, then standard deviation is equal to
(a) 0 (b) 1
(c) 2 (d) 3

13. If $n = 5$, $\Sigma x_i^2 = 125$, $\Sigma x_i = 25$, then variance σ_x^2 is equal to
(a) 3 (b) 2
(c) 1 (d) 0

14. Coefficient of variation is equal to
(a) $\dfrac{\overline{x}}{\sigma}\times 100$ (b) $\dfrac{\sigma}{\overline{x}}\times 100$
(c) $\dfrac{\sigma}{\sqrt{\overline{x}}}\times 100$ (d) $\dfrac{\sqrt{\sigma}}{\overline{x}}\times 100$

15. If $\overline{x} = 5, \sigma^2 = 100$, then coefficient of variation is equal to
(a) 50 (b) 5
(c) 100 (d) none of these

16. Mean deviation is minimum when calculated about
(a) Mean (b) Median
(c) Mode (d) Geometric mean

17. Quartile deviation is equal to
(a) $\dfrac{2}{3}$ standard deviation
(b) $\dfrac{4}{5}$ standard deviation
(c) $\dfrac{3}{2}$ standard deviation
(d) $\dfrac{5}{4}$ standard deviation

18. If standard deviation of a distribution is 3, then quartile deviation is
(a) 1 (b) 4.5
(c) 2.4 (d) 4.5

19. Mean deviation is equal to
(a) $\dfrac{2}{3}$ standard deviation
(b) $\dfrac{3}{2}$ standard deviation
(c) $\dfrac{4}{5}$ standard deviation
(d) $\dfrac{5}{4}$ standard deviation

20. If mean deviation of a distribution is 4 then standard deviation is equal to
(a) 4
(b) 5
(c) 6
(d) None of these

21. The mean of first n positive integer is
(a) $\dfrac{n+1}{2}$ (b) $\dfrac{n-1}{2}$
(c) $\dfrac{n}{2}$ (d) $n + 1$

22. The variance of first n positive integer is equal to
(a) $\dfrac{n^2+1}{12}$ (b) $\dfrac{n^2-1}{12}$
(c) $\dfrac{n^2+1}{2}$ (d) $\dfrac{n^2-1}{2}$

23. If the coefficient of variation and standard deviation of a frequency distribution are 5 and 2, then the mean of the distribution is
(a) 5 (b) 2
(c) 20 (d) 40

24. If $Q_1 = 14.95$, $Q_3 = 22.95$, then semi-interquartile range is equal to
(a) 2 (b) 3
(c) 4 (d) none of these

25. If the equations of regression lines are $y = 0.4\,x + 2$ and $x = 0.5y + 3$, the correlation coefficient is equal to

(a) 0.20 (b) $\sqrt{0.20}$

(c) –0.20 (d) $-\sqrt{0.20}$

26. If the correlation coefficient is +1, then two lines of regression are

(a) Parallel

(b) Perpendicular

(c) Inclined at 60° to each other

(d) Coincident

27. If two regression lines are perpendicular to each other, then correlation coefficient between x and y is

(a) 0 (b) –1

(c) 1 (d) 0.50

28. The correlation coefficient r_{xy} lies between

(a) –1, 0 (b) –1, 1

(c) 0, 1 (d) 1, 2

29. If regression coefficient of y on x is 2 and that of x on y is 4.5, then correlation coefficient is equal to

(a) 3 (b) 2

(c) 4 (d) 5

30. If $y = x + 1$ and $x = 3y - 7$ are the two lines of regression then $\overline{x}$ is equal to

(a) 2 (b) 3

(c) 4 (d) 1

31. If $y = x + 1$ and $x = 3y - 7$ are the two lines of regression then $\overline{y}$ is equal to

(a) 1 (b) 2

(c) 3 (d) 4

32. If $y = x + 1$ and $x = 3y - 7$ are the two lines of regression then correlation coefficient is

(a) 1 (b) 2

(c) $\sqrt{3}$ (d) 3

33. If x and y are independent variables then correlation coefficient is equal to

(a) 0 (b) 1

(c) –1 (d) $\frac{1}{2}$

34. If the two regression lines are perpendicular to each other then correlation coefficient is

(a) –1 (b) 0

(c) $\frac{1}{2}$ (d) 1

35. The point of intersection of the two regression lines is

(a) $\overline{x}$ (b) $\overline{y}$

(c) $(\overline{x}, \overline{y})$ (d) None of these

36. The moment coefficient of skewness is given by

(a) β_1 (b) $\sqrt{\beta_1}$

(c) β_2 (d) $\beta_2 - 3$

37. A frequency curve is said to be Mesokurtic if β_2 is equal to

(a) 0 (b) 1

(c) 2 (d) 3

38. When the variables are independent, the two lines of regression are

(a) Perpendicular

(b) Parallel

(c) Coincide to each other

(d) Inclined at 45° to each other

39. Degree of peakedness is measured by

(a) Central tendency (b) Dispersion

(c) Kurtosis (d) Skewness

40. Quartile deviation of skewness is equal to

(a) $\frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$ (b) $\frac{Q_3 - Q_1 + 2Q_2}{Q_3 - Q_1}$

(c) $\frac{Q_3 - Q_1}{2}$ (d) $\frac{Q_3 + Q_1}{2}$

ANSWERS

1.(a)	2.(b)	3.(b)	4.(a)	5.(b)	6.(a)	7.(a)	8.(c)	9.(b)	10.(a)
11.(d)	12.(a)	13.(d)	14.(b)	15.(a)	16.(b)	17.(a)	18.(a)	19.(c)	20.(b)
21.(a)	22.(b)	23.(d)	24.(c)	25.(b)	26.(d)	27.(a)	28.(b)	29.(a)	30.(a)
31.(c)	32.(c)	33.(a)	34.(b)	35.(c)	36.(b)	37.(d)	38.(a)	39.(c)	40.(a)

6 Probability and Distribution

6.1 INTRODUCTION

In this chapter we shall discuss some elementary definitions related to the probability theory such as addition rule of probability, joint probability, independent events, conditional probability, theorem of total probability, Baye's theorem and its application. We shall also discuss the discrete and continuous random variables and in the later part of the chapter some important discrete distributions such as binomial and poisson distributions and continuous distributions such as uniform, exponential, normal distributions along with their properties and related problems will be discussed.

Probability or chance is a word which is used by everyone in his/her lives. For example, we may say probably it may rain today, we may say it is more likely to have a good yield of wheat in district A than in district B, it means that we expect better yield of wheat from district A than from B. This expectation, of course comes from the knowledge about the conditions of whether in the month of particular season.

In fact we can say that the branch of mathematics which studies the influence of chance is known as theory of probability. Therefore, the probability is a concept which numerically measures the degree of uncertainty or uncertainty of occurrence or non-occurrence of events. Probability theory is quite useful in the study of subjects like engineering, social science, genetic, physics, chemistry, biology, medical sciences, etc.

In (1501–1576), an Italian mathematician Jerome Cardon wrote the first book on probability theory entitled "*Book on Game of Chance*". Later Pascal (1623–1662), Fermat (1601–1665), J. Bernoulli (1654–1705), De Moivre (1667–1754), Chebychev (1821–1894), A.A. Markov (1854–1922) and other mathematicians made out standing contributions in probability theory.

Before giving the proper and various definitions of probability, we shall define certain terms which will be used in the theory of probability.

6.2 TERMINOLOGY

(i) *Set*: A set is a collection of objects under study. Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc. For example, students of electrical engineering in a college, rivers of India, states of India, etc.

(ii) *Experiment*: An experiment is a physical process that is observed and whose result is noted. For example, turning a switch on or off, throwing a dice, etc. Experiments are of two types (a) deterministic and (b) non-deterministic experiment.

(a) *Deterministic experiment*: If we are sure about the outcome of the experiment before conducting the experiment, then it is known as deterministic experiment. For example, constructing a wall, throwing a stone upwards, etc.

(b) *Non-deterministic experiment*: An experiment is called a non-deterministic, random, probabilistic or stochastic if we are not sure that which of the possible outcomes will occur when an experiment is conducted. Here after an experiment means random experiment. For example, tossing a coin, throwing a dice, etc.

(iii) *Trial*: A single performance of an experiment is called a trial. For example, tossing a coin.

(iv) *Sample space or sample point*: A collection of all possible outcomes of a random experiment is called a sample space and usually denoted by S. The elements of a sample space are called sample points.

For example, if a dice is thrown, then any one of 1, 2, 3, 4, 5 or 6 will appear on the face. Hence, sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Therefore, set S is called sample space and 1, 2, 3, 4, 5 and 6 are called sample points of S.

(v) *Discrete sample space*: A sample space is called to be discrete, if it has finitely many or a countably infinite number of elements. For example, tossing of a coin the sample space $S = \{H, T\}$, throwing a dice, the sample space $S = \{1, 2, 3, 4, 5, 6\}$, etc.

(vi) *Continuous sample space*: A sample space is said to be continuous; if the elements of the sample space constitute a continuum. For example, all the points on a line segment.

(vii) *Event*: Any subset A of a sample space S is called an event. In tossing of a coin, we have $S = \{H, T\}$.

Here ϕ, $\{H\}$, $\{T\}$ and $\{H, T\}$ are sub sets of S and each of them is an event.

An event can further be divided into two categories (a) simple or elementary event and (b) compound event.

(a) *Simple or elementary event*: If an event has only one sample point and cannot be further divided into smaller events then it is known as simple or elementary event. For example, getting 1, 2, 3, 4, 5 and 6 in throwing a dice is an elementary event.

(b) *Compound event*: If an event has more than one sample point and can be obtained by combining the several elementary events is called a compound event. For example, when a coin is tossed twice then sample space $S = \{HH, HT, TH, TT\}$ and each event of S is a compound event.

(viii) *Favourable events*: The number of cases which favours an event in a trial is the number of favourable outcomes which entail the happening of the event. For example, in throwing two dice, the number of cases favourable of getting the sum 5 is $\{1, 4\}$, $\{2, 3\}$, $\{3, 2\}$ and $\{4, 1\}$, i.e., 4.

(ix) *Equally likely events*: If there is no reason to expect any one event in preference to any other event, then the events are called equally likely events. For example, in throwing a die, all the six faces are equally likely to occur.

(x) *Compliment of an Event*: Let S be a sample space and A be any event than compliment of event A denoted by A' is the set of all those points of S which are not in A. It is also denoted by $\bar{A}$.

$$\therefore \quad A' \text{ or } \bar{A} = S - A$$

(xi) *Mutually Exclusive Events*: Let A and B be two events then A and B are said to be mutually exclusive events, if there is no common point in both the events i.e., $A \cap B = \Phi$

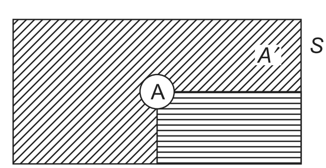


Fig. 6.1

For example, in throwing a dice

$$S = \{1, 2, 3, 4, 5, 6\}$$

and let A = set of odd numbers = $\{1, 3, 5\}$ and B = set of even numbers = $\{2, 4, 6\}$

$\therefore$ $A \cap B = \Phi$ and A, B are mutually exclusive events or some times we call them disjoint events.

(xii) *Exhaustive events*: The total number of possible outcomes in any trial of a random experiment is called exhaustive events.

(xiii) *Odd in favour of an event and odd against an event*: Let there are be m outcomes favourable to a certain event and n outcomes are not in favour to the event in a sample space S, then odd in favour of the event $= \dfrac{m}{n}$ and odd not in favour or against of the event = $\dfrac{n}{m}$.

(xiv) *Permutation*: A permutation is an arrangement of objects in a definite order. The number of permutations of n distinct objects used r at a time, denoted by

$$^nP_r = \frac{\lfloor n}{\lfloor n-r} \text{ or } \frac{n!}{(n-r)!}$$

(xv) *Combination*: A combination is a selection of objects regardless of any order. The number of combinations of n distinct objects selected r at a time denoted by

$$^nC_r \text{ or } \binom{n}{r} = \frac{\lfloor n}{\lfloor r \lfloor n-r} = \frac{n!}{r!\,(n-r)!}$$

(xvi) *Union of Events*: Let A and B be two events, then A union $B(A \cup B)$ is the event that consists all the points either in A or in B or in both A and B.

(xvii) *Intersection of Events*: Let A and B be two events then A intersection B $(A \cap B)$ is the set of all those points that are contained in both A and B.

Example 1 In how many ways can one make a first, second, third, and fourth choice among the ten firms leasing construction equipment.

Solution

$n = 10$ and $r = 4$ then the number of ways $= {}^nP_4 = {}^{10}P_4 = \dfrac{10!}{(10-4)!}$

$$= \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040 \text{ ways}$$

Example 2 An electronic controlling mechanism requires 4 identical memory chips. In how many ways can this mechanism be assembled by placing the 4 clips in the four positions within the controller?

Solution

$$n = 4 \text{ and } r = 4, \text{ then no. of ways } = {}^4P_4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Example 3 In how many ways can 3 of 20 laboratory assistants be chosen to assist with an experiment?

Solution

$$n = 20 \text{ and } r = 3, \text{ the number of ways } = {}^nC_r = \binom{n}{r}$$

$$= {}^{20}C_3 = \binom{20}{3} = \frac{20!}{3!(20-3)!} = \frac{20!}{3!17!} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140 \text{ ways}$$

Example 4 In how many different ways can the head of a research laboratory choose 2 chemists from among 7 applicants and 3 physicists from among 9 applicants?

Solution The 2 chemists can be chosen in $\binom{7}{2}$ ways = 21 ways and 3 physicists can be chosen in $\binom{9}{3}$ ways = 84 ways.

Total number of ways of choosing 2 chemists and 3 physicists

$$= 21 \times 84 = 1,764 \text{ ways.}$$

Example 5 Let $S = \{x|x = 1, 2, 3, 4, 5, \ldots\}$, where x is the time in hours required to complete a chemical reaction be a sample space which is used to model,

Let $A = \{x \mid 2 \le x < 15\}$ and

$B = \{x \mid 1 \le x \le 110\}$

be two events defined on S. Find

(a) $A \cup B$ (b) $A \cap B$ (c) A'

Also represent them by Venn diagrams.

Solution

(a) $A \cup B = \{x \mid 1 \le x \le 110\} = \{1, 2, ..., 110\}$

(b) $A \cap B = \{x \mid 2 \le x < 15\} = \{2, 3, ..., 14\}$

(c) $A' = \{x \mid x \notin A\} = \{x \mid x \ge 15\} = \{15, 16, ..., \}$

S, $A \cup B$, $A \cap B$ and A^1 are denoted by the following diagrams known as Venn diagrams.

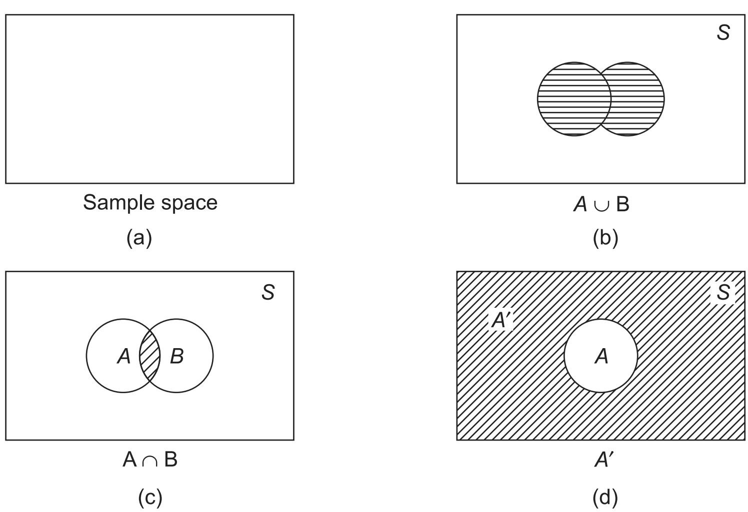
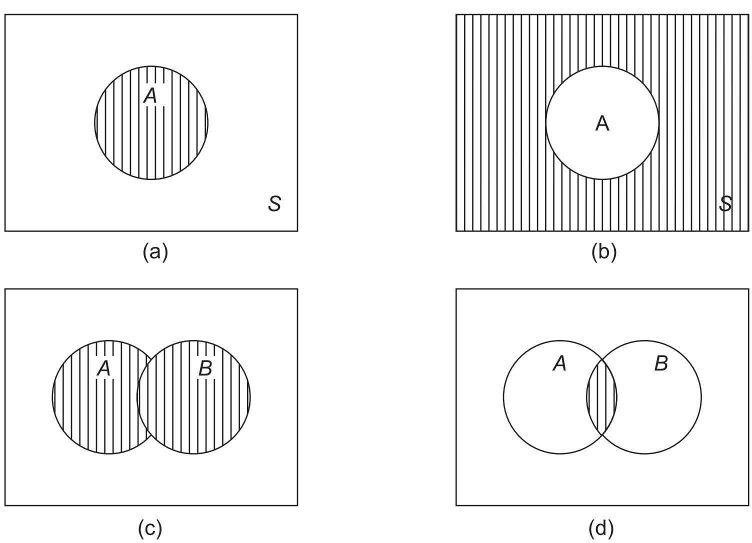


Fig. 6.2

EXERCISE 6.1

1. If there are 9 cars in a race, in how many different way can they be placed first, second and third?
2. A special purpose computer contains 3 switches, each of which can be set in 3 different way. In how many ways can the computer's bank of switches be set?
3. In an optics kit there are 6 convex lenses, 4 concave lenses and 3 prisms. In how many ways can one choose one of the convex lenses, concave lenses and one of the prisms?
4. In a small class, each of the students must write a report on one of 8 field trips. In how many different ways can they choose one of the field trips if:
   (a) Number 2 students may choose the same field trip;
   (b) There is no restrictions on their choice.
5. In how many ordered ways can a television director schedule 6 different commercials during the 6 times slots allocated to commercials during the telecast of the first period of a cricket match.
6. Determine the number of ways in which a manufacturer can choose 2 of 15 locations for a new warehouse?
7. A carton of 12 rechargeable batteries contain one that is defective. In how many ways can an inspector choose 3 of the batteries and
   (a) get the one that is defective;
   (b) not get the one that is defective
8. If A is the event that a certain student is taking a course in Engineering Mathematics and B is the event that the student is taking a course in Engineering Graphics. What events are represented by the shaded regions of four Venn diagram in Figs 6.3(a) and (b).
9. Find n if (i) $\binom{n}{2} = 21$ (ii) $\binom{n}{2} = 105$



(a) (b) (c) (d)

Fig. 6.3

10. Use Venn diagram to show that

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

Answers

1. 504		2. 6	
3. 72		4. (a) 1680	(b) 4096
5. 720		6. 105	
7. (a) 55	(b) 165		
8. (i) A	(ii) $\overline{A}$	(iii) $A \cup B$	(iv) $A \cap B$
9. (i) $n = 7$ (ii) $n = 15$			

6.3 DEFINITION OF PROBABILITY

The chance of happening of an event when expressed quantitiatively is called probability. We shall define probability in the following three ways.

(i) Classical definition of probability.
(ii) Empirical or statistical definition of probability.
(iii) Axiomatic definition of probability.

(i) Classical Definition of Probability Let a random experiment results 'n' mutually exclusive and equally likely outcomes and out of which 'm' outcomes are favourable to a particular event A, then the probability of A is denoted by

$$P(A) = \frac{m}{n} = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

Here m = number of favourable outcomes to A

$\therefore$ $n - m$ = non-favourable number of outcomes to A and set of non-favourable number of outcomes is denoted by $A' \Rightarrow P(A') = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$

$\Rightarrow$ $P(A) + P(A') = 1$

If $P(A)$ = Probability of success of an event $A = p$ and
$P(A')$ = Probability of failure of event $A = q$

then $$p = P(A) = \frac{m}{n}, 0 \le p \le 1$$

and $$q = P(A') = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p, 0 \le q \le 1$$

$\Rightarrow$ $p + q = 1$

Example 6 Find the probability of drawing a queen from a well-shuffled deck of 52 cards.

Solution

Number of queens = 4
Total number of cards = 52
Number of ways a queen can be drawn from four queens = ${}^4C_1 = 4 = m$

Number of ways a card can be drawn from 52 cards = ${}^{52}C_1 = 52 = n$

Let A be the event that drawn card is queen, then

$$P(A) = \frac{m}{n} = \frac{4}{52} = \frac{1}{13}$$

Example 7 If from a lottery of 30 tickets, marked 1, 2,…, 30, four tickets are drawn, what is the probability that the drawn tickets always marked with number 1 and 2 among them.

Solution When two tickets are always numbered 1 and 2, then remaining two tickets can be drawn in ${}^{28}C_2$ ways. Let A be the event that among drawn four cards 2 are marked 1 and 2.

$$\therefore \qquad n(A) = m = 1.1.{}^{28}C_2 = {}^{28}C_2$$

4 cards from 30 tickets can be drawn in 30_{c_2} ways = n

$$\therefore \qquad P(A) = \frac{m}{n} = \frac{{}^{28}C_2}{{}^{30}C_2} = \frac{2}{145}$$

Remark If $P(A) = 1$, then A is called sure event and if $P(A) = 0$, it is known as impossible event.

Limitations of classical definition of probability are as follows:

1. The elementary events must be equally likely, which is not always possible.
2. n, the number of exhaustive events must always be finite.

   If n is infinitely large, then we use Empirical or Statistical definition of probability.

(ii) Empirical or Statistical Definition of Probability

Let m = frequency of occurrence of event A

n = number of independent trials of a random experiment which are repeated under the same conditions.

Then $P(A) = \lim\limits_{n \to \infty} \dfrac{m}{n}$, provided limit is unique and finite. Where $\dfrac{m}{n} =$ relative frequency of the event A in n trials. If $n \to \infty$ then $\dfrac{m}{n}$ is very close to actual probability. This definition of probability is also known as relative frequency definition of probability.

Example 8 If records show that 800 of 1000 tested ceramic insulators were able to withstand a certain thermal shock. Find the probability that any one untested insulator will be able to withstand the thermal shock.

Solution Here $m = 800$, $n = 1000$. Let A be the event that any untested insulator will be able to withstand the thermal stock.

$$P(A) = \frac{800}{1000} = 0.80$$

Example 9 An electric engineer is studying the peak demand at a power plant. It is observed that 90 of the 100 days randomly selected for study from past records, the peak demand occurred between 6 and 7 p.m. What is the probability of occurring the peak demand between 6 and 7 p.m. on next day?

Solution Here, $m = 90$, $n = 100$

Let A be the event that demand will be on peak between 6 and 7 p.m. on next, then

$$P(A) = \frac{90}{100} = 0.90$$

Limitations of the Statistical Definition of Probability: Here the difference between the relative frequency $\frac{m}{n}$ and actual probability p will be smaller and smaller as n becomes larger and larger, i.e. $\frac{m}{n} \to p$ as $n \to \infty$, but in actual practice n is always a finite number however large. Therefore $\frac{m}{n}$ will not give an exact probability, it will always give an approximate value.

(iii) Axiomatic Definition of Probability If S is the sample space and A be any event of a random experiment, then

(a) $0 \le P(A) \le 1$ for each event $A \in S$
(b) $P(S) = 1$
(c) If A_1 and A_2 are two mutually exclusive events in S, then
$P(A_1 \cup A_2) = P(A_1) + P(A_2)$

(a), (b) and (c) are known as axioms of probability.

Example 10 Mr. X who is a broker feels that the probability that a given stock will go up in value during the day's trading is 0.6 and the probability that it will go down in value is 0.1. What is the probability that it will go up or down?

Solution Let A be the event that a given stock will go up in value and B be the event that a given stock will go down in value.

Given $P(A) = 0.6$ and $P(B) = 0.1$

$\therefore$ $P(A \cup B) = P(A) + P(B) = 0.6 + 0.1 = 0.7$ (Using (iii) axiom of probability)

Generalization of third axiom of probability: If $A_1, A_2, \ldots, A_n$ are mutually exclusive events in a sample space S, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

or
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

This can be proved using mathematical induction.

Example 11 A transport company needs tyres for its trucks and the probabilities are 0.18, 0.22, 0.02, 0.23, 0.30 and 0.05 that it will buy Goodyear tyres, Uniroyal tyres, Michelin tyres, General tyres, Goodrich or Armstrong tyres. Find the probability that it will buy

(i) Goodyear or General tyres.
(ii) Uniroyal, Michelin or Goodrich tyres.
(iii) Goodyear, Uniroyal, Armstrong or Goodrich tyres.

Solution Let A, B, C, D, E and F be the events that the transport company will buy Goodyear, Uniroyal, Michelin, General, Goodrich and Armstrong tyres respectively.

$\Rightarrow$ Given $P(A) = 0.18, P(B) = 0.22, P(C) = 0.02, P(D) = 0.23$

$P(E) = 0.30$ and $P(F) = 0.05$ (All events are mutually exclusive events).

(i) $P(A \text{ or } D) = P(A \cup D) = P(A) + P(D) = 0.18 + 0.23 = 0.41$

(ii) $P(B \text{ or } C \text{ or } E) = P(B \cup C \cup E) = P(B) + P(C) + P(E)$
$$= 0.22 + 0.02 + 0.30 = 0.54$$

(iii) $P(A \text{ or } B \text{ or } F \text{ or } E) = P(A \cup B \cup F \cup E)$
$$= P(A) + P(B) + P(F) + P(E)$$
$$= 0.18 + 0.22 + 0.05 + 0.30 = 0.75$$

Example 12 The distribution of blood type in the United States is roughly 41% type A, 9% type B, 4% type AB and 46% type O. An individual brought into emergency room and is to be one of above blood-typed, what is probability that the type will be *A*, *B*, or *AB*.

Solution Let *A*, *B*, *C*, *D* be the events that blood of an individual will be of type A, B, AB or O respectively, then

$$P(A) = 0.41,\ P(B) = 0.09,\ P(C) = 0.04 \text{ and } P(D) = 0.46$$

To find $P(A \cup B \cup AB) = P(A) + P(B) + P(C)$ ($\because$ *A*, *B* and *C* be are mutually exclusive events)
$$= 0.41 + 0.09 + 0.04 = 0.54$$

6.4 ADDITION LAW OF PROBABILITY OR THEOREM OF TOTAL PROBABILITY

Theorem 6.4.1

Let *A* and *B* be the mutually exclusive events, then prove that $P(A \cup B) = P(A) + P(B)$

***Proof*:** Let *n* be the total number of equally likely cases unit of which m_1 be the number that event *A* occurs and m_2 be the number that event *B* occurs then the number of favourable cases of occurring *A* or $B = m_1 + m_2$

$$\therefore \quad P(A \text{ or } B) = P(A \cup B) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n}$$

but $$P(A) = \frac{m_1}{n} \text{ and } P(B) = \frac{m_2}{n}$$

$$\therefore \quad P(A \cup B) = P(A) + P(B)$$

Theorem 6.4.2

Let *A* and *B* be two events then show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

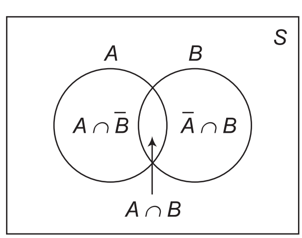


Fig. 6.4

***Proof*:** Using Venn diagram

$$A \cup B = A \cup (\bar{A} \cap B)$$

where A and $\bar{A} \cap B$ are mutually exclusive events.

$$\begin{aligned}\therefore \quad P(A \cup B) &= P(A) + P(\bar{A} \cap B) \\ &= P(A) + P(\bar{A} \cap B) + P(A \cap B) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B)\end{aligned}$$

$$\left[\begin{array}{l}\because \quad (\bar{A} \cap B) \text{ and } (A \cap B) \text{ and mutually exclusive} \\ \qquad\qquad \text{events and } B = (A \cap B) \cup (\bar{A} \cap B) \\ \therefore \quad P(B) = P(A \cap B) + P(\bar{A} \cap B)\end{array}\right]$$

$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, the result.

***Remark*:** If A and B are mutually exclusive events, then

$$P(A \cap B) = 0$$

and $P(A \cup B) = P(A) + P(B)$ (Which is same as theorem 6.4.1)

Theorem 6.4.3

If A, B and C are any three events then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

***Proof*:**

$$P(A \cup B \cup C) = P\left[A \cup (B \cup C)\right] = P[A] + P(B \cup C) - P[A \cap B \cup C]$$

(Using Theorem 6.4.2)

$$\begin{aligned}&= P(A) + P(B) + P(C) - P(B \cap C) - \left[P(A \cap B) \cup P(A \cap C)\right] \\ &= P(A) + P(B) + P(C) - P(B \cap C) - \left[P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)\right] \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)\end{aligned}$$

Hence, the result.

Theorem 6.4.4

Prove that $P(\Phi) = 0$

***Proof*:** $S \cup \Phi = S$ (S and Φ are mutually exclusive events)

$$\begin{aligned}\therefore \quad & P(S \cup \Phi) = P(S) \\ \Rightarrow \quad & P(S) + P(\Phi) = P(S) \\ \Rightarrow \quad & P(\Phi) = 0\end{aligned}$$

Theorem 6.4.5

Show that $P(A') = 1 - P(A)$

Proof: $A \cup A' = S$

(A and A^1 are mutually exclusive events)

$\therefore \quad P(A \cup A') = P(A) + P(A') = P(S) = 1 \qquad (\because \quad P(S) = 1)$

$\Rightarrow \quad P(A') = 1 - P(A)$

Hence, the result.

Example 13 A single dice is thrown once. Find the probability of getting *a* 3 or 5.

Solution In throwing a dice, the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$P(1) = P(2) = \cdots = P(6) = 1/6 \qquad \text{(all are mutually exclusive events)}$$

$$\therefore \quad P(3 \text{ or } 5) = P(3) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Example 14 Find the probability of drawing a king or a heart or both from a dack of cards?

Solution Let A be the event that a king is drawn B be the event that drawn card is a heart.

then $$P(A) = \frac{{}^{4}C_1}{{}^{52}C_1} = \frac{4}{52} = \frac{1}{13}$$

and $$P(B) = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{13}{52} = \frac{1}{4}$$

and $$P(A \cap B) = \frac{1}{52}$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4}{13}$$

Example 15 In throwing a pair of dice, find the probability of getting 3, 5 or 11.

Solution Total number of cases in throwing a pair of dice = 6 × 6 = 36.

Let A, B and C be the events that in throwing a pain of dice we get 3, 5, or 11.

then $$P(A) = \frac{2}{36}, P(B) = \frac{4}{36}, P(C) = \frac{2}{36}$$

$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$ ($\because$ A, B and C are mutually exclusive events)

$$= \frac{2}{36} + \frac{4}{36} + \frac{2}{36} = \frac{2}{9}$$

Example 16 A bag contains 15 balls numbered from 1 to 15. If a ball is drawn at random, what is the probability of having a ball with a number which is a multiple of 2 or 3.

Solution Let A be the event that drawing ball has a number of multiple of 2 and 3 be the event that drawing ball has a number of multiple of 3.

$$A = \{2, 4, 6, 8, 10, 12, 14\} \Rightarrow P(A) = 7/15$$

$$B = \{3, 6, 9, 12, 15\} \Rightarrow P(B) = 5/15$$

$$A \cap B = \{6, 12\} \Rightarrow P(A \cap B) = 2/15$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{7}{15} + \frac{5}{15} - \frac{2}{15} = \frac{10}{25} = \frac{2}{3}$$

6.5 CONDITIONAL PROBABILITY

In our daily life, it is not always possible that two or more events always happen independently but their occurrence may depend on one another, i.e., the probability of one event depends on the happening of another event.

Definition: Let A and B be two events in S, then conditional probability of A given B is

$$= P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) > 0.$$

Similarly probability of B given $A = P(B|A) = \dfrac{P(A \cap B)}{P(A)}, P(A) > 0.$

6.5.1 Theorem on Compound Probability of Multiplication Law of Probability

If A and B are any two events in sample space S, then

$$P(A \cap B) = P(AB) = P(A) \cdot P(B|A), \quad \text{if } P(A) > 0$$

$$= P(B) \cdot P(A|B), \quad \text{if } P(B) > 0$$

Similarly let A, B and C be three events then

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) P(C/A \cap B), \text{ provided } P(A), P(A \cap B) > 0.$$

6.5.2 Independent Events

Two events are said to be independent if happening or non-happening of the events do not depend on each other. If A and B are two events, then A and B are said to be independent if

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

or

$$P(AB) = P(A \cap B) = P(A) \cdot P(B)$$

This is called special multiplication rule for independent events.

***Remark*:** Therefore, we can say that two events A and B are independent, if and only if

(i) $P(A|B) = P(A)$, (ii) $P(B|A) = P(B)$ and (iii) $P(A \cap B) = P(A) \cdot P(B)$

and if $P(A \cap B) \neq P(A) \cdot P(B)$, then A and B are not independent events. (iii) can be extended for more than two events.

Definition: Let $E_1, E_2, \ldots, E_n$ be n events, then $E_1, E_2, \ldots, E_n$ are independent if and only if

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1) \cdot P(E_2) \cdots P(E_n)$$

Theorem

If A and B are independent events, then show that

(a) $\bar{A}$ and $\bar{B}$ are independent events.

(b) $\bar{A}$ and B are independent events.

(c) A and $\bar{B}$ are independent events.

Proof:

(a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\Rightarrow \quad P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) \quad$ [Using result $P(\bar{A}) = 1 - P(A)$]

$$\Rightarrow \quad P(\bar{A} \cap \bar{B}) = 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 1 - [P(A) + P(B) + P(A) \cdot P(B)] \quad (\because \ A \text{ and } B \text{ are independent events})$$
$$= 1[1 - P(A)] - P(B)[1 - P(A)]$$
$$= [1 - P(A)][1 - P(B)]$$
$$= P(\bar{A}) \cdot P(\bar{B})$$

Hence, the result.

(b) $B = (A \cap B) \cup (\bar{A} \cap B)$

$A \cap B$ and $\bar{A} \cap B$ are mutually exclusive events

$$\therefore \quad P(B) = P(A \cap B) + P(\bar{A} \cap B)$$
$$= P(A) \cdot P(B) + P(\bar{A} \cap B)$$
$$\Rightarrow \quad P(\bar{A} \cap B) = P(B)[1 - P(\bar{A})]$$
$$= P(\bar{A}) \cdot P(B)$$

$\therefore \quad \bar{A}$ and B are independent events.

(c) Same as (b).

Remark: Term 'independent' is defined in terms of probability of events whereas mutually exclusive is defined in terms of events. Moreover, mutually exclusive events never have an outcome common but independent events do have common outcome(s) provided each event is non-empty. Clearly 'independent' and 'mutually exclusive' do not have the some meaning.

Example 17 A dice is rolled. If the outcome is an even number what is the probability that it is prime?

Solution In rolling a dice, the sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let A be the event that the number is even = $\{2, 4, 6\}$ and B be the event that number is prime = $\{2, 3, 5\}$.

Then $A \cap B = \{2\} \Rightarrow P(A \cap B) = 2/6 = 1/3$

and $P(A) = 3/6 = 1/2$

$\therefore$ P(getting a prime number/getting an event number)

$$= \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

Example 18 A family has two children. What is the probability that both are girls, given that at least one of them is a girl?

Solution

The sample space $S = \{(b, b), (b, g), (g, b), (g, g)\}$,
where b = boy, g = girl.

Let A and B be the events that both are girls and at least one is girl respectively. Then

$$\therefore \qquad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Example 19 Find the probability of a 4 turning up at least once in two throws of a fair die.

Solution Let A be the event that a 4 turns up in first throw and B be the event that a 4 turns up in second throw.

$$\therefore \qquad P(A) = P(B) = 1/6$$

$$\therefore \qquad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

but A and B are independent events

$$\therefore \qquad P(A \cap B) = P(A) \cdot P(B)$$

$$\therefore \qquad P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$$

$$= \frac{1}{6} + \frac{1}{6} - \frac{1}{6} \cdot \frac{1}{6} = \frac{11}{36}$$

Example 20 The probability that a teacher will take an unannounced test during any class hour is $\frac{1}{6}$. If a student is twice absent, what is the probability that he will miss at least one test?

Solution Let A be the event that the first test held on his first day of absence and B be the event of second test held on second day of his absence. A and B are independent events. Probability that he will miss at least one test is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A) \cdot P(B)$$

$$= \frac{1}{6} + \frac{1}{6} - \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$$

$$= \frac{11}{36}$$

Example 21 If $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{3}$ and $P(A\cup B)=\frac{2}{3}$, then prove that $\bar{A}$ and $\bar{B}$ are independent events.

Solution

Given $$P(A)=\frac{1}{2}\Rightarrow P(\bar{A})=\frac{1}{2}$$

$$P(B)=\frac{1}{3}\Rightarrow P(\bar{B})=\frac{2}{3}$$

and $$P(A\cup B)=\frac{2}{3}$$

We know that

$$P(\bar{A}\cap\bar{B})=P(\overline{A\cup B})=1-P(A\cup B)=1-\frac{2}{3}=\frac{1}{3}$$

To prove $\bar{A}$ and $\bar{B}$ are independent events, we must have

$$P(\bar{A})\cdot P(\bar{B})=\frac{1}{3}$$

$$\therefore \quad \frac{1}{2}\cdot 2/3=\frac{1}{3}$$

Hence, the result.

Example 22 A bag contains 10 red and 15 white balls. Two balls drawn in succession. What is the probability that one of them is white and other is red?

Solution Let A be the event that drawn ball is white and B be the event that drawn ball is red.

Total number of cases of drawing two balls = ${}^{25}C_2$

$$P(A)=\frac{15}{25} \text{ and } P(B)=\frac{10}{24}$$

The probability that one of them is white and then is red

$$=P(A\cap B)=P(A)\cdot P(B)$$

$$=\frac{15}{25}\cdot\frac{10}{24}=\frac{1}{4}.$$

Example 23 A pair of dice is thrown, in which one is black and other is yellow and the events A and B are defined as follows:

A is the event that doubles is rolled and B is the event that yellow dice shown a 1 or 4. Are the events A and B independent?

Solution Number of favourable cases to A = 6

$$\therefore \quad P(A)=\frac{6}{36}=\frac{1}{6}$$

Number of favourables cases to B = 12

$\therefore \qquad P(B) = \frac{12}{36} = \frac{1}{3}$

and $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

$$\therefore \qquad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{18}\bigg|\frac{1}{3} = \frac{1}{6},\ P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{18}\bigg|\frac{1}{6} = \frac{1}{3}$$

$$P(A) \cdot P(B) = P(A \cap B) = 1/18$$

But $\qquad P(A) = 1/6 = P(A|B),\ P(B|A) = P(B)$, and

$$P(A \cap B) = P(A)P(B)$$

$\therefore$ A and B are independent events.

Example 24 If $P(A_1) = 0.5$, $P(A_2) = 0.4$ and $P(A_3) = 0.3$ and $P(A_1 \cap A_2 \cap A_3) = 0.06$. Are A_1, A_2 and A_3 independent events?

Solution For independence

$$P(A_1 \cap A_2 \cap A_3) = P(A) \cdot P(A_2) \cdot P(A_3)$$
$$= 0.5 \times 0.4 \times 0.3 = .06 = P(A_1 \cap A_2 \cap A_3)$$

Yes, A_1, A_2, A_3 are independent events.

Example 25 Find $P(A|B)$ if (i) $A \cap B = \Phi$, (*ii*) $A \subset B$, and (iii) $B \subset A$.

Solution

(i) Given $A \cap B = \Phi$

$$\therefore \quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\Phi)}{P(B)} = 0$$

(ii) Given $A \subset B \Rightarrow A \cap B = B \Rightarrow P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

(iii) Given $B \subset A \Rightarrow A \cap B = A$

$$\therefore \qquad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

Example 26 A and B take turns in throwing two dice and the person who throws 9 first is to be awarded a prize. Show that if A has the first turn, their chances of winning the prize are in the ratio of 9:8.

Solution Let E be the event of getting a sum of a when two dice are throwing. Following cases are favourable to the event E.

First dice	3	4	5	6	
Second dice	6	5	4	3	(i.e., 4)

$\therefore \qquad P(E) = 1/9 = p$(say)

and $\qquad P(\bar{E}) = 8/9 = q$(say)

According to the question, A will win if he throws 9 in first, third or fifth, ... throws.

$\therefore$ Chance of A' winning $A = p + q^2p + q^4p + \cdots$

$$= p(1+q^2+q^4+\cdots) = p\left(\frac{1}{1-q^2}\right) = \frac{p}{1-q^2} = \frac{1/9}{1-\left(\frac{8}{9}\right)^2}$$

$$= \frac{\frac{1}{9}}{17/81} = \frac{9}{17}$$

Chances of winning B is $= 1-\frac{9}{17} = \frac{8}{17}$

$\therefore$ Ratio of chances of winning A and B are $\frac{9}{17} : \frac{8}{17}$

i.e. 9 : 8

Example 27 Ram and Rahul appear in an interview for two vacancies of the same post. The probability of Ram's selection is $\frac{1}{7}$ and that of Rahul's selection is $\frac{1}{5}$. What is the probability that

(i) both of them will be selected?
(ii) only one of them will be selected?
(iii) none of them will be selected?

Solution Let A and B respectively be the event that Ram and Rahul will be selected.

Given $P(A) = \frac{1}{7}, P(B) = \frac{1}{5}$

(i) P(Both will be selected) $= P(A \cap B) = P(A)\cdot P(B) = \frac{1}{7}\cdot\frac{1}{5} = \frac{1}{35}$

(A and B are independent events)

(ii) P(only one of them will be selected) $= P(A \cap \bar{B} \text{ or } \bar{A} \cap B)$

$$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

($\because$ $(A \cap \bar{B})$ and $(\bar{A} \cap B)$ are mutually exclusive events)

$$= P(A)P(\bar{B}) + P(\bar{A})P(B)$$

$$= \frac{1}{7}\left(1-\frac{1}{5}\right) + \left(1-\frac{1}{7}\right)\left(\frac{1}{5}\right) = \frac{2}{7}$$

(iii) P (none of them will be selected) $= P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$

$$= \left(1-\frac{1}{7}\right)\left(1-\frac{1}{5}\right) = \frac{24}{35}$$

6.5.3 Extension of Multiplication Law of Probability to *n* Events

Let $A_1, A_2, \ldots, A_n$ be n events, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2), \cdots P(A_n | A_1 \cap A_2 \cap \cdots A_{n-1})$$

where $P(A_i | A_j \cap A_k \cap \cdots A_l)$ be the conditional probability of the event A_i given that the events A_j, $A_k \ldots, A_l$ have already happened.

(i) *Extension of Multiplication Law of Probability for n-independent Events*—Let $A_1, A_2, \ldots, A_n$ be n-independent events then

$$P(A_1 \cap A_2) \cap A_3 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdots P(A_n) \quad (1)$$

The above result can be proved using the concept of independence, if $A_1, A_2, \ldots, A_n$ are independent events then

$$P(A_2 | A_1) = P(A_2),\ P(A_3 | A_1 \cap A_2) = P(A_3), \cdots, P(A_n | A_1 \cap A_2 \cap \ldots \cap A_{n-1}) = P(A_n)$$

$$\Rightarrow \quad P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdots P(A_n)$$

(ii) *Pairwise Independent Events*—Let $A_1, A_2, \ldots, A_n$ defined on the same space so that $P(A_i) > 0$; $i = 1, 2, \ldots, n$, then these events are said to be pairwise independent, if every pair of two events is independent, i.e.,

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j),\ i \neq j = 1, 2, \ldots, n$$

(iii) *Mutually Independent Events*—Let S denote the sample space for a number of events. The events in S are said to be mutually independent if the probability of the simultaneous occurrence of any finite number of them is equal to the product of their separate probabilities, i.e. if $A_1, A_2, \ldots, A_n$ be n events in a sample space S, then they are said to be mutually independent if

$$P(A_{i1} \cap A_{i2} \cap \cdots \cap A_{ik}) = P(A_{i1})\, P(A_{i2}) \cdots P(A_{ik})$$

Hence, the events are mutually independent if they are independent by pairs, by triplets, and by quadruples and so on.

Example 28 Two fair dice thrown independently. Three events A, B and C are defined as follows:

(i) Odd face with the first dice.
(ii) Odd face with the second dice.
(iii) Sum of the number in the two dice are odd.
Are A, B and C mutually independent?

Solution A and B be the events in which odd face is with first and second dice respectively. When two dice are theorem then sample space S is as follows:

$$S = \begin{Bmatrix} (1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6);\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6) \\ (3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6);\ (4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6) \\ (5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6);\ (6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6) \end{Bmatrix}$$

Now $\quad P(A) = \dfrac{18}{36} = 1/2, \quad P(B) = 18/36 = 1/2$

The probability of sum of the numbers on the two dice is odd i.e., $P(C) = 1/2$

We have $P(A \cap B) = P(B \cap C) = P(A \cap C) = 1/4$

$$\therefore \quad P(A \cap B) = P(A) \cdot P(B)$$
$$P(A \cap C) = P(A) \cdot P(C)$$
$$P(B \cap C) = P(B) \cdot P(C)$$

Now we have $P(A \cap B \cap C) = 0$ because C can not happen when A and B happen.

Hence, $\quad P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$

$\therefore$ The events A, B and C are pairwise independent but not mutually independent.

Example 29 For any three events A, B and C, show that

$$P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$$

Solution We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P[(A \cap C) \cup (B \cap C)] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

Dividing both sides by $P(C)$, we get

$$P\left[\frac{(A \cap C) \cup (B \cap C)}{P(C)}\right] = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}, P(C) > 0$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad \frac{P[(A \cup B \cap C)]}{P(C)} = P(A|C) + P(B|C) - P(A \cap B|C)$$

$$\Rightarrow \quad P\left[(A \cup B)|C\right] = P(A|C) + P(B|C) - P(A \cap B|C)$$

Example 30 Let A_1, A_2 and A_3 be three events defined on a sample space S such that $A_2 \subset A_3$ and $P(A_1) > 0$, then prove that $P(A_2|A_1) \leq P(A_3|A_1)$.

Solution

$$P(A_3|A_1) = \frac{P(A_3 \cap A_1)}{P(A_1)} = \frac{P\left\{(A_2 \cap A_3 \cap A_1) \cup (\bar{A}_2 \cap A_3 \cap A_1)\right\}}{P(A_1)}$$

$$= \frac{P\left\{(A_2 \cap A_3 \cap A_1)\right\}}{P(A_1)} + \frac{P(\bar{A}_2 \cap A_3 \cap A_1)}{P(A_1)}$$

$\left[\because \quad (A_2 \cap A_3 \cap A_1) \text{ and } (\bar{A}_2 \cap A_3 \cap A_1) \text{ are mutually exclusive events})\right]$.

$$\Rightarrow \quad P(A_3|A_1) = P(A_2 \cap A_3|A_1) + P(\bar{A}_2 \cap A_3|A_1)$$

$$= P(A_2|A_1) + P(\bar{A}_2 \cap A_3|A_1) \quad \begin{bmatrix} \because & A_2 \subset A_3 \\ \Rightarrow & A_2 \cap A_3 = A_2 \end{bmatrix}$$

$$\Rightarrow \quad P(A_3|A_1) \geq P(A_2|A_1) \quad \left[\because \quad \bar{A}_2 \cap A_3|A_1 \geq 0\right]$$

Example 31 If A, B and C are mutually independent events then $A \cup B$ and C are also independent.

Solution We have to prove

$$P[(A \cup B) \cap C] = P(A \cup B) \cdot P(C)$$

$$\text{LHS} = P[(A \cap C) \cup B \cap C] \quad \text{(By distributive law)}$$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

$$= P(A) \cdot P(C) + P(B) \cdot P(C) - P(A) \cdot P(B) \cdot P(C)$$ [$\because$ A,B and C are mutually independent]

$$= P(C)[P(A) + P(B) - P(A \cap B)] = P(C) \cdot P(A \cup B) = \text{RHS}$$

Hence, proved.

$\Rightarrow$ $A \cup B$ and C are independent events.

Example 32 A book on statistics is independently reviewed by three reviewers favourably with probabilities $\frac{3}{5}, \frac{4}{7}$ and $\frac{2}{5}$ respectively. What is the probability that of the reviews:

(a) All will be favourable

(b) At least two reviews will be favourable

Solution Let A_1, A_2 and A_3 be the events that the book on statistics is reviewed favourably by first, second and third reviewer. Then we are given

$$P(A_1) = \frac{3}{5}, P(A_2) = \frac{4}{7} \text{ and } P(A_3) = \frac{2}{3}$$

(a) P(All three will be favourable) $= P(A_1 \cap A_2 \cap A_3)$

$$= P(A_1) \cdot P(A_2) \cdot P(A_3)$$ [$\because$ A_1, A_2 and A_3 are mutually independent]

$$= \frac{3}{5} \times \frac{4}{7} \times \frac{2}{5} = \frac{24}{175}$$

(b) P(At least two reviews will be favourable)

$$= P\left[(A_1 \cap A_2 \cap \bar{A}_3) \cup (A_1 \cap \bar{A}_2 \cap A_3) \cup (\bar{A}_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3)\right]$$

$$= P(A_1) \cdot P(A_2) \cdot P(\bar{A}_3) + P(A_1) \cdot P(\bar{A}_2) \cdot P(A_3) + P(\bar{A}_1) \cdot P(A_2) \cdot P(A_3) + P(A_1) \cdot P(A_2) \cdot P(A_3)$$

[$\because$ all events are mutually exclusive and A_1, A_2, A_3 are mutually independent]

$$= \frac{3}{5}\times\frac{4}{7}\times\frac{3}{5}+\frac{3}{5}\times\frac{3}{7}\times\frac{2}{5}+\frac{2}{5}\times\frac{4}{7}\times\frac{2}{5}+\frac{3}{5}\times\frac{4}{7}\times\frac{2}{5}$$

$$= \frac{94}{175}$$

EXERCISE 6.2

1. A committee consists of students, two of which are from 1^{st} year, three from 2^{nd} year and four from 3^{rd} year. Three students are removed at random. What is the chance that (i) the three students belong to different classes, (ii) two belong to the same class and third to the different classes, (iii) the three belong to the same class.

(V.T.U. 2002S)

2. What is the chance of that a non-leap year should have 53 Saturdays.

(Madras 2003)

3. A bag contains 7 red and 12 white balls. Find the probability of drawing a red ball.
4. In a single throw of two dice, find the probability of getting a total of 9 or 11.
5. Given $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$ and $P(A\cap B)=\frac{1}{4}$, find the value of $P(A\cup B)$.

(Burdwan 2003)

6. Let A and B be two events with $P(A)=\frac{1}{2}, P(B)=\frac{1}{2}$ and $P(A\cap B)=\frac{1}{4}$. Find $P(A/B)$, $P(A\cup B)$.

(Kurukshetra 2009, V.T.U. 2003S)

7. When a coin is tossed four times, find the probability of getting (i) exactly one head (ii) at most three heads and (iii) at least two heads.

(V.T.U. 2003)

8. Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.

(P.T.U. 2003)

9. Suppose 5 cards are drawn at random from a pack of 52 cards. If all cards are red, what is the probability that all of them are hearts?

(Mumbai 2005)

10. If the odds in favour of an event are 4 to 5. Find the probability that it will occur.
11. Find the probability of drawing either an ace or a spade or both from a pack of 52 cards.
12. In a class of 66 students 13 are boys and the rests are girls. Find the probability that a selected student will be a girl.
13. Given that a boy will pass an examination in $\frac{2}{5}$ and for a girl it is $\frac{2}{5}$. What is the probability that at least one of them passes examination?
14. A bag contains discs of which 4 are red, 3 are blue and 2 are yellow. A disc is drawn at random from the bag. Find the probability that it will be (i) red, (ii) yellow, (iii) blue (iv) not blue.
15. For any two events A and B, prove that

(i) $P(A\cap \overline{B}) = P(A) - P(A\cap B)$

(ii) $P(\overline{A}\cap B) = P(B) - P(A\cap B)$

16. If $B \subset A$, then show that

    (i) $P(A \cap \overline{B}) = P(A) - P(B)$

    (ii) $P(B) \le P(A)$

    (iii) $P(A \cap B) \le P(A)$ and

    (iv) $P(A \cap B) \le P(B)$

17. If A, B and C are three mutually exclusive events. In this assignment, are the following probabilities possible?
    $P(A) = 0.3$, $P(B) = 0.4$ and $P(C) = 0.5$
18. If $P(A) = 0.9$ and $P(B) = 0.6$, then show that $P(A \cap B) \ge 0.5$.
19. If $P(A) = 0.9, P(B) = 0.6$ and $P(A \cap B) = 0.5,$ determine (i) $P(A \cap \overline{B})$ and (ii) $P(\overline{A} \cap \overline{B})$.
20. Three groups of children contains 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys respectively. One child is selected at random from each group. What is the probability that among three selected children one is girl and 2 are boys.
21. Two computers C_1 and C_2 are to be marketed. A salesman who is assigned the job of finding customers for them has 60% and 40% chances respectively of succeeding in case of computers C_1 and C_2. The computers can be sold independently. Given that he was able to sell at least one computer, what is the probability that computer C_1 has been sold?
22. Amar can either take a course in computer or mathematics. If Amar takes the computer course, then he will get an A grade with probability $\frac{1}{2}$, if he takes the mathematics course then he will get an A grade with probability $\frac{1}{3}$. Amar decides to base his decision on the flip of a fair coin. What is the probability that Amar will get an A grade in mathematics?
23. A pair of dice is tossed twice. Find the probability of scoring 7 points (a) once (b) at least once (c) twice. **(Kurukshetra 2009S; V.T.U. 2004)**
24. A box A contains 2 white and 4 black balls. Another box B contains 5 white and 7 black balls. A ball is transferred from the box A to box B. Then a ball is drawn from the box B. Find the probability that it is white. **(V.T.U. 2004)**
25. A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd? **(Mumbai 2006)**
26. Two persons A and B toss an unbiased coin alternatively on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning. **(Madras 2000S)**
27. Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability p that the sum is odd if
    (i) the two cards are drawn together.
    (ii) the two cards are drawn one after the other without replacement.
    (iii) the two cards are drawn one after the other with replacement.
28. The odds that a book will be reviewed favourably by three independent critics are 5 to 2, 4 to 3 and 3 to 4. What is the probability that of the three reviews, a majority will be favourable? **(V.T.U. 2003)**

29. I can hit A target 3 times in 5 shots, B target 2 times in 5 shots and C target 3 times in 4 shots. They fire a volley. What is the probability that (i) two shots hit (ii) at least two shots hit?
(A.M.I.E.T.E, 2003, Madras 2000S)

30. A problem in mechanics is given to three students A, B and C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved?
(V.T.U. 2004)

31. The students in a class are selected at random one after the other for an examination. Find the probability p that the boys and girls in the class alternate if
    (i) the class consists of 4 boys and 3 girls.
    (ii) the class consists of 3 boys and 3 girls.
(J.N.T.U. 2003)

32. A purse contains 2 silver and copper coins and a second purse contains 4 silver and 4 copper coins. If a coin is selected at random from one of the two purses, what is the probability that it is a silver coin?
(Osmania 2002)

33. A box I contains 5 white balls and 6 black balls. Another box II contains 6 white balls and 4 black balls. A box is selected at random then a ball is drawn from it; (i) what is the probability that the ball drawn will be white? (ii) Given that the ball drawn is white, what is the probability that it came from box I.
(Mumbai 2006)

34. A speaks the truth in 75% cases and B in 80% cases. In what percentage of cases, are they like to contradict each other in stating the same fact?
(V.T.U. 2002S)

35. A student takes his examination in four subjects P, Q, R and S. He estimates his chances of passing in P as $\frac{4}{5}$, in Q as $\frac{3}{4}$ in R as $\frac{5}{6}$ and in S as $\frac{2}{3}$. To qualify, he must pass in P and at least two other subjects. What is the probability that he qualifies?
(Madras 2000S)

36. Let A_1 and A_2 be mutually exclusive events such that $P(A_1), P(A_2) > 0$. Show that these events are not independent.

37. Let A_1 and A_2 be independent events such that $P(A_1), P(A_2) > 0$. Show that these events are not mutually exclusive events.

38. If A, B and C are random events in a sample space and if A, B and C are pairwise independent and A is independent of $(B \cup C)$, then show that A, B and C are mutually independent.

39. Consider the example 32 and calculate the probability
    (a) Exactly one review will be favourable.
    (b) Exactly two reviews will be favourable.
    (c) At least one of the reviews will be favourable.

Answers

1.(i) $\frac{2}{7}$ (ii) $\frac{55}{84}$ (iii) $\frac{5}{84}$

2. $\frac{1}{7}$ 3. $\frac{7}{19}$ 4. $\frac{1}{6}$

5. 7/12 6. $P(A/B)=\frac{3}{4}, P(A\cup B)=\frac{7}{12}$

7. (i) $\frac{1}{4}$ (ii) $\frac{7}{8}$ (iii) $\frac{11}{16}$

8. $\frac{15}{1024}$ 9. 0.11 10. $\frac{4}{9}$

11. $\frac{4}{13}$ 12. $\frac{53}{66}$ 13. $\frac{19}{25}$

14. (i) $\frac{4}{9}$ (ii) $\frac{2}{9}$ (iii) $\frac{1}{3}$ (iv) $\frac{2}{3}$

17. No $\therefore\ P(A\cup B\cup C)=P(A)+P(B)+P(C)=1.2>1$

19. (i) 0.4 (ii) 0

20. $\frac{13}{22}$ 21. 0.7895 22. $\frac{1}{6}$

23. (a) $\frac{5}{18}$ (b) $\frac{11}{36}$ (c) $\frac{1}{36}$ 24. $\frac{16}{39}$

25. $\frac{1}{1+q}$ (where p is the probability of getting a head and q is the prob. of getting a tail).

26. Chance of A's winning = 2/3 and chance of B's winning is = $\frac{1}{3}$.

27. (i) p = 5/9 (ii) $p=\frac{5}{9}$ (iii) p = 1/2

28. $\frac{209}{343}$ 29.(i) 0.45 (ii) 0.63

30. $\frac{3}{4}$ 31.(i) $\frac{1}{35}$ (ii) $\frac{1}{10}$

32. $\frac{5}{12}$ 33.(i) $\frac{83}{110}$ (ii) $\frac{25}{83}$

34. $\frac{7}{20}$ 35. $\frac{61}{90}$

39. (a) $\frac{63}{175}$ (b) $\frac{70}{175}$ (c) $\frac{157}{175}$

6.6 BAYE'S THEOREM

Before we discuss Baye's theorem, we shall define the rule of elimination or theorem of total probability.

6.6.1 Rule of Elimination or Theorem of Total Probability

If the event A can occur along with event E. Suppose that event E occur in n mutually exclusive ways $E_1, E_2, \ldots, E_n$, then

$$P(A) = \sum_{i=1}^{n} P(E_i) \cdot P(A|E_i), \text{ provided } P(E_i) > 0 \; \forall \; i = 1, 2, ..., n.$$

Proof

$$A = A \cap E_1 \cup A \cap E_2 \cup \cdots \cup A \cap E_n$$

Events $(A \cap E_1), (A \cap E_2), \cdots (A \cap E_n)$ are mutually exclusive events.

$$\therefore \qquad P(A) = P\left[(A \cap E_1) \cup (A \cap E_2) \cup \cdots \cup (A \cap E_n)\right]$$

$$\Rightarrow \qquad P(A) = P(A \cap E_1) + P(A \cap E_2) + \cdots + P(A \cap E_n)$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

Using conditional probability

$$P(A \cap E_i) = P(E_i) \cdot P(A|E_i)$$

$$\therefore \qquad P(A) = \sum_{i=1}^{n} P(E_i) \cdot P(A|E_i)$$

6.6.2 Baye's Theorem

Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive events of which one of them must occur. Let A be any event, then

$$P(E_i|A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i) \cdot P(A|E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A|E_i)}$$

Example 33 A consulting firm rents cars from three agencies A_1, A_2 and A_3. 20% of the cars are rented from A_1, 20% from A_2 and remaining 60% from A_3. If 20% of the cars rented from A_1, 10% of cars rented from A_2 and 2% of cars rented from A_3 have bad tyres. What is the probability that a car rented from consulting firm will have bad tyres?

Solution Let E_1, E_2 and E_3 be the events that cars are rented from firms A_1, A_2 and A_3 respectively, and let A be the event that car has bad tyres.

Given $\qquad P(E_1) = 0.20, P(E_2) = 0.20$ and $P(E_2) = 0.60$

$$P(A|E_1) = 0.20, P(A|E_2) = 0.10, P(A|E_3) = 0.02$$

$$\therefore \qquad P(A) = \sum_{i=1}^{3} P(E_i) \cdot P(A|E_i)$$

$$= 20 \times 0.20 + 0.20 + 0.10 + 0.60 \times 0.02 = 0.072$$

Example 34 A bag contains 3 black and 4 white balls. Two balls are drawn at random one at a time without replacement. What is probability that second ball is white?

Solution Let E_1 and E_2 be the events that first ball is black and first ball is white respectively. Let A be the event that second ball is white.

then $$A = (A \cap E_1) \cup (A \cap E_2)$$

$$P(A \cap E_1) = P(E_1) \cdot P(A|E_1)$$

$$= \left(\frac{3}{7}\right)\left(\frac{4}{6}\right) = \frac{2}{7}$$

Similarly $P(A \cap E_2) = P(E_2) \cdot P(A/E_2)$

$$= \left(\frac{4}{7}\right)\left(\frac{3}{6}\right) = \frac{2}{7}$$

$$\therefore \qquad P(A) = P(A \cap E_1) + P(A \cap E_2) = \frac{4}{7}$$

Example 35 At an electronics firm, it is known from past experience that the probability a new worker who attended the company's programme meets the production quota is 0.90. The corresponding probability for a new worker who did not attend the training programme is 0.25. It is also known that 80% of all new workers attend the company's training programme. Find the probability that a new worker who met the production quota would have attended the company's training programme?

Solution Let E_1 be the event that a new worker attended the company's training programme and E_2 be the event that a new worker did not attend the company's training programme. Let A be the event that a new worker met the production quota.

$$\therefore \qquad P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)} \qquad \text{(Using Baye's Thoerem)}$$

where $$P(E_1) = 0.80,\ P(A|E_1) = 0.90$$

$$P(E_2) = 0.20,\ P(A|E_2) = 0.25$$

$$\therefore \qquad P(E_1|A) = \frac{0.80 \times 0.90}{0.80 \times 0.90 + 0.20 \times 0.25} = \frac{72}{77}$$

Example 36 There are 3 true coins and one false coin with 'head' on both sides. A coin is chosen at random and tossed 4 times. If 'head' occurs all the 4 times, what is the probability that the false coin has been chosen and used?

Solution Let T and F be the events of being true coin and false coin respectively, then

$$P(T)=\frac{3}{4},\ P(F)=\frac{1}{4}$$

Let A be the event of getting 4 heads in 4 losses, then

$$P(A|T)=\left(\frac{1}{2}\right)^4=\frac{1}{16},\text{ and } P(A|F)=1$$

$$\therefore\quad P(F|A)=\frac{P(F)\cdot(A|F)}{P(F)\cdot P(A|F)+P(T)\cdot P(A|T)}\qquad\text{(By Baye's Theorem)}$$

$$=\frac{\frac{1}{4}\cdot 1}{\frac{1}{4}\cdot 1+\frac{3}{4}\cdot\frac{1}{16}}=\frac{\frac{1}{4}}{19/64}=\frac{16}{19}$$

Example 37 In a certain college 25% of boys and 10% of girls are studying mathematics. The girls constitute 60% of the student body.

(i) What is the probability that mathematics is being studied.

(ii) If a student is selected at random and is found to be studying mathematics, find the probability that student is a girl?

Solution Let G be the event that student is girl and B be the event that student is boy. Let M be the event that mathematics being studied.

$$\therefore\quad P(G)=0.60,\ P(B)=1-0.60=0.40$$

The probability that mathematics is studied that a student is boy i.e., $P(M|B) = 0.25$

Similarly $P(G|B) = 0.10$

(i) The probability that mathematics being studied

$$=P(M)=P(G)\cdot P(M|G)+P(B)\,P(M|B)$$
$$=0.60\times 0.10+0.40\times 0.25$$
$$=0.06+0.10=0.16$$

(ii) The probability that a mathematics student is a girl

$$=P(G|M)=\frac{P(G)\cdot P(M|G)}{P(M)}$$
$$=\frac{0.60\times 0.10}{0.16}=\frac{.06}{.16}=\frac{3}{8}$$

EXERCISE 6.3

1. An urn I contains 3 white and 4 red balls and an urn II contains 5 white and 6 red balls. One ball is drawn at random from one of the urn and is found to be white, find the probability that it was drawn from urn I.

2. For example 36, if a student is selected at random and is found to be studying mathematics, find the probability that student is a boy.
3. There are three bags: first containing 1 white, 2 red, 3 green balls; second containing 2 white, 3 red, 1 green balls and third containing 3 white, 1 red and 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag. **(J.N.T.U. 2003)**
4. The contents of three urns are; 3 white, 2 red, 3 green balls; 2 white, 1 red, 1 green balls and 4 white, 5 red and 3 green balls. Two balls are drawn from an urn chosen at random. These are found to be one white and one green. Find the probability that the balls are drawn came from the third urn. **(Kurukshetra 2007)**
5. In a bolt factory, there are four machines *A*, *B*, *C*, *D* manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2% in the same order are defective bolts. A bolt is chosen at random from the factory's production and found defective. What is the probability that the bolt was manufactured by machine *A* or machine *D*. **(Hissar 2007, J.N.T.U. 2003)**
6. In a bolt factory, machines *A*, *B* and *C* manufactures 25%, 35% and 40% of the total of their inputs 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines *A*, *B* or *C*? **(V.T.U. 2006, Rohtak 2005, Madras 2000S)**

Answers

1. $\dfrac{33}{68}$
2. 5/8
3. $\dfrac{6}{11}$
4. $\dfrac{15}{59}$
5. 0.3175, 0.254
6. $\dfrac{25}{69}, \dfrac{28}{69}, \dfrac{16}{69}$

6.7 RANDOM VARIABLE

A random variable *X* is a function that associates a real number with each element in the sample space *S*.

or

Let *E* be a random experiment and *S* be a sample space associated with it, a function $X(s)$. Where $s \in S$ is called a random variable.

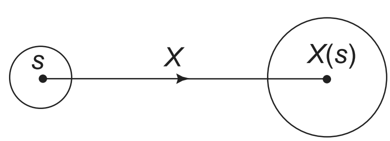


Fig. 6.5

or

If a real variable X is associated with the outcome of a random experiment is called a random variable. Suppose that our experiment E consists of tossing two fair coins, letting X denote the heads appearing then X is a random variable taking one of the values 0, 1, 2 with respective probabilities as follows:

$$P(X=0)=P(T,T)=\frac{1}{4}$$

$$P(X=1)=P\left[(H,T),(T,H)\right]=1/2$$

$$P(X=2)=P(H,H)=1/4$$

We have $P(X=0)+P(X=1)+P(X=2)=1$

Hence X, is a random variable taking values 0, 1, 2. As the values of X depend on chance, it is also called a chance variable, stochastic variable or simply a variable.

6.8 TYPES OF RANDOM VARIABLE

A random variables can be classified in two types:

(i) Discrete Random Variable, and

(ii) Continuous Random Variable.

6.8.1 Discrete Random Variable

A random variable X is said to be discrete, if its set of possible outcomes is countable or if it assumes only a finite number of countably infinite values of X. For example, a number of students who fails in a quiz marks obtained in an examination, number of telephone calls per unit time, number of accidents per month and number of complaints received at the office of police station in a week, etc.

6.8.2 Continuous Random Variable

A random variable X is said to be continuous if it takes the infinite number of values in an interval or when a random variable can take on values on a continuous scale. For example, height of a person, distance of two cities, price of a house, weight of a student, temperature of a room, etc.

6.9 DISCRETE PROBABILITY DISTRIBUTION

6.9.1 Probability Mass Function

A discrete random variable assumes each of its values with a certain probability. If X is a discrete random variable with distinct values $x_1, x_2, \ldots, x_n, \ldots$, then the function $p(x)$ is defined as

$$\begin{aligned} P(x) &= P(X=x_i)=p_i \quad \text{if } x=x_i \\ &= 0, \qquad\qquad\qquad \text{if } x\neq x_i,\ i=1,2,\ldots \end{aligned}$$

where (i) $p(x_i) \geq 0$ for all values of i and $\sum_{\forall i} P(x_i)=1$ is called the probability mass function (p.m.f.) of random variable X. Probability mass function is also known as discrete probability distribution or discrete density function.

The set of ordered pains $\{(x_1, p_1), (x_2, p_2), ..., (x_n, p_n), \cdots\}$ specified the probability distribution of the random variable X.

6.9.2 Distribution Function

If X is a discrete random variable, then the function $F(x)$ is defined as

$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$ where x is any integer is called distribution function of X, it is also known cumulative distribution function (CDF) of X. The domain of distribution function is $(-\infty, \infty)$ and its range is $[0, 1]$. The graph of $F(x)$ will be stair step form as shown in Fig. 6.6.

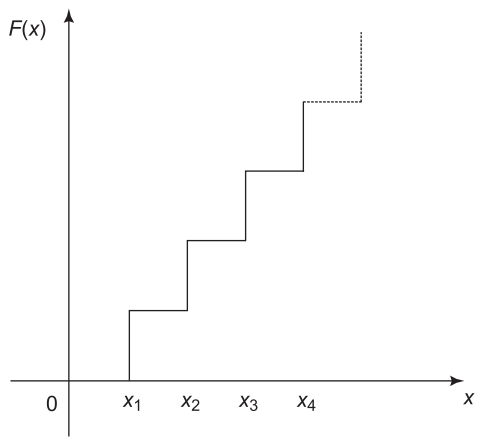


Fig. 6.6

Properties of F(x)

(i) The domain of $F(x)$ is $(-\infty, \infty)$ and range is $[0, 1]$

(ii) $0 \le F(x) \le 1$

(iii) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$ and $F(\infty) = \lim_{x \to \infty} F(x) = 1$

(iv) If $x < y$, then $P(x < x \le y) = F(y) - f(x)$

(v) If $x \le y$, then $F(x) \le f(y)$

(vi) If $x < y$, then

$$P(x \le X \le y) = F(y) - F(x) + P(X = x)$$

$$P(x < X < y) = F(y) - F(x) - P(X = x) \text{ and } P(x \le X < y) = F(y) - F(x) + P(X = x)$$

Example 38 If a pair of fair dice is rolled, find the probability distribution for getting their sum 2, 3, 4, …, 12.

Solution If a pair of fair dice is rolled, then

$$S = \{(1, 2, 3, 4, 5, 6) \times (1, 2, 3, 4, 5, 6)\}$$

$$= \left\{\begin{matrix}(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6);\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6); \\ (3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6);\ (4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6); \\ (5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6);\ (6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\end{matrix}\right\}$$

Let X be the random variable defined as the sum of two fair dice, then

$$\begin{aligned}
P(X=2) &= P(1,1) &= 1/36 \\
P(X=3) &= P[(1,2),(2,1)] &= 2/36 \\
P(X=4) &= P[(1,3),(2,2)\ (3,1)] &= 3/36 \\
P(X=5) &= P[(1,4),(2,3),(3,2),(4,1)] &= 4/36 \\
P(X=6) &= P[(1,5),(2,4),(3,3),(4,2),(5,1)] &= 5/36 \\
P(X=7) &= P[(1,6),(2,5),(3,4)\ (4,3),(5,2),(6,1)] &= 6/36 \\
P(X=8) &= P[(2,6,(3,5),(4,4),(5,3),(6,2)] &= 5/36 \\
P(X=9) &= P[(3,6),(4,5),(5,4),(6,3)] &= 4/36 \\
P(X=10) &= P[(4,6),(5,5),(6,4)] &= 3/36 \\
P(X=11) &= P[(5,6),(6,5)] &= 2/36 \\
P(X=12) &= P[(6,6)] &= 1/36
\end{aligned}$$

Hence the probability distribution of X is given as follows:

$X = x$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Example 39 Find the distribution function for the following probability distribution:

$X = x$	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Solution

$$F(x) = P(X \le x) = \sum_{i=0}^{x} p(x_i)$$

$$F(0) = \frac{1}{16},\ F(1) = p(0) + p(1) = 5/16,\ F(2) = p(0) + p(1) + p(2) = \frac{11}{16}$$

$$F(3) = p(0) + p(1) + p(2) + p(3) = \frac{15}{16},\ F(4) = p(0) + p(1) + p(2) + p(3) + p(4) = 1$$

$\therefore$ Distribution function in tabular form is given as follows:

x	0	1	2	3	4
$p(x)$	1/16	4/16	6/16	4/16	1/16
$F(x)$	1/16	5/16	11/16	15/16	1

Also distribution function can be written as:

$$\begin{aligned} F(x) &= 0, & x &< 0 \\ &= \frac{1}{16}, & 0 \le x &< 1 \\ &= \frac{5}{16}, & 1 \le x &\le 2 \\ &= \frac{11}{16}, & 2 \le x &< 3 \\ &= \frac{15}{16}, & 3 \le x &< 4 \\ &= 1, & x &\ge 4 \end{aligned}$$

Example 40 A random variable X has the following probability distribution:

$X = x$	–2	–1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$3k$	$2k$	0.3

(i) Determine k and hence,
(ii) Compute $P(X < 2)$, $P(X \ge 2)$,
(iii) Find the minimum value of k such that $P(X \le 1) > 0.32$.

Solution Using definition of probability distribution

(i) $$\sum_{i=-2}^{3} p(x_i) = 1 \Rightarrow 0.1 + k + 0.2 + 3k + 2k + 0.3 = 1$$

$$\Rightarrow \quad 6k = 0.4 \Rightarrow k = \frac{1}{15}$$

(ii) $$P(X < 2) = P(X \le 1) = F(1) = 0.1 + k + 0.2 + 3k = 0.3 + 0.4k = \frac{17}{30}$$

$$P(X \ge 2) = 1 - P(X < 2) = 1 - F(1) = 1 - \frac{17}{30} = \frac{13}{30}$$

(iii) $P(X \le 1) > 0.32$

$$\begin{aligned} F(1) &= 0.1 + k + 0.2 + 3k > 0.32 \\ &= 0.3 + 0.4\,k > 0.32 \\ &= 0.4k > 0.02 \\ &= k > \frac{0.02}{0.4} = 0.005 \end{aligned}$$

Minimum value of $k = 0.005$.

Example 41 From a lot of 10 items containing 3 defective, a sample of 4 items is drawn at random without replacement.

(i) Find the probability distribution of X, the number of defectives.

(ii) Compute $F(x)$ and (iii) Find $P(X \le 1)$, $P(X \ge 1)$, $P(0 < X < 2)$.

Solution

A sample of 4 can be drawn from a lot of 10 items is ${}^{10}C_4$ ways which are equally likely.

Let X be the number of defectives, then the number of defectives x out of 3 can be chosen in ${}^{3}C_x$ ways and remaining 4–x items can be chosen from 7 items in ${}^{7}C_{4-x}$ ways
where $x = 0, 1, 2, 3$, then

$$P(X = x) = \frac{{}^{7}C_{4-x} \cdot {}^{3}C_x}{{}^{10}C_4}$$

(i) Hence probability distribution of X:

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(ii) $F(x) = 0, x < 0$

$$= \frac{1}{6}, 0 \le x < 1$$

$$= \frac{2}{3}, 1 \le x < 2$$

$$= \frac{29}{30}, 2 \le x < 3$$

$$= 1, x \ge 3$$

(iii) $P(X \le 1) = F(1) = p(0) + p(1) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$

$$P(X \ge 1) = p(1) + p(2) + p(3) = 1 - p(0) = 1 - \frac{1}{6} = 5/6$$

$$P(0 < X < 2) = \frac{1}{2} - 0 = 1/2$$

6.10 CONTINUOUS PROBABILITY DISTRIBUTION

6.10.1 Probability Density Function

A continuous random variable has a probability zero of assuming exactly any of its values. The function $f(x)$ is said to be probability density function of a continuous random variable X, if it is defined as

(i) $f(x) \ge 0, -\infty < x < \infty$ (ii) $\int_{-\infty}^{\infty} f(x) = 1,$

(iii) $P(a < X < b) = \int_a^b f(x)\,dx$

6.10.2 Continuous Distribution Function

If X is a continuous random variable, then the function $f(x)$ is defined as:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, -\infty < x < \infty$$

is called the continuous distribution function and $f(x)$ is probability density function.

Properties of F(x)

(i) The domain of $F(x)$ is $(-\infty, \infty)$ and range $[0, 1]$

(ii) $F(x)$ is non-decreasing function of x on the right.

$$F'(x) = f(x) \ge 0$$

(iii) $F(x)$ is a continuous function of X on the right.

(iv) $F(-\infty) = \lim\limits_{x\to-\infty} \int_{-\infty}^{x} f(t)\,\mathrm{d}t = 0$

and $F(\infty) = \lim\limits_{x\to\infty} \int_{-\infty}^{-x} f(t)\,\mathrm{d}t = 1$

(v) $P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = P(a < X \le b) = F(b) - F(a)$

(vi) Since $F'(x) = f(x)$, we have

$$F'(x) = f(x) \Rightarrow \mathrm{d}F(x) = f(x) \qquad \left[\therefore\ F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x)\right]$$

$F(x)$ is known as probability differential of X.

Example 42 If a random variable X has the probability distribution as follows:

$$f(x) = \begin{cases} \frac{1}{4}; & -2 < x < 2 \\ 0; & \text{otherwise} \end{cases}$$

(i) Compute $P(X < 1)$ and (ii) $P[(2X+3) > 5]$

Solution

(i) $P(X < 1) = \int_{-\infty}^{1} f(x)\,\mathrm{d}x = \int_{-2}^{1} \frac{1}{4}\mathrm{d}x = \frac{1}{5}(x)_{-2}^{1} = \frac{3}{4}$

(ii) $P[(2X+3) > 5] = P[2x > 5-3] = P[X > 1] = \int_{1}^{\infty} f(x)\,\mathrm{d}x = 1 - P(X < 1)$

$$1 - \frac{3}{4} = \frac{1}{4}$$

Example 43 If X has the probability density

$$f(x) = \begin{cases} ke^{-3x}; & x > 0 \\ 0; & \text{otherwise} \end{cases}$$

Find the value of k.

Solution

To find $k\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^{\infty} ke^{-3x}dx = 1 \Rightarrow k = 3$

Example 44 The probability density of a random variable X is

$$f(x) = \begin{cases} x; & 0 < x \le 1 \\ 2 - x; & 1 \le x < 2 \\ 0; & x \ge 2 \end{cases}$$

Obtain the cumulative distribution function of X.

Solution

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

(i) For $x < 0$

$$F(x) = \int_{-\infty}^{0} f(t)\,dt = \int_{-\infty}^{0} 0 \cdot dt = 0$$

(ii) For $0 \le x < 1$

$$F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_0^x f(x)\,dx = 0 + \int_0^x x\,dx = \frac{x^2}{2}$$

(iii) For $1 \le x < 2$

$$F(x) = \int_{-\infty}^{2} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_0^1 x\,dx + \int_1^x (2-t)\,dt$$

$$= \frac{1}{2} + \left(2t - \frac{t^2}{2}\right)_1^x = 2x - \frac{x^2}{2} - 1$$

(iv) For $x \ge 2$

$$F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^{\infty} f(t)\,dt$$

$$= \int_{-\infty}^{0} 0 \cdot dt + \int_0^1 x\,dx + \int_1^2 (2-x)\,dx + \int_2^{\infty} 0 \cdot dx = 1$$

$$\therefore \quad F(x) = \begin{cases} 0; & x < 0 \\ \dfrac{x^2}{2}; & 0 \le x < 1 \\ 2x - \dfrac{x^2}{2} - 1; & 1 \le x < 2 \\ 1; & x \ge 2 \end{cases}$$

Example 45 Find a probability density function for the random variable X where distribution function is given by

$$F(x) = \begin{cases} 0; & x \le 0 \\ x; & 0 < x < 1 \\ 1; & x \ge 1 \end{cases}$$

Also, draw the graph.

Solution

To find the density function, we have

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x)$$

Then $$f(x) = \begin{cases} 0; & x \le 0 \\ 1; & 0 < x < 1 \\ 0; & x \ge 1 \end{cases}$$

Also we know that $F(x)$ is continuous from right and as $F(x)$ is not continuous at $x = 0$ and $x = 1$, we put $f(0)$ and $f(1)$ both equal to 0 as it does not matter how the probability density is defined at these two points.

Hence, we have

$$f(x) = \begin{cases} 1; & 0 < x < 1 \\ 0; & \text{otherwise} \end{cases}$$

Graph of $f(x)$:

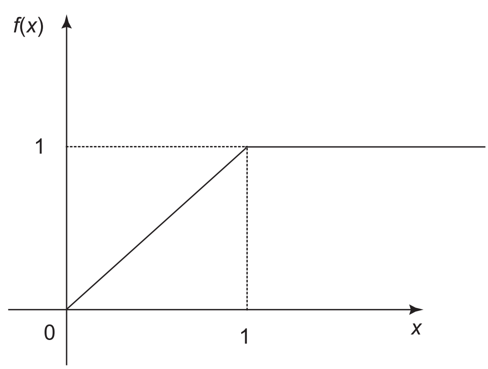


Fig. 6.7

6.11 EXPECTATION AND VARIANCE

6.11.1 Expectation

Let X be a random variable with probability distribution $f(x)$, then the mean or expected value of X is

$$\mu = E(x) = \sum_{\forall i} x_i\, f(x_i), \text{ if } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x, \text{ if } x \text{ is continuous.}$$

In general the mean or expected value of *a* function $g(x)$ of a random variable X is given by

$$E[g(x)] = \mu_{g(x)} = \sum_{\forall i} g(x_i)\, f(x_i), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} g(x)\, f(x)\,\mathrm{d}x, \text{ if } X \text{ is continuous.}$$

6.11.2 Variance

If a random variable X has probability distribution $f(x)$, then the variance of X is given by

$$E[(x-\mu)^2] = \sigma^2 = \sum_{\forall_i} (x_i - \mu)^2\, f(x_i), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} (x_i - \mu)^2\, f(x)\,\mathrm{d}x, \text{ if } X \text{ is continuous.}$$

The positive root of variance is called standard deviation (SD), i.e.

$$\text{SD} = \sigma = +\sqrt{E(X-\mu)^2}$$

6.11.3 Computational Formula for Variance

(i) $\sigma^2 = \sum_{\forall_i} x_i^2\, f(x_i) - \mu^2$, if X is discrete

(ii) $\sigma^2 = \int_{-\infty}^{\infty} x_i^2\, f(x)\,\mathrm{d}x - \mu^2$, if X is continuous

Proof:

(i) We know that

$$\sigma^2 = \sum_{\forall_i} (x_i-\mu)^2\, f(x_i) = \sum_i \left[(x_i^2 - 2x_i\,\mu + \mu^2)\, f(x_i)\right]$$

$$= \sum_i x_i^2\, f(x_i) - 2\mu \sum_{\forall_i} x_i\, f(x_i) + \mu^2 \sum_{\forall_i} f(x_i)$$

$$= \sum_{\forall_i} x_i^2\, f(x_i) - 2\mu\cdot\mu + \mu^2\cdot 1 \qquad \left[\begin{array}{l} \because \sum_{\forall_i} x_i\, f(x_i) = \mu \\ \text{and } \sum_{\forall_i} f(x_i) = 1 \end{array}\right]$$

$$= \sum x_1^2\, f(x_i) - \mu^2$$

Similarly (ii) can be proved for continuous random variable by replacing summation with integration.

Properties of Expectation and Variance

(i) If X be a random variable and c be a constant, then show that

(a) $E(c) = c$

(b) $E[cx] = c\,E(x)$

(c) $E[c + x] = c + E(x)$

***Proof*:**

(a) $$E(c) = \sum_{\forall x} c\ f(x) = c\sum_{\forall x} f(x) = c \qquad \because \quad \sum_{\forall c} f(x) = 1$$

(b) $$E(cx) = \sum_{\forall x} cx = f(x) = c\sum_{\forall x} xf(x) = cE(x)$$

(c) $$E(c + x) = \sum_{\forall x} (c + x)\, f(x) = \sum_{\forall c} c\, f(x) + \sum_{\forall x} x\, f(x)$$

$$= c + E(x)$$

***Note*:** For all the above three cases, we have proved for discrete random variable. Similar results can be obtained for continuous case by changing summation to integration.

(ii) Let X be a random variable and c be a constant, then

(a) Var $c = 0$

(b) Var$(cx) = c^2$ Var (x)

***Proof*:**

(a) To prove Var $c = 0$

By definition, Var $c = E(c^2) - [E(c)]^2$

$$= \sum_{\forall c} c^2 f(x) - c^2$$

$$= c^2 \sum_{\forall c} f(X) - c^2 = c^2 \cdot 1 - c^2 = 0$$

(b) Var $cX = E[(cx)^2] - [E(cx)]^2$

$$= E[c^2x^2] - \{E[cx]\}^2$$

$$= c^2 \sum_{\forall x} x^2 f(x) - c^2 \left[\sum_{\forall x} x\, f(x)\right]^2$$

$$= c^2 \left\{\sum_{\forall x} x^2 f(x) - \left[\sum xf(x)\right]^2\right\}$$

$$= c^2 \left\{E(x^2) - \left[E(x)\right]^2\right\} = c^2 \text{ Var } (x)$$

6.11.4 The r^{th} Moment about Origin

The r^{th} moment of a random variable X with probability distribution $f(x)$ is given by

$$\mu_r' = \sum_{\forall_i} x_i^r\ f(x_i), \quad \text{if } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} x_1^r\ f(x)\,\mathrm{d}x, \quad \text{if } x \text{ is continuous}$$

if $$r = 1 = \mu_1' = \mu = \sum_{\forall_i} x_i\ f(x_i) \text{ or } \int_{-\infty}^{\infty} x\ f(x)\,\mathrm{d}x = E(x)$$

i.e., mean or expected value of X is the first moment about origin.

6.11.5 The r^{th} Moment about Mean

The r^{th} moment of a random variable X with probability distribution $f(x)$ is given by

$$\mu_r = \sum_{\forall i} (x_i - \mu)^2 \; f(x_i), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} (x_i - \mu)^2 \; f(x)\,dx, \text{ if } X \text{ is discrete}$$

Let $r = 2$, then

$$\mu_2 = \sum_{i} (x_i - \mu)^2 \; f(x_i), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} (x_i - \mu)^2 \; f(x)\,dx, \text{ if } X \text{ is continuous}$$

$$= \sigma^2$$

i.e., variance of x is nothing but, it is a second moment about mean.

6.11.6 Mean Deviation from the Mean

The mean deviation from the mean of a random variable X is given by

$$= \sum_{\forall i} | x_i - \mu | \, f(x_i), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} | x_i - \mu | \, f(x)\,dx, \text{ if } X \text{ is continuous}$$

***Note*:** $\sigma^2 = \mu_2' - \mu^2$

Example 46 Find the mean and variance of the following probability distribution.

x	1	2	3	4	5	6
$f(x)$	1/6	1/6	1/6	1/6	1/6	1/6

Solution

Mean $$\mu = \sum_{i=1}^{6} x_i \, f(x_i) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6}$$

$$= \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$

Variance $$\sigma^2 = \sum x_i^2 \; f(x_i) = \mu^2$$

$$= 1^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{6} + 3^2\cdot\frac{1}{6} + 4^2\cdot\frac{1}{6} + 5^2\cdot\frac{1}{6} + 6^2\cdot\frac{1}{6} - \left(\frac{7}{2}\right)^2$$

$$= \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Example 47 *A random variable X has the following probability distribution:*

x	–2	–1	0	1	2	3
$f(x)$	0.1	0.1	0.2	0.2	0.3	0.1

Find the mean and variance of the distribution and hence compute the standard deviation.

Solution

Mean $$\mu = E(x) = \sum_{\forall_i} x_i \; f(x_i)$$

$$= (-2)(0.1)+(-1)(0.1)+(0)(0.2)+(1)(0.2)+(2)(0.3)+(3)(0.1)$$

$$= -(0.2)-(0.1)+0+(0.2)+(0.6)+(0.3) = 0.8$$

Variance $$\sigma^2 = \sum x_i^2 \; f(x_i) - \mu^2$$

$$= 0.1(-2)^2 + 0.1(-1)^2 + 0.2(0)^2 + (0.2)(1)^2 + (0.3)(2)^2 + 0.1(3^2) - (0.8)^2$$

$$= 0.4 + 0.1 + 0 + 0.2 + 1.2 + 0.9 - 0.64$$

$$= 2.80 - 0.64 = 2.16$$

S.D. $$\sigma = \sqrt{2\cdot 16} = 1.47$$

Example 48 If the probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} kx^2; & 0 < x < 1 \\ 0; & \text{otherwise} \end{cases}$$

Find the value of k and hence compute its mean and variance.

Solution To find k

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1 \Rightarrow \int_0^1 kx^2\mathrm{d}x = 1 \Rightarrow \frac{k}{3}(x^3)_0^1 = 1 \Rightarrow k = 3$$

$\therefore$ $$f(x) = \begin{cases} 3x^2; & 0 < x < 1 \\ 0; & \text{otherwise} \end{cases}$$

Mean $$\mu = \int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x = \int_0^1 x\cdot 3x^2\,\mathrm{d}x = \frac{3}{4}(x^4)_0^1 = \frac{3}{4}$$

Variance $$\sigma^2 = \int_{-\infty}^{\infty} x^2 \; f(x)\,\mathrm{d}x - \mu^2$$

$$= \int_0^1 x^2\cdot 3x^2\,\mathrm{d}x - \left(\frac{3}{4}\right)^2 = \frac{3}{5}(x^5)_0^1 - \frac{9}{16}$$

$$= \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

Example 49 The probability distribution of a random variable X is given by

X	–3	6	9
$f(x)$	1/6	1/2	1/3

Compute $E(X)$, $E(X)^2$ and hence find $E(3X + 2)^2$.

Solution

$$E(X) = \sum_{\forall_i} x_i\ f(x_i) = (-3)\frac{1}{6} + 6\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right) = \frac{11}{2}$$

$$E(X^2) = \sum_{\forall_i} x_i^2\ f(x_i) = (-3)^2 \cdot \frac{1}{6} + 6^2\left(\frac{1}{2}\right) + 9^2\left(\frac{1}{3}\right) = \frac{93}{2}$$

$$\therefore \quad E(3X + 2)^2 = E[9x^2 + 6x + 4] = 9E(x)^2 + 6E(x) + 4$$

$$= \left[9\left(\frac{93}{2}\right) + 6\left(\frac{11}{2}\right) + 4\right] = \frac{837}{2} + \frac{66}{2} + 4$$

$$= \frac{911}{2} = 455.50$$

Example 50 The probability distribution of a random variable x is given by

$$\begin{aligned} f(x) &= k\,x^3; && 0 \le x \le 1 \\ &= k\,(2 - x)^3; && 1 \le x \le 2 \\ &= 0; && \text{otherwise} \end{aligned}$$

Find the value of k and hence calculate its (i) mean (ii) variance, (iii) standard deviation and (iv) mean deviation about mean.

Solution Since the total probability is 1.

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1 \Rightarrow \int_0^1 k\,x^3\ \mathrm{d}x + \int_1^2 k\,(2 - x)^3\ \mathrm{d}x = 1$$

$$\therefore \quad k = 2$$

$$\therefore \quad \begin{aligned} f(x) &= 2x^3; && 0 \le x \le 1 \\ &= 2(2 \cdot x)^3; && 1 \le x \le 2 \\ &= 0; && \text{otherwise} \end{aligned}$$

(i) Mean $\mu = E(X) = \int_0^1 x \cdot 2x^3 + \int_1^2 x \cdot 2(2 - x)^3\ \mathrm{d}x = 1$

(ii) To find variance, compute

$$E(X^2) = \int_0^1 x^2 \cdot 2x^3 + \int_1^2 x^2 \cdot 2(2 - x)^3\ \mathrm{d}x = \frac{16}{15}$$

$$\therefore \quad \sigma^2 = E(X^2) - [E(X)]^2 = \frac{16}{15} - 1^2 = \frac{1}{15}$$

(iii) Standard deviation: $\sigma = \dfrac{1}{\sqrt{15}}$

(iv) Mean deviation about mean:

$$= \int_0^1 |x-1|\,2x^3\,dx + \int_1^2 |x-1|\,2\,(2-x)^3\,dx$$

$$= 2\left\{\int_0^1 (1-x)\,x^3\,dx + \int_1^2 (x-1)\,(2-x)^3\,dx\right\}$$

$$= 2\left\{\left(\frac{1}{4}-\frac{1}{5}\right)+\left(0+\frac{1}{20}\right)\right\} = \frac{1}{5}.$$

6.12 MOMENT GENERATING FUNCTION

The moment generating function (M.G.F.) of a random variable X is given by

$$M_X(t) = E[e^{tx}] = \sum_{\forall_i} e^{tx_i} f(x_i), \text{ if } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, \qquad \text{if } x \text{ is continuous}$$

and $$\mu_r' = \frac{d^r}{dt^r} M_X(t)\Big|_{t=0}$$

Example 51 Find the moment generating function of the following distribution:

$$f(x) = \frac{1}{2} e^{-\frac{x}{2}}; \qquad x > 0$$
$$= 0; \qquad \text{otherwise}$$

Hence find its mean, variance and standard deviation.

Solution

MGF $$M_X(t) = E[e^{tx}] = \int_0^{\infty} e^{tx}\cdot\frac{1}{2}e^{-\frac{x}{2}}\,dx = (1-2t)^{-1},\ t < \frac{1}{2}$$

$$= 1 + 2t + 4t^2 + 8t^3 + \cdots$$

$\therefore$ $$\mu_1^1 = \text{mean} = \frac{d}{dt} M_X(t)\Big|_{t=0} = 2 + 8t + 24t^2 + \cdots\Big|_{t=0} = 2$$

$$\mu_2^1 = E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = 8 + 48t + \cdots\Big|_{t=0} = 8$$

$\therefore$ Variance $\sigma^2 = \mu_2^1 - \mu_1^2 = 8 - (2)^2 = 4$

Standard deviation $\sigma = \sqrt{4} = 2$

EXERCISE 6.4

1. A random variable X has the following probability function:

Value of X	–2	–1	0	1	2	3
$p(x)$	0.1	k	0.2	$2k$	0.3	k

Find the value of k and calculate mean and variance.

(S.V.T.U. 2007, V.T.U. 2004, Madras 2003)

2. Determine whether the following can be the probability distribution of a random variable X which can take only 4 values 1, 2, 3 and 4.
   (i) $p(1) = 0.26 = p(2) = p(3) = p(4)$
   (ii) $p(1) = 0.15,\ p(2) = 0.28,\ p(3) = 0.29,\ p(4) = 0.28$
   (iii) $p(x) = \dfrac{x+1}{16},\ x = 1, 2, 3, 4$

3. A random variable X has the following probability distribution:

X	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

Find $F(x)$.

4. A random variable X is defined as the sum of the numbers on the faces when two unbiased dice are thrown. Find its mean.

5. A random variable X has the following probability distribution:

$X = x$	0	1	2	3	4	5	6	7
$P(X = x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Find (i) the value of k and hence calculate (ii) $P(X \le 5)$, (iii) $P(X > 5)$, and (iv) $P(0 < X < 6)$.

6. Following is the probability distribution of a random variable X:

X	0	1	2
$p(x)$	k	$2k$	$3k$

   (i) Find the value of k and hence write its $F(x)$.
   (ii) What is the smallest value of C for which $P(X \le C) > 1/2$.

7. Find whether the following function is a probability density function:

$$f(x) = \frac{x^2}{4}; \qquad -1 < x < 3$$
$$= 0; \qquad \text{otherwise}$$

8. Is the function defined as follows, a density function?

$$f(x) = \frac{1}{18}(3 + 2x); \qquad 2 \le x \le 4$$
$$= 0; \qquad \text{otherwise}$$

Also find the probability that a variate having this density falls in the interval $2 \le X \le 3$

9. Verify that

$$F(x) = \begin{cases} 0; & x < 0 \\ x^2; & 0 \le x < 1/2 \\ 1 - 3(1 - x^2); & \dfrac{1}{2} \le x < 1 \\ 1; & x \ge 1 \end{cases}$$

is a distribution function and derive density function $f(x)$ of X.

10. If a coin is tossed two times. Find the probability distribution for getting number of heads. Also find mean number of heads and its variance.

11. A random variable X has the following probability distribution:

$X = x$	0	1	2	3
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

Find (i) $P(X > 1)$ (ii) $P(X < 3)$

12. A dice is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of successes.

(V.T.U. 2011S, Rohtak 2004)

13. The probability density function of a variate X is

X	0	1	2	3	4	5	6
$p(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(i) Find $P(X < 4), P(X \ge 5), P(3 < X < 6)$

(ii) What will be minimum value of k so that $P(X \le 2) > \cdot 3$

(V.T.U. 2010)

14. X is a continuous random variable with probability density function given by

$$\begin{aligned} f(x) &= kx(0 \le x < 2) \\ &= 2k(2 \le x < 4) \\ &= -kx + 6k\ (4x < 6) \end{aligned}$$

Find k and mean value of X. **(J.N.T.U. 2003)**

15. The power reflected by an aircraft that is received by a radar can be described by an exponential random variable X. The probability density of X is given by

$$\begin{aligned} f(x) &= \frac{1}{x_0} e^{-\frac{x}{x_0}}; \qquad x \ge 0 \\ &= 0; \qquad x < 0 \end{aligned}$$

where x_0 is the average power received by the radar.

(i) What is the probability that the radar will receive power larger than the power received on the average?

(ii) What is the probability that the radar will receive power less than the power received on the average?

(Mumbai 2006)

16. The frequency function of a continuous random variable is given by $f(x) = y_0 x(2-x)$, $0 \le x \le 2$, find the value of y_0, mean and variance of X. **(Kerala 2005, J.N.T.U. 2003)**
17. A random variable gives measurements X between 0 and 1 with a probability function.

$$f(x) = 12x^3 + 21x^2 + 10x; \quad 0 \le x \le 1$$
$$= 0; \quad \text{otherwise}$$

(i) Find $P(X \le 1/2)$ and $P\left(X > \frac{1}{2}\right)$

(ii) Find a number k such that $P(X \le k) = \frac{1}{2}$

18. The probability density $f(x)$ of a continuous random variable is given by

$$f(x) = y_0 e^{-|x|}; -\infty < x < \infty$$

Prove that $y_0 = \frac{1}{2}$. Find the mean and variance of the distribution.

(S.V.T.U. 2008, Kurukshetra 2007, V.T.U. 2004)

19. Let X be a random variable with mean 30 and variance 5, then find $E(5x)$ and Var $(5x)$.
20. Let X be a random variable with mean 10 and variance 2 such that $E(cx) = 20$. Find the value of c and hence, calculate Var (cx).

Answers

1. $k = 1; \mu = 0.8, \sigma^2 = 2.232$
2. (i) No, $\sum_{\forall x} p(x) > 1$, (ii) Yes, $\sum_{\forall x} p(x) = 1$, (iii) No, $\sum_{\forall x} p(x) < 1$
3. $$F(x) = \begin{cases} 0; & x < 0 \\ \frac{1}{8}; & 0 \le x < 1 \\ \frac{4}{8}; & 1 \le x < 2 \\ \frac{7}{8}; & 2 \le x < 3 \\ 1; & x \ge 3 \end{cases}$$
4. $\mu = 7$
5. (i) $k = 0.1$, (ii) 0.81, (iii) 0.19, (iv) 0.81
6. (a) $k = \frac{1}{6}$ $$F(x) = \begin{cases} 0; & x < 0 \\ \frac{1}{6}; & 0 \le x < 1 \\ \frac{1}{2}; & 1 \le x < 2 \\ 1; & x \ge 2 \end{cases}$$

(b) $C = 2$

7. No.

8. Yes; $\frac{4}{9}$.

9. $f(x) = 2x; \quad 0 \le x < \frac{1}{2}$

$\quad = 6x; \quad \frac{1}{2} \le x < 1$

$\quad = 0;$

10. X = no. of heads

$X = x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

; mean $\mu = 1$; Variance $\sigma^2 = \frac{1}{2}$.

11. (i) $\frac{1}{4}$, (ii) 7/8

12. $\mu = 1; \sigma^2 = \frac{2}{3}$

13. (i) $P(X < 4) = \frac{16}{49}, P(X \ge 5) = \frac{24}{49}, P(3 < x < 6) = \frac{33}{49}$

(ii) Minimum value of $k = \frac{1}{30}$

14. $k = \frac{1}{8}; \mu = 3$

15. (i) 0.37 (ii) 0.63

16. $y_0 = 3/4$; Mean = 1; Variance = 1/5

17. (i) $\frac{9}{16}, \frac{7}{16}$, (ii) $k = 0.45$

18. 0.2.

19. $E(5x) = 150$, Var $(5x) = 125$

20. $c = 2$, Var$(cx) = 8$.

6.13 SOME IMPORTANT DISTRIBUTIONS

In this section, we shall be describing some important distributions such as Bernoulli, binomial and Poisson distributions which are discrete in nature. Some continuous distributions viz. uniform, exponential and normal will also be discussed. These distributions will be discussed one by one.

6.14 BERNOULLI DISTRIBUTION

When we are tossing a coin then the possible outcomes are two-head and tail which may be referred to success or failure. Each time a coin is tossed, the possible outcomes are head and tail. Such a process involves consisting of repeated trials whose results into only two mutually exclusive possibilities is often known as Bernoulli process and each trial of this process is called a Bernoulli trial.

Let X be a random variable which takes only two values 1 and 0 with probability p and q respectively termed as a Bernoulli variate and the distribution of X is called Bernoulli distribution which is given by

$$f(x) = P(X = x) = p^x q^{1-x};\ x = 0, 1 \text{ and } p + q = 1$$

6.14.1 Properties of Bernoulli Process

(i) Each trial results in an outcome that may be classified as a success or a failure.
(ii) The experiment consists of n repeated trials which are independent.
(iii) The probability of success remains constant from trial to trial.

6.14.2 Mean and Variance of Bernoulli Distribution

The mean $\mu = E(X) = p$

The variance $\sigma^2 = E(X^2) - [E(x)]^2 = pq$

6.15 BINOMIAL DISTRIBUTION

The concept of binomial distribution was first discovered by James Bernoulli in the year 1700. Actually binomial distribution is an extension of the multiplicative theorem of probability. Let X be the random variable denote the number of successes in n independent Bernoulli trials. Let p and q be the probabilities of success and failure in each trial which are constant. Then in n independent Bernoulli trials the probability there will be x successes and n-x failures are given by

$$P(X = x) = {}^nC_x p^x q^{n-x},\ x = 0, 1, 2, ..., n.$$

It is also denoted by

$$b(x; n \cdot p) = {}^nC_x p^x q^{n-x},\ x = 0, 1, 2, ..., n.$$

where $p + q = 1$, n and p are known as parameters of the distribution. The probability distribution of the random variable X is therefore given by

X	0	1	2	…	…	x	…	n
$p(X = x)$	${}^nC_0 p^0 q^{n-0}$	${}^nC_1 p^1 q^{n-1}$	${}^nC_2 p^2 q^{n-2}$	…	…	${}^nC_x p^x q^{n-x}$	…	${}^nC_n p^n q^{n-n}$

Hence the probability distribution is called binomial distribution because for $x = 0, 1, 2, \ldots, n$ the probabilities are the successive terms of the binomial expansion $(q + p)^n$.

***Note*:** For N set of n trials the successes $0, 1, 2, \ldots, x, \ldots, n$ are given by $N(q + p)^n$, which is called binomial frequency distribution.

6.15.1 Constants of Binomial Distribution

We shall calculate mean and variance of binomial distribution.

(a) Mean of Binomial Distribution: The First moment about origin or $E(x)$ or mean (μ) of binomial distribution is given by

$$\mu = E(x) = \mu_1' = \sum_{\forall x} x\, f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x\, q^{n-x}$$

$$= 0 \cdot q^n + 1 \cdot n_{C_1} pq^{n-1} + 2 \cdot n_{C_2} p^2 q^{n-2} + \cdots + np^n$$

$$= np\, q^{n-1} + n(n-1)p^2 q^{n-2} + \cdots + np^n$$

$$= np\left[q^{n-1} + (n-1)pq^{n-1} + \cdots + p^{n-1}\right]$$

$$= np\left[q^{n-1} + n-1_{C_1}\, pq^{n-2} + \cdots + n-1_{C_{n-1}}\, p^{n-1}\right]$$

$$= np[q+p]^{n-1} = np \cdot 1 = np$$

$\therefore$ $\boxed{\text{Mean } \mu = np}$

(b) Variance of Binomial Distribution

$$\mu_2 = \mu_2^1 - \mu_1^{12}$$

$$\mu^1{}_2 = \sum_{\forall x} x^2\, f(x) = \sum_{x=0}^{n} x^2 \cdot \binom{n}{x} p^x\, q^{n-x}$$

$$= \sum_{x=0}^{n} \left[x + x(x-1)\right] \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} + \sum_{x=0}^{1} x(x-1) \binom{n}{x} p^x q^{n-x} \quad \left[\because\ \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = \mu = np\right]$$

$$= np + \sum_{x=0}^{n} x(x-1) \frac{n(n-1)}{x(x-1)} \binom{n-2}{x-2} p^x q^{n-x}$$

$$= np + n(n-1)p^2 \sum_{x=0}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x}$$

$$= np + n(n-1)\, p^2 (q+p)^{n-2}$$

$$= np + n(n-1)\, p^2 \cdot 1 \qquad (\because\ q+p=1)$$

$$= np\left[1+(n-1)p\right] = np\left[1+np-p\right]$$

$$= np\left[q+np\right] = n^2p^2 + qnp \qquad (\because\ 1-p=q)$$

$$\therefore \qquad \mu_2 = \sigma^2 = \mu_2' - \mu^2 = n^2p^2 + npq - (np)^2 = npq$$

$\therefore$ $\boxed{\text{Variance } \sigma^2 = npq}$

(c) Other constants: Like μ_1', μ_2' and μ_2, we can calculate other moments of binomial distribution. Such as

$$\mu_3 = npq(q-p), \mu_4 = npq\,[1+3(n-2)\,pq]$$

Now,

Kurlosis $$(\beta_1) = \frac{\mu_3^2}{\mu_2^2} = \frac{(q-p)^2}{npq}$$

and $$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \left(\frac{1-6\,pq}{npq}\right)$$

$$\text{skewness} = \frac{1-2p}{\sqrt{npq}}$$

***Remark*:**

1. The mean of binomial distribution is greater than variance.
2. If skewness is zero then $p = q = 1/2$
3. If $p < \frac{1}{2}$, skewness is positive and if $p > \frac{1}{2}$, skewness is negative.

6.15.2 Moment Generation Function of Binomial Distribution

The M.G.F. of X is given by

$$M_X(t) = E[e^{tx}] = \sum_{\forall i} e^{tx} p(x) = \sum_{\forall x} e^{tx} \binom{n}{n} p^k q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x\, q^{n-x}$$

$$\boxed{M \propto (t) = (q + pe^t)^n}$$

Example 52 Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?

Solution Let p be the probability of getting 5 or 6 with one dice.

then $p = \frac{1}{3} \quad \therefore \quad q = 1 - \frac{1}{3} = \frac{2}{3}$

Given $n = 6$, $N = 729$

Then binomial distribution is

$$N(q+p)^n = 729\left(\frac{1}{3}+\frac{2}{3}\right)^6$$

$$= 729\left[{}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + {}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + {}^6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + {}^6C_6\left(\frac{1}{3}\right)^6\right]$$

$$= 233$$

Example 53 The mean and variance of a binomial distribution are 4 and 3 respectively. Find the probability of getting exactly six successes in this distribution.

Solution

Given $\mu = np = 4, \sigma^2 = npq = 3$

$\therefore$ $q = 3/4 \Rightarrow p = 1/4$ and hence $n = 16$

The probability of six successes = $P(X = 6) = \binom{16}{6}\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^{10}$.

Example 54 In 800 families with 4 children each, how many families would be expected to have

(i) 2 boys and 2 girls
(ii) at least 1 boy
(iii) no girl
(iv) at least 2 girls

Assuming that equal probabilities for girls and boys.

Solution Let p be the probability of having a boy and q be the probability of having a girl. Given $p = q = 1/2$

$$n = 4, N = 800$$

The binomial distribution is given by $N(q + p)^n$

$$= 800\left(\frac{1}{2}+\frac{1}{2}\right)^4$$

(i) Expected number of families having 2 boys and 2 girls

$$= 800 \cdot {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 300$$

(ii) Expected number of families having at least 1 boy.

$$= 800 \cdot \left[{}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4\right] = 750$$

(iii) The expected number of families having no girl.

$$= 800\ {}^4C_4\left(\frac{1}{2}\right)^4 = 50$$

(iv) The expected number of families having at least two girls.

$$= 800\left[{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4\right] = 550$$

Example 55 If 10% of the bolts produced by a machine are defective. Find the probability that out of 5 bolts selected at random, at most one will be defective.

Solution Let X be the number of defective bolts.

$\therefore$ $p = 0.10, x = 5$

$\therefore \quad P(X \leq 1) =$ The probability of at most one defective

$$= P(X=0)+P(X=1) = \binom{5}{0}(0.10)^0(0.90)^5 + \binom{5}{1}(0.10)\,(0.90)^4$$

$$= 5\cdot\left(\frac{9}{10}\right)^5 + 5\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^4$$

Example 56 The following data are the number of seeds germinating out by 10 on damp filter for 80 sets of seeds. Fit a binomial distribution to these data.

x	0	1	2	3	4	5	6	7	8	9	10	Total
f	6	20	28	12	8	6	0	0	0	0	0	80

Solution Given $n = 10$, $N = 80 = \Sigma f$

The mean $\quad \mu = \frac{\Sigma fx}{\Sigma f} = 0.6 + 1.20 + \cdots + 10.0 = \frac{174}{80}$

$\therefore \quad np = \frac{174}{80} \Rightarrow p = \frac{174}{80 \times 10} = 0.2175$

$\therefore \quad q = 1 - 0.2175 = 0.7825$

$\therefore$ The binomial distribution to be fitted to the data is

$N(q+p)^n = 80\,(0.7825 + 0.2175)^{10}$

Hence, the theoretical frequencies are as follows:

x	0	1	2	3	4	5	6	7	8	9	10
f	6.9	19.1	24	17.8	8.6	2.9	0.7	0.1	0	0	0

EXERCISE 6.5

1. Prove that sum of binomial probability distribution is 1.
2. A dice is tossed thrice. Getting an even number is considered success. What is the variance of the binomial distribution.
3. If moment generating function of a binomial variate is $(0.6 + 0.4e^t)^6$. Find its mean, variance and standard deviation.
4. The mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $P(X \geq 1)$.
5. The items produced by a firm are supposed to contain 5% defective items. What is the probability that a sample of 8 items will contain less than 2 defective items.
6. The probability that a pen manufactured by a company will be defective will be $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that
   (a) exactly two will be defective,
   (b) at least two will be defective, and
   (c) none will be defective. **(V.T.U. 2004, Burdwan 2003)**

7. In 256 sets of 12 tosses of a coin, in how many cases we can expect 8 heads and 4 tails. **(J.N.T.U. 2003)**
8. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.
9. Determine the binomial distribution for which mean = 2 and mean + variance = 3. Also find $P(X \leq 3)$. **(Kerala 2005)**
10. If the chance that one of the ten telephone lines is busy at an instant is 0.2.
    (a) What is the chance that 5 of the lines are busy?
    (b) What is the most probable number of busy lines and what is the probability of this number?
    (c) What is the probability that all the lines are busy? **(V.T.U. 2002S)**
11. If the probability that a new born child is a male is 0.6, find the probability that in a family of 5 children there are exactly 3 boys. **(Kurukshetra 2005)**
12. If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely. **(P.T.U. 2005)**
13. Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys (b) 5 girls (c) either 2 or 3 boys? Assume equal probabilities for boys and girls? **(V.T.U. 2004)**
14. A sortie of 20 aeroplanes is sent on an operational flight. The chance that an aeroplane fails to return is 5%. Find the probability that (i) one plane does not return (ii) at most 5 planes do not return, and (iii) what is the most probable number of returns? **(Hissar 2007)**
15. In a bombing action there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give a 99% chance or better to completely destroying the target. **(V.T.U. 2003S)**
16. 500 articles were selected at random out of a batch containing 10,000 articles, and 30 were found to be defective. How many defectives articles would you reasonably expect to have in the whole batch? **(J.N.T.U. 2003)**
17. Fit a binomial distribution to the following frequency distribution:

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4

(Kurukshetra 2009, S.V.T.U. 2007)

18. Fit a binomial distribution for the following data and compare the theoretical frequencies with the actual ones:

x	0	1	2	3	4	5
f	2	14	20	34	22	8

(Bhopal 2006)

Answers

2. $\frac{3}{4}$

3. $\mu = 2 \cdot 4; \sigma^2 = 1 \cdot 44; \sigma = 1 \cdot 2$

4. 0.99863

5. 1 (App.)

6. (a) 0.2301, (b) 0.3412 (c) 0.2833

7. 31

8. 323

9. $n = 4, p = \frac{1}{2} = q; \frac{15}{16}$

10. (a) 0.02579; (b) 0.04571; (c) 1.024×10^{-7}

11. 0.3456

12. $\frac{45927}{50000}$

13. (a) 250; (b) 25; (c) 500

14. (i) $20_{C_1}\left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^{19}$ (ii) $\sum_{x=0}^{5} 20_{C_x}\left(\frac{1}{20}\right)^x\left(\frac{19}{20}\right)^{20-x}$ (iii) 19

15. 11

16. 600

17. 200 $(0.554 + 446)^6$

18. $100(0.432 + 0.568)^5$.

6.16 POISSON DISTRIBUTION

In 1837, a French mathematician Simeon Poisson discovered the concept of Poisson distribution. Poisson distribution is a limiting form of the binomial distribution under the following constants:

(i) the number of trials is infinitely large, i.e. $n \to \infty$.

(ii) the probability of success for each trial is infinitely small, i.e. $p \to 0$.

(iii) np is a finite quantity say λ. Hence,

$$p = \frac{\lambda}{n}, q = 1 - p = 1 - \frac{\lambda}{n}$$

$\therefore$ The probability of x successes in a binomial distribution

$$P(X = x) = \binom{n}{x} p^x\, q^{n-x} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} p^x\, q^{n-x}$$

$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{x+1}{n}\right)(np)^x (1-p)^{n-x} / x!$$

Taking limit $n \to \infty$, $p \to 0$, such that $np = \lambda$, we have

$$P(X = x) = \frac{\lambda^x}{x!} \underset{\substack{n \to \infty \\ p \to 0}}{\text{limit}} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{x+1}{n}\right)\left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \underset{n \to \infty}{\text{limit}} \left(1 - \frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{x!} \underset{n \to \infty}{\text{limit}} \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \frac{e^{-\lambda}\lambda^x}{x!} \quad \left[\because \ \underset{n\to\infty}{\text{limit}}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \text{ and } \underset{n\to\infty}{\text{limit}}\left(1-\frac{\lambda}{n}\right)=1\right]$$

The probability of $x = 0, 1, 2, \ldots, x, \ldots$ successes are

$e^{-\lambda}, \frac{\lambda e^{-\lambda}}{\lfloor 1}, \frac{\lambda^2 e^{-\lambda}}{\lfloor 2}, \cdots \frac{\lambda^x e^{-\lambda}}{\lfloor x}, \cdots$ respectively and the sum of all these probabilities is 1.

∴ A random variable X is said to have a Poisson distribution if its probability mass function is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{\lfloor x}, \ x = 0, 1, 2, \ldots$$

6.17 CONSTANTS OF POISSON DISTRIBUTION

6.17.1 Mean

The first moment or mean (μ) or $E(x)$ is given by

$$\mu_1' = \mu = E(x) = \sum_{x=0}^{\infty}\frac{xe^{-\lambda}\lambda^x}{\lfloor x} = \lambda\cdot e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{\lfloor x-1}$$

$$= \lambda\cdot e^{-\lambda}\left[1+\frac{\lambda}{\lfloor 1}+\frac{\lambda^2}{\lfloor 2}+\cdots\right]$$

$$\mu = \lambda\cdot e^{-\lambda}\cdot e^{\lambda} = \lambda$$

$$\boxed{\text{Mean} = \mu = \lambda}$$

6.17.2 Variance

Variance $\sigma^2 = \mu_2^1 - \mu^2$

$$\mu_2^1 = \sum_{x=0}^{\infty} x^2\,\frac{e^{-\lambda}\lambda^x}{\lfloor x} = \sum_{x=0}^{\infty}\left[x(x-1)+x\right]\frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{\lfloor x} + \sum_{x=0}^{\infty}\frac{x\,e^{-\lambda}\,\lambda^x}{\lfloor x}$$

$$= e^{-\lambda}\lambda^2\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{\lfloor x-2} + \lambda \qquad \left(\because \ \sum_{x=0}^{\infty}\frac{xe^{-\lambda}\lambda^x}{\lfloor x} = \mu = \lambda\right)$$

$$\mu_2^1 = e^{-\lambda}\cdot\lambda^2\cdot e^{\lambda} + \lambda \qquad \left(\because \ \sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{\lfloor x-2} = e^{\lambda}\right)$$

$$= \lambda^2 + \lambda$$

$\therefore \qquad \mu_2 = \sigma^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

$\therefore \qquad \boxed{\sigma^2 = \text{Variance} = \lambda}$

***Note*:** The mean and variance of Poisson distribution is same.

$\therefore$ Standard deviation $\sigma = +\sqrt{\lambda}$.

6.17.3 Other Moments

Third moment about mean $\mu_3^1 = \lambda^3 + 3\lambda^2 + \lambda$

Third moment about mean $\mu_3 = \lambda$

Similarly, $\mu_4^1 = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$

and $\mu_4 = 3\lambda^2 + \lambda$

Now, Kurtosis $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{1}{\lambda}$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda}$$

skewness $\gamma_1 = \sqrt{\beta_1} = \dfrac{1}{\sqrt{\lambda}}$

$$\gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}$$

When $\lambda \to \infty$, $\beta_1 = 0$ and $\beta_2 = 3$

Hence, the Poisson distribution is always positively skewed.

6.17.4 Moment Generating Function of Poisson Distribution

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{\lfloor x} = e^{-\lambda} \sum_{x\to\infty}^{\infty} \left(\frac{\lambda e^t}{\lfloor x}\right)^x$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

$$\therefore \qquad \boxed{M_X(t) = e^{\lambda(e^t - 1)}}$$

The moments $\mu_1^1, \mu_2^1, \mu_3^1$ and μ_4^1 can be easily found either by simply expanding the expression of $M_X(t)$ given above or by differentiating it with respect to t and then putting $t = 0$.

6.17.5 Application of Poisson Distribution

Poisson distribution is used in many situations such as:

(i) Emission of radioactive particles.

(ii) Number of cars passing a certain street at a time t.

(iii) Number of faulty blades in a packet of 1000.
(iv) Number of printing mistakes at each chapter of the book.
(v) Number of suicides or deaths due to heart attack in one minute.
(vi) Number of children born blind per year in a city.
(vii) Number of accidents that take place on a busy road at a time t.
(viii) Number of telephone calls received at a particular switch board in one minute and many more.

Example 57 Six coins are tossed 6,400 times. Using the Poisson distribution, what is the appropriate probability of getting 6 heads x times.

Solution Let p be the probability of getting all the 6 heads in a throw of 6 coins.

$$\therefore \quad p = \left(\frac{1}{2}\right)^6 = \frac{1}{64}, \text{ given } n = 6400$$

$$\therefore \quad \lambda = np = \frac{1}{64} \times 6400 = 100$$

$$\therefore \quad P(X = x) = \frac{e^{-\lambda}\lambda^x}{\lfloor x} = \frac{e^{-100}(100)^x}{\lfloor x}, \; x = 0, 1, 2, \ldots$$

Example 58 In a Poisson distribution $P(x) = 0.1$ for $x = 0$. Find the mean given that $\log_e 10 = 2.3026$.

Solution

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{\lfloor x} \Rightarrow P(X = 0) = 0.1 = \frac{e^{-\lambda}\lambda^0}{\lfloor 0} = e^{-\lambda}$$

$$\therefore \quad e^{-\lambda} = 0.1 \Rightarrow e^{\lambda} = 10 \Rightarrow \lambda = \log_e 10 = 2.3026$$

$$\therefore \quad \text{Mean, } \lambda = 2.3026$$

Example 59 If the probability that an individual suffers a bad reaction from injection of a given serum is 0.001. Find the probability that out of 2000 individuals (i) exactly 3 (ii) more than 2 individuals and (iii) none of them suffer from bad reaction.

Solution Let X be the number of individual who suffer from bad reaction and p be the probability that an individual suffers a bad reaction from injection of a given serum.

Given $\quad p = 0.001, n = 2000$

$\therefore \quad \lambda = np = 2000 \times 0.001 = 2$

and $$P(X = x) = \frac{e^{-2}(2)^x}{\lfloor x}, \; x = 0, 1, 2, \ldots$$

(i) $$P(X = 3) = \frac{e^{-2}(2)^3}{\lfloor 3} = 0.18$$

(ii) $P(X > 2) = 1 - P(X \le 2) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$$= 1 - \left[e^{-2} + 2e^{-2} + \frac{4e^{-3}}{2}\right] = 0.323$$

(iii) $P(X = 0) = \dfrac{e^{-2}(2)^0}{\lfloor 0} = e^{-2} = 0.135$

Example 60 Fit a Poisson distribution to the following data and calculate theoretical frequencies:

Deaths (x)	0	1	2	3	4
Frequencies (f)	122	60	15	2	1

Solution

The Poisson distribution is $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{\lfloor x}$, $x = 0, 1, 2, \ldots$

Here mean $\lambda = \dfrac{\Sigma xf}{\Sigma f} = \dfrac{0.122 + 1.60 + 2.15 + 3.2 + 4.1}{200} = 0.5$

Hence, the theoretical frequencies of Poisson distribution

$$= \frac{Ne^{-\lambda}\lambda^x}{\lfloor x} = 200\frac{\left[e^{-0.5}(0.5)^x\right]}{\lfloor x}, \; x = 0, 1, 2, 3, 4$$

but $\qquad e^{-0.5} = 0.61$ (App.)

Therefore for $x = 0, 1, 2, 3, 4$, the theoretical frequencies are 122, 61, 15, 2 and 1 respectively.

EXERCISE 6.6

1. Show that for Poisson distribution $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{\lfloor x}$, $x = 0, 1, 2, \ldots = 1$

2. Show that for Poisson distribution with mean λ

$$\mu_{r+1} = r\lambda\mu_{r-1} + \lambda \cdot \frac{d\mu r}{dm}, \text{ where}$$

$$\mu_r = \sum_{x=0}^{\infty} (x - \lambda)^r \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

3. For Poisson distribution, prove that $P(x+1) = \dfrac{\lambda}{x+1}P(x)$, $x = 0, 1, 2, \ldots$

4. Using the above recurrence relation, if the variance of Poisson distribution is 2. Find the probability for $x = 1, 2, 3, 4$. Also find $P(X \ge 4)$.

5. Show that for a Poisson distribution $\gamma_1 \gamma_2 \sigma \lambda = 1$ when σ and λ are the standard deviation and mean respectively. **(S.V.T.U. 2008)**

6. If X is a Poisson variate such that $P(X = 2) = 9P(X = 4) + 90P(X = 6)$. Find the variance of X.

7. If X is a Poisson variate such that $3P(X = 3) = 4P(X = 4)$, find $P(X = 7)$.

8. If a random variable has a Poisson distribution such that $P(1) = P(2)$, find (i) mean of the distribution (ii) $P(4)$.

(V.T.U. 2003)

9. A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (i) on which there is no demand, (ii) on which demand is refused. $(e^{-1/5} = 0.2231)$

(Bhopal 2008S, J.N.T.U. 2003)

10. A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimeter is equal to 3. Ten 1 c.c., test tubes are filled with the liquid. Assuming that Poisson distribution is applicable, calculate the probability that all the test-tubes will show growth, i.e., contains at least 1 bacterium each. **(V.T.U. 2003)**

11. 10% of bolts produced by a certain machine turns out to be defective. Find the probability that in a sample of 10 bolts, selected at random two will be defective using (i) Poisson distribution and (ii) binomial distribution.

12. Wireless sets are manufactured with 25 soldered joints each on the average 1 joint in 500 defective. How many sets can be expected to be free from defective points in a consignment of 10000 sets.

13. Fit a Poisson distribution to the following:

x	0	1	2	3	4
f	46	38	22	9	1

(Kurukshetra 2009, Bhopal 2008; V.T.U. 2003S)

14. Fit a Poisson distribution to the following data given the number of yeast cells per square for 400 squares:

No. of cells per sq.	0	1	2	3	4	5	6	7	8	9	10
No. of squares	103	143	98	42	8	4	2	0	0	0	0

(S.V.T.U. 2007)

15. The frequency of accidents per shift in a factory is given in the following table:

Accidents per shift	0	1	2	3	4	Total
Frequency	192	100	24	3	1	320

Calculate the mean no. of accidents per shift and compare with actual frequencies.

Answers

4. $P(1) = 0.2706$, $P(2) = 0.2706$, $P(3) = 0.1804$; $P(4) = 0.0902$, $P(X \geq 4) = 0.1431$.

6. Variance $\sigma^2 = 1$.

7. $\dfrac{e^{-3}\, 3^7}{\lfloor 7}$

8. (i) $2; \frac{2}{3}e^{-2}$

9. (i) 0.2231; (ii) 0.1913

10. 0.6

11. (i) 0.184; (ii) 0.194

12. 9512

13. Theoretical frequencies are 44, 43, 21, 7, 1

14. Theoretical frequencies are 109, 142, 92, 40, 13, 3, 1, 0, 0, 0, 0.

15. Theoretical frequencies are 194, 97, 24, 4, 1.

6.18 UNIFORM DISTRIBUTION

A continuous random variable X is said to follow an uniform distribution on interval $[a, b]$ if its probability density function $f(x)$ is given by:

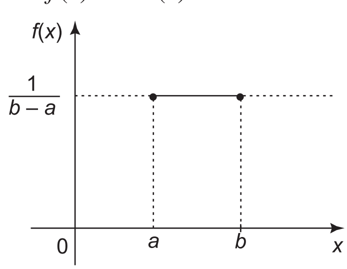
$$f(x) = \begin{cases} \dfrac{1}{b-a}; & a \le x \le b \\ 0; & \text{otherwise} \end{cases}$$

where a and b are known the parameters of uniform distribution and $b > a$. Uniform distribution is also known as Rectangular Distribution.

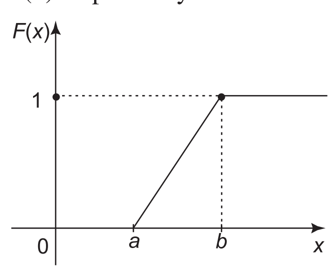
The distribution function $F(x)$ of X can be found easily as follows:

$$\begin{aligned} F(x) = {} & P(X \le x) = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t \\ & = 0, x < a \\ & = \int_{-\infty}^{a} f(t)\,\mathrm{d}t + \int_{a}^{x} f(t)\,\mathrm{d}t, \qquad \text{for } a \le x < b \\ & = 0 + \int_{a}^{x}\left(\frac{1}{b-a}\right)\mathrm{d}t = \frac{x-a}{b-a}, a \le x < b \\ & = \int_{-\infty}^{a} f(t)\,\mathrm{d}t + \int_{c}^{b} f(t)\,\mathrm{d}t + \int_{b}^{x} f(t)\,\mathrm{d}t, \qquad \text{for } x \ge b \\ \therefore \qquad & = 1 \\ & = \frac{x-a}{b-a}, \quad a \le x < b \\ & = 1, x \ge b \end{aligned}$$

The graphs of $f(x)$ and $F(x)$ are shown in Fig. 6.8(a) and (b) respectively.



(a) Graph of $f(x)$



(b) Graph of $F(x)$

Fig. 6.8

6.18.1 Mean and Variance of Uniform Distribution

(a) *Mean of Uniform Distribution*: The first moment about origin or mean (μ) or expected value of uniform random variable X is given by

$$\mu_1^1 = \mu = E(X) = \int_a^b x\, f(x)\,\mathrm{d}x$$

$$= \int_a^b x \cdot \left(\frac{1}{b-a}\right)\mathrm{d}x = \frac{1}{b-a}(b^2 - a^2) = \frac{b+a}{2}$$

(b) *Variance of Uniform Distribution*:

$$\sigma^2 = \mu_2^1 - \mu_1^2$$

$$\therefore \qquad \mu_2^1 = E(X^2) = \int_a^b x^2 f(x)\,\mathrm{d}x = \frac{1}{b-a}\int_a^b x^2\mathrm{d}x = \frac{1}{b-a}\left(\frac{b^3 - a^3}{3}\right)$$

$$\mu_2^1 = \frac{b^2 + a^2 + ab}{3}$$

$$\sigma^2 = \frac{b^2 + a^2 + ab}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{b^2 + a^2 - 2ab}{12} = \left(\frac{b-a}{12}\right)^2$$

6.18.2 Moment Generating Function of Uniform Distribution

$$M_X(t) = E[e^{tx}] = \int_c^b e^{tx} \cdot \left(\frac{1}{b-a}\right)\mathrm{d}x = \frac{e^{bt} - e^{at}}{t(b-a)}$$

6.19 EXPONENTIAL DISTRIBUTION

A continuous random variable is said to follow exponential distribution with parameter λ if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}; & x \geq 0, \lambda > 0 \\ 0; & \text{otherwise} \end{cases}$$

The distribution function $F(x)$ of an exponential variate is given by

$$F(x) = P(x \leq x) = \int_0^x \lambda e^{-\lambda x}\mathrm{d}x = 1 - e^{-\lambda x}$$

$$\therefore \qquad F(x) = \begin{cases} 0; & x < 0 \\ 1 - e^{-\lambda x}; & x \geq 0 \\ 1; & \text{otherwise} \end{cases}$$

6.19.1 Mean and Variance of Exponential Distribution

(a) The mean $\mu_1^1 = \mu = E(x) = \int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\mathrm{d}x$

$$= \lambda\int_0^\infty x\,e^{-\lambda x}\mathrm{d}x = \frac{\lambda\cdot\overline{|2}}{\lambda^2} \qquad \left[\begin{array}{c}\text{Using Gamma Function}\\ \int_0^\infty x^{n-1}e^{ax}\mathrm{d}x = \frac{\overline{|(n)}}{a^n}\end{array}\right]$$

$$\boxed{\mu = \frac{\lambda\cdot 1}{\lambda^2} = \frac{1}{\lambda}}$$

(b) The variance $\sigma^2 = \mu_2^1 - \mu_1'^2$

$$E(X^2) = \mu_2^1 = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x = \int_0^\infty x^2\cdot\lambda e^{-\lambda x}\mathrm{d}x = \lambda\int_0^\infty x^2 e^{-\lambda x}\mathrm{d}x$$

$$= \frac{\lambda\cdot\overline{|3}}{\lambda^3} = \frac{\lambda\cdot\underline{|2}}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\therefore \qquad \sigma^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

6.19.2 Moment Generating Function of Exponential Distribution

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx}\cdot f(x)\,\mathrm{d}x = \int_0^\infty e^{tx}\cdot\lambda e^{-\lambda x}\,\mathrm{d}x$$

$$= \lambda\int_0^\infty e^{-(\lambda-t)x}\mathrm{d}x = \frac{\lambda}{\lambda - t},\ t < \lambda$$

$$\therefore \qquad M_X(t) = \frac{\lambda}{\lambda - t},\ t < \lambda$$

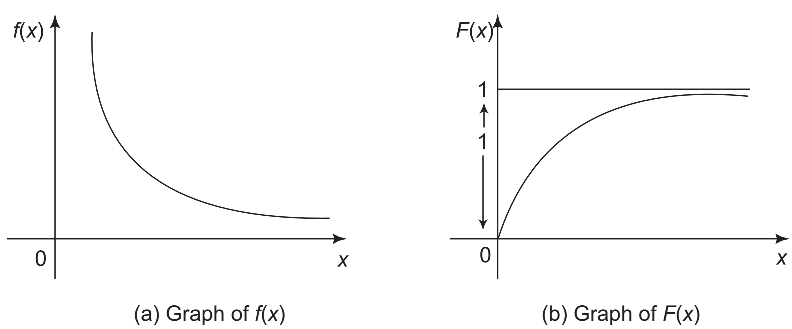


Fig. 6.9

Example 61 A random variable X has the uniform distribution with the density function given by

$$f(x) = \frac{1}{100}; \quad 0 < x < 100$$
$$= 0; \quad \text{otherwise}$$

Obtain (i) $P(X > 60)$ and (ii) $P(20 \le x \le 40)$.

Solution

(i) $$P(X > 60) = \int_{60}^{100} f(x)\,\mathrm{d}x = \frac{1}{100}\int_{60}^{100} 1 \cdot \mathrm{d}x = \frac{40}{100} = 0.40$$

(ii) $$P(20 \le x \le 40) = \int_{20}^{40} f(x)\,\mathrm{d}x = \frac{1}{100}\int_{20}^{40} 1 \cdot \mathrm{d}x = \frac{20}{100} = 0.20$$

The graphs of $f(x)$ and $F(x)$ of exponential distribution are given in Fig. 6.9(a) and (b) respectively.

Example 62 If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$. Find $P(X < 0)$.

Solution The uniform distribution is

$$f(x) = \begin{cases} \dfrac{1}{b-a}; & a \le x \le b \\ 0; & \text{otherwise} \end{cases}$$

The mean $$\mu = \frac{b+a}{2}$$

The variance $$\sigma^2 = \frac{1}{12}(b-a)^2$$

Given $$\mu = 1 = \frac{b+a}{2} \Rightarrow b + a = 2 \tag{2}$$

and $$\sigma^2 = \frac{4}{3} = \frac{1}{12}(b-a)^2 \Rightarrow 16 = (b-a)^2 \Rightarrow b - a = \pm 4 \tag{3}$$

By Eqs (2) and (3) $a = -1$ and $b = 3$

Now $$P(X < 0) = \int_{-1}^{0} f(x)\,\mathrm{d}x = \int_{-1}^{0} \frac{1}{3-(-1)}\mathrm{d}x$$

$$= \frac{1}{4}(x)_{-1}^{0} = \frac{1}{4}(0+1) = 1/4$$

Example 63 Find the distribution function for Example 61.

Solution The probability distribution function for Example 61 is as follows:

$$f(x) = \frac{1}{4}; \quad -1 \le x \le 3, \quad \text{as} \quad a = -1$$

$$= 0; \quad \text{otherwise} \qquad b = 3$$

$\therefore$ $F(x) = 0, x < -1$

$$= \frac{x+1}{4}; -1 \le x < 3$$
$$= 1; x \ge 1$$

Example 64 A random variable X has an exponential distribution with probability density function $f(x)$ given by

$$f(x) = \begin{cases} 2e^{-2x}; & x \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

What is the probability that X is not less than 2?

Solution

$$P(X \ge 2) = \int_2^\infty 2e^{-2x} \mathrm{d}x = \left| \frac{-2e^{-2x}}{2} \right|_2^\infty = e^{-4}.$$

Example 65 The sales-tax return of a sales person is exponentially distributed with the density function $f(x)$ is given by

$$f(x) = \begin{cases} \frac{1}{4} e^{-\frac{1}{4}x}; & x \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

What is the probability that his sale will exceed Rs. 10,000 assuming that sales-tax is levied at the rate of 5% on the sales?

Solution Given that

$$f(x) = \begin{cases} \frac{1}{4} e^{-\frac{1}{4}x}; & x \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

If sale exceeds Rs. 10,000, then the sales-tax will exceed

$$10{,}000 \times \frac{5}{100} = \text{Rs. } 500$$

The required probability is

$$P(X > 500) = \frac{1}{4} \int_{500}^\infty e^{-\frac{1}{4}} \mathrm{d}x = e^{-125}.$$

Example 66 Find the mean and variance of an uniform random variable X whose pdf is given by

$$f(x) = \begin{cases} \frac{1}{12}; & -4 \le x \le 8 \\ 0; & \text{otherwise} \end{cases}$$

Solution

Mean $$\mu = \frac{b+a}{2} = \frac{8-(-4)}{2} = 6$$

Variance $$\sigma^2 = \frac{(b-a)^2}{12} = \left(\frac{8-(-4)}{12}\right)^2 = 12$$

Example 67 For Example 65, find the mean and variance of X.

Solution

$$\mu = \int_0^\infty x\frac{1}{4}e^{-\frac{1}{4}}\,dx = \frac{1}{4}\int_0^\infty x e^{-\frac{1}{4}}\,dx = 4$$

Alter: Given $f(x)$ is a exponential distribution with $\lambda = \frac{1}{4}$

$\therefore$ $$\mu = \frac{1}{\lambda} = 4$$

and variance $\sigma^2 = \frac{1}{\lambda^2} = (4)^2 = 16.$

EXERCISE 6.7

1. Let X has an exponential distribution

$$f(x) = \begin{cases} \frac{1}{\sigma}e^{-x/\sigma}; & x \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

Find mean, variance and moment generating function of X.

2. The amount of time that a surveillance camera will run without having to be reset is a random variable X having exponential distribution with $\lambda = 50$ days. Find the probability that such a camera.
   (a) will have to be reset in less than 20 days
   (b) will not have to be reset at least 60 days.

3. Let $f(x) = \frac{1}{6}, -3 \le x \le 3,$ 0, otherwise, be the pdf of X. Graph the pdf $f(x)$ and *CDF* $F(x)$, and calculate, the mean and variance of X.

4. Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period. Let X equal the time within the 10 minutes that the customer arrived. If X is uniformly distributed over (0, 10), then find (i) the pdf of X, (ii) $P(X \ge 8)$, (iii) $P(2 \le X \le 8)$, (iv) $E(X)$ and (v) Var X.

5. If the moment generating function of X is

$$M_X(t) = \frac{e^{5t} - e^{4t}}{t}, \ t \neq 0 \text{ and } M_X(0) = 1$$

Find (a) $E(X)$, (b) Var (X) and (c) $P(4.2 \leq X \leq 4.7)$.

6. What are the pdf, the mean, and the variance of X, if the moment generating function of X is given by the following:

(a) $M_X(t) = \dfrac{1}{1-3t}, \quad t < 1/3$ (b) $M_X(t) = \dfrac{3}{3-t}, \quad t < 3$

Answers

1. Mean = Variance = σ; $M_X(t) = (1-\sigma t)^{-1}$
2. (a) 0.3297; (b) 0.3012
3. Mean $\mu = 3$; $\sigma^2 = 3$
4. (i) $f(x) = \dfrac{1}{10}, 0 < x < (10)$; 0, otherwise

   (ii) 0.2; (iii) 0.6; (iv) $\mu = 5$; (v) $\sigma^2 = \dfrac{25}{3}$
5. (a) $E(X) = \dfrac{9}{2}$ (b) $\text{Var}(X)\dfrac{1}{12}$; (c) 0.5
6. (a) $f(x) = \dfrac{1}{3} e^{-\frac{x}{3}}, 0 < x < \infty; \mu = 3, \sigma^2 = 9$

   (b) $f(x) = 3e^{-3x}, 0 < x < \infty; \mu = \dfrac{1}{3}, \sigma^2 = \dfrac{1}{9}$

6.20 NORMAL DISTRIBUTION

Normal distribution is one of the most widely used continuous probability distribution in application of statistical methods. First of all, English mathematician De Moivre discovered this distribution in 1723 as a limiting case of binomial distribution.

The normal distribution has wide application in the analysis and evaluation of every experimental data in Science, Engineering and Medicine. Normal distribution can also be obtained as limiting case of Poisson distribution with parameter $\lambda \to \infty$. Normal distribution is also known as Gaussian distribution.

A continuous random variable X is said to have the normal distribution, if its probability density function is given by

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{4}$$

$$-\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0,$$

where μ = mean and σ = standard deviation.

μ and σ are called parameters. A random variable X which has normal distribution with parameters μ and σ is denoted by $X \sim N(\mu, \sigma^2)$.

Equation (4) which is probability distribution function of normal distribution is the limiting case of binomial distribution when $n \to \infty$, neither p nor q being very small.

6.20.1 Properties of Normal Distribution

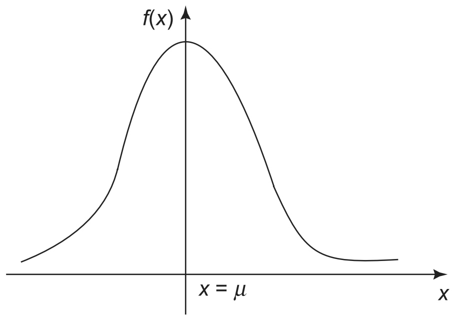


Fig. 6.10

The normal probability curve is given by

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

(i) Area under the curve is unity.

(ii) The mean, mode and median of the normal distribution coincide.

(iii) The curve is bell shaped and symmetric about the mean $x = \mu$. as shown in the figure.

(iv) It is unimodel with coordinates decreasing rapidly on both sides of the mean. The maximum ordinate is $\frac{1}{\sigma\sqrt{2\pi}}$, which we can get by putting $x = \mu$ in (4).

(v) The point of inflection are $\mu \pm \sigma$, which we get by putting $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$, which are equidistant from the mean on both the sides.

(vi) The mean deviation from the mean μ

$$= \int_{-\infty}^{\infty} |x-\mu| \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \sqrt{\frac{2}{\pi}} \text{ or } \frac{4}{5}\sigma \text{ (App.)}$$

(vii) The x-axis is an asymptote to the curve.

(viii) A linear combination of independent normal variates is also a normal variate.

(ix) The odd moments about the mean

$$\mu_{2n+1} = 0, n = 0, 1, 2, 3, \ldots$$

and even moments about the mean

$$\mu_{2n} = 1 \cdot 3 \cdot 5 \cdot ...(2n-1)\,\sigma^{2n},\ n = 0, 1, 2, 3, ...$$

$$\therefore \qquad \mu_2 = \sigma^2,\ \mu_4 = 3\sigma^4$$

Hence $\quad \beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = 0$ and $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = 3$

i.e. the coefficient of skewness is always zero and kurtosis is 3.

(x) As $f(x)$, the probability density is always non-negative, i.e. no portion of the curve lies below the x-axis.

(xi) The probability of X lying between a and b is defined by

$$P(a \le x \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{1}{2}z^2} \sigma\,\mathrm{d}z \quad \text{where } z = \frac{x-\mu}{\sigma},\ \mathrm{d}z = \frac{\mathrm{d}x}{\sigma},$$

$$z_1 = \frac{a-\mu}{\sigma},\ z_2 = \frac{b-\mu}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_2} e^{-\frac{1}{2}z^2} \mathrm{d}z - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{1}{2}z^2} \mathrm{d}z$$

$$= F_2(z) - F_1(z)$$

The values of each of the above integrals can be found by the table which gives the value of

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}z^2} \mathrm{d}z \text{ of various value of } z.$$

This integral is called the probability integral or the error function due to its use in the theory or errors and theory of sampling.

The variable $Z = \dfrac{X-\mu}{\sigma}$ is known as standard normal variate with mean 0 and variance 1. Z is denoted by $Z \sim N(0, 1)$ and its pdf is given by $y = f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},\ -\infty < z < \infty$ and its graph is shown in Fig. 6.11.

(xii) Using this table, we see that the area under the normal curve from $z = 0$ to $z = 1$ i.e., from $x = \mu$ to $\mu + \sigma$ is 0.3413. Therefore

(a) 68.27% area lies between the ordinates $x = \mu$ and $x = \mu + \sigma$, i.e. between $-1 < z < 1$, thus $\dfrac{2}{3}$ of the values lie within these limits.

(b) About 95.5% area lies between $x = \mu - 2\sigma$ and $\mu + 2\sigma$, i.e. $-2 < z < 2$, which implies that about 4.5% of the values lie outside these limits.

(c) 99.73% area lies between $x = \mu - 3\sigma$ and $\mu + 3\sigma$, i.e. $-3 < z < 3$.

(d) 95% are lies between $x = \mu - 1.96\sigma$ and $x = \mu + 1.96\sigma$, i.e. $-1.96 < z < 1.96$

(e) 99% area lies between $x = \mu - 2.58\,\sigma$ and $x = 2.58\,\sigma$, i.e. $-2.58 < z < 2.58$ and (*f*) 99.9% area lies between $x = \mu - 3.29\,\sigma$ and $x = \mu + 3.29\sigma$, i.e. $-3.29 < z < 3.29$.

Therefore, (a), (b), (c), (d), (e) and (f) can be written as

$$P(\mu - \sigma \le X \le \mu + \sigma) = P(-1 \le z \le 1) = .6827$$

$$P(\mu - 2\sigma \le X \le \mu + 2\sigma) = P(-2 \le z \le 2) = .9544$$

$$P(\mu - 3\sigma \le X \le \mu + 3\sigma) = P(-3 \le z \le 3) = .9973$$

$$P(\mu - 1.96\sigma \le X \le \mu + 1.96\sigma) = P(-1.96 \le z \le 1.96) = .9500$$

$$P(\mu - 2.58\,\sigma \le X \le \mu + 2.58\,\sigma) = P(-2.58 \le z \le 2.58) = .99$$

and $$P(\mu - 3.29\,\sigma \le X \le \mu + 3.29\,\sigma) = P(-3.29 \le z \le 3.29) = .999$$

These can be represented as follows in Fig. 6.11:

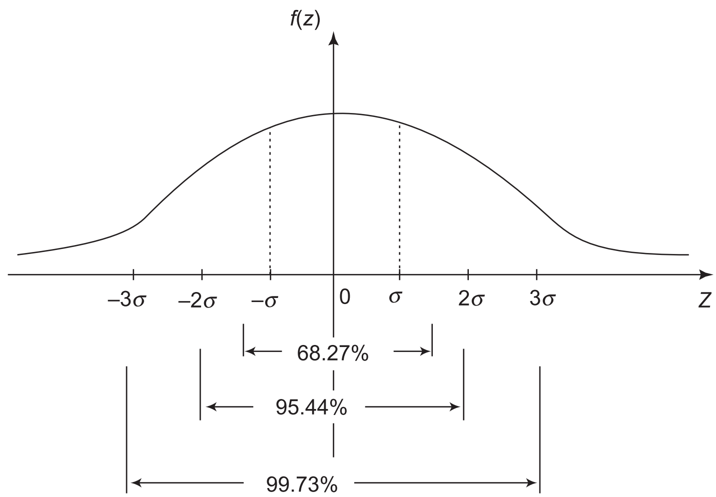


Fig. 6.11

(xiii) The moment generating function of a normal variate X is given by

$$M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

6.20.2 Normal Frequency Distribution

A normal curve can be fitted to any distribution. If N be the total frequency, μ be the mean and σ^2 be the variance of the given distribution, then the curve

$$y = f(x) = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

will fit the given distribution as best as the data allow. The frequency of the variable between x_1 and x_2 as given by the fitted curve, will be under $z = \frac{x-\mu}{\sigma}$ from x_1 to x_2.

6.20.3 Probable Error

All of us aware that any lot of goods manufactured always subject to some small errors i.e., it deviates from its required specification. There errors are generally of random nature, and therefore follow a normal distribution. Let probable error is denoted by β which is such that the probability of an error falling with the limits $\mu - \beta$ and $\mu + \beta$ is exactly equal to the chance of an error falling outside these limits, i.e. the probability of an error lying with $\mu - \beta$ and $\mu + \beta = 1/2$.

$$\therefore \qquad \frac{1}{2} = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\beta}^{\mu+\beta} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)} \mathrm{d}x \qquad \text{Let } z = \frac{x-\mu}{\sigma}$$

$$\sigma dz = dx$$

$$\Rightarrow \qquad \frac{1}{4} = \frac{1}{\sqrt{2\pi}} \int_0^{\beta/\sigma} e^{-\frac{z^2}{2}} \mathrm{d}z \Rightarrow \frac{\beta}{\sigma} = 0.6745 \text{ (By normal table)}$$

$$\text{or} \qquad \boxed{\beta = 0.67456}$$

Example 68 If random variable X has mean 10 and standard deviation 5, then find (i) $P(X \le 15)$ (ii) $P(X \ge 15)$ and (iii) $P(10 \le X \le 15)$.

Solution Given $X \sim N(\mu = 10, \sigma^2 = 25)$

$$\therefore \text{(i)} \quad P(X \le 15) = P\left(\frac{X-10}{5} \le \frac{15-10}{5}\right) \qquad \left(\because \quad Z = \frac{X-10}{5}\right)$$

$$= P(Z \le 1) = F(1) = 0.8413 \quad \text{(Using normal table)}$$

$$\text{(ii)} \quad P(X \ge 15) = 1 - P(X < 15)$$

$$= 1 - P\left(\frac{X-10}{5} < \frac{15-10}{5}\right)$$

$$= 1 - P(Z < 1) = 1 - .8413 = 0.1587$$

$$\text{(iii)} \quad P(10 \le X \le 15) = P\left(\frac{10-10}{5} \le \frac{X-10}{5} \le \frac{15-10}{5}\right)$$

$$= P(0 \le X \le 1) = F(1) - F(0)$$

$$= 0.8413 - 0.5000 = 0.3413 \qquad \text{(Using normal table)}$$

Example 69 Show that $F(-z) = 1 - F(z)$.

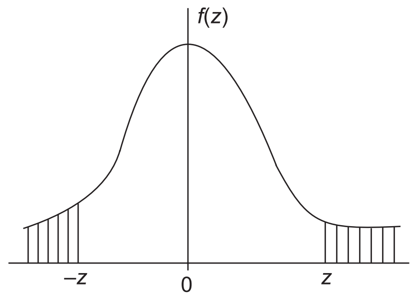


Fig. 6.12

Solution

$$F(-z) = P(Z \le -z) = P(Z > z) \quad \text{(Using symmetry)}$$
$$= 1 - P(Z < z) = 1 - F(z)$$

Hence the result.

Example 70 If X is normally distributed with mean 8 and standard deviation 4, find

(i) $P(5 \le X \le 10)$ (ii) $P(X \ge 15)$ (iii) $P(X \le 5)$

Solution Given $X \sim N(8, 16)$

(i) $$P(5 \le X \le 10) = P\left(\frac{5-8}{4} \le \frac{X-8}{4} \le \frac{10-8}{4}\right)$$

$$= P(-\cdot 75 \le Z \le \cdot 50) \qquad \left[\because \frac{X-8}{4} = Z\right]$$

$$= F(0\cdot 50) - F(-0\cdot 75)$$

$$= 0\cdot 5000 - 0\cdot 2266 = 0\cdot 2734 \qquad \text{(Using table)}$$

(ii) $$P(X \ge 15) = 1 - P(X < 15) = 1 - P\left(Z < \frac{15-8}{4}\right)$$

$$= 1 - F(1\cdot 75) = 1 - \cdot 9599 = 0.0411 \qquad \text{(Using table)}$$

(iii) $$P(X \le 5) = P\left(Z \le \frac{5-8}{4}\right) = F(-\cdot 75) = 0\cdot 2266 \quad \text{(Using table)}$$

Example 71 The marks of 1000 engineering students in a college are found to be normally distributed with mean 70 and variance 25. Find the number of students whose marks will be between (i) 60 and 75, (ii) more than 75, and (iii) less than 68.

Solution Let X denote the marks of the students.

$\therefore \qquad X \sim N(70, 25)$

$$Z = \frac{X-70}{25}$$

(i) $P(60 < X < 75) = P\left(\frac{60-70}{5} < \frac{X-70}{5} < \frac{75-70}{5}\right)$

$= P(-2 < Z < 1) = F(1) - F(-2)$

$= F(1) - 1 + F(2) = 0.8413 - 1 + 0.9772 = 1.8185 - 1 = 0.8185$ (Using table)

$\therefore$ Number of students whose marks are between 60 and 75

$= 1000 \times 0.8185 = 818.5 \cong 819$

(ii) $P(X > 75) = P(Z > 1) = 1 - P(Z \le 1) = 1 - F(1) = 1 - 0.8413 = 0.1587$ (Using table)

Number of students greater than 75 marks = 1000×0.1587

$= 158.7 \simeq 159$

(iii) $P(X < 68) = P\left(Z < \frac{68-70}{5}\right) = P(Z < -0.4)$

$= F(-0.4) = 1 - F(0.4) = 1 - 0.6554 = 0.3446$ (Using table)

Number of students < 68 marks = $1000 \times 0.3446 = 344.6 \simeq 345$.

Example 72 In a photographic process, the developing time of prints may be looked upon as a random variable X having normal distribution with mean 16.28 seconds and standard deviation 0.12 second. For what value is the probability 0.95 that it will be exceeded by the time it takes to develop one of the prints?

Solution Let X be the developing time

then $X \sim N\,[16.28, (0.12)^2]$

Let k be the number which exceeds the time

$$\therefore \quad P(X > k) = 0.95 = P\left(\frac{X-\mu}{\sigma} > \frac{k-16.28}{0.12}\right)$$

$$\Rightarrow \quad P\left(Z > \frac{k-16.28}{0.12}\right) = 0.95$$

$$\therefore \quad 1 - P\left(Z \le \frac{k-16.28}{0.12}\right) = 0.95$$

$$\Rightarrow \quad P\left(Z \le \frac{k-16.28}{0.12}\right) = 0.05$$

$$\Rightarrow \quad F\left(\frac{k-16.28}{0.12}\right) = 0.05$$

$$\Rightarrow \quad \frac{k-16.28}{0.12} = -1.645$$

$$k = 16.0826$$

Example 73 If the probability of committing an error of magnitude x is given by $y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$

Compute the probable error from the following data:

$$m_1 = 1.305;\ m_2 = 1.301;\ m_3 = 1.295;\ m_4 = 1.286;\ m_5 = 1.138$$
$$m_6 = 1.321;\ m_7 = 1.283;\ m_8 = 1.289;\ m_9 = 1.300;\ m_{10} = 1.286$$

(Kurukshetra 2005)

Solution Data is normally distributed. Therefore, the mean

$$\mu = \frac{1}{10}\sum_{i=1}^{10} m_i = \frac{12.984}{10} = 1.2984$$

and variance $$\sigma^2 = \frac{1}{10}\sum_{i=1}^{10}(m_i - \mu)^2$$
$$= \frac{1}{10}\left[(1.305 - 1.2984)^2 + \cdots + (1.286 - 1.2984)^2\right]$$
$$= 0.0001594$$

Hence, standard deviation $\sigma = \sqrt{.0001594} = 0.012625371$

$\therefore$ Probable error $= \frac{2}{3}\sigma = \frac{2}{3} \times 0.012625373$
$$= 0.0084169$$

Example 74 Fit a normal curve to the following distribution:

x	2	4	6	8	10
f	1	4	6	4	1

(V.T.U. 2001)

Solution

Mean $$\mu = \frac{\Sigma fx}{\Sigma f} = \frac{2\times 1 + 4\times 4 + 6\times 6 + 8\times 4 + 10\times 1}{16}$$
$$= \frac{2 + 16 + 36 + 32 + 10}{16} = \frac{96}{16} = 6$$

Variance $$\sigma^2 = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2 = \frac{1(2)^2 + 4(4)^2 + 6(6)^2 + 4(8)^2 + 1(10)^2 - (6)^2}{16}$$
$$= \frac{4 + 64 + 216 + 256 + 100}{16} - 36 = 40 - 36 = 4$$

$\therefore$ Standard deviation $\sigma = \sqrt{4} = 2$

$\therefore$ The equation of normal curve $\Rightarrow$

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-6}{2}\right)^2}$$

$$f(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{8}(x-6)^2} \tag{5}$$

Area under Eq. (5) in (x_1, x_2)

$$P(x_1 \le X \le x_2) = P\left(\frac{x_1-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{x_2-\mu}{\sigma}\right) \quad z_1 = \frac{x_1-2}{2}$$

$$z_2 = \frac{x_2-2}{2}$$

$$= P(z_1 \le Z \le z_2) = F(z_2) - F(z_1) \text{ and } \quad \because \frac{x-\mu}{\sigma} = Z$$

To evaluate the probabilities, we refer the table of normal distribution.

Mid point (x)	**(x_1, x_2)**	**(z_1, z_2)**	**Area under (x_1, x_2) or (z_1, z_2)**	**Expected Frequency = N(Area under x_1, x_2)**
2	(1, 3)	$z_1 = -2.5$ $z_2 = -1.5$	$F(-1.5) - F(-2.5)$ $= 0.0668 - .0062 = .0606$	$16 \times .0606 = 0.9696 \simeq 1$
4	(3, 5)	$z_1 = -1.5$ $z_2 = -0.5$	$F(-0.5) - F(-1.5)$ $= .3085 - .0668 = .2417$	$16 \times .2417 = 32.8672 \simeq 4$
6	(5, 7)	$z_1 = -0.5$.$z_2 = 0.5$	$.6915 - .3085$ $= F(0.5) - F(-5) = 0.383$	$16 \times .383 = 6.128 \simeq 6$
8	(7, 9)	$z_1 = 0.5$ $z_2 = 1.5$	$F(1.5) - F(0.5)$ $= .9332 - .6915 = .2417$	$16\times .2417 = 3.8672 \simeq 4$
10	(9, 11)	$z_1 = 1.5$ $z_2 = 2.5$	$F(2.5) - F(1.5)$ $= .9938 - .9332 = .0606$	$16 \times .0606 = .9696 \simeq 1$

Hence the expected (theoretical) frequencies are 1, 4, 6, 4 and 1, which are same to the given observed frequencies. This shows that normal curve is a proper fit to the given distribution.

EXERCISE 6.8

1. X is a normal variate with mean 30 and S.D. 5, find the probabilities that (i) $26 \le X \le 40$ (ii) $X \ge 45$ and (iii) $|X - 30| > 5$.

   (J.N.T.U. 2005)

2. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution.

   (V.T.U. 2009, S.V.T.U. 2008, Kurukshetra 2007S)

3. In a test on 2000 electric bulbs, it was found that the life a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for:
   (a) more than 2150 hours, (b) less than 1950 hours and
   (c) more than 1920 hours and but less than 2160 hours.

   (Bhopal 2008S, U.P.T.U. 2008)

4. If X is a normal variate with mean 12 and S.D. 4. Find (i) $P(X \geq 20)$, (ii) $P(X \leq 20)$ and (iii) $P(0 \leq X \leq 12)$.
5. For a certain normal distribution the first moment about 10 is 40 and the fourth moment about 50 is 48. What is mean and variance of the normal distribution.
6. Students of a class were given an aptitude test. Their marks were found to be normally distributed with mean 60 and SD 5. What percentage of students, scored more than 60 marks.
7. If the heights of 300 students are normally distributed with mean 64.5 inches and SD 3.3 inch, how many students have height.
   (i) less than 5 feet, (ii) between 5 feet and 5 feet 9 inches. Also find the height below which 99% of the students lie.
8. A random variable X is normally distributed with mean 12 and SD 2. Find $P(9{\cdot}6 < X < 13{\cdot}8)$.
9. The mean height of 500 students is 131 cm, and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students' height lie between 120 and 155 cm?

(Burdwan 2003)

10. In a certain examination, the percentage of candidates passing and getting distinctions were 45 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. (Assume that the distribution of marks to be normal).

(Kottayam 2005)

11. Assuming that the diameters of 1000 brass plugs taken consecutively from a machine, form a normal distribution with mean 0.7515 cm and standard deviation 0.0020 cm, how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm.

(Bhopal 2002)

12. The income of a group of 10,000 persons was found to be normally distributed with mean Rs. 750 p.m. and standard deviation of Rs. 50. Show that of this group, about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100.

(U.P.T.U. 2004S)

13. Given that the probability of committing an error of magnitude x is $y = \dfrac{h}{\sqrt{\pi}} e^{-h^2 x^2}$. Find the probable error.
14. If 10% of the items produced by a factory are defective. Find the standard deviation of the number of defectives and equation to the normal curve to represent the number of defectives.
15. Find the equation of the best fitting normal curve of the following data:

x	0	1	2	3	4	5
f	13	23	34	15	11	4

16. Fit a normal curve to the following frequency distribution:

Length of line (cm) x	6	7	8	9	10	11	12
Frequency	3	6	9	13	8	5	4

Answers

1. (i) 0.7653; (ii) 0.0014; (iii) 0.3174
2. Mean = 50; S.D. = 10.
3. (a) 67 (App.) (b) 184 (App.) (c) 1909 (App.)
4 (i) 0.0228; (ii) 0.9772; (iii) 0.4987
5. $\mu = 6.05$; $\sigma^2 = 6.26$
6. 50%
7. (i) 26; (ii) 248; 72.18 inch
8. 0.37
9. 294
10. 36.4
11. 52
12. Rs. 866
13. $0.4769/h$
14. $p = 0.1$, $q = 0.9$, $\sigma^2 = npq = 9 \Rightarrow$ S.D. $\sigma = 3$, $np = \mu = 10$, $n = 100$
    Equation of normal curve is

$$f(x) = \frac{100}{\sigma/\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-10}{3}\right)^2}$$

15. $f(x) = \dfrac{100}{\sqrt{3.4\pi}} e^{-\frac{(x-2)^2}{3\cdot 4}}; -\infty < x < \infty$

16. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, where $\mu = 9$, $\sigma = 7.543$.

6.21 NORMAL APPROXIMATION TO BINOMIAL DISTRIBUTION

Let X be a random variable having binomial distribution inch parameters n and p. The we can show that $P\left(\dfrac{X - np}{\sqrt{npq}} \le z\right) = P(Z \le z) = F(z)$ as $n \to \infty \Rightarrow$ when n is large, the binomial distribution can be approximated using normal distribution. The approximation is acceptable for values of n and p when either $p \le 0{\cdot}5$ and $np > 5$ or $p > 0{\cdot}5$ and $nq > 0{\cdot}5$.

Example 75 A manufacturer knows that on the average 2% of washing machines that he makes will require repair within 90 days after the machines are sold. Use normal approximation to the binomial distribution to determine the probability that among 1200 of these machines at least 30 will require repair within 90 days after they are sold 1.

Solution Let X be the number of machines that require repairs within 90 days after they are sold.

$\therefore$ X has binomial distribution with $n = 1200$, $p = 0.02$

$$\therefore \quad \mu = np = 1200 \times 0.02 = 24 \text{ and } \sigma^2 = npq = 23.52 \Rightarrow \sigma = \sqrt{23.52} = 4.85$$

$$\therefore \quad P(X \ge 30) = P\left(\frac{X.np}{\sqrt{npq}} \ge \frac{30 - 24}{4.84}\right) = P(Z \ge 1.24)$$

$$= 1 - P(Z < 1.24) = 1 - F(1.24) = 1 - 0.8925 = 0.1075 \qquad \text{(Using table)}$$

Correction of continuity: Since for continuous random variable $P(X \geq x) = P(X > x)$ or $P(X \leq x) = P(X < x)$, which is not true for discrete random variable. When we approximate binomial probability by normal probability, we must ensure that we do not lose the point. This is achieved by continuity of correction.

