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Concepts and Applications

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Futures and Options
Concepts and Applications

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To
My Parents
Savitri Parameswaran
and
Late A.S. Parameswaran

Foreword

It is a pleasure to introduce this excellent book on derivatives written by my old friend, student and colleague Professor Sunil Parameswaran. Sunil was a student of mine at the Fuqua School of Business, Duke University, from where he obtained his doctorate. Subsequently, he has been teaching courses on futures and options and risk management in India, and as a visiting Professor of Finance at La Trobe University, Melbourne.

Sunil's book has a number of attractive features that appeal to a seasoned reader like me. One of the most satisfying aspects is that it gives substantial attention to both futures, as well as, options markets. There are a number of authors who have expounded on these topics in varying degrees of detail. However, it has been my experience that most texts in this area cover either futures or options in sufficient detail, while paying only superficial attention to the other. The text makes no assumptions about prior knowledge on the part of readers and builds the concepts from basic principles. Professor Parameswaran has a unique and racy conversational style that gives a feeling to the reader that he is sitting across the table having a conversation with him/her, and explaining the concepts in simple language, the requisite mathematical details notwithstanding. He has also made an attempt to seamlessly integrate the features of the underlying products, such as bonds, foreign exchange, stock indices, and money market securities. The advantage for the reader is the book is a stand-alone text, and does not require the reader to necessarily refer to other treatises in order to comprehend the concepts and issues.

The book is fairly mathematical in parts, as is to be expected from a work on this subject. However, an appealing feature of the text is that the author has made extensive use of the simple and easy to understand binomial model for explaining the valuation of American options, and exotic options.

Some of the features which warrant a more extensive treatment are the valuation of American options, numerical methods in option pricing, and credit derivative products. I am sure that Sunil will do justice to these issues in a subsequent edition.

The book has a global flavour and focus, although most of the market specific details are with reference to the US derivatives market. Given its emphasis on concepts and principles, the book warrants the attention of readers at both the Bachelor's and Master's levels in all parts of the world, including Asia and Australia. Market professionals too are likely to find the book to be a useful resource for reference.

I think readers will find this text a very instructive introduction to derivatives which will help to demystify this very important topic. Beginners will find that it facilitates the development of a solid foundation on which they can build up their knowledge further with the aid of more advanced books. Readers with prior knowledge of derivatives will also find the book useful for it reinforces the fundamentals and provides an intermediate level perspective on the subject.

I thoroughly enjoyed reading the book and recommend the text without any reservations.

Tom Smith
Professor of Finance
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Preface

I began teaching a course on financial derivatives in 1996 at a time when the markets for such instruments did not exist in India. My interest in this fascinating subject was whetted by my experiences in the US as a Ph.D student where I had an opportunity to take courses in futures and options at both the Master's as well as doctoral levels. This book has grown out of my efforts to collate information about derivatives trading in international financial markets, and present them in a way that an inquisitive student of finance can relate to, not only in India, but also in other parts of Asia and the rest of the world. The fact that trading in financial futures and options is now booming even in developing countries such as India and China, has only served to make this book more pertinent and relevant for students as well as professionals.

It has been my experience that books which seek to cover both futures and options ultimately end up being unable to do justice to either of the two instruments and the markets that they trade in. This book, consequently makes an attempt to devote the required level of attention to both topics.

This book is intended primarily for second year MBA students. The entire book can be covered in a course on financial derivatives or risk management. Certain portions of the book should also be relevant for courses in international finance and fixed income securities. Market professionals should also hopefully find this book to be a one-stop reference on the subject.

The entire discussion is fairly technical and assumes an adequate level of mathematical knowledge on the part of the reader. All major results have been derived in adequate detail, for nothing is more frustrating for the reader than to have an author present a result with a statement that **'it should be obvious'**. Economics as a subject has become extremely quantitative over the years. I suppose it is inevitable as a subject makes a transition from the realm of humanities to become more of a **social science**. Finance, being an application of economics, is no exception to this tendency. The other advantage of having precise mathematical models is that they make theories culture-independent, and consequently portable across countries.

It must be pointed out that this book does not claim to offer insights into **making a fast buck** in the financial markets. It has been my experience that people who preach can rarely practise and vice versa[1], and keeping this maxim in view, I have stuck to my perceived core competency.

[1]Unless one happens to be a medical practitioner or a lawyer, as well as, an ordained priest!

This book has been written for the global market, and consequently the proper nouns are Western, and the examples are largely from the markets in developed countries, in order to ensure that students can relate to the book irrespective of their location. The final chapter however attempts to provide a reasonably detailed coverage of the Indian market.

The market for financial derivatives is growing rapidly in countries like India, as developing countries following the tenets of LPG (Liberalization, Privatization, and Globalization) are integrating with the world economy. This philosophy has led to an ever-increasing demand for knowledgeable finance professionals, and has also facilitated their mobility across countries and continents. Thus, knowledge of financial instruments such as futures and options is imperative for people aspiring for a career in this area, and such skill sets will certainly stand them in good stead.

I hope that this book will provide a firm platform for students and professionals wherever they may be, and facilitate their progress as they navigate their way through a career in finance.

Sunil K. Parameswaran

Acknowledgments

I owe a great intellectual debt to the researchers and academics who have explored the area of financial derivatives much before I began my study of the subject.

This book has developed over the years from my teaching notes. I have benefited immensely from the comments and feedback from my students at the T.A. Pai Management Institute (TAPMI), the Xavier Institute of Management (XIMB), the SDM Institute for Management Development (SDM-IMD), and La Trobe University. They have been the '**guinea pigs**', in the sense that students of each batch have patiently read through the draft of the book that was made available to them. I am very thankful to them for their incisive comments regarding both the content, as well as, the style. The expositional clarity that I believe I have been able to eventually achieve, is mainly due to their detailed feedback. I am also grateful to the library staff at all these institutions for ensuring that I readily got access to the required books and journal articles.

I wish to thank Tom Smith for taking the time to pen a foreword for this book, and to Shantaram Hegde of the University of Connecticut for his detailed comments on the text. I would like to thank Mr. R. Chandra Sekhar, Tata McGraw-Hill Education, New Delhi, and Ms. Pauline Chua, McGraw-Hill Education, Singapore, for strongly backing this project from the outset. The editorial team has provided wonderful support, and I am grateful for their relentless pressure. I am also thankful to them for making arrangements to process this manuscript in LaTeX which is something that I was very finicky about.

Finally, I wish to acknowledge my debt to my family members for their patience and moral support.

Sunil K. Parameswaran

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Author's Profile

Sunil K. Parameswaran is currently Professor of Finance at SDM-IMD, Mysore. He has taught at the T.A. Pai Management Institute, Manipal; University of Iowa; and the National University of Singapore, and has been a visiting faculty at the La Trobe University, Melbourne, and XIM, Bhubaneshwar. He has been regularly conducting training programs for leading IT companies in Bangalore, Chennai, Hyderabad and Pune. Dr. Parameswaran obtained his PGDM from IIM-Bangalore, and his Ph.D (Finance) from the Fuqua School of Business, Duke University, North Carolina. A prolific writer, Dr. Parameswaran has published 11 books with Tata McGraw-Hill, New Delhi, and John Wiley, Singapore. His areas of interest include securities markets, fixed income securities, derivatives, investment banking and commercial banking.

1

Introduction to Futures

1.1 Introduction

The focus of this book is on financial derivatives, securities that have inspired fascination as well as fear. They are termed as *Derivative Securities*, but a more appropriate term for them would be *Derivative Contracts*. These contracts confer upon the parties to them, certain rights or obligations. The reason why they are termed as *Derivatives*, is because they owe their existence to the presence of markets for an underlying asset or portfolio of assets, on which such agreements are written. If we consider the underlying asset to be the primary security, then these contracts which are derived from the underlying assets may be construed as derivative securities. The underlying asset may be a stock, a bond, a foreign currency, an agricultural commodity like wheat, a precious metal like gold, or a portfolio of assets such as a stock index.

Basic derivative securities may be classified into the following categories:

- Forward Contracts
- Futures Contracts
- Options Contracts
- Swaps

The book will begin with a detailed study of forward and futures contracts. At the outset we will briefly discuss options and swaps as well, primarily to facilitate a comparison between futures and options. The latter half of this book is devoted to an in-depth study of options, and the penultimate chapter contains a more elaborate exposition on swaps.

1.2 Cash/Spot versus Forward Contracts

Consider the case of a traveler who is flying from New York to Frankfurt. He may prior to his departure from JFK Airport like to convert the US dollars in his possession to euros. He will therefore go to a bank and buy the required number of euros by paying the equivalent amount in dollars. In this case as soon as the deal is struck between the buyer of euros (the traveler) and the seller (the bank), the buyer has to immediately hand over the payment for the asset being bought to the seller, who in turn is required to transfer the rights to the asset to the buyer. Such a transaction is called a *Cash* or *Spot* transaction.

Now consider a slightly different kind of transaction, which is illustrated in the example given below.

1.2.1 Example

Mind Tech, a software firm based in California, has entered into a contract with First National Bank to acquire 200,000 euros after 60 days at an exchange rate of $ 1.40 per euro. Consequently, 60 days after entering into the agreement, the company is required to pay $ 280,000 to the bank and accept 200,000 euros in lieu. The bank, as per the contract, is obligated to accept the equivalent amount in US dollars and deliver the euros.

The difference in the case of such a contract, is that the actual transaction does not take place when the agreement is reached between the buyer and the seller. In such cases at the time of negotiating the deal the two parties merely agree on the terms at which they will transact at a future point in time, including the price to be paid per unit of the underlying asset. The actual transaction per se will occur only at a future date that is decided at the outset. Consequently, unlike in the case of a cash transaction, no money changes hands when the two parties enter into such a contract.

Such a transaction is termed as a *forward contract*. In foreign exchange markets, traders make a distinction between three types of transactions—*Cash*, *Tom*, and *Spot*. A cash transaction is what we described at the beginning, where payment is made and delivery is received as soon as the deal is struck. *Tom* stands for tomorrow. In other words, a price is fixed in advance for a transaction that is scheduled to be completed on the next business day. The term *Spot* in such markets means that the terms are fixed in advance for a transaction that is scheduled to be consummated two business days later.[1] We will however use the terms *Cash* and *Spot* interchangeably, and by them we refer to transactions where payment and the corresponding delivery takes place as soon as a deal is negotiated between a buyer and a seller. *Futures Contracts* are in many respects similar to forward contracts, although there are some key differences which we will focus upon later.

In the case of both forward as well as futures contracts, having negotiated a contract for a transaction at a future date, both parties have a binding commitment to perform. In our example, if after 60 days Mind Tech declines to acquire the euros then it would tantamount to default on its part. On the other hand, if the bank refuses to deliver the euros as per the agreed upon rate, then it would be construed as default on its part. Consequently such instruments are known as *commitment* contracts.

[1]The term *business days* is used, because there could be intervening market holidays.

1.2.2 Long and Short Positions

Both forward and futures contracts are therefore agreements for the future delivery of a good at a pre-specified price at the end of a designated period of time. Thus there has to be a buyer of the underlying asset as well a seller of the underlying asset as per the specified terms. The party who commits himself to buy the underlying asset is said to have taken a *Long Position* and is known as the *Long*. The counterparty who is committed to sell the underlying asset is said to have taken a *Short Position* and he is known as the *Short*.

1.3 Options

Both forward as well as futures contracts, impose an obligation on the long as well as the short. On the other hand an option gives the buyer the right, but not the obligation, to go ahead with the transaction, subsequent to entering into an agreement with the seller. The option buyer is also referred to as the long, while the seller is known as the short.[2]

The difference between a right and an obligation is that a right need be exercised only if it is in the interest of its holder, and if he deems it appropriate. Thus the holder of an option is not obliged to go through with the transaction once he enters into such a contract. However, the counterparty, namely the short, always has an obligation. That is, if the long decides to exercise his right, the short is obliged to carry out his part of the deal.

The reader may wonder as to why both the long as well as the short cannot be given rights. The reason is the following. In any transaction which is scheduled for a future date, as per terms decided upon at the outset, one party will inevitably be at a loss by the time the transaction date arrives. Hence, if both parties are given rights then the party who has a loss will obviously refuse to perform. Thus, in such contracts, both the parties can have obligations imposed on them, which is what is done in the case of forward and futures contracts, or else one party can be given a right and the other party can have an obligation imposed on it, which is what is done in the case of an options contract.

1.3.1 Calls and Puts

Options give the holder the right to transact in the underlying asset. Such rights can obviously take on one of two forms. That is, the holder may be given the right to buy the underlying asset, or else he may be given the right to sell the underlying asset. An options contract which gives the holder the right, but not the obligation, to acquire the underlying asset is known as a *Call* option. In such cases, the seller of the call, has the obligation to deliver the asset if the buyer chooses to exercise his right.

On the other hand, an options contract which gives the holder the right, but once again not the obligation, to sell the underlying asset is known as a *Put* option.

[2]Option buyers are also referred to as option *Holders*, while option sellers are referred to as option *Writers*.

In such cases, if and when the put holder decides to exercise his option, the seller of the put has the obligation to acquire the underlying asset.

The difference between forward and futures contracts, and call and put options can be illustrated with the help of a simple table. In the table the short, in the case of options, is said to have a contingent obligation. The word *'contingent'* implies that he has an obligation if the long were to choose to exercise his right.

Table 1.1 Comparison of Futures and Forwards, and Options

Instrument	Nature of Commitment of the Long	Nature of Commitment of the Short
Forward/futures contract	Obligation to acquire the underlying asset	Obligation to sell the underlying asset
Call options	Right to acquire the underlying asset	Contingent obligation to deliver the underlying asset
Put options	Right to sell the underlying asset	Contingent obligation to accept delivery of the underlying asset

1.3.2 European and American Options

The holder of an option is given the right to transact in the underlying asset, on or before a pre-specified date. In the case of European options, the acquirer of the option can exercise it only at the time the right is scheduled to expire, which is known as the *Expiration Date* of the option. Obviously, if the holder were not to exercise at that point in time, then the contract itself will become null and void.

American options however, give the holder the right to transact at any point in time, between the time of acquisition of the right and the expiration date of the contract. Thus the expiration date of the option is the *only* point in time at which a European option can be exercised, and is the *last* point in time at which an American option can be exercised.

It must be emphasized that the terms American and European have nothing to do with the respective continents. Most options contracts that are traded on organized exchanges like the Chicago Board Options Exchange (CBOE), are American in nature. However, while introducing the concept, authors initially tend to focus more on European options. The reason is that, since such options can be exercised only at expiration, one needs to consider possible cash flows only at that instant, which makes the valuation of such instruments relatively simple as compared to American options with equivalent features.

1.3.3 Option Price versus Exercise Price

In the context of options, a reader will come across the terms *option price* and *exercise price*. The term *'option price'* or *'option premium'* refers to the amount payable by the buyer of an option at the outset, to the writer, for permitting him to acquire the right. Thus buyers of both call as well as put options are required to pay the writers of such options for giving them the right to transact at a subsequent point in time.

The term *'exercise price'*, or *'strike price'*, refers to the amount payable/receivable per unit of the underlying asset, if the option were to be exercised. It is the amount payable by the holder per unit of the underlying asset, if a call option were to be exercised. Equivalently, it is the amount receivable by the holder per unit of the underlying asset, if a put option were to be exercised.

The option premium may be perceived as a *'sunk cost'*. For, if the transaction were not to take place subsequently, the premium cannot be recovered. The exercise price, on the other hand, is payable/receivable only if the option holder were to choose to go ahead with the transaction. Since he has a right, he may or may not wish to exercise which obviously means that the exercise price may or may not be paid/received subsequently.

Why is it that buyers of options have to pay sellers, a price for entering into such contracts? The reason is that an option contract entails the giving of a right to the holder by the writer, who in turn is taking on an obligation to perform if the buyer were to exercise his right. Rights are never given free and are always accompanied by an economic price. In other words one has to pay to acquire them. Thus, option buyers are required to pay option writers for acquiring the right to transact in the underlying asset.

Futures and forward contracts are however different. Such contracts impose an equivalent obligation on both the long as well as the short. We will demonstrate subsequently, when we study the valuation of such contracts that, the futures price, which is the price at which the long is committed to acquire the underlying asset on a future date, will be set in such a way that the value of the futures contract at inception is zero, for both the long as well as the short. That is, a futures contract which entails two equivalent and opposite obligations, is structured in such a way that neither party has to pay the other at the outset.

An options contract may be perceived as an insurance contract. A party which owns an asset can protect himself from a potential price decline by acquiring a put option. For if the price were to decline subsequently, he can take advantage of his right to sell at the exercise price, which is what a put is essentially all about. Similarly, a firm that plans to buy an asset in the future can insure itself against a price rise by buying a call option. For if the price of the asset were to increase, it can exercise its right to buy at the strike price. Consequently, option premiums are comparable to insurance premiums. Option buyers buy protection against potential financial losses, while option writers sell insurance.

Illustration of a Call Kevin Stone has taken a long position in a European call option on American Express, with an exercise price of \$ 50, and six months to maturity. The option has been written by Kathy Scanlon, who consequently has a short position. Let us assume that the option premium is \$ 1.75 per share. Option prices are always quoted on a per share basis, although each contract on the exchange is for 100 shares. Thus the price payable by Kevin to Kathy at the outset is \$ 175.

If the spot price at the time of expiration of the contract, were to be greater than $ 50 per share, it would make sense for Kevin to exercise his right and acquire 100 shares at $ 50 each. Otherwise, he is in a position to buy the shares in the spot market at a price, which by assumption is lower than the exercise price. He has the freedom because he is under no compulsion to exercise the option, for it confers a right, and does not impose an obligation. It should be noted however, that the premium of $ 175 that he paid at the very outset cannot be recovered if the option is not exercised subsequently. Kathy on the other hand has a contingent obligation. If Kevin were to decide to exercise his right, she would have no choice but to deliver the shares at a price of $ 50 per share.

Illustration of a Put Ross Taylor has bought a European put option on IBM from Mike Robbins, with an exercise price of $ 85 and three months to maturity. The premium on a per share basis is $ 1.10. Thus Ross has to pay $ 110 to Mike at the outset. If the spot price at the time of expiration of the contract, were to be less than $ 85 per share, it would make sense for Ross to exercise the option and sell the shares at $ 85 each. Otherwise, he could simply forget the option, and sell the shares in the spot market at a price, which by assumption is higher than the exercise price. However, if Ross were to decide to exercise his right, Mike would have no option but to accept delivery of the shares at a price of $ 85 per share.

1.4 Swaps

A swap is an agreement between two parties to exchange cash flows calculated on the basis of two different criteria at predefined points in time. The word *'swap'* connotes that the two parties are exchanging or swapping cash flows.

The cash flows being exchanged represent interest payments on a specified principal, computed using two different yardsticks. For instance, one interest payment may be computed using a fixed rate of interest, while the other may be based on a variable benchmark such as the London Inter Bank Offer Rate (LIBOR). Such a swap where one payment is based on a fixed rate of interest, and the other on a floating rate, is referred to as a *Coupon Swap*.

Swaps where both cash flows are denominated in the same currency are referred to as *interest rate swaps*. In the case of such swaps there is obviously no need to exchange the principal amount since both interest streams are computed in the same currency. Nevertheless, we need to specify a principal amount to enable the computation of interest. Thus, the underlying principal, in the case of such swaps, is referred to as a *notional principal*.

Also, in the case of interest rate swaps, it would obviously make no sense to have a fixed rate-fixed rate swap. This is because the party who is required to pay a higher rate will always be paying, while the other party will always be receiving. No one will obviously agree to such a transaction. However, we can have floating rate-floating rate swaps, where each of the rates is based on a different benchmark. For instance, one leg of the swap could be based on LIBOR, while the other could be based on the US T-bill rate. Such swaps are called *Basis Swaps*.

There do exist swaps where the two cash flows being swapped are denominated in two different currencies. These are known as *currency swaps*. In the case of such swaps, in addition to fixed-floating, and floating-floating arrangements, we could also have fixed-fixed deals. Such swaps entail the exchange of the principal amount, denominated in two different currencies, at expiration, and at times at inception as well.

1.5 Forward Contracts versus Futures Contracts

By now readers would have noted one obvious similarity between forward contracts and futures contracts. That is, both types of contracts require the long to acquire the underlying asset on a future date, and the short to deliver the underlying asset at the same point in time. Thus, in both cases, there is an obligation on the long as well as the short. However, there is one vital difference between the two types of contracts. Futures contracts are *standardized* contracts that are traded on organized exchanges, whereas forward contracts are *customized* contracts that are traded over-the-counter (OTC).

The next issue that needs to be addressed is the clarification of the terms 'standardization' and 'customization'?

In contracts of this nature, which are based on an underlying asset, certain terms and conditions need to be clearly defined at the very beginning to avoid ambiguities and potential conflicts. The major terms which should be made explicit are the following:

- How many units of the underlying asset is the contract based on. That is, what is the quantity that the long is required to acquire, or in other words, what is the quantity that the short has to deliver?
- In the event of assets characterized by multiple grades, what is the acceptable grade, or in certain cases, what are the acceptable grades of the underlying asset that is/are allowable for delivery?
- What is the specified location at which the short is required to make delivery? In delivery allowed only at a particular location, or do one or both parties have a choice of locations?
- What is the delivery date? Is delivery permitted only on a particular day, or is there a specified period during which it can occur?

In the case of customized contracts, the applicable terms and conditions have to be decided based on bilateral negotiations between the buyer and the seller of the contract. Thus, in such cases, the two parties are free to incorporate any features, which they can mutually agree upon. However, in the case of standardized contracts, there is a third party which will specify the allowable terms and conditions. In such cases, the long and the short have limited freedom in the sense that they can design contracts within the boundaries specified by such a party. However, they cannot incorporate features other than those that are specifically allowed. The third party which we are referring to in the case of futures contracts is nothing but the futures exchange. A futures exchange is a securities exchange akin to a stock exchange, and is an arena where trading in futures contracts takes place.

We will illustrate the difference between customization and standardization with the help of an example.

1.5.1 Example

Consider the wheat futures contract that is listed for trading on the Kansas City Board of Trade. The terms specified by the exchange are such that each contract requires the delivery of 5,000 bushels of wheat.[3] As per the specifications, the allowable grades are No. 1, No. 2, and No. 3, and the allowable locations for delivery are Kansas City and Hutchinson. The exchange has stipulated that delivery can be made at any time during the expiration month.

Let us first take the case of Paul Pollock, a wholesale dealer in wheat, who wants to acquire 50,000 bushels of No. 2 wheat in Kansas City during the last week of the month. Assume that there is another party, Vince King, a farmer, who is interested in delivering 50,000 bushels of No. 2 wheat in Kansas City during the same period. In this case, the objectives of the two parties are such that the futures contracts that are listed on the exchange are suitable for both. Consequently, if they were to meet on the floor of the exchange at the same time, a trade could be negotiated between them for ten futures contracts, at a price of say $ 3.35 per bushel. It must be remembered that the price that is agreed upon for the underlying asset is one feature that is not specified by the exchange. This has to be negotiated between the two parties through bilateral negotiations, and is a function of demand and supply conditions.

Having demonstrated the suitability of futures contracts in the above case, let us now consider a slightly different scenario. Assume that Paul wishes to acquire 47,750 bushels of No. 2 quality wheat in Topeka during the last week of the month, and that Vince is looking to sell the same quantity of wheat in Topeka during that period. The terms of the contract that are being sought by the two parties are such that the futures contracts being permitted by the exchange in Kansas City are not suitable for them. However, in principle, the two men can iron out an agreement between them which incorporates the features that they desire. Such an agreement, characterized by features arrived at by a process of consensus would be a customized agreement, that is obviously tailor made to their needs. Such an agreement is what we have been calling a forward contract.

Thus futures contracts are products that are traded on an organized exchange just like equity shares and bonds, whereas forward contracts are private contracts.

While discussing the terms that are required to be spelt out in a futures contract we have mentioned that certain contracts may permit delivery of more than one

[3] A bushel is a unit of measure. It is used primarily as a unit of mass. A standard weight is assigned for each commodity that is measured in bushels. A bushel of wheat is 60 lb or 27.215 kilos.

specified grade, and/or at multiple locations. In such cases the issue is who gets to decide as to where, and what to deliver. Traditionally, the right to choose the location and the grade, has always been given to the short.

Also, the right to initiate the process of delivery, has traditionally been given to the short. That is, the delivery process will commence with the expression of the intention to deliver by a party with a short position. Thus, in practice, an investor with a long position, cannot demand delivery. What this also means is that, those investors with a long position who do not wish to take delivery, will exit the market prior to the commencement of the delivery period, by taking an opposite or offsetting position. For, once the delivery period specified by the exchange commences, they can always be called upon to take delivery without having the right to refuse.

What is offsetting? The term essentially means taking a counter-position. Thus if a party has originally gone long, it should subsequently go short in order to offset and vice versa. The effect of offsetting is to cancel an existing long or short position in a contract.

The second difference between forwards and futures is that although both types of contracts set forth terms for delivery, futures contracts are usually not settled by delivery. Generally, only a very small percentage of outstanding contracts are delivery settled. Others are simply offset by taking counter-positions on the exchange. Forward contracts, however, are usually settled by delivery.

Third, futures contracts are *Marked to Market* on a daily basis, whereas forward contracts are not. What this means is that, in the case of a futures contract, the profit or loss is calculated on a daily basis, and is added to/subtracted from the *Margin Account* of the trader. The margin account is one in which a trader keeps *good faith money or collateral*. Hence, futures contracts are subject to interim cash flows. In a forward contract, however, there is only one cash flow, that is, on the delivery date of the contract.

Fourth, the parties to a forward contract are exposed to *Credit Risk* because either party may default. In the case of futures contracts, credit risk is minimized because of the existence of an entity called a *Clearinghouse* which is associated with the exchange on which the contract is traded. The clearinghouse guarantees the other side of the transaction.[4]

Finally, it is easier to offset a position in a futures contract than in a forward contract. A forward contract by definition is a customized private contract between two parties. Hence, if a party to a forward contract were to desire to cancel the original agreement, he would have no option but to seek out the counter-party with whom he had entered into a deal and have the agreement canceled. In other words, without the concurrence of the counterparty with whom the deal was originally negotiated, a forward contract cannot be abrogated.

Canceling a futures contract is a lot easier in practice. This is because, a futures contract between two parties, say Paul and Vince, to transact in wheat at the end of a particular month, will be identical to a similar contract between two

[4]Closely related to this is the concept of margins.

other parties, say Kathy and Pauline. This is because both the contracts would have been designed in accordance with the framework specified by the exchange. Besides, once Paul and Vince enter into a contract with each other, they have both effectively entered into a contract with the clearinghouse, and the link between them is broken. So if Paul, who has entered into a long position, wants to get out of his position, he does not need to seek out Vince. All he has to do in practice is to go back to the floor of the futures exchange and express his desire to take a short position in a similar contract. This time the opposite position may be taken by a new party, say Doug. Thus, by taking a long position initially with Vince, and a short position subsequently with Doug, Paul can ensure that he has a nil net position, and therefore is effectively out of the market. This would obviously mean that he has no further obligations. As far as the records of the clearinghouse are concerned, Paul would have been recorded as having bought and sold an identical futures contract, and his net position would be reflected as zero. This is the meaning of offsetting.

In practice the futures price that was prevailing when a trader enters into a position, will obviously be different from the price prevailing at the time of offsetting. Thus offsetting will entail a profit or a loss which will be equal to the difference between the futures price that was prevailing at the time the original position was taken, and the price at the time the position is offset.

Except for these differences, we will find that the analysis of futures and forward contracts is fairly similar. Hence, we will first study the valuation of forward contracts in detail[5], and will then apply the results to futures markets. In fact, we will show later that when the riskless rate of interest is constant, and is the same for all maturities, then, the forward price will be the same as the futures price, for contracts on a given asset, with a specified delivery date.

1.6 Standardization and the Role of the Exchange

Futures contracts are standardized instruments. When developing a new contract, the exchange must specify the following features:

1.6.1 The Underlying Asset

The asset must be clearly mentioned. That is, the contract should clearly state as to whether it is corn, or wheat, or oranges that it is based on. If there are different grades or varieties of the commodity, the contract must specify the acceptable grade or grades. In the case of contracts where the underlying asset has multiple varieties, a standard grade will be established, which will usually be the most important grade of the good traded in the spot market. This is called the *Par* grade

[5]They are easier to analyze, because there are no interim cash flows, due to the absence of the marking to market mechanism.

and is deliverable at the prevailing futures price at the expiration of the contract.[6] Other grades of the commodity may be deliverable at specified premiums to or discounts from the price for the par grade. This price adjustment may be either additive or multiplicative, as we shall see later.

1.6.2 The Contract Size

The size of the contract specifies the amount of the underlying asset that has to be delivered as per the contract. Here, the exchange has to balance two factors. If the size is too large, traders who wish to transact small quantities, will be unable to do so. While, if the size is too small, more number of contracts will be required for a given trade, and trading will become more expensive due to greater transactions costs.

1.6.3 Delivery Arrangements

The contract must state the location where delivery is to be made. Sometimes, alternative locations may also be specified. The price paid by the long will be adjusted according to the location where delivery is made.

Delivery of the good is at the option of the short, with respect to the date of delivery[7], the grade of the good delivered, and the location of delivery. That is, *the short has the option of deciding when, where, and what grade of the asset he would like to deliver*. Delivery usually begins before trading ceases and continues after it has ceased.

1.6.4 The Delivery Month

A futures contract is referred to by the delivery month. For instance, a June contract, refers to a contract expiring in June.

At a given point in time, you will find that contracts are being traded for the closest delivery month and a number of subsequent delivery months. For example, most currency futures on the Chicago Mercantile Exchange have delivery months of March, June, September, and December. The exchange has to determine, as to how far into the future contracts must extend. It also has to decide as to when trading should begin in a particular contract, and has to stipulate the last day of trading as well.

The selection of the allowable expiration months varies from asset to asset. Kroll and Paulenoff (1993) give the following example regarding wheat futures.

Example "Futures trading in Chicago wheat is actively conducted in March, May, July, September and December. July is the first month of the new crop of Winter wheat and the July contract allows people to hedge the purchase of wheat in the spot market. The new crop of Spring wheat comes in September and can be hedged in that delivery month. The December contract facilitates

[6] You will understand this as we study the delivery process in greater detail.
[7] At any time during the delivery period specified by the exchange.

hedging against grain stored during the months in which navigation is affected by frozen waterways in the Mid West of the US. The March contract represents the mid point between heavy winter storage and heavy spring consumption, while the May contract can reflect either old or new crop fundamentals, depending on which is more dominant during a particular year."

The fact that futures contracts are standardized, helps reduce transactions costs in such markets. The cost of negotiating a customized contract, such as a forward contract, is typically much higher. Secondly, as we have seen, only certain kinds of futures contracts (with the features specified by the exchange), are eligible for trading. Thus futures markets tend to be highly liquid.

1.7 The Clearinghouse

A clearinghouse is an institution that is associated with a futures exchange.[8] The clearinghouse provides a guarantee to both parties to a contract, that they need not be concerned with the possibility of the other party defaulting. The way it achieves this in practice, is by positioning itself as the effective counter-party for each of the two original parties to the trade. In other words, once a trade is consummated, the clearinghouse effectively becomes the buyer for the party who seeks to sell and the seller for the party that seeks to buy. Hence the two parties need to worry only about the financial strength and integrity of the clearinghouse, and not of their respective counterparties. It must be noted that neither party actually trades with the clearinghouse. The clearinghouse enters the picture only after a trade is effected between the two parties.

Why is it that the intervention of a clearinghouse is required in practice? A futures contract imposes an obligation on both parties to the trade. By the time the contract expires, depending on the movement of prices in the interim, it will be in the interest of one of the two parties to go through with the transaction. However, a price move in favor of one party would obviously be tantamount to a loss for the other. Thus, given an opportunity, one of the parties would like to default on the expiration date, although at the outset we would not know who it would be. We will illustrate this with the help of an example.

1.7.1 Example

Let us take the case of two traders, Paul and Keith. Assume that Paul has gone long in a futures contract to buy an asset five days hence at a price of \$ 75, and that Keith has taken the opposite side of the transaction. We will first take the case where the spot price of the asset five days later is \$ 80.

Consider a situation where Keith already owns the asset. As per the contract, he is obliged to deliver it for \$ 75, thereby foregoing an opportunity to sell it in the spot market at \$ 80. If he were not have the asset, he is required to acquire it

[8]It may be a wing of the exchange or else it may be a separate corporation.

by paying \$ 80, and then subsequently deliver it to Paul for \$ 75. Quite obviously, Keith will choose to default unless he has impeccable moral values.