Example 76 Consider Example 75 and use continuity correction to calculate probability.

$$P(X \geq 30) = P\left(X \geq 30 - \frac{1}{2}\right) \quad \left(\text{Subtract } \frac{1}{2} \text{ for continuity}\right)$$

$$= P\left(Z \geq \frac{29.5 - 24}{4.85}\right) = P(Z \geq 1.13)$$

or $1 - P(Z < 1.13) = 1 - F(1.13) = 1 - 0.878 = 0.122$ (Using table)

Example 77 A safety engineer feels that 30% of all industrial accidents in his plant are caused by failure of employees to follow instructions. Find approximately the probability that among 84 industrial accidents any where from 20 to 30 (both inclusive) will be able to failure of employees to follow instructions.

Solution Let X be the number of accidents due to failure of employees to follow instructions:

$$n = 84,\ p = 0.30 \Rightarrow np = \mu = 25.2$$

and $$\sigma^2 = npq = 17.64 \Rightarrow \sigma = 4.2$$

$$\therefore\ P(20 \leq X \leq 30) = P\left(20 - \frac{1}{2} \leq X \leq 30 + \frac{1}{2}\right)$$

(For continuity correction subtract $\frac{1}{2}$ from L.H.S. and add $\frac{1}{2}$ in the right hand side)

$$\therefore\ P(20 \leq X \leq 30) = P\left(\frac{19.5 - 25.2}{4.2} \leq Z \leq \frac{30.5 - 25.2}{4.2}\right)$$

$$= P(-1.36 \leq Z \leq 1.26)$$

$$= F(1.26) - F(-1.36)$$

$$= F(1.26) - 1 + F(1.36) = 0.8962 - 1 + 0.9131 = 0.8093 \quad \text{(Using table)}$$

6.22 CHEBYSHEV'S INEQUALITY

If a probability distribution has mean μ and standard deviation σ, the probability of getting a value which deviates from μ by at least $k\sigma$ is at must $\frac{1}{k^2}$ i.e., $P\left[|X - \mu| \geq k\sigma\right] \leq \frac{1}{k^2}$ or $P\left[|X - \mu| < k\sigma\right] \geq 1 - \frac{1}{k^2}$

$P\left[|X-\mu| \geq k\sigma\right]$ is the probability associated with the set of outcomes for which x, the value of random variable having the given probability distribution is such that $|x-\mu| \geq k\sigma$.

Example 78 The number of customers who visit a car dealer's show room on a day is a random variable with $\mu = 18$ and $\sigma = 2.5$. With that probability can we assert that there will be between 8 and 28 customers?

Solution Given $\mu = 18$, $\sigma = 2.5$

$$\therefore \qquad k = \frac{28-18}{25} = \frac{18.8}{2.5} = 4$$

$\therefore$ The probability is at least $1-\dfrac{1}{k^2} = \dfrac{15}{16}$

EXERCISE 6.9

1. Let X be a random variable having binomial distribution with $n = 20$ and $p = 0.3$. Use the normal approximation to approximate each of the following:
   (i) $P(X \leq 3)$
   (ii) $P(3 \leq X \leq 6)$
   (iii) $P(X \geq 4)$
2. If 20% of the memory chips made in a certain plant are defective. Use normal approximation to binomial distribution to find the probabilities that in a lot of 100 randomly chosen for inspection.
   (a) at most 15 will be defective
   (b) exactly 15 will be defective.
3. Find the probability of getting between 3 heads to 6 heads in 10 tosses of a fair coin using the normal approximation to binomial distribution.
4. Show that for 40,000 flips of a balanced coin, the probability is at least 0.99 that the probability of heads will fall between 0.475 and 0.525.
5. The tensile strength of some synthetic fibres can be determined by tying two strands together and putting until either the right or left hand strand breaks. If 144 pairs of strands will be broken, what does Chebychev's inequality with $k = 4$ tell in about the number of cases where the left hand strand broke?

Answers

1. (i) 0.1093; (ii) 0.5512; (iii) 0.7704
2. (a) 0.1344 (b) 0.0461
3. 0.7718
5. The probability of getting more than 95 or less than 49 left hand breaks is $\leq \dfrac{1}{16}$.

SUMMARY

Following topics have been discussed in this chapter:

1. How to calculate mathematical, statistical axiomatic probabilities. We also defined important terms such as:
Sample space, mutually exclusive events, event, permutation, combination, impossible event, etc.

2. Additive law of Probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

if A and B are mutually exclusive events
then $P(A \cup B) = P(A) + P(B)$.

3. Conditional probability

$$P(A|B) = \frac{(A \cap B)}{P(B)}, \ P(B) > 0$$

if A and B are independent events then

$$P(A \cap B) = P(A) \cdot P(B).$$

4. Define Random variable (Discrete and continuous): How to calculate mean and variance of a random variable X.

$$\mu = E(X) = \sum_{\forall x} x\,p(x), \text{ if } x \text{ is discrete, } \int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x, \text{ if } x \text{ is continuous}$$

$$\sigma^2 = E(X^2) - [E(X)]^2 = \mu_2^1 - \mu^2$$

$$E(X^2) = \sum_{\forall x} x^2 p(x), \text{ if } x \text{ is discrete and } \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x, \text{ if } X \text{ is continuous.}$$

Moment generating function:

$$M_X(t) = E[e^{tx}] = \sum_{\forall x} e^{tx} p(x), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,\mathrm{d}x, \text{ if } X \text{ is continuous}$$

$$\mu_r' = \frac{\mathrm{d}^r}{\mathrm{d}t^r} M_X(t)\bigg|_{t=0}$$

5. Bernoulli, Binomial and Poisson distributions as a special case of discrete random variable.

(a) For Bernouli's Distribution:

$$f(x) = p^x (1-p)^{1-x}, x = 0, 1; \mu = p, \sigma^2 = pq, M_X(t) = q + pe^t$$

(b) For Binomial Distribution:

$$f(x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, 2, \ldots, n; \mu = np, \sigma^2 = npq, M_X(t) = (q + pe^t)^n$$

(c) For Poisson Distribution:

$$f(x) = \frac{e^{-\lambda}\lambda^x}{\underline{|x}}, x = 0, 1, 2, \ldots, n; \mu = \lambda = \sigma^2, M_X(t) = e^{\lambda(e^t - 1)}$$

6. For continuous random variable, uniform, exponential and normal distributions are discussed.

   (i) For uniform distribution:

$$f(x) = \frac{1}{b-a}, a \le x \le b; 0, \text{otherwise}$$

$$\mu = \frac{b+a}{2}; \sigma^2 = \frac{(b-a)^2}{12}, M_X(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$$

   (ii) For Exponential Distribution:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$
$$0, \qquad \text{otherwise}$$

$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}, M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} = \frac{\lambda}{\lambda - t}, t < \lambda$$

   (iii) For Normal Distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, - x < \infty < x.$$

$$E(X) = \mu = \mu, \text{Var}(X) = \sigma^2, M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}},$$

$$Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$$

$$F(-z) = 1 - F(z)$$

7. Normal Approximation to Binomial Distribution:
8. Chebychev's inequality:

$$P[|X - \mu| \ge k\sigma] \le 1 - \frac{1}{k^2}$$

OBJECTIVE TYPE QUESTIONS

1. The probability that A happens is $\frac{1}{4}$. The odd against happening of A are
(a) 2 : 2 (b) 2 : 1
(c) 3 : 1 (d) 5 : 1

2. The odds in favour of an event B are 6 to 4. The probability of success of B is
(a) $\frac{4}{6} = \frac{2}{3}$ (b) $\frac{4}{10} = \frac{2}{5}$
(c) $\frac{6}{10} = \frac{3}{5}$ (d) None of these

3. If $P(A) = \frac{2}{3}$, then $P(\bar{A})$ is
(a) $\frac{1}{3}$ (b) $\frac{1}{2}$
(c) $\frac{2}{3}$ (d) $\frac{1}{4}$

4. If Φ is an impossible event and S be sample space, then $P(\Phi)$ is
(a) 1 (b) 0.5
(c) 0 (d) None of these

5. $P(A/B)\cdot P(B)$ is equal to

(a) $P(A\cap B)$ (b) $P(A\cup B)$

(c) $P(A'\cup B')$ (d) $P(A'\cap B')$

6. If A and B are mutually exclusive events, such that $P(\bar{A})=0.5, P(\bar{B})=0.6,$ then $P(A\cup B)$ is equal to

(a) 0.8 (b) 0.9

(c) 0.7 (d) None of these

7. The probability of drawing any one spade card from a pack of cards is

(a) $\frac{1}{52}$ (b) $\frac{1}{13}$

(c) $\frac{4}{13}$ (d) $\frac{1}{4}$

8. The probability of drawing one white ball from a bag containing 6 red, 8 black 10 yellow and 1 green balls is

(a) $\frac{1}{25}$ (b) 0

(c) 1 (d) $\frac{24}{25}$

9. A coin is tossed three times in succession. The number of sample points in sample space is

(a) 6 (b) 9

(c) 3 (d) 8

10. In the simultaneous tossing of two prefect coins, the probability of having at least one head is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{3}{4}$ (d) 1

11. In the simultaneous tossing of two prefect dice, the probability of obtaining 4 as the sum of the resultant face is

(a) $\frac{1}{3}$ (b) $\frac{1}{12}$

(c) $\frac{1}{4}$ (d) $\frac{1}{6}$

12. A single letter is selected at random from the word 'probability'. The probability that it is a vowel is

(a) $\frac{3}{11}$ (b) $\frac{2}{11}$

(c) $\frac{4}{11}$ (d) 0

13. A number is chosen at random among first 120 natural numbers. The probability of the number chosen being a multiple of 5 or 15 is

(a) $\frac{1}{5}$ (b) $\frac{1}{8}$

(c) $\frac{1}{16}$ (d) $\frac{1}{9}$

14. If A and B are mutually exclusive events, then

(a) $P(A\cup B)=P(A)\cdot P(B)$

(b) $P(A\cup B)=P(A)+P(B)$

(c) $P(A\cup B)=0$

(d) None of these

15. If $P(A\cap B)=\frac{1}{2}, P(\bar{A}\cap\bar{B})=\frac{1}{3}$ and $P(A) = P(B) = p$, then the value of p is

(a) $\frac{1}{2}$ (b) $\frac{7}{8}$

(c) $\frac{1}{3}$ (d) $\frac{7}{12}$

16. A and B are independent events such that $P(A\cap\bar{B})=\frac{3}{25}$ and $P(\bar{A}\cap B)=\frac{8}{25}$. If $P(A) < P(B)$, then $P(A)$ is

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

17. A and B are independent events such that $P(\bar{A})=0.7, P(\bar{B})=k$ and $P(A\cup B) = 0.8$, then k is

(a) $\frac{5}{7}$ (b) 1

(c) $\frac{2}{7}$ (d) None of these

18. If the events S and T have equal probability and are independent with $P(S\cap T)=p>0$, then $P(S)$ is

(a) $\sqrt{p}$ (b) p^2

(c) $\dfrac{p}{2}$ (d) p

19. Events S and T are independent with $P(S) < P(T)$, $P(S \cap T) = \dfrac{6}{25}$ and $P(S/T) + P(T/S) = 1$. Then $P(S)$ is

(a) $\dfrac{1}{25}$ (b) $\dfrac{1}{5}$

(c) $\dfrac{6}{25}$ (d) $\dfrac{2}{5}$

20. An unbiased dice is thrown two independent times. Given that the first throw result in an even number, the probability that the sum obtained is 8 is

(a) $\dfrac{5}{36}$ (b) $\dfrac{1}{6}$

(c) $\dfrac{4}{21}$ (d) $\dfrac{7}{36}$

21. If $P(A \cap B) = \dfrac{1}{2}, P(\bar{A} \cap B) = \dfrac{1}{2}$ and $2P(A) = P(B) = p$, then the value of p is

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{2}$

(c) $\dfrac{1}{3}$ (d) $\dfrac{2}{3}$

22. The probability that a leap year would have 53 Mondays is

(a) $\dfrac{1}{7}$ (b) $\dfrac{2}{7}$

(c) $\dfrac{3}{7}$ (d) $\dfrac{4}{7}$

23. The probability of getting 3, 4 or 6 from a single dice is

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{3}$

(c) $\dfrac{2}{3}$ (d) $\dfrac{2}{5}$

24. The mean of a Bernoulli distribution with probability of success 0.5 is

(a) 0.6 (b) 0.4
(c) 0.5 (d) 0.8

25. The mean of a binomial distribution with $n = 100$ and $p = 0.4$ is

(a) 40 (b) 4
(c) 60 (d) 6

26. The variance of a binomial distribution with $n = 50$ and $p = 0.6$ is

(a) 30 (b) 20
(c) 12 (d) 10

27. If the mean of a binomial distribution is 20 and $n = 100$ then probability of success p is

(a) 0.6 (b) 0.4
(c) 0.2 (d) 0

28. If $n = 20$ and $p = 0.6$, then moment generating function of binomial distribution is

(a) $(0.4 + 0.6\,e^{t})^{20}$ (b) $(0.6 + 0.4\,e^{t})^{20}$

(c) $(0.4 + 0.6\,e^{2t})^{20}$ (d) $(0.4e^{2t} + 0.6)^{20}$

29. If the mean of a Poisson distribution is 20, then its variance is

(a) 10 (b) 15
(c) 20 (d) 25

30. If the mean of a Poisson distribution is 10, then its standard deviation is

(a) $\sqrt{10}$ (b) 10
(c) 20 (d) $\sqrt{20}$

31. The standard deviation of binomial distribution is

(a) np (b) $\sqrt{np}$

(c) npq (d) $\sqrt{npq}$

32. In a Poisson distribution if $P(X=2) = 2P(X=1)$, then the mean is

(a) –1 (b) 0
(c) 2 (d) 4

33. If the probability of hitting a target by one shot is $p = 0.8$, then the probability that out of 10 shots, seven will hit the target

(a) 0.1188 (b) 0.1288
(c) 0.1388 (d) 0.1488

34. If X has Poisson distribution such that $P(X = 1) = P(X = 2)$, then the variance of X is

(a) 0 (b) 1
(c) 2 (d) 3

35. If A and B are independent events such that $P(A) = 0.14$ and $P(B/A) = 0.24$, then $P(B)$ is

(a) 0.24 (b) 0.14
(c) 0.30 (d) None of these

36. If A and B are independent events such that $P(\overline{B}) = 0.75$ and $P(A/B) = 0.15$ then $P(\overline{A})$ is
(a) 0.15 (b) 0.75
(c) 0.25 (d) 0.85

37. If the mean of a normal distribution is 2, then its median is
(a) 1 (b) 2
(c) 3 (d) 4

38. If the median of normal distribution is 4, then its mode is
(a) 1 (b) 2
(c) 3 (d) 4

39. If mode of the normal distribution 5, then its mean is
(a) 2 (b) 3
(c) 4 (d) 5

40. The distribution of whose mean, mode and median are same is known is
(a) Binomial (b) Poisson
(c) Uniform (d) Normal

41. The discrete distribution for which mean is always greater than its variance is known
(a) Binomial (b) Poisson
(c) Uniform (d) Normal

42. For a normal distribution β_1 is
(a) 0 (b) 1
(c) 2 (d) 3

43. For a normal distribution β_2 is
(a) 0 (b) 1
(c) 2 (d) 3

44. If a random variable X has Poisson distribution with mean 2, then its moment generating function $M_X(t)$ is
(a) $e^{2(e^t-1)}$ (b) e^2
(c) $e^{(e^t-1)}$ (d) none of these

45. If a normal distribution has mean 2 and variance 6 then its moment generating function is
(a) e^{2t+3t^2} (b) e^{t+t^2}
(c) e^{3t+2t^2} (d) e^{2t+6t^2}

46. If moment generating function of a random variable X having normal distribution is e^{4t+6t^2} then its mean is
(a) 4 (b) 6
(c) 12 (d) 8

47. The number of ways in which 5 persons can be lined up to get on a bus are
(a) 100 (b) 120
(c) 140 (d) 150

48. The probability of getting a total of 5 when a pair of fair dice is tossed is
(a) $\frac{1}{9}$ (b) $\frac{2}{9}$
(c) $\frac{3}{9}$ (d) $\frac{1}{36}$

49. If $P(A) = 0.81$ and $P(A \cap B) = 0.18$, then $P(B/A)$ is
(a) $\frac{2}{9}$ (b) $\frac{1}{9}$
(c) $\frac{3}{9}$ (d) $\frac{4}{9}$

50. If two unbiased dice are thrown simultaneously, the probability that the sum of the numbers on them is at least 10 is
(a) $\frac{4}{6}$ (b) $\frac{3}{6}$
(c) $\frac{2}{6}$ (d) $\frac{1}{6}$

51. Let X be a random variable having Poisson distribution such that $P(X = 2) = P(X = 3)$, then $P(X = 0)$
(a) e^3 (b) e^{-3}
(c) e^2 (d) e^{-2}

52. If $X \sim N$ ($\mu = 10$ and $\sigma^2 = 4$), then $P(X \le 10)$ is
(a) 0.69 (b) 0.85
(c) 0.50 (d) none of these

53. If A and B are independent events such that $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$, then $P(A \cap B)$ is
(a) $\frac{1}{6}$ (b) $\frac{1}{3}$
(c) $\frac{1}{2}$ (d) 1

54. If X is binomial with $n = 50$ and $p = \frac{1}{5}$ then its standard deviation is
(a) $2\sqrt{2}$ (b) $\sqrt{2}$
(c) 3 (d) $\sqrt{3}$

55. If A and B are mutually exclusive events then $P(A \cap B)$ is

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) $\frac{1}{4}$

56. The probability of selecting 3 white balls from a bag containing 5 white and 5 red balls

(a) ${}^5C_3/{}^{10}C_3$ (b) 5/10

(c) 3/10 (d) none of these

57. Let X be a binomial distribution, under which condition(s) X tends to Poisson distribution

(a) $n \to \infty$

(b) $p \to 0$

(c) np constant

(d) $p \to 0$ $n \to \infty$ so that np is fixed

58. If the mean and variance of binomial variate are 12 and 4 respectively, then n is

(a) 12 (b) 18

(c) 24 (d) 36

59. If A and B are mutually exclusive events such that $P(A) = 0.29$, $P(B) = 0.43$, then $P(A \cup B)$ is

(a) 0.29 (b) 0.42

(c) 0.72 (d) none of these

60. If A and B are mutually exclusive events such that $P(A) = 0.43$ and $P(B) = 0.29$, then $P(A' \cap B)$ is

(a) 0.1653 (b) 0.1563

(c) 0.1356 (d) none of these

61. If A and B are two events such that $P(A) = \frac{1}{3}, P(B) = \frac{1}{2}$ and $P(A \cap B) = 1/4$, then $P(B/A)$ is

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) none of these

62. The chance of throwing 7 in a single throw of two dice is

(a) $\frac{1}{6}$ (b) $\frac{2}{6}$

(c) $\frac{3}{6}$ (d) $\frac{4}{6}$

63. The probability distribution of binomial distribution is

(a) $\binom{n}{x} p^x q^{n-x}; x = 0, 1, 2, \ldots, n$

(b) $p^x q^{n-x}$

(c) $\binom{n}{x} pq$

(d) none of these

64. If X be a random variable having uniform distribution on (–4, 4), then its probability distribution function is given by

(a) $\frac{1}{8}$ (b) $\frac{1}{4}$

(c) $\frac{2}{8}$ (d) $\frac{3}{8}$

65. The cumulative distribution function of a uniform random variable on (–2, 2) is

(a) $\begin{cases} 0, & x < -2 \\ \frac{x+2}{4}, & -2 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$

(b) $\frac{x-2}{4}$

(c) $\frac{x+2}{2}$

(d) none of these

66. The mean of a random variable which is uniformly distribution over (–3, 3) is

(a) 0 (b) 1

(c) 2 (d) 3

67. The variance of a random variable which is uniformly distributed over (–3, 3) is

(a) 0 (b) 1

(c) 2 (d) 3

68. The mean of a uniform random variable on $[a, b]$ is

(a) $\frac{b+a}{2}$ (b) $\frac{b-a}{2}$

(c) $\frac{a}{2}$ (d) $\frac{b}{2}$

69. The variance of a uniform distribution on $[a, b]$ is

(a) $\dfrac{b+a}{2}$ (b) $\dfrac{b-a}{2}$

(c) $\dfrac{(b-a)^2}{12}$ (d) $\dfrac{(b-a)^2}{2}$

70. The probable error is

(a) $\dfrac{4}{5}\sigma$ (b) $\dfrac{2}{3}\sigma$

(c) $\dfrac{3}{4}\sigma$ (d) none of these

where σ is standard deviation.

71. A card is drawn from well-shuffled pack of 52 cards, then the probability of this card being a red coloured queen is

(a) $\dfrac{1}{26}$ (b) $\dfrac{2}{26}$

(c) $\dfrac{3}{26}$ (d) $\dfrac{4}{26}$

72. If $P(X=1)=P(X=2)$ of a Poisson distribution, then its variance is

(a) 0 (b) 1

(c) 2 (d) 3

73. If X be Poisson variate such that $P(X = 1) = 0.3$ and $P(X = 2) = 0.2$, then $P(X = 0)$ is

(a) e^{-4} (b) e^{-3}

(c) e^{-2} (d) $e^{-4/3}$

74. If A and B are two events such that $P(A)=\dfrac{2}{3}, P(A\cup B)=\dfrac{3}{4}$, and $P(A\cap B)=\dfrac{1}{4}$, then $P(\bar{B})$ is

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{4}$

(c) $\dfrac{2}{3}$ (d) $\dfrac{2}{4}$

75. If A and B are two events such that $P(A)=\dfrac{2}{3}, P(A\cup B)=\dfrac{3}{4}$, and $P(A\cap B)=\dfrac{1}{4}$, then $P(B)$ is

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{4}$

(c) $\dfrac{2}{3}$ (d) $\dfrac{2}{4}$

76. If the mean of Poisson distribution is 4 then its variance is

(a) 4 (b) 8

(c) 2 (d) 6

77. If $f(x)=\begin{cases} kx, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$ then k is equal to

(a) 1 (b) 2

(c) 3 (d) 4

78. If $f(x)=\begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$; then $E(X)$ is

(a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$

(c) $\dfrac{1}{4}$ (d) none of these

79. If $f(x)=\begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$; then μ_2^1 is

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{2}$

(c) $\dfrac{1}{4}$ (d) 2/3

80. If $f(x)=\begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$; then σ^2 is

(a) 0 (b) $\dfrac{1}{2}$

(c) $\dfrac{1}{3}$ (d) $\dfrac{1}{18}$

81. If $V(X) = 2$, then Var $(2X + 2)$ is

(a) 2 (b) 4

(c) 6 (d) 8

82. If $E(X) = 5$, then $E(2X + 5)$ is

(a) 5 (b) 10

(c) 15 (d) 20

83. The probability distribution function of exponential distribution with mean 2 is

(a) $\dfrac{1}{2}e^{-\frac{1}{2}}, x>0$ (b) $2e^{-2x}, x>0$

(c) $e^{-\frac{1}{2}x}, x>0$ (d) $e^{-2x}, x>0$

84. If X has exponential distribution with pdf $\begin{matrix} 3e^{-3x}, & x>0, \\ 0, & \text{otherwise} \end{matrix}$ then its mean is

(a) 3 (b) $\dfrac{1}{3}$

(c) $\frac{2}{3}$ (d) 1

85. If X has exponential distribution with pdf $\frac{1}{4}e^{-\frac{1}{4}}, x > 0,$ otherwise then its variance is

(a) 4 (b) 8

(c) 16 (d) $\frac{1}{16}$

86. If $f(x) = x + \frac{2}{c}, x = 1, 2, 3, 4, 5$ then the value of c is

(a) $\frac{1}{7}$ (b) $-\frac{1}{7}$

(c) $\frac{5}{7}$ (d) –5/7

87. If $f(x) = \begin{cases} kx^3, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$, then k is equal to

(a) 1 (b) 2

(c) 3 (d) 4

88. If $\lambda e^{-\lambda x}, x > 0$ is pdf of exponential distribution then $E(X)$ is

(a) λ (b) λ^2

(c) $\frac{1}{\lambda}$ (d) $\frac{1}{\lambda^2}$

89. If $\lambda e^{-\lambda x}, x > 0$ is pdf of exponential distribution then σ^2 is

(a) λ (b) λ^2

(c) $\frac{1}{\lambda}$ (d) $\frac{1}{\lambda^2}$

90. If X is a continuous random variable having uniform distribution on [–5, 5] then its cdf $F(x)$

(a) $\frac{x+5}{10}, -5 \le x \le 5$

(b) $\frac{x-5}{10}, -5 \le x \le 5$

(c) $\frac{x+5}{5}, -5 \le x \le 5$

(d) $\frac{5-x}{10}, -5 \le x \le 5$

91. If $Z \sim N(0, 1)$ then $E(Z)$ is

(a) 0 (b) 1

(c) 2 (d) $\frac{1}{2}$

92. If X be a normal variance with mean 2 and variance 9 then Z is

(a) $\frac{X-2}{9}$ (b) $\frac{X-3}{3}$

(c) $\frac{X-2}{3}$ (d) $\frac{X-3}{2}$

93. If Z be a standard normal variate then its variance is

(a) 0 (b) 1

(c) 2 (d) 3

94. For a standard normal variate $F(-z)$ is

(a) $1 - F(z)$ (b) $F(z)$

(c) $1 - F\left(\frac{1}{z}\right)$ (d) none of these

95. The marks obtained by students were found normally distributed with mean 75 and variance 100. The percentage of students who scored more than 75 marks is

(a) 25% (b) 30%

(c) 40% (d) 50%

96. The standard normal variate is represented by

(a) $N(0, 1)$ (b) $N(0, 0)$

(c) $N(1, 1)$ (d) $N(1, 0)$

97. The points of inflexion of $X \sim N(\mu, \sigma^2)$ are

(a) $\mu \pm 3\sigma$ (b) $\mu \pm 2\sigma$

(c) $\mu \pm \sigma$ (d) $\mu \pm 4\sigma$

98. The area under the whole normal curve is

(a) 1 (b) 2

(c) 3 (d) 4

99. The normal distribution if variance is 25, then its mean deviation is

(a) 4 (b) 5

(c) 20 (d) 25

100. To fit a normal distribution, the parameter(s) which required are

(a) mean only

(b) variance only

(c) mean and variance both

(d) none of these

ANSWERS

1.(c)	2.(c)	3.(a)	4.(c)	5.(a)	6.(b)	7.(a)	8.(b)	9.(d)	10.(c)
11.(b)	12.(c)	13.(a)	14.(b)	15.(d)	16.(a)	17.(c)	18.(a)	19.(d)	20.(b)
21.(d)	22.(b)	23.(a)	24.(c)	25.(a)	26.(c)	27.(c)	28.(a)	29.(c)	30.(a)
31.(d)	32.(d)	33.(b)	34.(c)	35.(a)	36.(d)	37.(b)	38.(d)	39.(d)	40.(d)
41.(a)	42.(a)	43.(d)	44.(a)	45.(a)	46.(a)	47.(b)	48.(a)	49.(a)	50.(d)
51.(b)	52.(c)	53.(a)	54.(a)	55.(a)	56.(a)	57.(d)	58.(b)	59.(c)	60.(a)
61.(a)	62.(a)	63.(a)	64.(a)	65.(a)	66.(a)	67.(d)	68.(a)	69.(c)	70.(b)
71.(a)	72.(c)	73.(d)	74.(c)	75.(a)	76.(a)	77.(a)	78.(b)	79.(b)	80.(d)
81.(d)	82.(c)	83.(a)	84.(b)	85.(c)	86.(d)	87.(d)	88.(c)	89.(d)	90.(a)
91.(a)	92.(c)	93.(b)	94.(a)	95.(d)	96.(a)	97.(c)	98.(a)	99.(a)	100.(c)

7 Sampling and Inference and Testing of Hypothesis

7.1 INTRODUCTION

Before giving the notion of sample, we shall first define 'population'. The group of individuals under study is called 'population or universe'. Thus in statistics, population is an aggregate of objects, animals or inanimate which are under study. The population may be finite or infinite.

It is obvious that for any statistical investigation complete enumeration of the population is rather impractical. For example, if we are interested to know the average per capita (monthly) income of the citizens of India, we will have to enumerate all the earning individuals in the country, which is a very difficult task to do.

If the population is infinite, complete enumeration is impossible. Also, if the units are destroyed in the course of inspection (i.e., inspection of crackers, explosive materials, etc.), 100% inspection, though it is possible but not at all desirable. But even if the population is finite or the inspection is not destructive, 100% inspection is not taken recourse to be cause of multiplication of causes, viz., administrative and financial implications, time factor, etc., and we take the help of 'sampling'.

A finite subset of statistical individuals in a population is called a 'sample' and the number of individuals in a sample is called 'sample size'.

Sampling is quite often used in our day-to-day practical life. For example, in a shop we assess the quality of sugar, wheat or any other commodity by taking a handful of it from the bag and then divide to purchase it or not. A housewife normally tastes the cooked food to find if it is cooked properly or not and contains the required quantities of salt, etc.

7.2 TYPES OF SAMPLING

Some of commonly known and frequently used types of sampling are as follows:

(i) Purposive sampling
(ii) Random sampling
(iii) Stratified sampling, and
(iv) Systematic sampling

We shall focus on random sampling in the upcoming section.

7.2.1 Random Sampling

A random sample is the one in which each unit of population has an equal chance of being included in it. Suppose, we take a sample of size n from a finite population of size N. Then there are N_{C_n} possible samples. A sampling techniques in which each of the ${}^{N}C_n$ samples has an equal chance of being selected is known as 'random sampling' and the sample obtained by this technique is known as random sample.

7.2.2 Simple Sampling

Simple sampling is the random sampling method in which each unit of population has an equal chance, say p, of being included in the sample and this probability is independent of the previous drawings. Thus, a simple sample of size n from a population may be identified with a series of n independent trials with constant probability of success 'p' for each trial.

Remark: It may be noted that random sampling is not necessarily a simple sampling, though its converse is always true. To ensure that sampling is simple, it must be done with replacement, if population is finite. However in case of infinite population, no replacement is necessary.

7.3 PARAMETERS AND STATISTICS

The statistical constants such as mean (μ), variance (σ^2), etc., of a population are called parameters and the statistical constants such as sample mean $(\overline{x})$, sample variance (s^2), etc., calculated based on samples are known as statistics.

7.4 STATISTICAL INFERENCE

As we know that it is difficult to calculate the statistical constants of a population, so to gather maximum information about parameters with least cost, time and efforts, we compute the statistical constants based on samples. Therefore, the objective of sampling studies is to obtain the best possible values of the parameters under specific conditions and such a generalization from sample to population is called 'statistical inference'.

7.5 SAMPLING DISTRIBUTION

If we draw a sample of size n from a finite population of size n, then the total number of possible samples is as follows:

$$^{N}C_n = \frac{N!}{n!(N-n)!} = k(\text{say})$$

For each of these k-samples, we can compute some statistic $t = t(x_1, x_2, \cdots, x_n)$, in particular sample mean $\overline{x}$ and sample variance $s_1{}^2$ etc., as given in the table:

Sample no.		1	2	3	…	k
Statistic	t	t_1	t_2	t_3	…	t_k
	$\overline{x}$	$\overline{x}_1$	$\overline{x}_2$	$\overline{x}_3$	…	$\overline{x}_k$
	s^2	s_1^2	s_2^2	s_3^2	…	s_k^2

A set of these values of the statistic so obtained, one for each sample, constitutes what is called the 'sampling distribution' of the statistic. For the example, the values of t_1, t_2, …, t_k determines the sampling distribution of the statistic t. In other words, statistic t may be regarded as a random variable which can take the values t_1, t_2, …, t_k and we can compute various statistical constants like mean, variance, skewness, kurtosis, etc., for its distribution. Therefore, the mean and variance of sampling distribution of statistic t are given by

$$\bar{t} = \frac{1}{t}(t_1 + t_2 + \cdots + t_k) = \frac{1}{k}\sum_{i=1}^{k} t_i$$

and $$\text{var}(t) = \frac{1}{k}\sum_{i}(t_1 - \bar{t})^2$$

7.6 STANDARD ERROR

The standard deviation of the sampling distribution of a statistic is called the 'standard error' (S.E.). For example, S.E. of the sampling distribution of means is called S.E. of means. Standard error is used to know the discrepancy between the observed and expected value of a statistic. The reciprocal of S.E. is known as precision.

If $n \geq 30$, the sample is called large, otherwise small. For large values of n, the sampling distribution of a statistic is normal.

i.e., $$\frac{t - E(t)}{\sqrt{V(t)}} = Z \sim N(0, 1)$$

Remark: S.E. of a statistic may be reduced by increasing the sample size, but this results in corresponding increase in cost, labour and time, etc.

7.7 TESTING A HYPOTHESIS

Based on sample information, to reach decisions about populations, certain assumptions are made about the population, such assumptions which may or may not be true, are called statistical hypothesis. With the help of testing of hypothesis we accept or reject that hypothesis. We make a hypothesis about the population parameter which is taken is correct and by a method we calculate the probability of getting the observed value of statistic based on sample. If this probability is less than some reassigned value, the hypothesis is rejected otherwise accepted.

7.8 ERRORS

When we test a hypothesis based on a sample, then either we reject or accept. But if a hypothesis is rejected when it should have been accepted, then an error has been committed by us, known a Type I error. On the other hand, if a hypothesis is accepted while it should have been rejected, we say that another error has been made and this error is known as Type-II error. The aim of statistical testing hypothesis is limiting the Type-I error to a preassigned value (say: 1% or 5%) and to minimize the Type-II error. Both types of errors can be reduced by increasing the sample size.

7.9 NULL AND ALTERNATE HYPOTHESIS

A statement given for the parameter of the population to reject under the assumption that it is true, is called null hypothesis and denoted by H_0. The hypothesis which is to be established based on the sample is known as alternative hypothesis and denoted by H_1. To test whether one procedure is better than another, then assumption is made that there is no difference between the procedures. Similarly, suppose we are interested to find relationship between two variables, the assumption is made that they are independent. By accepting the hypothesis, we mean that the hypothesis is accepted based on this sample, while it may be wrong. Similarly, rejection of a hypothesis does not mean that it is wrong.

7.10 LEVEL OF SIGNIFICANCE

The level of significance α is the probability level below which we reject a hypothesis, and the region in which the value of the statistic, calculated based on the sample falls and rejected in known as 'critical region'. Generally, two critical regions which cover 5% and 1% areas of the normal curves are taken.

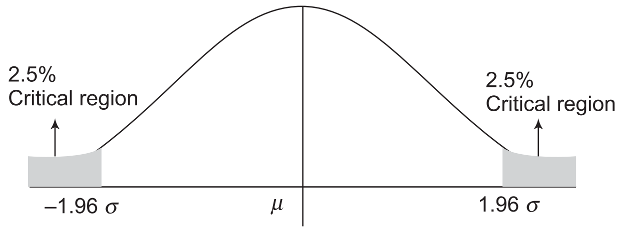


Fig. 7.1

The shaded portion in the figure corresponds to 5% level of significance.

Thus the 'level of significance' is the probability of the value of the variable falling in the critical region.

One-tailed or Two-tailed test: In one tailed test only the area on the right of an ordinate (known as right tailed test) or area on the left of an ordinate (known as left tailed test) is taken into consideration, and it depends on the nature of the problem. For two-tailed test, we consider the areas of both the tests. For example, to test whether a coin is biased or not, two tailed test should be used, because a biased coin can give more number of tails than heads.

7.11 TESTS OF SIGNIFICANCE

The procedure which tells us whether we accept or reject a hypothesis is called test of significance. In test of significance, we list whether there is any difference between the sample values and the population values or the difference between the values of two samples, and they are so large that they provide significant evidences against the hypothesis or these differences are so small by which it is not possible to know the fluctuations of sampling.

7.12 CONFIDENCE LIMITS

J Neyman developed the modern theory and terminology of confidence limits. According to him, let a statistic t having sampling distribution which is normal in nature, has mean μ and variance σ^2, then 95% times the statistic t lies between ($\mu - 1.96\ \sigma$ and $\mu + 1.96\ \sigma$) and that is why ($t - 1.96\sigma, t + 1.96\sigma$) is known as 95% confidence interval for estimating the value of μ. The end points of this interval ($t \pm 1.96\ \sigma$) are called 95%. Confidence limits or fiducial limits for t. Similarly, $t \pm 2.58$ are called 99% confidence limits. The number 1.96, 2.58, etc., are called confidence coefficients. The values of confidence coefficients for the different values of level of significance can be found from the normal table.

7.13 SIMPLE SAMPLING OF ATTRIBUTES

Let a population be defined by a particular characteristic whether it has or not that characteristic, i.e., we can possess the particular attribute or not. If we draw a simple sample of n items from this population then each item will be independent having the same probability of success (say p). Let X be the number of successes, then X will have a binomial distribution with mean np and variance npq. Then

(i) Mean proportion of successes = $\dfrac{np}{n} = p$

(ii) Variance of proportion of success = $\dfrac{npq}{n^2} = \dfrac{pq}{n}$ and standard error of the proportion of success = $\sqrt{pq/n}$ and

(iii) Precision of proportion of success = $\sqrt{\dfrac{n}{pq}}$, as p and q are constants the precision of proportion of success varies when n varies, and becomes more and more, when increase the value of n.

7.14 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

All of us are aware that for large values of n, a binomial distribution tends to normal distribution. Suppose we want to test that in each trial the probability of success is p, we assume that it is true, then the sampling distribution (S.D.) of a number of successes it will have a mean $\mu = np$ and variance $\sigma^2 = npq$, i.e., S.D. $\sigma = \sqrt{npq}$.

So for a normal distribution, 99% members lie within $\mu \pm 2.58\ \sigma$ and 95% lie with $\mu \pm 1.96\ \sigma$. Let x be the number of successes in the sample, then $\dfrac{x-\mu}{\sigma} = z$ will have a normal distribution with mean = 0 and variance = 1 i.e., it is a standard normal variate.

Therefore, we have the tests of significance which is as follows:

(a) If $|z| < 1.96$, the difference between the observed and expected number of successes is not significant at 5%.

(b) If $|z| > 1.96$, the difference between the observed and expected number of successes is significant at 5% level of significance.

(c) If $|z| < 2.58$, the difference between observed and expected number of successes is not significant at 1% level of significance, and

(d) If $|z| > 2.58$, the difference between observed and expected number of successes is significant at 1% of level of significance.

In general, we can say that ($|z| > z_1$) or ($|z| < z_1$) where z_1 can be found from normal distribution table for a given level of significance.

Example 1 A dice is thrown 9,000 times and a throw of 3 or 4 is observed 3,240 times. Show that the dice cannot be regarded as unbiased at 5% level of significance.

Solution Let the dice be unbiased.

Then the probability of getting 3 or 4 = 2/6 = 1/3 = p

Let x = number of getting a 3 or 4 = 3,240; n = 9,000

then the mean can be calculated as following:

$$np = 9000 \times \frac{1}{3} = 3000$$

Variance $npq = 2000$

$$\therefore \qquad z = \frac{x-np}{\sqrt{npq}} = \frac{3240-3000}{\sqrt{2000}} = 5.36$$

$\therefore \quad |z| = 5.36 > 1.96$, null hypothesis is rejected

$\Rightarrow$ Therefore dice is biased.

Example 2 A random sample of 500 apples was taken from a large assignment and 60 were found to be bad. Obtain the 98% confidence limits for the percentage of bad apples in the consignment.

Solution Let p be the proportion of bad apples in the consignment

i.e., $$p = \frac{60}{500} = 0.12 \text{ and } q = 1 - 0.12 = 0.88$$

98% confidence limits for p are as follows.

$$0.12 \pm 2.33\sqrt{\frac{0.12 \times 0.88}{500}} = 0.12 \pm (2.33)\,(0.01453)$$

$$= (0.08615, 0.15385)$$

Hence, 98% confidence limits for percentage of bad apples in consignment are (8.61, 15.39).

Example 3 A coin was tossed 1,600 times and the tailed turned up 864 times. Test the hypothesis that the coin is unbiased at 1% level of significance.

Solution Let the coin be unbiased.

and x be the number of tails turned up = 864

Let p be the probability of coming a tail = ½

$\therefore$ Expected number of successes = $\frac{1}{2} \times 1600 = 800$

and the observed values of successes = 864

Variance = $npq = 1600 \times \frac{1}{2} \times \frac{1}{2} = 400$

S.D. $\sqrt{npq} = \sqrt{400} = 20$

$$\therefore \quad z = \frac{x - np}{\sqrt{npq}} = \frac{864 - 800}{20} = \frac{64}{20} = 3.2$$

as $z > 2.58$, the hypothesis is rejected at 1% level of significance i.e., we conclude that the coin is biased.

Example 4 A random sample of 500 pineapples was taken from a large consignment and 65 were found to be bad. Show that standard error of the proportion of bad ones in a sample of this size is 0.015. Also, find 99% confidence limits for proportion of bad pineapples.

Solution Here, $n = 500$

Let X be the number of bad pineapples in the sample = 65

and p be proportion of bad pineapples = $\frac{65}{500} = 0.13 = p$

$$\therefore \quad q = 1 - 0.13 = 0.87$$

Therefore standard error of bad proportion = $\sqrt{pq/n}$

$$= \sqrt{\frac{0.13 \times 0.89}{500}} = 0.015$$

99% confidence limits for proportion of bad pineapples.

$$= 0.13 \pm 2.58\ (0.015)$$
$$= 0.13 \pm 0.0387 = (0.0913, 0.1687)$$

7.15 COMPARISON OF LARGE SAMPLES

Let p_1 and p_2 be the proportions of characteristic $A^{'s}$ of two large samples of sizes n_1 and n_2 taken from two populations.

(a) We want to test that both the population are same with respect to characteristic A. Let p be the proportion of both the combined samples then

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

Let E_1 and E_2 be the standard errors in the two samples, then

$$E_1^2 = \frac{pq}{n_1} \text{ and } E_2^2 = \frac{pq}{n_2}$$

If E be the standard error of the difference between p_1 and p_2, this

$$E^2 = E_1^2 + E_2^2 = \frac{pq}{n_1} + \frac{pq}{n_2} = \left(\frac{1}{n_1} + \frac{1}{n_2}\right)(pq)$$

$$\therefore \qquad z = \frac{p_1 - p_2}{E}$$

$$\Rightarrow \qquad z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n}\right)}} \sim N(0, 1)$$

(i) If $z < 2$, the difference between p_1 and p_2 may be due to the fluctuation of simple sampling.

(ii) If $z > 3$, then p_1 and p_2 are different

(iii) If $2 < z < 3$, then difference between p_1 and p_2 is significant at 5% level of significance.

(b) To test that both the population are not same with respect to characteristic A, then standard error (E) of $p_1 - p_2$ is given by

$$E = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$$

and

$$z = \frac{p_1 - p_2}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} \sim N(0, 1)$$

If $z < 3$, the difference between p_1 and p_2 is due to fluctuations of simple sampling.

Example 5 Random samples 400 men and 600 women were taken whether they would like to have a flyover near their residence. Around 200 men and 325 women were in favour of the proposal. Is the proportions of men and women are really in favour of the proposal.

Solution Let p_1 and p_2 be the proportions of men and women in favour of flyover near their residence,

then $p_1 = \dfrac{200}{400} = 0.500,\ p_2 = \dfrac{325}{600} = 0.541$

$$z = \frac{p_1 - p_2}{\sqrt{\dfrac{p_1q_1}{n_1} + \dfrac{p_2q_2}{n_2}}} = \frac{0.500 - 0.541}{\sqrt{\dfrac{0.50 \times 0.50}{400} + \dfrac{541(0.459)}{600}}} = -\frac{0.041}{0.0323} = -1.269$$

$\because$ $z < 3$, hence, the proportion of men and women are really in favour of the proposal.

Example 6 A cigarette manufacturing firm claims that its brand A of the cigarettes out sells its brand B. If it is found that 42 out of a sample of 200 smokers prefer brand A and 18 out of another random sample of 100 smokers prefer brand B. Test whether the difference is a valid claim. (Use 5% level of significance).

Solution $n_1 = 200,\ n_2 = 100$

Let x_1 be the number of smokers who use brand $A = 42$
and x_2 be the number of smokers who use brand $B = 18$

p_1 = proportion of smoker who use brand $A = \dfrac{42}{200} = 0.21$

p_2 = proportion of smoker who use brand $B = \dfrac{18}{100} = 0.18$

$$z = \frac{p_1 - p_2}{\sqrt{E}} = \frac{0.21 - 0.18}{\sqrt{\dfrac{p_1q_1}{n_1} + \dfrac{p_2q_2}{n_2}}} = \frac{0.03}{0.048} = 0.625$$

$$z = 0.625$$

The value z at 5% level is 1.96 from the liable

$\therefore$ $z = 0.625 < 1.96$

Hence, the difference between the two brands is a valid claim.

Example 7 In two large populations, there are 40% and 30%, respectively, are blue-eyed people. Is this difference likely to be hidden in sample of 1,000 and 800, respectively, from the two populations?

Solution Here, $n_1 = 1000,\ n_2 = 800$

Let p_1 = proportion of blue-eyed people in the first population = 40%
$= 0.40$

and p_2 = proportion of blue-eyed people in the second population = 0.30 = 3.0%

$\therefore$ $q_1 = 1 - p_1 = 0.60,\ q_2 = 1 - p_2 = 0.70$

$$z = \frac{p_1 - p_2}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} = \frac{0.40 - 0.30}{0.022} = \frac{0.10}{0.022} = 4.55$$

Hence, it is likely that the real difference is hidden.

EXERCISE 7.1

1. A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased at 5% level of significance. **(V.T.U. 2007)**
2. A dice was thrown 9,000 times and a throw of 5 of 6 was obtained 3,240 times. On the assumption of random throwing, does the data indicate an unbiased dice? **(V.T.U. 2010)**
3. A dice is tossed 960 times and if falls 5 upwards 184 times. Is it dice biased? **(V.T.U. 2006)**
4. In a sample of 1,000 people in Maharashtra; 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% level of significance?
5. In two large populations, there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1,200 and 900, respectively, from the two populations? **(Coimbatore 2001)**
6. An experience shows that 20% of a manufactured product is of top quality. In one day's production of 400 articles, only 50 are of top quality. Show that either the production of the day taken was not a representative sample or the hypothesis of 20% was wrong.
7. By a mobile court checking in certain buses it was found that out of 1,000 people checked on a certain day in a city, 10 persons were found to be ticket-less travellers. If daily 1 lakh passengers travel by the buses, find out the estimated limits to the ticket-less travellers.
8. In a random sample of 100 men taken from a village A, 60 were found to be consuming alcohol. In another sample of 200 men taken from village B, 100 were found to be consuming alcohol. Do the two villages differ significantly in respect of the proportion of men who consume alcohol?
9. In a referendum submitted to the students body at a university 850 men and 566 women voted. 530 of the men and 304 of the women voted yes. Does this indicate a significant difference of opinion on the matter at 1% level of significance, between men and women students?
10. In a city A, 20% of a random sample of 900 school boys had a certain slight physical defect. In another city B, 18.5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant? **(V.T.U. 2003S)**
11. A machine produces 16 imperfect articles in a sample of 500. After machine overhauled, it produces 3 imperfect articles in a batch of 100. Has the machine been improved? **(Rohtak 2005, Madras 2003)**
12. In a sample of 600 men from a certain city, 450 are found smokers. In another sample of 900 men from another city 450 are smokers. Do the data indicate that the cities are significantly different with respect to the habit of smoking among men? **(J.N.T.U. 2003)**
13. In a locality containing 10,000 families, a sample of 600 families were selected at random. Of these 600 families, 150 families were found to have a monthly income of \$500 or less. It is desired to estimate how many out of 10,000 families have a monthly income of \$500 or less. Within what limits would you place your estimate?

14. In a large city A, 20% of a random sample of 900 school children had defective eye-sight. In the other city B, 15% of random sample of 1,600 children had the same defect. Is this difference between the two proportions significant? Obtain 15% confidence limits for the difference in the population proportions.

Answers

1. $z = 1.6 < 1.96$ i.e., coin is unbiased.
2. $z = 5.4 > 2.58$ i.e., dice is biased.
3. Dice is biased
4. $z = 2.532 < 2.58 \Rightarrow$ wheat and rice an equally popular in the state.
5. $z \cong 2.5 \Rightarrow$ It is unlikely that real difference is hidden.
6. $z = 3.75$
7. 997 to 1003.
8. $z = 1.85$
9. $z = 3.2$, significant
10. $z = 0.37 < 1$, the difference between the proportions is not significant.
11. No
12. No
13. 19.70% and 30.30% app.
14. $$z = \frac{p_1 - p_2}{\sqrt{E}} = \frac{20 - 0.15}{\sqrt{\frac{900\times 0.20 + 1600 \times 0.15}{900+1600}\left[\frac{1 - 900\times 0.20 + 1600\times 0.15}{900+1600}\right]\left[\frac{1}{900}+\frac{1}{1600}\right]}}$$

    $z = 3.21 > 1.96 \Rightarrow$ It is significant at 5% level.

    95% confidence limits are

    $(p_1 - p_2) \pm 1.96$ E = (0.019 and 0.081)

7.16 SAMPLING OF VARIABLES

In this section, we will discuss in detail the sampling of variables such as age, income, height, weight, etc. In case of sampling of variables, each member of the population provides the value of the variable and aggregate of these values form the frequency distribution of the population . From the population, a random sample of size n is drawn which will be equal to the n values of the variables from those of the distribution.

7.17 SAMPLING DISTRIBUTION OF THE MEAN

Let X be a normal population with mean μ and variance σ^2, suppose a random sample of $x_1, x_2, \ldots, x_n$ is taken from X, then each X_i ($i = 1, 2, \ldots, n$) will be independent and will have a normal distribution with mean μ and variance σ^2. The sample mean $\overline{x}$ and variance of $\overline{x}$ will also have normal distribution with mean μ and variance σ^2/n i.e.,

If $X \sim N(\mu, \sigma^2)$, then

$$X_i \sim N(\mu, \sigma^2),\ i = 1, 2, \ldots, n \text{ and independent, then } E(\overline{X}) = \mu$$

and $\operatorname{var}(\overline{X}) = \sigma^2/n$

i.e., $\overline{X} \sim N(\mu, \sigma^2/n)$

Standard error of $\overline{X} = \sigma/\sqrt{n}$ and precision $= \dfrac{\sqrt{n}}{\sigma}$, if n increases then precision of a sample mean also increases.

7.18 CENTRAL LIMIT THEOREM

Central limit theorem plays an important role in the distribution of the mean of a sample, if the population from which sample is taken is not normal and size of the sample n is large.

Let X has a distribution, which is not normal with a mean μ and variance σ^2 then for large values of n.

$\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} = z$ as $n \to \infty$ has a standard normal distribution with mean = 0 and variance = 1. The result hold good if $n \geq 25$.

So we can say, if $X \sim N(\mu, \sigma^2)$ then $\overline{X} \sim N(\mu, \sigma^2/n)$ but some results holds good for $n > 25$ even X is not normal. Therefore, normal distribution has a wide applications in the theory of statistics.

7.19 CONFIDENCE LIMITS FOR UNKNOWN MEAN

Let a random variable X which represents a population having a mean μ and variance σ^2, μ is unknown. Let a random sample $x_1, x_2, \ldots, x_n$ of size n is drawn from X. We are interested to find a range of values of μ for which the observed value of sample mean $\bar{x}$ of the sample is significant at any assigned level of probability.

The relative deviation of sample mean $\bar{x}$ from population mean μ is

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}},\ \text{if } \bar{x} \text{ is not significant at 5\% level of significance, then}$$

$$\left|(\bar{x}-\mu)\right| \sigma \left|\sqrt{n}\right| < 1.96$$

$$\Rightarrow \qquad \bar{x} - 1.96\,\sigma \left|\sqrt{n}\,\right| < \mu < \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$$

Therefore, the 95% confidence or fiducial limits for the mean of population mean μ to the given n sample are $\bar{x} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$. Similarly, 99% confidence limits for μ are $\bar{x} \pm 2.58\dfrac{\sigma}{\sqrt{n}}$ or in general $\bar{x} \pm Z\alpha/2\dfrac{\sigma}{\sqrt{n}}$ (where α is the level of significance).

Example 8 A sample of 625 members has a mean of 3.50 cm. Can it be reasonably regarded as a truly random sample from a large population with mean = 3.20 and variance = 2.25 at 5% level of significance.

Solution Here $n = 625$, $\bar{x} = 3.50$, $\mu = 3.20$, $\sigma^2 = 2.25$

$$\therefore \qquad z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{3.50-3.20}{\sqrt{2.25/625}} = \frac{0.30}{0.06} = 5.0$$

$z = 5.0 > 1.96$, the deviation of the sample mean from the population mean is significant at 5% level of significance. Hence, it cannot be a regarded as a random sample.

Example 9

(a) A sample of 900 members has a mean = 3.4 cm and standard deviation = 2.61 cm. Is the sample from a large population of mean = 3.25 cm and S.D. = 2.61 cm? (use $\alpha = 0.05$).

(b) If the population is normal and its mean is unknown, find the 95% and 98%. Confidence limits for true mean μ.

Solution Here $n = 900$, $\mu = 3.25$, $\sigma = 2.61$, $\bar{x} = 3.40$

then $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) = \dfrac{3.40 - 3.25}{2.61/\sqrt{900}} = 1.73$

$\therefore$ $|z| = 1.73 < 1.96$, the deviation of sample mean $\bar{x}$ from the population mean is not significant at 5%. Hence, it can be regarded as a random sample.

(b) μ is unknown, then

(i) 95% confidence limits for μ are as follows:

$$\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}} = 3.40 \pm 1.96\frac{(2.61)}{\sqrt{900}}$$

$$= 3.40 \pm 0.1705$$

$$= (3.5705 \text{ and } 3.2295)$$

(ii) 98% confidence limits for μ are

$$\bar{x} \pm 2.33\frac{\sigma}{\sqrt{n}} = 3.40 \pm 2.33\left(\frac{2.61}{\sqrt{900}}\right)$$

$$= 3.40 \pm 0.2027$$
$$= (3.6027 \text{ and } 3.1973)$$

Example 10 As an application of central limit theorem, show that if E is such that $P(|\bar{x} - \mu| < E) > 0.95$, then the minimum sample size n is given by $n = \dfrac{(1.96)^2\sigma^2}{E^2}$ where μ and σ^2 are the mean and variance, respectively, of the population and $\bar{x}$ is the mean of the random sample.

Solution By central limit theorem, we know that

$$\bar{x} \sim N(\mu_1 \sigma^2/n), \text{ for large } n$$

$$\therefore \quad z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1), \text{ for large } n$$

$$\Rightarrow \quad P\left[\left|\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right| \le 1.96\right] = 0.95$$

$$\text{or} \quad P\left[|\bar{x} - \mu| \le 1.96\frac{\sigma}{\sqrt{n}}\right] = 0.95 \tag{1}$$

We are given that $P[|\bar{x} - \mu| < E] > 0.95$ (2)

From Eqs (1) and (2),

$$E > 1.96\frac{\sigma}{\sqrt{n}} \Rightarrow n \geq \frac{(1.96)^2 \sigma^2}{E^2} = \frac{3.84\sigma^2}{E^2}$$

Hence, minimum sample size for estimating μ with 95%. Confidence coefficient is given by $\frac{3.84\,\sigma^2}{E^2}$, where E = permissible error.

Example 11 The mean muscular endurance score of a random sample of 60 subjects was found to 145 with a variance 1600. Construct a 95% confidence interval for the true mean. Assume the sample size to be large enough for normal approximation. What size of sample is required to estimate the mean with 5 the true mean with a 95% confidence?

Solution Here $n = 60$, $\bar{x} = 145, \sigma^2 = 1600, \sigma = 40$

95% confidence limits for μ are

$$\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}} = 145 \pm 1.96\frac{40}{\sqrt{60}} = 145 \pm 10.12$$

$$= 134.88 \text{ and } 155.12$$

$\therefore$ 95% confidence interval for μ is (134.88, 155.12) to find n

$$n = \frac{(1.96 \times 4)^2}{5} = 245.86 \simeq 246$$

minimum sample size = 246

7.20 TEST OF SIGNIFICANCE FOR DIFFERENCE OF MEANS

Let $\bar{x}_1$ be the sample mean by sample of size n_1 from a population with mean μ_1 and variance σ_1^2 and let $\bar{x}_2$ be the mean of an independent random sample of size n_2 from another population with mean μ_2 and variance σ_2^2. Then

$$\bar{x}_1 \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right) \text{ and } \bar{x}_2 \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$$

$\because$ sample sizes are large.

Also $\bar{x}_1 - \bar{x}_2$, being the difference of two independent normal variates is also a normal variate with mean $\mu_1 - \mu_2$ and variance $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$.

$$\therefore \qquad z = \frac{(\bar{x}_1 - \bar{x}_2)(\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1) \; (\because \;\; \mu_1 - \mu_2 = 0)$$

because we want to test the difference between two sample mean $\bar{x}_1$ and $\bar{x}_2$ is significance.

$\therefore$ $\mu_1 - \mu_2 = 0$ and also $\text{cov}(X_1, X_2) = 0$ as both the samples are independent of each other.

Therefore, for large values of n_1 and n_2

Test of significance is as follows:

If $z > 1.96$, then the difference between $\bar{x}_1$ and $\bar{x}_2$ is significant at 5% level of significance.

In general if

$|z| > z_{\alpha/2}$, the difference between $\bar{x}_1$ and $\bar{x}_2$ is significant at α level of significance, $z_{\alpha/2}$ can be calculated from normal table.

Remark: If $\sigma_1^2 = \sigma_2^2$ i.e., if both the samples are drawn from the same population then

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0, 1)$$

Example 12 The means of simple samples of sizes 400 and 400 are 250 and 220 respectively. Can the sample be drawn from the two populations having s.d. 40 and 55 respectively having the same population means at 5% level of significance.

Solution

$$z = \frac{\bar{x}_1 \cdot \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{250 - 220}{\sqrt{\frac{40^2}{400} + \frac{55^2}{400}}} \simeq 8.82$$

$\therefore$

$|z| = 8.82 > 1.96$. Therefore, the mean of two populations are not same.

Example 13 The means of sample sizes 400 and 1600 are 70.0 and 65.0 cm, respectively. Can the sample be regarded as drawn from the same population of variance 4 at 1% level of significance.

Solution Here, we have $n_1 = 400$, $n_2 = 1600$

$$\bar{x}_1 = 70.0, \bar{x}_2 = 65.0, \sigma = \sqrt{4} = 2$$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{70 - 65}{2\sqrt{\frac{1}{400} + \frac{1}{1600}}} = 44.72$$

$\therefore$

$|z| = 44.72 > 2.58$. There significant difference between the means of two samples. Thus the sample can not be regarded as drawn from the same population.

EXERCISE 7.2

1. The means of simple samples of sizes 1000 and 2000 are 67.5 and 68.0 cm, respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 cm. (use $\alpha = 0.05$) **(Madras 2002)**
2. An insurance agent has claimed that the average age of policy holders who insure through him is not equal to the average for all agents, which is 30.5 years.

Age last birthday:	16–20	21–25	26–30	31–35	36–40
Number of persons:	12	22	20	30	16

Calculate the mean and standard deviation of this distribution and use these values to test his claim at the 5% level of significance.

3. The standard deviation of a population is 2.70 cm. Find the probability that in a random sample of size 66, the sample mean will differ from the population mean by 0.75 or more.
4. A normal population has a mean of 0.1 and standard deviation 2.1. Show that the probability that mean of a sample of size 900 will be negative is 0.0764.
5. The guaranteed average life of a certain type of electric light bulb is 1000 hours with a standard deviation of 125 hours. It is decided to sample the output so as to ensure that 90 percent of the bulbs do not fall short of the guaranteed average of more than 2.5 percent. What must be minimum size of the sample?
6. The average hourly wage of sample of 150 workers in a plant *A* was Rs. 2.56 with a standard deviation of Rs. 1.08. The average hourly wage of a sample of 200 workers in plant *B* was Rs. 2.87 with a standard deviation of Rs. 1.28. Can an applicant safely assume that the hourly wages paid by both the plants do not differ at 5% level of significance.
7. The mean height of 50 male students who showed above average participation in college athletics was 68.2 inches with a standard deviation of 2.5 inches, while 50 male students who showed no interest in such participation had a mean height of 67.5 inches with a standard deviation of 2.8 inches. Can the samples be regarded as drawn of from the same population? (use $\alpha = 0.05$)
8. In a random sample of 500, the mean is found to be 20. In another independent sample of 400 the mean is 15. Could the samples have been drawn from the same population with standard deviation 4 at 5% level of significance.
9. A sample of heights of 6,400 Englishmen has a mean of 67.85 inches and S.D. 2.56 inches, while a sample of heights of 16,00 Australian has a mean of 68.55 inches and S.D. of 2.52 inches. Do the data indicate that Australians, on the average are different to the Englishmen. (use $\alpha = 0.05$)
10. From the population of 169 units it is desired to choose a simple random sample of size n. If the population standard deviation is 2, determine the smallest n for which the probability that the sample mean differs from the population mean by more than 0.95 controlled at 0.05.
11. An economist would like to estimate the mean income μ in a large city. He has decided to use the sample mean as an estimate of μ and would like to ensure that the error in estimation is not more than 100 with probability 0.95. How large a sample should be taken if the standard deviation is known to be 1000?

Answers

1. $z = 5.1 > 1.96$. Samples can not be regarded as drawn from the same population.
2. $z = -2.681 \Rightarrow |z| = 2.681 > 1.96$. Significant at 5% level of significance.
3. 0.0246.
5. Minimum $n = 41$.
6. $|z| = 2.46 > 1.96$, significant at 5% level.
7. $z = 1.32 < 1.96$, not significant at 5% level.
8. $|z| = 18.63 > 1.96$, significant at 5% level.
9. $|z| = 9.2$, significant at 5% level.
10. $n = 6147$.
11. $n = 271$

7.21 SAMPLING OF VARIABLES – SMALL SAMPLES

We have seen that if a sample of large size is taken from any population, then using central limit theorem, the sampling distribution of the statistic computed from the sample always approaches normal distribution and the values of statistic are always considered the best estimates of the unknown parameters of the population from which the sample is drawn. But it is always not possible, suppose a sample of small size is drawn, then instead of using central limit theorem in which statistic follows approximately a normal distribution a new technique is available, which involves the concept of degree of freedom, which is explain in the upcoming section.

7.21.1 Number of Degrees of Freedom

Let $x_1, x_2, \ldots, x_n$ be a sample of size n then the degrees of freedom (d.f.) is $(n-1)$. In particular let $x_1 + x_2 + x_3 + x_4 = 20$, we can assign any arbitrary values to $x_1 + x_2 + x_3$ and estimate the value of x_4. Therefore, in this case $x_1 + x_2 + x_3$ are free and independent choices to find the value of x_4.

Hence, these are degrees of freedom, and here *d.f.* is 3. So, we can say the number of degrees of freedom is nothing but is the number of values in a sample which are assigned arbitrary. Let $x_1, x_2, \ldots x_n$ be random sample of size n from a population, then to calculate sample mean $(\overline{x})$, one degree of freedom is used therefore to estimate the population variance based on this sample $(n-1)$ degrees of freedom are left.

7.22 STUDENT'S *t*-DISTRIBUTION

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of small size n from a normal population μ and variance σ^2. If $\overline{x}$ and s^2 be the sample mean and sample variance, respectively, then the statistic 't'. which is defined as following:

$$t = \frac{\overline{x} - \mu}{s/\sqrt{n}}, \text{ where } \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

has t-distribution with $(n-1)$ *d.f.* and the distribution of t is given by

$$f(t) = \frac{A}{\left(1+\frac{t^2}{\nu}\right)^{\nu+\frac{1}{2}}}; -\infty < t < \infty$$

Where $\nu = n-1$ and A is a constant such that the area under the curve is unity.

7.22.1 Properties of *t*-Distribution

1. Since $f(-t) = f(t)$, the probability curve is symmetrical about the line $t = 0$. As t increases, $f(t)$ decreases rapidly and tends to zero as $t \to \infty$, so that t-axis is an asymptote to the curve.

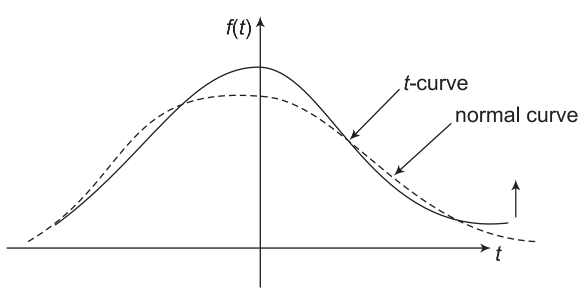


Fig. 7.2

The maximum value of t-curve also at $t = 0$, hence, the mode and mean of t-distribution are same.

2. When $n \to \infty$, then $f(t) = \dfrac{A \cdot e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}}, -\infty < + < \infty$, i.e., for large values of n, t-distribution tends to standard normal distribution. Therefore for large values of n instead of using t-table, we can use normal table.

3. $p[T > t_\alpha] = \int_\alpha^\infty f(t)\,dt$

   The values of t_α have been calculated for various values of p and different values of $\nu = n - 1$ (d.f.) from 1 to 30.

4. Since $f(t)$ is symmetrical about $t = 0$, then all the moments of odd order about origin/and mean are zero, i.e.,

$$\mu'_{2k+1} = 0, k = 0, 1, 2, ... = \mu_{2k+1}$$

i.e. μ'_1 = mean = 0

The moments of even order are given by

$$\mu_{2k} = \frac{nk(2k-1)(2k-3)\cdots 3.1}{(n-2)(n-4)\cdots(n-2k)}, \frac{n}{2} > k$$

In particular

$$\mu_2 = n \cdot \frac{1}{n-2} = \frac{n}{n-2}, (n > 2)$$

$$\mu_4 = \frac{n^2 \cdot 3.1}{(n-2)(n-4)} = \frac{3n^2}{(n-2)(n-4)}, n > 4$$

$$\therefore \quad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} = 3\frac{(n-2)}{(n-4)}, n > 4$$

Remark:

1. As $n \to \infty$, $\beta_1 = 0$ and $\beta_2 = 3$
2. For $n > 2$, $\mu_2 > 1$ i.e., the variance of t-distribution is greater than the variance of standard normal distribution.
3. For $n > 4$, $\beta_2 = 3$, therefore, t-distribution is more flat on the top than normal curve.
4. $p\left[|t| > t_{\nu,\alpha}\right] = \alpha$

   $\Rightarrow \quad p\left[|t| \le t_{\nu,\alpha}\right] = 1 - \alpha$
5. The t-distribution is generally used to test the hypothesis about the mean when the variance of the population is unknown.
6. Population from which sample is drawn is always taken normal.

7.22.2 Significance Test of a Sample Mean

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a normal population with a specified population mean μ_0. To test the hypothesis $\mu = \mu_0$ or there is no significance difference between sample mean $\overline{x}$ and population mean μ_0,

(i) Compute the statistic

$$t = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}, \text{ where } s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 .$$

(ii) Find the value of P for the given degrees of freedom from the table.

(a) if $t_{cal} > t_{.05}$, then the difference between sample mean $\overline{x}$ and specified population mean μ_0 is significant at 5% level of significance; otherwise not significant.

(b) if $t_{cal} > t_{.01}$, then the difference between sample mean $\overline{x}$ and specified population mean μ_0 is significant at 1% level of significance otherwise not significant.

(c) In general if $t_{cal} > t_{\alpha}$, then the difference between sample mean $\overline{x}$ and population mean μ_0 is significant at level of significance α.

Example 14 The mean weekly sales of soap bars in departmental stores was 146.3 bars per store. After an advertising campaign the mean weekly sales in 22 stores for a typical week increased to 153.7 and showed a standard deviation of 17.2. Was the advertising campaign successful at 5% level of significance.

Solution Here, $n = 22, \overline{x} = 153.7, s = 17.2, \mu_0 = 146.3$

$$t = \frac{\overline{x} - \mu_0}{s/\sqrt{v}}, \text{ where } v = n - 1 = 21$$

$$= \frac{153.7 - 146.3}{17.2/\sqrt{21}} = 9.03$$

t_{tab} at $v = 21 = 2.08$

$t_{cal} = 9.03 > 2.08 \Rightarrow$ It is highly significant. We conclude that the advertising campaign was highly successful.

Example 15 The heights of 10 males of a given locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64 and 66 inches. Is it reasonable to believe that the average height is greater than 64 inches?

Test at 5% level of significance assuming that for 9 degrees of freedom $P(t > 1.83) = 0.05$.

Solution

$$n = 10, \overline{x} = \frac{\sum x}{n} = \frac{660}{10} = 66, \mu_0 = 64$$

$$s^2 = \frac{1}{n-1}\sum(x_i - \overline{x})^2 = \frac{90}{9} = 10$$

$$\therefore \quad t = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} = \frac{66 - 64}{\sqrt{10/10}} = 2$$

For 9 d.f. t_{tab}, .05 = 2.26

$\therefore \quad t_{cal} < 2.26$. Average height is not greater than 6 inches.

Example 16 A random sample of 10 boys had the following IQs: 70, 120, 110, 101, 88, 83, 95, 98, 107 and 100. Do these data support the assumption of a population mean IQ of 100? Find a reasonable range in which most of the means IQ values of samples of 60 boys lie.

Solution Here $n = 10$, $\mu_0 = 100$

$$\bar{x} = \sum \frac{x_i}{n} = 97.2$$

$$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2 = 203.73$$

$$\therefore \quad t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{97.2 - 100}{\sqrt{203.73/10}} = 0.62$$

$|t| = 0.62$

t_{tab} at 5% = 2.262 > 0.62 $\Rightarrow$ the data is consistent with the assumption of mean IQ of 100 in the population.

95% confidence limits are as follows:

$$\bar{x} \pm t_{.05}\, s/\sqrt{n} = 97.2 \pm 2.262 \sqrt{\frac{203.73}{10}}$$

$$= 97.2 \pm 10.21 = 107.41 \text{ and } 86.99$$

$\therefore$ 95% C.I is [86.99, 107.41]

Example 17 A random sample of 16 values from a normal population showed a mean of 41.5 inches and the sum of squares of deviation from this mean is equal to 135 square inches. Show that the assumption of mean of 43.5 inches for the population is not reasonable. Obtain 95% fiducial limits for the same.

Solution Here $n = 16, \bar{x} = 41.5$ inches and $\sum_i (x_i - \bar{x})^2 = 135$ sq. inch

$$\therefore \quad s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2 = \frac{1}{15}(135) = 9,\ s = 3$$

$\mu_0 = 43.5$ inches

$$\therefore \quad t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{41.5 - 43.5}{3/\sqrt{16}} = 2.669$$

$|t| = 2.667$

$t_{.05}$ for 15 d.f. = 2.131

$|t|_{\text{cal}} = 2.667 > 2.131$ and we conclude that the mean 43.5 inches for the population is not reasonable 95% fiducial limits are as follows:

$$\bar{x} \pm t_{.05} \times \frac{s}{\sqrt{n}} = 41.5 \pm 2.131 \frac{s}{\sqrt{n}} = 4.15 \pm 1.598$$

$\Rightarrow \quad 39.902 < \mu < 43.098$

7.23 SIGNIFICANT TEST OF DIFFERENCE BETWEEN TWO SAMPLES

(i) Suppose two independent samples $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ of sizes n_1 and n_2 have been drawn from two normal populations having means μ_X and μ_Y, respectively, and same variance $\sigma^2 = \sigma^2_X = \sigma_Y^2$, that statistic is

$$t = \frac{(\bar{x} - \bar{y}) - (\mu_X - \mu_Y)}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}},$$

where $\bar{x} = \sum_i x_i / n_1$; $\bar{y} = \sum_j y_j / n_2$

$$s^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum_i (x_i - \bar{x})^2 + \sum_j (y_j - \bar{y})^2\right]$$

has a t-distribution with $n_1 + n_2 - 2$ d.f.

If $t_{cal} > t_{.05}$, the difference between sample means $\bar{x}$ and $\bar{y}$ is significant at 5% level of significance, otherwise not significant.

If $t_{cal} > t_{.01}$, then the difference between sample means $\bar{x}$ and $\bar{y}$ is significant at 1% level of significance, otherwise not significant .

In general if $t_{cal} > t_\alpha$, then the difference between the sample means $\bar{x}$ and $\bar{y}$ is significant α% level of significance, otherwise not.

(ii) Paired t-test for difference between two samples.

If (a) $n_1 = n_2 = n$ (say) and

(b) Two samples are not independent but the sample observations are paired together i.e., (x_i, y_i) $(i = 1, 2, \ldots, n)$ observations corresponds to the same i^{th} sample

Then the problem is to test if sample means differ significantly or not i.e. increments are due to fluctuations of sampling. The statistic $t = \dfrac{\bar{d}}{s/\sqrt{n}}$,

where $\bar{d} = \dfrac{1}{n}\sum di = \dfrac{1}{n}\sum (x_i - y_i)$, where $d_i = x_i = y_i$.

and $s^2 = \dfrac{1}{n-1}\sum (di - \bar{d})^2$, has t-dist. with $n - 1$ d.f.

then find t_α, if $t > t_\alpha$. The difference is significant, otherwise not.

Example 18 Given is the gain in weights (in kg) of pigs fed on two diet A and B.

Gain in weight

Diet A: 25, 32, 30, 34, 24, 14, 32, 24, 30, 31, 35, 25

Diet B: 44, 34, 22, 10, 47, 31, 40, 30, 32, 35, 18, 21, 25, 29, 22

Test, if the two diets differ significantly as regards their effect on increase in weight.

Solution Here $n_1 = 12$, $n_2 = 15$

$$\bar{x} = \sum \frac{x_i}{12} = \frac{336}{12} = 28, \quad \bar{y} = \sum \frac{y_j}{15} = \frac{450}{15} = 30$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum(x_i - \overline{x})^2 + \sum(y_j - \overline{y})^2\right] = 71.6$$

$$\therefore \qquad t = \frac{\overline{x} - \overline{y}}{s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{28.30}{\sqrt{71.6\left(\frac{1}{12} + \frac{1}{15}\right)}} = -0.609$$

$$|t| = 0.609$$

T_{cal} for $(n_1 + n_2 - 2)$ d.f. $= t_{.05}$ for $(12 + 15 - 2)$ $t_{.05}$ for 25 d.f. = 2.06

$|t|_{\text{lab}} = 6.09 < 2.06 \Rightarrow$ two diets not differ significantly as 5% level of significance.

Example 19 A certain stimulus administrated to each of the 12 patients resulted in the following increase of blood pressure:

5, 2, 8, –1, 3, 0, –2, 1, 5, 0, 4 and 6

Can it be concluded that the stimulus will, in general, be accompanied by an increase in blood pressure. **(V.T.U. 2007)**

Solution We have to test that stimulus will increase the blood pressure. To test this $t = \frac{\overline{d}}{s/\sqrt{n}}$ has a t-dist. with $n - 1$ d.f.

Here $n = 12$,

$d = x_i - y_i$	5	–2	8	–1	3	0	–2	1	5	0	4	6
d_i^2	25	4	64	1	9	0	4	1	25	0	16	36

$$\therefore \qquad \overline{d} = \sum\frac{d_i}{n} = \frac{31}{12} = 2.583$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}(d_i - \overline{d})^2 = 9.5382$$

$$\therefore \qquad t = \frac{\overline{d}}{s/\sqrt{n}}\ \frac{2.58}{\sqrt{9.5382}/\sqrt{12}} = 2.89$$

t_{tab} at 5% with (12 – 1) d.f. = 2.20

$|t|_{\text{cal}} = 2.89 > 2.20 \Rightarrow$ The stimulus does not increase the blood pressure

Example 20 In a certain experiment to compare two types of animal foods A and B, the following results in weights were observed in animals:

Animal number		1	2	3	4	5	6	7	8	Total
Increase weight in lb	Food A	49	53	51	52	47	50	52	53	407
	Food B	52	55	52	53	50	54	54	53	423

(i) Assuming that the two samples of animals are independent, can we conclude that there is no difference in Food A and B. (Use $\alpha = 0.05$)

(ii) Also, examine the case when the same set of eight animals were used in both the food. (Use $\alpha = 0.05$)

Solution (i) $t = \dfrac{\overline{x} - \overline{y}}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

$$\overline{x} = \sum_i \frac{x_i}{n} = \frac{407}{8} = 50.875, \ \overline{y} = \sum \frac{y_i}{n} = \frac{423}{8} = 52.875$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum_i (x_1 - \overline{x})^2 + (y_i - \overline{y})^2\right] = 3.41$$

$$\therefore \quad t = \frac{50.875 - 52.875}{\sqrt{3.41\left(\dfrac{1}{8} + \dfrac{1}{8}\right)}} = -2.17$$

$|t| = 2.17$

$t_{.05}$ with 14 d.f. 2.14

$|t|_{\text{cal}} = 2.17 > 2.14 \Rightarrow$ food A and B do not differ significantly at 5% level of significance with regards to their effect on increase in weight.

(ii) $t = \dfrac{\overline{d}}{s/\sqrt{n}}$ where $\overline{d} = \dfrac{\sum d_i}{n} = \sum\left(\dfrac{x_i - y_i}{n}\right)$

x_i	49	53	51	52	47	50	52	53	Total 407
y_i	52	55	52	53	50	54	54	53	423
d_i	–3	–2	–1	–1	–3	–4	–2	0	–16
d_i^2	9	4	1	1	9	16	0	0	44

$$\overline{d} = \sum \frac{d_i}{n} = -\frac{16}{8} = -2$$

$$s^2 = \frac{1}{n-1}\sum\left[(d_i - \overline{d})^2\right] = 1.7143$$

$$\therefore \quad |t| = \left|\frac{\overline{d}}{\sqrt{s^2/\sqrt{n}}}\right| = \frac{2}{\sqrt{1.7143/8}} = 4.32$$

tab $t_{.05}$ for $(8 - 1) = 7$ dif $f = 2.36$

$|t|_{\text{cal}} > 2.36 \Rightarrow$ 't' is significant at 5% level of significance and we conclude that both foods A and B do not differ.

EXERCISE 7.3

1. Nine items of a sample holds the following values: 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these differ significantly from the assumed mean of 47.5? **(V.T.U. 2010)**
2. A mechanic is making engine parts with axle diameter of 0.7 inch. A random sample of 10 parts show mean diameter 0.742 inch with a standard deviation of 0.04 inch. On the basis of this sample would you say that the work is inferior. **(V.T.U. 2009)**
3. Prove that 95% confidence limits for the population mean μ are $\bar{x} \pm \frac{\sigma}{\sqrt{n}} t_{0.05}$.
4. A random sample of 10 boys had the following IQ.
70, 120, 110, 101, 88, 83, 95, 98, 107, 100
Do these data support the assumption of a population mean IQ of 100 at 5% level of significance? **(V.T.U. 2006, Coimbatore 2001)**
5. A random sample of size 25 from a normal population has the mean $\bar{x} = 47.5$ and s.d. s = 8.4. Does this information refute the claim that the mean of the population is $\mu = 49.1$. **(J.N.T.U. 2003)**
6. The means of two random samples of sizes 9 and 7 are 196.42 and 198.82, respectively. The sum of squares of the deviations from the means are 26.94 and 18.73, respectively. Can the sample considered to have been drawn from the same normal population? **(Mumbai 2004)**
7. For a random sample of 10 pigs, fed on a diet *A*, the increase in weights in a certain period were:
10, 6, 16, 17, 13, 13, 12, 8, 14, 15, 9 lbs
For another random sample of 12 pigs fed on diet *B*, the increase in the same period were.
7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 lbs.
Find if the two samples are significantly different regarding the effect of diet. (Use $\alpha = 0.05$)
8. Eleven students were given a test in statistics. They were given a month's further tuition and a second list of equal difficulty was held at the end of it. Do the marks give evidence that students have been benefited by extra coaching?

Boys	1	2	3	4	5	6	7	8	9	10	11
Marks I test:	23	20	19	21	18	20	18	17	23	16	19
Marks II test:	24	19	22	18	20	22	20	20	23	20	17

(V.T.U. 2011S)

9. Test runs with 6 models of an experimental engine showed that they operated for 24, 28, 21, 23, 32 and 22 minutes with a gallon of fuel. If the probability of a type I error is at numbers 0.01, is this an evidence against a hypothesis that with this average, engine will operated for at least 29 minutes per gallon of the same fuel. Assume normality. **(J.N.T.U. 2003)**
10. Two houses *A* and *B* were tested according to the time (in seconds) to run a particular race with the following results:

House A:	28	30	32	33	33	29 and 34
House B:	29	30	30	24	27 and 29	

Test whether you can discriminate between two houses? **(Rohtak 2005, Coimbatore 2001)**

11. A group of 10 rats fed on a diet A and another group of 8 rats fed on a different diet B, recorded the following increase in weight:
Diet A: 5, 6, 8, 1, 12, 4, 3, 9, 6, 10 gms
Diet B: 2, 3, 6, 8, 10, 1, 2, 8 gms

Does it show the superiority of diet A over diet B? **(Madras 2003)**

12. The average number of articles produced by two machines per day are 200 and 250 with standard deviations 20 and 25, respectively, on the basis of records of 25 days production. Can you regard both the machines equally efficient at 5% level of significance.
13. A company is interested in knowing if there is a difference in the average salary received by foremen in two divisions. Accordingly samples of 12 foremen in the first division and 10 foremen in the second division are selected at random. Based upon experience foremen's salaries are known to be approximately normally distributed and the standard deviations are about the same.

Sample size	First division	Second division
	12	10
Average weekly salary of foremen (Rs)	1050	980
Standard deviation of salaries (Rs)	68	74

The table value of t for 20. d.f. at 5% level of significance is 2.09.

14 A random sample of size 25 from a normal population has mean $\overline{x}$ = 47.5 and s.d. s = 8.4. Does this information refute the claim that the mean of the population is $\mu = 42.1$ **(J.N.T.U. 2003)**

Answers

1. $t = 1.83$, $t_{.05}$ for 8 d.f. = 2.31, not significant.
2. $t = 3.16$.
3. $t = 0.62$, yes.
4. Refute the claim.
5. No.
6. $t = 1.51$.
7. $t = 1.48$.
8. Accept null hypothesis.
9. Yes with 75% confidence.
10. No.
11. $|t| = 7.65$.
12. $t = 2.2$.

14. Refute the claim as $t = 3.21$.

7.24 CHI-SQUARE (χ^2) TEST

Introduction: The square of a standard normal variable $Z \sim N(0, 1)$ is known as a Chi-square variate with let f.

$\therefore$ If $X \sim N(\mu, \sigma^2)$, then $Z = \dfrac{X-\mu}{\sigma} \sim N(0.1)$ and

$$Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2 \text{ is a Chi-square variate with 1 d.f.}$$

In general, if $X_1, X_2, \ldots, X_n$ are n independent normal variates with means $\mu_1, \mu_2, \ldots, \mu_n$ and variance $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2$, respectively, then

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu_i}{\sigma_i}\right)^2 = \chi^2, \text{ is a Chi-square variate with n.d.f.}$$

The p.d.f. of χ^2 is given by

$$f(\chi^2) = \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\left\lceil\frac{n}{2}\right.}\left[\exp\left(-\frac{1}{2}\chi^2\right)(\chi^2)^{\frac{n}{2}-1}\right]$$

$$= \frac{1}{2^{\frac{n}{2}}\sqrt{\frac{x}{2}}}\left[\exp\left(-\frac{1}{2}\chi^2\right)\right](\chi^2)^{\frac{n}{2}-1}, 0 \le \chi^2 < \infty$$

or

$$f(\chi^2) = C\, e^{-\frac{1}{2}\chi^2}\, (\chi^2)^{\frac{(\nu-1)}{2}} \tag{3}$$

where C is a constant and $\nu = n - 1$

The graph of the equation of χ^2 – curve which is given in eq. (3) is shown in Fig. 7.3.

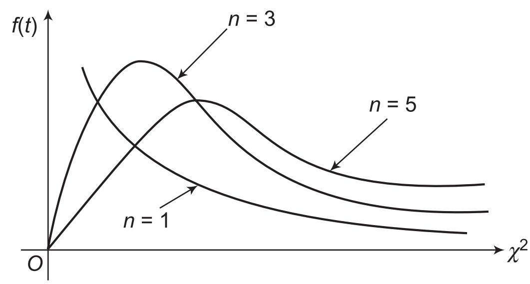


Fig. 7.3

7.24.1 Properties of χ^2-distribution

1. If $\nu = 1$, then Eq. (3) becomes $y = Ce^{-\frac{1}{2}\chi^2}$, which is exponential distribution.
2. The mean of a χ^2 distribution with ν d.f. is ν and various is 2ν.
3. Mode of χ^2 distribution with ν d.f. $= n - 2$
4. If $\nu = 1$, the curve of χ^2-variate is tangential to x-axis at origin and positively skewed.
5. The probability P that the value of χ^2 from a random sample will exceed χ_α^2 is given by

$$P\left[\chi^2 \ge \chi_\alpha^2\right] = \int_{\chi^2}^{\infty} f(\chi^2)\, dx = P$$

   and the values of χ_α^2 have been tabulated for various values of P and for values of degrees of freedom ν from 1 to 30.

   If $\nu > 30$, then χ^2-distributin fallows normal distribution and we use normal distribution tables for significant values of χ^2.
6. As the equation of χ^2-curve does not involve any of the parameters of the population, therefore χ^2-distribution does not depend on the distribution of the population and hence very useful in statistical theory.

7.24.2 Chi-Square (χ^2) Test of Goodness of Fit

A very powerful test for testing the significance of the discrepancy between theory and experiment was given by Prof. Karl Pearson in 1900 and is known as "Chi-Square test of goodness of fit". It enables us to find if the deviation of the experiment from theory is just by chance or is it really due to the inadequacy of the theory to fit the observed data.

Let $o_i (i = 1, 2, \ldots, n)$ be a set of observed (experimental) frequencies and $e_i (i = 1, 2, \ldots, n)$ be the corresponding set of expected (theoretical or hypothetical) frequencies, then

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{o_i - e_i}{e_i} \right)^2 \quad \text{has a } \chi^2\text{-distribution with } (n - 1) \text{ d.f.}$$

Procedure to test significance and goodness of fit:

(a) Set up a 'null hypothesis'.

(b) Compute $\chi^2 = \sum_{i=1}^{n} \left(\frac{o_i - e_i}{e_i} \right)^2$

(c) Find the degrees of freedom and also from the table of χ^2, find the value of χ^2 at desired level of significance with corresponding *d.f.*

(d) Also, find the probability P corresponding to the calculated value of χ^2 for the given d.f. from χ^2 table.

(e) If $P < 0.01$, the value is significant at 1% level of significance
If $P < 0.05$, the value is significance at 5% level of significance
If $P > 0.05$, the value is not significant and it is a good fit at 5% level of significance. Similarly
If $P > 0.01$, the value is not significant and it is a good fit at 1% level of significance
In general, we can say if
P > α, the value is not significant at a given level of significance α and it is a good fit.
If $P \leq \alpha$ then value is significant.

Remark: As we don not make assumptions on the population distribution, the test of χ^2 is also known as non-parametric test

Example 21 The demand for a particular spare part in a factory was found to vary from day to day. In a sample study the following information was obtained:

Days	Mon	Tue	Wed	Thu	Fri	Sat
No. of parts demanded	1124	1125	1110	1120	1126	1115

Test the hypothesis that the number of parts does not depend on the day of the week.

Solution The null hypothesis Ho. The number of parts demanded does not depend on the day of week.

Under the null hypothesis the expected frequencies of the spare part demanded on each of the six days would be

$$\frac{1}{6}(1124 + 1125 + 1110 + 1120 + 1126 + 1115) = \frac{6720}{6} = 1120 \text{ calculated by } \chi^2$$

Days:	Mon	Tue	Wed	Thu	Fri	Sat	Total
Observed frequencies (o_i)	1124	1125	1110	1120	1126	1115	6720
Expected Frequencies (e_i)	1120	1120	1120	1120	1120	1120	6720
$(o_i - e_i)^2$	16	25	100	0	36	25	202
$\frac{(O_i - e_i)^2}{e_i}$	0.014	0.022	0.089	0	0.032	0.022	0.179

Number of d.f. = (6 – 1) = 5

Tabulated value of $\chi^2_{.05}$ with 5 d.f. = 11.07

$\chi^2_{\text{cal}} = 0.179 < 11.07 \Rightarrow$ It is not significant and null hypothesis is accepted at 5% level of significance, Hence, we conclude that the number of parts demanded are same over the 6 days period.

Example 22 A sample analysis of examination results of 200 engineering students was made. It was found that 46 students failed, 68 secured a third division, 62, secured a second division and rest were placed in a first division. Are these figures commensurate with the general examination result which is in the ratio of 4 : 3 : 2 : 1 for various categories respectively?

Solution Null hypothesis: The observed figures do not differ significantly from the hypothetical frequencies which are in the ratio of 4 : 3 : 2 : 1. In three words, the given data are commensurate with the general examination result which is in the ratio of 4 : 3 : 2 : 1.

$$\therefore \quad e_1 = \frac{1}{10}(200 \times 4) = 80,\ e_2 = \frac{1}{10}(200)\cdot 3 = 60,\ e_3\ \frac{1}{10}(200)\cdot 2 = 40, \text{ and } e_4 = \frac{1}{10}(200)\cdot 1 = 20$$

Calculation for χ^2

Category	Frequency		$(o_i - e_i)^2$	$\left(\frac{o_i - e_i}{e_i}\right)^2$
	Observed (o_i)	*Expected* (e_i)		
Failed	46	80	1156	14.450
III division	68	60	64	1.069
II division	62	40	484	12.100
I division	24	20	16	0.800
Total	200	200		28.417

d.f = 4 – 1 = 3, Tab $\chi^2_{0.05}$ for 3 d.f. = 7.815

χ^2_{cal} = 28.417 > 7.815. It is significant and null hypothesis is rejected. Hence, we may conclude that data are not commensurate with the general examination.

Example 23 When the first proof of 392 pages of a book of 1200 pages were read, the distribution of printing mistakes were found to be as follows:

No. of mistakes in a page (x)	0	1	2	3	4	5	6
No. of pages (o_i)	275	72	30	7	5	2	1

Fit a Poisson distribution data and test the goodness of the given.

Solution *Null hypothesis:* The given data fit the Poisson distribution

Here $$\bar{x} = -\frac{1}{N}\sum_{1=0}^{6} f_i\, x_i = \frac{189}{392} = 0.482$$

The frequency of x mistakes per page is given by the Poisson law as follows:

$$e(x): N\,p(x) = \frac{392.\, e^{0.482}\ (0.482)^x}{x!},\ x = 0, 1, 2, \cdots, 6$$

$\therefore$ $$e_o = 242.1,\ e_1 = 116.7,\ e_2 = 28.1,\ e_3 = 4.5,\ e_4 = 0.5,\ e_5 = 0.1,\ e_6 = 0$$

Calculation of χ^2

Mistakes per page x	Frequency Observed (o_i)	Frequency Expected (e_i)	$(o_i - e_i)^2$	$\left(\frac{o_i - e_i}{e_i}\right)^2$
0	275	242.1	1082.41	4.471
1	72	116.7	1998.09	17.121
2	30	28.1	3.61	0.128
3	7	4.5		
4	5	0.5		
5	2	0.1		
6	1	00		
3–6	15	5.1	98.01	19.217
Total	392	392		40.937

d.f. = 7 – 1 – 1 – 3 = 2

($\because$ One d.f. being lost because of calculating $\bar{x}$ and 3.d.f. are lost because of pooling the last four expected cell frequencies, which are less than 5).

Tabulated value of χ^2 for 2 d.f. at 5% level = 5.99

$\chi^2_{\text{cal}} = 40.937 > 75.98$. It is highly significant. Hence, Poisson distribution is not good fit.

Note:

1. If any parameter(s) is(are) calculated based on sample data, the degrees of freedom will be reduced to the no. of parameter(s) computed
2. If any cell frequency is <5 then add these cell frequencies to the proceeding or succeeding cell and number of degrees will be reduced the number of those cells for which frequencies are added to another cell.
3. In above example $\bar{x}$ is calculated from sample data and 3 frequencies are added to their proceeding cell there fore number of d.f.

$$= (n - 1) - 1 - 3 = (7 - 1) - 1 - 3 = 2.$$

EXERCISE 7.4

1. A set of 5 similar coins is tossed 320 lines and the result is

Number of heads	0	1	2	3	4	5
Frequencies	6	27	72	112	71	32

Test the hypothesis that the data follows a binomial distribution

(Kottayam, 2005: P.T.U., 2005; V.T.U. 2004)

2. Fit a Poisson distribution to the following data and test the goodness of fit at a level of significance 0.05.

x	0	1	2	3	4
f	419	352	154	56	19

(V.T.U., 2008)

3. The following table gives the number of aircrafts accidents that occurred during the various days of the week. Whether the occident are uniformly distributed over the week?

Days	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
Number of accident	14	16	08	12	11	9	14	84

(Hissar, 2005)

4. A survey of 800 families with four children each revealed the following distribution:

Number of boys	0	1	2	3	4
Number of girls	4	3	2	1	0
Number of families	32	178	290	236	64

Is this result consistent with the hypothesis that male and female births one equally probable?

5. The following figure show the distribution of digits in numbers chosen at random from a telephone directory:

Digits	0	1	2	3	4	5	6	7	8	9	Total
Frequency	1026	1107	997	966	1075	933	1107	972	964	853	10,000

Test whether the digits may be taken to occur equally frequently in the directory.

6. Fit a normal distribution to the following data of weights of 100 students of an university and test the goodness of its

Weight (kg)	60-62	63-65	66-68	69-71	72-74
Frequency	5	18	42	27	8

7. Fit a Poisson distribution to the following data and test the goodness of fit.

x	0	1	2	3	4	5	6	7
f	305	366	210	80	28	9	2	1

8. A dice was thrown 60 times and the following frequency distribution was observed. Test whether the dice in unbiased?

Faces	1	2	3	4	5	6	Total
f	15	6	4	7	11	17	60

Answers

1. $\chi^2_{\text{cal}} = 78.68$, data do not follow binomial distribution at 5%.
2. $\chi^2_{\text{cal}} = 5.748$, data fit the Poisson distribution at 5%.
3. Yes, accidents are uniformly distributed.
4. $\chi^2_{\text{cal}} = 19.63$, male and female births are not equally probable at 5%.
5. $\chi^2_{\text{cal}} = 58.542$, digits are not uniformly distributed.
6. $\chi^2_{\text{cal}} = 0.8362$ (here d.f. 2 as mean and variance are computed.
7. $\chi^2 = 3.097$, Poisson distribution gives a good fit at 5% level.
8. Significant at 5%.

7.25 *F*-DISTRIBUTION

F-distribution introduced by English statistician R.A. Fisher is defined as follows:

Let X and Y are two independent Chi-square variates with v_1 and v_2 d.f., respectively, then F-statistics is given by

$$F = \frac{X/v_1}{Y/v_2}$$

In other words, F is defined as the ratio of two independent Chi-square variates divided by their respective degrees of freedom and devoted by $F(v_1, v_2)$.

7.25.1 Application of *F*-distribution

F-distribution has wide applications in the theory of statistics. One of them is F-test for equality of two population variances, which is defined as follows: Let $x_1, x_2, \ldots, x_{n_1}$, and $y_1, y_2, \ldots, y_{n_2}$ be the two independent random samples drawn from the normal populations with the same variance σ^2. Let $\bar{x}$ and $\bar{y}$ be the sample means and s_x^2 and s_y^2 be the sample variances of two samples, and

$$\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n_1} x_i,\ \bar{y} = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j,\quad s_x^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_i-\bar{x})^2,\ s_y^2 = \frac{1}{n_2-1}\sum_{j=1}^{n_2}(y_j-\bar{y})^2. \text{ Then } F = \frac{s_x^2}{s_y^2} \text{ has}$$

F-distribution with $n_1 - 1 = v_1$ and $n_2 - 1 = v_2$ degrees of freedom. The larger variance is always taken as numerator, so that the values of F is always positive.

7.25.2 Properties of *F*-distribution

1. The graph of F-distribution is shown as follows in Fig. 7.4

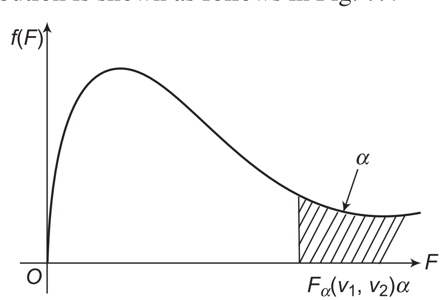


Fig. 7.4

2. The curve of F-distribution always lie in first quadrant as $F > 0$ (always).
3. F-distribution is unimodel and its mode is always < 1.
4. F-distribution depends on v_1 and v_2 not on the population variants σ^2 i.e, it is independent of σ^2.
5. v_1, v_2 are known as degrees as degrees of freedom of numerator and denominator, respectively.
6. $P\left[F(v_1, v_2) \geq F_\alpha\ (v_1, v_2)\right] = \alpha.$
7. $F_\alpha(v_1, v_2)\ \dfrac{1}{F_{1-\alpha}\ (v_2, v_1)}.$

7.25.3 Significance Test

F tables gives 5% and 1% points of significance for F. 1% points of F mean that area under the F-curve to the right of the ordinate at a value of F, is 0.01. The value of at 1% significance level is more that at 5%. F-distribution has wide applications and has a base for analysis of variance.

Example 24 In one sample of 8 observations, the sum of squares of derivations of the sample values from the sample mean was 84.4 and in the other sample of 10 observations it was 102.6. Test whether this difference is significant at 5% level of significance.

Solution Here $n_1 = 8$, $n_2 = 10$, $\sum_{i=1}^{8}(x_i - \overline{x})^2 = 84.4, \sum_{j=1}^{10}(y_j - \overline{y})^2 = 102.6$

$$\therefore \quad s_x^2 = \frac{1}{n_1 - 1}\sum(x_i - \overline{x})^2 = \frac{1}{7}(84.4) = 12.057$$

$$\text{and} \quad s_y^2 = \frac{1}{n_2 - 1}\sum(y_j - \overline{y})^2 = \frac{1}{9}(102.6) = 11.4$$

$$\therefore \quad F = \frac{s_x^2}{s_y^2} = \frac{12.057}{11.4} = 1.0576$$

Tabulated value

$F_{\cdot 0.5}(7, 9) = 3.29$ (from F-table)

$F_{\text{cal}} = 1.0576 < 3.29$. It is not significant

7.26 FISHER'S Z-DISTRIBUTION

If we put $z = \dfrac{1}{2}\log_e F$ or $F = e^{2z}$ in the F-distribution, we get Fisher's Z-distribution.

Z-distribution is more symmetrical then F-distribution. For various values of v_1 and v_2 degrees of freedom, a table showing the values of z that will be exceeded in simple sampling with probabilities 0.01 and 0.05.

7.26.1 Significance Test

In Z-table, we have critical values as $P[Z \geq Z_d] = \alpha$

Therefore, the 1% or 5% points of Z imply that the area to the right of ordinate $Z_{.05}$ or $Z_{\cdot 01}$ is 0.05 or 0.01. In other words, 1% and 5% points of Z-correspond to 2% and 10% level of significance respectively. As we generally use two-tailed test.

Example 25 Two gauge operations are tested for precision in making measurements. One operator completes a set of 26 readings with a standard deviation of 1.34 and the other does 34 readings with a standard deviation 0.98. What is the level of significance of this difference. (We can use $Z_{.05} = 0.305$, $Z_{.01} = 0.432$) for $v_1 = 25$ and $v_2 = 33$)

Solution Here $n_1 = 26, n_2 = 34, s_x = 1.34, s_y = 0.98$

$\therefore$ $s_x^2 = (1.34)^2$

and $s_y^2 = (0.98)^2$

$$\therefore \quad F = \frac{s_x^2}{s_y^2} = \frac{(1.34)^2}{(0.98)^2} = 1.8696$$

$$\therefore \quad Z = \frac{1}{2}\log_e F = 1.1513 \log_{10}(1.8696) = 0.31286$$

Calculated value of $Z > .305$ but less than 0.432

$\therefore$ level of significance lies between 1% and 5% which is closer to 5%

EXERCISE 7.5

1. Two samples of sizes 9 and 8 give the sum of squares of deviation from their means equal to 160 inches^2 and 91 inches^2, respectively. Can these be regarded as drawn from the same normal population. **[V.T.U. 2002]**
2. Pumpkins were grown under two experimental conditions. Two random samples of 11 and 9 pumpkins show the sample standard deviations of their weights as 0.8 and 0.5 respectively. Assuming that the weight distributions are normal, the hypothesis that two variances are equal against the alternative that they are not at 5% level.
3. Two random samples gave the following results.

Sample	Size	Sample mean	Sum of squares of deviation from the mean
1.	10	15	90
2.	12	14	108

Test whether the samples come from the same normal population at 5% level of significance.

4. The following are the values in thousands of an inch obtained by two engineers in 10 and 9 successive measurements with the same micrometer. Are both engineers significant of each other.

Engineer *A*	503	505	497	505	495	502	499	493	510	501
Engineer *B*	502	497	492	498	499	495	497	496	498	

5. Two random samples of sizes 8 and 11, drawn from the two normal populations are characterized as follows:

Sample	Size	Sum of observation	Sum of squares of observations
1	8	9.6	61.52
2	11	16.5	73.26

Test whether two populations can be taken to have the same variance.

6. Two samples of sizes 9 and 8 have variances 1101.1 and 319.7, respectively. Is the variance 1101.1 significantly greater than variance 319.7.

7. The IQ's of 25 students from one institute showed a variance of 20 and those of an equal number from the other institute had a variance of 10. Discuss whether there is any significant difference in variability of intelligence.
8. Show how you would use Fisher's Z test to decide whether the two sets of observations 15, 25, 16, 23, 25, 27, 25, 21, 15 and 14, 14, 18, 14, 18, 15, 13 and 19 indicate samples from the same universe.

Answers

1. $F \cong 1.54$, $F_{.05}(8.7) = 3.73$, not significant.
2. $F = 2.5$, $F_{.05}(10, 8) \cong 3.35$, not significant.
3. $F = 1.018$, $F_{0.05}(11, 9) = 3.10$, not significant.
4. F = 2.4.
5. $F = 1.47$.
6. $F = 3.44$, $F_{.05}(8, 7) = 3.73$ and $F_{.01}(8, 7) = 6.84$.
   It is significant at both the level of significance.
7. $F = 2$.
8. $F = 4.25$, $z = 0.72346$.

SUMMARY

In this chapter, the following topics have been discussed:

1. Population and sample.
2. Population parameters: μ and σ^2 etc.
3. Sample statistic: $\bar{x}, s^2$ etc, $\bar{x} = \sum \frac{x_i}{n}$, $s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$
4. Standard error: Standard deviation of sampling variance s^2
5. Null and alternate hypothesis = H_0 and H_1
6. P(Type I error) = P(rejecting null hypothesis when null hypothesis is true)
P(Type II error) = P(accepting null hypothesis when it is wrong)
7. P(Type I error) and level of significance are generally same.
8. Confidence limits for μ

$$\bar{x} \pm z_\alpha \frac{\sigma}{\sqrt{n}} \text{ (σ is known)}$$

$$\bar{x} \pm t_\alpha \frac{\sigma}{\sqrt{n}} \text{ (σ is unknown and σ is estimated based on sample)}$$

9. Central limit theorem: $\frac{X - E(x)}{\sqrt{\text{var }(x)}} = z \sim N(0, 1)$

For large $n \geq 30$, any distribution of X will follow standard normal distribution.

10. $z = \frac{\bar{x} - \mu}{\sigma \sqrt{n}}$ and $t = \frac{\bar{x} - \mu}{s \sqrt{n}}$

11. $F = \dfrac{s_x^2}{s_y^2}$

12. $\chi^2 = \sum_{i=1}^{n} \left(\dfrac{\chi_i - \mu}{\sigma} \right)^2 \sim \chi_{n-1}^2$

13. $z = \dfrac{1}{2} \log_e F$

OBJECTIVE TYPE QUESTIONS

1. If $s_1{}^2 = 10$ and $s_2{}^2 = 5$, then F is equal to
(a) 2 (b) 3
(c) 1 (d) 2.5

2. If S^2 be sample variance of n observations, then it is equal to

(a) $\dfrac{1}{n} \sum_i (x_i - \overline{x})^2$

(b) $\dfrac{1}{n-1} \sum_i (x_i - \overline{x})^2$

(c) $\dfrac{1}{n} \sum_i (x_i + \overline{x})^2$

(d) $\dfrac{1}{n-1} \sum_i (x_i + \overline{x})^2$

3. If a random variable X has χ^2 distribution, with 4 d.f. then its mean is equal to
(a) 4 (b) 3
(c) 2 (d) 1

4. If a random variable X has χ^2-distribution with 2 d.f., then its standard deviation is
(a) 4 (b) 3
(c) 2 (d) 1

5. If $p = 0.13$, $n = 500$, then standard error is equal to
(a) 0.011 (b) 0.012
(c) 0.013 (d) 0.015

6. X is normally distributed with mean 5 and variance 25. If a random sample of size 25 is drawn from X then variance of $\overline{X}$ is equal to
(a) 1 (b) 2
(c) 5 (d) none of these

7. If standard error of $\overline{X} = 0.2$, then its precision is equal to
(a) 0.5 (b) 1
(c) 2 (d) 5

8. If $n = 625$, $\overline{X} = 3.50$, $\mu = 3.20$ and $\sigma^2 = 2.25$, then z is equal to
(a) 5 (b) 2
(c) 1 (d) 0.5

9. If t distribution has mean zero, then its variance is
(a) > 12 (b) < 12
(c) $= 12$ (d) none of these

10. Variance of t distribution is always
(a) Greater than 1 (b) Less than 1
(c) Equal to 1 (d) Greater than 0.5

11. If we reject null hypothesis when it is correct we get _____ error

12. To use t-statistic, the population from which sample is drawn should have _____ distribution.

13. If the standard deviation of χ^2-distribution is 10, then its degree of freedom is
(a) 25 (b) 50
(c) 75 (d) 100

14. The test statistic, $F = \dfrac{s_x^2}{s_y^2}$ is used when

(a) $s_x^2 > s_y^2$ (b) $s_x^2 < s_y^2$

(c) $s_x^2 = s_y^2$ (d) none of these

15. In a t-distribution of sample size 18, the degree of freedom are
(a) 18 (b) 17
(c) 16 (d) 15

16. Fisher's z transformation is equal to

(a) $z = \frac{1}{2}\log_e F$ (b) $z = \frac{1}{2}\log_{10} F$

(c) $z = \log_e F$ (d) $\log_{10} F$

17. The range of t-statistic is

(a) $(0, \infty)$ (b) $(-\infty, \infty)$

(c) $(-\infty, 0)$ (d) $(2, \infty)$

18. If mean of χ^2-distribution is 4, then its mode is

(a) 4 (b) 3

(c) 2 (d) 1

19. If 6, 27, 72, 112, 71 and 32 are the observed frequencies; and 10, 50, 100, 100, 50 and 10 are respectively the expected frequencies of an experiments, respectively, then the value of χ^2 is

(a) 78.68 (b) 76.88

(c) 68.78 (d) 86.78

20. If $\bar{x}_1 = 67.5, \bar{x}_2 = 68.0, n_1 = 100, n_2 = 2000$ and $r = 2$, then z is

(a) 4.9 (b) 5.0

(c) 5.1 (d) 5.2

ANSWERS

1. (a)	2. (b)	3. (a)	4. (c)	5. (d)	6. (a)	7. (d)	8. (a)	9. (d)	10. (a)
11. Type I	12. Normal		13. (b)	14. (a)	15. (b)	16. (a)	17. (b)	18. (c)	19. (a)
20. (c)									

8 Finite Differences and Interpolation

8.1 INTRODUCTION

Numerical analysis is a branch of Mathematics in which we analyse the computational methods for solving scientific and engineering problems by applying basic arithmetic operations. The results obtained by using numerical methods are usually approximate to the true results. These approximations to the true results represent/involve errors but can be made more accurate up to some extent. There can several reasons behind the approximations, when we use a method to solve a problem may not be exact.

For example, cos x can be solved by expressing it as an infinite power series. This series has to be truncated to the finite number of terms. This truncation introduces an error in the calculated result.

8.2 FLOATING POINT REPRESENTATIONS

In computational calculations, a very small numbers such as size of an electron or very big numbers such as velocity of light occur frequently. These numbers cannot be represented in a usual manner. There are two ways of representing these numbers, called fixed point and floating point.

In a fixed-point representation all numbers are given with a fixed number of decimal places. For example, 62.358, 0.015, 1.000 and 1.001; all correctly expressed up to third decimal places.

In a floating-point representation, the number of significant digits is kept fixed (whereas the decimal point is *floating* as seen from the exponent). Examples are 6.236×10^3, 1.306×10^{-3} which are all given as four significant figures.

Significant Digits

The concept of significant digits has been introduced primarily to indicate the accuracy of a numerical value.

Significant digit of a number K is any given digit of K, except possibly for zeros to the first non-zero digit that serve only to fix the position of the decimal point.

Following rules are applied when zeros are encountered in the numbers:

(i) Zeros placed after other digits but behind a decimal point are significant; 3.80 has three significant digits.

(ii) Zeros placed before other digits are not significant; 0.038 has two significant digits.

(iii) Zeros placed between other digits are always significant; 3007 has four significant digits.

(iv) Zeros at the end of a number are significant if they are behind a decimal point as in Example 1, Part (iii) and (v).

Example 1 Find the accuracy of the following numbers:

(i) 83.234
(ii) 0.0037
(iii) 3800.00
(iv) 38
(v) 7200
(vi) 0.0536000

Solution

(i) The number 83.234 has five significant digits/figures.
(ii) The number 0.0037 has two significant digits.
(iii) Number 3800.00 has six significant digits because zeros were made significant by writing 0.00 after 3800.
(iv) The number 38 has only two significant figures.
(v) In the number 7200, it is not clear if the zeros are significant or not. The number of significant digits in 7200 are at least two but it could be three or four. To find the accuracy we use the scientific notation to place significant zeros behind a decimal point.

7.2×10^3 has two significant digits

7.20×10^3 has three significant digits

7.200×10^3 has four significant digits

(vi) The number 0.0536000 has six significant digits.

8.3 ROUNDING-OFF AND CHOPPING

All the non-exact numbers can be approximated with a finite number of digits of precision in the following manners:

If n digits are used to represent a non-terminating number then the simplest process is to kept the first n digits and chop off all remaining digits.

We know that $\frac{22}{7} = 3.1415926$

$$\sqrt{2} = 1.4142134, e = 2.71828182$$

The digits on the right are approximating to the exact value of the numbers on the left.

To round off a number to n significant digits, discard all digits to the right of n^{th} digit and if this discarded number is

(i) less than 5 in $(n + 1)^{\text{th}}$ place, leave the n^{th} digit unchanged. For example, the number 3.892 round off to 3 significant digit is 3.89.
(ii) greater than 5 in $(n + 1)^{\text{th}}$ place, add 1 to the n^{th} digit. For example, the number 8.3466 round off to 3 significant digits is 8.35.
(iii) exactly 5 in $(n + 1)^{\text{th}}$ place, add 1 to the n^{th} digit if it is an odd otherwise leave n^{th} digit unchanged. For example, $12.375 \to 12.38$ and $12.765 \to 12.76$ correct to 4 significant digits.

8.4 ERROR

An error is the difference between the true value or actual value and the approximate value from the numerical computation or from the experimental observations.

Let the true value be X and the approximate value be X_a, then

Error $(\in) = X - X_a$

8.4.1 Type of Errors

In numerical calculations the types of errors are following:

(a) *Round-off errors*: An error is caused by chopping (i.e., discarding all decimals from some decimal on) or rounding. This error is called rounding off error.

(b) *Inherent errors*: These errors are already present in the statement of a problem before its solution and such errors are called inherent errors. These errors are present either due to the given data being approximated or due to the limitations of mathematical measurements.

(c) *Absolute, relative and percentage errors*: The absolute error of measurement, number or calculation is the numerical difference between the true value of the quantity and its approximate value as given or calculated.

Relative error is the absolute error divided by the true value of the quantity. The percentage error is 100 times of the relative error.

Let X be the true value and X_a be the approximate value, then

Absolute error $(E_a) = |X - X_a|$

Relative error $(E_r) = \dfrac{|X - X_a|}{X}$

Percentage error $(E_P) = E_a \times 100$

(d) *Truncation error*: These errors are caused by the usage of a closed form such as the first few terms of an infinite series to express a quantity defined by limiting process.

For example, consider, use of a finite number of terms in the infinite series expansion of cos x or sin x by using Taylor's or Maclaurin's series expansion, such type of errors are called the truncation errors.

***Remark*:**

1. The relative and percentage errors are independent of the units while the absolute error is expressed in terms of units used.
2. If the relative error in an approximate number is less than $\dfrac{1}{(k+1)\times 10^{n-1}}$, the number is correct to n significant digits.
3. If the relative error of any number is not greater than $\dfrac{1}{2\times 10^{-n}}$, the number is certainly correct to n significant digits.
4. If the first significant figure of a number is k, and the number is correct to n significant figures, then the relative error $< \dfrac{1}{k \times 10^{n-1}}$.
5. Let X be a real number and X_1 be a another number having non-terminal decimal expansion, then X_1 represents X rounded to k decimal places if

   $|X - X_1| \leq \dfrac{1}{2}10^{-k}$; where k is a positive integer.

Example 2 Round off the following numbers to four significant figures:

(i) 30.0567
(ii) 0.042514
(iii) 0.0049125
(iv) 0.00010125
(v) 3.14159

Solution We have to retain first four significant figures:

(i) 30.0567 to 30.06
(ii) 0.042514 to 0.04251
(iii) 0.0049125 to 0.004912 (because the digit in 4^{th} place is even)
(iv) 0.00010125 to 0.0001012 (because the digit in 4^{th} place is even)
(v) 3.14159 to 3.142

Example 3 If $\pi = 3.14159265$, then find out to how many decimal places the approximate value of $\frac{22}{7}$ is accurate?

Solution

$$\text{Error} = \left|\pi - \frac{22}{7}\right| = 0.00126449 = 1.26\cdots \times 10^{-2}$$

Since $\quad 0.00126449 < 0.005 = \frac{1}{2} \times 10^{-2}$

Hence, $\quad k = 2$

So, we conclude that the approximation is accurate to two decimal places or three significant digits.

Example 4 If 0.333 is the approximate value of $\frac{1}{3}$, then find absolute, relative and percentage errors.

[U.P.T.U. 2002]

Solution Here, we have $X = \frac{1}{3}$ and $X_a = 0.333$.

$$\therefore \quad \text{Absolute error } (E_a) = |X - X_a|$$

$$= \left|\frac{1}{3} - 0.333\right|$$

$$= \left|\frac{1}{3} - \frac{333}{1000}\right|$$

$$= \left|\frac{1000 - 999}{3000}\right|$$

$$= \left|\frac{1}{3000}\right|$$

$$= 0.00033$$

Relative error $(E_r) = \dfrac{E_a}{X}$

$$= \frac{0.00033}{\frac{1}{3}}$$

$$= 0.00099$$

Now, percentage error $(E_p) = E_r \times 100$

$$= 0.00099 \times 100$$

$$= 0.099$$

Example 5 Given $x = 0.005998$, if x is rounded off to three decimal digits, find absolute, relative and percentage errors. **[U.P.T.U. 2004]**

Solution Number x rounded of three decimal digits is 0.006

Now, error $= 0.005998 - 0.006$

$$= -0.000002$$

Now, Absolute error $(E_a) = |-0.000002|$

$$= 0.000002$$

Relative error $(E_r) = \dfrac{E_a}{x}$

$$= \frac{0.000002}{0.005998}$$

$$= 0.0033344$$

and Percentage error $(E_p) = E_r \times 100$

$$= 0.0033344 \times 100$$

$$= 0.33344$$

Example 6 Find the sum $S = \sqrt{3} + \sqrt{5} + \sqrt{7}$ to four significant figures. Also find the absolute, relative and percentage errors. **[U.T.U. 2012, U.P.T.U. 2004]**

Solution We know that

$$\sqrt{3} = 1.732, \sqrt{5} = 2.236, \text{ and } \sqrt{7} = 2.646$$

$\therefore$ $S = \sqrt{3} + \sqrt{5} + \sqrt{7}$

$$= 1.732 + 2.236 + 2.646$$

$$= 6.614$$

Since, $\sqrt{3}, \sqrt{5}$ and $\sqrt{7}$ each has 3 decimal places, so each has the error $= \frac{1}{2} \times 10^{-3} = 0.0005$

$\therefore$ The absolute error $(E_a) = 0.0005 + 0.0005 + 0.0005$

$$= 0.0015$$

$$= 0.15 \times 10^{-2}$$

This absolute error shows that the sum of error is correct to 2 decimal places. Thus, the sum(S) is correct to 3 significant digits only.

$\therefore$ We assume $S = 6.61$

The relative error $(E_r) = \frac{E_a}{S}$

$$= \frac{0.15 \times 10^{-2}}{6.61}$$

$$= 0.0002$$

Now, percentage error $(E_p) = E_r \times 100$

$$= 0.0002 \times 100$$

$$= 0.02$$

Example 7 Round off the number 75462 to four significant figures also find the absolute, relative and percentage errors. **[U.P.T.U. (CO) 2004]**

Solution The number 75462 is round off to four significant figures = 75460

Now Absolute error $(E_a) = |75462 - 75460|$

$$= 2$$

Relative error $(E_r) = \frac{E_a}{\text{number}}$

$$= \frac{2}{75462}$$

$$= 0.0000265$$

and percentage error $(E_p) = E_r \times 100$

$$= 0.0000265 \times 100$$

$$= 0.00265$$

8.5 GENERAL ERROR FORMULA

Let $u = f(x_1, x_2, x_3, \cdots, x_n)$ be a function of independent variables $x_1, x_2, x_3, \cdots, x_n$ and let the errors $\Delta x_1, \Delta x_2, \Delta x_3, \cdots, \Delta x_n$ in $x_1, x_2, x_3, \cdots, x_n$.

Then the error Δu in u is given by

$$u + \Delta u = f(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3, \cdots, x_n + \Delta x_n)$$

In order to obtain an expression for Δu, we write Taylor's expansion of the right hand side, Thus,

$$u + \Delta u = f(x_1, x_2, \ldots, x_n) + \left(\Delta x_1 \frac{\partial f}{\partial x_1} + \Delta x_2 \frac{\partial f}{\partial x_2} + \Delta x_3 \frac{\partial f}{\partial x_3} + \cdots \Delta x_n \frac{\partial f}{\partial x_n}\right)$$

$$+ \frac{1}{2!}\left[(\Delta x_1)^2 \frac{\partial^2 f}{\partial x_1^2} + (\Delta x_2)^2 \frac{\partial^2 f}{\partial x_2^2} + \cdots + (\Delta x_n)^2 \frac{\partial^2 f}{\partial x_n^2}\right] + \cdots$$

Since, the errors $\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n$ are assumed to be so small that their squares, products and highest powers can be neglected, we have

$$u + \Delta u = u + \Delta x_1 \frac{\partial f}{\partial x_1} + \Delta x_2 \frac{\partial f}{\partial x_2} + \cdots \Delta x_n \frac{\partial f}{\partial x_n}$$

or

$$\Delta u = \Delta x_1 \frac{\partial f}{\partial x_1} + \Delta x_2 \frac{\partial f}{\partial x_2} + \cdots + \Delta x_n \frac{\partial f}{\partial x_n}$$

We observe that this formula has the same form as that for the total differential of u. Thus the relative error is given by

$$E_r = \frac{\Delta u}{u}$$

$$= \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{u} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{u} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{u}$$

8.6 ERRORS IN NUMERICAL COMPUTATIONS

(i) *Error in Addition of Numbers*: Let $u = x_1 + x_2 + x_3 + \cdots + x_n$ be a function. Suppose $\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n$ be the errors in $x_1, x_2, x_3, \cdots, x_n$ respectively.

Then $$u + \Delta u = (x_1 + \Delta x_1) + (x_2 + \Delta x_2) + \cdots + (x_n + \Delta x_n)$$

$$u + \Delta u = (x_1 + x_2 + \cdots + x_n) + (\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n)$$

$\therefore$ $$\Delta u = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n$$

Thus, the relative error $(E_r) = \dfrac{\Delta u}{u}$

$$= \frac{\Delta x_1}{u} + \frac{\Delta x_2}{u} + \cdots + \frac{\Delta x_n}{u}$$

Maximum relative error is

$$\left|\frac{\Delta u}{u}\right| \le \left|\frac{\Delta x_1}{u}\right| + \left|\frac{\Delta x_2}{u}\right| + \left|\frac{\Delta x_3}{4}\right| + \cdots + \left|\frac{\Delta x_n}{u}\right|$$

(ii) *Error in Subtraction of Numbers*: Let $u = x_1 - x_2$, then

$$u + \Delta u = (x_1 + \Delta x_1) - (x_2 + \Delta x_2)$$

$$u + \Delta u = (x_1 - x_2) + (\Delta x_1 - \Delta x_2)$$

$\therefore$ $$\Delta u = \Delta x_1 - \Delta x_2$$

Now, absolute error is $|\Delta u| = |\Delta x_1 - \Delta x_2|$

$$\leq |\Delta x_1| + |\Delta x_2|$$

Absolute relative error is

$$\left|\frac{\Delta u}{u}\right| \leq \left|\frac{\Delta x_1}{u}\right| + \left|\frac{\Delta x_2}{u}\right|$$

(iii) *Error in division of numbers*: Let $u = \frac{x_1}{x_2}$

$$\Delta u = \frac{\partial u}{\partial x_1}\Delta x_1 + \frac{\partial u}{\partial x_2}\Delta x_2$$

$$\therefore \quad \frac{\Delta u}{u} = \frac{\Delta x_1}{u}\frac{\partial u}{\partial x_1} + \frac{\Delta x_2}{u}\frac{\partial u}{\partial x_2} \tag{1}$$

Now, $\frac{\partial u}{\partial x_1} = \frac{1}{x_2}$ and $\frac{\partial u}{\partial x_2} = -\frac{x_1}{x_2^2}$

Equation (1) becomes

$$\frac{\Delta u}{u} = \frac{\Delta x_1}{\frac{x_1}{x_2}}\cdot\left(\frac{1}{x_2}\right) + \frac{\Delta x_2}{\frac{x_1}{x_2}}\cdot\left(-\frac{x_1}{x_2^2}\right)$$

$$\frac{\Delta u}{u} = \frac{\Delta x_1}{x_1} - \frac{\Delta x_2}{x_2}$$

Thus, the relative error is

$$\left|\frac{\Delta u}{u}\right| \leq \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right|$$

(iv) *Error in Product of Numbers*: Let $u = x_1 \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_n$ be a function.

Then $$\Delta u = \frac{\partial u}{\partial x_1}\Delta x_1 + \frac{\partial u}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial u}{\partial x_n}\Delta x_n \tag{2}$$

Now,

$$\frac{\partial u}{\partial x_1} = x_2 \cdot x_3 \cdot \cdots \cdot x_n$$

$$\frac{\partial u}{\partial x_2} = x_1 \cdot x_3 \cdot \cdots \cdot x_n$$

$$\frac{\partial u}{\partial x_3} = x_1 \cdot x_2 \cdot x_4 \cdot \cdots \cdot x_n$$

$$\vdots \qquad \vdots$$

$$\frac{\partial u}{\partial x_n} = x_1 \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_{n-1}$$

Putting the values of $\dfrac{\partial u}{\partial x_1}, \dfrac{\partial u}{\partial x_2}, \cdots, \dfrac{\partial u}{\partial x_n}$ in Eq. (2), we get

$$\Delta u = \Delta x_1 \cdot (x_2\, x_3 \cdots x_n) + \Delta x_2 (x_1 \cdot x_3 \cdots\cdots x_n) + \cdots + \Delta x_n (x_1 x_2 x_3 \cdots x_{n-1})$$

$$\therefore \qquad \frac{\Delta u}{u} = \frac{\Delta x_1}{u}(x_2 \cdot x_3 \cdots\cdots x_n) + \frac{\Delta x_2}{u}(x_1 \cdot x_3 \cdots\cdots x_n) + \cdots + \frac{\Delta x_n}{u}(x_1 x_2 \cdots x_{n-1})$$

or
$$\frac{\Delta u}{u} = \frac{\Delta x_1 \cdot (x_2\, x_3 \cdots x_n)}{(x_1\, x_2\, x_3 \cdots x_n)} + \frac{\Delta x_2 \cdot (x_1\, x_3 \cdots x_n)}{(x_1\, x_2\, x_3 \cdots x_n)} + \cdots + \frac{\Delta x_n (x_1 x_2 \cdots x_{n-1})}{(x_1 x_2 \cdots x_{n-1} . x_n)}$$

or
$$\frac{\Delta u}{u} = \frac{\Delta x_1}{x_1} + \frac{\Delta x_2}{x_2} + \cdots + \frac{\Delta x_n}{x_n}$$

$\therefore$ The maximum relative error is

$$\left|\frac{\Delta u}{u}\right| \le \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right| + \cdots + \left|\frac{\Delta x_n}{x_n}\right|$$

8.7 ERROR IN A SERIES APPROXIMATION

The Taylor's series expansion for $f(x)$ at $x = a$ with a remainder after n terms is given.

$$f(x) = f[(a) + (x-a)] = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n(x),$$

where $R_n(x) = \dfrac{(x-a)^n}{n!} f^n(\theta);\ a < \theta < x$

The last term $R_n(x)$, which is called the remainder term. If the series is convergent then $R_n(x) \to 0$ as $n \to \infty$. Thus, if $f(x)$ is approximated by the first n terms of this series, then the maximum error committed in this approximation is given by the remainder term.

Example 8 Compute the percentage error in the time period $T = 2\pi\sqrt{l/g}$ for $l = 1$ M, if the error in measurement of $l = 0.01$. **[G.E.U. 2015]**

Solution We have

$$T = 2\pi\sqrt{l/g} \tag{3}$$

Taking log both sides of Eq. (3), we get

$$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

Differentiating both sides, we obtain

$$\frac{1}{T}\Delta T = \frac{1}{2l} \cdot \Delta l$$

$\therefore$ Percentage error is

$$\frac{1}{T}\Delta T \times 100 = \frac{1}{2l}\Delta l \times 100$$

$$= \frac{1}{2\times 1}\times 0.01\times 100 = 0.05\%$$

Example 9 Given that $u = \frac{5xy^2}{z^3}$, Δx, Δy and Δz denote the errors in x, y and z respectively, such that $x = y = z = 1$ and $\Delta x = \Delta y = \Delta z = 0.001$. Find the maximum relative error in u. **[U.P.T.U. 2007]**

Solution Given
$$u = \frac{5xy^2}{z^3} \tag{4}$$

We know that

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z \tag{5}$$

From Eq. (4), we have

$$\frac{\partial u}{\partial x} = \frac{5y^2}{z^3}, \frac{\partial u}{\partial y} = \frac{10\,xy}{z^3} \text{ and } \frac{\partial u}{\partial z} = \frac{-15xy^2}{z^4}$$

Taking above values in Eq. (5), we get

$$\Delta u = \frac{5y^2}{z^3}\Delta x + \frac{10xy}{z^3}\Delta y + \frac{-15xy^2}{z^4}\Delta z$$

$\therefore$
$$(\Delta u)_{\max} = \left|\frac{5y^2}{z^3}\Delta x\right| + \left|\frac{10\,xy}{z^3}\Delta y\right| + \left|\frac{-15xy^2}{z^4}\Delta z\right|$$

$$= |5\times 0.001| + |10\times 0.001| + |-15\times 0.001|$$

$$= 0.005 + 0.010 + 0.015$$

$$= 0.030$$

The maximum relative error is given by

$$(E_r)_{\max} = \frac{(\Delta u)_{\max}}{u} = \frac{0.030}{5} = 0.006$$

Example 10 Given that $u = \frac{4x^2y^3}{z^4}$ and the errors in x, y and z be 0.001, compute the maximum relative error in u when $x = y = z = 1$. **[U.P.T.U. 2003]**

Solution Do same as above example,

$$(E_r)_{\max} = 0.009$$

Example 11 The error in the measurement of the area of a circle is not allowed to exceed 0.1%. How accurately should the diameter be measured?

Solution The area of a circle with diameter (d) is

$$A = \frac{\pi d^2}{4}$$

Taking log both sides, we get

$$\log A = \log \pi + 2\log d - \log 4$$

Differentiating both sides, we have

$$\frac{\Delta A}{A} = \frac{2\Delta d}{d}$$

Percentage error in diameter is

$$\frac{\Delta A}{A} \times 100 = 2\frac{\Delta d}{d} \times 100$$

$$\frac{\Delta d}{d} \times 100 = \frac{1}{2}\left(\frac{\Delta A}{A} \times 100\right) = \frac{1}{2} \times 0.1 = 0.05$$

Example 12 In a ΔABC, $a = 30$ cm, $b = 80$ cm and $\angle B = 90°$. Find the maximum possible error in the computed value of area of ΔABC, if possible errors in a and b are $\frac{1}{3}\%$ and $\frac{1}{4}\%$ respectively.

Solution Given $a = 30$ cm, $b = 80$ cm

$$\Delta a = \frac{a}{100} \times \frac{1}{3} = \frac{30}{300} = 0.1$$

$$\Delta b = \frac{b}{100} \times \frac{1}{4} = \frac{80}{400} = 0.2$$

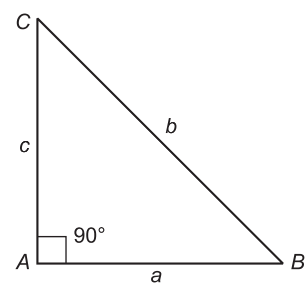


Fig. 8.1

Area of ΔABC, is given by

$$A = \frac{1}{2}(a \times c)$$

or $$A = \frac{a}{2}\sqrt{(b^2 - a^2)}$$

$$\frac{\partial A}{\partial a} = \frac{1}{2}\frac{b^2 - 2a^2}{\sqrt{(b^2 - a^2)}}, \quad \frac{\partial A}{\partial b} = \frac{1}{2}\frac{ab}{\sqrt{(b^2 - a^2)}}$$

$$\Delta A = \frac{\partial A}{\partial a}\Delta a + \frac{\partial A}{\partial b}\Delta b$$

$\therefore$ $$|\Delta A| = \left|\frac{\partial A}{\partial a}\Delta a\right| + \left|\frac{\partial A}{\partial b}\Delta b\right|$$

$$= \left|\frac{b^2 - 2a^2}{2\sqrt{b^2 - a^2}}\Delta a\right| + \left|\frac{ab}{2\sqrt{b^2 - a^2}}\Delta b\right|$$

$$= \left|\frac{80^2 - 2\times 30^2}{2\sqrt{80^2 - 30^2}} \times 0.1\right| + \left|\frac{30\times 80}{2\sqrt{80^2 - 30^2}} \times 0.2\right|$$

$$= 6.33748$$

Example 13 Find the number of terms of the exponential series such that their sum gives the value of e^x correct to six decimal places at $x = 1$.

Solution We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + R_n(x) \tag{6}$$

where $$R_n(x) = \frac{x^n}{n!} e^{\theta}; \; 0 < \theta < x$$

$\therefore$ The maximum absolute error at $\theta = x$ is

$$= \frac{x^n}{n!} e^x$$

and the maximum relative error = $\dfrac{x^n}{n!}$

Thus, $$(E_r)_{\max} \text{ at } (x = 1) = \frac{1}{n!}$$

For a six decimal accuracy at $x = 1$, we have

$$\frac{1}{n!} < \frac{1}{2} \times 10^{-6}$$

or $$n! > 2 \times 10^6 \Rightarrow n = 10$$

Hence, we need 10 terms of the series in Eq. (6) in order that its sum is correct to 6 decimal places.

Example 14 Use the series of $\log_e\left(\dfrac{1+x}{1-x}\right) = 2\left[x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right]$ to compute the value of log (1.2) correct to 7 decimal places and find the number of terms retained. **[U.P.T.U. 2002]**

Solution Given that

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1}\right] + R_n(x)$$

If we retain n terms then

$$R_n(x) = 2\frac{x^{2n+1}}{2n+1} \log\left(\frac{1+\theta}{1-\theta}\right); \; 0 < \theta < x$$

Maximum absolute error at $\theta = x$

$$= 2\frac{x^{2n+1}}{2n+1} \cdot \log\left(\frac{1+x}{1-x}\right)$$

and maximum relative error

$$= \frac{2}{2n+1}x^{2n+1}$$

Let $\qquad f(x) = \frac{1+x}{1-x} = 1.2 \Rightarrow 1+x = 1.2(1-x)$

or $\qquad x = \frac{0.2}{2.2} = \frac{1}{11}$

Hence, $\quad (E_r)_{\max}$ at $\left(x = \frac{1}{11}\right) = \frac{2}{2n+1}\left(\frac{1}{11}\right)^{2n+1}$

For accuracy of seven decimal places,

$$\frac{2}{2n+1}\left(\frac{1}{11}\right)^{2n+1} < \frac{1}{2}\times 10^{-7}$$

$$(2n+1)\,(11)^{2n+1} > 4\times 10^{7}$$

which gives $n > 2$

Thus, retaining the first three terms of the given series, we get

$$\log(1.2) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right)$$

$$= 0.1823215; \text{ at } x = \frac{1}{11}$$

EXERCISE 8.1

1. The true value of x is $\frac{10}{3}$ and approximate value is 3.33. Find the absolute and relative errors.

2. Round off the following numbers to 4 significant digits.
   (i) 48.3668 (ii) 8.4155
   (iii) 0.80012 (iv) 0.0049125
   (v) 0.00020215

3. Consider the number $\frac{2}{3}$. Its floating point representation rounded to 5 decimal places is 0.66667. Find out to how many decimal places the approximate value of $\frac{2}{3}$ is accurate?

4. Round the number $x = 2.2554$ to 3 significant digits. Find the absolute and relative error.
5. What is the relative error in the computation of $x - y$, where $x = 0.3721448693$ and $y = 0.3720214371$ with 5 decimal place of accuracy?
6. Compute the relative error in computations $x - y$ for $x = 12.05$, $y = 8.02$ having absolute errors $\Delta x = 0.005$ and $\Delta y = 0.001$.
7. Find the number of correct digits in the number x given its relative error is 0.3 and $x = 386.4$.
8. If $u = 2x^6 - 5x$, find the percentage error in u at $x = 1$, if the error in u is 0.05.

9. The function $f(x) = \cos x$ can be expanded as $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots +$. Compute the number of terms required to estimate $\cos\left(\frac{\pi}{4}\right)$ so that the result is correct to at least two significant figures.

10. If $u = \frac{4xy^2}{z^3}$ and errors in x, y, z be 0.001, show that the maximum relative error at $x = y = z = 1$ is 0.006.

11. If $u = \frac{4x^2y^3}{z^4}$ and errors in x, y, z be 0.001, find the maximum relative error in u, when $x = y = z = 1$.

12. If $r = 3h(h^6 - 2)$, find the percentage error in r at $h = 1$, if the percentage error in h is 5.

13. If $\sqrt{29} = 5.385$ and $\sqrt{11} = 3.317$, correct to four significant digits. Find the relative error in their sum and difference.

14. Calculate the value of $x - x\cos\theta$ correct to 3 significant figures if $x = 10.2$ cm and $\theta = 5°$. Find the permissible errors in x and θ.

15. Determine the number of terms required in the series for $\log_e(1 + x)$ to evaluate log (1.2) correct to six decimal places.

16. If $u = 10x^3y^2z^2$ and errors in x, y, z are 0.03; 0.01, 0.02 respectively at $x = 3$, $y = 1$ and $z = 2$. Compute absolute and percentage relative error in u.

Answers

1. $E_a = 0.003333$, $E_r = 0.000999$
2. (i) 48.37 (ii) 8.416 (iii) 0.8001 (iv) 0.004912 (v) 0.0002022
3. $k = 5$
4. $E_a = 0.0054$, $E_r = 0.0024$
5. $E_r = 1.3 \times 10^{-5}$
6. 0.00029
7. 1
8. –11.667%
9. $n = 3$
11. 0.009
12. 25%
13. 1.149×10^{-4}, 4.835×10^{-4}
14. 0.0656, 0.00028
15. $n = 10$
16. 75.6, 7%

8.8 FINITE DIFFERENCES

The calculus of finite differences is an interesting topic and has wide applications in various fields. Using this concept, we deal with the changes that take place in the value of the function, the dependent variable due to finite changes in the independent variable.

Suppose a table of values (x_i, y_i), $i = 1, 2, 3, \dots, n$, of any function $y = f(x)$, the values of x being equally spaced, i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \dots, n$. We are required to obtain the values of $f(x)$ for some intermediate values of x or to obtain the derivative of $f(x)$ for some x in the range $x_0 \le x \le x_n$. Following three types of differences are found useful:

(i) *Forward Differences*: If a function $y = f(x)$ is tabulated for the equally spaced arguments $x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh$ giving the functional values $y_0, y_1, y_2, \dots, y_n$.
The constant difference between two consecutive arguments (x) is called the interval of differencing and is denoted by h.
The forward difference operator Δ is defined as

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2 \text{ and so on } \Delta y_n = y_{n+1} - y_n$$

$\Delta y_0, \Delta y_1, \Delta y_2, \cdots, \Delta y_n$ are called first forward differences.

The differences of the first forward differences are called 2nd forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, \cdots, \Delta^2 y_n$, defined as

$$\begin{aligned}\Delta^2 y_0 &= \Delta(\Delta y_0)\\ &= \Delta(y_1 - y_0)\\ &= \Delta y_1 - \Delta y_0\\ &= (y_2 - y_1) - (y_1 - y_0)\\ &= y_2 - y_1 - y_1 + y_0\\ \Delta^2 y_0 &= y_2 - 2y_1 + y_0\end{aligned}$$

Similarly, we can define the 3rd forward differences, 4th forward differences, etc.

Thus $\Delta^3 y_0 = \Delta^2(\Delta y_0) = \Delta^2(y_1 - y_0)$

$$\begin{aligned}&= \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)\\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

and $\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$

$$\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

In general,

$$\Delta^n y_n = \Delta^{n-1} y_{n+1} - \Delta^{n-1} y_n$$

The following table shows how the forward differences of all orders can be formed.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

(ii) *Backward Differences*: If a function $y = f(x)$ is tabulated for the equally spaced arguments $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ giving the functional values $y_0, y_1, y_2, \ldots, y_n$. The backward difference operator ∇ is defined as

$$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \cdots, \nabla y_n = y_n - y_{n-1}$$

are called the first backward difference operator.

In a similar way, we can define the backward differences of higher orders.

Thus, $\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1)$

$$= \nabla y_2 - \nabla y_1$$

$$= (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$$\nabla^3 y_3 = y_3 - 3y_2 + 3y_1 - y_0, \text{ etc.}$$

The following table shows how the backward differences of all orders can be formed:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
x_4	y_4	∇y_4			

(iii) *Central Differences*: Some times it is a very useful to employ another system of differences known as central differences. The central difference operator δ(delta) is defined by the relations

$$y_1 - y_0 = \delta y_{\frac{1}{2}}, \; y_2 - y_1 = \delta y_{3/2}, \cdots$$

$$y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}$$

Similarly, the central differences of higher orders are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1$$

$$\delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2; \cdots$$

$$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

Central Difference Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta_{y_{1/2}}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta_{y_{3/2}}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta_{y_{5/2}}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta_{y_{7/2}}$			
x_4	y_4				

It is clear that, when we use forward, backward or central differences.
Thus, we have

$$\Delta y_0 = \nabla y_1 = \delta y_{1/2}$$

$$\Delta^3 y_2 = \nabla^3 y_5 = \delta^3 y_{7/2} \text{ etc.}$$

(iv) *Shift operator*: The shift operator E is defined as

$$Ef(x) = f(x+h) \text{ or } Ey_x = y_{x+h}$$

$$E^2 f(x) = f(x+2h) \text{ or } E^2 y_x = y_{x+2h}$$

$$\vdots \quad \vdots$$

$$E^n f(x) = f(x+nh) \text{ or } E^n y_x = y_{x+nh}$$

The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h) \text{ or } E^{-1} y_x = y_{x-h}$$

$$E^{-2} f(x) = f(x-2h) \text{ or } E^{-2} y_x = y_{x-2h}$$

$$\vdots \quad \vdots$$

$$E^{-n} f(x) = f(x-nh) \text{ or } E^{-n} y_x = y_{x-nh}$$

(v) *Averaging operator*: The averaging operator μ(mu) is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

Remark

1. Pascal's triangle: It is start from 11 and is defined as

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

and so on.

The forward and backward differences of different orders are defined as

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x) \text{ and so on}$$

$$\Delta^5 f(x) = f(x+5h) - 5f(x+4h) + 10f(x+3h) - 10f(x+2h) + 5f(x+h) - f(x)$$

Now, $\nabla f(x) = f(x) - f(x-h)$

$$\nabla^2 f(x) = f(x) - 2f(x-h) + f(x-2h)$$

$$\nabla^3 f(x) = f(x) - 3f(x-2h) + 3f(x-3h) - f(x-4h) \text{ etc.}$$

2. In the difference calculus, the operators Δ, ∇, δ and μ can be expressed in terms of shift operator E.

8.9 RELATION BETWEEN OPERATORS

(i) *Relation between* Δ, ∇ and E

We know that

$$\Delta y_x = y_{x+h} - y_x$$

$$= Ey_x - y_x$$

$$\Delta y_x = (E-1)y_x$$

or $\Delta \equiv E - 1$

or $E = 1 + \Delta$ (7)

Now, $\nabla y_x = y_x - y_{x-h}$

$$= y_x - E^{-1}y^x$$

or $\nabla = 1 - E^{-1}$

or $E = \dfrac{1}{1-\nabla}$ (8)

From Eqs (7) and (8), we have

$$1 + \Delta = \frac{1}{1-\nabla}$$

or $\Delta = \dfrac{1}{1-\nabla} - 1 = \dfrac{\nabla}{1-\nabla}$

$\Rightarrow$ $\Delta = \dfrac{\nabla}{1-\nabla}$

(ii) *Relation between* δ and E

We know that

$$\delta y_x = y_{x+h/2} - y_{x-h/2}$$

$$= E^{1/2}y_x - E^{-1/2}y_x$$

$$\delta y_x = (E^{1/2} - E^{-1/2})y_x$$

or $$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

(iii) *Relation between* μ and E

We know that

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{\frac{1}{2}}f(x) + E^{-\frac{1}{2}}f(x)\right]$$

or $$\mu f(x) = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]f(x)$$

or $$\mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$$

(iv) *Relation between* Δ, ∇, E and δ

We know that

$$E(\Delta f(x)) = E(f(x) - f(x-h))$$

$$= Ef(x) - Ef(x-h)$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

$\Rightarrow$ $$E\nabla = \Delta \tag{9}$$

Now $$\nabla(Ef(x)) = \nabla f(x+h)$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

$\Rightarrow$ $$\nabla E = \Delta \tag{10}$$

$$\delta E^{\frac{1}{2}}f(x) = \delta f(x+h/2) = f(x+h) - f(x)$$

$$= \Delta f(x)$$

$\Rightarrow$ $$\delta E^{1/2} = \Delta \tag{11}$$

From Eqs (9), (10) and (11), we obtain,

$$\Delta = E\nabla = \nabla E = \delta E^{1/2}$$

Similarly, we can prove that

(i) $E = E^{hD}$

(ii) $\Delta - \nabla = \delta^2$

(iii) $\Delta = \dfrac{\delta^2}{2} + \delta\sqrt{1+\dfrac{\delta^2}{4}}$

(iv) $\mu\delta = \dfrac{1}{2}\Delta E^{-1} + \dfrac{\Delta}{2}$

(v) $\Delta \cdot \nabla = \Delta - \nabla$

Theorem (Differences of a Polynomial)

The n^{th} differences of a polynomial of n^{th} degree are constant and all higher order differences are zero when the values of the independent variable are at equal interval.

[U.P.T.U. 2002, 2004, M.C.A. 2004]

***Proof*:**

Let the rational integral function of the n^{th} degree in x be.

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l$$

where n is a positive integer, a, b, c, …, l are constants and $a \neq 0$.

$$\therefore \quad f(x+h) = a(x+h)^n + b(x+h)^{n-1} + c(x+h)^{n-2} + \cdots + k(x+h) + l$$

Now,
$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \cdots + kh\end{aligned}$$

or
$$\Delta f(x) = anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + k'x + l' \tag{12}$$

where $b', c', ..., l'$ are new constants coefficients.

Thus, the first difference of a rational integral function/polynomial of n^{th} degree is a rational integral function of degree $(n – 1)$

The 2^{nd} difference of $f(x)$ is

$$\begin{aligned}\Delta^2 f(x) &= \Delta[\Delta f(x)]\\ &= \Delta[f(x+h) - f(x)]\\ &= \Delta f(x+h) - \Delta f(x)\end{aligned}$$

$$\Delta^2 f(x) = an(n-1)h^2 x^{n-2} + b''x^{n-3} + \cdots + k'' \text{ [using Eq. (12)]}$$

Thus, the 2^{nd} difference is a polynomial of degree $(n – 2)$.

By continuing in this manner, we arrive at the result.

$$\begin{aligned}\Delta^n f(x) &= \Delta^n (ax^n)\\ &= a[n(n-1)(n-2)...1]h^n \cdot x^{n-n}\\ &= ah^n \cdot n!\end{aligned}$$

The n^{th} difference is constant, and all higher differences are zero, that is, the $(n + 1)^{\text{th}}$ and higher differences of a polynomial of the n^{th} degree are zero.

Remark

The converse of the above theorem is also true, i.e.

If the n^{th} differences of a tabulated function are constant when the values of the independent variable are taken at equal intervals, the function is a polynomial of degree 'n'.

Example 15 Evaluate $\Delta^3(1-x)(1-2x)(1-3x)$.

Solution

Let
$$f(x) = (1-x)(1-2x)(1-3x)$$
$$f(x) = -6x^3 + 11x^2 - 6x + 1$$

So that the polynomial $f(x)$ is of degree 3.

$$\therefore \quad \Delta^3 f(x) = \Delta^3(-6x^3 + 11x^2 - 6x + 1)$$
$$= \Delta^3(-6x^3) + \Delta^3(11x^2) - \Delta^3(-6x) + \Delta^3(1)$$
$$= (-6)(3!)$$
$$= -36$$

Example 16 Prove that $e^x = \left(\dfrac{\Delta^2}{E}\right)e^x \cdot \dfrac{Ee^x}{\Delta^2 e^x}$ the intervals of differencing being h.

[G.E.U. 2015]

Solution We know that

$$\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$$
$$\Delta^2 e^x = \Delta(\Delta e^x)$$
$$= \Delta[e^x(e^h - 1)]$$
$$= [e^{x+h} - e^x](e^h - 1)$$
$$= e^x \cdot (e^h - 1)^2$$
$$Ee^x = e^{x+h}$$

Now,
$$\left(\frac{\Delta^2}{E}\right)e^x = (\Delta^2 E^{-1})e^x$$
$$= \Delta^2 e^{x-h}$$
$$= e^{-h}\Delta^2 e^x = e^{-h}e^x(e^h - 1)^2$$
$$\text{R.H.S.} = e^{-h}e^x(e^h - 1)^2 \times \frac{e^{x+h}}{e^x(e^h - 1)^2}$$

$$= e^{x-h+x+h-x} \cdot \frac{(e^h-1)^2}{(e^h-1)^2}$$

$$= e^x$$

$$= \text{L.H.S}$$

Hence, proved.

Example 17 Construct a forward difference table for the data given below: **[Gulbarga 1993]**

x	10	20	30	40
y	1.1	2.0	4.4	7.9

Solution Forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	1.1			
		0.9		
20	2.0		1.5	
		2.4		−0.4
30	4.4		1.1	
		3.5		
40	7.9			

Thus $\Delta^3 y_{10} = -0.4$

Example 18 If $u_0 = 3$, $u_1 = 12$, $u_2 = 81$, $u_3 = 2000$, $u_4 = 100$, calculate $\Delta^4 u_0$. **[Kerala 1990]**

Solution The forward difference table is

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$
0	3				
		9			
1	12		60		
		69		1790	
2	81		1850		–7549
		1919		–5669	
3	2000		–3819		
		–1900			
4	100				

Thus $\Delta^4 u_0 = -7459$

Example 19 Evaluate $\nabla^3 \log 40$ and $\nabla^4 \log 50$ for the data given below: **[G.E.U. 2015]**

x	10	20	30	40	50
y* = log *x	1	1.3010	1.4771	1.6021	1.6990

Solution Backward difference table is given:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	1				
		0.3010			
20	1.3010		−0.1249		
		0.1761		$\boxed{0.0738}$	
30	1.4771		−0.0511		$\boxed{-0.0508}$
		0.1250		0.0230	
40	1.6021		−0.0281		
		0.0969			
50	1.6990				

From the given table, we have

$$\nabla^3 \log 40 = 0.0735 \text{ and}$$

$$\nabla^4 \log 50 = -0.0508$$

Example 20 Construct a backward difference table from the data given:

sin 30° = 0.5, sin 35° = 0.5736, sin 40° = 0.6428 and sin 45° = 0.7071. Assuming 3rd difference to be constant, find the value of sin 25°. **[G.E.U. 2012]**

Solution Backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
25	0.4225			
		0.0775		
30	0.5000		−0.0039	
		0.0736		$\boxed{-0.0005}$
35	0.5736		−0.0044	
		0.0692		−0.0005
40	0.6428		−0.0049	
		0.0643		
45	0.7071			

Since, the 3rd differences are constant

$$\nabla^3 y_{40} = -0.0005$$

$$\nabla^2 y_{40} - \nabla^2 y_{35} = -0.0005$$

$$-0.0044 - \nabla^2 y_{35} = -0.0005$$

or $$\nabla^2 y_{35} = -0.0039$$

$$\nabla y_{35} - \nabla y_{30} = -0.0039$$

$$0.0736 - \nabla y_{30} = -0.0039 \Rightarrow \nabla y_{30} = 0.0775$$

Again $$y_{30} - y_{25} = 0.0775 \Rightarrow y_{25} = -0.0775 + 0.5$$

$$= 0.4225$$

$\therefore$ $$\sin 25° = 0.4225$$

Example 21 With usual notation, prove that,

$$\Delta^n\left(\frac{1}{x}\right)=(-1)^n\frac{n!h^n}{x(x+h)(x+2h)...(x+nh)}$$

Solution

$$\Delta^n\left(\frac{1}{x}\right)=\Delta^{n-1}\,\Delta\left(\frac{1}{x}\right)$$

$$=\Delta^{n-1}\left[\frac{1}{x+h}-\frac{1}{x}\right]=\Delta^{n-1}\left[\frac{-h}{x(x+h)}\right]$$

$$=(-h)\Delta^{n-2}\,\Delta\left[\frac{1}{x(x+h)}\right]$$

$$=(-1)\,\Delta^{n-2}\left[\Delta\left(\frac{1}{x}-\frac{1}{x+h}\right)\right]$$

$$=(-1)\,\Delta^{n-2}\left[\left(\frac{1}{x+h}-\frac{1}{x}\right)-\left(\frac{1}{x+2h}-\frac{1}{x+h}\right)\right]$$

$$=(-1)\,\Delta^{n-2}\left[\frac{2}{x+h}-\frac{1}{x}-\frac{1}{x+2h}\right]$$

$$=(-1)\Delta^{n-2}\left[\frac{-2h^2}{x(x+h)(x+2h)}\right]$$

$$=(-1)^2\,\Delta^{n-2}\left[\frac{2!\,h^2}{x(x+h)(x+2h)}\right]$$

or
$$=(-1)^3\;\Delta^{n-3}\left[\frac{3!\,h^3}{x(x+h)(x+2h)(x+3h)}\right]$$

$$\vdots \qquad \vdots$$

$$\Delta^n\left(\frac{1}{x}\right)=(-1)^n\frac{n!h^n}{x(x+h)(x+2h)...(x+nh)}$$

Hence, proved.

Example 22 Evaluate $\Delta^{10}(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)$ if $h=1$.

[U.P.T.U. (A.g.) 2004, G.E.U. 2011]

Solution Here $f(x)=(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)$

$$=abcd\;x^{10}-bcd\;x^9-acd\;x^8+(cd-abd)\,x^7$$
$$+(bd-abc)x^6+(bc+ad)x^5+(ac-d)x^4$$
$$+abx^3-(b+c)x^2-ax+1$$

$\therefore\quad \Delta^{10}f(x)=abcd(10)!$ (since, maximum degree of polynomial $f(x)$ is 10 and coefficient of x^{10} is $abcd$).

Example 23 Calculate y_6, given that $y_0 = -3$, $y_1 = 6$, $y_2 = 8$, $y_3 = 12$; the 3rd difference being constant.

Solution Prepare the difference table from the given data:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	–3			
		9		
1	6		–7	
		2		9
2	8		2	
		4		
3	12			

Now
$$y_{0+6} = E^6 y_0 = (1+\Delta)^6 y_0$$
$$= (1+6\Delta+15\Delta^2+20\Delta^3)y_0$$
[∵ $\Delta^3 y_0 = 9$ and the higher order differences will be zero]
$$= y_0 + 6\Delta y_0 + 15\Delta^2 y_0 + 20\Delta^3 y_0$$
$$= -3 + 6\times 9 + 15 - 7 + 20\times 9$$
$$= 126$$

Example 24 Evaluate $\Delta^n[\sin(ax+b)]$.

Solution
$$\Delta\sin(ax+b) = \sin[a(x+h)+b] - \sin(ax+b)$$
$$= 2\sin\frac{ah}{2}\cos\left[a\left(x+\frac{h}{2}\right)+b\right]$$
$$= 2\sin\frac{ah}{2}\sin\left[ax+b+\frac{ah+\pi}{2}\right]$$

Now,
$$\Delta^2\sin(ax+b) = \Delta\left[2\sin\frac{ah}{2}\sin\left(ax+b+\frac{ah+\pi}{2}\right)\right]$$
$$= 2\sin\frac{ah}{2}\cdot 2\sin\frac{ah}{2}\sin\left[ax+b+\frac{ah+\pi}{2}+\frac{ah+\pi}{2}\right]$$
$$= \left(2\sin\frac{ah}{2}\right)^2\sin\left[ax+b+2\left(\frac{ah+\pi}{2}\right)\right]$$

Continuing in the same manner, we get
$$\Delta^3\sin(ax+b) = \left(2\sin\frac{ah}{2}\right)^3\sin\left[ax+b+3\left(\frac{ah+\pi}{2}\right)\right]$$
$$\vdots \qquad \vdots$$
$$\Delta^n\sin(ax+b) = \left(2\sin\frac{ah}{2}\right)^n\sin\left[ax+b+n\left(\frac{ah+\pi}{2}\right)\right]$$

Example 25 Evaluate $\Delta^n[\cos(ax+b)]$.

Solution Do same as in Example 24,

$$\Delta^n \cos(ax+b) = \left(2\sin\frac{ah}{2}\right)^n \cos\left[ax+b+n\left(\frac{ah+\pi}{2}\right)\right]$$

Example 26 Prove that $y_n = (1+\Delta)^n y_0$

$$= y_0 + {}^nC_1\Delta y_0 + {}^nC_2\Delta^2 y_0 + \cdots + \Delta^n y_0$$

Solution Let $y_0, y_1, y_2, ..., y_n$ be the entries of $y = f(x)$ at $x = x_i; i = 0,1,2,...,n$.

From the definition of forward difference, we have

$$\Delta y_0 = y_1 - y_0 \Rightarrow y_1 = y_0 + \Delta y_0$$

or $$y_1 = (1+\Delta)y_0 \tag{13}$$

$$\Delta y_1 = y_2 - y_1 \Rightarrow y_2 = y_1 + \Delta y_1$$

$$y_2 = (1+\Delta)y_1$$

or $$y_2 = (1+\Delta)(1+\Delta)y_0 \quad \text{[using Eq. (13)]} \tag{14}$$

$$\Delta y_2 = y_3 - y_2 \Rightarrow y_3 = y_2 + \Delta y_2$$

$$= (1+\Delta)y_2$$

$$y_3 = (1+\Delta)^3 y_0 \qquad \text{[using Eq. (14)]} \tag{15}$$

$$y_4 = (1+\Delta)^4 y_0 \qquad \text{[using Eq. (15)]} \tag{16}$$

Continuing in the same manner, we get

$$y_n = (1+\Delta)^n y_0$$

$$= [1 + {}^nC_1\Delta + {}^nC_2\Delta^2 + {}^nC_3\Delta^3 + \cdots + \Delta^n]y_0$$

or $$y_n = y_0 + {}^nC_1\Delta y_0 + {}^nC_2\Delta^2 y_0 + {}^nC_3\Delta^3 y_0 + \cdots + \Delta^n y_0$$

Hence, proved.

8.10 FACTORIAL NOTATION

The product of factors of which the first factor is x and the successive factors decrease by a constant difference is called a factorial function or polynomial and is denoted by "$x^{(n)}$, n being a positive integer and is read as x raised to the power n factorial". In general, the interval of differencing is h.

The factorial polynomial $x^{(n)}$ is defined as

$$x^{(n)} = x(x-h)(x-2h)...[x-(n-1)h]$$

In particular

$$x^{(0)} = 1, x^{(1)} = x, x^{(2)} = x(x-h),$$

$$x^{(3)} = x(x-h)(x-2h), \text{ etc.}$$

Differences of $x^{(n)}$

The first difference of $x^{(n)}$ is

$$\begin{aligned}
\Delta x^{(n)} &= (x+h)^{(n)} - x^{(n)} \\
&= (x+h)(x)(x-h)(x-2h)\ldots[x-(n-2)h] - x(x-h)(x-2h)\ldots[x-(n-1)h] \\
&= x(x-h)(x-2h)\ldots[x-(n-2)h]\,[(x+h)-(x-(n-1)h)] \\
&= x(x-h)(x-2h)\ldots[x-(n-2)h]\cdot nh \\
\Delta x^{(n)} &= nhx^{(n-1)}
\end{aligned}$$

Similarly,
$$\begin{aligned}
\Delta^2 x^{(n)} &= \Delta[\Delta x^{(n)}] \\
&= \Delta[nhx^{(n-1)}] \\
&= n(n-1)h^2 x^{(n-2)}
\end{aligned}$$

Continuing, this process r times, we have

$$\Delta^r\, x^{(n)} = n(n-1)(n-2)\ldots(n-r+1)h^r\, x^{(n-r)},$$

where r is a positive integer and $n > r$.

In particular, if $h = 1$, then

$$\Delta^n x^{(n)} = n! \text{ and } \Delta^{n+1} x^{(n)} = 0$$

8.11 RECIPROCAL FACTORIAL

If h is the interval of differencing, then the reciprocal factorial function $x^{(-n)}$ is defined as

$$\begin{aligned}
x^{(-n)} &= \frac{1}{(x+h)(x+2h)\ldots(x+nh)} \\
&= \frac{1}{(x+nh)^{(n)}}
\end{aligned}$$

In particulars, $x^{(-1)} = \dfrac{1}{(x+h)}$, $x^{(-2)} = \dfrac{1}{(x+2h)^{(2)}}$

or $$x^{(-2)} = \frac{1}{(x+h)(x+2h)}$$

Differences of a reciprocal factorial

$$\begin{aligned}
\Delta x^{(-n)} &= (x+h)^{(-n)} - x^{(-n)} \\
&= \frac{1}{(x+2h)(x+3h)\ldots[x+(n+1)h]} - \frac{1}{(x+h)(x+2h)\ldots[x+(nh)]} \\
&= \frac{(x+h)-[x+(n+1)h]}{(x+h)(x+2h)..(x+nh)[x+(n+1)h]} \\
&= -nhx^{(-n-1)}
\end{aligned}$$

Similarly,

$$\Delta^2 x^{(-n)} = (-n)(-n-1)h^2 x^{(-n-2)}$$
$$= (-1)^2 n(n+1)h^2 x^{(-n-2)}$$

and in general

$$\Delta^r x^{(-n)} = (-1)^r n(n+1)(n+2)...(n+r-1)h^r \ x^{(-n-r)}; n > r$$

8.12 EXPRESSION OF ANY POLYNOMIAL *f*(*x*) IN FACTORIAL NOTATION

Let $f(x)$ be a polynomial of degree n, that is

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$$

where $a_0, a_1, a_2, ..., a_n$ are constants and $a_0 \neq 0$.

Thus, the polynomial $f(x)$ can be expressed in the factorial notation as

$$f(x) \equiv A_0 x^{(n)} + A_1 x^{(n-1)} + A_2 x^{(n-2)} + \cdots + A_{n-1} x^{(1)} + A_n x^{(0)} \tag{17}$$

Our aim is to find the values of $A_0, A_1, A_2, \ldots, A_{n-1}, A_n$ by applying the following process.

Dividing the R.H.S. of Eq. (17) by x, the remainder is A_n and dividing the quotient again by $(x - h)$, the remainder is A_{n-1} and then dividing the quotient again by $(x - 2h)$, the remainder is A_{n-2} and so on. After getting $A_n, A_{n-1}, A_{n-2}, A_0$ putting these values in Eq. (17), we obtain the required expression in factorial form.

Example 27 Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$.

Solution Let $f(x)$ be the required function given: $\Delta f(x) = 2x^3 + 3x^2 - 5x + 4$

Write $\Delta f(x)$ in factorial notation form, we have

$$2x^3 + 3x^2 - 5x + 4 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D \tag{18}$$

Using method of synthetic division, we divide by $x, (x-1), (x-2)$, successively, then,

1	2	3	–5	4 = D
	0	2	5	
2	2	5	0 = C	
	0	4		
3	2	9 = B		
	0			
	2 = A			

Putting the values of A, B, C and D in Eq. (18), we get

$$\Delta f(x) = 2x^{(3)} + 9x^{(2)} + 4 \tag{19}$$

Integrate both sides of Eq. (19), we obtain

$$f(x) = 2\frac{x^{(4)}}{4} + 9\frac{x^{(3)}}{3} + 4x^{(1)} + k$$

$$= \frac{1}{2}[x(x-1)(x-2)(x-3)] + [x(x-1)(x-2) + 4x] + k$$

Example 28 Express $f(x) = x^4 - 12x^2 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence, show that $\Delta^5 f(x) = 0$.

Solution Let $f(x) = Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$ (20)

Using the method of synthetic division, we divide by $x, (x-1), (x-2)$ and $(x-3)$, then

1	1	–12	24	–30	$9 = E$
	0	1	–11	13	
2	1	–11	13	$-17 = D$	
	0	2	–18		
3	1	–9	$-5 = C$		
	0	3			
4	1	$-6 = B$			
	0				
	$1 = A$				

Putting the values of A, B, C, D and E in Eq. (20), we get

$$f(x) = x^{(4)} - 6x^{(3)} - 5x^{(2)} - 17x^{(1)} + 9$$

$$\therefore \quad \Delta f(x) = 4x^{(3)} - 18x^{(2)} - 10x^{(1)} - 17$$

$$\Delta^2 f(x) = 12x^{(2)} - 36x^{(1)} - 10$$

$$\Delta^3 f(x) = 24x^{(1)} - 36$$

$$\Delta^4 f(x) = 24$$

and $\Delta^5 f(x) = 0$ **Hence, proved.**

Example 29 Find the missing values in the given table

x	$x_0 = 45$	$x_1 = 50$	$x_2 = 55$	$x_3 = 60$	$x_4 = 65$
y	$y_0 = 3$	$y_1 = ?$	$y_2 = 2$	$y_3 = ?$	$-24 = y_4$

Solution

Method-I

The total entries of y are 5, but 3 entries y_0, y_2, y_4 are given; so we assume the 3rd difference is zero.

$$\Delta^3 y_x = 0; x = 0,1$$

$$(E-1)^3 y_x = 0$$

$$(E^3 - 3E^2 + 3E - 1)y_x = 0$$

or $$E^3 y_x - 3E^2 y_x + 3Ey_x - y_x = 0$$

or $$y_{x+3h} - 3y_{x+2h} + 3y_{x+h} - y_x = 0 \quad (21)$$

Pascal triangle

		1		1		
	1		2		1	
1		3		3		1

Putting $x = 0$ and $h = 1$ in Eq.(21), we get

$$y_3 - 3y_2 + 3y_1 - y_0 = 0$$

or $$y_3 - 6 + 3y_1 - 3 = 0$$

or $$3y_1 + y_3 = 9 \quad (22)$$

Again putting $x = 1$ and $h = 1$ in Eq. (21), we get

$$y_4 - 3y_3 + 3y_2 - y_1 = 0$$

or $$-24 - 3y_3 - y_1 + 3 \times 2 = 0$$

or $$(y_1 + 3y_3) = -18 \quad (23)$$

Solving Eqs (22) and (23) for y_1 and y_3, we get

$$y_1 = \frac{45}{8} \text{ and } y_3 = -\frac{63}{8}$$

Method-II

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$y_0 = 3$			
		$y_1 - 3$		
50	y_1		$-2y_1 + 5$	
		$2 - y_1$		$y_3 + 3y_1 = 9$
55	$y_2 = 2$		$y_3 + y_1 = -4$	
		$y_3 - 2$		$3y_3 + y_1 = -18$
60	y_3		$-2y_3 - 22$	
		$-24 - y_3$		
65	$y_4 = -24$			

From the difference table, we get

$$3y_1 + y_3 = 9$$

$$y_1 + 3y_3 = -18$$

on solving, we obtain $y_1 = \dfrac{45}{8}$ and $y_3 = -\dfrac{63}{8}$

Example 30 Estimate the production for the year 1964 and 1966 from the following data:

Year (x)	1961	1962	1963	1964	1965	1966	1967
Production (y)	200	220	260	-	350	-	430

[G.E.U. 2015]

Solution Since 5 entries are known, we assume the 5th order differences is zero. Since two entries are unknown, we need two equations to determine.

Thus, $\Delta^5 y_0 = 0$ and $\Delta^5 y_1 = 0$

$$(E-1)^5 y_0 = 0 \text{ and } (E-1)^5 y_1 = 0$$

$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0 = 0$$

and $$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_1 = 0$$

or $$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

and $$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$$

Pascal triangle

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Putting the known values of y, we get

$$y_5 + 10y_3 = 3450 \tag{24}$$

$$-5y_5 - 10y_3 = -5010 \tag{25}$$

Solving Eqs (24) and (25) for y_3 and y_5, we obtain

$$y_3 = 306 \text{ and } y_5 = 390$$

Hence, the production for year 1964 = 306 and production for year 1966 = 390.

Example 31 Find the missing value of the following data:

x	1	2	3	4	5
y	7	-	13	21	37

[U.P.T.U. 2002]

Solution Since the four entries are known and one value is unknown so we assume all the 4th order differences is zero.

Hence, $$\Delta^4 y_1 = 0$$

$$(E-1)^4 y_1 = 0$$

or $$(E^4 - 4E^3 + 6E^2 - 4E + 1)y_1 = 0$$

or $$y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 0 \tag{26}$$

Substituting the known values in Eq. (26), we get

$$37 - 4 \times 21 + 6 \times 13 - 4y_2 + 7 = 0$$

or $$38 - 4y_2 = 0$$

or $$y_2 = 9.5$$

EXERCISE 8.2

1. Evaluate the following taking $h = 1$:

   (i) $\Delta \log x$ (ii) $\Delta \tan^{-1} ax$

   (iii) $\Delta(ab^{cx})$ (iv) $\Delta^2 \sin x$

   (v) $\Delta^n (e^{ax})$ (vi) $\Delta^2 \left[\dfrac{5x+12}{x^2+5x+6}\right]$

2. Given $u_0 = 3$, $u_1 = 12$, $u_2 = 81$, $u_3 = 200$, $u_4 = 100$ and $u_5 = 8$, find $\Delta^5 u_0$.
3. Prove the relation:

   (i) $(1+\Delta)(1-\nabla) = 1$ (ii) $(\Delta + \nabla) = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

   where Δ and $\nabla$are forward and backward difference operators respectively.
4. Obtain the first term of the series whose 2nd and subsequent terms are as follows 8, 3, 0, –1, 0.
5. Given $u_0 + u_8 = 1.9243$, $u_1 + u_7 = 1.9590$, $u_2 + u_6 = 1.9823$, $u_3 + u_5 = 1.9956$, then evaluate u_4.
6. Construct a forward difference table from the given data:

x	0	1	2	3	4
y	1	2	4	8	14

   Hence, evaluate $\Delta^3 y_1$ and y_x
7. A 3rd degree polynomial passes through the points (0, –1), (1, 1), (2, 1) and (3, –2). Find the polynomial.
8. Express the function $2x^3 - 3x^2 + 3x - 10$ in factorial notation and find its differences.
9. Evaluate the values of $\Delta^2 y_{10}, \Delta y_{20}, \Delta^3 y_{15}$ and $\Delta^5 y_{10}$, from the following set of values:

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

10. y_x is a function of x for which fifth differences are constant and

    $y_1 + y_7 = -786,\ y_2 + y_6 = 686,\ y_3 + y_5 = 1088$ find y_4.

11. Given log 100 = 2, log 101 = 2.0043, log 103 = 2.0128, log 104 = 2.0170. Find log 102.
12. Find the missing terms in the following table:

x	2	2.1	2.2	2.3	2.4	2.5	2.6
y	0.135	–	0.111	0.100	–	0.082	0.074

[U.P.T.U. 2004]

13. Express $x^3 - 2x^2 + x - 1$ in to factorial notation. Hence, show that $\Delta^4 f(x) = 0$.
14. Express $f(x) = 3x^4 - 4x^3 + 6x^2 + 2x + 1$ as a factorial polynomials and find differences of all orders.
15. Evaluate

    (i) $(\Delta+\nabla)^2 (x^2 + x); (h = 1)$ (ii) $\left(\dfrac{\Delta^2}{E}\right) x^3$ (iii) $\dfrac{\Delta}{E} \sin x$

16. Assuming that the following values of y belong to a polynomial of degree 4, find the next three terms.

x	0	1	2	3	4	5	6	7
y	1	–1	1	–1	1	–	–	–

17. Compute $f(6)$ and $f(7)$, from a table of differences for the function:

$$f(x) = x^3 + 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5$$

18. Prove that with the usual notations, $(E^{\frac{1}{2}} + E^{-\frac{1}{2}})(1+\Delta)^{\frac{1}{2}} = 2 + \Delta$

19. Show that

(i) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^2)(1-4x^4)] = 24 \times 10! \times 2^{10}$ if $h = 2$

(ii) $\Delta^4[(1-x)(1-2x)(1-3x)] = 0$ if $h = 1$

20. Obtain the function whose first difference is $x^3 + 3x^2 + 5x + 12$.

Answers

1. (i) $\log\left(1+\frac{1}{x}\right)$ (ii) $\tan^{-1}\left(\frac{a}{1+a^2x+a^2x^2}\right)$

(iii) $a(b^c - 1)b^{cx}$ (iv) $2(\cosh x - 1)\sin(x+h)$

(v) $(e^{ah}-1)^n e^{ax}$ (vi) $\frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$

2. 755

4. $y_1 = 15$

5. $u_4 = 0.99996$

6. $\Delta^3 y_1 = 0,\ y_x = 1 + \frac{13}{12}x - \frac{11}{24}x^2 + \frac{5}{12}x^3 - \frac{1}{24}x^4$

7. $\frac{1}{6}(x^3 + 3x^2 - 16x + 6)$

8. $y = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10,\ \Delta y = 6x^{(2)},\ \Delta^2 y = 12\,x^{(1)} + 6,\ \Delta^3 y = 12.$

9. $\Delta^2 y_{10} = -0.58$, $\Delta y_{20} = 1.05$, $\Delta^3 y_{15} = -0.01$ and $\Delta^5 y_{10} = 0.72$.

10. 570.90

11. log (102) = 2.0086

12. $y_{2.1} = 0.123,\ y_{2.4} = -0.0904$

15. (i) 8 (ii) $6xh^2$ (iii) $2(\cos h - 1)\sin x$

16. 124, 222, 351

17. 239, 371

20. $\frac{1}{4}x^{(4)} + 2x^{(3)} + \frac{9}{2}x^{(2)} + 125x^{(1)} + K$

8.13 INTERPOLATION

Interpolation is an interesting topic and has wide applications in various fields. It is the process of finding the value of a function for any value of arguments or nodes within an interval for which some values are given. Suppose the experimental or observed data is in the form a set of say $(n + 1)$ ordered pairs (x_i, y_i); $i = 0, 1, 2, \ldots, n$ which is tabular form of an unknown function $y = f(x)$. The process of finding the value of y for any $x \in [x_0, x_n]$ is called an interpolation. The method for solving this problem which attempts to find a polynomial $P(x)$ passing through the $(n + 1)$ given arguments such that

$$y_i = P(x_i);\ i = 0,1,2,3,\ldots,n,$$

where y_i are the given values at x_i is known as interpolation technique and the polynomial is called interpolation polynomial.

Extrapolation is the process of finding the value of a function outside an interval $[x_0, x_n]$ for which some values are given.

8.14 NEWTON'S-GREGORY FORWARD INTERPOLATION FORMULA

Suppose the values of $y_i = f(x_i)$ are given for equally spaced values of the arguments or the independent variable $x_i = x_0 + i\,h$ for $i = 0, 1, 2, \ldots, n$. Here h known as the size/differencing interval or spacing is constant. We have

x	x_0	$x_0 + h$	$x_0 + 2h$	…	$x_0 + n\,h$
$y = f(x)$	$f(x_0)$	$f(x_0 + h)$	$f(x_0 + 2h)$	…	$f(x_0 + n\,h)$

Suppose we want to compute the functional value $P(x)$ for $x = x_0 + uh$, where $-1 < u < 1$.

We have,

$$P(x) = f(x_0 + uh), \text{ where } u = \frac{x - x_0}{h}$$

Using shift operator and binomial theorem, we have

$$\begin{aligned} P(x) &= E^u f(x_0) \\ &= (1+\Delta)^u f(x_0) \\ &= \left[1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \cdots\right] f(x_0) \end{aligned}$$

$$P(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots$$

which is called Newton's-Gregory forward difference interpolation formula.

***Note*:** Newton's forward difference interpolation formula is used for interpolating the values of the function near the beginning of a set of tabulated values.

8.15 NEWTON'S-GREGORY BACKWARD INTERPOLATION FORMULA

Suppose the values of $y_i = f(x_i)$ are given for equally spaced values of the arguments $x_i = x_0 + i\,h$ for $i = 0, 1, 2, \ldots, n$. Here h known as the size or spacing of interval is constant. We want to calculate the functional value $P(x)$ for $x = x_n + uh, -1 < u < 1,$ we have

$$P(x) = f(x_n + uh); \text{ where } u = \frac{x - x_n}{h}$$

Using shift operator and binomial theorem, we have

$$P(x) = E^u f(x_n) = (1-\nabla)^{-u} f(x_n) \qquad [\because\ E^{-1} = 1 - \nabla]$$

$$P(x) = \left[1 + u\nabla + \frac{u(u+1)}{2!}\nabla^2 + \frac{u(u+1)(u+2)}{3!}\nabla^3 + \cdots\right] f(x_n)$$

$$= f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \cdots$$

which is known as Newton's-Gregory backward interpolation formula.

***Note*:** Newton's backward interpolation formula is used for interpolating the values of the function near the end of a set of tabulated values.

8.16 ERRORS IN NEWTON'S INTERPOLATION POLYNOMIAL

Suppose we have $(n + 1)$ values can be represented by a unique polynomial of degree $\leq n$. Hence, the Newton forward and backward differences interpolation formulae are basically the same, and have the same error. Both formulae give the nth degree polynomial $P(x)$ passing through $(n + 1)$ given points (x_i, y_i); $i = 0, 1, 2, 3, \ldots, n$.

Hence, the error involved is the same as that described for the Larange's interpolation. (Explain in next section).

The maximum absolute error is given by

$$E_a = \max\left|\Pi_0^n (x - x_i)\right| \max\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}\right|; x_0 \leq \xi \leq x_n,$$

where $\qquad x \in [x_0, x_n]$ and

$$\Pi_0^n (x - x_i) = (x - x_0)(x - x_1)\ldots(x - x_n).$$

Example 32 Fit a polynomial of degree three which takes the following values:

x	0	1	2	3
$y = f(x)$	1	2	1	10

Solution The Newton's-Gregory forward interpolating polynomial takes the given set of points (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) and is given by

$$P(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) \tag{27}$$

where $$u = \frac{x - x_0}{h} = \frac{x - 0}{1} = x \text{ and } h = 1 \text{ (given)}$$

The difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		–2	
		–1		12
2	1		10	
		9		
3	10			

Substituting the values of $f(x_0)$, $\Delta f(x_0)$, $\Delta^2 f(x_0)$, $\Delta^3 f(x_0)$ and $u = x$ in Eq. (27), we obtain

$$P(x) = 1 + x \times 1 + \frac{x(x-1)}{2} \times -2 + \frac{x(x-1)(x-2)}{6} \times 12$$

$$P(x) = 1 + 6x - 7x^2 + 2x^3$$

or $$P(x) = 2x^3 - 7x^2 + 6x + 1$$

***Note*:** If we applying Newton's backward interpolating polynomial, we obtain the same polynomial of degree three.

Therefore, Newton's-Gregory backward interpolation formula.

$$P(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n)$$

$$= 10 + (x-3) \times 9 + \frac{(x-3)(x-2)}{2} \times 10 + \frac{(x-3)(x-2)(x-1)}{6} \times 12$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Example 33 From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for maturing at the age of 63:

Age	45	50	55	60	65
Premium (In rupees)	114.84	96.16	83.32	74.48	68.48

Solution Since, we want to finding the premium at the age of 63 year, which is lies between 60 and 65 (from the given table), so here we applying Newton's backward interpolation formula, for this $h = 5$ and

$$u = \frac{x - x_n}{h} = \frac{63 - 65}{5} = -\frac{2}{5} = -0.4$$

The difference table is

Age (x)	Premium (in ₹)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
45	114.84				
		–18.68			
50	96.16		5.84		
		–12.84		–1.84	
55	83.32		4		0.68
		–8.84		–1.16	
60	74.48		2.84		
		–6			
65	68.48				

By Newton's backward difference interpolation formula

$$P(63) = y_{65} + u\nabla y_{65} + \frac{u(u+1)}{2!}\nabla^2 y_{65} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{65} + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{65}$$

$$= 68.48 + (-0.4)(-6) + \frac{(-0.4)(0.6)}{2}\times 2.84 + \frac{(-0.4)(0.6)(1.6)}{6}\times -1.16 + \frac{(-0.4)(0.6)(1.6)(2.6)}{24}\times 0.68$$

$$= 70.5852$$

Example 34 From the above example, estimate the premium for policies maturing at age of 46. The difference table is:

Age (x)	Premium (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	114.84				
		–18.68			
50	96.16		5.84		
		–12.84		–1.84	
55	83.32		4		0.68
		–8.84		–1.16	
60	74.48		2.84		
		–6			
65	68.48				

Solution Since, the premium for policies maturing at the age of 46, which is lies between age 45 and 50. Thus, we applying the Newton's forward difference interpolation formula is

$$P(46) = y_{45} + u\Delta y_{45} + \frac{u(u-1)}{2!}\Delta^2 y_{45} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{45} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_{45}$$

$$= 114.84 + (0.2)(-18.68) + \frac{(0.2)\times(0.2-1)}{2}\times 5.84 + \frac{(0.2)(0.2-1)(0.2-2)}{6}\times(-1.84) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{24}\times(0.68)$$

$$= 110.5256$$

Hence, the premium for policies maturing at the age of 46 is ₹ 110.5256.

Example 35 Commute $y(2.2)$ using forward differencing and linear, quadratic and cubic interpolation from the following table:

x	1	2	3	4	5	6	7	8
y	2.105	2.808	3.614	4.604	5.857	7.451	9.467	11.985

Solution We form the forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	2.105			
		0.703		
2	2.808		0.103	
		0.806		0.081
3	3.614		0.184	
		0.990		
4	4.604			

If two nearest points are used ($n = 1$), i.e. linear interpolation, then $x_1 = 2$, $y_1 = 2.808$
Let $x_0 = 2$, y_0 =2.808 (which is near by 2.2)

$$\therefore \qquad y(x) = y_0 + \left(\frac{x - x_0}{h}\right)\Delta y_0$$

$$y(2.2) = 2.808 + 0.2 \times 0.806 = 2.9692$$

Again, if 3 nearest points are used ($n = 2$), i.e., the quadratic interpolation, then we have

$$x_0 = 1,\ y_0 = 2.105,\ x_1 = 2,\ y_1 = 2.808,\ x_2 = 3,\ y_2 = 3.614$$

Then $$u = \frac{2.2 - 1}{1} = 1.2$$

$$\therefore \qquad y(2.2) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{21}\Delta^2 y_0$$

$$= 2.105 + 1.2 \times 0.703 + \frac{(1.2)(0.2)}{2} \times 0.103$$

$$= 2.961$$

Again, if 4 nearest points are used ($n = 3$) i.e., cubic interpolation, then we take $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, and $x_4 = 4$.

Then $$u = \frac{2.2 - 1}{1} = 1.2$$

$$\therefore \qquad y(2.2) = 2.105 + 1.2 \times 0.703 + \frac{1.2 \times .2}{2} \times 0.103 + \frac{1.2 \times 0.2 \times -0.8}{6} \times 0.081$$

$$y(2.2) = 2.958$$

Example 36 The following are the numbers of deaths in four successive ten year age groups. Find the number of deaths at 45–50 and 50–55.

Age group	25–35	35–45	45–55	55–65
Deaths	13229	18139	24225	31496

Solution Difference table of cumulative frequencies:

Age upto (x)	No. of deaths $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
35	13229			
		18139		
45	31368		6086	
		24225		1185
55	55593		7271	
		31496		
65	87089			

Here $h = 10$, $x_0 = 35$, $u = \dfrac{x - x_0}{h} = \dfrac{50-35}{10} = 1.5$

By Newton's forward difference formula.

$$P(50) = y_{35} + u\Delta y_{35} + \frac{u(u-1)}{2!}\Delta^2 y_{35} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{35}$$

$$= 13229 + 1.5\times 18139 + \frac{1.5\times .5}{2}\times 6086 + \frac{1.5\times .5\times -0.5}{6}\times 1185$$

$$= 42645.6875 \approx 42646$$

$\therefore$ Deaths at the age between 45–50 is $y_{50} - y_{45}$ = 42646 – 31368 = 11278

Deaths at the age between 50–55 is $y_{55} - y_{50}$ = 55593 – 42646 = 12947.

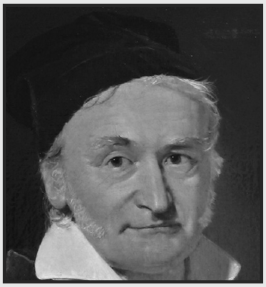
8.17 CENTRAL DIFFERENCE INTERPOLATION FORMULAE

The Newton's forward and backward difference formulae are not appropriate for approximating $f(x)$ when x lies near the centre of the table because neither will permit the highest order difference to have x_0 close to x. In the present section, we will discuss the central difference formulae which are most suited for interpolation near the middle of a tabulated set of values.

Central difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	$\boxed{y_0}$		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

8.17.1 Gauss's Central Difference Formulae



Carl Friedrich Gauss was born on 30 April 1777 in Germany. He was a great mathematician, who worked in various fields including number theory, statistics, analysis, differential geometry and geophysics, electrostatics, astronomy and geodesy, but he excelled most in the field of mathematics. Due to his impressive contributions and portrayal of pure brilliance in the subject, he is known as the 'Prince of mathematics' and the 'Greatest Mathematician since Antiquity'. His teachers straightaway spotted the potential when he added up the 1 to 100 integers by noticing that the result was 50 pairs of numbers the answer of each sum was 101. He attended the University of Göttingen from 1795 to 1798. During this time he worked and solved many theorems. In 1796, he solved the major construction problems by proving that Fermat Prime Polygon can be constructed by a compass and ruler. Furthermore, Gauss constructed a heptadecagon. He went deeper into modular arithmetic to further simplify the number theory. In 1809, he published a book on the theory of the motion of the planets called the 'Theoria Motus Corporum Coelestium'. He used the method of least squares which is also used today to calculate the values from existing annotations. He was also the first to develop Non-Euclidean geometry. Gauss was made a member of the Royal Swedish Academy of Sciences in 1821. He collaborated with Wilhelm Weber (a Physics professor) to develop new ideas in magnetism and circuit laws in electricity. His work the 'Dioptrische Untersuchungen' was published in 1840. A perfectionist and hard worker by nature, Gauss accomplished a lot in the field of mathematics by making important discoveries. He died on 23 February 1855.

(i) Gauss's Forward Interpolation Formulae

Let y_x be a function which takes the values .. $y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, \dots$ for equally spaced values and with unit intervals of u, i.e. $-3, -2, -1, 0, 1, 2, 3, \dots$, where $u = \dfrac{x - x_0}{h}$.

The Newton's forward difference interpolation formula is

$$y_u = y_0 + u_{c_1}\Delta y_0 + u_{c_2}\Delta^2 y_0 + u_{c_3}\Delta^3 y_0 + \cdots \tag{28}$$

But $\quad \Delta^2 y_0 = \Delta^2 E y_{-1} = \Delta^2(1+\Delta)y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

$\quad \Delta^3 y_0 = \Delta^3 E y_{-1} = \Delta^3(1+\Delta)y_{-1} = \Delta^3 y_{-1} + \Delta^4 y_{-1}$

and $\quad \Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$, etc.

In general $\quad \Delta^r y_0 = \Delta^r y_{-1} + \Delta^{r+1} y_{-1}$

$$\therefore \quad y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{u(u-1)(u-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$
$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \left[\frac{u(u-1)}{2!} + \frac{u(u-1)(u-2)}{3!}\right]\Delta^3 y_{-1}$$
$$+ \left[\frac{u(u-1)(u-2)}{3!} + \frac{u(u-1)(u-2)(u-3)}{4!}\right]\Delta^4 y_{-1} + \cdots$$

or $$y_u = y_0 + u_{C_1}\Delta y_0 + u_{C_2}\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-1} + {}^{u+1}C_4\,\Delta^4 y_{-1}$$
$$ {}^{u+1}C_5\Delta^5 y_{-1} + {}^{u+1}C_6\,\Delta^6 y_{-1} + \cdots \tag{29}$$

Again, expanding the 4[th] and higher order differences of y at $u = -1$ in Eq. (29) using $\Delta^r y_{-1} = \Delta^r y_{-2} + \Delta^{r+1} y_{-2}$, we have

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)\,u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)\,(u+1)u(u-1)\,(u-2)}{5!}\Delta^5 y_{-2} + \cdots$$

or $$y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(u^2-1)\,u(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u^2-1)(u^2-4)u}{5!}\Delta^5 y_{-2} + \cdots \tag{30}$$

Equation (30) is known as Gauss's forward formula for equal intervals.

Remark-1 This formula be central differences notation as:

$$y_u = y_0 + u\,\delta y_{1/2} + \frac{u(u-1)}{2!}\delta^3 y_0 + \frac{u(u^2-1)}{3!}\delta^3 y_{1/2} + \frac{u(u^2-1)(u-2)}{4!}\delta^2 y_0 + \cdots$$

Remark-2 This formula is useful when $0 < u < 1$, measured forwardly from the origin.

Remark-3 It is follow the different order of differences, which is shown as

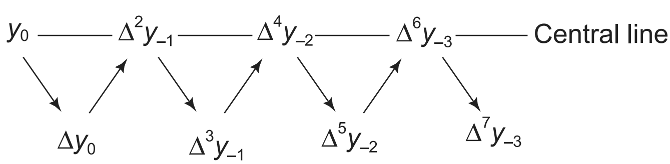


Fig. 8.2

(ii) Gauss's Backward Difference Interpolation Formulae

Due to Gauss's may be derived in a similar manner, we have

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$
$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3\, y_{-1},\ \Delta^3 y_0 = \Delta^3\, y_{-1} + \Delta^4\, y_{-1},\ \text{etc.}$$

In general $$\Delta^r y_0 = \Delta^r y_{-1} + \Delta^{r+1} y_{-1},\ \text{and}$$
$$\Delta^r y_{-1} = \Delta^r y_{-2} + \Delta^{r+1} y_{-2}$$

Substituting in Eq. (28) and grouping the terms, we get

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)\,u(u-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2}$$
$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-3}+\cdots \tag{31}$$

Equation (31) is known as Gauss's backward interpolation formula.

Remark-1 It is follow the different order of differences, which is shown as

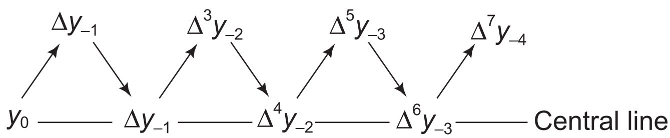


Fig. 8.3

Remark-2 This formula is useful when $-1 < u < 0$.

Remark-3 It is also written in central differences notation

$$y_u = y_0 + u_{C_1}\,\delta y_{-1/2} + {}^{u+1}C_2\,\delta^2 y_0 + {}^{u+1}C_3\,\delta^3 y_{-1/2} + {}^{u+2}C_4\,\delta^2 y_0 + \cdots$$

(iii) Stirling's Formula

Taking the average of the Gauss's forward and backward interpolation formulae, i.e. Eqs (30) and (31), we obtain

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1}$$
$$+\frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2}+\cdots \tag{32}$$

Equation (32) is known as Stirling's formula for equal intervals.

Remark This formula is more accurate, when $-\frac{1}{4} \le u \le \frac{1}{4}$.

(iv) Bessel's Formula

The Gauss's backward interpolation formula is

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+2)(u^2-1)}{4!}\Delta^4 y_{-2}+\cdots \tag{33}$$

Shift the origin to 1 by replacing u by $u-1$ and adding 1 to each arguments 0, –1, –2, –3, … in the above formula Eq. (33), we get

$$y_u = y_1 + (u-1)\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \frac{u(u-2)(u^2-1)}{4!}\Delta^4 y_{-1}+\cdots \tag{34}$$

Taking the average of Eq. (34) and Gauss's forward difference formula in Eq. (30), we get

$$y_u = \left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{u(u-1)\left(u - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \cdots \tag{35}$$

Equation (35) is known Bessel's interpolation formula for equal intervals.

Remark This formula is more accurate if $\frac{1}{4} \le u \le \frac{3}{4}$

(v) Laplace-Everett's Interpolation Formula

The Laplace-Everett's formula can be written as

$$y_u = \left[u\, y_1 + {}^{u+1}C_3\, \Delta^2 y_0 + {}^{u+2}C_5\, \Delta^4 y_{-1} + \cdots\right] + \left[v\, y_0 + {}^{v+1}C_3\, \Delta^2 y_{-1} + {}^{v+2}C_5\, \Delta^4 y_{-2} + \cdots\right]$$

where $v = 1 - u$

This formula is convenient especially when using tables in which only differences of even order are tabulated. It is more accurate, when $0 < u < 1$.

8.18 GUIDELINES FOR THE CHOICE OF INTERPOLATION

The choice of an interpolation formula, depends on the position of the interpolated value is given tabulated data:

Following rules will be found useful:

Rule 1: If interpolation is desired near the beginning of a table, then we use Newton's forward interpolation formula.

Rule 2: If interpolation is desired near the end of a table, then we use Newton's backward interpolation formula.

Rule 3: If interpolation near the middle of a table, then we can use either Stirling's formula gives the most accurate result for $-\frac{1}{4} \le u \le \frac{1}{4}$ or Bessel's formula is most efficient near $u = \frac{1}{2}; \left(\frac{1}{4} \le u \le \frac{3}{4}\right)$ or use Everett's formula.

Rule 4: If interpolation is required for u lying between 0 and 1, then we apply the Gauss's forward formula.

Rule 5: If interpolation is required for u lying –1 and 0, then we apply the Gauss's backward formula

Example 37 Use Gauss's forward interpolation formula to find the value of y when $x = 3.75$ for the following data:

x	2.5	3.0	3.5	4.0	4.5	5.0
$y = f(x)$	24.145	22.043	20.225	18.644	17.262	16.047

Solution Since, $x = 3.75$ lying between 3.5 and 4.0 so we take 3.5 as the origin, then

$$u = \frac{x - x_0}{h} = \frac{3.75 - 3.50}{0.5} = 0.5 \qquad \text{(Here } h = 0.5\text{)}$$

$u = 0.5$ lying between 0 and 1 so we apply the Gauss's forward formula.

The difference Table is:

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
–2	2.5	24.145					
			–2.102				
–1	3.0	22.043		0.284			
			–1.818		–0.047		
0	$\boxed{3.5}$	$\boxed{20.225}$		$\boxed{0.237}$		$\boxed{0.009}$	
			$\boxed{-1.581}$		$\boxed{-0.038}$		$\boxed{-0.003}$
1	4.0	18.644		0.199		0.006	
			–1.382		–0.032		
2	4.5	17.262		0.167			
			–1.215				
3	5.0	16.047					

The Gauss's forward interpolation formula is

$$P(x) = y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(u+1)\,u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)\,(u+1)\,u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \cdots$$

$$\therefore \quad P(3.75) = y_{0.5} = 20.225 + (0.5) \times (-1.581) + \frac{(0.5)\,(0.5-1)}{2} \times 0.237$$
$$+ \frac{(0.5+1)(0.5)(0.5-1)}{6} \times 0.038$$
$$+ \frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{24} \times 0.009$$
$$+ \frac{(0.5+2)(0.5+1)(0.5)(0.5-1)(0.5-2)}{120} \times (-0.003)$$
$$= 20.225 - 0.7905 - 0.029625 + 0.00238 + 0.0023750 + 0.0002106$$
$$= 19.41 \text{ (approximate)}$$

Hence, the estimated value of y when

$x = 3.75$ is 19.41

Further, we can find the value of the function $y = f(x)$ at $x = 3.75$, we use Gauss's backward formula.

$$P(x) = y_u = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1}\ \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{(u+2)(u+1)\,u(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)\,u(u-1)(u-2)}{5!}\Delta^5 y_{-3} + \cdots$$

$$P(3.75) = 20.225 + (0.5)(-1.818) + \frac{(0.5)(0.5+1)}{2}\times 0.237$$

$$+\frac{(0.5)(0.5+1)(0.5-1)}{6}\times(-0.047) + \frac{(0.5+2)(0.5+1)(0.5)(0.5-1)}{24}\times(0.009)+0$$

$$= 20.225 - 0.909 + 0.088875 + 0.00294 - 0.00035$$

$$= 19.40747 \approx 19.41 \quad \text{(approximate)}$$

Hence, $y_{0.5} = 19.41$ by Gauss's backward interpolations formula.

Example 38 Using Gauss's backward interpolation formula find the population for the year 1936 for the following data:

Year (x)	1901	1911	1921	1931	1941	1951
Population (y) (in thousand)	12	15	20	27	39	52

Solution Since 1936 lies between 1931 and 1941 we take 1931 as the origin and $h = 10$, then

$$u = \frac{x - x_0}{h} = \frac{1936-1931}{10} = \frac{5}{10} = 0.5$$

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
–3	1901	12					
			3				
–2	1911	15		2			
			5		0		
–1	1921	20		2		3	
			$\boxed{7}$		$\boxed{3}$		$\boxed{-10}$
0	1931	$\boxed{27}$		$\boxed{5}$		$\boxed{-7}$	
			12		–4		
1	1941	39		1			
			13				
2	1951	52					

The Gauss's backward formula is

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{u\,(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{(u+2)\,(u+1)\,u\,(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)\,(u+1)\,u(u-1)(u-2)}{5!}\Delta^5 y_{-3} + \cdots$$

$$\therefore \quad y_{0.5} = 27 + 0.5\times 7 + \frac{1.5\times .5}{2}\times 5 + \frac{1.5\times .5\times -0.5}{6}\times 3 + \frac{2.5\times 1.5\times 0.5\times -0.5}{24}\times 7$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2734 - 0.1172 + \frac{2.5 \times 1.5 \times 0.5 \times -0.5 \times -1.5}{120} \times -10$$

$$= 2.3427 \text{ thousands.}$$

Hence, the estimated population for the year 1936 is 32.3427 thousands.

Example 39 Using Gauss's forward interpolation formula to find y at $x = 30$ from the following table:

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3432	15.5154

Solution Since $x = 30$ lies between 29 and 33.

We take 29 as the origin and $h = 4$

$$\therefore \qquad u = \frac{x - x_0}{h} = \frac{30 - 39}{4} = 0.25 \qquad (0 < u < 1)$$

Thus, here we apply the Gauss's forward formula.

The difference table is

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
–2	21	18.4708				
			–0.6564			
–1	25	17.8144		–0.0510		
			–0.7074		–0.0074	
0	29	$\boxed{17.1070}$		$\boxed{-0.0564}$		$\boxed{-0.0022}$
			$\boxed{-0.7638}$		$\boxed{-0.0076}$	
1	33	16.3432		–0.0640		
			–0.8272			
2	37	15.5154				

The Gauss's forward formula is

$$P(x) = y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4\, y_{-2}$$

$$P(30) = y_{0.25} = 17.1070 + (0.25)\,(-0.7638) + \frac{(0.25)(-0.75)}{2} \times (-0.0564)$$

$$+ \frac{(1.25)(0.25)(-0.75)}{6} \times (-0.0076) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24} \times (-0.0022)$$

$$= 17.1070 - 0.19095 + 0.00529 + 0.0003 - 0.00004$$

$$= 16.9216 \quad \text{(approximate)}$$

Example 40 Using Gauss's backward formula to obtain y_{28}; given $y_{20} = 49225$, $y_{25} = 48316$ $y_{30} = 47236$, $y_{35} = 45926$ and $y_{40} = 44306$

Solution Since $x = 28$ lying between 25 and 30 so, we take 30 as the origin and interval difference $h = 5$.

$$\therefore \qquad u = \frac{28 - 30}{5} = -0.4$$

The difference table is

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
–2	20	49225				
			–909			
–1	25	48316		–171		
			$\boxed{-1080}$		$\boxed{-59}$	
0	30	$\boxed{47236}$		$\boxed{-230}$		$\boxed{-21}$
			–1310		–80	
1	35	45926		–310		
			–1620			
2	40	44306				

Now, the Gauss's backward formula is

$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)\,u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)\,u(u+1)\,u(u-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{36}$$

Putting the values in Eq. (36), we get

$$y_{28} = 47236 + (-0.4)\times(-1080) + \frac{(-0.4+1)(-.4)}{2}\times -230 + \frac{(-0.4+1)(-0.4)(0.4-1)}{6}\times -59 + \frac{(-0.4+2)(-0.4+1)(-0.4)(-0.4-1-1)}{24}\times(-21)$$

$$= 47686$$

Example 41 Using Stirling's interpolation formula to final y at $x = 32$ from the given table:

x	20	30	40	50
y	512	439	346	243

Solution Since $x = 32$ lying between 30 and 40. So we take 30 as the origin and $h = 10$.

$$\therefore \qquad u = \frac{x - x_0}{h} = \frac{32-30}{10} = 0.2$$

Here $u = 0.2$ which lies between $-\frac{1}{4} \le u \le \frac{1}{4}$.

Hence, we apply the Stirling's formula

The difference table is

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
–1	20	512			
			$\boxed{-73}$		
0	30	$\boxed{439}$		$\boxed{-20}$	$\boxed{10}$
			$\boxed{-93}$		
1	40	346		–10	
			–103		
2	50	243			

The Stirling's formula is

$$P(x) = y_u = y_0 + \frac{u}{2}\left(\Delta y_0 + \Delta y_{-1}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$\therefore \quad P(35) = y_{0.2} = 439 + 0.2\left(\frac{-93-73}{2}\right) + \frac{0.04}{2}(-20) + 0$$

$$= 439 - 16.60 - 0.40$$

$$= 422$$

Example 42 Using Bessel's interpolation formula to find y_{25}; given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$ and $y_{32} = 3992$.

Solution Since $x = 25$ lying between 24 and 28, so that we take 24 as the origin and $h = 4$.

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{25-24}{4} = \frac{1}{4} = 0.25$$

Here $u = \frac{1}{4}$, which lies in $\frac{1}{4} \le u \le \frac{3}{4}$.

Hence, we apply Bessel's formula.

The difference table is

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
–1	20	2854			
			308		
0	24	$\boxed{3160}$		$\boxed{74}$	
			$\boxed{382}$		$\boxed{-8}$
1	28	$\boxed{3544}$		$\boxed{66}$	
			448		
2	32	3992			

The Bessel's formula is

$$y_u = \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$= \frac{1}{2}(3160 + 3544) + \left(\frac{1}{4} - \frac{1}{2}\right) \times 382 + \frac{1}{4}\left(\frac{1}{4} - 1\right)\left[\frac{74+66}{2}\right] + \frac{\left(\frac{1}{4} - \frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4} - 1\right)}{3!} \times -(8)$$

$$= 3353 - 95.5 - 6.5625 - 0.0625$$

$$= 3250.8750$$

Hence the value of $y_{25} = 3250.8750$

Example 43 Use Laplace–Everett's interpolation formula to find y_{25} using the data and table of Example 42.

Solution Here $u = 0.25$ lies in $0 < u < 1$ and $v = 1 - 0.25 = 0.75$

$\therefore$ Everett's formula is

$$y_u = \left[uy_1 + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0 + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots\right]$$

$$+\left[vy_0 + \frac{v(v^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{v(v^2-1^2)(v^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots\right]$$

$$\therefore \quad y_{0.25} = \left[0.25\times 3544 + \frac{0.25[(0.25)^2-1]}{6}\times 6.6 + 0\right]$$

$$+\left[0.75\times 3162 + \frac{0.75[(0.75)^2-1]}{6}\times 74 + 0\right]$$

$$= 886 - 2.578125 + 2371.5 - 4.046875$$

$$= 3250.8750 \text{ (approximate).}$$

Example 44 If the third differences are constant, prove that

$$y_{x+\frac{1}{2}} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_{x-1} + \Delta^2 y_x)$$

Solution Bessel's formula up to 3rd difference is

$$y_x = \frac{1}{2}(y_0 + y_1) + \left(x - \frac{1}{2}\right)\Delta y_0 + \frac{x(x-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{\left(x-\frac{1}{2}\right)x(x-1)}{3!}\Delta^3 y_{-1} \quad (37)$$

Putting $x = \frac{1}{2}$ in Eq. (37), we get

$$y_{1/2} = \frac{1}{2}(y_0 + y_1) - \frac{1}{16}(\Delta^2 y_{-1} + \Delta^2 y_0)$$

Changing the origin to x, we have

$$\boxed{y_{x+1/2} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_{x-1} + \Delta^2 y_x)}$$

Hence, proved.

Example 45 Using a suitable formula to compute $y(12.2)$ form the following table ($y(x) = 1 + \log_{10}\sin x$):

x°	10	11	12	13	14
$10^5 y(x)$	23,367	28,060	31,788	35,209	38,368

Solution Since $x = 12.2$ lies between 12 and 13. So that we take 12 as the origin and $h = 1$.

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{12.2 - 12}{1} = 0.2$$

Thus, $u = 0.2$, which lies between $-\frac{1}{4}$ and $\frac{1}{4}$.

Hence, we apply the Stirling's formula will be quite suitable.

Stirling's formula is

$$y_u = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{38}$$

The difference table is

u	x°	$10^5 y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
–2	10	23,967				
			0.04093			
–1	11	28,060		–0.00365		
			$\boxed{0.03728}$		$\boxed{0.00058}$	
0	12	$\boxed{31,788}$		$\boxed{-0.00307}$		$\boxed{-0.00013}$
			$\boxed{0.03421}$		$\boxed{-0.00045}$	
1	13	35,209		–0.00062		
			0.03159			
2	14	38,368				

$$\therefore \quad y_{0.2} = 0.31788 + 0.2\left[\frac{0.03728 + 0.03421}{2}\right] + \frac{(0.2)^2}{2} \times (-0.00307)$$

$$+ \frac{(0.2)\,[(0.2)^2 - 1]}{6}\left[\frac{0.00058 - 0.00045}{2}\right] + \frac{(0.2)^2\,[(0.2)^2 - 1]}{24}[-0.00013]$$

$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002$$

$$= 0.32497 \quad \text{(approximate).}$$

EXERCISE 8.3

Using Newton's interpolation formulae, evaluate the following:

1. The population of a city (in thousands) were as under. Estimate the population for the year 1965.

Year	1961	1971	1981	1991	2001
Population (in thousand)	46	66	81	93	101

2. Given sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8197 and sin 60° = 0.8660, find sin 52°.
3. From the given data, find the number of students whose weight is between 60 and 70.

Weight in kg	0 – 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of students	250	120	100	70	50

4. Find the polynomial of least degree passing through the points (0, –1), (1, 1), (2, 1) and (3, –2)
5. In an examination the number of students who obtained marks between certain limits were as follows:

Marks	0 – 19	20 – 39	40 – 59	60 – 79	80 – 99
No. of students	41	62	65	50	17

Estimate the number of students who obtained fewer than 70 marks.

6. If the following table, values of y are consecutive terms of a series of which 23.6 is the 6^{th} term. Find the I^{st} and 10^{th} terms of the series.

x	3	4	5	6	7	8	9
y	4.8	8.4	14.6	23.6	36.2	52.8	73.9

(M.D.U. 2004)

7. Using Gauss's forward interpolation formula to obtain $y(32)$ given that $y(25) = 0.2707$, $y(35) = 0.3386$, $y(30) = 0.3027$ and $y(40) = 0.3794$.
8. If $f(x)$ is a polynomial of degree 4 and given that $f(4) = 270$, $f(5) = 648$, $\Delta f(5) = 682$, $\Delta^3 f(-4) = 132$. Find $f(5.8)$ using Gauss's backward interpolation formula.
9. Use Stirling's formula to find y_{28}; given

$$y_{20} = 49225,\ y_{25} = 48316,\ y_{30} = 47236,\ y_{35} = 45926 \text{ and } y_{40} = 44306.$$

10. Use Stirling's formula to find y_{25}, given

$$y_{20} = 24,\ y_{24} = 32,\ y_{28} = 35 \text{ and } y_{32} = 40$$

11. Use Bessel's interpolation formula to obtain $y(9)$; given that

x	4	6	8	10	12	14
y	3.5460	5.0753	6.4632	7.7217	8.8633	9.8986

12. Using Laplace-Everett's interpolation formula to find y_{25} for the following data:

$$y_{20} = 2854,\ y_{24} = 362,\ y_{28} = 3544 \text{ and } y_{32} = 3992.$$

13. Use Gauss's interpolation formula to obtain y_{41} with the help of following data:

$$y_{30} = 3678.2,\ y_{35} = 2995.1,\ y_{40} = 2400.1,\ y_{45} = 1876.2 \text{ and } y_{50} = 1416.3$$

14. From the following table, find the value of $f(0.5437)$ by Gauss's Stirling's, Bessel's and Everett's interpolation formula.

x	0.51	0.52	0.53	0.54	0.55	0.56	0.57
$f(x)$	0.529244	0.537895	0.546464	0.554939	0.663323	0.571616	0.579816

15. Using Stirling's formula to obtain f(1.22) from the following data:

x	1.0	1.1	1.2	1.3	1.4
$f(x)$	0.841	0.891	0.932	0.963	0.985

16. Evaluate $f(0.5437)$ for the probability integral function $f(x) = \dfrac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$, using the same data in Q. 14.
17. Given the following table, construct a difference table and from it estimate y when (i) $x = 0.15$ and (ii) $x = 0.35$

x	0	0.1	0.2	0.3	0.4
y	1	1.095	1.129	1.251	1.310

18. Using Stirling's formula to show that tan 16° = 0.2867; given that

q	0°	5°	10°	15°	20°	25°	30°
tan θ	0.0000	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

19. From the following table, find the value of $\log_{10}337.5$ by Gauss's, Stirling's, Bessel's and Everett's interpolation formulae.

x	310	320	330	340	350	360
$\log_{10}x$	2.4913617	2.5051500	2.5185139	2.5314789	2.5440680	2.5563025

Answers

1. 54.8528 thousands.
2. 0.7880032
3. Number of students whose weight is between 60 and 70

$$= y(70) - y(60)$$
$$= 424 - 370$$
$$= 54$$

4. $-\frac{1}{6}(x^3 + 3x^2 - 6x + 6)$
5. 201
6. 3.1, 100
7. $y_{0.4} = 0.3165$
8. $f(5.8) = 1162.944$
9. $y_{28} = 47692$
10. $y_{25} = 32.9453$
11. $y(9) = 7.1078$
12. $y_{25} = 3250.875$
13. $y_{41} = 2290.1$
14. 0.558052
15. 0.9389968
16. 0.5580
17. $f(0.15) = 1149, f(0.35) = -1.282$
19. 2.5282738 by each of the formula

8.19 INTERPOLATION FOR UNEQUAL INTERVALS

In the preceding section, we have discussed the interpolation formulae with equally spaced values of the arguments or nodes.

In the present section, we shall discuss the interpolation formulae with unequal spaced values of the arguments.

8.20 LAGRANGE'S INTERPOLATING POLYNOMIALS

To determine a polynomial of degree one through the two distinct points (x_0, y_0) and (x_1, y_1) is the same as approximating a function f for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of a first degree polynomial interpolating with, the values of f at the given points. Using this polynomial for approximation within the interval given by the end points is called polynomial interpolation.

Now, we define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

The linear Lagrange's interpolating polynomial through the points (x_0, y_0) and (x_1, y_1) is given by

$$P_1(x) = \sum_{i=0}^{1} L_i(x)\, f(x_i) = L_0(x)\, f(x_0) + L_1(x) f_1(x)$$

or
$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)}\, f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)}\, f(x_1) \tag{39}$$

But
$$L_i(x_j) = \begin{cases} 1; & \text{if } \quad i = j \\ 0; & \text{if } \quad i \neq j \end{cases} \tag{40}$$

i.e., $L_0(x_0) = 1$, $L_0(x_1) = 0$, $L_1\,(x_0) = 0$ and $L_1(x_1) = 1$.

$\therefore$ $P_1(x_0) = 1 \cdot f(x_0) + 0 f(x_1) = f(x_0) = y_0$

and $P_1(x_1) = 0 \cdot f(x_0) + 1\, f(x_1) = f(x_1) = y_1$

Hence, $P(x)$ is the unique polynomial of degree at most one.

In a similar way, the Lagrange polynomial of degree two passes through three points (x_0, y_0), (x_1, y_1) and (x_2, y_2) is given as

$$P_2(x) = \sum_{i=0}^{2} L_i(x)\, f(x_i)$$

or
$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\, f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\, f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \cdot f(x_2) \tag{41}$$

To generalize the concept of linear interpolation, Let we construct a polynomial interpolation of degree at most n that passes through the $(n + 1)$ points (x_0, y_0), $(x_1, y_1) \ldots (x_n, y_n)$. A sketch of the graph $P_n(x)$ when n is even is shown in Fig. 8.4.

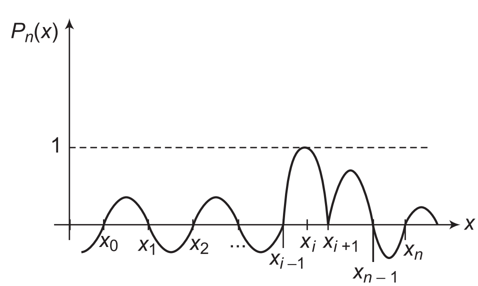


Fig. 8.4

Thus, $$P_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x_i) \tag{42}$$

where, for each $i = 0, 1, 2, 3 \ldots, n$

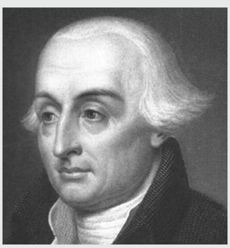
$$L_i(x) = \frac{(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

$$= \prod_{\substack{k=0 \\ i \neq k}}^{n} \frac{(x - x_k)}{(x_i - x_k)} \tag{43}$$

where $L_i(x)$ are polynomials in x of degree n.

Since $P(x_j) = f(x_j)$ for $j = 0, 1, 2, \ldots, n$ and also satisfy Eq. (40).

Equation (42) is called Lagrange's interpolation polynomial, the coefficients $L_i(x)$ defined in Eq. (43) are called Lagrange interpolation coefficients.



Joseph Louis Lagrange was born in 25 January 1736 in Turin, Italy. Lagrange was raised as a Catholic, but he soon turned Agnostic when he joined college. He discovered his passion for mathematics at the age of 17 when he fortuitously came across a paper on the subject written by Edmund Halley. He was largely self-taught and pushed himself to reach the top of the ladder during the early 18th century. One of the most proficient mathematicians at the time, Lagrange was responsible for developing many theories related to mechanics and also studied the number system. He studied a wide variety of topics and was a published astronomer who studied the solar system in great depths. His consistent works in the field of radicals, permutations, and fluid mechanics made him a pioneer of his time. He was appreciated by Napoleon for his efforts and was awarded the Legion of Honor. Lagrange was also given the honor of becoming a member of the Academy of Sciences. With his wide range of admirers and an unbeatable passion for work, Joseph Louis Lagrange went on to become one of the first and definitely, one of the most influential faces in the history of mathematics and science. After a brief stint as a mathematics professor in an artillery school, Lagrange decided to concentrate work on a specific division of mathematics that interested him the most; calculus. He developed theories in 1754 and also engaged himself in the subject of classical mechanics. Along with the help of his pupils in 1758, he established a guild called the Turin Academy of Sciences where he wrote elaborate papers and dissertations called the 'Miscellanea Taurinensia' and even discussed calculus at great lengths. His passion to understand the dynamics of the solar system got him to discover various solutions that came to be known as the 'Lagrangian points'. He defied traditional science and contributed heavily to even 'Newtonian Mechanics' which were later renamed as 'Lagrangian Mechanics'. Life started to get back on track in 1794 when Lagrange was appointed professor at the 'Ecole Polytechnique' where he was admired and venerated by his pupils. He was given the honor of the 'Mathematical chair' at the new institution called 'Ecole Normale'. After much deliberation and research, Lagrange's theory on the decimal subdivision was finally accepted by the French commission in 1799. Joseph Louis Lagrange died on 10 April 1813, in Paris just a week after he was awarded the 'Grande Croix'.

8.20.1 Another Form of Lagrange's Polynomial

Suppose, we have $(n + 1)$ values of the function $y = f(x)$ be $f(x_0), f(x_1), f(x_2) \ldots, f(x_n)$ corresponding to the arguments $x_0, x_1, x_2, \ldots, x_n$.

Let $P_n(x)$ is a polynomial in x of degree at most n.

Then $$P_n(x) = A_0(x - x_1)\, x - x_2) \ldots (x - x_n) + A_1(x - x_0)(x - x_1) \ldots (x - x_n)$$
$$+ A_2(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) + \ldots$$
$$A_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \tag{44}$$

where $A_0, A_1, A_2, \ldots, A_n$ are constants. We determine the $(n + 1)$ constants. So as to make

$$P_n(x_0) = f(x_0),\ P_n(x_1) = f(x_1),\ \ldots,\ P_n(x_n) = f(x_n)$$

To determine A_0, put $x = x_0$ and $P_n(x_0) = f(x_0)$ in (43), we get

$$f(x_0) = A_0\,(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

or $$A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

Similarly, $$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

$$\vdots \qquad \vdots$$

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

Substituting the values of $A_0, A_1, A_2, \ldots, A_n$ in Eq. (44), we obtain

$$P_n(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} f(x_1)$$
$$+ \cdots + \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} f(x_n)$$

Which is known as Lagrange interpolation formula.

Note:

(i) The Lagrange's interpolating formula can also used whether the values of arguments $[x_i : i = 0(1)n]$ are equally spaced or not.

(ii) This formula can also be used to split the given function into partial fractions.

(iii) The main drawback is that if another interpolation value is inserted, then the interpolation coefficient are required to be calculated.

8.21 ERROR IN LAGRANGE'S INTERPOLATION FORMULA

Suppose $x_0, x_1, x_2, \ldots, x_n$ are distinct arguments in the interval $[a, b]$. Then for each $x \in [a, b]$, a number $\in (x)$ between $x_0, x_1, \ldots, x_n$ and hence in $[a, b]$ exists with

$$f(x) - P_n(x) = R_n(x) = \frac{f^{(n+1)}(\in)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n) \tag{45}$$

where $P_n(x)$ is the interpolating polynomial given in Eq. (42)

Then, the estimate error

$$E_L = \underset{[a,b]}{\text{Max}} \left|R_n(x)\right| \tag{46}$$

If we assume that

$$\left|f^{(n+1)}(\xi)\right| \le M_{n+1}; \; a \le \xi \le b \tag{47}$$

Then

$$E_L \le \frac{M_{n+1}}{(n+1)!} \underset{[a,b]}{Max} \prod_{i=0}^{n} (x - x_i)$$

Example 46 If $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$, to find the second degree Lagrange's interpolation polynomial for $f(x) = 1/x$.

Solution Let the 2nd degree Lagrange polynomial.

$$P(x) = \sum_{i=0}^{2} L_i(x)\, f(x_i)$$

$$= L_0(x)\, f(x_0) + L_1(x) f(x_1) + L_2(x)\, f(x_2) \tag{48}$$

Now, to determine the Lagrange's coefficients $L_0(x)$, $L_1(x)$ and $L_2(x)$

$\therefore$ $$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{2}{3}(x-2.75)(x-4)$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4)$$

and $$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{2}{5}(x-2)(x-2.75)$$

Also, $$f(x_0) = f(2) = \frac{1}{2},\; f(x_1) = f(2.75) = \frac{4}{11} \text{ and } f(x_2) = f(4) = \frac{1}{4}$$

Thus Eq. (48) becomes

$$P(x) = \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75)$$

$$= \frac{x^2}{22} - \frac{35x}{88} + \frac{49}{44}$$

Example 47 Using Lagrange's interpolation formula, to determine $f(10)$ from the following table:

x	5	6	9	11
$f(x)$	12	13	14	16

Solution Here $x_0 = 5$, $x_1 = 6$, $x_2 = 9$ and $x_3 = 11$

Also $f(x_0) = f(5) = 12$, $f(x_1) = f(6) = 13$, $f(x_2) = f(9) = 14$ and $f(x_3) = f(11) = 16$

Suppose the 3rd degree Lagrange's interpolation polynomial is

$$P(x) = \sum_{i=0}^{3} L_i(x)\, f(x_i)$$

$$= L_0(x)\, f(x_0) + L_1(x) f(x_1) + L_2(x)\, f(x_2) + L_3(x) f(x_3) \tag{49}$$

Where the Lagrange's coefficients

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} = -\frac{1}{24}(x-6)(x-9)(x-11)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} = \frac{1}{15}(x-5)(x-9)(x-11)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} = -\frac{1}{24}(x-5)(x-6)(x-11)$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} = +\frac{1}{60}(x-5)(x-6)(x-9)$$

Substituting the values of L_0, L_1, L_2, L_3, $f(x_0)$, $f(x_1)$, $f(x_2)$ and $f(x_3)$ in Eq. (49), we get

$$P(x) = \frac{-(x-6)(x-9)(x-11)}{24}\times 12 + \frac{(x-5)(x-9)(x-11)}{15}\times 13$$

$$- \frac{(x-5)(x-6)(x-11)}{24}\times 14 + \frac{(x-5)(x-6)(x-9)}{60}\times 16$$

$$= -\frac{1}{2}(x-6)(x-9)(x-11) + \frac{13}{15}(x-5)(x-9)(x-11)$$

$$-\frac{7}{12}(x-5)(x-6)(x-11) + \frac{4}{15}(x-5)(x-6)(x-9) \tag{50}$$

Putting $x = 10$ in Eq. (50), we obtain

$$P(10) = -\frac{1}{2}(10-6)(10-9)(10-11) + \frac{13}{15}(10-5)(10-9)(10-11)$$

$$-\frac{7}{12}(10-5)(10-6)(10-11) + \frac{4}{15}(10-5)(10-6)(10-9)$$

$$P(10) = 14.66666667$$

Example 48 Using Lagrange's interpolation formula, prove that $y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$ approximately.

Solution If we shift the origin to 5, then the values of y are y_0, y_2, y_8 and y_{10},

So, we are obtain y_6.

Using Lagrange interpolation formula, we have

$$y_6 = \frac{(6-2)(6-8)(6-10)}{(0-2)(0-8)(0-10)} \cdot y_0 + \frac{(6-0)(6-8)(6-10)}{(2-0)(2-8)(2-10)} \cdot y_2$$

$$+ \frac{(6-0)(6-2)(6-10)}{(8-0)(8-2)(8-10)} \cdot y_8 + \frac{(6-0)(6-2)(6-8)}{(10-0)(10-2)(10-8)} \cdot y_{10}$$

or $$y_6 = -0.2\, y_0 + 0.5\, y_2 + y_8 - 0.3\, y_{10}$$

Now shifting back the origin, we obtain

$$\boxed{y_1 = y_3 - 0.3\,(y_5 - y_{-3}) + 0.2\,(y_{-3} - y_3)}$$ Hence proved.

Example 49 Using Lagrange's interpolation formula to express the function

$$\frac{x^2 + 6x - 1}{(x-1)(x+1)(x-4)(x-6)}$$ as a sum of partial fractions.

Solution

Let $$P_n(x) = x^2 + 6x - 1$$

$$x_0 = 1,\ x_1 = -1,\ x_2 = 4 \text{ and } x_3 = 6$$

Then $$P_n(x_0) = P_n(1) = 6$$

$$P_n(x_1) = P_n(-1) = -6$$

$$P_n(x_2) = P_n(4) = 39$$

$$P_n(x_3) = P_n(6) = 71$$

$$\therefore \quad \frac{x^2 + 6x - 1}{(x-1)(x+1)(x-4)(x-6)} = \frac{6}{(x-1)\cdot 2 \times -3 \times -5} + \frac{6}{(x+1) \times -2 \times -5 \times -7}$$

$$+ \frac{39}{(x-4) \times 3 \times 5 \times -2} + \frac{71}{(x-6) \times 5 \times 7 \times 2}$$

$$= \frac{1}{5(x-1)} + \frac{3}{35(x+1)} - \frac{13}{10(x-4)} + \frac{71}{70(x-6)}$$

Which is the required partial fraction.

Example 50 The function $f(x) = \sin x$ is defined on the interval [1, 3], find the Lagrange's linear interpolation polynomial in this interval and find the bound on the truncation error. Also find the approximate values of $f(1.5)$ and $f(2.5)$.

Solution

Let $$x_0 = 1 \text{ and } x_1 = 3$$

$$f(x_0) = f(1) = \sin 1 = 0.8415$$

$$f(x_1) = f(3) = \sin 3 = 0.1411$$

and $$f''(x) = -\sin x.$$

Suppose the Lagrange's linear polynomial on [1, 3] is

$$P(x) = \sum_{i=0}^{1} L_i(x) f(x_i) = L_0(x)\, f(x_0) + L_1(x)\, f(x_1) \tag{51}$$

Now $$L_0(x) = \frac{x-3}{1-3} = -\frac{1}{2}(x-3)$$

$$L_1(x) = \frac{x-1}{3-1} = -\frac{1}{2}(x-1)$$

Putting the values of $L_0(x)$ and $L_1(x)$ in Eq. (51), we get

$$P(x) = -\frac{1}{2}(x-3) \times 0.8415 + \frac{1}{2}(x-1) \times 0.1411$$
$$= -0.3502x + 1.1917$$

$$\left|E_L(x)\right| \le \frac{1}{2} \max_{[1,3]} \left|(x-1)(x-3)\right| \cdot \max_{[1,3]} \left|\sin x\right| = 0.5$$

Now, $f(1.5) = P(1.5) = 0.6664$ and $f(2.5) = P(2.5) = 0.3152$.

8.22 DIVIDED DIFFERENCES

Divided difference methods introduced in this section are used to successively generate the polynomials themselves.

Suppose $f(x_0), f(x_1), f(x_2) \ldots, f(x_n)$ be the $(n + 1)$ values corresponding to the arguments $x_0, x_1, x_2, \ldots, x_n$, where the interval space/difference are not equal. Then the first divided differences of f(x) for the two arguments x_0, x_1; x_1, x_2, etc. are defined as

$$\underset{x_1}{\Delta} f(x_0) = [x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

Similarly, $$\underset{x_2}{\Delta} f(x_1) = [x_1, x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$ and so on

The second divided difference of $f(x)$ for three arguments x_0, x_1, x_2 is defined as

$$\underset{x_1, x_2}{\Delta^2} f(x_0) = [x_0, x_1, x_2] = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2} = \frac{[x_0, x_1] - [x_1, x_2]}{x_0 - x_2}$$

Similarly, $$\underset{x_2, x_3}{\Delta^2} f(x_1) = [x_1, x_2, x_3] = \frac{f(x_1, x_2) - f(x_2, x_3)}{x_1 - x_3} = \frac{[x_1, x_2] - [x_2, x_3]}{x_1 - x_3}$$

Now the 3rd divided difference for x_0, x_1, x_2, x_3 is defined as

$$\underset{x_1 x_2 x_3}{\Delta^3} f(x_0) = [x_0, x_1, x_2, x_3] = \frac{[x_0, x_1, x_2] - [x_1, x_2, x_3]}{(x_0 - x_3)}$$

In general, the nth divided difference for $x_0, x_1, \ldots, x_n$ is defined as

$$\underset{(x_1, x_2, \ldots, x_n)}{\Delta^n} f(x_0) = [x_0, x_1, x_2, \ldots, x_n]$$

$$= \frac{f(x_0, x_1, \cdots, x_{n-1}) - f(x_1, x_2, \cdots, x_n)}{x_0 - x_n}$$

Divided Difference Table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$			
		$[x_0,x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$		
x_1	$f(x_1)$		$[x_0,x_1,x_2] = \frac{f(x_0,x_1) - f(x_1,x_2)}{x_0 - x_2}$	
		$[x_1,x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$		$[x_0,x_1,x_2,x_3] = \frac{f(x_0,x_1,x_2) - f(x_1,x_2,x_3)}{x_0 - x_3}$
x_2	$f(x_2)$		$[x_1,x_2,x_3] = \frac{f(x_1,x_2) - f(x_2,x_3)}{x_1 - x_3}$	
		$[x_2,x_3] = \frac{f(x_2) - f(x_3)}{x_2 - x_3}$		$[x_1,x_2,x_3,x_4] = \frac{f(x_1,x_2,x_3) - f(x_2,x_3,x_4)}{(x_1 - x_4)}$
x_3	$f(x_3)$		$[x_2,x_3,x_4] = \frac{f(x_2,x_3) - f(x_3,x_4)}{x_2 - x_4}$	
		$[x_3,x_4] = \frac{f(x_3) - f(x_4)}{x_3 - x_4}$		$[x_2,x_3,x_4,x_5] = \frac{f(x_2,x_3,x_4) - f(x_3,x_4,x_5)}{x_2 - x_5}$
x_4	$f(x_4)$		$[x_3,x_4,x_5] = \frac{f(x_3,x_4) - f(x_4,x_5)}{x_3 - x_5}$	
		$[x_4,x_5] = \frac{f(x_4) - f(x_5)}{x_4 - x_4}$		
x_5	$f(x_5)$			

8.22.1 Properties of Divided Differences

Property 1: The n^{th} divided differences of a polynomial of the nth degree are constant.

Property 2: Divided difference is a symmetrical function of all the arguments involved, i.e., $[x_0, x_1] = [x_1, x_0]$, etc.

Property 3: The divided difference of the algebraic sum of any number of functions is the algebraic sum of their separate divided differences. i.e.,

$$\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$$

Property 4: The divided difference of the product of a constant and a function is the product of the constant and the divided difference of the functions,

i.e. $$\Delta[\alpha \cdot f(x)] = \alpha \, \Delta f(x)$$

Property 5: The n^{th} divided difference can be expressed as the quotient of two determinants, each of order $(n + 1)$.

8.22.2 Newton's Divided Differences Formula

Suppose $f(x_0)$, $f(x_1)$, …, $f(x_n)$ be the values of the function $f(x)$ for the values of the arguments $x_0, x_1, \ldots, x_n$ respectively which are not equally spaced.

We know that the first divided difference of $f(x)$ is given by

$$\Delta f(x) = [x, x_0] = f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

or $$f(x) = f(x_0) + (x - x_0) f(x, x_0) \tag{52}$$

Also, the 2nd divided difference is given by

$$\Delta^2 f(x) = f(x, x_0, x_1) = [x, x_0, x_1] = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\Rightarrow \qquad f(x, x_0) = f(x_0, x_1) + (x - x_1)\, f(x, x_0, x_1)$$

$$\frac{f(x) - f(x_0)}{x - x_0} = f(x_0, x_1) + (x - x_1)\, f(x, x_0, x_1)$$

or $$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)\,(x - x_1)\, f(x_0, x_1, x_2) \tag{53}$$

Similarly, $$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2)$$
$$+ (x - x_0)\,(x - x_1)\,(x - x_2)\, f(x, x_0, x_2, x_2) \tag{54}$$

Continuing in the same manner, we have

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)\,(x - x_2)\, f(x_0, x_1, x_2, x_3) + \ldots \\ &+ (x - x_0)(x - x_1) \ldots (x - x_{n-1})\, f(x_0, x_1, x_2 \ldots x_n) \\ &+ (x - x_0)(x - x_1) \ldots (x - x_n)\, f(x, x_0, x_1) \ldots, x_{n+1}) \end{aligned} \tag{55}$$

Since, the function $f(x)$ is a polynomial of degree n, then $f(x, x_0, x_1, \ldots, x_{n+1}) = 0$

Hence, Eq. (55) becomes,

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)\,(x - x_1)\, f(x_0, x_1, x_2) \\ &+ \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1})\, f(x_0, x_1, x_2, \ldots, x_n) \end{aligned}$$

or $$\begin{aligned} f(x) = f(x_0) &+ (x - x_0)\ \Delta f(x_0) + (x - x_0)\,(x - x_1)\ \Delta^2 f(x_0) \\ &+ (x - x_0)\,(x - x_1)\,(x - x_2)\ \Delta^3 f(x_0) \\ &+ (x - x_0)\,(x - x_1)\,(x - x_2) \cdots (x - x_{n-1})\ \Delta^n f(x_0) \end{aligned}$$

Which is called Newton's divided general interpolation formula.

Note: Newton's divided differences interpolation formula reduces to Newton's Gregory forward difference formula if the values of the arguments are equally spaced.

Example 51 Find the 3rd difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

[U.P.T.U. 2005]

Solution Given $f(x) = x^3 - 2x$.

$\therefore$ $f(2) = 4, f(4) = 56, f(9) = 711$ and $f(10) = 980$

The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		$\dfrac{4-56}{2-4} = 26$		
4	56		$\dfrac{26-131}{2-9} = 15$	
		$\dfrac{56-711}{4-9} = 131$		$\dfrac{15-23}{2-10} = 1$
9	711		$\dfrac{131-269}{4-10} = 23$	
		$\dfrac{711-980}{9-10} = 269$		
10	980			

Hence, the 3rd divided difference

$$\Delta^3 f(x) = 1$$

Example 52 Prove that $\underset{bcd}{\Delta^3}\left(\dfrac{1}{a}\right) = -\dfrac{1}{abcd}$

Solution We have $f(x) = \dfrac{1}{x}$ and the arguments a, b, c and d.

The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\dfrac{1}{a}$			
		$\dfrac{\frac{1}{a}-\frac{1}{b}}{a-b} = -\dfrac{1}{ab}$		
b	$\dfrac{1}{b}$		$\dfrac{-\frac{1}{ab}+\frac{1}{bc}}{a-c} = (-1)^2\dfrac{1}{abc}$	
		$\dfrac{\frac{1}{b}-\frac{1}{c}}{b-c} = -\dfrac{1}{bc}$		$\dfrac{(-1)^2\left[\frac{1}{abc}-\frac{1}{bcd}\right]}{a-d} = (-1)^3\dfrac{1}{abcd}$
c	$\dfrac{1}{c}$		$\dfrac{-\frac{1}{bc}+\frac{1}{cd}}{b-d} = (-1)^2\dfrac{1}{bcd}$	
		$\dfrac{\frac{1}{c}-\frac{1}{d}}{c-d} = -\dfrac{1}{cd}$		
d	$\dfrac{1}{d}$			

From the table, we obtain the 3rd divided difference

$$\underset{bcd}{\Delta}^3\left(\frac{1}{a}\right) = (-1)^3\,\frac{1}{abcd}$$

$$= -\frac{1}{abcd}$$

Hence, proved.

Example 53 Using Newton's divided difference formula to calculate $f(8)$ and $f(15)$ from the following data:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

[U.P.T.U. 2004, 2006, G.E.U. 2010]

Solution To prepare the divide difference tables

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	$\boxed{48}$				
		$\frac{48-100}{4-5}=\boxed{52}$			
5	100		$\frac{52-97}{4-7}=\boxed{15}$		
		$\frac{100-294}{5-7}=97$		$\frac{15-21}{4-7}=\boxed{1}$	
7	294		$\frac{97-202}{5-10}=27$		$\frac{1-1}{4-11}=\boxed{0}$
		$\frac{294-900}{7-10}=202$		$\frac{21-27}{5-11}=1$	
10	900		$\frac{202-310}{7-11}=27$		$\frac{1-1}{5-13}=0$
		$\frac{900-1210}{10-11}=310$		$\frac{27-33}{7-13}=1$	
11	1210		$\frac{310-409}{10-13}=33$		
		$\frac{1210-2028}{11-13}=409$			
13	2028				

The Newton's divided formula is

$$f(x) = f(x_0) + (x-x_0)\,\Delta f(x_0) + (x-x_0)(x-x_1)\,\Delta^2 f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x_0) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\,\Delta^4 f(x_0)$$

$$= 48 + (x-4)\times 52 + (x-4)(x-5)\times 15 + (x-4)(x-5)(x-7)\times 1 + 0$$

$$\therefore \quad f(8) = 48 + (8-4)\times 52 + (8-4)(8-5)\times 15 + (8-4)(8-5)(8-7)\times 1$$

$$= 48 + 208 + 180 + 12$$

$$= 448$$

and $f(15) = 48 + (15-4) \times 52 + (15-4)(15-5) \times 15 + (15-4)(15-5)(15-7) \times 1$

$= 48 + 572 + 1650 + 880$

$= 3152$

Example 54 Using the Newton's divided difference formula to find a polynomial from the following data:

x	–4	–1	0	2	5
$f(x)$	1245	33	5	9	1355

[U.P.T.U. 2004, M.D.U. 2007, R.T.U. 2008]

Solution To prepare the divided difference table as

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
–4	1245	$\dfrac{1245-33}{-4+1} = -404$			
			$\dfrac{-404+28}{-4-0} = 94$		
–1	33	$\dfrac{33-5}{-1-0} = -28$		$\dfrac{94-10}{-4-2} = -14$	
0	5		$\dfrac{-28-2}{-1-2} = 10$		$\dfrac{-14-13}{-4-5} = 3$
		$\dfrac{5-9}{0-2} = 2$		$\dfrac{10-88}{-1-5} = 13$	
2	9		$\dfrac{2-442}{0-5} = 88$		
5	1355	$\dfrac{9-1355}{2-5} = 442$			

Using the Newton's divided formula

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x-x_0)(x-x_1)\,\Delta^2 f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x_0) + (x-x_0)(x-x_1)(x-x_3)\,\Delta^4 f(x_0)$$

$$= 1245 + (x+4) \times -404 + (x+4)(x+1) \times 94 + (x+4)(x+1)(x-0) \times -14$$

$$+ (x+4)(x+1)(x-0)(x-2) \times 3$$

$$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

Which is the required polynomial.

Example 55 Find the polynomial of the lowest possible degree which takes the values 3, 12, 15, –21. When the arguments (x) are 3, 2, 1 –1 respectively. **[U.P.T.U. 2008]**

Solution Suppose $x_0 = -1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

Also $f(x_0) = -21, f(x_1) = 15, f(x_2) = 12, f(x_3) = 3$

Now, to prepare the divided difference table as

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
–1	–21			
		$\dfrac{-21-15}{-1-1}=18$		
			$\dfrac{18+3}{-1-2}=-7$	
1	15			
		$\dfrac{15-12}{1-2}=-3$		$\dfrac{-7+3}{-1-3}=1$
2	12			
			$\dfrac{-3+9}{1-3}=-3$	
		$\dfrac{12-3}{2-3}=-9$		
3	3			

Now, using Newton's divided difference formula, we have

$$f(x) = f(x_0) + (x - x_0)\ \Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3\ f(x_0)$$

$$= -21 + (x + 1) \times 18 + (x + 1)(x - 1) \times -7 + (x + 1)(x - 1)(x - 2) \times 1$$

$$= -21 + 18x + 18 - 7x^2 + 7 + x^3 - 2x^2 - x + 2$$

$$f(x) = x^3 - 9x^2 + 17x + 6$$

8.23 INVERSE INTERPOLATION

Inverse interpolation is the process to calculate the value of the argument corresponding to a given value of the function (which is not in the table) is known inverse interpolation. For the case of inverse interpolation, we can apply the several methods like Lagrange's formula, Method of successive approximation, method of reversion of series etc., but in this section we shall discuss only Lagrange's formula in Section (8.20) in which the variables x and y are interchanging so, we obtain

$$P_n(y) = \sum_{i=0}^{n} L_i(y)\ f(y_i),\ \text{where for each } i = 0, 1, 2, \ldots, n$$

where $$L_i(y) = \frac{(y - y_0)(y - y_1)(y - y_2)\cdots(y - y_{i-1})(y - y_{i+1})\cdots(y - y_n)}{(y_i - y_0)(y_i - y_1)(y_i - y_2)\cdots(y_i - y_{i-1})(y_i - y_{i+1})\cdots(y_i - y_n)}$$

Example 56 Using Lagrange's formula to obtain the value of x corresponding to $y = 12$, from the following table:

x	1.2	2.1	2.8	4.1	4.9	6.2
y	4.2	6.8	9.8	13.4	15.5	19.6

Solution Here $x_0 = 1.2,\ x_1 = 2.1,\ x_2 = 2.8,\ x_3 = 4.1$

$x_4 = 4.9,\ x_5 = 6.2$ and $y_0 = 4.2,\ y_1 = 6.8$

$y_2 = 9.8,\ y_3 = 13.4,\ y_4 = 15.5,\ y_5 = 19.6$

Suppose the Lagrange's polynomial is

$$P(y) = \sum_{i=0}^{5} L_i(y)\, f(y_i) \qquad [\text{Here } x_i = f(y_i)]$$

At $y = 12$, we have

$$P(12) = \sum_{i=0}^{5} L_i(12)\, f(y_i) = L_0(12)\, f(y_0) + L_1(12)\cdot f(y_1)$$

$$+ L_2(12)\, f(y_2) + L_3(12)\, f(y_3) + L_4(12)\, f(y_4) + L_5(12)\, f(y_5) \tag{56}$$

Where the Lagrange's coefficients are

$$L_0(12) = \frac{(12-y_1)(12-y_2)(12-y_3)(12-y_4)(12-y_5)}{(4.2-y_1)(4.2-y_2)(4.2-y_3)(4.2-y_4)(4.2-y_5)}$$

$$= \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)(4.2-19.6)}$$

$$L_0(12) = 0.01833$$

Similarly $L_1(12) = -0.11143$, $L_2(12) = 0.44714$, $L_3(12) = 0.83390$, $L_4(12) = -0.19673$,

$L_5(12) = 0.00887$

Hence Eq. (56) becomes

$$P(12) = 0.01833 \times 1.2 - 0.11143 \times 2.1 + 0.44714 \times 2.8 + 0.83390 \times 4.1$$

$$- 0.19673 \times 4.9 + 0.00887 \times 6.2$$

$$P(12) = 3.55$$

8.24 HERMITE INTERPOLATION POLYNOMIAL

If f is a continuous function on $[a, b]$ and $x_0, x_1, \ldots, x_n \in [a, b]$ are distinct, then the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \ldots, x_n$ is the Hermite polynomial of degree at most $(2n + 1)$ given by

$$H_{2n+1}(x) = \sum_{i=0}^{n} H_i(x)\, f(x_i) + \sum_{i=0}^{n} \hat{H}_i(x)\, f'(x_i)$$

where $H_i(x) = \left[1 - 2(x - x_i)L_i'(x_i)\right] L_i^2(x)$

and $\hat{H}_i(x) = (x - x_i)\, L_i^2\,(x)$

Also, $L_i(x)$ denoting the i^{th} Lagrange's coefficient of degree n.

Example 57 Using Hermite formula to obtain the values of $f(x)$ for $x = 0.5$ and $x = -0.5$ from the following table:

x	–1	0	1
$f(x)$	1	1	3
$f'(x)$	–5	1	7

Solution We first compute the Lagrange's polynomials and their derivatives, we have

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_1)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2} = \frac{1}{2}(x^2-x)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(0+1)(0-1)} = -(x^2-1) = 1-x^2$$

and $$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{1}{2}(x^2+x)$$

Now, $$L'_0(x) = \frac{1}{2}(2x-1),\ L'_1(x) = -2x \text{ and } L'_2(x) = \frac{1}{2}(2x+1)$$

To compute the polynomials $H_i(x)$ and $\hat{H}_i(x); i = 0, 1, 2$

$$H_i(x) = [1-2(x-x_i)\,L'_i(x_i)]\,[L_i(x)]^2$$

$$H_0(x) = [1-2(x-x_0)\,L'_0(x_0)]\,[L_0(x)]^2 = \left[1-2\,(x+1)\left(-\frac{3}{2}\right)\right]\left[\frac{1}{2}(x^2-x)\right]^2$$

$$H_0(x) = \frac{1}{4}\left[(3x+4)\cdot(x^2-x)^2\right]$$

$$H_1(x) = [1-2(x-x_1)\,L'_1(x_1)]\,[L_1(x)]^2 = [1-2(x-0)\cdot 0]\cdot(1-x^2)^2$$

$$H_1(x) = (1-x^2)^2$$

and $$H_2(x) = [1-2(x-x_2)\,L'_2(x_2)]\,[L_2(x)]^2 = [1-2(x-1)\cdot(3/2)]\left[\frac{x^2+x}{2}\right]^2$$

$$H_2(x) = \frac{1}{4}(4-3x)(x^2+x)^2$$

$$\hat{H}_i(x) = (x-x_i)\,[L_i(x)]^2$$

$$\hat{H}_0(x) = (x-x_0)\,[L_0(x)]^2 = (x+1)\cdot\frac{1}{4}(x^2-x)^2$$

$$\hat{H}_1(x) = (x-x_1)\,[L_1(x)]^2 = (x-0)\,(1-x^2)^2 = x(1-x^2)^2$$

$$\hat{H}_2(x) = (x-x^2)[L_2(x)]^2 = (x-1)\cdot\frac{1}{4}(x^2+x)^2 = \frac{1}{4}(x-1)(x^2+x)^2$$

Here $n = 2$, then the required Hermite polynomial

$$H_5(x) = \sum_{i=0}^{2} H_i(x)\,f(x_i) + \sum_{i=0}^{2} \hat{H}_i(x)\,f'(x_i)$$

$$= [H_0(x)\,f(x_0) + H_1(x)\,f(x_1) + H_2(x)\,f(x_2)]$$

$$+\left[\hat{H}_0(x)\,f'(x_0) + \hat{H}_1(x)\,f'(x_1) + \hat{H}_2(x)\,f'(x_2)\right]$$

$$H_5(x) = \left[\frac{1}{4}(3x+4)(x^2-x)^2 \times 1 + (1-x^2)^2 \times 1 + \frac{1}{4}(4-3x)(x^2+x)^2 \times 3\right]$$

$$+\left[\frac{1}{4}(x+1)(x^2-x)^2 \times (-5) + x(1-x^2)^2 \times 1 + \frac{1}{4}(x-1)(x^2+x)^2 \times 7\right]$$

$$\therefore \qquad H_5(0.5) = \left[\frac{1}{4}(3\times0\cdot5+4)(.5^2-.5)^2 + (1-0.5^2)^2 + \frac{3}{4}(4-3\times0\cdot5)\times(0.5^2+0.5)^2\right]$$

$$+\left[-\frac{5}{4}(0\cdot5+1)(0.5^2-0.5)^2 + 0.5(1-0.5^2)^2 + \frac{7}{4}(0.5-1)(0.5^2+0.5)^2\right]$$

$$H_5(0.5) = 1.37500$$

Similarly, $H_5(-0.5) = 0.37500$

EXERCISE 8.4

1. Using Lagrange's formula; to compute $f(5)$ and $f(6)$ given that $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(4) = 16$, $f(7) = 128$.
2. The function $y = f(x)$ is given at the points (7, 3), (8, 1), (9, 1) and (10, 9). Find the value of y for $x = 9.5$, applying Lagrange's formula. **[U.P.T.U. 2004]**
3. Apply Lagrange's formula; evaluate $f(4)$ for the following table:

x	0	1	2	5
y	2	5	7	8

4. Find the value of $f(x)$ at $x = 2.5$, from the following data:

x	1	2	3	4
$f(x)$	1	8	27	64

[U.P.T.U. 2001]

5. Find the cubic Lagrange's polynomial for the given data:

x	0	1	2	5
$f(x)$	2	3	12	147

[U.P.T.U. 2004]

6. Find a polynomial of degree 2 for the following points (1, 1), (3, 27) and (4, 64). **[U.P.T.U. 2003]**
7. Using Lagrange's interpolation formula to compute a polynomial from the following data:

x	0	1	3	4
$f(x)$	–12	0	12	24

[M.D.U. 2003]

8. Prove that

(i) $\underset{y}{\Delta} x^2 = x + y$ (ii) $\underset{y,z}{\Delta}^2 x^3 = x + y + z$ (iii) $\underset{y,z}{\Delta}^2 x^2 = 1$

9. Using Newton's divided difference formula; to compute $f(6)$ for the following table:

x	1	2	7	8
$f(x)$	1	5	5	4

[U.P.T.U. 2002]

10. Using Newton's divided formula, find $\log_{10}301$, from the following table:

x	300	304	305	307
$y = \log_{10}x$	2.4771	2.4829	2.4843	2.4871

[U.P.T.U. 2001]

11. Using Newton's divided formula, find $f(4)$ from the following table:

x	1.5	3	6
$f(x)$	−0.25	2	20

12. Find $f(x)$ as a polynomial in x, from the table given below:

x	5	6	9	11
$f(x)$	12	13	14	16

Also, find $f(10)$.

13. Using Hermite formula to find a polynomial from the following data:

x	0	1	2
$f(x)$	0	1	0
$f'(x)$	0	0	0

14. Using Hermite polynomial; to find $f(1.5)$ from the following data:

x	1.3	1.6	1.9
$f(x)$	0.6200860	0.4554022	0.2818186
$f'(x)$	−0.5220232	−0.5698959	−0.5811571

Answers

1. $f(5) = 33, f(6) = 67$
2. $f(9.5) = 3.625$
3. $f(4) = 8.4$
4. $f(2.5) = 15.625$
5. $x^3 + x^3 - x + 2$
6. $8x^2 - 19x - 12$
7. $x^3 - 6x^2 + 17x - 12$
9. $f(6) = 6.2$
10. 2.4786
11. 6
12. $f(10) = 14.6666$
13. $H(x) = (x^2 - 2x)^2$
14. $H_5(1.5) = -0.5118277$

SUMMARY

Following topics have been discussed in this chapter:

1. Significant Digits

The concept of significant digits has been introduced primarily to indicate the accuracy of a numerical value.

Significant digit of a number K is any given digit of K, except possibly for zeros to the first non-zero digit that serve only to fix the position of the decimal point.

Following rules are applied when zeros are encountered in the numbers.

(i) Zeros placed after other digits but behind a decimal point are significant; 3.80 has three significant digits.

(ii) Zeros placed before other digits are not significant; 0.038 has two significant digits.

(iii) Zeros placed between other digits are always significant; 3007 has four significant digits.

(iv) Zeros at the end of a number are significant if they are behind a decimal point as in (i).

2. Rounding-Off and Chopping

All the non-exact numbers can be approximate with a finite number of digits of precision in the following manners.

If n digits are used to represent a non-terminating number then the simplest process is to kept the first n digits and chop off all remaining digits.

We know that $\frac{22}{7} = 3.1415926$

$$\sqrt{2} = 1.4142134, \; e = 2.71828182$$

The digits on the right are approximated to the exact value of the numbers on the left.

To round off a number to n significant digits, discard all digits to the right of n^{th} digit and if this discarded number is

(i) less than 5 in $(n + 1)^{th}$ place, leave the n^{th} digit unchanged.

(ii) greater than 5 in $(n + 1)^{th}$ place, add 1 to the n^{th} digit.

(iii) exactly 5 in $(n + 1)^{th}$ place, add 1 to the n^{th} digit if it is an odd otherwise leave n^{th} digit unchanged.

3. Error

An error is the difference between the true value or actual value and the approximate value from the numerical computation or from the experimental observations.

Let the true value be X and the approximate value be X_a, then

$$\text{Error } (E) = X - X_a$$

Type of errors: Following are the type of errors:

(a) ***Round-off errors*:** An error is caused by chopping (i.e., discarding all decimals from some decimal on) or rounding. This error is called rounding off error.

(b) ***Inherent Errors*:** This error is already present in the statement of a problem before its solution is called inherent errors. Such errors are present either due to the given data being approximated or due to limitations of mathematical measurements.

(c) ***Absolute, Relative and Percentage Errors*: ** The absolute error of measurement, number or calculation is the numerical difference between the true value of the quantity and its approximate value as given or calculate.

The relative error is the absolute error divided by the true value of the quantity. The percentage error is 100 times of the relative error.

Let X be the true value and X_a be the approximate value, then

$$\text{Absolute error } (E_a) = |X - X_a|$$

$$\text{Relative error } (E_r) = \frac{|X - X_a|}{X}$$

$$\text{Percentage error } (E_P) = E_a \times 100$$

(d) ***Truncation Error*: ** This error is caused by the usage of a closed form such as the first few terms of an infinite series to express a quantity defined by limiting process.

For example, consider, use of a finite number of terms in the infinite series expansion of cos x or sin x by using Taylor's or Maclaurin's series expansion, such type of errors are called the truncation errors.

4. Error in a Series Approximation

The Taylor's series expansion for $f(x)$ at $x = a$ with a remainder after n terms is given.

$$f(x) = f[(a + (x-a)] = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n(x)$$

where $\quad R_n(x) = \dfrac{(x-a)^n}{n!} f^n(\theta);\ a < \theta < x$

The last term $R_n(x)$, which is called the remainder term. If the series is convergent then $R_n(x) \to 0$ as $n \to \infty$. Thus, if $f(x)$ is approximated by the first n terms of this series, then the maximum error committed in this approximation is given by the remainder term.

5. Finite Differences

The calculus of finite differences is an interesting topic and has wide applications in various fields. Using this concept, we deal with the changes that take place in the value of the function, the dependent variable due to finite changes in the independent variable.

Suppose a table of values (x_i, y_i), $i = 1, 2, 3, \ldots, n$, of any function $y = f(x)$, the values of x being equally spaced i.e., $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$. We are required to obtain the values of $f(x)$ for some intermediate values of x or to obtain the derivative of $f(x)$ for some x in the range $x_0 \le x \le x_n$. The following three types of differences are found useful.

(i) ***Forward Differences*: ** If a function $y = f(x)$ is tabulated for the equally spaced arguments x_0, $x_0 + h$, $x_0 + 2h$, …, $x_0 + nh$ giving the functional values $y_0, y_1, y_2, \ldots, y_n$.

The constant difference between two consecutive arguments (x) is called the interval of differencing and is denoted by h.

The forward difference operator Δ is defined as

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2 \text{ and so on } \Delta y_n = y_{n+1} - y_n$$

$\Delta y_0, \Delta y_1, \Delta y_2, \cdots, \Delta y_n$ are called first forward differences.

The differences of the first forward differences are called 2nd forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, \cdots, \Delta^2 y_n$, defined as

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

Similarly, we can define the 3rd forward differences, 4th forward differences, etc.

Thus $$\Delta^3 y_0 = \Delta^2(\Delta y_0) = \Delta^2(y_1 - y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

and $$\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

In general,

$$\Delta^n y_n = \Delta^{n-1} y_{n+1} - \Delta^{n-1} y_n$$

(ii) ***Backward Differences*****:** If a function $y = f(x)$ is tabulated for the equally spaced arguments x_0, $x_0 + h$, $x_0 + 2h$, ..., $x_0 + nh$ giving the functional values $y_0, y_1, y_2, \ldots, y_n$. The backward difference operator ∇ is defined as

$$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \cdots, \nabla y_n = y_n - y_{n-1}$$

are called the first backward difference operator.

In a similar way, we can define the backward differences of higher orders.

Thus, $$\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = y_2 - 2y_1 + y_0$$

$$\nabla^3 y_3 = y_3 - 3y_2 + 3y_1 - y_0, \text{ etc.}$$

(iii) ***Central Differences*****:** Some times it is a very useful to employ another system of differences known as central differences. The central difference operator δ(delta) is defined by the relations

$$y_1 - y_0 = \delta y_{y_2}, y_2 - y_1 = \delta y_{3/2}, \cdots$$

$$y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}$$

Similarly, the central differences of higher orders are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1$$

$$\delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2; \cdots$$

$$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

(iv) ***Shift operator*****:** The shift operator E is defined as

$$Ef(x) = f(x+h) \text{ or } Ey_x = y_{x+h}$$

$$E^2 f(x) = f(x+2h) \text{ or } E^2 y_x = y_{x+2h}$$

$$\vdots \qquad \vdots$$

$$E^n f(x) = f(x+nh) \text{ or } E^n y_x = y_{x+nh}$$

The inverse operator E^{-1} is defined as

$$E^{-1}f(x) = f(x-h) \text{ or } E^{-1}y_x = y_{x-h}$$

$$E^{-2}f(x) = f(x-2h) \text{ or } E^{-2}y_x = y_{x-2h}$$

$$\vdots \quad \vdots$$

$$E^{-n}f(x) = f(x-nh) \text{ or } E^{-n}y_x = y_{x-nh}$$

(v) ***Averaging operator*****:** The averaging operator μ(mu) is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

6. Relation between Operators

(i) Relation between Δ, ∇ and E is

$$\Delta = \frac{\nabla}{1-\nabla}$$

(ii) *Relation between* δ and E

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

(iii) *Relation between* μ and E

$$\mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$$

(iv) *Relation between* Δ, ∇, E and δ is

$$\Delta = E\nabla = \nabla E = \delta E^{1/2}$$

7. Factorial Notation

The product of factors of which the first factor is x and the successive factors decrease by a constant difference is called a factorial function or polynomial and is denoted by $x^{(n)}$, n being a positive integer and is read as x raised to the power n factorial". In general, the interval of differencing is h.

The factorial polynomial $x^{(n)}$ is defined as

$$x^{(n)} = x(x-h)(x-2h)...[x-(n-1)h]$$

In particular

$$x^{(0)} = 1, x^{(1)} = x, x^{(2)} = x(x-h),$$

$$x^{(3)} = x(x-h)(x-2h), \text{etc.}$$

Differences of $x^{(n)}$

The first difference of $x^{(n)}$ is

$$\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

Or $$\Delta x^{(n)} = nhx^{(n-1)}$$

Similarly, $$\Delta^2 x^{(n)} = \Delta[\Delta x^{(n)}]$$

$$= \Delta[nhx^{(n-1)}]$$

$$= n(n-1)h^2 x^{(n-2)}$$

Continuing, this process r times, we have

$$\Delta^r x^{(n)} = n(n-1)(n-2)...(n-r+1)h^r x^{(h-r)},$$

where r is a positive integer and $n > r$.

In particular, if $h = 1$, then

$$\Delta^n x^{(n)} = n! \text{ and } \Delta^{n+1} x^{(n)} = 0$$

8. Reciprocal Factorial

If h is the interval of differencing, then the reciprocal factorial function $x^{(-n)}$ is defined as

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)...(x+nh)} = \frac{1}{(x+nh)^{(n)}}$$

In particulars, $x^{(-1)} = \dfrac{1}{(x+h)}, x^{(-2)} = \dfrac{1}{(x+2h)^{(2)}}$

or $$x^{(-2)} = \frac{1}{(x+h)(x+2h)}$$

Differences of a reciprocal factorial

$$\Delta x^{(-n)} = (x+h)^{(-n)} - x^{(-n)}$$

$$= \frac{1}{(x+2h)(x+3h)...[x+(\overline{n+1})h]} - \frac{1}{(x+h)(x+2h)...[x+(nh)]}$$

$$= \frac{(x+h)-[x+(n+1)h]}{(x+h)(x+2h)..(x+nh)[x+(n+1)h]}$$

$$= -nhx^{(-n-1)}$$

Similarly, $\Delta^2 x^{(-n)} = (-n)(-n-1)h^2 x^{(-n-2)}$

$$= (-1)^2 n(n+1)h^2 x^{(-n-2)}$$

and in general

$$\Delta^r x^{(-n)} = (-1)^r n(n+1)(n+2)...(n+r-1)h^r x^{(-n-r)}; n > r$$

9. Interpolation

Interpolation is an interesting topic and has wide application in various fields. It is the process of finding the value of a function for any value of arguments or nodes within an interval for which some values are given. Suppose the experimental or observed data is in the form a set of say $(n + 1)$ ordered pairs (x_i, y_i); $i = 0, 1, 2, \ldots, n$ which is tabular form of an unknown function $y = f(x)$. The process of finding the value of y for any $x \in [x_0, x_n]$ is called an interpolation. The method for solving this problem which attempts to find a polynomial $P(x)$ passing through the $(n + 1)$ given arguments such that

$$y_i = P(x_i); i = 0,1,2,3,...,n$$

where y_i are the given values at x_i is known as interpolation technique and the polynomial is called interpolation polynomial.

Extrapolation is the process of finding the value of a function outside an interval $[x_0, x_n]$ for which some values are given.

10. Newton's-Gregory Forward Interpolation Formula

Newton's-Gregory forward difference interpolation formula given as following:

$$P(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots$$

11. Newton's-Gregory Backward Interpolation Formula

Newton's-Gregory backward interpolation formula given as following:

$$P(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \cdots$$

12. Central Difference Interpolation Formulae

(i) Gauss's Central Difference Formulae

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{u(u^2-1)\,u(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u^2-1)(u^2-4)u}{5!}\Delta^5 y_{-2} + \cdots$$

(ii) Gauss's Backward Difference Interpolation Formula:

$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)\,u(u-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{(u+2)\,(u+1)\,u(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)\,u(u-1)(u-2)}{5!}\Delta^5 y_{-3} + \cdots$$

(iii) Stirling's Formula:

Taking the average of the Gauss's forward and backward interpolation formulae, we obtain

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots$$

Which is known as Stirling's formula for equal intervals.

(iv) Bessel's Formula:

Bessel's interpolation formula for equal intervals is given below:

$$y_u = \left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$
$$+ \frac{u(u-1)\left(u-\frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \cdots$$

(v) Laplace-Everett's Interpolation Formula:

The Laplace-Everett's formula can be written as

$$y_u = \left[u\, y_1 + {}^{u+1}C_3\, \Delta^2 y_0 + {}^{u+2}C_5\, \Delta^4 y_{-1} + \cdots\right] + \left[v\, y_0 + {}^{v+1}C_3\, \Delta^2 y_{-1} + {}^{v+2}C_5\, \Delta^4 y_{-2} + \cdots\right]$$

where $v = 1 - u$

13. Lagrange's Interpolating Polynomials

The Lagrange's interpolating formula is given below

$$P_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x_i)$$

where, for each $i = 0, 1, 2, 3 \ldots, n$

$$L_i(x) = \frac{(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

$$= \prod_{\substack{k=0\\ i\neq k}}^{n} \frac{(x - x_k)}{(x_i - x_k)}$$

where $L_i(x)$ are polynomials in x of degree n.

Since $P(x_j) = f(x_j)$ for $j = 0, 1, 2, \ldots, n$.

The coefficients $L_i(x)$ are called Lagrange interpolation coefficients.

Another Form of Lagrange's Polynomial

The Lagrange's interpolating formula is given below

$$P_n(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)} f(x_1)$$

$$+ \cdots + \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})} f(x_n)$$

14. Divided Differences

Divided difference methods introduced in this section are used to successively generate the polynomials themselves.

Suppose $f(x_0), f(x_1), f(x_2) \ldots, f(x_n)$ be the $(n + 1)$ values corresponding to the arguments $x_0, x_1, x_2, \ldots, x_n$, where the interval space/difference are not equal. Then the first divided differences of f(x) for the two arguments x_0, x_1; x_1, x_2, etc. are defined as

$$\underset{x_1}{\Delta} f(x_0) = [x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

Similarly, $\underset{x_2}{\Delta}^{2} f(x_1) = [x_1, x_2] = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$ and so on

The second divided difference of $f(x)$ for three arguments x_0, x_1, x_2 is defined as

$$\underset{x_1, x_2}{\Delta} f(x_0) = [x_0, x_1, x_2] = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

$$= \frac{[x_0, x_1] - [x_1, x_2]}{x_0 - x_2}$$

Similarly $$\underset{x_2, x_3}{\Delta^2} f(x_1) = [x_1, x_2, x_3] = \frac{f(x_1, x_2) - f(x_2, x_3)}{x_1 - x_3} = \frac{[x_1, x_2] - [x_2, x_3]}{x_1 - x_3}$$

Now the 3$^{\text{rd}}$ divided difference for x_0, x_1, x_2, x_3 is defined as

$$\underset{x_1\, x_2\, x_3}{\Delta} f(x_0) = [x_0, x_1, x_2, x_3] = \frac{[x_0, x_1, x_2] - [x_1, x_2, x_3]}{(x_0 - x_3)}$$

In general, the nth divided difference for x_0, x_1, …, x_n is defined as

$$\underset{(x_1, x_2, \ldots, x_3)}{\Delta} f(x_0) = [x_0, x_1, x_2, \ldots, x_n]$$

$$= \frac{f(x_0, x_1, \cdots, x_{n-1}) - f(x_1, x_2, \cdots, x_n)}{x_0 - x_n}$$

(i) Newton's Divided Differences Formula

Newton's divided general interpolation formula is given as following:

$$f(x) = f(x_0) + (x - x_0)\ \Delta\ f(x_0) + (x - x_0)(x - x_1)\ \Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\ \Delta^3 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})\ \Delta^n f(x_0)$$

15. Inverse Interpolation

The Inverse interpolation is given as following:

$$P_n(y) = \sum_{i=0}^{n} L_i(y)\, f(y_i), \text{ where for each } i = 0, 1, 2, \ldots, n$$

$$L_i(y) = \frac{(y - y_0)(y - y_1)(y - y_2)\cdots(y - y_{i-1})(y - y_{i+1})\cdots(y - y_n)}{(y_i - y_0)(y_i - y_1)(y_i - y_2)\cdots(y_i - y_{i-1})(y_i - y_{i+1})\cdots(y_i - y_n)}$$

16. Hermite Interpolation Polynomial

If f is continuous on $[a, b]$ and $x_0, x_1, \ldots, x_n \in [a, b]$ are distinct, then the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \ldots, x_n$ is the Hermite polynomial of degree at most $(2n + 1)$ given by

$$H_{2n+1}(x) = \sum_{i=0}^{n} H_i(x)\, f(x_i) + \sum_{i=0}^{n} \hat{H}_i(x)\, f'(x_i)$$

where $$H_i(x) = \left[1 - 2(x - x_i) L_i'(x_i)\right] L_i^2(x)$$

and $$\hat{H}_i(x) = (x - x_i)\, L_i^2\,(x)$$

Also, $L_i(x)$ denoting the i^{th} Lagrange's coefficient of degree n.

OBJECTIVE TYPE QUESTIONS

1. The value of $E^n f(x)$ will be
(a) $f(x-h)$ (b) $f(x-nh)$
(c) $f(x+nh)$ (d) $f(x)$

2. If degree of $f(x) = n$ then $\Delta^n f(x)$ is
(a) constant (b) zero
(c) 1 (d) $(n-1)!$

3. If $\Delta f(x) = x(x-1)$, then $f(x)$ will be
(a) $x(x-1)(x-2)+c$
(b) $x(x-2)+c$
(c) $(x-1)(x-2)+c$
(d) $\frac{1}{3}x(x-1)(x-2)+c$

4. If $f(0) = -3, f(1) = 6, f(2) = 8$ and $f(3) = 12$, then $\Delta^3 f(0)$ is
(a) 9 (b) 6
(c) 3 (d) 5

5. If $f(x) = 2x^3 - 3x^2 + 3x - 10$ then $\Delta^3 f(x)$ is
(a) 3 (b) 6
(c) 2 (d) 12

6. If $y = a(2)^x + bx(2)^x$ and $h = 1$ then $(\Delta^2 - 2\Delta + 1)y$ is
(a) 0 (b) 1
(c) 2 (d) 8

7. The value of $(E^{\frac{1}{2}} + E^{-\frac{1}{2}})(1+\Delta)^{\frac{1}{2}}$ is
(a) Δ (b) $2+\Delta$
(c) $2-\Delta$ (d) $\Delta \times \nabla$

8. The value of $\nabla^2 y_5$ for the following data

x	1	2	3	4	5
y	2	5	10	17	26

(a) 1 (b) 2
(c) –2 (d) 4

9. For the values of $f(0) = 3, f(1) = 6, f(2) = 11, f(3) = 18$ and $f(4) = 27$, then the function $f(x)$ will be
(a) $x^2 + 3x + 3$ (b) $2x^2 + 3x + 1$
(c) $x^2 + 2x + 3$ (d) $x^2 - 2x - 1$

10. A second degree polynomial $P(x)$ passes through the following data

x	0	1	2	3
y	1	3	7	13

(a) $x^2 + x + 1$ (b) $x^2 + 3x + 4$
(c) $x^2 - 2x + 5$ (d) $3x^2 - x - 1$

11. The value of

$$\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1+dx^4)\right] \text{ is}$$

(a) $abcd(8!)$ (b) $abc(10!)$
(c) $abcd(10!)$ (d) 0

12. If $f(x) = 3x^3 - 2x^2 + 1$, then $\Delta^3 f(x)$ will be
(a) 18 (b) 20
(c) –18 (d) 41

13. If $f(x) = x^3$, then $\underset{a^3,b^3,c^3}{\Delta^3} f(x)$ is
(a) $a^3 + b^3 + c^3$ (b) 0
(c) $a + b + c$ (d) $a^2 + b^2 + c^2$

14. The Lagrange's interpolation formula can be used when the values $x_0, x_1, x_2, \ldots, x_n$ are
(a) zero spaced only
(b) unequally spaced only
(c) equally spaced or not
(d) none of these

15. If $f(x) = \frac{1}{x}$ at $x = a, b, c, d$; then $\underset{bcd}{\Delta^3} f(x)$ is
(a) $abcd$ (b) $-\frac{1}{abcd}$
(c) $\frac{1}{abcd}$ (d) $(-1)^2 \frac{1}{abc}$

16. Which is correct
(a) $\Delta \equiv 1 + E^{-1}$ (b) $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$
(c) $\Delta \equiv 1 + E$ (d) all the above

17. A polynomial $P(x)$ satisfies the following $P(1) = P(3) = P(5) = 1, P(2) = P(4) = -1$. The minimum degree of such a polynomial is
(a) 1 (b) 2
(c) 3 (d) 4

[GATE (CS) 2000]

18. The values of a function $f(x)$ are tabulated below

x	0	1	2	3
f(x)	1	2	1	10

Using Newton's forward difference formula, the cubic polynomial that can be fitted to the above data is
(a) $2x^3 + 7x^2 - 6x + 2$
(b) $2x^3 - 7x^2 + 6x - 2$
(c) $x^3 - 7x^2 - 6x + 1$
(d) $2x^3 - 7x^2 + 6x + 1$ **[GATE (ME) 2004]**

19. Match the correct combination

P	Gauss–Siedel method	1.	Interpolation
Q	Newton's forward formula	2.	Non-linear differential equations
R	Runge–Kutta method	3.	Numerical integration
S	Simpson's rule	4.	Linear algebraic equations

(a) P-1, Q-4, R-3, S-2
(b) P-1, Q-4, R-2, S-3
(c) P-1, Q-3, R-3, S-4
(d) P-4, Q-1, R-2, S-2

[GATE (ME) 2006]

20. $\Delta^2 E^{-2}$ is equal to
(a) $1 - 2E^{-1} + E^{-2}$ (b) $1 + 2E^{-1} + E^{-2}$
(c) $2E^{-1} + E^{-2} - 1$ (d) $1 + 2E + E^2$

21. If $E^2 u_x = x^2$ and $h = 1$, then u_x is equal to
(a) $(x - 2)^2$ (b) $(x + 2)^2$
(c) $(x + 2)^3$ (d $\dfrac{1}{(x-2)^2}$

22. The value of $\left(\dfrac{\Delta^2}{E}\right)x^3$ is $(h = 1)$
(a) $6x$ (b) $-6x$
(c) $3x$ (d) $6x^2$

23. The value of $(\Delta - \nabla)\, x^2$ is $(h = 1)$
(a) 2 (b) 3
(c) 1 (d) 4

24. The value of $\Delta^2\,(ab^x)$ is
(a) $a(b^h - 1)b^x$ (b) $a(b^h - 1)^2.b^x$
(c) 0 (d) None of these

25. The value of $\Delta[\log f(x)]$ is
(a) $\log\left(1+\dfrac{\Delta f(x)}{f(x)}\right)$ (b) $1+\dfrac{\Delta f(x)}{f(x)}$
(c) $\log\left(1-\dfrac{\Delta f(x)}{f(x)}\right)$ (d) 0

ANSWERS

1. (c)	2. (a)	3. (d)	4. (a)	5. (d)	6. (a)	7. (b)	8. (b)	9. (c)	10. (a)
11. (c)	12. (a)	13. (c)	14. (c)	15. (b)	16. (b)	17. (d)	18. (d)	19. (d)	20.(a)
21.(a)	22.(a)	23.(a)	24.(b)	25.(a)					

9 Numerical Solution of Equations

9.1 INTRODUCTION

Numerical methods hold great importance in scientific and engineering problems for determining the roots of an equation

$$f(x) = 0 \tag{1}$$

Algebraic formulae are used to determine the roots of Eq. (1), when it is a quadratic, cubic or a biquadratic. On the other hand when Eq. (1) is a polynomial of higher degree or an expression involving transcendental functions algebraic methods are not available, then we use the numerical methods.

In this section, we shall discuss the basic terminology of the equations and some useful numerical method like Bisection, Regula Falsi, Iteration and Newton Raphson methods.

Polynomial An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$, where all a_i's are constants, provided $a_0 \neq 0$ and n is a positive integer called a polynomial in x of degree n.

Polynomial or algebraic equation An equation of the form $f(x) = 0$ is algebraic if it contains power of x, i.e., $f(x)$ is a polynomial.

For example, $x^2 + x + 3 = 0$, $3x = 7$, $x^7 = x(1+3x)$, etc.

Transcendental equation Equation (1) is called transcendental equation, if it contains power of x and some other functions such as hyperbolic, trigonometric, logarithmic, exponential, etc.

For example, $x + \sin x = 0$, $e^{\sqrt{x}} = x^2$, $\tan x = x$, etc.

Root of equation The value of x which satisfies Eq. (1) say x = a is called root of Eq. (1).

Geometrically, a root of Eq. (1) is that value of x, where the graph $y = f(x)$ cuts the x-axis.

9.2 SOME BASIC PROPERTIES OF AN EQUATION

Some of the basic properties of an equation are as follows:

(i) The total number of roots of an algebraic equation is the same as its degree.

(ii) If $f(x)$ is exactly divisible by $(x - a)$, then $x = a$ is a root of equation $f(x) = 0$.

(iii) If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2x^{n-2} + \cdots + a_n$ have the roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, then

$$\text{Sum of roots} = \sum_i \alpha_i = -\frac{a_1}{a_0};$$

Sum of the product of two roots $= \sum_{i<j} \alpha_i \alpha_j = \frac{a_2}{a_0}$; and product of the roots $= \underset{i}{\pi}\, \alpha_i = (-1)^n \frac{a_n}{a_0}$.

(iv) Every equation of the odd degree has at least one real root.

(v) If $\alpha + i\beta$ is a root of the equation $f(x) = 0$, then $\alpha - i\beta$ must also be its root.

(vi) An algebraic equation can have at most as many positive roots as the number of changes of sign in the coefficients of $f(x)$.

(vii) An algebraic equation can have at most as many negative roots as the number of changes of sign in the coefficient of $f(-x)$.

(viii) If an algebraic equation of degree 'n' has at the most α_P positive roots and at the most α_n negative roots, then the equation has at least $(n - \alpha_p - \alpha_n)$ imaginary roots.

(ix) *Intermediate Value Theorem*: If $f(x)$ is a continuous function on $[a, b]$ and the sign of $f(a)$ is different from the sign of $f(b)$; that is $f(a) \cdot f(b) < 0$, then there exists a point c, in the interval (a, b) such that $f(c) = 0$. Hence, any value $c \in (a, b)$ can be taken as an initial approximation to the root.

Note 1. Using the above theorem, the equation $f(x) = 0$ has at least one real root or an odd number of real roots in (a, b).

2. Interval in the form of may be integer values or fractional values where the given function have opposite signs.

Example 1 Equation $f(x) \equiv x^5 - 6x^2 - 3x + 4 = 0$.

Solution Changes of sign in $f(x)$: $+ - - + = 2$. Hence, the number of positive roots = 2

Changes of sign in $f(-x)$: $- - + + = 1$

Hence, the number of negative roots = 1

Thus, the number of imaginary roots

= degree of equation – (sum of positive and negative roots)

$= 5 - (2 + 1)$

$= 5 - 3$

$= 2.$

Example 2 Find the initial approximation to the root of the equation

$2x - \log_{10} x = 7.$

Solution Let $f(x) \equiv 2x - \log_{10} x - 7 = 0$.

The values of $f(x)$ are as given:

x	1	2	3	4
$f(x)$	–5	–3.301	–1.477	0.397

Since $f(3) \cdot f(4) < 0$. Hence, any value between $x = 3$ and $x = 4$, i.e., (3, 4) can be taken as an initial approximation.

9.3 BISECTION METHOD

This is one of the simplest method and is based on the Intermediate value theorem. This method is also known as the bisection method of Bolzano or binary-search method.

This bisection method is defined as follows:

(i) Using Intermediate value theorem, find an interval (a, b) if $f(a)\,f(b) < 0$, then the root lies in (a, b).

(ii) The first approximation to the root is $x_1 = \frac{a+b}{2}$. If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$, otherwise.

(iii) Use the Intermediate value theorem to decide whether the root lies in (a, x_1) or (x_1, b).

(iv) Repeat the step using the interval either (a, x_1) or (x_1, b).

(v) The procedure is repeated while a length of the last interval is less than the desired accuracy. The mid-point of this last interval is the required root of the given equation $f(x) = 0$.

9.3.1 Geometrical Interpretation

The function $f(x)$ has a desired root between a and b, since function $f(x)$ intersects the X-axis. The midpoint of a and b is x_1, i.e. $x_1 = \frac{a+b}{2}$. It is also observed that the function $f(x)$ intersects the x-axis in between a and x_1. Again find $x_2 = \frac{a+x_1}{2}$ and we observed that the function $f(x)$ intersects the X-axis in between x_1 and x_2, then find $x_3 = \frac{x_1+x_2}{2}$. Thus, we observe that after every bisection the length of interval is reduced and we approaches the desired root of the equation $f(x) = 0$, as shown in Fig. 9.1.

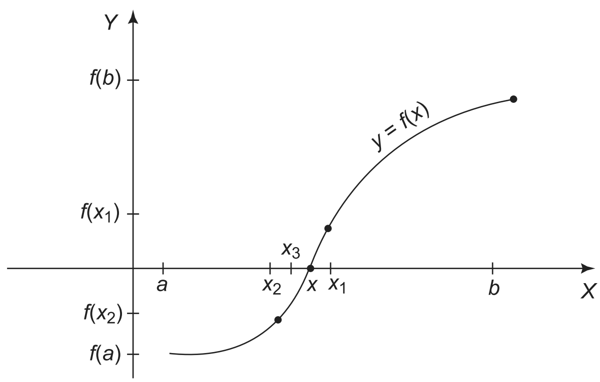


Fig. 9.1 *Geometrical interpretation of Bisection method*

9.3.2 Error Analysis

The maximum error after the n^{th} iteration using $\in_n \geq \frac{|b-a|}{2^n}$

Taking logarithms on both sides and on simplification, we obtain

$$n \geq \frac{\left[\log_e (b-a) - \log_e \in_n\right]}{\log_e 2} \tag{2}$$

Inequality in Eq. (2) gives the number of iteration required to achieve an accuracy $\in$, we have $\frac{\in_{n+1}}{\in_n} = \frac{1}{2}$. Thus, this method converges linearly.

Example 3 Find a real root of the equation $f(x) = 2^x - 3x = 0$ in interval $0 \le x \le 2$ by the bisection method.

Solution Since $f(0) = 1 > 0$ and $f(2) = -2 < 0$, hence, a root lies between 0 and 2.

Iteration 1: Let $a = 0$ and $b = 2$, then the first approximation $x_1 = \frac{a+b}{2} = \frac{0+2}{2} = 1$

Now $f(x_1) = f(1) = 2^1 - 3\times 1 = -1 < 0$ and $f(0)\,f(1) < 0$

$\therefore$ The root lies between 0 and 1.

Iteration 2: The 2nd approximation

$$x_2 = \frac{0+1}{2} = 0.5$$

Now $f(x_2) = f(0.5) = -0.0857 < 0$ and $f(0)\,f(0.5) < 0$

$\therefore$ The root lies between 0 and 0.5

Iteration 3: The 3rd approximation

$$x_3 = \frac{0+0.5}{2} = 0.25$$

Now $f(x_3) = f(0.25) = 0.439 > 0$

and $f(0.25)\cdot f(0.5) < 0$

$\therefore$ The root lies between 0.25 and 0.5

Iteration 4: $x_4 = \frac{0.25+0.5}{2} = 0.375$

Now $f(x_4) = f(0.375) = 0.172 > 0$ and $f(0.25)\cdot f(0.172) < 0$

The root lies between 0.172 and 0.25.

The successive approximations by bisection method are tabulated in the following table:

Iteration	a	b	$x = \frac{a+b}{2}$	$f(x)$
1	0	2	1	–1
2	0	1	0.5	–0.857
3	0	0.5	0.25	0.4392
4	0.25	0.5	0.375	0.1718
5	0.375	0.5	0.4375	0.04176
6	0.4375	0.5	0.46875	–0.0223
7	0.4375	0.46875	0.45313	0.009627
8	0.45313	0.46875	0.46094	–0.006377
9	0.5313	0.46094	0.45703	0.001620
10	0.45703	0.46094	0.45898	–0.002379
11	0.45703	0.45898	0.458007	–0.0003798

Hence, the solution of this equation is 0.458 after 11 iterations.

Example 4 Find a real root of the equation $x \log_{10}x = 1.2$ by bisection method correct to 4 decimal places. **[G.E.U. 2015]**

Solution

Let $f(x) = x \log_{10}x - 1.2$

Since $f(2.74) = -0.000563 < 0$

and $f(2.75) = 0.0081649 > 0$

Hence, a root lies between 2.74 and 2.75.

Iteration 1: Let $a = 2.74$ and $b = 2.75$, then the first approximation $x_1 = \dfrac{2.74+2.75}{2} = 2.745$

Now $f(x_1) = f(2.745) = 0.003798 > 0$ and $f(2.74)\cdot f(2.745) < 0$

Hence, root lies between 2.74 and 2.745.

Iteration 2: The 2nd approximation $x_2 = \dfrac{2.74+2.745}{2} = 2.7425$

Now $f(x_2) = f(2.7425) = 0.001617 > 0$

and $f(2.74)\cdot f(2.7425) < 0$

$\therefore$ The root lies between 2.74 and 2.7425.

Iteration 3: The 3rd approximation $x_3 = \dfrac{2.74+2.7425}{2} = 2.74125$

Now $f(x_3) = f(2.74125) = 0.0005267 > 0$

and $f(2.74)\cdot f(2.74125) < 0$

$\therefore$ The root lies between 2.74 and 2.74125.

Iteration 4: The 4th approximation

$$x_4 = \frac{2.74+2.74125}{2} = 2.740625$$

Now $f(x_4) = f(2.740625) = -0.00001839 < 0$

and $f(2.74125)\cdot f(2.740625) < 0$

$\therefore$ The root lies between 2.740625 and 2.74125

Iteration 5: The 5th approximation

$$x_5 = \frac{2.740625+2.74125}{2} = 2.7409375$$

Now $f(x_5) = f(2.7409375) = 0.000254 > 0$

and $f(2.7409375)\cdot f(2.740625) < 0$

$\therefore$ The root lies between 2.7409375 and 2.740625.

Iteration 6: The 6th approximation

$$x_6 = \frac{2.7409375+2.740625}{2} = 2.74078125$$

Now $\qquad f(x_6) = f(2.74078125) = 0.0001178 > 0$

and $\qquad f(2.740625)\cdot f(2.74078125) < 0$

$\therefore$ The root lies between 2.740625 and 2.74078125.

Iteration 7: The 7th approximation is

$$x_7 = \frac{2.740625 + 2.74078125}{2} = 2.740703125$$

Now $\qquad f(x_7) = f(2.740703125) = 0.00004973 > 0$

and $\qquad f(2.740625)\cdot f(2.740703125) < 0$

$\therefore$ The root lies between 2.740625 and 2.740703125.

Iteration 8: The 8th approximation is

$$x_8 = \frac{2.740625 + 2.740703125}{2} = 2.740664063$$

Now $\qquad f(x_8) = f(2.740667063) = 0.00001567 > 0$

and $\qquad f(2.740625)\cdot f(2.740664063) < 0$

$\therefore$ The root lies between 2.740625 and 2.740664063.

Iteration 9: The 9th approximation is

$$x_9 = \frac{2.740625 + 2.740664063}{2} = 2.7406445$$

Since x_8 and x_9 are equal up to 4 decimal places.

Hence, the required root of the given equation is $x = 2.7406$.

Example 5 Find a positive real root of the equation $x - \cos x = 0$ by bisection method correct up to 4 decimal places between 0 and 1. **[U.P.T.U. 2002]**

Solution

Let $\qquad f(x) = x - \cos x = 0$

Since $\qquad f(0) = 0 - \cos 0 = -1 < 0$

and $\qquad f(1) = 1 - \cos 1 = 0.45970 > 0$

Now, perform the same iteration as in example 4, after iteration 7 or 8, we reach to the desired root up to 4 decimal places.

Hence, the root $x = 0.7391$.

9.4 FIXED POINT ITERATION METHOD

A fixed point α for a function $\phi(x)$ is a number, when the value of the function does not change.

Fixed point iteration or successive approximation method find a root of an Eq. (1).

The first step of this method is to write the Eq. (1) is of the form

$$x = \phi(x) \tag{3}$$

Let x_0 be an initial approximation to the root of Eq. (1), then the first approximation is $x_1 = \phi(x_0)$.

The successive approximations are as follows:

$$x_2 = \phi(x_1)$$
$$x_3 = \phi(x_2)$$
$$x_4 = \phi(x_3)$$

In general, $x_{n+1} = \phi(x_n); n = 0, 1, 2, 3, ...$ is called the fixed-point iteration.

9.5 GEOMETRICAL INTERPRETATION OF ITERATION METHOD

Let $f(x) = 0$ is an equation and it can be rewritten in to the form of $x = \phi(x)$. Now, the intersection of the curve $y = \phi(x)$ and the line $y = x$ gives the root of the equation $f(x) = 0$. Let the initial arbitrary root $x = x_0$. If the point (x_0, y_1) does not lie on the line $y = x$, then we consider the another point (x_1, y_1) and check if it is on the line or not. The process will continue and if it is converges, then we obtain a desired root.

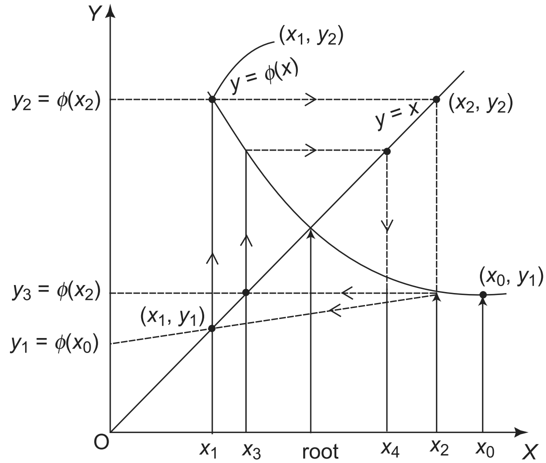


Fig. 9.2

9.5.1 Fixed-point Theorem

Let $\phi \in C[a, b]$ be such that $\phi(x) \in [a, b]$, for all $x \in [a, b]$. Suppose $\phi'(x)$ exists on (a, b) and constant $0 < k < 1$ exists with

$$|\phi'(x)| \le k, \forall\ x \in (a, b)$$

Then, for any $x_0 \in [a, b]$, the sequence

$$x_{n+1} = \phi(x_n); n = 0, 1, 2, 3, ...$$

Converges to the unique fixed point x in $[a, b]$.

Remarks

(i) The round off errors are minimized where as these errors are computed by other methods.

(ii) The iteration method for $x = \phi(x)$ is convergent if $|\phi'(x)| < 1$. If $|\phi'(x)| = 1$ and $\phi'(x) < 1$, the number of iterations would be large. But $\phi'(x)$ is very small, then the number of iterations will be reduced.

(iii) If $|\phi'(x)| > 1$, then the iterative process is divergent.

Example 6 Find a real root of an equation $2x - \log_{10}x = 7$ correct to four decimal places using iteration method.

[U.P.T.U. 2004]

Solution Given

$$f(x) \equiv 2x - \log_{10} x - 7 = 0 \tag{4}$$

Now, find opposite sign function, we have

$$f(3) = 6 - \log_{10}3 - 7 = -1.4471$$

and $$f(4) = 8 - \log_{10}4 - 7 = 0.3980$$

$\therefore \quad f(3)\cdot f(4) < 0$

Hence, a root lies between 3 and 4.

Now, Eq. (4), rewritten in to the form of

$$x = \frac{1}{2}(\log_{10} x + 7) = \phi(x) \qquad \text{(say)}$$

$$\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right) \qquad \left[\because \quad \log_{10} x = \log_e x \cdot \log_{10} e\right]$$

$\therefore \quad |\phi'(x)| < 1;\ x \in (3, 4)$ and $\log_{10}e = 0.4343$

Since $|f(4)| < |f(3)|$, so the required root is near to 4.

Hence, iteration method is applicable.

Let the initial approximation $x_0 = 3.6$, then the successive approximations are as follows:

$$x_1 = \phi(x_0) = \frac{1}{2}\left[\log_{10} x_0 + 7\right]$$

or $$x_1 = \phi(3.6) = \frac{1}{2}\left[\log_{10} 3.6 + 7\right] = 3.77815$$

$$x_2 = \phi(x_1) = \frac{1}{2}\left[\log_{10} 3.77815 + 7\right] = 3.78863$$

$$x_3 = \phi(x_2) = \frac{1}{2}\left[\log_{10} 3.78863 + 7\right] = 3.78924$$

$$x_4 = \phi(x_3) = \frac{1}{2}\left[\log_{10} 3.78924 + 7\right] = 3.78927$$

Since x_3 and x_4 are equal. Hence, the required root is $x = 3.7892$, correct to 4 decimal places.

Example 7 Obtain a real root of the equation $1 + \cos x = 3x$ correct to three decimal places using iteration method.

Solution We have

$$f(x) \equiv \cos x - 3x + 1 = 0 \tag{5}$$

To find an interval for which the function $f(x)$ have the opposite signs.

Now, $$f(0) = \cos 0 - 3 \times 0 + 1 = 2$$

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \frac{3\pi}{2} + 1 = \left(1 - \frac{3\pi}{2}\right) = -3.71239$$

$$\therefore \qquad f(0)\cdot f\left(\frac{\pi}{2}\right) < 0$$

Hence, the required root lies in $\left(0, \frac{\pi}{2}\right)$

Equation (5) can be rewritten into the form of

$$x = \frac{1}{3}(1 + \cos x) \equiv \phi(x) \quad \text{(say)}$$

We have $\qquad \phi'(x) = -\dfrac{\sin x}{3}$

and

$$|\phi'(x)| = \left|-\frac{1}{3}\sin x\right|$$

$$= \frac{1}{3}|\sin x|$$

$$|\phi'(x)| < 1; \quad x \in \left(0, \frac{\pi}{2}\right)$$

Since, $\qquad |f(0)| > \left|f\left(\dfrac{\pi}{2}\right)\right|$

Thus, the required root is near to 0.
Let the initial approximation $x_0 = 0$.
Then the successive approximations are as follows:

$$x_1 = \phi(x_0) = \phi(0) = \frac{1}{3}(1 + \cos 0) = \frac{2}{3} = 0.6667$$

$$x_2 = \phi(x_1) = \phi(0.6667) = \frac{1}{3}(1 + \cos 0.6667) = 0.5953$$

$$x_3 = \phi(x_2) = \frac{1}{3}(1 + \cos 0.5953) = 0.6093$$

$$x_4 = \phi(x_3) = \frac{1}{3}(1 + \cos 0.6093) = 0.6067$$

$$x_5 = \phi(x_4) = \frac{1}{3}(1 + \cos 0.6067) = 0.6072$$

$$x_6 = \phi(x_5) = \frac{1}{3}(1 + \cos 0.6072) = 0.6071$$

Since x_5 and x_6 are almost equal.
Hence, the required root is $x = 0.607$ correct to 3 decimal places.

***Note*:** The another interval is (0, 1), where $f(0) = 2$ and $f(1) = -1.4597$.

Example 8 Compute a real root of the equation $10(x-1) = \sin x$ using the iteration method.

Solution Given

$$f(x) \equiv \sin x - 10x + 10 = 0 \tag{6}$$

Obtain an interval for which the function $f(x)$ have opposite signs.

Now, $f(1) = \sin 1 - 10 \times 1 + 10 = 0.84147$

$$f(2) = \sin 2 - 10 \times 2 + 10 = -9.09070$$

$\therefore$ $f(1)\,f(2) < 0$

Hence, the required root lies in (1, 2)

We can rewrite Eq. (6) as

$$x = 1 + \frac{\sin x}{10} \equiv \phi(x)$$

Now $$\phi'(x) = \frac{\cos x}{10}$$

$\therefore$ $$|\phi'(x)| = \frac{1}{10}|\cos x| < 1; \quad x \in (1, 2)$$

Thus, the iteration method is applicable.

Since, $|f(1)| < |f(2)|$, so the root is near to 1. Assume initial value of x is 1. Then the successive approximations are as follows:

$$x_1 = \phi(x_0) = 1 + \frac{1}{10}\sin(1) = 1.08415$$

$$x_2 = \phi(x_1) = 1 + \frac{1}{10}\sin(1.08415) = 1.08839$$

$$x_3 = \phi(x_2) = 1 + \frac{\sin(1.08839)}{10} = 1.08859$$

$$x_4 = \phi(x_3) = 1 + \frac{\sin(1.08859)}{10} = 1.08860$$

$$x_5 = \phi(x_4) = 1 + \frac{\sin(1.08860)}{10} = 1.08860$$

Since, x_4 and x_5 are almost equal.

Hence, the required root is $x = 1.08860$.

9.6 ITERATION METHOD FOR THE SYSTEM OF NON-LINEAR EQUATIONS

Consider the non-linear system of equation are as follows:

$$\left.\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned}\right\} \tag{7}$$

whose real roots are to be required within the given degree of accuracy.

Let us assume

$$\left.\begin{aligned} x &= \phi(x, y) \\ y &= \psi(x, y) \end{aligned}\right\} \tag{8}$$

where the functions ϕ and ψ satisfy the conditions

$$\left|\frac{\partial \phi}{\partial x}\right| + \left|\frac{\partial \phi}{\partial y}\right| < 1 \text{ and } \left|\frac{\partial \psi}{\partial x}\right| + \left|\frac{\partial \psi}{\partial y}\right| < 1 \tag{9}$$

in the neighbourhood of the root.

Suppose (α, β) be the exact roots of Eq. (7) and let (x_0, y_0) be the initial approximations.

Then from Eq. (8), the successive approximations are as follows:

$$x_1 = \phi(x_0, y_0), \quad y_1 = \psi(x_0, y_0)$$

$$x_2 = \phi(x_1, y_1), \quad y_2 = \psi(x_1, y_2)$$

$$x_3 = \phi(x_2, y_2), \quad y_3 = \psi(x_2, y_2)$$

and so on, we get following two iterative formulae:

$$x_{n+1} = \phi(x_n, y_n) \text{ and } y_{n+1} = \psi\,(x_n, y_n)$$

If these formulae converges, then we get

$$\alpha = \phi(\alpha, \beta) \text{ and } \beta = \psi(\alpha, \beta)$$

Hence, (α, β) gives the root of system of Eq. (7).

Example 9 Using iteration method, find a real root of the system of equations

$$x = 0.2x^2 + 0.8$$

$$y = 0.3xy^2 + 0.7$$

Solution Let us assume that

$$\left.\begin{aligned} \phi(x, y) &= 0.2x^2 + 0.8 \\ \psi(x, y) &= 0.3xy^2 + 0.7 \end{aligned}\right\} \tag{10}$$

Now, $$\frac{\partial \phi}{\partial x} = 0.4x, \quad \frac{\partial \phi}{\partial y} = 0$$

and $$\frac{\partial \psi}{\partial x} = 0.3y^2, \quad \frac{\partial \psi}{\partial y} = 0.6\,xy$$

Initially, we choose $x_0 = \frac{1}{2}, y_0 = \frac{1}{2}$. Then

$$\left|\frac{\partial \phi}{\partial x}\right|_{\left(\frac{1}{2}, \frac{1}{2}\right)} + \left|\frac{\partial \phi}{\partial y}\right|_{\left(\frac{1}{2}, \frac{1}{2}\right)} = 0.2 < 1$$

and $$\left|\frac{\partial \psi}{\partial x}\right|_{\left(\frac{1}{2}, \frac{1}{2}\right)} + \left|\frac{\partial \psi}{\partial y}\right|_{\left(\frac{1}{2}, \frac{1}{2}\right)} = \frac{0.3}{4} + \frac{0.6}{4}$$

$$= \frac{0.9}{4} < 1$$

Thus, conditions in Eq. (9) are satisfied.

Hence, the successive approximations are as follows:

$$x_1 = \phi(x_0, y_0) = \phi\left(\frac{1}{2}, \frac{1}{2}\right) = (0.2)\left(\frac{1}{2}\right)^2 + 0.8 = 0.85$$

$$y_1 = \psi(x_0, y_0) = \psi\left(\frac{1}{2}, \frac{1}{2}\right) = (0.3)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + 0.7 = 0.74$$

$$x_2 = \phi(x_1, y_1) = \phi(0.85, 0.74) = (0.2)(0.85)^2 + 0.8 = 0.9445$$

$$y_2 = \psi(x_1, y_1) = \psi(0.85, 0.74) = (0.3)(0.85)(0.74)^2 + 0.7 = 0.8396$$

$$x_3 = \phi(x_2, y_2) = (0.2)(0.9445)^2 + 0.8 = 0.9784$$

$$y_3 = \psi(x_2, y_2) = (0.3)(0.9445) + (0.8396)^2 + 0.7 = 0.8997$$

From these three approximations, we conclude that the root converges to (1, 1).

Also from Eq. (10), we get

$$\phi(1, 1) = 1 \text{ and } \psi(1, 1) = 1$$

9.7 NEWTON'S METHOD

Newton–Raphson (or Newton's) method is one of the most powerful and well-known numerical method.

Let x_0 be an approximate root of the equation $f(x) = 0$ and let $(x_0 + \Delta x)$ be an exact root.

Therefore, $f(x_0 + \Delta x) = 0$ (11)

Expanding Eq. (11) in Taylor's series, we obtain

$$f(x_0) + \Delta x\, f'(x_0) + \frac{(\Delta x)^2}{2!} f''(x_0) + \cdots = 0$$

Neglecting the second and higher powers of Δx, we have

$$f(x_0) + \Delta x\, f'(x_0) = 0$$

or
$$\Delta x = -\frac{f(x_0)}{f'(x_0)}$$

Hence, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ is an approximation.

Thus, the successive approximations is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; n = 0, 1, 2, 3, \ldots$$

which is the Newton's–Raphson formula.

9.7.1 Geometrical Interpretation

In the Fig. (9.3), starting with the initial approximation x_0, the approximation x_1 is the x-intercept of the tangent line to the curve $y = f(x)$ at the point $A[x_0, f(x_0)]$. Now the approximation x_2 is the x-intercept of the tangent line to the curve $y = f(x)$ at the point $B[x_1, f(x_1)]$ and so on.

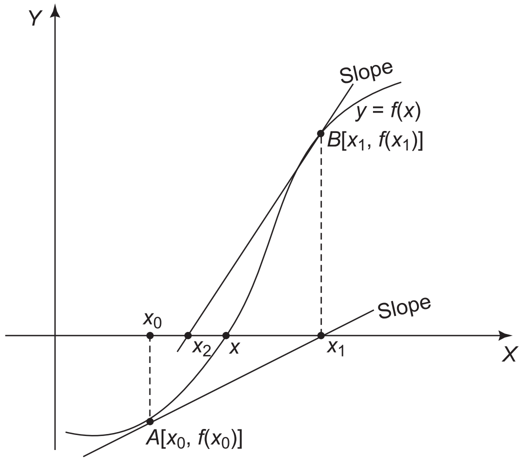


Fig. 9.3

9.7.2 Steps of Newton's Method

Following are the steps for Newton's Method:

1. Find the interval, where the function $f(x)$ have opposite signs.
2. Take initial approximation (x_0) in the interval.
3. Find $f(x_0)$ and $f'(x_0)$
4. Compute the approximations using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; n = 0, 1, 2, 3, \ldots$$

5. Continuing this process till, when the two successive approximations are almost same. i.e., $|x_{n+1} - x_n| < \text{error}; \ n = 0, 1, 2, \ldots$

9.7.3 Some Important Points for Newton's Method

Some important points related Newton's Method are as follows:

1. It is useful, when the curve $y = f(x)$ crossing the x-axis is nearly vertical.
2. It converges, provided the initial approximation is chosen sufficiently close to the root.
3. It can be used to compute complex roots.
4. It gives better results as compared to other methods.
5. It is conditionally convergent.
6. It converges, if $|f(x).f'(x)| < |f'(x)|^2$.
7. It has a quadratic convergence.
8. It has a linear convergence for the double root.

Special Note The Newton's method does not always converge to a simple root. It has

(i) divergent sequence for the function $f(x) = xe^{-x}$ with $x_0 = 2$.

(ii) divergent oscillating sequence for the function $f(x) = \tan^{-1} x$ with $x_0 = 1.44$ and

(iii) cyclic sequence for the function $f(x) = x^3 - x - 3$ with $x_0 = 0$.

Example 10 Find a real root of the equation $3x = \cos x + 1$, using Newton's method.

Solution

Let $f(x) \equiv 3x - \cos x - 1 = 0$ (12)

$f'(x) = 3 + \sin x$ (13)

Now $f(0) = 3 \times 0 - \cos 0 - 1 = -2 < 0$

and $f(1) = 3 \times 1 - \cos 1 - 1 = 1.4597 > 0$

Since $f(0)$ and $f(1)$ are opposite signs. Hence, the root lies in [0, 1].

Let us take $x_0 = 0.6$.

The Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

or
$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}; n = 0, 1, 2, 3, \ldots \tag{14}$$

The first approximation to the root is

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0.6 \sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}$$

$$= 0.6071$$

The second approximation to the root is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{(0.6071)\sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$

$$x_2 = 0.6071$$

Since $x_1 = x_2$, hence, the root is 0.6071 correct to four decimal places.

Example 11 Using Newton's method to find a real root of the equation $x \log_{10} x = 1.2$ correct to four decimal places. **[U.P.T.U. 2004, M.D.U. 2004, 2006, G.E.U. 2010]**

Solution

Let $f(x) \equiv x \log_{10} x - 1.2 = 0$ (15)

$$f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e \qquad \left[\because \quad \log_{10} x = \log e^x \cdot \log_{10} e\right]$$

$$= \log_{10} x + \log_{10} e = \log_{10} x + 0.4343 \tag{16}$$

Now $f(1) = -1.2 < 0$

$f(2) = 2\log_{10}2 - 1.2 = -0.5979 < 0$

and $f(3) = 3\log_{10}3 - 1.2 = 0.2314 > 0$

Since $f(2)$ and $f(3)$ are opposite sign, therefore root lies in [2, 3].
Let the initial approximation $(x_0) = 2$
Now, the Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.4343}$$

or
$$x_{n+1} = \frac{0.4343\, x_n + 1.2}{\log_{10} x_n + 0.4343};\quad n = 0, 1, 2, 3, \ldots \tag{17}$$

Putting $n = 0$ in Eq. (17), we get the first approximation is

$$x_1 = \frac{0.4343\, x_0 + 1.2}{\log_{10} x_0 + 0.4343} = \frac{(0.4343)\times 2 + 1.2}{\log_{10} 2 + 0.4343} = 2.8133$$

Putting $n = 1$ in Eq. (17), the second approximation is

$$x_2 = \frac{0.4343\, x_1 + 1.2}{\log_{10} x_1 + 0.4343} = \frac{(0.4343)(2.8133) + 1.2}{\log_{10}(2.8133) + 0.4343} = 2.7411$$

Putting $n = 2$ in Eq. (17), the third approximation

$$x_3 = \frac{0.4343\, x_2 + 1.2}{\log_{10} x_2 + 0.4343} = \frac{(0.4343)(2.7411) + 1.2}{\log_{10}(2.7411) + 0.4343} = 2.7408$$

Putting $n = 3$ in Eq. (17), the fourth approximation is

$$x_4 = \frac{0.4343\, x_3 + 1.2}{\log_{10} x_3 + 0.4343} = \frac{(0.4343)(2.7408) + 1.2}{\log_{10}(2.7408) + 0.4343} = 2.7408$$

Since $x_3 = x_4$, hence, the root is 2.7408 correct to four decimal places.

Example 12 Find the smallest root of the equation $e^{-x} = \sin x$, up to four decimal places.

Solution

Let $f(x) \equiv e^{-x} - \sin x = 0$ (18)

$f'(x) = -e^{-x} - \cos x$ (19)

Now, $f(0) = e^{-0} - \sin 0 = 1 > 0$

$f(1) = e^{-1} - \sin 1 = -0.47359 < 0$

Since, $f(0)$ and $f(1)$ are opposite sign, so the root lies in [0, 1].
Let us take $x_0 = 0.6$.
Using Newton–Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)};\quad n = 0, 1, 2, 3, \ldots$$

$$x_{n+1} = x_n - \frac{e^{-x_n} - \sin x_n}{-\cos x_n - e^{-x_n}}$$

$$x_{n+1} = x_n + \frac{e^{-x_n} - \sin x_n}{e^{-x_n} + \cos x_n}; \quad n = 0, 1, 2, \ldots \tag{20}$$

Putting $n = 0$ in Eq. (20), we get the first approximation

$$x_1 = x_0 + \frac{e^{-x_0} - \sin x_0}{e^{-x_0} + \cos x_0}$$

$$x_1 = 0.6 + \frac{e^{-0.6} - \sin(0.6)}{e^{-0.6} + \cos(0.6)} = 0.58848$$

Putting $n = 1$ in Eq. (20), we get the 2nd approximation

$$x_2 = x_1 + \frac{e^{-x_1} - \sin x_1}{e^{-x_1} + \cos x_1} = 0.58848 + \frac{e^{-0.58848} - \sin(0.58848)}{e^{-0.58848} + \cos(0.58848)}$$

$$x_2 = 0.58853$$

Since x_1 and x_2 are almost same, hence, the smallest root of the given question is 0.5885 up to 4 decimal places.

Example 13 Using Newton–Raphson method to find a real root of the equation $xe^x = 1$.

Solution Do same as Example 12. In this the interval is [0, 1] and we take $x_0 = 1$.

$$x = 0.5671.$$

9.8 REGULA FALSI METHOD

The Regula falsi method is also known as method of false position. With the help of this method, we can compute the real roots of the equation $f(x) = 0$. The graphical interpretation of this method is shown in the Fig. (9.4).

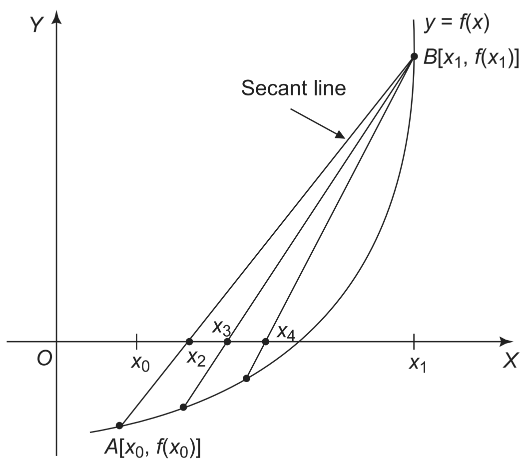


Fig. 9.4

This method follows a test to ensure that the root is always bracketed between successive approximations. Start two initial approximations x_0 and x_1 ($x_0 < x_1$) with $f(x_0)\,f(x_1) < 0$. Then the graph of $y = f(x)$ cuts the x-axis at some point between x_0 and x_1. Therefore, the equation of the chord joining two points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f_1(x_0)}{x_1 - x_0}(x - x_0) \tag{21}$$

Since Eq. (21) intersects the X-axis, where $y = 0$.

Thus, $$-f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

or $$x = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

Hence, the first approximation to the root is

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} \cdot f(x_0) \tag{22}$$

In general

$$x_{n+1} = x_{n-1} - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \cdot f(x_{n-1}) \tag{23}$$

Provided that at each step $f(x_{n-1}) \cdot f(x_n) < 0$.

Now, to decide which secant line to use to find the approximation x_3, consider, if $f(x_2) \cdot f(x_1) < 0$, then x_1 and x_2 bracket a root. x_3 is the x-intercept of the line joining the points $[x_1, f(x_1)]$ and $[x_2, f(x_2)]$.

If not, choose x_3 as the x-intercept of the line joining the points $[x_0, f(x_0)]$ and $[x_2, f(x_2)]$ and then interchange the indices on x_0 and x_1 in Eq. (22). Continuing this process until we get the root to desired accuracy.

9.9 SECANT METHOD

Newton's method is a powerful technique, but it has a weakness, the need to compute the value of f' at each approximation. To avoid computation of f', $f'(x_n)$ is replaced by $\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$ and we obtained secant method as

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot f(x_n) \tag{24}$$

This technique is called the secant method. This method starting two initial approximations x_0 and x_1, the approximation x_2 is the x–intercept of the secant line joining two points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$. The approximation x_3 is the x-intercept of the secant line joining two points $[x_1, f(x_1)]$ and $[x_2, f(x_2)]$ and continuing this process until we get the root to desired accuracy.

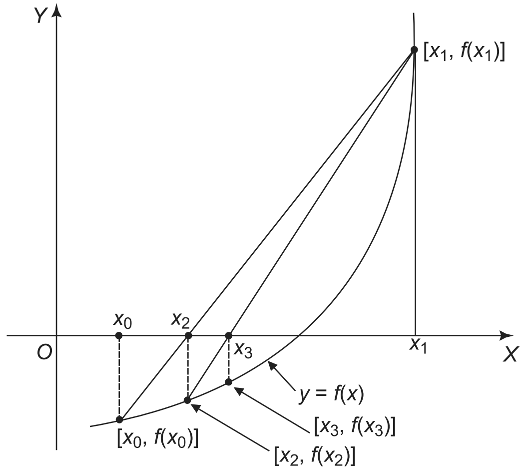


Fig. 9.5

Example 14 Using Regula-Falsi method, find a real root of the equation $x \log_{10} x = 1.2$ correct to four decimal places.

Solution

Let $\quad f(x) \equiv x \log_{10} x - 1.2 = 0 \quad (25)$

Now $\quad f(2) = -0.59794 < 0$

$f(3) = 0.23136 > 0$

Since, $f(2)$ and $f(3)$ are opposite sign, so a root lies between 2 and 3.

Iteration 1: Using Regula–Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_0)$$

$$x_2 = 2 - \frac{(3-2)}{0.23136 + 0.59794} \times -0.59794$$

$x_0 = 2 \quad x_2 \quad x_1 = 3$

$$= 2 + \frac{0.59794}{0.82930}$$

$$x_2 = 2.72102$$

$\therefore \quad f(x_2) = -0.01709 < 0$

Now, root lies between 2.72102 and 3.

Iteration 2:

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} \times f(x_2)$$

$x_2 \quad x_3 \quad x_1$

or
$$x_3 = 2.72102 - \frac{3 - 2.72102}{0.23136 + 0.01709} \times -0.01709$$

$$= 2.72102 + \frac{0.27898}{0.24845} \times 0.07709$$

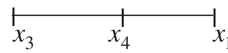
$$x_3 = 2.74021$$

$\therefore \qquad f(x_3) = -0.00038 < 0$

Hence, root lies between 2.74021 and 3.

Iteration 3:

$$x_4 = x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} \times f(x_3)$$



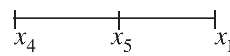
$$x_4 = 2.74021 - \frac{(3 - 2.74021)}{(0.23136 + 0.00038)} \times -0.00038$$

$$x_4 = 2.74064$$

$\therefore \qquad f(x_4) = -0.00001 < 0$

Iteration 4:

$$x_5 = x_4 - \frac{x_1 - x_4}{f(x_1) - f(x_4)} \times f(x_4)$$



$$= 2.74064 - \frac{3 - 2.74064}{(0.23136 + 0.00001)} \times -0.00001$$

$$x_5 = 2.74064$$

Since $x_4 = x_5$, hence, the required root correct to four decimal places is 2.7406.

Example 15 Using Regula-Falsi method to find the smallest positive root of the equation $x - e^{-x} = 0$.

Solution

Let $\qquad f(x) = x - e^{-x}$

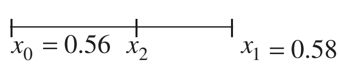
Since $\quad f(0.56) = -0.01121$ and $f(0.58) = 0.201$.

Hence, root lies in (0.56, 0.58).

Let $\qquad x_0 = 0.56$ and $x_1 = 0.58$.

Iteration 1: Using Regula-Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$



$$x_2 = 0.56 - \frac{(0.58 - 0.56)}{(0.201 + 0.01121)} \times -0.01121$$

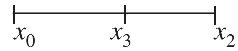
$$x_2 = 0.56716$$

$\therefore \qquad f(x_2) = 0.00002619 > 0$

Now, the root lies in (0.56, 0.56716).

Iteration 2: We have

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} \cdot f(x_0)$$



$$= 0.56 - \left(\frac{0.56716 - 0.56}{0.00002619 + 0.01121}\right) \times (-0.01121) = 0.56714.3$$

Since $\quad |x_3 - x_2| = |0.567143 - 0.56716| = 0.00002 < 0.005$

Hence, the required root is 0.567.

Example 16 Compute the real root of the equation $x^3 - 5x + 3 = 0$ in the interval [1, 2] by the Secant method by performing four iterations.

Solution

Let $\quad f(x) = x^3 - 5x + 3 \quad$ (26a)

In the interval, $f(1) = -1$ and $f(2) = 1$, take $x_0 = 1$ and $x_1 = 2$.

Hence, one root lies in [1, 2]

Using the iterative formula

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n) \tag{26b}$$

Iteration 1: Putting $n = 1$ in Eq. (26b), we get

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_1)$$

$$x_2 = 2 - \frac{2-1}{(1+1)} \times 1 = 1.5$$

$\therefore \quad f(x_2) = -1.125 < 0$

Iteration 2: Put $n = 2$ in Eq. (27), we get

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) \qquad [\because \quad x_1 = 1.5, x_2 = 2]$$

$$x_3 = 2 - \left(\frac{2 - 1.5}{1 + 1.125}\right) \times 1 = 1.7647$$

$\therefore \quad f(x_3) = -0.3279036 < 0$

Iteration 3: Put $n = 3$ in Eq. (26b), we get

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} \cdot f(x_3)$$

$$x_4 = 1.7647 - \left(\frac{1.7647 - 2}{-0.3279 - 1}\right) \times -0.3279 \qquad [\because \quad x_2 = 2, x_3 = 1.7647]$$

$$x_4 = 1.8228$$

$$f(x_4) = 0.2090399 > 0$$

Similarly *Iteration* 4: $x_5 = 1.8312$

9.10 CONVERGENCE FOR ITERATIVE METHODS

Let $\left[x_n\right]_{n=0}^{\infty}$ is a sequence that converges to x, with $x_n \neq x$ for all n. If the positive constants λ and μ exist with

$$\lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n - x|^{\mu}} = \lambda,$$

then $\left[x_n\right]_{n=0}^{\infty}$ converges to x of order μ, with asymptotic error λ.

In an iterative method of the form $x_n = \phi(x_{n-1})$ or $x_{n+1} = \phi(x_n)$ is said to be of order μ if the sequence $\left\{x_n\right\}_{n=0}^{\infty}$ converges to the solution $x = \phi(x)$ of order μ.

- If $\mu = 1$ and $\lambda < 1$, then the sequence is linearly convergent
- If $\mu = 2$, then the sequence is quadratically convergent.

9.10.1 Rate of Convergence of Bisection Method

In Bisection method the error in x_{n+1}^{th} approximation is bounded by one half of the error in x_n^{th} approximation. In other words

$$|\in_{n+1}| \leq \frac{1}{2}|\in_n|$$

or $\qquad \in_{n+1} \propto \in_n$

Hence, the Bisection method is linearly convergent.

9.10.2 Rate of Convergence of Iteration Method

Suppose $f(x) = 0$ is an equation, which can be expressed as $x = \phi(x)$. The iterative formula for solving this equation is

$$x_{n+1} = \phi(x_n) \tag{27}$$

If α is the root of the equation $x = \phi(x)$ lies in (a, b) such that $\alpha = \phi(\alpha)$.

The formula in Eq. (27) may also be written as

$$x_{n+1} = \phi(x + \overline{x_n - \alpha})$$

Using mean value theorem, we have

$$x_{n+1} = \phi(\alpha) + (x_n - \alpha)\,\phi'(c_n); \quad c_n \in (a, b)$$

But $\phi(\alpha) = \alpha$

$$\therefore \qquad x_{n+1} = \alpha + (x_n - \alpha)\,\phi'(c_n)$$

or $$x_{n+1} - \alpha = (x_n - \alpha)\,\phi'(c_n) \tag{28}$$

Now, if $\in_{n+1}$ and $\in_n$ are the error for the approximations x_{n+1} and x_n, then

$$\in_{n+1} = x_{n+1} - \alpha \quad \text{and} \quad \in_n = x_n - \alpha$$

Using these, then Eq. (28) becomes

$$\in_{n+1} = \in_n \phi'(c_n)$$

Since, $\phi(x)$ is a continuous function, therefore it is bounded.

$$\therefore \qquad |\phi'(c_n)| \le \lambda; \quad \lambda \in (a, b)$$

or $$\in_{n+1} \le \lambda \in_n$$

Since, the index of $\in_n$ is one, hence the rate of convergence of the iterative method is linear.

9.10.3 Rate of Convergence of Newton's Method

We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{29}$$

Also, we have

and $$\left.\begin{aligned} \in_n &= x_n - \alpha \text{ or } x_n = \in_n + \alpha \\ x_{n+1} &= \in_{n+1} + \alpha \end{aligned}\right\} \tag{30}$$

From Eq. (29) and (30), we have

$$(\in_{n+1} + \alpha) = (\in_n + \alpha) - \frac{f(\in_n + \alpha)}{f'(\in_n + \alpha)}$$

or $$\in_{n+1} = \in_n - \frac{f(\in_n + \alpha)}{f'(\in_n + \alpha)}$$

On expanding by Taylor's Theorem, we get

$$\in_{n+1} = \in_n - \frac{f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + \in_n f''(\alpha) + \frac{\in_n^2}{2!} f'''(\alpha) + \cdots}$$

Neglecting the third and higher order, we get

$$\in_{n+1} = \in_n - \frac{f(\alpha) + \in_n f'(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}$$ Since α is a root of $f(x) = 0$ $[\because \ f(\alpha) = 0]$

$$= \in_n - \frac{\in_n f'(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}$$

$$= \frac{\in_n^2 f''(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}$$

or $$\in_{n+1} = \frac{\in_n^2 f''(\alpha)}{f'(\alpha)} \left[1 + \in_n \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$= \frac{\in_n^2 f''(\alpha)}{f'(\alpha)} \left[1 - \in_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]$$ [Using Binomial theorem]

$$\in_{n+1} = \in_n^2 \frac{f''(\alpha)}{f'(\alpha)}$$

or $$\in_{n+1} = M \in_n^2, \text{ where } M = \frac{f''(\alpha)}{f'(\alpha)}$$

or $$\in_{n+1} \propto \in_n^2$$

This shows that the subsequent error is proportional to the square of the previous error. Hence, the rate of convergence of Newton's method is quadratic.

9.10.4 Rate of Convergence of Regula-Falsi Method

Let $x = \alpha$ is the exact root of $f(x) = 0$ and $\in_{n-1}$, $\in_n$ and $\in_{n+1}$ are the errors in x_{n-1}, x_n and x_{n+1} approximation respectively, then

$$\left.\begin{aligned} \in_{n-1} &= x_{n-1} - \alpha \\ \in_n &= x_n - \alpha \\ \in_{n+1} &= x_{n+1} - \alpha \end{aligned}\right\} \tag{31}$$

Using Regula-Falsi method, we have

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n) \tag{32}$$

From Eqs (31) and (32), we get

$$\alpha + \in_{n+1} = (\alpha + \in_n) - \frac{\in_n - \in_{n-1}}{f(\alpha+\in_n) - f(\alpha+\in_{n-1})} \cdot f(\alpha+\in_n)$$

or $$\in_{n+1} = \in_n - \frac{(\in_n - \in_{n-1})}{f(\alpha+\in_n) - f(\alpha+\in_{n-1})} \cdot f(\alpha+\in_n)$$

Expand by Taylor's Theorem, we get

$$\in_{n+1} = \in_n - \frac{(\in_n - \in_{n-1})\left[f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha) + \cdots\right]}{\left[f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha) + \cdots\right] - \left[f(\alpha) + \in_{n-1} f'(\alpha) + \frac{\in_{n-1}^2}{2!} f''(\alpha) + \cdots\right]}$$

Neglecting the 3rd and higher order terms of $\in_n$ and $\in_{n-1}$, we get

$$\in_{n+1} = \in_n - \frac{\in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha)}{f'(\alpha) + \left(\frac{\in_n + \in_{n-1}}{2}\right) f''(\alpha)}$$

After simplifications, we get

$$\in_{n+1} = \frac{1}{2} \in_{n-1} \cdot \in_n \frac{f''(\alpha)}{f'(\alpha)} + O(\in_n^2) \tag{33}$$

Now, we shall find a constant μ such that

$$\in_{n+1} = \lambda \in_n^{\mu} \text{ and } \in_n = \lambda \in_{n-1}^{\mu}$$

or
$$\in_{n-1} = \lambda^{-\frac{1}{\mu}} \in_n^{1/\mu}$$

Thus, from Eq. (33), we have

$$\lambda \in_n^{\mu} = M \lambda^{\frac{1}{\mu}} \in_n^{1/\mu} \cdot \in_n, \text{ where } M = \frac{f''(\alpha)}{2f'(\alpha)}$$

or
$$\lambda \in_n^{\mu} = M \lambda^{1/\mu} \cdot \in_n^{\left(\frac{1}{\mu}+1\right)}$$

Equating both sides, the powers of $\in_n$, we get

$$\mu = \frac{1}{\mu} + 1 \text{ or } \mu^2 - \mu - 1 = 0 \quad (34)$$

Solving Eq. (34), we have

$$\mu = \frac{1 \pm \sqrt{5}}{2}$$

The positive value of μ is $\frac{1}{2}(1+\sqrt{5}) = 1.618$

Hence, $\in_{n+1} = \lambda \in_n^{1.618}$

Thus, the order of convergence of Regula-Falsi method is 1.618.

EXERCISE 9.1

1. Using Bisection method, find a real root of the following equations:
   (i) $x^3 - x - 1 = 0$ which lies between 1 and 2. **[U.P.T.U. 2004]**
   (ii) $x^3 - 4x - 9 = 0$ which lies between 2 and 3.
   (iii) $3x - \sqrt{(1 + \sin x)} = 0$.
   (iv) $3x + \sin x - e^x = 0$ **[U.P.T.U. 2005]**
   (v) $e^x = 3x$
   (vi) $x = e^{-x}$ which lies between 0 and 1.
   (vii) $x^3 - 5x + 3 = 0$
   (viii) $x^3 - 9x + 1 = 0$ in [2, 4] **[U.P.T.U. 2005]**
   (ix) $x = \sqrt{28}$
   (x) $\sin x = \frac{1}{x}$ which lies in [1, 1.5] **[V.T.U, 2003]**
2. Using iteration method, find a real root of the following equations:
   (i) $x = 0.21 \sin(0.5 + x)$ with $x_0 = 0.12$ **[U.P.T.U. 2003]**
   (ii) Find $x = \sqrt[3]{15}$ in (2, 3).
   (iii) $e^{-x} = 10\,x$

(iv) $x = \dfrac{1}{(x+1)^2}$

(v) $xe^x = 1$.

(vi) $2x = 7 + \log_{10}x$ in (3, 4)

(vii) $1 - x + \dfrac{x^2}{(2!)^2} - \dfrac{x^3}{(3!)^2} + \dfrac{x^4}{(4!)^2} - \dfrac{x^5}{(5!)^2} + \cdots = 0.$

(viii) $3x - \log_{10}x = 6$ correct to four significant digits.

3. Find a real root of the equation $x^3 + x^2 - 1 = 0$ on [0, 1] with an accuracy up to four places of decimal.
4. Using Newton's method to find a root of the equation $x^3 - 2x - 5 = 0$. **[U.P.T.U. 2005]**
5. Using Newton-Raphson's method to find the real root of the equation $x^5 - 5x + 2 = 0$.
6. Using Newton's method, find a real root of the following equations:
   (i) $\log x - \cos x = 0$
   (ii) $x^4 - x - 10 = 0$
   (iii) $x = e^{-x}$
   (iv) $x^2 - 25 = 0$
   (v) $1 - x\, e^{1-x} = 0$
7. Find a positive value of $(17)^{\frac{1}{3}}$ correct to four decimal places by Newton's method. **(U.P.T.U. 2003]**
8. Find a positive root of $\cos x - xe^x = 0$, correct to four decimal places by Newton's method.
9. Using Regula-Falsi method, to find a positive root of the equation $xe^x = 2$. **[U.P.T.U. 2007]**
10. Find the smallest positive root of the equation $x - e^{-x} = 0$ using false position method.
11. Find the root of the equation $xe^x = \cos x$ in the interval (0, 1) using Regula-Falsi method correct to four decimal places.
12. Solve $x^6 - x^4 - x^3 - 1 = 0$ by False-position method method in the interval (1.4, 1.41).
13. Find the real root of the equation $x^3 - 9x + 1 = 0$ by Regula–Falsi method.
14. Solve $x \tan x = -1$, using False–position method starting with $x_0 = 2.5$ and $x_1 = 3$ correct to 3 decimal places.
15. Using Secant method, find the root of the following equations:
   (i) $x^2 e^{-x/2} = 1$ in the interval [0, 2]
   (ii) $x^2 - 2x - 3\cos x = 0$
   (iii) $\cos x - xe^x = 0$
16. Find an Newton iterative formula to find $\sqrt{n}$, where n is a positive number and hence, find $\sqrt{12}$ correct to 4 decimal places.
17. Using Newton's method to find a positive root of the equation $x^4 - x = 10$ correct to 3 decimal places.
18. Using the iterative method, to find a root of the equation $3x - \log_{10}x = 6$ correct to 4 significant digits.
19. The equation $x^2 + ax + b = 0$ has two real roots α and β. Show that the iteration method $x_{n+1} = -\left(\dfrac{x_n^2 + b}{a}\right)$ is convergent near $x = \alpha$ if $2|\alpha| < |\alpha + \beta|$.

Answers

1. (i) 1.325	(ii) 2.7066	(iii) 0.39	(iv) 0.3604
(v) 1.5121	(vi) 0.5671477	(vii) 0.657	(viii) 2.94282
(ix) 5.2915	(x) 1.11719		
2. (i) 0.12242	(ii) 2.466	(iii) 0.91276	(iv) 0.4656
(v) 0.5671477	(vi) 3.7892	(vii) 1.44	(viii) 2.108
3. 0.7549			
4. 2.2790			
5. 4.5616			
6. (i) 1.303	(ii) 1.856	(iii) 0.5671	(iv) 5
7. 2.5713		8. 0.517757	
9. 0.852605		10. 0.567	
11. 0.5177		12. 1.4036	
13. 2.9428		14. 2.798	
15. (i) 1.429	(ii) 1.728	(iii) 0.5177574	
16. 3.4641		17. 1.856	
18. 2.108			

9.11 LINEAR SYSTEM OF EQUATIONS

Linear system of equations arises in the study of many fields, both directly in modelling of physical problem and indirectly in the numerical solution of other mathematical models. To determine the numerical solution of a system of linear equations is an important part of the study. Numerical methods for solving linear algebraic systems can be divided into two methods, namely, direct iterative or indirect methods. In this section, we shall discussed the direct methods, namely, Gauss's elimination, Triangularisation Gauss's Jordan and Crout's methods and the iterative methods, namely, Gauss's–Jacobi and Gauss's–Seidel methods.