Now let us consider a second situation where the price of the asset five days hence is \$ 70. In this case, if Keith already has the asset, he would be perfectly willing to deliver it for \$ 75, for the alternative scenario for him is a sale in the spot market at a price of \$ 70. In such a situation, even if he were not to have the asset, he would be delighted to acquire it for \$ 70 in the spot market, and deliver it to Paul in return for a payment of \$ 75.

The problem in the second case is that Paul will refuse to pay \$ 75 for the asset, if he can get away with it. If he were not to require the asset, taking delivery at \$ 75 would mean a subsequent sale in the spot market at \$ 70, and consequently a loss of \$ 5. On the other hand, even if he were to require the asset, he would rather buy it in the spot market for \$ 70.

The function of a clearinghouse is to ensure that such defaults are prevented. In practice, a clearinghouse ensures protection for both counterparties, by requiring them to post a performance bond or collateral called a margin. Both the long as well as the short have to deposit margins with their respective brokers, once the trade is consummated. Their brokers in turn are required to deposit margins with the clearinghouse. The margin deposited by a broker with the clearinghouse is known as *Clearing Margin*. Every day, the collateral is adjusted to take into account any profit or loss for either party, as compared to the previous day, based on the price movement during the day. In the process, the clearinghouse effectively takes away the incentive for a party to default as you shall shortly see.

1.8 Margins and Marking to Market

As we have just seen, any contract requiring performance as per the agreement entered into at a prior point of time, is fraught with the possibility of default. Futures exchanges, ensure compliance by requiring both the parties to the trade to deposit collateral with their respective broker, in an account known as the *Margin Account*. This margin deposit is therefore a good-faith deposit or a performance guarantee.

The amount of collateral that is required to be deposited is obviously related to the potential loss that each party is likely to incur. Since both parties to a futures contract have an obligation, it is therefore necessary to collect collateral from both of them. The rationale behind requiring them to deposit such a performance guarantee is that, once such potential losses are collected, the incentive to default is effectively taken away. This is because, even if the party that ends up on the losing side were to fail to perform its obligation, the collateral collected from it would be adequate to take care of the interests of the other party.

It is important to appreciate that the loss for one of the parties or, the equivalent gain for the other, will not arise all of a sudden at the time of expiration of the contract. As the futures price fluctuates in the market from trade to trade, one of the two parties to an existing futures contract will experience a gain, while the other will experience an equivalent loss. Hence, the total loss or gain from the time of getting into a futures position till the time the position is terminated is the sum of these small losses/profits corresponding to each observed price in the interim. The point of termination of an existing contract is the time of expiration of the contract or else the time that it is offset by taking a counter-position, if that were to happen earlier,

The term *'Marking to Market'* refers to the process of calculating the loss for one party, or equivalently the corresponding gain for the other, at specified points in time, with reference to the futures price that was prevailing at the time the contract was previously marked to market. When a futures contract is entered into, the broker will mark it to market for the first time at the end of the day. Thereafter, he will mark it to market every day until the position is either offset or else the contract itself expires. The party who has incurred a profit will have the amount credited to his margin account, while the other party, who would obviously have incurred an identical loss, will have his margin account debited. *Thus, a futures contract is effectively settled and rewritten on a daily basis. Hence a futures contract is like a series of one day forward contracts.*

Before we illustrate the marking to market mechanism, let us first define the concept of a *settlement price*.

1.8.1 Settlement Price

We have just seen that futures contracts are marked to market on a daily basis. The profit or loss is equal to difference between that day's price and the price that was used when the contract was previously marked to market. The settlement price is nothing but the price that is used to compute the daily gains and losses while marking a contract to market.

A common practice is to set the settlement price equal to the observed closing price for the day. However, due to the occurrence of heavy trading towards the close of the day, some exchanges choose to set the settlement price equal to the average of the observed futures prices, in the last half hour or hour of trading.[9] At the other extreme, if there were to be no trades at the end of the day, the exchange may set the settlement price equal to the average of the observed 'bid' and 'ask' quotes.

What do the terms 'bid' and 'ask' connote? A market maker, who is essentially a broker who stands ready to buy and sell on his own account, will offer a two-way quote at any point in time. One side of the quote will be the price at which he is prepared to buy from a potential customer, while the other price represents the rate at which he is willing to sell to a potential client. The bid will always be less than the ask[10], for obviously the market maker will seek to buy low and sell high in order to make a profit.

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[9] The average may be volume weighted.

[10] We will discuss an exception when we study the principle of 'Indirect Quotes' in foreign exchange markets.

In futures markets, the 'bid' represents the futures price at which a customer can take a short position, while the 'ask' is obviously the price at which he can take a long position.

We will now give a detailed illustration as to how profits and losses arise in the process of marking to market, and will highlight the way in which the corresponding changes are reflected in the margin accounts of the respective parties.

Let us take the case of Paul who has gone long in a futures contract expiring five days hence with Keith, at a futures price of \$ 75. We will assume that the price at the expiration of the contract is \$ 82.50, and that the prices at the end of each day prior to expiration are as follows.

We will also assume that as per the contract, Paul is committed to buying 100 units of the asset, and that at the time of the trade, both the parties had to deposit \$ 1,000 as collateral in their margin accounts. The collateral that an investor is required to deposit at the time of entering into a futures contract, is referred to as the *Initial Margin*.

Table 1.2 **End of the Day Futures Prices**

Day	Futures Price
0	75.00
1	78.50
2	73.50
3	71.00
4	79.50
5	82.50

Day '0' denotes the time the contract was entered into, and the corresponding price is the futures price at which the deal was struck. Day 't' represents the end of that particular day, and the corresponding price is the prevailing futures price at that instant.

At the end of the first day the futures price is \$ 78.50. This means that if a trader were to enter into a contract at the end of the that day, the applicable price per unit of the underlying asset would be \$ 78.50. If Paul were to offset the position that he had entered into earlier that day, he would obviously have to do so by agreeing to sell 100 units at \$ 78.50 per unit. If so, he would earn a profit of \$ 3.50 per unit, or \$ 350 in all. In the process of marking Paul's position to market, the broker will behave as though he were offsetting. Thus, he would calculate his profit as \$ 350, and would credit this amount to his margin account. However, remember that Paul has not actually expressed a desire to offset. Consequently, taking cognizance of this fact, the broker would act as if Paul were re-entering into a long position at the prevailing futures price of \$ 78.50.

At the end of the second day, the prevailing futures price is \$ 73.50. Thus, when the contract is marked to market on this day, Paul will make a loss of \$ 500. It must be remembered, that his contract was re-written the previous day at a price of \$ 78.50, and if the broker were to now behave as if he were offsetting at

$ 73.50, the loss would amount to $ 5 per unit, or $ 500 in all. Having marked the contract to market, the broker would once again establish a new long position for Paul, this time at a price of $ 73.50.

This process will continue every day either until the delivery date, when Paul will actually take possession of the asset, or until the day that he chooses to offset his position, if that were to happen earlier. As should be obvious from this illustration, rising futures prices lead to profits for traders with a long position, whereas falling futures prices lead to losses.

Now let us consider the situation from Keith's perspective. At the end of the first day, when the futures price is $ 78.50, marking to market would mean a loss of $ 350 for him. This can be understood as follows. His earlier contract to sell at $ 75 will be effectively offset by making him buy at $ 78.50. The corresponding loss for 100 units is $ 350. Once this is done, a new short position would be established for him at $ 78.50. Applying the same logic, at the end of the second day, his margin account will be credited with a profit of $ 500. As should be obvious from Keith's perspective, rising futures prices lead to losses for traders with short positions, whereas declining futures prices lead to profits.

Thus, the profit/loss for a trader with a long position is identical to the loss/profit for one with a short position. Hence, futures contracts are called *Zero Sum Games*. One man's gain is another man's loss.

As you can see, by the time the contract expires, in this instance after five days, the loss incurred by one of the two parties, in this case the short, has been totally recovered. In our example, Paul's account would have been credited with $ 750. This amount represents the difference between the terminal futures price and the initial futures price, multiplied by the number of units of the underlying asset. This money would have come from Keith's account which would have been debited by $ 750. Thus, at expiration, if Keith were to refuse to deliver the asset, Paul would not be affected at all. Since he has already realized a profit of $ 750, he can take delivery in the spot market at the terminal spot price of $ 82.50 per unit, in lieu of taking delivery under the futures contract.[11] Thus, effectively, Paul will get possession of the asset at a price of $ 75 per unit, which is what had been specified as per the terms of the original futures contract.

The role of the clearinghouse in these series of transactions is essentially that of a banker. It will debit the margin account of the broker whose client has suffered a loss, and simultaneously credit the margin account of the broker whose client has made a profit. Hence, the margin accounts maintained by brokers with the clearinghouse are adjusted daily for profits and losses, in exactly the same way that a client's margin account with the broker is adjusted.

Forward contracts are different from futures contracts in this respect. That is, they are not periodically marked to market. As a consequence, both the parties to such contracts are exposed to credit risk, which is the risk that the counterparty may default. Thus the parties to a forward contract tend to be large and well

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[11] You will see shortly that at the time of expiration of the contract, the futures price must be equal to the prevailing spot price.

known, such as banks, financial institutions, corporate houses, and brokerage firms. The reason is that the creditworthiness of such parties is easier to appraise as compared to that of individual investors.

At the time of entering into a futures contract, both parties have to deposit a performance bond with their brokers, which we have termed as the Initial Margin. If the markets were to subsequently move in favor of a party to the contract, the balance in his margin account will increase, else if the market were to move against him, the balance will stand depleted. It is very important for the broker to ensure that a client always has adequate funds in his margin account. Otherwise the entire purpose of requiring clients to maintain margins can be defeated. As a consequence, the broker will specify a threshold balance for the margin account called the *Maintenance Margin*, which will be less than the *Initial Margin*. If due to one or more adverse price movements, the balance in the margin account were to decline below the level of the Maintenance Margin, the broker will immediately ask the client to deposit additional funds, to take the balance in his account back to the level of the Initial Margin. Such requests for more collateral are termed as *Margin Calls*. The additional funds deposited by a client to top up his account, in compliance with a Margin Call, are referred to as *Variation Margin*.

We will now give a detailed example to illustrate the concepts that we have just discussed. Let us once again consider the case of Paul, who went long in a contract for 100 units of the asset at a price of $ 75 per unit, and deposited $ 1,000 as collateral for the same. Let us assume that the broker fixes a maintenance margin of $ 750 for the contract. Assuming that the contract lasts for a period of five days, and that the futures prices on the subsequent days are as shown in Table 1.2, the effect on the margin account will be as summarized in Table 1.3.

Table 1.3 Changes in the Margin Account Over the Course of Time

Day	Futures Price	Daily Gain/Loss	Cumulative Gain/Loss	Account Balance	Margin Call
0	75.00			1,000	
1	78.50	350	350	1,350	
2	73.50	(500)	(150)	850	
3	71.00	(250)	(400)	600	400
4	79.50	850	450	1,850	
5	82.50	300	750	2,150	

Numbers in parentheses denote losses

We will analyze in detail a few of the entries in Table 1.3 to illustrate the concepts that we have expounded. Consider the second row. As compared to the time the contract was entered into, the price has increased by $ 3.50 per unit or $ 350 for 100 units. Thus, Paul, who has entered into a long position, has gained $ 350, which is required to be credited to his margin account. After this amount is credited, the margin account which had an opening balance of $ 1,000, has an end-of-the day balance of $ 1,350.

The settlement price at the end of the second day is \$ 73.50. Thus, as compared to the position at the end of the previous day, Paul has suffered a loss of \$ 5 per unit or \$ 500 for 100 units. When this loss is debited to his margin account, the balance in the account falls to \$ 850. The settlement price at the end of the next day is \$ 71, which implies that Paul has suffered a further loss of \$ 250. When this loss is factored into the margin account, by debiting it with \$ 250, the balance in the account falls to \$ 600, which is less than the maintenance margin requirement of \$ 750. Therefore at this point in time, a margin call will be issued for \$ 400, which is the amount required to take the balance back to the initial margin level of \$ 1,000. In response to the call, Paul will be expected to pay a variation margin of \$ 400.

It is not necessary for every trader to meet the initial margin requirement by depositing cash. It is an acceptable practice in many markets to offer securities such as Treasury-bills and equity shares as collateral. However, the value assigned to these assets will be less than their market values at the time of submission. This is because the broker would like to protect himself against a sudden sharp decline in the value of the collateral. For instance, if the required initial margin is \$ 90, the broker may ask the client to deposit securities with a market value of \$ 100. Technically speaking, we would say that the broker has applied a *Haircut* of 10%.

However, in practice variation margins, must always be paid in cash. The reason for this is the following. Initial margins represent performance guarantees, and can consequently be offered in the form of cash or marketable securities. A variation margin, which is offered in response to a margin call, is however a manifestation of actual losses suffered by the client, and consequently must be paid in cash.

1.8.2 Value At Risk (VaR)

We have just seen that once the potential loss for the parties to a futures contract is estimated and collected in advance, the likelihood of either of them defaulting reduces considerably. In practice, if the quantum of the margin is sufficiently high, then the possibility of default is fairly remote. Thus in order to preclude default, the margin requirement specified by the futures exchange should be based on an estimate of the potential loss.

However, no exchange can perfectly forecast as to what the magnitude of loss is likely to be from one day to the next. In practice, it can at best quantify the loss as an amount that is unlikely to be exceeded with a given level of probability. Such a forecasting approach is precisely the rationale behind the concept of Value at Risk or VaR.

The value at risk of a position may be defined as the potential loss of the asset(s) over a specified time horizon.[12]. Thus, if we were to state that the 95% VaR of an asset over a one day horizon is \$ 3,000, the implication would be that the loss in terms of the value of the asset over a one day holding period, is expected to exceed \$ 3,000, with a probability of 5% only. The calculated VaR is a function of both the probability level as well as the holding period. Thus in order to facilitate the interpretation of a calculated VaR number, it is necessary to specify both these

[12]See Linsmeier and Pearson (2000).

parameters. For a given asset(s), changing one or both parameters, can lead to significantly different estimates of VaR.

As Darbha (2001) points out, VaR may be interpreted as *the best of the worst possible scenarios or equivalently as the worst of the best possible scenarios*. Readers should also note that the calculated VaR, does not connote the maximum possible loss that a portfolio can suffer over the holding period. In principle, the value of any asset or portfolio can always go to zero, and consequently, the maximum loss that an investor may suffer is equal to the entire portfolio value. The second feature to be noted is that the VaR does not measure the intensity of potential losses. For instance, consider two portfolios with the same 95% VaR of say $ 5,000. Although the VaR is identical in the two cases, the magnitude of losses in excess of the VaR on the remaining 5% of the days could be considerably different for the two portfolios.

1.8.3 Futures Commission Merchants

Futures Commission Merchants (FCMs) are brokers who are authorized to open trading accounts on behalf of parties who wish to take positions in futures contracts. An FCM will collect margin money from the client, and record and report all the trades routed through him. It is important to emphasize that all brokers in the futures markets cannot be categorized as FCMs. Some clients prefer to deal with *Introducing Brokers*. Such intermediaries, perform the function of getting a client acquainted with an FCM. However, while they are authorized to accept orders and route them through an FCM, they are not allowed to maintain margin accounts.

All FCMs need not in practice be members of the clearinghouse. Those that are, are referred to as *Clearing Members*. Only those FCMs who are clearing members are authorized to maintain clearing margins with the clearinghouse and clear transactions through it. Thus, if a client were to initiate a trade through a broker who is not a clearing member, the broker in turn must route the order through a clearing member.

1.8.4 Gross versus Net Margining

Let us assume that there are two FCMs, Alpha and Beta, and that both are authorized to clear through the clearinghouse.[13] Alpha has two clients A and B. A has a long position in 100 futures contracts while B has a short position in 80 futures contracts. Beta also has two clients C and D. C has a long position in 80 contracts while D has a short position in 100 contracts.

The FCMs have to collect margins from their clients and the clearinghouse has to collect margins from the FCMs. We will make the following assumptions.

- The clearinghouse collects margins from the clearing members on a *Net* basis.

[13]This section is based on an illustration in Edwards and Ma (1992).

- The clearing members collect margins from their clients on a *Gross* basis.[14]
- The initial margin required by the clearinghouse is \$ 4 per contract.
- The initial margin required by the clearing members from their clients is also \$ 4 per contract.
- Contracts are marked to market daily and variation margins are paid or withdrawn the next morning, by both clearing members (to or from the clearinghouse) as well as by the clients (to or from the clearing members).
- There is a daily price limit of \$ 4 in either direction.

Now, what is this concept of a price limit?

Price Limits Exchanges often impose limits on the maximum daily price change. These limits are measured from the previous day's settlement price and apply in both directions. We will clarify this with the help of an example. Let us assume that the settlement price for Soybeans is \$ 6 a bushel today and that there is a limit of 30 cents on the maximum daily price change. So tomorrow's price limits will be \$ 6.30 and \$ 5.70. If the price moves down to the lower limit, then the contract is said to be *limit down*, whereas if it moves up to the upper limit, then it is said to be *limit up*. A move to either the upper or the lower bound, is called a *limit move*. When the price reaches one of the limits during the day, trading will slow down and may even come to a halt. Let us suppose that prices have hit limit up. It means that the buyers outnumber the sellers. If there are no sellers at the limit, then trading will cease. Sometimes, fresh news could filter in, causing prices to move up from the lower limit or down from the upper limit. If so, trading may resume. The exchange is authorized to intervene and change the limit if it deems it necessary.

It should now be obvious as to why we have assumed a \$ 4 price limit in the above example. This is because the initial margin is \$ 4, and we have not assumed a maintenance margin. Hence a limit move should be such that the margin account cannot go below zero for either the long or the short.[15]

FCM Alpha will collect \$ 400 from A and \$ 320 from B. That is, in all it will collect \$ 720. Similarly FCM Beta would also collect a total of \$ 720 from its two clients. This is the meaning of *Gross Margin*. If the clearinghouse were also to collect margins on a gross basis, then both the FCMs would have to deposit the entire amount collected by them with the clearinghouse.

In this case the clearinghouse is collecting on a net basis. It will therefore collect \$ 80 from Alpha and an equal amount from Beta. This is because Alpha has a net long position of 20 contracts with the clearinghouse, while Beta has a net short position of 20 contracts with it.

Now, suppose that the futures price rises by \$ 4. The longs will have a profit of \$ 4 per contract, while the shorts will have a loss of \$ 4. FCM Alpha will require an amount of \$ 400 to pay client A, while FCM Beta will have to pay \$ 320 to

[14]The meanings of the terms Net and Gross will become clear to you shortly.
[15]Remember the margin account cannot show a negative balance.

client C. In case the customers want to hold on to their positions,[16] then client B will have to pay $ 320 to FCM Alpha and client D would have to pay $ 400 to FCM Beta. FCM Beta will use $ 320 to pay client C. The balance $ 80 will be paid by Beta to the clearinghouse, which will pass it on to Alpha. Now, Alpha would have received $ 320 from client B. So in total it will collect $ 400, which will be just adequate for it to pay client A.

Thus, the clearing members perform a banking function, by transferring funds from one customer to another. The clearinghouse also performs a banking function by facilitating the transfer of funds from one FCM to another.

As long as the magnitude of the price change does not exceed the initial margin, the deposits held by the clearinghouse and the clearing members will be enough to protect both the buyers and the sellers. In the above case, let us suppose that the price goes up by $ 4 and that the shorts are unable to pay variation margin. If so, the profits of the longs, that is, $ 720, could be paid by the margins already posted by the shorts. Alpha already has $ 320 that it has collected from client B. Beta has $ 400 that it has collected from client D. It requires $ 320 to pay client C but has collected $ 400. Thus it has a surplus. Alpha requires $ 400 to pay client A but has only $ 320. The balance will be passed on to it by the clearinghouse. This amount of $ 80, represents the margin deposited on a net basis by Beta with the clearinghouse. Thus, the profits of either the longs or the shorts, are protected up to the amount of margin posted. In the event of a default, the clearinghouse and the clearing members will liquidate the defaulter's position by entering into an offsetting transaction, thereby eliminating any further liabilities.

Therefore, in the event of a default, customers with profits will look to their respective clearing members to obtain their dues and the clearing members will look to the clearinghouse for what is owed to them on their net positions with other clearing members. Notice that in the above case, if the shorts default, the clearinghouse would only guarantee the 20 contracts that represent Alpha's net long position with it. Alpha itself would be responsible for paying the amount due on the other 80 long contracts that have been routed through it. This is a feature of net margining. Of course, if the clearinghouse itself were to operate on a gross margin basis, then it could guarantee all the contracts.

Edwards and Ma (1992), point out that if the clearinghouse were to collect margins on a gross basis, customers would no longer be concerned with the financial health of their clearing members. This may not only reduce the credit-worthiness of clearing members, but can also lead to an increase in the cost of operations, because the clearinghouse will now have to guarantee the financial health of all its clearing members.

1.8.5 Default

Default in the case of futures contracts can manifest itself in two ways. First, the trader may not respond to a margin call issued by the FCM. Second, he may refuse

[16]Remember the margin accounts of the shorts have gone to a zero balance level, and they must raise the balance to the initial margin level.

to take delivery at maturity if he is a long with an open position, or else may refuse to give delivery if he is a short with an open position. Let us first consider the case where a client defaults before maturity, and illustrate it using the data in Table 1.3.

At the end of day 3, when the balance in the margin account falls to \$ 600, a margin call will be issued for \$ 400. If the client were to fail to respond positively by paying the required variation margin, the broker is at liberty to actually offset his position. In our example, Paul has taken a long position. Thus, an offsetting transaction initiated by the broker will entail the assumption of a short position at the prevailing market price. In our case, the price at the time the margin call was issued was \$ 71.00. Let us assume, that by the time the broker is able to offset the contract, the price has declined further to \$ 70.25. Paul, in this case would have incurred a further loss of \$ 75 per contract. The broker will deduct this amount, along with any transactions charges incurred by him, from the balance of \$ 600 that is available in Paul's margin account. The balance will be refunded to the client. Exactly the same procedure will be followed by a clearinghouse, if an FCM were to fail to respond to a margin call issued by it.

The second point in time at which default may occur is at the time of expiration of the futures contract. Let us first consider the case where the short fails to deliver the asset, which would tantamount to default on his part. In such a situation, the broker will acquire the good in the spot market and deliver it to the long. On the other hand if the default were to be on the part of the long, the broker will acquire the good from the short and sell it in the cash market. In either case, he will deduct his costs and losses from the balance in the defaulting party's margin account.

1.9 Arbitrage

Arbitrage entails the locking in of a cost-less riskless profit by simultaneously entering into transactions in two or more markets. The key phrase here is '*cost-less and riskless*'. The logic is as follows. If you get into a risky strategy entailing a cost, you should get a risk adjusted expected rate of return. This is the approach taken by models like the CAPM. If your strategy is riskless but requires an initial investment, then you should get the riskless rate of return. However, if you do not have to invest anything, and face no risk, then logically you should get no returns.[17] The presence of a positive return in such a circumstance, is referred to as an arbitrage opportunity, and the people who seek to exploit such opportunities are referred to as arbitrageurs. As the book progresses, we will see time and again that, it is the activities of arbitrageurs that keep prices in alignment and help to maintain equilibrium in the market.

The principle of arbitrage can best be explained with the help of a numerical example. Let us take the case of a share that trades on both the NYSE and the Chicago Stock Exchange. Assume that the price is \$ 75 on the NYSE and \$ 80

[17] Another manifestation of arbitrage would be the existence of returns in excess of the riskless rate for an investment that does not entail any risk.

on the Chicago Exchange. Consider the case of an investor who is in a position to borrow \$ 7,500,000. He is obviously in a position to acquire 100,000 shares on the NYSE. Having bought the shares, he can immediately sell them for \$ 8,000,000 on the Chicago Exchange. After repaying his loan, he will be left with a profit of \$ 500,000 which was made without his having to invest any money of his own, and without taking any perceptible risks. Such a situation is essentially a manifestation of an arbitrage opportunity.

These opportunities cannot prevail for extended periods of time. Our investor is unlikely to be the only one who has detected such an opportunity. As others begin to perceive the significance of this opportunity and rush to buy shares on the NYSE, the price there will rise. At the same time, the arbitrageurs will begin to unload their shares on the Chicago Exchange, which will cause the price there to fall. Taken together, these two factors will usually quickly eliminate any such opportunities for cost-less riskless profits.

In our illustration we have assumed the absence of transactions costs like bid-ask spreads and brokerage fees. For retail investors, such costs will be significant in practice, and may therefore preclude them from exploiting perceived arbitrage opportunities. However, institutional investors will face much lower costs, and will consequently exploit such opportunities to the hilt. It must also be pointed out that if the two exchanges across which the arbitrage strategy is being implemented, have an identical settlement cycle, an arbitrageur will require a stockpile of shares as well as adequate cash to execute such a strategy. This is because, in such a situation, he will be unable to take delivery at an exchange early enough so as to meet the deadline for delivery at the other.

1.10 Spot-Futures Convergence at Expiration

When a futures contract is scheduled to expire, the futures price must be the same as the spot price in order to preclude arbitrage. The rationale for this is the following. A futures position is a contract to transact at a future point in time. At the point of expiration, any futures position that it is entered into at that instant, must lead to an immediate transaction because the contract is scheduled to expire immediately. Thus an investor who is entering into a futures contract at its point of expiration is effectively entering into a spot contract. Therefore, if the futures price at expiration were to be different from the spot price, there will be an obvious arbitrage opportunity.

Let us denote the futures price at expiration by F_T and the spot price at that point in time by S_T. To rule out arbitrage, the two need to converge, that is F_T must be equal to S_T. We will now examine strategies to realize arbitrage profits, if F_T were to be greater than S_T or if F_T were to be less than S_T.

1. $F_T > S_T$
   In such circumstances an arbitrageur will implement the following strategy. He will take a long spot position at a price of S_T and will simultaneously

go short in a futures contract. Since the contract is scheduled to expire immediately, he can at once deliver for a price of F_T. Hence, the price differential, $F_T - S_T$, which by assumption is positive, is obviously an arbitrage profit for such an investor.

1.10.1 Numerical Illustration

Assume that the futures price of an asset at the time of expiration is \$ 75, whereas the spot price at the same point in time is \$ 72. The contract size for the futures position is 100 units. An arbitrageur will acquire 100 units of the asset in the spot market at \$ 72 per unit, and will immediately take a short position in a futures contract. Since the futures contract is expiring, he will immediately deliver at \$ 75, thereby making an arbitrage profit of \$ 3 per unit. The total profit is $\$ 3 \times 100 = \$ 300$.

2. $F_T < S_T$

   Now let us consider a situation where the spot price at the time of expiration is greater than the prevailing futures price. An arbitrageur will exploit this situation by taking a long position in a futures contract. Since it is about to expire, he will take immediate delivery at a price of F_T, and will then sell in the spot market for S_T. In this case, $S_T - F_T$, which by assumption is positive, represents an arbitrage profit.

1.10.2 Numerical Illustration

Assume that the futures price of an asset at the time of expiration is \$ 75, whereas the spot price is \$ 77.50. An arbitrageur will take a long futures position, which will entail taking immediate delivery at \$ 75 per unit. The acquired goods can then be sold in the spot market for a price of \$ 77.50 per unit. Hence, there is an arbitrage profit of \$ 2.50 per unit which is equivalent to a total profit of $\$ 2.50 \times 100 = \$ 250$.

1.11 Delivery

As you would have understood by now, under a futures contract, the long has an obligation to take delivery, while the short has an obligation to make delivery. There are three ways of settling a futures contract:

- Physical Delivery
- Cash Settlement
- Exchange for Physicals

1.11.1 Physical Delivery

This is the most cumbersome way of liquidation. The period during which delivery can be made, as was explained earlier, is defined by the exchange and varies from contract to contract. The decision as to when to deliver during the delivery period is made by the short.

The actual delivery process spans a three day period. In practice, delivery will commence with an expression of a desire to deliver by the short. This is called the *Position Day*. At this point in time, the short's broker will convey his client's intention to the clearinghouse by giving a notice of delivery. The notice will state, as to how many contracts are being delivered, the location of delivery in case the contract permits delivery at multiple locations, and the grade of the underlying asset being delivered in the case of contracts that permit the delivery of more than one grade. On the next business day, which is known as the *Notice Day*, the exchange will normally select the person with the oldest outstanding long position to accept delivery.[18] Finally, on the third day, known as the *Delivery Day*, the long will make the required payment to the short, and in return will get a warehouse receipt granting him title to the underlying asset.

It must be noted that cessation of trading is not a pre-condition for delivery to commence. In most futures markets, the first day on which a short can indicate his intention to deliver, or the first position day is before the last day of trading, whereas the last notice day, is after the day on which trading finally ceases. For illustrative purposes, the details of the delivery schedule for the corn futures contract on the CME Group are given in Table 1.4.

As we have mentioned, the normal practice for the exchange is to match the short who has expressed a desire to deliver with the party with the oldest outstanding long position. The rationale for this is the following. Assume that Keith traded with Paul a month ago by assuming a short position. However, today when Keith announces his intention to deliver, it is not necessary that he be in a position to deliver to the original counterparty, for Paul may have already exited from the market by taking an offsetting position. Hence the exchange will give the notice of Keith's intention to deliver, to the party with the oldest outstanding long position.

Table 1.4 Delivery Schedule for Corn Futures on the CME Group

First Notice Day	Last Notice Day	Last Trading Day
Last business day prior to the delivery month	Business day following the last trading day	Business day prior to the 15th calendar day of the Delivery Month

In the case of contracts settled by delivery of the underlying asset, the price paid per unit will be the spot price prevailing at that time. We will now analyze the logic behind this. Futures contracts are subject to marking to market on every business day during their lifetime. Thus, by the time of expiration, both the long and the short would have realized the profit or loss from their respective positions. Therefore, in order to ensure that the long gets to acquire the asset at the price that was agreed upon at the outset, he has to be asked to pay the prevailing futures price at expiration. This price of course, due to the no-arbitrage condition will

[18]The rationale for this will be explained shortly.

be the same as the prevailing spot price at expiration. We will illustrate the logic using symbols, as well as with the help of a numerical example.

Let us take the case of a contract that was entered into on *day 0* at a price F_0, and which expires on *day T*. We will denote the price at expiration by F_T. Such a contract will be marked to market on days 1,2,3 . . . . up to day T. The cumulative profit or loss for the long, due to the marking to market procedure, is:

$$(F_T - F_{T-1}) + (F_{T-1} - F_{T-2}) + (F_{T-2} - F_{T-3}) + \cdots + (F_2 - F_1) + (F_1 - F_0) = (F_T - F_0) \tag{1.1}$$

Thus, in order to ensure that the long is able to acquire the asset at the original price of F_0, which is as per the conditions of the original contract, he must be asked to pay a price P at the time of delivery, such that:

$$P - (F_T - F_0) = F_0$$
$$\Rightarrow P = F_T = S_T \tag{1.2}$$

In other words, in order to ensure that the long is able to effectively take delivery at the price specified in the futures contract, he must be asked to pay the prevailing futures price, or equivalently the prevailing spot price, at the time of delivery.

Forward contracts, however, are not marked to market. Consequently there is no adjustment of profits/losses on a daily basis. Hence, in the case of such contracts, the price paid by the long, must be the same as the price that was agreed upon originally. That is:

$$P = F_0 \tag{1.3}$$

Numerical Illustration Consider a futures contract on corn, that was entered into at a price of \$ 6.50 per bushel. We will assume that the contract lasts for a period of 5 days, and that the movement in the futures price over that period is as depicted below.

Table 1.5 **Marking a Contract to Market**

Day	Futures Price	Profit From Marking to Market
0	6.50	
1	6.25	(0.25)
2	6.00	(0.25)
3	6.30	0.30
4	6.70	0.40
5	7.10	0.40
	Total	0.60

In this case, $F_0 = 6.50$ and $F_T = 7.10$. An investor who had taken a long futures position at time *0* at a price of \$ 6.50, would have to pay the terminal futures price of \$ 7.10 at the time of delivery. After factoring in the profit of \$ 0.60 due to marking to market, he would have effectively paid \$ 6.50 for the asset, which is nothing but the initial futures price.

However, an investor who takes a long forward position at $ 6.50, will not have his position marked to market, and consequently would have to pay $ 6.50 at the time of taking delivery.

When delivery is made, the clearinghouse will be informed and it will cancel the obligations of the corresponding FCMs on its books. The FCMs will then delete the names of the individual clients from their own books.