A linear system of n equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form:

$$\left.\begin{aligned} &E_1 : a_{11}\, x_1 + a_{12} x_2 + \cdots + a_{1n}\, x_n = b_1 \\ &E_2 : a_{21}\, x_1 + a_{22}\, x_2 + \cdots a_{2n}\, x_n = b_2 \\ &\ldots \\ &\ldots \\ &E_n : a_{n_1}\, x_1 + a_{n_2} x_2 + \cdots + a_{nn} x_n = b_n \end{aligned}\right\} \tag{35}$$

We can write Eq. (35) in matrix form as

$$AX = B \tag{36}$$

where, the coefficient matrix $A = \left[a_{ij}\right]_{n\times n}$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

The column vector $X = \left[x_1, x_2, \cdots, x_n\right]^T$ and

the constant matrix $B = \left[b_1, b_2, \cdots, b_n\right]^T$

If $B = 0$, then the system Eq. (35) is called homogeneous; otherwise it is non-homogeneous.

The augmented matrix C of the system Eq. (35) is

$$C = \left[A|B\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \tag{37}$$

A solution of Eq. (35) is a set of numbers $x_1, x_2, \ldots, x_n$ that satisfy all the n equations, and a solution vector of Eq. (35) is a vector X whose components constitute a solution of Eq. (35).

9.12 GAUSSIAN ELIMINATION METHOD

This method is applied on the linear system Eq. (35) and we write the augmented matrix

$$\left[A|B\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

The entries in the $(n + 1)^{\text{th}}$ column are the values of B, i.e., $a_{i,\,n+1} = b_i$ for each $i = 1, 2, \ldots, n$. Provided $a_{11} \neq 0$, now we perform the operations corresponding to

$E_j \to \left(E_j - \left(\dfrac{a_{j1}}{a_{11}}\right)E_1\right)$ for each $j = 2, 3, \ldots, n$, to eliminate the coefficient of x_1 in each rows.

The entries in rows 2, 3, …, n are change, for each of notation we again denote the entry in the i^{th} row and the j^{th} column by a_{ij}.

We follow a sequential procedure for $i = 2, 3, \ldots(n - 1)$ and perform the operation

$E_j \to \left(E_j - \left(\dfrac{a_{ji}}{a_{ii}}\right)E_i\right)$ for each $j = i + 1, i + 2, \ldots, n$, provided $a_{ii} \neq 0$ and we have a resulting matrix is of the form

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,\,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,\,n+1} \\ \vdots & \ddots\ \vdots & & \vdots & \vdots \\ 0 & \ldots\ 0 & & a_{nn} & a_{n,\,n+1} \end{array}\right] \tag{38}$$

Now, we write Eq. (38) in equation form as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= a_{1,\,n+1} \\ a_{22}x_2 + \cdots + a_{2n}x_n &= a_{2,\,n+1} \\ \ddots \quad \vdots \qquad & \qquad \vdots \\ a_{nn}x_n &= a_{n,\,n+1} \end{aligned}$$

Using back substitution find $x_1, x_2, \ldots, x_n$.
We define the above procedure with the help of the following example.

Example 17 Solve the following equations by Gauss's elimination method:

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 3 \\ 3x_1 + 2x_2 - 2x_3 &= -2 \\ x_1 - x_2 + x_3 &= 6 \end{aligned}$$

Solution We write the given system in matrix form as

$$\begin{bmatrix} 2 & 4 & 1 \\ 3 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$$

or $$AX = B \tag{39}$$

where $$A = \begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 4 & 1 & 3 \\ 3 & 2 & -2 & -2 \\ 1 & -1 & 1 & 6 \end{array}\right] \tag{40}$$

Interchanging the first and third row in Eq. (40), we have

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 3 & 2 & -2 & -2 \\ 2 & 4 & 1 & 3 \end{array}\right]$$

$$\begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 5 & -5 & -20 \\ 0 & 6 & -1 & -9 \end{array}\right]$$

$$R_2 \to \frac{1}{5} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 6 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ -9 \end{bmatrix}$$

$$R_3 \to R_3 - 6R_2 \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 5 & 15 \end{array}\right] \tag{41}$$

Using back substitution, we get

$$\begin{aligned} x_1 - x_2 + x_3 &= 6 \\ x_2 - x_3 &= -4 \\ 5x_3 &= 15 \end{aligned}$$

or $\qquad x_3 = 3, x_2 = -1, x_1 = 2$

$\therefore \qquad X = (x_1, x_2, x_3)^T = (2, -1, 3)^T$

Example 18 Solve the following equations by Gauss's elimination method:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 9 \\ x_1 + x_2 + x_3 &= 6 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

Solution

We write the given system in matrix form

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$A X = B \tag{42}$$

where $\qquad A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

Now, the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{array}\right]$$

Interchanging first and third row, we have

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 9 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ \\ R_3 \to R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

$$R_2 \to \frac{1}{2} R_2 \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Now, we write the above system in equations form as

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_2 &= 2 \\ x_3 &= 5 \end{aligned}$$

Using back substitution, $x = 1$, $x_2 = 2$ and $x_3 = 5$.

9.13 GAUSS'S–JORDAN METHOD

This method follow the same procedure of Gauss's elimination method, but also from $E_1, E_2, \ldots, E_{i-1}$. The augmented matrix $[A|B]$ reducing as

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & 0 & \cdots & 0 & a_{1,n+1}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & 0 & a_{2,n+1}^{(2)} \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn}^{(n)} & a_{n,n+1}^{(n)} \end{array}\right]$$

The solution is obtained by $x_i = \dfrac{a_{i,n+1}^{(i)}}{a_{ii}^{(i)}}$, for each $i = 1, 2, 3, \ldots, n$.

Now, we explain the above procedure with the help of the following example:

Example 19 Solve the system of linear equations, using Gauss's–Jordan method:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 + 3x_3 &= 10 \\ 3x_1 - x_2 + 2x_3 &= 13 \end{aligned}$$

Solution We write the given system in matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix} \tag{43}$$

or $$AX = B \tag{44}$$

where $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

Now, the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ \\ R_3 \to R_3 - 3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right]$$

$$R_3 \to R_3 - 7R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$$R_3 \to \left(-\frac{1}{8}\right)R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - R_3 \\ \\ R_1 \to R_1 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 \to R_1 + 2R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now the solution of given system is obtain by

$$x_i = \frac{a_{i,n+1}^{(i)}}{a_{ii}^{(i)}};\ n = 3$$

$$\therefore \qquad x_1 = \frac{a_{1,4}^{(1)}}{a_{11}^{(1)}} = \frac{2}{1} = 2$$

Similarly $x_2 = -1$ and $x_3 = 3$.
Hence, $x_1 = 2$, $x_2 = -1$ and $x_3 = 3$.

9.14 CROUT'S METHOD

Crout's method determining the numerical solution of a system of linear equations and it has an advantage over the Gauss's elimination method in that it requires the number of less computations. Suppose, the systems of linear equations is of the form $AX = B$

where $A = \left[a_{ij}\right]_{m\times n}$ for each $i = 1(1)m$ and $j = 1(1)n$,

$$B = \left[b_1, b_2, ..., b_m\right]^T, X = \left[x_1, x_2, ..., x_m\right]^T$$

Then
$$C = [A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Now, we define an Auxiliary matrix

$$C' = [A'|B']$$

Using C', we determine the solution column X.

> *Prescott Durand Crout* was an American mathematician born on 28 July 28 1907 in Ohio. In the year of 1929, he finished his MIT class. His PhD thesis entitled "*The Approximation of Functions and Integrals by a Linear Combination of Functions*" was completed under the supervision of George Rutledge. He was the member of the Faculty of Mathematics from 1934 to 1973 and emeritus from 1973 till his death in 1984. He belonged to the Radiation Laboratory staff from 1941 to 1945. His students at the MIT were Francis Hildebrand (1940), Carl Nordling (1941), Frank Bothwell (1946), Norman Painter (1947), Merle Andrew (1948), Frederick Holt (1950), and Carl Steeg, Jr. (1952). He died, aged 77, in Lexington, Middlesex, Massachusetts. Prescott Durand Crout wrote the book "*The Determination of Fields Satisfying Laplace's Poisson's, and Associated Equations by Flux Plotting*". He is probably the inventor of the Crout matrix decomposition method.

9.14.1 Rules for Determining the Auxiliary Matrix C'

Rule 1: The elements in first column of matrix C' are equal with the corresponding elements in first column of matrix C and the remaining elements of each row and column as follows:

$$a'_{11} = a_{11}, a'_{21} = a_{21} \cdots, a'_{n1} = a_{n1}$$

$$a'_{12} = {}^{a_{12}}\!/_{a_{11}}, a'_{13} = {}^{a_{13}}\!/_{a_{11}}, \cdots, a'_{1n} = {}^{a_{1n}}\!/_{a_{11}}$$

$$b'_1 = {}^{b_1}\!/_{a_{11}}$$

Rule 2: The elements on or below the principal diagonal of C' are calculated by subtracting from the corresponding elements of C, the inner product of its row and column in C' when all uncalculated elements are consider as zero; using

$$a'_{ij} = a_{ij} - \sum_{k=1}^{j-1} a'_{ik}\, a'_{kj} \quad \text{for } i \geq j$$

and
$$a'_{ij} = \frac{1}{a'_{ii}}\left[a_{ij} - \sum_{k=1}^{i-1} a'_{ik}\, a'_{kj}\right] \quad \text{for } i < j$$

Rule 3: The elements of B' are evaluated by subtracting from the corresponding elements of B, the inner product of column B' and its row in A' and then divided by the diagonal elements of its row in A' i.e.,

$$b'_i = \frac{1}{a'_{ii}}\left[b_i - \sum_{k=1}^{i-1} a'_{ik}\, b'_k\right]$$

Rule 4: The solution of given system calculated in the order of $x_n, x_{n-1}, x_{n-2}, \ldots, x_2, x_1$ from bottom to top as using the formula.

$$x_i = b'_i - \sum_{k=i+1}^{n} a'_{ik}\, x_k$$

Explain the above procedure with the help of the following example:

Example 20 Using Crout's method, solve the following system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 + x_2 - 3x_3 &= 5 \\ x_1 - 2x_2 - 5x_3 &= 10 \end{aligned}$$

Solution We have, augmented matrix

$$C = [A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 5 \\ 1 & -2 & -5 & 10 \end{array}\right]$$

Suppose the auxiliary matrix

$$C' = [A'|B'] = \left[\begin{array}{ccc|c} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

Now, our aim to determine the values of x_1, x_2 and x_3 using (R_4), we have

$$x_3 = b'_3,\; x_2 = b'_2 - a'_{23}\, x_3 \text{ and}$$

$$x_1 = b'_1 - a'_{12}\, x_2 - a'_{13}\, x_3$$

For this we calculate matrix C'.

According (R_1 to R_3) $a_{11} = 1 = a'_{11},\ a'_{12} = \dfrac{a_{12}}{a_{11}} = 1,\ a'_{13} = \dfrac{a_{13}}{a_{11}} = 1$ and $b'_1 = \dfrac{b_1}{a_{11}} = 1.$

Also, $\quad a'_{21} = a_{21} = 3,\ a'_{31} = a_{31} = 1$

Now, for $\quad a'_{22} = a_{22} - a'_{21}\, a'_{12} = 1 - 3.1 = -2$

$$a'_{23} = \frac{1}{a'_{22}}\left[a_{23} - a'_{21}\, a'_{12}\right] = -\frac{1}{2}\left[-3 - 3 \times 1\right] = 3$$

$$b'_2 = \frac{1}{a'_{22}}\left[b_2 - a'_{21}\, b'_1\right] = -\frac{1}{2}\left[5 - 3 \times 1\right] = -1$$

Also, $\quad a'_{32} = a_{32} - a'_{31}\, a'_{12} = -2 - 1 \times 1 = -3$

$$a'_{33} = a_{33} - a'_{31}\, a'_{13} - a'_{32}\, a'_{23}$$

$$a'_{33} = -5 - 1 \times 1 - (-3) \cdot 3 = 3$$

and $\quad b'_3 = \dfrac{1}{a'_{33}}\left[b_3 - (a'_{31}\, b'_1 + a'_{32}\, b'_2)\right]$

$$= \frac{1}{3}\left[10 - (1 \times 1 + (-3) \times (-1)\right] = 2$$

Thus, the auxiliary matrix

$$C' = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & -2 & 3 & -1 \\ 1 & -3 & 3 & 2 \end{array}\right]$$

Now, $\quad x_3 = b'_3 = 2,\ x_2 = b'_2 - a'_{23}\, x_3 = -1 - 3 \times 2 = -7$

and $\quad x_1 = b'_1 - a'_{12}\, x_2 - a'_{13}\, x_3$

$$= 1 - 1 \times -7 - 1 \times 2 = 6$$

$\therefore \quad x_1 = 6,\ x_2 = -7,\ x_3 = 2.$

which is the required solution.

Example 21 Using Crout's method, solve the following system of equations:

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 2 \\ 10x_1 + 3x_2 + 4x_3 &= 16 \\ 3x_1 + 6x_2 + x_3 &= -6 \end{aligned}$$

Solution

We have $\quad C = \left[A \middle| B\right] = \left[\begin{array}{ccc|c} 2 & 3 & 2 & 2 \\ 10 & 3 & 4 & 16 \\ 3 & 6 & 1 & -6 \end{array}\right]$

To determine the values of x_3, x_2 and x_1 and

$$x_3 = b_3',\ x_2 = b_2' - a_{23}'\,x_3 \ \text{ and}$$

$$x_1 = b_1' - a_{12}'\,x_2 - a_{13}'\,x_3$$

Using rule R_1 to R_3, we obtain the auxiliary matrix

$$C' = \left[\begin{array}{ccc|c} 2 & \frac{3}{2} & 1 & 1 \\ 10 & -12 & \frac{1}{2} & -\frac{1}{2} \\ 3 & \frac{3}{2} & -\frac{11}{4} & 3 \end{array}\right]$$

Thus, we get

$$x_1 = 1,\ x_2 = -2 \text{ and } x_3 = 3.$$

9.15 LU DECOMPOSITION METHOD

The Gauss's elimination method is the principal tool in the direct solution of linear system of equations. The LU decomposition method has the property that the matrix decomposition step can be performed independent of the right hand side vector. In this method the coefficient matrix A is expressed as the product of lower triangular matrix (L) and an upper triangular matrix (U), that is

$$A = LU \tag{45}$$

Then, the linear system of equations,

$$AX = B \text{ becomes}$$

$$LUX = B \tag{46}$$

Let $\quad UX = Y, \qquad (47)$

then Eq. (46) becomes

$$LY = B \tag{48}$$

Now, first we solve the lower triangular system Eq. (48) for Y. Once Y is known, then solve the upper triangular system Eq. (47) for X.

We shall discussed the following three approaches of decomposition using 3×3 matrices.

9.15.1 Doolittle Decomposition

In this method we choose $l_{ii} = 1$ for $i = 1, 2, 3$ and we write the given system as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

9.15.2 Crout's Decomposition

In this method, we choose $u_{ii} = 1$ for $i = 1, 2, 3$ and we write the given system as

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

9.15.3 Cholesky's Method

This method is applicable for a symmetric and positive definite matrix A (i.e., $A^T = A$ and $X^T AX > 0$ for all $X \neq 0$), then, according to this method matrix A can be written as the product of two triangular matrix such that

$$A = LL^T,$$

where L is lower triangular matrix and L^T is an upper triangular matrix.

$$\therefore \quad A^{-1} = (L\,L^T)^{-1}$$
$$= (L^T)^{-1}\, L^{-1}$$
$$= (L^{-1})^T\, L^{-1}$$
$$A^{-1} = S^T S; \quad \text{where } S = L^{-1}$$

which gives the inverse of matrix A.

We illustrate the above procedure with the help of an example.

Example 22 Using LU Decomposition method to solve the equations:

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 3x_2 + 4x_3 = 20$$
$$3x_1 + 4x_2 + x_3 = 14$$

Solution

The coefficient matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$, which is symmetric

We write $A = L\,U$, with $u_{ii} = 1$,

$$\therefore \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}\,u_{12} & l_{11}\,u_{13} \\ l_{21} & l_{21}\,u_{12} + l_{22} & l_{21}\,u_{13} + l_{22}\,u_{23} \\ l_{31} & l_{31}\,u_{12} + l_{32} & l_{31}\,u_{13} + l_{32}\,u_{23} + l_{33} \end{bmatrix}$$

Compare both side the corresponding elements, we have

$$l_{11} = 1,\ l_{21} = 2,\ l_{31} = 3,\ l_{11}\,u_{12} = 2 \Rightarrow u_{12} = 2$$

$$l_{11}\, u_{13} = 3 \Rightarrow u_{13} = 3,\; l_{21}\, u_{12} + l_{22} = 3 \Rightarrow l_{22} = -1$$

$$l_{21}\, u_{13} + l_{22}\, u_{23} = 4 \Rightarrow 2 \times 3 + (-1)\, u_{23} = 4$$

$$\Rightarrow \qquad u_{23} = 2$$

$$l_{31}\, u_{12} + l_{32} = 4 \Rightarrow 3 \times 2 + l_{32} = 4$$

$$\Rightarrow \qquad l_{32} = -2$$

$$l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33} = 1$$

or $\quad 3 \times 3 + (-2) \times 2 + l_{33} = 1 \Rightarrow l_{33} = 1 - 5 = -4$

Hence, $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & -2 & -4 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Now $\quad LUX = B \qquad (49)$

Let $\quad UX = Y \qquad (50)$

Using Eqs (50) and (49) becomes

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & -2 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 20 \\ 14 \end{bmatrix}$$

$$y_1 = 14,\; 2y_1 - y_2 = 20 \Rightarrow y_2 = 8 \quad \text{[Using forward substitution]}$$

$$3y_1 - 2y_2 - 4y_3 = 14$$

$$42 - 16 - 4y_3 = 14$$

or $\quad 4y_3 = 22 - 14$

$$4y_3 = 8$$

$$y_3 = 2$$

$\therefore \quad Y = [14, 8, 2]^T$

Now from Eq. (50), we have

$$UX = Y$$

or $\quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 2 \end{bmatrix}$

Using back substitution, we have

$$x_3 = 2;\; x_2 + 2x_3 = 8 \Rightarrow x_2 = 4$$

and $x_1 + 2x_2 + 3x_3 = 14 \Rightarrow x_1 = 0$

Hence, $X = [0, 4, 2]^T$ is the required solution of the given linear system of equations.

Example 23 Determine the Cholesky decomposition or factorization ($L L^T$) of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$$

Solution Using Cholesky decomposition method, we write the given matrix

$$A = L L^T$$

or
$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{11} l_{21} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{11} l_{31} & l_{21} l_{31} + l_{22} l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Compare both side the corresponding elements, we have

$$l_{11}^2 = 4 \Rightarrow l_{11} = 2, \; l_{11} l_{21} = -1 \Rightarrow l_{21} = -0.5$$

$$l_{11} l_{31} = 1 \Rightarrow l_{31} = 0.5, \; l_{21}^2 + l_{22}^2 = 4.25 \Rightarrow l_{22} = 2$$

$$l_{21} l_{31} + l_{22} l_{32} = 2.75 \Rightarrow l_{32} = 1.5, l_{31}^2 + l_{32}^2 + l_{33}^2 = 3.5 \Rightarrow l_{33} = 1.$$

Thus

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}$$

which is the required Cholesky decomposition.

Example 24 Determine the Crout's factorization of the linear system of equations:

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 1 \end{aligned}$$

Hence, find the solution.

Solution We write the given system in matrix form, we have

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

i.e., $AX = B$ (51)

Since, the coefficient matrix A is symmetric, we write

$$A = L\,U$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}\,u_{12} & l_{11}\,u_{13} & l_{11}\,u_{14} \\ l_{21} & l_{21}\,u_{12} + l_{22} & l_{21}\,u_{13} + l_{22}\,u_{23} & u_{21}\,u_{14} + l_{22}\,u_{24} \\ l_{31} & l_{31}\,u_{12} + l_{32} & l_{31}\,u_{13} + l_{32}\,u_{23} + l_{33} & l_{31}\,u_{14} + l_{32}\,u_{24} + l_{33}\,u_{34} \\ l_{41} & l_{41}\,u_{12} + l_{42} & l_{41}\,u_{13} + l_{42}\,u_{23} + l_{43} & l_{41}\,u_{14} + l_{41}\,u_{24} + l_{43}\,u_{34} + l_{44} \end{bmatrix}$$

Comparing both sides, the corresponding elements, we have

$$l_{11} = 2,\ l_{11}\,u_{12} = -1 \Rightarrow u_{12} = -\frac{1}{2}$$

$$l_{21} = -1,\ l_{21}\,u_{12} + l_{22} = 2 \Rightarrow l_{22} = -\frac{3}{2}$$

$$l_{22}\,u_{23} = -1 \Rightarrow u_{23} = -\frac{2}{3},\ l_{32} = -1,\ l_{41} = 0,\ l_{31} = 0$$

$$l_{32}\,u_{23} + l_{33} = 2 \Rightarrow l_{33} = \frac{4}{3},\ l_{33}\,u_{34} = -1 \Rightarrow u_{34} = -\frac{3}{4}$$

$$l_{43} = -1,\ l_{43}\,u_{34} + l_{44} + l_{41}\,u_{14} + l_{41}\,u_{24} = 0 \Rightarrow l_{44} = \frac{5}{4}$$

$\therefore$ The Crout's factorization

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU$$

Now $$LY = B \Rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which gives $Y = \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1\right]^T$

and $$UX = Y \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

which gives $X = [1, 1, 1, 1]^T$.

9.16 ITERATIVE METHODS

In this section, we shall discuss the iterative or indirect methods, an iterative method repeats its process over and each time using the current approximation to produce a better approximation for the exact solution, until the current approximation is sufficiently close to the exact solution that is we expect that $x^{(k)}$ to be close to x if $\left|x^{(k)} - x^{(k-1)}\right|$ is very small.

An iterative methods to solve the $n \times n$ linear system $AX = B$ start with an initial approximation $x^{(0)}$ to the solution x and generates a sequence of vectors $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ that converge to x. Now, we describe the Jacobi and the Gauss–Seidel iterative methods.

9.16.1 Jacobi's Method

This method converges, if the coefficient matrix A strictly diagonally dominant; that is

$$|a_{ii}| > \sum_{i \neq j} |a_{ij}| \quad \text{for } i = 1, 2, 3, \ldots, n. \tag{52}$$

The iterative formula for the linear system of equations $AX = B$ by Jacobi method can be written as

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[-\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\, x_j^{(k)} + B_i \right] \tag{53}$$

for $i = 1, 2, 3, \ldots, n$; provided $a_{ii} \neq 0$.

Working Procedure

Step-I: Check the condition in Eq. (52) for the coefficient matrix A.

Step-II: Write the given system as in the form of Eq. (53).

Step-III: Assume initial solution is $x_i^{(0)} = 0$ (if not given) for $i = 1, 2, \ldots, n$.

Step-IV: Using Step-III, compute $x_i^{(k)}$ and continuing this process till if $\left|x_i^{(k)} - x_i^{(k-1)}\right| < \in$.

Example 25 Solve the following linear system of equations by Jacobi method:

$8x_1 - 3x_2 + 2x_3 = 20$; $4x_1 + 11x_2 - x_3 = 33$; $6x_1 + 3x_2 + 12x_3 = 36$.

Solution

Step-I:

$$|a_{ii}| = |a_{11}| + |a_{22}| + |a_{33}| = 8 + 11 + 12 = 31$$

$$\sum_{i \neq j} |a_{ij}| = |a_{12}| + |a_{13}| + |a_{21}| + |a_{23}| + |a_{31}| + |a_{32}|$$

$$= |-3| + |2| + |4| + |-1| + |6| + |3|$$

$$= 19$$

$$\therefore \qquad |a_{ii}| > \sum_{i \neq j} |a_{ij}|$$

Step-II: We write the given equations as

$$x_1^{(k+1)} = \frac{1}{8}\left[20 + 3x_2^{(k)} - 2x_3^{(k)}\right] \tag{54}$$

$$x_2^{(k+1)} = \frac{1}{11}\left[33 - 4x_1^{(k)} + x_3^{(k)}\right] \tag{55}$$

$$x_3^{(k+1)} = \frac{1}{12}\left[36 - 6x_1^{(k)} - 3x_2^{(k)}\right] \tag{56}$$

Step-III: Assume the initial solution is

$(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0, 0, 0)$.

Step-IV:

$$x_1^{(1)} = 2.5000$$

$$x_2^{(1)} = 3.0000$$

$$x_3^{(1)} = 3.0000$$

Using $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$, then Eqs (54), (55) and (56) give

$$x_1^{(2)} = 2.8750, \quad x_2^{(2)} = 2.3636, \quad x_3^{(2)} = 1.0000$$

Continuing this process, after 10 iterations we obtain the approximate solution

$(x_1^{(10)}, x_2^{(10)}, x_3^{(10)}) = (3.0000, 1.9999, 0.9999)$ as shown in the following table:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
1	2.5000	3.0000	3.0000
2	2.8750	2.3636	1.0000
3	3.1364	2.0455	0.9716
4	3.0241	1.9478	0.9205
5	3.0003	1.9840	1.0010
6	2.9938	2.0000	1.0038
7	2.9990	2.0026	1.0031
8	3.0002	2.0006	0.9998
9	3.0003	1.9999	0.9997
10	3.0000	1.9999	0.9999

Example 26 Solve the following equations by Jacobi's method:

$$27x_1 + 6x_2 - x_3 = 85$$
$$6x_1 + 15x_2 + 2x_3 = 72$$
$$x_1 + x_2 + 54x_3 = 110$$

Solution

Step-I:
$$|a_{ii}| = |a_{11}| + |a_{22}| + |a_{33}| = 96$$
$$\sum_{i \neq j} |a_{ij}| = |a_{12}| + |a_{13}| + |a_{21}| + |a_{23}| + |a_{31}| + |a_{32}|$$
$$= |6| + |-1| + |6| + |2| + |1| + |1| = 17$$
$$\therefore \quad |a_{ii}| > \sum_{i \neq j} |a_{ij}|$$

Step-II: We write the given equations as

$$x_1^{(k+1)} = \frac{1}{27}\left[85 - 6x_2^{(k)} + x_3^{(k)}\right] \tag{57}$$

$$x_2^{(k+1)} = \frac{1}{15}\left[72 - 6x_1^{(k)} - 2x_3^{(k)}\right] \tag{58}$$

$$x_3^{(k+1)} = \frac{1}{54}\left[110 - x_1^{(k)} - x_2^{(k)}\right] \tag{59}$$

Step-III: Assume the initial solution is

$$\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0, 0, 0)$$

Step-IV: Using initial solution, Eqs (57), (58) and (59) give

$$x_1^{(1)} = 3.150,\ \ x_2^{(1)} = 4.800,\ \ x_3^{(1)} = 2.040$$

Using $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$, then Eqs (57), (58) and (59) give

$$x_1^{(2)} = 2.160,\ x_1^{(2)} = 3.270,\ x_1^{(2)} = 1.890$$

Continuing this process, after 7 iterations, we obtain the approximate solution $\left(x_1^{(7)}, x_2^{(7)}, x_3^{(7)}\right) = (2.426, 3.572, 1.926)$ as shown in the following table:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
1	3.150	4.800	2.040
2	2.160	3.270	1.890
3	2.491	3.684	1.936
4	2.401	3.545	1.923
5	2.432	2.583	1.927
6	2.423	3.570	1.926
7	2.426	3.572	1.926

Another Form of the Jacobi's Method

To solve the linear system of equations $AX = B$, where

$$A = \left[a_{ij}\right]_{n\times n};\ X = \left[x_1, x_2, ..., x_n\right]^t; B = [b_1, b_2, \cdots, b_n]^t$$

Now, the matrix A is split in to

$$A = L + D + U$$

where $$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$ be

the diagonal part of A, whose diagonal elements are the diagonal elements of A.
L is the strictly lower-triangular part of A and U is strictly upper triangular part of A.

$$L = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{bmatrix},\quad U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Therefore, we have

$$(L + D + U)\ X = B$$

or $$DX = -(L + U)X + B$$

Since, $a_{ii} \neq 0$ for each i, that is D^{-1} exists, then

$$D^{-1} = \operatorname{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \cdots, \frac{1}{a_{nn}}\right)$$

$$\therefore \qquad X = -D^{-1}(L+U)X + D^{-1}B$$

$$\therefore \qquad X^{(k+1)} = -D^{-1}(L+U)X^{(k)} + D^{-1}B$$

$$\boxed{X^{(k+1)} = M_j X^{(k)} + C_j}$$

where $M_j = -D^{-1}(L+U)$ and $C_j = D^{-1}B$

The matrix M_j is called the iteration matrix.

Example 27 Using Jacobi matrix method to solve the following linear equations:

$x_1 + x_2 - x_3 = 0, -x_1 + 3x_2 = 2, x_1 - 2x_3 = -3$

Assume the initial solution $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0.8, 0.8, 2.1)$.

Solution

Suppose that the matrix representation of matrix A in the form of

$$A = L + D + U = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, $\qquad M_j = -D^{-1}(L+U)$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

and $\qquad C_j = D^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{3}{2} \end{bmatrix}$

$$\therefore \qquad X^{(k+1)} = M_j\, X^{(k)} + C_j; \;\; X = (x_1, x_2, x_3)^t$$

$$X^{(k+1)} = \begin{bmatrix} 0 & -1 & 1 \\ 1/3 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} X^{(k)} + \begin{bmatrix} 0 \\ 2/3 \\ 3/2 \end{bmatrix}$$

$$X^{(1)} = \begin{bmatrix} 0 & -1 & 1 \\ 1/3 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.8 \\ 2.1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2/3 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0.9333 \\ 1.9 \end{bmatrix}$$

Again

$$X^{(2)} = \begin{bmatrix} 0 & -1 & 1 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} X^{(1)} + \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{3}{2} \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 0 & -1 & 1 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1.3 \\ 0.9333 \\ 1.9 \end{bmatrix} \begin{bmatrix} 1.3 \\ \frac{2}{3} \\ \frac{3}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 0.9667 \\ 1.1000 \\ 2.1500 \end{bmatrix}$$

Continuing this process, after 9 iterations, we obtain the solution of the given system is $X = (1, 1, 2)^t$.

9.16.2 Gauss–Seidal Method

The iterative formula for the linear system of equations $AX = B$ by Gauss–Seidal method can be written as

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k)} + B_i \right] \tag{60}$$

for $i = 1, 2, 3, \ldots, n$; provided $a_{ii} \neq 0$.

Note-1 This method converges, if the matrix A is symmetric and positive definite.

Note-2 If this method does not converges, then the Jacobi method does not converge. But if Jacobi method converges then Gauss–Seidel method converges and the convergence of this method is faster.

Note-3 Putting the initial value $x_j^{(0)}$ for the unknown on the RHS of Eq. (60), we obtain $x_j^{(1)}$. Again putting the new value $x_j^{(1)}$ on the RHS of Eq. (60) and obtain $x_j^{(2)}$ and continuing this process until $x_j^{(k)}$ is equal to $x_j^{(k+1)}$ with in the required accuracy.

The Gauss–Seidel matrix formulation, for the linear system $AX = B$ is

$(L + D + U)\, X = B$ or $(L + D)X = -UX + B$

or
$$X = -(L+D)^{-1} UX + (L+D)^{-1} B$$
$$X = M_S X + C_S,$$

where
$$M_S = -(L+D)^{-1} U \text{ and } C_S = (L+D)^{-1} B.$$

∴ The Gauss–Seidel matrix iteration is

$$X^{(k+1)} = M_S X^{(k)} + C_S;\ k = 0, 1, 2, 3, \ldots$$

Example 28 Solve the following system of equations:

$x + y - z = 0,\ -x + 3y = 2,\ x - 2z = 3$ by the Gauss–Seidel method. Also, write its matrix form. Assume the initial solution $X = (0.8, 0.8, 2.1)^t$.

Solution The iterative process for Gauss–Seidel method is given by

$$x^{(k+1)} = -y^{(k)} + z^{(k)} \tag{61}$$

$$y^{(k+1)} = \frac{1}{3}(2 + x^{(k+1)}) \tag{62}$$

$$z^{(k+1)} = \frac{1}{2}(3 + x^{(k+1)}) \tag{63}$$

Using initial solution $y^{(0)} = 0.8$ and $z^{(0)} = 2.1$, then Eq. (61) gives

$$x^{(1)} = 1.3$$

Now putting $x^{(1)} = 1.3$ in Eq. (62), we have

$$y^{(1)} = 1.1$$

Putting $x^{(1)} = 1.3$ in Eq. (63), we get $z^{(1)} = 2.15$

Again putting $y^{(1)} = 1.1$ and $z^{(1)} = 2.15$ in Eq. (61), we get

$$x^{(2)} = 1.05$$

Put $x^{(1)} = 1.3$ in Eq. (62) and (63), we get

$$y^{(2)} = 1.01667,\ z^{(2)} = 2.025$$

Similarly, $x^{(3)} = 1.00833$, $y^{(3)} = 1.00278$ and $z^{(3)} = 2.004165$

$$x^{(4)} = 1.001385,\ \ y^{(4)} = 1.00046,\ \ z^{(4)} = 2.00069$$

$$x^{(5)} = 1.00023,\ \ y^{(5)} = 1.000077,\ \ z^{(5)} = 2.000115$$

Since $\left|x^{(5)} - x^{(4)}\right| = 0.0012, \left|y^{(5)} - y^{(4)}\right| = 0.0038$ and $\left|z^{(5)} - z^{(4)}\right| = 0.00057$.

The error after 5th iterations correct to two place accuracy, hence, the solution of the given system is $X = (1, 1, 2)$.

The matrix form can be written as:

$$X^{(k+1)} = -\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X^{(k)} + \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 0 & -6 & 6 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{bmatrix} X^{(k)} - \frac{1}{6}\begin{bmatrix} 0 \\ -4 \\ -9 \end{bmatrix}$$

Starting the initial solution $X^{(0)} = (0.8, 0.8, 2.1)$, we have $X = (1, 1, 2)$.

Example 29 Using Gauss–Seidel method to solve the following system of linear equations;

$8x-3y+2z=20,\ 4x+11y-z=33,\ 6x+3y+12z=36.$

Solution The iteration process for the Gauss–Seidel method is given as

$$x^{(k+1)}=\frac{1}{8}\left[20+3y^{(k)}-2z^{(k)}\right] \tag{64}$$

$$y^{(k+1)}=\frac{1}{11}\left[33-4x^{(k+1)}+z^{(k)}\right] \tag{65}$$

$$z^{(k+1)}=\frac{1}{12}\left[36-6x^{(k+1)}-3y^{(k+1)}\right] \tag{66}$$

Let the initial solution $X^{(0)}(x^{(0)}, y^{(0)}, z^{(0)})=(0, 0, 0)$

Using $y^{(0)}=0, z^{(0)}=0$ in Eq. (64), we get

$x^{(1)}=2.5000$

Using $x^{(1)}=2.5000, z^{0}=0$ in Eq. (65), we have

$y^{(1)}=2.0909$

Using $x^{(1)}=2.5000, y^{(1)}=2.0909$ in Eq. (66), we have

$z^{(1)}=1.2273$

Similarly, using $y^{(1)}$ and $z^{(1)}$ in Eq. (64), we have

$x^{(2)}=2.9773$

Using $x^{(2)}$ and $z^{(1)}$ in Eq. (65), we have

$y^{(2)}=0.0289$

Using $x^{(2)}$ and $y^{(2)}$ in Eq. (66), we have

$z^{(2)}=1.0041$

Continuing this process, after 6 iteration, we obtain the solution $X=(3, 2, 1)^{t}$ as shown in the following table:

k	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
1	2.5000	2.0909	1.2273
2	2.9773	2.0289	1.0041
3	3.0098	1.9968	0.9959
4	2.9998	1.9997	1.0002
5	2.9998	2.0001	1.0001
6	3.0000	2.0000	1.0000

EXERCISE 9.2

1. Using Gauss elimination method to solve the following linear system of equations:

$$x_1 + x_2 + x_3 = 3,$$
$$4x_1 + 3x_2 + 4x_3 = 8$$
$$9x_1 + 3x_2 + 4x_3 = 7.$$

2. Use Gauss elimination method to solve

$10x_1 - 7x_2 = 7$; $-3x_1 + 2.099\, x_2 + 6x_3 = 3.901$; $5x_1 - x_2 + 5x_3 = 6$.

3. Solve the following system by Gauss elimination method

$$5x_1 + x_2 + x_3 + x_4 = 4, \quad x_1 + 7x_2 + x_3 + x_4 = 12,$$
$$x_1 + x_2 + 6x_3 + x_4 = -5, \quad x_1 + x_2 + x_3 + 4x_4 = -6.$$

4. Solve the following systems using the Gauss elimination method:

(i)
$$3x_1 + 2x_2 + 3x_3 = 5$$
$$x_1 + 4x_2 + 2x_3 = 4$$
$$2x_1 + 4x_2 + 8x_3 = 8$$

(ii)
$$x_1 - x_2 + x_3 = 0$$
$$2x_1 + 3x_2 + x_3 - 2x_4 = -7$$
$$3x_1 + x_2 - x_3 + 4x_4 = 12$$
$$3x_2 - 5x_3 + x_4 = 9$$

5. Using *LU*–decomposition method to solve the following system:

(i)
$$2x_1 + x_2 + x_3 = 5$$
$$x_1 + 3x_2 + 2x_3 = 4$$
$$-x_1 + x_2 + 6x_3 = 4$$

(ii)
$$4x_1 + x_2 + 2x_3 = 3.6$$
$$x_1 + 3x_2 + x_3 = 2.5$$
$$2x_1 + x_2 + 2x_3 = 4.0$$

6. Solve the following system of linear equations using *LU*–decomposition method:

$$6x_1 - 2x_2 = 14$$
$$9x_1 - x_2 + x_3 = 21$$
$$3x_1 - 7x_2 + 5x_3 = 9.$$

7. Using Gauss–Jordan method to solve the following system of linear equations:

(i)
$$x_1 + x_2 + x_3 = 9,$$
$$2x_1 - 3x_2 + 4x_3 = 13,$$
$$3x_1 + 4x_2 + 5x_3 = 40.$$

(ii)
$$10x_1 + x_2 + x_3 = 12,$$
$$2x_1 + 10x_2 + x_3 = 13,$$
$$x_1 + x_2 + 5x_3 = 7$$

8. Using Crout's method to solve the following system of equations:

$$3x_1 + 2x_2 + 7x_3 = 4, 2x_1 + 3x_2 + x_3 = 5, 3x_1 + 4x_2 + x_3 = 7.$$

9. Solve the following system of equations by

(i) Crout's method (ii) Cholesky's method

(a) $x_1 + x_2 + 2x_3 = 7$; $3x_1 + 2x_2 + 4x_3 = 13$; $4x_1 + 3x_2 + 2x_3 = 8$.

(b) $10x_1 + x_2 + 2x_3 = 13$; $3x_1 + 10x_2 + x_3 = 14$; $2x_1 + 3x_2 + 10x_3 = 15$.

(c) $2x_1 - 6x_2 + 8x_3 = 24;\ 5x_1 + 4x_2 - 3x_3 = 2;\quad 3x_1 + x_2 + 2x_3 = 16.$

(d) $x_1 + x_2 + x_3 = 3;\ 2x_1 - x_2 + 3x_3 = 16;\ 3x_1 + x_2 - x_3 = -3.$

10. Solve the following system of linear equations using
 (i) Jacobi's method (ii) Gauss–Seidel method
 (a) $8x_1 + x_2 + x_3 = 8; 2x_1 + 4x_2 + x_3 = 4; x_1 + 3x_2 + 3x_3 = 5$
 (b) $2x_1 - x_2 + x_3 = -1; x_1 + 2x_2 - x_3 = 6; x_1 - x_2 + 2x_3 = -3$
 (c) $20x + y - 2z = 17; 3x + 20y - z = -18; 2x - 3y + 20z = 25$
 (d) $10x_1 - 2x_2 - x_3 - x_4 = 3; -2x_1 + 10x_2 - x_3 - x_4 = 15;$
 $-x_1 - x_2 + 10x_3 - 2x_4 = 27; -x_1 - x_2 - 2x_3 + 10x_4 = -9$

Answers

1. $x_1 = -\frac{1}{5}, x_2 = 4, x_3 = -\frac{4}{5}.$
2. $x_1 = 0, x_2 = -1, x_3 = 1.$
3. $X = (1, 2, -1, -2)^t.$
4. (i) $X = \left(1, \frac{1}{2}, \frac{1}{2}\right)^t$; (ii) $X = (1, -1, -2, 2)^t.$
5. (i) $X = (2, 0, 1)^t$; (ii) $X = (0.3, 0.4, 1)^t.$
6. $X = (2, -1, 2)^t.$
7. (i) $X = (1, 3, 5)^t$, (ii) $X = (1, 1, 1)^t.$
8. $X = (x_1, x_2, x_3)^t = \left(\frac{7}{8}, \frac{9}{8}, \frac{1}{8}\right)^t.$
9. (a) $X = (-1, 2, 3)^t$ (b) $X = (1, 1, 1)^t$
 (c) $X = (1, 3, 5)^t$ (d) $X = (1, -2, 4)^t.$
10. (a) $X = (0.83, 0.32, 1.07)^t$ (b) $X = (1, 2, -1)^t$
 (c) $X = (1, -1, 1)^t$ (d) $X = (1, 2, 3, 0)^t.$

9.17 MATRIX INVERSION

Let $A = \left[a_{ij}\right]_{n\times n}$ is a non-singular square matrix of order n if an $n \times n$ matrix A^{-1} exists with $AA^{-1} = I = A^{-1}A$. Then the matrix A^{-1} is called the inverse of A. In this section, we shall discuss the following methods to find the inverse of a given square matrix.

(i) Gauss elimination method
(ii) Gauss–Jordan's method
(iii) Crout's method
(iv) Cholesky's method

We explain the above methods with the help of an example.

Example 30 Find the inverse of matrix A, where

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}_{3\times 3}$$

Solution

(i) Gauss–Elimination method: In this method we write the given matrix A as $A = [A|I]$, where I is the identity matrix of same order of A and perform the elementary operations reduce the LHS in to an identity matrix and identity matrix in A^{-1} such that $I = A\,A^{-1}$.

Now, we write A as

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 3 & 10 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2 \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 3R_3 \\ R_1 \to R_1 - 6R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 18 & -6 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

$$I = A\,A^{-1}$$

$$\therefore \qquad A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

(ii) Gauss–Jordan's method: This method is similar to the Gauss–elimination method in which the a_{11} entry should be unit if not divide itself or interchange any two rows.

Now, writing the given matrix A side by side with the identity matrix of order 3, we have

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 2 & 5 & 15 & 0 & 1 & 0 \\ 6 & 15 & 46 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 3 & 10 & -6 & 0 & 1 \end{array}\right] \quad [\because \quad a_{11} = 1]$$

$$R_3 \to R_3 - 3R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - 3R_3 \\ R_1 \to R_1 - 6R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 18 & -6 \\ 0 & 1 & 0 & -2 & 10 & -3 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{array}\right]$$

$$R_1 \to R_1 - 2R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 10 & -3 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{array}\right]$$

Hence, the inverse of the given matrix is

$$A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

Example 31 Find the inverse of matrix A,

For $A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 6 \\ 6 & 2 & 2 \end{bmatrix}$

Solution Writing the matrix A side by side with the identity matrix of order 3, we have

$$[A|I] = \left[\begin{array}{ccc|ccc} 0 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 6 & 0 & 1 & 0 \\ 6 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \quad [\because \quad a_{11} = 0]$$

$$R_1 \leftrightarrow R_2 \sim \left[\begin{array}{ccc|ccc} 2 & 4 & 6 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 6 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \quad [\because \quad a_{11} = 2]$$

$$R_1 \to \frac{1}{2}R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 6 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \quad [\because \quad a_{11} = 1]$$

$$R_3 \to R_3 - 6R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & -10 & -16 & 0 & -3 & 1 \end{array}\right]$$

$$R_3 + R_3 + 5R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 5 & -3 & 1 \end{array}\right]$$

$$R_3 \to \left(\frac{1}{4}\right)R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{4} & -\frac{3}{4} & \frac{1}{4} \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - 4R_3 \\ R_1 \to R_1 - 3R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{15}{4} & \frac{11}{4} & -\frac{3}{4} \\ 0 & 2 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{4} & -\frac{3}{4} & \frac{1}{4} \end{array}\right]$$

$$R_1 \to R_1 - R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 2 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{4} & -\frac{3}{4} & \frac{1}{4} \end{array}\right]$$

$$R_2 \to \frac{1}{2}R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & -2 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{4} & -\frac{3}{4} & \frac{1}{4} \end{array}\right]$$

Hence, the inverse of matrix A is

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -2 & \frac{3}{2} & -\frac{1}{2} \\ \frac{5}{4} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

(i) *Crout's method*: In this method, we write the matrix A as

$$A = L\,U \text{ with } u_{ii} = 1. \tag{67}$$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Now Eq. (67) gives

$$A^{-1} = (LU)^{-1} = U^{-1}\,L^{-1} \tag{68}$$

We want to find U^{-1}, $U^{-1} = X$, where X is an upper triangular matrix.

$\therefore \quad UX = I$

Now, find L^{-1}, $L^{-1} = Y$, where γ is a lower triangular matrix.

$\therefore \quad LY = I$

For the matrix $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$

$$A = LU$$

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}\,u_{12} & u_{13}\,l_{11} \\ l_{21} & l_{21}\,u_{12} + l_{22} & l_{13}\,l_{21} + l_{22}\,u_{23} \\ l_{31} & l_{31}\,u_{21} + l_{32} & l_{31}\,u_{13} + l_{32}\,u_{23} + l_{33} \end{bmatrix}$$

$\therefore \quad l_{11} = 1,\ l_{21} = 2,\ l_{31} = 6$

$$l_{11}\,u_{12} = 2 \Rightarrow u_{12} = 2$$

$$l_{21}\,u_{12} + l_{22} = 5 \Rightarrow l_{22} = 1$$

$$l_{31}\,u_{12} + l_{32} = 15 \Rightarrow l_{32} = 3$$

$$l_{13}\,l_{11} = 6 \Rightarrow l_{13} = 6$$

$$l_{13}\,l_{21} + l_{22}\,u_{23} = 15 \Rightarrow u_{23} = 3$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 46 \Rightarrow l_{33} = 1$$

$$\therefore \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

To find U^{-1}, $U^{-1} = X \Rightarrow UX = I$

$$\therefore \qquad \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} & x_{12} + 2x_{22} & x_{13} + 2x_{23} + 6x_{33} \\ 0 & x_{22} & x_{23} + 3x_{33} \\ 0 & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad x_{11} = 1, \quad x_{22} = 1, \quad x_{33} = 1$$

$$x_{12} + 2x_{22} = 0 \Rightarrow x_{12} = -2, \qquad x_{23} + 3x_{33} = 0 \Rightarrow x_{23} = -3$$

$$x_{13} + 2x_{23} + 6x_{33} = 0 \Rightarrow x_{13} = 0$$

$$\therefore \qquad U^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, to find L^{-1}, $L^{-1} = Y$ or $LY = I$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_{11} & 0 & 0 \\ y_{21} & y_{22} & 0 \\ y_{31} & y_{32} & y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} y_{11} & 0 & 0 \\ 2y_{11} + y_{21} & y_{22} & 0 \\ 6y_{11} + 3y_{21} + y_{31} & 3y_{22} + y_{32} & y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad y_{11} = 1, \; y_{22} = 1, \; y_{33} = 1$$

$$2y_{11} + y_{21} = 0 \Rightarrow y_{21} = -2$$

$$3y_{22} + y_{32} = 0 \Rightarrow y_{32} = -3$$

$$6y_{11} + 3y_{21} + y_{31} = 0 \Rightarrow y_{31} = 0$$

$$\therefore \qquad L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Hence, $A^{-1} = U^{-1} L^{-1}$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

(ii) *Cholesky's method*: This method is discussed in the Section 9.16.3, we have $A^{-1} = S^T S$, where $S = L^{-1}$ with $l_{ii} = 1$.

For the matrix $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$

Clearly, matrix A is symmetric and positive definite. We have

$$A = L\, L^T$$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & l_{21} & l_{31} \\ l_{21} & l_{21}^2 + 1 & l_{21}\, l_{31} + l_{32} \\ l_{31} & l_{31}\, l_{21} + l_{32} & l_{31}^2 + l_{32}^2 + 1 \end{bmatrix}$$

We obtain $l_{21} = 2,\ l_{31} = 6$

$l_{21}\, l_{31} + l_{32} = 15 \Rightarrow l_{32} = 3.$

Thus, $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}$

Now, to find L^{-1}, $L^{-1} = X$ or $LX = I$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} x_{11} & 0 & 0 \\ 2x_{11} + x_{21} & x_{22} & 0 \\ 6x_{11} + 3x_{21} + x_{31} & 3x_{22} + x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad x_{11} = 1,\ x_{22} = 1,\ x_{33} = 1$$

$$2x_{11} + x_{21} = 0 \Rightarrow x_{21} = -2$$

$$6x_{11} + 3x_{21} + x_{31} = 0 \Rightarrow x_{31} = 0$$

$$3x_{22} + x_{32} = 0 \Rightarrow x_{32} = -3$$

Thus $$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = S \quad \text{(say)}$$

$\therefore$ $$S^T = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $$A^{-1} = S^T S = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

EXERCISE 9.3

1. Find the inverse of matrix A by
   (i) Gauss-Elimination method
   (ii) Gauss–Jordan's method
   (iii) Crout's method
   (iv) Cholesky's method

   (a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$

2. Find the inverse of the matrix A by Jordan's method, where

$$A = \begin{bmatrix} 50 & 107 & 36 \\ 25 & 54 & 20 \\ 31 & 66 & 21 \end{bmatrix}$$

3. Using Gauss-Jordan's method, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

4. Find the inverse of the matrix

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

5. Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Answers

1. (a) $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 7 & -5 & 1 \\ -5 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (b) $A^{-1} = \begin{bmatrix} -\dfrac{5}{12} & \dfrac{1}{4} & \dfrac{1}{3} \\ \dfrac{7}{12} & \dfrac{1}{4} & -\dfrac{2}{3} \\ \dfrac{1}{12} & -\dfrac{1}{4} & \dfrac{1}{3} \end{bmatrix}$

2. $A^{-1} = \begin{bmatrix} -186 & 129 & 196 \\ 95 & -66 & -100 \\ -24 & 17 & 25 \end{bmatrix}$ 3. $A^{-1} = \begin{bmatrix} 3 & 1 & \dfrac{3}{2} \\ -\dfrac{5}{4} & -\dfrac{1}{4} & -\dfrac{3}{4} \\ -\dfrac{1}{4} & -\dfrac{1}{4} & -\dfrac{1}{4} \end{bmatrix}$

4. $A^{-1} = \begin{bmatrix} \dfrac{1}{8} & -\dfrac{1}{8} & \dfrac{3}{8} \\ -\dfrac{1}{8} & \dfrac{1}{8} & \dfrac{5}{8} \\ \dfrac{3}{8} & \dfrac{5}{8} & \dfrac{23}{8} \end{bmatrix}$ 5. $A^{-1} = \begin{bmatrix} \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} \\ \dfrac{3}{20} & \dfrac{1}{20} & -\dfrac{7}{20} \\ \dfrac{1}{20} & \dfrac{7}{20} & \dfrac{11}{20} \end{bmatrix}$

9.18 EIGEN VALUE PROBLEMS

Let A is $n \times n$ matrix with elements a_{ij}, then A defines a linear transformation.

Consider the transformation

$AX = \lambda X$, where λ is a scalar and X is an eigen vector

or $$(A - \lambda I)X = 0 \tag{69}$$

Equation (69) is a system of homogeneous linear algebraic equations, it will have a non-trivial solution if

$$\det(A - \lambda I) = 0 \tag{70}$$

Equation (70) is called nth order characteristic equation in λ and if solved, we obtain n values of λ, which are called characteristic roots or latent roots or eigen values or invariant roots corresponding to n values of λ, in general, there will be n-non-zero solutions of Eq. (69) which are called eigen vectors or characteristic vectors or invariant vectors.

9.18.1 Properties

Some important properties of eigen values and eigen vectors are as follows:

1. If λ is an eigen value of matrix A, then $\dfrac{1}{\lambda}$ is the eigenvalue of A^{-1}.
2. The eigen values of A and A^T are same but in general, not the eigen vector's.
3. If λ is an eigen value of A nearest to a number p, then $(\lambda - p)$ shall be the eigen value of the matrix $(A - pI)$.
4. If λ is an eigen value of an orthogonal matrix, then $\dfrac{1}{\lambda}$ is also its eigen value.
5. If λ is an eigen value of matrix A, then λ^r is an eigen values of A^r.
6. All the eigen values of symmetric matrix are real.
7. The sum of all the eigen values of matrix A is equal to the trace of A.
8. The product of all the eigen values of matrix A is equal to the determinant of A.

9.18.2 Critical Points of Linear Systems

We can use the eigen value problem in the Section 9.19 to investigate the critical points of a linear system $AX = B$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

$$X = [x_1, x_2]^T \text{ and } B = [\mathrm{b}_1, b_2]^T.$$

The nature of the isolated critical point (0, 0) depends on the two non-zero eigen values λ_1 and λ_2 of the matrix A are:

(i) if the eigen values are real and unequal with the same sign, then the critical point is an improper node.

(ii) if the eigen values are real and unequal with opposite signs, then the critical point is unstable saddle point.

(iii) if the eigen values are real and equal, then the critical point is a proper node.

(iv) if the eigen values are pure imaginary, then it is a center.

(v) if the eigen values are complex conjugate, then the critical point is a spiral point.

9.18.3 Power Method

This method is applicable to determine the largest eigen-value (in magnitude). Let matrix A has a complete set of eigen vectors $X_1, X_2, X_3, \ldots, X_n$ corresponding to eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$, such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|, \tag{71}$$

where λ_1 is the dominant eigen value.

Since the n vectors may be expressed as a linear combination.

Thus, let a vector

$$y_0 = \sum_{r=1}^{n} \alpha_r X_r \tag{72}$$

Construct the iterative scheme

$$Z_{i+1} = A\,y_i \quad \text{and} \quad y_{i+1} = \frac{Z_{i+1}}{c_i};\ i = 0, 1, 2, \ldots \tag{73}$$

where c_i is largest (in magnitude) component of Z_{i+1}.

Therefore, $y_1 = \dfrac{Z_1}{c_0} = \dfrac{1}{c_0} A\,y_0$; c_0 is the largest element of $A\,y_0$.

$$y_2 = \frac{1}{c_1} A\,y_1 = \frac{1}{c_0 c_1} A^2 y_0; \ c_1 \text{ is largest element of } A\,y_1$$

In general,

$$y_k = \frac{1}{c_{k-1}\,c_{k-2}\ldots c_0} A^k y_0$$

$$= \frac{A^k}{c_{k-1}\,c_{k-2}\ldots c_0} \sum_{r=1}^{n} \alpha_r X_r$$

$$= \frac{1}{c_{k-1}c_{k-2}\ldots c_0} \sum_{r=1}^{n} \alpha_r (A^k X_r)$$

$$= \frac{1}{c_{k-1}\,c_{k-1}\ldots c_0} \sum_{r=1}^{n} \alpha_r\ \lambda_r^k X_r$$

$$= \frac{1}{c_{k-1}c_{k-2}\cdots c_0} \left[\alpha_1\,\lambda_1^k X_1 + \alpha_2\,\lambda_2^k X_2 + \cdots + \alpha_n \lambda_n^k X_n\right]$$

$$y_k = \frac{\lambda_1^k}{c_{k-1}c_{k-2}\cdots c_0} \left[\alpha_1 X_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k \alpha_2 X_2 + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k \alpha_n X_n\right]$$

Since $\left|\dfrac{\lambda_i}{\lambda_1}\right| < 1$ for $i = 2, 3, 4, \ldots, n$, then

$$\lim_{k\to\infty} \left|\frac{\lambda_i}{\lambda_1}\right|^k \to 0 \ \text{ for } i = 2, 3, 4, \ldots, n$$

$$\therefore \qquad y_k = \frac{\lambda_1^k}{c_{k-1}c_{k-2}\cdots c_0} \alpha_1 X_1 \qquad (\because \quad \alpha_1 \neq 0) \tag{74}$$

Also, $$y_{k+1} = \frac{\lambda_1^{k+1}}{c_k c_{k-1} \cdots c_0} \alpha_1 X_1 \tag{75}$$

The j^{th} element of y_k and y_{k+1} are as follows:

$$y_k(j) = \frac{(\lambda_1)^k}{c_{k-1} c_{k-2} \cdots c_0} \alpha_1 x_{j1}, \text{ where } x_{ji} \text{ is the } j^{\text{th}} \text{ element of } X_1 \tag{76}$$

and $$y_{k+1}(j) = \frac{(\lambda_1)^{k+1}}{c_k c_{k-1} \cdots c_0} \alpha_1 x_{j1} \tag{77}$$

From Eqs (76) and (77), we at once obtain

$$\lambda_1 = \lim_{k\to\infty} \frac{y_{k+1}(j)}{y_k(j)}; \ j = 1, 2, 3, ..., n$$

Example 32 Find the largest eigen value and corresponding the eigen vector of the following matrix using power method correct to 2 decimal places, where $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

Solution

Suppose $y_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the initial arbitrary vector.

Then $$Z_1 = A y_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \ y_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$Z_2 = A y_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2y_2$$

$$Z_3 = A y_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = 4\, y_3$$

$$Z_4 = A y_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5 \begin{bmatrix} 0.714 \\ -1 \\ 0.714 \end{bmatrix} = 3.5\, y_4$$

$$Z_5 = A y_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.714 \\ -1 \\ 0.714 \end{bmatrix} = 3.42 \begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = 3.42\, y_5$$

$$Z_6 = A\,y_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = 3.416\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix} = 3.416\,y_6$$

$$Z_7 = A\,y_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix} = 3.414\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix} = 3.414\,y_7$$

For 2 decimal places

$$\left|\lambda_1^{(k+1)} - \lambda_1^k\right| \le \frac{1}{2}\times 10^{-2} = 0.005$$

Here $\lambda_1^7 = 3.414$ and $\lambda_1^6 = 3.416$

$$\therefore \quad \left|\lambda_1^7 - \lambda_1^6\right| = |3.414 - 3.416| = 0.002 < 0.005$$

Hence, the largest eigen value $\lambda_1 = 3.41$ and eigen vector is $\begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix} \sim [1 \quad -1.414 \quad -1]^T$.

Example 33 Find the largest eigen value and its corresponding eigenvector for the matrix A, where

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

Solution Suppose the initial arbitrary vector be

$$y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and $\quad Z_{i+1} = A\,y_i,\ y_{i+1} = \dfrac{1}{c_i}(Z_{i+1});\ i = 0, 1, 2, 3, \ldots$ where c_i is the largest element of Z_{i+1}.

Now $\quad Z_1 = A\,y_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix}$

$$= 25\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = 25\,y_1$$

$$Z_2 = A\,y_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = 25.2\,y_2$$

$$Z_3 = A\,y_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \begin{bmatrix} 25.1778 \\ 1.1332 \\ 1.7337 \end{bmatrix} = 25.1778\begin{bmatrix} 1 \\ 0.0450 \\ 0.0688 \end{bmatrix} = 25.1778\,y_3$$

$$Z_4 = A\,y_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0450 \\ 0.0688 \end{bmatrix} = \begin{bmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{bmatrix} = 25.1826\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = 25.1826\,y_4$$

$$Z_5 = A\,y_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1323 \\ 1.7260 \end{bmatrix} = 25.1821\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = 25.1821\,y_5$$

Since $Z_4 \approx Z_5$, hence the largest eigen value is $\lambda_1 = 25.182$ and eigen vector is

$$\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

***Note*:** For 3 decimal places $\left|\lambda_1^{k+1} - \lambda_1^k\right| \le \frac{1}{2}\times 10^{-3} = 0.0005$

Here $\lambda_1^4 = 25.1826$, $\lambda_1^5 = 25.1821$ $\quad\therefore\quad |25.1821 - 25.1826| = 0.0005$.

SUMMARY

Following topics have been discussed in this chapter:

1. Some Basic Properties of an Equation

(i) The total number of roots of an algebraic question is the same as its degree.

(ii) If $f(x)$ is exactly divisible by $(x - a)$, then $x = a$ is a root of equation $f(x) = 0$.

(iii) If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ have the roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, then

Sum of roots = $\sum_i \alpha_i = \frac{a_1}{a_0}$;

Sum of the product of two roots = $\sum_{i<j}\alpha_i\alpha_j = \frac{a_2}{a_0}$; and product of the roots = $\pi_i\,\alpha_i = (-1)^n\,\frac{a_n}{a_0}$.

(iv) Every equation of the odd degree has at least one real root.

(v) If $\alpha + i\beta$ is a root of the equation $f(x) = 0$, then $\alpha - i\beta$ must also be its root.

(vi) An algebraic equation can have at most as many positive roots as the number of changes of sign in the coefficients of $f(x)$.

(vii) An algebraic equation can have at most as many negative roots as the number of changes of sign in the coefficient of $f(-x)$.

(viii) If an algebraic equation of degree 'n' has at the most α_p positive roots and at the most α_n negative roots, then the equation has at least $(n - \alpha_p - \alpha_n)$ imaginary roots.

(x) *Intermediate Value Theorem*: If $f(x)$ is a continuous function on $[a, b]$ and the sign of $f(a)$ is different from the sign of $f(b)$; that is $f(a) \cdot f(b) < 0$, then there exists a point c, in the interval (a, b) such that $f(c) = 0$. Hence, any value $c \in (a, b)$ can be taken as an initial approximation to the root.

***Note*:** Using the above theorem, the equation $f(x) = 0$ has at least one real root or an odd number of real roots in (a, b).

2. Bisection Method

This is one of the simplest method and is based on the Intermediate value theorem. This method is also known as the bisection method of Bolzano or Binary or Binary-search method. This bisection method is defined as following:

(i) Using Intermediate value theorem, find an interval (a, b) if $f(a)\,f(b) < 0$, then the root lies in (a, b).

(ii) The first approximation to the root is $x_1 = \dfrac{a+b}{2}$. If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$, otherwise.

(iii) Use the Intermediate value theorem to decide whether the root lies in (a, x_1) or (x_1, b).

(iv) Repeat the step using the interval either (a, x_1) or (x_1, b).

(v) The procedure repeated while an length of the last interval is less than the desired accuracy. The mid-point of this last interval is the required root of the given equation $f(x) = 0$.

3. Fixed Point Iteration Method

A fixed point α for a function $\phi(x)$ is a number, when the value of the function does not change.

Fixed point iteration or successive approximation method find a root of an equation

$$f(x) = 0.$$

The first step of this method is to write the above equation in the form

$$x = \phi(x)$$

Let x_0 be an initial approximation to the root of $f(x) = 0$, then the first approximation is $x_1 = \phi(x_0)$.

The r^{th} successive approximations are as follows:

$x_{n+1} = \phi(x_n)$; $n = 0, 1, 2, 3, \ldots$ is called the fixed-point iteration.

4. Newton's Method

Newton–Raphson (or Newton's) method is one of the most powerful and well-known numerical method.

The successive approximations is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; n = 0, 1, 2, 3, \ldots$$

is known the Newton's–Raphson formula.

5. Regula Falsi Method

The Regula falsi method is also known as method of false position with the help of this method, we compute the real roots of the equation $f(x) = 0$.

Hence, the first approximation to the root is $x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} \cdot f(x_0)$
In general

$$x_{n+1} = x_{n-1} - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \cdot f(x_{n-1})$$

Provided that at each step $f(x_{n-1}) \cdot f(x_n) < 0$.

6. Secant Method

Newton's method is a powerful technique, but it has a weakness, the need to compute the value of f' at each approximation. To avoid computation of f', $f'(x_n)$ is replaced by $\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$ and we obtained secant method as

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot f(x_n)$$

This technique is called the secant method. This method starting two initial approximations x_0 and x_1, the approximation x_2 is the x–intercept of the secant line joining two points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$. The approximation x_3 is the x-intercept of the secant line joining two points $[x_1, f(x_1)]$ and $[x_2, f(x_2)]$ and continuing this process until we get the root to desired accuracy.

7. Rate of Convergence for Iterative Methods

(i) Rate of Convergence of Bisection Method is linearly convergent.
(ii) Rate of Convergence of Iteration Method is linear.
(iii) Rate of Convergence of Newton's Method is quadratic.
(iv) The order of convergence of Regula-Falsi method is 1.618.

8. Gaussian Elimination Method

This method is applied on the linear system and we write the augmented matrix

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

The entries in the $(n + 1)^{\text{th}}$ column are the values of B i.e., $a_{i,n+1} = b_i$ for each $i = 1, 2, \ldots, n$.

Provided $a_{11} \neq 0$, now we perform the operations corresponding to $E_j \to \left(E_j - \left(\frac{a_{j1}}{a_{11}}\right)E_1\right)$ for each $j = 2, 3, \ldots, n$, to eliminate the coefficient fx_1 in each rows.

The entries in rows 2, 3, …, n are change, for each of notation we again denote the entry in the i^{th} row and the j^{th} column by a_{ij}.

We follow a sequential procedure for $i = 2, 3, \ldots(n - 1)$ and perform the operation

$E_j \to \left(E_j - \left(\frac{a_{ji}}{a_{ii}}\right)E_i\right)$ for each $j = i + 1, i + 2, \ldots, n$, provided $a_{ii} \neq 0$ and we have a resulting matrix is of the form

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \ddots\ \vdots & & \vdots & \vdots \\ 0 & \ldots\ 0 & & a_{nn} & a_{n,n+1} \end{array}\right]$$

Now, we write the above equation form as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= a_{1,n+1} \\ a_{22}x_2 + \cdots + a_{2n}x_n &= a_{2,n+1} \\ \ddots \qquad \vdots \qquad & \quad \vdots \\ a_{nn}x_n &= a_{n,n+1} \end{aligned}$$

Using back substitution find $x_1, x_2, \ldots, x_n$.

9. Gauss's–Jordan Method

This method follow the same procedure of Gauss's elimination method, but also from $E_1, E_2, \ldots, E_{i-1}$. The augmented matrix $[A|B]$ reducing as

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & 0 & \cdots & 0 & a_{1,n+1}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & 0 & a_{2,n+1}^{(2)} \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn}^{(n)} & a_{n,n+1}^{(n)} \end{array}\right]$$

The solution is obtained by $x_i = \frac{a_{i,n+1}^{(i)}}{a_{ii}^{(i)}}$, for each $i = 1, 2, 3, \ldots, n$.

Now, we explain the above procedure with the help of the following example:

10. Crout's Method

Crout method is used to determine determining the numerical solution of a system of linear equations and it has an advantage over the Gauss's elimination method in that it requires the number of less computations. Suppose, the systems of linear equations is of the form $AX = B$

where $A = \left[a_{ij}\right]_{m\times n}$ for each $i = 1(1)m$ and $j = 1(1)n$,

$$B = \left[b_1, b_2, ..., b_m\right]^T, X = \left[x_1, x_2, ..., x_m\right]^T$$

Then $$C = \left[A|B\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Now, we define an Auxiliary matrix

$$C' = \left[A'|B'\right]$$

Using C', we determine the solution column X.

11. LU Decomposition Method

The Gauss's elimination method is the principal tool in the direct solution of linear system of equations. The LU decomposition method has the property that the matrix decomposition step can be performed independent of the right hand side vector. In this method the coefficient matrix A is expressed as the product of lower triangular matrix (L) and an upper triangular matrix (U), that is

$$A = LU$$

Then, the linear system of equations,

$$AX = B \text{ becomes}$$

$$LUX = B$$

Let $$UX = Y,$$

then the given equation $\alpha UX = B$ becomes

$$LY = B$$

Now, first we solve the lower triangular system for Y. Once Y is known, then solve the upper triangular system for X.

We shall discussed the following approaches of decomposition using 3 × 3 matrices.

(i) Doolittle Decomposition

In this method, we choose $l_{ii} = 1$ for $i = 1, 2, 3$ and we write the given system as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

(ii) Crout's Decomposition

In this method, we choose $u_{ii} = 1$ for $i = 1, 2, 3$ and we write the given system as

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(iii) Cholesky's Method

This method is applicable for a symmetric and positive definite matrix A, (i.e., $A^T = A$ and $X^T AX > 0$ for all $X \neq 0$), then, according to this method matrix A can be written as the product of two triangular matrix such that

$$A = LL^T,$$

where L is lower triangular matrix and L^T is an upper triangular matrix.

$$\begin{aligned} \therefore \qquad A^{-1} &= (L\,L^T)^{-1} \\ &= (L^T)^{-1}\, L^{-1} \\ &= (L^{-1})^T\, L^{-1} \\ A^{-1} &= S^T S; \quad \text{where } S = L^{-1} \end{aligned}$$

which gives the inverse of matrix A.

12. Iterative Methods

In this section, we shall discuss the iterative or indirect methods, an iterative method repeats its process over and over, each time using the current approximation to produce a better approximation for the exact solution, until the current approximation is sufficiently close to the exact solution that is we expect that $x^{(k)}$ to be close to x if $\left|x^{(k)} - x^{(k-1)}\right|$ is very small. An iterative methods to solve the $n \times n$ linear system $AX = B$ start with an initial approximation $x^{(0)}$ to the solution x and generates a sequence of vectors $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ that converge to x. Now, we describe the Jacobi and the Gauss–Seidel iterative methods.

(i) Jacobi's Method

This method is converges if the coefficient matrix A strictly diagonally dominant; that is

$$\left|a_{ii}\right| > \sum_{i \neq j} \left|a_{ij}\right| \quad \text{for } i = 1, 2, 3, \ldots, n.$$

The iterative formula for the linear system of equations $AX = B$ by Jacobi method can be written as

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[-\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\, x_j^{(k)} + B_i \right]$$

for $i = 1, 2, 3, \ldots, n$; provided $a_{ii} \neq 0$.

(ii) Gauss–Seidal Method

The iterative formula for the linear system of equations $AX = B$ by Gauss–Seidal method can be written as

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k)} + B_i \right]$$

for $i = 1, 2, 3, \ldots, n$; provided $a_{ii} \neq 0$.

OBJECTIVE TYPE QUESTIONS

1. The iteration formula to find the square root of a positive real number b using the Newton's–Raphson method is

(a) $x_{k+1} = 3(x_k + b)/2x_k$

(b) $x_{k+1} = 3(x_{k+b})/2x_k$

(c) $x_{k+1} = x_k - 2x_k/(x_k^2 + b)$

(d) none of the above

[GATE (CS) 1995]

2. Newton's–Raphson iteration formula for finding $\sqrt[3]{c}, c > 0$ is

(a) $x_{n+1} = \dfrac{2x_n^3 + \sqrt[3]{c}}{3x_n^2}$

(b) $x_{n+1} = \dfrac{2x_n^3 - \sqrt[3]{c}}{3x_n^2}$

(c) $x_{n+1} = \dfrac{2x_n^3 + c}{3x_n^2}$

(d) $x_{n+1} = \dfrac{2x_n^3 - c}{3x_n^2}$

[GATE (CS) 1996]

3. We wish to start $x^2 - 2 = 0$ by Newton's–Raphson technique. Let the initial guess. $x_0 = 1.0$ subsequent estimate of x (i.e., x_1) will be

(a) 1.414 (b) 1.5

(c) 2.0 (d) none of these

[GATE (ME) 1998]

4. If $a > 0$, the reciprocal value of $\dfrac{1}{a}$ by Newton–Raphson algorithm for $f(x) = 0$ is

(i) The Newton–Raphson algorithm for the function will be

(a) $x_{k+1} = \dfrac{1}{2}\left(x_k + \dfrac{a}{x_k}\right)$

(b) $x_{k+1} = x_k + \dfrac{1}{2}a x_k^2$

(c) $x_{k+1} = 2x_k - ax_k^2$

(d) $x_{k+1} = x_k - \dfrac{a}{2}x_k^2$

(ii) For $a = 7$ and starting with $x_0 = 0.2$, the first two iterations will be

(a) 0.11, 0.1299 (b) 0.12, 0.1392

(c) 0.12, 0.1416 (d) 0.13, 0.1428

[GATE (CE) 2005]

5. Starting from $x_0 = 1$, one step of Newton–Raphson method in solving the equations $x^3 + 3x - 7 = 0$ gives the next value (x_1) is

(a) 0.5 (b) 1.406

(c) 1.5 (d) 2

[GATE (ME) 2005]

6. Consider the series $x_{n+1} = \dfrac{x_n}{2} + \dfrac{9}{8x_n}$, $x_0 = 0.5$ obtained from the Newton–Raphson method. The series converges to

(a) 1.5 (b) $\sqrt{2}$

(c) 1.6 (d) 1.4

[GATE (ECE) 2005, (CS) 2008]

7. For $k = 0, 1, 2, \ldots$, the steps of Newton–Raphson method for solving a non-linear equation is given by

$$x_{k+1} = \frac{2}{3}x_k + \frac{5}{3}x_k^{-2}$$

Starting from a suitable initial choice and $k \to \infty$ the iterate x_K tends to

(a) 1.7099 (b) 2.2361

(c) 3.1251 (d) 5.0000

[GATE (AIE) 2006]

8. Identify the Newton–Raphson iteration scheme for finding the square root of 2 is

(a) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$

(b) $\dfrac{1}{2}\left(x_n - \dfrac{2}{x_n}\right)$

(c) $x_{n+1} = \dfrac{2}{x_n}$

(d) $x_{n+1} = \sqrt{(2 + x_n)}$

[GATE (AIE) 2007]

9. The following equation needs to be numerically solved by using the Newton–Raphson method

$x^3 + 4x - 9 = 0$. The iterative equation for this purpose is (k indicates the iteration level)

(a) $x_{k+1} = \dfrac{2x_k^3 + 9}{3x_k^2 + 4}$

(b) $x_{k+1} = \dfrac{3x_k^2 + 4}{2x_k^2 + 9}$

(c) $x_{k+1} = x_k - 3x_k^2 + 4$

(d) $x_{k+1} = \dfrac{4x_k^2 + 3}{9x_k^2 + 2}$

[GATE (CE) 2007]

10. The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using the Newton–Raphson method. If $x = 2$ is taken as the initial approximation of the solution, then the next approximation using the method will be

(a) $\dfrac{2}{3}$ (b) $\dfrac{4}{3}$

(c) 1 (d) $\dfrac{3}{2}$

[GATE (ECE) 2007]

11. The recursion relation to solve $x = e^{-x}$ using Newton–Raphson method is

(a) $x_{n+1} = e^{-x_n}$

(b) $x_{n+1} = x_n - e^{-x_n}$

(c) $x_{n+1} = (1 + x_n)\dfrac{e^{-x_n}}{1 + e^{-x_n}}$

(d) $x_{n+1} = \dfrac{x_n^2 - e^{-x_n}(1 + x_n) - 1}{x_n - e^{-x_n}}$

[GATE (ECE) 2008]

12. The Newton–Raphson iteration

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

can be used to compute the

(a) square of R (b) reciprocal of R

(c) square root of R (d) logarithms of R

[GATE (CE) 2008]

13. Let $x^2 - 117 = 0$. The iterative steps for the solution, using Newton's–Raphson method is given by

(a) $x_{k+1} = \dfrac{1}{2}\left(x_k + \dfrac{117}{x_k}\right)$

(b) $x_{k+1} = x_k - \dfrac{117}{x_k}$

(c) $x_{k+1} = x_k - \dfrac{x_k}{117}$

(d) $x_{k+1} = x_k - \dfrac{1}{2}\left(x_k + \dfrac{117}{x_k}\right)$

[GATE (EE) 2009]

14. A numerical solution of the equation $f(x) = x + \sqrt{x-3} = 0$ can be obtained using Newton–Raphson method. If the starting value is $x = 2$ for the iteration, the value of x that is to be used in the next step is

(a) 0.306 (b) 0.739

(c) 1.694 (d) 2.306

[GATE (ECE) 2011]

15. The bisection method is applied to compute a zero of the function $f(x) = x^4 - x^3 - x^2 - 4 = 0$ in the interval $\{1, 9\}$. The method converges to a solution after iterations

(a) 1 (b) 3

(c) 5 (d) 7

[GATE (CS) 2012]

16. When the Newton–Raphson method is applied to solve the equation $f(x) = x^3 + 2x - 1 = 0$, the solution at the end of the first iteration with the initial guess value as $x_0 = 1.2$ is

(a) –0.82 (b) 0.49

(c) 0.705 (d) 1.69

[GATE (EE) 2013]

17. The function $f(x) = e^x - 1$ is to be solved using Newton's–Raphson method. If the initial value of x_0 is taken as 1.0, then the absolute error observed at second iteration is _____.

[GATE (EE) 2014]

18. Newton–Raphson method is used to find the roots of the equation $x^3 + 2x^2 + 3x - 1 = 0$. If the initial guess is $x_0 = 1$, then the value of x after 2nd iteration is ______.

[GATE (ME) 2015]

19. The iteration step in order to solve for the cube roots of a given number N using the Newton–Raphson method is

(a) $x_{k+1} = x_k + \frac{1}{3}(N - x_k^3)$

(b) $x_{k+1} = \frac{1}{3}\left(2x_k + \frac{N}{x_k^2}\right)$

(c) $x_{k+1} = x_k - \frac{1}{3}(N - x_k^3)$

(d) $x_{k+1} = \frac{1}{3}\left(3x_k - \frac{N}{x_k^2}\right)$

[GATE (IN) 2014]

20. The order of convergence in Newton–Raphson method is

(a) 0 (b) 2
(c) 3 (d) None of these

21. The order of convergence of iterative formula

$x_{n+1} = \frac{x_n}{2}\left(3 - \frac{x_n^2}{a}\right)$ of $\sqrt{a}$ is

(a) $\frac{3}{2}$ (b) 0
(c) 2 (d) 1

22. Newton's-Raphson method is useful in case of large values of

(a) $f'''(x)$ (b) $f''(x)$
(c) $f'(x)$ (d) $f(x)$

23. The order of convergence of the iterative method

$$x_{n+1} = x_0 f(x_n) - \frac{x_n \cdot f(x_0)}{f(x_n) - f(x_0)}$$

for finding a simple root of the equation $f(x) = 0$ is

(a) 1 (b) 2
(c) 3 (d) 4

24. The rate of convergence in bisection method is

(a) 1 (b) 2
(c) 3 (d) 4

25. The bisection method for finding the roots of an equation $f(x) = 0$ is

(a) $x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$

(b) $x_{k-1} = \frac{1}{2}(x_{k+1} - x_k)$

(c) $x_k = \frac{1}{2}(x_{k-1} - x_{k+2})$

(d) None of these

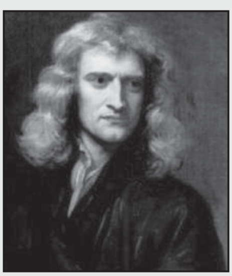
ANSWERS

1. (d)	2.(c)	3. (a)	4. (i-c) (ii-b)	5. (b)	6. (a)	7. (a)
8. (a)	9. (a)	10. (d)	11.(c)	12. (c)	13. (a)	14. (1.694)
15. (b)	16. (0.705)	17. (0.06)	18. (0.3043)	19. (b)	20.(b)	21.(a)
22.(c)	23.(a)	24.(b)	25. (a)			

10 Numerical Differentiation and Integration

10.1 NUMERICAL DIFFERENTIATION

Numerical differentiation is concerned with the method of finding the successive differentiations of a function at the given arguments, that is, it is the process of determining the value of the derivative of a function $y = f(x)$ at some assigned values of independent variable from the given set of values (x_i, y_i). If the values of the arguments are equally spaced and if the derivative is to be found near the beginning of the table, then we apply Newton's forward formula, if it is required near the end of the table, then we apply the Newton's backward formula. If we want to find the derivative near the middle of the given set of values, then we apply any one of the central difference formula. If the values of the arguments are



Isaac Newton was born on 4 January 1643 in Lincolnshire, England. He was one of the greatest and most influential man to have contributed to numerous fields such as physics, mathematics, astronomy, philosophy and theology. Newton began a life that was to become an inspiration for the scientific world for centuries to come. After his early education from 'The King's School' in Grantham, Newton tried his luck for Trinity College of the Cambridge University. Succeeding in his attempt he entered Trinity College in June 1661 where he acquired knowledge of the modern philosophers and astronomers like Descartes, Galileo, Kepler and Copernicus. He showed immense aptitude in mathematics and physics and soon he was working on binomial theorems and developing mathematical theories that were later to form a branch of mathematics called infinitesimal calculus. Newton's Law of Gravitation was conceived in 1665. Newton worked in all branches of mathematics portraying pure brilliance in each one. He particularly advanced calculus by giving solutions to problems of analytical geometry by using differentiation and integration. Newton's method, Newton's Identities, binomial theorem, improvements to the theory of finite differences and solution to Diophantine Equations are all credited to Newton. His work in the field of Optics also resulted in major advances that gave clarity to previous vague theories. Newton investigated through various experiments, known as the 'Experimentum Crucis', the refraction of light showing that when white light passes through a glass prism, it forms into spectra of different colors with each color refracting at a different angle. He used various media like oil, water and even soap bubble to work more on his color theory. His work 'Opticks' is a model of his theories of Optics, published in 1704. Newton was the master of all sciences. He died on 20 March 1726 in Kensington, Middlesex, England.

not equally spaced, then for determining the derivative by using Newton's divided difference formula or Lagrange's interpolation formula.

To determine the maximum or minimum value of a tabulated function, then we find the necessary differences from the given table and put them in an appropriate interpolation formula. Now, we compute the first derivative of the function obtained from the interpolation formula and put the derivative equal to zero and obtain the values of stationary point(s) and then determine the maximum or minimum value of the function.

10.2 NUMERICAL DIFFERENTIATION USING THE INTERPOLATION FORMULAE

10.2.1 Newton's Forward Interpolation Formula

Newton's forward interpolation formula is given by

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \tag{1}$$

where $$u = \frac{x - x_0}{h}$$

Differentiating Eq. (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \cdots \tag{2}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \frac{4u^3 - 18u^2 + 22u - 6}{4!}\Delta^4 y_0 + \cdots\right] \tag{3}$$

At $x = x_0$, $u = 0$, then (3) becomes

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right].$$

Similarly, higher order derivatives

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

and $$\left(\frac{d^3 y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

and so on.

10.2.2 Newton's Backward Interpolation Formula

Newton's backward interpolation formula is given by

$$y = y_n + \nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots \tag{4}$$

where $$u = \frac{x - x_n}{h} \tag{5}$$

Differentiating Eq. (4) w.r.t. u and Eq. (5) w.r.t. x, we obtain

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^3 y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

and so on.

10.2.3 Stirling's Central Difference Formula

To compute the values of the derivatives of the function near the middle of the given set of arguments. We have

$$y = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-2^2)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right] + \cdots \tag{6}$$

where $$u = \frac{x - x_0}{h}. \tag{7}$$

Differentiating Eq. (6) w.r.t. u and Eq. (7) w.r.t. x; we obtain

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) - \cdots\right]$$

$$\left[\frac{d^2 y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots\right]$$

and so on.

10.2.4 Bessel's Central Difference Formula

The Bessel's formula is

$$y = \left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{u(u-1)\left(u - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)$$

$$+ \frac{(u+1)u(u-1)(u-2)\left(u - \frac{1}{2}\right)}{5!}\Delta^5 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)(u-3)}{6!}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \cdots \tag{8}$$

where $$u = \frac{x - x_0}{h} \tag{9}$$

Differentiating Eq. (8) w.r.t. u and Eq. (9) w.r.t. x and we have

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{60}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \cdots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) - \frac{1}{2}\Delta^3 y_{-1} - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \frac{1}{24}\Delta^5 y_{-2} + \frac{1}{90}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \cdots\right]$$

and so on.

10.2.5 Newton's Divided Difference Formula

We have

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 f(x_0) + \cdots \tag{10}$$

Differentiating Eq. (10) w.r.t. 'x' we have

$$f'(x) = \Delta f(x_0) + \left[2x - (x_0 + x_1)\right]\Delta^2 f(x_0) + \left[3x^2 - 2x(x_0 + x_1 + x_2) + (x_0 x_1 + x_1 x_2 + x_2 x_0)\right]\Delta^3 f(x_0) + \cdots$$

10.2.6 Lagrange's Interpolation Formula

This formula is applied in both cases, that is if the arguments are equally spaced or unequally spaced. The Lagrange's interpolation formula is as follows:

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1)+\cdots \tag{11}$$

Differentiating Eq. (11) w.r.t. x one or more times and we obtain the derivatives of $f(x)$.

10.2.7 Maxima and Minima of a Function given the Tabulated Values

To determine the value(s) of arguments x at which the curve $y = f(x)$ is maxima or minima can be obtained by equating $\frac{dy}{dx}$ to zero. The same procedure can be used to find maxima or minima for the tabulated function by differentiating the interpolating function. Explain the detailed procedure with the help of following example.

Example 1 Compute the minimum value of y from the following table:

x	0	1	2	3	4	5
y	0	0.25	0	2.25	16.00	56.25

Solution Since the values of arguments x are equally spaced. Also, clearly the minimum value of y occurs at $x = 0$ or $x = 2$. Here we apply Newton's forward interpolation formula. The difference table is given as following:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		0.25			
1	0.25		–0.50		
		–0.25		3	
2	0		2.50		6
		2.25		9	
3	2.25		11.50		6
		13.75		15	
4	16.00		26.50		
		40.25			
5	56.25				

$$\frac{dy}{dx}=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\cdots$$

For minimum of y,

$$\frac{dy}{dx}=0\Rightarrow 0+u\times\frac{1}{4}+\frac{u(u-1)}{2}\times\left(-\frac{1}{2}\right)=0$$

$\Rightarrow \quad u - u^2 + u = 0$

$\Rightarrow \quad u(u - 2) = 0 \Rightarrow u = 0;\ u = 2$

At $u = 0$

$\therefore \quad x = x_0 + uh \Rightarrow x = 0 + 0 \times 1 \Rightarrow x = 0$

and at $u = 2$

$$x = x_0 + uh \Rightarrow x = 0 + 2 \times 1 = 2$$

Also $$\left(\frac{d^2y}{dx^2}\right)_{x=0}=\frac{1}{h^2}\left[\Delta^2 y_0-\Delta^3 y_0+\cdots\right]$$

$$= [-0.50 - 3] = -3.50 < 0$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x=2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \cdots\right]$$

$$= [2.25 - 9] = -6.75 < 0$$

Hence, $x = 0$ and $x = 2$ are minimum.
Therefore, $y = 0$ is the minimum value at $x = 0$ and $x = 2$.

Example 2 From the following table find $f'(6)$ and the maximum value of $f(x)$.

x	0	2	3	4	7	9
$y = f(x)$	4	26	58	112	466	922

Solution

Since the arguments are not equally spaced, therefore we can use either Newton's divided difference formula or Lagrange's interpolation formula. Here we apply Newton's formula.

Divided difference table is as follows:

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		56		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

Since the third divided differences is constant, so the Newton's divided difference formula is as follows:

$$f(x) = f(x_0) + (x - x_0)\ \Delta f(x_0) + (x - x_0)(x - x_1)\ \Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\ \Delta^3 f(x_0) + \cdots$$
$$= 4 + (x - 0)\times 11 + (x - 0)(x - 2)\times 7 + (x - 0)(x - 2)(x - 3)\times 1$$
$$= x^3 + 2x^2 + 3x + 4.$$

Therefore, $$f'(x) = 3x^2 + 4x + 3$$

$$f'(6) = 3(6)^2 + 4\times 6 + 3$$
$$= 135.$$

For maxima of $f(x)$, putting $f'(x) = 0$

$$\therefore \quad 3x^2 + 4x + 3 = 0 \Rightarrow x = -\frac{2}{3} \pm \frac{\sqrt{5}}{3} i$$

Since, roots are imaginary.
Therefore, there is no extreme value in the range. In fact, it is an increasing curve.

Example 3 Using the following data, compute $f'(5)$:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	4			
		11		
2	26		7	
		32		1
3	58		11	
		54		1
4	112		16	
		118		1
7	466		22	
		228		
9	922			

Solution

Since, the arguments are not equally spaced, we use Newton's divided difference formula, we have

$$f(x) = f(x_0) + (x - x_0)\ \Delta f(x_0) + (x - x_0)(x - x_1)\ \Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\ \Delta^3 f(x_0) + \cdots \tag{12}$$

Differentiating Eq. (12) w.r.t. x, we get

$$f'(x) = \Delta f(x_0) + \left[2x - (x_0 + x_1)\right]\ \Delta^2 f(x_0) + \left[3x^2 - 2x(x_0 + x_1 + x_2)(x_0x_1 + x_1x_2 + x_2x_0)\right]$$
$$\times\ \Delta^3 f(x_0) \tag{13}$$

Put $\quad x = 5,\ x_0 = 0,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4$

$\therefore \quad f'(5) = 11 + 56 + 31$

$\quad = 98.$

Example 4 Following table reveals the velocity v of a body during the time t. Find its acceleration at $t = 1.1$.

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

[UPTU 2002; GEU 2015]

Solution

Since, the arguments are equally spaced. Therefore, we use the Newton's forward interpolation formula.

The difference table is:

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1.0	43.1				
		4.6			
$\boxed{1.1}$	$\boxed{47.7}$		–0.2		
		$\boxed{4.4}$		0.1	
1.2	52.1		$\boxed{-0.1}$		0.1
		4.3		$\boxed{0.2}$	
1.3	56.4		0.1		
		4.4			
1.4	60.8				

Here $x_0 = 1.1$, $v_0 = 47.7$ and $h = 0.1$.

The acceleration at $t = 1.1$ is given by

$$\left(\frac{dv}{dt}\right)_{t=1.1} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right]$$

$$= \frac{1}{0.1}\left[4.4 - \frac{1}{2}\times(-0.1) + \frac{1}{3}(0.2)\right] = 45.1667$$

Hence, the acceleration is 45.1667.

Example 5 A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of time t (in seconds)

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.20	4.67

Compute the angular velocity and angular acceleration of the rod at time $t = 0.6$ sec.

[U.P.T.U. 2003, 2004]

Solution Since the arguments are equally spaced. We compute the velocity and acceleration at $t = 0.6$ so we use any central difference interpolation formula.

The difference table is

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
$\boxed{0.6}$	$\boxed{1.12}$		$\boxed{0.27}$		0
		$\boxed{0.9}$		$\boxed{0.01}$	
0.8	2.02		$\boxed{0.28}$		0
		1.18		0.01	
1.0	3.20		0.29		
		1.47			
1.2	4.67				

Here $t_0 = 0.6$, $\theta_0 = 1.12$ and $h = 0.2$

Hence, we apply Stirling central difference formula.

Angular velocity at $t = 0.6$ sec is given by

$$\left(\frac{d\theta}{dt}\right)_{t=0.6} = \frac{1}{h}\left[\left(\frac{\Delta\theta_0 + \Delta\theta_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3\theta_{-1} + \Delta^3\theta_{-2}}{2}\right) + \cdots\right]$$

$$= \frac{1}{0.2}\left[\left(\frac{0.9 + 0.63}{2}\right) - \frac{1}{6}\left(\frac{0.01 + 0.01}{2}\right)\right]$$

$$= 3.8167 \text{ rad/sec}$$

and angular acceleration

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.6} = \frac{1}{h^2}\left[\Delta^2\theta_{-1} - \frac{1}{12}\Delta^4\theta_{-2} + \cdots\right]$$

$$= \frac{1}{(0.2)^2}[0.27\}$$

$$= 6.75 \text{ rad/s}^2$$

Example 6 The distance covered by an athlete for the 50 metre race is given in the following table:

Time (sec)	0	1	2	3	4	5	6
Distance (metre)	0	2.5	8.5	15.5	24.5	36.5	50

Determine the speed of athlete at $t = 5$ second.

Solution Since the arguments are equally spaced and we find the derivative at $t = 5$ which is near the end of the table.

Hence, we shall apply Newton's backward difference formula.

The backward difference table is as follows:

Time (t)	**Distance(s)**	∇s	$\nabla^2 s$	$\nabla^3 s$	$\nabla^4 s$	$\nabla^5 s$	$\nabla^6 s$
0	0						
		2.5					
1	2.5		3.5				
		6		–2.5			
2	8.5		1		3.5		
		7		1		$\boxed{-3.5}$	
3	15.5		2		$\boxed{0}$		1
		9		$\boxed{1}$		–2.5	
4	24.5		$\boxed{3}$		–2.5		
		$\boxed{12}$		–1.5			
5	36.5		1.5				
		13.5					
6	50						

Now, the speed of athlete at $t = 5$ s is given by

$$\left[\frac{ds}{dt}\right]_{t=5} = \frac{1}{h}\left[\nabla s + \frac{1}{2}\nabla^2 s + \frac{1}{3}\nabla^3 s + \frac{1}{4}\nabla^4 s + \frac{1}{5}\nabla^5 s\right]$$

$$= \frac{1}{1}\left[12 + \frac{1}{2}\times 3 + \frac{1}{3}\times 1 + \frac{1}{4}\times 0 + \frac{1}{5}\times -3.5\right]$$

$$= 13.13333 \text{ metre/second or } 13.13333 \text{ m/s}$$

Example 7 Determine $f'(1.1)$ and $f''(1.1)$ from the following table:

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0	0.1280	0.5440	1.2960	2.4320	4.0000

[U.P.T.U. 2004, 2006]

Solution Since, the arguments are equally spaced.

i.e., $h = 0.2$. To find the derivatives at $x = 1.1$ which lies between given arguments $x = 1.0$ and 1.2. So, we use Newton's forward interpolation formula. The difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0				
		0.1280			
1.2	0.1280		0.2880		
		0.4160		0.0480	
1.4	0.5440		0.3660		0
		0.7520		0.0480	
1.6	1.2960		0.3840		0
		1.1360		0.0480	
1.8	2.4320		0.4320		
		1.5680			
2.0	4.0000				

Therefore, Newton's forward formula.

$$f(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots \tag{14}$$

where $$u = \frac{x - x_0}{h} = \frac{x-1}{0.2} = 5(x-1) \tag{15}$$

Differentiating Eq. (14) w.r.t. 'x', we have

$$f'(x) = \left[\Delta f(x_0) + \frac{(2u-1)}{2!}\Delta^2 f(x_0) + \frac{3u^2 - 6u + 2}{3!}\Delta^3 f(x_0)\right]\frac{du}{dx}$$

$$f'(x) = 5\left[\Delta f(x_0) + \frac{(24-1)}{2}\Delta^2 f(x_0) + \frac{3u^2 - 6u + 2}{6}\Delta^3 f(x_0)\right] \tag{16}$$

$$\left[\text{From (15)}, \frac{du}{dx} = 5\right]$$

At $x = 1.1$, $u = 5(1.1 - 1) = 0.5$

$$\therefore \quad f'(1.1) = 5\left[0.1280 + \frac{2 \times .5 - 1}{2} \times (.2880) + \frac{3(.5)^2 - 6 \times .5 + 2}{6} \times .0480\right]$$

$$= 5[0.1280 + 0 - 0.0020]$$

$$f'(1.1) = 0.6300$$

Now, differentiating Eq. (16), w.r.t. x, we have

$$f''(x) = 5\left[\Delta^2 f(x_0) + \frac{64-6}{6}\Delta^3 f(x_0)\right]\frac{du}{dx}$$

$$\therefore \quad f''(1.1) = 25[0.2880 + (0.5-1) \times 0.0480]$$

$$f'' = 6.600$$

Example 8 Following table gives the result of an observations. θ is the observed temperature in degree centigrade of a vessel of cooling water, t is the time in minutes from the beginning of the observations:

t	1	3	5	7	9
θ	85.3	74.5	67.0	60.5	54.3

Find the approximate rate of cooling at $t = 3$ and 3.5.

Solution Since, arguments are equally spaced, so we use the Newton's forward interpolation formula.

The difference table:

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
1	85.3				
		–10.8			
3	74.5		3.3		
		–7.5		–2.3	
5	67.0		1.0		1.6
		–6.5		–0.7	
7	60.5		0.3		
		–6.2			
9	54.3				

When $t = 3$, $\theta_0 = 74.5$, $h = 2$

Therefore, rate of cooling

$$\left(\frac{d\theta}{dt}\right)_{t=3} = \frac{1}{h}\left[\Delta\theta_0 - \frac{1}{2}\Delta^2\theta_0 + \frac{1}{3}\Delta^3\theta_0 - \frac{1}{4}\Delta^4\theta_0\right]$$

$$= \frac{1}{2}\left[-7.5 - \frac{1}{2}\times 1 + \frac{1}{3}\times -0.7\right]$$

$$= -4.1167 \text{ °C/min}$$

Since, $t = 3.5$ is non-tabular value, so, Newton's forward formula

$$\frac{d\theta}{dt} = \frac{1}{h}\left[\Delta\theta_0 + \left(\frac{2u-1}{2}\right)\Delta^2\theta_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3\theta_0 + \left(\frac{2u^3-9u^2+11u-3}{12}\right)\Delta^4\theta_0 + \cdots\right] \tag{17}$$

Here, $t = 3.5$, $h = 2$ and $t_0 = 3$, then $u = \dfrac{3.5-3}{2} = 0.25$

$\therefore$ Equation (17), becomes

$$\left(\frac{d\theta}{dt}\right)_{t=3.5} = \frac{1}{2}\left[-7.5 + \left(\frac{2\times .25-1}{2}\right)\cdot 1 + \left(\frac{3(.25)^2 - 6\times .25 + 2}{6}\right)\times(-0.7)\right]$$

$$= -3.9151 \text{ °C/min.}$$

EXERCISE 10.1

1. Compute $f'(1.5)$ and $f''(1.5)$ from the following data:

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

2. Find $f'(2.5)$ from the following table:

x	1.5	1.9	2.5	3.2	4.3	5.9
$f(x)$	3.375	6.059	13.625	29.368	73.907	196.579

3. A slider in a machine moves along a fixed straight rod. Its distance (x) in cm along the rod is given at various times (t) in seconds:

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.28	31.43	32.98	33.54	33.97	33.48	32.13

Calculate $\frac{dx}{dt}$ at $t = 0.1$ and 0.5.

4. From the following table, calculate $f'(10)$:

x	3	5	11	27	34
$f(x)$	–13	23	899	17315	35606

5. Using the divided differences, computer $f'(8)$; given that $f(6) = 1.556$,
$f(7) = 1.690, f(9) = 1.908, f(12) = 2.158$.
6. The following data given below corresponding values of pressure and specific values of a super heated steam:

V	2	4	6	8	10
P	105	42.7	25.3	16.7	13

Calculate the rate of change of pressure (P) with respect to volume (V), where $V = 2$.

7. A curve passes through the points (0, 18), (1, 10), (3, –18) and (6, 90) find the slope of the curve at $x = 2$.

8. The population of a certain town is given below. Find the rate of growth of the population in 1941 and 1961.

Year	1931	1941	1951	1961	1971
Population in lakhs	40.62	60.80	79.95	103.56	132.65

9. Find the first, second and third derivatives of the function tabulated below, at the point $x = 1.5$:

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

10. Find the gradient of the road the middle point of the elevation above a datum line of seven points of road which are given below:

x	0	300	600	900	1200	1500	1800
y	135	149	157	183	201	205	193

11. Find x for which y is maximum and find y for value of x:

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

12. From the following table, for what value of x, y is minimum. Also find y at this value:

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.260	0.250	0.224

13. Compute $f'(50)$ from the following data:

x	50	55	60	65
$f(x)$	1.6990	1.7404	1.7782	1.8129

14. Compute $f'(1)$ and $f''(1)$ using Newton's forward difference formula from the following data:

x	1	2	3	4	5
$f(x)$	2	4	10	18	27

15. Using Newton's forward difference formula compute $f'(1.1)$ and $f''(1.1)$ from the following data:

x	1	1.2	1.4	1.6
$f(x)$	1.00	1.17	1.32	1.55

16. The values of pressure (p) and specific volume (v) of a superheated steam are given in the following table:

v	2	4	6	8	10
p	105	42.7	25.3	16.7	13

Find the rate of change of (i) pressure with respect to volume when $v = 2$ and (ii) volume with respect to pressure when $p = 105$.

17. The population of a certain town is shown in the following table:

year	1951	1961	1971	1981	1991
Population (in thousand)	19.96	39.65	58.81	77.21	94.61

Compute the population in the years 1996 and 1993. Also find the rate of growth of population in the year 1981.

18. Find $f'(0.4)$ and $f''(0.4)$ from the following data:

x	0.1	0.2	0.3	0.4
$f(x)$	1.10517	1.2214	1.34986	1.49182

19. Find the first and second derivative of the function $y = \sqrt{x}$ at $x = 15$ from the following data:

x	15	17	19	21	23	25
$y(x)$	3.873	4.123	4.359	4.583	4.796	5.000

20. The angular displacement (θ) radians at different interval of time (t) seconds given as follows:

θ	0.052	0.105	0.168	0.242	0.327	0.408	0.489
t	0	0.02	0.04	0.06	0.08	0.10	0.12

Estimate the angular velocity at $t = 0.06$ seconds.

Answers

1. $f'(1.5) = 4.750, f''(1.5) = 9$
2. $f'(2.5) = 16.750$
3. 32.44166 cm/s; –24.05833 cm/s
4. 233
5. 0.1086
6. 52.4
7. –16
8. 1.8378; 2.6553
9. 4.750; 9.000; 6.000
10. $\left(\dfrac{dy}{dx}\right)_{x=900} = 0.0852$
11. $x = 1.576,\ y_{\max}(1.576) = 0.9999$
12. $x = 5.6875,\ y_{\min}(5.6875) = 0.2628.$
13. 0.0087
14. $-\dfrac{11}{12}$ and $\dfrac{83}{12}$
15. 0.003 and 3.92
16. (i) –52.4 (ii) –0.0191
17. 49.3, 97.68, 1.8 thousands per year
18. 1.4913, 1.4770
19. 0.1789, –0.004
20. 4.054.

10.3 NUMERICAL INTEGRATION

In many physical problems, the process to compute the definite integral $\int_{x_0}^{x_n} f(x)\,dx,$ where $f(x)$ may be known explicitly or as a tabulated data (equally or unequally spaced) is called numerical integration.

If the integrand is a function of one variable, the process is called mechanical quadrature and if the integrand is a function of two independent variables, the process of double integration is called mechanical cubature. We divide the given interval into a large number of subintervals of equal size (h) and replace the function tabulated at the points of subdivision by any one of the interpolated polynomial.

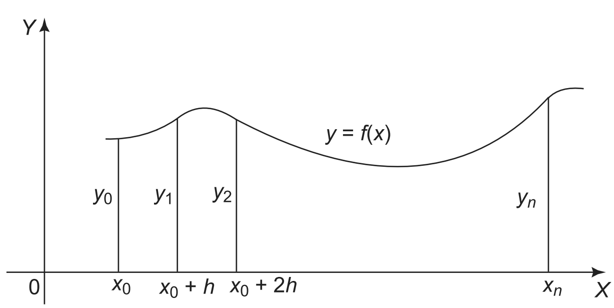


Fig. 10.1

10.4 NEWTON–COTE'S QUADRATURE FORMULA

Let $I = \int_{x_0}^{x_n} f(x)\,dx$, where $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots, x_n$. The interval (x_0, x_n) divided into n equal subintervals with equal step size (h) shown in the following table:

x	x_0	$x_1 = x_0 + h$	$x_2 = x_0 + 2h$	$\ldots$	$x_n = x_0 + nh$
$y = f(x)$	y_0	y_1	y_2	$\ldots$	y_n

Now
$$I = \int_{x_0}^{x_n} f(x)\,dx \qquad [\text{put } x = x_0 + rh,\ dx = h\,dr]$$

$$= h\int_0^n f(x_0 + rh)\,dr$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \right] dr$$

[Using Newton's forward interpolation formula]

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \tag{18}$$

which is known as the Newton–Cote's quadrature formula. A number of formulae can be deduced from Eq. (18) by putting $n = 1, 2, 3, \ldots$.

10.4.1 Trapezoidal Rule

Putting $n = 1$ in Eq. (18) and taking the curve through two points (x_0, y_0) and (x_1, y_1) as a straight line, we have

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right] = \frac{h}{2}(y_0 + y_1) \qquad [\because \quad \Delta y_0 = y_1 - y_0]$$

Similarly

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1 \right) = \frac{h}{2}(y_1 + y_2)$$

$\ldots$

$\ldots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[y_{n-1} + y_n \right]$$

Adding these n integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$$= \frac{h}{2}\left[(\text{Sum of first and last terms}) + 2 \times \text{Sum of remaining terms}\right]$$

which is known as Trapezoidal rule.

10.4.2 Simpson's One-third Rule

Putting $n = 2$ in Eq. (18) and taking the curve through the points (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$\cdots$

$\cdots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right], \text{ when } n \text{ is even.}$$

Adding all these integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]; n \text{ is even}$$

$$= \frac{h}{3}\ [(\text{sum of first and last terms}) + 4\ (\text{sum of odd terms}) + 2\ (\text{sum of even terms})]$$

This formula is known as Simpson's one-third rule.

Thomas Simpson was born on 20 August 1710 in Leicestershire. His father was a weaver and he owed his education to his own efforts. His mathematical interests were first aroused by the solar eclipse which took place in 1724, and with the aid of a fortune-telling pedlar he mastered Cocker's Arithmetic and the elements of algebra. He then gave up his weaving and became an usher at a school, and by constant and laborious efforts improved his mathematical education, so that by 1735 he was able to solve several questions which had been recently proposed and which involved the infinitesimal calculus. In 1743, he was appointed as a professor of mathematics at Woolwich. The works published by Simpson prove him to have been a man of extraordinary natural genius and extreme industry. The most important of them are his Fluxions, 1737 and 1750, with numerous applications to physics and astronomy his Laws of Chance and his Essays, 1740; his theory of Annuities and Reversions, 1742; his Dissertations, 1743, in which the figure of the earth, the force of attraction at the surface of a nearly spherical body, the theory of the tides, and the law of astronomical refraction are discussed; his Algebra, 1745; his Geometry, 1747; his Trigonometry, 1748, in which he introduced the current abbreviations for the trigonometrical functions; his Select Exercises, 1752, containing the solutions of numerous problems and a theory of gunnery; and lastly, his Miscellaneous Tracts, 1754. The work last mentioned consists of eight memoirs, and these contain his best known investigations. The first three papers are on various problems in astronomy; the fourth is on the theory of mean observations; the fifth and sixth on problems in fluxions and algebra; the seventh contains a general solution of the isoperimetrical problem; the eighth contains a discussion of the third and ninth sections of the Principia, and their application to the lunar orbit. In this last memoir Simpson obtained a differential equation for the motion of the apse of the lunar orbit similar to that arrived at by Clairaut, but instead of solving it by successive approximations, he deduced a general solution by indeterminate coefficients. The result agrees with that given by Clairaut. Simpson solved this problem in 1747, two years later than the publication of Clairaut's memoir, but the solution was discovered independently of Clairaut's researches, of which Simpson first heard in 1748 and he died on 14 May 1761.

10.4.3 Simpson's Three-eight Rule

Putting $n = 3$ in Eq. (18) and taking the curve through the points (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) as a polynomial of third order, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly $\displaystyle\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$ and so on.

Adding these expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we have

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1})$$

$$+ 2(y_3 + y_6 + y_9 + \cdots y_{n-3})\Big]$$

$$= \frac{3h}{8} \text{ [(sum of first and last term) + 3 (sum of remaining terms) + 2 (sum of multiple of 3)]}$$

which is known as Simpson's three-eight rule.

10.4.4 Boole's Rule

Putting $n = 4$ in Eq. (18) and taking the curve through the points (x_i, y_i); $i = 0(1).3$ as a polynomial of order four, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\left[\begin{array}{r} 7y_0 + 32y_1 + 12y_2 + 32y_3 + (7y_4 + 7y_4) + 32y_5 + 12y_6 + 32y_7 + 7y_8 + \cdots + 7y_{n-4} \\ + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n \end{array}\right]$$

which is known as a Boole's rule.

10.4.5 Weddle's Rule

Putting $n = 6$ in Eq. (18) and taking the curve through the points (x_i, y_i); $i = 0(1)5$ as a polynomial of order six, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left[\begin{array}{l} y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + (y_6 + y_6) + 5y_7 \\ \qquad + y_8 + 6y_9 + y_{10} + 5y_{11} + (y_{12}) + \cdots \\ \qquad + y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + 5y_{n-1} + y_n \end{array}\right]$$

$$= \frac{3h}{10}\sum_{i=0}^{n} \lambda y_i$$

where $\lambda = 1, 5, 1, 6, 1, 5, 1, 1, 5, 1, 6, 1, 5, 1,$ etc.
which is known as Weddle's rule.

***Note*:**

1. In Trapezoidal rule, the shape of each strip between any consecutive points is taken to trapezium. Area of every strip compute separately that the area bounded by $y = f(x)$, $x = x_0$ and $x = x_n$ is approximately equal to the sum of the areas of 'n' trapezium.
2. In Simpson's one-third rule, the interval must be divided into even number of equal sub-intervals. This rule is also known as parabolic rule.
3. In Simpson's three-eight rule, the number of sub-intervals should be taken as multiple of 3.
4. In Boole's rule, the number of sub-intervals should be taken as a multiple of 4.
5. In Weddle's rule, the number of sub-intervals should be taken as a multiple of 6.
6. Weddle's rule is generally more accurate than any of the other rules as discussed above while, among two Simpson's rules, the one-third is better than the three-eight rule.

10.5 ERROR'S IN QUADRATURE FORMULAE

(i) *Truncation error in Trapezoidal Rule*: Let $y = f(x)$ be a continuous function and have continuous derivatives in $[x_0, x_n]$. Expanding y in a Taylor's series about $x = x_0$, we obtain the error

$$E = -\frac{1}{12}h^3 n\, y''(x) \quad \text{in } [x_0, x_1]$$

$$= -\frac{(b-a)}{12}h^2 y''(x);\ nh = b-a$$

where $y''(x)$ is the largest value of the n quantities.

(ii) *Error in Simpson's one-third rule*: The error in $[x_0, x_2]$ is $E = -\frac{(b-a)}{180}h^4\, y^{iv}(x)$, where $y^{iv}(x)$ is the largest value of the 4th derivative.

(iii) *Error in Simpson's three-eight rule*: The error in $[x_0, x_3]$ is

$$E = -\frac{3}{80}h^5 y^{iv}(x)$$

(iv) *Error in Boole's rule*: The error in $[x_0, x_4]$ is $E = -\frac{8}{945}h^7\, y^{vi}(x)$

(v) *Error in Weddle's rule*: The error in $[x_0, x_6]$ is $E = -\frac{1}{140}h^7\, y^{vi}(x)$

Example 9 Evaluate the integral $\int_4^{5.2} \log x\, dx$ using Trapezoidal rule.

Solution Divide the interval (4, 5.2) in to six equal parts, each of width $h = \frac{5.2-4}{6} = 0.2$. Then the value of $y = \log x$ for each points of sub-division are given below:

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$y = \log x$	1.38629 y_0	1.43508 y_1	1.48160 y_2	1.52605 y_3	1.56861 y_4	1.60943 y_5	1.64865 y_6

Then by Trapezoidal rule

$$\int_4^{5.2} \log x\, dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.2}{2}\left[(1.38629 + 1.64865) + 2(1.43508 + 1.48160 + 1.52605 + 1.56861 + 1.60943)\right]$$

$$= 0.1 \times 18.27648$$

$$= 1.827648$$

Example 10 Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ by using

(i) Trapezoidal rule
(ii) Simpson's one-third rule
(iii) Simpson's three-eight rule
(iv) Weddle's rule. **[U.P.T.U. 2002, 2004, G.E.U. 2015]**

Solution Divide the interval (0, 6) in to six equal parts, each of width $h = \frac{6-0}{6} = 1$. Then the values of $y = \frac{1}{1+x^2}$ for each points of sub-division are given below:

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.05882	0.03846	0.02703
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) Using Trapezoidal Rule:

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[(1 + 0.02703) + 2(0.5 + 0.2 + 0.1 + 0.05882 + 0.03846)\right]$$

$$= 1.41080$$

(ii) Using Simpson's one-third rule

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{1}{3}\left[(1 + 0.02703) + 4(0.5 + 0.1 + 0.03846) + 2(0.2 + 0.05882)\right]$$

$$= \frac{1}{3}\left[1.02703 + 2.55384 + 0.51764\right]$$

$$= 1.36617$$

(iii) Using Simpson's three-eight rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$= \frac{3\times 1}{8}\left[(1 + 0.02703) + 3(0.5 + 0.2 + 0.05882 + 0.03846) + 2(0.1)\right]$$

$$= 1.35708$$

(iv) Using Weddle's rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$= \frac{3}{10}\left[1 + 5(0.5) + 0.2 + 6(0.1) + 0.05882 + 5(0.03846) + 0.02703\right]$$

$$= 1.37345$$

Example 11 From the following table, find the area bounded by the curve $y = f(x)$ and the x-axis from $x = 7.47$ to $x = 7.52$:

x	7.47	7.48	7.49	7.50	7.51	7.52
y	1.93	1.93	1.98	2.01	2.03	2.06
	y_0	y_1	y_2	y_3	y_4	y_5

Solution We know that the area bounded by the curve $y = f(x)$ and x-axis is

$$= \int_{7.47}^{7.52} y\,dx \text{ and interval width } h = 0.01.$$

Using Trapezoidal Rule,

$$\text{Area} = \int_{7.47}^{7.52} y\,dx$$

$$= \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right] \quad \left[\text{From the given table}\right]$$

$$= \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.93 + 1.98 + 2.01 + 2.03)\right]$$

$$= \frac{0.01}{2}[19.89]$$

$$= 0.10 \approx 0.0995$$

Example 12 Evaluate the integral $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\,dx$

Using:
(i) Trapezoidal rule
(ii) Simpson's one-third rule
(iii) Simpson's three-eight rule
(iv) Weddle's rule

Solution Divide the interval (0.2, 1.4) into 12 sub-intervals with each equal width $h = \frac{1.4 - 0.2}{12} = 0.1$.

Then, the values of the function $y = \sin x - \log_e x + e^x$ are given in the following table:

x	$\sin x$	$\log_e x$	e^x	$y = \sin x - \log_e x = e^x$
0.2	0.19867	–1.60943	1.22140	$3.02950 = y_0$
0.3	0.29552	–1.20397	1.34986	$2.84935 = y_1$
0.4	0.38942	–0.91629	1.49182	$2.79753 = y_2$
0.5	0.47943	–0.69315	1.64872	$2.82130 = y_3$
0.6	0.56464	–0.51083	1.82212	$2.89759 = y_4$
0.7	0.64422	–0.35667	2.01375	$3.01464 = y_5$

0.8	0.71736	–0.22314	2.22554	$3.16604 = y_6$
0.9	0.78333	–0.10537	2.45960	$3.34830 = y_7$
1.0	0.84147	0.00000	2.71828	$3.55975 = y_8$
1.1	0.89121	0.09530	3.00417	$3.80008 = y_9$
1.2	0.93204	0.18232	3.32012	$4.06984 = y_{10}$
1.3	0.96356	0.26236	3.66930	$4.37050 = y_{11}$
1.4	0.98545	0.33647	4.05520	$4.70418 = y_{12}$

(i) Using Trapezoidal rule, we have

$$\int_{0.2}^{1.4} y\,dx = \frac{h}{2}\left[(y_0 + y_{12}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11}\right]$$

$$= \frac{0.1}{2}\left[(3.02950 + 4.70418) + 2(2.84935 + 2.79753 + 2.82130 + 2.89759 + 3.01464 + 3.16604 + 3.34830 + 3.55975 + 3.80008 + 4.06984 + 4.37050)\right]$$

$$= \frac{0.1}{2}\left[7.73368 + 2(36.69492)\right]$$

$$= 4.05617$$

(ii) Using Simpson's one-third rule, we have

$$\int_{0.2}^{1.4} y\,dx = \frac{h}{3}\left[(y_0 + y_{12}) + 4(y_1 + y_3 + y_5 + y_7 + y_9 + y_{11}) + 2(y_2 + y_4 + y_6 + y_8 + y_{10})\right]$$

$$= \frac{0.1}{3}\left[7.73368 + 4(20.20417) + 2(16.49075)\right]$$

$$= 4.05106$$

(iii) Using Simpson's three-eight rule, we have

$$\int_{0.2}^{1.4} y\,dx = \frac{3h}{8}\left[(y_0 + y_{12}) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + y_{10} + y_{11}) + 2(y_3 + y_6 + y_9)\right]$$

$$= \frac{3 \times 0.1}{8}\left[7.73368 + 3(26.90750) + 2(9.78742)\right]$$

$$= 4.05106$$

(iv) Using Weddle's rule, we have

$$\int_{0.2}^{1.4} y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

$$= \frac{3\times 0.1}{10}\big[3.02950+5(2.84935)+2.79753+6(2.82130)$$
$$+2.89759+5(3.01464)+2(3.16604)$$
$$+5(3.34830)+3.55975+6(3.80008)$$
$$+4.06984+5(4.37050)+(4.70418)\big]$$
$$= 4.05098.$$

Example 13 A solid of revolution is formed by rotating about x-axis, the area between the x-axis, the lines $x = 0$ and $x = 1$ and a curve through the points with the following co-ordinates:

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

Compute the volume of the solid formed using Simpson's one-third rule.

Solution Here interval width $h = 0.25$ and the values of y are as follows:

$$y_0 = 1.0000,\ y_1 = 0.9896,\ y_2 = 0.9589,\ y_3 = 0.9089 \text{ and } y_4 = 0.8415.$$

Then, the required volume of the solid generated is

$$\int_0^1 \pi\, y^2\, dx = \pi\,\frac{h}{3}\left[(y_0^2+y_4^2)+4(y_1^2+y_3^2)+2(y_2^2)\right]$$
$$= \frac{0.25\,\pi}{3}\left[(1+0.8415^2)+4(0.9896^2+0.9089^2)+2(0.9589^2)\right]$$
$$= \frac{0.25\times 3.1416}{3}[1.7081+7.2216+1.8390]$$
$$= 2.8192$$

Example 14 The velocity (v) of a particle at distance (s) from a point on its path is given by the following table;

Distance (s) (metres)	0	10	20	30	40	50	60
Velocity (v) (m/sec)	47	58	64	65	61	52	38

Calculate the time taken to travel the distance of 60 metres by using Simpson's one-third rule.

Solution Here, the interval width $h = 10$ and we know that velocity (v) = $\frac{ds}{dt}$

$$\therefore \qquad dt = \frac{ds}{v}$$

To compute the time taken to travel the distance of 60 metres, we have to find

$$\int_0^{60} dt = \int_0^{60} \frac{ds}{v}$$

$$= \int_0^{60} y\,ds, \quad \text{where } y = \frac{1}{v}$$

The table values of y for different values of S are given below:

S	0	10	20	30	40	50	60
$y = \frac{1}{v}$	0.0213 y_0	0.0172 y_1	0.0156 y_2	0.0154 y_3	0.0164 y_4	0.0192 y_5	0.0263 y_6

Using Simpson's '$\frac{1}{3}$' rule; we have

$$\int_0^{60} y\,ds = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$= \frac{10}{3}\left[(0.0213 + 0.0263) + 2(0.0156 + 0.0164) + 4(0.0172 + 0.0154 + 0.0192)\right]$$

$$= 1.0627$$

Hence, the time taken to travel 60 metres is 1.0627 seconds.

Example 15 A car is moving at the speed of 30 m/sec, if suddenly brakes are applied. The speed of the car per second after 't' seconds is given below:

Time (t)	0	5	10	15	20	25	30	35	40	45
Speed (v)	30	24	19	16	13	11	10	8	7	5

Using Simpson's three-eight rule to compute the distance covered by the car in 45 seconds.

[U.P.T.U. 2003]

Solution If the car covered the distance (s) metres in t seconds, then

$$\frac{ds}{dt} = v \Rightarrow [s]_{t=0}^{45} = \int_0^{45} v\,dt \tag{19}$$

From the given table the width $h = 5$ and the values of v are $v_0 = 30$, $v_1 = 24$, $v_2 = 19$, $v_3 = 16$, $v_4 = 13$, $v_5 = 11$, $v_6 = 10$, $v_7 = 8$, $v_8 = 7$, $v_9 = 5$.

Using Simpson's three-eight rule, we have

$$\int_0^{45} v dt = \frac{3h}{8}\left[(v_0 + v_9) + 3(v_1 + v_2 + v_4 + v_5 + v_7 + v_8 + 2(v_3 + v_6)\right]$$

$$= \frac{3 \times 5}{8}\left[(30 + 5) + 3(24 + 19 + 13 + 11 + 8 + 7) + 2(16 + 10)\right]$$

$$= 624.3750 \text{ metres.}$$

Hence, the distance covered by car in 45 seconds is 624.3750 metres.

Example 16 Evaluate the integral $\int_0^4 \frac{dx}{1+x^2}$, using Boole's rule taking (i) $h = 0.5$, (ii) $h = 1$.

Compare the result with the actual value and compute the error in both cases.

Solution Divide the interval (0, 4) in to 8 equal. Sub-intervals with each equal with $h = \frac{4-0}{8} = 0.5$.

Then the values of $y = \frac{1}{1+x^2}$ are given in the following table:

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$y = \frac{1}{1+x^2}$	1 y_0	0.8 y_1	0.5 y_2	0.3077 y_3	0.2 y_4	0.1379 y_5	0.1 y_6	0.0755 y_7	0.0588 y_8

Using Boole's rule, we have

$$\int_0^4 y\,dx = \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8\right]$$

$$= \frac{2\times.5}{45}\begin{bmatrix} 7\times1+32\times0.8+12\times0.5+32\times0.3077+14\times0.2 \\ +32\times0.1379+12\times.1+32\times0.0755+7\times0.0588 \end{bmatrix}$$

$$= 1.326373$$

But the actual value is

$$\int_0^4 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^4 = \tan^{-1}(4) = 1.325818$$

Now, the error is $= \frac{1.325818 - 1.326373}{1.325818} \times 100$

$$= 0.0419\%$$

(ii) Divide the interval (0, 4) into 4 equal sub-interval with width (h) = 1. The values of $y = \frac{1}{1+x^2}$ are given in the following table:

x	0	1	2	3	4
$y = \frac{1}{1+x^2}$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4

Using Boole's rule, we have

$$\int_0^4 \frac{dx}{1+x^2} = \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right]$$

$$= \frac{2\times1}{45}[7\times1+32\times0.5+12\times0.2+32\times0.1+7\times0.0588]$$

$$= \frac{2}{45}(29.0116)$$

$$\int_0^4 \frac{dx}{1+x^2} = 1.289412$$

But the actual value is

$$\int_0^4 \frac{dx}{1+x^2} = 1.325818$$

Now, the error is

$$E = \frac{1.325818 - 1.289412}{1.325818} \times 100 = 2.746\%$$

Example 17 Evaluate the integral $\int_{0.5}^{0.7} \sqrt{x}\, e^{-x} dx$ approximately by using a suitable formula.

Solution Divide the interval (0.5, 0.7) in to 4 equal sub-intervals each of width $h = \frac{0.7-0.5}{4} = 0.05$.

The values of $y = \sqrt{x}\, e^{-x}$ for each point of sub-interval are given in the following table:

x	0.5	0.55	0.60	0.65	0.7
$y = \sqrt{x}\, e^{-x}$	0.4288818 y_0	0.4278774 y_1	0.4251076 y_2	0.4208867 y_3	0.4154730 y_4

Using Simpson's one-third rule, we have

$$\int_{0.5}^{0.7} \sqrt{x}\, e^{-x}\, dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)\right]$$

$$= \frac{0.05}{3}\left[(0.4288818 + 0.4154730) + 4(0.4278774 + 0.4208867) + 2(0.4251076)\right]$$

$$= \frac{0.05}{3}(5.0896264) = 0.0848271$$

Example 18 Evaluate $\int_1^6 \left[2 + \sin(2\sqrt{x})\right] dx$; using

(i) Trapezoidal rule and (ii) Simpson's one-third rule with 11 points.

Solution Divide the interval (1, 6) in to 10 equal sub-intervals each of width $h = \frac{6-1}{10} = 0.5$. The values of $y = \left[2 + \sin(2\sqrt{x})\right]$ for each point of sub-interval are given in the following table:

x	1	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
y	2.90930 y_0	2.63816 y_1	2.30807 y_2	1.97932 y_3	1.68305 y_4	1.43530 y_5	1.24320 y_6	1.10832 y_7	1.02872 y_8	1.00024 y_9	1.01736 y_{10}

(i) Using Trapezoidal rule, we have

$$\int_1^6 \left[2+\sin(2\sqrt{x})\right] dx = \frac{h}{2}\left[(y_0+y_{10})+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\right]$$

$$= \frac{0.5}{2}\left[\begin{array}{r}(2.90930+1.01736)+2(2.63816+2.30807+1.97932+1.68305\\ +1.43530+1.24320+1.10832+1.02872+1.00024)\end{array}\right]$$

$$= 8.19386$$

(ii) Using Simpson's one-third rule, we have

$$\int_1^6 \left[2+\sin(2\sqrt{x})\right] dx = \frac{h}{3}\left[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{0.5}{3}\left[\begin{array}{r}(2.90930+1.01736)+4(2.63816+1.97932+1.43530+1.10832\\ +1.00024)+2(2.30807+1.68305+1.24320+1.02872)\end{array}\right]$$

$$= \frac{0.5}{3}[3.92666+32.64536+12.52608]$$

$$= 8.18302$$

EXERCISE 10.2

1. Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3} dx$, using Simpson's rule and hence find the value of $\log_e 3$. **(U.P.T.U. 2006)**

2. Evaluate $\int_3^5 \frac{4}{2+x^2} dx$ by dividing the range into 8 equal parts.

3. Evaluate $\int_{0.4}^{1.5} \frac{x}{\sinh x} dx$ by dividing the 12 sub-intervals, using Weddle's rule.

4. Using Boole's rule to evaluate $\int_0^{\pi/2} \sqrt{\sin x}\, dx$. **[U.P.T.U. 2008]**

5. A river is 80 metres wide. The depth (y) of the river at a distance (x) from bank to given by the following table:

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Using Boole's rule find the approximate area of cross-section of the river. **[U.P.T.U. 2004]**

$$\left[\textbf{Hint:} \text{ Area of cross-section of the river} = \int_0^{80} y\,dx\right]$$

6. Evaluate $\int_0^6 \frac{e^x}{1+x} dx$ using Simpson's three-eight rule. **[U.P.T.U. 2006]**

7. Evaluate the integral $\int_0^5 \frac{dx}{4x+5}$ using Weddle's rule.

8. Evaluate the integral $\int_0^{1.5} \frac{x^3}{e^x - 1} dx$, using Weddle's rule.

9. Evaluate the integral $\int_2^{10} \frac{dx}{1+x}$ using Simpson's one-third rule.

10. Evaluate the integral $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal rule. **[U.P.T.U. 2005]**

11. Using Simpson's three-eight rule to find the value of $\int_0^{0.3} (1-8x^3)^{1/2} dx$.

12. Evaluate $\int_0^1 \sqrt{(\sin x + \cos x)}\, dx$ correct to five decimal places using 7 ordinates.

13. Using trapezoidal and Simpson's rules to evaluate $I = \int_0^1 \sqrt{1-x^2}\, dx$.

14. The velocities of a train at intervals of minutes are given below:

Time in minutes	0	2	4	6	8	10	12
Velocity in Km/hr	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the train.

15. Evaluate the integral $\int_3^7 x^2 \log x\, dx$ by taking 4 strips.

16. A reservoir discharging through sluices at a depth (h) below the water surface has a surface area (A) for various values of h as given below:

***h* in ft**	10	11	12	13	14
***A* in sq. ft**	950	1070	1200	1550	1530

If t denotes the time in minutes, the rate of fall of the surface is given by

$$\frac{dh}{dt} = -\frac{48\sqrt{h}}{A}$$

Estimate the time taken for the water level to fall from 14 ft to 10 ft above the sluices.

17. A rocket is launched from the ground vertically upwards. Its acceleration (a) is registered during the first 80 seconds and is given in the table

Time (s)	0	10	20	30	40	50	60	70	80
$a(m/s^2)$	30.00	31.00	33.44	35.47	37.75	40.33	43.29	46.69	50.67

Estimate the velocity and the height of the rocket at time $t = 80$ seconds.

18. Prove that the trapezoidal and Simpson's rules are convergent.
19. Use Simpson's one-third rule to prove that $\log_e 7$ is approximately 1.95.
20. When do we need numerical integration?
21. How does truncation error relate the accuracy of numerical integration.
22. What are the difference between iterative and non-iterative methods of numerical integration for (i) Trapezoidal (ii) Simpson's rules?
23. Will numerical integration method useful in numerical differential equation?
24. Are numerical integration methods iterative?
25. What is Simpson's rule for numerical integration? What is the order of error in Simpson's rule?

Answers

1. 0.23108
2. 26.716
3. 1.01019
4. 1.18062
5. 708
6. 70.1652
7. 0.4023
8. 0.6155
9. 1.29962
10. 0.69413
11. 0.29159
12. 1.13935
13. 0.784567, 0.785073
14. 3.556
15. 177.4830
16. 29.0993
17. Speed/Velocity = 3087 m/s, height = 112.75 Km.

SUMMARY

Following topics have been discussed in this chapter:

1. Newton's Forward Interpolation Formula

(i) $$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

(ii) $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

(iii) $$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

and so on.

2. Newton's Backward Interpolation Formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

and so on.

3. Stirling's Central Difference Formula

Differentiating the above equation w.r.t. u and x respectively we obtain

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) - \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots\right]$$

and so on.

4. Bessels's Central Difference Formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\right.$$
$$\left. - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{60}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) - \frac{1}{2}\Delta^3 y_{-1} - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\right.$$
$$\left. + \frac{1}{24}\Delta^5 y_{-2} + \frac{1}{90}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \cdots\right]$$

and so on.

5. Newton's Divided Difference Formula

We have

$$f(x) = f(x_0) + (x - x_0)\ \Delta f(x_0) + (x - x_0)(x - x_1)\ \Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\ \Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\ \Delta^4 f(x_0) + \cdots$$

Differentiating the above equation w.r.t. 'x' we have

$$f'(x) = \Delta f(x_0) + \left[2x - (x_0 + x_1)\right] \Delta^2 f(x_0) + \left[3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)\right] \Delta^3 f(x_0) + \cdots$$

6. Lagrange's Interpolation Formula

This formula applied in both the cases, that is if the arguments are equally spaced or unequally spaced. The Lagrange's interpolation formula is given below:

$$f(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)} f(x_1) + \cdots$$

Differentiating the above equation w.r.t. x one or more times and we obtain the derivatives of $f(x)$.

7. Maxima and Minima of a Function given the Tabulated Values

To determine the value(s) of arguments x at which the curve $y = f(x)$ is maxima or minima can be obtained by equating $\frac{dy}{dx}$ to zero. The same procedure can be used to find maxima or minima for the tabulated function by differentiating the interpolating function.

8. Newton–Cote's Quadrature Formula

$$\int_{x_0}^{x_n} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

which is known as the Newton–Cote's quadrature formula. A number of formulae can be deduced from the above equation by putting $n = 1, 2, 3, \ldots$.

Newton-Cote's quadrature formula gives the following numerical formulae to evaluate the integration:

9. Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$$= \frac{h}{2}\left[(\text{Sum of first and last terms}) + 2 \times \text{Sum of remaining terms}\right]$$

10. Simpson's One-third Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$

$= \frac{h}{3}$ [(sum of first and last terms) + 4 (sum of odd terms) + 2 (sum of even terms)]

11. Simpson's Three-eight Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots y_{n-3})\right]$$

$= \dfrac{3h}{8}$ [(sum of first and last term) + 3 (sum of remaining terms) + 2 (sum of multiple of 3)]

12. Boole's Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\begin{bmatrix} 7y_0+32y_1+12y_2+32y_3+14y_4+32y_5+12y_6+32y_7 \\ +7y_8+\cdots+7y_{n-4}+32y_{n-3}+12y_{n-2}+32y_{n-1}+7y_n \end{bmatrix}$$

13. Weddle's Rule

Putting $n = 6$ in the above equation and taking the curve through the points (x_i, y_i); $i = 0(1)5$ as a polynomial of order six, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\begin{bmatrix} y_0+5y_1+y_2+6y_3+y_4+5y_5+2y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+ \\ y_{12}+\cdots+y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1}+y_n \end{bmatrix}$$

which is known as Weddle's rule.

OBJECTIVE TYPE QUESTIONS

1. Simpson's rule for integration gives exact result when $f(x)$ is a polynomial of degree

(a) 1 (b) 2
(c) 3 (d) 4

[GATE (CS) 1993]

2. The Trapezoidal rule for integration gives exact result when the integrand is a polynomial of degree

(a) 0 but not 1 (b) 1 but not 0
(c) 0 or 1 (d) 2

[GATE (CS) 1995]

3. The value of $\int_1^2 \frac{1}{x}dx$, computed using Simpson's rule with a step size of $h = 0.25$ is

(a) 0.69430 (b) 0.69385
(c) 0.69325 (d) 0.69415

[GATE (EE) 1998]

4. The order of error is the Simpson's rule for numerical integration with step size h is

(a) h (b) h^2
(c) h^3 (d) h^4

[GATE (ME) 1997]

5. The accuracy of Simpson's rule quadrature for a step size h is

(a) $O(h^2)$ (b) $O(h^3)$
(c) $O(h^4)$ (d) $O(h^5)$

[GATE (ME) 2003]

6. The integration $\int_1^3 \frac{1}{x}dx$, evaluated by using Simpson's $\frac{1}{3}$ rule, on two equal subintervals each of length 1, equals

(a) 1.000 (b) 1.098
(c) 1.111 (d) 1.120

[GATE (ME) 2011]

7. The magnitude as the error (correct of two decimal places) in the estimation of integral

$\int_0^4 (x^4 + 10)\,dx$, using Simpson's $\frac{1}{3}$ rule is

[GATE (CE) 2013]

8. The estimate of $\int_{0.5}^{1.5} \frac{1}{x}\,dx$ obtained and using Simpson's rule with three-point function evaluation exceeds the exact value by

(a) 0.235 (b) 0.068
(c) 0.024 (d) 0.012

[GATE (CE) 2012]

9. The value of the integral $\int_{0.1}^{0.5} e^{-x^3}\,dx$ evaluated by Simpson's rule using 4 sub intervals (up to 3 digits after the decimal point) is _______

[GATE (CH) 2013]

10. Function $f(x)$ is known at the following points

x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0
$f(x)$	0	0.09	0.36	0.81	1.44	2.25	3.24	4.41	5.76	7.29	9.0

The value of $\int_0^3 f(x)\,dx$ computed using the trapezoidal rule is

(a) 8.983 (b) 9.003
(c) 9.017 (d) 9.045

[GATE (CS) 2013]

11. Match the correct pairs

Numerical Integration scheme	Order of fitting polynomial
P Simpson's $\frac{3}{8}$ rule	1. First
Q Trapezoidal rule	2. Second
R Simpson's $\frac{1}{3}$ rule	3. Third

(a) P-2, Q-1, R-3 (b) P-3, Q-2, R-1
(c) P-1, Q-2, R-3 (d) P-3, Q-1, R-2

[GATE (ME) 2013]

12. Using the trapezoidal rule and dividing the interval of integration in to 3 equal sub-intervals, the definite integral $\int_{-1}^{1} |x|\,dx$ is _______

[GATE (ME) 2014]

13. The value of $\int_{2.5}^{4} \log x\,dx$ calculated using the Trapezoidal rule with five sub-intervals is _______

[GATE (ME) 2014]

14. For step-size, $\Delta x = 0.4$, the value of following integral using Simpson's $\frac{1}{3}$ rule is _______

$$\int_0^{0.8} \left[0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5\right] dx$$

[GATE (CE) 2015]

15. The integral $\int_{x_1}^{x_2} x^2\,dx$ with $x_2 > x_1 > 0$ is evaluated analytically as well as numerically using a single application of the trapezoidal rule. If I is the exact value of the integral obtained analytically and J is the approximate value obtained using the trapezoidal rule. Which of the following statements is correct about their relationship?

(a) $J > I$
(b) $J < I$
(c) $J = I$
(d) Insufficient data to determine the relationship

[GATE (CE) 2015]

16. Using a unit size, the value of integral $\int_1^2 x \log x\,dx$ by trapezoidal rule is _______

[GATE (ME) 2015]

17. Simpson's $\frac{1}{3}$ rule is used to integrate the function $f(x) = \frac{3}{5}x^2 + \frac{9}{5}$ between $x = 0$ and $x = 1$, using the least number of equal sub-intervals. The value of the integral is _____

[GATE (ME) 2015]

18. The values of function $f(x)$ at 5 discrete points are given below

x	0	0.1	0.2	0.3	0.4
$f(x)$	0	10	40	90	160

Using Trapezoidal rule step size of 0.1, the value of $\int_0^{0.4} f(x)\,dx$ is _______

[GATE (ME) 2015]

19. Simpson's $\frac{1}{3}$ rule is a special case of Newton's cotes quadrature formula, when n is equal to

(a) 3 (b) 5
(c) 1 (d) 2 or multiple of 2

20. Error in the Simpson's $\frac{3}{8}$ rule is

(a) $-\frac{3}{80}h^4 f^{iv}(x)$ (b) $-\frac{3}{80}h^3 f^{iv}(x)$
(c) $-\frac{3}{80}h^6 f^{iv}(x)$ (d) $-\frac{3}{80}h^5 f^{iv}(x)$

21. In Trapezoidal rule $f(x)$ is a

(a) constant function
(b) polynomial of 3rd degree
(c) polynomial of 2nd degree
(d) linear polynomial of x

22. Error in the Trapezoidal rule is of the order

(a) h^4 (b) h^3
(c) h^2 (d) h

23. Simpson's $\frac{3}{8}$ rule is a special case of quadrature formula, when n is equal to

(a) 1
(b) 2
(c) 3 or multiple of 3
(d) 3

24. Error in the Simpson's $\frac{1}{3}$ rule is of the order

(a) h (b) h^3
(c) h^2 (d) h^4

25. The value of $\int_0^1 \frac{dx}{1+x^2}$ by Simpson's $\frac{1}{3}$ rule $\left(h=\frac{1}{4}\right)$ is _______

(a) 0.7854 (b) 0.0854
(c) 1.7854 (d) 0.0135

ANSWERS

1.(b)	2.(b)	3.(c)	4.(d)	5.(d)	6.(c)	7.(0.53)	8.(d)
9.(0.3849)	10.(d)	11.(d)	12.(1.11)	13.(1.7533)	14.(1.3674)	15.(a)	16.(0.693)
17.(2)	18.(22)	19.(d)	20.(d)	21.(d)	22.(c)	23.(c)	24.(d)
25.(a)							

11 Numerical Solution of Ordinary Differential Equations

11.1 INTRODUCTION

Most of the problems in science and engineering can be formulated in terms of differential equations. A differential equation is an equation which involves the differential coefficients of dependent variable with respect to independent variables. If a differential equation involves only one independent variable then it is called an ordinary differential equation (ODE). The solution of this differential equation is the relation between dependent and independent variables. When this relation is expressed numerically then the solution is known as numerical.

11.1.1 Solution of the ODE

Many ordinary differential equations can be solved using analytical method. However, a majority of differential equations appearing in physical problems cannot be solved analytically. Thus, it becomes imperative to discuss their solution by numerical methods.

In this chapter, we shall discuss some of the methods for computing numerical solution of the

(i) First order and first degree ordinary differential equation $y' = f(x, y)$ subject to the condition $y(x_0) = y_0$.

(ii) Simultaneous first order ordinary differential equations $\dfrac{dy}{dx} = f(x, y, z)$ and $\dfrac{dz}{dx} = \phi(x, y, z)$ subject to the condition $y(x_0) = y_0$ and $z(x_0) = z_0$.

(iii) Second order ordinary differential equations $\dfrac{d^2y}{dx^2} = f\left(x, y, \dfrac{dy}{dx}\right)$ subject to the conditions $y(x_0) = y_0$; $y'(x_0) = y'_0$.

11.2 INITIAL AND BOUNDARY VALUE PROBLEMS

Problems in which all the initial conditions are specified only at the initial points are known as an initial value problems (IVP). Thus, in an IVP, all the auxiliary conditions are specified at a point.

For example, $\dfrac{dy}{dx} = y' = x - y^2$; $y(0) = 1$ and

$$y' = y;\ y(0) = 1.$$

are an IVP.

Problems involving second and higher order differential equations in which auxiliary conditions are specified at two or more points are known boundary value problems (BVP).

For example $\quad y'' = xy$; $y(0) = 0$, $y(3) = 2$ is a BVP.

11.3 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

11.3.1 Picard's Method of Successive Approximations

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

With the initial condition $y(x_0) = y_0$.

Integrating Eq. (1) between the limits x_0 to x and the corresponding limits y_0 to y, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

or $$y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

or $$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \tag{2}$$

Equation (2) in which the unknown function y appears under the integral sign, is called an integral equation. We solve Eq. (2) by successive approximations. Thus, if the first approximation to y is obtained by putting y_0 for y on the R.H.S of Eq. (2) and we write

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx \tag{3}$$

The integrand in Eq. (3) is now a function of x alone and can be solved in general.

The second approximation is obtained by substituting.

$y^{(1)}$ for y in the R.H.S of Eq. (2), we get

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

Similarly, third approximation is as follows:

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$$

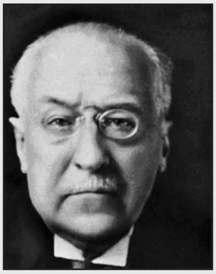
and the n^{th} approximation is as follows:

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

Thus, we obtain a sequence of approximate solutions

$$y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(n)}$$

Each giving a better result than the preceding one.



Charles-Émile Picard was born on 24 July 1856 in Paris, France. He was a French mathematician whose theories did much to advance research in analysis, algebraic geometry, and mechanics. Picard became a lecturer at the University of Paris in 1878 and a professor at the University of Toulouse in the year from 1881 to 1898 he held various posts with the University of Toulouse and the École Normale Supérieure, and in 1898 he was appointed a professor at the University of Paris. In 1917, he was elected permanent secretary for the mathematical sciences in the French Academy of Sciences. After World War I he led a decade-long movement to boycott German scientists and mathematicians. Picard made his name in 1879 when he proved that an entire function takes every finite value, with one possible exception. Then, inspired by Niels Henrik Abel of Norway and Bernhard Riemann of Germany, he generalized Riemann's work to complex functions of two variables. His study of the integrals attached to algebraic surfaces and the related topological questions developed into an important part of algebraic geometry, with varied applications to topology and functional analysis. Picard also worked on Fuchsian and Abelian functions and on the allied theories of discontinuous and continuous groups of transformation. Picard successfully revived the method of successive approximations to prove the existence of solutions to differential equations. He also created a theory of linear differential equations, analogous to the Galois theory of algebraic equations. His studies of harmonic vibrations, coupled with the contributions of Hermann Schwarz of Germany and Henri Poincaré of France, marked the beginning of the theory of integral equations. He died on 11 December 1941 in Paris, France.

Remark The solution of Eq. (1) is evaluated under the following assumptions:

(i) If $f(x, y)$ is a single valued and continuous function is the domain D, defined by $|x - x_0| \le k,\ -\infty < y < \infty$

(ii) $f(x, y)$ is bounded in domain D.

(iii) $|f(x, y^{(n)}) - f(x, y^{(n+1)})| \le L|y^{(n)} - y^{(n+1)}|$

where L is a positive number and it is called a Lipschitz constant for f.

Example 1 Use Picard's method to find $y(0.2)$. Given $\dfrac{dy}{dx} = x - y$ with initial condition $y = 1$, when $x = 0$.

Solution Here $f(x, y) = x - y$; $x_0 = 0$ and $y_0 = 1 = y(0)$.

Using Picard's formula

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx \tag{4}$$

Put $n = 1$ is Eq. (4), we get the first approximation

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx$$

$$= 1 + \int_{0}^{x} (x - 1)\, dx$$

$$y^{(1)} = 1 - x + \frac{x^2}{2}$$

$$\therefore \quad y^{(1)}(0.2) = 0.82000$$

Put $n = 2$ in Eq. (4), we get the second approximation

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx = 1 + \int_0^x (x - y^{(1)})\, dx$$

$$= 1 + \int_0^x \left[x - 1 + x - \frac{x^2}{2} \right] dx$$

$$= 1 - x + x^2 - \frac{x^3}{6}$$

$\therefore$ $y^{(2)}(0.2) = 0.83867$

Put $n = 3$ in Eq. (4), we get the third approximation

$$y^{(3)} = y_0 + \int_0^x f(x, y^{(2)})\, dx$$

$$= 1 + \int_0^x [x - y^{(2)}]\, dx$$

$$= 1 + \int_0^x \left[x - 1 + x - x^2 + \frac{x^3}{6} \right] dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

$\therefore$ $y^{(3)}(0.2) = 0.83740$

Similarly, fourth approximation

$$y^{(4)} = y_0 + \int_0^x f(x, y^{(3)})\, dx$$

$$= 1 + \int_0^x (x - y^{(3)})\, dx$$

$$= 1 + \int_0^x \left[x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24} \right] dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

$\therefore$ $y^{(4)}(0.2) = 0.83746$

Fifth approximation

$$y^{(5)} = 1 + \int_0^x [f(x, y^{(4)})]\, dx = 1 + \int_0^x [x - y^{(4)}]\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}$$

$\therefore \qquad y^5(0.2) = 0.83746$

Since, $y^4(0.2) = 0.83746 = y^{(5)}(0.2)$, i.e., two successive approximations are same, hence

$$y(0.2) = 0.83746.$$

Example 2 Use Picard's method, to find $y(0.1)$.

Given that $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$ with $y(0) = 1$.

Solution Here $f(x, y) = \dfrac{y-x}{y+x}$; $x_0 = 0,\ y_0 = y^{(0)} = 1$.

Using Picard's method.

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx \tag{5}$$

The first approximation

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx = y_0 + \int_0^x \left[\frac{y^{(0)} - x}{y^{(0)} + x}\right] dx$$

$$= 1 + \int_0^x \left[\frac{1-x}{1+x}\right] dx = 1 + [2 \log(1+x) - x]_0^x$$

$$y^{(1)} = 1 + 2\log(1+x) - x$$

$\therefore \qquad y^{(1)}(0.1) = 1 + 2\log(1.1) - 0.1 = 0.9828.$

Second approximation

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$= y_0 + \int_0^x \left[\frac{y^{(1)} - x}{y^{(1)} + x}\right] dx$$

$$= 1 + \int_0^x \left[\frac{1 + 2\log(1+x) - 2x}{1 + 2\log(1+x)}\right] dx$$

$$= 1 + x - 2\int_0^x \frac{x}{1 + 2\log(1+x)}\, dx$$

$$= 1 + x - 2\int_0^t \frac{e^{2t}}{1+2t}\, dt + 2\int_0^t \frac{e^t}{1+2t}\, dt,$$

where $t = \log(1+x)$. When $x = 0$, $t = 0$.

Which cannot be integrated. Hence, $y(0.1) = 0.9828$

Example 3 Find the solution of $\frac{dy}{dx} = 1 + xy,\ y(0) = 1$ which passes through (0, 1) in the interval (0, 0.5) such that the value of y is correct to 3 decimal places. Taking $h = 0.1$.

Solution Here $f(x, y) = 1 + xy$; $x_0 = 0$ and $y_0 = y^{(0)} = 1$.

Using Picard's formula.

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx \tag{6}$$

$$n = 1, 2, 3, \ldots$$

First approximation is as follows:

$$y^{(1)} = y_0 + \int_0^x f(x, y^{(0)})\, dx = 1 + \int_0^x [1 + x\,y^{(0)}]\, dx$$

$$y^{(1)} = 1 + \int_0^x [1 + x \cdot 1] = 1 + x + \frac{x^2}{2}$$

Second approximation is as follows:

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx$$

$$= 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$y^{(2)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

Third approximation is as follows:

$$y^{(3)} = y_0 + \int_0^x f(x, y^{(2)})\, dx$$

$$y^{(3)} = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right] dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

Fourth approximation

$$y^{(4)} = 1 + \int_0^x [1 + x\,y^{(3)}]\, dx$$

$$= 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}\right)\right] dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{105} + \frac{x^8}{384}$$

Given $x = 0, y = 1.000$

When $x = 0.1, y^{(1)} = 1.105, y^{(2)} = 1.1053$

Since $y^{(1)} = 1.105 = y^{(2)}$

$\therefore$ $y(0.1) = 1.105$

When $x = 0.2, y^{(1)} = 1.220, y^{(2)} = 1.223, y^{(3)} = 1.223$

Since $y^{(2)} = y^{(3)}$

$\therefore$ $y(0.2) = 1.223$

Similarly,

When $x = 0.3, y^{(2)} = y^{(3)} = 1.355$

$\therefore$ $y(0.3) = 1.355$

When $x = 0.4, y^{(2)} = 1.505 = y^{(3)}$

$\therefore$ $y(0.4) = 1.505$

when $x = 0.5, y^{(3)} = y^{(4)} = 1.677.$

$\therefore$ $y(0.5) = 1.677.$

Example 4 Use Picard's method to obtain the value of y when $x = 0.1$ given that

$$\frac{dy}{dx} = 3x + y^2 \text{ with } y(0) = 1.$$

Solution Here $f(x, y) = 3x + y^2$, $x_0 = 0$ and $y_0 = y^{(0)} = 1$.

Using Picard's formula

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx : n = 1, 2, 3, \cdots \tag{7}$$

First approximation is as follows:

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx$$

$$= 1 + \int_0^x [3x + (y^{(0)})^2)\, dx$$

$$= 1 + \int_0^x [3x + 1]\, dx = 1 + x + \frac{3x^2}{2}$$

$\therefore$ $y^{(1)}(0.1) = 1.1150$

Second approximation is as follows:

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx = 1 + \int_0^x \left[3x + \left(1 + x + \frac{3x^2}{2}\right)^2 \right] dx$$

$$= 1 + x + \frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20}$$

$\therefore \quad y^{(2)}(0.1) = 1.1264$

Third approximation is as follows:

$$y^{(3)} = 1 + \int_0^x [3x + (y^{(2)})^2]\, dx$$

$$= 1 + x + \frac{5x^2}{2} + 2x^3 + \frac{23}{12}x^4 + \frac{25x^5}{12} + \frac{68x^6}{45} + \frac{1157x^7}{1260} + \frac{17x^8}{32} + \frac{47x^9}{240} + \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}$$

$\therefore \quad y^{(3)}(0.1) = 1.1272$

Since $\quad y^{(2)} = 1.127 = y^{(3)}$

Thus, $y(0.1) = 1.127$ correct to 3 decimal places.

EXERCISE 11.1

1. Obtain Picard's second approximate solution of the initial value problem,
$$\frac{dy}{dx} = \frac{x^2}{1+y^2};\ y(0) = 0$$
2. Employ Picard's method to find correct to four decimal places the solution of the equation
$$\frac{dy}{dx} = x^2 + y^2 \text{ for } x = 0.1 \text{ with } y(0) = 0$$
3. Using Picard's method to find $y(0.2)$ of the differential equation $\frac{dy}{dx} = \log(x+y);\ y(0) = 1$
4. Solve numerically $\frac{dy}{dx} = 2x - y;\ y(0) = 0.9$ and find $y(0.4)$ by Picard's method with three iterations and compare the result with the exact value.
5. Find an approximate value of y when $x = 0.1$, if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$, using Picard's method.

Answers

1. $y^{(2)} = \frac{x^3}{3} - \frac{x^9}{81} + \frac{x^{15}}{1215}$
2. $y(0.1) = 0.0214$
3. $y(0.2) = 1.0082$
4. $y(0.4) = 0.7432;\ 0.7439$
5. $y(0.1) = 0.9138$

11.3.2 Taylor's Series Method

Consider $\dfrac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ (8)

is a first order initial value problem.

Suppose $y = \phi(x)$ is the exact solution of Eq. (8), then $\phi(x)$ expanding as a Taylor's series about $x = x_0$, we have

$$y(x) = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots \tag{9}$$

Equation (9) gives a power series for y. The values of $y_0', y_0'', y_0''', \cdots$ can be obtain from Eq. (8).

We can write Eq. (8) as

$$y' = f(x, y),\ y'' = f' = f_x + f_y y' = f_x + f_y \cdot f$$

Similarly,

$$y''' = f'' = f_{xx} + f_{xy} \cdot f + f[f_{yx} + f_{yy} \cdot f] + f_y\, [f_x + f_y \cdot f]$$

$$y''' = f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f^2 f_y$$

and so on.

This method is explained with the help of following examples:

Example 5 Using Taylor's series method to solve the initial value problem $y' = x + y^2$ with $y(0) = 0$ and hence evaluate $y(0.2)$.

Solution

Here $\dfrac{dy}{dx} = y' = x + y^2;\ x_0 = 0,\ y_0 = 0$

$\therefore \qquad y_0' = x_0 + y_0^2 = 0 + 0 = 0$

Differentiating the given ordinary differential equation successively, we get

$$y'' = 1 + 2yy',\ y_0'' = 1 + 2{\cdot}0{\cdot}0 = 1.$$

$$y''' = 2(y')^2 + 2yy'',\ y_0''' = 2(0)^2 + 2\cdot 0\cdot 1 = 0$$

$$y^{\text{iv}} = 4y'y'' + 2y'y'' + 2yy''',\ y_0^{\text{iv}} = 4\cdot 0\cdot 1 + 1{\cdot}0{\cdot}1 + 1{\cdot}0{\cdot}0 = 0$$

$$y^{\text{v}} = 6(y'')^2 + 6y'y''' + 2y'y''' + 2yy^{\text{iv}},\ y_0^{\text{v}} = 6(1)^2 + 6{\cdot}0{\cdot}0 + 2{\cdot}0{\cdot}0 + 2{\cdot}0{\cdot}0 = 6$$

and so on.

The Taylor's series expansion about $x = 0$ is

$$y(x) = y_0 + (x - 0)\, y_0' + \frac{(x-0)^2}{2!} y_0'' + \frac{(x-0)^3}{3!} y_0''' + \frac{(x-0)^4}{4!} y_0^{\text{iv}} + \frac{(x-0)^5}{5!} y_0^{\text{v}} + \cdots$$

$$= 0 + x\cdot 0 + \frac{x^2}{2!}\cdot 1 + \frac{x^3}{3!}\cdot 0 + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 6 + \cdots$$

or $$y(x) = \frac{x^2}{2!} + \frac{6x^5}{5!} + \cdots$$

or $$y(x) = \frac{x^2}{2} + \frac{x^5}{20} + \cdots$$

$\therefore$ $$y(0.2) = 0.020016.$$

Example 6 Using Taylor's series, find $y(2.1)$ correct to 5 decimal places. Given that $xy' = x - y$ with $y(2) = 2$.

Solution

Here $y' = \dfrac{x-y}{x} = 1 - \dfrac{y}{x}$ and $y(2) = 2$

$\therefore$ $$y_0' = 1 - \frac{y_0}{x_0} = 1 - \frac{2}{2} = 1 - 1 = 0$$

Differentiating the given differential equation successively, we get

$$y'' = \frac{y}{x^2} - \frac{y'}{x}, \; y_0'' = \frac{1}{2}$$

$$y''' = -\frac{y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3}, \quad y_0''' = -\frac{3}{4}$$

$$y^{\text{iv}} = -\frac{y'''}{x} + \frac{3y''}{x^2} - \frac{6y'}{x^3}, + \frac{6y}{x^4}, \quad y_0^{\text{iv}} = \frac{3}{2}$$

and so on.

The Taylor's series expansion about $x = 2$ is

$$y(x) = y_0 + (x-2)\, y_0' + \frac{(x-2)^2}{2!} y_0'' + \frac{(x-2)^3}{3!} y_0''' + \frac{(x-2)^4}{4!} y_0^{\text{iv}} + \cdots$$

or $$y(x) = 2 + (x-2)\cdot 0 + \frac{(x-2)^2}{2!} \times \frac{1}{2} + \frac{(x-2)^3}{6!} \times -\frac{3}{4} + \frac{(x-2)^4}{24!} \times \frac{3}{2} + \cdots$$

$$y(x) = 2 + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{1}{16}(x-2)^4 - \cdots$$

At $x = 2.1$, we get

$$y(2.1) = 2.00238.$$

Example 7 Using Taylor's series method, solve $y' = x + y$ numerically, Start with $x = 1$, $y = 0$ and carry to $x = 1.2$ with $h = 0.1$. Compare the final result with the value of the explicit solution $e^{0.2} = 1.221$.

Solution

Given

$$y' = x + y;\ x_0 = 1,\ y_0 = 0$$

$\therefore$ $$y_0' = x_0 + y_0 = 1 + 0 = 1$$

$$y'' = 1 + y', \qquad y_0'' = 1 + y_0' = 1 + 1 = 2$$

$$y''' = y'', \qquad y_0''' = y_0'' = 2$$

$$y^{iv} = y''', \qquad y_0^{iv} = y_0''' = 2$$

$$y^{v} = y^{iv}, \qquad y_0^{v} = y_0^{iv} = 2$$

and so on.

The Taylor's series expansion is as follows:

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots$$

$$y_1 = y(1+h) = 0 + (0.1)\cdot 1 + \frac{(0.1)^2}{2}\times 2 + \frac{(0.1)^3}{6}\times 2 + \frac{(0.1)^4}{24}\times 2 + \cdots$$

or $\quad y(1.1) = 0.1103081 = 0.110$ (approximate)

Also $\quad x_1 = x_0 + h = 1 + 0.1 = 1.1$

Now, $\quad y_1' = x_1 + y_1 = 1.1 + 0.110 = 1.21$

$$y_1'' = 1 + y_1' = 1 + 1.21 = 2.21$$

$$y_1''' = 0 + y_1'' = 2.21$$

$$y_1^{iv} = y_1''' = 2.21$$

$$y_1^{v} = y_1^{iv} = 2.21$$

Therefore,

$$y_2 = y(x_1 + h) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots$$

$$y(1.1 + 0.1) = y(1.2) = 0.11 + (0.1)\,(1.21) + \frac{(0.1)^2}{2}\times(2.21)$$
$$+ \frac{(0.1)^3}{6}(2.21) + \frac{(0.1)^4}{24}\times 2.21 + \cdots$$

$\therefore \quad y(1.2) = 0.242$ (approximate).

Now, the exact solution of the given ODE is

$$y = -x - 1 + 2e^{x-1}$$

At $x = 1.2$, we get

$$y = -1.2 - 1 + 2\,e^{1.2-1}$$
$$= -2.2 + 2\,e^{0.2}$$
$$= -2.2 + 2(1.221)$$
$$= 0.242$$

Thus the approximation solution of y at $x = 1.2$ is approximately equal to $y(1.2)$.

EXERCISE 11.2

1. Using Taylor's series method solve $\frac{dy}{dx} = x^2 - y;\ y(0) = 1$ at $x = 0.1, 0.2, 0.3$ and 0.4. Also, compare the values with exact solutions.
2. Using Taylor's series method to evaluate y at $x = 0.1$ and $x = 0.2$ to five places of decimals from $\frac{dy}{dx} = x^2 y - 1;\ y(0) = 1$.
3. Solve $y' = y - \frac{2x}{y};\ y(0) = 1$ by Taylor's series method and also find $y(0.1)$.
4. Given $y' = 1 + xy;\ y(0) = 1,$ obtain the Taylor's series for $y(x)$. Compute $y(0.1)$ correct to four places of decimal.

Answers

1. $y(0.1) = 0.905125$, $y(0.2) = 0.8212352$.
   $y(0.3) = 0.7491509$, $y(0.4) = 0.6896519$
2. $y(0.1) = 0.90033$, $y(0.2) = 0.80227$
3. $y(0.1) = 1.0954$
4. $y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \cdots;\ y(0.1) = 1.1053$

11.3.3 Euler's Method

Euler's method is one of the oldest and simplest numerical method. It is generally used to find the numerical solution of the differential equations, because it provides a simple procedure for computing approximations to the exact solutions.

Consider the differential equation of first order is as follows:

$$\frac{dy}{dx} = f(x, y) \tag{10}$$

With initial condition $y(x_0) = y_0$.

The general approximations of Eq. (10) is given by

$$y_{n+1} = y_n + h\,f(x_n, y_n), \text{ where } h = \frac{x_n - x_0}{n} \tag{11}$$

Equation (11) is called Euler's algorithm.

Thus, starting from x_0 when $y = y_0$ and we construct a table for y for given steps of h in x.

Note If the value of h is small, this method is very slow. However, if h is large then this method is inaccurate.

Example 8 Using Euler's method, find the value of $y(0.1)$.

Given $\frac{dy}{dx} = x + y + xy;\ y(0) = 1$ and step size $h = 0.025$.

Solution Here $f(x, y) = x + y + xy$, $x_0 = 0$, $y_0 = 1$ and $h = 0.025$.

Euler's algorithm is

$$y_{n+1} = y_n + h\,f(x_n, y_n),\ n = 0, 1, 2, \ldots \tag{12}$$

Put $n = 0$ in Eq. (12), we have

$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= 1 + 0.025\,f(0, 1) \\ &= 1 + 0.025\,(0 + 1 + 0 \times 1) = 1.025 \end{aligned}$$

Next, we get $\quad x_1 = x_0 + h = 0 + 0.025 = 0.025$

Again put $n = 1$ in Eq. (12), we have

$$\begin{aligned} y_2 &= y_1 + h\,f(x_1, y_1) \\ &= 1.025 + 0.025\,f(0.025, 1.025) \\ &= 1.025 + 0.025\,(0.025 + 1.025 + 0.025 \times 1.025) \\ y_2 &= 1.0518 \end{aligned}$$

Next, we get $x_2 = x_1 + h = 0.025 + 0.025 = 0.05$

Again put $n = 2$ in Eq. (12), we have

$$\begin{aligned} y_3 &= y_2 + h\,f(x_2, y_2) \\ &= 1.0518 + 0.025\,(0.05 + 1.0518 + 0.05 \times 1.0518) \\ &= 1.0806 \end{aligned}$$

Next, we get $x_3 = x_2 + h = 0.05 + 0.025 = 0.075$

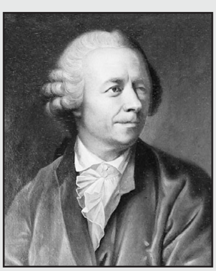
Again put $n = 3$ in Eq. (12), we have

$$\begin{aligned} y_4 &= y_3 + h\,f(x_3, y_3) \\ &= 1.0806 + 0.025\,(0.075 + 1.0806 + 0.075 \times 1.0806) \\ y_4 &= 1.1115 \end{aligned}$$

Next, we get $x_4 = x_3 + h = 0.075 + 0.025 = 0.1$

When $x = x_4 = 0.1$, then

$\therefore \qquad y = 1.1115$, when $x = 0.1$.



Leonhard Euler was born on 15 April 1707, in Basel, Switzerland. Euler showed an early aptitude and propensity for mathematics, and thus, after studying with Johan Bernoulli, he attended the University of Basel and earned his master's degree during his teens. Moving to Russia in 1727, Euler served in the navy before joining the St. Petersburg Academy as a professor of physics and later heading its mathematics division. Euler was one of the most pioneering thinkers of mathematics, establishing a career as an academy scholar and contributing greatly to the fields of geometry, trigonometry and calculus, among many others. He released hundreds of articles and publications during his lifetime, and continued to publish after losing his sight. In 1736, he published his first book of many, Mechanica. In the mid-1740s, Euler was appointed as the mathematics director of the newly created Berlin Academy of Science and Beaux Arts, taking on a variety of management roles as well becoming head of the organization itself for a time starting in 1759. He was a revolutionary thinker in the fields of geometry, trigonometry, calculus, differential equations, number theory and notational systems—including the utilization of π and $f(x)$—among a legion of other accomplishments. He died on 18 September 1783.

11.3.4 Modified Euler's Method

Modified Euler's method gives a better improvement in accuracy over the original Euler's method. In this method, we use a line through (x_0, y_0) whose slope is the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$, where $y_1^{(1)} = y_0 + h\,f(x_0, y_0)$.

Thus, we obtain a generalization form of Euler's modified method as following:

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]; n = 0, 1, 2, \cdots \tag{13}$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

The iteration formula in Eq. (13) can be started by choosing $y_1^{(1)}$ from Euler's formula Eq. (11)

i.e., $\quad y_1^{(1)} = y_0 + h\,f(x_0, y_0)$

Equation (13) is also known as Single step predictor-corrector method.

Example 9 Using modified Euler's method to obtain a solution of the differential equation $\frac{dy}{dx} = x + \left|\sqrt{y}\right|$, with initial conditions $y = 1$ at $x = 0$ for the range $0 \le x \le 0.6$ in steps of 0.2.

Solution Here $f(x, y) = x + \left|\sqrt{y}\right|$; $x_0 = 0$, $y_0 = 1$ and $h = 0.2$, we have

$$\begin{aligned} y_1^{(1)} &= y_0 + h\,f(x_0, y_0) \\ &= 1 + 0.2\,f(0, 1) \\ y_1^{(1)} &= 1 + 0.2\left[0 + \left|\sqrt{1}\right|\right] = 1.2 \end{aligned}$$

Next, we get $x_1 = x_0 + h = 0 + 0.2 = 0.2$.

Using Eq. (13), we get the second approximation as following:

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0)\ f(x_1, y_1^{(1)})\right] \\ &= 1 + \frac{0.2}{2}\left[f(0, 1) + f(0.2, 1.2)\right] \\ &= 1 + \frac{0.2}{2}\left[\left(0 + \left|\sqrt{1}\right|\right) + \left(0.2 + \left|\sqrt{1.2}\right|\right)\right] \\ y_1^{(2)} &= 1.2295 \end{aligned}$$

Again using Eq. (13), we get the third approximation as following:

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 1 + \frac{0.2}{2}\left[f(0, 1) + f(0.2, 1.2295)\right] \\ &= 1 + \frac{0.2}{2}\left[f(0, |\sqrt{1}|) + (0.2, +|\sqrt{1.2295}|)\right] = 1.2309 \end{aligned}$$

Again, using Eq. (13), we get the fourth approximation as following:

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + \frac{0.2}{2}[f(0,1) + f(0.2, 1.2309)]$$
$$= 1.2309$$

Since $\qquad y_1^{(3)} = y_1^{(4)}.$

Hence, $\qquad y_1 = 1.2309$ at $x = 0.2$.

Now, we find y for $x = 0.4$, with $x_1 = 0.2$, $y_1 = 1.2309$ and $h = 0.2$.
Initially, find

$$y_2^{(1)} = y_1 + h\,f(x_1, y_1)$$
$$= 1.2309 + (0.2)\left[0.2 + \left|\sqrt{1.2309}\right|\right]$$
$$y_2^{(1)} = 1.4927$$

Next, we get $\qquad x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

Now, using Eq. (13), we get

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$
$$= 1.2309 + \frac{0.2}{2}\left[f(0.2, 1.2309) + f(0.4, 1.4927)\right]$$
$$= 1.2309 + 0.1\left[(0.2 + |\sqrt{1.2309}\,|) + (0.4 + |\sqrt{1.4927}\,|)\right]$$
$$y_2^{(2)} = 1.5240$$

Now, $\qquad y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$

$$= 1.2309 + \frac{0.2}{2}\left[(0.2 + |\sqrt{1.2309}\,|) + (0.4 + |\sqrt{1.5240}\,|)\right]$$
$$= 1.5253$$

Similarly,

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$
$$= 1.5253$$

Since, $\qquad y_2^{(3)} = y_2^{(4)}.$

Hence, $\qquad y_2 = 1.5253$ at $x = 0.2$.

Now, we find y for $x = 0.6$; with $x_2 = 0.4$, $y_2 = 1.5253$ and $h = 0.2$

Initially, find $\quad y_3^{(1)} = y_2 + h\,f(x_2, y_2)$

$$= 1.5253 + 0.2\,f(0.4,\ 1.5253)$$
$$y_3^{(1)} = 1.8523.$$

Next, we get $\qquad x_3 = x_2 + h = 0.4 + 0.2 = 0.6.$

Now, Using Eq. (13), we get

$$y_3^{(2)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, x_3^{(1)})\right]$$

$$= 1.5253 + \frac{0.2}{2}\left[(0.4 + |\sqrt{1.5253}|) + (0.6 + |\sqrt{1.8523}|)\right]$$

$$= 1.8849$$

Similarly, $y_3^{(3)} = 1.8861 = y_3^{(4)}$

Since, $y_3^{(3)} = y_3^{(4)}$

Hence, $y_3 = 1.8861$ at $x = 0.6$.

Example 10 Using modified Euler's method find $y(0.4)$ correct to 3 decimal places.

Given $\frac{dy}{dx} = x - y^2$ with initial condition $y(0.2) = 0.02$. and step size $(h) = 0.2$.

Solution Here $f(x, y) = x - y^2$; $x_0 = 0.2$, $y_0 = 0.02$ and $h = 0.2$.

Next, we get $x_1 = x_0 + h = 0.2 + 0.2 = 0.4$

Initially find $y_1^{(1)} = y_0 + h\,f(x_0, y_0)$

$$= 0.2 + 0.2\,f(0.2, 0.02)$$

$$= 0.2 + 0.2\,[0.2 - (0.02)^2]$$

$$= 0.060$$

Now, we find y for $x = 0.4$ by modified Euler's method in Eq. (13), we have

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 0.02 + \frac{0.2}{2}[f(0.2, 0.02) + f(0.4, 0.060)]$$

$$= 0.02 + 0.1 \quad [(0.2 - 0.02^2) + (0.4 - 0.060^2)]$$

$$y_1^{(2)} = 0.080$$

Again, using Eq. (13), we get

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 0.02 + \frac{0.2}{2}[f(0.2, 0.02) + f(0.4, 0.080)]$$

$$= 0.079$$

Similarly,

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 0.02 + \frac{0.2}{2}[f(0.2, 0.02) + f(0.4, 0.079)]$$

$$= 0.079$$

Since $\quad y_1^{(4)} = y_1^{(3)}$

Hence, $\quad y = 0.079$ at $x = 0.4$.

EXERCISE 11.3

1. Using Euler's method find $y(0.6)$ of $y' = 1 - 2xy$. Given that $y(0) = 0$, by taking $h = 0.2$.
2. Solve $y' = 1 - y;\ y(0) = 0$ in the range $0 \le x \le 0.3$ by taking $h = 0.1$ by Euler's modified method.
3. Given that $y' = 2 + \sqrt{xy}$ and $y(1) = 1$. Find $y(2)$ in steps of 0.2 using Euler's method. **(Gulbarga 1996, Warangal 1995)**
4. Solve $y' = x^2 + y;\ y(0) = 1$ for $x = 0.02$, 0.04 and 0.06 using Euler's modified method.
5. Compute $y(0.5)$ for the differential equation $\dfrac{dy}{dx} = y^2 - x^2$ with $y(0) = 1$ using Euler's method.
6. Given $y' = \log_{10}(x + y)$ with $y(0) = 1$. Find $y(0.2)$ and $y(0.5)$ using Euler's modified method.

Answers

1. $y(0.6) = 0.4748$
2. $y(0.1) = 0.095$, $y(0.2) = 0.180975$, $y(0.3) = 0.2587823$
3. 5.051
4. $y(0.02) = 1.0202$, $y(0.04) = 1.0408$, $y(0.06) = 1.0619$
5. $y(0.5) = 1.76393$
6. $y(0.2) = 1.0082$; $y(0.5) = 1.0490$

11.3.5 Runge–Kutta Method of Fourth Order

As we know, the Euler's method is less efficient in practical problems because it requires "h" to be small for obtaining reasonable accuracy. The Runge–Kutta (RK) method are developed to find the greater accuracy and they possess the advantage of requiring only the function values at some selected points on the subinterval.

The fourth order RK method is most commonly used in practice and is often referred to as the Runge–Kutta method only without any reference to the order.

To solve $\dfrac{dy}{dx} = f(x, y);\ y(x_0) = y_0$ by RK method, compute

$$K_1 = h\,f(x_r, y_r),\ K_2 = h\,f\left(x_r + \frac{h}{2},\ y_r + \frac{K_1}{2}\right)$$

$$K_3 = h\,f\left(x_r + \frac{h}{2},\ y_r + \frac{K_2}{2}\right)$$

$$K_4 = h\,f(x_r + h,\ y_r + K_3)$$

and
$$y_{r+1} = y_r + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4); \quad r = 0, 1, 2, \ldots \tag{14}$$

Which is the required fourth order RK method. The inherent error in the fourth order RK method is of order h^5.

Example 11 Use RK method to find $y(1.2)$ in step size $h = 0.1$, given that

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(1) = 1.5.$$

Solution

Here $f(x, y) = x^2 + y^2$; $x_0 = 1$, $y_0 = 1.5$ and $h = 0.1$

Using Eq. (14), we have

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Now,

$$K_1 = h\,f(x_0, y_0) = (0.1)\,f(1, 1.5) = (0.1)\,[1^2 + 1.5^2]$$

$$= 0.3250$$

$$K_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.1)\,f(1 + 0.05, 1.5 + 0.1625)$$

$$= (0.1)\,f(1.05, 1.6625)$$

$$K_2 = (0.1)\,[(1.05)^2 + (1.6625)^2] = 0.3866$$

$$K_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.1)\,f(1.05, 1.6933)$$

$$= (0.1)\,[(1.05)^2 + (1.6933)^2] = 0.3969$$

and
$$K_4 = h\,f(x_0 + h, y_0 + K_3) = (0.1)\,f(1 + 0.1, 1.5 + 0.3969)$$

$$= (0.1)\,[(1.1)^2 + (1.8969)^2] = 0.4808$$

$$\therefore \quad y_1 = y(1.1) = 1.5 + \frac{1}{6}[0.3250 + 0.7732 + 0.7938 + 0.4808]$$

$$y(1.1) = 1.8954$$

Again, using Eq. (14), we have; $x_1 = 1.1$ and $y_1 = 1.8954$.

$$y_2 = y(x_1 + h) = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

Now,
$$K_1 = h\,f(x_1, y_1) = (0.1)\,f(1.1, 1.8954)$$

$$= (0.1)\,[(1.1)^2 + (1.8954)^2] = 0.4802$$

$$K_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = (0.1)\,f(1.1 + 0.05, 1{,}8954 + 0.2401)$$

$$= (0.1)\,[(1.15)^2 + (2.1355)^2] = 0.5882.$$

$$K_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= (0.1)\,f(1.15, 2.1895) = (0.1)\,[(1.15)^2 + (2.1895)^2]$$

$$K_3 = 0.6116$$

and $\quad K_4 = h\,f(x_1 + h, y_1 + K_3) = (0.1)\,f(1.1 + 0.1, 1.8954 + 0.6116)$

$= (0.1)\,[(1.2)^2 + (2.5070)^2] = 0.7725$

$\therefore \quad y_2 = y(1.1 + 0.1) = 1.8954 + \frac{1}{6}[0.4802 + 1.1764 + 1.2232 + 0.7725]$

or $\quad y(1.2) = 2.5041.$

Example 12 Using RK method to obtain $y(0.1)$ and $y(0.2)$; given that $\frac{dy}{dx} = x^2 - y$ with initial conditions $y(0) = 1$.

Solution Here $f(x, y) = x^2 - y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

Now $\quad K_1 = h\,f(x_0, y_0) = (0.1)\,f(0, 1)$

$= (0.1)\,[0^2 - 1^2] = -0.1.$

$$K_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.1)\,f(0 + 0.05, 1 - 0.05)$$

$= (0.1)\,f(0.05, 0.95)$

$= (0.1)\,[(0.05)^2 - 0.95] = -0.09475$

$$K_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.1)\,f(0.05, 0.952625)$$

$= (0.1)\,[(0.05)^2 - 0.952625] = -0.0950125$

and $\quad K_4 = h\,f(x_0 + h, y_0 + K_3)$

$= (0.1)\,f(0.1, 0.9049875)$

$= (0.1)\,[(0.1)^2 - 0.9049875] = -0.0894987$

Using Eq. (14), we have

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$\therefore \quad y(0.1) = 1 + \frac{1}{6}[-0.1 + 2(-0.09475) + 2(-0.0950125 - 0.0894987]$$

$y(0.1) = 0.9051627$

Now, we find $y(0.2)$, taking $x_1 = 0.1$, $y_1 = 0.9051627$

Again using Eq. (14), we have

$$y_2 = y(x_1 + h) = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

Now $\quad K_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 0.9051627)$

or $\quad K_1 = (0.1)\,[(0.1)^2 - 0.9051627] = -\,0.0895162$

$$K_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$= (0.1)\,f(0.15, 0.8604046)$

$$= (0.1)\,[(0.15)^2 - 0.8604046]$$

$$= -0.0837904$$

Similarly, $$K_3 = h\,f\left(x_1 + \frac{h}{2},\, y_1 + \frac{K_2}{2}\right)$$

$$= (0.1)\,f(0.15,\, 0.8632674)$$

$$= -0.0840767$$

and $$K_4 = h\,f(x_1 + h,\, y_1 + K_3)$$

$$= (0.1)\,f(0.2,\, 0.8210859)$$

$$= -0.0781085$$

$\therefore$ $$y(0.2) = 0.9051627 + \frac{1}{6}\,[-\,0.895162 + 2(-\,0.0837904) + 2(-\,0.0840767) - 0.0781085]$$

$$y(0.2) = 0.8212695$$

EXERCISE 11.4

1. Using RK method of fourth order find $y(0.1)$, $y(0.2)$ and $y(0.3)$ given that $y' = xy + y^2$; $y(0) = 1$.
2. Using RK fourth order method to find an approximate values of y when $x = 0.2$ given that $y' = x + y$; $y(0) = 1$. **[Calicut 1992, Madras 1991, Rewa 1990]**
3. Using RK method, compute $y(0.2)$. Given $y' = 3x + y/2$; $y(0) = 1$ taking $h = 0.1$ **[Mysore 1994, 1997, Osmania 1995]**
4. Using RK method find $y(0.2)$ in steps of 0.1, if $y' = x + y^2$; $y(0) = 1$.
5. Using RK method of fourth, find $y(0.2)$ for the equation $y' = \dfrac{y - x}{y + x}$; $y(0) = 1$ and $h = 0.2$.

Answers

1. $y(0.1) = 0.11689$, $y(0.2) = 1.27739$, $y(0.3) = 1.50412$
2. $y(0.2) = 1.2428$
3. $y(0.2) = 1.1749$
4. $y(0.2) = 1.2736$
5. $y(0.2) = 1.1678$

11.3.6 Predictor-Corrector Method

The general solution for this method for the solution of an ordinary differential equation is defined as follows:

First step is to predict a solution for the given differential equation using an explicit method and then correct the predicted value using another implicit formula of it. Continue this process of prediction and correction till the solution reaches within the error limit.

11.3.6.1 Milne's Predictor-Corrector Method

It is a multi-step predictor-corrector method based on two formulae

(i) Milne's formula for prediction is as follows:

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_n - f_{n-1} + 2f_{n-2}) \tag{15}$$

Equation (15) is used to predict the value of y_{n+1}, when y_n, y_{n-1}, y_{n-2} and y_{n-3} are known.

(ii) Simpson's formula for correction is as follows:

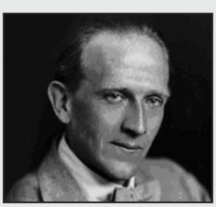
$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}) \tag{16}$$

Equation (16) is used to correct the value of y_{n+1}, where y_{n+1}, y_n and y_{n-1} are known.

Also $f_{n+1} = f(x_{n+1}, y_{n+1})$ is used in eqs (15) and (16) for compute y_{n+1}.

***Note*:**

1. The convergence, stability and accuracy of this method depend only on the correction formula.
2. It is a multi-step method, the truncation or discretization error can be computed, which improves the accuracy at each step.
3. The speed of convergence is improved by a suitable chosen step-size.
4. Milne's method is simple with truncation error $O(h^5)$. In certain cases, it is unstable and its error does not goes to zero even if h is made small. If step-size is too big to give the desired degree of accuracy.



Alan Alexander Milne was born in London on 18 January 1882. His father was a headmaster of a small private school. Milne was very good in mathematics when he was a boy. Later, he attended Trinity College, Cambridge, to study that subject. Milne was educated at Westminster School in London and at the University of Cambridge's Trinity College. While at Cambridge, he studied mathematics and also edited and wrote for the student magazine *Granta*. On realizing that writing was his true vocation, he moved to London after his graduation in 1903. He began writing for the literary magazine *Punch* in 1906, and his essays and humorous poetry were published in the magazine through 1914. He died on 31 January 1956 in Pooh, Piglet.

Example 13 Using Milne's predictor-corrector method to find $y(2)$ if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x + y)$, given that $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$.

Solution Given

$$y_0 = 2,\ y_1 = 2.636,\ y_2 = 3.595 \text{ and } y_3 = 4.968 \text{ for } x_0 = 0,\ x_1 = 0.5,\ x_2 = 1 \text{ and } x_3 = 1.5.$$

Now, we find y_4 corresponding to $x_4 = 2$

Using prediction formula in Eq. (15), we have

$$y_{n+1} = y_{n-3} + \frac{4h}{3}[2f_n - f_{n-1} + 2f_{n-2}]$$

$$\therefore \quad y_4 = y_0 + \frac{4h}{3}[2f_3 - f_2 + 2f_1] \tag{17}$$

Here $h = 0.5$ and $f_{n+1} = f(x_{n+1}, y_{n+1})$.

$\therefore$ For $n = 0, 1, 2$, we get

$$f_1 = f(x_1, y_1) = f(0.5, 2.636) = \frac{1}{2}(0.5 + 2.636) = 1.568$$

$$f_2 = f(x_2, y_2) = 2.2975$$

$$f_3 = f(x_3, y_3) = 3.234.$$

Therefore Eq. (17), gives

$$y_4 = 2 + \frac{4 \times 0.5}{3}[2 \times 3.234 - 2.2975 + 2 \times 1.568]$$

$$y_4 = 6.871.$$

But $\quad f_4 = f(x_4, y_4) = \frac{1}{2}(x_4 + y_4) = 4.4355$

Now, we have corrected this value.

Using Eq. (16), we have

$$y_{n+1} = y_{n-1} + \frac{h}{3}[f_{n+1} + 4f_n + f_{n-1}]$$

$$\therefore \quad y_4 = y_2 + \frac{h}{3}[f_4 + 4f_3 + f_2]$$

$$y_4 = 3.595 + \frac{0.5}{3}[4.4355 + 4 \times 3.234 + 2.2975]$$

$$y_4 = 6.8732 \equiv y_4' \text{ (say)}$$

Now, $\quad f_4' = f(x_4, y_4') = \frac{1}{2}(x_4 + x_4') = 4.4365\,.$

Again using corrector formula, we get the second corrected value y_4^2.

$$y_4^2 = y_2 + \frac{h}{3}[f_4' + 4f_3 + f_2]$$

$$= 3.595 + \frac{0.5}{3}[4.4365 + 4 \times 3.234 + 2.2975]$$

$$= 6.8733$$

Hence $\quad y_4 = y(2) = 6.873$ correct to 3 decimal places.

Example 14 Using Milne's method to find $y(0.3)$ from the differential equation $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. Find the initial values $y(-0.1)$, $y(0.1)$ and $y(0.2)$ from the Taylor's series method.

Solution

Given $$y' = x^2 + y^2 \tag{18}$$

Differentiating Eq. (18) w.r.t. 'x', successively, we get

$$y'' = 2x + 2yy',\ y''' = 2 + 2\,[yy'' + (y')^2]$$

At $x_0 = 0$, $y_0 = 1$, we have

$$y'(0) = 1,\ y''(0) = 2,\ y'''(0) = 8$$

The Taylor's series for $y(x)$ about $x = 0$ is given by

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0)$$

$$y(x) = 1 + x + x^2 + \frac{4x^3}{3} + \cdots$$

$$\therefore \quad y(-0.1) = 1 - 0.1 + (-0.1)^2 + \frac{4(-0.1)^3}{3} - \cdots = 0.9087$$

$$y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{4(0.1)^3}{3} - \cdots = 1.1113$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \frac{4(-0.2)^3}{3} + \cdots = 1.2506$$

Now $\quad y_{-1} = 0.9087$, $y_1 = 1.1113$, $y_2 = 1.2506$ for $x = -0.1, 0.1, 0.2$ and $h = 0.1$

Also, $\quad f_{n+1} = f(x_{n+1}, y_{n+1})$

$$\therefore \quad f_0 = f(x_0, y_0) = f(0, 1) = 1$$

$$f_1 = f(x_1, y_1) = f(0.1, 1.1113) = 1.2449$$

$$f_2 = f(x_2, y_2) = f(0.2, 1.2506) = 1.6040$$

Now using Milne's predictor formula in Eq. (15), we have

$$\therefore \quad y_3 = y_{-1} + \frac{4h}{3}[2f_2 - f_1 + 2f_0]$$

$$= 0.9087 + \frac{4 \times 0.1}{3}[2 \times 1.6040 - 1{,}2449 + 2 \times 1]$$

$$y_3 = 1.4371 \quad \text{for} \quad x_3 = 0.3$$

$$f_3 = f(x_3, y_3) = 2.1552$$

Using Milne's corrector formula in Eq. (16), we have

$$y_3 = y_1 + \frac{h}{3}[f_3 + 4f_2 + f_1]$$

$$= 1.1113 + \frac{0.1}{3}[2.1552 + 6.4160 + 1.2449]$$

$$y_3 = 1.4385 \equiv y_3' \quad \text{(say)}$$

Now $\qquad f_3' = f(x_3, y_3') = f(0.3, 1.4385) = 2.1593$

Again using corrector formula, we get the second corrected value y_3^2

$$y_3^2 = y_1 + \frac{h}{3}[f_3' + 4f_2 + f_1] = 1.1113 + \frac{0.1}{3}[2.1593 + 6.4160 + 1.2449]$$

$$= 1.4386$$

Hence, $y(0.3) = 1.4386$.

EXERCISE 11.5

1. Given $y' = x^2(1+y)$ with $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, compute $y(1.4)$ by Milne's predictor-corrector method.
2. Solve $y' = x - y^2$ at $x = 0.8$ using Milene's predictor-corrector method. Given that $y(0) = 0$, $y(0.2) = 0.0200$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$.
3. Solve $y' = y - x^2$ by Milne's predictor-corrector method with the starting values are $y(0) = 1$, $y(0.2) = 1.12186$, $y(0.4) = 1.4682$, $y(0.6) = 1.7379$, and find $y(0.8)$.
4. Using Milne's predictor-corrector method to solve $y' = -xy^2$ at $x = 0.8$, given that $y(0) = 2$, $y(0.2) = 1.923$, $y(0.4) = 1.724$, $y(0.6) = 1.471$.
5. Solve $\frac{dy}{dx} = x + y$, $y(0) = 0$ for $0.4 \le x \le 1.0$ with $h = 0.1$ using Milne's method.

Answers

1. $y(1.4) = 2.5750$
2. $y(0.8) = 0.3049$
3. $y(0.8) = 2.0111$
4. $y(0.8) = 1.219$
5. 0.0918, 0.1487, 0.2221, 0.3138, 0.4255, 0.5596, 0.7183.

11.3.6.2 Adam's-Bashforth Method (or Adam's-Moulton Formula)

It is a very popular and effective fourth-order predictor corrector multi-step method.

We require four starting values of y which are calculated by some other techniques (such as Taylor's, RK, Euler's or Picard's method). This method is based on the following two formulae:

(i) predictor formula known as Adams-Bashforth predictor

(ii) corrector formula known as Adams-Moulton corrector.

To solve the initial value problem $\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0$.

We compute $y_1 = y(x_0 + h),\ y_2 = y(x_0 + 2h),\ y_3 = y(x_0 + 3h)$, by RK method, Euler's method or Taylor's series, etc.

Next we determine

$$f_1 = f(x_0 + h, y_1),\ f_2 = f(x_0 + 2h, y_2),\ f_3 = f(x_0 + 3h, y_3)$$

Now using Adams-Bashforth predictor formula

$$y_4^p = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0] \tag{19}$$

In general Eq. (19) can be written as

$$y_{n+1}^p = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$$

and Adams-Bashforth corrector formula

$$y_4^c = y_3 + \frac{h}{24}[9f_4 + 19f_3 - 5f_2 + f_1], \tag{20}$$

where $f_4 = f(x_0 + 4h, y_4^p)$

In general Eq. (20) can be written as

$$y_{n+1}^c = y_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$$

The improved value of y_4 is calculated and then used corrected formula in Eq. (20) to find a better value of y_4 and continue this process till y_4 remains unaltered and then proceed to find y_5 as above.

Example 15 Use Adams-Bashforth method to find $y(1.4)$.

Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$.

Solution Here $f(x, y) = x^2(1 + y)$ and the starting values of the Adams-Bashforth method with $h = 0.1$ are as follows:

$$x_0 = 1,\ x_1 = 1.1,\ x_2 = 1.2,\ x_3 = 1.3;\ y_0 = 1.000,\ y_1 = 1.233,\ y_2 = 1.548,\ y_3 = 1.979$$

$$\therefore \quad f_0 = f(x_0, y_0) = f(1, 1) = 1^2(1+1) = 2$$

$$f_1 = f(x_1, y_1) = f(1.1, 1.233) = (1.1)^2(1+1.233) = 2.702$$

$$f_2 = f(x_2, y_2) = f(1.2, 1.548) = 3.669$$

$$f_3 = f(x_3, y_3) = f(1.3, 1.979) = 5.035$$

Using the predictor formula in Eq. (19), we have

$$y_4^p = 1.979 + \frac{0.1}{24}[55 \times 5.035 - 59 \times 3.669 + 37 \times 2.702 - 9 \times 2]$$

$$y_4^p = 2.572$$

Now $f_4 = f(x_0 + 4h, y_4^p) = f(1.4, 2.572) = 7.001$

Using corrector formula in Eq. (20), we have

$$y_4^c = 1.979 + \frac{0.1}{24}[9 \times 7.001 + 19 \times 5.035 - 5 \times 3.669 + 2.702] = 2.575$$

Hence, $y(1.4) = 2.575$.

Example 16 Using Adams-Bashforth predictor–corrector formula to evaluate $y(1.4)$ if $y(x)$ satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{1-xy}{x^2}$, given that

$$y(1) = 1,\ y(1.1) = 0.996,\ y(1.2) = 0.986,\ y(1.3) = 0.972.$$

Solution

Here $f(x, y) = \dfrac{1-xy}{x^2}$ and the starting values of the Adams method with $h = 0.1$ are as follows:

$$x_0 = 1,\ x_1 = 1.1,\ x_2 = 1.2,\ x_3 = 1.3;\ y_0 = 1,\ y_1 = 0.996,\ y_2 = 0.986,\ y_3 = 0.972$$

$$\therefore \qquad f_0 = f(x_0, y_0) = f(1,1) = \frac{1-1}{1^2} = 0$$

$$f_1 = f(x_1, y_1) = f(1.1, 0.996) = -0.7902$$

$$f_2 = f(x_2, y_2) = f(1.2, 0.986) = -0.1273$$

$$f_3 = f(x_3, y_3) = f(1.3, 0.972) = -0.156$$

Using Adams-predictor formula in Eq. (19), we have

$$y_4^p = 0.972 + \frac{0.1}{24}[55 \times -0.156 - 59 \times 0.1273 + 37 \times -0.7902 - 9 \times 0]$$

$$y_4^p = 0.8457$$

Now $\qquad f_4 = f(x_0 + 4h, y_4^p) = f(1.4, 0.8457) = -0.09387$

Using Adams-corrector formula in Eq. (20), we have

$$y_4^c = 0.972 + \frac{0.1}{24}[9 \times -0.9387 + 19 \times -0.156 - 5 \times -0.1273 + (-0.7902)]$$

$$= 0.955$$

Hence, $\qquad y(1.4) = 0.955.$

EXERCISE 11.6

1. Find $y(0.1), y(0.2), y(0.3)$ from $\dfrac{dy}{dx} = x^2 - y;\ y(0) = 1$, by Taylor's method and hence obtain $y(0.4)$ using Adam's-Bashforth method.
2. Using Adam's-Bashforth Method to find $y(0.4)$ given the equation $y' = \dfrac{xy}{2};\ y(0) = 1,\ y(0, 1) = 1.0025$, $y(0.2) = 1.0101$, $y(03) = 1.0228$.
3. Find $y(1)$ by Adam's-Bashforth method given, $\dfrac{dy}{dx} = x^2 - y^3;\ y(0) = 1,\ y(0.25) = 0.821028$, $y(0.5) = 0.741168$, $y(0.75) = 0.741043$.
4. Solve $y' = 1 + xy^2;\ y(0) = 1$ for $x = 0.4$ by Adam's method, given $y(0.1) = 1.105$, $y(0.2) = 1.223$, $y(0.3) = 1.355$.

Mixed Problems

5. Solve the Ricatti equation $\frac{dy}{dx} = x^2 + y^2$, using Taylor's series method for the initial condition $y(0) = 0$, where $0 \le x \le 0.4$ and $h = 0.2$.
6. Using RK method of fourth order to obtain $y(1.1)$; given that $y' = x - y$ with the initial condition $y(1) = 1$ and take $h = 0.1$.
7. Using modified Euler's method to find $y(0.25)$; given that $\frac{dy}{dx} = 2xy$ with the initial condition $y(0) = 1$.
8. Obtain the value of $y(0.4)$ using Milne's method, given that $y' = xy + y^2$, $y(0) = 1$, using Taylor's series to get the values of $y(0.1)$, $y(0.2)$ and $y(0.3)$. **[Delhi 1989]**
9. For the solution of the first order differential equation $y' = f(x, y)$, obtain the predictor formula of the form
$$y_{n+1} = a_1 y_n + h(a_2 y'_n + a_3 y'_{n-1} + a_4 y'_{n-2})$$ and its corresponding corrector formula.
10. Given $y¢ = 1 + y^2$ with boundary condition $y(0) = 0$ in $0 \le x \le 1$, obtain y as a series in powers of x. **[Nagpur 1998]**
11. Using RK method of fourth order to compute $y(0.1)$; given that $y' = 3x + y^2$ with the condition $y(0) = 1$.
12. Let $x = x(t)$ satisfy the differential equation $\frac{dx}{dy} = f(x, t)$, the RK method of order 4 with step-size h is
where $K_1 = hf(t, x)$, $K_2 = hf\left(t + \frac{h}{2}, x + \frac{K_1}{2}\right)$, $K_3 = hf\left(t + \frac{h}{2}, x + \frac{K_2}{2}\right)$ and
$K_4 = hf(t + h, x + K_3)$.
13. From Q. 10, compute y at $x = 0.2$, 0.4 and 0.6 take $h = 0.2$.
14. Using Milne's method compute $y(4.4)$; given that $5xy' + y^2 - 2 = 0$, also $y(4) = 1$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$.
15. Using Milne's method, compute $y(0.4)$; given $y' = (1 + x^2)\frac{y^2}{2}$ also $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$ and $y(0.3) = 1.21$.

Answers

1. $y(0.4) = 0.6896522$
2. $y(0.4) = 1.0408$
3. $y(1) = 0.8145958$
4. $y(0.4) = 1.5$
5. $y(0.4) = 0.021352$
6. $y(1.1) = 1.0040458$
7. $y(0.25) = 1.0645$
8. $y(0.4) = 1.83698$
10. $y(1) = 1.5460$
11. $y(0.1) = 1.127259$
12. $x(0.1) = x_0 + \Delta x = 0.0048375$
$x(0.2) = x_1 + \Delta x = 0.0187305$
13. $y(0.2) = 0.2027067$, $y(0.4) = 0.4228$ and $y(0.6) = 0.6481$
14. $y(4.4) = 1.01874$
15. $y(0.4) = 1.2798$

11.4 NUMERICAL SOLUTION OF SIMULTANEOUS FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

In this section we shall discuss the numerical solution of simultaneous ordinary differential equations using Picard's method and RK method of fourth order.

11.4.1 Picard's Method

Let $\dfrac{dy}{dx} = f(x, y, z)$ and $\dfrac{dz}{dx} = g(x, y, z)$

be the first order simultaneous ordinary differential equations with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. The Picard's method gives

$$\left.\begin{aligned} y_{n+1} &= y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\,dx \\ \text{and} \quad z_{n+1} &= z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\,dx \end{aligned}\right\} \tag{21}$$

11.4.2 Runge–Kutta Method of Fourth Order

Let $\dfrac{dy}{dx} = f(x, y, z)$ and $\dfrac{dz}{dx} = g(x, y, z)$

be the simultaneous first order ODEs with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Starting from (x_0, y_0, z_0) and the increments k and l in y and z are given by the following formula:

$$\begin{array}{l|l} k_1 = hf(x_r, y_r, z_r); & l_1 = hg(x_r, y_r, z_r) \\ k_2 = hf\left(x_r + \dfrac{h}{2}, y_r + \dfrac{k_1}{2}, z_r + \dfrac{l_1}{2}\right); & l_2 = hg\left(x_r + \dfrac{h}{2}, y_r + \dfrac{k_1}{2}, z_r + \dfrac{l_1}{2}\right) \\ k_3 = hf\left(x_r + \dfrac{h}{2}, y_r + \dfrac{k_2}{2}, z_0 + \dfrac{l_2}{2}\right); & l_3 = hg\left(x_r + \dfrac{h}{2}, y_r + \dfrac{k_2}{2}, z_r + \dfrac{l_2}{2}\right) \\ k_4 = hf(x_r + h, y_r + k_3, z_r + l_3); & l_4 = hg(x_r + h, y_r + k_3, z_r + l_3) \\ k = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); & l = \dfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \end{array} \tag{22}$$

Hence,

$$y_{r+1} = y_r + k \text{ and } z_{r+1} = z_r + l;\ r = 0, 1, 2, 3, \ldots, n \tag{23}$$

Example 17 Find the approximate values of y and z at $x = 0.1$, using Picard's method for the solution of the differential equations

$$\frac{dy}{dx} = z, \frac{dz}{dx} = x^3(y + z) \text{ given that } y(0) = 1 \text{ and } z(0) = \frac{1}{2}.$$

Solution

Here, $\quad x_0 = 0, y_0 = 1, z_0 = \dfrac{1}{2}$

and $$\frac{dy}{dx} = f(x,y,z) = z; \frac{dz}{dx} = g(x,y,z) = x^3(y+z)$$

Using Picard's method in Eq. (21), we have

First approximation ($n = 0$)

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)dx = 1 + \int_0^x \left(\frac{1}{2}\right)dx = 1 + \frac{x}{2}$$

$$z_1 = z_0 + \int_{x_0}^{x} g(x, y_0, z_0)dx = \frac{1}{2} + \int_0^x x^3\left(1 + \frac{1}{2}\right)dx = \frac{1}{2} + \frac{3x^4}{8}$$

At $x = 0.1$; $y_1 = 1.05$ and $z_1 = 0.5000375$

Second approximation ($n = 1$)

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)dx = 1 + \int_0^x \left[\frac{1}{2} + \frac{3x^4}{8}\right]dx = 1 + \frac{x}{2} + \frac{3x^5}{40}$$

$$z_2 = z_0 + \int_{x_0}^{x} g(x, y_1, z_1)dx = \frac{1}{2} + \int_0^x x^3\left[1 + \frac{x}{2} + \frac{1}{2} + \frac{3x^4}{8}\right]dx$$

$$z_2 = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}$$

At $x = 0.1$, $y_2 = 1.500008$ and $z_2 = 0.5000385$

Third approximation ($n = 2$)

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)dx = 1 + \int_0^x \left[\frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right]dx$$

$$y_3 = 1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{x^6}{60} + \frac{x^9}{192}$$

$$z_3 = z_0 + \int_{x_0}^{x} g(x, y_2, z_2)\,dx = \frac{1}{2} + \int_0^x \left[x^3\left(1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right)\right]dx$$

$$z_3 = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^9}{360} + \frac{x^{12}}{256}$$

At $x = 0.1$, $y_3 = 1.500008$ and $z_3 = 0.5000385$

Since $y_2 = y_3$ and $z_2 = z_3$

Hence, $y(0.1) = 1.500008$

$z(0.1) = 0.5000385$.

Example 18 Using Picard's method to find approximate value of y and z at $x = 0.1$ of the differential equations $\frac{dy}{dx} = x + z$ and $\frac{dz}{dx} = x - y^2$ with initial condition $y(0) = 2$, $z(0) = 1$.

Solution

Here $x_0 = 0, y_0 = 2, z_0 = 1$

and
$$\frac{dy}{dx} = f(x, y, z) = x + z; \frac{dz}{dx} = g(x, y, z) = x - y^2$$

Using Picard's method in Eq. (21), we have

First approximation ($n = 0$)

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)dx = 2 + \int_0^x (x+1)dx = 2 + x + \frac{x^2}{2}$$

$$z_1 = z_0 + \int_{x_0}^{x} g(x, y_0, z_0)dx = 1 + \int_0^x (x-4)dx = 1 - 4x + \frac{x^2}{2}$$

At $x = 0.1$, $y_1 = 2.105, z_1 = 0.605$

Second approximation ($n = 1$)

$$y_2 = y_0 \int_{x_0}^{x} f(x, y_1, z_1)dx = 2 + \int_0^x \left[x + 1 - 4x + \frac{x^2}{2}\right]dx$$

$$= 2 + x - \frac{3x^2}{2} + \frac{x^3}{6}$$

$$z_2 = z_0 + \int_{x_0}^{x} g(x, y_1, z_1)\,dx = 1 + \int_0^x \left[x - \left(2 + x + \frac{x^2}{2}\right)^2\right]dx$$

$$= 1 - 4x + \frac{3x^2}{2} - x^3 - \frac{x^4}{4} - \frac{x^5}{20}$$

At $x = 0.1, y_2 = 2.085, z_2 = 0.584$

Third approximation ($n = 2$)

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)dx = 2 + \int_0^x \left[1 - 3x + \frac{3x^2}{2} - x^3 - \frac{x^4}{4} - \frac{x^5}{20}\right]dx$$

$$= 2 + x - \frac{3x^2}{2} - \frac{x^3}{2} - \frac{x^4}{2} - \frac{x^5}{20} - \frac{x^6}{120}$$

$$z_3 = z_0 + \int_{x_0}^{x} g(x, y_2, z_2)dx = 1 + \int_0^x \left[x - \left(2 + x - \frac{3x^2}{2} + \frac{x^3}{6}\right)^2\right]dx$$

$$= 1 - 4x - \frac{3x^2}{2} + \frac{5x^3}{2} + \frac{7x^4}{12} - \frac{31x^5}{60} + \frac{x^6}{12} - \frac{x^7}{252}$$

At $x = 0.1$,

$$y_3 = 2.085,\ z_3 = 0.587$$

Since $y_2 = y_3$ and $z_2 = z_3$

Hence, $y(0.1) = 2.085$ and $z(0.1) = 0.587$.

Example 19 Using RK method of fourth order to find $y(0.1)$ and $z(0.1)$ from the system of equations $\frac{dy}{dx} = x + z;\ \frac{dz}{dx} = x - y^2$ with initial conditions $f(0) = 2$, $z(0) = 1$

Solution Here, $x_0 = 0$, $y_0 = 2$, $z_0 = 1$, $h = 0.1$ and $\frac{dy}{dx} = f(x, y, z) = x + z$ and $\frac{dz}{dx} = g(x, y, z) = x - y^2$. Using Eq. (22), we get

$$\begin{aligned}
k_1 &= (0.1)\, f(0, 2, 1) \\
&= 0.1 \\
k_2 &= (0.1) f(0.05, 2.05, 0.8) \\
&= (0.1)(0.05 + 0.8) \\
&= 0.085 \\
k_3 &= (0.1) f(0.05, 2.0425, 0.79238) \\
&= (0.1)(0.05 + 0.79238) \\
&= 0.084238 \\
k_4 &= (0.1) f(0.1, 2.084238, 0.5878) \\
&= (0.1)[0.1 + 0.5878] \\
&= 0.06878
\end{aligned}$$

$$\begin{aligned}
l_1 &= (0.1) g(0, 2, 1) = (0.1)[0 - 2^2] \\
&= -0.4 \\
l_2 &= (0.1) g(0.05, 2.05, 0.8) \\
&= (0.1)[0.05 - (2.05)^2] \\
&= -0.41525 \\
l_3 &= (0.1) g(0.05, 2.0425. 0.79238) \\
&= (0.1)[0.05 - (2.0425)^2] \\
&= -0.4122 \\
l_4 &= (0.1) g(0.1, 2.084238, 0.5878) \\
&= (0.1)[0.1 - (2.084238)^2] \\
&= -0.4244
\end{aligned}$$

$\therefore$

$$k = \frac{1}{6}[0.1 + 2(0.085 + 0.084238) + 0.06878]$$

$$k = 0.0845$$

and

$$l = \frac{1}{6}[-0.4 - 2(0.41525 + 0.4122) - 0.4244)]$$

$$l = -0.4132$$

$$y_1 = y_0 + k = 2 + 0.0845 = 2.0845$$

$$z_1 = z_0 + l = 1 - 0.4132 = 0.58678$$

Hence, $y(0.1) = 2.0845$ and $z(0.1) = 0.58678$.

Example 20 Using RK method of fourth order to find $y(0.1)$ and $z(0.1)$ from the system of equations $\frac{dy}{dx} = x + yz;\ \frac{dz}{dx} = xz + y$ with the initial conditions $y(0) = 1$, $z(0) = -1$.

Solution Here, $x_0 = 0$, $y_0 = 1$, $z_0 = -1$, $h = 0.1$

and $\frac{dy}{dx} = f(x, y, z) = x + yz,\ \frac{dz}{dx} = g(x, y, z) = xz + y$

Using Eq. (21), we get

$$k_1 = (0.1)f(0, 1, -1) = -0.1 \qquad l_1 = (0.1)g(0, 1, -1) = 0.1$$

$$k_2 = (0.1)f(0.05, 0.95, -0.95) = -0.08525 \qquad l_2 = (0.1)g(0.05, 0.95, -0.95) = 0.09025$$

$$k_3 = (0.1)f(0.05, 0.957375, -0.954875); = -0.0864173 \qquad l_3 = (0.1)g(0.05, 0.957375, -0.954875) = -0.090963$$

$$k_4 = (0.1)f(0.1, 0.9135827, -0.9090369); = -0.073048 \qquad l_4 = (0.1)\,g(0.1, 0.9135827, -0.9090369) = 0.082279$$

Now,

$$k = \frac{1}{6}[-0.1+2(-0.08525)+2(-0.0864173) - 0.073048]$$
$$= -0.0860637$$

and

$$l = \frac{1}{6}[0.1+2(0.09025)+2(0.090963) + 0.0822679]$$
$$= -0.090782$$

$\therefore$

$$y_1 = y(0.1) = y_0 + k = 1 - 0.0860637 = 0.9139363$$
$$z_1 = z(0.1) = z_0 + l = -1 + 0.0907823 = -0.909218$$

EXERCISE 11.7

Using Picard's method solve the following (1 to 3):

1. Find the third approximation of the solution of the equations $\frac{dy}{dx} = z; \frac{dz}{dx} = x^2z + x^4y$ with initial conditions $y(0) = 5$, $z(0) = 1$.
2. Find $y(0.3)$ and $z(0.3)$ for the system of equation $\frac{dy}{dx} = 1 + xz; \frac{dz}{dx} = -xy$ with initial conditions $y(0) = 0$, $z(0) = 1$.
3. Approximate y and z for the system of equations $\frac{dy}{dx} = 2x + z; \frac{dz}{dx} = 3xy + x^2z$ with $y(0) = 2$, $z(0) = 0$ up to 3rd approximation.
4. Using RK method of fourth order find y and z at $x = 0.1, 0.2$ for the differential equations $\frac{dy}{dx} = x + z; \frac{dz}{dx} = x - y$ given that $y(0) = 0$, $z(0) = 1$.
5. In question number (2) find y and z at $x = 0.3$ using RK method of fourth order.
6. Using RK method of fourth order to find $y(1.1)$ and $z(1.1)$; given that $y' = xyz, z' = \frac{xy}{z}$ with $y(1) = \frac{1}{3}, z(1) = 1$.
7. Using RK method of fourth order to find y and z at $x = 0.2, 0.4$ for the system of equations $y' = -xz, z' = y^2$ taking step size $h = 0.2$.

Answers

1. $y_3 = 5 + x + \frac{x^4}{12} + \frac{x^6}{6} + \frac{2x^7}{63} + \frac{x^9}{72}$

   $z_3 = 1 + \frac{x^3}{3} + x^5 + \frac{2x^6}{9} + \frac{x^8}{8} + \frac{11x^9}{224} + \frac{7x^{11}}{264}$

2. $y(0.3) = 0.3448$, $z(0.3) = 0.99$

3. $y_3 = 2 + x^2 + x^3 + \frac{3}{20}x^5 + \frac{1}{10}x^6$

   $z_3 = 3x^2 + \frac{3x^3}{4} + \frac{6x^5}{5} + \frac{3x^7}{20} + \frac{3x}{40}$

4. $y(0.1) = 0.1050$, $z(0.1) = 0.9998$
   $y(0.2) = 0.2199$, $z(0.2) = 0.9986$

5. $y(0.3) = 0.34483$, $z(0.3) = 0.98999$

6. $y(1.1) = 0.3707$, $z(1.1) = 1.03615$

7. $y(0.2) = 0.9774$, $z(0.2) = 1.1971$
   $y(0.4) = 0.9$, $z(0.4) = 1.375$

11.5 NUMERICAL SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

In this section, we shall discuss the numerical solution of second order ODE by Picard's method, Runge–Kutta method of fourth order and Milne's method.

11.5.1 Picard's Method

Consider the second order differential equation $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$ (24)

Putting $\frac{dy}{dx} = z$ and $\frac{d^2y}{dx^2} = \frac{dz}{dx}$, then Eq. (24) can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z \text{ and } \frac{dz}{dx} = f(x, y, z).$$

These equations can be solved as discussed above in Section 11.4.1.

Example 21 Using Picard's method obtain the second approximation to the solution of $y'' - x^3y' - x^3y = 0$ with initial condition $y(0) = 1$, $y'(0) = \frac{1}{2}$.

Solution

Given $y'' - x^3y' - x^3y = 0$ (25)

Putting $y' = z \quad \therefore \quad y'' = z'$, then Eq. (25) becomes

$$z' - x^3z - x^3y = 0$$

$$\therefore \qquad \frac{dy}{dx} = z; \frac{dz}{dx} = x^3 z + x^3 y$$

Now solve it in the same way as in Example 17.

Example 22 Using Picard's method to approximate y at $x = 0.1$ given that $y'' + 2xy' + y = 0$ with the initial condition $y(0) = 0.5$, $y'(0) = 0.1$..

Solution

Put $y' = z$ so that $y'' = z'$, then the given differential equation becomes

$$z' + 2xz + y = 0;\ y(0) = 0.5,\ z(0) = 0.1$$

Now the problem becomes $\frac{dy}{dx} = z \equiv f(x, y, z)$ and $\frac{dz}{dx} = -(2xz + y)$; $x_0 = 0$, $y_0 = 0.5$, $z_0 = 0.1$, $h = 0.1$.

Now using Eq. (20), we get

The first approximation is as follows:

$$y_1 = 0.5 + \int_0^x (0.1)\,dx = 0.5 + (0.1)x$$

$$z_1 = 0.1 - \int_0^x [(0.2)x + 0.5]\,dx = 0.1 - (0.5)x + (0.1)x^2$$

Second approximation is as follows:

$$y_2 = 0.5 + \int_0^x z_1 dx = 0.5 + \int_0^x [0.1 - 0.5x + 0.1x^2]\,dx$$

$$= 0.5 + (0.1)x - \frac{0.5x^2}{2} - \frac{0.1x^3}{3}$$

$$z_2 = 0.1 - \int_0^x [2xz_1 + y_1]\,dx$$

$$= 0.1 - \int_0^x [2x(0.1 - 0.5x - 0.1x^2) + (0.5 + 0.1x)]\,dx$$

$$= 0.1 - (0.5)x - \frac{(0.3)x^2}{2} + \frac{x^3}{3} + \frac{(0.2)x^4}{4}$$

Similarly, third approximation is as follows:

$$y_3 = 0.5 + (0.1)x - \frac{(0.5)x^2}{2} - \frac{(0.1)x^3}{2} + \frac{x^4}{12} + \frac{(0.1)x^5}{10}$$

$$z_3 = 0.1 - (0.5)x + \frac{(0.3)x^2}{2} - \frac{(2.5)x^3}{6} + (0.2)x^4 + \frac{2}{15}x^5 + \frac{0.1}{6}x^6$$

Hence, At $x = 0.1$

$$y_1 = 0.51,\ y_2 = 0.50746667,\ y_3 = 0.50745933.$$

$$\therefore \qquad y(0.1) = 0.5075 \text{ (correct to 4 decimal places).}$$

11.5.2 Runge–Kutta Method for Second Order O.D.E.

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = \phi\left(x, y, \frac{dy}{dx}\right); \ y(x_0) = y_0, \ y'(x_0) = y_0' \tag{26}$$

Let $\frac{dy}{dx} = z$ so that $\frac{d^2y}{dx^2} = \frac{dz}{dx}$, than Eq. (26) becomes

$$\frac{dz}{dx} = \phi(x, y, z); \ y(x_0) = y_0, \ z(x_0) = z_0$$

Therefore, the problem reduces to two simultaneous ordinary differential equations

$$\left.\begin{aligned} &\frac{dy}{dx} = z = f(x, y, z) \\ \text{and} \quad &\frac{dz}{dx} = \phi(x, y, z) \end{aligned}\right\} \tag{27}$$

Subject to the conditions $y(x_0) = y_0, \ z(x_0) = z_0$.

Equation (27) shows the two simultaneous ODEs and this system can be solved in the Section 11.4.2.

Example 23 Using RK method of fourth order to find the approximate value of y at $x = 0.1$ of the differential equation $y'' = xy' - y$ with initial conditions $y(0) = 3$, $y'(0) = 0$.

Solution

Given $$y'' = xy' - y \tag{28}$$

with $y(0) = 3$ and $y'(0) = 0$

Let $y' = z$ so that $y'' = z'$

Therefore Eq. (26) reduces to

$$\left.\begin{aligned} &y' = z = f(x, y, z) \\ \text{and} \quad &z' = xz - y = \phi(x, y, z) \end{aligned}\right\} \tag{29}$$

subject to the conditions $y(0) = 3$ and $z(0) = 0$

i.e., $x_0 = 0, \ y_0 = 3$ and $z_0 = 0$, taking $h = 0.1$

Now using Eq. (22) in Section 11.4.2, we have

$$\begin{aligned} k_1 &= hf(x_0, y_0, z_0) \\ &= (0.1)f(0,3,0) \\ &= 0 \\ k_2 &= (0.1)f(0.05, 3, -0.15) \\ &= -0.015 \\ k_3 &= (0.1)f(0.05, 2.9925, -0.150375) \\ &= -0.0150375 \\ k_4 &= (0.1)f(0.1, 2.9849625, -0.3000018) \\ &= -0.0300001 \end{aligned}$$

$$\begin{aligned} l_1 &= h\phi(x_0, y_0, z_0) \\ &= (0.1)\phi(0,3,0) \\ &= -0.3 \\ l_2 &= (0.1)\phi(0.05, 3, -0.15) \\ &= -0.30075 \\ l_3 &= (0.1)\phi(0.05, 2.9925, -0.150375) \\ &= -0.3000018 \\ l_4 &= (0.1)\phi(0.1, 2.9849625, -0.3000018) \\ &= -0.3014962 \end{aligned}$$

Again using Eq. (23) in Section 11.4.2, we get

$$y_1 = y(0.1) = y_0 + k$$

$$= 3+\frac{1}{6}[k_1+2(k_2+k_3)+k_4]$$

$$= 3+\frac{1}{6}[0+2(-0.015-0.0150375)-0.0300001]$$

$$y_1 = -0.0150125$$

and
$$z_1 = z(0.1) = z_0 + l = 0+\frac{1}{6}[l_1+2(l_2+l_3)+l_4]$$

$$= 0+\frac{1}{6}\left[-0.3+2(-0.30075-0.3000018)-0.3014962\right]$$

$$= -0.3004999.$$

11.5.3 Milne's Method for Second Order O.D.E.

In the section (11.5.2) the reduced system (2) can be solved by Milne's method, which is discussed as follows:

We applying Milne's predictor formula, first to z and then to y, we have

$$\left.\begin{aligned} z_{n+1} &= z_{n-3}+\frac{4h}{3}\left[2z'_n - z'_{n-1}+2z'_{n-2}\right] \text{ and} \\ y_{n+1} &= y_{n-3}+\frac{4h}{3}\left[2y'_n - y'_{n-1}+2y'_{n-2}\right] \end{aligned}\right\} \tag{30}$$

Then find z'_{n+1} and y'_{n+1} at $x = x_1 + nh$ by using (31).

Now, applying Milne's corrector formula, we have

$$\left.\begin{aligned} z_{n+1} &= z_{n-1}+\frac{h}{3}\left[z'_{n+1}+4z'_n+z'_{n-1}\right] \text{ and} \\ y_{n+1} &= y_{n-1}+\frac{h}{3}\left[y'_{n+1}+4y'_n+y'_{n-}\right] \end{aligned}\right\} \tag{31}$$

Example 24 Using Milne's method to find $y(0.4)$, given $y''+xy'+y=0,\ y(0)=1,\ y'(0)=0$ and find $y(x)$ at $x = 0.1, 0.2, 0.3$ by Taylor's series method.

Solution Given

$$y''+xy'+y=0 \tag{32}$$

with $y(0) = 1, y'(0) = 0$

Putting $y' = z$ so that $y'' = z'$

Therefore equation (32) reduces to

$$\left.\begin{aligned} y' &= z = f(x, y, z) \text{ and} \\ z' &= -(xz+y) = \phi(x, y, z) \end{aligned}\right\} \tag{33}$$

subject to the conditions $y(0) = 1$, and $z(0) = 0$

Taking $h = 0.1$

Differentiating (32) n times by Leibnitz theorem, we get

$$y_{n+2} + x\,y_{n+1} + n\,y_n + y_n = 0$$

At $x = 0$, $y^{(0)}_{n+2} = -(n + 1)\,y_n(0)$

$$\left.\begin{aligned} \therefore\quad & y(0) = 1, \text{ then } y_2(0) = -1,\ y_4(0) = 3,\ y_6(0) = -15, \text{ and} \\ & y_1(0) = 0, \text{ then } y_3(0) = y_5(0) = y_7(0) = \cdots = 0 \end{aligned}\right\} \tag{34}$$

Using Taylor's series method, $y(x)$ is

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots$$

$$\therefore \qquad y(n) = 1 - \frac{x^2}{2!} + \frac{3}{4!}x^4 - \frac{15}{6}x^6 + \cdots \qquad \text{[Using (3)]} \tag{35}$$

and $\qquad z(x) = y'(x)$

$$= -x + \frac{x^3}{2} - \frac{x^5}{8} + \cdots$$

$$= -x\left(1 - \frac{x^2}{2!} + \frac{3}{4!}x^4 - \cdots\right)$$

$$z(x) = -xy \tag{36}$$

From (35), we obtain

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8}(0.1)^4 - \cdots = 0.995$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \cdots = 0.9802$$

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \cdots = 0.956$$

Using $y(0.1)$, $y(0.2)$ and $y(0.3)$ then equation (5) gives

$$z(0.1) = -(0.1)\cdot y(0.1) = -0.1 \times 0.995 = -0.0995$$

$$z(0.2) = -0.196,\ z(0.3) = -0.2863$$

From (33), $\quad z'(x) = -(xz + y)$

Therefore $\quad z'(0.1) = -(0.1 \times -0.0995 + 0.995) = 0.985$

$$z'(x) = -0.941 \text{ and } z'(0.3) = -0.87$$

Applying Milne's predictor formula (32) in Art (11.5.3), we get for $n = 3$

$$z(0.4) = z_0 + \frac{4 \times 0.1}{3}\left[2z'(0.3) - z'(0.2) + 2z'(0.1)\right]$$

$$= 0 + \frac{0.4}{3}\left[2 \times -0.87 + 0.941 + 2 \times 0.985\right]$$

$$= -0.3692 \qquad\qquad \therefore \qquad y'(0.4) = -0.3692$$

and $$y(0.4) = y(0) + \frac{4\times 0.1}{3}\left[2y'(0.3) - y'(0.2) + 2y'(0.1)\right]^4$$

$$= 0 + \frac{0.4}{3}[-0.5736 + 0.196 - 0.199]$$

$$y(0.4) = 0.9231 \qquad \therefore \qquad y$$

Using (33) $z'(x) = -(xz + y)$

$\therefore$ $$z'(0.4) = -[0.4 \times -0.3692 + 0.9231]$$

$$z'(0.4) = -0.7754$$

Now applying Milne's corrector formula (34) in Art (11.5.3), we obtain

$$z(0.4) = z(0.2) + \frac{h}{3}\left[z'(0.4) + 4z'(0.3) + z'(0.2)\right]$$

$$= -0.196 + \frac{0.1}{3}[-0.7754 - 3.48 - 0.941]$$

$$= -0.3692$$

and $$y(0.4) = y(0.2) + \frac{h}{3}\left[y'(0.4) + 4y'(0.3) + y'(0.2)\right]$$

$$= 0.9802 + \frac{0.1}{3}[-0.3692 - 1.1452 - 0.196]$$

$$= 0.9232$$

Hence $y(0.4) = 0.9232$ and $z(0.4) = -0.3692$.

Example 25 Using Milne's method compute $y(0.4)$, given $y'' = xy' - y$ with initial conditions $y(0) = 3$, $y'(0) = 0$. Find $y(0.1)$, $y(0.2)$ and $y(0.3)$ by using Taylor's series method. (taking $h = 0.1$)

Solution Given

$$y'' - xy' + y = 0 \tag{37}$$

with $y(0) = 3$ and $y'(0) = 0$

Put $y' = z$ so that $y'' = z'$

Therefore equation (37) reduces to

$$\left.\begin{aligned} y' &= z = f(x, y, z) \text{ and} \\ z' &= xz - y = \phi(x, y, z) \end{aligned}\right\} \tag{38}$$

subject to the conditions $y(0) = 3$ and $z(0) = 0$

Differentiating (37) n times by Leibnitz theorem, we get

$$y_{n+2} - x\,y_{n+1} - ny_n + y_n = 0$$

At $$x = 0,\ y_{n+2}^{(0)} = (n-1)\,y_n(0)$$

$$y(0) = 3 \text{ so } y_2(0) = -3,\ y_4(0) = -3,\ y_6(0) = 3 \times -3, \ldots$$

and $$y_1(0) = 0 \text{ so } y_3(0) = 0 = y_5(0) = y_7(0) = \cdots = 0$$

Using Taylor's series method, expand $y(x)$

$$y(x) = y_0 + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots$$

$$= 3 - \frac{3}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{80}x^6 - \cdots \tag{39}$$

Also $$z(x) = y'(x) = 3x - \frac{x^3}{2} - \frac{3}{40}x^5 \cdots \tag{40}$$

At $x = 0.1$, 0.2 and 0.3, Eq. (39) gives

$$y(0.1) = 3 - \frac{3(0.1)^2}{2} - \frac{(0.1)^4}{8} - \frac{(0.1)^6}{80} = 2.985$$

$y(0.2) = 2.940$ and $y(0.3)$ Eq. (40) gives

At $x = 0.1$, 0.2 and 0.3 Eq. (40) gives

$$z(0.1) = 3\times(0.1) - \frac{(0.1)^3}{2} - \frac{3}{40}(0.1)^5 \cdots = 0.299$$

$z(0.2) = 0.596$ and $z(0.3) = 0.886$

Using (38) $z'(x) = xz - y$

$\therefore$ $z'(0.1) = (0.1)\,z(0.1) - y(0.1) = 0.1 \times 0.299 - 2.985 = -2.955$

$z'(0.2) = 0.2 \times 0.596 - 2.940 = -2.821$

$z'(0.3) = 0.3 \times 0.886 - 2.864 = -2.598$

Applying Milne's predictor formula (37), we get

$$z(0.4) = z_0 + \frac{4\times 0.1}{3}\left[2z'(0.3) - z'(0.2) + 2z'(0.1)\right]$$

$$= 0 + \frac{0.4}{3}[2\times -2.958 + 2.821 + 2\times -2.955]$$

$$= -1.105$$

and $$y(0.4) = y_0 + \frac{4\times 0.1}{3}\left[2y'(0.3) - y'(0.2) + 2y'(0.1)\right]$$

$$= 3 + \frac{0.4}{3}[2\times 0.886 - 0.596 + 2\times 0.299]$$

$y(0.4) = 3.237$

Now, using (38) $z'(x) = xz - y$

$\therefore$ $z'(0.4) = 0.4 \times -1.105 - 3.237 = -3.679$

Now, applying Milne's corrector formula (38), we get

$$z(0.4) = z(0.2) + \frac{h}{3}\left[z'(0.4) + 4z'(0.3) + z'(0.2)\right]$$

$$= 0.596+\frac{0.1}{3}[-3.679+4\times -2.598-2.821]$$

$$z(0.4) = 0.033 \qquad [\because \quad y' = z]$$

and $$y(0.4) = y(0.2)+\frac{h}{3}\left[y'(0.4)+4y'(0.3)+y'(0.2)\right]$$

$$= 2.940+\frac{0.1}{3}[0.033+4\times 0.886+0.596]$$

$$y(0.4) = 3.079$$

Hence, $y(0.4) = 3.079$ and $z(0.4) = 0.033$.

EXERCISE 11.8

1. Using Picard's method to obtain the second approximation to the solution of $y''-x^3y'-x^3y=0$ with $y(0)=1;\ y'(0)=\frac{1}{2}$.
2. Using Picard's method, find the third approximation of the initial value problem $y''-xy=1$ with $y(0)=1;\ y'(0)=0$.
3. Use Picard's method to solve the equation $y''+4y=xy$ with $y(0)=3; y'(0)=0$ to approximate y when $x = 0.1$.
4. Using RK method, solve the equation $y''=x(y')^2-y^2$ for $x = 0.2$ correct to four decimal places with initial conditions $y(0) = 1$ and $y'(0)=0$.
5. Using RK method, to find $y(0.1)$ to the equation $y''=-y$ with $y(0) = 1$ and $y'(0) = 0$.
6. Use RK method to find $y(0.1)$ given $y''=y^3$ with $y(0)=10$ and $y'(0)=5$.
7. Solve $y''-x^2y'-2xy=0$ given that $y(0) = 1;\ y'(0)=0$ for $y(0.1)$ using RK method.

Answers

1. $y_2=1+\frac{x}{2}+\frac{3x^5}{40}$
2. $y_3=1+\frac{x^2}{2}+\frac{x^3}{6}$
3. $y(0.1) = 2.9399$
4. $y(0.2) = 0.9801$
5. $y(0.1) = 1.0204$
6. $y(0.1) = 17.42$
7. $y(0.1) = 1.005334$

SUMMARY

Following topics have been discussed in this chapter:

(i) *Picard's Method of Successive Approximations*

For the differential equation

$$\frac{dy}{dx} = f(x, y)$$

With the initial condition $y(x_0) = y_0$.

Then the n^{th} approximation is

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

Thus, we obtain a sequence of approximate solutions

$$y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(n)}$$

each giving a better result than the preceding one.

(ii) *Taylor's Series Method*

For the differential equation $\dfrac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$.

Suppose $y = f(x)$ is the exact solution of the given equation, then $f(x)$ expanding as a Taylor's series about $x = x_0$, we have

$$y(x) = y_0 + (x - x_0)\; y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

If the values $y_0', y_0'', y_0''', \cdots$ are known, then the above equation gives a power series for y. $y_0', y_0'', y_0''', \cdots$ can be obtain from the above equation.

We can write the above equation as

$$y' = f(x, y),\; y'' = f' = f_x + f_y y' = f_x + f_y \cdot f$$

Similarly,

$$y''' = f'' = f_{xx} + f_{xy} \cdot f + f[f_{yx} + f_{yy} \cdot f] + f_y\,[f_x + f_y \cdot f]$$

or $$y''' = f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f^2 f_y$$

and so on.

(iii) *Euler's Method*

Consider the differential equation of first order is as follows:

$$\frac{dy}{dx} = f(x, y)$$

With initial condition $y(x_0) = y_0$.

The general approximations of the given equation is

$$y_{n+1} = y_n + h\,f(x_n, y_n), \text{ where } h = \frac{x_n - x_0}{n}$$

This is called Euler's algorithm.

Thus, starting from x_0 when $y = y_0$ and we construct a table for y for given steps of h in x.

Note If the value of h is small, this method is very slow. However, if h is large then this method is inaccurate.

(iv) *Modified Euler's Method*

Modified Euler's method gives a better improvement in accuracy over the original Euler's method. In this method, we use a line through (x_0, y_0) whose slope is the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$, where $y_1^{(1)} = y_0 + h\,f(x_0, y_0)$.

Thus, we obtain a generalization form of Euler's modified method as follows:

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right];\ n = 0, 1, 2, \cdots$$

where $y_1(n)$ is the nth approximation to y_1.

The iteration formula can be started by choosing $y_1^{(1)}$ from Euler's formula

i.e., $y_1^{(1)} = y_0 + h\,f(x_0, y_0)$.

Therefore the above formula is also known as Single step predictor-corrector method.

(v) *Runge–Kutta (RK) Method of Fourth Order*

The fourth order Runge–Kutta (RK) method is most commonly used in practice and is often referred to as the RK method only without any reference to the order.

To solve $\dfrac{dy}{dx} = f(x, y);\ y(x_0) = y_0$ by RK method, compute

$$K_1 = h\,f(x_r, y_r),\ K_2 = h\,f\left(x_r + \frac{h}{2}, y_r + \frac{K_1}{2}\right)$$

$$K_3 = h\,f\left(x_r + \frac{h}{2}, y_r + \frac{K_2}{2}\right)$$

$$K_4 = h\,f(x_r + h,\ y_r + K_3);$$

$$\text{and } y_{r+1} = y_r + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4);\ r = 0, 1, 2 \ldots$$

Which is the required fourth order RK method. The inherent error in the fourth order RK method is of order h^5.

(vi) *Predictor-Corrector Method*

The general solution for this method for the solution of an ordinary differential equation is defined as following:

Step one is to predict a solution for the given differential equation using an explicit method and then correct the predicted value using another implicit formula of it. Continue this process of prediction and correction till the solution reaches within the error limit.

(a) Milne's Predictor-Corrector Method It is a multi-step predictor-corrector method based on two formulae

(i) Milne's formula for prediction is as follows:

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_n - f_{n-1} + 2f_{n-2})$$

It is used to predict the value of y_{n+1}, when y_n, y_{n-1}, y_{n-2} and y_{n-3} are known.

(ii) Simpson's formula for correction is as follows:

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1})$$

It is used to correct the value of y_{n+1}, where y_{n+1}, y_n and y_{n-1} are known.

Also $f_{n+1} = f(x_{n+1}, y_{n+1})$ is used in the given equations to compute y_{n+1}.

(B) Adam's-Bashforth Method (or Adam's-Moulton Formula) It is a very popular and effective fourth-order predictor corrector multi-step method.

We require four starting values of y which are calculated by some other techniques (such as Taylor's, RK, Euler's or Picard's method). This method is based on the following two formulae:

(i) Predictor formula known as Adams-Bashforth predictor

(ii) corrector formula known as Adams-Moulton corrector.

To solve the initial value problem $\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0$.

We compute $y_1 = y(x_0 + h),\ y_2 = y(x_0 + 2h),\ y_3 = y(x_0 + 3h),$ by RK method, Euler's method or Taylor's series, etc.

Next we determine

$$f_1 = f(x_0 + h, y_1),\ f_2 = f(x_0 + 2h, y_2),\ f_3 = f(x_0 + 3h,\ y_3).$$

Now using Adams-Bashforth predictor formula

$$y_4^p = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

It can be written as

$$y_{n+1}^p = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$$

and Adams-Bashforth corrector formula

$$y_4^c = y_3 + \frac{h}{24}[9f_4 + 19f_3 - 5f_2 + f_1]$$

where $f_4 = f(x_0 + 4h,\ y_4^p)$

The above equation can be written as

$$y_{n+1}^c = y_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$$

The improved value of y^4 is calculated and then use corrected formula to find a better value of y^4 and continue this process fill y^4 remains unaltered and the proceed to find y^5 as above.

1. Numerical Solution of Simultaneous First Order Ordinary Differential Equations

(i) *Picard's Method*

Let $\dfrac{dy}{dx} = f(x, y, z)$ and $\dfrac{dz}{dx} = g(x, y, z)$

be the first order simultaneous ordinary differential equations with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. The Picard's method gives

$$\left.\begin{aligned} y_{n+1} &= y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\,dx \\ \text{and} \qquad z_{n+1} &= z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\,dx \end{aligned}\right\}$$

(ii) *Runge–Kutta Method of Fourth Order*

Let $\dfrac{dy}{dx} = f(x, y, z)$ and $\dfrac{dz}{dx} = g(x, y, z)$

be the simultaneous first order ODEs with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Starting from (x_0, y_0, z_0) and the increments k and l in y and z are given by the following formula:

$$\begin{array}{l|l}
k_1 = hf(x_r, y_r, z_r); & l_1 = hg(x_r, y_r, z_r) \\
k_2 = hf\left(x_r + \dfrac{h}{2}, y_r + \dfrac{k_1}{2}, z_r + \dfrac{l_1}{2}\right); & l_2 = hg\left(x_r + \dfrac{h}{2}, y_r + \dfrac{k_1}{2}, z_r + \dfrac{l_1}{2}\right) \\
k_3 = hf\left(x_r + \dfrac{h}{2}, y_r + \dfrac{k_2}{2}, z_0 + \dfrac{l_2}{2}\right); & l_3 = hg\left(x_r + \dfrac{h}{2}, y_r + \dfrac{k_2}{2}, z_r + \dfrac{l_2}{2}\right) \\
k_4 = hf(x_r + h, y_r + k_3, z_r + l_3); & l_4 = hg(x_r + h, y_r + k_3, z_r + l_3) \\
k = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); & l = \dfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)
\end{array}$$

Hence,

$$y_{r+1} = y_r + k \text{ and } z_{r+1} = z_r + l;\ r = 0, 1, 2, 3, \ldots, n.$$

2. Numerical Solution of Second Order Ordinary Differential Equations

(i) *Picard's Method*

Consider the second order differential equation $\dfrac{d^2y}{dx^2} = f\left(x, y, \dfrac{dy}{dx}\right)$

Putting $\dfrac{dy}{dx} = z$ and $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$, then, the above equation can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z \text{ and } \frac{dz}{dx} = f(x, y, z).$$

(iii) *Runge–Kutta Method for Second Order (ODE)*

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = \phi\left(x, y, \frac{dy}{dx}\right); \; y(x_0) = y_0, \; y'(x_0) = y'_0$$

Let $\frac{dy}{dx} = z$ so that $\frac{d^2y}{dx^2} = \frac{dz}{dx}$, then the equation becomes:

$$\frac{dz}{dx} = \phi(x, y, z); \; y(x_0) = y_0, \; z(x_0) = z_0.$$

Therefore, the problem reduces to two simultaneous ordinary differential equations

$$\left.\begin{aligned} &\frac{dy}{dx} = z = f(x, y, z) \\ \text{and} \quad &\frac{dz}{dx} = \phi(x, y, z) \end{aligned}\right\}$$

Subject to the conditions $y(x_0) = y_0, \; z(x_0) = z_0$

Given equation shows the two simultaneous ordinary differential equations and this system can be solved in the Section 11.4.2 in the chapter.

OBJECTIVE TYPE QUESTIONS

1. Picard's method to find the solution of $\frac{dy}{dx} = f(x, y); \; y(x_0) = y_0$ is

(a) $y = y_0 + \int_{x_n}^{x_0} f(x, y_{n-1})\,dx$

(b) $y = y_0 + \int_{x_0}^{x_n} f(x, y_{n-1})\,dx$

(c) $y = y_n + \int_{x_0}^{x_n} f(x, y_{n-1})\,dx$

(d) $y = x_0 + \int_{x_0}^{x_n} f(x_n, y_n)\,dx$

2. Second order RK method is applied to the initial value problem $\frac{dy}{dx} = -y; \; y(x_0) = y_0$ with step size h, then $y(x)$ is

(a) $y(x) = y_1 + \frac{1}{3}(K_1 + K_2)$

(b) $y(x) = y_0 + hf(x_0, y_0)$

(c) $y(x) = hf(x_1, y_0 + K_1)$

(d) None of these

3. Using Euler's method to solve $y' = -xy$ with $y(0) = 1$, $h = 0.05$, then $y(0.5)$ is

(a) 1 (b) –1

(c) 0 (d) 0.9975

4. When $y' = x + y^2$ and $y_0 = 1$, then using Picard's method y_1 is

(a) $1-x+\frac{x^2}{2}$ (b) $1-x-\frac{x^2}{2}$

(c) $1+x-\frac{x^2}{2}$ (d) $1-x-\frac{x^3}{3}$

5. If $y' = 2xy$ with $y(0) = 1$, using modified Euler's method, $y(0.2)$ is (taking $h = 0.25$)

(a) 1.0625 (b) 0.3421

(c) 3.2506 (d) 0.0625

6. Runge–Kutta method is a self-starting method

(a) False (b) Do not know

(c) True (d) None of these

7. Given the differential equation $y' = x - y$ with $y(0) = 0$. The value of $y(0.1)$, calculated numerically upto the third place of decimal by the second order Runge–Kutta method with step size $h = 0.1$ is ______.

[GATE (CS) – 1993]

8. Backward Euler's method for solving the differential equation $y' = f(x, y)$ is specified by

(a) $y_{n+1} = y_n + hf(x_n, y_n)$

(b) $y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$

(c) $y_{n+1} = y_{n-1} + 2hf(x_n, y_n)$

(d) $y_{n+1} = 1 + hf(x_{n-1}, y_{n-1})$

[GATE (CS) – 1994]

9. Gauss–Seidel iterative method can be used for solving a set of

(a) Linear differential equation

(b) Linear algebraic equation

(c) Both linear and non-linear algebraic equations

(d) Both linear and non-linear differential equations

[GATE (EE) – 1997]

10. Match the correct combination

A Newton–Raphson method	1. Solving non-linear equations
B Runge–Kutta method	2. Solving linear simultaneous equations
C Simpson's rule	3. Solving ordinary differential equations
D Gauss method	4. Numerical Integration
	5. Interpolation
	6. Calculation of eigen values

(a) A-6, B-1, C-5, D-3

(b) A-1, B-6, C-4, D-3

(c) A-1, B-3, C-4, D-2

(d) A-5, B-3, C-4, D-1

[GATE (EC) – 2005]

11. Match the items in column I and II

Column-I	Column-II
A Gauss–Seidel method	1. Interpolation
B Newton's forward formula	2. Non-linear differential equations
C Runge–Kutta method	3. Numerical integration
D Trapezoidal rule	4. Linear algebraic equations

(a) A-1, B-4, C-3, D-2

(b) A-1, B-4, C-2, D-3

(c) A-1, B-3, C-3, D-4

(d) A-4, B-1, C-2, D-3

[GATE (ME) – 2006]

12. While numerically solving the differential equation $y' + 2xy^2 = 0$, $y(0) = 1$, using Euler's predictor-corrector (improved Euler–Cauchy) method with a step size of 0.2 the value of y after the first step is

(a) 1 (b) 1.03

(c) 0.97 (d) 0.96

[GATE (IN) – 2013]

13. The second order Runge–Kutta method is

(a) Euler's method

(b) Modified Euler's method

(c) Taylor's series method

(d) None of these

14. Using Euler's method, solve $\frac{dy}{dx} = \frac{y-2x}{y}$, $y(0) = 1$, then $y(0.1)$ is

(a) 1.1818 (b) 2.0183

(c) 1.6870 (d) None of these

15. Taylor's series method will be useful to give some starting values of

(a) Milne's method

(b) Runge–Kutta method

(c) Euler's method

(d) None of these

16. Using RK fourth order method, the approximate value of $y(0.2)$; given that $y' = x + y$, $y(0) = 1$ is

(a) 1.3678 (b) 1.2468

(c) 0.2468 (d) 2.3678

17. Given y_0, y_1, y_2, y_3, using Milne's corrector formula to find y_4 for $y' = f(x, y)$ is

(a) $y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

(b) $y_4 = y_1 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

(c) $y_4 = y_2 - \frac{h}{3}(f_2 + 4f_3 + f_4)$

(d) $y_4 = y_2 + \frac{h}{3}(f_2 - 4f_3 - f_4)$

18. Using Euler's method, find $y(0.4)$ for $y' = -y$ with $y(0) = 1$, $h = 0.01$ is

(a) 0.9606 (b) 0.1345
(c) 1.9606 (d) None of these

19. Using RK fourth order to $y' = x\,y^{\frac{1}{3}}$; $y(1) = 1$, value of $y(1.1)$ is

(a) 1.10682 (b) 0.10682
(c) 0.10684 (d) 2.10682

20. Taylor's series method for $y' = f(x, y)$ is single step method

(a) True (b) False
(c) Wrong (d) None of these

ANSWERS

1.(b)	2.(d)	3.(a)	4.(a)	5.(a)	6.(a)	7.(0.005)		8.(a)	9.(b)
10.(c)	11.(c)	12.(d)	13.(b)	14.(a)	15.(a)	16.(b)	17.(a)	18.(a)	19.(a)
20.(a)									

12 Numerical Solution of Partial Differential Equations

12.1 INTRODUCTION

Partial differential equation (PDE) plays an important role in the field of technology, science and many branches of applied mathematics (like in Fluid dynamics, Human body, electromagnetic fields, Debluring problem in image processing, Quantum computation, Study of displacement of a vibrating string, Heat transfer quantum mechanics and Study of diffusion of mater, etc.). Most of the physical problems can be modelled mathematically by partial differential equations, but only few of them can be solved by analytical method. So in most of the cases, we apply the numerical method to approximate solution. In this chapter, we shall discuss the explicit, implicit and iterative methods. The method of finite differences (explicit) is most commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations.

12.2 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Let $u(x, y)$ be a continuous function which maps R^2 in to R. The general second order partial differential equation is of the form

$$\phi\left(x, y, z, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0$$

or

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \tag{1}$$

where A, B, C, D, E, F and G are all functions of x and y. Equation (1) can be classified into three types according to the numerical value of the discriminant $\Delta = B^2 - 4\,AC$ which are as follows:

(i) If $\Delta < 0$ at a point in the (x, y) plane, then the Eq. (1) is called elliptic.
(ii) If $\Delta = 0$ at a point in the (x, y) plane, then the Eq. (1) is called parabolic.
(iii) If $\Delta > 0$ at a point in the (x, y) plane, then the Eq. (1) is called hyperbolic.

12.3 SOME STANDARD PDE's

In this section, we shall list some standard PDEs that play a very important role in various applications of science and engineering.

12.3.1 Heat Equation

The heat equation represents the flow of heat in a conductor and is defined as

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2},$$

where C is a constant and u is temperature at time t at a point.

12.3.2 Wave Equation

The wave equation represents vibrations of an elastic body or a particle and is defined as

$$\frac{\partial^2 u}{\partial t^2} = K^2 \frac{\partial^2 u}{\partial x^2},$$

where K^2 is velocity of wave propagation which is constant, u is displacement at time t of a particle in space or line.

12.3.3 Laplace's Equation

The Laplace's equation represents the steady-state distribution of heat in a body. It is defined as

(i) In $2D$, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ or $\nabla^2 u = 0$,

where the Laplace operator in $2D$ is $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

(ii) In $3D$, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ or $\nabla^2 u = 0$,

where the Laplace operator in $3D$, is $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

12.3.4 Poisson's Equation

Poisson's equation represents potential in $2D$ or $3D$ and is defined as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \text{ or } \nabla^2 u = f(x, y) \text{ in } 2D,$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z) \text{ or } \nabla^2 u = f(x, y, z) \text{ in } 3D,$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

12.3.5 Potential Equation

This equation is a special form of Poisson's equation and it is represents electromagnetic or gravitational fields.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-\rho(x, y) \text{ or } \nabla^2 u=-\rho(x, y).$$

12.3.6 Schrödinger's Equation

For a particle of mass equal to $\frac{1}{2}$ is $\frac{\partial u}{\partial t}=i\frac{\partial^2 u}{\partial x^2}$,

where units of length and time have been chosen in such a way that Planck's constant becomes 1.

The Schrödinger's equation in quantum mechanics for the scattering of a wavepacket by 1-D potential $V(x)$ is used for the evaluation of a complex quantity ψ and it is described as

$$i\frac{\partial \psi}{\partial t}=-\frac{\partial^2 \psi}{\partial x^2}+V(x)\psi,$$

where $i=\sqrt{-1}$ and t is time.

***Note*:** In stationary case:

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}-V(x)\psi=0.$$

12.3.7 Navier–Stokes Equation

It is used in the application of fluid mechanics, when the fluid is incompressible and viscous, then the system of partial differential equation is defined as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2},$$

where u and v are the components of the velocity vector in a fluid flow and p is the function of pressure.

12.3.8 Cauchy–Riemann Equation

The Cauchy–Riemann equation is defined as

$$\frac{\partial u}{\partial t}+i\frac{\partial u}{\partial x}=0,$$

where $i=\sqrt{-1}$.

12.3.9 Boundary and Initial Value Problem

A solution of a partial differential equation, satisfying the differential equation also satisfies certain conditions known as boundary condition and initial condition.

The boundary condition can be as following:

(i) If the dependent variable (u) specified on each point of the boundary set T (Dirichlit's condition) and then the problem is called a Dirichlet's problem.

(ii) If $\frac{\partial u}{\partial n}$, the normal derivative is specified for each point T-the problem with such condition is called Neuman's problem.

(iii) If u is specified on some part of T and $\frac{\partial u}{\partial n}$ on the other part of T is called mixed boundary value problem.

Besides' the boundary condition, in a time dependent problem the solution also satisfies the condition at time $t = 0$ is called an initial value problem (or Cauchy problems).

A differential equation or problem is said to be well-posed, if sufficient number of boundary and initial conditions are given on the solution.

12.4 FINITE DIFFERENCE METHOD

Let a rectangular region R in the (x, y) plane be divided into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in Fig. (12.1) by drawing the sets of lines.

$$\left.\begin{aligned} x &= ih; \quad i = 0, 1, 2, \ldots \\ y &= jk; \quad j = 0, 1, 2, \ldots \end{aligned}\right\} \tag{2}$$

The points of intersection of Eq. (2) are called mesh points, nodal points, lattice points or grid points.

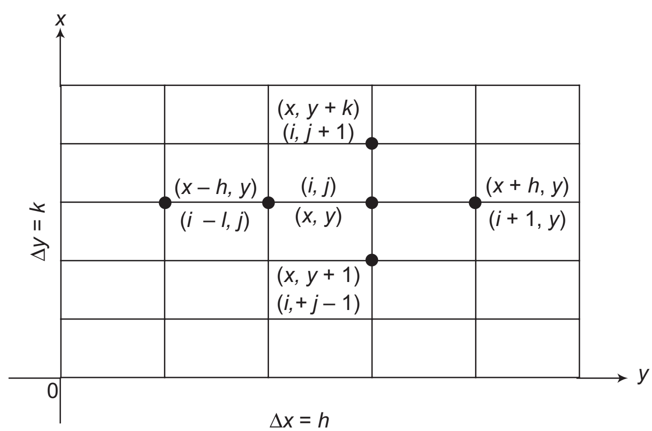


Fig. 12.1 *Geometrical representation of partial difference quotients*

Now, to obtain the finite difference approximations to the derivatives by using Taylor's series, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots \tag{3}$$

or $$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2!} f''(x) + \cdots$$

or $$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h) \text{ (error)} \tag{4}$$

Equation (4) is the forward difference approximation for $f'(x)$. Similarly, we expand $f(x - h)$ by Taylor's series, we have

$$f(x-h) = f(x) = -hf'(x) + \frac{h^2}{2!} f''(x) - \cdots \tag{5}$$

or $$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h) \tag{6}$$

Equation (6) is the backward difference approximations for $f'(x)$.

Now subtracting Eq. (5) from Eq. (3), we get

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!} f'''(x) + \cdots$$

or $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \quad \text{(error term)} \tag{7}$$

Clearly, Eq. (7) gives a better approximation to $f'(x)$ comparison to Eqs (4) and (6).

Now, adding Eqs (3) and (5), we get

$$f(x+h) + f(x-h) = 2f(x) + 2\frac{h^2}{2!} f''(x) + \frac{2h^4}{4!} f^{iv}(x) + \cdots$$

or $$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \frac{h^2}{12} f^{iv}(x) + \cdots$$

or $$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2) \tag{8}$$

Equation (8) is the forward difference approximations. Next, the approximations to the partial derivatives for the function $u = u(x, y)$ at (x_i, y_i), for $i = 1, 2, 3, \ldots$ and $u(x_i, y_i) = u_{ij}$.

Therefore,

$$\left(\frac{\partial u}{\partial x}\right)_{(x_i, y_i)} = \frac{u_{i+1, j} - u_{i-1, j}}{2h} \tag{9}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i, y_i)} = \frac{u_{i+1, j} - 2u_{i, j} + u_{i-1, j}}{h^2} \tag{10}$$

Similarly, the other first and second order partial derivatives are as follows:

$$\left(\frac{\partial u}{\partial y}\right)_{(x_i, y_i)} = \frac{u_{i, j+1} - u_{i, j-1}}{2k} \tag{11}$$

and $$\left(\frac{\partial^2 u}{\partial y^2}\right)_{(x_i, y_i)} = \frac{u_{i, j+1} - 2u_{i, j} + u_{i, j-1}}{k^2} \tag{12}$$

Now using Eqs (9), (10), (11) and 12 in the given partial differential equations and these equations converted to difference equations. The solution of difference equations is the solution of the given partial differential equations.

12.5 PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

The parabolic partial differential equation, we consider is the heat, or diffusion equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; 0 < x < l, t > 0 \tag{13}$$

Subject to the conditions

$$u(0, t) = 0 = u(l, t), t < 0$$

and $$u(x, 0) = f(x), 0 \le x \le l$$

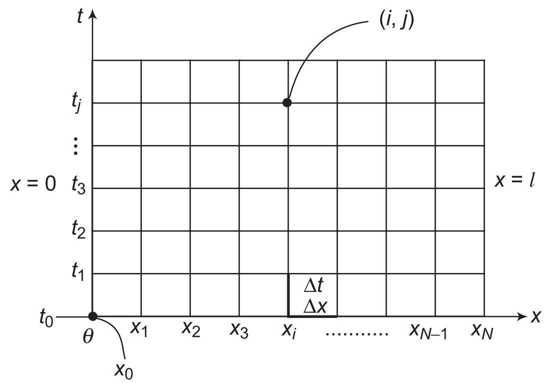


Fig. 12.2

Consider *Xt*-plane, the *X*-axis along the rod and *t*-axis through *O* to it. Now we divide the interval [0, *l*] into *N* equal parts of length $\Delta x = h$ each. Thus $x_i = i\Delta x = ih$

$$i = 0, 1, 2, 3, \ldots (N-1), N$$

Also on *t*-axis, the step size $\Delta t = k$ each.

Thus $$t_j = j\Delta t = jk; \ j = 0, 1, 2, \ldots$$

Draw lines parallel to *X*-axis through the points t_j on *t*-axis and lines parallel to *t*-axis through the point x_i on *X*-axis.

To converting the region $0 \le x \le l$, $t > 0$ in to a mesh point. The point intersection of lines are called nodes or mesh point or grid point. The intersection of the lines $x = x_i$ and $t = t_j$ is denoted by $(i\Delta x, j\Delta t)$ or (x_i, t_j) or simply (i, j) and the temperature is $u(i, j)$ or u_i^j or $u_{i,j}$. We assume that *u* is known upto the time level t_j and we want to find *u* for the unknown time step t_{j+1}.

12.5.1 Explicit Scheme (Finite Difference Method)

Substituting the finite difference relations defined by Eqs (10) and (11) in Eq. (13), we get

$$\frac{u_{i,\,j+1} - u_{i,\,j}}{\Delta t} = \alpha^2 \frac{u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}}{(\Delta x)^2} \quad \begin{bmatrix} \text{Here } \Delta t = k \\ \Delta x = h \end{bmatrix}$$

or $$u_{i,\,j+1} = \alpha^2 \frac{\Delta t}{(\Delta x)^2}(u_{i+1,\,j} + u_{i-1,\,j}) + \left(1 - 2\alpha^2 \frac{\Delta t}{(\Delta x)^2}\right) u_{i,j}$$

or $$u_{i,\,j+1} = r\,(u_{i+1,\,j} + u_{i-1,\,j}) + (1-2r)\,u_{i,\,j}, \tag{14}$$

where $$r = \alpha^2 \frac{\Delta t}{(\Delta x)^2}; \quad j = 0, 1, 2, 3, \ldots$$

$$i = 1, 2, 3, \ldots, (N-1)$$

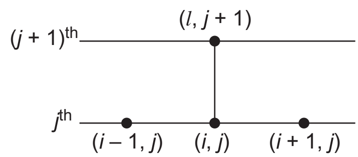


Fig. 12.3

Computational molecule or Forward difference Stencil for the finite difference method.

Equation (14) is valid when $0 < r \le \frac{1}{2}$. This is an explicit scheme is expressed u at unknown $(j+1)^{\text{th}}$ time level in terms of the known values of u at j^{th} time level.

For $r = \frac{1}{2}$, Eq. (14) reduces to

$$u_{i,\,j+1} = \frac{1}{2}[u_{i+1,\,j} + u_{i-1,\,j}] \tag{15}$$

Equation (15) is called the *Bender–Schmidt recurrence equation/scheme*.

12.5.2 Implicit Scheme

In the explicit scheme, the time step (Δt) has to be necessarily very small. The explicity scheme is valid only for $0 \le r \le \frac{1}{2}$, i.e., $r = \alpha^2 \frac{\Delta t}{(\Delta x)^2}$.

Now, the partial differential Eq. (13) is discretized by replacing the time derivative by backward difference and the space second derivative by a central difference at the node $(i, j+1)$, we have

$$\frac{u_{i,\,j+1} - u_{i,\,j}}{\Delta t} + O(\Delta t) = \frac{\alpha^2}{(\Delta x)^2}[u_{i-1,\,j+1} - 2u_{i,\,j+1} + u_{i+1,\,j+1}] + O(\Delta x)^2$$

or $$u_{i,\,j+1} - u_{i,\,j} = \alpha^2 \frac{\Delta t}{(\Delta x)^2}[u_{i-1,\,j+1} - 2u_{i,\,j+1} + u_{i+1,\,j+1}] \quad \text{[Neglecting the error terms]}$$

or $$(1+2r)\,u_{i,\,j+1} - r\,u_{i-1,\,j+1} - r\,u_{i+1,\,j+1} = u_{i,\,j}, \tag{16}$$

where $$r = \alpha^2 \frac{\Delta t}{(\Delta x)^2}$$

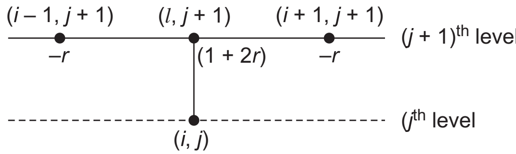


Fig. 12.4 *Implicit Stencil*

From Eq. (16), all the values of u on left are unknown and on the right $u_{i,\,j}$ is known.

12.5.3 Implicit Scheme Second or Crank–Nicolson Scheme or Method

This scheme is based on numerical approximations for the solution of Eq. (13) at the node $(i, j + 1)$.

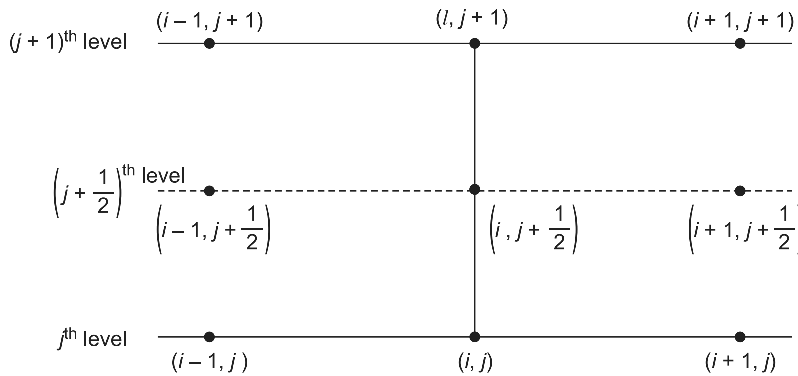


Fig. 12.5

In implicit scheme, the time derivative $\dfrac{\partial u}{\partial t}$ is replaced by a backward difference at $(i, j + 1)$, i.e.

$$\frac{\partial u}{\partial t} = \frac{u_{i,\,j+1} - u_{i,\,j}}{\Delta t} + O(\Delta t)$$

In this scheme error terms are mixed order, i.e. $\{O(\Delta t) + O(\Delta x)^2\}$.

For having the error terms as $[O(\Delta t)^2 + O(\Delta x)^2]$, we replace the time derivative at a fictitious node $\left(i,\, j + \dfrac{1}{2}\right)$ in the mid of (i, j) and $(i, j + 1)$ by a central difference. Then we have a space second derivative $\dfrac{\partial^2 u}{\partial x^2}$ at the node $\left(i,\, j + \dfrac{1}{2}\right)$.

We write the average of the approximate at $(i, j + 1)$ and (i, j) so Eq. (13) becomes

$$\frac{u_{i,\,j+1} - u_{i,\,j}}{2\dfrac{\Delta t}{2}} + O(\Delta t) = \frac{\alpha^2}{2(\Delta x)^2}\left[u_{i+1,\,j+1} - 2u_{i,\,j+1} + u_{i-1,\,j+1} + u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}\right] + O(\Delta x)^2$$

or $$2u_{i,\,j+1}-2u_{i,\,j}=r\left[u_{i+1,\,j+1}-2u_{i,\,j+1}+u_{i-1,\,j+1}+u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right]+\{O(\Delta t)^2+O(\Delta x)^2\},$$

where $$r=\frac{\alpha^2\Delta t}{(\Delta x)^2}$$

Neglecting the term's of $[O(\Delta t)^2+O(\Delta x)^2]$, we have

$$-r\,u_{i-1,\,j+1}+(2+2r)\,u_{i,\,j+1}-r\,u_{i+1,\,j+1}=r\,u_{i-1,\,j}+(2-2r)\,u_{i\,,\,j}+r\,u_{i+1,\,j}$$

or $$-r\,u_{i-1,\,j+1}+(2+2r)\,u_{i,\,j+1}-r\,u_{i+1,\,j+1}=b_{i,\,j},\tag{17}$$

where $$b_{i,\,j}=r\,u_{i-1,\,j}+(2-2r)u_{i,\,j}+r\,u_{i+1,\,j}$$

$$j=0,\,1,\,2,\,3,\,\ldots$$
$$i=1,\,2,\,3,\,\ldots,\,(N-1).$$

Equation (17) is valid any value of r, however r is taken small further the LHS of Eq. (17) involves 3 values of u at unknown step $(j+1)$.

Now, we write the Eq. (17) for $i = 1, 2, 3, \ldots (N-1)$.

We have

For $i = 1$

$$-r\,u_{0,\,j+1}+(2+2r)\,u_{1,\,j+1}-r\,u_{2,\,j+1}=b_{1,\,j}$$

But from the boundary condition $u_{0,\,j}=0\ \ \forall\, j$

$$\therefore\quad (2+2r)\,u_{1,\,j+1}-r\,u_{2,\,j+1}=b_{1,\,j}\tag{18}$$

For $i = 2$

$$-r\,u_{1,\,j+1}+(2+2r)\,u_{2,\,j+1}-r\,u_{3,\,j+1}=b_{2,\,j}\tag{19}$$

$$\vdots\qquad\qquad\vdots$$

For $i = N-1$

$$-r\,u_{N-2,\,j+1}+(2+2r)\,u_{N-1,\,j+1}-r\,u_{N,\,j+1}=b_{N-1,\,j}$$

The boundary condition $u_{N,\,j}=0\ \ \forall\, j$

$$\therefore\quad -r\,u_{N-2,\,j+1}+(2+2r)u_{N-1,\,j+1}=b_{N-1,\,j}\tag{20}$$

Thus, we obtain a set of $(N-1)$ linear simultaneous equations with $(N-1)$ unknown's and the coefficient matrix being tridiagonal, i.e.,

$$\begin{bmatrix}(2+2r) & -r & 0 & \cdots & 0\\ -r & (2+2r) & \ddots & & 0\\ 0 & \ddots & \ddots & \ddots & -r\\ 0 & \cdots & 0 & -r & (2+2r)\end{bmatrix}\begin{bmatrix}u_{1,\,j+1}\\ u_{2,\,j+1}\\ u_{3,\,j+1}\\ u_{N-1,\,j+1}\end{bmatrix}=\begin{bmatrix}b_{1,\,j}\\ b_{2,\,j}\\ b_{3,\,j}\\ b_{N-1,\,j}\end{bmatrix}$$

This system may be solved by either Gaussian elimination method or LU method.

For $r = 1$, the scheme i.e., Eq. (17) gives

$$-u_{i-1,\,j+1} + 4u_{i,\,j+1} - u_{i+1,\,j+1} = b_{i,\,j} \tag{21}$$

where $$b_{i,\,j} = (u_{i-1,\,j} + u_{i+1,\,j})$$

for $$i = 1, 2, 3, \ldots (N-1),\ j = 1, 2, 3, \ldots$$

This scheme is known as *Crank–Nicolson Scheme*.

12.5.4 Three Time Level Scheme or Richardson's Scheme or Method

In the partial differential Eq. (13), we discretize at the node (i, j) by replacing both the derivatives by a central difference taking Δt as one time interval.

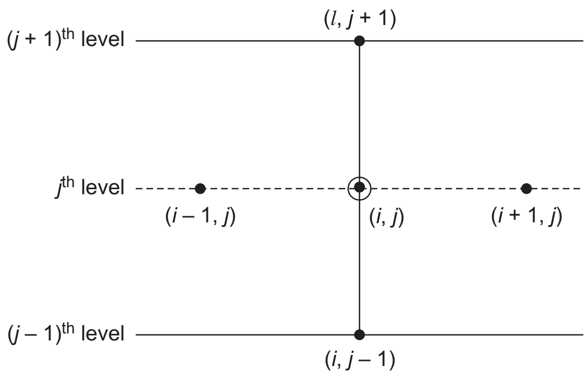


Fig. 12.6

Thus we get,

$$\frac{u_{i,\,j+1} - u_{i,\,j-1}}{2\Delta t} = \frac{\alpha^2}{(\Delta x)^2}\left[u_{i-1,\,j} - 2u_{i,\,j} + u_{i+1,\,j}\right]$$

or $$u_{i,\,j+1} = 2r\left[u_{i-1,\,j} - u_{i,\,j} + u_{i+1,\,j}\right] + u_{i,\,j-1}, \tag{22}$$

where $$r = \frac{\alpha^2 \Delta t}{(\Delta x)^2};\ i = 1, 2, 3, \ldots, (N-1)$$

$$j = 0, 1, 2, 3, \ldots$$

This scheme is a three time level scheme in which the unknown $u_{i,\,j+1}$ is explicitly expressed in terms of known values of u at the previous known $(j-1)^{\text{th}}$ and j^{th} time level's.

Example 1 One dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0, t) = u(5, t) = 0$ and $u(x, 0) = x^2(25 - x^2)$, using the explicit scheme to find the solution.

for $\quad x = i\Delta x\ (i = 0, 1, 2, ..., 5;\ h = \Delta x = 1)$ and

$$t = j\Delta t\left(j = 0, 1, 2, ..., 6;\ k = \Delta t = \frac{1}{2}\right).$$

Solution

Here $\quad \Delta x = 1,\ \Delta t = \dfrac{1}{2}$, then $r = \dfrac{\Delta t}{(\Delta x)^2} = \dfrac{1}{2\cdot(1)^2} = \dfrac{1}{2}$

We apply the explicit scheme. (Bender–Schmidt scheme)

$$u_{i,\,j+1} = \frac{1}{2}\left[u_{i+1,\,j} + u_{i-1,\,j}\right] \tag{23}$$

Also $u(0, 0) = 0,\ u(1, 0) = 24,\ u(2, 0) = 2^2(25 - 2^2) = 84,$

$$u(3, 0) = 3^2\ (25-3^2) = 144,\ u(4, 0) = 4^2(25-4^2) = 144$$

$$u(5, 0) = 5^2(25-5^2) = 0$$

Now, putting $j = 0$ in Eq. (23), we get

$$u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right] \tag{24}$$

Put $i = 1$ in Eq. (24), we get

$$u_{1,1} = \frac{1}{2}\left[u_{2,0} + u_{0,0}\right] = \frac{1}{2}[84+0] = 42$$

Put $i = 2$ in Eq. (24), we get

$$u_{2,1} = \frac{1}{2}\left[u_{3,0} + u_{1,0}\right] = \frac{1}{2}[144+24] = 84$$

Put $i = 3$ in Eq. (24), we get

$$u_{3,1} = \frac{1}{2}\left[u_{4,0} + u_{2,0}\right] = \frac{1}{2}[144+84] = 114$$

Put $i = 4$ in Eq. (24), we get

$$u_{4,1} = \frac{1}{2}\left[u_{5,0} + u_{3,0}\right] = \frac{1}{2}[0+144] = 72$$

Thus, the second row is filled as given in the table.

Similarly, putting $j = 1, 2, 3, 4, 5$ in Eq. (23) and the other rows are filled.

The following table gives the values of $u_{i,j}$

$j \downarrow$ \ $i \rightarrow$	0	1	2	3	4	5
0	0	24	84	144	144	0
1	0	42	84	114	72	0
2	0	42	78	78	57	0
3	0	39	60	67.5	39	0
4	0	30	53.25	49.50	33.75	0
5	0	26.6	39.75	43.5	24.75	0
6	0	19.88	35.06	32.25	21.75	0

Example 2 Solve the given equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Subject to the condition $u(x, 0) = \sin \pi x,\ 0 \le x \le 1;$

$u(0, t) = u(1, t) = 0$, using

(i) Schmidt scheme

(ii) Richardson's scheme/method

(iii) Crank–Nicolson method/scheme

by taking $\Delta x = \frac{1}{3}, \Delta t = \frac{1}{36}$.

Solution

Here $\Delta x = \frac{1}{3}, \Delta t = \frac{1}{36}$ so that

$$r = \frac{\Delta t}{(\Delta x)^2} = \frac{1}{36\left(\frac{1}{3}\right)^2} = \frac{9}{36} = \frac{1}{4}$$

Also $$u(1, 0) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$u(2, 0) = \sin\frac{2\pi}{3} = \sqrt{3}/2$$

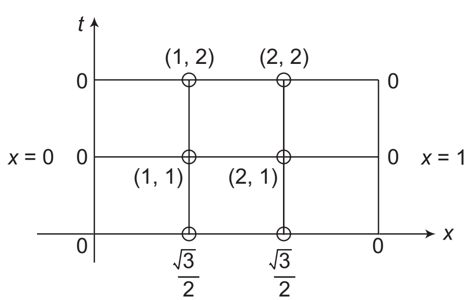


Fig. 12.7

(i) The explicit Schmidt scheme in Eq. (14) is

$$u_{i,\, j+1} = r\left[u_{i+1,\, j} + u_{i-1,\, j}\right] + (1-2r)\, u_{i,\, j}$$

For $r = \frac{1}{4}$, Eq. (14) becomes

$$u_{i,\, j+1} = \frac{1}{4}\left[u_{i+1,\, j} + u_{i-1,\, j}\right] + \frac{1}{2} u_{i,\, j}$$

$$= \frac{1}{4}\left[u_{i+1,\, j} + u_{i-1,\, j} + 2\, u_{i,\, j}\right] \tag{25}$$

Putting $j = 0$ in Eq. (25), we get

$$u_{i,1} = \frac{1}{4}\left[u_{i+1,0} + u_{i-1,0} + 2\, u_{i,0}\right] \tag{26}$$

Now putting $i = 1, 2$, in Eq. (26), we obtain

$$u_{1,1} = \frac{1}{4}\left[u_{2,0} + u_{0,0} + 2\, u_{1,0}\right] = \frac{1}{4}\left[\frac{\sqrt{3}}{2} + 0 + 2 \cdot \frac{\sqrt{3}}{2}\right] = 0.65$$

$$u_{2,1} = \frac{1}{4}\left[u_{3,0} + u_{1,0} + 2\, u_{2,0}\right] = \frac{1}{4}\left[0 + \frac{\sqrt{3}}{2} + 2 \cdot \frac{\sqrt{3}}{2}\right] = 0.65$$

Again, putting $i = 1, 2$ and $j = 1$ in Eq. (25), we get

$$u_{1,2} = \frac{1}{4}\left[u_{2,1} + u_{0,1} + 2\, u_{1,1}\right]$$

$$= \frac{1}{4}\left[0.65 + 0 + 2 \times 0.65\right] = 0.49$$

$$u_{2,2} = \frac{1}{4}\left[u_{3,1} + u_{1,1} + 2u_{2,1}\right] = \frac{1}{4}\left[0 + 0.65 + 2 \times 0.65\right]$$

$$= 0.49$$

(ii) The Richardson's scheme is given by

$$u_{i,\, j+1} = \frac{1}{2}\left[u_{i-1,\, j} - 2\, u_{i,\, j} + u_{i+1,\, j}\right] + u_{i,\, j-1} \qquad \left[\because \ r = \frac{1}{4}\right] \tag{27}$$

To starts the calculations, with the help of $u_{1,\,1}$ and $u_{2,\,1}$.
From the above scheme, we have

$$u_{1,\,1} = 0.65 \text{ and } u_{2,\,1} = 0.65$$

Putting $i = 1, 2$ and $j = 1$ in Eq. (27), we get

$$u_{1,2} = \frac{1}{2}\left[u_{0,1} - 2u_{1,1} + u_{2,1}\right] + u_{1,0}$$

$$= \frac{1}{2}\left[0 - 2 \times 0.65 + 0.65\right] + \frac{\sqrt{3}}{2}$$

$$u_{1,\,2} = 0.25$$

and $$u_{2,2} = \frac{1}{2}\left[u_{1,1} - 2\,u_{2,1} + u_{3,1}\right] + u_{2,0}$$

$$= \frac{1}{2}[0.65 - 2 \times 0.65 + 0] + \frac{\sqrt{3}}{2}$$

$$u_{2,2} = 0.54$$

(iii) *Crank–Nicolson method/scheme*: The Eq. (17) becomes

$$-\frac{1}{4}u_{i-1,\,j+1} + \frac{5}{2}u_{i,\,j+1} - \frac{1}{4}u_{i+1,\,j+1} = b_{i,\,j} \tag{28}$$

where $$b_{i,\,j} = \frac{1}{4}u_{i-1,\,j} + \frac{3}{2}u_{i,\,j} + \frac{1}{4}u_{i+1,\,j}$$

Putting $i = 1, 2$ and $j = 0$ in Eq. (28), we obtain

$$-\frac{1}{4}u_{0,1} + \frac{5}{2}u_{1,1} - \frac{1}{4}u_{2,1} = b_{1,0}$$

$$= \frac{1}{4}u_{0,0} + \frac{3}{2}u_{1,0} + \frac{1}{4}u_{2,0}$$

or $$-u_{0,1} + 10\,u_{1,1} - u_{2,1} = u_{0,0} + 6\,u_{1,0} + u_{2,0}$$

or $$10\,u_{1,1} - u_{2,1} = \frac{7\sqrt{3}}{2} \tag{29}$$

and $$-u_{1,1} + 10\,u_{2,1} - u_{3,1} = u_{1,0} + 6u_{2,0} + u_{3,0}$$

or $$-u_{1,1} + 10\,u_{2,1} = \frac{7\sqrt{3}}{2} \tag{30}$$

Solving Eqs (29) and (30), we get

$$u_{1,1} = 0.67 \text{ and } u_{2,1} = 0.67$$

Again putting $i = 1, 2$, and $j = 1$ in Eq. (28), we obtain

$$10\,u_{1,2} - u_{2,2} = 4.69 \tag{31}$$

and $$-u_{1,2} + 10u_{2,2} = 4.69 \tag{32}$$

Solving Eqs (31) and (32), we get

$$u_{1,2} = 0.52 \text{ and } u_{2,2} = 0.52$$

Example 3 Solve the heat conduction equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. Subject to the conditions $u(0, t) = 0$ and $u(1, t) = 10$; $u(x, 0) = 10\,x$ for $0 \le x \le 1$. Taking $\Delta x = \frac{1}{4}$, $\Delta t = \frac{1}{18}$ and $r = 1$ using Crank–Nicolson method.

Solution Using the Crank–Nicolson scheme, we have

$$-u_{i-1,\,j+1}+4u_{i,\,j+1}-u_{i+1,\,j+1}=b_{i,\,j} \tag{33}$$

where $\quad b_{i,\,j}=(u_{i-1,\,j}+u_{i+1,j})$

$\therefore$ For $j = 0$, then Eq. (33) becomes

$$-u_{i-1,1}+4u_{i,1}-u_{i+1,1}=b_{i,0}$$

where $\quad b_{i,0}=(u_{i-1,0}+u_{i+1,0})$

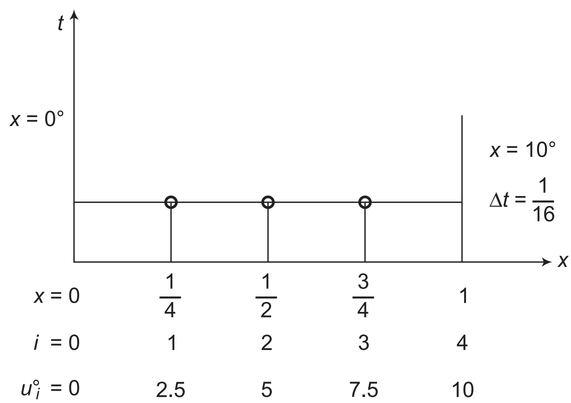


Fig. 12.8

Now, writing the scheme for $i = 1, 2, 3$, we have

$$4u_{1,1}-u_{2,1}=b_{1,0}+u_{0,1}$$
$$=(u_{0,0}+u_{1,0})+u_{0,1}$$
$$=(0+2.5)+0$$
$$4\,u_{1,1}-u_{2,1}=2.5 \tag{34}$$
$$-u_{1,1}+4\,u_{2,1}-u_{3,1}=(u_{1,0}+u_{3,0})=2.5+7.5=10 \tag{35}$$
$$-u_{2,1}+4\,u_{3,1}=b_{3,0}+u_{4,1}$$
$$=(u_{2,0}+u_{4,0})+u_{4,1}$$
$$=(5+10)+10$$
$$-u_{2,1}+4\,u_{3,1}=25 \tag{36}$$

writing Eqs (34), (35) and (36) in Matrix form

$$\begin{bmatrix}4 & -1 & 0\\ -1 & 4 & -1\\ 0 & -1 & 4\end{bmatrix}\begin{bmatrix}u_{1,1}\\ u_{2,1}\\ u_{3,1}\end{bmatrix}=\begin{bmatrix}2.5\\ 10\\ 25\end{bmatrix}$$

Solving the above matrix system by Gaussian elimination method, so we obtain the equations

$$-u_{1,1} + 4u_{2,1} - u_{3,1} = 10$$

$$u_{2,1} - 0.27\, u_{3,1} = 2.83$$

$$3.73\, u_{3,1} = 27.83$$

$\therefore$ $$u_{1,1} = 1.90,\ u_{2,1} = 4.84 \text{ and } u_{3,1} = 7.46$$

EXERCISES 12.1

1. Solve the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < 5$ by Crank–Nicolson scheme.

   Subject to the conditions

   $u(x, 0) = 20$

   $u(0, t) = 0,\ u(5, t) = 100; \quad t > 0$

   Taking $h = 1$ and $k = 1$

2. Using Crank–Nicolson method, solve

   $u_{xx} = 16\, u_t;\ 0 < x < 1,\ t > 0$

   Subject to the conditions

   $u(x, 0) = 0,\ u(0, t) = 0,\ u(1, t) = 50\, t$

   Find u for two steps in t direction taking $h = \dfrac{1}{4}$.

3. Solve the heat equation $u_t = u_{xx}$, subject to the conditions

   $u(0, t) = 0,\ u(1, t) = 0$ and $u(x, 0) = x(1 - x)$

   Taking $h = 0.1$ and $t = 0, 1, 2$.

4. Solve $u_t = \dfrac{1}{2} u_{xx}$ subject to the conditions $u(0, t) = 0 = u(4, t)$ and $u(x, 0) = x(4 - x)$ taking $h = 1$ by using Bender–Schmidt scheme. Continue the solution through 10 time steps.

5. Solve $u_t = u_{xx}$, $0 \le x \le 1$, $t \ge 0$ under the conditions $u(0, t) = 0 = u(1, t)$

   and $$u(x, 0) = \begin{cases} 2x & \text{for } 0 \le x \le \dfrac{1}{2} \\ 2(1 - x) & \text{for } \dfrac{1}{2} \le x \le 1 \end{cases}.$$

Answers

1.

0	20	20	20	100
0	9.80	30.72	59.92	100

2.

$j \downarrow$ \ $i \rightarrow$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0.89285	3.5714	13.39285	50
2	0	1.7857	7.1429	26.7857	100

3.

$j \downarrow$ \ $i \rightarrow$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0.09	0.16	0.21	0.24	0.25	0.24	0.21	0.16	0.09	0
2	0	0.08	0.15	0.20	0.23	0.24	0.23	0.20	0.15	0.08	0

4.

$j \downarrow$ \ $\rightarrow i$	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0
6	0	0.375	0.5	0.375	0
7	0	0.25	0.375	0.25	0
8	0	0.1875	0.25	0.1875	0
9	0	0.125	0.1875	0.125	0
10	0	0.094	0.125	0.094	0

5.

$j \downarrow$ \ $i \rightarrow$	0	0.2	0.4	0.6	0.8	1.0	0.8	0.6	0.4	0.2	0
0.1	0	0.1936	0.3689	0.5400	0.6461	0.6921	0.6461	0.5400	0.3689	0.1936	0
0.02	2	0.1989	0.3956	0.5834	0.7381	0.7691	0.7381	0.5834	0.3956	0.1989	0

12.6 SOLUTION OF HYPERBOLIC EQUATIONS

In this section, we shall discuss the numerical solution to the wave equation, an example of a hyperbolic partial differential equation.

Consider, 1-D wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; 0 \le x \le l, t > 0 \tag{37}$$

Subject to the conditions

$$u(0, t) = u(l, t) = 0 \text{ for } t > 0$$

$u(x, 0) = f(x)$ and $u_t(x, 0) = \frac{\partial u}{\partial t}(x, 0) = g(x)$ for $0 \le x \le l$, where α is a constant dependent on the physical conditions of the given problem.

The finite differences of

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,\, j+1} - 2\, u_{i,\, j} + u_{i,\, j-1}}{(\Delta t)^2} + O(\Delta t)^2$$

and
$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1\, j} - 2\, u_{i,\, j} + u_{i-1,\, j}}{(\Delta x)^2} + O(\Delta x)^2$$

Therefore Eq. (37) becomes,

$$\frac{u_{i,\, j+1} - 2\, u_{i,\, j} + u_{i,\, j-1}}{(\Delta t)^2} + O(\Delta t)^2 = \alpha^2 \frac{u_{i+1,\, j} - 2u_{i,\, j} + u_{i-1,\, j}}{(\Delta x)^2} + O(\Delta x)^2$$

or
$$\frac{u_{i,\, j+1} - 2\, u_{i,\, j} + u_{i,\, j-1}}{(\Delta t)^2} = \alpha^2 \left[\frac{u_{i+1,\, j} - 2u_{i,\, j} + u_{i-1,\, j}}{(\Delta x)^2}\right] + \left[O(\Delta t)^2 + O(\Delta x)^2\right]$$

Neglecting the error term $\left[O(\Delta t)^2 + O(\Delta x)^2\right]$, we get

$$\frac{u_{i,\, j+1} - 2\, u_{i,\, j} + u_{i,\, j-1}}{(\Delta t)^2} = \frac{\alpha^2}{(\Delta x)^2}\left[u_{i+1,\, j} - 2u_{i,\, j} + u_{i-1,\, j}\right]$$

or
$$u_{i,\, j+1} - 2u_{i,\, j} + u_{i,\, j-1} = \left(\frac{\alpha\, \Delta t}{\Delta x}\right)^2 \left[u_{i+1,\, j} - 2\, u_{i,\, j} + u_{i-1,\, j}\right]$$

or
$$u_{i,\, j+1} - 2u_{i,\, j} + u_{i,\, j-1} = r^2 \left[u_{i+1,\, j} - 2\, u_{i,\, j} + u_{i-1,\, j}\right], \tag{38}$$

where
$$r = \frac{\alpha \Delta t}{\Delta x}$$

Consider the approximations in both rows j and $j - 1$ are known.

Equation (38) can be used to compute $u_{i,\, j+1}$ for $i = 1, 2, 3, 4, \ldots, N - 1$ and $j = 1, 2, 3, \ldots$

$$u_{i,\, j+1} = 2\,(1 - r^2)\, u_{i,\, j} + r^2\, (u_{i+1,\, j} + u_{i-1,\, j}) - (u_{i,\, j-1}) \tag{39}$$

for $i = 1, 2, 3, 4, \ldots, N - 1$, $j = 1, 2, 3, \ldots$. The boundary conditions gives $u_{0,\, j} = 0 = u_{N,\, j}$ for each $j = 1, 2, 3, \ldots$

and the initial condition implies that

$$u_{i,0} = f(x_i) \text{ for each } i = 1, 2, ..., N-1$$

To find the initial velocity condition

$$\frac{\partial u}{\partial t}(x, 0) = g(x); 0 \le x \le l$$

The partial derivative $\frac{\partial u}{\partial t}$ is replace by a forward difference approximation $\frac{\partial u}{\partial t} = \frac{u_{i,\,j+1} - u_{i,0}}{\Delta t}$, we obtain $u_{i,1} = u_{i,0} + g(x_i)$ for $i = 1, 2, ..., N-1$.

We observe that the 4 known values on the RHS of Eq. (39), which are used to find $u_{i,\,j+1}$ can be shown in Fig. 12.9.

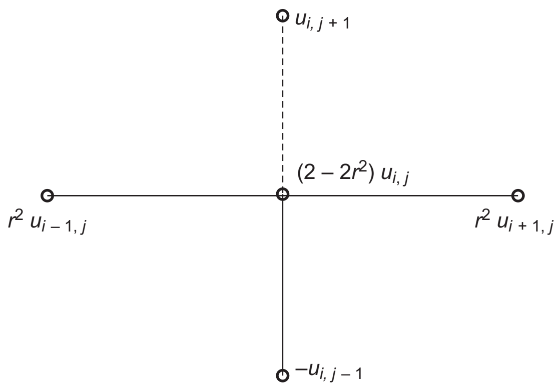


Fig. 12.9

Note**:** If $r = 1$, then Eq. (39) becomes

$$u_{i,\,j+1} = u_{i+1,\,j} + u_{i-1,\,j} - u_{i,\,j-1} \tag{40}$$

Example 4 Solve $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ with the boundary conditions $u(0, t) = 0 = u(4, t), \frac{\partial u}{\partial t}, (x, 0) = 0$ and $u(x, 0) = x(4 - x); 0 \le x \le 4$.

Solution Here $\alpha^2 = 4 \Rightarrow \alpha = 2$

Taking $\Delta x = 1$ so we get $\Delta t = \frac{\Delta x}{\alpha} = \frac{1}{2} = 0.5$

$$\therefore \quad r = \frac{\alpha \Delta t}{\Delta x} = \frac{2 \times 1}{2 \times 1} = 1$$

Subject to the conditions (In difference form) are as follows:

$$u_{0,\,j} = 0 \text{ and } u_{4,\,j} = 0 \text{ for each } j$$

Now $\quad u_t(x, 0) = 0 \Rightarrow \frac{1}{\Delta t}\left[u_{i,\,j+1} - u_{i,\,j}\right] = 0$

when $\quad j = 0,\ u_{i,\,1} = u_{i,\,0}$ for all i (41)

$$u(x, 0) = x(4 - x)$$

$$\Rightarrow \quad u(0, 0) = 0, u(1, 0) = 3, u(2, 0) = 4, u(3, 0) = 3, u(4, 0) = 0$$

In difference form

$$u_{i,0} = u(i, 0) = i(4 - i) \text{ for } i = 0, 1, 2, 3, 4$$

Now, using the formula in Eq. (40)

$$u_{i,\, j+1} = u_{i+1,\, j} + u_{i-1,\, j} - u_{i,\, j-1} \tag{42}$$

Put $j = 1$ in Eq. (42), we get

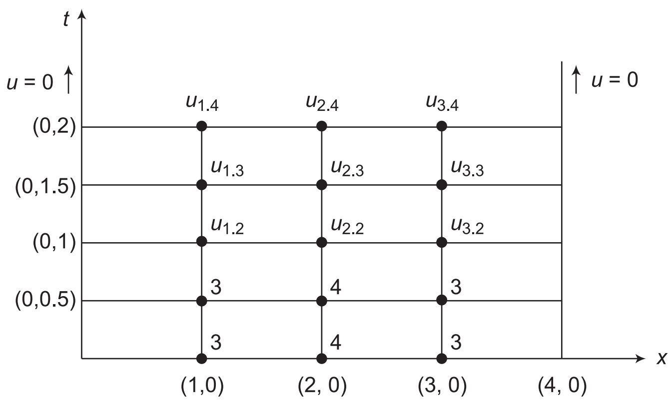
$$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0} \tag{43}$$

Putting $i = 1, 2, 3$ in Eq. (43), we get

$$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0} = 4 + 0 - 3 = 1 \quad \text{[using Eq. (41)]}$$

$$u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0} = 3 + 3 - 4 = 2 \quad \text{[using Eq. (41)]}$$

$$u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0} = 0 + 4 - 3 = 1 \quad \text{[using Eq. (41)]}$$



[Grid representation]

Fig. 12.10

i.e., the 3rd row is filled up.

In a similar manner we can fill up the remaining rows as shown in the following table:

$j \downarrow$ \ $i \rightarrow$	**0**	**1**	**2**	**3**	**4**
0	0	3	4	3	0
1	0	3	4	3	0
2	0	1	2	1	0
3	0	–1	–2	–1	0
4	0	–3	–4	–3	0

Example 5 Using finite difference method to solve $u_{tt} = u_{xx}$ with the conditions

$$u(0, t) = u(1, t) = 0,\ u(x, 0) = \frac{1}{2}x(1-x) \text{ and}$$
$$u_t(x, 0) = 0 \text{ taking } \Delta x = \Delta t = 0.1 \text{ for } 0 \le t \le 0.4.$$

Solution

Here $\alpha^2 = 1 \Rightarrow \alpha = 1$

$$r = \alpha\frac{\Delta t}{\Delta x} = 1\cdot\frac{0.1}{0.1} = 1$$

Subject to the conditions are

$$u_{0,\,j} = 0 \text{ and } u_{1,\,j} = 0 \text{ for each } j$$

Now $\quad u(x, 0) = \frac{1}{2}x(1-x),\ u(i, 0) = \frac{1}{2}i(1-i)$

$\therefore \quad u(i, 0) = 0, 0.045, 0.08, 0.105, 0.120, 0.125, 0.120, 0.105$

for $\quad i = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6.$

Now, using the formula in Eq. (40)

$$u_{i,\,j+1} = u_{i+1,\,j} + u_{i-1,\,j} - u_{i,\,j-1} \tag{44}$$

Since $\quad u_t(x, 0) = 0 \Rightarrow \frac{1}{\Delta t}[u_{i,\,j+1} - u_{i,\,j}] = 0$

when $\quad j = 0,\ u_{i,\,1} = u_{i,\,0}$ for all i

Putting $\quad j = 0$ in Eq. (44), we get

$$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}$$
$$= u_{i+1,0} + u_{i-1,0} - u_{i,1} \qquad [\because \quad u_{i,1} = u_{i,-1}]$$

or $\quad 2u_{i,\,1} = u_{i+1,\,0} + u_{i-1,\,0}$

or
$$u_{i,\,1} = \frac{1}{2}[u_{i+1,0} + u_{i-1,0}] \tag{45}$$

Putting $i = 1, 2, 3, 4, 5, 6$ in Eq. (45), we obtain

$$u_{1,1} = \frac{1}{2}[u_{2,0} + u_{0,0}] = \frac{1}{2}[0.080 + 0] = 0.040$$

$$u_{2,1} = \frac{1}{2}[u_{3,0} + u_{1,0}] = \frac{1}{2}[0.105 + 0.045] = 0.075$$

$$u_{3,1} = \frac{1}{2}[u_{4,0} + u_{2,0}] = \frac{1}{2}[0.120 + 0.08] = 0.100$$

$$u_{4,1} = \frac{1}{2}[u_{5,0} + u_{3,0}] = \frac{1}{2}[0.125 + 0.105] = 0.115$$

$$u_{5,1} = \frac{1}{2}[u_{6,0} + u_{4,0}] = \frac{1}{2}[0.120 + 0.120] = 0.120$$

$$u_{6,1} = \frac{1}{2}[u_{7,0} + u_{5,0}] = \frac{1}{2}[0.105 + 0.125] = 0.115$$

Putting $j = 1$ in Eq. (44), we get

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0} \tag{46}$$

Now put $i = 1, 2, 3, 4, 5$ in Eq. (46), we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 0.075 - 0.045 = 0.03$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 0.040 + 0.100 - 0.08 = 0.060$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 0.075 + 0.115 - 0.105 = 0.085$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 0.100 + 0.120 - 0.120 = 0.100$$

$$u_{5,2} = u_{4,1} + u_{6,1} - u_{5,0} = 0.115 + 0.115 - 0.125 = 0.105$$

In a similar manner we can fill up the remaining rows as shown in the following table:

		0	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**
	j \ i	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	0.045	0.080	0.105	0.120	0.125	0.120
0.1	1	0	0.040	0.075	0.100	0.115	0.120	0.115
0.2	2	0	0.030	0.060	0.085	0.100	0.105	
0.3	3	0	0.020	0.040	0.060	0.075	0.080	
0.4	4	0	0.010	0.020	0.030	0.040	0.048	

EXERCISE 12.2

1. Solve the equation $u_{tt} = 25\, u_{xx}$ for one half period of oscillation taking $h = 1$ subject to the conditions

$$u(0, t) = 0 = u(5, t);\ u_t(x, 0) = 0$$

and $$u(x, 0) = \begin{cases} 2x & \text{for } 0 \le x \le 2.5 \\ 10 - 2x & \text{for } 2.5 \le x \le 5 \end{cases}$$

2. Solve the wave equation $u_{tt} = u_{xx}$ for $x = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ and $t = 0, 0.1, 0.2$, subject to the conditions.

$$u(x, 0) = \frac{1}{8}\sin \pi x,\ u(0, t) = 0 = u(1, t),\ t \ge 0$$

and $u_t(x, 0) = 0;\ 0 \le x \le 1.$

3. Solve $16\, u_{xx} = u_{tt}$ under the conditions

$$u(0, t) = 0 = u(5, t),\ u_t(x, 0) = 0 \text{ and}$$

$$u(x, 0) = x^2(x - 5). \text{ Taking } h = 1$$

Answers

1.

j \ i	0	1	2	3	4	5
0	0	2	4	4	2	0
0.2	0	2	4	4	2	0
→ 0.4	0	2	2	2	2	0
0.6	0	0	0	0	0	0
0.8	0	–2	–2	–2	–2	0
1.0	0	–2	–4	–4	–2	0

2.

$t\downarrow$ \ $x\rightarrow$	0	0.1	0.2	0.3	0.4	0.5
0	0	0.037	0.070	0.096	0.113	0.119
0.1	0	0.031	0.059	0.082	0.096	0.101
0.2	0	0.023	0.043	0.059	0.07	0.074

3.

j \ i	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	4	12	18	16	0
2	0	8	10	10	2	0
3	0	6	6	–6	–6	0
4	0	–2	–10	–10	–8	0
5	0	–16	–18	–12	–4	0

12.7 NUMERICAL SOLUTION OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

An elliptical partial differential is of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ or } \nabla^2 u=0 \tag{47}$$

where
$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Equation (47) is called the Laplace equation.

Substituting the derivative differences in Eq. (47) from Eqs (10) and (12), we get

$$\frac{u_{i+1,\,j}-2u_{i,\,j}+u_{i-1,\,j}}{h^2}+\frac{u_{i,\,j+1}-2\,u_{i,\,j}+u_{i,\,j-1}}{k^2}=0 \tag{48}$$

Putting $h = k$ for a square mesh in Eq. (48) from the above expression, we get

$$u_{i,\,j}=\frac{1}{4}\left[u_{i+1,\,j}+u_{i-1,\,j}+u_{i,\,j+1}+u_{i,\,j-1}\right] \tag{49}$$

Equation (49) shows that, the value of u is the average of its values at the four neighbouring mesh points to the left, right, below and above. This formula is known as the standard five point formula (SFPF) and it is represented in the Fig. (12.11).

Now, if we rotate the co-ordinate axes through 45°, then the Laplace equation remains unaltered. Therefore, we may use the values at the diagonal points in place of the neighbouring points. Then the formula in Eq. (49) can be written as

$$u_{i,\,j}=\frac{1}{4}\left[u_{i-1,\,j-1}+u_{i+1,\,j-1}+u_{i+1,\,j+1}+u_{i-1,\,j+1}\right] \tag{50}$$

Equation (50) shows that, the value of u is the average of its values at the 4 neighbouring diagonal mesh points.

This formula is known as the diagonal five point formula (DFPF) and it is represented in Fig. 12.12.

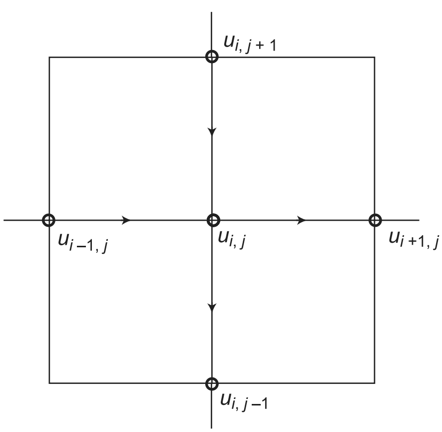


Fig. 12.11 *Standard five point formula*

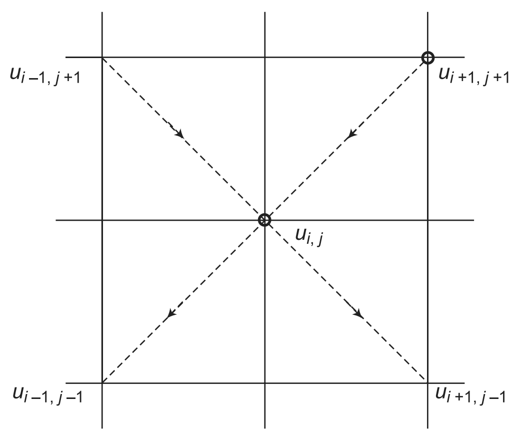


Fig. 12.12 *Diagonal five point formula*

The problems concerning study of Poisson's equation, equilibrium stress in elastic structures, viscous flow, etc., are the elliptic type of equations.

The accuracy of the values of $u_{i,\,j}$ (which are obtained from the Eqs (49) and (50) is improved by either of the following iterative methods.

12.7.1 Point Jacobi's Method

Let $u_{i,\,j}^{(n)}$ be the n^{th} iterative value of $u_{i,\,j}$. Then the iterative formula to solve (49) is

$$u_{i,\,j}^{(n+1)}=\frac{1}{4}\left[u_{i+1,\,j}^{(n)}+u_{i-1,\,j}^{(n)}+u_{i,\,j+1}^{(n)}+u_{i,\,j-1}^{(n)}\right] \tag{51}$$

for the interior mesh points. This procedure is known as the point Jacobi's method.

12.7.2 Gauss–Seidal Method

The iterative formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)}\right] \tag{52}$$

It uses the latest iterative values available and scans the mesh points systematically from left to right along successive rows.

It can be shown that the Gauss–Seidel scheme converges twice as fast as the Jacobi's method.

***Note 1*:** The accuracy of computations depends on the mesh-size; if mesh-size h is small then accuracy is better. But if h is too small; it may increase rounding-off error's.

***Note 2*:** We iterate all the mesh or grid points systematically from left to right along successive rows by the iterative formula (which obtain from Eq. (49) is given as:

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$$

This formula is called *Liebmann's iterative formula*.

Example 6 Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with the boundary values as shown in the Fig. 12.13. Iterate until the difference between two successive values at any point is less than 0.001.

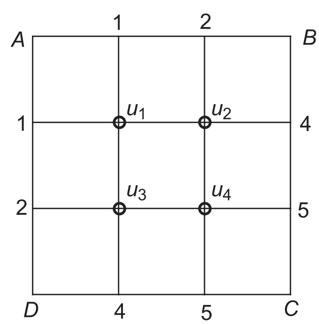


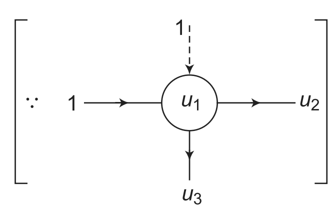
Fig. 12.13

Solution Since the given Fig. 12.13 is symmetrical about the line AC.

Therefore $u_2 = u_3$

Let us assume $u_2 = 0$

Now using standard five point formula in Eq. (49), we get



$$u_1^{(0)} = \frac{1}{4}[1+0+1+0] = 0.5$$

$$u_4^{(0)} = \frac{1}{4}[0+5+0+5] = 2.5$$

$$u_2^{(0)} = \frac{1}{4}[0.5+2+4+2.5] = 2.25$$

Since $u_3^{(0)} = u_2^{(0)} = 2.25$

Now, using the iterative scheme (51), we get the equations u_1, u_2 and u_4 as:

$$u_1^{(n+1)} = \frac{1}{4}\left[u_2^{(n)} + 1 + 1 + u_3^{(n)}\right]$$

$$= \frac{1}{4}\left[u_2^{(n)} + 2 + u_2^{(n)}\right] \qquad \left[\because \; u_3^{(n)} = u_2^{(n)}\right]$$

$$u_1^{(n+1)} = \frac{1}{2}\left[1 + u_2^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[4 + u_1^{(n+1)} + 2 + u_4^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[6 + u_1^{(n+1)} + u_4^{(n)}\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[5 + u_3^{(n+1)} + u_2^{(n+1)} + 5\right]$$

$$= \frac{1}{2}\left[5 + u_2^{(n+1)}\right] \qquad \left[\because \; u_3 = u_2\right]$$

Iteration-1 (For $n = 0$)

$$u_1^{(1)} = \frac{1}{2}\left[1 + u_2^{(0)}\right] = \frac{1}{2}[1 + 2.25] = 1.625$$

$$u_2^{(1)} = \frac{1}{4}[6 + 1.625 + 2.5] = 2.53125$$

$$u_4^{(1)} = \frac{1}{2}\left[5 + u_2^{(1)}\right] = \frac{1}{2}[5 + 2.53125] = 3.765625$$

Iteration-2 (For $n = 1$)

$$u_1^{(2)} = \frac{1}{2}\left[1 + u_2^{(1)}\right] = \frac{1}{2}[1 + 2.53125] = 1.765625$$

$$u_2^{(2)} = \frac{1}{4}\left[6 + u_2^{(1)} + u_4^{(1)}\right] = \frac{1}{4}[6 + 1.765625 + 3.765625] = 2.8828125$$

$$u_4^{(2)} = \frac{1}{2}[2.8828125 + 5] = 3.9414063$$

Iteration-3 (For $n = 2$)

$$u_1^{(3)} = \frac{1}{2}[1 + 2.8828125] = 1.9414063$$

$$u_2^{(3)} = \frac{1}{4}[6 + 1.9414063 + 3.9414063] = 2.9707031$$

$$u_4^{(3)} = \frac{1}{2}[5 + 2.9707031] = 3.9853516$$

Iteration-4 (For $n = 3$)

$$u_1^{(4)} = \frac{1}{2}[1 + 2.9707031] = 1.9853516$$

$$u_2^{(4)} = \frac{1}{4}[6 + 1.9853516 + 3.9853516] = 2.9926758$$

$$u_4^{(4)} = \frac{1}{2}[5 + 2.9926758] = 3.9963379$$

Iteration-5 (For $n = 4$)

$$u_1^{(5)} = \frac{1}{2}[1 + 2.9926578] = 1.9963289$$

$$u_2^{(5)} = \frac{1}{4}[6 + 1.9963289 + 3.9963379] = 2.9981667$$

$$u_4^{(5)} = \frac{1}{2}[5 + 2.9981667] = 3.9990834$$

Iteration-6 (For $n = 5$)

$$u_1^{(6)} = \frac{1}{2}[1 + 2.9981667] = 1.9990834$$

$$u_2^{(6)} = \frac{1}{4}[6 + 1.9990834 + 3.9990834] = 2.9995417$$

$$u_4^{(6)} = \frac{1}{2}[5 + 2.9995417] = 3.9997709$$

Thus, $u_1^{(6)} = u_1^{(5)} \Rightarrow u_1 = 1.999$

$u_2^{(6)} = u_2^{(5)} \Rightarrow u_2 = 2.999$

and $u_4^{(6)} = u_4^{(5)} \Rightarrow u_4 = 3.999$

Example 7 Using Gauss–Seidel method to solve the Laplace equation $\nabla^2 u = 0$ in the domain of the following Fig. 12.14.

Solution

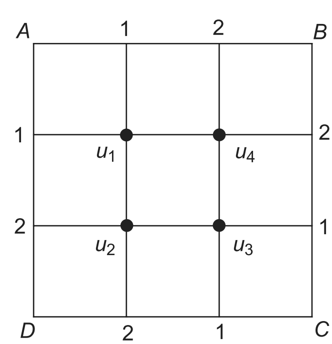


Fig. 12.14

Solution

Initially $u_1^{(0)} = u_2^{(0)} = u_3^{(0)} = u_4^{(0)} = 0$

Now, using the iterative scheme (52), we obtain the following iterations:

Iteration-1 (for $n = 0$)

$$u_1^{(1)} = \frac{1}{4}[1+0+1+0] = 0.5$$

$$u_2^{(1)} = \frac{1}{4}[2+0+2+0.5] = 1.125$$

$$u_3^{(1)} = \frac{1}{4}[1+1.125+1+0] = 0.781$$

$$u_4^{(1)} = \frac{1}{4}[2+0.781+2+0.5] = 1.320$$

Iteration-2 (for $n = 1$)

$$u_1^{(2)} = \frac{1}{4}[1+1+1.125+1.320] = 1.111$$

$$u_2^{(2)} = \frac{1}{4}[2+2+1.111+0.781] = 1.473$$

$$u_3^{(2)} = \frac{1}{4}[1+1+1.473+1.320] = 1.198$$

$$u_4^{(2)} = \frac{1}{4}[2+2+1.111+1.198] = 1.577$$

Iteration-3 (for $n = 2$)

$$u_1^{(3)} = \frac{1}{4}[1+1+1.473+1.577] = 1.263$$

$$u_2^{(3)} = \frac{1}{4}[2+2+1.263+1.198] = 1.615$$

$$u_3^{(3)} = \frac{1}{4}[1+1+1.615+1.577] = 1.298$$

$$u_4^{(3)} = \frac{1}{4}[2+2+1.263+1.298] = 1.640$$

Iteration-4 (for $n = 3$)

$$u_1^{(4)} = \frac{1}{4}[1+1+1.615+1.640] = 1.314$$

$$u_2^{(4)} = \frac{1}{4}[2+2+1.314+1.298] = 1.653$$

$$u_3^{(4)} = \frac{1}{4}[1+1+1.653+1.640] = 1.323$$

$$u_4^{(4)} = \frac{1}{4}[2+2+1.314+1.323] = 1.659$$

Iteration-5 (for $n = 4$)

$$u_1^{(5)} = \frac{1}{4}[1+1+1.653+1.659] = 1.328$$

$$u_2^{(5)} = \frac{1}{4}[2+2+1.328+1.323] = 1.663$$

$$u_3^{(5)} = \frac{1}{4}[1+1+1.663+1.659] = 1.331$$

$$u_4^{(5)} = \frac{1}{4}[2+2+1.328+1.331] = 1.665$$

Iteration-6 (for $n = 5$)

$$u_1^{(6)} = \frac{1}{4}[1+1+1.663+1.665] = 1.333$$

$$u_2^{(6)} = \frac{1}{4}[2+2+1.332+1.331] = 1.666$$

$$u_3^{(6)} = \frac{1}{4}[1+1+1.666+1.665] = 1.333$$

$$u_4^{(6)} = \frac{1}{4}[2+2+1.332+1.333] = 1.666$$

Since, Iteration-6 is approximately equal to iteration-5.

Hence, $u_1 = 1.333$, $u_2 = 1.666$, $u_3 = 1.333$ and $u_4 = 1.666$.

Example 8 Using the iterative scheme to solve the Laplace equation $\nabla^2 u = 0$, find u_1, u_2, u_3 and u_4 in the Fig. 12.15.

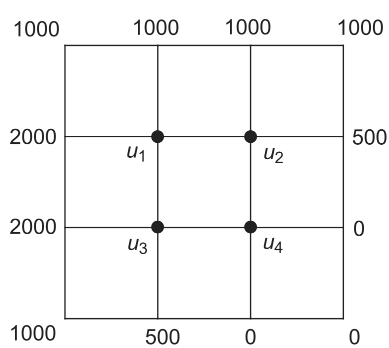


Fig. 12.15

Solution To find the initial values of u_1, u_2, u_3 and u_4, we assume $u_4 = 0$.

Then; using diagonal formula in Eq. (50), we get

$$u_1 = \frac{1}{4}[1000+0+1000+2000] = 1000$$

and using standard point formula, we have

$$u_2 = \frac{1}{4}[1000+500+1000+0] = 625$$

$$u_3 = \frac{1}{4}[2000+0+1000+500] = 875$$

$$u_4 = \frac{1}{4}[875+0+625+0] = 375$$

Now, using the successive Gauss–Seidel iterative method (52), we have

$$u_1^{(n+1)} = \frac{1}{4}\left[2000+u_2^{(n)}+1000+u_3^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)}+500+1000+u_4^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[2000+u_4^{(n)}+u_1^{(n+1)}+500\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[u_3^{(n+1)}+0+u_2^{(n+1)}+0\right]$$

Iteration-1 (for $n = 0$)

$$u_1^{(1)} = \frac{1}{4}[2000+625+1000+875] = 1125$$

$$u_2^{(1)} = \frac{1}{4}[1125+500+1000+375] = 750$$

$$u_3^{(1)} = \frac{1}{4}[2000+375+1125+500] = 1000$$

$$u_4^{(1)} = \frac{1}{4}[1000+0+750+0] = 438$$

Iteration-2 ($n = 1$)

$$u_1^{(2)} = \frac{1}{4}[2000+750+1000+1000] = 1188$$

$$u_2^{(2)} = \frac{1}{4}[1188+500+1000+438] = 782$$

$$u_3^{(2)} = \frac{1}{4}[2000+438+1188+500] = 1032$$

$$u_4^{(2)} = \frac{1}{4}[1032+0+782+0] = 454$$

Similarly,

Iteration-3 (for $n = 2$)

$$u_1^{(3)} = 1204, u_2^{(3)} = 789, u_3^{(3)} = 1040, u_4^{(3)} = 458$$

Iteration-4 (for $n = 3$)

$$u_1^{(4)} = 1207, u_2^{(4)} = 791, u_3^{(4)} = 1041, u_4^{(4)} = 458$$

and *Iteration*-5 (for $n = 4$)

$$u_1^{(5)} = 1208, u_2^{(5)} = 791.50, u_3^{(5)} = 1041.50, u_4^{(5)} = 458.25$$

Thus, there is a very small difference between the 4$^{\text{th}}$ and 5$^{\text{th}}$ iteration values.

$\therefore$ $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$.

EXERCISE 12.3

1. Solve $u_{xx} + u_{yy} = 0$ in $0 \le x \le 4, 0 \le y \le 4$ given that $u(0, y) = 0; u(4, y) = 8 + 2y;$ $u(x, 0) = \dfrac{x^2}{2}$ and $u(x, 4) = x^2$. Take $h = k = 1$ and obtain the result correct to two decimal places.
2. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in the Fig. 12.16:

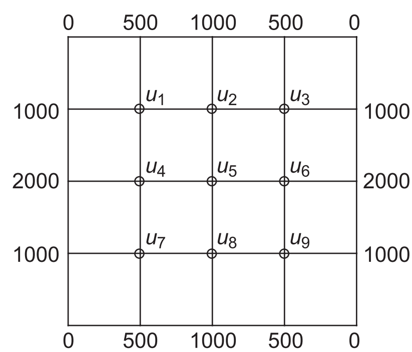


Fig. 12.16

3. Solve $u_{xx} + u_{yy} = 0$ for the following square meshes Figures (12.17 a and b) with boundary conditions.

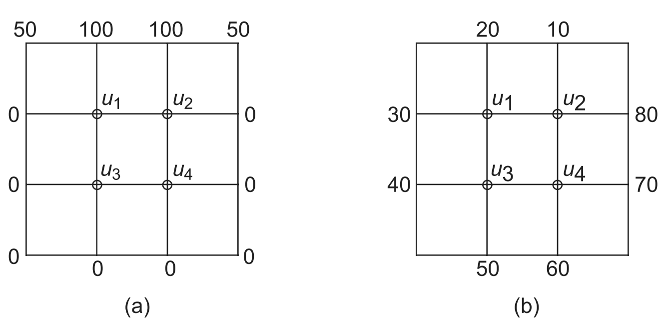


Fig. 12.17

4. Solve $u_{xx} + u_{yy} = 8x^2y^2$ for square mesh given $u = 0$ on the four boundaries dividing the square in to 16 sub-squares of length one unit.
5. The function u satisfies Laplace's equation at all points within the square in the following Fig. 12.18 and has the boundary values indicated. Compute a solution correct to two decimal places.

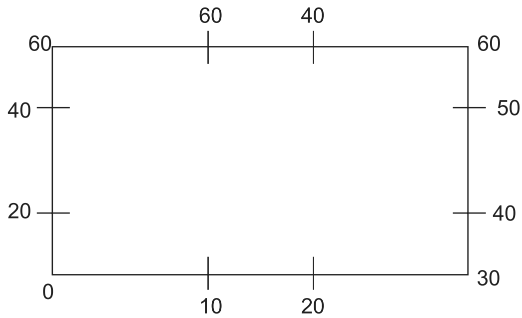


Fig. 12.18

Answers

1. $u_1 = 1.99$, $u_2 = 4.91$, $u_3 = 8.99$, $u_4 = 2.06$, $u_5 = 4.69$, $u_6 = 8.06$, $u_7 = 1.57$, $u_8 = 3.71$ and $u_9 = 6.57$.
2. $u_1 = 939$, $u_2 = 1001$, $u_3 = 939$, $u_4 = 1251$, $u_5 = 1126$, $u_6 = 1251$, $u_7 = 939$, u_8, 1001, $u_9 = 939$.
3. (a) $u_1 = 37.5$, $u_4 = 37.5$, $u_2 = 12.5$, $u_3 = 12.5$
   (b) $u_1 = 34.986$, $u_2 = 44.993$, $u_3 = 44.993$, $u_4 = 54.996$.
4. –3, –2, –3, –2, –2, –2, –3, –2, –3.
5. $u_1 = 26.66$, $u_2 = 33.33$, $u_3 = 43.33$ and $u_4 = 46.66$.

12.8 SOLUTION OF POISSON'S EQUATION

The elliptic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \text{ or } \nabla^2 u = f(x, y), \tag{53}$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

where $f(x, y)$ is a given function of x and y. Eq. (53) is called the *Poisson's equation*.

Equation (53) can be solved by replacing the derivatives by differences expressions at the points $x = ih$ and $y = jh$.

$$\therefore \quad \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j+1}}{h^2} + O(h^2) + O(h^2) = f(ih, jh)$$

Neglecting the error term, we get

$$u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = h^2 f(ih, jh) \tag{54}$$

Using Eq. (54) at each interior mesh point, we obtain the linear equations in the nodal values $u_{i,j}$ and these equations can be solved by Gauss–Seidel scheme.

Example 9 Solve the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10\,(x^2 + y^2 + 10)$$

over the square with sides $x = 0$, $y = 0$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length is 1.

Solution Here $x = 0 = y$ and $x = 3 = y$, also the mesh length $h = 1$. See Fig. (12.18).

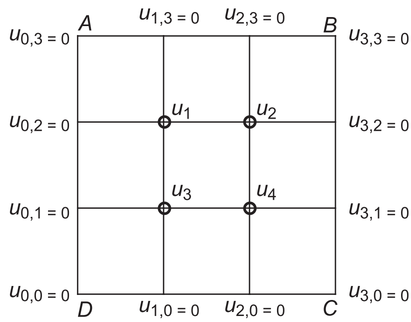


Fig. 12.19

$$f(x, y) = -10(x^2 + y^2 + 10).$$

Using standard formula Eq. (54), we have

$$u_{i+1,\,j} + u_{i-1,\,j} - 4\,u_{i,\,j} + u_{i,\,j+1} + u_{i,\,j-1} = h^2 f(ih,\,jh) \tag{55}$$

For u_1, putting $i = 1$ and $j = 2$ in Eq. (55), we get

$$u_{2,\,2} + u_{0,\,2} - 4u_{1,\,2} + u_{1,\,3} + u_{1,\,1} = f(1,\,2)$$

or $$u_2 + 0 - 4u_1 + 0 + u_3 = -10(1 + 4 + 10)$$

or $$u_1 = \frac{1}{4}\left[u_2 + u_3 + 150\right]$$

Now, for u_2, putting $i = 2 = j$ in Eq. (55), we get

$$u_{3,\,2} + u_{1,\,2} - 4u_{2,\,2} + u_{2,\,3} + u_{2,\,1} = f(2,\,2)$$

or $$0 + u_4 - 4u_2 + 0 + u_4 = -180$$

or $$u_2 = \frac{1}{4}\left[u_1 + u_4 + 180\right]$$

For u_3, putting $i = 1 = j$ in Eq. (55), we get

$$u_{2,\,1} + u_{0,\,1} - 4u_{1,\,1} + u_{1,\,2} + u_{1,\,0} = f(1,\,1)$$

or $$u_3 = \frac{1}{4}\left[u_1 + u_4 + 120\right]$$

For u_4, putting $i = 2$ and $j = 1$ in Eq. (55) we get

$$u_4 = \frac{1}{4}\left[u_2 + u_3 + 150\right]$$

Since $u_1 = u_4$

$$\therefore \quad u_1 = \frac{1}{4}\left[u_2 + u_3 + 150\right]$$

$$u_2 = \frac{1}{4}\left[u_1 + u_4 + 180\right] = \frac{1}{2}[u_1 + 90]$$

$$u_3 = \frac{1}{4}\left[u_1 + u_4 + 120\right] = \frac{1}{2}[u_1 + 60]$$

To improve the values of u by Gauss–Siedel method.
Let $u_2 = 0 = u_3$
Iteration-1

$$u_1^{(1)} = \frac{1}{4}\left[u_2 + u_3 + 150\right] = \frac{1}{4}[0 + 0 + 150] = 37.5$$

$$u_2^{(1)} = \frac{1}{2}[37.5 + 90] = 63.75$$

$$u_3^{(1)} = \frac{1}{2}[37.5 + 60] = 48.75$$

Iteration-2

$$u_1^{(2)} = \frac{1}{4}\left[u_2^{(1)} + u_3^{(1)} + 150\right] = \frac{1}{4}\left[63.75 + 48.75 + 150\right] = 65.625$$

$$u_2^{(2)} = \frac{1}{2}\left[u_1^{(2)} + 90\right] = \frac{1}{2}\left[65.625 + 90\right] = 77.8125$$

$$u_3^{(2)} = \frac{1}{2}\left[65.625 + 60\right] = 62.8125$$

Iteration-3

$$u_1^{(3)} = \frac{1}{4}\left[u_2^{(2)} + u_3^{(2)} + 150\right] = \frac{1}{4}\left[77.8125 + 62.8125 + 150\right] = 72.65625$$

$$u_2^{(3)} = \frac{1}{2}[72.65625 + 90] = 81.328125$$

$$u_3^{(3)} = \frac{1}{2}[72.65625 + 60] = 66.32815$$

Iteration-4

$$u_1^{(4)} = \frac{1}{4}\left[u_2^{(3)} + u_3^{(3)} + 150\right] = \frac{1}{4}[81.328125 + 66.32815 + 150]$$
$$= 74.414063$$

$$u_2^{(4)} = \frac{1}{2}[74.414063 + 90] = 82.207031$$

$$u_3^{(4)} = \frac{1}{2}[74.414063 + 60] = 67.207031$$

Iteration-5

$$u_1^{(5)} = \frac{1}{4}\left[u_2^{(4)} + u_3^{(4)} + 150\right] = \frac{1}{4}[82.207031 + 67.207031 + 150]$$
$$= 74.853516$$

$$u_2^{(5)} = \frac{1}{2}[74.853516 + 90] = 82.426758$$

$$u_3^{(5)} = \frac{1}{2}[74.853516 + 60] = 67.426758$$

Iteration-6

$$u_1^{(6)} = \frac{1}{4}[82.426758 + 67.426758 + 150] = 74.963379$$

$$u_2^{(6)} = \frac{1}{2}[74.963379 + 90] = 82.481689$$

$$u_3^{(6)} = \frac{1}{2}[74.963379 + 60] = 67.481689$$

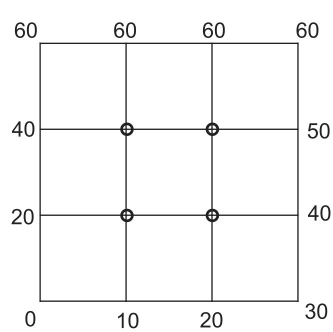
Since, on the iterations 5^{th} and 6^{th}, the values are nearly same and the required solution is

$$u_1 = 74.9,\ u_2 = 82.5 \text{ and } u_3 = 67.5$$

$$\therefore \qquad u_4 = u_1 = 74.9$$

EXERCISE 12.4

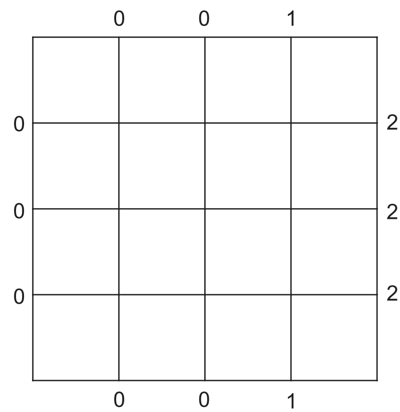
1. Using the successive iteration scheme to solve the Laplace equation $\nabla^2 u = 0$ for the following figure: **[V.T.U. 2000]**



2. Solve the Laplace equation $\nabla^2 u = 0$ with the conditions $u(0, y) = 0$, $u(4, y) = 12 + y$ for $0 \le y \le 4$; $u(x, 0) = 3x$ and $u(x, 4) = x^2$ for $0 \le x \le 4$.
3. Using Bendre–Schmidt's scheme to solve the equation $u_t = 4u_{xx}$ with the boundary conditions $u(0, t) = u(8, t) = 0$ and $u(x, 0) = \frac{1}{2}x(8 - x)$ at the points

$$x = i;\ i = 0, 1, 2, 3, \ldots, 8 \text{ and } t = \frac{1}{8}j;\ j = 0, 1, 2, 3, 4, 5.$$

4. Solve $\nabla^2 u = 0$ with the boundary values as shown in the figure given below:



5. Using Bendre–Schmidt method to solve the boundary value problem $u_t = u_{xx}$, under the conditions $u(0, t) = 0 = u(1, t)$ and $u(x, 0) = \sin \pi x$; $0 \le x \le 1$ (Take $\Delta x = h = 0.2$ and $\Delta t = k = 0.02$). **[Madras 1997, Rohtak 2003, V.T.U. 2003]**
6. The transverse displacement u of a point at a distance x from one end at any time t of a vibrating string satisfies the partial differential equation $u_{tt} = 25\, u_{xx}$, under the conditions $u(0, t) = 0 = u(5, t)$ and the initial conditions $u(x, 0) = \begin{cases} 20\,x & \text{for } 0 \le x \le 1 \\ 5(5 - x) & \text{for } 1 \le x \le 5 \end{cases}$ and $u_t(x, 0) = 0$. (Take $h = 1$, $k = 0.2$).
7. Explain the nature of parabolic, hyperbolic and elliptic equations.
8. Define the finite difference scheme.
9. What is the difference between Jacobi's and Gauss–Seidel methods.
10. Define the boundary and initial value problems.

Answers

1. $u_1 = 26.66$, $u_2 = 33.33$, $u_3 = 43.33$ and $u_4 = 46.66$.
2. 2.37, 5.60, 9.87, 2.89, 6.14, 9.89, 3.02, 6.17 and 9.51.

3.

$j\downarrow$ \ $i\rightarrow$	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.62	6	6.5	6	4.62	2.5	0
4	0	2.31	4.25	5.56	6	5.56	4.25	2.31	0
5	0	2.12	3.94	5.12	5.56	5.12	3.94	2.12	0

4. $u_1 = 10.188$, $u_2 = 0.5$, $u_3 = 1.188$, $u_4 = 0.25$, $u_5 = 0.625$, $u_6 = 1.25$

5.

$j\downarrow$ \ $i\rightarrow$	0	1	2	3	4	5
0	0	0.59	0.95	0.95	0.59	0
1	0	0.475	0.77	0.77	0.475	0
2	0	0.38	0.62	0.62	0.38	0
3	0	0.31	0.50	0.50	0.31	0
4	0	0.25	0.41	0.41	0.25	0
5	0	0.20	0.33	0.33	0.20	0

6.

$j\downarrow$ \ $i\rightarrow$	0	1	2	3	4	5
0	0	20	15	10	5	0
1	0	7.5	15	10	5	0
2	0	–5	2.5	10	5	0
3	0	–5	–10	–2.5	5	0
4	0	–5	–10	–15	–7.5	0
5	0	–5	–10	–15	–20	0

SUMMARY

Following topics have been discussed in this chapter:

1. Classification of Partial Differential Equation

Let $u(x, y)$ be a continuous function which maps R^2 in to R. The general second order partial differential equation is of the form

$$\phi\left(x, y, z, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0 \text{ or}$$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

where A, B, C, D, E, F and G are all functions of x and y. Above equation can be classified into three types according to the numerical value of the discriminant $\Delta = B^2 - 4\,AC$.

(i) If $\Delta < 0$ at a point in the (x, y) plane, then the given equation is called elliptic.
(ii) If $\Delta = 0$ at a point in the (x, y) plane, then the given equation is called parabolic.
(iii) If $\Delta > 0$ at a point in the (x, y) plane, then the given equation is called hyperbolic.

2. Finite Difference Method

The approximations to the partial derivatives for the function $u = u(x, y)$ at (x_i, y_i), for $i = 1, 2, 3, \ldots.$ and $u(x_i, y_i) = u_{ij}$.

Therefore

$$\left(\frac{\partial u}{\partial x}\right)_{(x_i, y_i)} = \frac{u_{i+1, j} - u_{i-1, j}}{2h}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i, y_i)} = \frac{u_{i+1, j} - 2u_{i, j} + u_{i-1, j}}{h^2}$$

Similarly, the other first and second order partial derivatives are as follows:

$$\left(\frac{\partial u}{\partial y}\right)_{(x_i, y_i)} = \frac{u_{i, j+1} - u_{i, j-1}}{2k}$$

and

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{(x_i, y_i)} = \frac{u_{i, j+1} - 2u_{i, j} + u_{i, j-1}}{k^2}$$

Now using the given equations give in partial differential equations and these equations converted to difference equations. The solution of difference equations is the solution of the given partial differential equations.

3. Parabolic Partial Differential Equations

The parabolic partial differential equation, we consider is the heat, or diffusion equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; 0 < x < l, t > 0$$

Subject to the conditions

$$u(0, t) = 0 = u(l, t), t < 0$$

and

$$u(x, 0) = f(x), 0 \le x \le l$$

(i) *Explicit Scheme (Finite Difference Method)* Substituting the finite difference relations, we get

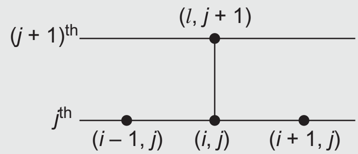
$$\frac{u_{i,\,j+1}-u_{i,\,j}}{\Delta t}=\alpha^2\,\frac{u_{i+1,\,j}-2u_{i,\,j}+u_{i-1,\,j}}{(\Delta x)^2}\qquad\left[\begin{aligned}\text{Here } \Delta t&=k\\ \Delta x&=h\end{aligned}\right]$$

or $$u_{i,\,j+1}=\alpha^2\frac{\Delta t}{(\Delta x)^2}(u_{i+1,\,j}+u_{i-1,\,j})+\left(1-2\alpha^2\frac{\Delta t}{(\Delta x)^2}\right)u_{ij}$$

or $$u_{i,\,j+1}=r\,(u_{i+1,\,j}+u_{i-1,\,j})+(1-2r)\,u_{i,\,j}$$

where $$r=\alpha^2\frac{\Delta t}{(\Delta x)^2};\quad j=0,1,2,3,\ldots$$

$$i=1,2,3,\ldots,(N-1)$$



Computational molecule or Forward difference Stencil for the finite difference method.

The derived equation is valid when $0<r\le\frac{1}{2}$. This is an explicit scheme is expressed u at unknown $(j+1)^{\text{th}}$ time level in terms of the known values of u at j^{th} time level.

For $r=\frac{1}{2}$, so the derived equation reduces to

$$u_{i,\,j+1}=\frac{1}{2}[u_{i+1,\,j}+u_{i-1,\,j}]$$

Above equation is called the *Bender–Schmidt recurrence equation/scheme*.