1.11.2 Cash Delivery

In the case of certain futures contracts, physical delivery is not permissible. What happens in such cases is that the marking to market procedure will be followed till the last day of trading, and subsequently all the positions will be declared closed. This is referred to as cash settlement. In such circumstances therefore both the parties to the contract will exit the market, with their profit/loss. In the above example, the long would have made a profit of $ 0.60, and he would exit the market with this amount, but without taking delivery of the underlying asset. In principle, this is no different from delivery based settlement. This is because, if he were to desire to have physical possession of the asset, he can acquire it in the spot market for $ 7.10, which would effectively mean that he had paid $ 6.50 for the asset.

Cash settlement is the prescribed mode of settlement all over the world for Stock Index Futures. These are futures contracts which are based on popular stock indices such as the Dow Jones Industrial Average, The Standard & Poor's 500, and the Nikkei 225. In order to form a portfolio that mimics an index, known as a tracking or mimicking portfolio, the investor would have to acquire all the component securities in exactly the same proportions as they are present in the index. It should be obvious that physical delivery under such circumstances will be an extremely cumbersome affair.

In certain markets, where there is fear of price manipulation, the regulator may specify cash delivery as the required procedure, as a mechanism to guard against the possibility of traders creating artificial shortages in the underlying asset.

1.11.3 Exchange for Physicals (EFPs)

Physical delivery, as per contract specifications, can be made only during the delivery period specified by the exchange. The short is also required to adhere to the terms and conditions specified in the contract. That is, delivery must be at one of the allowable locations, and the grade that is being delivered must be one of the permissible grades. There are times, however, when two parties with opposite positions can transact prior to the delivery period specified by the exchange. Such an arrangement is called Exchange for Physicals or EFP. Under such a procedure, a long and a short can get together and agree on a transaction that would close out their respective positions. An EFP, is an *Off-The-Exchange* transaction. The reason why exchanges permit EFPs is that they offer the parties greater flexibility, and make physical delivery a more attractive option. The mechanism can best be illustrated with the help of an example.

Example Tom King has a short position in corn futures, while Mike Casey has an equivalent long position in futures contracts on the same asset. However, the grade of corn that Tom has, is different from the grade specified in the contract. Under normal circumstances, Tom will be unable to deliver this wheat, even if Mike is willing to accept it. However, under an EFP, once Tom and Mike mutually agree on a price, the two can transact in the commodity after reporting the transaction to the exchange. The exchange, in such circumstances, will simply treat the transaction, as though each party had offset his position with the other.

There is another procedure for delivering the underlying asset on terms other than those specified in the contract. This is known as an *Alternative Delivery Procedure* or ADP. The difference between an EFP and an ADP, is that in the case of an EFP a party desirous of transacting has to locate a suitable counterparty. An ADP however takes place after the exchange has matched a long and a short, in response to a short's declaration of his intention to deliver. In an ADP transaction, the transaction can be on terms different from those specified in the contract. However, the parties are required to notify the clearinghouse of the same.

1.12 Trading Volume versus Open Interest

The number of futures contracts that are traded during the course of a day, on a particular underlying asset, is referred to as the trading volume for that day. There is another term used in derivative markets, namely the *Open Interest*. The open interest at any point in time, is the total number of outstanding contracts or the number of open positions at that point in time. Every long position must be matched by a corresponding short position. Consequently open interest may be measured either as the number of open long positions at a point in time, or equivalently, as the number of open short positions. There is a relationship between the trading volume for a day and the change in the open interest from the close of trading on the previous day. It depends on the nature of the transactions that take place during the course of the day, and can best be illustrated with the help of an example.

Let us assume that a new futures contract on a product has just opened for trading and that five trades have taken place on the first day as depicted below.

Table 1.6 Trade Details for the First Day

Time	Trade	No. of Contracts
10:30 a.m	Monica goes long and Alfred goes short	150
1 p.m	Rachel goes long and Violet goes short	200
2 p.m	Paul goes long and Ross goes short	150
3 p.m.	Eric goes long and Veronica goes short	200
4 p.m.	Mike goes long and Phil goes short	300

The trading volume for the day is obviously 1,000 contracts. Notice that nobody has offset any contracts during the course of the day. Thus the number of open positions at the end of the day is 1,000 contracts. In other words the open interest is 1,000 contracts.

Now consider the following scenarios for the next day.

1. **Case A**

   Monica goes long in 150 contracts and Phil goes short.

   Both these parties are entering into a trade that increases their open positions. As a consequence of the trade, the number of long positions has increased by 150 as has the number of short positions. The trading volume for the day is obviously 150 contracts. The number of open positions at the end of the day is 1,000 plus 150 or 1,150 contracts. Hence, the change in the open interest as compared to the previous day is +150 contracts. Thus, if a trade involves two parties who are establishing new positions by entering into a contract with each other, the open interest will rise.

2. **Case B**

   Monica goes long in 150 contracts and Mike goes short.

   The trading volume for the day is once again 150 contracts. But what about the open interest at the end of the day? In this case, no new positions have been created. What has happened, is that Mike who had a long position in 300 contracts has partially offset by taking a counter-position in 150 contracts, and his place has been taken by Monica. Thus the number of long or short positions has not changed as compared to the previous day, and the number of open positions at the end of the day continues to remain at 1,000 contracts. Hence, the change in the open interest as compared to the previous day is zero. Thus, if a trade involves one party taking a counter-position by trading with another party who is opening a position, then the open interest will remain unchanged.

3. **Case C**

   Monica goes short in 150 contracts and Phil goes long.

   As before, the trading volume for the day is 150 contracts. The change in the open interest can be derived as follows. Monica has exited the market by going short in 150 contracts. At the same time the trade has resulted in a reduction of 150 in Phil's short position. The net result is that the number of open contracts has reduced by 150 and the change in the open interest as compared to the previous day is - 150 contracts. Consequently the open interest at the end of the second day is only 850 contracts. Therefore, if a trade involves one party taking a counter-position by trading with another party who is also taking a counter-position, then the open interest will fall.

   Trading volumes are indicators of the level of liquidity in the market. On the other hand, open interest is a barometer of future liquidity. If the open interest at the end of a day is high, it is an indication that there is more scope for counter-positions on subsequent days, and consequently it can be construed as a signal that future volumes are likely to be high.

1.13 Conversion Factors When There Are Multiple Deliverable Grades

Certain futures contracts give the short the flexibility to deliver more than one grade of the underlying asset. Thus the short can choose the grade that he wishes to deliver. The exchange will in such cases, designate one grade as the *Par* grade. If the short were to deliver the par grade, he will receive the prevailing futures price at expiration, F_T, for reasons that we have already explained. However, if he were to deliver a more valuable grade, he will receive a premium, whereas if he were to deliver a less valuable grade, he would have to do so at a discount. There are two ways of incorporating the premium or discount for grades other than the par grade, known as *Multiplicative Adjustment* and *Additive Adjustment*. We will examine each of these techniques.

1.13.1 Multiplicative Adjustment

In the case of contracts which permit more than one grade of the underlying asset to be delivered, and which specify a multiplicative system of price adjustment, the short will receive an amount equal to $a_i F_T$ at the time of delivery, if he were to deliver grade i. For premium grades, a_i will be greater than 1.0, whereas for discount grades, it will be less than 1.0.

We will denote the spot price of grade i at expiration by $S_{i,T}$. Hence, the profit for the short if he were to deliver grade i is:

$$a_i F_T - S_{i,T} \tag{1.4}$$

Grade i will be preferred to another grade j if

$$a_i F_T - S_{i,T} > a_j F_T - S_{j,T} \tag{1.5}$$

At expiration, in order to preclude arbitrage, the profit from delivering the most preferred grade must be zero. If we denote this grade as grade i, it must be the case that

$$a_i F_T - S_{i,T} = 0$$

$$\Rightarrow F_T = \frac{S_{i,T}}{a_i} \tag{1.6}$$

In the case of all other grades, it must be the case that

$$a_j F_T - S_{j,T} < 0$$

$$\Rightarrow F_T < \frac{S_{j,T}}{a_j} \tag{1.7}$$

Thus, the grade that will be chosen for delivery will obviously be the one for which $\frac{S}{a}$ is the lowest. Such a grade is called the *Cheapest to Deliver Grade* and $\frac{S}{a}$ is called the *Delivery Adjusted Spot Price*. Thus, the cheapest to deliver grade, will be the one with the lowest delivery adjusted spot price. At expiration, therefore, the futures price must converge to the delivery adjusted spot price of the cheapest to deliver grade.

Example Gold futures contracts allow for the delivery of gold within a certain weight range and with varying degrees of fineness.[19] The price received by the short = weight × fineness × futures price. Thus, per ounce of gold delivered, the short will receive fineness × F. The conversion factor in this case is the fineness and the par grade is 100% fine.

Assume that gold is available with either 99% fineness or with 100% fineness. Let the spot price of 99% fine gold be \$ 495 per ounce and that of 100% fine gold be \$ 505 per ounce. The delivery adjusted spot prices are $\frac{495}{.99} = 500$, and $\frac{505}{1} = 505$ respectively. Thus the 99% fine gold is the cheapest to deliver grade.

1.13.2 Additive Adjustment

Now let us consider the case of contracts, which permit more than one grade to be delivered, but use an additive system of price adjustment. In the case of such contracts, the short will receive $F_T + a_i$, if he were to deliver grade i. For a premium grade, a_i will be positive, whereas for a discount grade, it will be negative.

The profit from delivering grade i, will be

$$F_T + a_i - S_{i,T} \tag{1.8}$$

and grade i will be preferred to another grade j if

$$\begin{aligned} F_T + a_i - S_{i,T} &> F_T + a_j - S_{j,T} \\ \Rightarrow S_{i,T} - a_i &< S_{j,T} - a_j \end{aligned} \tag{1.9}$$

Hence, the cheapest to deliver grade is the one for which $S - a$ is the lowest. That is, grade i will be the cheapest to deliver grade if

$$S_{i,T} - a_i < S_{j,T} - a_j \forall j \tag{1.10}$$

To rule out arbitrage, the profit from delivering the cheapest to deliver grade must be zero. That is

$$\begin{aligned} F_T + a_i - S_{i,T} &= 0 \\ \Rightarrow F_T &= S_{i,T} - a_i \end{aligned} \tag{1.11}$$

In this case $S - a$ is the delivery adjusted spot price, and once again, the futures price will converge to the delivery adjusted spot price of the cheapest to deliver grade.

[19] See Siegel and Siegel (1990).

It must be noted that irrespective of whether the multiplicative or the additive system is used, the cheapest to deliver grade need not be the one with the lowest spot price. For example, consider the following data for corn.

Table 1.7 **An Illustration of Additive Price Adjustment**

Grade	Spot Price	Conversion Factor	Delivery Adjusted Spot Price
No. 1	6.51	0.015	6.495
No. 2	6.50	0	6.500
No. 3	6.49	−0.015	6.505

The par grade is obviously No. 2. But the cheapest to deliver grade is No. 1, which incidentally has the highest spot price.

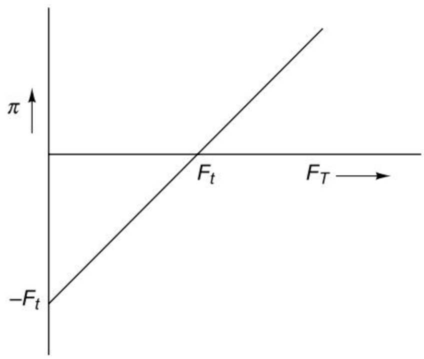
1.14 Profit Profiles

Investors with long positions in futures contracts stand to gain if the futures price were to rise after they have taken a position, whereas they stand to lose if the price were to decline subsequently. For the shorts it is the opposite. That is, they profit from declining futures prices and stand to lose if the futures price were to rise.

Thus the profit for a long futures position may be expressed as $F_T - F_t$, where t represents the point of time at which the contract is initiated, and T is the point of time at which the contract either expires, or is offset. Therefore, for every dollar increase in the terminal futures price the profit is one dollar more, while for every dollar decrease in the terminal futures price, the profit is one dollar less.

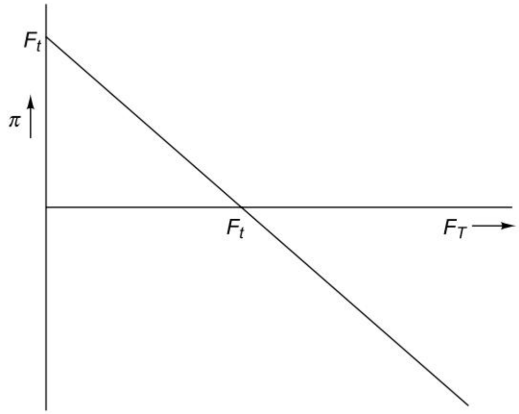
If we plot the profit from a long position versus the terminal futures price, the graph will be linear, and upward sloping as can be seen from the depiction below.

Figure 1.1 **Profit Profile: Long Futures**



Let us interpret the above diagram. π represents the profit, which is shown along the Y-axis. F_T is the terminal futures price which is shown along the X-axis. The maximum loss occurs when $F_T = 0$, and is equal to F_t in magnitude. The maximum profit is unlimited since F_T has no upper bound. The position breaks even if the terminal futures price is equal to the initial futures price, or in other words the price remains unchanged. Thus holders of long positions face the specter of potentially infinite profits, but finite losses. Investors with long positions are obviously bullish about the market, for they will gain if prices were to rise.

Figure 1.2 **Profit Profile: Short Futures**



For an investor with a short position in a futures contract, the profit may be depicted as $F_t - F_T$. The profit diagram for a short futures position is therefore also linear, but downward sloping, as depicted in Fig. 1.2. In this case, the maximum profit occurs when $F_T = 0$, and is equal to F_t in magnitude. The maximum loss is obviously unlimited. Thus holders of short positions are confronted with the prospect of finite profits but infinite losses. Such investors are obviously bearish in nature, for they stand to make profits if the prices were to decline.

1.15 Types of Assets Underlying Futures Contracts

On international markets, futures contracts are available on a number of assets such as food grains, oil seeds, fiber, meat, livestock, metals, petroleum, foreign currencies, stock indices, and interest rates. Until a few decades ago, most of the contracts were based on agricultural commodities. But subsequently, trading in contracts based on financial instruments such as bonds, stock indices, and foreign currencies, has outstripped the demand for commodity based contracts. One of the latest additions to the financial futures family, is *single stock* or *individual stock* futures, that is, futures contracts on the common stocks of companies.

In the US, food grains and oil seeds on which contracts are available, include corn, oats, soybeans, rice and wheat. Livestock and meat contracts are available on hogs, cattle and pork bellies. Cocoa, coffee, cotton, sugar, orange juice, butter and milk are some of the food and fiber products on which contracts are traded. Contracts are also traded on metals like copper, gold, and silver, and on petroleum products such as crude oil, heating oil and gasoline.

Contracts on a number of foreign currencies are traded in the US.[20] These include Australian and Canadian Dollars, Pound Sterling, the Euro, Japanese Yen and Swiss Francs. Popular stock indices on which contracts are traded include the Dow Jones Industrial Average, the Nikkei 225, the NASDAQ 100, and the Standard and Poor's (S&P)500. Interest rate futures contracts are available on Treasury bills, notes and bonds. T-bills, notes and bonds are all government securities. They differ with respect to their original time to maturity. If the time to maturity is less than or equal to one year, the instrument is called a T-bill; if it is between one to ten years, it is called a T-note; and if it is over ten years, it is called a T-bond.

1.16 Futures Exchanges

Two of the largest futures exchanges in the world are in Chicago. These are, the Chicago Board of Trade (*CBOT*) and the Chicago Mercantile Exchange (*CME*). They have now merged to form the CME Group. The *IMM* or the *International Monetary Market* is a division of the CME and conducts futures trading in foreign currencies.

Some of the other major futures exchanges in the world, are EUREX, Euronext.Liffe, and the Korea Futures Exchange. Futures exchanges are also rapidly growing in the developing countries. Examples of such *'emerging markets'* are the Dalian Commodity Exchange in China, the National Stock Exchange in India, and the Bolsa de Mercadorias e Futuros (BM&F) in Brazil.

1.17 Hedgers and Speculators

These are the two key types of players in the derivatives markets, besides arbitrageurs. They have contrasting attitudes to risk, and the active participation of both types of investors is essential for the success of a futures market. We will now define each one of them.

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[20]We are using the word *foreign* as seen from an American perspective.

1.17.1 Hedgers

Hedgers are traders who seek to protect themselves against unfavorable movements in the price of the underlying asset. Obviously, a hedger already has a position in the underlying asset prior to entering the futures market. Let us consider the case of a refinery which is planning to acquire crude oil after a month. It will obviously be worried that the price of crude oil may rise during the period, and may like to lock in a price today for the commodity. *Thus, a hedger is a person whose objective is to reduce his risk.* Futures and options contracts can help such entities.

1.17.2 Speculators

Unlike a hedger who essentially wants to avoid being exposed to adverse price movements in the spot market, a speculator is an investor who deliberately wishes to take a position in the market. That is, he wants to consciously take risk hoping to profit from subsequent price changes. Such a person is either betting that the price will rise in which case we would say that he is bullish, or else he is hoping that it will fall, in which case we would categorize him as bearish. Futures and options can be used by speculators irrespective of their views about the direction of the market.

1.18 Leverage

A strategy is said to be *Levered* or *Geared*, if a fairly small market movement tends to have a disproportionately large impact on the funds deposited. Futures and options provide leverage to traders who take positions in them.

Consider a person who has gone long in a corn futures contract at a price of $ 6 per bushel, by depositing a margin of $ 1,000. As we have seen earlier, each contract is for 5,000 bushels. If the price were to move up to $ 6.15, the investor would make a profit of $ 750, which is 75% of the initial deposit. On the contrary, had he chosen to go long in the spot market at a price of $ 6 a bushel, he would have procured 5,000 bushels by paying $ 30,000, and a profit of $ 750 would have meant a return of only 2.50%. However, leverage is a double-edged sword. If the futures price were to fall to $ 6.85 at the end of the day, the investor would make a loss of $ 750, which is equivalent to a 75% erosion of his margin deposit. However, had he chosen to buy the corn in the spot market, a loss of $ 750 would tantamount to a loss of only 2.50% of his initial investment.

Options also similarly provide leverage. Consider a share which is selling at a price of $ 75. Assume, that European call options with an exercise price of $ 75 are available at a premium of $ 6. We will first consider the case where the share price at the time of expiration of the option is $ 82.50. If the investor were to have bought a share, he could sell it for a profit of $ 7.50, which is equivalent to a 10% return on investment. On the contrary, if he had chosen to buy a call option, he would get a payoff of $ 7.50 by exercising, which represents a 25% return on an investment of $ 6. However, if the stock price at the time of expiration of the options contract were to be $ 67.50, the option holder would have to forego the entire premium amounting to a loss of 100%, since he will not exercise. On the contrary, had he chosen to acquire the share at the outset, he would now incur a loss of only 10%.

1.19 The Role of Futures and Options Markets

Derivative contracts can help traders in a number of ways. Some of the major benefits provided by derivatives are the following.

1.19.1 Re-allocation of Risk

All investors are not alike from the standpoint of their attitude towards risk. On one hand there is a category of traders whom we have labeled as 'Hedgers'. These are people who seek to avoid risk. On the other hand, we have another category of traders whom we have labeled as 'Speculators'. Unlike hedgers, speculators consciously seek to take calculated risks. Derivative contracts can be used to transfer or re-allocate risk from those who seek to avoid it to those who are willing to bear it. Very often, if a hedger were to take a position in a derivative security, the odds are that the counterparty is a speculator. This is however not necessarily always the case. Both parties to a contract may be hedgers or speculators, albeit with different price expectations.

1.19.2 Price Discovery

In a free market economy, fair and accurate prices are imperative for ensuring the correct allocation of resources. For unlike in the case of a command economy, which is usually characterized by a centralized planning agency, it is the freedom to trade and the capacity of the market to correct supply-demand imbalances through the pricing mechanism, that helps resources to flow in a free and fair manner. In practice, supply and demand information tends to percolate derivative markets more easily, and consequently such markets help facilitate the dissemination of such information.

Why is it that information permeates the derivatives markets more easily? Taking a long derivatives position entails the depositing of a small margin, whereas to take a spot position the investor is required to pay the full price. Similarly, an investor who anticipates a bear market can more easily take a short position in derivatives than sell the asset short. Short selling entails the sale of an asset which the investor does not own. In practice this is achieved by borrowing it from another investor or broker. While long positions are used by bullish investors to take positions consistent with their views on prices, the freedom to short sell is critical for those with a bearish viewpoint. However, short selling is not always freely possible. Even in those markets where it is permitted, the investor is required to deposit the entire proceeds with the broker who will pay a low or nil rate of interest. Thus, from the standpoints of both the longs as well as the shorts, trading in derivatives is attractive. This has two major consequences which further fuel the level of activity in such markets. First, transactions costs tend to be lower in derivatives markets as compared to spot markets. Secondly, derivatives markets are characterized by a high degree of liquidity.

Let us consider the issue of liquidity, and analyze as to why it is important. *Liquidity refers to the ability of market participants, to transact quickly at prices which are close to the true or fair value of the asset. It refers to the ability of buyers and sellers, to discover each other quickly and without having to induce a transaction by offering a large premium or discount*. Quite obviously, an investor, irrespective of his motive or market view, would like to trade without having to make major price concessions. In an illiquid market large trades can have a significant price impact. Large buy orders, placed by traders desirous of taking long positions, have the potential to send prices rising, whereas large orders to sell, placed by traders seeking to go short, can cause prices to decline rapidly. We have referred to the bid-ask spread earlier. Illiquid or thin markets are characterized by large bid-ask spreads.

1.19.3 Market Efficiency

There is obviously a relationship between the price of a futures or options contract, and the spot price of the underlying asset on which it is based.[21] Since derivatives trading is relatively easier and cheaper, the flow of new information into such markets may temporarily manifest itself as an arbitrage opportunity between the derivatives and cash markets. Such asynchronous prices will immediately lead to the implementation of the relevant strategies by arbitrageurs. Thus, the presence of a derivatives market helps ensure an efficient spot asset market.

1.19.4 Ease of Speculation

Speculation is a sine qua non for the efficient functioning of a capital market. A market that does not facilitate speculation will not have the volumes required for it to function as an efficient trading arena. Derivatives markets enable speculators to easily take positions by depositing small amounts of collateral.

1.20 Reasons for the Rapid Growth of Derivative Markets

Until the late 1960's, most of the activity in derivatives trading was restricted to commodities and to the leading capitalist economies like the US. Financial derivatives as a concept became significant only in the 1970's and 80's. The explosion of trading in financial futures and options has not only manifested itself

[21] Readers will appreciate this better when we cover the valuation of derivatives in subsequent chapters.

by way of higher observed trading volumes in the leading markets of the world, but has lead to the establishment of newer exchanges with more sophisticated technologies, in both the developed as well as the developing world.

- For a period of time in the last century, currencies were based on what was termed the *Gold Exchange Standard*. In this system, every currency had a value in terms of the US dollar, which in turn had a value linked to gold. By the late 1960s, however, the foreign dollar liabilities of the US were much higher than its gold reserves. In 1971, this chapter of international economics came to a close. After the collapse of this system, also known as the *Bretton Woods* system, the major economies of the world switched from fixed exchange rate regimes to floating rate mechanisms. Consequently, currency risk and its management became very important, leading to growth and innovations in the market for Forex derivatives.
- There was a major war in the Middle East in 1973. After this, petroleum prices became highly volatile and unpredictable. This had far reaching effects on the prices of all commodities, since the transportation costs of all goods is directly linked to the price of crude oil. This gave a further impetus to the commodity derivatives markets. Futures contracts on crude oil, heating oil, and gasoline were introduced to facilitate the hedging of risk posed by volatile oil prices. The importance of this can be appreciated even in today's world where wildly fluctuating oil prices continue to be an issue of global concern.
- Beginning with the US. Federal Reserve, major central banks began to abandon their policies of keeping interest rates stable. The focus shifted to adjustments in the levels of money supply, and interest rates became market determined. The resultant volatility in interest rates was addressed by the introduction of interest rate derivatives. Volatility of interest rates has implications for the risk of a business as a whole, since businesses thrive on borrowed money. Thus it has ramifications not just for interest rate derivatives but for all risk management tools.
- The three pronged strategy of *'Liberalization, Privatization, and Globalization (LPG)'* increasingly began to gain currency worldwide towards the end of the last century. Many countries across the globe began to liberalize their economies. With the removal of restrictions, capital began to move freely across borders, and markets became more integrated. Not surprisingly, risks multiplied and became a common matter of concern, for the flow of capital is inevitably accompanied by the transmission of the attendant risks.
- Significant market reforms were implemented in the traditionally free market economies with respect to the brokerage and banking industries. In October 1986 the London Stock Exchange (LSE) eliminated fixed brokerage commissions. This event came to be known as the '*Big Bang*'. From February of the same year, the LSE had started admitting foreign brokerage firms as full members. These changes were intended to make

London an attractive international financial market.[22] London is ideally located geographically, and serves as a middle link between markets in the US and those in the Far East, thereby facilitating 24 hours trading. Prior to World War II, London was the most active financial market in the world. However after the war, which devastated the British economy, the focus of international business shifted across the Atlantic to New York. Realizing the need for a vibrant market in London, the LSE sought to project itself as an institution of global importance.

Similar changes were effected in the United States in 1975 and in Japan in 1999. Today, in most countries commissions are negotiable between the broker and the client. There are however countries where government or exchange regulations specify fixed commission rates that ought to be charged by a broker. For instance, until early 2003 the minimum commission in Hong Kong was mandated to be 0.25% of the trade value.[23]

In a deregulated brokerage industry, commissions vary substantially from broker to broker depending on the extent and quality of services provided. Prior to the regulatory changes, institutional investors were paying more than what they should have, considering the volumes of their trades, while retail investors were paying less than what was warranted. In other words, institutional investors were subsidizing the retail investors. When the brokerage industry was deregulated, retail brokerage charges initially rose sharply. However soon, a new industry called *Discount Broking* was born. Today, in developed markets we have three categories of brokers. On one hand, we have *Full-service* brokers who charge the maximum commissions, but offer considerable value added services and investment advice. On the other hand, there exist *Discount* and *Deep discount* brokers who charge the least by way of commissions, but whose only function is to execute trades. In other words they do not provide research reports or other investment advice to their clients. Some of these discount brokers like Charles Schwab in the US have grown rapidly in size and are now counted among the major financial institutions. The difference between discount brokers and deep discount brokers is that while both do not offer investment advice, deep discount brokers require their clients to place orders of considerable size in order to avail of even lower brokerage rates.

- The rapid growth in *Information Technology* and *Telecommunications* that has been witnessed over the past three decades has been a key factor in the development of derivatives exchanges. There was a time when hardware was expensive and software was primitive. Computers have now come within the reach of a middle class investor in most societies. From streamlining back-end operations in the exchanges and brokerage houses, to facilitating Stock Index Arbitrage, computers have played a pivotal role in the growth of these markets. Similarly, modern telecommunications

[22] See Resnick (1996).
[23] See Harris (2003).

facilities, both voice as well as data, are indispensable for the growth of modern financial markets. Today prices and news from the leading exchanges in the world are flashed across the globe in a matter of seconds. Markets are now extremely closely linked and cross-border trading and arbitrage is an essential part of global economics. Institutions and high net worth (HNI) investors have come to realize that while it is true that one should not put all his eggs in one basket, there is no reason why the basket should be domestic. Mutual funds, pension funds, and other investors routinely diversify their wealth by investing across borders.

Suggestions for Further Reading

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Concept Check

***State whether the following statements are True or False*.**

1. Futures contracts are usually settled by delivery.
2. A person who is long in a put option, has the right to sell.

3. Everything else being the same, an American option will always be worth at least as much as a European option.
4. A floating-rate to floating-rate swap is known as a Basis swap.
5. Once a futures contract is entered into, the clearinghouse effectively becomes the buyer for every seller and the seller for every buyer.
6. The margin deposited when a futures contract is entered into is known as the Variation Margin.
7. Futures contracts have linear profit diagrams.
8. Both futures as well as options provide leverage.
9. A bullish investor may go long in call options or short in put options.
10. It is usually easier to go short in a futures contract than to short sell the underlying asset.
11. The decision as to when to deliver, where to deliver, and what grade to deliver, is always made by the short.
12. Every FCM is empowered to deal with the clearinghouse.
13. A short can give notice of intention to deliver only after trading has ceased in a contract.
14. Stock index futures contracts are usually delivery settled.
15. An EFP is an off-the-exchange transaction.
16. An EFP transaction is possible only after an exchange has matched a long with a short.
17. High open interest on a given day is a signal of greater liquidity on subsequent days.
18. When you take delivery under a futures contract, the price paid for the asset is the futures price that was prevailing when the contract was initiated.
19. If a trade involves one party closing out its position by trading with a party who is opening a position, then the open interest will remain unchanged.
20. The cheapest to deliver grade will always have the lowest spot price.

QUESTIONS AND PROBLEMS

Question-I

Compare and contrast Forward Contracts and Futures Contracts.

Question-II

What is Value at risk? Discuss.

Question-III

What is the economic role played by derivatives markets?

Question-IV

‘Trading in derivatives was primarily restricted to contracts on agricultural commodities until the 1970s, when the demand for contracts on financial products suddenly increased.’ Comment.

Question-V

Paul Easley has taken a short position in a gold futures contract. Each contract is for 100 ounces of gold and the futures price at the time of entering into the contract is $ 500 per ounce. The initial margin is $ 10,000 and the maintenance margin is 80% of the initial margin. The contract is entered into in the morning of March 1, 2006 and is held for a period of 10 days.

The following are the settlement prices at the end of everyday (per ounce of gold).

Day	Price
1	515
2	525
3	550
4	510
5	490
6	475
7	460
8	480
9	500
10	480

Assuming that balances in the margin account in excess of the initial margin level are not withdrawn, draw up a detailed table showing the daily gain/loss, the cumulative gain/loss, the balance in the margin account and the variation margins paid.

Question-VI

What do we mean by the term 'Exchange for Physicals'? When is it usually undertaken?

Question-VII

What is the difference between Trading Volume and Open Interest? Is the trading volume for a day always equal to the change in the open interest from the previous day?

Question-VIII

The Bendigo futures exchange has just been inaugurated and trading has commenced in gold futures. The following transactions are observed on the first day of trading.

Long Position	# of Contracts	Short Position
Ann George	100	Janet Turner
Mike Ramon	200	Robin Chandler
Arnold Getty	100	Mike Ramon
Victor King	100	Ann George
Greg Chapman	200	Randy Timken
Mike Ramon	100	Ginger Rogers
Randy Timken	100	Greg Chapman
Steven Ingo	200	Ann George
Sally Ramirez	250	Mike Ramon
Rennie Dennison	100	Janet Turner

1. What is the trading volume for the day?
2. What is the open interest at the end of the day? Clearly explain your logic at every step.

Question-IX

The following grades are eligible for delivery under a rice futures contract. The corresponding spot prices at expiration and the conversion factors are given. An additive system of price adjustment is used.

Grade	Adjustment Factor	Spot Price
Sona Mussoorie	−2	18.5
BT Quality 2	−1.5	19.25
BT Quality 1	−.75	20.5
Dehradun Special	+.75	22
Basmati Ordinary	–	23.5
Basmati Superfine	+2	24.5

1. Which is the par grade?
2. Calculate the delivery adjusted spot price for each of the grades.
3. What should be the futures price at expiration?

Question-X

Assume that multiple grades are eligible for delivery as per a futures contract, and that the multiplicative adjustment system is being used. The following are the spot prices of various grades at the time of expiration, along with their respective adjustment factors.

Grade	Spot Price	Adjustment Factor
A	25.00	0.95
E	27.50	1.025
I	22.50	0.925
M	24.00	0.94
R	28.00	1.04
W	22.00	0.90
Z	30.00	1.05

1. Which is the cheapest to deliver grade?
2. What will be the futures price at the time of expiration of the contract?

2

Valuation of Futures Contracts

2.1 Introduction

Futures prices should accurately reflect the price of the underlying asset. The concept of arbitrage is critical for understanding as to how spot prices and futures and forward prices are linked. If the postulated relationship between the two prices is not satisfied, arbitrageurs will exploit the resulting profit opportunities until they are eliminated.