(ii) *Implicit Scheme* In the explicit scheme, the time step (Δt) has to be necessarily very small. The explicity scheme is valid only for $0\le r\le\frac{1}{2}$, i.e., $r=\alpha^2\frac{\Delta t}{(\Delta x)^2}$.

Now, the partial differential is discretized by replacing the time derivative by backward difference and the space second derivative by a central difference at the node $(i, j+1)$, we have

$$(1+2r)u_{i,\,j+1}-r\,u_{i-1,\,j+1}-r\,u_{i+1,\,j+1}=u_{i,\,j}$$

where $$r=\alpha^2\frac{\Delta t}{(\Delta x)^2}$$

From the given equation, all the values of u on left are unknown and on the right $u_{i,\,j}$ is known.

(iii) *Implicit Scheme Second or Crank–Nicolson Scheme or Method* This scheme is based on numerical approximations for the solution at the node $(i, j+1)$.

$$-r\,u_{i-1,\,j+1}+(2+2r)\,u_{i,\,j+1}-r\,u_{i+1,\,j+1}=b_{i,\,j}$$

where $$b_{i,j} = r\,u_{i-1,j} + (2-2r)u_{i,j} + r\,u_{i+1,j}$$

$$j = 0, 1, 2, 3, \ldots$$

$$i = 1, 2, 3, \ldots, (N-1).$$

For $r = 1$, the scheme, i.e. the derived equation gives

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = b_{i,j}$$

where $$b_{i,j} = (u_{i-1,j} + u_{i+1,j})$$

for $$i = 1, 2, 3, \ldots (N-1),\ j = 1, 2, 3, \ldots$$

This scheme is known as *Crank–Nicolson Scheme*.

(iv) *Three Time Level Scheme or Richardson's Scheme or Method* In the partial differential, we discretize at the node (i, j) by replacing both the derivatives by a central difference taking Δt as one time interval.

Thus we get,

$$\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t} = \frac{\alpha^2}{(\Delta x)^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

or $$u_{i,j+1} = 2r\,[u_{i-1,j} - u_{i,j} + u_{i+1,j}] + u_{i,j-1},$$

where $$r = \frac{\alpha^2 \Delta t}{(\Delta x)^2};\ i = 1, 2, 3, \ldots, (N-1)$$

$$j = 0, 1, 2, 3, \ldots$$

This scheme is a three time level scheme in which the unknown $u_{i,j+1}$ is explicitly expressed in terms of known values of u at the previous known $(j-1)^{\text{th}}$ and time level's.

4. Solution of Hyperbolic Equations

In this section, we shall discuss the numerical solution to the wave equation, an example of a hyperbolic partial differential equation.

Consider, 1-D wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2};\ 0 \le x \le l,\ t > 0$$

Subject to the conditions

$$u(0, t) = u(l, t) = 0 \text{ for } t > 0$$

$u(x, 0) = f(x)$ and $u_t(x, 0) = \dfrac{\partial u}{\partial t}(x, 0) = g(x)$ for $0 \le x \le l$, where α is a constant dependent on the physical conditions of the given problem.

The finite differences of

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2\,u_{i,j} + u_{i,j-1}}{(\Delta t)^2} + O(\Delta t)^2$$

and $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1\,j} - 2\,u_{i,\,j} + u_{i-1,\,j}}{(\Delta x)^2} + O(\Delta x)^2$$

Using the above finite differences, the equation becomes

$$u_{i,\,j+1} - 2u_{i,\,j} + u_{i,\,j-1} = r^2\left[u_{i+1,\,j} - 2\,u_{i,\,j} + u_{i-1,\,j}\right]$$

where $$r = \frac{\alpha \Delta t}{\Delta x}$$

Consider the approximations in both rows j and $j - 1$ are known.

The above equation can be used to compute $u_{i,\,j+1}$ for $i = 1, 2, 3, 4, \ldots, N - 1$ and $j = 1, 2, 3, \ldots$

$$u_{i,\,j+1} = 2\,(1 - r^2)\,u_{i,\,j} + r^2\,(u_{i+1,\,j} + u_{i-1,\,j}) - (u_{i,\,j-1})$$

for $i = 1, 2, 3, 4, \ldots, N - 1$, $j = 1, 2, 3, \ldots$. The boundary conditions gives $u_{0,\,j} = 0 = u_{N,\,j}$ for each $j = 1, 2, 3, \ldots$

and the initial condition implies that

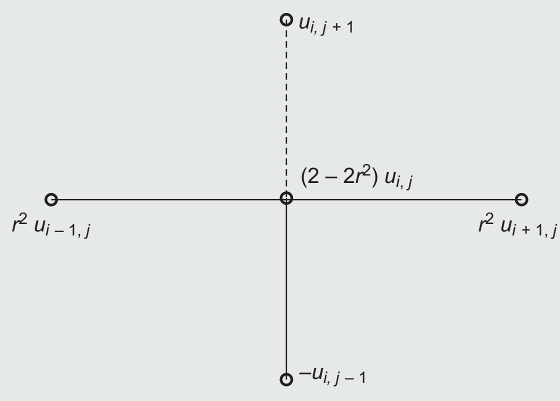
$$u_{i,0} = f(x_i) \text{ for each } i = 1, 2, \ldots, N - 1$$

To find the initial velocity condition

$$\frac{\partial u}{\partial t}(x, 0) = g(x);\ 0 \le x \le l$$

The partial derivative $\frac{\partial u}{\partial t}$ is replaced by a forward difference approximation $\frac{\partial u}{\partial t} = \frac{u_{i,\,j+1} - u_{i,\,0}}{\Delta t}$, we obtain $u_{i,1} = u_{i,0} + g(x_i)$ for $i = 1, 2, \ldots, N - 1$

We observe that the 4 known values on the RHS of given equation, which are used to find $u_{i,\,j+1}$ can be shown in the given figure.



***Note*:** If $r = 1$, then (25) becomes

$$u_{i,\,j+1} = u_{i+1,\,j} + u_{i-1,\,j} - u_{i,\,j-1}$$

5. Numerical Solution of Elliptic Partial Differential Equations

An elliptical partial differential is of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ or } \nabla^2 u=0$$

where $$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Above equation is called the Laplace equation.

Substituting the derivative differences in given equation, we get

$$\frac{u_{i+1,\,j}-2u_{i,\,j}+u_{i-1,\,j}}{h^2}+\frac{u_{i,\,j+1}-2\,u_{i,\,j}+u_{i,\,j-1}}{k^2}=0$$

Putting $h = k$ for a square mesh from the above expression, we get

$$u_{i,\,j}=\frac{1}{4}\left[u_{i+1,\,j}+u_{i-1,\,j}+u_{i,\,j+1}+u_{i,\,j-1}\right]$$

Above equation (3) shows that, the value of u is the average of its values at the four neighbouring mesh points to the left, right, below and above. This formula is known as the standard five point formula (SFPF).

Now, if we rotate the co-ordinate axes through 45°, then the Laplace equation remains unaltered. Therefore, we may use the values at the diagonal points in place of the neighbouring points. Then the formula can be written as

$$u_{i,\,j}=\frac{1}{4}\left[u_{i-1,\,j-1}+u_{i+1,\,j-1}+u_{i+1,\,j+1}+u_{i-1,\,j+1}\right]$$

Above formula shows that, the value of u is the average of its values at the 4 neighbouring diagonal mesh points.

The problems concerning study of Poisson's equation, equilibrium stress in elastic structures, viscous flows etc. are the elliptic type of equations.

The accuracy of the values of $u_{i,\,j}$ (which are obtained from the given equations is improved by either of the following iterative methods.

6. Point Jacobi's Method

Let $u_{i,\,j}^{(n)}$ be the n^{th} iterative value of $u_{i,\,j}$. Then the iterative formula to solve the equation is

$$u_{i,\,j}^{(n+1)}=\frac{1}{4}\left[u_{i+1,\,j}^{(n)}+u_{i-1,\,j}^{(n)}+u_{i,\,j+1}^{(n)}+u_{i,\,j-1}^{(n)}\right]$$

for the interior mesh points. This procedure is known as the point Jacobi's method.

7. Gauss–Seidal Method

The iterative formula is

$$u_{i,\,j}^{(n+1)}=\frac{1}{4}\left[u_{i-1,\,j}^{(n+1)}+u_{i+1,\,j}^{(n)}+u_{i,\,j+1}^{(n+1)}+u_{i,\,j-1}^{(n)}\right]$$

It uses the latest iterative values available and scans the mesh points systematically from left to right along successive rows.

It can be shown that the Gauss–Seidel scheme converges twice as fast as the Jacobi's method.

***Note 1*:** The accuracy of computations depends on the mesh-size; if mesh-size h is small then accuracy is better. But if h is too small; it may increase rounding-off error's.

***Note 2*:** We iterate all the mesh or grid points systematically from left to right along successive rows by the iterative formula (which obtain from derived equation) is given as:

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$$

This formula is called *Liebmann's iterative formula.*

9. Solution of Poisson's Equation

The elliptic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \text{ or } \nabla^2 u = f(x, y),$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

where $f(x, y)$ is a given function of x and y. Above equation is called the *Poisson's equation.*

Above equation can be solved by replacing the derivatives by differences expressions at the points $x = ih$ and $y = jh$ we obtain

$$u_{i+1,j} - 4\,u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = h^2 f(ih, jh)$$

Using the above equation at each interior mesh point, we obtain the linear equations in the modal values $u_{i,j}$ and these equations can be solved by Gauss–Seidel scheme.

OBJECTIVE TYPE QUESTIONS

1. The differential equation $4_{xx} + 3u_{xy} + u_{yy} = 0$ is

(a) Elliptic (b) Hyperbolic
(c) Parabolic (d) none of these

2. The explicit scheme $u_{i,j+1} = r(u_{i+1,j} + u_{i-1,j}) + (1 - 2r)u_{i,j}$ is valid for

(a) $r = 2$ (b) $r \geq \frac{1}{2}$
(c) $r = 1$ (d) $0 < r \leq \frac{1}{2}$

3. The formula $u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right]$ is known as

(a) explicit (b) implicit
(c) Bender–Schmidt (d) none of these

4. The formula $b_{i,j} = 4u_{i,j+1} - u_{i+1,j+1} - u_{i-1,j+1}$; where $b_{i,j} = (u_{i-1,j} + u_{i+1,j})$ is called

(a) Bender–Schmidth formula
(b) Crank–Nicolson formula
(c) Standard five-point formula
(d) Diagonal five-point formula

5. The scheme

$$u_{i,j+1} = 2\frac{\alpha^2 \Delta t}{(\Delta x)^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] + u_{i,j-1}$$

is called

(a) C–N scheme
(b) Richardson's scheme
(c) Bender–Schmidt scheme
(d) None of these

6. The formula

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right]$$

is called
(a) Standard five-point formula
(b) Diagonal five-point formula
(c) C–N formula
(d) None of above

7. The formula

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right]$$

is known
(a) C–N scheme
(b) Bender–Schmidt formula
(c) diagonal 5-point formula
(d) none of these

8. The formula

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$$

is known
(a) Liebmann's iterative formula
(b) C–N scheme
(c) Richardson's scheme
(d) Gauss–Siedel formula

9. The equation $u_{xx} + u_{yy} = f(x, y)$ is known
(a) Bendre–Schmidt equation
(b) Poisson's equation
(c) Laplace equation
(d) None of the above

10. Which of the following is a step by step method
(a) Taylor's
(b) Picard's
(c) Adams–Basforth
(d) None of the above

11. As soon as a new value of a variable is found by iteration, it is used immediately in the following equations, this method is called
(a) Relaxation method
(b) Jacobi's method
(c) Gauss–Jordan method
(d) Gauss–Seidel method

12. Schmidt's finite difference scheme to solve the equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is

(a) $u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right]$

(b) $u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j} + 2u_{i,j} + u_{i-1,j}\right]$

(c) $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

(d) None of these

13. The finite difference scheme

$$u_{i,j+1} = r(u_{i+1,j} + u_{i-1,j}) + (1-2r)\,u_{i,j}$$

where $r = \alpha^2 \frac{\Delta t}{(\Delta x)^2}$ is apply on
(a) Parabolic equation
(b) Hyperbolic equation
(c) Elliptic equation
(d) Above all

14. The fast iteration method for the solution of the partial differential equation is
(a) Gauss–Seidel method
(b) Jacobi's method
(c) Successive over relaxation method
(d) None of these

15. The equation $u_{xx} + u_{yy} = -\rho(x, y)$ is called
(a) Potential equation
(b) Schrödinger's equation
(c) Laplace equation
(d) Poisson's equation

ANSWERS

1.(b)	2.(d)	3.(c)	4.(b)	5.(b)	6.(a)	7.(c)	8.(a)	9.(b)	10.(c)
11.(d)	12.(a)	13.(a)	14.(c)	15.(a)					

13 Linear Programming

13.1 INTRODUCTION

In this chapter, we shall be dealing with the problems, which are linear in nature. To solve these problems, various methods such as graphical method, simplex method, artificial variable method will be used. Duality and dual simplex method used to solve linear programming problems will be discussed here in this chapter. In the end of the chapter, transportation problem and assignment problem which are special cases of linear programming problem will also be discussed.

In 1941 G.B. Dantzig designed linear programming to solve optimization problem in which objective function and all constraints are linear. In fact, linear programming is a technique in which all constraints and objective functions are quantitative in nature. In other words, we can say that linear programming is the analysis of a problem which, having a linear function of a number of variables called objective function. This needs to be optimized (maximized or minimized), when these variables are subject to a number of constraints to the mathematical linear inequalities.

Therefore a linear programming is nothing, it is a mathematical programming in which objective function and all constraints are linear function of decision variables.

For example,

$$\begin{aligned} \text{Maximize } & z = 2x_1 + 3x_2 + 6x_3 \\ \text{Subject to } & x_1 + 4x_2 + 3x_3 \leq 15 \\ & 2x_1 + 6x_2 + 10x_3 \geq 80 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The above problem is a linear programming problem (LPP) because here the objective function z and all the constraints are linear function of variables x_1, x_2 and x_3.

13.2 GENERAL FORM OF LINEAR PROGRAMMING

Following is the general form of a linear programming problem:

Maximize or Minimize $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ (1)

Subject to constraints

$$\left.\begin{aligned} & \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n \leq \text{ or } \geq b_1 \\ & \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n \leq \text{ or } \geq b_2 \\ & \vdots \\ & \alpha m_1x_1 + \alpha m_2x_2 + \cdots + \alpha_{mn}x_n \leq \text{ or } \leq b_m \end{aligned}\right\} \quad (2)$$

$$x_1, x_2, ..., x_n \geq 0 \tag{3}$$

In above, Eq. (1), the function which is to be maximized or minimized is called *objective function*, Eq. (2) are called constraints and Eq. (3) are called non-negative restrictions (NNR), where as $x_1, x_2, \dots, x_n$ are known as decision variables.

Now question arises, how to convert a given problem in above form. This is answered in the following article.

13.3 FORMULATION OF MODEL

The most important thing for the effective usage of linear programming in real life problem needs proper and correct formulation of the model, which consists of an objective function (to be maximized or minimized), constraints and non-negativity restrictions. Following steps are required to formulate a linear programming problem.

Step 1: Identify the decision variables.

Step 2: Identify the objective of the problem. (Minimization or Maximization) underlying in it.

Step 3: Construct the objective function.

Step 4: Construct the constraints.

Step 5: Write down the non-negative restrictions, i.e., all decisions variables will have the values greater than or equal to zero.

Example 1 A company produces two products X and Y. Production of both the products require the same processes P_1 and P_2. The production of Y also results in a by-product Z at no extra cost. The product X can be sold at a profit of ₹40 per unit and Y at a profit of ₹60 per unit. Some of these by-products can be sold at a unit profit of ₹20, the remainder has to be destroyed and the destruction cost is₹10 per unit. The company gets 4 units of Z for each unit of Y produced and only up to 6 units of Z can be sold. The manufacturing times are 4 hours per unit of X on both the processes while Y takes 5 hours and 6 hours per unit on processes P_1 and P_2, respectively. Because of the product Z results from producing Y, no time is required to produce Z. The available times are 20 and 25 hours for processes P_1 and P_2, respectively. Formulate this problem as LLP to determine the quantity of X and Y which should be produced, keeping Z in mind to make the maximum total profit to the company.

Solution

Step 1: Let x_1 = number of units of product X to be produced

x_2 = number of units of product Y to produced

x_3 = number of units of product Z to produced

and x_4 = number of units of product Z to be destroyed

Step 2: Objective is to maximize the profit

Step 3: Construction of objective function

$$\text{Maximize } z = 40x_1 + 60x_2 + 20x_3 - 10x_4$$

Step 4: Construction of constraints

$$4x_1 + 5x_2 \leq 20$$

$$4x_1 + 6x_2 \leq 25$$

$$x_3 \leq 6$$

$$-4x_2 + x_3 + x_4 = 0$$

Step 5: Non-negative restrictions:

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

Combining all above steps, we get the following LPP.

Maximize $z = 40x_1 + 60x_2 + 20x_3 - 10x_4$

Subject to

$$4x_1 + 5x_2 \leq 20$$

$$4x_1 + 6x_2 \leq 25$$

$$x_3 \leq 6$$

$$-4x_2 + x_3 + x_4 = 0$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Example 2 A company produces and sells two different products under the brand names blue and white. The profit per unit on these products is ₹ 50 and ₹ 70, respectively. Both the products employ the same manufacturing process, which has a fixed total capacity of 50,000 man hours. As per the estimates of marketing research department of the company, there is a market demand for maximum 8,000 units of blue and 12,000 units of white. Subject to over all demand, the products can be sold in any possible combinations. If it takes 5 hours to produce one unit of blue and 4 hours to produce one unit of white, formulated the above as linear programming to maximize the profit of the company.

Solution Let x_1 and x_2 be the number of units of blue and white products, respectively, produced by the company. Then LPP is as follows:

Maximize $z = 50\,x_1 + 70\,x_2$

Subject to

$$5x_1 + 4x_2 \leq 50{,}000$$

$$x_1 \leq 8{,}000$$

$$x_2 \leq 12{,}000$$

$$x_1, x_2 \geq 0$$

Example 3 A company has three operational departments weaving, processing and packaging with capacity to produce three different types of clothes namely suitings, shirtings and woolens yielding a profit ₹ 10, ₹ 12 and ₹ 15 per metre, respectively. One metre suitings requires 4 minutes in weaving, 3 minutes in processing and 2 minutes in packing respectively. One metre of shirting require 5 minutes in weaving, 2 minutes in processing and 4 minutes in packing. One metre of woollens require 4 minutes in all weaving processing and packing departments. In a week, total run time of each department is 70, 50 and 90 hours; respectively, for weaving, processing and packing departments. Formulate the above as LPP to maximize the profit.

Solution The data of the above problem can be summarized as given in the following table:

Clothes	Departments			Profit (₹ per meter)
	Weaving	Processing	Packing	
Suitings	4 minutes	3 minutes	2 minutes	10
Shirtings	5 minutes	2 minutes	4 minutes	12
Woollens	4 minutes	4 minutes	4 minutes	15
Availability of time in minutes	$70 \times 60 = 4200$	$50 \times 60 = 3{,}000$	$90 \times 60 = 5400$	

Let x_1 be the metres of suitings produced, x_2 be the metres of shirtings produced and x_3 be the metres of woollens produced by company. Then the LPP is as the follows:

Maximize $z = 10x_1 + 12x_2 + 15x_3$

Subject to

$$4x_1 + 5x_2 + 4x_3 \leq 4{,}200$$
$$3x_1 + 2x_2 + 4x_3 \leq 3{,}000$$
$$2x_1 + 4x_2 + 4x_2 \leq 5{,}400$$
$$x_1, x_2, x_3 \geq 0$$

Example 4 David wants to decide the constraints of a diet which will fulfill his daily requirements of protein, fats and carbohydrates at a minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given the in following table.

Food type	Yield/Unit			Cost per unit (in ₹)
	Proteins	Fats	Carbohydrates	
I	5	4	6	60
II	6	4	5	50
III	8	7	7	80
IV	8	6	5	75
Minimum requirement	1000	350	750	

Formulate the above as LPP.

Solution Let x_1, x_2, x_3 and x_4 be the unite of food type I, II, III and IV respectively. Then the LPP is

Minimize $z = 60x_1 + 50x_2 + 80x_3 + 75x_4$

Subject to

$$5x_1 + 6x_2 + 8x_3 + 8x_4 \geq 1000$$
$$4x_1 + 4x_2 + 7x_3 + 6x_4 \geq 350$$
$$6x_1 + 5x_2 + 7x_3 + 5x_4 \geq 750$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Example 5 A furniture manufacturer wants to determine the number of tables and chains to be made by him in order to optimize the use of his available resources. These products utilize two different types of timber and he has on hand 4,000 board feet of first type and 2,500 board feet of the second type. He has 1500 man hours available for the total job. Each table and chair requires 5 and 3 board feet respectively, of the first type timber and 3 and 4 board feet respectively of the second type timber, 6 man hours are required to make a table and 5 main hours required to make a chair. The manufacturer makes a profit of ₹500 on a table and ₹400 on a chair. Formulate the above as LPP to maximize the profit.

Solution Let x_1 and x_2 be the number of tables and chains manufactured. Then LPP is as follows:

Maximize $z = 500\, x_1 + 400\, x_2$

Subject to

$$5x_1 + 3x_2 \leq 4{,}000$$
$$3x_1 + 4x_2 \leq 2{,}500$$
$$x_1, x_2 \geq 0 \text{ and integers}$$

EXERCISE 13.1

1. A company manufactures two types of sandal soaps. To manufacture these, the company requires three different inputs; we call these inputs A, B and C. The sandal soap of first type requires 30 grams of input A, 20 grams of input B and 6 grams of input C. On the other hand, one sandal soap of second type requires 25, 5 and 15 grams of inputs A, B and C respectively. The maximum availability of input A, B and C are 6000, 3000 and 3000 grams, respectively.

   The selling prices of sandal soaps of types 1 and 2 are ₹40 and ₹50 per piece, respectively. How many sandal soaps of types 1 and 2 should be manufactured within the available inputs so that the sales revenue is maximized. Assume the all soaps manufactured are sold. Formulate the LPP.
2. A man who wants to keep some hens has ₹20,000 with him. The young hens which are available for ₹1,000 each lay 5 egs per day. The old hens which are available for ₹500 lay 3 eggs per day. He has the capacity to keep 30 hens. Each egg is sold at ₹4.00. The feed for young and old hens costs ₹100 and ₹60 per week. How many young and old hens should he buy so that the profit per week is maximized?
3. The manager of a milk dairy decides that each cow should get at least 15 units, 20 units and 24 units of nutrients A, B and C daily respectively. Two varieties of feed are available. In feed of variety 1 (Variety 2), the contents of nutrients A, B and C are respectively 1(3), 2(2), 3(2) units per kg. The costs of units 1 and 2 are ₹20 and ₹30 per kg respectively. How much of feed of each variety should be purchased to feed a cow daily so that the expenditure is minimized. Formulate the above as LPP.
4. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket and a profit of ₹ 300 is made on each economy class ticket. The airlines reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class then by first class. How many tickets of each class must be sold in order to maximize profit for airlines. Formulate the problem as an LP model. **(Rohtak, 2006)**
5. A company produces two types of models M_1 and M_2. Each M_1 model requires 8 hours of grinding and 4 hours of polishing, where as each M_2 model requires 4 hours of grinding and 10 hours of polishing. The company has 4 grinders and 6 polishers. Each grinder works 80 hours a week and each polisher works 120 hours a week. Profit on models M_1 and M_2 are ₹60 and ₹80 respectively. Whatever is produced in a week is sold in the week. Formulate the above as LPP to maximize the profit.

6. Food F_1 contains 9 units of vitamin A per gram and 10 units of vitamin B per gram and cost 18 paise per gram. Food F_2 contains 12 units of vitamin A per gram and 18 units of vitamin B per gram and cost 30 paise per gram. The daily minimum requirements of vitamins A and B are 150 units and 180 units respectively, formulate the above as LP model to minimize the total cost.
7. A firm manufactures 3 producers P_1, P_2 and P_3. The profits are₹ 30,₹ 20 and₹ 40 on each unit of the products P_1, P_2 and P_3 respectively. The firm has 2 machines M_1 and M_2 and below is the required processing time in minutes for each machine on each product.

		Product		
		P_1	P_2	P_3
Machine	M_1	4	3	5
	M_2	2	2	4

Machines M_1 and M_2 have 2000 and 2500 machine minutes, respectively. The firm must manufacture 100 $P_1'^s$, 200 $P_2'^s$ and 50 $P_3'^s$ but no more 150 $P_3'^s$. Formulate the above as LPP to maximize the profit.

8. Three grades of coal A, B and C contain ash and phosphorus as impurities. In a particular industrial process, a fuel obtained by blending the above grades containing not more 25% ash and 0.03% phosphorus is required. The maximum demand of the fuel is 100 tonnes. Percentage impurities and costs of various grades of coals are shown in the table. Assuming that there is an unlimited supply of each grade of coal and there is no loss in blending. Formulate this as an LPP to minimize the cost.

Coal Grade	% ash	% phosphorus	Cost per ton in ₹
A	30	0.02	240
B	20	0.04	300
C	25	0.03	280

Answers

1. Let x_1 and x_2 be the number of sandal soaps of types 1 and 2 respectively, then LPP is
Maximize $z = 40x_1 + 50x_2$
Subject to the constraints

$$\begin{aligned} 30x_1 + 25x_2 &\le 6000 \\ 20x_1 + 5x_2 &\le 3000 \\ 6x_1 + 15x_2 &\le 3000 \\ x_1, x_2 &\ge 0 \text{ and integers} \end{aligned}$$

2. Let x_1 and x_2 be the number of young and old hens purchased, respectively, by a man then LPP is

Maximize $z = 4(5x_1 + 3x_2) - 100x_1 - 60x_2$

Subject to the constraints

$$x_1 + x_2 \leq 30$$
$$1000\, x_1 + 500\, x_2 \leq 20000$$
$$x_1, x_2 \geq 0 \text{ and integers}$$

3. Let x_1 and x_2 be the amounts of feed 1 and 2, respectively, purchased daily to feed a cow. Then LPP is
Minimize $z = 20\, x_1 + 30\, x_2$
Subject to the constraints

$$x_1 + 3x_2 \geq 15$$
$$2x_1 + 2x_2 \geq 20$$
$$3x_1 + 2x_2 \geq 24$$
$$x_1, x_2 \geq 10$$

4. Let x_1 and x_2 be the number of first class and economy class tickets, respectively. Then LPP is
Maximize $z = 400\, x_1 + 300\, x_2$
Subject to

$$x_1 + x_2 \leq 200$$
$$x_1 \geq 200,\ x_2 \geq 4x_1$$
$$x_1, x_2 \geq 0$$

5. Maximize $z = 60\, x_1 + 80\, x_2$
Subject to

$$8x_1 + 4x_2 \leq 320$$
$$4x_1 + 10x_2 \leq 720$$
$$x_1, x_2 \geq 0$$

6. Minimize $z = 18\, x_1 + 30\, x_2$
Subject to

$$9x_1 + 12x_2 \geq 150$$
$$10x_1 + 18x_2 \geq 180$$
$$x_1, x_2 \geq 0$$

7. Maximize $z = 30\, x_1 + 20\, x_2 + 40\, x_3$
Subject to

$$4x_1 + 3x_2 + 5x_3 \leq 2000$$
$$2x_1 + 2x_2 + 4x_3 \leq 2500$$
$$100 \leq x_1 \leq 150$$
$$0 \leq x_2 \leq 200$$
$$0 \leq x_3 \leq 50$$

8. Minimize $z = 240\,x_1 + 300\,x_2 + 280\,x_3$
   Subject to

$$x_1 - x_2 + 2x_3 \le 0$$
$$-x_1 + x_2 \le 0$$
$$x_1 + x_2 + x_3 \le 100$$
$$x_1, x_2, x_3 \ge 0$$

where x_1, x_2 and x_3 are tonnes of grades A, B and C respectively.

13.4 STANDARD FORM OR EQUATION FORM OF LINEAR PROGRAMMING PROBLEM

We can solve a linear programming problem if it is in standard form which is known as equation form of LPP. In this section, we shall define the standard form of a LPP and how it is to be converted in equation form if it is not in standard form.

(i) The objective function is either to be maximized or minimized, is linear in nature.
(ii) All constraints are in equations form i.e., with equal to = sign.
(iii) right hand side of all the constraints $b_i'^s$ are greater than equal to zero i.e., all $b_i'^s \ge 0$.
(iv) all decision variables are non-negative, i.e., ≥ 0.

So, a LPP can be written in standard or equation form as:

Maximize or minimize $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to the constraints

$$\alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = b_1$$
$$\alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n = b_2$$
$$\vdots$$
$$\alpha_{m1}\;x_1 + \alpha_{m2}\;x_2 + \cdots + \alpha_{mn}\;x_n = b_m$$
$$x_1, x_2, \ldots, x_n \ge 0$$

or the LLP can be written in matrix form as:

Maximize or minimize $z = c^T X$

Subject to constraints

$$AX = b$$
$$X \ge 0$$

where
$$c = (c_1, c_2, \ldots, c_n)^T$$
$$X = (x_1, x_2, \ldots, x_n)^T$$
$$b = (b_1, b_2, \ldots, b_m)^T \ge 0$$

are column vectors and $A = (\alpha_{ij})$ is an $m \times n$ matrix.

Therefore, a LPP can be written as follows:

Maximize or minimize $z = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} (x_1, x_2, \ldots, x_n)$

Subject to constraints

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{m1} & \alpha_{mn} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$x_1, x_2, ..., x_n \geq 0$$

All LPP can be converted into standard form in the following way:

(i) *Addition of a slack variable:* If an i^{th} constraint of a LPP is ≤ type then by adding a non-negative variable s_i called slack variable to the i^{th} constraint can be converted to the equation form. For example: constraint $2x_1 + 4x_2 + 3x_3 \leq 80$ can be converted into equation form by adding a variable s_1 ($s_1 \geq 0$) known slack variable as $2x_1 + 4x_2 + 3x_3 + s_1 = 80$

(ii) *Addition of a surplus variable:* If an i^{th} constraint of a LPP is ≥ type then by subtracting a non-negative variable s_i, called surplus variable to i^{th} constraint can be converted to the equation form. For example: constraint $4x_1 + 3x_2 + 5x_3 \geq 60$ can be converted into equation form by adding variable ($-s_1$), $s_1 \geq 0$ known as surplus variable as $4x_1 + 3x_2 + 5x_3 - s_1 = 60$

Example 6 onvert the following LPP in standard form.

Maximize $z = 3x_1 - 4x_2 + 6x_3$

Subject to the constraints

$$-x_1 + x_2 + 2x_3 \geq -4$$
$$x_1 - 2x_2 + 3x_3 \geq 17$$
$$2x_1 - 3x_2 = 6$$
$$x_1, x_2, x_3 \geq 0$$

Solution We have the following constraint:

$$-x_1 + x_2 + 2x_3 \geq -4 \Rightarrow x_1 - x_2 - 2x_3 \leq 4$$

Now, if we want to convert this constraint into equation then we have to add slack variable s_1 ($s_1 \geq 0$), then constraint becomes in equation form

$$x_1 - x_2 - 2x_3 + s_1 = 4$$

Similarly Second constraint

$x_1 - 2x_2 + 3x_3 \geq 17$ can be converted into equation form by subtracting surplus variable s_2($s_2 \geq 0$) and we get constraint in equation form as follows:

$$x_1 - 2x_2 + 3x_3 - s_2 = 17$$

Third constraint is already in equation form. So given LPP can be written in equation form or standard form as

Maximize $z = 3x_1 - 4x_1 + 6x_3$

Subject to

$$x_1 - x_2 - 2x_3 + s_1 = 4$$
$$x_1 - 2x_2 + 3x_3 + s_2 = 17$$
$$x_1 - x_2 = 6$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Sometimes instead of all the decision variables which are non-negative in nature, it is given that either all or some of them are unrestricted in sign, then how to convert that LPP in equation form, this can be illustrated with the following example.

Example 7 Convert the following LPP in standard form.

Maximize $z = 3x_1 - 4x_2 + 6x_3$

Subject to the constraints

$$2x_1 + 2x_2 + 3x_3 \geq -3$$
$$x_1 - x_2 + 2x_3 \geq 7$$
$$x_1 - x_2 = 1$$
$$x_1, x_3 \geq 0 \text{ and } x_2 \text{ unrestricted in sign.}$$

Solution Now since x_2 is unrestricted in sign that is x_2 can take any of the values > 0, < 0 or 0. To convert the given LPP in equation, x_2 can be written as a combination of two variables x_2^+ and x_2^- as following:

$$x_2 = x_2^+ - x_2^- \text{ where } x_2^+, x_2^- \geq 0$$

$\Rightarrow$ If $x_2^+ \geq x_2^-$ then $x_2 \geq 0$ and if $x_2^+ \leq x_2^-$, then $x_2 \leq 0$

Now by replacing x_2 by $x_2^+ - x_2^-$, the standard form of given LPP becomes:

Maximize $z = 3x_1 - 4(x_2^+ - x_2^-) + 6x_3$

Subject to the constraints

$$2x_1 - 2\,(x_2^+ - x_2^-) - 3x_3 + s_1 = 3$$
$$x_1 - (x_2^+ - x_2^-) + 2x_3 - s_2 = 7$$
$$x_1 - (x_2^+ - x_2^-) = 1$$
$$x_1, x_2^+, x_2^-, x_3, s_1, s_2 \geq 0$$

After seeing above examples, we can say a LPP can be converted into a standard (equation) form of LPP by introducing slack, surplus or a combination of two variables of the variable(s) with unrestricted in sign.

EXERCISE 13.2

Convert the following problems in standard form.

1. Minimize $z = 2x_1 - x_2 + \dfrac{x_3}{2}$

   Subject to

$$x_1 + x_2 - x_3 \leq 5$$
$$2x_1 + 3x_3 \geq 6$$
$$x_1 + 3x_2 \leq -7$$
$$x_1, x_2, x_3 \geq 0$$

2. Maximize $z = 3x_1 - x_2 + 7x_3$
Subject to
$$2x_1 - x_2 - x_3 \le 7$$
$$x_1 - 2x_2 + x_3 \ge -3$$
$$x_1, x_2, x_3 \ge 0$$

3. Minimize $z = -x_1 + 3x_2 + 4x_3$
Subject to
$$x_1 - 7x_2 + 3x_3 \le 24$$
$$-x_1 + 4x_2 - 5x_3 \ge -12$$
$$x_1 \ge 2,\ x_2 \text{ unrestricted in sign},\ x_3 \ge 0$$

4. Maximize $z = 3x_1 - 2x_2 + 4x_3$
Subject to
$$2x_1 + x_2 + 2x_3 \le 12$$
$$x_1 - 2x_2 - x_3 \ge -6$$
$$3x_2 - 2x_3 \le 1$$
$$x_1, x_2, x_3 \ge 0$$

5. Linearize the following objective function.
Minimize $z = \text{Max.}\{|x_1 - 2x_2 + 3x_3|, |-2x_1 + 3x_2 - 2x_3|\}$.

Answers

1. Minimize $z = 2x_1 - x_2 + \dfrac{x_3}{2}$
Subject to
$$x_1 + x_2 - x_3 + s_1 = 5$$
$$2x_1 + 3x_3 - s_2 = 6$$
$$-x_1 - 3x_2 - s_3 = 7$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

2. Maximize $z = 3x_1 - x_2 + 7x_3$
Subject to
$$2x_1 - x_2 - x_3 + s_1 = 7$$
$$-x_1 + 2x_2 - x_3 + s_2 = 3$$
$$x_1, x_2, x_3, s_1\ s_2 \ge 0$$

3. Minimize $z = -y_1 + 3(x_2^+ - x_2^-) + 4x_3 - 2$
Subject to
$$y_1 - 7(x_2^+ - x_2^-) + 3x_3 + s_1 = 22$$
$$y_1 - 4(x_2^+ - x_2^-) + 5x_3 + s_2 = 10$$
$$\text{all variables} \ge 0$$
($\because$ Put $y_1 = x_1 - 2 \Rightarrow x_1 = y_{1+2}$ and $x_2 = (x_2^+ - x_2^-)$)

4. Maximize $z = 3x_1 - 2x_2 + 4x_3$
Subject to

$$2x_1 + x_2 + 2x_3 + s_1 = 12$$
$$-x_1 + 2x_2 + x_3 + s_2 = 6$$
$$\text{all variables} \geq 0$$

5. Let Max. $\{|x_1 - 2x_2 + 2x_3|, |-2x_1 + 3x_2 - 2x_3|\} = y$
Then minimize $z = y$

and either
$$|x_1 - 2x_2 + 2x_3| \leq y \tag{4}$$
$$|-2x_1 + 3x_2 - 2x_3| \leq y \tag{5}$$

From Eq. (4)
$$x_1 - 2x_2 + 2x_3 \leq y$$
$$-x_1 + 2x_2 - 2x_3 \leq y$$

and from eq (5)
$$-2x_1 + 3x_2 - 2x_3 \leq y$$
$$2x_1 - 3x_2 + 2x_3 \leq y$$

Then the problem becomes, minimize $z = y$
Subject to
$$x_1 - 2x_2 + 2x_3 \leq y$$
$$-x_1 + 2x_2 - 2x_3 \leq y$$
$$-2x_1 + 3x_2 - 2x_3 \leq y$$
$$2x_1 - 3x_2 + 2x_3 \leq y, \quad x_1, x_2, x_3, y \geq 0$$

13.5 SOME IMPORTANT TERMS

(i) *Solution*: Values of decision variables of a linear programming model is called solution of LPP.

(ii) *Basic solution*: Some variables of a LPP have zero values while others are non-zero. The variables with zero values are known as non-basic while the variables with non-zero values are called basic variables. Let there are m equations in n unknowns ($n \geq m$), a solution obtained by selling $n - m$ variables as zero and solving the remaining m equations in m unknown is called basic solution.

(iii) *Feasible solution*: A solution that satisfies all the constraints of a given LPP is called a feasible solution.

(iv) *Basic feasible solution*: A feasible solution which is basic is called basic feasible solution i.e., a solution which satisfies all the constraints including non-negative restrictions is called basic feasible solution (BFS) of a LPP.

(v) *Optimal Feasible Solution*: A BFS which optimizes the objective function of given LPP is called optimal feasible solution.

(vi) *Infeasible solution or No-feasible solution*: The solution of a LPP which does not satisfy all the constraints or non-negative restrictions is called infeasible or no-feasible solution of LPP.

(vii) *Degenerate solution*: A basic solution if one or more basic variable have value zero is called degenerate solution of LPP.

13.6 SOLUTION OF A LINEAR PROGRAMMING PROBLEM

A given LPP can be solved by the following two methods:

(i) Graphical method, and
(ii) Algebraic method

We shall discuss both the methods in detail in the upcoming section. First we shall discuss graphical method.

13.6.1 Graphical Method

If a given LPP contains two variables, it can be solved graphically. The following steps are involved in solving a LPP by graphical method:

Step 1: Formulate the problem mathematically.

Step 2: Plot the points in two dimensional axes and draw the lines accordingly of each constraint.

Step 3: Identify the solution area.

Step 4: Identify each of the extreme points known as corner points of feasible reason.

Step 5: Calculate the values of objective function at each corner point of feasible region.

Step 6: The optimal solution of the LPP occurs at that corner point with gives maximum (minimum) value of the objective function in case of maximization (minimization) problem.

Example 8 Use graphical method to solve the following LPP:

Maximize $z = 5x_1 + 4x_2$

Subject to the constraints

$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2 \ge 0$$

Solution

Step 1: Given problem already in mathematical form.

Step 2: Now construct the graph by constructing x_1 as horizontal axis and x_2 as vertical axis.

Step 3: Consider the first constraint $3x_1 + 5x_2 \le 15$, which will graph as $3x_1 + 5x_2 = 15$. To plot this line, find two points which satisfy the equation and draw a straight line through them.

Now, when $x_1 = 0 \Rightarrow x_2 = 3$ and when $x_2 = 0 \Rightarrow x_1 = 5$. The two points are (0, 3) and (5, 0). Connect these two points by a line, but the given constraint is $3x_1 + 5x_2 \le 15$.

So any point that falls on the line or the area below it satisfies the given constraint.

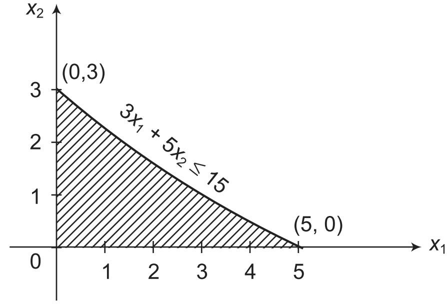


Fig. 13.1

The shaded area in (Fig. 13.1) satisfy the constraint $3x_1 + 5x_2 \le 15$.

Step 4: Similarly plot the second constraint $5x_1 + 2x_2 \leq 10$ as follows:

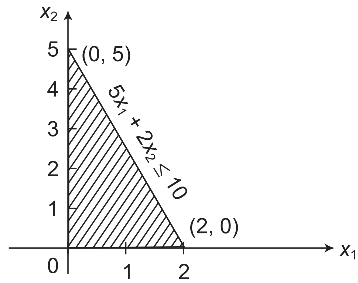


Fig. 13.2

The shaded area in (Fig. 13.2) satisfies the constraint $5x_1 + 2x_2 \leq 10$ combining both the figures (Figs. 13.1 and 13.2) we get the area bounded by all these two constraints, called feasible region, is shown in Fig. 13.3 by shaded area *OABC*.

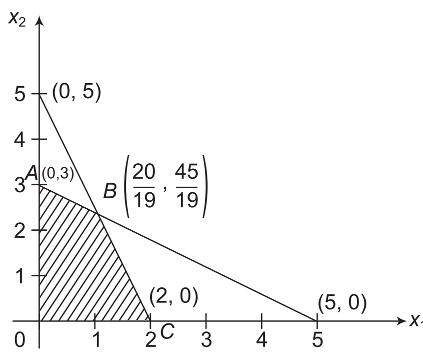


Fig. 13.3

The coordinates of extreme points of feasible region are as follows:

$$O = (0, 0),\ A = (0, 3),\ B = \left(\frac{20}{19}, \frac{45}{19}\right) \text{ and } C = (0, 3)$$

Step 5: Calculate the value of objective function at each extreme point of the feasible reason.

Extreme (corner) point	Coordinates (x_1, x_2)	$Z = 5x_1 + 4x_2$
O	(0, 0)	$5 \times 0 + 4 \times 0 = 0$
A	(0, 3)	$5 \times 0 + 4 \times 3 = 12$
B	$\left(\frac{20}{19}, \frac{45}{19}\right)$	$5 \times \frac{20}{19} + 4 \times \frac{45}{19} = \frac{280}{19}$
C	(2, 0)	$5 \times 2 + 4 \times 0 = 10$

Therefore maximum $Z = \dfrac{280}{19}$, which occurs at point B.

$\Rightarrow$ The solution of given LPP is as follows:

$$x_1 = \frac{20}{19},\ x_2 = \frac{45}{19} \text{ and } z = \frac{280}{19}$$

Example 9 Maximize $z = 6x_1 + 5x_2$

Subject to the constraints

$$x_1 + x_2 \le 46$$
$$x_1 + 2x_2 \le 60$$
$$x_1, x_2 \ge 0$$

Solution Plot on a graph each constraint by first treating as a linear equation in the same way as explained in Example 8. We get the feasible region as shown in Fig. 13.4.

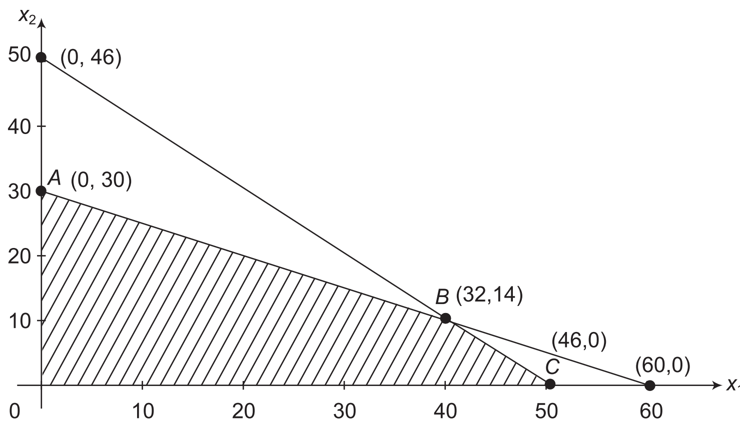


Fig. 13.4

The coordinates of extreme points $O = (0, 0)$, $A = (0, 30)$, $B = (32, 14)$, and $C = (46, 0)$.
Calculating the value of objective function $z = 6x_1 + 5x_2$ at all these points we get

$$z = 6 \times 0 + 5 \times 0 = 0 \text{ at } O$$
$$z = 6 \times 6 \times 0 + 5 \times 30 = 150 \text{ at } A$$
$$z = 6 \times 32 + 5 \times 14 = 262 \text{ at } B$$

and $$z = 6 \times 46 + 5 \times 0 = 276 \text{ at } C$$

Maximum of z occurs at point C.

$\Rightarrow$ Solution of given LPP is as follows:

$$x_1 = 6,\ x_2 = 0 \text{ and } z = 276.$$

Example 10 Minimize $z = 50x_1 + 60x_2$

Subject to the constraints

$$2x_1 + 3x_2 \le 1500$$
$$3x_1 + 2x_2 \ge 1500$$
$$x_1 \le 400$$
$$x_1, x_2 \ge 0$$

Solution Plot the constraints on (x_1, x_2) axis and find the feasible region which is given in Fig. 13.5 by shaded area A, B, C.

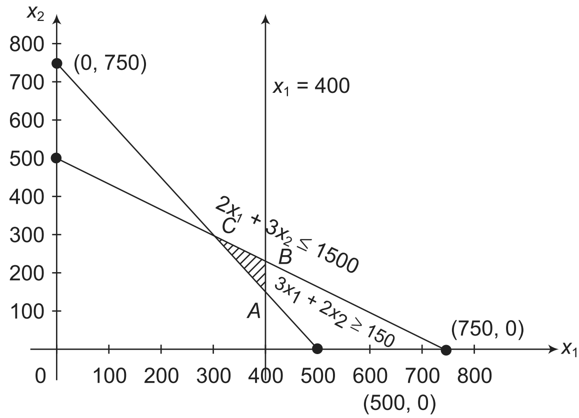


Fig. 13.5

The coordinates of $A = (400, 150)$, $B = \left(400, \dfrac{700}{3}\right)$, and $C = (300, 300)$ calculating the value of objective function $z = 50x_1 + 60x_2$ at A, B and C we get respectively 29,000, 34,000 and 33,000. Therefore minimum of z occurs at point A which is equal to 29,000

$\Rightarrow$ Solution of LPP is $x_1 = 400$, $x_2 = 150$ and $z = 29,000$.

13.6.2 Special Cases

(i) Unbounded Solution In a situation in which the values of objective function and decision variables increase infinitely without violating the feasibility conditions, the solution is called an unbounded solution of the given LPP.

Example 11 Find the maximum value of

$$z = 6x_1 + 2x_2$$

Subject to the constraints

$$2x_1 + x_2 \geq 3$$
$$-x_1 + x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

Solution: Plot the graph in the usual manner, which is shown in the following figure (Fig. 13.6).

The feasible region shown in Fig. 13.6 is unbounded in nature as the values of objective function at points $B(1, 1)$ and $A(0, 3)$ are 8 and 6 respectively. As the given problem is a maximization problem, there are infinite number of points in feasible region at which the values of objective function is more than 8. If we make the values of both the variables randomly large then the value of z will also increase. Hence, the given LPP has unbounded solution. But if the objective function is changed from maximization to minimization, then the value of $z = 6$ which occurs at point A and solution of problem becomes $x_1 = 0$, $x_2 = 3$ and $z = 6$.

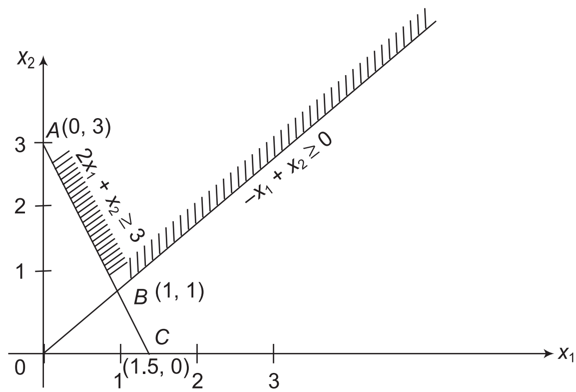


Fig. 13.6

(ii) Infeasible Solution If the feasible region is empty i.e., if there is no common region for all the constraints or no variable satisfies all the constraints then we have no feasible or unfeasible solution of the LPP or we call that feasible solution of the given LPP does not exist.

Example 12 Solve the following LPP graphically

Maximize $z = 4x_1 + 5x_2$

Subject to the constraints

$$
\begin{aligned}
4x_1 + 2x_2 &\leq 8 \\
4x_1 &\geq 8 \\
x_2 &\geq 6 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Solution Plot the graph as usual and region is shown in Fig. 13.7.

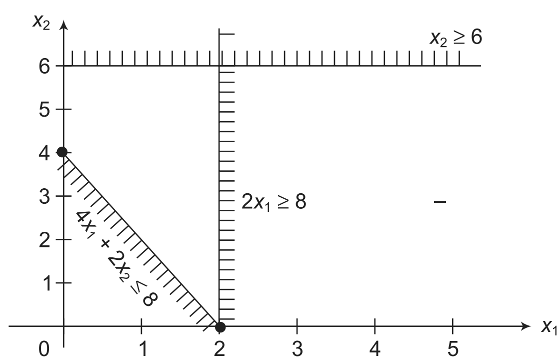


Fig. 13.7

If we see Fig. 13.7, we find that there is no common region which is intersection of all the constraints. Therefore, the given LPP has no feasible solution.

***Remark*:** If all the constraints are less than equal type then given LPP will always have a feasible solution. This situation occurs only if at least one of the constraints or all the constraints are greater than equal to type.

(iii) Alternate Solution When a given LPP has more than one solution i.e., if optimal value of objective function occurs at more than one extreme point than all the optimal solutions are known as alternate optimal solutions of the given LPP.

Example 13 Maximize $z = x_1 + x_2$

Subject to the constraints

$$x_1 + 2x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Solution Plotting the graph, we get the following feasible region shown in Fig. 13.8.

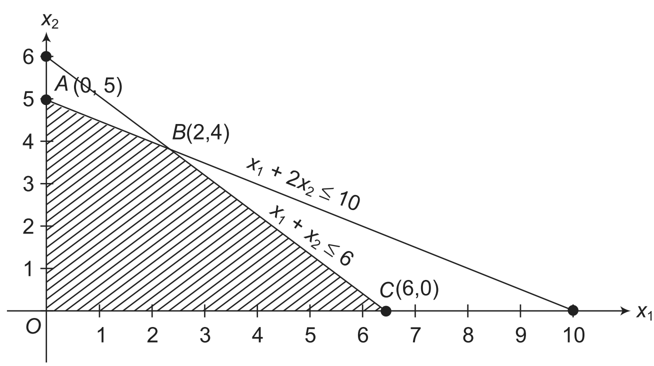


Fig. 13.8

Feasible region is given by $OABC$ whose coordinates are $O(0, 0)$, $A(0, 5)$, $B(2, 4)$, $C(6, 0)$ and value of z at these points are 0, 5, 6 and 6 respectively. Maximum value of z is 6 which occurs at two points B and C. Therefore, this problem has an alternate solution. Solutions of problem are as follows:

(a) $x_1 = 2, x_2 = 4, z = 6$ (b) $x_1 = 6, x_2 = 0, z = 6$

***Remark*:** If a LPP has optimal solution at more than one point then all the points lying on the line joining those points will also be solutions of the problem. Therefore we can say LPP has infinitely many solution. For example, if X_1 and X_2 are two solutions (optimal) of the given problem then

$$X = \alpha X_1 + (1 - \alpha) X_2, 0 \leq \alpha \leq 1, \alpha \geq 0$$

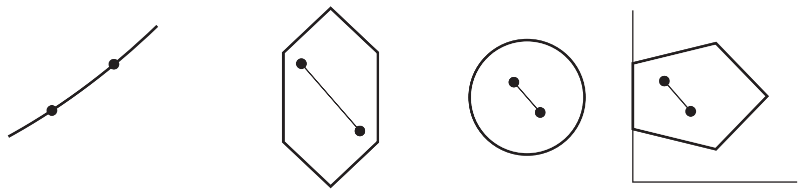
This will also be solution of the given LPP for all values of $0 \leq \alpha \leq 1$ and $\alpha \geq 0$. In this case X is called convex linear combination of the points X_1 and X_2.

(iv) Degenerate Solution If one or more than one basic variable has value equal to zero, then solution of the given LPP is called degenerate solution.

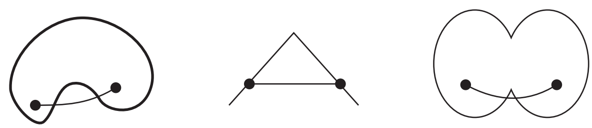
Example 14 In previous example, we see that one of the optimal solution of the given LPP is $x_1 = 6$ and $x_2 = 0$ with $z = 6$. Here x_2 is a basic variable which is equal to zero. It implies that the optimal solution is a degenerate solution.

***Remark*:** A region (set) of points is said to be convex if the line joining any two of its points, lies completely in the region (sets).

For example,



are convex while



are not convex sets.

EXERCISE 13.3

Use graphical method to solve the following LP problems.

1. Maximize $z = 3x_1 + 6x_2$,
   Subject to
   $$x_1 + 2x_2 \le 2000,\ x_1 + x_2 \le 1500,\ x_2 \le 600 \text{ and } x_1, x_2 \ge 0$$ **(Rohtak 2004)**
2. Minimize $z = 20x_1 + 30x_2$
   Subject to
   $$x_1 + 2x_2 \le 40,\ 3x_1 + x_2 \ge 30,\ 4x_1 + 3x_2 \ge 60,\ x_1, x_2 \ge 0$$
   [Kurukshetra 2009S, Mumbai 2004, V.T.U. 2004]
3. Maximize $z = 6x_1 + 4x_2$
   Subject to
   $$2x_1 + x_2 \ge 1,\ 3x_1 + 4x_2 \ge \frac{3}{2},\ x_1, x_2 \ge 0$$ **[Bombay 2004]**
4. Minimize $z = x_1 - 2x_2$
   Subject to
   $$-x_1 + x_2 \le 1,\ 2x_1 + x_2 \le 2,\ x_1, x_2 \ge 0$$
5. Minimize $z = -4x_1 + x_2$
   Subject to
   $$\begin{aligned} x_1 - 2x_2 &\le 2 \\ -2x_1 + x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$
6. Maximize $z = x_1 + 2x_2$
   Subject to
   $$x_1 - x_2 \le 1,\ x_1 + x_2 \ge 3,\ x_1, x_2 \ge 0$$

7. Minimize $z = 5x_1 + 3x_2$
Subject to

$$x_1 + x_2 \leq 6$$
$$2x_1 + 3x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

8. Maximize $z = 4x_1 + 3x_2$
Subject to

$$2x_1 + x_2 \leq 1000,\ x_1 + x_2 \leq 800,\ x_1 \leq 400,\ x_2 \leq 700,\ x_1, x_2 \geq 0$$

9. Minimize $z = x_1 - 2x_2$
Subject to

$$x_1 + x_2 \leq 3,\ 4x_1 + x_2 \geq 4,\ 2x_1 - x_2 \geq 1;\ x_1, x_2 \geq 0$$

10. Minimize $z = x_1 - x_2$
Subject to

$$-x_1 + x_2 \geq 3,\ x_1 + x_2 \geq 6,\ 2x_1 + 3x_2 \leq 6;\ x_1, x_2 \geq 0$$

Answers

1. $x_1 = 1000, x_2 = 500$; max $z = 5500$
2. $x_1 = 15, x_2 = 0$; max $z = 300$
3. Unbounded solution
4. $x_1 = 0, x_2 = 1$; Min. $z = -2$
5. Unbounded solution
6. Infeasible solution
7. $x_1 = x_2 = 3$; Min. $z = 24$
8. $x_1 = 200, x_2 = 600$; Max. $z = 2600$
9. $x_1 = \frac{4}{3}, x_2 = \frac{5}{3}$; Min. $z = -2$
10. Infeasible solution

13.7 ALGEBRAIC SOLUTION OF A PROGRAMMING PROBLEM

If an LPP has more than two decision variables then it can not be solved by graphical method. We solve the LPP algebraically following steps are used to solve the given LPP:

Step 1: Convert the LPP into standard form by introducing slack or surplus variables.

Step 2: Find all the basic solutions of the given LPP. If after converting the LPP in standard form, number of decision variables are n and number of constraints are m, then number of basic solutions are equal to ${}^nC_m$. This is done by putting $n–m$ variables equal to zero known as non-basic variables.

Step 3: Identify the basic feasible solutions, which are $\leq {}^nC_m$.

Step 4: Calculate the value of objective function corresponding to all basic feasible solution.

Step 5: Identify that basic feasible solution(s) which optimize(s) the objective function and this will be optimum solution.

Example 15 Find all basic solutions of the following LPP

Maximize $z = x_1 + 3x_2 + 3x_3$

Subject to the constraints

$$x_1 + 2x_2 + 3x_3 = 4$$
$$2x_1 + 3x_2 + 5x_3 = 7$$
$$x_1, x_2, x_3 \geq 0$$

Also, find which of the basic solutions are as follows:

(a) Basic feasible solution

(b) Non-degenerate basic feasible solution

(c) Optimal basic feasible solution.

Solution Here $n = 3$, $m = 2$

$\Rightarrow$ we have ${}^3C_2 = 3$ basic solutions, which we shall get by putting $n - m = 3 - 2 = 1$ variable equal to zero. These solutions can be shown in the following table.

S.No.	Basic variables	Non-Basic variable	Values of Basic variables	Value of z	Feasible	Non degener-ate	Optimal
1	x_1, x_2	$x_3 = 0$	$x_1 = 2, x_2 = 1$	5	Yes	Yes	Yes
2	x_1, x_3	$x_2 = 0$	$x_1 = 1, x_3 = 1$	4	Yes	Yes	Yes
3	x_2, x_3	$x_1 = 0$	$x_2 = -1, x_3 = 2$	3	No	No	No

From the above table, we see that first two solutions are non-degenerate feasible solutions while the third one is non-degenerate, non-feasible solution and the first solution is optimal feasible solution.

Therefore, the optimal solution of the given LPP is as follows:

$$x_1 = 2, x_2 = 1, x_3 = 0 \text{ and } z = 5$$

Example 16 Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables: $2x_1 + x_2 + 4x_3 = 11, 3x_1 + x_2 + 5x_3 = 14$.

[Mumbai 2004, V.T.U. 2003S]

Also identify the degenerate or non-degenerate basic solutions and hence, calculate basic feasible solution.

Solution Here $n = 3$, $m = 2$, total number of basic solution are ${}^3C_2 = 3$, number of basic variables = $m = 2$, number of non-basic variable = $n - m = 3 - 2 = 1$ whose value is equal to zero. All the solutions can be shown in the following table.

S.No.	Basic variables	Non-Basic variable	Values of Basic variables	Feasible	Non-degenerate
1	x_1, x_2	$x_3 = 0$	$x_1 = 3$ $x_2 = 5$	Yes	No
2	x_1, x_3	$x_2 = 0$	$x_1 = 0.5$ $x_3 = 2.5$	Yes	Yes
3	x_2, x_3	$x_1 = 0$	$x_2 = 3$ $x_3 = -1$	No	No

The basic-feasible solutions are as following

(a) $x_1 = 3, x_2 = 5, x_3 = 0$ (b) $x_1 = 0.5, x_2 = 0, x_3 = 2.5$ and they are also non-degenerate solutions.

EXERCISE 13.4

1. Show that the following system of linear equations has two degenerate feasible basic solutions and the non-degenerate basic solution is not feasible

$$2x_1 + x_2 - x_3 = 2, 3x_1 + 2x_2 + x_3 = 3$$ **[Kurukshetra 2007S]**

2. Find all the basic solutions to the following problem
Maximize $z = x_1 + 3x_2 + 3x_3$
Subject to

$$x_1 + 2x_2 + 3x_3 = 4, 2x_1 + 3x_2 + 5x_3 = 7, \text{ and } x_1, x_2, x_3 \geq 0$$

which of the basic solutions are (a) non-degenerate basic feasible (b) optimal basic feasible?
[Kurukshetra 2009S, Mumbai 2003]

3. Show that the following LPP
Maximize $z = 2x_1 + 3x_2 + 4x_3 + 7x_4$
Subject to
$2x_1 + 3x_2 - x_3 + 4x_4 = 8, x_1 - 2x_2 + 6x_3 - 7x_4 = -3, x_1, x_2, x_3, x_4 \geq 0$ has optimal solution

$$x_1 = x_2 = 0, x_3 = \frac{44}{17}, x_4 = \frac{45}{17} \text{ and max } z = \frac{419}{17}$$

4. Show that the LPP
Minimize $z = x_1 - 2x_2$
Subject to
$-x_1 + x_2 \leq 1, 2x_1 + x_2 \leq 2, x_1, x_2 \geq 0$ has optimal solution $x_1 = 0, x_2 = 1$ and minimize $z = -2$.

5. Show that the maximum of LPP $z = 2x_1 - 3x_2$
Subject to
$3x_1 + 2x_2 \geq 6, -x_1 + x_2 \geq 1, -x_1 + x_2 \leq 2, x_1, x_2 \geq 0$ occurs at (0, 1) and hence, compute the value of max z.

Answers

2. Basic solutions are (i) $x_1 = 2, x_2 = 1$ (basic) and $x_3 = 0$
(ii) $x_1 = x_3 = 1$ (basic) and $x_2 = 0$; (iii) $x_2 = -1, x_3 = 2$ (basic) and $x_1 = 0$
(a) First two solutions are non-degenerate basic feasible solution
(b) Optimal solution is $x_1 = 2, x_2 = 1$ and max $z = 5$

13.8 SIMPLEX METHOD

When n and m are large then it is very difficult to find the solutions of a given LPP algebraically. For example, if $m = 10$ and $n = 4$ then number of basic solutions are equal to ${}^{10}C_4 = 210$, so it is very difficult to find 210 basic solution, then to find solution of such problems a method was designed by G.B. Dontzig in 1947 known as simplex method. This method involves steps of algorithm until an optional solution is reached. This is also known as iterative method. In this method, we find first a starting basic feasible solution, then move to another basic feasible solution in such a way that the value of objective function improves and continue this until we get an optimal solution. The various steps involved in simplex method are as follows:

Step 1: Convert the given LPP in standard form.

Step 2: The matrix A should have an identity matrix I as a submatrix. The variables corresponding to I give the starting basic feasible solution as b the right hand side of the LPP.

Step 3: The objective function must be represented in terms of non-basic variables that is in z-row, the entries below basic variables are zero.

Step 4: Write the starting table which gives the starting basic feasible solution (BFS).

Step 5: Decide the entering variable.

For maximization problem, the variable with most negative entry in z-row enters as basic variable while for minimization problem the variable with most positive entry in z-row enters as basic variable. This ensures largest improvement in the objective function.

Step 6: Decide the leaving variable, which can be found by the minimum ratio rule $\min\left(\dfrac{\text{solution}}{\text{entering column} > 0}\right)$ and mark the pivot element which is corresponding to entering variable and leaving row.

Step 7: Now make pivot element is equal to 1 and other elements in that column i.e., column corresponding to entering variable equal to zero by row operation.

Step 8: Continue till the optimal solution is reached i.e., for maximization problem all variables in z-row become ≥ 0 and minimization problem all variables in z-row ≤ 0.

***Remark*:**

1. In case of tie in entering variable, choose arbitrarily.
2. In case of tie in leaving variable, choose arbitrarily but it may be noted in this case new basic feasible solution would be degenerate solution.

Example 17 Solve the following LPP by simplex method:

Maximize $z = 12x_1 + 15x_2 + 14x_3$

Subject to the constraints

$$-x_1 + x_2 \leq 0$$
$$-x_1 + 2x_3 \leq 0$$
$$x_1 + x_2 + x_3 \leq 100$$
$$x_1, x_2, x_3 \geq 0$$

Solution

Step 1: Convert the given LPP in standard form, i.e,

Maximize $z = 12x_1 + 15x_2 + 14x_3 + 0.s_1 + 0.s_2 + 0.s_3$

Subject to the constraints

$$-x_1 + x_2 + s_1 = 0$$
$$-x_1 + 2x_3 + s_2 = 0$$
$$x_1 + x_2 + x_3 + s_3 = 100$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

where s_1, s_2 and s_3 are non-negative slack variables with cost coefficient zero. Now, the given LPP can be written as follows:

$$\text{Maximize } z = (12, 15, 14, 0, 0, 0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

Subject to the constraints

$$\begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

where $c^T = (12 \quad 15 \quad 14 \quad 0 \quad 0 \quad 0)$

$$X = (x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2 \quad s_3)^T$$

$$A = \begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}$$

Step 2: Matrix A has an identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ corresponding to variables}$$

s_1, s_2 and $s_3 \Rightarrow s_1, s_2$ and s_3 are starting basic variables and x_1, x_2, x_3 are non-basic variables. Therefore, the starting BFS is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix} \Rightarrow s_1 = 0 = s_2 \text{ and } s_3 = 100$$

Step 3: Now the objective function becomes as follows:

$$z = 12x_1 - 15x_2 - 14x_3 - 0s_1 - 0s_2 - 0s_3 = 0$$

Step 4: Write the starting simplex table as follows, which gives the starting BFS (always write basic variables in the end).

Basic variable	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	–12	–15	–14	0	0	0	0
s_1	0	–1	$\boxed{1}$	0	1	0	0	0
s_2	0	–1	0	2	0	1	0	0
s_3	0	1	1	1	0	0	1	100

Step 5: To decide entering variable, we see that the value in z-row below variable x_2 is most negative i.e., –15 (as it is maximization case), x_2 enters.

Step 6: To decide the leaving variable, we find the $\min\left(\frac{0}{1}, \frac{100}{1}\right) = 0$ and it is corresponding to variable $s_1 \Rightarrow s_1$ leaves and the element corresponding to x_2-column and s_1-row is 1 and it is pivot element we mark it by $\boxed{}$.

Step 7: Now we have to apply row-operation to mark a pivot element 1 and all other elements in that column as 0 as follows:

$$R_2(\text{new}) = R_2(\text{old}) - R_1(\text{new}) \times 0$$
$$R_3(\text{new}) = R_3(\text{old}) - R_1(\text{new}) \times 1$$
$$z - \text{row}(\text{new}) = z - \text{row}(\text{old}) - R_1(\text{new}) \times (-15)$$

and we get the new simplex table as given in the following table.

Basic variables	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	–27 ↓	0	–14	15	0	0	0
x_2	0	–1	1	0	1	0	0	0
s_2	0	–1	0	2	0	1	0	0
s_3	0	$\boxed{2}$	0	1	–1	0	1	100

New continue this operation till we get all entries in z-row ≥ 0.

Basic variables	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	0	0	$-\frac{1}{2}$ ↓	3/2	0	$\frac{27}{2}$	1350
x_2	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	50
← s_2	0	0	0	$\boxed{5/2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	50
x_1	0	1	0		$-\frac{1}{2}$	0	$\frac{1}{2}$	50

x_3 enters and $\min\left(\frac{50}{1/2}, \frac{50}{5/2}, \frac{50}{1/2}\right) = 20$, s_2 leaves and we get the following table:

Basic variables	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	0	0	0	7/5	1/5	68/5	1360
x_2	0	0	1	0	3/5	–1/5	2/5	40
x_3	0	0	0	1	–1/5	2/5	1/5	20
x_1	0	1	0	0	–2/5	–1/5	2/5	40

The value of all variables in z-row ≥ 0. Hence, the above table is optimal. Therefore, the optimal solution of the given LPP is $x_1 = 40, x_2 = 40, x_3 = 20$ and $z_{\max} = 1360$.

Example 18 Solve the following LPP by simplex method.

Minimize $z = x_1 - 3x_2 + 2x_3$

Subject to the constraints

$$3x_1 - x_2 + 2x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Solution We can proceed in the similar manner to solve the given LPP as example.

Minimize $z = x_1 - 3x_2 + 2x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to the constraints

$$3x_1 - x_2 + 2x_3 + s_1 = 7$$
$$-2x_1 + 4x_2 + s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Basic variables	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	–1	3↓	–2	0	0	0	0
s_1	0	3	–1	2	1	0	0	7
$\leftarrow s_2$	0	–2	$\boxed{4}$	0	0	1	0	12
s_3	0	–4	3	8	0	0	1	10
z	1	$\frac{1}{2}\downarrow$	0	–2	0	–3/4	0	–9
$\leftarrow s_1$	0	$\boxed{5/2}$	0	2	1	$\frac{1}{4}$	0	10
x_2	0	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	3

Basic variables	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
s_3	0	–5/2	0	8	0	–3/4	1	1
z	1	0	0	–12/5	–1/5	–4/5	0	–11
x_1	0	1	0	4/5	2/5	$\frac{1}{10}$	0	4
x_2	0	0	1	2/5	1/5	$\frac{3}{10}$	0	5
s_3	0	0	0	10	1	$-\frac{1}{2}$	1	11

all entries in z-row ≤ 0, hence the above table is optimal. Therefore optimal solution of the given LPP is

$$x_1 = 4, x_2 = 5, x_3 = 0 \text{ and } \min z = -11$$

Example 19 Use simplex method to solve the following LPP.

Minimize $z = -12x_1 - 15x_2$

Subject to the constraints

$$4x_1 + 3x_2 \leq 12$$
$$2x_1 + 5x_2 \leq 10$$
$$x_1, x_2 \geq 10$$

Solution Proceed in the same manner as examples 17 and 18 to obtain the optimal solution of the problem.

Basic variables	z	x_1	x_2	s_1	s_2	Solution
z	1	12	15↓	0	0	0
s_1	0	4	3	1	0	12
s_2	0	2	$\boxed{5}$	0	1	10
z	1	6	0	0	–3	–30
s_1	0	$\boxed{14/5}$ ↓	0	1	–3/5	6
x_1	0	$\frac{2}{5}$	1	0	$\frac{1}{5}$	2
z	1	0	0	–15/9	–12/7	–300/7
x_1	0	1	0	5/14	–3/14	15/7
x_2	0	0	1	–1/7	2/7	8/7

All entries in z-row ≤ 0. Hence, this is the optimal table as shown in the above table. Therefore optimal solution of the given LPP is as follows:

$$x_1 = \frac{15}{7}, x_2 = 8/7, \text{ Minimize } z = \frac{-300}{7}$$

EXERCISE 13.5

Use simplex method to solve the following LP problems:

1. Maximize $z = x_1 + 3x_2$
   Subject to
   $$x_1 + 2x_2 \leq 10, 0 \leq x_1 \leq 5, 0 \leq x_2 \leq 4$$ **[Kurukshetra 2009, V.T.U. 2003]**
2. Maximize $z = 3x_1 + 2x_2 + 5x_3$
   Subject to
   $$x_1 + 2x_2 + x_3 \leq 430, 3x_1 + 2x_3 \leq 460, x_1 + 4x_2 \leq 420, x_1, x_2, x_3 \geq 0$$
   [Mumbai 2004]
3. Minimize $z = 3x_1 + 5x_2 + 4x_3$
   Subject to
   $$2x_1 + 3x_2 \leq 8, 2x_2 + 5x_3 \leq 10, 3x_1 + 2x_2 + 4x_3 \leq 15, x_1, x_2, x_3 \geq 0$$
   [Mumbai 2004S]
4. Minimize $z = x_1 - 3x_2 + 2x_3$
   Subject to
   $$3x_1 - x_2 + 2x_3 \leq 7, -2x_1 + 4x_2 \leq 12, -4x_1 + 3x_2 + 8x_3 \leq 10, x_1, x_2, x_3 \geq 0$$
   [Karnataka 1992, IAS 1993, Madras 2006]
5. Maximize $z = 4x_1 + 10x_2$
   Subject to
   $$2x_1 + x_2 \leq 50, 2x_1 + 5x_2 \leq 100, 2x_1 + 3x_2 \leq 90, x_1, x_2 \geq 0$$
   [Kurukshetra 2006]
6. Maximize $z = 3x_1 + 2x_2$
   Subject to
   $$x_1 + x_2 \leq 4, x_1 - x_2 \leq 2, x_1, x_2 \geq 0$$ **[Calicut 1990 IAS 1992]**
7. Maximize $z = 2x_1 + 3x_2$
   Subject to
   $$x_1 + x_2 \leq 4, -x_1 + x_2 \leq 1, x_1 + 2x_2 \leq 5; x_1 \geq 0, x_2 \geq 0$$ **[Anantapur 1990]**
8. Maximize $z = 5x_1 + 3x_2$
   Subject to
   $$x_1 + x_2 \leq 2, 5x_1 + 2x_2 \leq 10, 3x_1 + 8x_2 \leq 12; x_1, x_2 \geq 0$$ **[Purvanchal 1996]**
9. Maximize $z = 5x_1 + 3x_2$
   Subject to
   $$x_1 \leq 4, x_2 \leq 3, x_1 + 2x_2 \leq 18, x_1 + x_2 \leq 9; x_1, x_2 \geq 0$$ **[Nagarjuna 1995]**
10. Maximize $z = 2x_1 + 4x_2 + x_3 + x_4$
    Subject to
    $$x_1 + 3x_2 + x_4 \leq 4, 2x_1 + x_2 \leq 3, x_2 + 4x_3 + x_4 \leq 3, x_1, x_2, x_3, x_4 \geq 0$$
    [Madras 1996]

11. Maximize $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$
Subject to

$$x_1 + x_2 + x_3 + x_4 \le 15,\ 7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$$
$$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100;\ x_1, x_2, x_3, x_4 \ge 0$$

[Delhi 1999, Bhartidesan 1995]

12. Maximize $z = 3x_1 + 2x_2$
Subject to

$$x_1 + x_2 \le 6,\ 2x_1 + x_2 \le 6,\ x_1, x_2 \ge 0$$ **[Madurai 1993]**

13. Maximize $z = 4x_1 + 10x_2$
Subject to

$$2x_1 + x_2 \le 50,\ 2x_1 + 5x_2 \le 100,\ 2x_1 + 3x_2 \le 90;\ x_1, x_2 \ge 0$$ **[Jodhpur 1990]**

Answers

1. $x_1 = 2, x_2 = 4$; max $z = 14$
2. $x_1 = 0, x_2 = 100, x_3 = 230$; max $z = 1350$
3. $x_1 = \frac{89}{41}, x_2 = \frac{50}{41}, x_3 = \frac{62}{41}$; max $z = 765/41$
4. $x_1 = 4, x_2 = 5, x_3 = 0$; min $z = -11$
5. $x_1 = 0, x_2 = 20$; max $z = 200$
6. $x_1 = 3, x_2 = 1$; max $z = 11$
7. (i) $x_1 = 4, x_2 = 0$; max $z = 8$
   (ii) $x_1 = 1, x_2 = 2$; max $z = 8$

Alternate solution

8. $x_1 = 2, x_2 = 0$; max $z = 10$
9. $x_1 = 4, x_2 = 3$; max $z = 29$
10. $x_1 = 1, x_2 = 1, x_3 = \frac{1}{2}, x_4 = 0$; max $z = \frac{13}{2}$
11. $x_1 = \frac{50}{7}, x_2 = 0 = x_4, x_3 = \frac{55}{7}$; max $z = \frac{695}{7}$
12. $x_1 = 0, x_2 = 6$; max $z = 12$
13. $x_1 = 0, x_2 = 20$, max $z = 200$

13.9 ARTIFICIAL VARIABLE TECHNIQUE

We have seen that to solve a LPP by simplex method, the identity matrix must be present in the LPP after adding slack variables. So far we considered only these LPP problem in which we added only slack variables and by adding slack variables, we got identity matrix. But this is not possible always. For example, consider the LPP.

Example 20

$$\left.\begin{aligned} \text{Minimize } z &= 3x_1 + 2x_2 \\ \text{Subject to the constraints} & \\ x_1 + x_2 &\geq 2 \\ x_1 + 3x_2 &\leq 3 \\ x_1 - x_2 &= 1 \\ x_1, x_2 &\geq 0 \end{aligned}\right\} \quad (6)$$

Converting the given LPP into standard form, we get

$$\left.\begin{aligned} \text{Minimize } z &= 3x_1 + 2x_2 \\ \text{Subject to the constraints} & \\ x_1 + x_2 - s_1 &= 2 \\ x_1 + 3x_2 + s_2 &= 3 \\ x_1 - x_2 &= 1 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}\right\} \quad (7)$$

Note that solution of Eq. (7) is also solution of Eq. (6) as objective function of both are same, the optimal solution of Eq. (7) is the optimal solution of Eq. (6). But the standard form of Eq. (7) does not have an identity matrix as a sub-matrix, so simplex method can not be applied. Now an identity matrix can be created artificially by adding artificial variables r; to the system Eq. (7) as follows:

$$\left.\begin{aligned} \text{Minimize } z &= 3x_1 + 2x_2 \\ \text{Subject to} & \\ x_1 + x_2 - s_1 + r_1 &= 2 \\ x_1 + 3x_2 + s_2 &= 3 \\ x_1 - x_2 + r_3 &= 1 \\ x_1, x_2, s_1, s_2, r_1, r_3 &\geq 0 \end{aligned}\right\} \quad (8)$$

As for slack (surplus) variables, we shall denote the artificial variables added to i^{th} constraint by r_i. In the above $r_i \geq 0$, so that the above system satisfies the requirements of applications of simplex method.

In system Eq. (8) the identity matrix is given by r_1, s_2 and r_3 columns. Now, we can apply simplex method to system Eq. (8).

The difficulty is that a solution of Eq. (8) need not be a solution of Eq. (7), because the constraints equations of Eq. (8) and Eq. (7) are not same. But a solution of Eq. (8) with all artificial variables equal to zero is a solution of Eq. (7). In a LPP, we are interested in BFS. The above holds for BFS also. Thus our aim now is to get a BFS of Eq. (8) in which all artificial variables r_i are zero. This will be a BFS of Eq. (7) also. Therefore we seek methods which give optimal BFS of Eq. (8) with all $r_i = 0$. This will be an optimal solution of Eq. (7). There are two methods to set this. One of them is known as M-method or method of penalty and second one Two-phase method. We shall discuss them one by one.

13.9.1 M-Method or Method of Penalty

In this we modify the objective function of a maximization problem by adding $-M$ times sum of all artificial variables to of where $M > 0$ a big number. In the case of minimization problem, we add to objective function M times sum of artificial variables, where $M > 0$, a big number. This problem is called modified problem. The modified problem of Eq. (8) is

$$\left.\begin{aligned} \text{Minimize } z = 3x_1 + 2x_2 + Mr_1 + Mr_3 \\ \text{Subject to the constraints} \\ x_1 + x_2 - s_1 + r_1 = 2 \\ x_1 + 3x_2 + s_2 = 3 \\ x_1 - x_2 + r_3 = 1 \\ \text{all variables} \geq 0 \end{aligned}\right\} \tag{9}$$

We apply simplex method to system Eq. (9). The above is a minimization problem and hence the appropriate rules of simplex method will minimize the objective function of Eq. (9). It is clear that the rules of simplex method will automatically derive r_1 and r_3 to zero provided the original problem has a solution, because M is a big positive number and all artificial variables ≥ 0. M is the penalty of r_i to be in the basis at a positive level. Big M will drive artificial variables from basis. To understand this let us suppose that the optimal solution of Eq. (8) is $x_1 = \frac{3}{2}$, $x_2 = \frac{1}{2}$, $s_2 = 0$, $s_1 = 0$, Min $z = \frac{11}{2}$. This is also a BFS of Eq. (9) with $r_1 = 0 = r_3$. So the optimal solution of Eq. (9) is $x_1 = \frac{3}{2}$, $x_2 = \frac{1}{2}$, $s_1 = s_2 = r_1 = r_3 = 0$, Min $z = \frac{11}{2}$. The optimal solution of Eq. (9) can not have an artificial variable > 0, otherwise the objective function will become sufficiently large due to M. Thus, if the given problem has a solution, then the modified problem will have a solution with $r_i = 0$. The simplex method will yield this solution.

For applying simplex method to system (9) we make the following table.

Basic variables	z	x_1	x_2	s_1	r_1	s_2	r_3	Solution
z	1	–3	–2	0	–M	0	–M	0
r_1	0	1	1	–1	1	0	0	2
s_2	0	1	3	0	0	1	0	3
r_3	0	1	–1	0	0	0	1	1

We can not apply simplex method to above table as the objective function is not expressed in terms of non-basic variables (there must be zeros below r_1, s_2 and r_3). The above is called initial table. The entry 0 on the right at the top of given table is the right hand side (RHS) and not the solution. To get the starting table of simplex method, we bring zeros below r_1 and r_3. (which are basic variables) in z-row by row operations. Adding M times r_1 row and M times r_3 row to z-row, we get the starting table and then use simplex method. Now all values below basic variables in z-row are zero. The iterations are shown in the given table.

Basic variables	z	x_1	x_2	s_1	r_1	s_2	r_3	Solution
z	1	–3+2M↓	–2	–M	0	0	0	3 M
r_1	0	1	1	–1	1	0	0	2
s_2	0	1	3	0	0	1	0	3
$\leftarrow r_3$	0	$\boxed{1}$	–1	0	0	0	1	1
z	1	0	–5+2 M	–M	0	0	3–2M	3 + M
$\leftarrow r_1$	0	0	$\boxed{2}$	–1	1	0		1
s_2	0	0	4	0	0	1		2
x_1	0	1	–1	0	0	0		1
z	1	0	0	–5/2	5/2 –M	0		11/2
x_2	0	0	1	$-\frac{1}{2}$		0		1/2
s_2	0	0	0	2		1		0
x_1	0	1	0	$-\frac{1}{2}$		0		3/2

The given table is the optimal table. The optional solution is

$$x_1 = \frac{3}{2},\ x_2 = \frac{1}{2},\ \text{Minimize } z = 11/2$$

***Remark*:**

1. Whenever there is a tie among leaving variables with one as artificial variable then always prefer artificial variable because our aim to bring artificial variables to zero.
2. An artificial variable is never considered for re-entry into the basis due to obvious reasons. Once it leaves the basis, its purpose is served. Therefore their columns are not calculated in next table.
3. If in the optimal table, all artificial variables are not zero, then the original problem has no solution. This is because then Eq. (8) has no solution with all $r_i = 0$, otherwise big positive penalty M for r_i would have given it. That original LP(7) has no solution.

Example 21 Use M-method to solve the following LPP.

Maximize $z = -x_1 + 3x_2$
Subject to the constraints

$$\begin{aligned} x_1 + 2x_2 &\geq 2 \\ 3x_1 + x_2 &\leq 3 \\ x_1 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

Maximize $z = -x_1 + 3x_2 - Mr_1$
Subject to the constraints

$$x_1 + 2x_2 - s_1 + r_1 = 2$$
$$3x_1 + x_2 + s_2 = 3$$
$$x_1 + s_3 = 4$$
$$\text{all variables} \geq 0$$

Basic variables	z	x_1	x_2	s_1	r_1	s_2	s_3	Solution	
z	1	1	–3	0	M	0	0	0	
r_1	0	1	2	–1	1	0	0	2	initial table
s_2	0	3	1	0	0	1	0	3	
s_3	0	1	0	0	0	0	1	4	
z	1	1–M	$-3-2M\downarrow$	M	0	0	0	–2 M	
$\leftarrow r_1$	0	1	$\boxed{2}$	–1	1	0	0	2	starting table
s_2	0	3	1	0	0	1	0	3	
s_3	0	1	0	0	0	0	1	4	
z	1	5/2	0	$-\frac{3}{2}\downarrow$	$\frac{3}{2}+M$	0	0	3	
x_2	0	$\frac{1}{2}$	1	$-\frac{1}{2}$		0	0	1	
s_2	0	5/2	0	$\boxed{\frac{1}{2}}$		1	0	2	
s_3	0	1	0	0		0	1	4	
z	1	10	0	0		3	0	9	
x_2	0	3	1	0		1	0	3	optimal table
s_1	0	5	0	1		2	0	4	
s_3	0	1	0	0		0	1	4	

The optimal solution is

$$x_1 = 0, x_2 = 3, \text{Maximize } z = 9$$

13.9.2 Two-Phase Method

In M-method, it is not fixed how big M is, M-method therefore, can not be used in computer. This necessitates the need for another method which would be suitable for use in computer. Dantizig, Orden and others developed the two-phase method. Consider again Example 21. The aim is to find a BFS of Eq. (8) with all artificial variables equal to zero. We will use this as starting BFS of Eq. (7) and apply simplex method to solve the problem. This is achieved in two phases. In phase-I, we obtain a BFS of Eq. (8) with all artificial variables equal to zero. In phase-II, using the BFS of Phase-I as starting BFS of the given problem, we obtain the optimal solution of the original problem.

Phase-I: We formulate the following auxiliary problem

$$\text{Min } r = \text{sum of artificial variables}$$

$$\left.\begin{aligned} &= r_1 + r_3 \\ &\text{Subject to the same constraints} \\ x_1 + x_2 - s_1 + r_1 &= 2 \\ x_1 + 3x_2 + s_2 &= 3 \\ x_1 - x_2 + r_3 &= 1 \\ \text{all variables} &\geq 0 \end{aligned}\right\} \tag{10}$$

Clear the minimum of above problem is $r_1 = r_3 = 0$ and Min $r = 0$. So if we use simplex method to the above auxiliary problem we will get a BFS of constraints of the problem Eq. (7) with $r_1 = r_3 = 0$. This is a BFS of the given problem. Thus in Phase-I by introducing an appropriate objective function, we use simplex method to the advantage of a getting a BFS of given system. Now we use simplex method to solve systems Eq. (10).

The z-row is

x_1	x_2	s_1	r_1	s_2	r_3	**RHS**
0	0	0	–1	0	–1	**0**

Initial Table (IT) and Starting Table (ST) are shown in the given table. To get the starting table of simplex method, that is the table in which the entries below basic variables in z-row are zero, we add r_1 and r_3 rows to z-row. Now apply simplex method to set the optimal solution of the LPP.

Basic variables	r	x_1	x_2	s_1	r_1	s_2	r_3	**Solution**	
r	1	0	0	0	–1	0	–1	0	
r_1	0	1	1	–1	1	0	0	2	initial table
s_2	1	1	3	0	0	1	0	3	
r_3	0	1	–1	0	0	0	1	1	
r	1	2	0	–1	0	0	0	3	
r_1	0	1	1	–1	1	0	0	2	starting table
s_2	0	1	3	0	0	1	0	3	
r_3	0	$\boxed{1}$	–1	0	0	0	1	1	
r	1	0	2	–1	0	0	–2	1	
r_1	0	0	$\boxed{2}$	–1	1	0		1	
s_2	0	0	4	0	0	1		2	

Basic variables	r	x_1	x_2	s_1	r_1	s_2	r_3	Solution	
x_1	0	1	–1	0	0	0		1	
r	1	0	0	0	–1	0		0	
x_2	0	0	1	–1/2		0		1/2	optimal table
s_2	0	0	0	2		1		0	
x_1	0	1	0	1/2		0		3/2	

The given table is the optimal table of Phase-I, giving a BFS of given problem as

$x_1 = \dfrac{3}{2}, x_1 = \dfrac{1}{2}, s_2 = 0$.

Remark:

1. Remarks 1 and 2 of M-method hold for Phase-I also.
2. As in M-method, if original problem has a solution, then, Equation (10) has a solution with $r_1 = 0 = r_3$.
3. If in the optimal table of Phase-I, all artificial variables are not zero, clearly Min $r \neq 0$, then original problem has no solution.

Phase-II: In the above problem, x_2, s_2 and x_1 rows in optimal table of Phase-I gives a BFS of given problem. We now this use as the starting BFS to get optimal solution of given problem. We now replace r-row by z-row in the optimal table of Phase-I by the given objective function, that is by z-row and omit artificial variables. This gives initial table. Then make all entries corresponding to basic variables x_2, s_2 and x_1 zero by multiplying x_1 and x_2 rows by 3 and 2 and adding to z-row.

Basic variables	z	x_1	x_2	s_1	s_2	Solution	
z	1	–3	–2	0	0	0	
x_2	0	0	1	–1/2	0	1/2	initial table
s_2	0	0	0	2	1	0	
x_1	0	1	0	1/2	0	3/2	
z	1	0	0	–5/2	0	11/2	
x_2	0	0	1	–1/2	0	1/2	starting table
s_2	0	0	0	2	1	0	
x_1	0	1	0	1/2	0	3/2	

Above table is the starting simple table of Phase-II. The above table is also optimal table of phase-II of the problem as all entries in z row ≤ 0 (minimization problem). Hence, the optional solution of the LPP is

$$x_1 = 3/2, x_2 = 1/2, \text{minimize } z = 11/2 .$$

Example 22 Solve the following LPP by Two-phase method.

Maximize $z = -4x_1 - 3x_2 - 9x_3$

Subject to the constraints

$$2x_1 + 4x_2 + 6x_3 \geq 15$$
$$6x_1 + x_2 + 6x_3 \geq 12$$
$$x_1, x_2, x_3 \geq 0$$

Phase-I: Minimize $r = r_1 + r_2$

Subject to the constraints

$$2x_1 + 4x_2 + 6x_3 - s_1 + r_1 = 15$$
$$6x_1 + x_2 + 6x_3 - s_2 + r_2 = 12$$
$$\text{all variables} \geq 0$$

Basic variables	r	x_1	x_2	x_3	s_1	s_2	r_1	r_2	**Solution**	
r	1	0	0	0	0	0	–1	–1	0	
r_1	0	2	4	6	–1	0	1	0	15	initial table
r_2	0	6	1	6	0	–1	0	1	12	
r	1	8	5	12↓	–1	–1	0	0		
r_1	0	2	4	6	–1	0	1	0	15	starting table
$\leftarrow r_2$	0	6	1	$\boxed{6}$	0	–1	0	1	12	
r	1	–4	3↓	0	–1	1	0		3	
r_1	0	–4	$\boxed{3}$	0	–1	1	1		3	
x_3	0	1	$\frac{1}{6}$	1	0	$-\frac{1}{6}$	0		2	
r	1	0	0	0	0	0			0	
x_2	0	–4/3	1	0	–1/3	1/3			1	
x_3	0	11/9	0	1	1/18	–2/9			11/6	

Phase-II: Phase-II is shown in the following table

Basic variables	z	x_1	x_2	x_3	s_1	s_2	**Solution**	
z	1	4	3	9	0	0	0	
x_2	0	–4/3	1	0	–1/3	1/3	1	initial table
x_3	0	11/9	0	1	1/18	–2/9	11/6	
z	1	–3↓	0	0	1/2	1	–39/2	
x_2	0	–4/3	1	0	–1/3	1/3		starting table
$\leftarrow x_3$	0	$\boxed{11/9}$	0	1	1/18	2/9		

Basic variables	z	x_1	x_2	x_3	s_1	s_2	Solution	
z	1	0	0	$\frac{27}{11}$	$\frac{7}{11}$	$\frac{5}{11}$	–15	
x_2	0	0	1	$\frac{12}{11}$	$-\frac{3}{11}$	$\frac{1}{11}$	3	optimal table
x_1	0	1	0	$\frac{9}{11}$	$\frac{1}{22}$	$-\frac{2}{11}$	3/2	

Optimal solution of the given LPP is

$$x_1 = \frac{3}{2}, x_2 = 3, x_3 = 0, \text{ Maximize } z = -15$$

***Remark*:** In phase-I of Two-phase method, the objective function is always minimization in nature which is sum of all artificial variables (In case of both types, i.e., minimization and maximization problem).

13.10 EXCEPTIONAL CASES IN LPP

We will discuss some important exceptional cases which are often encountered in the simplex method. The cases discussed are as following:

(i) Non-existing feasible solution or no-feasible solution
(ii) Degeneracy
(iii) Unbounded solution
(iv) Alternate optimal solution

13.10.1 Non-existing Feasible Solution or No Feasible Solution

This case arises when no point satisfies all the constraints. If the number of variables and constraints is large, it will not be possible to recognize this situation before hand. If the standard form of the problem does not involve any artificial variable, this case will not occur. If use of artificial variable is necessary the problem has no feasible solution, if

(a) An artificial variable is non-zero in the optimal table when M-method is used.
(b) Minimum value of the sum of artificial variables obtained after the phase-I of the two-phase method is not zero.

The following example will explain (a) and (b) respectively (solve by both M-method and Two-phase method).

Example 23 Min $z = x_1 - x_2$

Subject to the constraints

$$x_1 + x_2 \leq 1$$
$$2x_1 + 3x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Solution M-method

Minimize $z = x_1 - x_2 + Mr_2$

Subject to the constraints

$$x_1 + x_2 + s_1 = 1$$
$$2x_1 + x_2 - s_2 + r_2 = 6$$
$$\text{all var.} \geq 0$$

Basic variables	z	x_1	x_2	s_2	s_1	r_2	Solution	
z	1	–1	1	0	0	–M	0	
s_1	0	1	1	0	1	0	1	initial table
r_2	0	2	1	–1	0	1	6	
z	1	–1+2M	1+2M↓	–M	0	0	6 M	
$\leftarrow s_1$	0	1	$\boxed{1}$	0	1	0	1	starting table
r_2	0	2	1	–1	0	1	6	
z	1	–2	0	–M	–1–2M	0	–1+ 4 M	
x_2	0	1	1	0	1	0	1	optimal table
r_2	0	1	0	–1	–1	1	5	

In the above table $r_2 \neq 0 \Rightarrow$ LPP has no feasible solution.

Two-Phase method

Phase-I: Min $r = r_2$

Subject to the constraints

$$x_1 + x_2 + s_1 = 1$$
$$2x_1 + x_2 - s_2 + r_2 = 6$$
$$\text{all variables} \geq 0$$

Basic variables	r	x_1	x_2	s_2	s_1	r_2	Solution	
r	1	0	0	0	0	–1	0	
s_1	0	1	1	0	1	0	1	initial table
r_2	0	2	1	–1	0	1	6	
r	1	2 ↓	1	–1	0	0	6	
$\leftarrow s_1$	0	$\boxed{1}$	1	0	1	0	1	starting table
r_2	0	2	1	–1	0	1	6	
r	1	0	–1	–1	–2	0	4	
x_1	0	1	1	0	1	0	1	optimal table
r_2	0	0	–1	–1	–2	1	4	

$\rightarrow r_2 \neq 0$ in the above Table, hence, the given LPP has no-feasible solution.

13.10.2 Degeneracy

A basic solution is called degenerate, if the value of at least one of the basic variables is zero. Otherwise, the basic solution is called non-degenerate. If, in the simplex method there is a tie between two or more variables to leave the basis, we can select any one of them to leave the basis but the new basic solution thus obtained will have the remaining such variables at zero level and so this new solution will be degenerate three cases arise:

(a) We may obtain a degenerate optimal solution
(b) We may obtain a non-degenerate optimal solution (This is called temporary degeneracy)
(c) After applying some iterations to a degenerate solution, the same solution may appear again and the optimal solution is never obtained. This is called cycling. There are methods for tackling this problem. But we will not consider these here.

Example 24 Maximize $z = -x_1 + 4x_2 - 2x_3$

Subject to the constraints

$$x_1 + 2x_2 - 4x_3 \le 4$$
$$2x_1 + 2x_2 - 6x_3 \le 4$$
$$x_1 + x_2 + 2x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0$$

Solution Convert the LPP in standard form and then solve by simplex method.

Maximize $z = -x_1 + 4x_2 - 2x_3$

Subject to

$$x_1 + 2x_2 - 4x_3 + s_1 = 4$$
$$2x_1 + 2x_2 - 6x_3 + s_2 = 4$$
$$x_1 + x_2 + 2x_3 + s_3 = 3$$
$$\text{all variables} \ge 0$$

Basic variables	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
z	1	1	$-4 \downarrow$	2	0	0	0	0	
$\leftarrow s_1$	0	1	$\boxed{2}$	–4	1	0	0	4	
s_2	0	2	2	–6	0	1	0	4	
s_3	0	1	1	2	0	0	1	3	
z	1	3	0	$-6 \downarrow$	2	0	0	8	
x_2	0	$\frac{1}{2}$	1	–2	$\frac{1}{2}$	0	0	2	
s_2	0	1	0	–2	–1	1	0	0	degenerate solution
$\leftarrow s_3$	0	$\frac{1}{2}$	0	$\boxed{4}$	$-\frac{1}{2}$	0	1	1	
z	1	15/4	0	0	5/4	0	3/2	19/2	

(*Contd.*)

Basic variables	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
x_2	0	$\frac{3}{4}$	1	0	$\frac{1}{4}$	0	$\frac{1}{2}$	5/2	optimal table
s_2	0	5/4	0	0	–5/4	1	$\frac{1}{2}$	$\frac{1}{2}$	
x_3	0	$\frac{1}{8}$	0	1	$-\frac{1}{8}$	0	$\frac{1}{4}$	1/4	

If you see, it is a case of temporary degeneracy as in second table basic variable $s_2 = 0$ but optimal solution is non-degenerate i.e., $x_1 = 0,\ x_2 = 5/2, x_3 = \frac{1}{4}$ and maximize $z = \frac{19}{2}$.

13.10.3 Unbounded Solution

Unbounded solution can occur when the solution space in unbounded. The solution space is unbounded if while applying simplex method, it is observed that in some simplex table all the entries in a column corresponding to a non-basic variable are non-positive in all the rows except the z-row and entry in the z-row is non-zero. This happens because this variable can enter the basis at an arbitrary level without violating the constraints. The optimal solution may or may not be finite. However, in the following situations the optimal solution will be unbounded.

If the entry referred to above in z-row in a maximization (minimization) problem is negative (positive) i.e., if all the entries of the column of entering variable ≤ 0, then LPP has unbounded solution.

Example 25 Minimize $z = -4x_1 + x_2$

Subject to

$$x_1 - 2x_2 \le 2$$
$$-2x_1 + x_2 \le 2$$
$$x_1, x_2 \ge 0$$

Solution Standard form of the given LPP is as follows:

Minimize $z = -4x_1 + x_2$

Subject to

$$x_1 - 2x_2 + s_1 = 2$$
$$-2x_1 + x_2 + s_2 = 2$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Solve the above problem by simplex method.

Basic variables	z	x_1	x_2	s_1	s_2	Solution
z	1	4 ↓	–1	0	0	0
← s_1	0	$\boxed{1}$	–2	1	0	2
s_2	0	–2	1	0	1	2

Basic variables	z	x_1	x_2	s_1	s_2	Solution
z	1	0	7	–4	0	–8
x_1	0	1	–2	1	0	2
s_2	0	0	–3	2	1	6

In the given table x_2 is to enter in the basis but all the entries in that column are negative so no variable can leave the basis, hence, the LPP has unbounded solution.

13.10.4 Alternative Optimal Solution

If in the optimal table, coefficient of a non-basic variable in z-row is zero, there are alternative optimal solutions. Another basic optimal solution can be obtained by making such non-basic variable enter the basis. Any convex linear combation of basic optimal solution is an optimal solution through it is not basic.

Example 26 Maximize $z = 6x_1 - 3x_2$

Subject to

$$\begin{aligned} x_1 + x_2 &\le 1 \\ 2x_1 - x_2 &\le 1 \\ -x_1 + 2x_2 &\le 1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution Write the given LPP in standard form and then solve by simplex method.

Maximize $z = 6x_1 - 3x_2$

Subject to

$$\begin{aligned} x_1 + x_2 + s_1 &= 1 \\ 2x_1 - x_2 + s_2 &= 1 \\ -x_1 + 2x_2 + s_3 &= 1 \\ \text{all variables} &\ge 0 \end{aligned}$$

Basic variables	z	x_1	x_2	s_1	s_2	s_3	Solution	
z	1	–6	3	0	0	0	0	
s_1	0	1	1	1	0	0	1	
s_2	0	$\boxed{2}$	–1	0	1	0	1	
s_3	0	–1	2	0	0	1	1	
z	1	0	0	0	3	0	3	

(*Contd.*)

Basic variables	z	x_1	x_2	s_1	s_2	s_3	Solution	
s_1	0	0	$\boxed{3/2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	
x_1	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	optimal table
s_3	0	0	3/2	0	$\frac{1}{2}$	1	3/2	

In the above table (optimal) x_2 is non-basic variable but its coefficient in z-row is zero, so the given LPP has alternate solution. If we enter x_2 variable in the basis then s_1 leaves and we get the following simplex table which is optimal.

Basic variables	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	0	0	0	3	0	3
x_1	0	0	1	2/3	$-\frac{1}{3}$	0	1/3
x_1	0	1	0	1/3	$\frac{1}{3}$	0	2/3
s_3	0	0	0	–1	1	1	1

The optimal solutions of the LPP are as follows:

(i) $x_1 = \frac{1}{2}, x_2 = 0, \max z = 3$

(ii) $x_1 = \frac{2}{3}, x_2 = \frac{1}{3}, \max z = 3$

(iii) $x_1 = \lambda\left(\frac{1}{2}\right) + (1-\lambda)\frac{2}{3}, 0 \le \lambda \le 1$

$x_2 = \lambda(0) + (1-\lambda)\frac{1}{3}, 0 \le \lambda \le 1$

$\Rightarrow$ $X = \lambda(X_1) + (1-\lambda)X_2$ is called convex linear combination of X_1, X_2 for $0 \le \lambda \le 1$.

EXERCISE 13.6

Solve the following LP Problem by M-method

1. Maximize $z = 5x_1 - 2x_2 + 3x_3$
   Subject to

$$2x_1 + 2x_2 - x_3 \ge 2,\ 3x_1 - 4x_2 \le 3,\ x_2 + 3x_3 \le 5,\ x_1, x_2, x_3 \ge 0$$

[Mumbai 2004]

2. Maximize $z = x_1 + 2x_2 + 3x_3 - x_4$

   Subject to

   $$x_1 + 2x_2 + 3x_3 = 15,\ 2x_1 + x_2 + 5x_3 = 20,\ x_1 + 2x_2 + x_3 + x_4 = 10,\ x_1, x_2, x_3, x_4 \ge 0$$

   [Madras 2003, IAS 1995, Meerut 1996, Delhi 1997]

3. Maximize $z = 6x_1 + 4x_2$

   Subject to

   $$2x_1 + 3x_2 \le 30,\ 3x_1 + 2x_2 \le 24,\ x_1 + x_2 \ge 3;\ x_1, x_2 \ge 0$$

   Is the solution unique? If not give two different solutions. **[Jodhpur 1993]**

4. Maximize $z = 3x_1 + 2x_2$

   Subject to

   $$2x_1 + x_2 \le 2,\ 3x_1 + 4x_2 \ge 12,\ x_1, x_2 \ge 0$$

   [Madurai 1989, Bharathidesan 1995]

5. Minimize $z = 12x_1 + 20x_2$

   Subject to

   $$6x_1 + 8x_2 \ge 100,\ 7x_1 + 12x_2 \ge 120,\ x_1, x_2 \ge 0$$ **[Bharthidesan 1990, 1993]**

6. Minimize $z = 5x_1 - 6x_2 - 7x_3$

   Subject to

   $$x_1 + 5x_2 - 3x_3 \ge 15,\ 5x_1 - 6x_2 + 10x_3 \ge 0,\ x_1 + x_2 + x_3 = 5,\ x_1, x_2, x_3 \ge 0$$

   [Delhi 1996]

7. Maximize $z = 3x_1 + 2x_2 + x_3$

   Subject to

   $$-3x_1 + 2x_2 + 2x_3 + 2x_3 = 8,\ -3x_1 + 4x_2 + x_3 = 7,\ x_1, x_2, x_3 \ge 0$$

   [Karnataka 1994]

8. Maximize $z = 2x_1 + 3x_2$

   Subject to

   $$x_1 + 2x_2 \le 4,\ x_1 + x_2 = 3;\ x_1, x_2 \ge 0$$ **[North Bengal 1990]**

Use Two-Phase method to solve the following LP problems:

9. Minimize $z = x_1 + x_2$

   Subject to

   $$2x_1 + x_2 \ge 4$$
   $$x_1 + 7x_2 \ge 7$$
   $$x_1, x_2 \ge 0$$

   [Rajasthan 2005, Bharthidesan 1994]

10. Maximize $z = 5x_1 + 3x_2$

    Subject to

    $$2x_1 + x_2 \le 1,\ x_1 + 4x_2 \ge 6,\ x_1, x_2 \ge 0$$ **[Kottayam 2005]**

11. Maximize $z = 5x_1 - 4x_2 + 3x_3$

    Subject to

    $$2x_1 + 2x_2 - x_3 \ge 2,\ 3x_1 - 4x_2 \le 3,\ x_2 + x_3 \le 5,\ x_1, x_2, x_3 \ge 0$$

    [Mumbai, 2009]

12. Minimize $z = 5x_1 + 8x_2$
Subject to

$$3x_1 + 2x_2 \geq 3,\ x_1 + 4x_2 \geq 4,\ x_1 + x_2 \leq 5,\ x_1, x_2 \geq 0$$ **[Meerut, 1995]**

13. Minimize $z = x_1 + x_2 + x_3$
Subject to

$$x_1 - 3x_2 + 4x_3 = 5,\ x_1 - 2x_2 \leq 3,\ 2x_2 + x_3 \geq 4$$

$x_1 \geq 0$, $x_2 \geq 0$ and x_3 unrestricted in sign.

[Delhi, 1995]

14. Maximize $z = 12x_1 + 15x_2 + 9x_3$
Subject to

$$8x_1 + 16x_2 + 12x_3 \leq 250,\ 4x_1 + 8x_2 + 10x_3 \geq 80,\ 7x_1 + 9x_2 + 8x_3 \geq 105$$

$$x_1, x_2, x_3 \geq 0$$

[Delhi, 1994]

15. Maximize $z = 5x_1 - 4x_2 + 3x_3$
Subject to

$$2x_1 + x_2 - 6x_3 = 20,\ 6x_1 + 5x_2 + 10x_3 \leq 76,\ 8x_1 - 3x_2 + 6x_3 \leq 50$$

$$x_1, x_2, x_3 \geq 0$$

[Dibrugarh, 1994]

Show that the following LP problem have unbounded solutions.

16. Maximize $z = 15x_1 + 6x_2 + 9x_3 + 2x_4$
Subject to

$$2x_1 + x_2 + 5x_3 + 0.6x_4 \leq 10$$

$$3x_1 + x_2 + 3x_3 + 0.25x_4 \leq 12$$

$$7x_1 + x_4 \leq 35,\ x_1, x_2, x_3, x_4 \geq 0$$

[Calicut, 1991]

17. Maximize $z = 107x_1 + x_2 + 2x_3$
Subject to

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$

$$16x_1 + x_2 - 6x_3 \leq 5$$

$$3x_1 - x_2 - x_3 \leq 0$$

$$x_1, x_2, x_3, x_4 \geq 0$$

[Meerut, 1992]

Answers

1. $x_1 = \dfrac{23}{3}, x_2 = 5, x_3 = 0$; max $z = 85/3$
2. $x_1 = x_2 = x_3 = \dfrac{5}{2}, x_4 = 0$; max $z = 15$
3. Problem has alternate solutions, they are

(i) $x_1 = 8, x_2 = 0$; max $z = 48$ (ii) $x_1 = \dfrac{12}{5}, x_2 = \dfrac{42}{5}$; max $z = 48$

4. Non-feasible solution
5. $x_1 = 15, x_2 = 5/4$; min $z = 205$
6. Non-feasible solution
7. Unbounded solution
8. $x_1 = 2, x_2 = 1$, max $z = 7$
9. $x_1 = \frac{21}{13}, x_2 = \frac{10}{13}$; max $z = \frac{31}{13}$
10. Non-feasible solution
11. $x_1 = \frac{23}{3}, x_2 = 5, x_3 = 0$; max $z = \frac{85}{3}$
12. $x_1 = 0, x_2 = 5$; min $z = 40$
13. $x_1 = 0, x_2 = \frac{21}{5}, x_3 = \frac{22}{5}$; min $z = \frac{43}{5}$
14. $x_1 = 6, x_2 = 7, x_3 = 0$; max $z = 177$
15. $x_1 = \frac{55}{7}, x_2 = \frac{30}{7}, x_3 = 0$; max $z = \frac{155}{7}$

13.11 DUALITY IN LINEAR PROGRAMMING

With every LPP we shall be associating another LPP. The first is called primal problem, while the second is called dual problem. We shall see that the solution of primal and dual are closely related. To be specific, it is sufficient to solve either of them. We solve the one which has smaller number of constraints. Besides this there are many other advantages of duality, which will be discussed. To define the dual problem of a LPP, we first define canonical form of a LPP.

13.11.1 Canonical Form of a LPP

A LPP is said to be in canonical form, if

(a) it is a minimization problem
(b) all constraints are ≥ type (no restriction on $b_i'^{s}$)
(c) all decision variables ≥ 0

Thus the LPP
Minimize $z = c^T X$
Subject to

$$AX \geq b,\ X \geq 0 \text{ is in canonical form.}$$

We know that

(a) Maximize $z = c^T X$ can be converted into
Minimize $z = -c^T X$

(b) $\alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n \leq b_i$ can be converted into $-\alpha_{i1}x_1 - \alpha_{i2}x_2 - \cdots - \alpha_{in}x_n \geq -b_i$ and

(c) $\alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n = b_i$ can be replaced by two constraints

$$\alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n \geq b_i$$

$$\alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n \leq b_i \Rightarrow -\alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n \geq -b_i$$

This suggests that every LPP can be converted into another LPP in canonical form.

Example 27 Transform the following LPP in canonical form.

Maximize $z = 3x_1 - 2x_2 + 4x_3$
Subject to

$$x_1 + x_2 + x_3 \leq 230,\ 2x_1 + 3x_2 - 4x_3 \geq 170,\ x_1 - x_2 - x_3 = 50$$

$$x_1, x_2, x_3 \geq 0$$

Solution Canonical form of the given LPP:

Minimize $z = -3x_1 + 2x_2 - 4x_3$

Subject to

$$-x_1 - x_2 - x_3 \ge -230,\ 2x_1 + 3x_2 - 4x_3 \ge 170$$
$$x_1 - x_2 - x_3 \ge 50, -x_1 + x_2 + x_3 \ge -50$$
$$x_1, x_2, x_3 \ge 0$$

13.11.2 Dual of a LPP

As seen above, every LPP can be transformed into canonical form we shall now define dual of a LPP. Let a LPP be in canonical form

$$\left.\begin{array}{l}\text{Minimize } z = c^T X \\ \text{subject to } AX \ge b,\ X \ge 0\end{array}\right\} \tag{11}$$

The dual of Eq. (11) is the LPP

$$\left.\begin{array}{l}\text{Maximize } \quad w = b^T Y \\ \text{subject to } A^T Y \le c, Y \ge 0 \\ \qquad Y = (y_1, y_2, \ldots, y_m)^T\end{array}\right\} \tag{12}$$

Eq. (12) is called dual and Eq. (11) is called primal.

We notice that dual is a maximization problem obtained from a minimization problem in canonical form, objective function is obtained by multiplying b^T with $Y = (y_1, y_2, \ldots, y_m)^T$ where m is the number of constraints of primal LPP and $y_1, y_2, \ldots, y_n$ are known as dual variables associated with each primal constraint. Constraints of dual are $A^T y \le c$. Thus if primal has n-variables and m constraints its dual will have n constraints and m variables.

Example 28 Formulate the dual of the LPP

Maximize $z = 5x_1 + 6x_2$

Subject to

$$x_1 + 9x_2 \ge 60$$
$$2x_1 + 3x_2 \le 45$$
$$x_1, x_2 \ge 0$$

Solution The canonical form of the above LPP is

Minimize $z = -5x_1 - 6x_2$

Subject to

$$x_1 + 9x_2 \ge 60$$
$$-2x_1 - 3x_2 \ge -45$$
$$x_1, x_2 \ge 0$$

Let y_1 and y_2 be the dual variables associated with first and second constraint respectively. Then dual of given LPP is

Maximize $w = 60\, y_1 - 45\, y_2$

Subject to

$$y_1 - 2y_2 \le -5$$
$$9y_1 - 3y_2 \le -6$$
$$y_1, y_2 \ge 0$$

We shall now prove an interesting result, which says that dual of a dual is primal itself.

***Theorem*:** The dual of dual is primal.

***Proof*:** Let primal (in canonical form) be

$$\left.\begin{array}{l} \text{Minimize} \quad z = c^T X \\ \text{subject to: } AX \ge b_1,\ X \ge 0 \end{array}\right\} \qquad (13)$$

Dual of Eq. (13) is

$$\left.\begin{array}{l} \text{Maximize } w = b^T Y \\ \text{subject to}: A^T Y \le c,\ y \ge 0 \end{array}\right\} \qquad (14)$$

To write dual of Eq. (14), write this in canonical form, which is as follows:

$$\left.\begin{array}{l} \text{Minimize} \quad w = -b^T Y \\ \text{subject to}: -A^T Y \ge -c,\ y \ge 0 \end{array}\right\} \qquad (15)$$

Now dual of Eq. (15) is

$$\left.\begin{array}{l} \text{Maximize} \quad v = -c^T Z \\ \text{subject to: } (-A^T)^T Z \le -b^T,\ Z \ge 0 \end{array}\right\} \qquad (16)$$

Eq. (16) can be written as

$$\left.\begin{array}{l} \text{Minimize } v = c^T Z \\ \text{subject to: } AZ \ge b,\ Z \ge 0 \end{array}\right\} \qquad (17)$$

Eq. (17) and Eq. (13) are same except for the variable notation Z. Hence, the result

Thus we find that if primal is Eq. (13), its dual is Eq. (14), and if primal is Eq. (14), its dual is Eq. (13). In other words we can say that, if primal is

(a) Minimization problem with constraints $\ge$ type

(b) Maximization problem with constraints $\le$ type, then dual of (a) is of (b) type and dual of (b) is of (a) type. This suggests that dual can be written directly without converting it into canonical form.

Example 29 Write the dual of

Maximize $z = 5x_1 + 6x_2$

Subject to

$$x_1 + 9x_2 \ge 60$$
$$2x_1 + 3x_2 \le 45$$
$$x_1, x_2 \ge 0$$

Solution The given LPP can be written as

Maximize $z = 5x_1 + 6x_2$

Subject to

$$-x_1 - 9x_2 \leq -60$$
$$2x_1 + 3x_2 \leq 45$$
$$x_1, x_2 \geq 0$$

Let y_1, y_2 be the dual variables associated with first and second constraints respectively, then its dual is

Minimize $w = -60y_1 + 45y_2$

Subject to

$$-y_1 + 2y_2 \geq 5$$
$$-9y_1 + 3y_2 \geq 6$$
$$y_1, y_2 \geq 0$$

Let us now take an example, in which there is constraint with = sign.

Example 30 Write the dual of

Minimize $z = 2x_1 + 3x_2 + 4x_3$

Subject to

$$2x_1 + 2x_2 + 3x_3 \leq 4$$
$$3x_1 + 4x_2 + 5x_3 \geq 5$$
$$x_1 + x_2 + x_3 = 7$$
$$x_1, x_2, x_3 \geq 0$$

Solution The given LPP can be written as

Minimize $z = 2x_1 + 3x_2 + 4x_3$

Subject to

$$-2x_1 - 2x_2 - 3x_3 \geq -4$$
$$3x_1 + 4x_2 + 5x_3 \geq 5$$
$$x_1 + x_2 + x_3 \geq 7$$
$$-x_1 - x_2 - x_3 \geq -7$$
$$x_1, x_2, x_3 \geq 0$$

Let y_1, y_2, y_3 and y_4 be dual variables associated with I, II, III and IV constraints. Then dual of given LPP is

Maximize $w = -4y_1 + 5y_2 + 7y_3 - 7y_4$

Subject to

$$-2y_1 + 3y_2 + y_3 - y_4 \leq 2$$
$$-2y_1 + 4y_2 + y_3 - y_4 \leq 3$$
$$-3y_1 + 5y_2 + y_3 - y_4 \leq 4$$
$$y_1, y_2, y_3, y_4 \geq 0$$

In the above dual problem, we find that $y_3 - y_4$ appears as one identity, so, if we replace $y_3 - y_4 = y_5$, then dual problem becomes.

Minimize $w = -4y_1 + 5y_2 + 7y_5$

Subject to

$$-2y_1 + 3y_2 + y_5 \le 2$$
$$-2y_1 + 4y_2 + y_5 \le 3$$
$$-3y_1 + 5y_2 + y_5 \le 4$$
$$y_1, y_2 \ge 0$$

but y_5 is unrestricted in sign as $y_5 = y_3 - y_4$ may have any sign as $y_3, y_4 \ge 0$. Thus we see that dual can directly be written without converting an equal type constraints into $\le$ and $\ge$ type constraints. Only one thing we have to keep in mind is that the dual variable associated with = type constraint have to be kept unrestricted in sign.

EXERCISE 13.7

Write the dual of the following problems (1 –5)

1. Minimize $z = 3x_1 - 3x_2 + x_3$
   subject to
   $$2x_1 - 3x_2 + x_3 \le 5,\ 4x_1 - 2x_2 \ge 9,\ -8x_1 + 4x_2 + 3x_3 = 8$$
   $x_1, x_2 \ge 0$ and x_3 unrestricted in sign. **[Mumbai 2004]**
2. Maximize $z = 5x_1 - 7x_2$
   Subject to
   $$x_1 - 9x_2 \le 9,\ -2x_1 + 3x_2 \le 6,\ x_1 + x_2 \le 12,\ x_1 \ge 2,\ x_1, x_2 \ge 0$$
3. Maximize $z = 5x_1 + 12x_2 + 4x_3$
   Subject to
   $$x_1 + 2x_2 + x_3 \le 5,\ 2x_1 - x_2 + 3x_3 = 2,\ x_1, x_2, x_3 \ge 0$$
4. Maximize $z = 5x_1 + 3x_2$
   Subject to
   $$3x_1 + 5x_2 \le 15,\ 5x_1 + 2x_2 \le 10,\ x_1, x_2 \ge 0$$
   [Meerut 1991]
5. Minimize $z = x_1 + x_2 + x_3$
   Subject to
   $$x_1 - 3x_2 + 4x_3 = 5,\ x_1 - 2x_2 \le 3,\ 2x_2 - x_3 \ge 4$$
   $x_1, x_2 \ge 0$, x_3 unrestricted **[Madras 2000, Jodhpur 1993, Meerut 1994]**
6. Write the dual of the following LPP
   Minimize $z = 4x_1 + 6x_2 + 18x_3$
   Subject to
   $$x_1 + 3x_2 \ge 3,\ x_2 + 2x_3 \ge 5,\ x_1, x_2 \ge 0$$
   and show that dual of dual is primal it self.

7. Show that the dual of the dual in case of following LPP is given LPP it self.
   Maximize $z = 6x_1 + 4x_2 + 18x_3$
   Subject to
   $$x_2 + 3x_3 \geq 6,\ x_1 + 2x_2 \leq 4,\ x_1, x_2 \geq 0$$

Answers

1. Maximize $w = -5y_1 + 9y_2 + 8y_3$
   Subject to
   $$-2y_1 + 4y_2 - 8y_3 \leq 3,\ 3y_1 - 2y_2 + 4y_3 \leq -2,\ -y_1 + 3y_3 = 1,$$
   $$y_1, y_2 \geq 0 \text{ and } y_3 \text{ unrestricted}$$

2. Minimize $w = 9y_1 + 6y_2 + 12y_3 - 2y_4$
   Subject to
   $$y_1 - 2y_2 + y_3 - y_4 \geq 5$$
   $$-9y_1 + 3y_2 + y_3 \geq -7$$
   $$y_1, y_2, y_3, y_4 \geq 0$$

3. Minimize $w = 5y_1 + 2y_2$
   Subject to
   $$y_1 + 2y_2 \geq 5$$
   $$2y_1 - y_2 \geq 12$$
   $$y_1 + 3y_2 \geq 4$$
   $$y_1 \geq 0,\ y_2 \text{ unrestricted}$$

4. Minimize $w = 15\,y_1 + 10y_2$
   Subject to
   $$3y_1 + 5y_2 \geq 5,\ 5y_1 + 2y_2 \geq 3,\ y_1, y_2 \geq 0$$

5. Maximize $w = 5y_1 - 3y_2 + 4y_3$
   Subject to
   $$y_1 - y_2 \leq 1, -3y_1 + 2y_2 + 2y_3 \leq 1,\ 4y_1 - y_3 = 1$$
   $$y_1 \text{ unrestricted},\ y_2, y_3 \geq 0$$

13.11.3 Duality Principle

Now we shall move towards establishing relations between the solutions of primal and dual.

Let us consider the primal as

Minimize $z = c^T X$

Subject to

$$AX \geq b,\ x \geq 0$$
$$X = (x_1, x_2, \ldots, x_n)^T;\ b = (b_1, b_2, \ldots, b_m)^T$$
$$c = (c_1, c_2, \ldots, c_n)^T;\ A = (\alpha_{ij})_{m\times n}$$

and its dual as
Maximize $w = b^T Y$
Subject to

$$A^T Y \le c, Y \ge 0 \text{ where } Y = (y_1, y_2, ..., y_m)^T$$

***Theorem* 1:** If X is a feasible solution of primal and Y be the feasible solution of dual, then

$$\text{Minimize } z \ge \text{maximize } w \text{ i.e., } c^T X \ge b^T Y.$$

***Theorem* 2:** If X^* is a feasible solution of the primal and Y^* a feasible solution of the dual such that minimize z = maximize w then X^* and Y^* are optimal solutions of primal and dual respectively.

***Theorem* 3:** If primal has an optimal solution, then its dual also has an optimal solution.

***Remark*:** It is to mention that if

	Primal	then	Dual
1.	Has bounded optimal solution.		Has bounded optimal solution.
2.	Has unbounded solution.		Has no feasible solution.
3.	Has no feasible solution.		Has either unbounded solution or no feasible solution.

Now, we state an interesting theorem without proof known as complementary slackness theorem which brings closer the solution of primal and dual.

***Theorem* 4:** (complementary slackness theorem)

(a) If in a primal, any slack/surplus variable, say, s_k, appears as the basic variable in optimal solution then corresponding dual variable y_k is zero in the optimal dual solution.

(b) If, in a primal, any decision variable, say, x_k, appears as the basic variable in the optimal solution then the corresponding k^{th} constraint in dual holds an equality i.e., $k^{-\text{th}}$ dual slack/surplus variable is zero.

Example 31 Let the LPP be

$$\left.\begin{aligned} &\text{Maximize } z = 5x_1 + 6x_2 \\ &\text{subject to} \\ &x_1 + 9x_2 \ge 60 \\ &2x_1 + 3x_2 \le 45 \end{aligned}\right\} \tag{18}$$

Solve the LPP and its dual by M-method.

Solution: The dual of given LPP is

$$\left.\begin{aligned} &\text{Minimize } w = -60y_1 + 45y_2 \\ &\text{subject to} \\ &-y_1 + 2y_2 \ge 5 \\ &-9y_1 + 3y_2 \ge 6 \\ &y_1, y_2 \ge 0 \end{aligned}\right\} \tag{19}$$

We solve now both the problems by M-method one by one.

The primal and dual in standard form can be written as

$$\left.\begin{aligned} &\text{Maximize } z = 5x_1 + 6x_2 - Mr_1 \\ &\text{subject to} \\ &\qquad x_1 + 9x_2 - s_1 + r_1 = 60 \\ &\qquad 2x_1 + 3x_2 + s_2 = 45 \\ &\qquad\quad \text{all variables} \geq 0 \end{aligned}\right\} \tag{20}$$

and

$$\text{Minimize } w = -60y_1 + 45y_2 + Mr_1 + Mr_2$$

Subject to

$$\left.\begin{aligned} -y_1 + 2y_2 - s_1 + r_1 &= 5 \\ -9y_1 + 3y_2 - s_2 + r_2 &= 6 \\ \text{all variables} &\geq 0 \end{aligned}\right\} \tag{21}$$

Apply simplex method to find solution of primal and dual respectively.
Solution of primal:

Basic variables	z	x_1	x_2	s_1	r_1	s_2	Solution	
z	1	–5	–6	0	M	0	0	
r_1	0	1	9	–1	1	0	60	initial table (primal) IPT
s_2	0	2	3	0	0	1	45	
z	1	–5–M	–6–9 M↓	M	0	0	–60 M	
$\leftarrow r_1$	0	1	$\boxed{9}$	–1	1	0	60	Starting primal table (SPT)
s_2	0	2	3	0	0	1	45	
z	1	$-\frac{13}{3}\downarrow$	0	–2/3	$\frac{2}{3}+M$	0	40	
x_2	0	$\frac{1}{9}$	1	$-\frac{1}{9}$	$\frac{1}{9}$	0	20/3	PT_1 (Primal Table 1)
$\leftarrow s_2$	0	$\boxed{5/3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	25	
z	1	0	0	1/5	–1/5 +M	13/5	105	
x_2	0	0	1	–2/15	2/15	–1/15	5	OPT (optimal table)
x_1	0	1	0	1/5	–1/5	3/5	15	

Solution is: $x_1 = 15$, $x_2 = 5$; max $z = 105$

Solution of dual is

Basic variables	w	y_1	y_2	s_1	s_2	r_1	r_2	Solution	
w	1	60	–45	0	0	$-M$	$-M$	0	
r_1	0	–1	2	–1	0	1	0	5	initial dual table
r_2	0	–9	3	0	–1	0	1	6	
w	1	$60-10M$	$-45+5M\downarrow$	$-M$	$-M$	0	0	$11M$	
r_1	0	–1	2	–1	0	1	0	5	(SDT) Starting primal table
r_2	0	–9	$\boxed{3}$	0	–1	0	1	6	
w	1	$-75+5M\downarrow$	0	$-M$	$-15+\frac{2}{3}M$	0	$-15+\frac{5}{3}M$	$90+M$	
r_1	0	$\boxed{5}$	0	–1	2/3	1	–2/3	1	DT_1 (Primal Table 1)
y_2	0	–3	1	0	$-\frac{1}{3}$	0	1/3	2	
w	1	0	0	–15	–5	$15-M$	$5-M$	105	
y_1	0	1	0	–1/5	2/15	1/5	–2/15	$\frac{1}{5}$	(ODT) optimal dual table
y_2	0	0	1	–3/5	1/15	3/5	–1/15	$\frac{13}{5}$	

Solution is $y_1 = \frac{1}{5}, y_2 = \frac{13}{5}$, minimize $w = 105$

From the above solutions of primal and dual, we note the following:

1. In any simplex table, the matrix under starting basis is B^{-1} at that step.

In PT_1: $\begin{bmatrix} \frac{1}{9} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 3 & 1 \end{bmatrix}^{-1}$

In OPT: $\begin{bmatrix} \frac{2}{15} & -\frac{1}{15} \\ -\frac{1}{5} & 3/5 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 3 & 2 \end{bmatrix}^{-1}$

In DT_1: $\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1}$

In ODT: $\begin{bmatrix} \frac{1}{5} & -\frac{2}{15} \\ \frac{3}{5} & -\frac{1}{15} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -9 & 23 \end{bmatrix}^{-1}$

Before discussing further, we define, simplex multiplier as $\pi = (\pi_1, \pi_2, ..., \pi_m)^T = c_B^T B^{-1}$ at any step. It is clear that if LPP has a bounded solution, then the simplex multiplier of optimal table is the required solution of the dual.

2. In PT1: $\pi = (6, 0)\begin{bmatrix} \frac{1}{9} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} = \left(\frac{2}{3}, 0\right)$

   In OPT: $\pi = (6, 5)\begin{bmatrix} \frac{2}{15} & -\frac{1}{15} \\ -\frac{1}{5} & 3/5 \end{bmatrix} = \left(-\frac{1}{5}, \frac{13}{5}\right)$

   In DT_1 : $\pi = (M, 45)\begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix} = \left[M,\ 15 - \frac{2}{3}M\right]$

   In ODT: $\pi = [-60, 45]\begin{bmatrix} 1/5 & -2/15 \\ 3/5 & -1/15 \end{bmatrix} = (15, 5)$

   Thus, π of OPT is an optimal solution of dual with sign of y_1 changed because b_1 is negative in primal. Similarly, π of ODT is an optimal solution of primal. Of course optimal value of both primal and dual are 105.

3. The simplex multiplier is used in many more ways. Entries in z-row.
   For example entry below variable x_i in z-row

$$Z_i - c_i = c_B^T\ B^{-1}\ A_i - c_i = \pi\ A_i - c_i\ .$$

   In PT 1:

   below x_1: $\left(\frac{2}{3}, 0\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} - (5) = -\frac{13}{3}$

   below x_2: $\left(\frac{2}{3}, 0\right)\begin{pmatrix} 9 \\ 3 \end{pmatrix} - 6 = 0$

   below s_1: $\left(\frac{2}{3}, 0\right)\begin{pmatrix} -1 \\ 0 \end{pmatrix} - 0 = -2/3$

   below s_2: $\left(\frac{2}{3}, 0\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = 0$

   below r_1: $\left(\frac{2}{3}, 0\right)\begin{pmatrix} +1 \\ 0 \end{pmatrix} - (-M) = \frac{2}{3} + M$

In OPT:

below x_1: $\left(-\frac{1}{5}, \frac{13}{5}\right)\begin{pmatrix}1\\2\end{pmatrix} - 5 = 0$

below x_2: $\left(-\frac{1}{5}, \frac{13}{5}\right)\begin{pmatrix}9\\3\end{pmatrix} - 6 = 0$

below s_1: $\left(-\frac{1}{5}, \frac{13}{5}\right)\begin{pmatrix}-1\\0\end{pmatrix} - 0 = \frac{1}{5}$

below s_2: $\left(-\frac{1}{5}, \frac{13}{5}\right)\begin{pmatrix}0\\1\end{pmatrix} - 0 = \frac{13}{5}$

below r_1: $\left(-\frac{1}{5}, \frac{13}{5}\right)\begin{pmatrix}+1\\0\end{pmatrix} - (-M) = -\frac{1}{5} + M$

same thing can be seen easily on dual tables.

4. As stated earlier, π of optimal table of primal is an optimal solution of the dual. But π to any step of simplex iteration need not be feasible solution of dual.

   In PTI: $\pi = \left(\frac{2}{3}, 0\right)$

   1st constraint: $-y_1 + 2y_2 \geq 5 \Rightarrow -\frac{2}{3} \geq 5$

   2nd constraint: $-9y_1 + 3y_2 \geq 6 \Rightarrow -6 \geq 6$

   Thus it does not even satisfy the dual constraint.

5. Solution at any step is $c_B^T B^{-1} b = \pi b$

   In PT1: $\left(\frac{2}{3}, 0\right)\begin{pmatrix}60\\45\end{pmatrix} = 40$

   In OPT: $\left(-\frac{1}{5}, \frac{13}{5}\right)\begin{pmatrix}60\\45\end{pmatrix} = 105$

   In DT1: $\left(M, 15 - \frac{2}{3}M\right)\begin{pmatrix}5\\6\end{pmatrix} = 90 + M$

   In ODT: $(15, 5)\begin{pmatrix}5\\6\end{pmatrix} = 105$

Example 32 Using duality solve the following LPP.

Minimize $z = 10x_1 + 5x_2 + 5x_3$
Subject to

$$5x_1 - 5x_2 - 3x_3 \geq 1$$
$$-x_1 + x_2 \geq -3$$
$$x_1 - x_3 \geq -7$$

$$-4x_1 + 4x_2 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution The dual of given LPP is

Maximize $w = y_1 - 3y_2 - 7y_3 + 5y_4$

Subject to

$$5y_1 - y_2 + y_3 - 4y_4 \leq 10$$
$$-5y_1 + y_2 + 4y_4 \leq 5$$
$$-3y_1 - y_3 + y_4 \leq 5$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Now solve dual by simplex method.

Basic variables	w	y_1	y_2	y_3	y_4	s_1	s_2	s_3	Solution
w	1	–1	3	7	–5 ↓	0	0	0	0
s_1	0	5	–1	1	–4	1	0	0	10
← s_2	0	–5	1	0	$\boxed{4}$	0	1	0	5
s_3	0	–3	0	–1	1	0	0	1	5
w	1	–29/4	17/4	7	0	0	5/4	0	25/4
s_1	0	0	0	1	0	1	1	0	15
y_4	0	–5/4	$\frac{1}{4}$	0	1	0	$\frac{1}{4}$	0	5/4
s_3	0	–7/4	–1/4	–1	0	0	–1/4	1	15/4

In the above table y_1 to enter but no variable is to leave as the entries in the corresponding column of y_1 are ≤ 0. Therefore the dual has unbounded solution and hence primal has no feasible solution.

EXERCISE 13.8

Using duality solve the following problems (1 – 3)

1. Maximize $z = 2x_1 + x_2$
   Subject to
   $$x_1 + 2x_2 \leq 10,\ x_1 + x_2 \leq 6, x_1 - x_2 \leq 2,\ x_1 - 2x_2 \leq 1,\ x_1, x_2 \geq 0$$
   [Andhra 2006, Karnataka 1992]
2. Minimize $z = 2x_1 + 9x_2 + x_3$
   Subject to
   $$x_1 + 4x_2 + 2x_3 \geq 5,\ 3x_1 + x_2 + 2x_3 \geq 4 \text{ and } x_1, x_2 \geq 0$$
   [J.N.T.U. 2001]
3. Maximize $z = 2x_1 + 4x_2 + 4x_3 - 3x_4$
   Subject to:
   $$x_1 + x_2 + x_3 = 4,\ x_1 + 4x_2 + x_4 = 8,\ x_1, x_2, x_3, x_4 \geq 0$$
   [Delhi 2002]

4. Maximize $z = 3x_1 + 2x_2 + 5x_3$
Subject to

$$x_1 + 2x_2 + x_3 \le a_1, 3x_1 + 2x_3 \le a_2, x_1 + 4x_2 \le a_3$$

where a_1, a_2 and a_3 are constraints. For specific values of a_1, a_2 and a_3 the optimal solution is

Basic variables	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	4	0	0	c_1	c_2	0	1350
x_2	0	b_1	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	0	b_2	0	1	0	$\frac{1}{2}$	0	c_3
s_3	0	b_3	0	0	–2	1	1	20

where b_1, b_2, b_3, c_1, c_2 and c_3 are constants. Determine
(a) the values of a_1, a_2 and a_3 that yield the given optimal situation.
(b) the values of b_1, b_2, b_3 and c_1, c_2, c_3 in the optimal table.
(c) the optimal dual solution. **[Delhi 1996]**

Write down the dual of the following LP problems (5–6) and solve them.

5. Maximize $z = 15x_1 + 10x_2$
Subject to

$$3x_1 + 5x_2 \ge 5, 5x_1 + 2x_2 \ge 3, x_1, x_2 \ge 0$$

[Meerut 1991]

6. Maximize $z = 5x_1 + 2x_2$
Subject to

$$6x_1 + x_2 \ge 6, 4x_1 + 3x_2 \ge 12, x_1 + 2x_2 \ge 4, x_1, x_2 \ge 0$$ **[Sambhalpur 1994]**

7. Use duality to solve the LPP
Minimize $z = x_1 - x_2 + 3x_3 + 2x_4$
Subject to

$$x_1 + x_2 \ge -1, x_1 - 3x_2 - x_3 \le 7, x_1 + x_3 - 3x_4 = -2 \; x_1, x_2, x_3, x_4 \ge 0$$

[Delhi 1995]

Answers

1. $x_1 = 4, x_2 = 2$; max $z = 10$

2. $x_1 = 0 = x_2, x_3 = \frac{5}{2}$; min $z = 5/2$

3. $x_1 = 0, x_2 = 2, x_3 = 2, x_4 = 0$; max $z = 16$

4. (a) $a_1 = 430, a_2 = 460, a_3 = 480$ (b) $b_1 = -\frac{1}{4}, b_2 = \frac{3}{2}, b_3 = -4$
$c_1 = 1, c_2 = 2, c_3 = 230$

(c) $y_1 = 1, y_2 = 2$; min $w = 1350$

5. $y_1 = \frac{5}{10}, y_2 = \frac{16}{9}$; min $w = \frac{235}{19}$

6. Unbounded solution.

7. Unbounded solution.

13.12 DUAL SIMPLEX METHOD

In situations when a particular basic solution is infeasible, that is one or more basic variables are negative, but satisfies optimality criterion and it is possible to move towards feasibility maintaining optimality. This is what dual simplex method does. This method developed by Lemke applies to problems in which optimality condition is satisfied. It is called dual simplex method because the rules for leaving and entering variables are derived from the dual problem but are used in the primal problem.

The procedure to be followed in dual simplex method is as follows:

1. (a) Express the problem in the standard form as following:

   Minimize or maximize $z = c^T X$

   Subject to $AX = b, X \geq 0$,

   where at least one of the $b_i'^s$ is negative and A contains identity matrix as a submatrix.

   (b) Express z in terms of non-basic variables: The dual simplex method is applicable if optimality condition is satisfied in above format. If so, make starting dual simplex table which is same as starting simplex table except that at least one entry of solution column (not excluding z-row) is negative.

2. The above gives a basic solution (not feasible). The leaving variable x_r is given by

$$x_r = \min_{1 \leq i \leq m} (x_i, x_i < 0)$$

   This ensures that the basic non-feasible solution is forced towards feasibility.

3. The entering variable is x_k is selected by

$$\theta = \left| \frac{Z_k - c_k}{\alpha_r^k} \right| = \min_j \left\{ \left| \frac{Z_j - c_j}{\alpha_r^j} \right|; \alpha_r^j < 0 \right\}$$

   This ensures that optimality condition will remain satisfied. If all α_r^j are non-negative, the problem has no feasible solution.

4. The next dual simplex table is obtained by same procedure as in regular simplex method.
5. When a basic solution is reached in which all variables are non-negative, we had obtained the optimal solution.

Example 33 Use dual simplex method to solve the following LPP.

Minimize $z = 2x_1 + 2x_3 + 2x_5 + 4s_1 + 6s_2 + 8s_3$

Subject to

$$\begin{aligned} -3x_1 - 2x_2 - x_3 + s_1 &= -1500 \\ -x_2 - x_4 - 2x_5 + s_2 &= -1000 \\ -x_3 - x_4 + s_3 &= -3000 \\ \text{all variables} &\geq 0 \end{aligned}$$

The starting dual simplex table is

Basic	z	x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	Solution
z	1	–14	–14	–14	–14	–14	0	0	0	–36000
s_1	0	–3	–2	–1	0	0	1	0	0	–1500
s_2	0	0	–2	0	–1	–2	0	1	0	–1000
s_3	0	0	0	–1	–1	0	0	0	1	–3000

In given table optimality condition is satisfied and hence dual simplex method is applicable. Most negative basic variable s_3 is the leaving variable. To find the entering variable, take the negative entries in the row of leaving variable and calculate the ratio of these with corresponding entries in z-row. The entering variable is the one corresponding to the minimum of absolute values of these ratios. Here

$\min\left\{\left|-\dfrac{14}{-1}\right|, \left|-\dfrac{14}{1}\right|\right\} = 14 \Rightarrow$ either x_3 or x_4 is the entering variable. The details are

shown in following table:

Basic	z	x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	Solution
z	1	–14	–14	0	0	–14	0	0	–14	6000
s_1	0	–3	–2	0	1↓	0	1	0	–1	1500
$\leftarrow s_2$	0	0	–1	0	$\boxed{-1}$	–2	0	1	0	–1000
x_3	0	0	0	1	1	0	0	0	–1	3000
z	1	–14	–14	0	0	–14	0	0	–14	6000
s_1	0	–3	–3	0	0	–2	1	1	–1	500
x_4	0	0	1	0	1	2	0	–1	0	1000
x_3	0	0	–1	1	0	–2	0	1	–1	2000

In above table the basic solution has become feasible and hence it is the optimal solution. The optimal solution is

$$x_1 = x_2 = x_5 = 0,\ x_3 = 2000,\ x_1 = 1000;\ \min z = 6000$$

***Remark*:**

1. In dual simplex method we first decide leaving variable and then entering variable. Both for minimization and maximization problems, the rules regarding leaving and entering variables are same.
2. If dual simplex method is applicable then LP can not have unbounded solution. If there is no negative entry in the row of leaving variable then LPP has no feasible solution.

Example 34 Solve the following by dual simplex method:

Maximize $z = -4x_1 - 6x_2 - 18x_3$

Subject to

$$x_1 + 3x_3 \geq 3$$
$$x_2 + 2x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution Write the given LPP in standard form by introducing slack variables and then use dual simplex method to solve it.

Standard form is
Maximize $z = -4x_1 - 6x_2 - 18x_3$
Subject to

$$-x_1 - 3x_3 + s_1 = -3$$
$$-x_2 - 2x_3 + s_2 = -5$$
$$\text{all variables} \geq 0$$

Basic	z	x_1	x_2	x_3	s_1	s_2	Solution	
z	1	4	6	18	0	0	0	
s_1	0	–1	0	–3↓	1	0	–3	
s_2	0	0	–1	$\boxed{-2}$	0	1	–5	
z	1	4	–3	0	0	9	–45	
s_1	0	–1	$\boxed{3/2}$	0	1	–3/2	9/2	Apply regular simplex method
x_3	0	0	$\frac{1}{2}$	1	0	–1/2	5/2	
z	1	2	0	0	2	6	–36	
x_2	0	–2/3	1	0	2/3	–1	3	Optimal table
x_3	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	1	

The optimal solution is

$$x_1 = 0,\ x_2 = 3,\ x_3 = 1 \text{ and max } z = -36$$

EXERCISE 13.9

Using dual simplex method, solve the following problems:

1. Minimize $z = 2x_1 + x_2$
   Subject to

   $$3x_1 + x_2 \geq 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \leq 3,\ x_1, x_2 \geq 0$$ **[Kurukshetra 2007]**

2. Minimize $z = 2x_1 + 2x_2 + 4x_3$
Subject to
$$2x_1 + 3x_2 + 5x_3 \geq 2,\ 3x_1 + x_2 + 7x_3 \leq 3,\ x_1 + 4x_2 + 6x_3 \leq 5\ x_1, x_2, x_3 \geq 0$$
[Kurukshetra 2009, Kerala 2005]

3. Maximize $z = -2x_1 - x_2$
Subject to
$$3x_1 + x_2 \geq 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \leq 3,\ x_1, x_2 \geq 0$$
[Madras 1999, Meerut 1994, Madurai 1992]

4. Minimize $z = x_1 + x_2$
Subject to
$$2x_1 + x_2 \geq 4,\ x_1 + 7x_2 \geq 7;\ x_1, x_2 \geq 0$$
[Kerala 1991]

5. Minimize $z = 4x_1 + 9x_2$
Subject to
$$x_1 + x_2 \geq 6$$
$$2x_1 + 3x_2 \geq 18$$
$$x_1, x_2 \geq 0$$

6. Maximize $z = -15x_1 - 5x_2 - 6x_3$
Subject to
$$5x_1 - 2x_2 + 3x_3 \geq 6$$
$$10x_1 + 3x_2 + 3x_3 \geq 15$$
$$x_1 + x_2 + x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Answers

1. $x_1 = \frac{3}{5},\ x_2 = 6/5;\ \min z = 12/5$
2. $x_1 = 0,\ x_2 = 2/3,\ x_3 = 0;\ \min z = 4/3$
3. $x_1 = \frac{3}{5},\ x_2 = 6/5;\ \max z = -12/5$
4. $x_1 = \frac{41}{26},\ x_2 = \frac{11}{13};\ \min z = \frac{63}{26}$
5. $x_1 = 9,\ x_2 = 0;\ \min z = 36$
6. $x_1 = 0 = x_2,\ x_3 = 2;\ \max z = 12$

13.13 TRANSPORTATION PROBLEM

13.13.1 Introduction

This problem first arose in optimizing the transportation cost of shipment of goods from various sources to different destinations. These problems when formulated or say when mathematical models of these problems were formulated, then it was noticed that these are LPPs of special type. Later, any LPP when had that character was also termed as transportation problem.

13.13.2 Formulation of Transportation Problem

Let there be m sources, namely $S_1, S_2, \ldots, S_m$ and n destinations $D_1, D_2, \ldots, D_n$. The destinations could be shops, godown, warehouses, etc. and the sources could be industries, warehouses, cargo offices, distributors etc. The goods are to be transported from S_i to D_j ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$). The cost of transportation per unit from S_i to D_j is denoted by c_{ij}. Let a_i be the quantity available at i^{th} source ($i = 1, 2,\ldots, m$) and b_j be the quantity required D_j destination ($j = 1, 2, \ldots, n$). Then the above data can be represented in tabular form as following table.

	D_1	D_2	D_3	….			D_n	**Availability**
s_1	c_{11} x_{11}	c_{12} x_{12}	c_{13} x_{13}	└	└	⋯	c_{1n} x_{1n}	a_1
s_2	c_{21} x_{21}	c_{22} x_{22}	c_{23} x_{23}	└	└	⋯	c_{2n} x_{2n}	a_2
⋮								⋮
S_m	c_{m1} x_{m1}	c_{m2} x_{m2}	c_{m3} x_{m3}	└	└	⋯	c_{mn} x_{mn}	a_m
Requirement	b_1	b_2	b_3	…			b_n	

Let x_{ij} be the quantity of goods transported from source S_i to destination Dj. then transportation problem is

$$\left.\begin{aligned}
&\text{Minimize } z = \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}x_{ij}\\
&\text{Subject to}\\
&\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m,\ a_i > 0\\
&\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n,\ b_j > 0
\end{aligned}\right\} \tag{22}$$

$$x_{ij} \geq 0 \text{ and integer}$$

Any LPP of type Eq. (22) is called transportation problem. It has $m + n$ constraints and mn decision variables.

13.13.3 Balanced Transportation Problem

If in a transportation problem $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ i.e., total availability is equal to the total requirements then the problem is called a balanced transportation problem (TP). Also above relation gives

$$\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Which allows to eliminate one of the constraints in terms of others in the constraints of (22). Thus in case of balanced transportation problem having m sources and n destinations, there are mn variables and $m + n - 1$ linearly independent constraints.

13.13.4 Unbalanced Transportation Problem

If in a transportation problem $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then it is called an unbalanced TP. An unbalanced TP can be made balanced TP as follows:

***Case* I:** If $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, i.e., availability is more than requirement, which normally term as short demand. In such case we create a dummy or a fictitious destination D_F with requirement $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ and cost from S_i to DF i.e., C_{iF} will be taken zero.

***Case* II:** If $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$ i.e., availability is less than requirement, which is normally term as short supply. In such case we introduce a dummy or fictitious source S_F with availability $\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$ and cost from S_F to D_j i.e., C_{Fj} will be taken zero.

***Remark*:**

1. If problem requires no transportation from S_i to D_j, then c_{ij} is replace by big-M($M > 0$) or ∞.
2. If problem requires transportation to D_j from S_i only, then c_{ij} is replaced by zero to find optimal solution but for optimal cost we add $c_{ij}x_{ij}$.
3. In case of short supply i.e., $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$, we introduce S_F with cost zero to each D_i. There may arise two situations:
   (i) Definition(s) may impose penalties for a short supply on each item supplied short. If at destination D_j, the penalty is P_j then c_{Fj} is to be taken P_j.
   (ii) Demand at D_j must be fulfilled. This case may be treated as S_F does not supply any thing to D_j i.e., c_{Fj} is to be taken as big M(M > 0) or ∞.
4. In case of short demand i.e., $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, we introduce D_F with cost zero from each S_i. There may arise two situations.
   (i) Source(s) would be left with some quantities of goods which were sent to D_F i.e., which were untransported. If sources are going to spend some amount on their storage, then it has to be included. If source S_i occurs an expense of q_i on each item then c_{iF} is to be replaced by q_i.
   (ii) If problem requires that source S_i must transport all its goods, then we assume that S_i does not supply to D_F i.e., C_{iF} is to be taken as big M ($M > 0$) or ∞.

Thus each transportation problem, after in corporating the above remarks, can be converted into a balanced one. We shall now discuss how a balanced transportation problem can be solved.

13.13.5 Solution of a Balanced Transportation Problem

A TP having m sources and n destinations (balanced) will have a solution consisting of basic and non-basic variables. Each basic feasible solution (BFS) would have at most $m + n – 1$ non-zero variables. If BFS has less than $m + n - 1$ non-zero variables then BFS would be degenerate solution.

To find optimal solution of a TP, first we would have starting BFS and then will move towards optimality.

(a) Basic Feasible Solution

A BFS can be found by the following three methods:

(i) North-West Corner Method
(ii) Least Cost Entry Method
(iii) Vogal's Approximation Method

(i) North-West Corner Method (NWCM) We shall select variables which would be our basic variables and assign them values. In this method, we shall start from North-West corner variables i.e., x_{11} then proceed to $x_{12}, x_{13}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}$.

Assigning a value to x_{ij} would mean to determine the quantity of goods to be transported from S_i to D_j which will be Min (a_i, b_j). After assigning x_{ij}, say the value A, we would decrease the value of the requirement at D_j by A and the availability at S_i by A because of the natural reasons and move to next variable.

When all the availabilities and requirements reduce to zero, i.e., get filled, we stop and the boxes inside in which some entry is made become basic variables and where the entries are not made are called non-basic variables.

Thus we arrive at a BFS, which is starting BFS and the value of objective function is obtained by adding the product of basic variables with their cost. We illustrate this concept (method) by an example:

Example 35 Find a BFS for the following transportation problem using NWCM.

	D_1	D_2	D_3	D_4	
S_1	1	5	6	4	90
S_2	3	2	3	3	50
S_3	2	6	1	5	60
S_4	6	4	2	2	60
	70	60	60	70	

Solution It is a balanced TP i.e., $\sum_{i=1}^{4} a_i = 260 = \sum_{j=1}^{4} b_j$

To find BFS by NWCM, we start from the variable x_{11} which can take value Min (70, 90) = 70, reduced a_1 by 70 it becomes 20 and b_1 is satisfied completely. Continue this process, till all a_i and b_j reduced to zero and the following BFS is obtained.

	D_1	D_2	D_3	D_4	
S_1	$\lfloor 1$ 70	$\lfloor 5$ 20	$\lfloor 6$	$\lfloor 4$	~~90~~ 20
S_2	$\lfloor 3$	$\lfloor 2$ 40	$\lfloor 3$ 10	$\lfloor 3$	~~50~~ 10
S_3	$\lfloor 2$	$\lfloor 6$	$\lfloor 1$ 50	$\lfloor 5$ 10	~~60~~ 10
S_4	$\lfloor 6$	$\lfloor 4$	$\lfloor 2$	$\lfloor 2$ 60	~~60~~
	~~70~~	~~60~~ 40	~~60~~ 50	~~70~~ 60	

Starting BFS is

$$x_{11} = 70, x_{12} = 20, x_{22} = 40, x_{23} = 10, x_{33} = 50, x_{34} = 10$$

and x_{44} = 60 (These variables are known as basic while remaining variables are called non-basic variables). The value of the function

$$\min z = \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}x_{ij} = 70\times1+20\times5+40\times2+10\times3+50\times1+10\times5+60\times2 = 500$$

(ii) Least Cost Entry Method (LCM) It also works in the same manner as NWCM except that we do not start from NW Corner i.e., x_{11} and proceed to $x_{12}, \ldots, x_{mn}$ but at each step, we pick the box with the least cost. In case of tie, we pick the one where maximum number can be assigned. In case of further tie, break arbitrarily.

Example 36 Find the BFS of following TP by LCM.

	D_1	D_2	D_3	
S_1	$\lfloor 1$	$\lfloor 5$	$\lfloor 6$	90
S_2	$\lfloor 3$	$\lfloor 2$	$\lfloor 3$	10
S_3	$\lfloor 2$	$\lfloor 6$	$\lfloor 0$	20
S_4	$\lfloor 5$	$\lfloor 4$	$\lfloor 2$	40
	60	50	50	

Solution The given TP is balanced i.e., $\sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j = 160$. To find BFS by LCM, we check the least cost which is 0 corresponding to the box (3, 3), so the maximum allocation can be made in this box is min (20, 50) = 20 i.e., x_{33} = 20 and continue this process till all availabilities and requirements reduce to zero and we get the following solution.

	D_1	D_2	D_3	
S_1	60 [1]	30 [5]	[6]	90
S_2	[3]	10 [2]	[3]	10
S_3	[2]	[6]	20 [0]	20
S_4	[5]	10 [4]	30 [2]	40
	60	50	50	

Thus the BFS is

$$x_{11} = 60,\ x_{12} = 30,\ x_{22} = 10,\ x_{33} = 20,\ x_{42} = 10,\ x_{43} = 30$$

and Transportation cost = $60 \times 1 + 30 \times 5 + 10 \times 2 + 20 \times 0 + 10 \times 4 + 30 \times 2 = 330$

Let no solve the same example 36 by NWCM.

	D_1	D_2	D_3	
S_1	60 [1]	30 [5]	[6]	90
S_2	[3]	10 [2]	[3]	10
S_3	[2]	10 [6]	10 [0]	20
S_4	[5]	[4]	40 [2]	40
	60	50	50	

The BFS is

$x_{11} = 60, x_{12} = 30, x_{22} = 10, x_{32} = 10, x_{33} = 10, x_{43} = 40$ and transportation cost = 370. Thus it emerges that LCM gives better solution then NWCM.

(iii) Vogel's Approximation Method (VAM) VAM works on the principle of taking into account the penalty (extra cost) one has to pay, if least cost is not chosen. We first define the penalty.

Penalty of row (column) is the difference between the lowest cost and the next higher cost in that row (column). If the least cost appears at two places in a row (column), then to calculate penalty of that row (column), we shall take only the next higher cost i.e., penalty would not be zero, unless all costs in that row (column) is same.

We shall follow the following steps in VAM:

1. Calculate the penalty for each row (column) and write them on the right of each row and below each column.
2. Choose a row (column) having largest penalty and all of the maximum in the least cost box.
3. Reduce the availability and requirement.
4. Calculate the penalty again after suppressing row (column) satisfied.
5. Continue till all the rows (columns) are satisfied.

Remark

1. If penalty (largest) at any step is not unique then select the one having the least cost box, otherwise break the tie arbitrarily.
2. If the least cost in a row (column) is not unique, choose the box where maximum allocation can be made, otherwise break the tie arbitrarily.

Example 37 Find a BFS for the following transportation problem by VAM.

	D_1	D_2	D_3	
S_1	1	5	6	90
S_2	3	2	3	10
S_3	2	6	0	20
S_4	5	4	1	40
	60	50	50	

Solution It is a balanced TP i.e., $\sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j = 160$.

To find BFS by VAM, calculate the penalty for each row (column) at write then on the right of each availability for row and below requirement for each column.

	D_1	D_2	D_3		Row penalty
S_1	1 60	5	6	~~90~~ 30	4
S_2	3	2	3	10	1
S_3	2	6	0	20	2
S_4	5	4	2	40	2
	0 ~~60~~	50	50		
column penalty	1	2	2		

The largest penalty is 4 corresponding to row 1, the least cost in row 1 is 1 so maximum allocation can be made in box (1, 1) in min (60, 90) = 60, suppress first column and row by 60, we see that column 1 is satisfied, now for next iteration we need not to consider column 1 for calculating penalties. Continue this process till all the availabilities and requirements are satisfied.

	D_1	D_2	D_3		Row penalty
S_1	1 60	5 30	6	~~90~~ 30	4 1 1 5
S_2	3	2 10	3	10	1 1 1 2
S_3	2	6	0 20	20	2 6 – –
S_4	5	4 10	2 30	~~40~~ 10	2 2 2 4
column penalty	~~60~~	~~50~~	~~50~~ ~~30~~		
	0	20			
	1	2	2		
	–	2	2		
	–	2	1		
	–	2	–		

The BFS is as follows:

$x_{11} = 60, x_{12} = 30, x_{22} = 10, x_{33} = 20, x_{42} = 1, x_{43} = 30$ and the transportation cost is 330.

***Remark*:**

1. The above problem is solved by all the three methods and we find that LCM and VAM gave the same BFS but better than NCWM, but actually, it is mere a coincide. In fact VAM gives the best result in comparison of LCM and NWCM. So, generally we apply VAM to find BFS of a TP unless otherwise the method is specified.
2. Whenever shipment of all the goods from source S_i is over, we say i^{th} row is satisfied. It also means i^{th} constraint of availability is satisfied. Similarly, if shipment to destination D_j is over, we say j^{th} column is satisfied. It also means that j^{th} constraint of requirement constraint is satisfied.
3. Whenever a row and column are satisfied simultaneously then it would mean two constraints are satisfied simultaneously and it would amount to reduction of a basic variable. Therefore in such case we assume that only one either row or column is satisfied and the other row (column) still has to dispatch (receive) 0 items. This ultimately would mean that in one box, we would enter '0' or say one basic variable would be zero or say that BFS would be degenerate.

Now we take examples of unbalanced problems, and illustrate how to obtain starting BFS.

Example 38 Let there be 3 destinations and 3 sources, cost of transportation, availabilities and requirements are given by the following table. Find a starting BFS.

	D_1	D_2	D_3	
S_1	1	5	6	90
S_2	3	2	3	10
S_3	2	6	1	20
	60	50	50	

Solution It is an unbalanced problem, $\sum_{i=1}^{3} a_i = 120$ and $\sum_{j=1}^{3} b_j = 160$. To find BFS of the given TP, introduce fictitious source S_F with availability 160 – 120 = 40 and $C_{Fj} = 0, j = 1, 2, 3$, now use VAM to find BFS of the problem.

	D_1	D_2	D_3		Row penalty
S_1	60 (1)	30 (5)	(6)	90	4 1 1 1 1
S_2	(3)	10 (2)	(3)	10	1 1 1 1 1
S_3	(2)	(6)	20 (1)	20	1 5 – – –

S_F	0	0 10	0 30	40	0 0 0 0 –
column penalty	60	50	50		
	1	2	1		
	–	2	1		
	–	2	3		
	–	2	–		
	–	3	–		

The BFS is as follows:

$$x_{11} = 60,\ x_{12} = 30,\ x_{22} = 10,\ x_{23} = 20,\ x_{F2} = 10,\ x_{F3} = 30$$

Transportation cost = 250.
As $x_{F2} = 10$ and $x_{F3} = 30$, so the short supply to D_2 and D_3 are 10 and 30 units respectively.

Example 39 Consider the Example 38 with an additional condition that for short supply there is a penalty by D_j, say 2, 4 and 6 respectively by D_1, D_2 and D_3. Find the BFS using VAM.

Solution The cost from fictitious source to destinations D_1, D_2 and D_3 c_{Fj} (j = 1, 2, 3) in the Example 38 are to be replaced by 2, 4 and 6 from 0 and then apply VAM to find BFS as follows:

	D_1	D_2	D_3		Row penalty
S_1	1 60	5 10	6 20	~~90~~ 30	4 1 1 1
S_2	3	2	3 10	10	1 1 1 –
S_3	2	6	1 20	~~20~~	1 5 – –
SF	2	4 40	6	40	2 2 2 2
column penalty	0	10	20		~~50~~ ~~30~~ 20
	1	2	1		
	–	2	2		
	–	2	3		
	–	2	0		

The BFS is as follows:

$$x_{11} = 60, x_{12} = 10, x_{13} = 20, x_{23} = 10, x_{33} = 20, x_{F2} = 40 \text{ and TC} = 440.$$

There is short supply of 40 units to D_2.

(b) Optimal Solution

After getting basic feasible solution of a TP, we move towards optimality. Following steps are taken.

1. Allocate variables $u_1, u_2, \ldots, u_m$ to each row and $v_1, v_2, v_3, \ldots, v_n$ to each column.
2. Find the values of each u_i ($i = 1$ to m) and v_j($j = 1$ to n), using the formula $u_i + v_j - c_{ij}$ for basic variables. As we know that in any TP, with m sources and n destinations, there will be $m + n - 1$ basic variables having values ≥ 0. So for $m + n - 1$ variables $u_i + v_j - c_{ij} = 0$ but total no. of $u_i + v_j = m + n$. So to find values of $u_i + v_j$, put one of the u_i and $v_j = 0$ and then find the values of remaining $m + n - 1$ variables u_i and v_j.
3. Calculate $u_i + v_j - c_{ij}$ for all non basic variables, if all these values ≤ 0, then the table is optimal, else go to step 4.
4. In order to get optimality, we need entering variable, leaving variable and do iteration(s). To decide the entering variable, we pick the box for which $u_i + v_j - c_{ij}$ is most positive, the corresponding variable will be entering variable.
5. In order to allocate maximum possible goods in this box, at least one of the box is going to empty. One of the box, or say variable attached to it, is taken to be the leaving variable.

In order to do this, after deciding entering variable, we find the basic variables (filled boxes) which are going to be affected by it. This is done by constructing a loop by the following method.

Start from the box where entry is to be made vertically or horizontally and return to the starting point with the following restrictions.

(i) Loop would have straight lines as edges with corners.
(ii) Corners of the loop should be only in the box of a basic variable i.e., a box where entry has already been made.
(iii) The direction of loop is immaterial.
(iv) The loop could be self intersecting.

Now assume 'θ' allocating in the starting box, then move in one direction to the corner this θ amount should be reduced in this box, continuing in the same direction, go to next corner, θ amount to be increased at this corner and continue till we reach to the starting point. Now pick the boxes where θ has been reduced. Find 'θ' which is minimum of the entries in these boxes, so that entry in all the corner. This gives the maximum allocation in this starting box. The box which becomes empty becomes leaving variable.

Caution: If more than one box becomes empty, then we make only one box empty and leave '0' in other box(es). Now allocate the amount so obtained in the starting box and reduce increase at other corner boxes. This completes the iteration.

Again assign values to the variables u_i and v_j and calculate afresh the values of $u_i + v_j - c_{ij}$ for non-basic variable and continue the process till all $u_i + v_j - c_{ij} \leq 0$.

Example 40 Consider the Example 36 and find its optimal solution.

Solution The starting BFS of Example 36 is given by NWCM which is as follows:

	D_1	D_2	D_3	
S_1	60 (1)	30 (5)	(6)	90
S_2	(3)	10 (2)	(3)	10
S_3	(2)	10 (6)	10 (0)	20
S_4	(5)	(4)	40 (2)	40
	60	50	50	

To find optimal solution, we allocate $u_i, (i = 1 \text{ to } 4)$ to each row and $v_j (j = 1 \text{ to } 3)$ to each column and calculate values of each u_i and v_j. For the purpose, we allot '0' to one of u_i's and v_j's. Let $v_2 = 0$.

For basic variables $x_i + v_j = c_{ij}$

i.e.,

$$u_1 + v_1 = c_{11} = 1 \Rightarrow v_1 = -4$$
$$u_1 + v_2 = c_{12} = 5 \Rightarrow u_1 = 5$$
$$u_2 + v_2 = c_{22} = 2 \Rightarrow u_2 = 2$$
$$u_3 + v_2 = c_{32} = 6 \Rightarrow u_3 = 6$$
$$u_3 + v_3 = c_{33} = 0 \Rightarrow v_3 = -6$$
$$u_4 + v_3 = c_{43} = 2 \Rightarrow u_4 = 8$$

Also calculate $u_i + v_j - c_{ij}$ for all non-basic variables and put them in south west corner of each box. These entries are shown in the following table.

		D_1 $v_1 = -4$	D_2 $v_2 = 0$	D_3 $v_3 = -6$
$u_1 = 5$	S_1	1 60	5 30	6 $\boxed{-7}$
$u_2 = 2$	S_2	3 $\boxed{-5}$	2 10	3 $\boxed{-7}$
$u_3 = 6$	S_3	2 $\boxed{0}$	10 6 $\boxed{0}$	0 10
$u_4 = 8$	S_4	5 $\boxed{-1}$	4 $\boxed{4}$	3 40

Remark: If a non-basic variable box has '0' in south-west corner, it means for this transportation problem, there is an alternate solution.

The given table is not optimal and south-west corner corresponding to variable x_{42} is maximum, hence, variable x_{42} enters. Now to decide leaving variable, we constrict a loop starting box x_{42} and whose edges are at boxes corresponding to boxes x_{32}, x_{33} and x_{43} as follows and enter θ in box x_{42} and reduce θ in boxes x_{32} and x_{43} and increase x_{33}.

$10 - \theta$	$10 + \theta$
θ	$40 - \theta$

Max. θ = Min. (10, 40) = 10

So we allocate $\theta = 10$ in x_{42} and reduce and increase accordingly and get the following next BFS in the given table.

		D_1 $v_1 =$	D_2 $v_2 =$	D_3 $v_3 =$
$u_1 =$	S_1	1 60	5 30	6
$u_2 =$	S_2	3	2 10	3
$u_3 =$	S_3	2	6	0 20
$u_4 =$	S_4	5	4 10	2 30

Compute the values of u_i and v_j afresh and also calculate $u_i + v_j - c_{ij}$ for non-basic variables and we get the following table.

	$v_1 = -4$	$v_2 = 0$	$v_3 = -2$
$u_1 = 5$	$\lfloor 1$ 60	$\lfloor 5$ 30	$\lfloor 6$ $\overline{-3}\rceil$
$u_2 = 2$	$\lfloor 3$ $\overline{-5}\rceil$	$\lfloor 2$ 10	$\lfloor 3$ $\overline{-3}\rceil$
$u_3 = 2$	$\lfloor 2$ $\overline{-4}\rceil$	$\lfloor 6$ $\overline{-4}\rceil$	$\lfloor 0$ 20
$u_4 = 4$	$\lfloor 5$ $\overline{-5}\rceil$	$\lfloor 4$ 10	$\lfloor 2$ 30

all $u_i + v_j - c_{ij}$ in above table for non-basic variables are ≤ 0. Hence, the above table is optimal table. Hence, optimal solution is $x_{11} = 60, x_{12} = 30, x_{22} = 10, x_{32} = 20, x_{42} = 10, x_{43} = 30$ and Min TC = 330.

Example 41 Solve the following transportation problem. Use LCM to find starting basic feasible solution.

	D_1	D_2	D_3	
S_1	5	1	0	20
S_2	3	2	4	10
S_3	7	5	2	15
S_4	9	6	0	15
	5	10	15	

Demand at D_1 must met from S_4 only.

Solution It is an unbalanced problem, i.e., $\sum_{i=1}^{4} a_i = 60$ and $\sum_{j=1}^{3} b_j = 30$, introduce a dummy destination D_F with demand 60 – 30 = 30 units with cost $C_{iF} = 0$ (i = 1, 2, 3, 4). Further it is given that demand at D_1 must satisfied by S_4 only i.e., the cost c_{11}, c_{21} and c_{32} should be replaced by big $M(M > 0)$. Now, find starting BFS by LCM which is given in following table:

		D_1 $v_1 = 9$	D_2 $v_2 = 6$	D_3 $v_3 = 5$	DF $v_4 = 0$	
$u_1 = -5$	S_1	M $\boxed{4-M}$	1 5	0 15	0 $\boxed{-5}$	20
$u_2 = 0$	S_2	M $\boxed{9-M}$	2 $\boxed{4}$	4 $\boxed{1}$	0 10	10
$u_3 = 0$	S_3	M $\boxed{9-M}$	5 $\boxed{1}$	2 $\boxed{3}$	0 15	15
$u_4 = 0$	S_4	9 $\boxed{5}$	6 5	0 $\boxed{5}$	0 5	15
		5	10	15	30	TC = 80

Above table is not optimal. x_{43} enters. Loop is

$$\begin{array}{ccc} 5+\theta & & 15-\theta \\ & & \\ 5-\theta & & \theta \end{array} \qquad \theta = 5,\ x_{22} \text{ leaves}$$

New BFS is given in following table, and also we check its optimality:

		$v_1 = 9$ D_1	$v_2 = 1$ D_2	$v_3 = 0$ D_3	$v_4 = 0$ D_F	
$u_1 = 0$	S_1	M $\boxed{9-M}$	1 10	0 10	0 $\boxed{0}$	20
$u_2 = 0$	S_2	M $\boxed{9-M}$	2 $\boxed{-1}$	4 $\boxed{-4}$	0 10	10
$u_3 = 0$	S_3	M $\boxed{9-M}$	5 $\boxed{-4}$	2 $\boxed{-2}$	0 15	15
$u_4 = 0$	S_4	9 5	6 $\boxed{-5}$	0 5	0 5	15
		5	10	15	30	

all entries for non basic variables in above table ≤ 0.

Hence it is optimal table. Therefore optimal solution is

$$x_{12} = 10,\ x_{13} = 10, x_2F = 10,\ x_3F = 15,\ x_{41} = 5,\ x_{43} = 5,\ x_{4F} = 5 \text{ and TC} = 55.$$

***Note*:** The problem has alternate solution as $u_1 + v_4 - c_{iF} = 0$, so we can enter variable x_{iF}.

13.13.6 Degeneracy in Transportation Problem

The variable(s) attached to box(es) where allocation is made is a basic variable. If the value of a basic variable at any step is equal to zero, then solution is said to be degenerate solution. Degeneracy occurs in two cases.

(i) If in NWCM, LCM or VAM, a row and a column is satisfied simultaneously we get one variable short in box and there a basic variable with value '0'. For this purpose, in this case, we assume that only one, either row or a column is satisfied and at the at the other there are '0' items to be dispatched or received.

(ii) If in the iteration by forming a loop, two or more boxes become empty at the same time, it also yields one variable short in basis, thus a degenerate solution. In this case, we assume that only one box has become empty and in the other boxes, there are still '0' allocation and proceed as usual.

In case of a degenerate solution, while performing the iteration, which has '0' in one of the boxes and is not an optimal solution, a situation may arise that one may have the value $\theta = 0$, which should be in corporated. This is the case where allocation '0' has shifted from a basic variable to a non-basic variable (empty box) without effecting the transportation cost at this step though the solution, thus an alternate solution.

13.13.7 Alternate Solution

In a transportation problem, situations of alternate solution arise in the following cases:

(i) As mentioned above, while at any step of iteration an allocation $\theta = 0$ is shifted from one box to another.

(ii) In an optimal table, an empty box has '0' in south west corner i.e., the value of $u_i + v_j - c_{ij}$ for a non-basic variable is zero, we can force this variable to enter.

13.13.8 Special Cases

(i) In a balanced transportation problem, if each box in all rows has cost c_i, then optimal transportation cost is equal to $\sum_{i=1}^{m} c_i a_i$.

(ii) In a balanced transportation problem, if each box in all columns has cost c_j, then optimal transportation cost is equal to $\sum_{j=1}^{n} c_j b_j$.

EXERCISE 13.10

1. Obtain the initial basic feasible solution of the following problem by
   (i) North-West Corner Method
   (ii) Least Cost Entry Method, and
   (iii) Vogel's Approximation Method

	D_1	D_2	D_3	D_3	
S_1	21	16	25	13	11
S_2	17	18	14	23	13
S_3	32	27	18	41	19
	6	10	12	15	

2. Consider the following transportation problem:

4	2	3	2	6	8
5	4	5	2	1	12
6	5	4	7	3	14
4	4	6	8	8	

Find the initial basic feasible solution of the above problem by

(a) NWCM (b) LCM (c) VAM

3. A company has three factories I, II and III and four warehouses A, B, C and D. The transportation cost (in ₹) per unit from each factory to each warehouse, the availability of goods at each factory and requirements of each warehouse are given below:

	Warehouse				
	A	B	C	D	Availability
I	42	48	38	37	160
II	40	49	52	51	150
III	39	38	40	43	190
Requirement	80	90	110	160	

Determine the optimal schedule to minimize the transportation cost. (use VAM to find initial BFS) and answer the following:

(i) Does the problem have degenerate solution.

(ii) Does alternate solution exist for the problem, if yes, find all the solutions.

4. Using VAM, find initial BFS of the following transportation problem, show that the initial BFS itself is optimal solution.

6	8	4	14
4	0	8	12
1	2	6	5
6	10	15	

5. A person has three factories I, II and III which supply goods to three godowns G_1, G_2 and G_3. Daily factory capacities are 10, 80 and 15 units respectively while the daily requirements of godowns are 75, 25 and 45 units respectively. Unit shipping cost (in ₹) are given below:

		Godown		
		G_1	G_2	G_3
	I	5	1	7
Factory	II	6	4	6
	III	3	2	5

The penalty cost for not satisfying demand on godowns G_1, G_2 and G_3 are ₹5, ₹3 and ₹2 per unit respectively. Determine the optimal shipping schedule to minimize the cost.

6. Solve the following transportation problem. The demand at destination 2 must be shipped from source 2 only. The entries are transportation cost per unit. Does problem has an alternate solution also.

5	1	0	20
3	2	4	10
6	5	0	15
5	10	15	

7. Consider the following transportation problem:

	D_1	D_2	D_3	
S_1	2	6	7	90
S_2	4	3	4	10
S_3	3	7	2	20
	60	50	50	

Let the penalty cost per unit for unsatisfied demand for destination D_1, D_2 and D_3 respectively are 7, 5 and 3 respectively. Find the optimal solution. (Use VAM to find initial BFS).

8. In Exercise 7, let there be no penalties but the demand at D_3 must be satisfied exactly. Find the optimal solution.

9. There are four villages V_1, V_2, V_3 and V_4 which are affected by floods. Food grain is to be dropped in these villages by three aircrafts A_1, A_2 and A_3. The following matrix is given

	V_1	V_2	V_3	V_4	a_i
A_1	9	7	5	2	60
A_2	7	9	4	2	40
A_3	1	4	8	9	50
b_j	30	50	60	40	

In above matrix, a_i denote, total number of trips that air craft.
A_i can make in a day and b_j denote the number of trips required to village V_j in one day, the cost c_{ij} denote the amount of food grains that aircraft A_i can carry to village V_j in one trip. Find the number of trips that aircraft A_i should make to village V_j so that the total quantity of food dropped in a day is maximized. (use VAM to find initial BFS)
***Hint*:** As it is maximized problem from convent all c_{ij} to $-c_{ij}$ to make it minimization problem and then solve in usual manner.

Answers

1. (i) $x_{11} = 6, x_{12} = 5, x_{22} = 5, x_{23} = 8, x_{33} = 4, x_{34} = 15$; TC = 1095

   (ii) $x_{14} = 11, x_{21} = 1, x_{23} = 12, x_{31} = 5, x_{32} = 10, x_{34} = 4$; TC = 922

   (iii) $x_{11} = 11, x_{21} = 6, x_{22} = 3, x_{24} = 4, x_{32} = 7, x_{33} = 12$; TC = 796

2 (a) $x_{11} = 4, x_{12} = 4, x_{23} = 6, x_{24} = 6, x_{34} = 2, x_{35} = 8, x_{3F} = 6$; TC = 104

(b) $x_{14} = 2, x_{1F} = 6, x_{24} = 4, x_{25} = 8, x_{31} = 4, x_{32} = 4, x_{33} = 6, x_{34} = 2$; TC = 102

(c) $x_{12} = 4, x_{14} = 4, x_{24} = 4, x_{25} = 8, x_{31} = 4, x_{33} = 6, x_{3F} = 4$; TC = 80

3. $x_{14} = 160, x_{21} = 80, x_{22} = 10, x_{32} = 80, x_{33} = 110$; TC = 17050
4. $x_{13} = 14, x_{21} = 1, x_{22} = 10, x_{23} = 1, x_{31} = 5$; TC = 73
5. $x_{12} = 10, x_{21} = 60, x_{22} = 15, x_{23} = 5, x_{31} = 15$ and $x_{F3} = 40$; TC = Rs. 505
6. Yes, problem has alternate solution.
7. $x_{11} = 60, x_{12} = 30, x_{22} = 10, x_{33} = 20, x_{F2} = 10, x_{F3} = 30$
8. $x_{11} = 60, x_{12} = 0, x_{13} = 30, x_{22} = 10, x_{33} = 20, x_{F2} = 40$; TC = 400
9. $x_{11} = 30, x_{12} = 10, x_{13} = 20, x_{22} = 40, x_{33} = 10, x_{34} = 40$

   V_3 falls short of 30 trips.
   Maximize value = 1240

13.14 ASSIGNMENT PROBLEM

13.14.1 Introduction

Assignment problem is nothing but it is a special case of transportation problem in which the objective is to assign a number of sources to the equal number of destinations at a minimum cost.

Assignment problem is a completely degenerate form of transportation problem. The units available at each source is 1 and also the units required at each destination is equal to 1. It means in each row and each column there will be exactly one cell not like transportation problem ($n + n - 1 = 2n - 1$ cell).

13.14.2 Mathematical Formulation of Assignment Problem

Let there are n workers and n jobs and the problem is to assign n jobs to n workers in such a way that total cost is minimum. The cost matrix $\{c_{ij}\}$ is given below:

		Jobs				
		J_1	J_2	…	J_n	Availability
	W_1	c_{11}	c_{12}	…	c_{1n}	1
Workers	W_2	c_{21}	c_{22}	…	c_{2n}	1
	⋮	⋮				⋮
	W_n	c_{n1}	c_{n2}	…	c_{nn}	1
Requirement		1	1	…	1	

Let x_{ij} is the assignment of i^{th} worker to j^{th} job, such that

$$x_{ij} = 0, \text{ if } i^{th} \text{ worker is assigned job } j$$
$$= 0, \text{ otherwise}$$

Then mathematical formulation of assignment problem is

Minimize $z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}$

Subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \ \forall \ j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \ \forall \ i = 1, 2, \ldots, n$$

$$x_{ij} = 0 \text{ or } 1 \ \forall \ i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, n$$

As we have mentioned that assignment problem is a special case of transportation problem, but if we assignment problem is solved by the methods discussed ahead to solve the transportation problem, we will get highly degenerate solution. Therefore an assignment problem is solved by a method known as Hungarian Algorithm.

13.14.3 Hungarian Algorithm to Solve an Assignment Problem

An efficient method to solve an assignment problem was developed by Hungarian mathematician D. Konig, which is given as below:

***Step* 1:** From the cost matrix, check whether the number of sources is equal to the number of destinations or not. If yes, go to the next step otherwise add a dummy source or dummy destination with cost zero to make the number of sources and destinations equal.

***Step* 2:** After making number of sources and destinations equal, identify the smallest element in each row and subtract the same from each element of row. This will give you at least one zero in each row.

***Step* 3:** In the reduced matrix obtained in step 2, identify the smallest element in each column and subtract the same form each element of the each column. This assumes at least one zero in each column.

***Step* 4:** By step 2 and 3, we get a reduced matrix in which there is at least one zero in each row and each column. Now we can go for optimal assignment as follows:

- (i) Examine each row and see which row has a single zero. Enrectangle this zero $(\square)$ and cross off all the zeros in its column. Continue until all the rows have been taken care of.
- (ii) Similarly, repeat the procedure 4(i) for each column of reduced matrix.
- (iii) If a row or and a column has two or more than two zeros and one can not be chosen by inspection then assign arbitrarily any one of these rows and cross off all other zeros of the row/column.
- (iv) Repeat (i) to (iii) above until the chain of assignment $(\square)$ or cross (X) ends.

***Step* 5:** If the number of assignment $(\square)$ is equal to n (the order of cost matrix), then it is the optimal solution. If the number of assignment is less than n, then go to the next step 6.

***Step* 6:** Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of reduced matrix. This can be done by using the following procedure.

(i) Mark (V) rows that do not have any assigned zero.
(ii) Mark (V) columns, that have assigned zeros in the marked rows.
(iii) Mark (V) rows that have assigned zeros in the marked columns.
(iv) Repeat (ii) and (iii) above until the chain of marking is completed.
(v) Draw lines through all the unmarked rows and marked columns.

***Step* 7:** Find the new reduced cost matrix as follows:

(i) Find the smallest element of the reduced matrix not covered by any of two lines.
(ii) Subtract this element from all the uncovered elements and all the same to all the elements lying at the intersection of any two lines.

***Step* 8:** Go to step 5 and repeat the procedure until we got optimal solutions

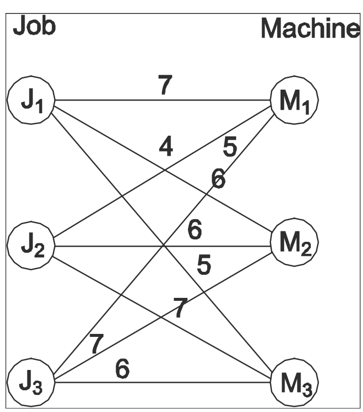
Example 42 An engineer wants to assign 3 jobs J_1, J_2, J_3 to 3 machines M_1, M_2 and M_3 in such a way that each job is assigned to some machine and no machine works on move than one job. The cost of assigning job i to machine is given below:

		Machine		
		M_1	M_2	M_3
	J_1	7	6	5
Job	J_2	4	6	7
	J_3	5	7	6

(a) Draw the associated network
(b) Formulate the network LPP

Solution

(a) Network



(b) LPP is:

Minimize $z = 7x_{11} + 6x_{12} + 5x_{13} + 4x_{21} + 6x_{22} + 7x_{23} + 5x_{31} + 7x_{32} + 6x_3$

Subject to

$$x_{11} + x_{12} + x_{13} = 1$$
$$x_{21} + x_{22} + x_{23} = 1$$
$$x_{31} + x_{32} + x_{33} = 1$$
$$x_{11} + x_{21} + x_{31} = 1$$
$$x_{12} + x_{22} + x_{32} = 1$$
$$x_{13} + x_{23} + x_{33} = 1$$
$$x_{ij} = 0 \text{ or } 1,\ i = 1, 2, 3 \text{ and } j = 1, 2, 3$$

Example 43 Solve the following assignment problem by using Hungarian method. J_1, J_2, J_3, J_4 are jobs and W_1, W_2, W_3, W_4 are workers.

The cost matrix is given in ₹ 100 in the following table:

Jobs

Workers		J_1	J_2	J_3	J_4
	W_1	7	9	11	3
	W_2	11	5	10	8
	W_3	10	8	8	7
	W_4	8	6	9	10

Solution

***Step* 1:** To get the reduce matrix we subtract smallest element of each row, they are 3, 5, 7 and 6 of row I, II, III and IV, subtract them from each row respectively, we get the following reduced matrix in the following table

	J_1	J_2	J_3	J_4
W_1	4	6	8	0
W_2	6	0	5	3
W_3	3	1	1	0
W_4	2	0	3	4

***Step* 2:** Subtract smallest element of each column which are 2, 0, 1, 0 from the respective element and get the next reduced matrix as table

	J_1	J_2	J_3	J_4
W_1	2	6	7	0
W_2	4	0	4	3
W_3	1	1	0	0
W_4	0	0	2	4

***Step* 3:** Now make assignment as follows:

(i) Consider the first row, there is only one zero, enrectangle it $\left(\boxed{}\right)$ and cross out (X) all the zero in column.

(ii) Enrectangle zero of row 2 and cross out (X) zero in the corresponding column 4.

(iii) Enrectangle zero of third row.

(iv) enrectangle zero of IV row.

By doing these all zeros are either enrectangle or crossed and number of assignments are 4 which are shown in table

	J_1	J_2	J_3	J_4
W_1	2	6	7	$\boxed{0}$
W_2	4	$\boxed{0}$	4	3
W_3	1	1	$\boxed{0}$	$\cancel{0}$
W_4	$\boxed{0}$	$\cancel{0}$	2	4

Solution The following assignments are optimal solution made:

$$W_1 \to J_4$$
$$W_2 \to J_2$$
$$W_3 \to J_4$$
$$W_4 \to J_1$$

Minimum cost = $100\,(3+5+7+6+2+0+1+0) =$ ₹ 2400

13.14.4 Special Cases in Assignment Problem

(a) Maximization Case In some cases the objective function is maximization in nature instead of minimization. To solve such problems we can still use the same Hungarian method by changing the objective function to minimization from maximization i.e., multiply each element of the given matrix by –1 and then solve it in usual manner by the Hungarian algorithm but to find the value of objective function we take positive c_{ij} instead of $-c_{ij}$.

(b) Prohibited Assignment Sometimes due to some reason, we can put a condition that a particular person (machine) cannot be assigned a particular job (activity). In such cases the cost of performing that particular job by a particular person is considered to be very large (i.e., M a big positive number) as large to prohibit the entry of this pair.

Example 44 The following is the cost matrix of assigning 4 persons to 4 jobs. Find the optimal assignment if person 1 cannot be assigned to job A.

Person	Jobs			
	A	B	C	D
1	–	6	3	1
2	5	8	6	7
3	6	9	5	4
4	4	7	7	3

What is the minimum total cost?

Solution Reduce the cost matrix by subtracting the smallest element of each row (column) from the corresponding row (column). In the reduced matrix make assignments in rows and columns that have single row. Draw the minimum number of lines to cover all the zero of reduced matrix as given in following table

M	2	1	$\cancel{0}$
$\boxed{X}$	$\cancel{0}$	$\cancel{0}$	$\cancel{2}$
$\cancel{2}$	$\cancel{2}$	$\cancel{0}$	$\cancel{0}$
1	1	3	$\cancel{0}$

Note: *Cross out* 2^{nd}, 3^{rd} *rows and* 4^{th} *column.*

Modify the reduced cost matrix by subtracting 1 from all the elements not covered by the lines and adding 1 at the intersecting two lines. We get Table 2.

M	1	$\cancel{0}$	$\boxed{0}$
$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{3}$
$\cancel{2}$	$\cancel{2}$	$\boxed{0}$	$\cancel{1}$
0	$\boxed{0}$	2	$\cancel{0}$

The optimum solution is

Person 1 $\to$ Job D		Person 1 $\to$ Job D
Person 2 $\to$ Job A	or	Person 2 $\to$ Job B
Person 3 $\to$ Job C		Person 3 $\to$ Job C
Person 4 $\to$ Job B		Person 4 $\to$ Job A

Minimum cost = 18

Example 45 A company wishes to assign 4 jobs to 3 machines. The estimates of the times (in minutes) each machine would take to complete a job is given below. How should the jobs should be allocated to the machines so that the total cost is minimum.

Jobs	Machines		
	M_1	M_2	M_3
I	8	25	14
II	12	26	5
III	34	19	14
IV	17	29	19

Solution Since the given problem is unbalanced, we add a dummy machine M_4 with all the entries zero and use Hungarian algorithm to find optimal solution.

Now reduce the balanced cost matrix and make assignment in rows and columns having single rows. We have the following table:

	M_1	M_2	M_3	M_4
I	$\boxed{0}$	6	9	$\cancel{0}$
II	4	7	$\boxed{0}$	$\cancel{0}$
III	26	$\boxed{0}$	9	$\cancel{0}$
IV	9	10	14	$\boxed{0}$

The optimal assignment is:

$I \to M_1$, $II \to M_3$, $III \to M_2$, $IV \to M_4$ i.e., job IV remains incomplete. The minimum time is $8 + 5 + 19 = 32$ minutes.

EXERCISE 13.11

Solve the following assignment problems (1–4)

1.

	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

[Madras 1991]

2.

	A	B	C	D
1	10	25	15	20
2	15	30	5	15
3	35	20	12	24
4	17	25	24	20

[IAS 1990, Anantapur 1990]

3.

	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

[Bharthidesan 1995]

4.

	M_1	M_2	M_3	M_4
J_1	5	8	3	2
J_2	10	7	5	8
J_3	4	10	12	10
J_4	8	6	9	4

[Sambhalpur 1994]

5. A dean wants four tasks to be performed by four heads of the department. The heads differ in efficiency, and the tasks differ in their intrinsic difficulty. His estimate of time each head would take to perform each task is given as follows:

		Heads			
		A	B	C	D
Tasks	I	20	28	19	13
	II	15	30	16	28
	III	40	21	20	17
	IV	21	28	26	12

How should the tasks be allocated one to a head, so as to minimize the total man hour?

6. Solve the following assignment problem by Hungarian algorithm

		Men		
		1	2	3
Tasks	I	9	26	15
	II	13	27	6
	III	35	20	15
	IV	18	30	20

[Dyalbagh 1989, IAS 1993]

7. Find the optimal assignment schedule for the following problem.

		Markets			
		M_1	M_2	M_3	M_4
	I	81	71	76	73
Salesmen	II	76	76	81	86
	III	79	79	83	79

What is the total maximum sale?

Answers

1. I $\rightarrow$ A, II $\rightarrow$ C, III $\rightarrow$ B, IV $\rightarrow$ D; minimum cost = 21
2. 1 $\rightarrow$ A, 2 $\rightarrow$ C, 3 $\rightarrow$ B, 4 $\rightarrow$ D; minimum cost = 55
3. A $\rightarrow$ 2, B $\rightarrow$ 3, C $\rightarrow$ 4, D $\rightarrow$ 1; minimum cost = 38
4. $J_1 \rightarrow M_4$, $J_2 \rightarrow M_3$, $J_3 \rightarrow M_1$, $J_2 \rightarrow M_2$; minimum cost = 17
5. I $\rightarrow$ C, II $\rightarrow$ A, III $\rightarrow$ B, IV $\rightarrow$ D, Minimum man hours = 67
6. I $\rightarrow$ 1, II $\rightarrow$ 3, III $\rightarrow$ 2, IV $\rightarrow$ 4, which is dummy man.
   The minimum cost is 35 hours.
7. I $\rightarrow M_2$, II $\rightarrow M_1$, III $\rightarrow M_4$, IV $\rightarrow M_3$ (IV salesman is dummy)
   Maximum sales = 226.

SUMMARY

Following topics have been discussed in this chapter:

1. Definition of Linear Programming Problem (LPP).
2. Formulation of LPP.
3. Solution of LPP
   (i) Graphical Method
   (ii) Algebraic Method
   (iii) Simplex Method
   (iv) Artificial Variable Method
      (a) M-Method
      (b) Two Phase Method
4. Exceptional cases in LPP.
5. Construction of a dual problem
6. Relation between primal and a dual problem
7. Dual simplex Method
8. Transportation Problem and its solution
   $\left.\begin{array}{l}\text{(i) NWCM (North-West) corner Method)}\\ \text{(ii) LCM (Least Cost Entry Method)}\\ \text{(iii) VAM (Vogal's Approximation Method)}\end{array}\right\}$ For initial BFS
   (iv) u–v method for optimal solution
9. Assignment Problem and its solution by Hungarian Algorithm.

OBJECTIVE TYPE QUESTIONS

1. Which of the following is not correct?
Linear Programming must have an
(a) objective that we aim to maximize or minimize
(b) constraints that we need to specify
(c) decision variables that we need to determine
(d) decision variables are to be unrestricted

2. Which of the following is not correct in LPP?
(a) All constraints must be linear
(b) Objective function must be linear
(c) All the constraints and decision variables must be of either '≤' or '≥' type
(d) All decision variables must be non-negative

3. Which of the following is not associated with an LPP?
(a) Proportionality
(b) Uncertainty
(c) Additivity
(d) Divisibility

4. Which is of the following is correct?
(a) LPP takes into consideration the effect of time and uncertainty
(b) An LPP can have only two decision variables
(c) Decision variables in an LPP may be more or less than the number of constraints
(d) Linear programming deals with problems involving only a single objective

5. Which of the following is not correct?
(a) Feasible solution of an LPP is independent of the objective function
(b) The feasible region of an LPP must be a convex set
(c) The feasible region for a constraint is restricted if its '≥' or '≤' sign is replaced by $a' = '$ sign
(d) It is not possible to obtain feasible solution of an LPP by graphical method

6. Using graphical method, the optimum solution of the LPP of maximizing $z = 10x_1 + 15x_2$, subject to: $2x_1 + x_2 \le 26$, $x_1 + 2x_2 \le 28$, $x_2 - x_1 \le 5$; $x_1, x_2 \ge 0$ is obtained as
(a) $x_1 = 8, x_2 = 10$
(b) $x_1 = 6, x_2 = 1$
(c) $x_1 = 6, x_2 = 10$
(d) $x_1 = 8, x_2 = 8$

7. Which of the following is correct?
(a) An LPP always has unbounded feasible region
(b) An LPP has only two constraints $2x_1 + x_2 \le 40$ and $x_1 \ge 30$, besides $x_1 \ge 0$ and $x_2 \ge 0$. It would have a feasible solution
(c) A constraint $x_1 + 2x_2 \le 36$ of an LPP is replaced by the constraint $x_1 + 2x_2 \le 24$. This would make the LPP more restrictive in nature
(d) An LPP cannot have more than one redundant constraint

8. Given an LPP to maximize $z = -5x_2$; subject to $x_1 + x_2 \le 1$, $\frac{1}{2}x_1 + 5x_2 \ge 0$ and $x_1, x_2 \ge 0$.
Using graphical method, we have
(a) No feasible solution
(b) Unbounded solution
(c) Unique optimal solution
(d) Multiple optimum solution

9. An LPP is in standard form, if
(a) the constraints are strict equation
(b) the constraints are inequalities of '≤' type
(c) the constraints are inequalities of '≥' type
(d) all the division variables are unrestricted in sign

10. Given a system of m simultaneous linear equation in n unknowns ($m < n$), the number of basic variables will be
(a) m (b) n
(c) $n - m$ (d) $n + m$

11. The basic solutions of an LPP having m constraints and n unknowns ($m < n$) are
(a) $\binom{n}{m}$ (b) $\le \binom{n}{m}$
(c) m (d) $\ge \binom{n}{m}$

12. A necessary and sufficient condition for a basic feasible solution to a minimization LPP to be an optimum is that all entries in z-row is
(a) ≥ 0 (b) ≤ 0
(c) $= 0$ (d) < 0 or > 0

13. At any iteration of usual simplex method, if there is at least one basic variable in the basis at zero level and all entries in z-row ≥ 0, the current solution is
(a) Infeasible (b) Unbounded
(c) Non-degenerate (d) Degenerate

14. In a maximization LPP, if at least one artificial variable is in the basis, but not at zero level and the coefficient of M in z-row is non-negative then we have
(a) a feasible solution
(b) infeasible solution
(c) an unbounded solution
(d) an optimum solution

15. In a LPP
(a) number of BFS $\leq$ number of vertices
(b) number of BFS = number of vertices
(c) number of BFS $\geq$ number of vertices
(d) number of BFS < number of vertices

16. In some simplex table of a maximization LPP, the column corresponding to variable x_j is $(3; -2, -1, 0)^T$. Then this show that
(a) feasible region is bounded
(b) feasible region is unbounded in x_j-direction
(c) solution is unbounded
(d) none of the above

17. Consider the system

$$x_1 + x_2 - x_3 - 2x_4 - 5x_5 = 2$$
$$x_2 + x_3 + 5x_4 - 5x_5 = 2$$

The solution $x_1 = x_3 = x_4 = 0$, $x_2 = 7$ and $x_5 = 1$ of the above system is
(a) a basic solution
(b) a basic feasible solution
(c) not a basic solution
(d) none of the above

18. If in a simplex table the coefficient in z-row for a non-basic variable, then there exists an alternate optimal solution provided
(a) it is a starting simplex table
(b) it is a optimal simplex table
(c) it can be any simplex table
(d) none of the above

19. The value in z-row of a basic variable is always
(a) negative (b) positive
(c) zero (d) non-negative

20. Phase-I of simplex method
(a) optimizes the objective function of a given problem
(b) gives a starting BFS
(c) is required if a variable is unrestricted in sign
(d) none of the above

21. In a LPP in standard form there are 6 variables and 4 constraints. Then the number of BFS are
(a) 15 (b) ≤ 15
(c) ≥ 15 (d) all of the above

22. If in Phase-I of the simplex method an artificial variable remains at positive level in the optimal table of Phase-I then
(a) the solution is unbounded
(b) there exists an optimal solution
(c) there exists no solution
(d) there is an alternate solution

23. In any simplex iteration if there is a tie between two leaving variables then the next iteration will be
(a) optimal solution
(b) degenerate solution
(c) unbounded solution
(d) an alternate solution

24. If a variable x_j is unrestricted in sign in the primal LPP, then the corresponding (that is j^{th}) dual constraint is
(a) with $\leq$ sign if the primal is a minimization problem
(b) with $\geq$ sign if the primal is a maximization problem
(c) with equality sign
(d) all the above

25. If the j^{th} constraint in the primal LPP is an equality then the corresponding (that is j^{th}) dual variable is
(a) unrestricted in sign
(b) restricted to ≥ 0
(c) restricted to ≤ 0
(d) strictly = 0

26. If the primal has an unbounded situation, then the dual problem
(a) has an optional solution
(b) has no solution
(c) has an unbounded solution
(d) none of the above

27. In dual simplex method
(a) the iterations move towards feasibility maintaining optimality
(b) the iterations move towards optimality maintaining feasibility
(c) the iterations maintain both feasibility and optimality
(d) none of the above

28. Let the primal be minimization. Let a feasible solution, which is not optimal, of primal has value 25. Then which of the following can be the value of a dual at a feasible solution of dual
(a) 25 (b) 24.5
(c) 26 (d) 28

29. In a balanced TP with m sources and n destinations, the number of linearly independent constraint is
(a) $m + n$ (b) $m + n + 1$
(c) $m + n - 1$ (d) $m - n$

30. In the optimal table of a TP a zero in the south-west corner show that
(a) the optimal solution is degenerate
(b) an alternate optimum solution exists
(c) no feasible solution
(d) none of the above

31. In an assignment problem with m jobs and m persons, the number of assignment will be
(a) m (b) $m + 1$
(c) $m - 1$ (d) $2m - 1$

32. The assignment problem is a special type of transportation problem in which the number of sources
(a) equals the number of destinations
(b) is greater than the number of destination
(c) is less than the number of destinations
(d) is less than or equal to the number of destinations

33. The minimum number of lines covering all zeros in a reduced cost matrix of order n can be
(a) at most n (b) at least n
(c) $n - 1$ (d) $n + 1$

34. In an assignment problem involving four workers and three jobs, total number of assignments possible are
(a) 4 (b) 3
(c) 7 (d) 12

35. An assignment problem can be solved if only the number of rows are equal to
(a) number of columns
(b) greater than number of columns
(c) less than number of columns
(d) greater than or equal to number of columns

ANSWERS

1.(d)	2.(c)	3.(b)	4.(c)	5.(d)	6.(a)	7.(c)	8.(a)	9.(a)	10.(a)
11.(a)	12.(b)	13.(d)	14.(b)	15.(c)	16.(c)	17.(c)	18.(b)	19.(c)	20.(b)
21.(b)	22.(c)	23.(b)	24.(c)	25.(a)	26.(b)	27.(a)	28.(b)	29.(c)	30.(b)
31.(a)	32.(a)	33.(a)	34.(b)	35.(a)					

14 Calculus of Variations

14.1 INTRODUCTION

Calculus of variations plays an important role in modern developments in analysis, geometry and physics. Originating as a study of certain maximum and minimum problems not treatable by the methods of elementary calculus, variational calculus in its present form provides powerful methods for the treatment of differential equations, theory of invariants, existence theorems in geometric function theory, variational principles in mechanics. It is also important to the applications to boundary value problems in partial differential equations and in the numerical calculation of many types of problems which can be stated in variational form. No literature representing these diverging viewpoints is to be found among standard texts on calculus of variations, and in this course an attempt will be made to do justice to this variety of problems. The subject matter with which calculus of variations is concerned is a class of extremum (i.e., maximum or minimum) problems which can be considered as an extension of the familiar class of extremum problems dealt with by elementary differential calculus. In the elementary problems, one seeks extremal values of a function of one or more (but in any case a finite number) real variables in the more general problems considered by calculus of variations, the functions to be extremized, sometimes called functionals, have functions as independent variables. In this chapter, we will study various method of variational with fixed boundaries

14.2 FUNCTION

Let X and Y be any two non-empty sets and there be a correspondence or association between the elements of X and Y such that for every elements of X and Y, there exists a unique element $y \in Y$ written as $y = f(x)$ then we say that y is a mapping or function from X to Y and written as $f: X \to Y$ such that $y = f(x)\ \forall\ x \in X, y \in X$.

14.3 FUNCTIONAL

Let there be functional belonging to class of functions. Then a variable quantity denoted by $I[y(x)]$ is a functional if to each function belonging to the class of functions there is definite value of I.

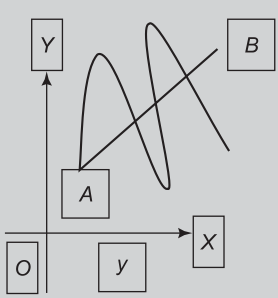
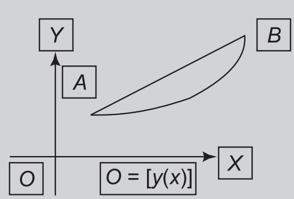
Thus functional is a function between the set of functions and set variable quantity. Obviously, domain of the functional is a set of functions for which the functional have been defined as following:

(i) $\int_{x_1}^{x_2} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$ determines the length of arc between the points (x_1, y_1) and (x_2, y_2) on the curve $y = f(x)$. Length of the arc is determined by the choice of functions. Thus, we see corresponding to functions through (x_1, y_1) and (x_2, y_2) determine the length of arc between these points. Therefore it is a clear example of functional. Thus we note $I\left[y(x)\right] = \int_{x_1}^{x_2} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx.$

We can also write it as $I\left[y(x)\right] = \int_{x_1}^{x_2} f\left[x, y(x), y'(x)\right]dx.$

(ii) Area S of a surface bounded by a given curve C is determined by the surface $z = z(x, y)$ and given by $\iint_D \sqrt{\left(1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right)}dxdy$ where D is the projection of area bounded by curve on xy-plane.

14.4 DIFFERENCE BETWEEN FUNCTION AND FUNCTIONAL

S.No	Function	Functional
1.	The variable y is called a function of a variable x, in writing as $y = y(x)$.	The variable I is called a functional depending on a function $y = y(x)$, in writing as $I = I[y(x)]$.
2.	If to each value of x from a certain domain there corresponds a certain value of y, to a given number x, there corresponds a number y.	If to each function $y(x)$, from a certain class of function there corresponds a certain value of I, to a given function $y(x)$, there corresponds a number I.
3.	The increment Δx of the argument x of a function $y(x)$ is the difference of two values of this argument $\Delta x = x - x_1$. If x is the independent variable, then the differential of x coincides with its increment $dx = \Delta x$.	The increment or the variation δy of the argument $y(x)$ of a functional $I[y(x)]$ is the difference of two functions $\delta y = y(x) - y_1(x)$. It is assumed that the argument is $y(x)$ runs through a certain class of functions.
4.	A function $y(x)$ is said to be a continuous function, if small variations of x always lead to small variations of the functions $y(x)$. 	A functional $I[y(x)]$ is said to be a continuous function, if small variations of $y(x)$ always lead to small variations of the functional $I[y(x)]$. 

5.	The function $l(x)$ is called linear if $l(cx) = cl(x)$.	The functional $L(x)$ is called linear, if $L(cx) = cL(x)$.
6.	If the increment $\Delta y = y = y(x + \Delta x) - y(x)$ is of the form $\Delta y = A(x)\Delta x + \beta(x, \Delta x).\Delta x$.	If the increment $\Delta I = I\left[y(x)+\delta y)\right] - I\left[y(x)\right]$ of a functional is of the form $\Delta I = L\left[y(x), \delta x\right] + \beta(y(x), \delta y).\max\left\|\delta y\right\|$.
7.	The differential of a function $y(x)$ is given by $\left(\dfrac{\partial}{\partial \alpha} y(x+\alpha.\Delta x)\right)_{\alpha=0}$.	The variation of a functional $I[y(x)]$ is given by $\left(\dfrac{\partial}{\partial \alpha} I\left[y(x+\alpha.\delta y)\right]\right)_{\alpha=0}$.

14.5 CLOSENESS OF CURVES

In the calculus of real variable we have defined the continuity of function at $x = a$. As the variable x is becoming closer to a, then $f(x)$ becomes closer to $f(a)$.

Here in case of functional, domain is the class of functions and here idea of closeness of $y(x)$ and $y_1(x)$ is to say that the absolute value of their difference, i.e. $|y(x) - y_1(x)|$ is small for all x for which $y(x)$ and $y_1(x)$ are defined.

If it happens then we say $y(x)$ is close to $y_1(x)$ in the sense of zero order proximity. $I\left[y(x)\right] = \int_{x_1}^{x_2} f\left[x, y(x), y'(x)\right]dx$. Curve $y(x)$ and $y_1(x)$ are said to be close in the sense of n^{th} order proximity if $|y(x) - y_1(x)|$, $\left|y'(x) - y_1{}'(x)\right| \cdots \left|y^n(x) - y_1^n(x)\right|$ are small for the values of for x which these functions are defined.

14.6 CONTINUITY OF FUNCTIONAL

Domain of functional is the set of functions, therefore continuity of functional at some function is defined as following: Functional $I[y(x)]$ is said to be continuous at $y = y_0(x)$ in the sense of n^{th} order proximity if given for any positive number ε, $\exists\ \delta > 0$ such that $\left|I\left[y(x)\right] - I\left[y_0(x)\right]\right| < \varepsilon$

for $\left|y(x) - y_0(x)\right| < \delta, \left|y'(x) - y_0{}'(x)\right| < \delta \cdots \left|y^n(x) - y_0^n(x)\right| < \delta$

Linear Property Let the functional $I[y(x)]$ be defined in the linear space M of the functions $y(x)$. Functional $I[y(x)]$ is said to be linear if it satisfies the following conditions

(a) $I\left[c\ y(x)\right] = c\ I\left[y(x)\right]$ where c is any arbitrary constant.

(b) $I\left[\ y_1(x) + y_2(x)\right] = I\left[y_1(x)\right] + I\left[y_2(x)\right]$ where $y_1(x)$ and $y_2(x) \in M$

Example 1 Functional $I\left[y(x)\right] = \int_a^b \left(y' + 2y\right)dx$ defined in the space which consists of the continuous function possessing continuous derivatives of first order.

Solution

Given Functional $$I\left[y(x)\right]=\int_a^b\left(y'+2y\right)dx$$

(a) $$I\left[c\, y(x)\right]=\int_a^b\left[(cy)'+2cy\right]dx=\int_a^b\left(cy'+2cy\right)dx=c\int_a^b\left(y'+2y\right)dx=c\, I\left[\, y(x)\right]$$

(b) $$I\left[\, y_1(x)+y_2(x)\right]=\int_a^b\left[\left(y_1+y_2\right)'+2\left(y_1+y_2\right)\right]dx$$

$$=\int_a^b\left(y_1'+2y_1\right)dx+\int_a^b\left(y_2'+2y_2\right)dx=I\left[y_1(x)\right]+I\left[y_2(x)\right]$$

Functional $I[y(x)]$ is said to be linear.

14.7 VARIATION OF FUNCTIONAL

In calculus of real variable, in case of the function $y = y(x)$ of single variable. We have studied that incremental ratio $\frac{\Delta y}{\Delta x}$ tends to $f'(x)$, when Δx tends to zero suggesting thereby $\frac{\Delta y}{\Delta x}$ differs from function of $f'(x)$ by a small quantity α, where α is function of Δx and tends to zero as $\Delta x \to 0$. $\frac{\Delta y}{\Delta x}=f^1(x)+\alpha(x,\delta x) \Rightarrow \Delta y=\left(f^1(x)+\alpha(x,\delta x)\right)\Delta x=A(x)\Delta x+\alpha(x,\delta x)\Delta x,$ where $A(x)$ is a function of x. Principal part of the increment Δx, i.e. $f'(x)\Delta x$ is known as differential in y. Likewise in case of functional $I[y(x)]$, the increment ΔI is given as $\Delta I=I\left[y(x)+\delta y(x)\right]-I\left[y(x)\right]$ (domain of functional is functional and $\delta(y(x))$ denotes the corresponding change) which can be written as

$$\Delta I=L\left[y(x),\delta y(x)\right]+\alpha\left[y(x),\delta y\right]\max\left|\delta y\right|,$$

where $L\left[y(x),\delta y(x)\right]$ denotes the functional linear in δy and $\alpha(y(x),\delta y)$ tends to zero as the maximum value of $\delta y\to 0.$ Principal part of the increment Δy i.e. $L\left[y(x),\delta y(x)\right]$ is called the variation of the functional and is denoted by δI.

Theorem 1 Variation of the functional $I[y(x)]$ is equal to $\frac{\partial}{\partial\beta}I\left[y(x)+\beta\ \delta y(x)\right]$ at $\beta = 0$, for fixed y and δy and different values of parameter β.

Proof Increment ΔI can be written as

$$\Delta I=I\left[y(x)+\beta\ \delta y\right]-I\left[y(x)\right]=L\left[y(x),\beta\ \delta y(x)\right]+\alpha\left[y(x),\beta\delta y\right]\max\left|\beta\delta y\right|$$

$$=L\left[y(x),\beta\ \delta y(x)\right]+\alpha\left[y(x),\beta\delta y\right]\left|\beta\right|\max\left|\delta y\right|$$

Now derivative of $I\left[y(x)+\beta\ \delta y\right]$ with respect to β at $\beta = 0$.

$$=\lim_{\Delta\beta\to 0}\frac{\Delta I}{\Delta\beta}=\lim_{\beta\to 0}\frac{\Delta I}{\beta}\quad (\beta \text{ Change from 0 to } \beta, \text{ since } \Delta\beta=\beta)$$

$$= \lim_{\Delta\beta\to 0} \frac{L[y(x), \beta\ \delta y] + \alpha[y(x), \beta\ \delta y]|\beta| \max|\delta y|}{\beta}$$

$$= \lim_{\beta\to 0} \frac{L[y(x), \beta\ \delta y]}{\beta} \quad \text{as } \alpha \to 0 \text{ as } \beta \to 0$$

$$= \lim_{\beta\to 0} \frac{\beta L[y(x),\ \delta y]}{\beta} \qquad \text{(By property of linearity)}$$

$\therefore \quad \dfrac{\partial}{\partial\beta} I[y(x) + \beta\ \delta y]$ is equal to δI, i.e. variation of functional $I[y(x)]$.

14.8 MAXIMA OR MINIMA OF FUNCTIONALS

(i) A functional $I[y(x)]$ is said to have maximum of functional on a curve $y = y_0(x)$ if the value of the functional on any curve close to $y = y_0(x)$ is less than or equal to that of $I[y(x)]$. In other words, $\Delta I = I[y(x)] - I[y_0(x)] \le 0$.

(ii) If $\Delta I < 0$ and is zero only on $y = y_0(x)$ then we say that strict maxima is obtained on $y = y_0(x)$.

(iii) Functional $I[y(x)]$ attains minimum on a curve $y = y_0(x)$ if the values of the functional on any curve close to $y_0(x)$ is greater than to that of obtained on $y = y_0(x)$, i.e. $\Delta I = I[y(x)] - I[y_0(x)] \ge 0$.

(iv) If $\Delta I > 0$ is zero only on $y = y_0(x)$, then we say that strict minima is obtained on $y = y_0(x)$.

Definition

Distance ρ between two curves $y = y_1(x)$ and $y = y_2(x)$ is obtained by $\rho = \rho(y_1, y_2)$ for $x_0 \le x \le x_1$ and is defined as $\rho(y_1, y_2) = \max\limits_{x_0 \le x \le x_1} |y_1(x) - y_2(x)|$ in reference to concept of zero order proximity:

Similarly with reference to n^{th} order proximity, it is taken as

$$\rho = \rho(y_1, y_2) = \sum_{m=1}^{n} \max_{x_0 \le x \le x_1} \left| y_1^{(m)}(x) - y_2^{(m)}(x) \right|.$$

Example 2 Investigate the closeness of the following curves $y(x) = \dfrac{\sin nx}{n^2}$ and $y_1(x) = 0$ on $[0, \pi]$.

Solution

Consider $|y(x) - y_1(x)| = \left|\dfrac{\sin nx}{n^2} - 0\right| \le \dfrac{1}{n^2}$

or $\quad |y(x) - y_1(x)| \le \dfrac{1}{n^2} \to 0$ as $n \to \infty$

also $\quad y'(x) = \dfrac{\cos nx}{n}$, $y_1'(x) = 0 \quad |y(x) - y_1(x)| = \left|\dfrac{\cos nx}{n} - 0\right| \le \dfrac{1}{n}$

or $\quad |y(x) - y_1(x)| \le \dfrac{1}{n} \to 0$ as $n \to \infty$

Hence, curves are close in first order proximity for large value of n.

Example 3 Find the distance ρ between the curves $y_1(x) = x$ and $y_2(x) = x^2$ on the interval [0, 1].

Solution

Consider $\rho(y_1, y_2) = \max\limits_{x_0 \le x \le x_1} \left|y_1(x) - y_2(x)\right| = \max\limits_{0 \le x \le 1} \left|x - x^2\right|.$

Let $y(x) = x - x^2,\ y^1(x) = 1 - 2x = 0 \Rightarrow x = \frac{1}{2},\ y''(x) = -2 < 0$ (maxima)

$$y\left(\frac{1}{2}\right) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow \rho = \frac{1}{4}.$$

14.9 FUNDAMENTAL LEMMA OF CALCULUS OF VARIATION

If for every continuous function $\eta(x)$, $\int\limits_{x_1}^{x_2} \phi(x)\eta(x)dx = 0$, where the function $\phi(x)$ is continuous in the closed interval $[x_1, x_2]$, then $\phi(x) = 0$ in the closed interval $[x_1, x_2]$.

Remarks

(i) Maximum or Minimum of the functional on a curve $y = y_0(x)$ means the largest or smallest value on the functional is relative to the value of the functional close to the curve $y = y_0(x)$.

(ii) Maximum or Minimum obtained by the functional is said to be strong on the curve $y = y_0(x)$, if $|y(x) - y_0(x)|$ is small with respect to all the curves $y = y(x)$.

(iii) If functional attains maximum on the curve $y = y_0(x)$ with respect to all the curves $y = y(x)$, such that $|y(x) - y_0(x)|$ and $|y'(x) - y'_0(x)|$ are both small then maximum or minimum is said to be weak.

(iv) If a strong maximum or minimum of functional is obtained on the curve $y = y_0(x)$, then weak maximum or minimum is also obtained on the same curve.

(v) If two curves are close with respect to first order proximity then they are close with respect to zero order proximity.

(vi) If a function $I[y(x)]$ attains a maximum or minimum on $y = y_0(x)$, where the domain of definition of functional is same class of functions then at $y = y_0(x)$, $\delta I = 0$.

14.10 EXTREMAL

One of the main problems of the calculus of variation is to determine that curve connecting two given points which either maximizes or minimizes some given integral. Consider the curve $y = y(x)$, where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for some given known function $F(x, y, y')$ of variables x, y, y', the integral $I\left[y(x)\right] = \int\limits_{x_1}^{x_2} f\left(x, y(x), y'(x)\right)dx$ is either maximum or minimum also called an extremum or stationary value. A curve $y = y(x)$ which satisfies this property is called an extremal.

14.11 SOME IMPORTANT LEMMAS

***Lemma* 1** If $\alpha(x)$ is continuous in $[a, b]$, and if $\int\limits_a^b \alpha(x)h(x)\,dx = 0$ for every function $h(x) \in (a, b)$ such that $h(a) = h(b) = 0$, then $\alpha(x) = 0$ for all $x \in [a, b]$.

Proof Suppose the function $\alpha(x)$ is non-zero value at some point in $[a, b]$ then $\alpha(x)$ is also positive in some interval $[x_1, x_2]$ contained in $[a, b]$.

Given $\int_a^b \alpha(x)h(x)dx = 0$ for every function $h(x) \in (a, b)$ such that $h(a) = h(b) = 0$.

To show that $a(x) = 0$ for all $x \in [a, b]$.

If we set $h(x) = (x - x_1)(x_2 - x)$ for all $x \in [a, b]$, and $h(x) = 0$ otherwise, then $h(x)$ satisfies the conditions of the lemma.

$$\int_a^b \alpha(x)h(x)dx = \int_a^b \alpha(x)(x - x_1)(x_2 - x)\, dx > 0,$$

Since the integrand is positive (except at x_1 and x_2).

It means that our assumption is wrong $\alpha(x) = 0$ for all $x \in [a, b]$.

***Lemma* 2** If $\alpha(x)$ is continuous in $[a, b]$, and if $\int_a^b \alpha(x)h'(x)\,dx = 0$ for every function $h(x) \in D_1(a, b)$ such that $h(a) = h(b) = 0$, then $\alpha(x) = c$ for all $x \in [a, b]$, where c is a constant.

Proof Suppose c is a constant defined by the condition $\int_a^b \left[\alpha(x) - c\right]dx = 0$ and let $h(x) = \int_a^x \left[\alpha(t) - c\right]dt = 0,$ so that $\int_a^b \alpha(x)h'(x)dx = 0$ $h(x) \in D_1(a, b)$ and satisfies the condition $h(a) = h(b) = 0$ then on the one hand, $\int_a^b \left[\alpha(x) - c\right]h'(x)dx = \int_a^b \alpha(x)\,h'(x)dx - c\left[h(a) - h(b)\right] = 0.$ While on the other hand, $\int_a^b \left[\alpha(x) - c\right]h'(x)\,dx = \int_a^b \left[\alpha(x) - c\right]^2 dx.$

It follow that $\alpha(x) - c = 0$, i.e. $\alpha(x) = c$ for all $x \in [a, b]$.

***Lemma* 3** If $\alpha(x)$ is continuous in $[a, b]$, and if $\int_a^b \alpha(x)h''(x)\,dx = 0$ for every function $h(x) \in (a, b)$ such that $h(a) = h(b) = 0$ and $h'(a) = h'(b) = 0$, then $\alpha(x) = c_0 + c_1 x$ for all $x \in [a, b]$ where c_0 and c_1 are constants.

Proof Let c_0 and c_1 be a constant defined by the condition

$$\int_a^b \left[\alpha(x) - c_0 - c_1 x\right]dx = 0 \text{ and } \int_a^b \left\{\int_a^x \left[\alpha(t) - c_0 - c_1 t\right]dt\right\}dx = 0.$$

Let $h(x) = \int_a^x \left(\int_a^t \left[\alpha(z) - c_0 - c_1 z\right]dz\right)dt,$ such that $h(x)$ automatically belongs to (a, b) and satisfies the conditions $h(a) = h(b) = 0$, $h'(a) = h'(b) = 0$. Then

$$\int_a^b \left[\alpha(x)-c_0-c_1x\right]h''(x)\,dx = \int_a^b \alpha(x)h''(x)\,dx - c_0\left[h'(b)-h'(a)\right]-c_1\int_a^b xh''(x)\,dx$$

$$= -c_1\left[bh'(b)-ah'(a)\right]-c_1\left[h(b)-h(a)\right]=0.$$

While on the other hand, $\int_a^b \left[\alpha(x)-c_0-c_1x\right]h''(x)\,dx = \int_a^b \left[\alpha(x)-c_0-c_1x\right]^2 dx = 0.$ It follows $\alpha(x)-c_0-c_1x=0 \Rightarrow \alpha(x)=c_0+c_1x\ \forall\ x\in[a,b]$.

***Lemma* 4** If $\alpha(x)$ and $\beta(x)$ are continuous in $[a, b]$, and if $\int_a^b \left[\alpha(x)h(x)+\beta(x)h'(x)\right]dx = 0$ for every function $h(x) \in (a, b)$ such that $h(a) = h(b) = 0$, then $\beta(x)$ is differentiable, and $\beta'(x) = \alpha(x)$ for all $x \in [a, b]$.

Proof Given Let $\alpha(x)$ and $\beta(x)$ are continuous in $[a, b]$, and if

$$\int_a^b \left[\alpha(x)h(x)+\beta(x)h'(x)\right]dx = 0 \tag{1}$$

for every function $h(x) \in (a, b)$ such that

$$h(a) = h(b) = 0 \tag{2}$$

To show that $\beta(x)$ is differentiable, and $\beta'(x)= \alpha(x)$ for all $x \in [a, b]$.

Now,
$$A(x) = \int_a^x \alpha(t)\,dt \Rightarrow \int_a^b \left[\alpha(x)h(x)\right]dx = -\int_a^b \left[A(x)h'(x)\right]dx \tag{3}$$

Equation (1) can be rewritten as
$$\int_a^b \left[-A(x)+\beta(x)\right]h'(x)\,dx = 0 \tag{4}$$

But According to lemma 2, we get $-A(x) + b(x) =$ constant $\Rightarrow b(x)$ is differentiable and hence by the definition of $A(x)$, for all $x \in [a, b]$.

14.12 EULER'S EQUATION

Let us examine the functional

$$I\left[y(x)\right] = \int_{x_1}^{x_2} F\left[x, y(x), y'(x)\right]dx. \tag{5}$$

Suppose that F is a given function which is twice differentiable with respect to any combination of its arguments (i.e., x, y, y'). Again suppose that an admissible curve $y(x)$ exists which is twice differentiable on $[x_1, x_2]$.

Also, satisfies the fixed boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ (6)

And which extremizes (i.e. minimizes or also maximizes) the functional $I[y(x)]$.

Let $\eta(x)$ be any function with the properties that $\eta'(x)$ is continuous and

$$\eta(x_1) = \eta(x_2) = 0. \tag{7}$$

If λ is a small parameter then $\bar{y}(x) = y(x) + \lambda.\eta(x)$ (8)

represents one parameter family of curves. The other curve in this family from the extremizing curve $y = y(x)$ is $\lambda.\eta(x)$ as shown in Fig. (14.1).

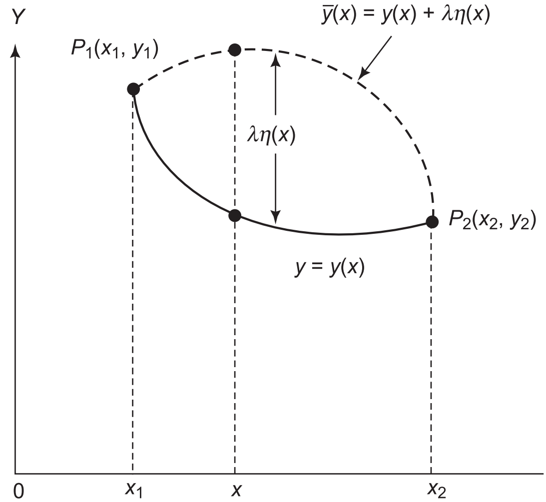


Fig. 14.1

The difference $\bar{y}(x) - y(x) = \lambda.\eta(x)$ is known as the variation of the function $y(x)$ and is denoted by $\delta y(x)$. This notation can be developed into a useful formula and is the source of the name calculus of variation. From Eq. (8), we note that for each family of this type, i.e. for each choice of the function $\eta(x)$, the extremizing curve $y = y(x)$ belongs to the family and corresponds to the value of the parameter $\lambda = 0$.

From Eq. (4),

$$\bar{y}'(x) - y'(x) = \lambda.\eta'(x) \tag{9}$$

Now, with $\eta(x)$ fixed, replacing y and y' by $\bar{y}(x)$ and $\bar{y}'(x)$ respectively in the functional Eq. (5), we get a functional of α, $I[\lambda] = \int_{x_1}^{x_2} F\left(x, \bar{y}, \bar{y}'\right) dx.$ (10)

Since setting $\lambda = 0$ has the effect of replacing $\bar{y}(x)$ and $\bar{y}'(x)$ in Eq. (10) by the function $y = y(x)$ and its derivatives $y'(x)$, if $\lambda = 0$ then $I[\lambda]$ must take on its extreme value.

By elementary calculus, a necessary condition is given by $I'[\lambda] = 0$. (11)

Using Leibnitz's rule of differentiation under the integral sign, (10) yields

$$I'[\lambda] = \int_{x_1}^{x_2} \frac{\partial}{\partial \lambda} F(x, \bar{y}, \bar{y}')\, dx \tag{12}$$

By the chain rule for differentiating functions of several variables, we get

$$\frac{\partial}{\partial \lambda} F\left(x, \bar{y}, \bar{y}'\right) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \lambda} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \lambda} = 0 + \frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \lambda} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \lambda} \text{ by Eqs (8) and (9)}$$

Hence, (8) reduces to

$$I'[\lambda]=\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \lambda}+\frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \lambda}\right)dx.$$

Using Eq. (7), we get

$$\lim_{\Delta\lambda\to 0}\left(\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \lambda}+\frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \lambda}\right)dx\right)=0 \tag{13}$$

If $\lambda = 0$ then $\bar{y}(x) = y(x)$ and $\bar{y}'(x) = y'(x)$ using in Eq. (13), we get

$$\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial \bar{y}}\eta(x)+\frac{\partial F}{\partial \bar{y}'}\eta'(x)\right)dx=\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta(x)\right)dx+\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y'}\eta'(x)\right)dx=0$$

$$\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\cdot\eta(x)\right)dx+\left(\frac{\partial F}{\partial y'}\cdot\eta'(x)\right)_{x_1}^{x_2}-\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\,dx=0 \tag{14}$$

Using Eq. (3) in Eq. (10), we get

$$\int_{x_1}^{x_2}\left\{\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}.\eta(x)\,dx=0 \tag{15}$$

The fundamental lemma of the calculus of variations, a necessary condition for the functional to have an extremum value is that the extremizing function $y = y(x)$ must satisfy the differential equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0. \quad x_1 < x < x_2 \tag{16}$$

is known as Euler's equation.

14.13 ALTERNATIVE FORMS OF EULER'S EQUATION

14.13.1 Derivation of Second Forms Euler's Equation

Euler's Equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \tag{1}$$

Since F is a function of x, y, y', we have

By total derivatives $dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial y'}dy'$

$$\frac{dF}{dx}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial y'}\frac{dy'}{dx}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}y'+\frac{\partial F}{\partial y'}y'' \tag{18}$$

Again, $\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)=y'\frac{d}{dx}\left(\frac{\partial F}{\partial y}\right)+\frac{\partial F}{\partial y'}\,y''$

$$\Rightarrow \qquad \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)-y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=\frac{\partial F}{\partial y'}\,y'' \tag{19}$$

Using Eq. (19) in Eq. (18), we get

$$\Rightarrow \qquad \frac{dF}{dx}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}y'+\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)-y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow \qquad \frac{dF}{dx}-\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}y'-y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow \qquad \frac{d}{dx}\left(F-y'\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial x}=y'\left\{\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}=y'\cdot 0 \quad \text{(using Eq. 17)}$$

$$\Rightarrow \qquad \frac{d}{dx}\left(F-y'\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial x}=0$$ is an alternative second form of Euler's equation.

14.3.2 Derivation of Third Forms Euler's Equation

Since $\frac{\partial F}{\partial y'}$ is a function of x, y, y', we have

by total derivatives $d\left(\frac{\partial F}{\partial y'}\right)=\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)dx+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)dy+\frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)dy'$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}+\frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{dy'}{dx}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=\frac{\partial^2 F}{\partial x\partial y'}+y'\frac{\partial^2 F}{\partial y\partial y'}+y''\frac{\partial^2 F}{\partial y'^2} \tag{20}$$

Using Eq. (20) in Eq. (17), we get

$$\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial x\partial y'}-y'\frac{\partial^2 F}{\partial y\partial y'}-y''\frac{\partial^2 F}{\partial y'^2}=0$$ is an alternative third form of Euler's equation.

Example 4 On what curves can the functional $\int_0^1 (y'^2+12\,xy)\,dx$, $y(0)=0$, $y(1)=1$, be extremized?

Solution

Comparing the given functional with $\int_0^1 F(x,y,y')\,dx$, we get

$$F(x, y, y')=(y'^2+12xy) \tag{21}$$

Given condition $y(0) = 0$, $y(1) = 1$, (22)

Euler's equation $$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \tag{23}$$

$$\frac{\partial F}{\partial y} = 12x, \frac{\partial F}{\partial y'} = 2y', \text{ and } \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$$

Using in Eq. (23). We get $12x - 2y'' = 0 \Rightarrow y'' = 6x \Rightarrow y' = 3x^2 + c_1$

$\Rightarrow$ $$y = x^3 + c_1 x + c_2 \tag{24}$$

Using Eq. (22) in Eq. (24). We get $\Rightarrow 0 = c_2$ and $1 = 1 + c_1 + c_2$ so that $c_1 = c_2 = 0$ an extremum can be achieved only on $y = x^3$.

Example 5 On what curves can the functional $\int_0^{\frac{\pi}{2}} (y'^2 - y^2 + 2xy)\, dx$, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 0$ be extremized?

Solution Comparing the given functional with $\int_0^{\frac{\pi}{2}} F(x, y, y')dx$, we get

$$F(x, y, y') = y'^2 - y^2 + 12\, xy \tag{25}$$

Given condition $y(0) = 0, y\left(\frac{\pi}{2}\right) = 0$ (26)

Euler's equation $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{27}$$

$\frac{\partial F}{\partial y} = -2y + 2x, \frac{\partial F}{\partial y'} = 2y', \text{ and } \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$ using in Eq. (27), we get

$$-2y + 2x - 2y'' = 0 \Rightarrow y'' + y = 6x \Rightarrow (D^2 + 1)\, y = 6x, \text{ where } D = \frac{d}{dx}$$

Auxiliary equating Eq. (28) is

$$\Rightarrow (D^2 + 1) = 0 \text{ so that } D = \pm i \tag{28}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$\text{PI} = \frac{1}{(D^2 + 1)} 6x = 6x$$

Solution is

$\Rightarrow$ $$y = c_1 \cos x + c_2 \sin x + 6x \tag{29}$$

Using Eq. (26) in Eq. (29), we get $\Rightarrow 0 = c_1$ and $0 = \frac{6\pi}{2} + c_2$, so that $c_1 = 0$ and $c_2 = -3\pi$.

An extremum can be achieved only on $y = x - 3\pi \sin x$.

Example 6 Find the extremal for the functional $I[y(x)] = \int_0^{\frac{\pi}{2}} (y^2 - y'^2 - 2y \sin x)\, dx$, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$.

Solution Comparing the given functional with $\int_0^{\frac{\pi}{2}} F(x, y, y')\,dx$, we get

$$F(x, y, y') = (y^2 - y'^2 - 2y\sin x) \tag{30}$$

Given condition

$$y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 1 \tag{31}$$

Euler's equation $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{32}$$

$\dfrac{\partial F}{\partial y} = 2y - 2\sin x,\ \dfrac{\partial F}{\partial y'} = -2y'$ and $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 2y''$ using in Eq. (3), we get

$$2y - 2\sin x + 2y'' = 0 \Rightarrow y'' + y = \sin x$$

$$\Rightarrow \qquad (D^2 + 1)y = \sin x, \text{ where } D = \frac{d}{dx} \tag{33}$$

Auxiliary Eq. (33) is $\Rightarrow (D^2 + 1) = 0$ so that $D = \pm i$, CF $= c_1 \cos x + c_2 \sin x$

PI of Eq. (33) $= \dfrac{1}{D^2 + 1}\sin x = -\dfrac{x}{2}\cos x$.

Then the solution is

$$\Rightarrow \qquad y = c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x. \tag{34}$$

Using Eq. (31) in Eq. (34), we get $\Rightarrow 0 = c_1$ and $1 = c_2$.

An extremum can be achieved only on $y = \sin x - \left(\dfrac{x}{2}\right)\cos x$.

It is the curve for two given fixed points (0, 0) and $\left(\dfrac{\pi}{2}, 1\right)$ for which the given functional will be extremum.

***Case* I** If F is independent of y' so that $\dfrac{\partial F}{\partial y'} = 0$, then Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ reduce to $\dfrac{\partial F}{\partial y} = 0$ which is a finite equation, and not a differential equation. The solution of $\dfrac{\partial F}{\partial y} = 0$ does not contain any arbitrary constants and does not satisfy the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

In general, there is no solution for this variational problem. Only if the curve $\dfrac{\partial F}{\partial y} = 0$ passes through the boundary points (x_1, y_1) and (x_2, y_2) does there exist a curve on which an extremum can be attained.

Example 7 Test for an extremum of the functional $I[y(x)] = \int_{x_1}^{x_2} y^2 dx,\ y(x_1) = y_1 = y(x_2) = y_2$.

Solution Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y')\,dx$ we get

$$F(x, y, y') = y^2 \tag{35}$$

Given condition $y(x_1) = y_1, y(x_2) = y_2$ (36)

Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$ (37)

$\frac{\partial F}{\partial y} = 2y, \frac{\partial F}{\partial y'} = 0$ and $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ using in Eq. (37), we get

$$2y = 0 \Rightarrow y = 0. \tag{38}$$

The extremal $y = 0$ passses through the points only at $y_1 = 0$ and $y_2 = 0$. If at least one of y_1 and y_2 is not zero, then the functional is not minimized on continuous functions.

***Case* II** If F is linearly dependent on y' such that $F(x, y, y') = M(x, y) + N(x, y)y'$, then

$$\frac{\partial F}{\partial y} = \frac{\partial M}{\partial y} + \frac{\partial N}{\partial y}y', \frac{\partial F}{\partial y'} = N(x, y), \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{dN(x, y)}{dx}$$

The Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ becomes $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial y}y' - \frac{dN(x, y)}{dx} = 0$

$$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial y}y' - \left(\frac{\partial N}{\partial x} + \frac{\partial N}{\partial y}\frac{dy}{dx}\right) = 0 \text{ so that } \frac{\partial M}{\partial y} - \frac{\partial N}{\partial y} = 0 \tag{39}$$

Which is a finite equation, and not a differential equation. So the curve given by Eq. (39) does not satisfy the given boundary condition $y(x_1) = y_1$, and $y(x_2) = y_2$. Hence, in general, the variational problem does not a Solution.

The given functional $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx = \int_{x_1}^{x_2} Mdx + Ndy$ is independent of the path of integration, the value of the functional $I[y(x)]$ is constant on admissible curves. In such cases, the variational problems become meaningless.

Example 8 Test for an extremum of the functional $I[y(x)] = \int_0^1 (y^2 + x^2 y')\,dx,\ y(0) = 0, y(1) = b$.

Solution Comparing the given functional with $\int_0^1 F(x, y, y')\,dx$, we get

$$F(x, y, y') = (y^2 + x^2 y') \tag{40}$$

Given condition $y(0) = 0, y(1) = b$ (41)

Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{42}$$

$$\frac{\partial F}{\partial y} = 2y, \frac{\partial F}{\partial y'} = x^2 \text{ and } \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2x \text{ using in Eq. (42), we get}$$

$$2y = 0 \Rightarrow 2y - 2x = 0 \text{ or } y = x \tag{43}$$

Equation (43) satisfies the first boundary condition $y(0) = 0$. However, the second boundary condition $y(1) = b$ is satisfied by Eq. (43) only if $b = 1$. But if $b \neq 1$ then there is no extremal that satisfy the given boundary conditions.

***Case* III** The function F is independent of x and y, i.e., F is dependent only on y' such that $F = F(y')$

An alternative third form of Euler's equation

$$\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - y'\frac{\partial^2 F}{\partial y \partial y'} - y''\frac{\partial^2 F}{\partial y'^2} = 0 \tag{44}$$

Since $$F = F(y') \Rightarrow \frac{\partial F}{\partial y} = 0 \text{ and } \frac{\partial F}{\partial x} = 0 \text{ hence} \Rightarrow \frac{\partial^2 F}{\partial x \partial y'} = 0 \text{ and } \frac{\partial^2 F}{\partial y \partial y} = 0$$

Equation (44) reduces to

$$y''\frac{\partial^2 F}{\partial y'^2} = 0 \Rightarrow y'' = 0 \text{ or } \frac{\partial^2 F}{\partial y'^2} = 0$$

If $y'' = 0$ then $y = c_1 x + c_2$ (which is a two parameter family of straight lines)

If $\dfrac{\partial^2 F}{\partial y'^2} = 0$ has one or several real roots $y' = \beta_n$, then $y = \beta_n + C$ (which is a one parameter family of straight lines contained in the two parameter family). In this case, extremals are all possible straight lines.

Example 9 Show that the shortest distance between two fixed points in the Euclidean xy-plane is a straight line.

Solution Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points on a plane. Let $P(x, y)$ be any point on any curve joining A and B and let arc (AP) = s, then arc length (AB) is given by

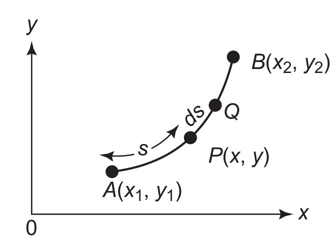


Fig. 14.2

$$arc(AB) = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} (1+y'^2)^{\frac{1}{2}}\, dx, \tag{45}$$

where $\quad y(x_1) = y_1$, and $y(x_2) = y_2$ (46)

Comparing Eq. (45) with $\int_{x_1}^{x_2} F(x, y, y')\, dx$, we have

$$F(x, y, y') = (1+y'^2)^{\frac{1}{2}} \tag{47}$$

Euler's equation is $\dfrac{\partial F}{\partial y} - \dfrac{\partial^2 F}{\partial x \partial y'} - y'\dfrac{\partial^2 F}{\partial y \partial y'} - y''\dfrac{\partial^2 F}{\partial y'^2} = 0$ (48)

From Eq. (47) $\Rightarrow \dfrac{\partial F}{\partial y} = 0, \; \dfrac{\partial F}{\partial x} = 0,$ so that $\dfrac{\partial^2 F}{\partial x \partial y'} = 0$ and $\dfrac{\partial^2 F}{\partial y \partial y'} = 0.$

Hence, Eq. (48) reduces to $-y''\dfrac{\partial^2 F}{\partial y'^2} = 0$ (49)

Since $\dfrac{\partial^2 F}{\partial y'^2} \neq 0$, Eq. (49) reduces to $y'' = 0$ or $\dfrac{d^2 y}{dx^2} = 0 \Rightarrow y = c_1 x + c_2$. (50)

This is a straight line. Since Eq. (50) passes through $A(x_1, y_1)$ and $B(x_2, y_2)$. We get

$$y_1 = c_1 x_1 + c_2 \tag{51}$$

and $$y_2 = c_1 x_2 + c_2 \tag{52}$$

Subtracting Eqs (51) and (52), $\quad y_2 - y_1 = c_1(x_2 - x_1)$ (53)

Subtracting Eqs (51) and (50), $y - y_1 = c_1(x - x_1)$ (54)

Dividing (54) by (53), we obtain $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{(x - x_1)}{(x_2 - x_1)}$ or $y - y_1 = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$

This is a straight line. Joining the given points of A and B. Hence, the shortest between A and B in a plane is a straight line.

***Case* IV** The function F is independent of y such that $F = F(x, y')$ then $\dfrac{\partial F}{\partial y} = 0$ and

Euler's Equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0.$ It reduces to $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0.$

Its integration yields $\dfrac{\partial F}{\partial y'} = C$, where C is a constant. Since this relation is independent of y, it can be solved for y' as a function of x. Another integration, to a solution involving two arbitrary constants which can be obtained by using the boundary condition $y(x'_1) = y_1$ and $y(x_2) = y_2$.

Example 10 Show that the extremal of $I[y(x)] = \int_0^2 \left(\dfrac{y'^2}{x}\right) dx$ with $y(0) = 0$ and $y(2) = 1$ is a parabola.

Solution Comparing the given functional with $\int_0^2 F(x, y, y')\,dx$, we get

$$F(x, y, y') = \left(\frac{y'^2}{x}\right) \tag{55}$$

Given condition $y(0) = 0$ and $y(2) = 1$ (56)

Euler's Equation is $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0.$ (57)

$$\frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial y'} = \frac{2y'}{x} \text{ and Eq. (57) } \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = c_1 \Rightarrow \frac{2y'}{x} = c_1$$

$$\Rightarrow \qquad \frac{dy}{dx} = \left(\frac{c_1}{2}\right)x$$

$$\Rightarrow \qquad dy = \left(\frac{c_1}{2}\right)x\,dx \Rightarrow y = \left(\frac{c_1}{4}\right)x^2 + c_2 \tag{58}$$

Using given condition in Eq. (58), we get

$\Rightarrow$ $0 = c_2$ and $1 = c_1 + c_2$ such that $c_1 = 1, c_2 = 0$

By Eq. (58), an extremum can be attained only on the curve $y = \left(\dfrac{x^2}{4}\right)$ is a parabola.

***Case* V** The function F is dependent of y and y' alone such that $F = F(y, y')$, i.e. F is independent of x such that $\dfrac{\partial F}{\partial x} = 0.$

Second from Euler's equation is $\dfrac{d}{dx}\left(F - y'\dfrac{\partial F}{\partial y'}\right) - \dfrac{\partial F}{\partial x} = 0.$

Since $\dfrac{\partial F}{\partial x} = 0,$ we have $\dfrac{d}{dx}\left(F - y'\dfrac{\partial F}{\partial y'}\right) = 0 \Rightarrow F - y'\dfrac{\partial F}{\partial y'} = c_1,$ (59)

where c_1 is a constant, since first order Eq. (59) does not contain x explicitly, it may be integrated by solving for y' and separating the variables, or by introducing a parameter.

In this case Euler's equation, first form can also be used.

Example 11 Find the extremal of the functional $I[y(x)] = \int_0^1 \left(\dfrac{1+y^2}{y'}\right)dx,$ through the origin and the point (1, 1).

Solution Comparing the given functional with $\int_0^1 F(x, y, y')\,dx$, we get

$$F(x, y, y') = \left(\frac{1+y^2}{y'}\right), \tag{60}$$

which is independent of x. Euler's equation (*Case* V)

$$F - y'\left(\frac{\partial F}{\partial y'}\right) = C_1 \tag{61}$$

$$\frac{\partial F}{\partial y'} = -\frac{(1+y^2)}{y'^2}$$

The required Euler's equation $\left(\dfrac{1+y^2}{y'}\right) + \left(\dfrac{1+y^2}{y'}\right) = C_1$

or $$y'' + y = 0 \Rightarrow \frac{dy}{1+y^2} = \frac{2}{C_1}\,dx \tag{62}$$

The general solution of Eq. (62) is

$$\tan^{-1} y = \left(\frac{2}{C_1}\right)x + C_2 \tag{63}$$

Using given conditions in Eq. (63), we get $C_1 = \dfrac{8}{\pi}$ and $C_2 = 0$.

An extremum can be attained only on the curve $y = \tan\left(\dfrac{\pi x}{4}\right)$.

Example 12 Find the curve passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ which when rotated about the x-axis gives a minimum surface area.

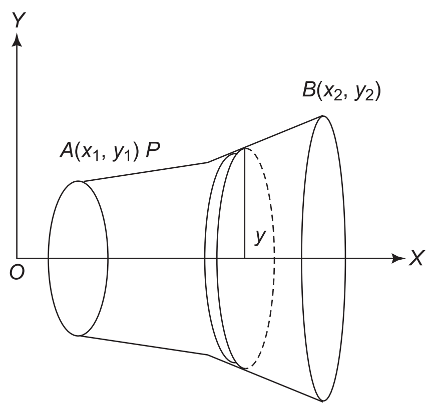


Fig. 14.3

Solution When a curve joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is revolved about x-axis, the area of the surface of revolution is given by $S[y(x)] = 2\pi \int_{x_1}^{x_2} y(1+y'^2)^{\frac{1}{2}}\,dx$.

Comparing this with $\int_{x_1}^{x_2} F(x, y, y')\, dx$ and

Omitting the irrelevant factor 2π, we have $F(x, y, y') = y(1+y'^2)^{\frac{1}{2}}$,

Since $F(x, y, y')$ $F(x, y, y')$ is function of y and y' only Euler's equation is (Case V) $F - y'\left(\frac{\partial F}{\partial y'}\right) = C_1$

$$\Rightarrow \quad y\times(1+y'^2)^{\frac{1}{2}} - y'\times\left(\frac{y}{2}\right)\times(1+y'^2)^{-\frac{1}{2}}\times 2y' = C_1$$

$$\Rightarrow \quad y\times(1+y'^2) - yy'^2 = C_1(1+y'^2)^{\frac{1}{2}} \Rightarrow y = C_1(1+y'^2)^{\frac{1}{2}} \Rightarrow y'^2 = \frac{y^2 - C_1^2}{C_1^2}$$

$$\Rightarrow \quad \frac{dy}{(y^2 - C_1^2)^{\frac{1}{2}}} = \frac{dx}{C_1} \Rightarrow \cosh^{-1}\left(\frac{y}{C_1}\right) = \frac{x}{C_1} + \frac{B}{C_1} \Rightarrow y = C_1\cosh\left(\frac{x+B}{C_1}\right),$$

where B and C_1 are two arbitrary constants. The constants B and C_1 can be obtained from the given boundary conditions depending on the positions of the points $A(x_1, y_1)$ and $B(x_2, y_2)$. There may be one, two or no solution.

Example 13 If $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ are two points in a plane then the length of the arc of the curve joining these points is given by $I[r(\theta)] = \int_{\theta_1}^{\theta_2} \sqrt{(r^2 + r'^2)}\, d\theta$, where $r' = \frac{dr}{d\theta}$. By minimizing the above integral obtain the equation of a straight line in polar co-ordinates.

Solution Comparing the given functional with $\int_{\theta_1}^{\theta_2} F(\theta, r, r')\, d\theta$, and we get

$$F(\theta, r, r') = \sqrt{(r^2 + r'^2)} \tag{64}$$

Euler's Equation is $\frac{\partial F}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial r}\right) = 0.$ (65)

$$\frac{\partial F}{\partial r} = \frac{r}{\sqrt{(r^2 + r'^2)}}, \frac{\partial F}{\partial r'} = \frac{r'}{\sqrt{r^2 + r'^2}} \text{ and } \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) = \frac{r''(r^2 + r'^2) - r'(rr' + r'r'')}{\sqrt{(r^2 + r'^2)}\times(r^2 + r'^2)}.$$

The required Euler's equation

$$\Rightarrow \quad \frac{r}{\sqrt{(r^2 + r'^2)}} - \frac{r''(r^2 + r'^2) - r'(rr' + r'r'')}{\sqrt{(r^2 + r'^2)}\times(r^2 + r'^2)} = 0$$

or $\quad \Rightarrow r.\, r'' - 2r'^2 + r^2 = 0$ (66)

Let $\quad p = r' = \dfrac{dr}{d\theta}, r'' = \dfrac{d^2r}{d\theta^2} = \dfrac{d}{d\theta}\left(\dfrac{dr}{d\theta}\right) = \dfrac{dp}{d\theta} = \dfrac{dp}{dr}\cdot\dfrac{dr}{d\theta} = p\cdot\dfrac{dp}{dr}$

Using in Eq. (66), we get

$$\Rightarrow \qquad r\cdot p\cdot\frac{dp}{dr} - 2p^2 + r^2 = 0 \Rightarrow \frac{dp}{dr} = 2\left(\frac{p}{r}\right) + \left(\frac{r}{p}\right) \tag{67}$$

Let $\left(\dfrac{r}{p}\right) = v$, i.e. $p = r\,v$ such that $\dfrac{dp}{dr} = v + r\left(\dfrac{dv}{dr}\right)$

Then using above value in Eq. (67) yields $v + r\left(\dfrac{dv}{dr}\right) = 2v + \left(\dfrac{1}{v}\right) \Rightarrow r\left(\dfrac{dv}{dr}\right) = v + \left(\dfrac{1}{v}\right)$

$$\Rightarrow \qquad \frac{2dr}{r} = \frac{2vdv}{1+v^2} \text{ such that } 2\log r - \log a^2 = \log(v^2+1) \Rightarrow v^2 + 1 = \frac{r^2}{a^2} \Rightarrow v = \frac{(r^2-a^2)^{\frac{1}{2}}}{a}$$

$$\Rightarrow \qquad \frac{p}{r} = \frac{(r^2-a^2)^{\frac{1}{2}}}{a} \Rightarrow p = \frac{r(r^2-a^2)^{\frac{1}{2}}}{a} \Rightarrow \frac{dr}{d\theta} = \frac{r(r^2-a^2)^{\frac{1}{2}}}{a}$$

$$\Rightarrow \qquad \theta - \alpha = \sec^{-1}\left(\frac{r}{a}\right) \Rightarrow \sec(\theta-\alpha) = \left(\frac{r}{a}\right)$$

$\Rightarrow r\cos(\theta - \alpha) = a$, , where α and a are arbitrary constants. (68)

Equation (68) is the polar of a straight line. Thus, the minimum distance between two points in a plane is a straight line given by Eq. (68).

14.14 VARIATIONAL PROBLEMS FOR FUNCTIONAL INVOLVING SEVERAL DEPENDENT VARIABLES

$$\int_{x_1}^{x_2} F\left[x, y_1(x), y_2(x), ... y_n(x), y_1'(x), y_2'(x), ... y_n'(x)\right] dx$$

14.14.1 Case of Two Dependent Variables

is given by $\left\{\dfrac{\partial F}{\partial y_1} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_1'}\right)\right\} = 0$ and $\left\{\dfrac{\partial F}{\partial y_2} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_2'}\right)\right\} = 0$

is called an Euler's equations.

14.14.2 Case of n^{th} Dependent Variables

The system of Euler's equations for finding extremals of the functional

$$\int_{x_1}^{x_2} F\left[x, y_1(x), y_2(x), ..., y_n(x), y_1'(x), y_2'(x), ..., y_n'(x)\right] dx$$

is given by $\left\{\dfrac{\partial F}{\partial y_i} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_i'}\right)\right\} = 0$ where $i = 1, 2, 3, 4, \ldots, n$

Example 14 Find the extremals of the functional $I\left[y(x), z(x)\right] = \int_0^{\frac{\pi}{2}} (y'^2 + z'^2 + 2yz)\,dx$, $y(0) = 0$, $y\left(\dfrac{\pi}{2}\right) = 1$, $z(0) = 0$ and $z\left(\dfrac{\pi}{2}\right) = -1$.

Solution Comparing the given functional with $\int_0^{\frac{\pi}{2}} F(x, y, z, y', z')\,dx$,

we have $F(x, y, z, y', z') = (y'^2 + z'^2 + 2yz)$. (69)

The system of Euler's equations is given by

$$\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\} = 0 \text{ and } \left\{\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)\right\} = 0$$

i.e., $2z - \dfrac{d}{dx}(2y') = 0$ and $2y - \dfrac{d}{dx}(2z') = 0$

i.e., $z - y'' = 0$ and $y - z'' = 0$

i.e., $$\frac{d^2y}{dx^2} = z \tag{70}$$

and $$\frac{d^2z}{dx^2} = y. \tag{71}$$

We shall now solve the system of simultaneous differential Eqs (70) and (71) as follows. Differentiating both side of Eq. (70) twice w.r.t. 'x', we get

$\dfrac{d^4y}{dx^4} = \dfrac{d^2z}{dx^2}$ or $\dfrac{d^4y}{dx^4} = y$, using (71)

$$(D^4 - 1)y = 0 \text{ where } D = \frac{d}{dx} \tag{72}$$

Auxiliary Eq. (72) is $(D^4 - 1) = 0 \Rightarrow D = 1,\ -1,\ i,\ -i$.

Hence the general solution of Eq. (72) is given by

$$y = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x \tag{73}$$

$$\frac{dy}{dx} = C_1e^x - C_2e^{-x} - C_3\sin x + C_4\cos x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = C_1e^x + C_2e^{-x} - C_3\cos x - C_4\sin x$$

Substituting this value of $\frac{d^2y}{dx^2}$ in Eq. (70), we have

$$z = \frac{d^2y}{dx^2} = C_1e^x + C_2e^{-x} - C_3\cos x - C_4\sin x \tag{74}$$

Given that when $x = 0$, $y = 0$ and $z = 0$. Hence Eqs (73) and (74) give

$$C_1 + C_2 + C_3 = 0 \tag{75}$$

and $$C_1 + C_2 - C_3 = 0 \tag{76}$$

Also, given that when $x = \frac{\pi}{2}$, $y = 1$ and $z = -1$. Hence Eqs (73) and (74) give

$$C_1e^{\frac{\pi}{2}} + C_2e^{-\frac{\pi}{2}} + C_4 = 1 \tag{77}$$

And $$e^{\frac{\pi}{2}} + C_2e^{-\frac{\pi}{2}} - C_4 = -1. \tag{78}$$

Subtracting Eqs (76) from (75), we have

$$C_3 = 0 \tag{79}$$

With $C_3 = 0$, Eq. (75) reduces to $C_1 + C_2 = 0$ (80)

Subtracting (78) from (77) $C_4 = 1$ (81)

Adding Eqs (77) and (78), we have $C_1e^{\frac{\pi}{2}} + C_2e^{-\frac{\pi}{2}} = 0$ (82)

Solving Eqs (80) and (82), we have $C_1 = C_2 = 0$ (83)

Using Eqs (79), (81) and (83). The required curves are given by $y = \sin x$ and $z = -\sin x$

Example 15 Find the differential equations of the lines of propagation of light in a optically non-homogeneous medium in which the speed of light is $v(x, y, z)$.

Solution According to well known Fermat's principle, light propagates from one point $A(x_1, y_1, z_1)$ and to another $B(x_2, y_2, z_2)$ along a curve, for which the Time T of passage of light will be least. If the equation of the required path of the light ray be $y = y(x)$ and $z = z(x)$, then we easily have

$$T = \int_{x_1}^{x_2} \frac{ds}{v(x, y, z)} = \int_{x_1}^{x_2} \frac{\sqrt{(1+y'^2+z'^2)}}{v}\,dx, \tag{84}$$

where ds is a line element of the path.

Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, z, y', z')\,dx$, we have

$$F(x, y, z, y', z') = \frac{\sqrt{(1+y'^2+z'^2)}}{v}. \tag{85}$$

The system of two Euler's equation is given by

$$\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\} = 0$$

and $$\left\{\frac{\partial F}{\partial z}-\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)\right\}=0$$

$$\frac{\partial v}{\partial y}\times\frac{\sqrt{(1+y'^2+z'^2)}}{v^2}+\frac{d}{dx}\left(\frac{y'}{v\times\sqrt{(1+y'^2+z'^2)}}\right)=0 \tag{86}$$

$$\frac{\partial v}{\partial z}\times\frac{\sqrt{(1+y^2+z^2)}}{v^2}+\frac{d}{dx}\left(\frac{z'}{v\times\sqrt{(1+y'^2+z'^2)}}\right)=0 \tag{87}$$

The system of Eqs (86) and (87) together determine the lines of light propagation.

14.15 FUNCTIONAL DEPENDENT ON HIGHER ORDER DERIVATIVES

14.15.1 Particular Case

The necessary condition for

$I[y(x)]=\int_{x_1}^{x_2} F(x, y, y', y'')\,dx$ to be extremum if $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0$ is known as *Euler–Poisson equation.*

14.15.2 General Case

The necessary condition for $I[y(x)]=\int_{x_1}^{x_2} F(x, y, y', y'', \cdots, y^{(n)})\,dx$ to be extremum, if

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)-\frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right)+\cdots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)=0$$

is known as *Euler–Poisson equation*.

Remark If the functional I is of the form

$$I[y(x), z(x)]=\int_{x_1}^{x_2} F\left(x, y, y', y'', \cdots y^{(n)}, z, z', z'', \cdots z^{(n)}\right)dx$$

then by varying only $y(x)$ and $z(x)$ to be fixed, we can easily prove that the extremizing function $y(x)$ and $z(x)$ must satisfies the Euler–Poisson equation

$\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)+\cdots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial^n F}{\partial y^n}\right)\right)=0$ by varying only $z(x)$ and $y(x)$ to be fixed, we can easily prove that the extremizing function $y(x)$ and $z(x)$ must satisfies the Euler–Poisson equation

$$\left(\frac{\partial F}{\partial z}-\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial z''}\right)+\cdots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial^n F}{\partial z^n}\right)\right)=0$$

Thus, the functions $y(x)$ and $z(x)$ must satisfy a system of two equations.

Example 16 Find the extremal of the functional $I[y(x)]=\int_0^1 (1+y''^2)\,dx,$ subjected to boundary condition $y(0) = 0$, $y'(0) = 1$, $y(1) = 1$ and $y'(1) = 1$.

Solution Comparing the given functional with $I[y(x)]=\int_{x_1}^{x_2} F(x, y, y', y'')\,dx,$ we get

$$F(x, y, y', y'') = (1 + y''^2) \tag{88}$$

Euler–Poisson equation is $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0$ (89)

$\frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial y'}=0, \frac{\partial F}{\partial y''}=2y''.$ Hence Eq. (89) reduces to $\frac{d^2}{dx^2}(2y'')=0 \Rightarrow \frac{d^4y}{dx^4}=0$

whose general solution is $y=C_1+C_2x+C_3\frac{x^2}{2}+C_4\frac{x^3}{6}.$ (90)

Using boundary conditions in Eq. (90), we get $C_1 = 0$, $C_2 = 1$, $C_3 = 0$ and $C_4 = 0$

The required extremum can be attained only on line $y = x$.

Example 17 Find the extremal of the functional $I[y(x)]=\int_0^1 (140y-y'''^2)\,dx,$ subjected to boundary condition $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$.

Solution Comparing the given functional with $I[y(x)]=\int_0^1 F(x, y, y', y'')\,dx,$ we get

$$F(x, y, y', y'')=(140y-y'''^2) \tag{91}$$

Euler–Poisson equation is $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)-\frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right)=0$ (92)

$\frac{\partial F}{\partial y}=480, \frac{\partial F}{\partial y'}=0, \frac{\partial F}{\partial y''}=0$ and $\frac{\partial F}{\partial y'''}=-2y'''$ hence (92) reduces to

$480-\frac{d^3}{dx^3}(-2y''')=0 \Rightarrow \frac{d^6y}{dx^6}=-240$ whose general solution is

$$y=-\frac{1}{3}x^6+\frac{C_1}{72}x^5+\frac{C_2}{24}x^4+\frac{C_3}{6}x^3+\frac{C_4}{2}x^2+C_5x+C_6 \tag{93}$$

Using boundary conditions in Eq. (93), we get $C_4 = 0$, $C_5 = 0$, $C_6 = 0$

The required extremum can be attained only on curve $y=-\frac{1}{3}x^6+\frac{C_1}{72}x^5+\frac{C_2}{24}x^4+\frac{C_3}{6}x^3$

14.16 FUNCTIONALS DEPENDENT ON THE FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES

Suppose to find the following functional for an extremum

$$I[z(x,y)] = \iint_D F(x, y, z, p, q)\,dxdy \tag{94}$$

The values of the function $z(x, y)$ are given on the boundary of the domain that is, a spatial path (or contour) is given, through which all permissible surfaces have to pass (see figure). In Eq. (94) $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. We assume that $\boldsymbol{F}$ is thrice differentiable.

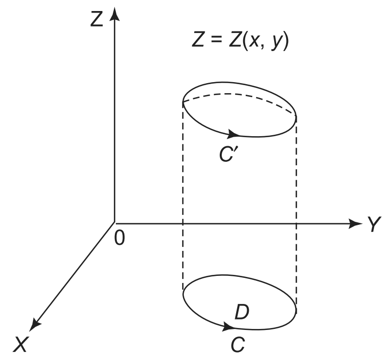


Fig. 14.4

Take some admissible surface $z = \overline{z}(x, y)$ close to $z = z(x, y)$ and include the surfaces $z = \overline{z}(x, y)$ close to $z = z(x, y)$ in a one-parameter family of surfaces

$$z(x, y, \alpha) = z(x, y) + \alpha\ \delta z, \tag{95}$$

where $\delta z = \overline{z}(x, y) - z(x, y)$.

For $\alpha = 0$, we get the surface $z = z(x, y)$.

For $\alpha = 1$, we have $z = \overline{z}(x, y). \delta z$ is called the variation of the function $z = z(x, y)$. On functions of the family $z(x, y, \alpha)$, the functional reduces to the function of α which has to have an extremum for $\alpha = 1$, hence, we have

$$\left[\frac{\partial I\left[z(x, y, \alpha)\right]}{\partial \alpha}\right]_{\alpha=0} = 0. \tag{96}$$

The derivative of $I[z, (x, y, \alpha)]$ with respect to α. For $\alpha = 0$, is known as the variation of the function and is denoted by δI. Accordingly, we have

$$\delta I = \left[\frac{\partial}{\partial \alpha} \iint_D F\big(x, y, z(x, y, \alpha), p(x, y, \alpha), q(x, y, \alpha)\big)dxdy\right]_{\alpha=0}$$

$$\delta I = \iint_D \big(F_z \delta z + F_p\ \delta p + F_q\ \delta q\big)dxdy = \iint_D \big(F_z \delta z\big)dxdy + \iint_D (F_p\ \delta p + F_q\ \delta q)\,dxdy \tag{97}$$

Equation (95), $z(x,y,\alpha) = z(x,y) + \alpha\ \delta z$

$$\Rightarrow \qquad p(x,\ y,\ \alpha) = \frac{\partial z(x,\ y,\ \alpha)}{\partial x} + \alpha\ \delta p \tag{98}$$

and $$q(x,\ y,\ \alpha) = \frac{\partial z(x,\ y,\ \alpha)}{\partial y} + \alpha\ \delta q \tag{99}$$

Now, we have

$$\frac{\partial(F_p \times \delta z)}{\partial x} = \frac{\partial F_p}{\partial x}\delta z + F_p\ \frac{\partial(\delta z)}{\partial x} = \frac{\partial F_p}{\partial x}\delta z + F_p\ \delta p \Rightarrow F_p\ \delta p = \frac{\partial(F_p \times \delta z)}{\partial x} - \frac{\partial F_p}{\partial x}\delta z \tag{100}$$

and $$\frac{\partial(F_q \times \delta z)}{\partial y} = \frac{\partial F_q}{\partial y}\delta z + F_q\ \frac{\partial(\delta z)}{\partial y} = \frac{\partial F_q}{\partial y}\delta z + F_q\ \delta q \Rightarrow F_q\ \delta q = \frac{\partial(F_q \times \delta z)}{\partial y} - \frac{\partial F_q}{\partial y}\delta z \tag{101}$$

Using Eq. (100) and (101), we have

$$\iint_D (F_p\ \delta p + F_q\ \delta q)\,dxdy = \iint_D \left\{\left(\frac{\partial(F_p \times \delta z)}{\partial x} - \frac{\partial F_p}{\partial x}\delta z\right) + \left(\frac{\partial(F_q \times \delta z)}{\partial y} - \frac{\partial F_q}{\partial y}\delta z\right)\right\} dxdy$$

$$\iint_D (F_p\ \delta p + F_q\ \delta q)\,dxdy = \iint_D \left\{\left(\frac{\partial(F_p \times \delta z)}{\partial x} - \frac{\partial(F_q \times \delta z)}{\partial y}\right)\right\} dxdy - \iint_D \left\{\left(\frac{\partial F_p}{\partial x} + \frac{\partial F_q}{\partial y}\right)\delta z\right\} dxdy, \tag{102}$$

where $\frac{\partial F_p}{\partial x}$ is known as total partial derivative with respect to x. While computing it, y is assumed to be fixed, but the dependence of z, p and q upon x is taken into account. Therefore, we have total derivative

$$dF_p = \frac{\partial F_p}{\partial x}dx + \frac{\partial F_p}{\partial z}dz + \frac{\partial F_p}{\partial p}dp + \frac{\partial F_p}{\partial p}dq \quad \text{and}$$

$$dF_q = \frac{\partial F_q}{\partial y}dy + \frac{\partial F_q}{\partial z}dz + \frac{\partial Fq}{\partial p}dp + \frac{\partial F_q}{\partial p}dq$$

$$\Rightarrow \qquad \frac{\partial F_p}{\partial x} = F_{px} + F_{pz}\frac{\partial z}{\partial x} + F_{pp}\frac{\partial p}{\partial x} + F_{pq}\frac{\partial q}{\partial x} \tag{103}$$

$$\Rightarrow \qquad \frac{\partial F_q}{\partial y} = F_{qy} + F_{qz}\frac{\partial z}{\partial y} + F_{qp}\frac{\partial p}{\partial y} + F_{qq}\frac{\partial q}{\partial y} \tag{104}$$

Using the well known Green's theorem** [** $\oint_C (M\ dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$], we have

$$\iint_D \left\{\left(\frac{\partial(F_p \times \delta z)}{\partial x} - \frac{\partial(F_q \times \delta z)}{\partial y}\right)\right\} dxdy = \iint_C \left\{F_p dy - F_q dx\right\}\delta z = 0 \tag{105}$$

The last integral is equal to zero, since on the contour C the variation $\delta z = 0$ because all permissible surfaces pass through one and the same spatial

Contour C'. Using Eqs (105) and (102) reduces to

$$\iint_D \left(F_p\ \delta p + F_q\ \delta q\right) dxdy = -\iint_D \left\{\left(\frac{\partial F_p}{\partial x} + \frac{\partial F_q}{\partial y}\right)\delta z\right\} dxdy \tag{106}$$

Using Eqs (106), (97) reduces to

$$\delta I = \iint_D (F_z \delta z) dxdy - \iint_D \left\{\left(\frac{\partial F_p}{\partial x} + \frac{\partial F_q}{\partial y}\right)\delta z\right\} dxdy = \iint_D \left\{\left(F_z - \frac{\partial F_p}{\partial x} - \frac{\partial F_q}{\partial y}\right)\delta z\right\} dxdy \tag{107}$$

The necessary condition $\delta I = 0$ for an extremum of the functional Eq. (94) takes the form

$$\iint_D \left\{\left(F_z - \frac{\partial F_p}{\partial x} - \frac{\partial F_q}{\partial y}\right)\delta z\right\} dxdy = 0 \tag{108}$$

It follows from the fundamental lemma of the calculus variation that on the extremizing surface $z = z(x, y)$, we must have

$$F_z - \frac{\partial F_p}{\partial x} - \frac{\partial F_q}{\partial y} = 0 \text{ or } \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0 \tag{109}$$

Is Euler's equation form extremal of functional Eq. (94), as before, a stationary function (if one exists) in an extremal that satisfies the given boundary conditions.

Thus the required extremizing function $z(x, y)$ is obtained from the Solution of the second order partial differential Eq. (109) which is known as Euler–Ostrogradsky equation.

Remarks

1. The functional

$$I[z(x_1, x_2, \cdots, x_n)] = \iint \ldots\ldots \int_D F(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n)\, dx_1 dx_2 \ldots dx_n$$

where $p_i = \dfrac{\partial z}{\partial x_i}$ in exactly similar way, the following Euler–Ostrogradsky equation $F_z - \displaystyle\sum_{i=1}^{n} \frac{\partial F_{p_i}}{\partial x_i} = 0$ which the function $z(x_1, x_2, \ldots, x_n)$ extremizing the functional I must satisfy.

2. If the integrand of the functional I depends on derivatives of higher order. Then, by applying several times the transformations used in driving the Euler–Ostrogradsky equation the necessary condition for an extremum, that the extremizing function must satisfy an equation similar to the Euler–Poisson equation

For Example The functional $I[z(x, y)] = \displaystyle\iint_D F(x, y, z, p, q, r, s, t)\, dxdy$ where

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, r = \frac{\partial^2 z}{\partial x^2}, \text{and } t = \frac{\partial^2 z}{\partial y^2}.$$

Euler–Poisson equation is

$$F_z - \frac{\partial F_p}{\partial x} - \frac{\partial F_q}{\partial y} + \frac{\partial^2 F_r}{\partial x^2} + \frac{\partial^2 F_s}{\partial x \partial y} + \frac{\partial^2 F_t}{\partial y^2} = 0.$$

Hence, the fourth-order partial differentiable equation must be satisfied by the function extremizing the functional

Example 18 Obtain the Euler–Ostrogradsky equation for

$$I\left[z(x,y)\right] = \iint_D \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 + 2zf(x,y)\right\} dxdy,$$

where the values of z are prescribed on the boundary C of the domain.

Solution Writing $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$, given functional is

$$I\left[z(x,y)\right] = \iint_D \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 + 2zf(x,y)\right\} dxdy \tag{110}$$

Comparing Eq. (110) with $I[z(x,y)] = \iint_D F(x, y, z, p, q)\, dxdy$, we have

$$F(x, y, z, p, q) = \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 + 2zf(x,y)\right\} \tag{111}$$

Euler–Ostrogradsky equation is

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0 \tag{112}$$

$$\frac{\partial F}{\partial y} = 2f(x,y), \frac{\partial F}{\partial p} = 2p = 2\left(\frac{\partial z}{\partial x}\right), \frac{\partial F}{\partial q} = 2q = 2\left(\frac{\partial z}{\partial y}\right).$$

Hence Eq. (112) reduces to

$$2f(x,y) - \frac{\partial}{\partial x}\left(2\left(\frac{\partial z}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(2\left(\frac{\partial z}{\partial y}\right)\right) = 0 \Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x,y)$$

is well known Poisson's equation. We are to find a solution continuous in d, of this equation that takes on prescribed values on the boundary C of the domain D.

Example 19 Obtain the Euler–Ostrogradsky equation for

$$I\left[u(x,y,z)\right] = \iiint_D \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\} dxdydz,$$

where the values of u are prescribed on the boundary C of the domain D.

Solution Writing $p_1 = \frac{\partial u}{\partial x}, p_2 = \frac{\partial u}{\partial y}, p_3 = \frac{\partial u}{\partial z}$, given functional is

$$I\left[u(x,y,z)\right] = \iiint_D \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right\} dxdydz \tag{113}$$

Comparing Eq. (113) with $I[u(x,y,z)] = \iiint_D F(x, y, z, u, p_1, p_2, p_3)dxdydz$

we have $$F(x, y, z, u, p_1, p_2, p_3) = \left\{p_1^2 + p_2^2 + p_3^2\right\} \tag{114}$$

Euler–Ostrogradsky equation is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial p_2}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial p_3}\right) = 0 \tag{115}$$

$$\frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial p_1} = 2p_1 = 2\left(\frac{\partial u}{\partial x}\right), \frac{\partial F}{\partial p_2} = 2p_2 = 2\left(\frac{\partial u}{\partial y}\right) \text{ and } \frac{\partial F}{\partial p_3} = 2p_3 = 2\left(\frac{\partial u}{\partial z}\right),$$

Hence Eq. (115) reduces to

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Example 20 Find the fourth-order partial differential equation satisfied by the function extremizing the functional $I\left[z(x,y)\right] = \iint_D \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \right\} dxdy$, where the values of z are prescribed on the boundary C of the domain D.

Solution Writing $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$ given functional is

$$I\left[z(x,y)\right] = \iint_D \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \right\} dxdy \tag{116}$$

Comparing Eq. (116) with $I[z(x,y)] = \iint_D F(x,y,z,p,q,r,s,t)\,dxdy$, we have

$$F(x, y, z, p, q, r, s, t) = \left\{r^2 + t^2 + 2s^2\right\} \tag{117}$$

Then, the partial differential equation satisfied by the function extremizing the functional (1) is given by

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial r}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial s}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial t}\right) = 0 \tag{118}$$

$$\frac{\partial F}{\partial z}=0,\ \frac{\partial F}{\partial p}=0,\ \frac{\partial F}{\partial q}=0,\ \frac{\partial F}{\partial r}=2r,\ \frac{\partial F}{\partial s}=4s \text{ and } \frac{\partial F}{\partial t}=2t$$

Hence, Eq. (118) reduces to

$$0-\frac{\partial}{\partial x}(0)-\frac{\partial}{\partial y}(0)+\frac{\partial^2}{\partial x^2}(2r)+\frac{\partial^2}{\partial x\partial y}(2s)+\frac{\partial^2}{\partial y^2}(2t)=0$$

$$\Rightarrow \qquad 2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 z}{\partial x^2}\right)+4\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2 z}{\partial x\partial y}\right)+2\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 z}{\partial y^2}\right)=0$$

$$\Rightarrow \qquad \frac{\partial^4 z}{\partial x^4}+2\frac{\partial^4 z}{\partial x^2\partial y^2}+\frac{\partial^4 z}{\partial y^4}=0 \text{ is well known biharmonic equation.}$$

Example 21 Obtain the surface of minimum area, stretched over a given closed curve C, enclosing the domain D in the xy plane.

Solution We know that the required given problem reduces to find the extremal of the functional is

$$S[z(x,y)]=\iint_D\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}dxdy \tag{119}$$

Writing $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$, comparing Eq. (119) with

$$S[z(x,y)]=\iint_D(1+p^2+q^2)\,dxdy=\iint_D F(x,y,z,p,q)\,dxdy,$$

we have $F(x, y, z, p, q, r, s, t)=\{r^2+t^2+2s^2\}$ (120)

Then, Euler–Ostrogradsky equation is $\frac{\partial F}{\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right)=0$ (121)

$$\frac{\partial F}{\partial z}=0,\ \frac{\partial F}{\partial p}=\frac{p}{(1+p^2+q^2)^{\frac{1}{2}}},\ \frac{\partial F}{\partial q}=\frac{p}{(1+p^2+q^2)^{\frac{1}{2}}},$$

Hence Eq. (121) reduces to

$$-\frac{\partial}{\partial x}\left(\frac{p}{(1+p^2+q^2)^{\frac{1}{2}}}\right)-\frac{\partial}{\partial y}\left(\frac{q}{(1+p^2+q^2)^{\frac{1}{2}}}\right)=0$$

$$\Rightarrow \qquad \frac{\partial}{\partial x}\left(\frac{z_x}{(1+z_x^2+z_y^2)^{\frac{1}{2}}}\right)+\frac{\partial}{\partial y}\left(\frac{z_y}{(1+z_x^2+z_y^2)^{\frac{1}{2}}}\right)=0 \tag{122}$$

where $z_x = \dfrac{\partial z}{\partial x}$ and $z_y = \dfrac{\partial z}{\partial y}$. We use the usual notations, $z_{xx} = \dfrac{\partial^2 z}{\partial x^2}$, $z_{yy} = \dfrac{\partial^2 z}{\partial y^2}$ and $z_{xy} = \dfrac{\partial^2 z}{\partial x \partial y}$. Then, Eq. (122) reduces to

$$\Rightarrow \quad \frac{z_{xx}}{(1+z_x^2+z_y^2)^{\frac{1}{2}}} - \frac{2z_x(z_x z_{xx} + z_y z_{yx})}{(1+z_x^2+z_y^2)^{\frac{3}{2}}} + \frac{z_{yy}}{(1+z_x^2+z_y^2)^{\frac{1}{2}}} - \frac{2z_y(z_x z_{xy} + z_y z_{yy})}{(1+z_x^2+z_y^2)^{\frac{3}{2}}} = 0$$

$$\Rightarrow \quad z_{xx}(1+z_x^2+z_y^2) - z_x(z_x z_{xx} + z_y z_{yx}) + z_{yy}(1+z_x^2+z_y^2) - 2z_y(z_x z_{xy} + z_y z_{yy}) = 0$$

$$\Rightarrow \quad z_{xx}(1+z_y^2) + z_{yy}(1+z_x^2) - 2z_x z_y z_{xy} = 0$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x^2}\left(1+\left(\frac{\partial z}{\partial y}\right)^2\right) + \frac{\partial^2 z}{\partial y^2}\left(1+\left(\frac{\partial z}{\partial x}\right)^2\right) - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y} = 0$$

is known as the desired minimal surface.

14.17 VARIATIONAL PROBLEMS IN PARAMETRIC FORM

Consider a functional of the form $I\left[x(t), y(t)\right] = \int_{t_1}^{t_2} F(t, x, y, \dot{x}, \dot{y})\, dt$, (123)

where $\dot{x} = \dfrac{dx}{dt}, \dot{y} = \dfrac{dy}{dt}$

Subject to given fixed boundary conditions at t_1 and t_2, where x and y both are functions of another parameter t. Proceeding as in previous articles, introduce parameter α in Eq. (123) so that

$$I(\alpha) = \int_{t_1}^{t_2} F(t, x+\alpha\eta_1, y+\alpha\eta_2, \dot{x}+\alpha\dot{\eta}_1, \dot{y}+\alpha\dot{\eta}_2)\, dt, \tag{124}$$

where η_1 and η_2 are functions of t and both at t_1 and t_2, $\dot{\eta}_1 = \dfrac{d\eta_1}{dt}$ and $\dot{\eta}_2 = \dfrac{d\eta_2}{dt}$. Now expanding the integrand is Eq. (124) in Taylor's series, we get

$$I(\alpha) = \int_{t_1}^{t_2}\left[F(t, x, y, \dot{x}, \dot{y}) + \alpha\eta_1\frac{\partial F}{\partial x} + \alpha\eta_2\frac{\partial F}{\partial y} + \alpha\dot{\eta}_1\frac{\partial F}{\partial \dot{x}} + \alpha\dot{\eta}_2\frac{\partial F}{\partial \dot{y}} + \text{higher power of } \alpha\right] dt,$$

Eliminating higher power of α

$$\therefore \quad \frac{d\,I(\alpha)}{d\alpha} = \int_{t_1}^{t_2}\left[\eta_1\frac{\partial F}{\partial x} + \eta_2\frac{\partial F}{\partial y} + \dot{\eta}_1\frac{\partial F}{\partial \dot{x}} + \dot{\eta}_2\frac{\partial F}{\partial \dot{y}}\right] dt, \tag{125}$$

Now, $\displaystyle\int_{t_1}^{t_2} \dot{\eta}_1\frac{\partial F}{\partial \dot{x}}dt = \left[\eta_1\frac{\partial F}{\partial \dot{x}}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\eta_1\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)dt = -\int_{t_1}^{t_2}\eta_1\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)dt \quad \{\because \quad \eta_1(t_1) = \eta_1(t_2) = 0\}$

and $\displaystyle\int_{t_1}^{t_2} \dot{\eta}_2\frac{\partial F}{\partial \dot{y}}\,dt = \left[\eta_2\frac{\partial F}{\partial \dot{y}}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\eta_2\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)dt = -\int_{t_1}^{t_2}\eta_2\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)dt \quad \{\because \quad \eta_2(t_1) = \eta_2(t_2) = 0\}$

Hence Eq. (125) may be

$$\therefore \qquad \frac{d\,I(\alpha)}{d\alpha}=\int_{t_1}^{t_2}\left[\eta_1\left\{\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right\}+\eta_2\left(\frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)\right)\right]dt,$$

Applying extremum condition $\dfrac{d\,I(\alpha)}{d\alpha}=0,$ we have

$$\int_{t_1}^{t_2}\left[\eta_1\left\{\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right\}+\eta_2\left\{\frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)\right\}\right]dt=0$$

Giving $$\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)=0 \tag{126}$$

and $$\frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)=0 \tag{127}$$

Equations (126) and (127) are the required Euler's Equation in parametric form. It should be noted that these equations are not equations are not independent, but one equation is a consequence of the other to obtain the extremals.

Remarks The Weirstrassian form of Euler's Eqs (126) and (127)

$$\frac{1}{r}=\frac{F_{x\,\dot{y}}-F_{y\,\dot{x}}}{F_1(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} \tag{128}$$

Here, r is the radius of curvature of the external, F_1 is the common values of the ratios

$$F_1=\frac{F_{\dot{x}\,\dot{x}}}{\dot{y}^2}=\frac{F_{\dot{y}\,\dot{y}}}{\dot{x}^2}=\frac{F_{\dot{x}\,\dot{y}}}{-\dot{x}\dot{y}}$$

Example 22 Find the externals of the functional

$$I\left[x(t),y(t)\right]=\int_{t_0}^{t_1}\left[\sqrt{\dot{x}^2+\dot{y}^2}+a^2(x\dot{y}-y\dot{x})\right]dt$$

Solution Putting $$F(x,y,\dot{x},\dot{y})=\left[\sqrt{\dot{x}^2+\dot{y}^2}+a^2(x\dot{y}-y\dot{x})\right] \tag{129}$$

We see that the function F is positive homogeneous of the first degree $\dot{x}$ and $\dot{y}$.

Taking advantage of the Weierstrassian form of Euler's equation, we have

$$F_{x\,\dot{y}}=a^2,\ F_{y\,\dot{x}}=-a^2,\ F_1=\frac{F_{\dot{x}\dot{y}}}{\dot{y}^2}=\frac{1}{(\dot{x}^2+\dot{y}^2)^{3/2}}$$

For this reason, Eq. (128) assumes in this case the form

$$\frac{1}{r}=\frac{F_{x\,\dot{y}}-F_{y\,\dot{x}}}{F_1(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}=2a^2 \Rightarrow \frac{1}{r}=2a^2 \text{ constant, the extremals are circles.}$$

Example 23 Find the externals of the functional

$$I\left[x(t), y(t)\right] = \int_0^{\frac{\pi}{4}} \left[2x^2 + 2y^2 + \dot{x}\dot{y}\right] dt\ ,\ \ddot{x} = \frac{d^2x}{dt^2} \text{ and } \ddot{y} = \frac{d^2y}{dt^2}.$$

Subject to the initial conditions at $t = 0$, $x = y = 0$; at $t = \frac{\pi}{4}$, $x = y = 1$;

Solution Comparing the given functional with $I\left[x(t), y(t)\right] = \int_0^{\frac{\pi}{4}} F(t, x, y, \dot{x}, \dot{y})\, dt$,

We have
$$F(t, x, y, \dot{x}, \dot{y}) = \left[2x^2 + 2y^2 + \dot{x}\dot{y}\right] \tag{130}$$

Euler's equation in parametric form are given by

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \tag{131}$$

and
$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0 \tag{132}$$

From Eq. (130), $\frac{\partial F}{\partial x} = 4x$, $\frac{\partial F}{\partial \dot{x}} = \dot{y}$ and $\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \ddot{y}$ where $\ddot{x} = \frac{d^2x}{dt^2}$ and $\ddot{y} = \frac{d^2y}{dt^2}$.

Using above equation in (131), we get

$$4x - \ddot{y} = 0 \Rightarrow \ddot{y} - 4x = 0 \tag{133}$$

From Eq. (130), $\frac{\partial F}{\partial x} = 4y$, $\frac{\partial F}{\partial \dot{x}} = \dot{x}$ and $\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \ddot{x}$

Using above equation in Eq. (131), we get

$$4y - \ddot{x} = 0 \Rightarrow \ddot{x} - 4y = 0 \tag{134}$$

Solve the system of Eqs (133) and (134) we get

$$(5) \Rightarrow \ddot{\ddot{x}} - 4\ddot{y} = 0 \Rightarrow \ddot{\ddot{x}} - 16x = 0 \Rightarrow (D^4 - 16)x = 0$$

where
$$D = \frac{d}{dt} \tag{135}$$

Auxiliary equation is $(D^4 - 16) = 0$ or $D = \pm 2i, \pm 2$

Hence the general solution of Eq. (135) is

$$x = C_1 \cosh 2t + C_2 \sinh 2t + C_3 \cos 2t + C_4 \sin 2t \tag{136}$$

Now differentiating Eq. (136), twice w. r . to 't', we have

$$\ddot{x} = 4(C_1 \cosh 2t + C_2 \sinh 2t - C_3 \cos 2t - C_4 \sin 2t) \tag{137}$$

Substituting this value of $\ddot{x}$ in Eq. (134), we have

$$y = (C_1 \cosh 2t + C_2 \sinh 2t - C_3 \cos 2t - C_4 \sin 2t) \tag{138}$$

Given that $x = y = 0$ when t -= 0. Hence Eqs (136) and (139) yield

$$C_1 + C_3 = 0 \text{ and } C_1 - C_3 = 0$$

So that $$C_1 = C_3 = 0 \tag{139}$$

Also, given that $x = y = 1$ when $t = \dfrac{\pi}{4}$ hence Eqs (136) and (139) yield

$$C_2 \sinh\left(\frac{\pi}{2}\right) + C_4 = 1 \text{ and } C_2 \sinh\left(\frac{\pi}{2}\right) - C_4 = 1$$

so that $$C_4 = 0 \text{ and } C_2 = \frac{1}{\sinh\left(\frac{\pi}{2}\right)} \tag{140}$$

Using Eqs (139) and (140) in Eq. (136) and Eq. (139) . The required extremal in parametric form

$$x = y = \frac{\sinh 2t}{\sin\left(\frac{\pi}{2}\right)}$$

14.18 ISOPERIMETRIC PROBLEMS

14.18.1 Working Rules for Isoperimetric Problem

It is necessary to make a given integral $I = \int\limits_{x_1}^{x_2} F(x, y, y')\,dx$ subject to the constraint $J = \int\limits_{x_1}^{x_2} G(x, y, y')\,dx$ constant, such problem involve one or more constraint conditions. The necessary condition for integral $\int\limits_{x_1}^{x_2} H dx$ to be an extremum is $\dfrac{\partial H}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial H}{\partial y'}\right) = 0$, where $H = F \pm \lambda G$, λ is called Lagrange multipliers.
We determine the Lagrange multiplier λ, together with the constant of Integration arising in the solution of Euler's equation and J having given constant value.

Example 24 (Dido's problem) Find the simple closed curve of a given parameter which encloses a maximum area.

Solution Let l be the fixed perimeter of a plane curve between the point P and Q with abscissas x_1 and x_2

Thus $$l = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx \text{ (constraints)} \tag{141}$$

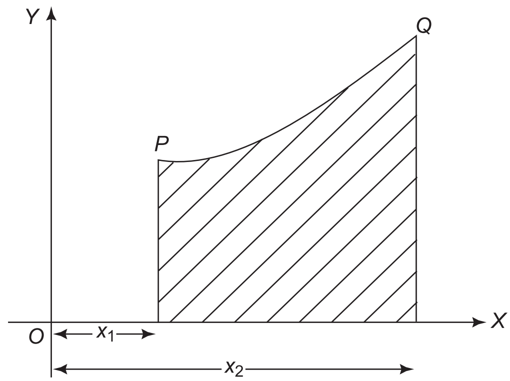


Fig. 14.5

If A is the area between $x = x_1, x_2$ and x-axis $A = \int_{x_1}^{x_2} y dx$ (142)

We have to maximize Eq. (142) subject to constraints Eq. (141).

Here $F = y$ and $G = \sqrt{1 + y'^2}$

$$\therefore \qquad H = F + \lambda G = y + \lambda\sqrt{1 + y'^2}$$

Now, H must satisfy the Euler's equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \tag{143}$$

$$\frac{\partial H}{\partial y} = 1 \text{ and } \frac{\partial H}{\partial y'} = \frac{\lambda y'}{\sqrt{1 + y'^2}}$$

Therefore, Eq. (143) becomes $1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1 + y'^2}}\right) = 0 \Rightarrow \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1 + y'^2}}\right) = 1$

$$\Rightarrow \qquad \lambda y' = (x + c)\left(\sqrt{1 + y'^2}\right) \Rightarrow \left[\lambda^2 - (x + c)^2 y'^2\right] = (x + c)^2$$

$$\Rightarrow \qquad y' = \frac{(x + c)}{\sqrt{\lambda^2 - (x + c)^2}} \Rightarrow dy = \frac{(x + c)dx}{\sqrt{\lambda^2 - (x + c)^2}} \tag{144}$$

Let $\lambda^2 - (x + c)^2 = t \Rightarrow -2(x + c)dx = dt \Rightarrow (x + c)dx = -\frac{1}{2}dt$

Then Eq. (144)

$$\Rightarrow \qquad y = \int -\frac{1dt}{2t^{1/2}} = -\frac{1}{2}\int t^{-1/2} dt = -t^{1/2} + b \Rightarrow (y - b)^2 = t = \lambda^2 - (x + c)^2$$

$$\Rightarrow \qquad (y - b)^2 + (x + c)^2 = \lambda^2$$

which is the equation of circle. Hence the curve is circle.

Example 25 Find extremals of the isoperimetric problem $I[y(x)] = \int_{x_0}^{x_1} y'^2 dx$ given that $\int_{x_0}^{x_1} y dx = C$ (Constant).

Solution Here $F = y^2$ and $G = y$

We write $H = F + \lambda G = y'^2 + \lambda y$. Now H must satisfy the Euler's equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \tag{145}$$

$\therefore \quad \frac{\partial H}{\partial y} = \lambda$ and $\frac{\partial H}{\partial y'} = 2y'$

Therefore, Eq. (145) becomes $\lambda - \frac{d}{dx}(2y') = 0$

Integrating $\lambda x - 2y' = a$, $y' = \frac{dy}{dx} = \frac{\lambda x - a}{2}$ or $y = \frac{\lambda x^2}{4} - \frac{ax}{2} + b$,

where λ, a, b are determined from the isoperimetric and boundary conditions.

EXERCISE 14.1

1. Find the extremals of the following functional

(i) $I[y(x)] = \int_{x_1}^{x_2} y'(1 + x^2 y')dx$

(ii) $I[y(x)] = \int_{x_1}^{x_2} \left(\frac{1}{y}\right)\sqrt{(1 + y'^2)}dx$

(iii) $I[y(x)] = \int_{x_1}^{x_2} (y^2 + 2xy\, y')dx$ subjected to condition $y(x_1) = y_1$, $y(x_2) = y_2$

(iv) $I[y(x)] = \int_{x_1}^{x_2} (y^2 + y'^2 + 2y \ \mathrm{sec}h\ x)\, dx$

(v) $I[y(x)] = \int_{0}^{\frac{\pi}{2}} (y'^2 - y^2 + 4y \sin^2 x)\, dx$ subjected to boundary condition $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = \frac{1}{3}$.

(vi) $I[y(x)] = \int_{x_1}^{x_2} (2xy + y'''^2)\, dx$

(vii) $I[y(x)] = \int_{x_1}^{x_2} (y'''^2 + y^2 - 2yx^3)\, dx$

(viii) $I[y(x), z(x)] = \int_{x_1}^{x_2} (2yz + y'^2 - 2y^2 + z'^2)\, dx$

(ix) $I[x(t), y(t)] = \int_0^1 (\dot{x}y + \dot{y}x - x^2 - y^2)dt$ subjected to boundary condition $x(0) = y(0) = 1$, $x(1) = y(1) = 0$, where $\dot{x} = \dfrac{dx}{dt}$, $\dot{y} = \dfrac{dy}{dt}$.

(x) $I[y(x)] = \int_1^e (xy'^2 + yy')dx;\ y(1) = 0,\ y(e) = 1$

2. Find the extremals and extremum value of the following functional

(i) $I[y(x)] = \int_{\frac{1}{2}}^{1} (x^2 y'^2)dx$

(ii) $I[y(x)] = \int_0^2 (x - y'^2)dx$

3. Write down the Euler–Ostrogradsky insulations for the following functionals.

(i) $I[z(x,y)] = \int_D\int \left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1\right]^{\frac{1}{2}} dxdy$

(ii) $I[z(x,y)] = \int_D\int \left(\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}\right)^2 dxdy$

(iii) $I[z(x,y)] = \int_D\int \left[\left(\dfrac{\partial^2 z}{\partial x^2}\right)^2 + \left(\dfrac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 - 2zf(x,y)\right] dxdy$

4. Find the extremal of the functional $\int_0^{\pi} (y'^2 - y^2)\, dx$ under the conditions $y(0) = 0$, $y(\pi) = 1$ and subject to the constraint $\int_0^{\pi} ydx = 1$.

5. Find a function $y(x)$ for which $\int_0^1 (x^2 - y'^2)\, dx$ is stationary given that $\int_0^1 y^2 dx = 2$, $y(0) = 0$, $y(1) = 0$.

Answers

(i) $y = \dfrac{C_1}{x} + C_2$

(ii) $(x-C_1)^2+y^2=C_2^2$

(iii) $y=\sinh(C_1x+C_2)$

(iv) $y=C_1\cosh x+C_2\sinh x+x\sinh x-\cosh x\,\log_e(\cosh x)$

(v) $y=\left(\dfrac{2\sin x-\cos 2x}{3}\right)$

(vi) $y=C_1+C_2x+C_3x^2+C_4x^3+C_5x^4+C_6x^5+\dfrac{x^7}{7!}$

(vii) $y=C_1e^x+C_2e^{-x}+e^{\frac{x}{2}}\left\{C_3\cos\left(\dfrac{x\sqrt{3}}{2}\right)+C_4\sin\left(\dfrac{x\sqrt{3}}{2}\right)\right\}$

$$+e^{-\frac{x}{2}}\left\{C_5\cos\left(\frac{x\sqrt{3}}{2}\right)+C_6\sin\left(\frac{x\sqrt{3}}{2}\right)\right\}+x^3$$

(viii) $y=(C_1+C_2x)\cos x+(C_3+C_4x)\sin x$ and

$$z=(C_1+C_2x)\cos x+(C_3+C_4x)\sin x-2C_2\sin x+2C_4\cos x$$

(ix) $y=\dfrac{\sinh(1-t)}{\sinh 1}$

(x) $y=\ln x$ or $\log x$

2. (i) $y=3-\left(\dfrac{1}{x}\right)$; Value $=1$

(ii) $y=x+\left(\dfrac{x^2}{2}\right)$; Value $=2$

3. (i) $\left(\dfrac{\partial z}{\partial x}\right)^2\left\{\left(\dfrac{\partial z}{\partial y}\right)^2+1\right\}-2\dfrac{\partial z}{\partial x}\times\dfrac{\partial z}{\partial y}\times\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial^2 z}{\partial y^2}\left\{\left(\dfrac{\partial z}{\partial x}\right)^2+1\right\}=0$

(ii) $\Delta\Delta r=0\Rightarrow\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}$

(iii) $\dfrac{\partial^4 z}{\partial x^4}+2\dfrac{\partial^4 z}{\partial x^2\partial y^2}+\dfrac{\partial^4 z}{\partial y^4}=f(x,y)$

4. $y=\dfrac{1}{2}(1-\cos x)+\dfrac{1}{2}(2-\pi)\sin x$

5. $y=\pm 2\sin n\pi x$, where n is an integer.

14.19 SUFFICIENT CONDITIONS FOR AN EXTREMUM

14.19.1 General Definitions

1. Proper Field A family of curve $y = y(x, c)$ is said to form a proper field in a given region D of the xy plane if one and only curve of the of the family passes through every point of the region D.

For example (i): Consider the circle $x^2 + y^2 = 1$. Inside the circle $x^2 + y^2 \leq 1$, the family of curve $y = mx + c$ is a proper field, because there passes one and only one curve of the family through any point of the given circle

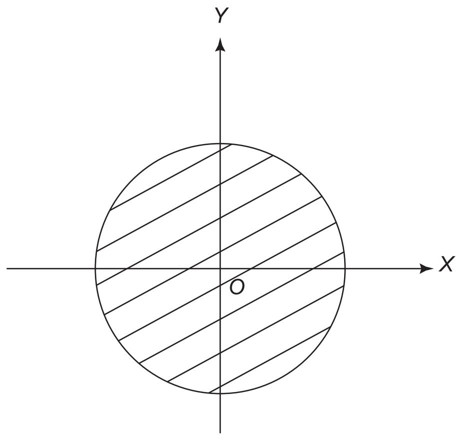


Fig. 14.6 *Proper field*

(ii) The family of Parabola $y = (x + c)^2$ inside the same circle $x^2 + y^2 = 1$ does not form a proper field since the parabola of this family inside the circle as shown in Fig. 14.7.

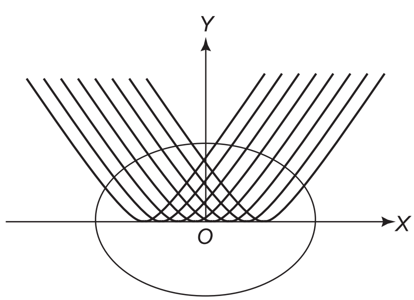


Fig. 14.7 *Not proper field*

2. Central Field If all the curves of the family $y = y(x, c)$ pass through a certain point (x_0, y_0) i.e. if they form a pencil of curves. them they do not form a proper field in the region D. if the centre of the pencil (x_0, y_0) belongs to D. If the curves of the pencil curve the centre region D and do not intersect anywhere in this region, with the exception of the centre of the pencil (x_0, y_0) then the family $y = y(x, c)$ is said to form a central field.

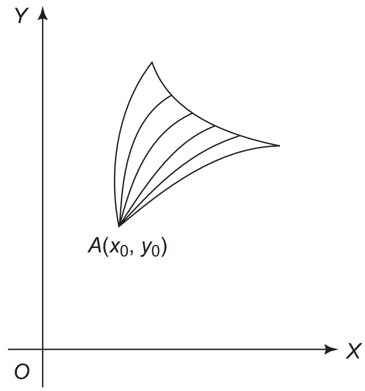


Fig. 14.8 *Central field*

For example The pencil of sinusoids $y = c \sin x$ for $o \le x \le a$, $a < \pi$ forms a central field. But the above mentioned pencil of sinusoids form a proper field in a sufficiently small neighborhood of the segment of x-axis for $\delta \le x \le a$ where $\delta > 0$, $a < \pi$ as shown in Fig. 14.8. Again the above mentioned pencil of sinusoids does not form a proper field in a neighborhood of the segment of the x-axis, for $o \le x \le a$, $a > \pi$ as shown in Fig. 14.9.

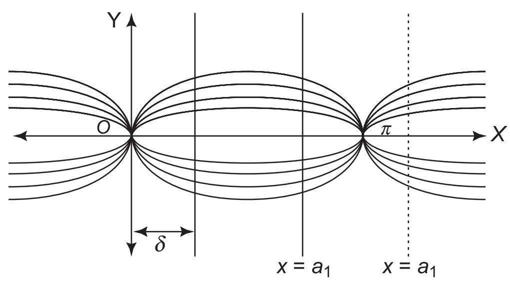


Fig. 14.9

***Remark*:** If they passes though a single point (x_0, y_0) Which is not in D then (x_0, y_0) is said to be "centre of the pencil of curves"

3. Field of Extremals If a proper field or a central field is formed by a family of extremal of a given variational problem, then it is known as an extremal field.

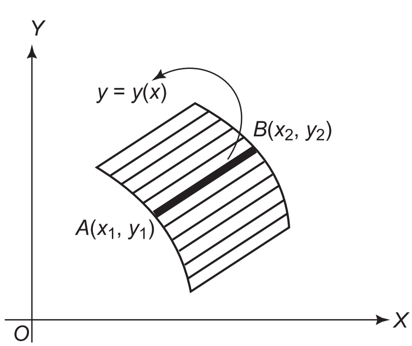


Fig. 14.10

(a) Embedding in a Field of Extremal Let $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx$ be a functional with external $y = y(x)$ and fixed and points (x_1, y_1) and (x_2, y_2) we can find a family of extremal $y = y(x, c)$ such that a field in formed by this family with extremising curve $y(x)$ as a member of this field for some particular value of $C = C_1$.

Let D be the domain in which $y = y(x, c)$ forms a field. If the extremal does not lie on the boundary of the domain D then extremal $y = y(x)$ is said to be embedded in an extremal field.

(b) Embedding in a Central Field Let a pencil of extremals be having centre at Point $A(x_1, y_1)$ forms a central field in the neighborhood of the extremal $y = y(x)$ which passes through A, then the extremal $y = y(x)$ is said to be embedded in central field in Fig. 14.11.

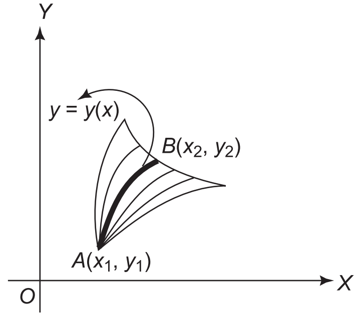


Fig. 14.11

***Remark*:** The slop of the tangents to the curve of the pencil at A may be chosen as a parameter of the family of extremals.

14.20 JACOBI CONDITION

Consider a one parameter family $F(x, y, c) = 0$ of planes. The C-discernment of $F(x, y, c) = 0$ is the focus of the points is given by

$$F(x, y, c) = 0 \text{ and } \frac{\partial F}{\partial c} = 0 \tag{146}$$

Here, the C-Discernment included the envelope of the above family of curves, the focus of the nodal points and locus of cusps. If we have a pencil of curves with centre at $P_1(x_0, y_0)$ then P_1 belongs to C-Discernment curve.

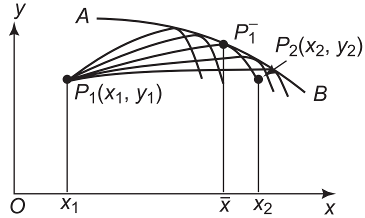


Fig. 14.12 (a)

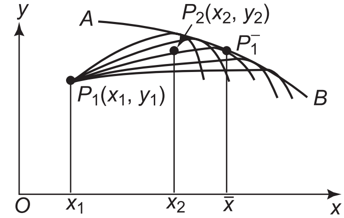


Fig. 14.12 (b)

Consider a pencil of extremals $y = y(x, c)$ passing through P_1 and get its C-Discernment curve $\phi(x, y) = 0$. Then the envelope AB of this pencil of curves belong to $F(x, y) = 0$. The point P_1^* at which the extremal $y = y(x)$ touches AB is called the conjugate point of P_1.

As shown in Figs (14.12 (a) and (b) we see that the following two situations may arise:

(i) When the conjugate point P_1^* of P_1 lies between P_1 and P_2 in this case the extrenals of the pencil close to P_1P_2 intersect and so the extremals P_1P_2 cannot be included in a central field. (Fig. 14.12(a)).

(ii) When the point $P(x_2, y_2)$ lies between P_1 and P_1^* in this case the extrenals of the pencil close to P_1P_2 do not intersect. Hence the extremals close to P_1P_2 form a central field which included the arc P_1P_2.

Therefore, to embed an arc P_1P_2 of the extremal in a central field of extremals, it is sufficient that the conjugate point of P_1 does not lie on the arc P_1P_2. This is known as Jacobi condition.

Example 26 Is the Jacobi condition fulfilled for extremal of the functional

$I[y(x)] = \int_0^a (y'^2 + y^2 + x^2)\mathrm{d}x$, that passes through the point $A(0, 0)$ and $B(a, 0)$?

Solution Comparing the given functional with $I[y(x)] = \int_0^a F(x, y, y')dx$

We have $F(x, y, y') = (y'^2 + y^2 + x^2)$ (147)

The Jacobi equation is given by

$$\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)\times u - \frac{d}{dx}(F_{y'y'} \times u') = 0, \tag{148}$$

where $u = \dfrac{\partial y(x,c)}{\partial c}$ and $y = y(x, c)$ in the equation of a pencil of extremals with centre at $A(0, 0)$. From Eq. (147), $F_{yy} = 2, F_{yy'} = 0$ and $F_{y'y'} = 2$ and so (2) yields $(2-0)\times u - \dfrac{d}{dx}(2u') = 0$

or
$$u'' - u = 0 \text{ or } (D^2 - 1)u = 0 \tag{149}$$

where $D = \dfrac{d}{dx}$. Auxiliary of Eq. (149) is $(D^2 - 1) = 0$ so that $D \pm 1$.

Therefore the general solution of (149) is given by

$$u = C_1 \cosh x + C_2 \sinh x \tag{150}$$

Using the condition $u(0)= 0$, (150) gives $C_1 = 0$. Then Eq. (150) reduces to

$$u = C_2 \sinh x \tag{151}$$

Clearly the curves of the pencil $u = C_2 \sinh x$ intersect the x-axis only at the point $x = 0$ hence the Jacobi condition is fulfilled for any value of a

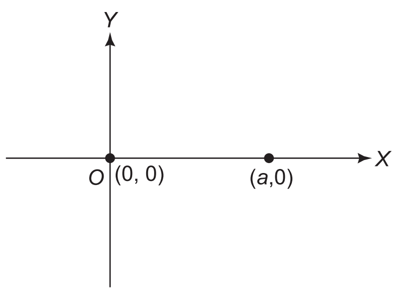


Fig. 14.13 (a)

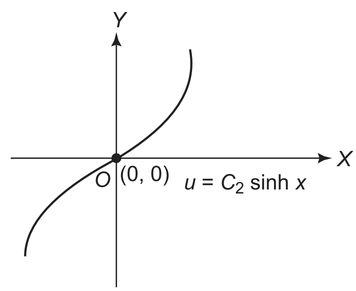


Fig. 14.13 (b)

Example 27 Show that the Jacobi Condition is fulfilled for the functional $\int_{-1}^{1}(12xy + y'^2 + x^2)\,dx$ with fixed boundaries $A(-1, -2)$ and $B(1, 0)$.

Solution Comparing the given functional with $I[y(x)] = \int_{-1}^{1} F(x, y, y')\,dx$

We have $F(x, y, y') = (12\,xy + y'^2 + x^2)$ (152)

The Euler's equation is $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0 \Rightarrow\ 12x - \dfrac{d}{dx}(2y') = 0$

$$y'' = 6x \Rightarrow y(x) = Ax + B + x^3$$

$$y(-1) = -2 \Rightarrow -2 = -A + B - 1 \Rightarrow B - A = -1$$

$$y(1) = 0 \Rightarrow 0 = A + B + 1 \Rightarrow B + A = -1$$

$\Rightarrow$ $A = 0$ and $B = -1$

Then the extremal $y(x)^3 = x^3 - 1$

The Jacobi condition

$$\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)\times u - \frac{d}{dx}(F_{y'y'} \times u') = 0 \tag{153}$$

$F_{yy} = 0$, $F_{yy'} = 0$ and $F_{y'y'} = 2$ then Eq. (153) gives

$$(0 - 0)\times u - \frac{d}{dx}(2u') = 0$$

Or $u'' = 0 \Rightarrow u(x) = C\,x + D$ (154)

$\Rightarrow$ $u(-1) = 0 \Rightarrow -C + D = 0 \Rightarrow C = D \Rightarrow u(x) = C(x + 1)$

Which implies that $u(x)$ is zero only at $x = -1$. Hence Jacobi condition is satisfied.

Example 28 Is the Jacobi condition fulfilled for the extremal passing through the condition the points $A(0, 0)$ and $B(a, 0)$ of the functional $I[y(x)] = \int_0^a (y'^2 - 4y^2 + e^{-x^2})\,dx$, $\left(a \neq \left(n - \dfrac{1}{2}\right)\pi\right)$

Solution Comparing the given functional with $I[y(x)] = \int_0^a F(x, y, y')\,dx$ then

We have $F(x, y, y') = (y'^2 - 4y^2 + e^{-x^2})$ (155)

Then the Jacobi condition is

$$\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)\times u - \frac{d}{dx}(F_{y'y'} \times u') = 0 \tag{156}$$

From Eq. (155), $F_{yy} = -8$, $F_{yy'} = 0$ and $F_{y'y'} = 2$ then Eq. (156) gives $(-8 - 0)\times u - \dfrac{d}{dx}(2u') = 0$

or $u'' + 4u = 0 \Rightarrow (D^2 + 1)u(x) = 0 \Rightarrow D^2 + 1 = 0 \Rightarrow D = \pm 2i$ (157)

Its general solution is $u(x) = C_1 \sin 2x + C_2 \cos 2x$ from the condition $u(0) = 0$ we find that $C_2 = 0$ so that $u(x) = C_1 \sin 2x$

If $a < \frac{\pi}{2}$, then the functional $u(x)$ does not vanish for $0 < x \le a$. and Jacobi condition is fulfilled. But if $a > \frac{\pi}{2}$ then the solution of the Jacobi equation $u(x) = C_1 \sin 2x$ vanishes at the point $x = \frac{\pi}{2}$, which lies in the interval $[0, a]$ and on the arc of the extremal $y = 0$, $(0 < x \le a)$ there is a point conjugate to the point $A(0, 0)$. Thus for $a > \frac{\pi}{2}$ there does not exist a central field of extremals that includes the given extremal.

Note The Jacobi condition is necessary for attaining an extremum of the functional $I[y(x)]$, i.e., On the extremal AB that realizes an extremum, the point conjugate to A cannot lie in the interval $x_0 < x < x_1$.

14.21 WEIERSTRASS FUNCTION

***E*(*x*, *y*, *p*, *y*′)** Consider the extremum of the functional

$$I[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\,dx,\ y(x_0) = y_0,\ y(x_1) = y_1 \tag{158}$$

Further, suppose that the extremal C that passes through the points $A(x_0, y_0)$ and $B(x_1, y_1)$ satisfies the Jacobi's condition. Hence the extremal 'C' can be included in the field whose slope in $P(x, y)$ as shown in the Fig. (14.13). To determine the sign of an increment ΔI of a functional I which obtains when passing from an extremal curve C to some other neighboring admissible extremal C^* we will transform the increment.

$$\Delta I = \int_{C^*} F(x, y, y')\,dx - \int_{C} F(x, y, y')\,dx, \tag{159}$$

where $\int_{C^*} F(x, y, y')\,dx$ and $\int_{C} F(x, y, y')\,dx$ denote the value of the functional

$I[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx$ taken along the arc of C^* or C respectively.

Consider the auxiliary functional $\int_{C^*} \left\{F(x, y, p) + (y' - p)F_p(x, y, p)\right\} dx$

If $$y' = \frac{dy}{dx} \tag{160}$$

which reduces to $\int_{C} F(x, y, y')\,dx$ on the extremal C, since $y' = \frac{dy}{dx} = p$ on extremals of the field. Now re-writing the functional Eq. (160), we have

$$\int_{C^*} F(x, y, p)dx + \int_{C^*} y'\, F_p(x, y, p)dx - \int_{C^*} p\, F_p(x, y, p)dx \tag{161}$$

which is the integral of an exact differential. In fact, the differential of the functional $I^*[x, y]$ into which the functional $I[y(x)]$ is transformed on the extremals of the field has the form

$$dI^* = \left\{F(x, y, y') - y'F_{y'}(x, y, y')\right\} dx + F_{y'}(x, y, y')\,dx$$

which differs form the integrand in Eq. (161) with y' replaced by p, the slope of the tangent line to the extremal of the field. Thus, on the extremal C the integral of (160) coincides of an exact differential and, hence, does not depend on the path of integration. It follows that

$$\int_C F(x,y,y')\,dx = \int_{C^*} \left\{F(x,y,p)+(y'-p)F_p(x,y,p)\right\} dx \tag{162}$$

not only for $C = C^*$ but for any choice of C^* as well.

Using Eqs (162), (159) may be re-written as

$$\Delta I = \int_{C^*} F(x,y,y')\,dx - \int_{C^*} \left\{F(x,y,p)+(y'-p)F_p(x,y,p)\right\} dx$$

or

$$\Delta I = \int_{C^*} \left\{F(x,y,y')-F(x,y,p)+(y'-p)F_p(x,y,p)\right\} dx$$

$$\Delta I = \int_{C^*} E(x,y,p,y')\,dx, \tag{163}$$

where

$$E(x,y,p,y') = \left\{F(x,y,y')-F(x,y,p)+(y'-p)F_p(x,y,p)\right\} \tag{164}$$

and $E(x,y,p,y')$ is known as the Weierstrass function, From (163), it clearly follows that

- **(i)** A sufficient condition for the functional I to attain a minimum on the extremal C is $E \geq 0$ since it implies $\Delta I \geq 0$ similarly
- **(ii)** A sufficient condition for a maximum is $E \leq 0$ since it implies $\Delta I \leq 0$.
- **(iii)** Further a sufficient condition for a weak minimum is $E \leq 0$ (or for a weak maximum a sufficient conditions is $E \geq 0$) is satisfied for values of x, y close to the values of x, y on the extremal C under consideration, and for the values of y' close to $p(x, y)$ on the same extremal.
- **(iv)** The sufficient condition for a strong minimum (or strong maximum) the same inequality must hold for the same x, y but now for arbitrary y'. Since in the case of a strong extremum, of close lying curves may have arbitrary direction of tangent lines and in the case of a weak extremum the values of y' on close lying curves arc close to the value of $y' = p$ on the extremal C. The following sufficient condition for the functional I to attain an extremum on the curve C.

14.21.1 For Weak Extremum

- **(i)** The curve C is an extremal satisfying the boundary conditions Eq. (159).
- **(ii)** The extremal C must be included in the field of extremals or Jacobi condition must be fulfilled.
- **(iii)** The Weierstrass function E does not change sign at point (x, y) close to the curve c or for values of y'close to $p(x, y)$ on the extremal. For a minimum, $E \geq 0$ and for a maximum $E \leq 0$.

14.21.2 For Strong Extremum

- **(i)** The curve C is an extremal satisfying the Boundary conditions in Eq. (159).
- **(ii)** The extremal C must be included in the field of extremals or Jacobi condition must be fulfilled.
- **(iii)** The Weierstrass function E does not change sign at any point (x, y) close to the curve C, and for a maximum $E \leq 0$.

Note We can show that the weierstrass condition is necessary. More precisely, if in a central field including the extremal C, the weierstrass function E has opposite signs at points of the extremal for contain y', then a strong extremum is not attained, if this property occurs for values of y'arbitrary close to p, then a weak extremum is not attained.

Example 29 Investigate for the functional $\int_0^1 (y'^2 + x^3)dx$, given $y(1) = 1$.

Solution Comparing the given functional by $I[y(x)] = \int_0^1 F(x, y, y')\,dx$

$$F(x, y, y') = (y'^2 + x^3) \tag{165}$$

from (1) $\dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial y'} = 2y'$

Then the Eulers equations $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0 \Rightarrow 0 - \dfrac{d}{dx}(2y') = 0 \quad \Rightarrow y'' = 0$

$$\Rightarrow \qquad y = C_1 x + C_2 \tag{166}$$

$$\text{given } y(1) = 1 \Rightarrow 1 = C_1 \cdot 1 + C_2 \tag{167}$$

$$(2) \Rightarrow p = \frac{dy}{dx} = C_1 \tag{168}$$

Therefore, the Weierstrass functional is

$$E(x, y, p, y') = \left\{F(x, y, y') - F(x, y, p) + (y' - p)F_p(x, y, p)\right\}$$

$$= y'^2 + x^3 - p^2 - x^3 - (y' - p)2p = (y'^2 - p^2) - (y' - p)2p$$

$$= (y' - p)(y' + p) - (y' - p)2p = (y' - p)(y' + 2p - p) = (y' - p)^2$$

$$= (y' - C_1)^2 \quad (\because \quad p = C_1)$$

$$\therefore \qquad E(x, y, p, y') \geq 0$$

Therefore, the given functional Eq. (165) has strong minimum on all its extremals

Example 30 Investigate the functional $\int_0^1 (1 + x)y'^2 dx$, given $y(0) = 0$, $y(1) = 1$.

Solution Comparing the given functional with $I[y(x)] = \int_0^1 F(x, y, y')\,dx$ then

$$F(x, y, y') = (1 + x)y'^2 \tag{169}$$

given $y(0) = 0$, $y(1) = 1$. Here $F(x, y, y')$ is independent of y.

Then the Eulers

$$\frac{\partial F}{\partial y'} = C \Rightarrow 2y'(1 + x) = C \ \Rightarrow \ y' = \frac{C}{2(1 + x)} = \frac{C_1}{(1 + x)} = p \quad \therefore \quad C_1 = \frac{C}{2} \tag{170}$$

Integrating equations (170)

$$y = C_1 \log(1 + x) \tag{171}$$

Given $\quad y(0) = 0 \Rightarrow 0 = C_1 \cdot \log(1+x) \Rightarrow 0 = 0$

$$y(1) = 1 \Rightarrow 1 = C_1 \log 2 \Rightarrow C_1 = \frac{1}{\log 2} \tag{172}$$

Therefore, the Weierstrass functional is

$$\begin{aligned} E(x, y, p, y') &= \left\{F(x, y, y') - F(x, y, p) + (y' - p)F_p(x, y, p)\right\} \\ &= (1+x)y'^2 - (1+x)p^2 - (y'-p)2p(1+x) \\ &= (1+x)\left\{y'^2 - p^2 - 2py' + 2p^2\right\} \\ &= (y'-p)^2(1+x) = \left(y' - \frac{C_1}{1+x}\right)^2 (1+x) \end{aligned}$$

$\Rightarrow \quad E \geq 0 \ \forall \ y'.$

Hence the given function has strong minimum.

Example 31 Investigate for an extremum the functional $I[y(x) = \int_0^1 \left(x + 2y + \frac{1}{2}y'^2\right)dx$ given $y(0) = 0$, $y(1) = 0$.

Solution Comparing the given functional with $I[y(x)] = \int_0^1 F(x, y, y')dx$ then

we have $F(x, y, y') = \left(x + 2y + \frac{1}{2}y'^2\right)$ (173)

Euler's equation is $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ (174)

From (1) $\dfrac{\partial F}{\partial y} = 2, \dfrac{\partial F}{\partial y'} = y'$ so that $2 - \dfrac{d}{dx}(y') = 0$ or $\dfrac{d^2y}{dx^2} = 2$ integrating $\dfrac{dy}{dx} = 2x + C_1$

So that $y = x^2 + C_1x + C_2$ (175)

Using the given boundary conditions $y(0) = 0$ and $y(1) = 0$.
Eq. (175) Yields $C_2 = 0$ and $C_1 + C_2 + 1 = 0$ so that $C_1 = -1$, $C_2 = 0$
Then from Eq. (175)

$$y = x^2 - x \tag{176}$$

Thus by Eq. (176), now the Jacobi's equation is given by

$$\left(F_{yy} - \frac{d}{dx}F_{yy'}\right) \times u - \frac{d}{dx}(F_{y'y'} \times u') = 0 \tag{177}$$

$F_{yy} = 0, F_{yy'} = 0, F_{y'} = y'$ and $F_{y'y'} = 1.$ Then Eq. (177) gives

Yields $-\dfrac{du'}{dx}=0$ or $\dfrac{d^2u}{dx^2}=0$, giving $u''=0 \Rightarrow u(x)=C\,x+D$ (178)

$\Rightarrow \quad u(0)=0 \Rightarrow D=0 \Rightarrow u(x)=C\cdot x, (C \neq 0)$

since $u(x) = C{\cdot}x$ does not vanish (for $C \neq 0$) anywhere in [0, 1] except at $x = 0$ we find that the Jacobi condition is satisfied and extremal (4) can be included in the centre field of extremals given by $y = x^2 + C_1x$ with centre at the origin $O(0, 0)$ as shown in Fig. 14.14.

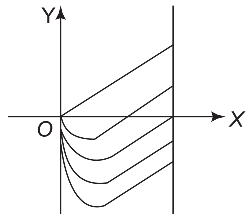


Fig. 14.14

Now, the Weierstrass function is given by

$$E(x,y,p,y')=\left\{F(x,y,y')-F(x,y,p)+(y'-p)F_p(x,y,p)\right\} \tag{179}$$

Form Eq. (173),

$$F(x,y,p)=x+2y+\frac{p^2}{2} \tag{180}$$

From Eq. (180),

$$F_p(x, y, p) = p \tag{181}$$

Using Eqs (173), (180) and (181), (179) reduces to

$$E(x,y,p,y')=x+2y+\frac{y'^2}{2}-\left(x+2y+\frac{p^2}{2}\right)-(y'-p)p$$

$$E(x,y,p,y')=\frac{y'^2}{2}-p\,y'+\frac{p^2}{2}=\frac{(y'-p)^2}{2}$$

Showing that $E \geq 0$ for arbitrary y'. Hence a strong minimum is attained on the extremal for using Eq. (176) we have

$$I[y(x)=\int_0^1\left(x+2y+\frac{1}{2}y'^2\right)dx=\int_0^1\left(x+2y+\frac{1}{2}(2x-1)^2\right)dx$$

$$=\int_0^1\left(4x^2-3x+\frac{1}{2}\right)dx \qquad (\because\ y=x^2-x \Rightarrow y'=2x-1)$$

$$=\left[4\left(\frac{x^3}{3}\right)-3\frac{x^2}{2}+\frac{x}{2}\right]_0^1=\frac{4}{3}-\frac{3}{2}+\frac{1}{2}=\frac{1}{3}.$$

Thus, the value of the strong minimum on the extremal u is $\frac{1}{3}$.

Example 32 Test for an extremal the functional $I[y(x) = \int_0^a (y'^3)\,dx$ given $y(0) = 0$, $y(a) = b$ $a > 0$, $b > 0$.

Solution Comparing the given functional with $\int_0^a F(x, y, y')dx$ we have

$$F(x, y, y') = (y'^3) \tag{182}$$

Euler's equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{183}$$

from $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial y'} = 3y'^2$ so that $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}(6y') \;= 6y''$

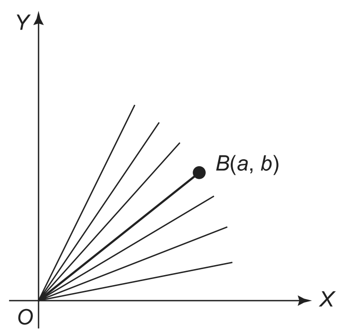


Fig. 14.15

with these values Eq. (183) gives

$$y'' = 0 \Rightarrow y(x) = C_1\, x + C_2 \tag{184}$$

using the boundary condition $y(0) = 0$ and $y(a) = b$.

$\Rightarrow$ $C_2 = 0$ and $C_1 a + C_2 = b$ so that $C_2 = 0$ and $C_1 = \frac{b}{a}$.

Then from (184)

$$\Rightarrow \qquad y(x) = \frac{b}{a}\cdot x \tag{185}$$

Thus an extremal and the given functional is attained only on the straight lines given by Eq. (185).

Here the pencil of straight lines $y(x) = \frac{b}{a}\cdot x$, centered at the point $O(0, 0)$ forms a central field that includes the extremals $y(x) = \frac{b}{a}\cdot x$, as shown in the Fig. 14.15 by straight line OB from Eq. (182).

$$F(x, y, y') = p^3 \text{ and so } F_p(x, y, p) = 3p^2 \tag{186}$$

Now the Weirestrass function is given by

$$E(x,y,p,y') = \left\{F(x,y,y') - F(x,y,p) + (y'-p)F_p(x,y,p)\right\} \tag{187}$$

Using (182), (186) and (187) reduces to

$$E(x,y,p,y') = y'^3 - p^3 - (y'-p)3p^2 = (y'-p)^2(y+2p) \tag{188}$$

On the extremal $y(x) = \frac{b}{a}\cdot x$, the slope of the field $p = \frac{b}{a} > 0$ and if y' assumes values close to $p = \frac{b}{a}$, then $E \geq 0$ and hence all the conditions that are sufficient for attaining a weak minimum are satisfied. Therefore a weak minimum in attained on the extremal $y(x) = \frac{b}{a}\cdot x$.

On the other hand. if y' takes on arbitrary, values them ($y' + 2p$) may have any sign and so the function E changes sign, and the condition sufficient for attaining a strong minimum are not satisfied. further, using this in weiestrass function on the straight line $y(x) = \frac{b}{a}\cdot x$.

Example 33 Test for an extremum the functional $I[y(x) = \int_0^a (6y'^2 - y'^4 + yy')\,dx$ given $y(0) = 0$, $y(a) = b$ $a > 0, b > 0$.

Solution

Comparing the given functional with $\int_0^a F(x,y,y')dx$ we have

$$F(x,y,y') = (6y'^2 - y'^4 + yy') \tag{189}$$

$$\frac{\partial F}{\partial y} = +y', \quad \frac{\partial F}{\partial y'} = 12y'^2 - 4y'^3 + y$$

Then Euler's equation gives

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{190}$$

$$\Rightarrow y' - \frac{d}{dx}(12y' - 4y'^3 + y) = 0 \Rightarrow -y' - 12y'' + 12y'^2 + y' = 0 \Rightarrow -12y'' + 12y'^2 y'' = 0$$

$$\Rightarrow y''(-12 + 12y'^2) = 0 \Rightarrow y'' = 0 \Rightarrow y(x) = C_1\, x + C_2 \tag{191}$$

The boundary condition hold of the line $y(x) = \frac{b}{a}\cdot x$, which can be included in the pencil $y(x) = C_1\cdot x$, of extremals that from a central field.

$$F(x,y,y') = p^3 \text{ and } F_p(x,y,p) = 3p^2 \tag{192}$$

Now the Weirestrass function is given by

$$E(x,y,p,y') = \left\{F(x,y,y') - F(x,y,p) + (y'-p)F_p(x,y,p)\right\} \tag{193}$$

$$E(x,y,p,y') = 6y'^2 - y'^4 + yy' - 6p^2 + p^4 - yp - (y'-p)(12p - 4p^3 + y)$$

$$= -(y'-p)^2\left\{y'^2 + 2py' - (6 - 3p^2)\right\}$$

Its sign is always opposite to the sign of the last factor $y'^2 + 2py' - (6 - 3p^2)$. This factor vanishes and may change its sign only when y' passes through the value $y' = -p \pm \sqrt{6 - 2p^2}$, if $6 - 2p^2 \le 0$ or $p \ge \sqrt{3}$.

Then for arbitrary y' we have $y'^2 + 2py' - (6 - 3p^2) \ge 0$ if $6 - 2p^2 > 0$ or $p < \sqrt{3}$ then the expression $y'^2 + 2py' - (6 - 3p^2)$ changes its sign. If at the same time y' in very close to p, then the latter expression for $p > 1$ retains the positive sign, and for $p < 1$, it retains the negative sign. Consequently if $p = b/a > 1$ or $b > a$, then there is a weak minimum. For $E > 0$, provides y' is sufficiently close to p. If $p = b/a > 1$ or $b > a$ the there is a weak minimum. If $p = b/a \ge 3$ there is a strong maximum, for this case $E \le 0$ regardless of y' when $p = b/a < \sqrt{3}$, then there is no strong extremum, neither minimum, nor maximum. Fig. 14.16.

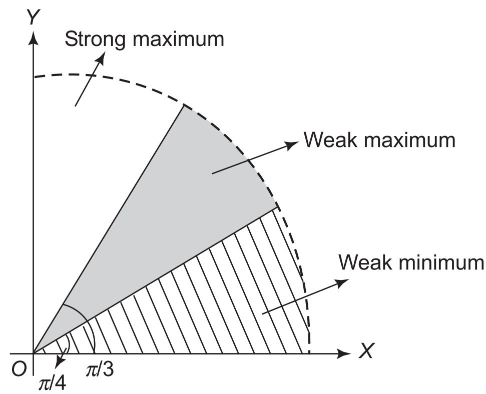


Fig. 14.16

14.22 LEGENDRE CONDITIONS

Consider the extremum of the functional

$$I[y(x)] = \int_{x_0}^{x_1} F(x, y, y')dx,\ y(x_0) = y_0,\ y(x_1) = y_1 \tag{194}$$

Further, suppose that the extremal C that passes through the points $A(x_0, y_0)$ and $B(x_1, y_1)$ satisfies the Jacobi's condition. Hence the extremal 'C' can be included in the field whose slope in $P(x, y)$ as shown in the figure. To determine the sign of an increment ΔI of a functional I which obtains when passing from an extremal curve C to some other neighboring admissible extremal C^* we will trans form the increment

$$\Delta I = \int_{C^*} F(x, y, y')\,dx - \int_{C} F(x, y, y')\,dx \tag{195}$$

where $\int_{C^*} F(x, y, y')\,dx$ and $\int_{C} F(x, y, y')\,dx$ denote the value of the functional $I[y] = \int_{x_0}^{x_1} F(x, y, y')dx$ taken along the arc of C^*or C respectively.