We will now go on to analyze how forward and futures prices are related to the spot price of the underlying asset. We will first focus on forward contracts because they are much easier to analyze than futures contracts. This, as we explained earlier, is because there are no intermediate cash flows in the case of such contracts, due to the absence of the marking to market mechanism. Later on we will demonstrate that forward and futures prices will be equal, when interest rates are either constant or are a known function of time. And finally, we will analyze the consequences of relaxing this assumption.

2.2 Notation

We will use the following symbols to denote the corresponding variables:

- $t \equiv$ the point in time at which we are standing, that is, today.
- $T \equiv$ the point in time at which the forward contract expires.
- $S_t \equiv$ current spot price of the asset underlying the forward contract.
- $K \equiv$ delivery price as per the forward contract.
- $F_t \equiv$ forward price of a contract initiated at time 't' and expiring at time 'T'.
- $f \equiv$ current value of a long forward contract.
- $r \equiv$ rate of interest for the period between 't' and 'T'.[1]

[1] Note, this is not an annualized rate.

2.3 Assumptions

In our analysis we will make the following assumptions:[2]

1. There are no information or transactions costs associated with buying or selling either the forward contract or the underlying asset.
2. Market participants have an unlimited ability to borrow and lend money at the rate r.
3. There is no credit risk in either forward or spot markets.
4. Commodities can be stored indefinitely without any change in their features such as quality.
5. There are no taxes.
6. Assets can be sold short with full use of proceeds.
7. Arbitrage opportunities are fully exploited as soon as they are perceived.

Let us first understand the difference between the *Forward Price* and the *Delivery Price*. The delivery price is the price that is specified in the forward contract. That is, it is the price at which the short is obliged to make delivery or equivalently it is the price at which the long is obliged to take delivery.

The forward price, at any point of time, is the applicable delivery price for a contract that is being negotiated at that particular instant. If a contract were to be sealed, based on the bilateral negotiations, the prevailing forward price would become its delivery price. However, a contract that were to be negotiated an instant later is unlikely to have the same forward price. In other words, the forward price will keep changing as new trades are negotiated.

Let us view the issue as follows. If a trader were to make a statement that she had taken a forward position at a prior point in time, the natural response would be “what was the delivery price?" and not “what was the forward price then?", although both would mean the same thing. However if we were to be confronted with an offer to get into a forward position, the question to ask would be “what is the forward price?". If the contract were to be sealed, the current forward price would become the delivery price of the contract, which would remain invariant for the life of the contract.

2.4 Forward Contracts on a Security that Provides No Income

Obvious examples of such securities are stocks that pay no dividends and zero coupon bonds. Securities that provide no income represent the simplest of assets that we will value, and in this initial case, we will illustrate our arguments using

[2] Some of them will be relaxed subsequently.

both discrete compounding / discounting as well as continuous compounding / discounting.[3]

2.4.1 Cash and Carry Arbitrage

From the perspective of the party who has a short position in a forward contract, it represents a commitment to make delivery. Consider the case of an investor with a short position in the contract, and who has a unit of the underlying asset with him. He would obviously have to fund the acquisition of the asset. It would entail an actual interest cost if he were to borrow the required amount, and an opportunity cost if he were to deploy his own funds. If the difference between the forward price, which is what he is committed to receive at expiration, and the prevailing spot price at the outset, which is the amount that he has financed, were to exceed the cost of carrying the asset until delivery, then clearly there would exist an arbitrage opportunity. In the case of an asset that pays no income, the only component of the carrying cost is the interest cost rS_t, where r is the riskless rate of interest and S_t is the prevailing spot price at the time of entering into the contract. Hence if the price differential were to exceed the interest cost, or

$$F_t - S_t > rS_t$$

then a person could exploit the situation by borrowing and buying the asset, and concurrently taking a short position in a forward contract.

Such a strategy, that is intended to realize an arbitrage profit, is called *Cash and Carry Arbitrage*. Hence, to rule out such arbitrage, we require that

$$F_t - S_t \leq rS_t \Rightarrow F_t \leq S_t(1 + r) \tag{2.1}$$

Numerical Illustration We will illustrate cash and carry arbitrage with the help of a suitable numerical example.

Assume that Coca Cola is currently selling for \$ 50 per share, and as per the forecast of investors, is not expected to pay any dividends for the next six months. We will assume that the price of a forward contract for one share of the company to be delivered after six months is \$ 54. Notice that the phrase *'security that provides no income'* refers to securities that are not expected to make any payouts during the life of the forward contract. It does not mean that the asset will never make payouts.

Let us take the case of an arbitrageur who is in a position to borrow funds at the rate of 6% per six monthly period. He can obviously borrow \$ 50 and acquire one share of the company. Assume that at the time of the transaction, he simultaneously goes short in a forward contract to deliver the share after six months for \$ 54. In other words he locks in the sale price of the asset at the very outset. The rate of return on his investment, for a six month horizon is:

$$\frac{(54 - 50)}{50} = 0.08 \equiv 8\%$$

whereas his funding cost is only 6%.

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[3]The method chosen in a particular textbook would depend on the personal preference of the author. It is often felt by theoreticians that continuous time methods yield more elegant solutions. Students of modern finance, should ideally be equally comfortable with both techniques.

Thus, the arbitrage strategy that has been implemented is obviously a profitable proposition. This is because the forward contract is overpriced, that is:

$$F_t > S_t(1 + r)$$

The rate of return for the arbitrageur in a cash and carry strategy is referred to as the *Implied Repo Rate(IRR)*. Obviously, such an arbitrage strategy would be profitable only if the Implied Repo Rate were to exceed the borrowing rate faced by the arbitrageur.

The net result of such a strategy may be perceived as follows. By engaging in such a transaction, the arbitrageur has ensured a payoff for himself of \$ 54 after six months, in return for an initial investment of \$ 50. Thus, it is as if he has bought a *Zero Coupon* debt security with a maturity value of \$ 54, by paying a price of \$ 50. Hence, a combination of a long position in the underlying asset and a short position in a forward contract is equivalent to a long position in a zero coupon security. Such a deep discount instrument, which is artificially generated using other assets, is referred to as a *Synthetic T-bill*. Hence we can express the relationship as

Spot - Forward = Synthetic T-bill

In this expression, the negative sign in front of the forward contract denotes that the arbitrageur has taken a short position in it. Thus, an investor who has taken natural positions in any two of the three assets, can artificially create a position in the third. One significant implication of the above equation is that although the spot and forward positions are exposed to price risk when held in isolation, their combination leads to a riskless position.

2.4.2 Reverse Cash and Carry Arbitrage

We have just seen that cash and carry arbitrage requires a short position in a forward contract and arises if the contract were to be overpriced. That is:

$$F_t > S_t(1 + r)$$

However, what if the forward price, F_t were to be less than $S_t(1 + r)$, or in other words the contract were to be underpriced. It turns out that such a situation too represents an arbitrage opportunity. However, to exploit the mispricing in this case, the arbitrageur would have to take a long position in the contract. The strategy to be deployed under such circumstances requires the investor to short sell the asset and invest the proceeds at the riskless rate of interest. He also needs to simultaneously take a long position in the forward contract.

This kind of an arbitrage strategy is called *Reverse Cash and Carry Arbitrage*, and is feasible in those circumstances where the difference between the forward price and the spot price is less than the carrying cost. In order to rule out such opportunities, we obviously require that:

$$F_t \geq S_t(1 + r) \tag{2.2}$$

Numerical Illustration Let us illustrate reverse cash and carry arbitrage with the help of a numerical example. We will assume once again that Coca Cola is selling for \$ 50 per share, and that the company is not expected to pay any dividends for the next six months. We will assume that the riskless rate of interest continues to be 6% for this period. However, the price of a forward contract for one share to be delivered after six months will be assumed to be \$ 52.50.

Consider the following strategy which can be deployed by an arbitrageur under such circumstances. He can short sell a share at \$ 50, invest the proceeds at a rate of 6% for six months, and simultaneously take a long position to reacquire the share after six months at a price of \$ 52.50.

One of the crucial assumptions that we have made in this illustration, is that the arbitrageur can lend the proceeds from the short sale at the market rate of interest. In practice, short selling entails the depositing of the proceeds with the broker. In addition, the short seller has to put up additional collateral to protect the broker against a subsequent rise in the price of the shares. The broker can obviously earn interest on the amount deposited with him. Institutional investors can, in a competitive market, demand that the broker share a part of the interest income with them. Such payments are referred to as *Short Interest Rebates*. However, even if the broker were to pay interest to the client, the effective rate of return earned by the short seller will be lower than the prevailing market rate.

The effective cost of borrowing for the short seller in our example is:

$$\frac{(52.50 - 50.00)}{50.00} = 0.05 \equiv 5\%$$

which is less than the lending rate of 6%.

Thus the arbitrage strategy leads to a profit for the investor. The reason why it is attractive is obviously because the contract is underpriced. That is:

$$F_t < S_t(1 + r)$$

The effective borrowing cost incurred by an arbitrageur who employs such a strategy is known as the *Implied Reverse Repo Rate(IRRR)*. Hence, reverse cash and carry arbitrage is profitable only if the Implied Reverse Repo Rate is less than the lending rate.

Let us analyze the strategy from a related perspective. By engaging in such a transaction, the investor has ensured the sale of what is akin to a zero coupon instrument with a face value of \$ 52.50, for a price of \$ 50. Thus, a combination of a short position in the stock and a long position in a forward contract is equivalent to a short position in an artificial zero coupon instrument, which is what we have previously termed as a synthetic T-bill. Hence we can express the relationship as: −Spot + Forward = −Synthetic T-bill.

Cash and carry arbitrage is feasible if the contract is overpriced, that is, $F_t \geq S_t(1 + r)$, while reverse cash and carry arbitrage is feasible if the contract is underpriced, that is, $F_t \leq S(1 + r)$. Thus, in order to rule out both forms of arbitrage, which is an indication that the contract is fairly priced, we require that:

$$F_t = S_t(1 + r) \tag{2.3}$$

In this example therefore, the no-arbitrage forward price should be $F_t = 50 \times (1 + .06) = \$\ 53$.

2.4.3 The Case of Continuous Compounding

Let us re-examine the above examples on cash and carry arbitrage and reverse cash and carry arbitrage, using continuous compounding.

Consider a person who buys one share of Coca Cola at $ 50 and enters into a forward contract to sell the share after six months. He does not have to pay anything to get into a forward contract, for its value at inception is zero. So the initial cost of this strategy is equal to the value of the share, which is $ 50. Now, according to the terms of the agreement, the person will deliver the share six months later and will receive the delivery price. So all he is doing, is paying $ 50 today in exchange for a cash flow, equal to the delivery price after six months. So to preclude arbitrage opportunities, the delivery price should equal $ 50 compounded at the borrowing rate r.

Therefore, $K = 50e^{r} = 50e^{.06} = 53.0918$

By definition, the forward price and the delivery price will be equal at inception.

Hence, $F_t = K = 53.0918$

Notice that the equilibrium forward price differs from what we got earlier because compounding continuously at 6%, differs from compounding discretely at 6%.

2.5 Repos and Reverse Repos

We have used the terms Repo rate and Reverse Repo rate in the above discussion. These are terms that are commonly used in the money market, and we will briefly discuss them.

2.5.1 Repos

The term 'Repo' is a short form for a Repurchase transaction, and represents collateralized borrowing by pledging securities. Using a repo, a party in need of funds, can borrow from another party who has funds to invest. A typical repo transaction is an overnight deal. That is, the borrower will sell securities to the lender at a price P_1, with a simultaneous commitment to buy them back a day later, at a higher price P_2. If we denote the overnight repo rate as R% per annum[4] then

$$P_2 = P_1 \times \left(1 + \frac{R}{360}\right) \tag{2.4}$$

The first leg of the repo is a spot transaction, while the second leg is a forward contract. Consequently, in some markets repos are known as ready-forward transactions. The difference between the price at which the securities are sold and that at which they are subsequently re-purchased, constitutes the interest for the lender.

[4]Repo rates are quoted on a 360 day year basis, as is the norm in the money market.

While most repos are done on an overnight basis, there exist what are known as *term repos*, which are undertaken for longer periods. There are dealers who make a market in these transactions. They will first locate a party with a surplus, such as a corporation or a Money Market Mutual Fund, and borrow from it. The funds will subsequently be lent out to a party which has a shortfall and hence seeks to borrow. The majority of such transactions in global markets, are done on the strength of government securities. In the U.S. however, other money market securities such as commercial paper may also be used as collateral.

2.5.2 Reverse Repos

A repurchase transaction is a collateralized arrangement for borrowing money. Such an agreement, as perceived from the lender's position is known as a '*Reverse Repo*'. Thus, by employing a reverse repo, a lender will grant a loan, on the strength of the securities delivered by the borrower. The securities will obviously be returned at a higher price at the end of the period for which the funds have been loaned. The difference between the sale price and the purchase price constitutes the interest earned by the lender. Reverse repo transactions are primarily overnight deals, although dealers do undertake *Term Reverses*.

The nature of a given transaction depends on the motive of the dealer who undertakes it. Dealers who seek to borrow do repos, whereas those who seek to lend engage in reverses. Often a dealer will do a repo at a particular rate and will immediately follow it up with a reverse repo at a higher rate. In the process he will obviously make a spread.

2.6 Short Selling

The ability to short sell the underlying asset is critical for the execution of a reverse cash and carry arbitrage strategy. Let us therefore take a closer look at the mechanics of a short sale. Long positions in shares are taken by what we term as *bulls*. Such investors acquire shares because they foresee an increase in their value over a period of time. If their views were to turn out to be accurate, they can obviously liquidate their long positions with a profit. Thus the ability to take long positions constitutes a speculative tool for those who are bullish about the market. The investment principle being followed in such cases is therefore, *'buy low and sell high'*.

However, it is not necessary that all investors should have a bullish outlook at a given point in time. There will always be those who expect the market to decline over a period of time. Such investors are referred to as *bears*. These traders too require a tool to facilitate speculation on their part. Short selling represents such a technique.

What is short selling and how is it accomplished? A short sale is carried out by selling an asset that does not belong to the seller. Quite obviously, to do so he must borrow it from another investor. In practice investors borrow from brokers. A broker may have the shares in his personal inventory or else he may borrow it from a client such as a financial institution. The securities which are borrowed in such fashion, have obviously to be bought back and returned. The process of acquisition of shares to close out an existing short position, is called *'covering the short position'*. Short sellers undertake such positions in the belief that they can re-acquire the asset at a lower value at the time of closing out the position. The principle is obviously *'sell high and buy low'*. At times, during the period of the short sale, the asset may make a payout. In the case of equity shares the company may declare a dividend, whereas in the case of bonds the issuer may pay a coupon. The dividend or coupon in such cases will go the current owner of the security. However, from the perspective of the party who has lent the security to facilitate the short sale, he has merely lent it and not sold it. If he had not parted with the asset he would have obviously received the payment. Hence, as per the terms of the short sale agreement, the borrower of the securities is required to compensate the lender for any lost income.

Similarly, there could be other corporate actions during the period of a short sale as a stock split. If so, the short seller has to make the necessary adjustment while returning the shares. For instance, if there were to be a 5:1 split, an investor who has borrowed one share prior to the split, is responsible for returning five shares after the split.

When a trader borrows a share from a broker and sells it, he has an obligation to buy it back eventually. From the standpoint of the broker there is always a risk that the security which the short seller thinks would depreciate in value, actually ends up appreciating. Consequently the broker needs to guard against the possibility of default. Thus, the broker in practice will retain the sale proceeds with him. Further, to protect himself against rising prices, he will require the party who is selling short to deposit additional collateral. The additional collateral as well as the sale proceeds will show up as a credit balance in the trader's account.

2.6.1 Numerical Illustration

Maureen decides to short sell 200 shares of Merck, which is currently quoting at \$ 55. Thus the proceeds from the sale will be \$ 11,000. These funds have obviously to be deposited with the broker. Regulation T, commonly known as Reg T, which is the regulation that covers short selling in the United States, requires the seller to deposit additional collateral equivalent to at least 50% of the proceeds from the short sale. Let us assume that Maureen therefore deposits an additional \$ 5,500 with the broker. Her account position may be represented as follows.

Figure 2.1

Liabilities	Assets
200 Shares @ \$ 55 = \$ 11,000	Credit Balance = \$ 16,500
Owner's Equity = \$ 5,500	

2.7 The Value of a Forward Contract

As we mentioned earlier, the value of a forward contract when it is first entered into is zero. Later on, it may have a positive or a negative value.

In the earlier example, the current equilibrium forward price, for a contract on Coca Cola, was \$ 53, which by definition, was equal to the delivery price of a contract being negotiated at that point in time. Let us consider another forward contract that was entered into a while earlier, with a delivery price of \$ 50. At that time, the forward price would have been equal to \$ 50. Today, say a week later, the delivery price as per the contract is still \$ 50, but the forward price is \$ 53. So the question is, what value will the old contract have today?

Consider the holder of a long forward contract with a delivery price of \$ 50. He will require \$ 50 after six months to pay and take delivery. So all he needs today, is the present value of \$ 50, which if invested at the prevailing rate of 6%, will be adequate for taking delivery after six months.

The present value of \$ 50 $= \frac{50}{1.06} = 47.1698$. Hence, if this person has \$ 47.1698 plus one long forward contract, then he will be entitled to one share of Coca Cola after six months. Now, consider another individual who pays \$ 50 now and buys one share of Coca Cola in the spot market. At the end of six months, he too will be in possession of one share of the company. Thus, the portfolios held by both these persons will be worth the same at the end of six months. If the future values of these portfolios are the same, then they must have identical present values.

Let f be the value of the forward contract. Therefore

$$f + 47.1698 = 50 \Rightarrow f = 2.8302.$$

Hence, the value of a forward contract with a delivery price of \$ 50 is \$ 2.8302. In symbolic terms

$$f = S_t - \frac{K}{(1+r)}$$

From the no-arbitrage condition we know that

$$S_t = \frac{F_t}{(1+r)}$$

Therefore

$$f = \frac{F_t}{(1+r)} - \frac{K}{(1+r)} = \frac{(F_t - K)}{(1+r)} \tag{2.5}$$

Thus the value of a long forward contract is the present value of the difference between the forward price and the delivery price.

Now, you are aware that forward contracts are customized contracts that are negotiated individually between the buyer and the seller. So you may be wondering how the holder of a long forward contract can possibly realize the value that we have calculated above? In order to realize the value of the above contract, all the holder has to do, is to enter into a new contract that offsets the original. We will illustrate this with the help of an example.

2.7.1 Example

Consider a person who is holding a long forward contract on IBM, with a delivery price of \$ 100. In order to get out of his position, he will have to go short in a fresh contract with a current delivery price of say \$ 105. At expiration therefore, he can buy the asset under the first contract at \$ 100 and sell it under the second at \$ 105. So he has a guaranteed payoff of \$ 5 waiting for him at expiration. If we assume that the riskless rate is 6%, then the value of his original contract is the present value of this future payoff. That is:

$$\text{Value} = \frac{5}{1.06} = 4.7170$$

2.7.2 Value of a Futures Contract

Having understood how to determine the value of a forward contract, let us now turn our attention to futures contracts. At the outset, neither the long nor the short has to pay to get into a position in either a forward or a futures contract. However, as the futures price changes subsequently, an open futures position will acquire value. There is however a difference between forwards and futures in terms of how this value is dealt with. In the case of futures contracts, due to the marking to market mechanism, the profit/loss is calculated at the end of every day and credit/debited to the margin account. The position is then re-initialized at the settlement price that is used to mark to market at the end of that particular day. This process of marking to market is nothing but a settlement of built up value. Thus, once the profit/loss is adjusted, the value of the futures contract will once again revert back to zero. Therefore the only time futures contracts accumulate value, is in the period between two successive settlement price calculations. Once the settlement of built up value occurs at the end of the day, the value of both long and short positions will go back to zero.

2.8 Forward Contracts on Assets that Provide a Known Cash Income

Let us now go on to consider the case of forward contracts on assets that provide a perfectly predictable cash income. Examples include stocks which pay known dividends and coupon paying bonds. If an investor were to get a cash inflow from the asset that he is holding, then he would obviously have a reduced carrying cost. The carrying cost on account of interest, as we have seen before, is rS_t. Let us denote the future value of the income from the asset, as calculated at the time of expiration of the forward contract, by I. Why do we need to calculate the future value of the income? The carrying cost is computed as at the point of expiration of the futures contract. In order to be consistent with the principles of time value of money, the value of the income too should be computed at the same point in time. Hence, if the income were to be received prior to the time that the contract matures, it has to be compounded for the remaining period using the riskless rate of interest. Consequently, in order to rule out cash and carry arbitrage we require that

$$F_t - S_t \leq rS_t - I \Rightarrow F_t \leq S_t(1 + r) - I \tag{2.6}$$

Now let us consider a reverse cash and carry arbitrage strategy. This strategy entails a short sale, and as we have mentioned earlier, the short seller is responsible for compensating the lender of the asset for any lost income. Consequently, the effective income obtained by a short seller by investing the proceeds from the sale will be reduced by the amount of payouts from the asset. Hence, reverse cash and carry arbitrage will be profitable only if

$$F_t - S_t < rS_t - I$$

Thus to rule out reverse cash and carry arbitrage we require that

$$F_t - S_t \geq rS_t - I \Rightarrow F_t \geq S_t(1 + r) - I \tag{2.7}$$

Hence, to preclude both forms of arbitrage it must be the case that

$$F_t = S_t(1 + r) - I \tag{2.8}$$

Let us now illustrate the two arbitrage strategies in the case of forward contracts on a share that pays a known dividend. We will continue with the Coca Cola example but will now assume that the stock will pay a dividend of \$ 2.50 after three months and another \$ 2.50 after six months. The second payment will be assumed to be made an instant prior to delivery under the forward contract. We will also assume that an arbitrageur can borrow or lend at the rate of 6% per semi-annual period.

Let the forward price be \$ 48.

2.8.1 Cash and Carry Arbitrage

Consider the following strategy. Buy one share of Coca Cola, by borrowing the \$ 50 required to finance the purchase. Simultaneously enter into a forward contract to sell one share after six months at \$ 48. During this period of six months, you will get a dividend of \$ 2.50 after three months, followed by another dividend of \$ 2.50 three months later. The first dividend can be invested until the expiry of the forward contract at 6% per six months. So at the end of six months, you will get \$ 48 when you deliver the share, plus interest and principal on account of the first dividend that you have invested, plus the second dividend. So the total final cash flow is

$$48 + 2.50\left(1 + \frac{1}{2} \times 0.06\right) + 2.50 = 53.075.$$

The rate of return is:

$$\frac{(53.075 - 50)}{50} \equiv 6.15\%$$

for six months. But the borrowing rate is only 6% per six-monthly period. Hence, it is obvious that there is an arbitrage opportunity. So the forward price of \$ 48 does not represent an equilibrium.

Let us consider another case where the forward price is $ 47. The above strategy, as you can verify, will not yield arbitrage profits. But it turns out that you can make arbitrage profits by reversing the above strategy.

2.8.2 Reverse Cash and Carry Arbitrage

This strategy can be executed as follows. Short sell one share of Coca Cola at $ 50 and go long in a forward contract to enable you to procure one share after six months. Three months later, you would require $ 2.50 to compensate the person who lent you the share because he would have received a dividend equal to this amount had he not parted with the share. This amount can be borrowed at 6% per half-yearly period. Once again, by the same logic, at the end of another three months you would be required to pay $ 2.50.

Thus your total outflow at the end of six months is

$$47 + 2.50\left(1 + \frac{1}{2} \times 0.06\right) + 2.50 = 52.075$$

By engaging in the above strategy, you have effectively taken a loan of $ 50, and paid back $ 52.075 six months later. The interest rate on your borrowing is

$$\frac{(52.075 - 50)}{50} = 4.15\%$$

However, the $ 50 that you receive when you short sell the stock, can be lent at 6% for six months. Thus, there clearly exists an arbitrage opportunity.

Once again, to eliminate arbitrage profits, the IRR must equal the IRRR, which must equal the borrowing and lending rates. Let F_t be the equilibrium forward price. Then:

$$\frac{(F_t + 5.075 - 50)}{50} = 0.06 \Rightarrow F_t = 47.925$$

The amount of $ 5.075 is the future value at the end of the six monthly period, of the payouts from the asset during the period, which we have denoted by I. Therefore, in symbolic terms

$$\frac{(F_t + I - S_t)}{S_t} = r \Rightarrow F_t = S_t(1 + r) - I$$

2.8.3 Value of the Forward Contract

Let the current forward price, as calculated above, be $ 47.925. Consider a forward contract that was entered into a week ago, with a delivery price of $ 45. A person who has a long position in one such contract and has the present value of $ 45 with him, will have one share of Coca Cola in his possession after six months. Now consider another person who owns a share of Coca Cola and has borrowed the present value of the first dividend as well as the present value of the second dividend.[5] The borrowings can be paid back with interest when the respective dividends are received. Thus at the end of six months, this individual too will

[5] The present values are calculated at the beginning of the six monthly period.

have one share of Coca Cola with him. Since both of them will have the same terminal wealth, the present values of their portfolios must be identical. Therefore, if f is the value of the forward contract

$$f + \frac{45}{(1.06)} = 50 - \frac{2.5}{(1.03)} - \frac{2.50}{(1.06)}$$

$$\Rightarrow f = 2.7615$$

Symbolically

$$f = S_t - \frac{K}{(1+r)} - \frac{I}{(1+r)}$$

$$F_t = S_t(1+r) - I \Rightarrow S_t = \frac{F_t}{(1+r)} + \frac{I}{(1+r)}$$

$$\text{Hence, } S_t - \frac{I}{(1+r)} = \frac{F_t}{(1+r)}$$

$$\Rightarrow f = \frac{F_t}{(1+r)} - \frac{K}{(1+r)} = \frac{(F_t - K)}{(1+r)} \tag{2.9}$$

2.9 Forward Contracts on Assets that Provide a Known Dividend Yield

What does the concept of a known dividend yield mean? It means that the dividend income expressed as a percentage of the asset price is known.

Assume that dividends are paid every quarter at an annual rate of 10%. What this means is the following. If the asset price when a dividend is paid is \$ 100, then the quantum of the dividend $= .10 \times \frac{1}{4} \times 100 = \$\ 2.50$. Similarly, if at the end of the next quarter, the asset price is \$ 120, then the dividend paid $= .10 \times \frac{1}{4} \times 120 = \$\ 3.00$.

Let us assume that dividends are paid m times in a year at fixed intervals and let the annual dividend rate be $d\%$. Consider two portfolios. Portfolio A, consists of a long forward contract expiring after $T - t$ days with a delivery price of K, plus cash equal to $\frac{K}{(1+r)}$.

Portfolio B consists of $\frac{1}{\left(1+\frac{d}{m}\right)^n}$ units of the asset, where n is the number of times that dividends are paid between t and T. The stipulation is that all dividend income received should be reinvested in the asset.

At the delivery date, portfolio A will obviously consist of one unit of the asset. What about portfolio B? Our holdings in this portfolio will also grow at an annual rate of d, as a consequence of the fact that dividends are being reinvested in the asset as soon as they are received. Thus, this portfolio also would consist of one unit of the asset at time T. This can be best illustrated with the help of an example.

2.9.1 Example

Consider an asset with a price of \$ 100. Let us assume that dividends are paid at the rate of 10% per annum, every quarter. So the current dividend is

$$.10 \times \frac{1}{4} \times 100 = \$\,2.50$$

Consider a person who has 10,000 shares. He will receive \$ 25,000. If he reinvests this money in the same asset, then he can buy 250 shares. So his holding will grow at the rate of

$$\frac{(10{,}250 - 10{,}000)}{10{,}000} \equiv 2.5\%.$$

A rate of 2.5% per quarter, is equivalent to an annual growth rate of 10% .

Now suppose that three months later when the next dividend is paid, the stock price is \$ 120. The dividend paid is

$$.10 \times \frac{1}{4} \times 120 \times 10{,}250 = \$\,30{,}750$$

With this amount, the investor can buy

$$\frac{30{,}750}{120} = 256.25 \text{ shares.}$$

Once again, the portfolio would grow by

$$\frac{(10{,}250 + 256.25)}{10{,}250} \equiv 2.5\%.$$

Hence, each time a dividend is paid, the portfolio will grow by 2.5%. So if you start with

$$\frac{1}{\left(1+\frac{d}{m}\right)^n} \text{ shares}$$

then every time a dividend is paid, the portfolio will grow at the rate of $\frac{d}{m}$. This will happen n times during the period $T - t$ and consequently you will have one share at time T.

So if both portfolio A and portfolio B consist of one share at time T, then they must have the same value today.

Therefore, if f is the value of the long forward contract, then

$$f + \frac{K}{(1+r)} = S\frac{1}{\left(1+\frac{d}{m}\right)^n}$$

The forward price F_t is that value of K that will make f equal to zero. Hence

$$F_t = \frac{S_t(1+r)}{\left(1+\frac{d}{m}\right)^n} \tag{2.10}$$

If F_t is greater than this, one can make arbitrage profits by going short in the forward contract and buying the stock. While, if F_t is less than the no-arbitrage value, one can make profits by going long in the forward contract and short selling the stock.

2.9.2 The Continuous Time Case

Let us consider the case where dividends are paid continuously at the rate of d% per annum. One may argue that in real life dividends are not paid continuously. This model is just an assumption to get an approximation to reality. It is a reasonable assumption for stock indices or portfolios of stocks, where different components pay dividends at different points in time and hence a model with continuous dividends provides a fairly good approximation.

It can be shown in this case that[6]

$$F_t = S_t e^{(R-d)\frac{(T-t)}{360}} \tag{2.11}$$

where R is the annual rate of interest.

2.10 Forward Contracts on Commodities

Thus far we have considered assets that provide no income like non dividend paying stocks and zero coupon bonds; assets which pay a known cash income like coupon paying bonds; and finally assets that can be assumed to provide a known dividend yield like stock indices.

Let us now consider the case of commodities. Commodities typically do not earn any income. On the contrary, it costs money to store them and to insure them against unforeseeable events. It turns out that we must make a distinction between commodities that are held mainly as investments by most investors (typical examples are precious metals) and those that are held for consumption purposes (like agricultural commodities).

2.10.1 Restrictions on Short Sales

For the pricing relationships that we have developed thus far to hold, both cash and carry as well as reverse cash and carry arbitrage should be freely possible. Let us recapitulate these strategies. The cash and carry strategy, requires that the arbitrageur buy the asset and take a short position in the forward contract. This per se, should pose no problems. But in the reverse cash and carry arbitrage strategy, the arbitrageur is required to short sell the asset and go long in a forward contract. The problem is that there are certain assets, which are held by people for reasons other than pure investment. People who choose to hold these assets get some special benefits from them and therefore may not be willing to lend them for short sales. In such cases, reverse cash and carry arbitrage will be infeasible.

[6]Readers should derive this expression and convince themselves.

We will first make a distinction between *Pure* and *Convenience* assets and then will go on to look at detailed examples, which will hopefully help clarify these and other related issues.

2.10.2 Pure Versus Convenience Assets

What is the definition of a short sale? It is a transaction wherein you hand over a commodity to a person, who immediately sells it and returns it intact to you at a future date. Any income that you would have received in the intervening period is given to you by the borrower of the asset, be it dividends or coupon payments. So, if you do not really require the asset during the period of the short sale, you are not foregoing anything.

The above argument is true if the owner of the asset is holding it for an investment purpose, that is, to earn capital gains and other income due to him. Such assets are called *Pure* assets or *Investment* assets. Financial assets such as stocks and bonds are examples of pure assets.

On the other hand, certain assets like wheat or rice are often held for reasons other than potential returns. Such assets are called *Convenience* or *Consumption* assets. The owners of such assets may not permit them to be sold short. This can best be clarified with the help of an example.

Example Let us consider the case of wheat. As everyone is aware, prices usually rise before a harvest season and fall thereafter. A person who hoards wheat before the harvest, not only has to incur storage costs but also faces the specter of a capital loss. Thus from an investment angle, it makes little sense for a person to hoard wheat during the harvest period. But a wheat mill owner may choose to hold the commodity for other reasons. For instance, he may want to avoid closing the mill during an anticipated temporary shortage, which may be due to a natural calamity or a rainfall failure. The value of such potential needs is called the *Convenience Value*.

The holder of a convenience asset, will possibly give it up for a short sale only if he is compensated for the convenience value. Thus, you may feel that the convenience value is similar in principle to a dividend. But there are differences. First, how do you quantify a convenience value? Second, the perception of value will differ from holder to holder. That is why, we do not have markets for short selling convenience assets.