Consider the auxiliary functional $\int_{C^*} \left\{F(x, y, p) + (y' - p)F_p(x, y, p)\right\}dx$ if $y' = \dfrac{dy}{dx}$ (196)

which reduces to $\int_C F(x,y,y')\,dx$ on the extremal C, since $y' = \dfrac{dy}{dx} = p$ on extremals of the field.

Now re-writing the functional (3), We have

$$\int_{C^*} F(x,y,p)\,dx + \int_{C^*} y'\,F_p(x,y,p)\,dx - \int_{C^*} p\,F_p(x,y,p)\,dx \tag{197}$$

which is the integral of an exact differential. In fact, the differential of the functional $I^*[x, y]$ into which the functional $I[y(x)]$ is transformed on the extremals of the field has the form

$$dI^* = \left\{F(x,y,y') - y'F_{y'}(x,y,y')\right\}dx + F_y(x,y,y')\,dx$$

which differs from the integrand in Eq. (197) with y'replaced by p, the field. Thus, on the extremal C the integral of Eq. (196) coincides of an exact differential and, hence, does not depend on the path of integration. It follows that

$$\int_C F(x,y,y')\,dx = \int_{C^*} \left\{F(x,y,p) + (y'-p)F_p(x,y,p)\right\}dx \tag{198}$$

not only for $C = C^*$ but for any choice of C^* as well.

Using Eqs (198), (192) may be re-written as

$$\Delta I = \int_{C^*} F(x,y,y')\,dx - \int_{C^*} \left\{F(x,y,p) + (y'-p)F_p(x,y,p)\right\}dx$$

or

$$\Delta I = \int_{C^*} \left\{F(x,y,y') - F(x,y,p) + (y'-p)F_p(x,y,p)\right\}dx$$

$$\Delta I = \int_{C^*} E(x,y,p,y')\,dx \tag{199}$$

where $$E(x,y,p,y') = \left\{F(x,y,y') - F(x,y,p) + (y'-p)F_p(x,y,p)\right\} \tag{200}$$

Let us assume that the function $F(x, y, y')$ has third derivative with respect to y', then by Taylor's formula we have

$$F(x,y,y') = F(x,y,p) + (y'-p)F_p(x,y,p) + \frac{1}{2!}(y'-p)^2 F_{y'y'}(x,y,q) \tag{201}$$

where q lies between y' and p.

Substituting the value of $F(x, y, y')$ given by (8) in Eq. (200) we get

$$E(x,y,p,y') = \frac{1}{2}(y'-p)^2 F_{y'y'}(x,y,q) \tag{202}$$

Showing that the function E does not change sign if $F_{y'y'}(x,y,q)$ does not. It follows that while investigating for a weak extremum the function $F_{y'y'}(x,y,q)$ must retain sign for values of x and y at points close to the points of the extremal under consideration, and for value of q close to $p(x, y)$, if $F_{y'y'}(x,y,y') \neq 0$ at points of the extremal C. Then by virtue of continuity of this derivative maintains sign both at points close to the curve C and for values of y' close to the values of y' on the curve C^*.

Thus for testing for a weak minimum, the condition $E \ge 0$ may be replaced by $F_{y'y'} > 0$ on the extremal C and for a weak maximum the condition $E \le 0$ may be replaced by $F_{y'y'} < 0$ on C this is known as the Legendre Condition.

On the other hand, while testing for a strong minimum the condition $E \ge 0$ may be replaced by the condition $F_{y'y'}(x, y, q) \ge 0$ at point (x, y) close to points of the curve C for arbitrary value of q. In this case, of course, we assume that the formula Eq. (201) holds good for any y'. Similarly while testing for a strong maximum the condition $E \le 0$ may be replaced the condition $F_{y'y'}(x, y, q) \le 0$ at point (x, y) close to points of the curve C for arbitrary values of q. Here again we assume that (8) holds good for any y'.

Example 34 Investigate the given functional $I[y(x)] = \int_0^2 (y'^4 + y'^2)\, dx,\ y(0) = 1,\ y(2) = 5.$

Solution Comparing the given functional with $\int_0^2 F(x, y, y')\, dx,$ we have

$$F(x, y, y') = (y'^4 + y'^2) \tag{203}$$

$\dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial y'} = 4y'^3 + 2y'.$ Then Euler's equation gives

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0 \tag{204}$$

$$0 - \frac{d}{dx}(4y'^3 + 2y') = 0 \Rightarrow -12y'^2 y'' + 2y'' = 0 \Rightarrow\ y''(-12y'^2 + 2)\ = 0$$

$$\Rightarrow \qquad y'' = 0 \Rightarrow\ y = C_1 x + C_2$$

The extremals are a straight lines $y = C_1x + C_2$. The descried extremal satisfying the given boundary conditions $y(0) = 1$, $y(2) = 5$.

$$y(0) = 1 \Rightarrow C_2 = 1 \quad \text{and } y(2) = 5 \Rightarrow C_1 + C_2 = 1 \text{ solving } C_1 = 2,\ C_2 = 1$$

Then extremal $y = 2x + 1$. In the given case $F_{y'y'} = 12y'^2 + 2$ and at all points of the extremal $y = 2x + 1$ we have $F_{y'y'} = 50 > 0$

Legendre strong condition is fulfilled and the extremal $y = 2x + 1$ may be included in the extremal field. This is also evident directly. The extremal $y = 2x + 1$ lies in the one-parameter family of extremals $y = 2x + \alpha$ (where α is a parameter) that form a proper field.

Example 35 The given functional $\int_{-1}^{1} (x^2 y'^2 + 12y^2)\, dx\ y(-1) = -1,\ y(1) = 1$ satisfies the Legendre Condition or not?

Solution Comparing the given functional with $\int_{-1}^{1} F(x, y, y')\, dx$ then

$$F(x, y, y') = (x^2 y'^2 + 12y^2) \tag{205}$$

$\dfrac{\partial F}{\partial y} = 24y,\ \dfrac{\partial F}{\partial y'} = 2x^2 y'.$ Then the Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

$$\Rightarrow \qquad 24y - \frac{d}{dx}(2xy^2 y') = 0 \Rightarrow 24y - (4xy' + 2x^2 y'') = 0 \Rightarrow\ x^2 y'' + 2xy' - 12y = 0$$

which is Homogenous linear different equation then its general Solution is $y = C_1x^3 + C_2x^4$.

The extremal $y = x^3$ satisfies the given boundary condition. It cannot be included in the field. The only one parameter family of extremals that contain it in the family $y = \alpha x^3$. This field does not cover the region containing the point with abscissa $x = a$ (the extremals of this family do not pass through points of the y-axis with ordinates different from zero

In the given case $F_{y'y'} = 2x^2$ and the Legendre Condition is not fulfilled for $x = 0$.

Example 36 Examine the extremal of the functional $I[y(x)] = \int_0^{x_1} \left(\dfrac{\sqrt{1+y'^2}}{\sqrt{y}} \right) dx,$

$y(0) = 0,\ y(x_1) = y_1$

Solution Comparing the given functional with $\int_0^{x_1} F(x, y, y')\, dx.$ Then

$$F(x, y, y') = \left(\frac{\sqrt{1+y'^2}}{\sqrt{y}} \right) \tag{206}$$

Euler's equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y} \right) = 0 \tag{207}$$

Since $F(x, y, y')$ is independent of x then Euler's equation reduced to

$$F - y' \frac{\partial F}{\partial y'} = \text{Constant} = C$$

$$\Rightarrow \quad \frac{1}{\sqrt{y}}(1+y'^2)^{1/2} - y' \times \frac{1}{2\sqrt{y}}(1+y'^2)^{-1/2} \times 2y' = C \quad \text{using Eq. (206)}$$

$$\Rightarrow \quad 1 + y'^2 - y'^2 = C(1+y'^2)^{1/2} \Rightarrow y(1+y'^2) = 2C_1, \tag{208}$$

$$\text{where } 2C_1 = \frac{1}{C^2} \text{ introduce the parameter by putting } y' = \frac{dy}{dx} = \cot t. \tag{209}$$

Then from Eq. (208)

$$y = \frac{C_1}{1+\cot^2 t} = C_1 \sin^2 t = \frac{C_1}{2}(1 - \cos 2t)$$

From Eq. (209)

$$dx = \frac{dy}{y'} = \frac{2C_1 \sin t \cos t\ dt}{\cot t} = 2C_1 \sin^2 t\ dt = C_1(1 - \cos 2t)\, dt$$

Integrating $$x = C_1\left(t - \frac{\sin 2t}{2} \right) + C_2 = \frac{C_1}{2}(2t - \sin 2t) + C_2.$$

Hence, the equation of the given curve in parametric form is given by

$$x - C_2 = \frac{C_1}{2} \times (2t - \sin 2t) \text{ and } y = \frac{C_1}{2}\ (1 - \cos 2t). \tag{210}$$

Taking $\frac{C_1}{2} = a$ and $2t = \phi$ then Eq. (210) reduce to

$$x - C_2 = a \times (\phi - \sin\phi) \text{ and } y = a(1 - \cos\phi) \tag{211}$$

Now given condition $y(0) = 0, 0 = a(1 - \cos\phi) \Rightarrow \phi = 0$ so $C_2 = 0$

Then Eq. (211) reduces to

$$x = a(\phi - \sin\phi),\ y = a(1 - \cos\phi) \tag{212}$$

which is equal of a family of eyeloids, where a in determined from the condition that Eq. (212) must pass through the second boundary point $B(x_1, y_1)$ if $x_1 < 2\pi\alpha$ as shown in the Fig. 14.17.

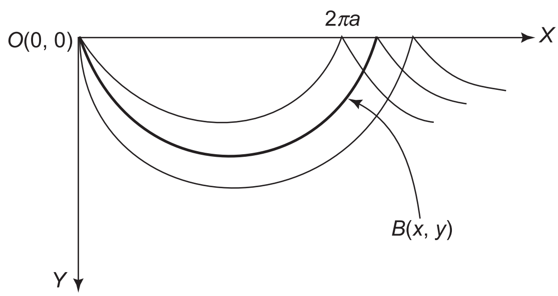


Fig. 14.17

Now from (206) we have

$$F_{y'} = \frac{y'}{\sqrt{y\sqrt{1 + y'^2}}} \text{ and so } F_{y'y'} = \frac{1}{y^{1/2}(1 + y'^2)^{\frac{3}{2}}} > 0 \text{ for any } y'$$

Hence, by Legendre condition, for $x_1 < 2\pi a$, a strong minimum is attained on the eyeloid Eq. (212).

Example 37 Examine the extremal of the functional $I[y(x)] = \int_0^a \left(\frac{y}{y'^2}\right) dx$, $y(0) = 1$, $y(a) = b$, $a > 0$, $0 < b < 1$.

Solution Comparing the given functional with $\int_0^a F(x, y, y')\, dx$

we have $F(x, y, y') = \frac{y}{y'^2}$ (213)

Since $F(x, y, y')$ is function of y and y' only then the

Euler's equation of the from $F - y' \frac{\partial F}{\partial y'} = \text{Constant} = C$

$$\Rightarrow \quad \frac{y}{y'^2} - y' \times \left(\frac{-2y'}{y'^3} \right) = C \Rightarrow \frac{3y}{y'^2} = \frac{3}{4C_1^2} \text{ where } C = \frac{3}{4C_1^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm 2C_1 y^{1/2} \Rightarrow \pm \frac{1}{2y^{-1/2}} dy = C_1 dx$$

Integrating $\pm y^{1/2} = C_1 x + C_2$ or $y = (C_1 x + C_2)^2$ (214)

Which is a family of parabolas using the given condition $y(0) = 1$, then Eq. (214) gives $C_2 = 1$. Now the Pencil of Parabolas $y = (C_1 x + 1)^2$ with centre $y = 0$ at $A(0, 1)$ has C_1-discriminant curve $y = 0$ as shown in the Fig. 14.18. Two parabola of pencil pass through the point $B(a, b)$. There is a conjugate point A^* to a point A, lying on the arc AB of one of these parabolas (L_1). The arc AB of the other parabola (L_2) contains no conjugate point to A. Therefore, the Jacobi condition for the arc L_2 holds and consequently this arc may give an extremum.

At a neighborhood of this extremal arc $F_{y'y'} = \frac{6y}{y'^4} > 0$ for arbitrary y'. But due to this fact we cannot say that a strong minimum can be attained on the arc L_2 since the function $F(x, y, y') = \frac{y}{y'^2}$ cannot be represented in the form

$$F(x, y, y') = F(x, y, p) + (y' - p)F_p(x, y, p) + \frac{1}{2!}(y' - p)^2 F_{y'y'}(x, y, q).$$

For arbitrary values of y' due to the presence of a discontinuity of the function $F(x, y, y')$ when $y' = 0$. However, we can only say that a weak minimum achieved on L_2, since for values of y' close to the slope of the field on the curve L_2 we have an expansion of the function $F(x, y, y')$ by Taylor's formula. A complete discussion of this function for an extremum involves the consideration of Weiestrass function given by

$$E(x, y, p, y') = \left\{ F(x, y, y') - F(x, y, p) + (y' - p)F_p(x, y, p) \right\}$$

i.e., $$E(x, y, p, y') = \frac{y}{y'^2} - \frac{y}{p^2} + \frac{2y}{p^3}(y' - p) = \frac{y(y' - p)^2 (2y' + p)}{y'^2 p^3}.$$

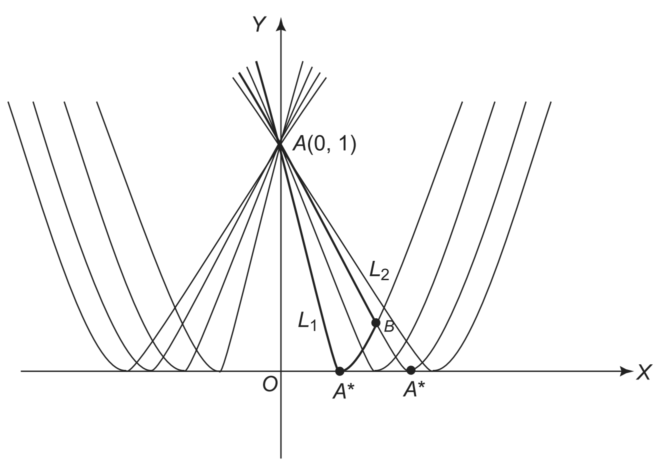


Fig. 14.18

Since the factor $(2y' + p)$ change sign for arbitrary y' hence with help "Note" of Weiestrass function Article. We conclude that a strong minimum cannot be attained on the arc L_2.

14.23 LEGENDRE CONDITION FOR QUADRATIC FUNCTION

A quadratic functional $L_2(h)$ specified on some normed space is said to be strong-positive if there exists a constant $k > 0$ such that $L_2(h) \geq k\|h\|^2$ for all h

A sufficient condition for a minimum. For a functional $I[y(x)]$ defined in a normed space to have a minimum at a stationary point $y = y_0$, it is sufficient that when $y = y_0$ the second variation be strong-positive, that the following condition be fulfilled:

$\delta^2 I[y_0, \delta y] \geq k\|\delta y\|^2$ where k = constant, $k > 0$. Let us seek the extremum of the functional

$$I\left[y_1, y_2, \ldots. y_n\right] = \int_{x_0}^{x_1} F(x, y, y_1 \ldots, y_n, y_1', y_2', \ldots y_n')\,dx \tag{215}$$

dependent on n-functions $y_1(x)$, $y_2(x)$, $\ldots y_n(x)$ with the boundary conditions

$$y_k(x_0) = y_{k0},\ y_k(x_1) = y_{k1} \qquad (k = 1, 2, \ldots, n) \tag{216}$$

Legendre's strong condition consists in the requirement that the following inequalities hold true

$$F_{y'y'} \geq 0, \begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} \\ F_{y_2'y_1'} & F_{y_2'y_2'} \end{vmatrix} \geq 0, \ldots \begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} & \cdots & F_{y_1'y_n'} \\ F_{y_2'y_1'} & F_{y_2'y_2'} & \cdots & F_{y_2'y_n'} \\ \vdots & \vdots & \vdots & \vdots \\ F_{y_n'y_1'} & F_{y_n'y_2'} & \cdots & F_{y_n'y_n'} \end{vmatrix} \geq 0 \tag{217}$$

at all points of the indicated extremal of the functional Eq. (215).

Jacobi's strong condition consists in the requirement that the interval $[x_0, x_1]$ not contain a point conjugate to the point x_0. Legendre's strong condition Eq. (217) in conjunction with Jacobi's strong condition ensures the existence of at least a weak minimum of the functional Eq. (215).

Example 38 Test for an extremum the functional $I\left[y, z\right] = \int_0^1 (y'^2 + z'^2)\,dx$ (218)

$$y(0) = 0,\ z(0) = 0 \text{ and } y(1) = 1, z(1) = 2 \tag{219}$$

Solution Euler's equations for the functional Eq. (215) are $y'' = 0$ and $z'' = 0$

$$y(x) = C_1 + C_2 x \text{ and } z(x) = C_3 + C_4 x$$

Utilizing the conditions Eq. (216) we get

$$C_1 = 0,\ C_2 = 1,\ C_3 = 0\ C_4 = 2$$

The desired extremal

$$y(x) = x \text{ and } z(x) = 2x \tag{220}$$

is a straight line passing through the origin.

We have

$$F_{y'y'} = 2,\ F_{y'z'} = 0,\ F_{z'y'} = 0,\ F_{z'z'} = 2$$

Legendre's strong condition is fulfilled:

$$F_{y'y'} = 2 \geq 0, \begin{vmatrix} F_{y'y'} & F_{y'z} \\ F_{z'y'} & F_{z'z'} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0. \tag{221}$$

Let us now check to see whether Jacobi's strong condition is fulfilled of not.

One of the definitions of a conjugate point is the following that. Suppose we have a family of extremals of the functional Eq. (221) issuing from an initial point (x_0, y_{10}, y_{20}, $y_{30, \ldots,}$ y_n) in close-lying but linearly independent directions.

The point $x^* \in [x_0, x_1]$ is said to be conjugate to the point x_0 if there exists a sequence of extremals issuing from an initial point and arbitrarily close to the given extremal such that each of these extremals intersects the given extremal, and abscissas of the intersection points converge to the point x^*.

In the given example, the extremals are the straight lines Eq. (220). All the extremals issuing from the point (0, 0, 0) intersect the extremal Eq. (220) only at that point. Hence, the interval [0, 1] of variation of x does not contain a point conjugate to the point $x_0 = 0$. Thus both Legendre's strong condition and Jacobi's strong condition are fulfilled, so that the extremal Eq. (220) gives the functional Eq. (218) a weak minimum.

EXERCISE 14.2

1. Examine the following functional are central and proper field.

(i) $y = (x - c)^3, \dfrac{x^2}{4} + \dfrac{y^2}{9} \leq 1$

(ii) $y = c(x^2 - 2x)$

(a) $0 \leq x \leq 1$ (b) $-1 \leq x \leq 3$ (c) $\dfrac{1}{2} \leq x \leq \dfrac{3}{2}$

(iii) $y = c \sin\left(x - \dfrac{\pi}{4}\right)$

(a) $\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3} \leq x \leq \pi$ (c) $\dfrac{\pi}{8} \leq x \leq 2\pi$

(iv) $y = e^{x+c}$, $x^2 + y^2 \leq 1$

2. Indicate the proper and central field of extremals for the following functionals

(i) $I[y(x)] = \int_0^a (y'^2 + y^2)\, dx \quad a > 0$

(ii) $I[y(x)] = \int_0^a (y'^2 - 2xy)\, dx,\ y(0) = y(1) = 0$

(iii) $I[y(x) = \int_0^a (2e^x y + y'^2)\, dx,\ y(0) = 1,\ y(1) = e$

(iv) $I[y(x)] = \int_0^a (y^2 - y'^2)\, dx,\ (a > 0,\ a \neq \pi k)\ y(0) = 0,\ y(a) = 0$

(v) $I[y(x) = \int_0^2 (y'^2 + x^2)\, dx,\ y(0) = 1,\ y(2) = 3$

3. Is the Jacobi condition fulfilled for the extremals.

(i) $I[y(x)] = \int_{-1}^{1} (12xy + y'^2 + x^2)\, dx,\ y(-1) = -2,\ y(1) = 0$

(ii) $I[y(x)] = \int_{0}^{a} (y'^2 + 9y^2 - 3x)\, dx \qquad y(0) = 0,\ y(a) = 0$

4. Check the possibility of including an extremal in the field for the following functional (Legendre Condition)

(i) $I[y(x)] = \int_{0}^{1} (y'^2 - yy')\, dx,\ y(0) = 0,\ y(1) = 0$

(ii) $I[y(x)] = \int_{0}^{a} y'^3\, dx,\ y(0) - 0,\ y(a) = b > 0$

(iii) $I[y(x)] = \int_{x_0}^{x_1} ny\left(\sqrt{1 + y'^2}\right) dx,\ y(x_0) = y_0,\ y(x_1) = y_1,\ y > 0$

(iv) $I[y(x)] = \int_{0}^{a} (6y'^2 - y'^4\, dx,\ y(0) = 0,\ y(a) = b,\ a > 0,\ b > 0$

5. Test the following functional for extremal.

(i) $I[y(x)] = \int_{0}^{3} (xy' + y'^2)\, dx,\ \ y(0) = 1,\ y(2) = 0$

(ii) $I[y(x)] = \int_{0}^{1} \left(\frac{1}{y'}\right) dx,\ y(0) = 0,\ y(x_1) = y,\ x_1 > 0,\ y_1 > 0$

(iii) $I[y(x)] = \int_{1}^{2} \left(\frac{x^3}{y'^2}\right) dx,\ y(1) = 1,\ y(2) = 4$

(iv) $I[y(x)] = \int_{1}^{3} (12xy + y'^2)\, dx,\ y(1) = 0 y(3) = 26$

(v) $I[y(x)] = \int_{0}^{\pi/4} (4y^2 - y'^2 + 6y \sin 2x)\, dx\ y(0) = 0,\ y(\pi/4) = 1$

(vi) $I[y(x)] = \int_{0}^{2} (y^2 - y'^2 - 2xy)\, dx,\ y(0) = 0,\ y(2) = 3$

6. Show that the extremal $y = 0$ of the variational problem

$$I[y(x)] = \int_{0}^{1} (y'^2 - yy'^3)\, dx,\ y(0) = y(1) = 0$$

give the functional a weak minimum.

7. Test the following functionals for extremal.

(i) $I[y(x), z(x)] = \int_{0}^{1} (\sqrt{1 + y'^2 + z'^2})\, dx,\ y(0) = 0,\ y(1) = 2,\ z(0) = 0,\ z(1) = 4$

(ii) $I[y(x), z(x)] = \int_{0}^{1} (y'^2 + z'^2 + 4z)\, dx,\ y(0) = 0,\ y(1) = 1,\ z(0) = 0,\ z(1) = 0$

8. Find the shortest distance between the points $A(1, 0, -1)$ and $B(0, -1, 1)$ lying on the surface $x + y + 2 = 0$.

Answer

1.
   (i) Proper field
   (ii) (a) Central field (b) a field is not formed (c) Proper field
   (iii) (a) Central field (b) Proper field (c) a field is not formed
   (iv) A field is not formed since this family of curves does not cover the entire domain D

2.
   (i) $y = C_1 \cosh x$ from a proper field of extremals
   (ii) The extremal $y = \frac{x}{6}(1 - x^2)$ is included in the central field of extremals $y = C_1 x - \frac{x^3}{6}$ with centre at the point $O(0, 0)$.
   (iii) The extremal $y = e^x$ may be included in the Proper field of extremals $y = e^x + c$.
   (iv) If $a < \pi$, then the extremal $y = 0$ may be included in the central field of extremals $y = c \sin x$ with centre at the point $O(0, 0)$ for $a > \pi$ the family of curves $y = c \sin x$ does not form a field.
   (v) The extremal $y = x + 1$ is included in the Proper field $y = x + c$.

3. (i) The Jacobi condition is fulfilled (ii) The Jacobi condition is fulfilled for any 'a'

4 . (i) Yes (ii) Yes (iii) Yes (iv) Yes but Legendre's condition is only fulfilled for $\frac{b}{a} < 1$

5. (i) A strong minimum is attained for $y = 1 - \frac{x^2}{4}$
   (ii) A weak minimum is attained for $y = \frac{xy_1}{x_1}$
   (iii) A weak minimum is attained for $y = x_2$
   (iv) A strong minimum is attained for $y = x^2 - 1$
   (v) A strong maximum is attained for $y = \sin 2x$
   (vi) A strong maximum is attained for $y = x + \frac{\sinh x}{\sinh 2}$

7. (i) On the extremal $y = 2x$, $z = 4x$ is attained a weak minimum.
   (ii) The extremal is the parabola $y = x$, $z = x^2 - x$, which is included in the central field of extremals

(8) $\sqrt{6}$

SUMMARY

Following topics have been discussed in this chapter:

1. Function

Let X and Y be any two non-empty sets and there be correspondence or association between the elements of X and Y such that for every elements of X and Y there exists a unique element $y \in Y$ written as $y = f(x)$ then we say that y is a mapping or function from X to Y and written as $f: X \to Y$ such that $y = f(x)\ \forall\ x \in X, y \in Y$

2. Functional

Let there be functional belonging to class of functions. Then a variable quantity denoted by $I[y(x)]$ is a functional if to each function belonging to the class of functions there is definite value of I.

3. Closeness of Curves

In the calculus of real variable we have defined the continuity of function at $x = a$. As the variable x is becoming closer to a, Then $f(x)$ becomes closer to $f(a)$.

Here in case of functional, domain is the class of functions and here idea of closeness of $y(x)$ and $y_1(x)$ is to say that the absolute value of their difference, i.e., $\left|y(x)-y_1(x)\right|$ is small for all x for which $y(x)$ and $y_1(x)$ are defined.

If it happens then we say $y(x)$ is close to $y_1(x)$ in the sense of zero order proximity. $I[y(x)] = \int_{x_1}^{x_2} f[x, y(x), y'(x)]\,dx.$ Curve $y(x)$ and $y_1(x)$ are said to be close in the sense of n^{th} order proximity if $\left|y(x)-y_1(x)\right|, \left|y'(x)-y_1'(x)\right| \cdots \left|y^n(x)-y_1^n(x)\right|$ are small for the values of for x which these functions are defined.

4. Continuity of Functional

Domain of functional is a set of functions therefore continuity of functional at some function is defined as following:

Functional $I[y(x)]$ is said to be continuous at $y = y_0(x)$ in the sense of n^{th} order proximity if for given any positive number ε, $\exists\ \delta > 0$ such that $\left|I\left[y(x)\right]-I\left[y_0(x)\right]\right| < \varepsilon$

for $\left|y(x)-y_0(x)\right| < \delta, \left|y'(x)-y_0{}'(x)\right| < \delta \ldots \left|y''(x)-y_0''(x)\right| < \delta$

Linear Property Let the functional $I[y(x)]$ be defined in the linear space M of the functions $y(x)$. Functional $I[y(x)]$ is said to be linear if it satisfies the following conditions

(a) $I\left[c\, y(x)\right] = c\, I\left[y(x)\right]$ where c is any arbitrary constant.

(b) $I\left[\, y_1(x)+y_2(x)\right] = I\left[y_1(x)\right]+I\left[y_2(x)\right]$, where $y_1(x)$ and $y_2(x) \in M$

5. Variation of Functional

In Calculus of real variable, in case of the function $y = y(x)$ of single variable. We have studied that incremental ratio $\frac{\Delta y}{\Delta x}$ tends to $f'(x)$, when Δx tends to zero suggesting there by $\frac{\Delta y}{\Delta x}$ differs from function of called $f'(x)$ by a small quantity α say where α is function of Δx and tends to zero as $\Delta x \to 0$.

$$\frac{\Delta y}{\Delta x} = f'(x)+\alpha(x, \delta x) \Rightarrow \Delta y = \big(f'(x)+\alpha(x, \delta x)\big)\Delta x = A(x)\Delta x+\alpha(x, \delta x)\,\Delta x$$

where $A(x)$ is a function of x. Principal part of the increment Δx, i.e. $f'(x)\Delta x$ is known as differential in y. Likewise in case of functional $I[y(x)]$, the increment ΔI is given as $\Delta I = I\left[y(x)+\delta y(x)\right]-I\left[y(x)\right]$

(Domain of functional is functional and $\delta(y(x))$ denotes the corresponding change) which can be written as

$$\Delta I = L\big[y(x), \delta y(x)\big] + \alpha\big[y(x), \delta y\big] \max\left|\delta y\right|$$

where $L[y(x), \delta y(x)]$ denotes the functional linear in δy and $a(y(x), \delta y)$ tends to zero as the maximum value of $\delta y \to 0$. Principal part of the increment Δy, i.e. $L[y(x), \delta y(x)]$ is called the variation of the functional and is denoted by δI.

6. Fundamental Lemma of Calculus of Variation

If for every continuous function $h(x)$, $\int_{x_1}^{x_2} \phi(x)\eta(x)\,dx = 0$, where the function $\phi(x)$ is continuous in the closed interval $[x_1, x_2]$, then $\phi(x) = 0$ in the closed interval $[x_1, x_2]$.

7. Extremal

One of the main problems of the calculus of variation is to determine that curve connecting two given point which either maximizes or minimizes some given Integral. Consider the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for some given known function $F(x, y, y')$ of variables x, y, y' the integral $I[y(x)] = \int_{x_1}^{x_2} f(x, y(x), y'(x))\,dx$ is either maximum or minimum also called an extremum or stationary value. A curve $y = y(x)$ which satisfies this property is called an extremal.

8. Euler's Equation

Examine the functional $I[y(x)] = \int_{x_1}^{x_2} F(x, y(x), y'(x))\,dx$

The fundamental lemma of the calculus of variations, a necessary condition for the functional to have an extremum value is that the extremizing function $y = y(x)$ must satisfy the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad x_1 < x < x_2$$ is known as Euler's equation.

9. Variational Problems for Functional Involving Several Dependent Variables

$$\int_{x_1}^{x_2} F\Big[x, y_1(x), y_2(x), \ldots. y_n(x), y_1{}'(x), y_2{}'(x), \ldots. y_n{}'(x)\Big]\,dx$$

(i) Case of two dependent variables

is given by $\left\{\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1{}'}\right)\right\} = 0$ and $\left\{\frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2{}'}\right)\right\} = 0$

is called an Euler's equations.

(ii) Case of n^{th} dependent variables: The system of Euler's equations for finding extremals of the functional

$$\int_{x_1}^{x_2} F\left[x, y_1(x), y_2(x), \ldots. y_n(x), y_1'(x), y_2'(x), \ldots. y_n'(x)\right] dx$$

is given by $\left\{\dfrac{\partial F}{\partial y_i} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_i'}\right)\right\} = 0$, where $i = 1, 2, 3, 4, \ldots, n$.

10. Functional Dependent on Higher Order Derivatives

(i) Particular case: The necessary condition for

$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y', y'')\, dx$ to be extremum if $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) + \dfrac{d^2}{dx^2}\left(\dfrac{\partial F}{\partial y''}\right) = 0$

is known as **Euler–Poisson equation.**

(ii) General case: The necessary condition for $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y', y'', \ldots y^{(n)})\, dx$ to be

extremum if $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) + \dfrac{d^2}{dx^2}\left(\dfrac{\partial F}{\partial y''}\right) - \dfrac{d^3}{dx^3}\left(\dfrac{\partial F}{\partial y'''}\right) + \cdots + (-1)^n \dfrac{d^n}{dx^n}\left(\dfrac{\partial F}{\partial y^{(n)}}\right) = 0$

is known as Euler–Poisson equation.

11. Isoperimetric Problems

(i) Working rules for isoperimetric problem: It is necessary to make a given integral $I = \int_{x_1}^{x_2} F(x, y, y')\, dx$ subject to the constraint $J = \int_{x_1}^{x_2} G(x, y, y')\, dx = \text{constant}$, such problem involve one or more constraint conditions. The necessary condition for integral $\int_{x_1}^{x_2} H dx$ to be an extremum is $\dfrac{\partial H}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial H}{\partial y'}\right) = 0$ where $H = F \pm \lambda G$, λ is called Lagrange multipliers.

We determine the Lagrange multiplier λ, together with the constant of Integration arising in the solution of Euler's equation and J having given constant value.

(ii) Analytic form of the Jacobi condition: The variational problem

$$I[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\, dx,\ y(x_0) = y_0,\ y(x_1) = y_1$$

If the Solution $u = u(x)$ of the Jacobi equation $\left(F_{yy} - \dfrac{d}{dx} Fyy'\right) u - \dfrac{d}{dx}(Fy'\, y'u') = 0$

That satisfies the condition $u(x_0) = 0$ vanishes at some other point of the interval $x_0 < x < x_1$. Then the point P_1^* that is conjugate to the point $P_1(x_1, y_1)$ lies on the arc AB of the extremal.

If there is a Solution $u = u(x)$ of the Jacobi equation that satisfies the condition $u(x_0) = 0$ and does not vanish at any point of the half-open interval $x_0 < x \le x_1$, then there are no points conjugate to P_1 on the arc AB. In that case we can include the arc AB of the extremal in the central field of extremals with centre at the point $P_1(x_1, y_1)$.

It in necessary, in place of $y(x)$, to put the right member of the functions $F_{yy}(x, y, y')$, $F_{yy'}(x, y, y')$ and $F_{y'y'}(x, y, y')$ in equation.

(iii) Sufficient condition of Legendre: A sufficient condition for including an extremal of the functional $I[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\,dx,\ y(x_0) = y_0,\ y(x_1) = y_1$ in the field of extremals in the fulfilment of Legendre strong conditions.

It consists in demanding the fulfilment of the in equality $F_{y'y'} > 0$ at all points of the extremal under consideration that is for all $x \in [x_0, x_1]$.

(iv) Sufficient Conditions for the extremum Consider functional $I[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\,dx$

Subject to condition $y(x_0) = (y_0),\ y(x_1) = y_1$

(v) Sufficient Conditions of Weierstrass: The Weierstrass fun. $E(x, y, p, y')$ is an function defined by the equation $E(x, y, p, y') = \{F(x, y, y') - F(x, y, p) + (y' - p)\,F_p(x, y, p)\}$,
where $p = p(x, y)$ is the slope of the extremal field of the variational problem under consideration at the point (x, y).

(vi) Sufficient conditions for a weak extremum: A curve C gives the functional $I[y(x)] = \int_{x_0}^{x_i} F(x, y, y')\,dx$ a weak extremum if:

(a) The curve C is an extremal of the functional $I[y(x)] = \int_{x_0}^{x_i} F(x, y, y')\,dx$ which satisfies the boundary conditions $y(x_0) = y_0,\ y(x_1) = y_1$ that is, for the functional $I[y(x)] = \int_{x_0}^{x_i} F(x, y, y')\,dx$; it is a solution of Euler's equation that satisfies the conditions $y(x_0) = y_0,\ y(x_1) = y_1$.

(b) The extremal C may be included in the field of extremals. This will occur in particular if the Jacobi condition is fulfilled.

(c) The Weierstrass function $E(x, y, p, y')$ must preserve sign at all points (x, y) close to the extremal C and for values of y' close to $p = p(x, y)$. The functional $I[y(x)]$ will have a maximum on C if $E \le 0$ and a minimum if $E \ge 0$.

(vi) Sufficient conditions for a strong extremum: A curve C gives the functional $I[y(x)] = \int_{x_0}^{x_i} F(x, y, y')\,dx$ a strong extremum if:

(a) The curve C is an extremal of the functional $I[y(x)] = \int_{x_0}^{x_i} F(x, y, y')\,dx$ that satisfies the boundary conditions $y(x_0) = y_0,\ y(x_1) = y_1$.

(b) The extremal C may be included in the field of extremals.

(c) The Weierstrass functions $E(x, y, p, y')$ preserves sign at all points (x, y) close to the maximum for $E \le 0$ and a minimum for $E \ge 0$.

(viii) Sufficient condition of Legendre: A function $F(x, y, y')$ have a continuous partial derivatives $F_{y'y'}$ and let the extremal C be included in the field of extremals.

If on the extremal C we have $F_{y'y'} > 0$, then a weak minimum is attained on the curve C, if $F_{y'y'} < 0$ on the extremal C, then a weak maximum of the given functional is attained on it. These condition are known as Legendre's strong conditions.

When $F_{y'y'} \ge 0$ at points (x, y) close to the extremal C for arbitrary value of y' we have a strong minimum when $F_{y'y'} \le 0$ for the indicated values of the arguments, we have a strong maximum.

(A) The list of sufficient conditions $I[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\, dx$

S.No.	Weak minimum	Strong minimum	Weak minimum
1.	$F_y - \frac{d}{dx} F_{y'} = 0$	$F_y - \frac{d}{dx} F_{y'} = 0$	$F_y - \frac{d}{dx} F_{y'} = 0$
2.	Jacobi condition	Jacobi condition	Jacobi condition
3.	$F_{y'y'} > 0$ Along the investigated extremal.	$F_{y'y'} \ge 0$ For those points (x, y) that are close to the extremal under examination and for arbitrary values of y' It is assumed here that $F(x, y, y')$ has third-order derivative with respect to y', for all y'	$E(x, y, p, y¢) \ge 0$ for all the points (x, y) sufficiently close to the extremal under examina-tion, and all y' sufficient-ly close to $p(x, y)$
To obtain sufficient conditions for a maximum we have only to change the sense of inequalities.			

(B) For a minimum of simplest functional $y(x_0) = y_0, y(x_1) = y_1$

S.No.	Weak minimum	Strong minimum	Weak minimum
1.	$F_y - \frac{d}{dx} F_{y'} = 0$	$F_y - \frac{d}{dx} F_{y'} = 0$	$F_y - \frac{d}{dx} F_{y'} = 0$
2.	Jacobi condition	There exists a field of extremals includ-ing the investigated extremal curve.	There exists a field of extremals including the investigated extremal curve.
3.	$E(x, y, p, y') \ge 0$ for all the the points (x, y) sufficiently close to the extremal under examination and arbitrary y'	$E(x, y, p, y') \ge 0$ For those points (x, y) sufficiently close to the extremal under examination and all y' sufficiently close to $p(x, y)$	$E(x, y, p, y') \ge 0$ for all the points (x, y) sufficiently close to the extremal under exami-nation and arbitrary y'
To obtain sufficient conditions for a maximum we have only to change the sense of inequalities.			

OBJECTIVE TYPE QUESTIONS

1. The Functional

$$\int_0^1 (y'^2 + 4y^2 + 8ye^x)\,dx,\ y(0) = -\frac{4}{3},\ y(1) = -\frac{4e}{3}$$

possesses **[GATE, 2012]**

(a) Strong minima on $y = -\frac{e^x}{3}$

(b) Strong minima on $y = -\frac{4e^x}{3}$

(c) Strong maxima on $y = -\frac{e^x}{3}$

(d) Strong maxima on $y = -\frac{4e^x}{3}$

2. The maximum value of the Function $f(x, y, z) = xyz$ subject to the constraint $xy + yz + zx - a = 0$, $a > 0$ is **[Gate, 2012]**

(a) $a^{\frac{3}{2}}$ (b) $\left(\frac{a}{3}\right)^{\frac{3}{2}}$

(c) $\left(\frac{3}{a}\right)^{\frac{3}{2}}$ (d) $\left(\frac{3a}{2}\right)^{\frac{3}{2}}$

3. Extremal of the isoperimetric problem

$$V[y(x)] = \int_1^4 y'^2 dx,\ y(1) = 3,\ y(4) = 24.$$

subject to the condition $\int_1^4 y\,dx = 36$ is

(a) straight line (b) a parabola
(c) a hyperbola (d) a circle

4. Equation $\frac{d}{dx}\left\{f - y'\frac{\partial f}{\partial y'}\right\} - \frac{\partial f}{dx}$ is

(a) Hamiltion's equation
(b) Euler's equation
(c) Liouville's Equation
(d) Bessel's Equation

5. The Solutions of Euler–Poission equation are called as

(a) trial solution
(b) extremals
(c) stationary solution
(d) functional

6. The extremals of the functional

$$\int_0^{\frac{\pi}{2}} (y''^2 - y^2 + x^2)\,dx$$

that satisfies the conditions

$$y(0) = 1,\ y'(0) = 0,\ y\left(\frac{\pi}{2}\right) = 0,\ y'\left(\frac{\pi}{2}\right) = -1 \text{ is}$$

(a) $y(x) = \cos x$ (b) $y(x) = \sin x$
(c) $y(x) = \tan x$ (d) $y(x) = \sec x$

7. A function $y(x)$ such that $\int_0^{\pi} y^2 dx = 1$ which makes $\int_0^{\pi} y''^2 dx = 1$ a minimum if $y(0) = y''(0) = 0$, $y(\pi) = y''(0) = 0$ is

(a) $y(x) = a_n \cos nx$, $n = 0\ 1, 2, 3, 4, \ldots$
(b) $y(x) = a_n \sin mx$, $m = 0, 1, 2, 3, 4, \ldots$
(c) $y(x) = a_n \sin nx$, $n = 0, 1, 2, 3, 4, \ldots$
(d) $y(x) = \frac{a_n}{\sin nx}$, $n = 0, 1, 2, 3, 4, \ldots$

8. The extremals of the functional $\int_{-a}^{a}\left(\lambda y + \frac{1}{2}\mu y''^2\right)dx$ that satisfies the conditions

$y(-a) = 0$, $y'(-a) = 0$, $y(a) = 0$, $y'(a) = 0$ is

(a) $y(x) = -\frac{\lambda}{\mu}(x^2 - a^2)$

(b) $y(x) = -\frac{\lambda}{24\mu}(x^2 - a^2)^2$

(c) $y(x) = \frac{\lambda}{24\mu}(x^2 - a^2)^2$

(d) $y(x) = -\frac{\lambda}{24\mu}(x^2 - a^2)$

9. For the boundary value problem $y''^2 + \pi^2 y = x,\ y(0) = 1,\ y(1) = -0.9$ value of $y(0.5)$ is if trial Solution is

$$y = 1 - 1.9x + c_1 x(1-x) + c_2 x^2(1-x).$$

(a) 0.05 (b) 0.005
(c) 0.50 (d) 0.55

10. Extremals of the isoperimetric problem $V[y(x)] = \int_{x_0}^{x_1} y'^2 dx$ given that $\int_{x_0}^{x_1} y dx = C$, a constant is

(a) $y(x) = \lambda x^2 + ax + b$
(b) $y(x) = \lambda x + b$
(c) $y(x) = b$
(d) $y(x) = \lambda x + bx^2$

11. Function $y(x)$ for which $\int_0^1 (x^2 + y'^2)\, dx$ is stationary, given that $\int_0^1 y^2\, dx = 2;\ y(0) = 0$ is

(a) $y(x) = \sin m\pi x$
(b) $y(x) = \pm 2 \sin m\pi x$
(c) $y(x) = \pm 4 \sin m\pi x$
(d) $y(x) = \pm 3 \sin m\pi x$

12. Solid figure of revolution which, for a given surface area, has maximum volume

(a) a circle (b) a sphere
(c) a ellipse (d) a parabola

13. Plane curve of fixed perimeter and maximum area is a

(a) circle (b) parabola
(c) straight line (d) hyperbola

14. The Solution of variational problem

$$\delta \int_1^2 \left[x^2 (y')^2 + 2y(x+y)\right] dx = 0, \text{ given}$$

$y(1) = y(2) = 0$

(a) $y(x) = \dfrac{1}{21}\left\{8 \log 2(x^2 - x^{-1}) + 7x \log x\right\}$

(b) $y(x) = \dfrac{1}{21}\left\{8 \log 3(x^{-2} - x) + 7x \log x^2\right\}$

(c) $y(x) = \dfrac{1}{21}\left\{8 \log 2\, (x^{-2} - x) + 7x \log x\right\}$

(d) None of the above

15. Curve on which the functional

$$\int_0^1 \left[(y')^2 - y^2 + 2xy\right] dy = 0 \text{ with}$$

$$y(0) = y\left(\frac{\pi}{2}\right) = 0, \text{ be extremized is}$$

(a) $y(x) = \dfrac{\pi}{2} \sin x$

(b) $y(x) = x - \dfrac{\pi}{2} \sin x$

(c) $y(x) = x^2 - \dfrac{\pi}{2} \sin x$

(d) $y(x) = -\dfrac{\pi}{2} \sin x$

16. Curve on which the functional $\int_0^1 \left[(y')^2 + 12xy\right] dy = 0$ with $y(0) = 0$ and $y(1) = 1$, be extremized is

(a) $y(x) = x^3$ (b) $y(x) = x^2$
(c) $y(x) = x$ (d) $y(x) = x^4$

17. An integral $\int_{x_1}^{x_2} f(x, y, y')\, dx$, such as, which assumes a definite value for functions of the type $y = y(x)$ is called as

(a) Euler's equation
(b) Extremum
(c) Functional
(d) Stationary function

18. The necessary condition for the integral $\int_{x_1}^{x_2} H\, dx$ to be an extremum is

(a) $\dfrac{\partial H}{\partial y} - \dfrac{\partial}{\partial x}\left(\dfrac{\partial H}{\partial y'}\right) = 0$

(b) $\dfrac{\partial H}{\partial x} - \dfrac{\partial}{\partial y}\left(\dfrac{\partial H}{\partial \dot{x}}\right) = 0$

(c) $\dfrac{\partial H}{\partial y'} - \dfrac{\partial}{\partial x}\left(\dfrac{\partial H}{\partial y}\right) = 0$

(d) $\frac{\partial H}{\partial y'} - \frac{d}{dx}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0$

19. Integral $J = \int_{x_1}^{x_2} f(y, y_x, x)\,dx$, with y_x has

(a) extremum values
(b) no Extremum values
(c) no Functional
(d) none of the above

20. Necessary condition for $J = \int_{x_1}^{x_2} F(t, x, \dot{x}, \ddot{x})\,dt$, to be an extremum is that

(a) $\frac{\partial F}{\partial x} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$

(b) $\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$

(c) $\frac{\partial F}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$

(d) $\frac{\partial F}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$

21. For $F\left[y(x)\right] = \int_b^a f[x, y(x), y'(x)]\,dx, F$ is called

(a) functional (b) range
(c) arc length (d) none of the above

22. Euler–lagrange equation is

(a) $\frac{\partial f}{\partial x'} - \frac{d}{dx}\left(\frac{\partial H}{\partial x}\right) = 0$

(b) $\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial H}{\partial y}\right) = 0$

(c) $\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial f}{\partial x'}\right) = 0$

(d) $\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$

23. On the interval [0, 1], Let y be a twice continuously differentiable function which is an external of the functional

$$J[y] = \int_0^1 \frac{\sqrt{1+2y'^2}}{x}\,dx \text{ with } y(0) = 1,\ y(1) = 2.$$

Then, for some arbitrary constant c, y satisfies **[GATE 2011]**

(a) $y'^2(2 - c^2x^2) = c^2x^2$
(b) $y'^2(2 + c^2x^2) = c^2x^2$
(c) $y'^2(1 - c^2x^2) = c^2x^2$
(d) $y'^2(1 + c^2x^2) = c^2x^2$

24. The extremal of the functional

$$\int_0^1 \left[y + x^2 + \frac{y'^2}{4}\right] dx,\ y(0) = 0,\ y(1) = 0$$

is extremized is **[GATE 2009]**

(a) $4(x^2 - x)$ (b) $3(x^2 - x)$
(c) $2(x^2 - x)$ (d) $x^2 - x$

25. Let $y = s(x)$ and $y(x) = s(x) + \varepsilon t(x)$ be two admissible curves of integral and $t(x)$ is an arbitrary function of x, independent of ε then the variation y is referred as

(a) strong variation
(b) weak variation
(c) open arc
(d) none of these

26. Let $y = s(x)$ and $y(x) = s(x) + \varepsilon t(x)$ be two admissible curves of integral, $I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx$, where ε is an arbitrary independent of y and x, $t(x)$ is an arbitrary function of x, independent of ε. with this restriction on $t(x)$, the ordinate y is referred as

(a) strong variation
(b) weak variation
(c) open arc
(d) none of these

27. The Solution to Euler's characteristic equation is referred as

(a) Extremals (b) Zeros
(c) Nulls (d) None of these

28. The integral $I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx$, whose end points are fixed, is stationary for weak variations, if y satisfies

(a) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

(b) $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=F$

(c) $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\geq 0$

(d) None of these

29. The integral $I=\int_a^b F(y, y')\,dx$, whose end points are fixed, is stationary for weak variations, if y satisfies

(a) $F-y'\left(\frac{\partial F}{\partial y'}\right)=C$, where C is an arbitrary constant

(b) $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$

(c) $F+y'\left(\frac{\partial F}{\partial y'}\right)=C$, where C is an arbitrary constant

(d) None of these

30. The functional $\int_0^1 (1+x)\,y^2 dx,\ y(0)=0,\ y(1)=1$ possesses **[GATE 2007]**

(a) Strong maxima (b) Strong minima

(c) Weak maxima but not Strong maxima

(d) Weak minima but not Strong minima

ANSWERS

1.(b)	2.(b)	3.(b)	4.(b)	5.(b)	6.(a)	7.(c)	8.(c)	9.(b)	10.(a)
11.(b)	12.(b)	13.(a)	14.(c)	15.(b)	16.(a)	17.(c)	18.(a)	19.(b)	20.(b)
21.(a)	22.(c)	23.(a)	24.(a)	25.(b)	26.(b)	27.(a)	28.(a)	29.(a)	30.(b)

15 Integral Equations

15.1 INTRODUCTION

Integral equations form one of the most useful techniques in many branches of pure analysis, such as the theories of functional analysis and stochastic process. It is one of the most important branches of mathematical analysis, particularly on account of its importance in boundary value problems in the theory of ordinary and partial differential equations. Integral equations occur in many field of mechanics and mathematical physics. They are also related with the problems in mechanical vibrations, theory of analytic function orthogonal systems, theory of quadratic forms of infinitely many variables. Integral equations arise in several problems of science and technology and may be obtained directly from physical problems, e.g., radiation transfer problem and neutron diffusion problem etc. They also arise as representation formulae for the solutions of differential equations. The differential equations can be replaced by an integral equation with the help of initial and boundary conditions. As such, each solution of the integral equation automatically satisfies these boundary conditions.

Integral equations have been encountered in mathematics for a number of years, originally in the theory of Fourier integrals. One of the first results which can be connected to integral equations were the Fourier inversion formulae

$$\phi(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(\xi) \cos \xi x \, d\xi \tag{1}$$

$$u(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi(\xi) \cos \xi x \, d\xi \tag{2}$$

It can be regarded that the Eq. (2) gives a solution of the integral Eq. (1), where $u(x)$ is an unknown function and $\phi(x)$ is a known function.

In 1826, another integral equation was obtained by Abel: A material point under the action of force of gravity moves in the vertical plane (ξ, η) along some smooth curve. It is required to define the curve such that material point, starting from rest at a point $P(x, y)$ of the curve with coordinate y, reaches the point $Q(\xi, \eta)$ at any instant t. The absolute value of velocity V of the moving point is as follows:

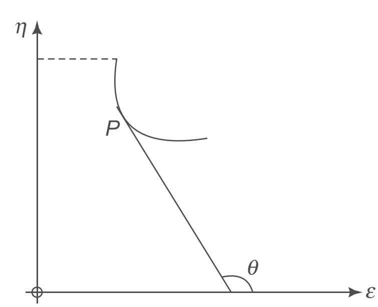


Fig. 15.1

$$V = \sqrt{2g(y - \eta)}$$

$$\frac{d\eta}{dt} = -\sqrt{2g(y-\eta)}\,\sin\theta,$$

where θ is the angle of slope of the tangent with ξ-axis

$$\Rightarrow \qquad dt = -\frac{d\eta}{\sqrt{2g(y-\eta)}\,\sin\theta}.$$

By integrating the limits from 0 to y, we have

$$\int_0^y \frac{\phi(\eta)\,d\eta}{\sqrt{y-\eta}} = -\sqrt{2g}\; f(y),$$

where $\phi(\eta) = (\sin\theta)^{-1}$ $F(y) = \int_a^y (y-\eta)^{-\frac{1}{2}}\,\phi(\eta)\,d\eta,\, F(a) = 0$ known as the Abel's integral equation, where $F(y)$ is a given continuous function and $\phi(\eta)$ is the unknown function. The time of descent can be computed, if the path is known.

The actual development of the theory of integral equations began only at the end of 19$^{\text{th}}$ century due to the works of the Italian mathematician V. Volterra (1896), and principally to the year 1900, in which the Swedish mathematician I. Fredholm published his work on the method of solution for the Dirichlet Problem.

15.2 INTEGRAL EQUATION

An integral equation is an equation in which an unknown function, to be determined, appears under one or more integral signs. If the derivatives of the functions are involved, it is called an integro–differential equation.

An integral equation is called linear if only linear operations are performed in it upon the known functions, i.e., the equation in which no non-linear functions of the unknown function are involved, e.g.,

$$\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi \tag{3}$$

$$\phi(x) = \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi \tag{4}$$

is a linear integral equation.

And
$$\phi(x) = \lambda \int_a^b K(x,\xi)\,[\phi(\xi)]^3\,d\xi \tag{5}$$

is a non-linear integral equation.

Here the function $\phi(x)$ in the Eqs. (3), (4) and (5) is the unknown function while all the other functions are known. These functions may be complex valued functions of the real variables x and ξ.

An equation of the form

$$\alpha(x)\,\phi(x) + F(x) + \lambda \int_\Omega K(x,\xi)\,\phi(\xi)\,d\xi = 0$$

is called the linear integral equation, where $\phi(x)$ is the unknown function; $\alpha(x)$, $F(x)$ and the kernel of the integral equation $K(x, \xi)$ are known functions, λ is a non-zero real or complex parameter, and the integration extends over the domain Ω of the auxiliary variable ξ.

Integral equations, which are linear, involve the integral operator.

$$L = \int_{\Omega} K(x, \xi)\, d\xi$$

having the kernel $K(x, \xi)$. It satisfies the linearity condition

$$L\left[C_1\phi_1(\xi)+C_2\phi_2(\xi)\right]=C_1L\left[\phi_1\ (\xi)\right]+C_2L\left[\phi_2(\xi)\right],$$

where $L\left[\phi(\xi)\right]=\int_{\Omega} K(x, \xi)\, \phi(\xi)\, d\xi$ and C_1, C_2 are constants.

Linear integral equations are classified into the following two basic types:

I. *Volterra Integral Equation* An integral equation is said to be a Volterra integral equation, if the upper limit of integration is a variable, e.g.,

$$\alpha(x)\, \phi(x) = F(x) + \lambda \int_a^x K(x, \xi)\, \phi(\xi)\, d\xi.$$

(i) When $\alpha = 0$, the unknown function ϕ appears only under the integral sign and nowhere else in the equation, then

$$F(x) = \lambda \int_a^x K(x, \xi)\, \phi(\xi)\, d\xi, \quad a > -\infty$$

is called the Volterra's integral equation of first kind.

(ii) When $\alpha = 1$, the equation involves the unknown function ϕ, both inside as well as outside the integral sign then

$$\phi(x) = F(x) + \lambda \int_a^x K(x, \xi)\, \phi(\xi)\, d\xi$$

is called the Volterra's integral equations of second kind.

(iii) When $\alpha = 1$, $F(x) = 0$, the equation reduces to

$$\phi(x) = \lambda \int_a^x K(x, \xi)\, \phi(\xi)\, d\xi$$

which is called the homogeneous Volterra's integral equation of second kind.

II. *Fredholm Integral Equations* An integral equation is said to be Fredholm integral equation, if the domain of integration Ω is fixed, e.g.,

$$\alpha(x)\, \phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi.$$

(i) When $\alpha \equiv 0$, the equation involves the unknown function ϕ only under the integral sign, then

$$F(x) = \lambda \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi, \qquad a \le x \le b$$

is called the Fredholm integral equation of first kind.

(ii) When $\alpha \equiv 1$, the equation involves the unknown function ϕ both inside as well as outside the integral sign, then

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi, \qquad a \le x \le b$$

is called the non-homogeneous Fredholm integral equation of second kind.

(iii) When $\alpha \equiv 1$, $F(x) = 0$, the equation reduces to

$$\phi(x) = \lambda \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi, \qquad a \le x \le b$$

called as the homogeneous Fredholm integral equation of second kind.

Linear integral equations of the first and second kind are the particular cases of the linear integral equations of the third kind, (which are of the form I and II) where α is not a constant but is a prescribed function of x, i.e.,

$$\alpha(x)\, \phi(x) = F(x) + \lambda \int_\Omega K(x, \xi)\, \phi(\xi)\, d\xi$$

III. *Singular Integral Equation* When one or both limits of integration become infinite or the kernel becomes infinite at one or more points within the range of integration, the integral equation is called singular integral equation. For example

$$\phi(x) = f(x) + \lambda \int_{-\infty}^{\infty} \exp\left\{-|x-\xi|\right\} \phi(\xi)\, d\xi$$

and

$$f(x) = \int_a^x (x-\xi)^{-\alpha}\, \phi(\xi)\, d\xi \qquad 0 < \alpha < 1$$

are called the singular integral equation. The second equation represents the Abel's integral equations for $\alpha = -\dfrac{1}{2}$.

IV. *Non-linear Integral Equation* If the unknown function appears under an integral sign to a power $n(n > 1)$, then the equation is said to be a non-linear integral equation. For example,

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, \phi^n(\xi)\, d\xi$$

V. *Convolution Integral* If the kernel $K(x, \xi)$ of the integral equation is defined as a function of the difference $(x - \xi)$, i.e.,

$$K(x, \xi) = K(x - \xi)$$

where K is a certain function of one variable, then the integral equation.

$$\phi(x) = F(x) + \lambda \int_a^x K(x-\xi)\, \phi(\xi)\, d\xi \tag{6}$$

and the corresponding Fredholm equation

$$\phi(x) = F(x) + \lambda \int_a^b K(x-\xi)\, \phi(\xi)\, d\xi \tag{7}$$

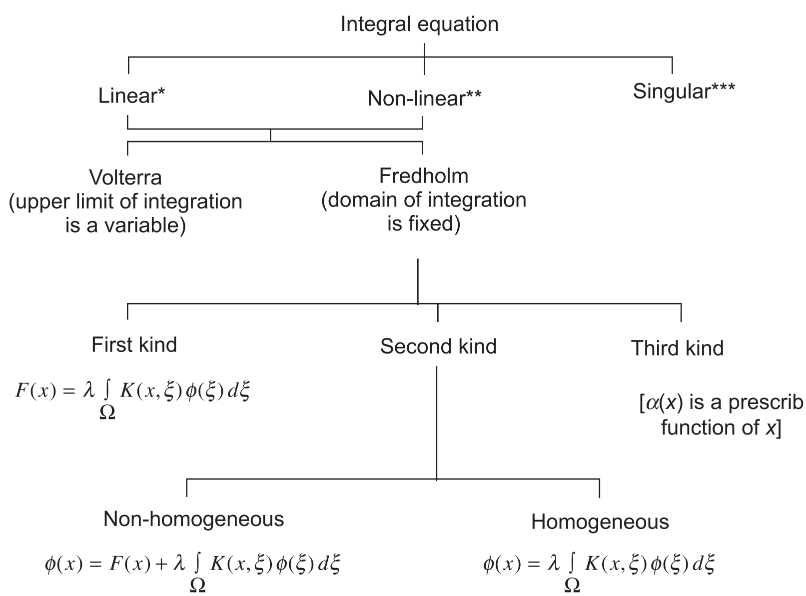
are called integral equation of the convolution type.

The function defined by the integral

$$\int_0^x K(x-\xi)\, \phi(\xi)\, d\xi = \int_0^x K(\xi)\, \phi(x-\xi)\, d\xi \tag{8}$$

is called the convolution or the faltung of the two functions K and ϕ and is known as the convolution integrals. In general, the convolution integrals may be defined as

$$\int_{-\infty}^{\infty} K(x-\xi)\, \phi(\xi)\, d\xi = \int_{-\infty}^{\infty} K(\xi)\, \phi(x-\xi)\, d\xi \tag{9}$$



* $$\alpha(x)\, \phi(x) + F(x) + \lambda \int_\Omega K(x, \xi)\, \phi(\xi)\, d\xi = 0$$

** $$\alpha(x)\, \phi(x) + F(x) + \lambda \int_\Omega K(x, \xi)\, \phi^n(\xi)\, d\xi = 0$$

*** $$\alpha(x)\, \phi(x) + F(x) + \lambda \int_{-\infty}^{\infty} \exp\left\{-|x-\xi|\right\} \phi(\xi)\, d\xi = 0$$

15.3 DIFFERENTIATION OF A FUNCTION UNDER AN INTEGRAL SIGN

Consider the function $I_n(x)$ defined by the relation

$$I_n(x) = \int_a^x (x-\eta)^{n-1} f(\eta)\, d\eta \tag{10}$$

where η is a positive integral and a is a constant. We know that

$$\frac{d}{dx}\int_{P(x)}^{Q(x)} F(x,\eta)\, d\eta = \int_P^Q \frac{\partial}{\partial x}\{F(x,\eta)\} d\eta + F[x, Q(x)]\frac{dQ}{dx} - F[x, P(x)]\frac{dP}{dx}$$

which is valid if F and $\dfrac{\partial F}{\partial x}$ are continuous functions of both x, η and the first derivative of $P(x)$ and $Q(x)$ are continuous.

Differentiating Eq. (10) under the integral sign, we have

$$\frac{d\, I_n}{dx} = (n-1)\int_a^x (x-\eta)^{n-2} f(\eta)\, d\eta + \left[(x-\eta)^{n-2} f(\eta)\right]_{\eta=x}$$

$$\frac{d\, I_n}{dx} = (n-1)\, I_{n-1},\ n>1 \tag{11}$$

From the Eq. (10), we have

$$I_1(x) = \int_a^x f(\eta)\, d\eta \Rightarrow \frac{d\, I_1}{dx} = f(x) \tag{12}$$

Differentiating Eq. (11) successively m times, we have

$$\frac{d^m I_n}{dx^m} = (n-1)(n-2)(n-3)\cdots(n-m)\, I_{n-m},\quad n>m$$

In particular, we have

$$\frac{d^{n-1} I_n}{dx^{n-1}} = (n-1)!\, I_1(x)$$

$$\frac{d}{dx}\left(\frac{d^{n-1} I_n}{dx^{n-1}}\right) = (n-1)!\frac{dI_1}{dx} \Rightarrow \frac{d^n I_n}{dx^n} = (n-1)!\, f(x) \tag{13}$$

Thus, we have

$$I_1(x) = \int_a^x f(x_1)\, dx_1$$

$$\frac{d\, I_2}{dx} = I_1 = \int_a^x f(x_1)\, dx_1$$

$$\Rightarrow \qquad I_2(x) = \int_a^x \int_a^{x_2} f(x_1)\, dx_1\, dx_2$$

In general, we have

$$I_n(x) = (n-1)! \int_a^x \int_a^{x_n} \cdots \int_a^{x_3} \int_a^{x_2} f(x_1)\, dx_1\, dx_2 \cdots dx_{n-1}\, dx_n \tag{14}$$

From the Eqs. (10) and (14), we have

$$\int_a^x \int_a^{x_n} \cdots \int_a^{x_3} \int_a^{x_2} f(x_1)\, dx_1\, dx_2 \cdots dx_{n-1}\, dx_n$$

$$= \frac{1}{(n-1)!} I_n(x) = \frac{1}{(n-1)!} \int_a^x (x-\eta)^{n-1} f(\eta)\, d\eta$$

This may be represented as the result of integrating the function f from a to x and then integrating $(n-1)$ times, we have

$$\int_a^x f(\eta)\, d\eta^n = \int_a^x \frac{(x-\eta)^{n-1}}{(\eta-1)!} f(\eta)\, d\eta \tag{15}$$

15.4 RELATION BETWEEN DIFFERENTIAL AND ITNEGRAL EQUATIONS

There is a fundamental relationship between integral equation and ordinary and partial differential equations with given initial values. Consider the differential equation of n^{th} order as

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + a_2(x)\frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n(x)\, y = F(x) \tag{16}$$

with continuous coefficients $a_i(x)$, $i = 1, 2, 3, \ldots, n$. The initial conditions are prescribed as follows:

$$y(0) = C_0,\ y'(0) = C_1,\ y''(0) = C_2, \ldots, y^{n-1}(0) = C_{n-1}, \tag{17}$$

where the prime denotes differentiation with respect to x.

Consider, $\dfrac{d^n y}{dx^n} = \phi(x)$

By integrating and using the initial conditions from Eq. (17), we have

$$\frac{d^{n-1} y}{dx^{n-1}} = \int_0^x \phi(\xi)\, d\xi + C_{n-1}$$

$$\frac{d^{n-2} y}{dx^{n-2}} = \int_0^x \phi(\xi)\, d\xi^2 + C_{n-1}\, x + C_{n-2}$$

$$\frac{d^{n-n} y}{dx^{n-n}}\, y = y = \int_0^x \phi(\xi)\, d\xi^n + C_{n-1}\frac{x^{n-1}}{(n-1)!} + C_{n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + C_0 \tag{18}$$

where $\int_0^x \phi(\xi)\, d\xi^n$ represents for a multiple integral of order n.

From the Eqs. (16) and (18), we obtain

$$\phi(x)+a_1(x)\int_0^x \phi(\xi)\,d\xi+a_2(x)\int_0^x \phi(\xi)\,d\xi^2+\cdots+a_n(x)\int_0^x \phi(\xi)\,d\xi$$

$$= F(x)+\sum_1^n i\,C_i\;\chi_i(x), \tag{19}$$

where
$$\chi_i(x)=a_i(x)+\frac{x}{1!}a_{i+1}(x)+\cdots+a_n(x)\frac{x^{n-i}}{(x-i)!} \tag{20}$$

$$\phi(x)+\int_0^x\left[a_1(x)+a_2(x)\,(x-\xi)+\cdots+a_n(x)\,\frac{(x-\xi)^{n-1}}{(n-1)!}\right]\phi(\xi)\,d\xi=C_i(x), \tag{21}$$

where
$$C_i(x)=F(x)+\sum_1^n i\,C_i\;\chi_i(x). \tag{22}$$

The Eq. (21) represents the non-homogeneous Volterra's integral equation of second kind.

Particular Case

Consider the linear differential equation of second order

$$\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=F(x) \tag{23}$$

with initial conditions

$$y(0)=C_0 \text{ and } y'(0)=C_1 \tag{24}$$

Consider
$$\frac{d^2y}{dx^2}=\phi(x)$$

By integrating and using the initial conditions in Eq. (24) we have

$$\frac{dy}{dx}=\int_0^x \phi(\xi)\,d\xi+C_1 \tag{25}$$

and
$$y=\int_0^x (x-\xi)\,\phi(\xi)\,d\xi+C_1x+C_0 \tag{26}$$

The given differential equation reduces to

$$\phi(x)+a_1(x)\left[\int_0^x \phi(\xi)\,d\xi+C_1\right]+a_2(x)\left[\int_0^x (x-\xi)\,\phi(\xi)\,d\xi+C_1(x)+C_0\right]y=F(x)$$

or
$$\phi(x)+\int_0^x\left[a_1(x)+a_2(x)\,(x-\xi)\right]\phi\,(\xi)\,d\xi$$

$$= F(x)-C_1a_1(x)-C_1x\,a_2(x)-C_0\;a_2(x)$$

or $$\phi(x) = f(x) + \lambda \int_0^x K(x, \xi)\, \phi(\xi)\, d\xi, \tag{27}$$

where $K(x, \xi) = a_1(x) + a_2(x)(x - \xi),\ \lambda = -1$

$$f(x) = F(x) - C_1\, a_1(x) - C_1 x\, a_2(x) - C_0\, a_2(x) \tag{28}$$

which represents the Volterra's integral equation of the second kind. Similarly, the boundary value problems in ordinary differential equations lead to Fredholm integral equations.

Example 1 Show that the function $\phi(x) = 1 - x$ is a solution of the integral equation

$$\int_0^x e^{x-\xi} \phi(\xi)\, d\xi = x$$

Solution

Since $\phi(x) = 1 - x$

$$\therefore \quad \int_0^x e^{x-\xi} \phi(\xi)\, d\xi = \int_0^x e^{x-\xi}\, (1 - \xi)\, d\xi$$

$$= \int_0^x e^x e^{-\xi} (1 - \xi)\, d\xi$$

$$= e^x \int_0^x e^{-\xi} - e^x \int_0^x \xi\, e^{-\xi} d\xi$$

$$= -e^x \left(e^{-\xi}\right)_0^x + e^x \left(\xi\, e^{-\xi} + e^{-\xi}\right)_0^x$$

$$= e^x \left(1 - e^{-x}\right) + e^x \left(x e^{-x} + e^{-x} - 1\right)$$

$$= x = \text{R.H.S.}$$

Hence, $\phi(x) = 1 - x$ is a solution of the integral equation.

Example 2 Show that the function $\phi(x) = 1$ is a solution of the Fredholm integral equation

$$\phi(x) + \int_0^1 x(e^{x\xi} - 1)\, \phi(\xi)\, d\xi = e^x - x$$

Solution

Since $\phi(x) = 1$

$$\therefore \quad \phi(x) + \int_0^1 x(e^{x\xi} - 1)\, \phi(\xi)\, d\xi = 1 + \int_0^1 x(e^{x\xi} - 1)\, d\xi = 1 + (e^{x\xi} - x\xi)_0^1$$

$$= 1 + (e^x - x - 1) = e^x - x = \text{R.H.S.}$$

Hence, $\phi(x) = 1$ is a solution of the integral equation.

Example 3 Show that the function $\phi(x) = e^x\left[2x - \left(\frac{2}{3}\right)\right]$ is a solution of the Fredholm integral equation

$$\phi(x) + \lambda\int_0^1 e^{x-\xi}\phi(\xi)\, d\xi = 2xe^x,\ \lambda = 2$$

Solution

Since $$\phi(x) = e^x\left[2x - \left(\frac{2}{3}\right)\right]$$

$$\therefore \quad \phi(x) + \lambda\int_0^1 e^{x-\xi}\phi(\xi)\, d\xi = e^x\left(2x - \frac{2}{3}\right) + 2\int_0^1 e^x\left(2\xi - \frac{2}{3}\right) d\xi$$

$$= e^x\left(2x - \frac{2}{3}\right) + 2e^x\left(\xi^2 - \frac{2\xi}{3}\right)_0^1$$

$$= e^x\left(2x - \frac{2}{3} + \frac{2}{3}\right)$$

$$= 2xe^x = \text{R.H.S.}$$

Hence, $\phi(x) = e^x\left[2x - \frac{2}{3}\right]$ is a solution of the given integral equation.

EXERCISE 15.1

1. Show that the function $\phi(x) = \left(1 + x^2\right)^{-\frac{3}{2}}$ is a solution of the Volterra integral equation

$$\phi(x) = \frac{1}{1+x^2} - \int_0^x \frac{\xi}{(1+x^2)}\phi(\xi)\, d\xi$$

2. Show that the function $\phi(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of the integral equation

$$\int_0^x \frac{\phi(\xi)}{\sqrt{(x-\xi)}} d\xi = 1$$

3. Show that the function $\phi(x) = \frac{x}{(1+x^2)^{5/2}}$ is the solution of integral equation

$$\phi(x) = \frac{3x + 2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x + 2x^3 - \xi}{(1+x^2)^2}\phi(\xi)\, d\xi$$

4. Verify that the given function $\phi(x) = \frac{1}{2}$ is the solution of the integral equation

$$\int_0^x \frac{d(\xi)}{\sqrt{(x-\xi)}}\, d\xi = \sqrt{x}$$

5. Verify that the given function $\phi(x) = e^x$ is the solution of the integral equation

$$\phi(x) = 1 - \int_0^1 \sin x\, \xi\, \phi(\xi)\, d\xi$$

Example 4 Form an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} + y = \cos x$$

with initial conditions

$$y(0) = 0,\ y'(0) = 1$$

Solution

Consider $$\frac{d^2y}{dx^2} = \phi(x)$$

Then $$\frac{dy}{dx} = \int_0^x \phi(\xi)\, d\xi + 1,$$

and $$y = \int_0^y (x-\xi)\, \phi(\xi)\, d\xi + x$$

Substituting the given relations in the given differential equation we have

$$\phi(x) + \int_0^x (x-\xi)\, \phi(\xi)\, d\xi + x = \cos x$$

or $$\phi(x) = (\cos x - x) - \int_0^x (x-\xi)\, \phi(\xi)\, d\xi$$

which is the desired integral equation.

Example 5 Form an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

with the initial conditions

$$y(0) = 1,\ y^1(0) = 0$$

Solution

Consider $$\frac{d^2y}{dx^2} = \phi(x)$$

Then $$\frac{dy}{dx} = \int_0^x \phi(\xi)\, d\xi + A$$

$$= \int_0^x \phi(\xi)\, d\xi, \text{ since } y'(0) = 0$$

and $$y = \int_0^x (x-\xi)\, \phi(\xi) + B$$

$$y = \int_0^x (x-\xi)\, \phi(\xi)\, d\xi + 1, \qquad \text{Since } y(0) = 1$$

Substituting the above relations in the given differential equation, we have

$$\phi(x) + x\int_0^x \phi(\xi)\, d\xi + \left\{\int_0^x (x-\xi)\, \phi(\xi)\, d\xi + 1\right\} = 0$$

$$\phi(x) = -1 - \int_0^x (2x-\xi)\, \phi(\xi)\, d\xi$$

which is the desired integral equation.

Example 6 Reduce the initial value problem

$$\phi''(x) + \lambda\phi(x) = F(x)$$

with $$\phi(0) = 1, \phi'(0) = 0$$

Solution

The differential equation is given as

$$\phi''(x) + \lambda\phi(x) = F(x)$$

$$\Rightarrow \qquad \phi''(x) = F(x) - \lambda\phi(x)$$

Integrating both the sides with regard to x, we have

$$\int_0^x \phi''(x)\, dx = \int_0^x \{F(x) - \lambda\phi(x)\} dx$$

$$\Rightarrow \qquad [\phi'(x)]_0^x = \int_0^x \{F(x) - \lambda\phi(x)\} dx$$

$$\Rightarrow \qquad \phi'(x) - \phi'(0) = \int_0^x \{F(x) - \lambda\phi(x)\} dx$$

$$\Rightarrow \qquad \phi'(x) = \int_0^x \{F(x) - \lambda\phi(x)\} dx, \quad \text{Since } \phi'(0) = 0$$

Integrating both the sides with regard to x, we have

$$\int_0^x \phi'(x)\,dx = \int_0^x \left\{F(x) - \lambda\phi(x)\right\} dx^2$$

$$\Rightarrow \qquad \phi(x) - \phi(0) = \int_0^x \left\{F(\xi) - \lambda\phi(\xi)\right\} d\xi^2$$

$$\Rightarrow \qquad \phi(x) = 1 + \int_0^x (x-\xi)\left\{F(\xi) - \lambda\phi(\xi)\right\} d\xi, \quad \text{Since } \phi(0) = 1$$

which reduces to a Volterra's integral equation of second kind.

EXERCISE 15.2

1. From an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

with the initial conditions $y(0) = 0$, $y'(0) = -1$.

2. Form an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} - \sin x\frac{dy}{dx} + e^x y = x$$

with the initial conditions $y(0) = 1$, $y'(0) = -1$.

3. Convert the differential equation

$$y''(x) - 3y'(x) + 2y(x) = 4\sin x$$

with the initial condition $y(0) = 1$, $y'(0) = -2$ into Volterra's integral equation of second kind.

4. Form an integral equation corresponding to the differential equation

$$\frac{d^3y}{dx^3} - 2xy = 0$$

with the initial condition $y(0) = \frac{1}{2}$, $y'(0) = 1 = y''(0)$.

5. Form an integral equation corresponding to the differential equation

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + (x^2 - x)y = xe^x + 1$$

with initial conditions $y(0) = 1 = y'(0)$, $y''(0) = 0$.

Answers

1. $\phi(x) = (6x-5) + \int_0^x (5-6x+6\xi)\,\phi(\xi)\,d\xi$

2. $\phi(x) = \left[x - \sin x + e^x(x-1)\right] + \int_0^x \left[\sin x - e^x(x-\xi)\right]\phi(\xi)\,d\xi$

3. $\phi(x) = 4(-2+x+\sin x) + \int_0^x \left[3-2(x-\xi)\right]\phi(\xi)\,d\xi$

4. $\phi(x) = x(x+1)^2 + \int_0^x x(x-\xi)^2\,\phi(\xi)\,d\xi$

5. $\phi(x) = xe^x + 1 - x(x^2-1) - \int_0^x \left[x + \frac{1}{2}(x^2-x)(x-\xi)^2\right]\phi(\xi)\,d\xi$

Example 7 Show that the integral equation

$$\phi(x) = \left(1 - x + \frac{1}{6}x^3\right) + \int_0^x \left[\sin\varepsilon - (x-\xi)(e^{\varepsilon} + \cos\xi)\right]\phi(\xi)\,d\xi$$

is equivalent to the differential equation

$$\phi''(x) - \sin x\,\phi'(x) + e^x\,\phi(x) = x,\ \phi(0) = 1,\ \phi'(0) = -1$$

Solution

Given
$$\phi(x) = \left(1 - x + \frac{1}{6}x^3\right) + \int_0^x \left[\sin\xi - (x-\xi)(e^{\varepsilon} + \cos\xi)\right]\phi(\xi)\,d\xi \tag{29}$$

Differentiating Eq. (29) with regard to x, we have

$$\phi'(x) = -1 + \frac{1}{2}x^2 + \frac{d}{dx}\int_0^x \left[\sin\varepsilon - (x-\xi)\,(e^{\varepsilon} + \cos\xi)\right]\phi(\xi)\,d\xi$$

$$\Rightarrow \quad \phi'(x) = -1 + \frac{1}{2}x^2 + \int_0^x \frac{\partial}{\partial x}\left[\sin\varepsilon - (x-\xi)\,(e^{\varepsilon} + \cos\xi)\right]\phi(\xi)\,d\xi + \left[\sin x - (x-x)(e^x + \cos x)\right]\phi(x)\cdot 1$$

$$\Rightarrow \quad \phi'(x) = -1 + \frac{1}{2}x^2 - \int_0^x (e^{\varepsilon} + \cos\xi)\,\phi(\xi)\,d\xi + \sin x\,\phi(x) \tag{30}$$

Differentiating Eq. (30) with regard to 'x' we have

$$\phi''(x) = x - \frac{d}{dx}\int_0^x (e^{\xi} + \cos\xi)\,\phi(\xi)\,d\xi + \cos x\,\phi(x) + \sin\phi'(x)$$

$$\Rightarrow \qquad \phi''(x) = x - \left[\int_0^x \frac{\partial}{\partial x}\left\{(e^{\xi} + \cos\xi)\,\phi(\xi)\right\} d\xi + (e^{x} + \cos x)\,\phi(x)\cdot 1\right] + \cos x\,\phi(x) + \sin x\,\phi'(x)$$

$$\Rightarrow \qquad \phi''(x) = x - (e^{x} + \cos x)\,\phi(x) + \cos x\,\phi(x) + \sin x\,\phi'(x)$$

$$\Rightarrow \qquad \phi''(x) - \sin x\,\phi'(x) + e^{x}\phi(x) = x$$

which is the required differential equation. Putting $x = 0$ in the Eqs. (29) and (30), we have

$$\phi(0) = 1 \text{ and } \phi'(0) = -1$$

Hence proved.

Example 8 Show that the integral equation

$$\phi(x) = (1 - x - 4\sin x) + \int_0^x \left[3 - 2(x-\xi)\right]\phi(\xi)\,d\xi$$

is equivalent to the differential equation

$$\phi''(x) - 3\phi'(x) + 2\phi(x) = 4\sin x,\ \phi(0) = 1,\ \phi'(0) = -2$$

Solution

Given

$$\phi(x) = (1 - x - 4\sin x) + \int_0^x \left[3 - 2(x-\xi)\right]\phi(\xi)\,d\xi \tag{31}$$

Differentiating Eq. (31) with regard to 'x' we have

$$\phi'(x) = -1 - 4\cos x + \frac{d}{dx}\int_0^x \left[3 - 2(x-\xi)\right]\phi(\xi)\,d\xi$$

$$\Rightarrow \qquad \phi'(x) = -1 - 4\cos x + \int_0^x \frac{\partial}{\partial x}\left[3 - 2(x-\xi)\right]\phi(\xi)\,d\xi + \left[3 - 2(x-x)\right]\phi(x)\cdot 1$$

$$\Rightarrow \qquad \phi'(x) = -1 - 4\cos x + 3\phi(x) - 2\int_0^x \phi(\xi)\,d\xi \tag{32}$$

Differentiating Eq. (32) with regard to 'x' we have

$$\phi''(x) = 4\sin x + 3\phi'(x) - 2\frac{d}{dx}\int_0^x \phi(\xi)\,d\xi$$

$$\Rightarrow \qquad \phi''(x) = 4\sin x + 3\phi'(x) - 2\phi(x)$$

$$\Rightarrow \qquad \phi''(x) - 3\phi'(x) + 2\phi(x) = 4\sin x$$

which is the required differential equation. Putting $x = 0$ in Eqs. (31) and (32), we have

$$\phi(0) = 1, \quad \phi'(0) = -2$$

Hence, proved.

EXERCISE 15.3

1. Show that the integral equation

$$\phi(x) = \int_0^x (x+\xi)\,\phi(\xi)\,d\xi + 1$$

is equivalent to the differential equation

$$\phi''(x) - 2x\phi'(x) - 3\phi(x) = 0, \phi(0) = 1, \phi'(0) = 0.$$

2. Show that the integral equation

$$\phi(x) = \int_0^x \xi(\xi - x)\,\phi(\xi)\,d\xi + \frac{1}{2}x^2$$

is equivalent to the differential equation

$$\phi''(x) + xy = 1,\ \phi(0) = \phi'(0) = 0.$$

15.5 SOLUTION OF NON-HOMOGENOUS VOLTERRA'S INTEGRAL EQUATION OF SECOND KIND BY THE METHOD OF SUCCESSIVE SUBSTITUTION

Consider the Volterra's integral equation of second kind as

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\,\phi(\xi)\,d\xi, \tag{33}$$

where

(i) the kernel $K(x, \xi) \not\equiv 0$ is real and continuous in the rectangle R: $a \le x \le b$, $a \le \xi \le b$. Consider $|K(x,\xi)| \le$ where P is the maximum value in R.

(ii) the function $F(x) \neq 0$ is real and continuous in an interval $a \le x \le b$. Consider $|F(x)| \le Q$, where Q is the maximum value in the interval.

(iii) λ is a non-zero numerical parameter.

Substituting the unknown function $\phi(\xi)$ under an integral sign from the Eq. (33) itself, we have

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\left[F(\xi) + \lambda \int_a^{\xi} K(\xi,\xi_1)\,\phi(\xi_1)\,d\xi_1\right] d\xi$$

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\,F(\xi) + \lambda^2 \int_a^x K(x,\xi) \int_a^{\varepsilon} K(\xi,\xi_1)\,\phi(\xi_1)\,d\xi_1\,d\xi$$

Performing the operation successively for $\phi(\xi)$, we have

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\, F(\xi)\, d\xi + \lambda^2 \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1)$$

$$\left[F(\xi_1)\ \lambda \int_a^{\xi_1} K(\xi,\xi_2)\, \phi(\xi_2)\, d\xi_2 \right] d\xi_1\, d\xi$$

$$\Rightarrow \qquad \phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\, F(\xi)\, d\xi + \lambda^2 \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1)\, F(\xi_1)$$

$$d\xi_1\, d\xi + \lambda^3 \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1) \int_a^{\xi_1} K(\xi_1,\xi_2)\, \phi(\xi_2)\, d\xi_2\, d\xi_1\, d\xi + \cdots$$

In general, we have

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\, F(\xi)\, d\xi +$$

$$\lambda^2 \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1)\, F(\xi_1)\, d\xi_1\, d\xi +$$

$$\lambda^3 \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1) \int_a^{\xi_1} K(\xi_1,\xi_2)\, F(\xi_2)\, d\xi_2\, d\xi_1\, d\xi + \cdots +$$

$$\lambda^n \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1) \int_a^{\xi_1} K(\xi_1,\xi_2) \cdots \int_a^{\xi_{n-2}} K(\xi_{n-2},\xi_{n-1})\, F(\xi_{n-1})$$

$$d\xi_{n-1} \cdots d\xi_1\, d\xi + \lambda^{n+1} \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1) \int_a^{\xi_1} K(\xi_1,\xi_2) \cdots$$

$$\int_a^{\xi_{n-1}} K(\xi_{n-1},\xi_n) F(\xi_n)\, d\xi_n \cdots d\xi_1\, d\xi \tag{34}$$

Consider the infinite series

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\, F(\xi)\, d\xi + \lambda^2 \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1)\, F(\xi_1)$$

$$d\xi_1\, d\xi + \lambda^3 \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1) \int_a^{\xi_1} K(\xi_1,\xi_2)\, \phi(\xi_2)\, d\xi_2\, d\xi_1\, d\xi + \cdots + \cdots \tag{35}$$

Let $$S_n(x) = \lambda^n \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1) \cdots \int_a^{\xi_{n-2}} K(\xi_{n-2},\xi_{n-1})$$

$$F(\xi_{n-1})\, d\xi_{n-1}\, d\xi_{n-2} \cdots d\xi_1\, d\xi$$

$$\left|S_n(x)\right| = \left|\lambda^n\right| \left|\int_a^x K(x,\xi)\right| \left|\int_a^{\xi} K(\xi,\xi_1)\right| \cdots \left|\int_a^{\xi_{n-2}} K(\xi_{n-2},\xi_{n-1})\right|$$

$$\left|F(\xi_{n-1})\right| \left|d\xi_{n-1}\right| \cdots \left|d\xi_1\right| \left|d\xi\right|$$

Since $\left|K(x,\xi)\right| \le P$ and $\left|F(x)\right| \le Q$

then $\left|S_n(x)\right| \le \left|\lambda^n\right| Q P^n \dfrac{(x-a)^n}{n!}$

$$\left|S_n(x)\right| \le \left|\lambda^n\right| Q \frac{[P(b-a)]^n}{n!}$$

It follows that the series is convergent for all values of λ, P, Q, $(b-a)$ and hence, the series Eq. (35) is absolutely and uniformly convergent.

Again, $\left|S_{n+1}(x)\right| \le \left|\lambda^{n+1}\right| M P^{n+1} \dfrac{(x-a)^{n+1}}{(n+1)!}$

$$\left|S_{n+1}(x)\right| \le \left|\lambda^{n+1}\right| M \frac{[P(b-a)]^{n+1}}{(n+1)!},$$

where M is the maximum value of the absolute value of the function $\phi(x)$ which is continuous in the interval I

$$a \le x \le b$$

$$\lim_{n\to\infty} S_{n+1}(x) = 0.$$

We notice that the function $\phi(x)$ which satisfies the Eq. (34) is the continuous function given by the infinite series in Eq. (35). The integral Eq. (33) has a unique continuous solution in the interval I.

15.6 SOLUTION OF NON-HOMOGENEOUS VOLTERRA'S INTEGRAL EQUATION OF SECOND KIND BY THE METHOD OF SUCCESSIVE APPROXIMATION

A Volterra integral equation of second kind

$$\phi(x) = F(x) + \lambda \int_0^x K(x,\xi)\,\phi(\xi)\,d\xi$$

has one and only one solution, given by the relation

$$\phi(x) = F(x) + \lambda \int_0^x K(x,\xi;\lambda)\,F(\xi)\,d\xi,$$

where the resolvent kernel $R(x, \xi; \lambda)$ is the sum of the series

$$R(x,\xi;\lambda) = K(x,\xi) + \sum_{V=1}^{\infty} \lambda^V K_V(x,\xi)$$

Convergent for all values of λ.

Consider the Volterra integral equation of second kind

$$\phi(x) = F(x) + \lambda \int_0^x K(x, \xi)\, \phi(\xi)\, d\xi \tag{36}$$

where the kernel $K(x, \xi)$ is a continuous function for $0 \le x \le a$, $0 \le \xi \le a$, and the function $F(x)$ is continuous for $0 \le x \le a$.

Consider an infinite power series in ascending powers of λ a

$$\phi(x) = \phi_0(x) + \lambda\, \phi_1(x) + \lambda^2 \phi_2(x) + \cdots + \lambda^n \phi_n(x) + \cdots \tag{37}$$

Let the series in Eq. (37) is a solution of the Eq. (36) then

$$\phi_0(x) + \lambda\phi_1(x) + \lambda^2 \phi_2(x) + \cdots + \cdots + \lambda^n \phi_n(x) + \cdots$$

$$= F(x) + \lambda \int_0^x K(x, \xi) \left[\phi_0(\xi) + \lambda\phi_1(\xi) + \lambda^2 \phi_2(\xi) + \cdots + \lambda^n\ \phi_n\ (\xi) \right] d\xi \tag{38}$$

Equating the coefficients of like power of λ, we get

$$\phi_0(x) = F(x)$$

$$\phi_1(x) = \int_0^x K(x, \xi)\, \phi_0(\xi)\, d\xi$$

$$\phi_2(x) = \int_0^x K(x, \xi)\, \phi_1(\xi)\, d\xi$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\phi_n(x) = \int_0^x K(x, \xi)\, \phi_{n-1}\ (\xi)\, d\xi \tag{39}$$

Thus, it yields a method for a successive approximation of the function $\phi_n(x)$. It may be shown that the series in Eq. (37) conveys uniformly in x and λ for any λ and $\chi \in [0, a]$, under these assumptions with regard to $F(x)$ and $K(x, \xi)$, its sun is a unique solution in Eq. (36). Further, from Eq. (39) it follows that

$$\phi_1(x) = \int_0^x K(x, \xi)\ F(\xi)\, d\xi$$

$$\phi_2(x) = \int_0^x K(x, \xi) \left\{ \int_0^\xi K(\xi, \xi_1)\ F(\xi_1)\, d\xi_1 \right\} d\xi$$

Here $\qquad \xi_1 = 0,\ \ \xi_1 = \xi;\ \ \xi = 0,\ \ \xi = x$

By interchanging the order of integration, we have

$$\phi_2(x) = \int_0^x F(\xi_1)\, d\xi_1 \left\{ \int_{\xi_1}^x K(x, \xi)\, K(\xi_0, \xi_1)\, d\xi \right\}$$

$$\Rightarrow \qquad \phi_2(x) = \int_0^x K_2(x, \xi_1)\, F(\xi_1)\, d\xi_1, \tag{40}$$

where $$K_2(x, \xi_1) = \int_{\xi_1}^x K(x, \xi)\, K(\xi, \xi_1)\, d\xi. \tag{41}$$

In general we have

$$\phi_n(x) = \int_0^x K_n(x, \xi)\, F(\xi)\, d\xi, \qquad n = 1, 2, \ldots \tag{42}$$

The functions $K_n(x, \xi)$ are called iterated kernels, which can readily be shown that

$$K_1(x, \xi) = K(x, \xi)$$

and $K_2(x, \xi)$, $K_3(x, \xi)$ etc., are defined recursively by the formula

$$K_{n+1}(x, \xi) = \int_{\xi}^x K(x, z)\, K_n\,(z, \xi)\, dz, \qquad n = 1, 2, \ldots \tag{43}$$

Equation (37) represents the solution of Eq. (36) can therefore be written as follows:

$$\phi(x) = F(x) + \sum_{V=1}^{\infty} \lambda^V \int_0^x K_V(x, \xi)\, F(\xi)\, d\xi \tag{44}$$

where $$R(x, \xi; \lambda) = K_1(x, \xi) + \lambda\, K_2(x, \xi) + \lambda^2\, K_3(x, \xi) + \cdots + \lambda^{n-1}\, K_n(x, \xi) + \lambda^n\, K_{n+1}(x, \xi) + \cdots \tag{45}$$

$$\Rightarrow \qquad R(x, \xi; \lambda) = \sum_{V=0}^{\infty} \lambda^V\, K_{V+1}\,(x, \xi)$$

The function $R(x, \xi; \lambda)$ is called the resolvent kernel or reciprocal kernel of Eq. (36).

The series converges absolutely and uniformly in the case of a continuous kernel $K(x, \xi)$. Iterated kernels and the resolvent kernel do not depend on the lower limit of an integral equation. The resolvent kernel $R(x, \xi; \lambda)$ satisfies the functional equation

$$R(x, \xi; \lambda) = K(x, \xi) + \lambda \int_{\xi}^x K(x, z)\, R(z, \xi; \lambda)\, dz$$

Thus, the solution of the Eq. (36) reduces to

$$\phi(x) = F(x) + \lambda \int_0^x R(x, \xi; \lambda)\, F(\xi)\, d\xi.$$

15.7 DETERMINATION OF SOME RESOLVENT KERNELS

(a) Consider that the kernel $K(x, \xi)$ is a polynomial of degree $(n-1)$ in ξ such that it may be expressed in the form

$$K(x,\xi) = a_0(x) + a_1(x)(x-\xi) + a_2(x)\frac{1}{2!}(x-\xi)^2 + \cdots + \frac{a_{n-1}(x)}{(n-1)!}(x-\xi)^{n-1} + \cdots + \tag{46}$$

where the coefficients $\sum_{V=0}^{n-1} a_V(x)$ are continuous in the interval $[0, a]$.

Let the auxiliary function be

$$\phi(x,\xi;\lambda) = \frac{1}{(x-1)!}(x-\xi)^{n-1} + \lambda\int_{\xi}^{x} R(t,\xi;\chi)\frac{(x-t)^{n-1}}{(n-1)!}dt \tag{47}$$

with the conditions

$$\phi = \frac{d\phi}{dx} = \cdots = \frac{d^{n-2}\phi}{dx^{n-2}} = 0 \text{ at } x = \xi \text{ and } \frac{d^{n-1}\phi}{dx^{n-1}} = 1 \text{ at } x = \xi. \tag{48}$$

In addition, we have

$$R(x,\xi;\lambda) = \frac{1}{\lambda}\frac{d^n}{dx^n}\phi(x,\xi;\lambda). \tag{49}$$

Since the resolvent kernel satisfies the functional equation

$$R(x,\xi;\lambda) = K(x,\xi) + \lambda\int_{\xi}^{x} K(x,z)\,R(z,\xi;\lambda)\,dz \tag{50}$$

From Eqs. (49) and (50), we have

$$\frac{d^n}{dx^n}\phi(x,\xi;\lambda) = \lambda\,K(x,\xi) + \lambda^2\int_{\xi}^{x} K(x,z)\cdot\frac{1}{\lambda}\frac{d^n}{dz^n}\phi(z,\xi;\lambda)\,dz \tag{51}$$

$$\frac{d^n}{dx^n}\phi(x,\xi;\lambda) = \lambda K(x,\xi) + \lambda\left[K(x,z)\cdot\frac{d^{n-1}\phi}{dz^{n-1}} - \frac{\partial K(x,z)}{\partial z}\frac{d^{n-2}\phi}{dz^{n-2}} + \cdots + \frac{d^{n-1}K}{dz^{n-1}}\phi\right]_{z=\xi}^{x} \tag{52}$$

Using Eqs. (46) and (48), the relation in Eq. (52) reduces to

$$D\phi = \frac{d^n\phi}{dx^n} - \lambda\left[a_0(x)\frac{d^{n-1}\phi}{dx^{n-1}} + a_1(x)\frac{d^{n-2}\phi}{dx^{n-2}} + \cdots + a_{n-1}(x)\phi\right] = 0. \tag{53}$$

The function $\phi(x, \xi; \lambda)$ is therefore the integral of the linear equation $D\phi = 0$ which satisfies the Cauchy conditions. Thus, we have an expression for the resolvent kernel as

$$R(x,\xi;\lambda) = \frac{1}{\lambda}\frac{d^n}{dx^n}\phi(x,\xi;\lambda) \tag{54}$$

(b) Further, assume that the kernel $K(x, \xi)$ is a polynomial of degree $(n - 1)$ in x such that it may be expressed in this form

$$K(x,\xi) = b_0(\xi) + b_1(\xi)(\xi - x) + b_2(\xi)\frac{1}{2!}(\xi - x)^2 + \cdots + \frac{b_{n-1}(\xi)}{(x-1)!}(\xi - x) + \cdots \tag{55}$$

where the coefficients $b_V(\xi)$ are continuous in interval [0, a].

Consider
$$R(x, \xi; \lambda) = -\frac{1}{\lambda}\frac{d^n\phi}{d\xi^n}(\xi, x; \lambda). \tag{56}$$

The auxiliary function $\phi(x, \xi; \lambda)$ satisfies the following conditions:

$$\phi = \frac{d\phi}{d\xi} = \cdots = \frac{d^{n-2}\phi}{d\xi^{n-2}} = 0 \text{ at } \xi = x; \ \frac{d^{n-1}\phi}{d\xi^{n-1}} = 1 \text{ at } \xi = x. \tag{57}$$

Therefore, the functional relation reduces to

$$\frac{d^n\phi}{d\xi_n} = -\lambda K(x, \xi) + \lambda \int_{\xi}^{x} K(z, \xi)\frac{d^n}{dt^n}\phi(x, z; \lambda)\, dt \tag{58}$$

Using the Eqs. (56) and (57) and integrating by parts to the integral on R.H.S, we have

$$D_1\phi = \frac{d^n\phi}{d\xi^n} + \lambda\left[b_0(\xi)\frac{d^{n-1}\phi}{d\xi^{n-1}} + b_1(\xi)\frac{d^{n-2}\phi}{d\xi^{n-2}} + \cdots + b^{n-1}(\xi)\phi\right] = 0 \tag{59}$$

Thus the auxiliary function $\phi(x, \xi; \lambda)$ is the integral of the linear equation $D_1\phi = 0$ which satisfies the Cauchy conditions.

$$R(x, \xi; \lambda) = -\frac{1}{\lambda}\frac{d^n}{d\xi^n}\phi(\xi, x; \lambda) \tag{60}$$

Example 9 Find the resolvent Kernel of the Volterra integral equation with Kernel: $K(x, \xi) = 1$.

Solution

We know that

$$K_1(x, \xi) = K(x, \xi) = 1$$

By the iterated Kernels, we have

$$K_2(x, \xi) = \int_{\xi}^{x} K(x, z)\, K_1(z, \xi)\, dz$$

$$\Rightarrow \quad K_2(x, \xi) = \int_{\xi}^{x} dz = (z)_{\xi}^{x} = (x - \xi)$$

or
$$K_3(x, \xi) = \int_{\xi}^{x} 1\cdot(z - \xi)\, dz = \frac{1}{2!}(x - \xi)^2$$

or
$$K_4(x, \xi) = \int_{\xi}^{x} 1\cdot\frac{(z - \xi)^2}{2!}\, dz = \frac{1}{3!}(x - \xi)^3$$

$$\cdots \quad \cdots \quad \cdots$$

or
$$K_V(x, \xi) = \int_{\xi}^{x} 1\cdot K_{V-1}(z, \xi)\, dx$$

or $$K_V(x, \xi) = \int_{\xi}^{x} 1 \cdot \frac{(z-\xi)^{V-2}}{(v-2)!} dz = \frac{1}{(v-1)!}(x-\xi)^{v-1}$$

Hence, the resolvent Kernel is determined as we get

$$R(x, \xi, \lambda) = \sum_{V=0}^{\infty} \lambda^V \; K_{v+1}(x, \xi)$$

or $$R(x, \xi, \lambda) = \sum_{V=0}^{\infty} \frac{\lambda^V (x-\xi)^v}{v!} = e^{\lambda(x-\xi)}$$

Example 10 With the aid of the resolvent Kernel find solution of the integral equation

$$\phi(x) = x + \int_0^x (\xi - x)\phi(\xi)\, d\xi$$

Solution

Here $$K_1(x, \xi) = K(x, \xi) = (\xi - x) \tag{61}$$

and $$K_v(x, \xi) = \int_{\xi}^{x} K(x, z)\, K_{v-1}(z, \xi)\, dz \tag{62}$$

Substituting $v = 1, 2, 3, 4, \ldots$ in the Eq. (62), we have

$$K_2(x, \xi) = \int_{\xi}^{x} K(x, z)\, K_1(z, \xi)\, dz$$

$\Rightarrow$ $$K_2(x, \xi) = \int_{\xi}^{x} (z-x)(\xi - z)\, dz = \frac{1}{3!}(\xi - x)^3$$

or $$K_3(x, \xi) = \int_{\xi}^{x} K(x, z)\, K_2(z, \xi)\, dz$$

$\Rightarrow$ $$K_3(x, \xi) = \int_{\varepsilon}^{x} (z-x)\left\{-\frac{1}{3!}(\xi - z)^3\right\} = \frac{1}{5!}(\xi - x)^5$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

By mathematical induction, we have

$$K_V(x, \xi) = (-1)^{v-1} \frac{1}{(2v-1)!}(\xi - x)^{2v-1}, \; v = 1, 2, 3, \ldots$$

The resolvent Kernel is defined as

$$R(x, \xi; \lambda) = \sum_{V=1}^{\infty} \lambda^{V-1} K_V(x, \xi) = \sum_{V=1}^{\infty} K_V(x, \xi); \lambda = 1$$

$$\Rightarrow \quad R(x, \xi; \lambda) = \frac{(\xi - x)}{1!} - \frac{(\xi - x)^3}{3!} + \frac{(\xi - x)^5}{5!} - \cdots$$

$$\Rightarrow \quad R(x, \xi; \lambda) = \sin(\xi - x)$$

The solution of the integral equation is determined as

$$\phi(x) = f(x) + \lambda \int_0^x R(x, \xi; \lambda)\, f(\xi)\, d\xi$$

$$\Rightarrow \quad \phi(x) = x + \int_0^x \xi \sin(\xi - x)\, d\xi$$

$$\Rightarrow \quad \phi(x) = \sin x$$

Example 11 Find the resolvent Kernel of integral equations with the following Kernels; ($\lambda = 1$)

(i) $K(x, \xi) = 2x$

(ii) $K(x, \xi) = 2 - (x - \xi)$

(iii) $K(x, \xi) = -2 + 3(x - \xi)$

Solution

(i) Here $K(x, \xi) = 2x$; $\lambda = 1$

Comparing with the relation (§ 15.6, 46), we have

$a_0(x) = 2x$, and all other $a_V(x) = 0$

The differential equation (§ 15.6, 53) reduces to

$$\frac{d\phi}{dx} - 2x\phi = 0, \ \phi = 1, \text{ at } x = \xi$$

$$\Rightarrow \quad \phi = A(\xi)\, e^{x^2} \tag{63}$$

From Eqs. (62) and (63), we have

$$\phi(x, \xi; 1) = e^{x^2 - \varepsilon^2}$$

Thus the resolvent Kernel is given by

$$R(x, \xi; 1) = \frac{1}{\lambda} \frac{d}{dx} \phi(x, \xi; 1) = 2xe^{x^2 - \xi^2}$$

(ii) Here $K(x, \xi) = 2 - (x - \xi)$; $\lambda = 1$

Comparing with the relation (§ 15.6, 46) we have

$a_0(x) = 2$, $a_1(x) = -1$, and all other $a_v(x) = 0$

Thus the equation (§ 15.6, 53) reduces to

$$\frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx} + \phi = 0 \tag{64}$$

with the conditions

$$\phi = 0 \text{ at } x = \xi, \frac{d\phi}{dx} = 1 \text{ at } x = \xi \tag{65}$$

The solution of Eq. (64) is given by

$$\phi = \left[A(\xi) + B(\xi)x\right]e^{x} \tag{66}$$

From Eqs. (65) and (66), we obtain

$$\phi(x, \xi; 1) = (x - \xi)\, e^{x-\xi}$$

Thus, the resolved Kernel is given by

$$R(x, \xi; 1) = \frac{1}{\lambda}\frac{d^2}{dx^2}\phi(x, \xi, 1) = (x - \xi + 2)e^{x-\xi}$$

(iii) Here $K(x, \xi) = -2 + 3(x - \xi)$; $\lambda = 1$
Comparing with the relation (§ 15.6, 46), we have

$$a_0(x) = -2,\ a_1(x) = 3; \text{ and all the other } a_V(x) = 0$$

The differential equation (§ 15.6, 53) reduces to

$$\frac{d^2\phi}{dx^2} + 2\frac{d\phi}{dx} - 3\phi = 0 \tag{67}$$

with the conditions

$$\phi = 0 \text{ at } x = \xi, \frac{d\phi}{dx} = 1 \text{ at } x = \xi \tag{68}$$

Equation (67) can be solved as follows:

$$\phi = A(\xi)e^{-3x} + B(\xi)\, e^{x} \tag{69}$$

From Eqs. (68) and (63), we obtain

$$\phi(x, \xi; 1) = \frac{1}{4}\, e^{(x-\xi)} - \frac{1}{4}e^{-3(x-\xi)}$$

Hence, the resolvent Kernel is given by

$$R(n, \xi; 1) = \frac{1}{\lambda}\frac{d^2}{dx^2}\phi(x, \xi; 1)$$

$$= \frac{1}{4}e^{(x-\xi)} - \frac{9}{4}e^{-3(x-\xi)}$$

Example 12 Solve the linear integral equation:

$$\phi(x) = 29 + 6x + \int_0^x \left[5 - 6(x - \xi)\right]\phi(\xi)\, d\xi$$

Solution

Here $f(x) = 29 + 6x$; $\lambda = 1$ and $K(x, \xi) = 5 - 6(x - \xi)$

From the relation (§, 15.6, 46) we get

$$a_0(x) = 5,\ a_1(x) = -6, \text{ and all the other } a_V(x) = 0$$

Thus, the equation reduces to

$$\frac{d^2\phi}{dx^2} - 5\frac{d\phi}{dx} + 6\phi = 0, \tag{70}$$

with the conditions

$$d = 0 \text{ at } x = \xi, \frac{d\phi}{dx} = 1 \text{ at } x = \xi. \tag{71}$$

Thus, the solution of Eq. (70) is given by

$$\phi(x_1\ \xi; 1) = A(x)\, e^{3x} + B(x)\, e^{2x} \tag{72}$$

From Eqs. (71) and (72) we have

$$\phi(x, \xi; 1) = e^{3(x-\xi)} - e^{2(x-\xi)}$$

Hence, the resolvant Kernel becomes

$$R(x, \xi; 1) = \frac{1}{\lambda}\frac{d^2}{dx^2}\phi(x, \xi; 1)$$

or $$R(x, \xi; 1) = 9e^{3(x-\varepsilon)} - 4e^{2(x-\xi)} \tag{73}$$

Thus, the solution of the integral equation is given by

$$\phi(x) = 29 + 6x + \int_0^x \left[9\, e^{3(x-\xi)} - 4e^{2(x-\xi)}\right](29 + 6\xi)\, d\xi$$

or $$d(x) = 29 + 6x + \left[-87\, e^{3(x-\xi)} + 58e^{2(x-\xi)} - 54\left\{\frac{1}{3}\xi\, e^{3(x-\xi)} + \frac{1}{9}e^{3(x-\xi)}\right\}\right.$$
$$\left. + 24\left\{\frac{1}{2}\,\xi\, e^{2(x-\xi)} + \frac{1}{4}\, e^{2(x-\xi)}\right\}\right]_{\xi=0}^{\xi=x}$$

or $$\phi(x) = 29 + 6x - 29 - 6x + 93e^{3x} - 64e^{2x}$$
$$= 93e^{3x} - 64e^{2x}$$

Example 13 Solve the Volterra integral equation of second kind, by using the method of successive approximation:

(i) $\phi(x) = (1 + x) - \int_0^x \phi(\xi)\, d\xi$, with $\phi_0(x) = 1$

(ii) $\phi(x) = x - \int_0^x (x - \xi)\, \phi(\xi)\, d\xi$, with $\phi_0(x) = 0$

Solution

(i) The integral equation is given as

$$\phi(x) = (1 + x) - \int_0^x \phi(\xi)\, d\xi$$

Here $\quad f(x) = 1 + x,\ K(x, \xi) = 1$, and $\lambda = -1$

The V^{th} order approximation is given by

$$\phi_V(x) = f(x) + \lambda \int_0^x K(x, \xi)\, \phi_{V-1}(\xi)\, d\xi$$

or
$$\phi_V(x) = (1 + x) - \int_0^x \phi_{V-1}(\xi)\, d\xi$$

Substituting $V = 1, 2, 3, \ldots$, we have

$$\phi_1(x) = (1 + x) - \int_0^x \phi_0(\xi)\, d\xi = (1 + x) - \int_0^x d\xi = 1$$

$$\phi_2(x) = (1 + x) - \int_0^x \phi_1(\xi)\, d\xi = (1 + x) - x = 1$$

$$\phi_3(x) = (1 + x) - \int_0^x \phi_2(\xi)\, d\xi = (1 + x) - x = 1$$

$$\phi_V(x) = (1 + x) - \int_0^x \phi_{V-1}(\xi)\, d\xi = 1$$

Hence, the solution of the integral equation is given by

$$\phi(x) = \lim_{v \to \infty} \phi_V(x) = 1$$

(ii) The integral equation is given as

$$\phi(x) = x - \int_0^x (x - \xi)\, \phi(\xi)\, d\xi,\ \phi_0(x) = 0$$

Here $\quad f(x) = x,\ K(x, \xi) = x - \xi$, and $\lambda = -1$

The V^{th} order approximation is given by

$$\phi_V(x) = f(x) + \lambda \int_0^x K(x, \xi)\, \phi_{V-1}(\xi)\, d\xi$$

$$\Rightarrow \quad \phi_V(x) = x - \int_0^x (x - \xi)\, \phi_{V-1}(\xi)\, d\xi$$

Substituting $v = 1, 2, 3, \ldots$, we have

$$\phi_1(x) = x - \int_0^x (x - \xi)\, \phi_0(\xi)\, d\xi = x$$

$$\phi_2(x) = x - \int_0^x (x - \xi)\, \phi_1(\xi)\, d\xi$$

$$= x - \int_0^x (x\,\xi - \xi^2)\, d\xi = x - \frac{x^3}{3!}$$

$$\phi_3(x) = x - \int_0^x (x-\xi)\,\phi_2(\xi)\, d\xi$$

$$= x - \int_0^x (x-\xi)\left(\xi - \frac{\xi^3}{6}\right) d\xi = x - \frac{x^3}{3!} - \frac{x^5}{5!}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

In general, we have

$$\phi_V = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{V-1}\frac{x^{2V-1}}{(2v-1)!}$$

Hence, the solution of the integral equation is given by

$$\phi(x) = \lim_{v\to\infty} \phi_V(x)$$

$$\Rightarrow \qquad \phi(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{V-1}\frac{x^{2V-1}}{(2v-1)!}$$

$$\Rightarrow \qquad \phi(x) = \sin x$$

EXERCISE 15.4

1. Find the resolvent Kernel of the Volterra integral equation with the following Kernels:

   (i) $K(x, \xi) = x - \xi$

   (ii) $K(x, \xi) = e^{x^2 - \xi^2}$

2. With the aid of the resolvent Kernel, find the solution of the integral equation

   (i) $\phi(x) = e^x + \int_0^x e^{x-\xi}\,\phi(\xi)\, d\xi$

   (ii) $\phi(x) = e^{x^2+2x} + 2\int_0^x e^{x^2-\xi^2}\,\phi(\xi)\, d\xi$

Answers

1. (i) $\frac{1}{\sqrt{\lambda}}\sinh\sqrt{\lambda}\,(x-\xi),\ \lambda > 0$ (ii) $e^{\lambda(x-\xi)}e^{(x^2-\xi^2)}$

2. (i) $\phi(x) = 3^x(1 - e^{-x})$ (ii) $\phi(x) = e^{x^2+2x}(1+2x)$

15.8 SOLUTION OF THE FREDHOLM INTEGRAL EQUATION BY THE METHOD OF SUCCESSIVE SUBSTITUTIONS

Consider the Fredholm integral equation of second kind as

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi, \tag{74}$$

where

(i) The kernel $K(x, \xi) \not\equiv 0$ is real and continuous in the rectangle R: $a \le x \le b$, $a \le \xi \le b$. Consider $K(x, \xi) \le P$, where P is the maximum value in R.

(ii) The function $F(x) \ne 0$ is real and continuous in an interval I: $a \le x \le b$. Consider $|F(x)| \le Q$, when Q is the maximum value in the interval.

(iii) λ is a non zero numerical parameter.

Since there exists a continuous solution $\phi(x)$ so substituting the unknown function under an integral sign from the Eq. (74) itself, we obtain

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi) \left\{ F(\xi) + \lambda \int_a^b K(\xi, \xi_1)\, \phi(\xi_1)\, d\xi_1 \right\} d\xi$$

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, F(\xi)\, d\xi + \lambda^2 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1)\, \phi(\xi_1)\, d\xi_1\, d\xi$$

Proceeding in this manner successively for $\phi(\xi)$, we get

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, F(\xi)\, d\xi + \lambda^2 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1) \Big\{ f(\xi_1) +$$

$$\lambda \int_a^b K(\xi_1, \xi_2)\, \phi(\xi_2)\, d\xi_2 \Big\} d\xi_1\, d\xi$$

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, F(\xi)\, d\xi + \lambda^2 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1)\, F(\xi_1)\, d\xi_1\, d\xi$$

$$+ \lambda^3 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1) \int_a^b K(\xi_1, \xi_2)\, \phi(\xi_2)\, d\xi_2\, d\xi_1\, d\xi + \cdots + \cdots$$

In general, we get

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, F(\xi)\, d\xi + \lambda^2 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1)\, F(\xi_1)\, d\xi_1, d\xi$$

$$+ \lambda^2 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1) \int_a^b K(\xi_1, \xi_2)\, \phi(\xi_2)\, d\xi_2\, d\xi_1\, d\xi + \cdots +$$

$$\lambda^n \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1) \cdots \int_a^b K(\xi_{n-2)}, \xi_{n-1})\, F(\varepsilon_{n-1})\, d\xi_{n-1}\, d\xi_1\, d\xi +$$

$$\lambda^{n+1} \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1) \cdots \int_a^b K(\xi_{n-1}, \xi_n)\, F(\xi_n)\, d\xi_n \cdots d\xi_1\, d\xi \tag{75}$$

Consider the infinite series

$$\phi(x) = F(x) + \lambda\int_a^b K(x,\xi)\,F(\xi)\,d\xi + \lambda^2\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\,F(\xi)\,d\xi_1\,d\xi + \cdots \tag{76}$$

As the kernel $K(x, \xi)$ and the known function $F(\xi)$ are real and continuous, so each term of the above series represents a continuous function in I, provided it converges uniformly in that interval.

Since $\quad |K(x,\xi)| \le P$ and $|F(x)| \le Q$

Contains the maximum value in R and I respectively.

Assume

$$S_n(x) = \lambda^n\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\cdots\int_a^b K(\xi_{n-2}\ \xi_{n-1})F(\xi_{n-1})\,d\xi_{n-1}\cdots d\xi_1\,d\xi$$

Then $\quad |S_n(x)| \le |\lambda|^n\,QP^n(b-a)^n$

It will converge only if

$$|\lambda|\,P(b-a) < 1 \Rightarrow |\lambda| < \frac{1}{P(b-a)} \tag{77}$$

Thus the series in Eq. (75) converges absolutely and uniformly when the relation in Eq. (76) holds.

Again, let $S_{n+1}(x) = \lambda^{n+1}\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\cdots\int_a^b K(\xi_{n-1},\xi_n)\,F(\xi_n)\,d\xi_n\cdots d\xi_1\,d\xi$

or, $\quad |S_{n+1}(x)| < |\lambda^{n+1}|\,MP^{n+1}\,(b-a)^{n+1}$

where M is the maximum value of the absolute value of the function $\phi(x)$ in I.

If $\quad |\lambda|\,P(b-a) < 1$, then $\underset{n\to\infty}{Lt}\ S_{n+1}(x) = 0$

Thus, we notice that the function $\phi(x)$ which satisfies Eq. (75) is the continuous function given by the series in Eq. (74). We can verify by direct substitution that the function $\phi(x)$ defined by Eq. (75) satisfies the integral Eq. (74). Multiplying Eq. (75) both the sides with $\lambda\,K(x, \varepsilon)$ and integrating term by term within the fixed domain, we have,

$$\lambda\int_a^b K(x,\xi)\,\phi(\xi)\,d\xi = \lambda\int_a^b K(x,\xi)\left[F(\xi) + \lambda\int_a^b K(\xi,\xi_1)\,F(\xi_1)\,d\xi_1 + \cdots + \right]d\xi$$

The R.H.S. may be expressed as

$$= \lambda\int_a^b K(x,\xi)\,F(\xi)\,d\xi + \lambda^2\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\,F(\xi_1)\,d\xi_1\,d\xi + \cdots$$

$$= \phi(x) - F(x)$$

$$\Rightarrow \quad \phi(x) = F(x) + \lambda\int_a^b K(x,\xi)\,\phi(\xi)\,d\xi$$

Hence, proved.

Particular Case

When $\lambda = 1$, the integral Eq. (74) reduces to

$$\phi(x) = F(x) + \int_a^b K(x, \xi)\,\phi(\xi)\,d\xi$$

The equation contains a continuous solution, even though the Eq. (77) is not satisfied, e.g., the integral equation

$$\phi(x) = \left(\frac{x}{2} - \frac{1}{3}\right) + \int_0^1 (x+\xi)\,\phi(\xi)\,d\xi$$

has the continuous solution $\phi(x) = x$ although $|\lambda|\,P(b-a) = 2 \not< 1$.

The method of solving integral equations of the second kind is based on an iterative technique which yields a sequence of approximations leading to an infinite series solution. This is known as Liouville Neumann series.

15.9 ITERATED KERNELS

The n^{th} iterated Kernel $K_n(x, \xi)$ is determined by the relation

$$K_n(x, \xi) = \int_a^b K_m(x, z)\,K_{n-m}(z, \xi)\,dz \qquad m < n$$

We knew that $K_1(x, \xi) = K(x, \xi$ (78)

and $$K_n(x, \xi) = \int_a^b K(x, z)\,K_{n-1}(z, \xi)\,dz,\ n = 2, 3, 4, \ldots \tag{79}$$

or, $$K_n(x, \xi) = \int_a^b K(x, z_1)\,K_{n-1}(z_1, \xi)\,dz_1 \tag{80}$$

Substituting $(n-1)$ for n in Eq. (79), we have

$$K_{n-1}(x, \xi) = \int_a^b K(x, z)\,K_{n-2}(z, \xi)\,dz$$

or, $$K_{n-1}(z_1, \xi) = \int_a^b K(z_1, z_2)\,K_{n-2}(z_2, \xi)\,dz_2 \tag{81}$$

Substituting $K_{n-1}(z_1, \xi)$ in the Eq. (80), we have

$$K_n(x, \xi) = \int_a^b K(x, z_1) \int_a^b K(z_1, z_2)\,K_{n-2}(z_2, \xi)\,dz_2\,dz_1$$

or $$K_n(x, \xi) = \int_a^b \int_a^b K(x, z_1)\,K(z_1, z_2)\,K_{n-2}(z_2, \xi)\,dz_2\,dz_1 \tag{82}$$

Substituting $(n - 1)$ for n in Eq. (81) we have

$$K_{n-2}(z_2, \xi) = \int_a^b K(z_2, z_3)\, K_{n-3}\,(z_3, \xi)\, dz_3 \tag{83}$$

From the relations in Eqs. (82) and (83), we have

$$K_n(x, \xi) = \int_a^b \int_a^b \int_a^b K(x, z_1)\, K(z_1, z_2)\, K(z_2, z_3) K_{n-3}(z_3, \xi)\, dz_3\, dz_2\, dz_1 \tag{84}$$

Performing this operation successively, we have

$$K_n(x, \xi) = \int_a^b \int_a^b \int_a^b \cdots \int_a^b K(x, z_1)\, K(z_1, z_2)\, K(z_2, z_3) ... K(z_{n-1}, \xi)\, dz_{n-1} \cdots dz_3\, dz_2\, dz_1$$

or,

$$K_n(x, \xi) = \underbrace{\int_a^b \int_a^b \int_a^b \cdots \int_a^b}_{(n-1)\text{th order integral}} K(x, z_1)\, K(z_1, z_2)\, K(z_2, z_3) ... K(z_{n-1}, z_n)\, K(z_n, z_{n+1}) ... \tag{85}$$

$$K(z_{n-1}, \xi)\, dz_{n-1} \cdots dz_3 dz_2\, dz_1$$

Similarity, we obtain

$$K_m(x, z) = \underbrace{\int_a^b \int_a^b \int_a^b \cdots \int_a^b}_{\substack{(m-1)\text{th order}\\ \text{integral}}} K(x, p_1)\, K(p_1, p_2)\, K(p_2, p_3) ... K(p_{m-1}, z)\, dp_{m-1} \cdots dp_3\, dp_2\, dp_1 \tag{86}$$

and

$$K_{n-m}(z, \xi) = \underbrace{\int_a^b \int_a^b \int_a^b \cdots \int_a^b}_{\substack{(n-m-1)\text{th order}\\ \text{integral}}} K(z, q_1)\, K(q_1, q_2)\, K(q_2, q_3) ... K(q_{n-m-1}, \xi)\, dq_{n-m-1} \cdots dq_3\, dq_2\, dq_1 \tag{87}$$

From the relations in Eqs (86) and (87) we have

$$\int_a^b K_m(x, z)\, K_{n-m}(z, \xi)\, dz = \int_a^b \Bigg[\Bigg\{ \underbrace{\int_a^b \int_a^b \int_a^b \cdots \int_a^b}_{\substack{(m-1)\text{th order}\\ \text{integral}}} K(x, p_1)\, K(p_1, P_2)$$

$$K(p_2, p_3)\, K(p_{m-1}, z)\, dp_{m-1} \cdots dp_3\, dp_2\, dp_1 \Bigg\} \times$$

$$\Bigg\{ \underbrace{\int_a^b \int_a^b \int_a^b \cdots \int_a^b}_{\substack{(n-m-1)\text{th order}\\ \text{integral}}} K(z, q_1)\, K(q_1, q_2)\, K(q_2, q_3) \cdots K(q_{n-m-1}) \xi\, dq_{n-m-1} \cdots dq_3 dq_2 dq_1 \Bigg\} \Bigg] dz$$

or

$$\int_a^b K_m(x, z)\, K_{n-m})(z, \xi)\, dz = \underbrace{\int_a^b \int_a^b \int_a^b \cdots \int_a^b}_{(n-1)\text{th order}} K(x, p_1)\, K(p_1, p_2) \cdots K(p_{m-1}, z)\, K(z, q_1)$$

$$K(q_1, q_2) \cdots K(q_{n-m-1}, \xi)\, dq_{n-m-1} \cdots dq_2\, dq_1\, dz dp_{m-1} \cdots dp_2\, dp_1 \tag{88}$$

Now without changing the limits of integration, the variables of integration are changed as follows:

$$p_1 p_2 p_3 \cdots p_{m-1}\ z\ q_1\ q_2\ q_3 \cdots q_{n-m-1}$$

$$z_1\ z_2\ z_3 \cdots z_{m-1}\ z_m\ z_{m+1}\ z_{m+2}\ z_{m+3} \cdots z_{n-1}$$

Thus we have

$$\int_a^b K_m(x, z)\, K_{n-m}(z, \xi)\, dz = \int_a^b \int_a^b \int_a^b \cdots \int_a^b K(x, z_1)\, K(z_1, z_2)\, K(z_2, z_3) \cdots K(z_{m-1}, z_m)$$

$$K(z_m, z_{m-1}) \cdots K(z_{n-1}, \xi)\, dz_{n-1}\, dz_m \cdots dz_2\, dz_1$$

Hence, from the relations in Eqs (86) and (88) we have

$$K_n(x, z) = \int_a^b K_m(x, z)\, K_{n-m}(z, \xi)\, dz, \quad m < n$$ **Hence, proved.**

15.10 SOLUTION OF THE FREDHOLM INTEGRAL EQUATION BY THE METHOD OF SUCCESSIVE APPROXIMATION

The method of successive approximations (iteration technique) applied in the solution of Volterra integral equation may be extended to the basic Fredholm equation of the second kind. Consider the Fredholm integral equation of second kind as

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi \tag{89}$$

where

(i) the Kernel $K(x, \xi) \not\equiv 0$ is real and continuous in the rectangle R for which $a \le x \le b$ and $a \le e \le b$;

(ii) $F(x) \ne 0$ is real and continuous in an interval I, for which $a \le x \le b$.

(iii) λ is a non-zero numerical parameter.

Consider the solution of the integral Eq. (89) is in the form of a power series in λ as

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots + \lambda^n\phi_n(x) + \cdots \tag{90}$$

where the function $\phi_0(x), \phi_1(x), \phi_2(x) \cdots$ and so on are real and continuous in an interval I.

Let the series (90) is a solution of the integral Eq. (89), then

$$\phi_0(x) + \lambda\phi_1(x) + \cdots + \lambda^n\phi_n(x) + \cdots$$

$$= F(x) + \lambda \int_a^b K(x, \xi) \left\{ \phi_0(\xi) + \lambda\, \phi_1(\xi) + \lambda^2 \phi_2(\xi) + \cdots + \lambda^n \phi_n(\xi) + \cdots \right\} d\xi \qquad (91)$$

Equating the coefficients of like powers of λ, we have

$$\phi_0(x) = F(x)$$

$$\phi_1(x) = \int_a^b K_1(x, \xi)\, \phi_0(\xi)\, d\xi = \int_a^b K_1(x, \xi)\, F(\xi)\, d\xi$$

$$\phi_2(x) = \int_a^b K_2(x, \xi)\, \phi_1(\xi)\, d\xi = \int_a^b K_2(x, \xi)\, F(\xi)\, d\xi$$

$$\phi_n(x) = \int_a^b K(x, \xi)\, \phi_{n-1}(\xi)\, d\xi = \int_a^b K_n(x, \xi)\, F(\xi)\, d\xi, \quad n \geq 1$$

where $K_1(x, \xi) = K(x, \xi)$

$$K_2(x, \xi) = \int_a^b K(x, z)\, K_1(z, \xi)\, dz$$

$$K_3(x, \xi) = \int_a^b K(x, z)\, K_2(z, \xi)\, dz$$

and in general,

$$K_n(x, \xi) = \int_a^b K(x, z)\, K_{n-1}(z, \xi)\, dz \qquad (n = 2, 3, \ldots) \qquad (92)$$

The functions $K_n(x, \xi)$ are called the successive iterated kernels of the kernel $K(x, \xi)$

$$K_n(x, \xi) = \int_a^b K_n(x, z)\, K_{n-m}(z, \xi)\, dz, \quad m < n \qquad (93)$$

Consider the series

$$R(x, \xi; \lambda) = K(x, \xi) + \lambda K_2(x, \xi) + \cdots + \lambda^{n-1} K_n(x, \xi) + \cdots \qquad (94)$$

$$\Rightarrow \qquad R(x, \xi; \lambda) = \sum_{V=0}^{\infty} \lambda^V K_{V+1}(x, \xi)$$

The series on the R.H.S. is called the Neumann series of the kernel $K(x, \xi)$. If M is an upper limit of $\left|K(x, \xi)\right|$ then the iterated kernel $\left|K_n(x, \xi) < M^n (b-a)^{n-1}\right|$ at every point of the Domain D. The series Eq. (94) is therefore uniformly convergent in this region, if the parameter $|\lambda| < \dfrac{1}{M}(b-a)$. Thus, the relation in Eq. (20) which gives the solution of the integral Eq. (91) assumes the form

$$\phi(x) = F(x) + \lambda \int_a^b R(x, \xi; \lambda)\, F(\xi)\, d\xi \qquad (95)$$

The function $R(x, \xi; \lambda)$ is called resolvent or the resolving kernel for Eq. (89). Equation (95) represents an explicit solution of the integral equation in terms of the resolvent.

In view of the relation in Eq. (94), it follows that the resolvent kernel $R(x, \xi; \lambda)$ satisfies the functional equation

$$R(x, \xi; \lambda) - K(x, \xi) = \lambda \sum_{V=0}^{\infty} \lambda^V K_{V+2}(x, \xi)$$

$$R(x, \xi; \lambda) - K(x, \xi) = \lambda \int K(x, z) \sum_{V=0}^{\infty} \lambda^V K_{V+1}(z, \xi)\, dz$$

$$R(x, \xi; \lambda) - K(x, \xi) = \lambda \int K(x, z)\, \lambda(z, \xi; \lambda)\, dz$$

$$R(x, \xi; \lambda) = K(x, \xi) + \lambda \int_a^b K(x, z)\, R(z, \xi; \lambda)\, dz$$

Similarly, we may obtain

$$R(x, \xi; \lambda) = K(x, \xi) + \lambda \int_a^b K(z, \xi)\, R(x, z; \lambda)\, dz \tag{96}$$

The first equation is established by transposing x by z in then multiplying both sides by $K(x, z)$ and integrating term by term in the interval (a, b). Similarly we may prove the other relation.

Now, if the resolvent kernel $R(x, \xi; \lambda)$ satisfies Eq. (96), then the function $f(x)$ in the Eq. (95) satisfies Fredholm integral equation. Substituting $\phi(x)$ from Eqs. (95) and (96) we have

$$F(x) + \lambda \int_a^b R(x, \xi; \lambda)\, F(\xi)\, d\xi = F(x) + \lambda \int_a^b \left\{ K(x, \xi) \times F(\xi) + \lambda \int_a^b R(\xi, z; \lambda)\, F(z)\, dz \right\} d\xi$$

By transposing ξ and z, we get

$$F(x) + \lambda \int_a^b R(x\, \xi; \lambda)\, F(\xi)\, d\xi = F(x) + \lambda \int_a^b \left\{ K(x, \xi + \lambda \int_a^b K(x, z)\, R(z, \xi; \lambda)\, dz\, F(\xi) d \right\} \tag{97}$$

From Eqs. (96) and (97), we see that the functional relation in Eq. (95) is the unique solution of the Fredholm integral Eq. (96) when the absolute value of the parameter λ is sufficiently small.

15.11 RECIPROCAL FUNCTIONS

If $K(x, \xi)$ is real and continuous in R then $\exists$ a reciprocal function $K(x, \xi)$ provided that $P(b - a) < 1$, where P is the maximum of $|K(x, \xi)|$ in R.

We know that the iterated Kernels of $K(x, \xi)$ are represented

$$K_n(x, \xi) = \int_a^b K_m(x, z)\, K_{n-m}(z, \xi)\, dz, \qquad m < n$$

and $\quad K_1(x, \xi) = K(x, \xi)$

Let $\quad k(x, \xi) = K_1(x, \xi) + K_2(x, \xi) + \cdots + K_n(x, \xi) + \cdots \qquad (98)$

When $K(x, \xi)$ is real and continuous in R, the infinite series for $k(x, \xi)$ is absolutely and uniformly convergent if $P(b-a) < 1$, thus, we have

$$-k(x, \varepsilon) - K(x, \xi) = K_2\ (x, \xi) + K_3(x, \xi) + \cdots + K_n(x, \xi) \cdots$$

or
$$k(x, \xi) - k(x, \xi) = \int_a^b K_1(x, z)\ K_1(z, \xi)\ dz + \cdots + \cdots + \int_a^b K_1(x, z)\ K_{n-1}\ (z, \xi)\ dz + \cdots$$

or
$$-k(x, \xi) - K(x, \xi) = \int_a^b K_1(x, z)\ K_1(z, \xi)\ dz + \cdots + \int_a^b K_{n-1}\ (x, z)\ K_1\ (z, \xi)\ dz + \cdots$$

or
$$-k(x, \xi) - K(x, \xi) = \int_a^b \left[K_1(x, z) + \cdots + \cdots K_{n-1}(x, z) + \cdots K_1(z, \xi)\ dz\right]$$

From Eq. (98) we have

$$K(x, \xi) + k(x, \xi) = \int_a^b k(x, z)\ K(z, \xi)\ dz \tag{99}$$

The functions $K(x, \xi)$ and $k(x, \xi)$ are said to be reciprocal if they are both real and continuous in R and satisfies the condition in Eq. (99). A function $k(x, \xi)$ reciprocal to $K(x, \xi)$ will exist, provided the series Eq. (98) converges uniformly.

15.12 VOLTERRA'S SOLUTION OF FREDHOLM'S EQUATION

A function $k(x, \xi)$ reciprocal to $K(x, \xi)$ exists where $K(x, \xi)$ is real and continuous in R, $K(x, \xi) \not\equiv 0$, $f(x)$ is real and continuous in I, $f(x) \not\equiv 0$, then the Fredholm's integral equation of second kind has one and only one continuous solution is the interval I, $a \le x \le b$.

Consider the Fredholm integral equation of second kind as

$$\phi(x) = f(x) + \int_a^b K(x, \xi)\ \phi(\xi)\ d\xi \tag{100}$$

where the reciprocal function $k(x, \xi)$ of $K(x, \xi)$ is known. Let the Eq. (100) has a continuous solution $\phi(x)$, then

$$\phi(\xi) = f(\xi) + \int_a^b K(\xi, \xi_1)\ \phi(\xi_1)\ d\xi_1$$

Multiplying by $k(x, \xi)$ and integrating over the interval (a, b) we have

$$\int_a^b k(x, \xi)\ \phi(\xi)\ d\xi = \int_a^b k(x, \xi)\ f(\xi)\ d\xi + \int_a^b \int_a^b k(x, \xi)\ K(\xi, \xi_1)\ \phi(\xi_1)\ d\xi_1\ d\xi$$

$$= \int_a^b k(x, \xi)\ \phi(\xi)\ d\xi = \int_a^b k(x, \xi)\ f(\xi)\ d\xi + \int_a^b \left\{K(x, \xi_1) + k(x, \xi_1)\right\} \phi(\xi)\ d\xi$$

$$0 = \int_a^b k(x, \xi)\ f(\xi)\ d\xi + \int_a^b K(x, \xi_1)\ \phi(\xi_1)\ d\xi_1$$

Again $$\int_a^b K(x,\xi)\,\phi(\xi_1)\,d\xi_1 = \phi(x) - f(x)$$

$$\Rightarrow \qquad \phi(x) = f(x) - \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi \tag{101}$$

It follows that the integral Eq. (100) has a continuous solution and is given by the Eq. (101) which is unique.

Example 14 Find the first two iterated Kernels of the Kernel

$$K(x,\xi) = (x-\xi)^2;\ a = -1,\ b = 1$$

Solution

We know that

$$K_1(x,\xi) = K(x,\xi) = (x-\xi)^2$$

or $$K_2(x,\xi) = \int_a^b K(x,z)\,K_1\,(z,\xi)\,dz$$

or $$K_2(x,\xi) = \int_{-1}^{1} (x-z)^2\,(z-\xi)^2\,dz$$

or $$K_2(x,\xi) = \frac{2}{3}(x+\xi)^2 + 2x^2\,\xi^2 + \frac{4}{3}x\xi + \frac{2}{5}$$

and $$K_3(x,\xi) = \int_a^b K(x,z)\,K_2(z,\xi)\,dz$$

or $$K_3(x,\xi) = \int_{-1}^{1} (x-z)^2 \left\{\frac{2}{3}(z+\xi)^2 + 2z^2\xi^2 + \frac{4}{3}z\xi + \frac{2}{5}\right\}$$

or $$K_3(x,\xi) = \frac{56}{45}(x^2+\xi^2) + \frac{8}{3}(x^2\,\xi^2) - \frac{32}{9}\,x\xi + \frac{8}{15}$$

Example 15 Solve the following linear integral equations:

$$\phi(x) = x + \int_0^{1/2} \phi(\xi)\,d\xi$$

Solution

Here, $K(x,\xi) = 1$, $\lambda = 1$ and $F(x) = x$

We know that

$$K_1(x_1\,\xi) = K(x_1\,\xi) = 1$$

or $$K_2(x, \xi) = \int_0^{1/2} dz = \frac{1}{2}$$

or $$K_3(x, \xi) = \frac{1}{2}\int_0^{1/2} dz = \left(\frac{1}{2}\right)^2$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

or $$K_n(x, \xi) = \left(\frac{1}{2}\right)^{n-1}$$

Thus, the resolvent Kernel is given by

$$R(x, \xi, \lambda) = \sum_{n=1}^{\infty}\left(\frac{\lambda}{2}\right)^{n-1} = \frac{2}{2-\lambda}$$

or $\quad R(x, \xi; 1) = 2$, as $\lambda = 1$

Hence, the solution of the integral equation is given by

$$\phi(x) = F(x) + \lambda\int_a^b R(x, \xi; \lambda)\, F(\xi)\, d\xi$$

or $$\phi(x) = x + \int_0^{1/2} 2\xi\, d\xi = x + \frac{1}{4} = x + \text{constant}$$

Example 16 Solve the integral equation

$$\phi(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 x\,\xi\,\phi(\xi)\, d\xi$$

Solution

Here $\quad K(x, \xi) = x\xi,\ \lambda = \frac{1}{2},\ F(x) = \frac{5x}{6}$

We know that

$$K_1(x, \xi) = K(x, \xi) = x\xi$$

or $$K_2(x, \xi) = \int_0^1 xz\, z\xi\, dz = \frac{1}{3}x\xi$$

or $$K_3(x, \xi) = \int_0^1 x\, z \cdot \frac{1}{3} z\xi\, dz = \left(\frac{1}{3}\right)^2 x\xi$$

$$\cdots \quad \cdots \quad \cdots$$

or $$K_n(x_1\ \xi) = \left(\frac{1}{3}\right)^{n-1} x\xi$$

Hence, the resolvent Kernel is given by

$$R(x, \xi; \lambda) = x\,\xi \sum_{n=1}^{\infty} \left(\frac{\lambda}{2}\right)^{n-1} = \frac{3x\,\xi}{3-\lambda}$$

or $$R\left(x, \xi; \frac{1}{2}\right) = \frac{6}{5}\,x\,\xi.$$

Thus the solution of the integral equation is given by

$$\phi(x) = \frac{5x}{6} + \frac{1}{2}\cdot\frac{6}{5}\int_0^1 x\,\xi\,\frac{5}{6}\,\xi\,dz = \frac{5x}{6} + \frac{x}{6} = x$$

EXERCISE 15.5

1. Find the iterated Kernels of the following Kernels:

 (a) $K(x, \xi) = \sin(x-\xi);\ a = 0, b = \dfrac{\pi}{2}$ for $n = 2, 3$

 (b) $K(x, \xi) = xe^{\xi};\quad a = 0, b = 1$

2. Construct the Resolvent Kernel of the following Kernels:

 (a) $K(x, \xi) = 1;\ a = 0, b = 1$

 (b) $K(x, \xi) = x\xi;\ a = -1, b = 1$

 (c) $K(x, \xi) = \sin x \cos\xi;\ a = 0, b = \dfrac{\pi}{2}$

 (d) $K(x, \xi) = x^2\,\xi^2;\ a = -1, b = 1$

Answer

1. (a) $K_2(x, \xi) = \dfrac{1}{2}\sin(x+\xi) - \dfrac{\pi}{4}\cos(x-\xi)$

 $K_3(x, \xi) = \dfrac{4-\pi^2}{16}\sin(x-\xi)$

 (b) $K_n(x, \xi) = xe^{\xi}$

2. (a) $R(x, \xi; \lambda) = 2$

 (b) $R(x, \xi; \lambda) = \dfrac{3x\xi}{3-2\lambda};\ |\lambda| < \dfrac{3}{2}$

 (c) $R(x, \xi; \lambda) = \dfrac{2\sin x\cos\xi}{2-\lambda};\ |\lambda| < 2$

 (d) $R(x, \xi; \lambda) = \dfrac{5x^2\,\xi^2}{5-2\lambda};\ |\lambda| < \dfrac{5}{2}$

Integral Equation of the Convolution Type

We shall discuss the convolution type integral equations whose Kernels have the form

$$K(x, \xi) = K(x - \xi)$$

which is a function only of the difference between the two coordinates x and ξ. The method of solution generally involves the use of integral transform.

Laplace Transform

The Laplace Transform $L(\phi)$ of a function $d(x)$ is defined as

$$L(\phi) = \Phi(u) = \int_0^\infty e^{-ux}\phi(x)\, dx \tag{102}$$

The basic result which enables us to solve integral equations of the convolution type is the convolution theorem which defines that if $K(u)$ is the Laplace Transform of $K(x)$ then $K(u)\ \Phi(u)$ is the Laplace Transform of

$$\int_0^x K(x - S)\ \phi(S)\ dS \tag{103}$$

The Laplace Transform of (103) is given as

$$\int_0^\infty e^{-ux} dx \int_0^x K(x - S)\ \phi(S)\ dS$$

Let $t = x - S$ then $dt = dx$

$$= \int_0^\infty e^{-u(t+s)} dt \int_0^\infty K(t)\ \phi(S)\ dS$$

$$= \int_0^\infty e^{-ut} K(t)\ dt \int_0^\infty e^{-uS} \phi(S) dS = K(u)\ \Phi(u) \tag{104}$$

The inverse Laplace transform is defined as

$$L^{-1}(\Phi) = \phi(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Phi(u)\ e^{ux} du \tag{105}$$

This transformation is also linear because

$$L(a\phi + b\psi) = aL(\Phi) + bL(\psi) \tag{106}$$

for two constants a and b.

Properties of the Laplace Transform

I. If $L\{f(S)\} = F(p)$ then $L\left[e^{aS} f(S)\right] = F(P - a)$

II. If $L\{f(S)\} = F(p)$ then $L\left[f(aS)\right] = \frac{1}{a} L\{f(S)\}_{P \to P/a}$

$$= \frac{1}{a} F\left(\frac{P}{a}\right)$$

III. If $L\{f(S)\} = F(P)$ then $L\{f(S-a)\,1-(S-a)\} = e^{-aP} F(P)$

This property may be expressed as follows:

If $L\{f(S)\} = F(P)$ and $g(S) = \begin{cases} f(S-a); & S > a \\ 0; & S < a \end{cases}$

Then $\quad L\{g(S)\} = e^{-aP} F(P)$

IV. $L\{f'(S)\} = P\,L\{f(S)\} - f(0)$

V. $L\{f''(S)\} = P^2\,L\{f(S)\} - Pf(0) - f'(0)$

VI. If $L\{f(S)\} = F(P)$ then $L[S\,f(S)] = -\dfrac{d}{dP} F(P)$

VII. If $L\{f(S)\} = F(P)$ then $L\{S^n f(S)\}(-1)^n \dfrac{d^n}{dp^n} F(P)$

VIII. If $L\{f(S)\} = F(P)$ then $L\left\{\dfrac{f(S)}{S}\right\} = \displaystyle\int_P^\infty F(P)\,dp$

IX. $\lim\limits_{S\to 0} f(S) = \lim\limits_{P\to\infty} Pf(P)$

$\lim\limits_{S\to\infty} f(S) = \lim\limits_{P\to 0} Pf(P)$

X (a) The convolution of two functions $f(S)$ and $g(S)$ is defined as

$$f * g = \int_0^S f(x)\,g(S-x)\,dx - \int_0^S f(S-x)\,g(x)\,dx$$

(b) If $L\{f(S)\} = F(P)$ and $L\{g(S)\} = G(P)$

Then $L^{-1}\{f(S)g(S)\} = \displaystyle\int_0^S f(x)\,g(S-x)\,dx = f * g$

or $\quad L^{-1}\{f(S)\cdot g(S)\} = \displaystyle\int_0^S f(S-x)\,g(x)\,dx = f * g$

Known as convolution theorem.

Example 17 Solve the integral equations:

(i) $\phi'(x) = x + \displaystyle\int_0^x \cos\xi\,\phi(x-\xi)\,d\xi,\ \phi(0) = 4$

(ii) $\phi'(x) = \sin x + \int_0^x \cos \xi \, \phi(x-\xi) \, d\xi, \phi(0) = 0$

Solution

(i) The integral equation is given by

$$\phi'(x) = x + \int_0^x \cos \xi \, \phi(x-\xi) \, d\xi \tag{107}$$

The integral Eq. (107) may be written as

$$\phi'(x) = x + \phi(x) * \cos x, \phi(0) = 4 \tag{108}$$

Taking Laplace Transform of the relation in Eq. (108), we have

$$L[\phi'(x)] = L(x) + L\{\phi(x)\} \cdot L(\cos x)$$

or $$PL\{\phi(x)\} - \phi(0) = \frac{1}{P^2} + L\{\phi(x)\}. \frac{P}{P^2+1}$$

or $$\left(1 - \frac{1}{P^2+1}\right) P \, L\{\phi(x)\} = \frac{1}{P^2} + 4$$

or $$L\{\phi(x)\} = \frac{P^2+1}{P^3}\left(\frac{1}{P^2} + 4\right) = \frac{4}{P} + \frac{5}{P^3} + \frac{1}{P^5}$$

Taking inverse Laplace Transform, we have

$$\phi(x) = 4 + \frac{5}{2!}x^2 + \frac{1}{4!}x^4 = 4 + \frac{5}{2}x^2 + \frac{1}{24}x^4$$

(ii) The integral equation is given by

$$\phi'(x) = \sin x + \int_0^x \cos \xi \, \phi(x-\xi) \, d\xi \tag{109}$$

The integral Eq. (109) may be expressed as

$$\phi'(x) = \sin x + \phi(x) * \cos x, \phi(0) = 0 \tag{110}$$

Taking Laplace Transform of the relation in Eq. (110), we have

$$L[\phi'(x)] = L(\sin x) + L[\phi(x) * \cos x]$$

or $$PL[\phi^*(x)] - \phi(0) = \frac{1}{P^2+1} + L\{\phi(x)\} \cdot L\{\cos x\}$$

or $$PL[\phi(x)] - \phi(0) = \frac{1}{P^2+1} + L\{\phi(x)\} \cdot \frac{P}{P^2+1}$$

or $$\left[1 - \frac{1}{P^2+1}\right] P \, L\{\phi(x)\} = \frac{1}{P^2+1} \Rightarrow \frac{P^3}{P^2+1} L\{\phi(x)\} = \frac{1}{P^2+1}$$

or $$L\{\phi(x)\} = \frac{1}{P^3}$$

Taking inverse Laplace Transform, we have

$$\phi(x) = L^{-1}\left(\frac{1}{P^3}\right) = \frac{1}{2}x^2$$

Example 18 Solve the integral equation

$$\phi(x) = 1 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi$$

Solution

The integral equations is given by

$$\phi(x) = 1 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi \tag{111}$$

The integral Eq. (111) may be expressed as

$$\phi(x) = 1 + \phi(x) \times \sin x$$

Taking the Laplace Transform, we have

$$L\big[\phi(x)\big] = L[1] + \big[\phi(x) \times \sin x\big]$$

$$L\big[\phi(x)\big] = \frac{1}{P} L\big\{\phi(x)\big\} \cdot \frac{1}{P^2+1}$$

$$\left(1 - \frac{1}{P^2+1}\right) L\,\big\{\phi(x)\big\} = \frac{1}{P}$$

$$L\big\{\phi(x)\big\} = \frac{P^2+1}{P^3} = \frac{1}{P} + \frac{1}{P^3}$$

Taking inverse Laplace Transform, we have

$$\phi(x) = L^{-1}\left(\frac{1}{P}\right) + L^{-1}\left(\frac{1}{P^3}\right)$$

or $$\phi(x) = 1 + \frac{1}{2}x^2$$

EXERCISE 15.6

1. Solve the integral equation

$$\int_0^x e^{x-\xi}\phi(\xi)\,d\xi = x$$

2. Solve the integral equation

$$\phi(x) = 1 - \int_0^x (x-\xi)\,\phi(\xi)\,d\xi$$

3. Solve the integral equation

$$\phi(x) = x^2 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi$$

4. Solve the integral equation

$$\phi(x) = x + 2\int_0^x \cos(x-\xi)\,\phi(\xi)\,d\xi$$

5. Solve the integral equation

$$\int_0^x \phi(\xi)\,\phi(x-\xi)\,d\xi = 16\sin 4x$$

Answer

1. $\phi(x) = 1 - x$
2. $\phi(x) = \cos x$
3. $\phi(x) = x^2 + \frac{1}{12}x^4$
4. $\phi(x) = 2e^x(x-1) + x + 2$
5. $\phi(x) = \pm 8J_0(4x)$

SUMMARY

In mathematics, an integral equation is an equation in which an unknown function appears under an integral sign

There is a close connection between differential and integral equations and some problems may be formulated either way.

The mast basic type of integral equation is called a Fredholm equation of the first type

$$f(x) = \int_a^b K(x_1, \xi)\,\phi(\xi)\,d\xi$$

Here ϕ is an unknown function, f is known function, and K is another known function of two variables often called the Kernel function. Note that the limits of integration are constant this is what characterizes a Fredholm euqation.

If the unknown function occurs both inside and outside of the integral equation is known as a Fredholm equation of the second type

$$\phi(x) = f(x) + \lambda \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi$$

The parameter, λ is an unknown factor which plays the same role as the eigen value in linear Algebra.

If one limit of integration is a variables the equation is called a Volterra equation. The followiong are called Volterra equations of the first and second types respectively.

$$f(x) = \int_a^x K(x, \xi)\, \phi(\xi)\, d\xi$$

$$\phi(x) = f(x) + \lambda \int_a^x K(x, \xi)\, \phi(\xi)\, d\xi$$

in all of the above, if the known function f is identically zero, the equation is called a homogeneous integral equation. If f is non-zero. It is called an inhomogeneous integral equation.

Applications of Fredholm Integral Equation

Fredholm equations arise naturally in the theory of Signal Processing most notably as the famous spectral concentration problem popularized by David Slepian. They also commonly arise in linear forward modeling and inverse problems. In physics, the solution of such integral equations allows for experimental spectra to be related to various underlying distributions, for instance the mass distribution of polymers in a polymeric melt, or the distribution of relaxation times in the system.

Application of Volterra Integral Equation

Volterra integral equations find application in demography, the study of viscoelastic materials and in insurance mathematics through the renewal equation.

OBJECTIVE TYPE QUESTIONS

1. The integral equation

$$g(x)y(x) = f(x) + \lambda \int_\alpha^\beta K(x, \xi)\, y(\xi)\, d\xi$$

with $f(x)$, $g(x)$ and $K(x, \xi)$ as known functions, α and β as known constants and λ as a known parameter, is a

(a) linear integral equation of Volterra type
(b) linear integral equation of Fredholm type
(c) non-linear integral equation of Volterra type
(d) non-linear integral equation of Fredholm type

2. The solution of the integral equation

$$\phi(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 x\xi\phi(\xi)\, d\xi \text{ satisfies}$$

(a) $\phi(0) + \phi(1) = 1$

(b) $\phi\left(\frac{1}{2}\right) + \phi\left(\frac{1}{3}\right) = 1$

(c) $\phi\left(\frac{1}{4}\right) + \phi\left(\frac{1}{2}\right) = 1$

(d) $\phi\left(\frac{3}{4}\right) + \phi\left(\frac{1}{4}\right) = 1$

3. For the Volterra type linear integral equation

$$\phi(x) = x + 2\int_0^x e^{x-\xi}\,\phi(\xi)\,d\xi$$

The first iterated Kernel of the Kernel $e^{x-\xi}$ is

(a) $(x-\xi)^2\, e^{2(x-\xi)}$ (b) $(x-\xi)\, e^{x-\xi}$

(c) $e^{3(x-\xi)}$ (d) $e^{(x-\xi)}$

4. For the Volterra type integral equation

$$\phi(x) = x + 2\int_0^x e^{x-\xi}\,\phi(\xi)\,d\xi,$$

The resolvent Kernel $R(x, \xi; 2)$ of the Kernel $e^{x-\xi}$ is

(a) $(x-\xi)^2\, e^{2(x-\xi)}$ (b) $(x-\xi)\, e^{x-\xi}$

(c) $e^{3(x-\xi)}$ (d) $e^{(x-\xi)}$

5. For the integral equation

$u(x) = f(x) + \lambda\int_a^b K(x,\xi)\,u(\xi)\,d\xi$ to have a continuous solution in the interval $a \le x \le b$, which of the following assumptions are necessary?

(a) $K(x, \xi) \not\equiv 0$ is real and continuous in the region $a \le x \le b$, $a \le \xi \le b$ with $|K(x, \xi)| \le M$

(b) $f(x) \not\equiv 0$, is real and continuous in the interval $a \le x \le b$

(c) λ is a constant

(d) $|\lambda| < \dfrac{1}{M(b-a)}$

6. The integral equation

$$g(x)\,y(x) = f(x) + \lambda\int_a^x K(x,\xi)\,y(\xi)\,d\xi$$

with $f(x)$, $g(x)$ and $K(x, \xi)$ as known functions, α and β as known constants and λ as a known parameter, is a

(a) linear integral equation of Volterra type

(b) linear integral equation of Fredholm type

(c) non-linear integral equation of Volterra type

(d) non-linear integral equation of Fredholm type

7. Solution of integral equation given below will be

$$3\sin 2x = y(x) + \int_0^x (x-\xi)\,y(\xi)\,d\xi$$

(a) $-2\sin x - \sin 2x$ (b) $\sin x + \sin 2x$

(c) $\sin x + \sin^2 x$ (d) None

8. For the integral equation

$$u(x) = f(x) + \lambda\int_a^x K(x,\xi)\,u(\xi)\,d\xi$$

to have a continuous solution in the interval $a \le x \le b$, which of the following assumptions are necessary?

(a) $K(x, \xi) \not\equiv 0$ is real and continuous in the region $a \le x \le b$, $a \le \xi \le b$ with $|K(x, \xi)| \le M$

(b) $f(x) \not\equiv 0$, is real and continuous in the interval $a \le x \le b$

(c) λ is constant

(d) $|\lambda| < \dfrac{1}{M(b-a)}$

9. Solution of $y(x) = x + \int_0^{1/2} y(\xi)\,d\xi$ is

(a) $x - \dfrac{1}{4}$ (b) $x + \dfrac{1}{4}$

(c) $x + \dfrac{1}{2}$ (d) $-x + \dfrac{1}{4}$

10. Solution of $y(x) = \sin x - \dfrac{x}{4} + \dfrac{1}{4}\int_0^{\pi/2} x\,\xi\,y(\xi)\,d\xi$ is a

(a) $\sin x$ (b) $\cos x$

(c) $-\sin x$ (d) None

ANSWERS

1. (b)	2. (a), (d)	3. (d)	4. (c)
5. (a), (b), (d)	6. (a)	7. (d)	8. (a), (b), (c)
9. (b)	10. (a)		

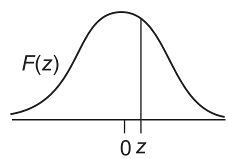
APPENDIX A

Statistical Tables

Table A.1 Ordinates of the Normal Probability Curve

$$Z = \frac{X - E(X)}{\sigma_X} = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2} dt, -\infty < z < \infty$$



$$F(-z) = 1 - F(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–5.0	0.0000003									
–4.0	0.00003									
–3.5	0.0002									
–3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
–3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0006	0.0003
–3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005
–3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
–3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
–2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
–2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
–2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
–2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
–2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
–2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084

(*Contd.*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
–1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0332	0.0314	0.0307	0.0301	0.0294
–1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
–1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
–1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
–1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
–1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
–1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3330	0.3264	0.3228	0.3192	0.3156	0.3221
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5973	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549

(*Contd.*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998									
4.0	0.99997									
5.0	0.9999997									

Table A.2 Values of |*t*| with probability *P* and degrees of freedom *n*

v \ P	0.50	0.10	0.05	0.02	0.01
1	1.000	6.34	12.71	31.82	63.66
2	0.816	2.92	4.30	6.96	9.92
3	0.765	2.35	3.18	4.54	5.84
4	0.741	2.13	2.78	3.75	4.60
5	0.727	2.02	2.57	3.36	4.03
6	0.718	1.94	2.45	3.14	3.71
7	0.711	1.90	2.36	3.00	3.50
8	0.706	1.86	2.31	2.90	3.36
9	0.703	1.83	2.26	2.82	3.25
10	0.700	1.81	2.23	2.76	3.17
11	0.697	1.80	2.20	2.72	3.11
12	0.695	1.78	2.18	2.68	3.06
13	0.694	1.77	2.16	2.65	3.01
14	0.692	1.76	2.14	2.62	2.98
15	0.691	1.75	2.13	2.60	2.95
16	0.690	1.75	2.12	2.58	2.92
17	0.689	1.74	2.11	2.57	2.90
18	0.688	1.73	2.10	2.55	2.88
19	0.688	1.73	2.09	2.54	2.86
20	0.687	1.72	2.09	2.53	2.84
21	0.686	1.72	2.08	2.52	2.83
22	0.686	1.72	2.07	2.51	2.82
23	0.685	1.71	2.07	2.50	2.81
24	0.685	1.71	2.06	2.49	2.80
25	0.684	1.71	2.06	2.48	2.79
26	0.684	1.71	2.06	2.48	2.78
27	0.684	1.70	2.05	2.47	2.77
28	0.683	1.70	2.05	2.47	2.76
29	0.683	1.70	2.04	2.46	2.76
30	0.683	1.70	2.04	2.46	2.75

Table A.3 Values of χ^2 with probability P and df v

v \ P	0.99	0.95	0.50	0.30	0.20	0.10	0.05	0.01
1	0.0002	0.004	0.46	1.07	1.64	2.71	3.84	6.64
2	0.20	0.103	1.39	2.41	3.22	4.60	5.99	9.21
3	0.115	0.35	2.37	3.66	4.64	6.25	7.82	11.34
4	0.30	0.71	3.36	4.88	5.99	7.78	9.49	13.28
5	0.55	1.14	4.35	6.06	7.29	9.24	11.07	15.09
6	0.87	1.64	5.35	7.23	8.56	10.64	12.59	16.81
7	1.24	2.17	6.35	8.38	9.80	12.02	14.07	18.48
8	1.65	2.73	7.34	9.52	11.03	13.36	15.51	20.09
9	2.09	3.32	8.34	10.66	12.24	14.68	16.92	21.67
10	2.56	3.94	9.34	11.78	13.44	15.99	18.31	23.21
11	3.05	4.58	10.34	12.90	14.63	17.28	19.68	24.72
12	3.57	5.23	11.34	14.01	15.81	18.55	21.03	26.22
13	4.11	5.89	12.34	15.12	16.98	19.81	22.36	27.69
14	4.66	6.57	13.34	16.22	18.15	21.06	23.68	29.14
15	5.23	7.26	14.34	17.32	19.31	22.31	25.00	30.58
16	5.81	7.96	15.34	18.42	20.46	23.54	26.30	32.00
17	6.41	8.67	16.34	19.51	21.62	24.77	27.59	33.41
18	7.02	9.39	17.34	20.60	22.76	25.99	28.87	34.80
19	7.63	10.12	18.34	21.69	23.90	27.20	30.14	36.19
20	8.26	10.85	19.34	22.78	25.04	28.41	31.41	37.57
21	8.90	11.59	20.34	23.86	26.17	29.62	32.67	38.93
22	9.54	12.34	21.34	24.94	27.30	30.81	33.92	40.29
23	10.20	13.09	22.34	26.02	28.43	32.01	35.17	41.64
24	10.86	13.85	23.34	27.10	29.55	33.20	36.42	42.98
25	11.52	14.61	24.34	28.17	30.68	34.68	37.65	44.31
26	12.20	15.38	25.34	29.25	31.80	35.56	38.88	45.64
27	12.88	16.15	26.34	30.32	32.91	36.74	40.11	46.96
28	13.56	16.93	27.34	31.39	34.03	37.92	41.34	48.28
29	14.26	17.71	28.34	32.46	35.14	39.09	42.56	49.59
30	14.95	18.49	29.34	33.53	36.25	40.26	43.77	50.89

Table A.4 5% and 1% points of *F*

v_2 \ v_1	1	2	3	4	5	6	8	12	24	∞
2	18.51	19.00	19.16	19.25	19.30	19.32	19.37	19.41	19.45	19.50
	98.49	99.00	99.17	99.25	99.30	99.33	99.36	99.42	99.46	99.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
	16.26	13.27	12.06	11.39	10.97	10.67	10.27	9.89	9.47	9.02
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.01	2.57
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.81
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39

APPENDIX

B

Some Basic Formulae

Complex Number

$z = x + iy = r(\cos\theta + i\sin\theta)$ and its modulus $|z| = |x + iy| = \sqrt{x^2 + y^2}$

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e., $\text{Re}(z) = \text{Im}(z) = 0$. (ii) $|\bar{z}|^2 = z\bar{z}$

(iii) $|z_1 z_2| = |z_1||z_2|$ (iv) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $z_2 \neq 0$.

(v) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1\,\bar{z}_2)$ (vi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\,(z_1\,\bar{z}_2)$

(viii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

Euler's Theorem

$\cos n\theta + i\sin n\theta = cis\theta = e^{i\theta}$

De-Moivers's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Series

Exponential Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

Logarithmic Series

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\infty; \qquad \log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\infty$$

Gregory Series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots; \qquad \tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Sine and Cosine Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots; \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Binomial Series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1+x)^{-1} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

$$(1-x)^{-1} = 1 + nx + \frac{n(n+1)}{2!}x^2 -+ \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

Circular Functions

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \tan x = \frac{1}{i}\left[\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\right], \cot x = i\left[\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}\right],$$

$$\operatorname{cosec} x = \frac{2i}{e^{ix} - e^{-ix}} \text{ and } \sec x = \frac{2}{e^{ix} + e^{-ix}}.$$

Hyperbolic Functions

For any real or complex 'x'

$$\sin hx = \frac{e^x - e^{-x}}{2}, \ \cosh x = \frac{e^x + e^{-x}}{2}, \tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \cot hx = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \operatorname{cosec} hx = \frac{2}{e^x - e^{-x}}$$

$$\text{and } \sec hx = \frac{2}{e^x + e^{-x}}.$$

Inverse Hyperbolic Functions

$$\cos h^{-1} z = \log\left[z + \sqrt{(z^2 - 1)}\right], \sin h^{-1} z = \log\left[z + \sqrt{(z^2 + 1)}\right] \text{ and } \tan h^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

Relation between Circular and Hyperbolic Function

(i) $\sin h(ix) = i \sin x$, $\cos h(ix) = \cos x$, $\tan h(ix) = i \tan x$.

(ii) $\sec h(ix) = \sec x$, $\operatorname{cosec} h(ix) = -i \operatorname{cosec} x$, $\cot h(ix) = -i \cot x$.

(iii) $\sin (ix) = i \sin h\, x$, $\tan (ix) = i \tan hx$, $\cos (ix) = \cos hx$.

Integration

$$\int \sqrt{(a^2 - x^2)}dx = \frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}; \quad \int \sqrt{(a^2 + x^2)}dx = \frac{x\sqrt{(a^2 + x^2)}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

$$\int \sqrt{(x^2 - a^2)}dx = \frac{x\sqrt{(x^2 - a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

$$\left[\int e^{ax} \sin bx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)\right] \text{ and } \left[\int e^{ax} \cos bx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)\right]$$

$$\int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2}; \quad \int_0^\infty e^{-ax} \cos bx dx = \frac{b}{a^2 + b^2}; \quad \int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$$

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R

S