2.11 Investment Assets

We will illustrate our arguments for the pricing of such assets using gold, which is primarily held for investment purposes. To keep matters simple, let us initially assume that storage costs are zero. If so, gold is similar to an asset that pays no income. The cash and carry and reverse cash and carry arguments, can then be used to show that

$$F_t = S_t(1 + r)$$

A cost is nothing but a negative income. For an asset which pays a known income, we have shown that

$$F_t = S_t(1 + r) - I$$

where I is the future value of payouts from the asset. Let us denote the future value of storage costs by Z. Therefore

$$F_t = S_t(1 + r) - (-Z) = S_t(1 + r) + Z \tag{2.12}$$

We will illustrate the no-arbitrage pricing condition for gold, using a suitable example.

2.11.1 Example

Let the spot price of gold be \$ 800 per ounce and the rate of interest be 7.50% per half-yearly period. We will assume that storage costs are \$ 10 per ounce per six monthly period, payable at the end of the period and that forward contracts are available for delivery six months into the future. If so:

$$F_t = 800 \times (1.075) + 10 = 870$$

Now let us consider mispriced forward contracts and the strategies for exploiting them.

Case A Consider the case where F_t = \$ 880. The contract is clearly overpriced. So to exploit this opportunity, a person can engage in cash and carry arbitrage. Consider the following strategy.

Borrow \$ 800 and buy one ounce of gold. Simultaneously, go short in a forward contract to sell the gold after six months at \$ 880. Six months later, when the gold is delivered, you will receive \$ 880 and will have to pay \$ 10 by way of storage costs. Thus your net inflow = \$ 870. The rate of return is:

$$\frac{(870 - 800)}{800} = 0.0875 \equiv 8.75\%$$

which is greater than the borrowing rate of 7.50% . So clearly, F_t cannot be greater than $S_t(1 + r) + Z$.

Case B Let F_t = \$ 860. The contract is underpriced. Conventional reverse cash and carry arbitrage arguments will entail the use of the following strategy.

Short sell the gold at \$ 800 per ounce and simultaneously go long in a forward contract to buy at \$ 860. The effective borrowing rate is:

$$\frac{860 - 800}{800} = 0.075 \equiv 7.50\%$$

which is the same as the lending rate. Thus, although the contract is obviously underpriced, a conventional reverse cash and carry strategy does not yield a profit. For such a strategy to yield profits to the arbitrageur, the lender of the gold would have to pass on at least a part of the storage cost saved by him, which is \$ 10 in this case. Let us assume that the lender does pass on \$ 7.50 to the arbitrageur. If so, the arbitrageur's net outflow will be \$ 852.50, which means that the implied reverse repo rate is 6.5625%, which is less than the lending rate of 7.50%.

In practice however, it is not usually possible to sell short in such a way that, the person lending the asset actually passes on the storage costs saved to the short seller. So does it mean that F_t can be less than $S_t(1 + r) + Z$, for an investment asset like gold?

The answer is no. If F_t were to be less than $S_t(1 + r) + Z$, then the mispricing can be exploited by a person who already owns gold. Consider the following strategy. A person who owns gold, but does not require it for six months, can sell it in the spot market and invest the proceeds. He can also go long in a forward contract to re-acquire the gold at the end of six months.

When he sells the gold, he will receive \$ 800. This will yield $800 \times (1.075) =$ \$ 860 at the end of six months. He will also have an additional \$ 10 with him, which represents the storage costs saved. After paying \$ 860 to re-acquire the asset, he will have \$ 10 with him, which represents an arbitrage profit.

The arguments that we have used above, represent a strategy known as *Quasi-Arbitrage*. The person who engages in such a strategy has not engaged in arbitrage in the conventional sense. Rather, he has liquidated his position in the asset and has employed a strategy which effectively ensures that he gets the asset back at the end of six months. In other words, it is as if he has effectively not parted with the asset. In the parlance of derivatives, we say that he has replaced an actual spot position with a *Synthetic Spot* position, or to put it differently he has *sold an asset without really selling it.* [7]

Thus quasi-arbitrage will help ensure that F_t cannot be less than

$$S_t(1 + r) + Z$$

Hence, to rule out both cash and carry arbitrage, as well as reverse cash and carry quasi-arbitrage, we require that

$$F_t = S_t(1 + r) + Z$$

2.12 Consumption Assets

Let us now consider the case of wheat. If

$$F_t > S_t(1 + r) + Z$$

then the resultant arbitrage opportunity can be exploited using a cash and carry strategy. But if

$$F_t < S_t(1 + r) + Z$$

then it may not be possible to make arbitrage profits. Conventional reverse cash and carry arbitrage may not be possible because short sales may not be feasible. Reverse cash and carry quasi arbitrage may also not be possible because the person who is holding the asset may be getting a convenience value.

[7] We will have more to say about quasi-arbitrage and synthetic positions, a little later.

Consequently, it may be the case that

$$F_t < S_t(1 + r) + Z$$

and that no one is able to exploit any opportunities for making arbitrage profits.

Assets such as wheat are usually consumption assets. In the case of such assets, all that we can say by way of a no-arbitrage pricing relationship is that

$$F_t \leq S_t(1 + r) + Z \tag{2.13}$$

The marginal convenience value Y is defined to be that value which satisfies the following equation, for convenience assets

$$F_t = S_t(1 + r) + Z - Y \tag{2.14}$$

The marginal convenience value is the lowest of the convenience values, as perceived by different market participants.[8]

Thus far, we have used arbitrage arguments to derive the relationship between the spot price and the forward price of an asset. Similar arguments can be used to derive the relationship between the prices of forward contracts that expire at different points in time.

2.13 Calendar Spreads and Arbitrage

Consider the following situation. Today, that is at time t, forward contracts on IBM which expire after three months (which we will denote by $T1$) are available with a delivery price of \$ 102.50 and contracts that expire after six months (which we will denote by $T2$) are available for \$ 105. The borrowing / lending rate on a three month loan to be made three months from today, is 2%.[9]

It turns out that we can make arbitrage profits in such a scenario by following the strategy described below.

2.13.1 Forward Cash and Carry Arbitrage

This strategy entails the following steps. At time t, go long in a three month forward contract and simultaneously go short in a six month contract. After three months, borrow \$ 102.50 and take delivery under the first contract. Three months later, sell the share under the second contract for \$ 105.

The forward implied repo rate is

$$\frac{(105 - 102.50)}{102.50} \equiv 2.44\%$$

which is greater than the borrowing rate of 2%. Thus, there is clearly an arbitrage opportunity.

[8]Remember, as we said earlier, it is not necessary that everyone should have an identical convenience value.

[9]This rate is assumed to be known at time t.

Now let us assume that the three month contract continues to be priced at \$ 102.50, while the six month contract is priced at \$ 104. In such a case, we can make arbitrage profits by engaging in the following strategy.

2.13.2 Forward Reverse Cash and Carry Arbitrage

This strategy entails the following sequence of steps. Go short in the three month contract and long in the six month contract. After three months, short sell the asset and deliver it under the first contract for 102.50. Three months later, take delivery under the second contract for \$ 104 and cover the short position.

The forward implied reverse repo rate is

$$\frac{(104 - 102.50)}{102.50} \equiv 1.46\%$$

which is less than the lending rate of 2%. Once again, we have an arbitrage opportunity.

If we assume that the three month contract is fairly priced, the no-arbitrage price for the six month contract, F, should be such that

$$\frac{(F - 102.50)}{102.50} = .02 \Rightarrow F = 104.55$$

In symbolic terms, if F_{T1} is the price of the short term contract, F_{T2} the price of the longer term contract and $r_{T1,T2}$ the interest rate for a loan between $T1$ and $T2$, then

$$\frac{(F_{T2} - F_{T1})}{F_{T1}} = r_{T1,T2} \Rightarrow F_{T2} = F_{T1}(1 + r_{T1,T2}) \quad (2.15)$$

2.13.3 The Case of Assets Making Payouts

If an asset makes payouts between $T1$ and $T2$, then the no-arbitrage pricing condition is

$$F_{T2} = F_{T1}(1 + r_{T1,T2}) - I \quad (2.16)$$

In this case is the future value (as calculated at $T2$) of all the payouts between $T1$ and $T2$. Let us suppose that the payout occurs at $t1$, where $T1 < t1 < T2$. Then I is the future value as at $T2$, with interest compounded for the period from $t1$ to $T2$, using the appropriate interest rate for the period.

2.14 Net Carry

Consider the case of an asset that pays a known income. The no-arbitrage pricing relationship for such assets, states that

$$F_t = S_t(1 + r) - I$$

This implies that

$$\frac{F_t}{S_t} = 1 + r - \frac{I}{S_t} \Rightarrow \frac{(F_t - S_t)}{S_t} = r - \frac{I}{S_t}$$

The *net carry* is defined as Interest minus the Future Value of Payouts, divided by the spot price of the asset. Therefore

$$\text{net carry} = \frac{(rS_t - I)}{S_t} = r - \frac{I}{S_t}. \tag{2.17}$$

Hence

$$\text{net carry} = \frac{(F_t - S_t)}{S_t} \Rightarrow F_t = S_t + (\text{net carry})S_t \tag{2.18}$$

If the net carry is positive, then the forward price will exceed the spot price, else it will be less than or equal to the spot price. In the case of physical commodities which are held for investment purposes, $I = -Z$, where Z represents the cost of storage. Therefore

$$\text{net carry} = r - \frac{-Z}{S_t} = r + \frac{Z}{S_t} \tag{2.19}$$

which is positive. Thus, the forward price for such commodities will be greater than the spot price.

However, in the case of consumption assets, where a convenience value is involved

$$F_t = S_t + (\text{net carry})S_t - Y \tag{2.20}$$

where Y is the convenience value. In this case, the relationship between the forward price and the spot price, would depend on the relative magnitudes of the net carry and the convenience value.

If $Y = 0$, we say that it is a *Full Carry* market, whereas if the convenience value is positive, we say that the market is not at full carry.

2.15 Backwardation and Contango

If the futures price exceeds the spot price or the price of the nearby futures contract is less than that of the more distant contract, then we say that there is a *Contango* market. Whereas if the futures price is less than the spot price or the price of the nearby futures contract is more than that of the distant contract, then we say that the market is in *Backwardation*. Consider the following data.

2.15.1 Example

The data given below represents hypothetical prices for corn. Case A illustrates a backwardation market whereas Case B is an illustration of a contango market. A particular commodity, need not continuously display the characteristics of either a backwardation or a contango market. That is, the market may switch from one mode to another.

Case A: Backwardation Market

Table 2.1 **Backwardation Market**

Contract	Price
Spot	6.95
March Futures	6.82
May Futures	6.65
July Futures	6.30
September Futures	6.15

Case B: Contango Market

Table 2.2 **Contango Market**

Contract	Price
Spot	6.75
March Futures	6.85
May Futures	6.95
July Futures	7.10
September Futures	7.25

What could be the possible reasons for a backwardation market in corn? Quite obviously, the market is not at full carry, implying that there is a convenience yield. In other words, reverse cash and carry arbitrage is not feasible.

The most likely reason for such a scenario, is that corn is in short supply. If so, people will not be willing to lend for the purpose of a short sale, or else indulge in quasi-arbitrage themselves. Under such circumstances, they may need it for fulfilling sale contracts entered into in advance or else for their own use. In practice it has been observed that markets tend to be in backwardation when spot prices are rising. And rising spot prices are an obvious indication of impending shortages.

For financial assets, the net carry can either be positive or negative, depending on the relationship between the financing cost, rS, and the future value of the payouts from the asset, I. If the financing cost were to exceed the value of the payouts, the net carry will be positive, and we will have a Contango market. Otherwise, the net carry will be negative, which will reveal itself as a market in Backwardation.

In the case of physical commodities, if the market is at full carry, then we will always have a Contango market. However, if the market is not at full carry, then we may either have a Backwardation or a Contango market. If the net carry is greater than the convenience value, there will be a contango market, else the market will be in backwardation.

2.16 Delivery Options

As you have learnt by now, futures contracts allow for the delivery of the asset at any time within a stipulated time period. As we have explained earlier, delivery can commence before the cessation of trading in the futures contract and continue after it. The option as to when to deliver during this period, is given to the short. The question that naturally arises therefore is, should a given futures contract be assumed to expire at the beginning, middle or end of the delivery period?[10]

Let us consider a futures contract on a commodity, priced at F_t, and let S_t be the corresponding spot price of the asset. We will denote the continuously compounded rate of interest by r, and the continuously compounded storage cost rate by z. It should be noted that neither r nor z represent annualized rates.

The no-arbitrage condition for a futures contract is given by:

$$F_t \leq S_t e^{(r+z)}$$
$$\Rightarrow F_t e^{y} = S_t e^{(r+z)} \qquad (2.21)$$

where y is the convenience value.

We have chosen to model the convenience yield as a continuously compounded variable, to be consistent with our assumptions regarding the interest rate and the storage cost. It is legitimate, since the convenience yield is nothing but a measure of an implicit dividend, that will equate the LHS and the RHS.

Let us consider the expression

$$F_t = S_t e^{(r+z-y)} = S_t e^{[r-(y-z)]}$$

For an investor with a short position, (y-z) represents the net benefit from holding the asset, while r represents the cost of financing the asset, which is either an actual interest outflow or else an opportunity cost.

In a Contango market, the futures price will be greater than the spot price. This implies that $r > (y - z)$. Thus, under such conditions, the financing cost is greater than the net benefit from holding on to the asset. In such circumstances therefore, the short would like to deliver the asset as early as possible. Thus, in a Contango market, the rational response for the short would be to deliver the asset right at the commencement of the delivery period.

In a Backwardation market however, the futures price will be less than the spot price. This implies that $r < (y - z)$. In this situation, the financing cost is less than the net benefit from holding on to the asset. Consequently, in these circumstances, the short would like to deliver the asset as late as possible. Thus, in such a market, the rational response for the short would be to deliver at the very end of the prescribed delivery period.

Our cost benefit analysis has implications for the pricing of futures contracts. The normal practice is to denote the expiration date of the futures contract by

[10]See Hull (2004) for a similar argument.

a point in time which is typically represented as T. In practice, however, for commodities, the delivery period is an interval of time and not a point in time. In such cases therefore the issue arises as to whether the value of T that is to be used for computing the no-arbitrage price should denote the start, the middle, or the end of the prescribed delivery period. From the above arguments, we can see that futures prices in a Contango market ought to be calculated under the assumption that delivery will take place right at the outset, whereas while computing the price in a Backwardation market, the logical assumption ought to be that delivery will take place at the end of the delivery period.

2.17 Imperfect Markets

Thus far, we have assumed away the existence of transactions costs. In real life, these can have a significant impact on observed futures prices. Transactions costs may be of several types. First, there exist *bid-ask spreads*. Second, the borrowing and lending rates are usually not equal. The borrowing rate is typically higher than the lending rate. Third, when an asset is sold short, the entire proceeds may not be made available to the arbitrageur for lending. That is, he may be able to lend only a fraction of the proceeds. The rest will have to be deposited with the broker from whom the share is borrowed and may or may not earn interest. Finally, there are brokerage fees involved. One has to pay these fees, while transacting in the spot market, as well as in the forward market. In addition, borrowing and lending activities also entail the payment of brokerage fees.

2.17.1 Notation

We will use the following notation for the ensuing discussion.

- $S_{b,t} \equiv$ bid price for the underlying asset in the spot market.
- $S_{a,t} \equiv$ ask price for the underlying asset in the spot market.
- $F_{b,t} \equiv$ bid price for the forward contract, that is, the price for going short.
- $F_{a,t} \equiv$ ask price for the forward contract, or the price for going long.
- $r_b \equiv$ the borrowing rate.
- $r_a \equiv$ the lending rate.
- $q \equiv$ fraction of the short sale proceeds that is available for lending.
- $C \equiv$ total brokerage fees.

2.17.2 Impact of Bid-Ask Spreads and Brokerage Fees

We will introduce transactions costs in steps. First, we will consider the impact of bid-ask spreads and brokerage fees. Then we will allow the borrowing and lending rates to be different. And finally, we will impose restrictions on the usage of proceeds from short sales.[11]

[11] We have taken an approach similar to Kolb (2003).

Let us go back to our IBM example. We will assume that the bid price in the stock market is \$ 99.75, while the ask price is \$ 100.25. So there exists a spread of 50 cents. The borrowing and lending rates are 5% for six months. The brokerage fees on a per share basis are assumed to be as follows: 15 cents in the stock market, 10 cents in the market for borrowing and lending funds and 5 cents in the forward market.

Cash and Carry Arbitrage Consider the following strategy. Buy one share of IBM at \$ 100.25, which is the ask price in the spot market by borrowing funds. Simultaneously, enter into a forward contract to sell the share after six months at $F_{b,t}$. These transactions entail the payment of brokerage fees on three occasions. When you buy the share, you will incur a cost of 15 cents. Let us call it C_1. Going short in the forward market will involve a cost C_2, which we have assumed to be 5 cents. Finally, when you borrow money to finance your purchase, you will have to pay a commission C_3, equal to 10 cents. We will assume that the cost of the share plus all commissions, can be financed by borrowing at 5% .

The initial outflow $= 100.25 + .15 + .05 + .10 = 100.55$. The final inflow $= F_{b,t}$. The implied repo rate is

$$\frac{F_{b,t} - 100.55}{100.55}$$

To preclude arbitrage, this should be less than the borrowing rate r. Thus, the no-arbitrage condition is

$$\frac{F_{b,t} - 100.55}{100.55} < .05 \Rightarrow F_{b,t} < 100.55 \times (1.05) = 105.5775$$

In symbolic terms, the condition can be expressed as

$$\frac{F_{b,t} - (S_{a,t} + C_1 + C_2 + C_3)}{(S_{a,t} + C_1 + C_2 + C_3)} < r$$

$$\text{Now, } C_1 + C_2 + C_3 = C$$

Therefore

$$\frac{F_{b,t} - (S_{a,t} + C)}{(S_{a,t} + C)} < r$$

$$\Rightarrow F_{b,t} < S_{a,t}(1 + r) + C(1 + r)$$

$C(1 + r)$ is the future value of the total transactions costs. Thus the no-arbitrage condition is that

$$F_{b,t} < S_{a,t}(1 + r) + \text{FV of } C \tag{2.22}$$

Reverse Cash and Carry Arbitrage This strategy works as follows. Short sell a share of IBM at \$ 99.75, which happens to be the bid price in the spot market. The funds received can be lent at the prevailing rate. Simultaneously, enter into a forward contract to buy the share six months later at $F_{a,t}$, which is the ask price in the forward market.

The total brokerage fees $= (C_1 + C_2 + C_3) = .30$. So the effective cash inflow from the short sale $= 99.75 - .30 = 99.45$. This amount can be lent out at 5% . To preclude arbitrage opportunities, the implied reverse repo rate should be greater than the lending rate. So we require that

$$\frac{F_{a,t} - 99.45}{99.45} > .05$$
$$\Rightarrow F_{a,t} > 104.4225$$

In symbolic terms, the condition may be expressed as

$$\frac{F_{a,t} - (S_{b,t} - C)}{(S_{b,t} - C)} > r$$
$$\Rightarrow F_{a,t} > S_{b,t}(1 + r) - C(1 + r)$$
$$\Rightarrow F_{a,t} > S_{b,t}(1 + r) - \text{FV of } C \tag{2.23}$$

In an active futures market, the bid-ask spread or the difference between $F_{a,t}$ and $F_{b,t}$ will be very small. So we can assume that $F_{a,t} = F_{b,t} = F_t$.

Thus for an arbitrage free market, the pricing relationship may be stated as

$$S_{b,t}(1 + r) - \text{FV of } C < F_t < S_{a,t}(1 + r) + \text{FV of } C \tag{2.24}$$

In the example given above, F_t should lie between 104.4225 and 105.5775 to preclude arbitrage.

2.17.3 Impact of Differential Rates for Borrowing and Lending

If the borrowing rate r_b is different from the lending rate r_l, then the arbitrage free pricing relationship may be expressed as

$$S_{b,t}(1 + r_l) - C(1 + r_l) < F_t < S_{a,t}(1 + r_b) + C(1 + r_b) \tag{2.25}$$

Let us assume that $r_b = .055$ and $r_l = .045$ with all the other variables having the same values as given earlier. If so, the price bounds are 103.9253 and 106.0803.

2.17.4 Impact of Restrictions on Short Sales

Finally, let us impose restrictions on short sales. It is obvious that short sales affect only the reverse cash and carry arbitrage strategy. Let q be the fraction of the proceeds from the short sale that the arbitrageur is allowed to lend. If so, then the effective lending rate will be $q \times r_l$. Thus, to preclude reverse cash and carry arbitrage, we require that

$$\frac{F_t - (S_{b,t} - C)}{(S_{b,t} - C)} > qr_l$$
$$\Rightarrow F_t - (S_{b,t} - C) > (S_{b,t} - C)qr_l$$

$$\Rightarrow S_{b,t}(1+qr_l) - C(1+qr_l) < F_t \qquad (2.26)$$

Let us assume that q = .80 with all the other variables having the same values as before. If so, the price bounds for the forward price are \$ 103.0302 and \$ 106.0803.

2.18 Synthetic Securities

Let us take another look at the cash and carry arbitrage argument. According to this strategy, if we buy the good in the spot market and simultaneously go short in a forward contract, it is as if we have gone long in a T-bill. Therefore

Spot − Forward[12] = Synthetic T-bill.

The terms can be repositioned to yield the following results:

- Spot − T-bill = Synthetic Forward.
- Forward + T-bill = Synthetic Spot.

Thus, if we have any two of the assets, we can artificially create the third.

Similarly, using forward cash and carry arbitrage, we can create synthetic forward positions. That is, by combining a long forward contract expiring at $T1$, with a short forward contract expiring at $T2$ (where $T1 < T2$) and a spot position between $T1$ and $T2$, one can create a synthetic T-bill between $T1$ and $T2$. That is,

Short Term Forward + Spot between $T1$ and $T2$ − Long Term Forward = Synthetic T-bill between $T1$ and $T2$.

It is very important to understand the principles behind the creation of synthetic positions, before we go on to take a detailed look at quasi-arbitrage.

2.19 Forward Prices Versus Futures Prices

In all the cases where we have derived the pricing relationships based on the no-arbitrage arguments, we have considered forward contracts and not futures contracts. This was because as we said at the outset, forward contracts are easier to analyze since they entail only one single cash flow at the end, whereas futures contracts are marked to market daily, and consequently involve intermediate cash flows.

We will now show that when the riskless rate of interest is a constant, and is the same for all maturities, then the forward price for a contract on a given asset for a specified delivery date, is the same as the futures price for a contract on the same asset, for the same delivery date. Therefore, under such conditions, the relationships that we have derived for the prices of forward contracts, are equally valid for the prices of futures contracts.

[12]The minus sign indicates a short position.

2.19.1 Proof

The arguments are similar to a proof given in Hull (2004), which is based on a strategy proposed by Cox, Ingersoll, and Ross (1981). However, Hull derives the results in continuous time, whereas we have used discrete compounding.

Consider a futures contract that lasts for n days. Let F_i be the futures price at the end of day i, where $0 \leq i \leq n$ and let δ be the constant rate of interest on a per day basis. Consider the following sequence of steps:

- Take a long futures position of $(1 + \delta)$ contracts at the end of day 0, that is, at the beginning of the contract.
- Increase the long position to $(1 + \delta)^2$ contracts at the end of day 1.
- Increase the long position to $(1 + \delta)^3$ contracts at the end of day 2.
- In general, increase your long position to $(1 + \delta)^i$ contracts, at the end of day $i - 1$.

When the contracts are marked to market at the end of day i, the profit or loss is

$$(F_i - F_{i-1})(1 + \delta)^i$$

This can be invested/borrowed until the end of day n, at a daily rate of δ. Thus, the future value of this cash flow, as calculated at the end of day n will be

$$(F_i - F_{i-1})(1 + \delta)^i(1 + \delta)^{n-i} = (F_i - F_{i-1})(1 + \delta)^n \tag{2.27}$$

On the final day, the cumulative value of all such gains and losses will be,

$$\begin{aligned}
&\sum_{i=1}^{n}(F_i - F_{i-1})(1 + \delta)^n \\
&= [(F_n - F_{n-1}) + (F_{n-1} - F_{n-2}) + \cdots (F_1 - F_0)](1 + \delta)^n \\
&= (F_n - F_0)(1 + \delta)^n
\end{aligned} \tag{2.28}$$

On the delivery date, F_n will be equal to the terminal spot price S_n. Therefore, the cumulative future value of daily cash flows, may be written as

$$(S_n - F_0)(1 + \delta)^n$$

Consider the case of an investor who besides indulging in the above strategy, also invests F_0 in a riskless bond on day 0. The terminal value of his overall portfolio will be

$$F_0(1 + \delta)^n + (S_n - F_0)(1 + \delta)^n = S_n(1 + \delta)^n \tag{2.29}$$

The initial value, as on day 0, of the portfolio will be F_0, since it should cost nothing to take a position in the futures market.

Now consider a person who goes long in $(1 + \delta)^n$ forward contracts on day 0. Let the delivery price be K_0. If this person also simultaneously invests K_0 in a riskless bond, then the initial value of his portfolio will be K_0. The terminal value will be

$$K_0(1+\delta)^n + (S_n - K_0)(1+\delta)^n = S_n(1+\delta)^n \tag{2.30}$$

Thus the portfolios of the two investors will have identical terminal values. Hence, to preclude arbitrage, they must have identical initial values. Therefore, it must be that

$$F_0 = K_0$$

That is, the forward and futures prices must be identical. Consequently, the pricing relationships that we have derived for forward contracts are equally valid for futures contracts.

2.20 Stochastic Interest Rates

In real life, interest rates are stochastic and hence forward prices need not equal futures prices. Let us consider a situation where interest rates and futures prices are positively correlated. If the futures price rises, the interest rate will also be high. When the price rises, the long will gain and in this case, will be able to invest his gains at a higher rate of interest. Correspondingly, in such a scenario, the short will be financing his losses[13] at a higher rate. Conversely, if the futures price falls, the interest rate will be low. This is a situation where the long will lose and the short will gain. Thus the long will be financing his losses at a lower rate, while the short will be investing his gains at a lower rate.

An investor with a long forward position, will however not be affected by interest rate movements in the interim, for his position will not be marked to market. Hence, an investor with a long futures position is clearly better off under such circumstances. Extending the argument along similar lines, a person with a short futures position is worse off than a person who has gone short in a forward contract. Since a long futures position is more attractive than a long forward position, the person going long in the futures contract should pay more for this advantage. In other words, the investor going short in the futures contract should receive more for this disadvantage. Hence, the futures price will exceed the forward price, when futures prices and interest rates are positively correlated.

Similar logical reasoning can be used to demonstrate that if interest rates and futures prices are negatively correlated, then the futures price will be less than the forward price.[14] These results were rigourously derived by Cox, Ingersoll and Ross (1981).

The question naturally arises as to which are the markets in which we can observe the effects predicted above. Arditti (1996) gives two examples, namely gold and T-bonds. Let us assume that interest rates rise because of higher expected inflation. We are all aware that gold is widely perceived as a hedge against

[13]It may either be in the form of an actual borrowing or as an opportunity cost.

[14]Readers should reason this out and convince themselves.

inflation. So gold prices will be expected to rise if inflation is expected to rise. Hence gold prices and interest rates should be positively correlated.

Interest rates on the other hand are negatively related to bond prices. So T-bond futures prices should be negatively correlated with interest rates. Consequently we would expect a gold futures contract to be worth more than a gold forward contract and a T-bond futures contract to be worth less than a T-bond forward contract.

However, just because we observe differences between futures and forward prices for a particular asset, we cannot conclude that it is due to a relationship between futures prices and interest rates. There could be other important factors at work. For instance, transactions costs could be different in the two markets. Also, futures contracts have a lower risk of default than forward contracts, due to the role of the clearinghouse and the marking to market mechanism. In addition, forward markets are usually less liquid than futures markets.

So in order to test the interest rate correlation hypothesis, we must look at a market where the other factors are insignificant. It has been argued that currency markets are fairly appropriate for this purpose.

2.21 Quasi-Arbitrage

The kind of arbitrage that we have discussed thus far, in connection with the cash and carry and reverse cash and carry strategies, is called *Pure Arbitrage*. Arbitrageurs, who engage in pure arbitrage, are forever on the lookout for mispriced securities and will exploit the profit opportunities till equilibrium is restored.

Quasi-arbitrage[15] is engaged in by investors who are not arbitrageurs in the conventional sense. The term refers to the use of cash and carry and reverse cash and carry techniques by investors, as alternative means of establishing a desired position in the market. For, as we have discussed in the section on synthetic securities, if we have two out of three securities, we can artificially take a position in the third.

We will illustrate this concept with the help of an example.

2.21.1 Example

Ralph Keating, is planning to make an investment in a riskless security for six months. One course of action that is available is to directly invest in a riskless asset such as a T-bill. The ask price for a six month T-bill with a face value of \$ 100,000, is \$ 98,875. The brokerage fee for an investment in bills with a face value of \$ 100 is 12.5 cents. So for a bill with a face value of \$ 100,000 Ralph will have to pay a commission of \$ 125. The total cost of acquisition is therefore

[15] For an elaborate discussion of quasi-arbitrage see Siegel and Siegel (1990).

\$ 99,000 and the corresponding rate of return is

$$\frac{100{,}000 - 99{,}000}{99{,}000} \equiv 1.0101\%$$

Now let us consider an arbitrage strategy. Assume that IBM shares are available at a price of \$ 90.25 per share. Futures contracts on the stock, maturing after six months are available for \$ 91.50. A pure arbitrageur can borrow at the rate of 1.25% for six months. The brokerage fees payable in the market are as follows. For every share that is bought or sold in the stock market, a commission of 12.50 cents is payable. For a transaction in futures contracts, a fee of 5 cents is payable per share. Finally, we will assume that while borrowing funds, the pure arbitrageur has to pay 12.50 cents for every share. Thus, the rate of return for a pure arbitrageur if he engages in a cash and carry strategy is

$$\frac{F_t - (S_{a,t} + C)}{(S_{a_t} + C)} = \frac{91.50 - (90.25 + 0.125 + 0.05 + 0.125)}{(90.25 + 0.125 + 0.05 + 0.125)} \equiv 1.0491\%$$

which is less than the borrowing rate of 1.25% . So clearly a pure arbitrageur cannot profitably engage in cash and carry arbitrage.

But in Ralph's case, instead of investing in T-bills, he can go long in the spot market and take a short position in the futures market, thereby creating an investment in a synthetic T-bill. His transaction costs are 12.5 cents for every share that he buys and five cents for every futures contract that he goes short in.[16] Consequently, his rate of return is

$$\frac{91.50 - (90.25 + 0.175)}{(90.25 + 0.175)} \equiv 1.1888\%$$

Thus, his rate of return if he follows this strategy, will be higher than what he would get if he were to invest in T-bills. Hence, a person like Ralph who is looking for a riskfree investment, would rather engage in a cash and carry strategy to buy synthetic T-bills. This is what we mean by quasi-arbitrage.

Notice certain key features inherent in the above argument. First, although pure arbitrage is not feasible, quasi-arbitrage is profitable. Second, a pure arbitrageur will compare the implied repo rate with the borrowing rate, whereas a person like Ralph, who is contemplating a quasi-arbitrage strategy, will compare the return from this strategy with his alternative lending rate.

Finally, there is one major issue that you must understand. A pure arbitrageur will exploit a perceived arbitrage opportunity till it vanishes.[17] A quasi-arbitrageur, is however, constrained by the amount of funds at his or her disposal. In the above example, if we assume that Ralph has \$ 1,000,000 with him, then his ability to buy synthetic T-bills will be restricted to this amount. If he wants to invest further in the cash and carry strategy, then he would have to borrow money, which would be tantamount to pure arbitrage and therefore

[16]Notice, that unlike the pure arbitrageur, he does not have to borrow.

[17]Remember that we have assumed that he can borrow or lend an unlimited amount of money.

will not be profitable. A consequence of this is that quasi-arbitrage opportunities are likely to persist longer than pure arbitrage opportunities, since each potential quasi-arbitrageur, faces his or her own funds constraint and hence, may not be able to exploit the opportunities for profit till they are completely eliminated.

2.22 Risk and Futures Prices

We shall now address the issue of the relationship between the current futures price and today's expectation of the future spot price? Let F_t be the current futures price. Let S_T be the spot price that is expected to prevail on day T, which is the expiration date of the contract. Notice that S_T is a random variable, and cannot therefore be observed today. Consequently, we can only have an expectation of it.

Telser (1958) and Gray (1961), argued that $F_t = E(S_T)$ and that therefore, $F_t = E(F_T)$, since at expiration, $F_T = S_T$. This is called the *Unbiased Expectations Hypothesis* or the *No Risk Premium Hypothesis*. It says that today's futures price is nothing but the market's expectation of the future spot price.

Keynes and Hicks however, argued that most hedgers are producers and hence will hedge by going short in futures. The reason why producers will go in for a short position, is because they are long in the spot market and consequently will be apprehensive that prices will fall. If the prices fall, such people will gain in the futures market.[18] This gain will help offset the loss made by them in the spot market.

If the hedgers take a short position in the futures market, they will require others to take a long position. Such people will obviously be speculators. In order to induce speculators, who by definition are taking risk, to go long in futures, they must be compensated by a risk premium or in other words, their expected gains must be positive. The expected gains will be positive for the longs only if the futures price is expected to rise. Therefore, according to the argument advanced by Keynes and Hicks, F_t should be less than $E(F_T)$, which means that F_t should be less than $E(S_T)$. This is called the theory of *Normal Backwardation*. According to this theory, hedgers buy insurance from speculators by transferring their unwanted risk. For this reason, speculators are entitled to an expected profit.

On the other hand, consider the case where the majority of hedgers take a long position in the futures market.[19] In such a situation, the speculators would have to take a short position, and must therefore be suitably compensated with a risk premium. Such speculators will get an expected gain, only if the futures prices are expected to fall. Thus F_t should be greater than $E(F_T)$, which means that F_t should be greater than $E(S_T)$. Such a situation is called *Normal Contango*.

[18]Remember, if the price declines, investors with a short position will gain.

[19]This would be the case if they were short in the spot market.

2.22.1 The Systematic Risk Theory

We will now focus on another explanation that has been advanced to explain the relationship between the futures price and the expected future spot price. According to this viewpoint, holders of futures contracts bear systematic risk and therefore must be suitably compensated for it.[20] The expected return demanded by the holder of the futures contract should therefore, be reflected by a difference between the futures price and the expected future spot price.

We have seen from the spot-futures equivalence relationship that:

Spot - T-bill = Synthetic Futures

We know that T-bills are riskless. Hence, the risk of a futures position is equal to the risk of the underlying asset. Thus, from the above relationship, we can deduce that the expected return from a long position in a futures position is equal to the expected return from a long position in the underlying asset less the riskless rate of return.

Why is it that a futures contract has the same inherent risk as the underlying asset, but offers a lower expected rate of return to an investor who takes a long position? The reason is the following. In order to take a long position in the underlying asset, the investor has to make an investment that is equal in magnitude to its spot price. Such an investment, which is risky in nature, must obviously earn the riskless rate of return plus a risk premium. Futures contracts, on the contrary do not require an investment on the part of an investor who is taking a long position. Thus, the expected return on a long futures position, should be equal to the risk premium on the underlying asset.

As per the Capital Asset Pricing Model (CAPM), the expected rate of return on asset i, which we will denote by $E(r_i)$, is given by:

$$E(r_i) = r_f + \beta_i[E(r_M) - r_f] \tag{2.31}$$

where r_f is the rate of return on a riskless asset, and r_M is the rate of return on the market portfolio. β_i is a measure of the systematic risk of asset i.

Hence a long futures position should yield an expected return of:

$$\beta_i[E(r_M) - r_f] \tag{2.32}$$

It is obvious that if the underlying asset were to have a positive beta, then a long futures position will have a positive expected return. On the other hand, in the case of short futures positions, if the beta of the asset were to be positive, then the expected return will be negative.

From the systematic risk theory, we can also conclude that the *'Unbiased Expectations Hypothesis'*, will be valid only if the underlying asset were to have a zero systematic risk.

[20]Those of you who are familiar with modern portfolio theory and the Capital Asset Pricing Model, will be aware of the concept of *Systematic or Non-diversifiable or Market Risk* and the logic as to why this is the only risk that is priced.

2.23 Risks Inherent in Arbitrage

Thus far, we have presented cash and carry arbitrage and reverse cash and carry arbitrage strategies, as positions designed to earn cost-less riskless profits. In practice, however, an arbitrageur who seeks to implement such strategies will have to confront a variety of risks.

Cash and carry arbitrage requires the investor to borrow at the riskless rate of interest in order to acquire the underlying asset. In practice, such investors may be unable to borrow at an interest rate that remains constant for the duration for which the position is in place, and hence they may have to borrow for shorter periods and roll over the loan. Therefore, the risk faced by such arbitrageurs is that interest rates may rise after they have implemented the strategy. Such risk is referred to as *Financing Risk*.

Second, in the case of assets which are scheduled to make payouts during the life of the contract, the potential to make arbitrage profits is critically linked to the arbitrageur's ability to forecast the payouts. If the asset does not make payouts as forecasted, then the profits that are expected at the outset may not materialize. Such risk is referred to as *Payout Risk* or *Dividend Risk*.

It should be obvious to the reader that such risks are factors in the case of reverse cash and carry arbitrage strategies as well.

2.23.1 Risk Arbitrage

Financing risk and payout risk are inherent in arbitrage strategies. However, there is a third element of risk as well, which is confronted by arbitrageurs who take long positions in contracts which specify multiple grades of the underlying asset as suitable for delivery. As we have seen, in the case of such contracts, the option as to which grade to deliver is always given to the short. The short will obviously opt to deliver the grade that is the cheapest at the time of delivery. The issue is that this grade need not correspond to the grade which has been sold short by an arbitrageur at an earlier point in time, as part of a reverse cash and carry arbitrage operation. If the grade that is delivered by the short is different from what the long requires, then the arbitrageur will have to acquire the grade which was originally short sold by him. The grade received from the short as a part of the futures contract, will obviously have to be disposed off in the spot market. The net result is that the ex-post implied reverse repo rate for the arbitrageur could be higher than the ex-ante implied reverse repo rate, and may at times be even higher than the lending rate. If so, the realized profit may be less than anticipated, or in a worse situation the arbitrageur may end up with a net loss. Hence, reverse cash and carry arbitrage under such circumstances is fraught with danger, and is more appropriately termed as *Risk Arbitrage*.

Before we go on to analyze the additional risk in the case of a reverse cash and carry transaction, let us first consider a cash and carry transaction. Thus far we have assumed the arbitrageur will take a spot-futures position in the ratio of

1:1. That is, if the futures contract is for 100 units of the underlying asset, he will short sell 100 units for every contract in which he goes long. However, while this is appropriate in the case of contracts which specify only one deliverable grade, the situation is different for cases where multiple deliverable grades have been specified, and a multiplicative price adjustment method has been specified. Assume that the arbitrageur goes long in one unit of grade i of the asset. We will assume that multiple grades have been specified for delivery, and that a multiplicative price adjustment system is in operation. Let us denote the required short position as h futures contracts. When he delivers the asset at expiration, he will receive $a_i F_T$. The profit from marking to market will be $h(F_t - F_T)$. For the strategy to be riskless, we require that

$$a_i F_T + h(F_t - F_T) = a_i F_t$$

$$\Rightarrow h = a_i \tag{2.33}$$

Thus the appropriate number of futures contracts is equal to the price adjustment factor of the grade in which the arbitrageur has taken a long position. It should be noted that if an additive system of price adjustment were to be used, then the arbitrageur would require a spot futures position of 1:1. This can be demonstrated as follows:

$$F_T + a_i + h(F_t - F_T) = F_t + a_i \Rightarrow h = 1$$

The very fact that a cash and carry arbitrage strategy was initiated, signifies that the ex-ante Implied Repo Rate is greater than the borrowing rate. That is:

$$\frac{a_i F_t - S_{i,t}}{S_{i,t}} > r \tag{2.34}$$

At the point of expiration of the futures contract, there are two possibilities. It may be the case that grade i is the cheapest to deliver grade. If so, the arbitrageur will deliver it, and his realized implied repo rate will be equal to what he has been anticipating from the very outset. However, what if, some other grade, j, has become the cheapest to deliver? If this were to be the case, the arbitrageur can sell the unit of grade i in his possession at its prevailing spot price, and acquire a_i units of grade j to deliver under the futures contract.[21]

The cash inflow will be

$$S_{i,T} + a_i \times (a_j F_T) + a_i(F_t - F_T)$$

This represents the inflow from three sources, namely, the cash flow from the sale of grade i in the spot market, the proceeds from the delivery of grade j under the futures contract, and the cumulative profit from the marking to market mechanism. The cash outflow, on account of the acquisition of grade j will be

$$a_i S_{j,T}$$

[21] We have used the symbol $S_{i,t}$ to denote the spot price of the i th asset at time t, and $S_{i,T}$ to denote its spot price at time T. Similarly, F_t and F_T denote the futures prices at times t and T respectively.

The net inflow will therefore be:

$$\begin{aligned} &S_{i,T} + a_i \times (a_j F_T) + a_i(F_t - F_T) - a_i S_{j,T} \\ &= S_{i,T} + a_i S_{j,T} + a_i(F_t - F_T) - a_i S_{j,T} \\ &= a_i F_t + a_i \left[\frac{S_{i,T}}{a_i} - \frac{S_{j,T}}{a_j} \right] > a_i F_t \end{aligned} \tag{2.35}$$

because if grade j is the cheapest to deliver by assumption, then its delivery adjusted spot price $\frac{S_{j,T}}{a_j}$ will be lower than that of grade i, and will be equal to the futures price, F_T. Thus we conclude that if a cash and carry arbitrage strategy is initiated because it looks profitable at the outset, it can only lead to a greater profit, if not the anticipated profit, at the time of expiration. In other words, the ex-post Implied Repo Rate will be greater than or equal to the ex-ante Implied Repo Rate, which by assumption was greater than the borrowing rate. Hence, there is no risk as such.

Now let us turn our attention to a reverse cash and carry strategy. The very fact that such a strategy is being initiated, implies that:

$$\frac{a_i F_t - S_{i,t}}{S_{i,t}} < r \tag{2.36}$$

If the short were to deliver grade i, the grade which has been short sold by the arbitrageur, at the end, then the anticipated arbitrage profit will be realized. However, what if the arbitrageur is forced to take delivery of another grade, j, because the short finds that it is the cheapest to deliver. If so, the arbitrageur would have to sell this grade in the spot market and acquire grade i at its prevailing spot price to cover his short position.

The outflow will be:

$$a_i(a_j F_T) + S_{i,T}$$

This represents the outflow on account of taking delivery under the futures contract, and the cost on account of the covering of the initial short position. The inflow will be:

$$a_i S_{j,T} + a_i(F_T - F_t)$$

This represents the cash flow on account of the sale of grade j, as well as the cumulative cash flow due to marking to market.

The net outflow is

$$\begin{aligned} &a_i(a_j F_T) + S_{i,T} - a_i S_{j,T} - a_i(F_T - F_t) \\ &= a_i S_{j,T} + S_{i,T} - a_i S_{j,T} - a_i(F_T - F_t) \\ &= a_i F_t + a_i \left[\frac{S_{i,T}}{a_i} - \frac{S_{j,T}}{a_j} \right] > a_i F_t \end{aligned} \tag{2.37}$$

Consequently, the ex-post Implied Reverse Repo Rate could be greater than the ex-ante Implied Reverse Repo Rate, and perhaps even greater than the lending rate. Therefore, to reiterate, what looked like a profitable arbitrage opportunity may end up as a reduced arbitrage profit or perhaps even as a loss. Thus arbitrage, under such circumstances, is accompanied by an additional element of risk.

The same logic holds for contracts on assets where multiple grades are permissible for delivery, but an additive system of price adjustment is used. We will consider the case of cash and carry arbitrage. The extension to reverse cash and carry arbitrage is straightforward.

We know that if an additive price adjustment mechanism is in place, then an arbitrageur will use a spot futures ratio of 1:1. Once again the very fact that a cash and carry strategy has been implemented signifies that the Implied Repo Rate is greater than the borrowing rate. That is

$$\frac{F_t + a_i - S_{i,t}}{S_{i,t}} > r$$

If at expiration, a different grade, grade j, has become the cheapest to deliver, then the arbitrageur will sell the unit of grade i in his possession and will acquire a unit of grade j to satisfy his delivery obligations. The inflow will be

$$S_{i,T} + F_T + a_j + (F_t - F_T)$$

The outflow will be $S_{j,T}$. The net inflow will therefore be

$$\begin{aligned} & S_{i,T} + F_T + a_j + (F_t - F_T) - S_{j,T} \\ & = S_{i,T} - a_i + F_t + a_i - (S_{j,T} - a_j) \\ & = F_t + a_i + [(S_{i,T} - a_i) - (S_{j,T} - a_j)] \\ & > F_t + a_i \end{aligned} \tag{2.38}$$

This is because since grade j by assumption is the cheapest to deliver grade, $S_{j,T} - a_j < S_{i,T} - a_i \forall i$. Hence if a cash and carry strategy were to be initiated, the ex-post Implied Repo Rate will be greater than or equal to the ex-ante Implied Repo Rate, which as per our assumption was greater than the borrowing rate.

2.23.2 Multiple Deliverable Grades and the No-arbitrage Futures Price Prior to Expiration

In the case of multiple deliverable grades, cash and carry arbitrage can be initiated with any of the allowable grades. Thus each grade will have its own Implied Repo Rate. Arbitrage activities will continue till there is no profit to be made from any of the deliverable grades. The cheapest to deliver grade prior to expiration will be the one that maximizes the IRR, which, in an arbitrage free setting will just equal the borrowing rate. If we call this grade, grade i then

$$\frac{F_t + a_i - S_{i,t}}{S_{i,t}} = r$$

$$\Rightarrow F_t = S_{i,t}(1 + r) - a_i \tag{2.39}$$

We will define $S_{i,t}(1+r)$ as the no-arbitrage futures price. Thus the futures price prior to expiration will be equal to the delivery-adjusted no-arbitrage futures price of the cheapest to deliver grade.

In the case of a multiplicative system of adjustment, the equivalent condition would be

$$\frac{a_i F_t - S_{i,t}}{S_{i,t}} = r$$

$$F_t = \frac{S_{i,t}(1+r)}{a_i} \tag{2.40}$$

Suggestions for Further Reading

1. Hull, J. *Fundamentals of Futures and Options Markets*. Prentice-Hall, 2004.
2. Kolb, R.W. *Futures, Options, and Swaps*. Blackwell, 2003.
3. Siegel, D.R. and D.F. Siegel *Futures Markets*. The Dryden Press, 1990.

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1. Arditti, F.D. *Derivatives*. Harvard Business School Press (1996).
2. Cox, J.C., J.E. Ingersoll, and S.A. Ross "The Relation Between Forward Prices and Futures Prices." *Journal of Financial Economics*, Vol. 9 (December 1981).
3. Gray, R.W. "The Search for a Risk Premium." *Journal of Political Economy*, Vol. 69 (June 1961).
4. Telser, L.G. Futures "Trading and the Storage of Cotton and Wheat." *Journal of Political Economy*, Vol. 66 (June 1958).

Concept Check

State whether the following statements are True or False.

1. In order to preclude arbitrage, the Implied Repo Rate should be less than the borrowing rate.
2. The value of a short position in a forward contract, is the present value of the difference between the delivery price and the forward price.
3. If the marginal convenience yield is positive, we say that the market is at Full Carry.
4. If the futures price exceeds the spot price, then the market is said to be in Backwardation.

5. A long position in the spot plus a long forward contract is equivalent to a synthetic T-bill.
6. If the riskless rate of interest is a constant and is the same for all maturities, then the forward price for a contract on an asset with a given delivery date, will be the same as the futures price for a contract on the same asset, for the same delivery date.
7. If futures prices and interest rates are positively correlated, then the forward price will exceed the futures price.
8. Quasi-arbitrage opportunities usually persist for a longer time than pure arbitrage opportunities.
9. If net hedgers are short hedgers, then futures prices should exceed expected future spot prices.
10. According to the systematic risk theory, futures prices are unbiased expectations of future spot prices, if the systematic risk of the asset is zero.
11. In the case of futures contracts that give a choice of grades that can be delivered, the profit from cash and carry arbitrage will be greater than or equal to the profit anticipated at the outset.
12. Interest rate risk and dividend risk have implications for both cash and carry as well as reverse cash and carry arbitrage.
13. In the absence of arbitrage opportunities, futures contracts on financial assets must always be in contango.
14. In the absence of arbitrage opportunities, futures contracts on physical assets that are held for investment purposes must always be in contango.
15. Restrictions on short sales have implications for both cash and carry as well as reverse cash and carry arbitrage.
16. In a backwardation market, contracts should be priced under the assumption that delivery will take place at the end of the specified period.
17. For a financial asset, the net carry may be positive or negative.
18. The marginal convenience value is the highest of the convenience values as perceived by market participants.
19. In the case of arbitrage using futures contracts that specify an additive system of price adjustment for premium and discount grades, the arbitrageur will use a spot-futures ratio of 1:1.
20. The maximum loss from a short sale is infinite in principle.

QUESTIONS AND PROBLEMS

Question-I

'At inception, the value of a long forward contract is zero, but subsequently it can have a positive or a negative value.' Explain.

Question-II

What is the difference between Pure Assets and Convenience Assets?

Question-III

If interest rates and futures prices are positively correlated, what will be the relationship between the forward price and the futures price for contracts on a given asset? Explain in detail.

What if interest rates and futures prices are negatively correlated?

Question-IV

What is the difference between arbitrage and quasi-arbitrage?

'Arbitrage opportunities usually disappear faster than quasi-arbitrage opportunities.' Explain.

Question-V

If net hedgers are short hedgers, what will be the relationship between the current futures price and today's expectation of the future spot price?

What if net hedgers are long hedgers?

Question-VI

What are synthetic securities? Explain how to create the following assets synthetically (illustrate the corresponding cash flows using symbols).

1. A Long position in a T-bill
2. A Long Forward Contract
3. A Long Spot Position
4. A Short position in a T-bill
5. A Short Forward Contract
6. A Short Spot Position

Question-VII

Forward contracts on palladium are available in New York. Each contract is for 100 ounces of the metal, deliverable after six months. The current spot price is \$ 300 per ounce and storage costs are \$ 5 per ounce per month, payable at the end of the contract. The riskless rate of interest is 6% per annum.

1. What should be the no-arbitrage forward price?
2. If the forward price is \$ 350 per ounce, how can one exploit the arbitrage opportunity?
3. If the forward price is \$ 330 per ounce, how can one exploit the arbitrage opportunity?

Question-VIII

Reliance has issued bonds with a face value of \$ 1,000 and five years to maturity. The bonds pay a coupon of 8% per annum on a semi-annual basis. The current YTM is 10% per annum.

Consider a forward contract that matures after one year, just after the bond makes a coupon payment. If the borrowing/lending rate is 10% per annum, what should be the no-arbitrage forward price? Suppose there exists a forward contract with a delivery price that is \$ 10 less than the forward price. What should be the value of such a contract?

Question-IX

A three month forward contract on IBM stock has a delivery price of \$ 105, whereas a 9 month forward contract on the same stock has a delivery price of \$ 110. The borrowing/lending rate for a six month loan to be made three months from today is 5.5%.

Is there a potential for making arbitrage profits? If so, explain your strategy in detail.

Question-X

Consider the following market information on July 1, 2008.

- Bid Price of McDonalds \$ 34.95
- Ask Price of McDonalds \$ 35.10
- Bid Price for 1 year McDonalds futures \$ 37.80
- Ask Price for 1 year McDonalds futures \$ 37.85
- The bid price for one year T-bills with a face value of \$ 1,000,000 is \$ 919,800 whereas the ask price is \$ 920,500.

The arbitrageur's normal borrowing rate is 9% and when he sells short, he must keep the proceeds with the broker who will pay him 85% of the borrowing rate as interest.

The transactions fees payable by investors are as follows.

- When a share is bought or sold short, .015% of the price of the share is payable as commission.
- When a long or a short position is taken in the futures market, .005% of the futures price is payable as commission.
- Whenever money is borrowed or lent, .0075% of the amount is payable as commission.
- Whenever a T-bill is bought or sold, .0075% of the face value of the bills is payable as commission.

Note: Ignore commissions on money borrowed to pay commissions.

1. Is cash and carry arbitrage feasible? Justify with numerical calculations.
2. Is reverse cash and carry arbitrage feasible? Justify.
3. Consider the case of Denise Ravenscroft, a portfolio manager with Morgan Stanley. She is currently holding 100,000 shares of McDonalds stock. Is she better off holding on to the shares or should she move to a synthetic stock position?

3

Hedging and Speculation

3.1 Introduction

Hedgers, by definition, are traders who seek to protect themselves from unfavorable movements in the price of an asset in which they have a commercial interest. People who seek to hedge would have already assumed a position in the underlying asset, before they attempt to implement a hedging strategy. One possibility is that the underlying asset may already be in their possession, in which case they would be said to have a long position in the spot market. In such cases, their uneasiness about unfavorable price movements, would stem from the fact that the price of the asset may decline subsequently. The other possibility is that they may have made a commitment to buy the underlying asset. Such investors would be said to have a established a short position in the spot market, and their desire to hedge may be attributed to their concern that the price of the asset may rise subsequently. In both situations, a desire to hedge demonstrates a desire on the part of the trader to avoid risk. Thus hedgers are traders who are uncomfortable facing a risk exposure and would like to minimize if not totally eliminate uncertainty.

A short position in an asset, is by definition a commitment to acquire an asset at a future point in time, and need not always be tantamount to a short sale. As we have seen earlier, a short sale entails the borrowing of an asset by the trader in order to sell it. Quite obviously the lender of the asset would expect that the asset will be bought back and returned at a future point in time. However, the definition of a short position is broader in scope. Consider the case of an Indian company which has imported goods from Germany and is required to pay the invoice amount in euros at the end of a stated credit period. The company being committed to acquire the euros at a future point in time is exposed to the risk of an appreciating euro. That is, it faces the risk that the rupee price of the euro may increase by the time the foreign currency is acquired. Therefore, like the short seller, a company in such a predicament would also be worried by the specter of a rise in the price of the underlying asset. Thus in a more general manner of speaking, a short position in an asset connotes a commitment to buy it at a future point in time, at a price that will not be revealed until the time of acquisition.

A hedging transaction helps reduce or under ideal circumstances, eliminate the price risk that an investor faces from either being long in the spot market (carrying unsold inventory) or from being short in the spot market (making forward sales without already procuring the commodity). The term *Price Risk* refers to the risk that prices will move in an adverse direction.

Thus a hedger will simultaneously take positions in the spot as well as the futures markets. If he is long in the spot market, then he will go short in the futures market and vice versa. The main concern for a hedger is the direction of change in the price difference between the spot and futures markets. The issue is, will one's profit / loss in the spot market, be more or less than the loss / profit in the futures market. As we shall see shortly, except in the rare case of a perfect hedge, risk cannot be completely eliminated. What a hedger does is to substitute the large and unpredictable price risk due to an open long or short position in the spot market, with a more acceptable risk of variation in the price difference between the spot price and the corresponding futures price. This price difference, called the *Basis*, is the key variable of interest in hedging and we will discuss it in detail shortly.

There are two types of hedges that are possible. One is a *Short Hedge* or a *Selling Hedge* while the other is a *Long Hedge* or a *Buying Hedge*. We will discuss each one of them with the help of suitable examples.

3.2 A Selling Hedge

A short hedge requires the investor to take a short position in the futures market. A person may go in for a short hedge if he already has the asset in his possession and expects to sell it at some time in the future. Such a hedge is called an *Inventory Hedge*.

It works as follows. If the price in the spot market were to decline, he can still sell at the original futures price, since the other party is under an obligation to buy at this price. We will illustrate this with the help of an example.

3.2.1 Illustration

Norman Davis, a corn farmer in Iowa knows that he will have 2,500,000 bushels to sell after a month. His worry is that there may be a glut in the market, as a consequence of which the spot price at the time of sale may be substantially lower than what he is anticipating right now. We know that futures contracts on corn are available at the CME Group, and that each contract is for 5,000 bushels. If the current futures price is \$ 5.25 per bushel, then by going short in 500 contracts, Norman can lock in a price of \$ 13,125,000 for the sale that he expects to make after a month. This amount is guaranteed to be received, and is independent of the actual price of corn that is likely to prevail in the spot market one month hence. Notice that while Norman is protected against a falling market, he will be unable to take advantage of the market conditions if the price of corn were to increase, due to perhaps the fact that the overall harvest was less plentiful than anticipated.

This, as we will discuss in more detail shortly, is a feature of hedges implemented using futures contracts.

Short hedges can also be used by people who do not own the asset at the time of initiating the hedge but know that it will be in their possession at some time in the future. Since it is taken out in anticipation of a future event, this type of a hedge is also known as an *Anticipatory Hedge*.

3.2.2 Illustration of an Anticipatory Hedge

Take the case of a company which has exported goods from India to France and knows that it will be paid in euros after 60 days. Its worry would be that the rupee may appreciate in the interim, or in other words that the rupee price of the euro may decline. Such a party may also hedge its risk by entering into a short position in a futures or a forward contract expiring after 60 days. This kind of a selling hedge is in anticipation of a future event, for the euros are not currently in stock but are expected to arrive in 60 days, and consequently is termed as an 'Anticipatory Hedge'.

3.3 A Buying Hedge

A long hedge requires the hedger to take a long position in the futures market. A person who needs to acquire an asset in the future and is concerned that the price will rise, will go in for such a hedge. This is another type of an anticipatory hedge. If the spot price were to rise subsequently, he can still buy at the original futures price since the other party is under an obligation to sell at this price.

3.3.1 Example

Kroger and Company has imported fruit juice worth 500,000 AUD from Golden Circle in Sydney, and is required to pay the counter-party after two months. Its worry would be that the US dollar may depreciate by then, or in other words, that the price per Australian dollar may go up. Currency futures contracts on the Australian dollar are available on the CME Group with a contract size of 100,000 AUD. The ask price for a contract, which is the relevant price for a long hedger is currently 0.6500.

If Kroger were to take a long position in five futures contracts, they can lock in a payable of $ 325,000.

3.4 Options and Hedging

Futures contracts are not the only derivative instruments that are suitable for hedging. Both call and put options can also be used for hedging. Call options give the holder of the option, the right to acquire the underlying asset at the exercise price. Thus, an investor with a short position in the underlying asset can protect himself against an upward movement in the spot price by buying a call. If the spot price of the underlying asset at the time of its acquisition were to exceed the exercise price, the investor can exercise the call and acquire the asset at the exercise price. However, while the option protects such a hedger against an upward price movement, it does not preclude him from taking advantage of a price decline. For, if the spot price at the time of acquisition of the asset is lower than the exercise price, the investor can simply forget the option and acquire the asset in the spot market. This is feasible for the option confers a right and not an obligation. We will illustrate this with the help of an example.

3.4.1 Numerical Illustration

Assume that Kroger, which has imported fruit juice from Australia, and consequently has a payable denominated in AUD, has bought 50 call options contracts on the Australian Dollar with an exercise price of $ 0.6500, where each options contract is for 10,000 AUD.

If the spot rate of exchange after two months were to be greater than $ 0.6500 per AUD, the company can exercise the contracts and acquire 500,000 AUD dollar by paying $ 325,000. However, unlike futures contracts, which if used for hedging, entail a commitment to buy at this price, options contracts give a choice to the hedger. For, if the spot rate after two months were to be less than $ 0.6500 per AUD, say $ 0.6000 per AUD, then the required foreign currency can be bought in the spot market for a consideration of $ 300,000, which represents a saving of $ 25,000. To reiterate, this is feasible because the option confers a right and does not impose an obligation.

Put options too can be used for hedging. Consider an individual who has a long position in the underlying asset, and is consequently perturbed by the possibility of a price decline. A put option by definition gives the holder the right to sell the underlying asset at the exercise price. Consequently, if the spot price of the underlying asset were to fall and reach a level below the exercise price at the time of sale, the option can be exercised and the long position can be unwound at the exercise price. It must be remembered that a put too, like a call, confers a right on the holder. Consequently, if the spot price at the time of sale were to be higher than the exercise price, the hedger can simply allow the option to expire unexercised and dispose off the underlying asset in the spot market. Let us illustrate this with the help of a numerical example.

3.4.2 Numerical Illustration

Charles, a trader based in Chicago has bought 100 put options contracts on GE with an exercise price of $ 45. Each contract is for 100 shares. Assume that Charles has an investment horizon of one month.

If the spot price of GE after a month were to be less than $ 45 per share, then he can exercise the options and deliver the shares for a total consideration of $ 450,000. However, if the spot price at that point in time were to be more than $ 45 per share, say $ 50, then the options contracts need not be exercised.

If the options are scheduled to expire at that time, then they can be allowed to do so. Else Charles can offset them by taking a counter-position at the prevailing premium. In the second instance, the effective proceeds received by him per share will be greater than the assumed spot price of $ 50.

3.5 Futures or Options

Futures contracts and options contracts, as we have just seen, both facilitate hedging. However they work differently, and hence cannot be said to be substitutes for each other.

Let us take the case of an investor who has a long position in the underlying asset and is contemplating the initiation of a hedge using derivative securities. If he were to use a futures contract, he would need to take a short position. On the other hand, if he were to use options, he would have to acquire a put options contract. A futures hedge would lock in a selling price for the underlying asset. Since the contract imposes an obligation on the investor he has no choice but to sell at the delivery price of the contract. Consequently, while he is protected from losses if the price of the asset were to decline subsequently, he will not be in a position to benefit from a price increase, if the price were to surge forward in the spot market. A hedging strategy using put options would however work differently. If the spot price at the time of sale were to exceed the spot price, the asset can be sold in the spot market after allowing the options contract to expire. However, if the spot price were to have declined below the exercise price of the option, the option can be exercised and the asset can be sold at the spot price.

Hedging strategies implemented by traders with short positions in the underlying asset, can also be undertaken using futures or options. A futures hedge would entail the assumption of a long position in a futures contract. This will lock in a value for the purchase price of the asset. Thus, while the hedger would be protected from a subsequent price increase in the value of the asset, he would be precluded from taking advantage of a declining market, for he is under an obligation to buy at the original futures price. An options hedge in this case, would entail the acquisition of a call option, and would work differently. If the price of the underlying asset at the time of acquisition were to exceed the exercise price of the option, the call can be exercised and the asset can be acquired at the strike price. Thus the maximum price payable is the exercise price of the option. On the contrary, if the price were to fall subsequently, the option can be allowed to expire and the asset can be acquired in the spot market at a price which by assumption is lower than the strike price.

Thus, futures contracts lock in the price at which an investor with a long position can sell the underlying asset, or the price at which an investor with a short position can acquire the same. However, the hedger will be unable to benefit from favourable price movements in either case. An options contract however, imposes a floor on the price at which a hedger with a long spot position can sell, and a ceiling at which an investor with a short spot position can buy. Thus, if options were to be used for hedging, the hedger is very much in a position to take advantage of a favourable price movement.

Now you may wonder that if options give protection to the hedger on one side, while permitting him to take advantage of favorable price movements in the other direction, why would anyone want to use futures contracts for hedging? The rationale is as follows.

Futures contracts do not entail the payment of a price or a premium by an investor, irrespective of whether he is taking a long or a short position. Traders do have to post margins. But, a margin deposit, it must be remembered, is a performance guarantee and not a cost. Thus the initial cost of a hedging strategy using futures contracts is nil, except of course for transactions related expenses such as brokerage fees.

However, options contracts require the payment of an option price at the outset by the buyer. Unlike a margin deposit, an option premium is an actual cost or expense and is non-refundable.

Thus an investor who is contemplating the use of derivative securities as a part of a hedging strategy, has a choice. If he were to use a futures contract, he need not pay at the outset. However the contract is inflexible in the sense that it locks in a price at which he can buy or sell the underlying asset. But, if he were to use an options contract for hedging, a price would have to be paid. In this case, if the price of the underlying asset were to move in an unfavorable direction, the option can be exercised and a transaction can be effected at the exercise price. Else, if the price were to move in favor of the hedger, the option need not be exercised and a transaction can be undertaken at the prevailing spot price. It should be obvious from this line of reasoning that futures and options are not interchangeable instruments from the standpoint of hedging. The choice of the instrument would therefore depend on the personal preferences of the individual who is seeking to hedge.

We will now illustrate as to why a hedging strategy using futures contracts will not dominate a strategy using options and vice versa. The word *'dominance'* in this context means having superiority under every conceivable outcome.

3.5.1 Numerical Illustration

Let us consider the case of Kroger, which has a payable of 500,000 AUD.

The exchange rate for a two month futures contract is 0.6500 USD/AUD. Call options with two months to expiration and an exercise price of 0.6500 USD/AUD are available at a premium of 0.015 USD/AUD or at 1.5 cents.

Let us first consider a hedge using futures contracts. Since Kroger has a payable in AUD it requires a long position. If the company goes long in 5 futures contracts written on Australian dollars, it is guaranteed a US dollar outflow of:

$$500{,}000 \times 0.6500 = \$\ 325{,}000$$

irrespective of the spot exchange rate prevailing at expiration.

Now let us see the difference if call options contracts were to be used for hedging. If we assume that contracts with an exercise price equal to the delivery price of the futures contracts are used, the payable in US dollars will be

$$500{,}000 \times 0.6500 + 500{,}000 \times 0.015 = 325{,}000 + 7{,}500 = \$\,332{,}500$$

if the options are exercised, that is, the spot rate at expiration is greater than 0.6500 USD/AUD.

However, if the spot rate at expiration, S_T, were to be less than 0.6500 USD/AUD, then the firm will choose to acquire the Australian currency in the spot market, and will allow the call options to expire. In this case, it will have to pay

$$500{,}000S_T + 7{,}500$$

If the spot exchange rate at the point of expiration of the derivative contracts were to be 0.6350 USD/AUD, then the consequences for the firm will be the same irrespective of whether it had chosen futures or options for hedging. This point of indifference can be computed by equating the outflows in both cases. That is:

$$325{,}000 = 500{,}000S_T^* + 7{,}500 \Rightarrow S_T^* = 0.6350$$

If the terminal exchange rate were to be lower, then the options hedge would yield a superior outcome. However, for values of S_T greater than 0.6350 USD/AUD, the payable in terms of US dollars would be less if a futures contract were to have been used for hedging.

Hence it cannot be categorically stated that a futures hedge dominates an options hedge or vice versa. Market participants will have differing perceptions about what the spot rate is likely to be at the time of expiration of the derivative contracts. Hence while some parties may be inclined to use options contracts for hedging, others may well prefer to use futures contracts.

3.6 Ex-post Regret

An investor who uses futures or options for hedging, may end up regretting his decision afterwards. Take the case of Kroger. If they were to have used futures contracts for hedging, they would have locked in a payable of \$ 325,000 at the rate of \$ 0.6500 USD/AUD. If the spot exchange rate at the end turns out to be greater than \$ 0.6500 USD/AUD, or in other words, if the US dollar depreciates, then their decision to hedge would be vindicated, for if they had chosen not to hedge, they would have ended up paying more in terms of the home currency.

However, if the dollar were to appreciate, which would mean a spot rate of less than \$ 0.6500, then their decision to hedge would certainly be perceived as unwise. This is because they could have bought the Australian currency at a lower cost, had they decided to stay unhedged.

A priori, a party like Kroger would not know whether the price of what they are seeking to acquire, will increase or decrease. So if they were to be risk averse, that is, if they were to be uncomfortable with the prospect of a depreciating home currency in this case, then they may very well decide to hedge, despite the possibility that their decision may turn out to be unsound, when an ex-post analysis is undertaken.

Thus, traders will hedge if they are uncomfortable leaving their price exposure open, despite the specter of ex-post regret. As Parameswaran and Hegde(2007) point out, *"Hindsight as they say is a 'Perfect Science'. A normal individual cannot be expected to be prescient, and if he were, he certainly would not need derivatives in order to hedge".*

While futures hedges can lead to ex-post regret, so can hedges that are implemented using options contracts. Let us consider the case where Kroger decides to hedge using call options with an exercise price of 0.6500 USD/AUD. The payoff in terms of dollars would be

$$332{,}500 \text{ if } S_T \geq 0.6500$$

and

$$500{,}000 S_T + 7{,}500 \text{ if } S_T \leq 0.6500$$

The payoff for an unhedged position would be:

$$500{,}000 S_T$$

It can be demonstrated that for values of S_T less than 0.6650 USD/AUD, it would have been better if Kroger had chosen to stay unhedged. This cutoff value can be derived by equating the payoffs in the two cases.

$$332{,}500 = 500{,}000 S_T \Rightarrow S_T = 0.6650$$

Once again the party cannot exactly predict the future exchange rate, which in fact is the very reason why it is contemplating a hedge. Consequently, the potential for ex-post regret is very much a factor for a hedged position that has been implemented using options contracts.

3.7 Cash versus Delivery Settlement

The above discussions have assumed that the futures contracts are settled by delivery. The reader may wonder as to whether our conclusions would be different if the contracts were to be cash settled. We will now demonstrate that our inferences are valid irrespective of how the contracts are settled.

Before we proceed further let us define the following variables which will be used in the ensuing discussion. Additional variables will be defined when required.

3.7.1 Notation

- $T \equiv$ time of expiration of the futures contract used for hedging.
- $t \equiv$ time the hedge is initiated.
- $t^* \equiv$ time the hedge is closed out. t^* will be before T or equal to it.
- $S_t \equiv$ spot price at the time the hedge is initiated.
- $F_t \equiv$ futures price at the time the hedge is initiated.
- $S_T \equiv$ spot price at the time of expiration of the contract.
- $F_T \equiv$ futures price at the time of expiration of the contract.

- S_{t*} ≡ spot price at the time the hedge is lifted.
- F_{t*} ≡ futures price at the time the hedge is lifted.
- Q ≡ quantity of the asset that is being hedged.
- Q_f ≡ quantity of the asset, underlying the futures position that is taken.
- b_t ≡ basis at the time the hedge is initiated.
- b_T ≡ basis at the time of expiration of the contract.
- b_{t^*} ≡ basis at the time the hedge is lifted.
- π ≡ profit

In the case of hedges created using futures contracts, the effective price that is locked in per unit of the underlying asset will be equal to the futures price prevailing at the time of initiation of the futures position. This is true for both long as well as short hedges, and is irrespective of whether the contract is delivery settled or cash settled.

Consider the case of a short hedge. If the contract is cash settled, the total profit from marking to market would be $F_t - F_T$. The underlying asset would have to be sold in the spot market at a price of S_T. The overall cash inflow will be

$$S_T + (F_t - F_T) = F_t$$

because, as we have seen earlier, at expiration the spot price must be equal to the futures price to preclude arbitrage. Notice that we have added the profit from the futures position to the terminal spot price, in order to determine the effective inflow. This is because if the cash flow from marking to market is a positive number, that is, it actually is a profit, then it will lead to a higher effective inflow. Else if it is a negative number, or in other words is a loss, then it will lead to a lower effective inflow.

3.7.2 Numerical Illustration

Take the case of Norman Davis who went short in 500 contracts on corn. The delivery price was \$ 5.25. Assume that the spot price of corn at the time of expiration is \$ 5.40. The cumulative profit from marking to market is:

$$500 \times 5{,}000 \times (5.25 - 5.40) = -\$\ 375{,}000$$

The sale proceeds from the spot market will be:

$$500 \times 5{,}000 \times 5.40 = \$\ 13{,}500{,}000$$

The effective amount received for 2,500,000 bushels of corn is therefore:

$$13{,}500{,}000 - 375{,}000 = \$\ 13{,}125{,}000$$

which amounts to \$ 5.25 per bushel, which is nothing but the initial futures price.

The same is true for a futures based long hedge. If the contract were to be cash settled, the hedger's cumulative profit from marking to market will be $F_T - F_t$. The underlying asset would have to be acquired in the spot market at a price of S_T. The overall cash outflow will be

$$-S_T + (F_T - F_t) = -F_t$$

Once again the profit from the futures position has been added to the outflow in the spot market in order to determine the effective outflow. This is because a profit, or an inflow from the futures market, will reduce the effective outflow, whereas a loss, or an outflow from the futures market, will increase the effective outflow.

3.7.3 Numerical Illustration

Take the case of Kroger which went long in 50 futures contracts at a price of \$ 0.6500 per AUD. Assume that the exchange rate at expiration is \$ 0.6750 per AUD. The cumulative profit from marking to market will be

$$50 \times 10{,}000 \times (0.6750 - 0.6500) = \$\ 12{,}500$$

The outflow on account of the spot purchase will be

$$500{,}000 \times 0.6750 = \$\ 337{,}500$$

The effective outflow is therefore

$$337{,}500 - 12{,}500 = \$\ 325{,}000$$

which corresponds to the delivery price of 0.6500.

3.8 A Perfect Hedge

A perfect hedge is defined as one which locks in a price for the hedger with absolute certainty. If the hedge were to be perfect, a party with a long spot position would face no uncertainty with regard to his sale price, and nor will a trader with a short spot position, face any risk with respect to his cost of acquisition. In the illustrations above, Norman Davis was able to lock in a selling price of \$ 5.25 per bushel of corn, while Kroger was able to lock in a rate of \$ 0.6500 per AUD as its cost of acquisition of Australian dollars. Consequently, both these hedges are perfect in nature. Thus if a hedge were to be perfect, the hedger can transform a risky spot position into a risk-free position.

3.8.1 Necessary Conditions

If a hedge is to be perfect, then the following conditions must hold.

1. The date of termination of the hedge, or in other words, the date on which the hedger wishes to buy or sell the underlying asset must coincide with the date of expiration of the futures contract that is being used for hedging.
2. The number of units of the underlying asset sought to be bought or sold by the hedger, must be an integer multiple of the contract size.
3. Futures contracts must be available on the commodity which the hedger is seeking to buy or to sell.

3.8.2 Importance of Condition-1

We will now show as to why it is important that the date on which the asset is bought or sold be the same as the expiration date of the futures contract, if the hedge is to be perfect. For ease of exposition, we will assume that the futures contract specifies a single delivery date.

Consider the case of a short hedger, who is desirous of selling his goods prior to the date of expiration of the futures contract. That is, he wishes to sell on day t^*, where $t < t^* < T$. Quite obviously, since the futures contract is no longer required, it would have to be offset on that day. The underlying asset will have to be disposed off in the spot market.

The cumulative profit of the short futures position due to marking to market will be $F_t - F_{t^*}$. The inflow from the spot market will be S_{t^*}. Thus the effective inflow on account of the sale transaction will be:

$$S_{t^*} + (F_t - F_{t^*}) = F_t + (S_{t^*} - F_{t^*}) \tag{3.1}$$

If the date of sale were to coincide with the contract's expiration date, that is, if t^* were to be the same as T, then S_{t^*} would be equal to F_{t^*}, due to the spot-futures convergence condition, and so the effective price received would be F_t. In other words, in such a situation, the hedge would be perfect for there would be no uncertainty regarding the effective sale price. However, if t^* were to be prior to the expiration of the futures contract, F_{t^*} will usually not be equal to S_{t^*}. In such a situation, the effective price received would depend on both F_{t^*}, the terminal futures price, and S_{t^*}, the terminal spot price. Therefore, since these are unknown variables until the end, there will always be uncertainty regarding the effective price that the hedger is likely to receive.

A similar line of argument can be advanced for long hedges. If the asset were to be bought on day t^* and the futures position were to be offset on that day, the effective outflow will be:

$$-S_{t^*} + (F_{t^*} - F_t) = -F_t - (S_{t^*} - F_{t^*}) \tag{3.2}$$

In this case too, the effective cost of acquisition of the asset will depend on both the terminal spot as well as futures prices, and consequently will be uncertain until the very end.

As can be seen, equality of spot and futures prices on the day of termination of the hedge, is imperative for there to be no prior uncertainty about the price that is being locked in. Such price convergence is assured only at the time of expiration of the futures contract, and consequently the date on which the transaction in the underlying asset takes place should be the same as the expiration date of the futures contract, if we are to ensure that the hedge is perfect.

3.8.3 Importance of Condition-2

In order for the hedge to be perfect, it is essential that the quantity being hedged should be an integer multiple of the size of the futures contract.

Assume that a farmer has 1,250 units of a commodity that he wishes to sell. Futures contracts on the commodity expiring exactly on the date of sale are available, and the contract size is 100 units. So theoretically a short position is required in 12.50 futures contracts. This is infeasible, and in practice, the farmer will have to go short in either 12 contracts or in 13. In either case, as we demonstrate below, the effective price received per unit will be uncertain.

Case-A: The farmer uses 12 contracts

The profit from marking to market will be

$$12 \times 100 \times (F_t - F_T)$$

The inflow on account of the sale of goods in the spot market will be $1,250S_T$. Thus the effective price received per unit of the good will be:

$$\frac{1,250S_T + 1,200(F_t - F_T)}{1,250} = \frac{50S_T + 1,200F_t}{1,250}$$

The effective price is dependent on S_T which is a random variable. Consequently the effective price that will be received per unit will be uncertain, and will be revealed only at the end. This is despite the fact that the date of termination of the hedge coincides with the expiration date of the futures contract.

Case-B: The farmer uses 13 contracts

The profit from marking to market will be

$$13 \times 100 \times (F_t - F_T)$$

The inflow from the spot market will be $1,250S_T$. The effective price received per unit of the good will be

$$\frac{1,250S_T + 1,300(F_t - F_T)}{1,250} = \frac{1,300F_t - 50S_T}{1,250}$$

Once again there will be uncertainty regarding the effective price that is to be received.

Importance of Condition-3

Take the case of an investor who wishes to hedge the risk inherent in a position that he has taken in an asset on which no futures contracts are available. Quite obviously, in such cases the hedger would have no choice but to use contracts on a related commodity, assuming such contracts were to be available. The use of a futures contract on a closely related commodity for the purpose of hedging is called *cross hedging*. The term *'closely related'*, implies that the prices of the two commodities move in tandem. In the case of such hedges, the greater the degree of positive correlation between the prices of the two commodities, the greater will be the effectiveness of the hedge.

A cross-hedge cannot be perfect in practice. We will demonstrate the truth of this assertion, by considering a short hedge for the purpose of illustration.

Let S_T be the spot price of the asset that the hedger is selling on day T. There are, however, no futures contracts on the asset, expiring on day T or thereafter. However, contracts are available that expire on that day, but are based on a related commodity. Let $\hat{F}_t$ be the initial futures price and $\hat{F}_T$ the terminal futures price. $\hat{S}_T = \hat{F}_T$, is the terminal spot price of the asset on which the futures contracts have been written.

On the day that the hedge is to be terminated, the hedger will obviously sell the asset in his possession at its prevailing spot price and collect his profit/loss from the futures market. The effective price received per unit will be

$$S_T + (\hat{F}_t - \hat{F}_T) = \hat{F}_t + (S_T - \hat{F}_T) \qquad (3.3)$$

In this case, however, S_T or the spot price of the asset, will in general not be equal $\hat{F}_T$ or the terminal futures price, because they represent the prices of two different commodities. Consequently, there will be uncertainty regarding the effective price that the hedger is likely to receive.

3.9 Basis Risk

One of the conditions for a perfect hedge is that the date of expiration of the futures contract should coincide with the date of sale of the underlying asset. If this condition were to be satisfied, and the position in the spot market is an integral multiple of the contract size, then the hedged position will enable the hedger to lock in the futures price that was prevailing at the inception of the hedge, assuming of course that contracts are available on the asset whose price is being sought to be hedged. In practice, even if the other required conditions are satisfied, we are unlikely to have a situation where the termination date coincides with the contract expiration date. Consequently, we would usually have to choose a futures contract that expires after the date on which we wish to terminate the hedge.[1]

We will assume that $Q = Q_f$. For instance, if Q = 100,000 bushels of wheat, and each futures contract is for 5,000 bushels, then we would use 20 contracts. We will also assume that the asset being hedged is the same as the asset underlying the futures contract.

Since by assumption, the futures contract is scheduled to expire after the transaction date, it would have to be offset on that day. For a short hedger the effective price received per unit of the underlying asset under such circumstances will be

$$F_t + (S_{t^*} - F_{t^*})$$

[1] If we use a futures contract that expires earlier then we would subsequently have a naked or uncovered position, which would mean that we would not be hedged or protected till the transaction date.

The uncertainty in this case arises due to the fact that on a date prior to the expiration date of the futures contract, the spot price S_{t^*} will not in general equal the futures price F_{t^*}. A similar line of reasoning holds for long hedges. Due to this reason, in practice hedgers face what is called *Basis Risk*. Let us first define the basis, which is critical for understanding hedging.

Basis = Spot Price of the Asset Being Hedged – Futures Price of the Contract that is Used for Hedging, In symbolic terms, at a point in time t, the basis is given by

$$b_t = S_t - F_t \tag{3.4}$$

At the time of expiration of the futures contract, due to the no-arbitrage condition, S_T is guaranteed to equal F_T, and we can be sure that the basis will be zero. At any point prior to expiration, t^*, however, we cannot make such an assertion. Thus, a hedger who is required to offset a futures contract prior to expiration faces uncertainty regarding the basis, or what is termed as *Basis Risk.*

Short Hedges and the Basis

Take the case of a person who has a long position in the spot market and a short position in the futures market. If this hedge is held till the expiration of the futures contract, then the profit from the spot market will be $S_T - S_t$ and the profit from the futures market will be $F_t - F_T$.

$$\text{So } \pi = (S_T - S_t) + (F_t - F_T) = (S_T - F_T) - (S_t - F_t) = b_T - b_t \tag{3.5}$$

We know that $S_T = F_T$ and therefore $b_T = 0$. So the profit from the short hedge = $-b_t$. The effective price received by the short hedger is

$$S_T + (F_t - F_T) = F_t + (S_T - F_T) = F_t + b_T = F_t \tag{3.6}$$

F_t is known at time t, when the hedge is initiated. Thus, there is no uncertainty about the effective price that will be received if the hedge is held till the date of expiration of the futures contract. In other words, risk can be completely neutralized.

Now, let us consider the case where the hedge is lifted at time t^*. The profit in this case, is

$$\pi = (S_{t^*} - S_t) + (F_t - F_{t^*}) = (S_{t^*} - F_{t^*}) - (S_t - F_t) = b_{t^*} - b_t \tag{3.7}$$

The effective price received is

$$S_{t^*} + (F_t - F_{t^*}) = F_t + b_{t^*} \tag{3.8}$$

If the basis remains unchanged, that is $b_{t^*} = b_t$, then the profit will be zero and the effective price received will be $F_t + (S_t - F_t) = S_t$. Thus, if the basis at the time the hedge is lifted is the same as the basis at the time the hedge was set up, the effective price received will be equal to the initial spot price. In general, the profit is, $b_{t^*} - b_t$. b_{t^*} is a random variable at the time the hedge is initiated and therefore there is uncertainty regarding the quantum of the profit or loss. The term *basis risk* refers to the uncertainty regarding as to how the basis will change. Thus an investor who chooses to hedge replaces price risk, which is the uncertainty regarding S_{t^*}, with basis risk, which is the uncertainty regarding b_{t^*}.

Illustration Assume that today is July 1, 2008. Victor Yallop a wheat farmer knows that he will have 100,000 bushels of wheat ready for sale in September. Futures contracts expiring on September 15 are available and each contract is for 5,000 bushels. The futures price is $ 4.00 per bushel.

If the date of sale of wheat were to coincide with the date of expiration of the futures contract, then obviously the hedge would be perfect. For instance, if the spot price of wheat on September 15 is $ 3.75 per bushel, then Yallop can sell the wheat for $ 375,000. The profit from the futures market will be $[4.00 - 3.75] \times 100{,}000 = \$\,25{,}000$. The total cash inflow will be $ 400,000, which means that the effective price locked in per bushel is $ 4.00, which is nothing but the initial futures price.

Now let us consider a situation where Yallop decides to sell the wheat on September 2. Assume that the spot price on that day is $ 3.75 per bushel, whereas the futures price for the September contract is $ 3.80. The proceeds from the sale of wheat in the spot market will be $ 375,000. The futures contracts will have to be offset. The profit will be $[4.00 - 3.80] \times 100{,}000 = \$\,20{,}000$. The total cash inflow will be $ 395,000, which means that the effective price received per bushel is $ 3.95.

$$3.95 = 4.00 + (3.75 - 3.80) \equiv F_t + (S_{t^*} - F_{t^*}) = F_t + b_{t^*}$$

Thus the effective price received is a function of the basis that is prevailing at the time of expiration of the hedge, and consequently cannot be predicted in advance. The hedge is obviously not perfect under such circumstances. The uncertainty arises due to the hedger's inability to accurately predict the value of the basis on the date of sale, which is what we have termed as basis risk.

A Long Hedge

Now let us take the case of a long hedger, that is a person with a short position in the spot market and a long position in the futures market. If such a hedge is held till expiration, the profit from the spot market will be $S_t - S_T$ and that from the futures market will be $F_T - F_t$. So

$$\pi = (S_t - S_T) + (F_T - F_t) = (S_t - F_t) - (S_T - F_T) = b_t - b_T \qquad (3.9)$$

But $b_T = 0$, since $S_T = F_T$. Therefore, the profit from the long hedge is equal to b_t. The effective price paid by the long hedger is

$$-S_T + (F_T - F_t) = -(F_t + b_T) = -F_t \qquad (3.10)$$

Why do we end up with a minus sign in front of F_t? It is because the effective price paid is an outflow. The quantum of the price paid is F_t. Thus once again, there is no uncertainty regarding the price. In other words we can completely neutralize risk and create a perfect hedge.

Let us now consider the case where the long hedge is lifted at time t^*. The profit is given by

$$\pi = (S_t - S_{t^*}) + (F_{t^*} - F_t) = (S_t - F_t) - (S_{t^*} - F_{t^*}) = b_t - b_{t^*} \qquad (3.11)$$

The effective price paid is

$$-S_{t^*} + (F_{t^*} - F_t) = -(F_t + b_{t^*}) \qquad (3.12)$$

If the basis were to remain unchanged the profit will equal zero, and the effective price paid will be $F_t + (S_t - F_t) = S_t$. Once again, like in the case of the short hedge, the hedger would have locked in the initial spot price. In general, of course, the profit will be equal to the change in the basis and the hedger would once again face basis risk.

Changes in the Basis

We have defined the basis as $S_t - F_t$.[2] If the market were to be in Contango, the basis as per our definition will be negative. If it were to become more negative as it evolves over time, we would say that it has widened. However, if it were to become less negative over time, it would be said to have narrowed.

On the other hand, if the market were to be in Backwardation, the basis as per our definition will be positive. If it were to become more positive as it evolves over time, we would say that it has widened. However, if it were to become less positive over time, it would be said to have narrowed. Hence, the terms *'widening* and *'narrowing'* are obviously used with reference to changes in the absolute value of the basis.

For a hedger with a short position in the futures market, the effective price received will be $F_t + b_{t^*}$. The higher the value of the basis on the date of termination, the greater will be his inflow. Thus such hedgers will benefit from a rising basis. On the other hand, for a hedger with a long position in the futures market, the effective outflow will be $F_t + b_{t^*}$. In this case, the lower the value of the basis, the smaller is the outflow. Consequently such hedgers stand to benefit from a declining basis.

Thus a rising basis benefits short hedgers, while a declining basis benefits long hedgers. While studying the features of long and short futures position, we have observed that traders with long positions benefit from rising prices while those with short positions gain from declining prices. Notice that the basis itself can be perceived as a price, for after all it is nothing but a difference of two prices. From this perspective, we can make the assertion that a short hedger is *'long the basis'*, while a long hedger is *'short the basis.'*

The Basis for a Cross Hedge

Thus far we have assumed that the asset whose price risk is being hedged, is the same as the asset on which the futures contracts being used are written. In the

[2] Some authors prefer to define it as the futures price minus the spot price.

case of a cross hedge, the two assets will be different. Let S be the spot price of the asset that we wish to hedge and $\hat{S}$ the spot price of the asset underlying the futures contract that we are using. The effective price that is paid or received is

$$S_{t^*} + (\hat{F}_t - \hat{F}_{t^*}) = \hat{F}_t + (\hat{S}_{t^*} - \hat{F}_{t^*}) + (S_{t^*} - \hat{S}_{t^*}) \qquad (3.13)$$

Thus, in such a case, the basis will consist of two components. The first component, $\hat{S}_{t^*} - \hat{F}_{t^*}$ is the basis that would exist, if the asset being hedged is the same as the asset underlying the futures contract. The second term, $S_{t^*} - \hat{S}_{t^*}$, is the basis that arises due the fact that the two assets are not the same.

Quantifying Basis Risk

The basis is defined as $S_t - F_t$. In Finance theory we measure risk by the variance of the random variable. So, using this yardstick

$$\sigma^2(b) = \sigma^2(S) + \sigma^2(F) - 2\rho(S, F)\sigma(S)\sigma(F) \qquad (3.14)$$

where $\sigma(S)$ is the standard deviation of the spot price, $\sigma(F)$ is the standard deviation of the futures price, and $\rho(S, F)$ is the correlation coefficient between the two variables.

The critical factor is ρ. The higher the correlation, the lower will be the basis risk. Hedging as we said earlier, substitutes price risk with basis risk. Hence, an investor will find it attractive to hedge, only if the basis risk is substantially lower than the price risk.

3.10 Selecting a Futures Contract

Once a person decides to hedge, the first thing he has to do is to decide as to which futures contract he is going to use. This is important because basis risk would depend on the contract that is chosen. The selection of the contract would be influenced by the following factors.

In general, if the commodity that a person wishes to hedge has futures contracts written on it then he would use such a contract. This is because the futures and spot prices are likely to be highly correlated, thereby reducing basis risk. Thus, if you have a position in cocoa and cocoa futures are available, then you would use such a contract. But what if cocoa futures are not available? In such a case, you would have no choice but to select a futures contract on a different commodity. This, as we have mentioned earlier, is called cross hedging. For cross hedging to succeed, one would need to select a futures contract on an asset whose price is highly positively correlated with the price of the asset that is being hedged. For instance, if you find that coffee and cocoa prices are highly positively correlated, then you may use a coffee futures contract for hedging a position in cocoa.

However, there may be a situation where you prefer to hedge cocoa using coffee futures, even though cocoa futures are available. This may be the case if cocoa futures contracts are not liquid. Liquidity is important, because when you wish to transact, you would like to both enter and exit the market quickly and at a fair price.

Another important factor to be kept in mind while selecting a contract is possible mispricing. An overpriced contract will benefit a short hedger because if prices fall subsequently he will gain. Similarly, an underpriced contract will benefit a long hedger.

The second major decision that a hedger has to take is as to which expiration month he should choose. Let us go back to our earlier example where Victor Yallop, the wheat farmer, was initiating a short hedge on July 1, 2008. The sale of wheat was scheduled for September 2008 and hence we said that he would use the September futures contract. Let us assume that Yallop had a choice of September 2008, December 2008, March 2009 and May 2009 contracts.

In practice, Yallop may prefer the December 2008 contract to the September 2008 contract. There are two reasons for this. The first is that in real life, futures prices can often exhibit unusual price movements in the month in which they expire. A hedger who is trying to close out his position, would not like to be exposed to such erratic movements.

Secondly, consider the case of a long hedger who too plans to terminate the hedge in September and hence goes in for a September futures contract. What happens if he is chosen for delivery[3]? Let us assume that Jenner and Company, a wheat mill, are the long hedgers. Their objective may be to procure the wheat in Pittstown, where they are based. To hedge against price risk, they may use futures contracts traded on the nearest exchange, which we will assume is in Kansas City. If they are chosen for delivery, they may be asked to take delivery in Kansas City, which is the location specified in the contract. Thus for them, taking delivery in Kansas City would entail the cost and inconvenience of taking the wheat to Pittstown. Thus a person whose long hedge expires in September may not like to use the September contract. In this case, if Jenner and Company had hedged using December futures, they could simply offset their position in September and buy the wheat at the prevailing spot price in Pittstown.

Another factor to be kept in mind is that, in general, the basis risk will be higher, the further away the expiration month of the contract is from the date of termination of the hedge. The reason is as follows. It must be remembered that both spot as well as futures prices of a commodity, are being influenced by the same economic factors. The difference is that spot prices represent today's market price, while the futures prices represent the price for a trade at some point of time in the future. If the expiration date of the contract is close to the date on which the hedge is being terminated, then both the spot and the futures markets will be discovering the price of the commodity for virtually the same point in time and therefore the prices may be expected to converge, such that the price difference, which is nothing but the basis, is more predictable. However, if the expiration date of the futures contract is much further from the date of lifting of the hedge, then the spot and futures prices are likely to be affected by different supply and

[3]Remember, the oldest outstanding long position is chosen.

demand forces, which reflect different sets of information, as a consequence of which we would expect to observe greater variability in the basis. Thus, in the case of Victor Yallop, the March 2009 contract is likely to entail greater basis risk than the December 2008 contract.[4]

Keeping these facts and factors in mind, Yallop is likely to use the December 2008 contract. *So the general rule is that the hedger should choose a contract with an expiration month that is as close as possible to the month in which the hedge is being terminated, but after it.*

However, it should be remembered that the longer the time to expiration of the contract, the less liquid it is likely to be. By less liquid, we mean that it may not be actively traded. Hence, while a hedger would generally choose an expiration month that comes after the month in which he wishes to terminate the hedge, this principle may have to be overruled due to the necessity of using a liquid contract. Thus, there may be cases where he will choose a contract with an expiration date that comes before the date on which the hedge is to be terminated. When such a contract moves into its expiration month, the futures position will be offset and a new position will be opened in the next available expiration month. Such a procedure is called *Rolling the Hedge Forward*.

3.11 A Rolling Hedge

Let us now assume that Victor Yallop wants to sell his wheat on 15 April 2009, and not in September 2008. In July 2008, he has a choice of contracts with different expiration months—September 2008, December 2008, March 2009 and May 2009, as we have seen earlier.

Instead of using the May 2009 contract, a hedger like Yallop may decide to roll the hedge forward. In practice, he could have two possible reasons for it. Firstly, the May 2009 contract may not be liquid. Secondly, it may not even be available on July 1, 2008, the date on which the hedge is initiated. We will now illustrate a rolling hedge with the help of the following example.

Example

Let us assume that Yallop wants to hedge 100,000 bushels of wheat and that each wheat futures contract is for 5,000 bushels. So he will require 20 contracts. Let the futures price of the September 2008 contract be $ 4.00 per bushel, on July 1, 2008 when the hedge is initiated. Consider the following sequence of actions that Yallop can take.

The reason the September contracts are offset on August 31 is that, as we said earlier, in general one would not like to hold a contract in its expiration month. The same logic applies for the other contracts. The gain in the futures market per bushel of wheat, may be calculated as follows.

[4]See Koontz and Purcell (1999).

Table 3.1 Illustration of a Rolling Hedge

Date	Action Taken
July 1, 2008	Short 20 September 2008 contracts at \$ 4.00
August 31, 2008	Offset the September contracts at \$ 3.75
	Go short in 20 December 2008 contracts at \$ 3.60
November 30, 2008	Offset the December contracts at \$ 3.25
	Go short in 20 March 2009 contracts at \$ 3.15
February 28, 2009	Offset the March contracts at \$ 3.30
	Go short in 20 May 2009 contracts at \$ 3.05
April 15, 2009	Offset the May contracts at \$ 2.95
	Sell the wheat in the spot market at \$ 2.80

- Profit from the September contracts = (4.00 – 3.75) = 0.25.
- Profit from the December contracts = (3.60 – 3.25) = 0.35.
- Profit from the March contracts = (3.15 – 3.30) = (0.15).
- Profit from the May contracts = (3.05 – 2.95) = 0.10.
- Total profit = 0.25 + 0.35 – 0.15 + 0.10 = \$ 0.55.

The effective price received = \$ 2.80 + \$ 0.55 = \$ 3.35 per bushel.

In practice, rolling the hedge forward too frequently may be expensive because every time you close out your position and take a fresh position, you will incur transactions costs.

3.12 The Hedge Ratio

The final decision that the hedger has to take while designing his hedging strategy, is how to choose the hedge ratio. The hedge ratio refers to the futures position taken, per unit exposure in the spot market. In other words, the hedge ratio is the ratio of the size of the position taken in the futures market to the size of the exposure in the spot market. The ideal ratio should be such that the futures profit / loss, exactly matches the spot loss /profit.

In our examples thus far, we have used a hedge ratio of one. For instance, to hedge 100,000 bushels of wheat, we assumed that Yallop would go short in 20 futures contracts, where each contract was for 5,000 bushels. But if the objective of the hedger is to minimize risk, then it can be demonstrated that a hedge ratio of one is not always optimal.

The Minimum Variance Hedge Ratio

We will denote the number of units of the underlying asset in which the investor has a long position by Q. The futures contract size will be denoted by N, and C will be used to denote the number of futures contracts that are being used for hedging.

Hence, the total number of units of the underlying asset being represented by the futures position is $Q_f = NC$.

The issue is, is it always optimal to set $Q_f = Q$, or in other words should we always use a hedge ratio of 1:1.

In order to answer this question, let us look at the effective revenue for a short hedger. It can be represented as

$$R = QS_{t^*} + (F_t - F_{t^*})Q_f \tag{3.15}$$

If $t^* = T$, or in other words the termination date were to coincide with the contract expiration date, then it would be ideal to set $Q_f = Q$, so that $R = QF_t$. In this case, we would have a perfect hedge, about whose revenue there would be no uncertainty.

However, when $t^* < T$, or the hedge is terminated before the contract expiration date, then a hedge ratio of 1:1 need not be optimal.

Let us define the hedge ratio as $h = \dfrac{Q_f}{Q}$ then $Q_f = hQ$.

A short hedger will receive an amount R when the hedge is lifted, such that

$$\begin{aligned}
R &= QS_{t^*} + (F_t - F_{t^*})hQ \\
&= QS_{t^*} - (F_{t^*} - F_t)hQ \\
&= QS_t - QS_t + QS_{t^*} - (F_{t^*} - F_t)hQ \\
&= QS_t + (S_{t^*} - S_t)Q - (F_{t^*} - F_t)hQ \\
&= QS_t + \triangle SQ - \triangle FhQ \\
&= QS_t + (\triangle S - h\triangle F)Q
\end{aligned}$$

We have denoted $(S_{t^*} - S_t)$ as $\triangle S$ and $(F_{t^*} - F_t)$ as $\triangle F$.

When the hedge is initiated, we will know only S_t and Q. The unknowns will be $\triangle S$ and $\triangle F$. If our objective is to minimize risk, then we should minimize the variance of R, using the only variable in our control namely h.

$$\begin{aligned}
\text{Var}\,(R) &= \text{Var}\,[QS_t + (\triangle S - h\triangle F)Q] \\
&= \text{Var}\,[(\triangle S - h\triangle F)Q] \\
&= Q^2\text{Var}\,[\triangle S - h\triangle F]
\end{aligned} \tag{3.16}$$

Let us denote Var $(\triangle S)$ by ${\sigma_s}^2$, Var $(\triangle F)$ by ${\sigma_f}^2$ and the correlation between $\triangle S$ and $\triangle F$ by ρ. Therefore

$$\text{Var}\,(R) = Q^2[{\sigma_s}^2 + h^2{\sigma_f}^2 - 2h\rho\sigma_s\sigma_f] \tag{3.17}$$

If we minimize the variance of R with respect to h, we will set

$$\begin{aligned}
\frac{d\,\text{Var}\,(R)}{dh} &= Q^2[2h{\sigma_f}^2 - 2\rho\sigma_s\sigma_f] = 0 \\
&\Rightarrow 2h{\sigma_f}^2 - 2\rho\sigma_s\sigma_f = 0 \\
&\Rightarrow h = \frac{\rho\sigma_s}{\sigma_f}
\end{aligned} \tag{3.18}$$

The Optimal Number of Futures Contracts We will denote the optimal hedge ratio by h^*. We know that

$$h^* = \frac{Q_f^*}{Q}$$

$$\Rightarrow Q_f^* = h^* Q \tag{3.19}$$

Therefore, the optimal number of contracts is

$$C^* = \frac{Q_f^*}{N} = \frac{h^* Q}{N} \tag{3.20}$$

Let us assume that the optimal hedge ratio has been calculated to be 0.85. If Q = 100,000 bushels of wheat, and N = 5,000 bushels, then the optimal number of futures contracts is

$$\frac{.85 \times 100,000}{5,000} = 17$$

3.13 Estimating the Hedge Ratio

Let us specify the following statistical relationship between the changes in the spot price of an asset and the changes in the corresponding futures price.

$$\triangle S = \alpha + \beta \triangle F + \epsilon \tag{3.21}$$

To estimate the parameters α and β, we can run a linear regression with $\triangle F$ as the independent variable and $\triangle S$ as the dependent variable. The slope coefficient β, is given by

$$\frac{\text{Cov}(\triangle S, \triangle F)}{\sigma^2(\triangle F)} = \frac{\rho\sigma_s\sigma_f}{{\sigma_f}^2} = \frac{\rho\sigma_s}{\sigma_f} \tag{3.22}$$

which is nothing but the minimum variance hedge ratio. We will illustrate the calculation of β using an example.

Example

Consider the following data. $\triangle S$ and $\triangle F$, are assumed to be measured over weekly non-overlapping time intervals. Let us denote $\triangle F$ by the variable x, and $\triangle S$ by the variable y (Table 3.2).

Calculations

$$\sum x_i = .0103; \sum y_i = .0123$$

$$\sum {x_i}^2 = .00007119; \sum {y_i}^2 = .00008491; \sum x_i y_i = .00007637$$

$$n = \text{number of observations} = 10.$$

Table 3.2 Data for Estimating. The Hedge Ratio

Week 'i'	$x_i = \Delta F$	$y_i = \Delta S$
1	.0032	.0038
2	.0040	.0034
3	.0025	.0030
4	.0035	.0035
5	.0037	.0048
6	−.0015	−.0012
7	−.0005	−.0005
8	−.0001	.0005
9	−.0025	−.0022
10	−.0020	−.0028

$$\sigma_f = \left[\frac{\sum {x_i}^2}{(n-1)} - \frac{(\sum x_i)^2}{n(n-1)}\right]^{\frac{1}{2}} = .002594459$$

$$\sigma_s = \left[\frac{\sum {y_i}^2}{(n-1)} - \frac{(\sum y_i)^2}{n(n-1)}\right]^{\frac{1}{2}} = .0027845$$

$$\rho = \frac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{[n\sum {x_i}^2 - (\sum x_i)^2]^{\frac{1}{2}}[n\sum {y_i}^2 - (\sum y_i)^2]^{\frac{1}{2}}}$$

$$= \frac{.00063701}{(.024613207)(.026416093)} = .97973689$$

$$h^* = \frac{\rho\sigma_s}{\sigma_f} = \frac{.97973689 \times .0027845}{.002594459} = 1.0515$$

Hedging Effectiveness The hedging effectiveness may be defined as:

$$1 - \frac{\sigma^2(b)}{\sigma^2(S)} \tag{3.23}$$

If the basis risk were to be equal to the price risk, then obviously there would be no risk reduction, and the hedging effectiveness will be zero. However, if the basis risk is zero, which would imply that we have a perfect hedge, the hedging effectiveness will be 1.0. In practice, when we run a linear regression of the form

$$\triangle S = \alpha + \beta \triangle F + \epsilon$$

the R^2 of the regression is a measure of the hedging effectiveness. The dependent variable can be written as the fitted value plus the residual. That is, $\triangle S$ can be written as

$$\triangle S = \triangle \hat{S} + \hat{e} \tag{3.24}$$

If we define the total sum of squares as

$$SST = \sum_{i=1}^{n} (\Delta S_i - \overline{\Delta S})^2 \tag{3.25}$$

the explained sum of squares as

$$SSE = \sum_{i=1}^{n} (\Delta \hat{S}_i - \overline{\Delta S})^2 \tag{3.26}$$

and the residual sum of squares as

$$SSR = \sum_{i=1}^{n} \hat{e}_i^2 \tag{3.27}$$

then $SST = SSE + SSR$. The R^2 is defined as $1 - \frac{SSR}{SST}$. The total sum of squares, SST is a measure of the risk of the unhedged position, while the residual sum of squares SSR, is a measure of the basis risk. Consequently, R^2 is consistent with our earlier definition of hedging effectiveness.

3.14 Tailing a Hedge

In all our discussions thus far, we have been treating a futures contract like a forward contract, that is, we have been ignoring the effects of marking to market. In real life, our hedging procedure should take into account this feature of futures markets. The process of adjusting the hedge ratio to take into account intermediate cash flows is called *Tailing*.

Let us assume that a hedge is set up today by selling futures contracts equivalent to Q_f units, at a price F_t. The hedge will expire at time t^* and in the interim, will be marked to market at two points in time, t_1 and t_2, where $t < t_1 < t_2 < t^*$. We will assume that all intermediate gains and losses can be invested / borrowed at the risk-less rate of interest r.

At time t_1, we will get a cash flow of $(F_t - F_{t_1})Q_f$. This can be reinvested / borrowed until the expiration of the hedge at the rate r, which for expositional convenience will be assumed to be a daily rate of interest.[5] Consequently at t^*, we will get a cash flow of $(F_t - F_{t_1})Q_f(1+r)^{t^*-t_1}$.

Similarly, when the contract is marked to market at t_2, there will be a cash flow of $(F_{t_1} - F_{t_2})Q_f$. This too will be reinvested or borrowed and will yield a corresponding cash flow of $(F_{t_1} - F_{t_2})Q_f(1+r)^{t^*-t_2}$ at time t^*.

And finally, when the hedge is lifted at time t^*, we will receive a cash flow of $(F_{t_2} - F_t^*)Q_f$.

Thus the total cash flow at time t^* is

$$(F_t - F_{t_1})Q_f(1+r)^{t^*-t_1} + (F_{t_1} - F_{t_2})Q_f(1+r)^{t^*-t_2} + (F_{t_2} - F_{t^*})Q_f \tag{3.28}$$

[5]Hence, all time periods will be measured in terms of number of days.

This in general, will not be equal to $(F_t - F_{t^*})Q_f$, which is what we would have received had we used a forward contract instead of a futures contract. The only way the two cash flows can be made equal is if we adjust the number of futures contracts used.

Let us now consider the following strategy. Assume that the hedge is initiated with $Q_{f1} = \frac{Q_f}{(1+r)^{t^*-t_1}}$ and that the futures position is subsequently changed as follows. At time t_1, let the number of futures contracts used be increased to $Q_{f2} = \frac{Q_f}{(1+r)^{t^*-t_2}}$. And finally, at time t_2, let the number of contracts used be increased to Q_f.

The total cash flow, at time t^* if the above procedure is used, will be

$$\begin{aligned}\frac{(F_t - F_{t_1})Q_f(1+r)^{t^*-t_1}}{(1+r)^{t^*-t_1}} &+ \frac{(F_{t_1} - F_{t_2})Q_f(1+r)^{t^*-t_2}}{(1+r)^{t^*-t_2}} + (F_{t_2} - F_{t^*})Q_f \\ &= (F_t - F_{t_1})Q_f + (F_{t_1} - F_{t_2})Q_f + (F_{t_2} - F_{t^*})Q_f \\ &= (F_t - F_{t^*})Q_f \end{aligned} \tag{3.29}$$

which is equal to the final payoff that would have been received, had a forward contract been used.

Thus tailing requires that the hedger adjust the number of futures contracts used or in other words the hedge ratio, every time there is an intermediate cash flow. If the number of futures contracts used were to be equivalent to Q_f units from the very beginning, then the hedger would be overhedging.

Let us illustrate the tailing procedure, with the help of an example.

Example

Assume that Victor Yallop, wants to hedge 10,000,000 bushels of wheat. Each contract is for 5,000 bushels, so if Yallop were to use a hedge ratio of 1, he would use 2,000 contracts from the outset. But if Yallop were to tail his hedge, then he would proceed as follows.

Let us assume that the daily interest rate is .1% and that the hedge is kept in place for three days. So at the time of initiating the hedge, Yallop would use Q_{f1} contracts, where $Q_{f1} = \frac{10{,}000{,}000}{(1.001)^2} = 9{,}980{,}030 \cong 9{,}980{,}000 \equiv 1{,}996$ contracts. When the contract is marked to market at the end of the first day, he will change the number of contracts used to, $Q_{f2} = \frac{10{,}000{,}000}{(1.001)} = 9{,}990{,}010 \cong 9{,}990{,}000 \equiv 1{,}998$ contracts. And finally, when the contract is marked to market at the end of the second day, Yallop will short $Q_f = 10{,}000{,}000 \equiv 2{,}000$ contracts.

Consider the following hypothetical futures prices (Table 3.3).

Yallop's profit / loss from his futures position, at the end of the third day, in such a scenario is

$$(4.00 - 3.75) \times 9{,}980{,}000 \times (1.001)^2 + (3.75 - 3.50) \times 9{,}990{,}000 \times (1.001) \\ +(3.50 - 3.25) \times 10{,}000{,}000$$

Table 3.3 Futures Settlement Prices

Time	Settlement Price
0	\$ 4.00
1	\$ 3.75
2	\$ 3.50
3	\$ 3.25

$$= 2{,}499{,}992.50 + 2{,}499{,}997.50 + 2{,}500{,}000 = 7{,}499{,}990 \cong \$\,7{,}500{,}000$$

Had he used a forward contract, his profit would have been the same, that is

$$(4.00 - 3.25) \times 10{,}000{,}000 = \$\,7{,}500{,}000$$

However, if Yallop had used a futures strategy with $Q_f = 10{,}000{,}000$ bushels, from the very beginning, his final payoff would have been

$$2{,}500{,}000 \times (1.001)^2 + 2{,}500{,}000 \times (1.001) + 2{,}500{,}000 = 7{,}507{,}503$$

Thus in this case, he would have overhedged. If the futures price had gone up instead of going down, overhedging would have lead to a bigger loss.

In practice, tailing will be feasible only if you have a large futures position. For instance, if Q_f had been 10,000 in the above case, then tailing would not have been possible, because you cannot buy or sell a fraction of a contract. The difference between Q_{f1} or Q_{f2} and Q_f would depend on the time left to expiration of the hedge and on the interest rate. The longer the time left or the greater the interest rate, the more will be the required adjustment.

It must be remembered that tailing is not free of costs. Every time the number of futures contracts is adjusted, there will be transactions costs. Hence you must weigh the pros and cons before deciding to tail. In real life, hedgers do not usually adjust their futures positions daily, that is, every time the contract is marked to market. Periodic adjustments are found to be adequate.

3.15 Hedging Processing Margins

In commodity markets, firms which process the raw materials into finished products, can hedge their gross refining margins or the difference between the price of the outputs and the cost of the inputs, by using futures contracts. Two common examples of such hedges are the Crack Spread in petroleum markets and the Crush Spread in the soybean market.

The Crack Spread

The crack spread is the differential between the price of crude oil and prices of the products that are generated from it. Put differently it is the processing margin for an oil refinery. The term *'cracking'* refers to the generation of shorter chain hydrocarbon products from the large chain hydrocarbons present in crude oil.

A barrel of crude oil generates many products such as gasoline, kerosene, diesel, heating oil, aviation fuel, and asphalt among other products. The mix of refined products can to an extent be customized to meet the requirements of the local market. Differences in demand depend on the relative need of fuel for purposes such as heating, cooking and transportation. The relative demand is also impacted by seasonal factors. The mix is also a function of the type of crude oil that is processed by a particular refinery as well as its refining capabilities.

The two major products of cracking are heating oil and gasoline. To keep matters simple, we will assume that the prices of other by products are highly correlated with those of heating oil and gasoline, and that therefore, their revenues can be hedged using futures contracts on heating oil and gasoline. The gross processing margin will obviously depend on the price of crude oil, the amount of heating oil and gasoline produced by each barrel of crude oil and the prices of the finished products. In practice refiners use crack ratios expressed as X:Y:Z where X is the number of barrels of the input, namely crude oil, and Y and Z are the number of barrels of gasoline and heating oil respectively, which are the primary outputs. Obviously X = Y + Z. Widely used spreads are 3:2:1, 5:3:2, and 2:1:1.

Exchanges have tailor-made options for trading crack spreads. The New York Mercantile Exchange (NYMEX) offers what is virtually a futures contract on the crack spread by treating the underlying contracts corresponding to a spread as a single transaction. The obvious benefit is that traders need only be concerned with the margin requirement for the spread as a whole, and not for the individual underlying positions.

We will illustrate in detail the use of a 3:2:1 spread. That is, we will assume that each barrel of oil, which consists of 42 gallons, can produce 14 gallons of heating oil, and 28 gallons of gasoline. Thus the gross refining margin, under our assumption, is equal to

14 × price per gallon of heating oil + 28 × price per gallon of gasoline − price per barrel of crude oil.

Consider the following time line.

Figure 3.1

We will assume that the futures contracts are entered into at time **t**. The crude oil is procured at time **t1** and the refined products are sold at time **t2**. **t2 – t1** represents the refining time. The gross refining margin per barrel of crude oil is

$$14S^{h}{}_{t2} + 28S^{g}{}_{t2} - S^{c}{}_{t1} \quad (3.30)$$

where the symbol S denotes the spot price and the superscripts h, g and c denote heating oil, gasoline and crude oil respectively.

Each crude oil futures contract is for 1,000 barrels or 42,000 gallons. Under our assumption, when the crude is refined, we will get 14,000 gallons of heating oil, and 28,000 gallons of gasoline. Thus corresponding to each crude oil contract, we require $\frac{14}{42}$ or $\frac{1}{3}$ heating oil contracts and $\frac{28}{42}$ or $\frac{2}{3}$ gasoline contracts. This ratio of 42:28:14 or 3:2:1 is called a 3-2-1 Crack Spread.

Consider the case of a refiner who wants to hedge the margin equivalent to 30,000 barrels of crude oil. He must therefore go long in 30 crude oil contracts, since he is going to procure the crude and simultaneously go short in 10 heating oil contracts and 20 gasoline contracts, to hedge the revenues from the sale of the finished products.[6]

The hedged refining margin is therefore,

$$\begin{aligned} &30{,}000 \times \text{ margin per barrel } + \text{ profit/loss on futures position} \\ &\quad = 30{,}000[14S^h_{t2} + 28S^g_{t2} - S^c_{t1}] + 10 \times 42{,}000\,(F^h_t - F^h_{t2}) \\ &\quad\quad +20 \times 42{,}000\,(F^g_t - F^g_{t2}) + 30 \times 1{,}000\,(F^c_{t1} - F^c_t) \end{aligned} \tag{3.31}$$

Notice one feature about the above equation. We are multiplying the change in the futures price by 42,000 in the case of heating oil and gasoline contracts but by 1,000 in the case of crude oil contracts. This is because futures prices for heating oil and gasoline are quoted on a per gallon basis, whereas prices for crude oil are quoted on a per barrel basis. As already mentioned, each contract is for 1,000 barrels and each barrel is equivalent to 42 gallons.

At the time of expiration of the futures contracts, the futures prices must converge to the spot prices. Therefore,

$S^h_{t2} = F^h_{t2}$, $S^g_{t2} = F^g_{t2}$, and $S^c_{t1} = F^c_{t1}$ Therefore, the hedged gross refining margin is

$$30{,}000[14F^h_t + 28F^g_t - F^c_t] \tag{3.32}$$

or $14F^h_t + 28F^g_t - F^c_t$ per barrel.

Example The following futures prices are observable.

July, 2008 crude oil: \$ 130.25/barrel.

August, 2008 heating oil: \$ 4.025/gallon.

August, 2008 gasoline: \$ 3.4125/gallon.

The 3-2-1 crack spread is therefore

$$14 \times 4.025 + 28 \times 3.4125 - 130.25 = \$21.65/\text{barrel}.$$

The Crush Spread

The Crush Spread refers to the gross processing margin in the case of soybeans, and is similar in principle to the crack spread.

[6]Remember that the ratio is 3:2:1, and futures contracts for all the three products are for 1000 barrels per contract.

A bushel of soybeans under normal conditions can be assumed to yield 11 pounds of oil, 48 pounds of soybean meal and one pound of hulls and waste. The hulls and waste are considered to be a processing loss and hence are valueless. Each bushel consists of 60 pounds of soybeans and 2,000 pounds constitute a metric ton. The price of soybean oil is quoted in terms of pounds, while that of soybean meal is quoted in terms of tons. Soybean prices are quoted in bushels.

So when a bushel of soybeans is crushed, the output is 11 pounds of oil and $\frac{48}{2{,}000} = .024$ tons of soybean meal. Thus the gross processing margin per bushel of soybeans is

11 × price per pound of oil + .024 × price per ton of meal − price per bushel of raw soybeans

$$= 11S^o_{t2} + .024S^m_{t2} - S^s_{t1} \tag{3.33}$$

where the superscripts o, m and s, represent oil, meal, and raw soybeans respectively. The processing margin can be hedged by going long in raw soybean contracts expiring at $t1$ and going short in oil and meal futures expiring at $t2$.

Each soybean futures contract is for 5,000 bushels of raw soybeans, whereas each soybean oil futures contract is for 60,000 pounds of oil. Futures contracts on soybean meal are for 100 tons.

5,000 bushels of raw soybeans will produce 55,000 pounds of oil, which will require $\frac{55{,}000}{60{,}000} = \frac{110}{120}$ contracts to hedge. The same quantity will also produce $5{,}000 \times .024 = 120$ tons of soybean meal, which will require $\frac{144}{120}$ contracts to hedge. Thus in order to hedge the processing margin, for every 120 soybean contracts that the producer goes long in, he must go short in 110 soybean oil contracts, and 144 soybean meal contracts.

The hedged processing margin for 5,000 bushels is therefore

$$\begin{aligned} 5{,}000[11S^o_{t2} + .024S^m_{t2} - S^s_{t1}] &+ \frac{110}{120} \times 60{,}000(F^o_t - F^o_{t2}) \\ &+ \frac{144}{120} \times 100(F^m_t - F^m_{t2}) + 5000(F^s_{t1} - F^s_t) \\ &= 5{,}000[11F^o_t + .024F^m_t - F^s_t] \\ &\equiv [11F^o_t + .024F^m_t - F^s_t] \text{ per bushel} \end{aligned} \tag{3.34}$$

Example The following prices are observable in the market.

August, 2008 Soybean futures: \$ 13.60.

September, 2008 Soybean Oil futures: \$ 0.6400.

September, 2008 Soybean Meal futures: \$ 332.15.

The crush spread = $11 \times 0.6400 + .024 \times 332.15 - 13.60 = \$1.4116$/bushel.

3.16 Speculation

Hedgers, as we have seen thus far, are investors who seek to mitigate if not eliminate price risk. Speculators, on the other hand, have a radically different attitude towards risk. Such investors, by definition, are people who consciously seek to take risk, hoping to profit from subsequent price movements. A speculator may be betting on a rising market, in which case he would be labeled as being bullish, or else he may be betting on a declining market, in which case he would be termed as being bearish. Contrary to what a lay person may believe, speculation is not tantamount to gambling.

Finance theory makes a clearcut distinction between speculation and gambling. Speculators are calculated risk takers. That is, before assuming a position in the market, such investors will evaluate the risk of an investment in its entirety, and weigh it against the anticipated return from it, prior to taking a position. Such investors will therefore take a position only if they were to be of the opinion, based on their analyses, that the anticipated return associated with the investment is adequate considering the risk that is being assumed. A gambler on the other hand will not make such an ex-ante risk-return tradeoff, and consequently is someone who takes a risk purely for the sheer thrill of taking it. The focus of such traders is solely on the associated risk, and the expected return is of no consequence while taking a decision to gamble.

Active speculation is a sine qua non for a deep and liquid market. A market characterized solely by the activities of hedgers, will in general not have the kind of trading volumes required to make it attractive. Very often, in practice, when a hedger seeks to take a position, the party taking a counter-position will be a speculator, although it must be clarified that he may also be a hedger albeit with an opposite view of the market. The success of the free market system critically hinges on the ability of market participants to actively trade based on their divergent points of view. Thus speculators play as important a role as hedgers and arbitrageurs.

3.17 Speculation and Futures

Let us take the case of an investor who is bullish about the market and who seeks to make a profit in the event of his price expectations being realized. One way of operationalizing his views, would be by taking a long position in the spot market. If his view of the future were to be realized, he can obviously offset his spot position at a potentially handsome profit.

However, such a strategy would entail incurring substantial costs. In addition, in the case of commodities, as opposed to financial assets, the investor has to incur costs associated with storage and possibly insurance. Such expenditure can be avoided if futures contracts were to be used for speculation. If the investor were to assume a long futures position, then he can lock in a price at which he can acquire the asset. In the event of his views being vindicated, he can take delivery at expiration, and offload the asset in the spot market at a price which by assumption is higher. The advantage of using futures contracts as a vehicle for such a speculative strategy is that the entire value of the asset need not be paid at the outset. These contracts require that a relatively small margin be deposited, or in other words offer leverage to the investor.

Numerical Illustration

Futures contracts on corn with one month to expiration are available at a price of \$ 5.25 per bushel. Andrew, a speculator, believes that the spot price after a month will be in excess of \$ 5.50 per bushel. Let us assume that he chooses to speculate by going long in 100 futures contracts, each of which is for 5,000 bushels.

If his hunch is correct, and the market were to rise, he can exit with a profit. Let us assume that the spot price after a month is \$ 5.80 per bushel. If so, Andrew can make a profit of:

$$100 \times 5{,}000 \times (5.80 - 5.25) = \$\ 275{,}000$$

There is however a risky dimension to this strategy. What if Andrew is wrong about the future, and that the price after a month is only \$ 4.75 per bushel. If so, he would have to acquire the corn at \$ 5.25 per bushel and sell it in the spot market at \$ 4.75.[7] The loss in this case will be

$$100 \times 5{,}000 \times (4.75 - 5.25) = -\$\ 250{,}000$$

Thus speculation using futures can be extremely profitable if one were to read the future state of the market correctly. However, in the event of an error of judgment, the losses can be substantial.

Bearish investors too can speculate using futures contracts. However, they would need to take a short position in order to do so. If their hunch turns out to be correct, and the market price were to decline, they can buy at the prevailing market price and sell it at the delivery price of the contract, which by assumption is higher.

Numerical Illustration

Unlike Andrew, Nick who is also a speculator, is of the opinion that after a month, the spot price of corn would have declined to below \$ 5.00 per bushel. He therefore decides to take a short position in 100 futures contracts.

If he were to be right and the market were to fall to a level of \$ 4.80 per bushel, he would end up making a profit of

$$100 \times 5{,}000 \times (5.25 - 4.80) = \$\ 225{,}000$$

However, if he were to read the market incorrectly, he faces the specter of a significant loss. Let us assume that the market price after a month is \$ 5.70 per bushel. In such a situation, Nick would incur a loss of

$$100 \times 5{,}000 \times (5.25 - 5.70) = -\$\ 225{,}000$$

Thus bears like bulls can use futures as a tool for speculation. However, they are advised to do so after taking cognizance of the fact that in their quest for substantial gains, there is always a risk that they could make substantial losses.

[7] Remember, a long futures position represents a commitment to buy.

3.18 Speculation and Options

Options can also be used as a speculative vehicle by investors. Bullish speculators can take a long position in call options or a short position in put options. If they were to use calls, and their hunch about the market turns out to be correct, they can exercise the options, acquire the asset at the exercise price, and sell it profitably at the prevailing market price, which by assumption is higher. If they were to use puts, and the market price at expiration were to exceed the strike price, then the counter-party will not exercise, and they can retain the premium they would have received at the outset, which constitutes the profit in this case.

Numerical Illustration

Call options on GE expiring after one month are available with an exercise price of \$ 45. The premium is \$ 1.95 per share. Each contract is for 100 shares. Nigel is bullish about the market and consequently takes a long position in 100 call option contracts.

Let us assume that his hunch about the market is right and that the price of GE after 30 days is \$ 49.50 per share. He can then exercise the options, acquire the shares at \$ 45 per share and immediately dispose them off for \$ 49.50 per share. After factoring in the premium paid at the outset, he will make a profit of

$$100 \times 100 \times (49.50 - 45) - 100 \times 100 \times 1.95 = \$\ 25{,}500$$

It may however turn out that he has read the market incorrectly, and that the spot price after a month is \$ 42.50 per share. In such a situation, he will simply refrain from exercising the options, since they constitute a right and not an obligation. The loss will be equal to the premium paid, which is \$ 19,500.

Similarly, bearish investors too can use options to speculate, either by buying put options, or else by writing call options. If the market does indeed fall as anticipated, and a long position in puts has been assumed, the investor can acquire the asset at the market price, and sell it at the strike price by exercising the option, thereby making a profit.

Numerical Illustration

Put options on GE with one month to expiration are available with an exercise price of \$ 45 per share. Each contract, is for 100 shares and the premium is \$ 0.90 per share. Nancy is bearish about the market and therefore decides to go long in 100 put option contracts.

If she were to read the market correctly, and the price does indeed decline to say to \$ 42 per share, she can acquire the shares in the spot market and deliver it under the contract at the exercise price of \$ 45. After factoring in the premium, her profit would be:

$$100 \times 100 \times (45 - 42) - 100 \times 100 \times 0.90 = \$\ 21{,}000$$

However, if her prediction about the market were to turn out to be wrong and the spot price after a month happens to be \$ 48 per share, then she will simply refrain from exercising the option. In this case, the option premium of $ 9,000 would constitute a loss.

Another way to speculate on a bearish market is by writing call options. If the investor is correct and the market price were to fall below the exercise price, then the counter-party will not exercise and the speculator can retain the premium that he receives at the outset.

3.19 Interchangeable?

Once again a question will arise in the minds of most readers. Are futures and options similar from the standpoint of speculation? And the answer, similar to our conclusion on hedging using futures and options, is once again no. Let us first take the case of the speculator who assumes a long position in a futures contract. If he reads the market correctly, and the market does indeed rise, he can make a substantial profit as we have just seen. However, there is every possibility that he could be wrong. If the market were to actually fall, he will make a loss which may be substantial, because the futures contract imposes an obligation on him to buy the underlying asset at the delivery price, and having done so, he would have to dispose it off in the spot market at a price which could be considerably lower. Similarly, a speculator who goes short in a futures contract, also faces the possibility of incurring substantial losses if his forecast of the market turns out to be wrong. This is because if the market were to rise, he would have no choice but to acquire the asset in the spot market and deliver it at the delivery price, as per the requirements of the contract.

The payoff possibilities in the case of speculative strategies using options contracts are clearly different. An investor who chooses to speculate by going long in call options, can make substantial profits if the market were to rise. However, if he were to be wrong and the market were to fall subsequently, he can simply refrain from exercising the option and let the contract expire worthless. This is feasible because an options contract gives a right to the holder and does not impose an obligation. In this case, his loss will be limited to the option premium that was paid at the outset. Speculating by buying put options is similar. If the speculator is right and there is a subsequent market decline, then he stands to benefit for he can acquire the asset at the spot price and sell it at the strike price. However, if it turns out that he was wrong, and the market were to subsequently rise, he can once again refrain from exercising the option.

Does this mean that all speculators will prefer to employ options to operationalize their strategies? The answer, as can be deduced from the logic presented thus far is no. This is because, while one can speculate using futures contracts by depositing a relatively small margin, speculation using long positions in options entails the upfront payment of a premium, which is irrecoverable if the option were not to be exercised. Options and futures are thus not interchangeable from the standpoint of speculation, and we cannot state that one strategy dominates the other.

Speculation by writing calls and puts is also different from speculating using futures contracts. A speculator who chooses to write options, whether calls or puts, cannot make a profit in excess of the premium that he receives at the outset, for the best thing that can happen from his perspective is that the contracts are not exercised subsequently by the counter-party. However, the maximum loss for such strategies can be substantial, particularly in the case of short positions in calls, for the asset may have to be acquired for delivery at a price which theoretically has no upper bound.

3.20 Developing Derivative Exchanges: Key Issues

Now that we have had a comprehensive look at the mechanics of futures markets, the valuation of futures contracts and risk management using futures contracts, we are in a position to discuss some of the salient features pertaining to the development of derivative exchanges, particularly from the standpoint of emerging markets.

As Tsetsekos and Varangis (2000) point out, derivative exchanges tend to be established for sound economic reasons as well as for national pride. A well functioning derivatives exchange facilitates the correct allocation of resources and the transfer of risk from those who seek to avoid it, to those who want to take it on. As Tsetsekos and Varangis (2000) point out, in many emerging markets, price discovery is difficult. Derivative markets tend to be more liquid and consequently information on supply and demand percolates faster into such markets. The result is more accuracy of prices in spot markets because as you have seen, spot and futures markets are inextricably linked by the process of arbitrage. Therefore, derivative markets have the potential to help economic agents in emerging markets to make better decisions about future production and trade.

However, in order to have an efficient derivatives exchange, it is imperative to have a cash market that is efficient and which attracts large volumes. It is also essential to have a sound legal system that facilitates enforcement of contracts between parties, as well as a well developed credit market.

A survey of derivatives exchanges undertaken by Tsetsekos and Varangis (1998), reveals that exchanges in emerging markets tend to introduce financial derivatives, rather than commodity derivatives. The main reason is that with liberalization and globalization, prices of commodities in the domestic markets of such countries are no longer insulated from international trends and price movements. Consequently, there tends to be a strong correlation between domestic spot prices in the emerging markets and futures prices on well established international exchanges in the developed countries. In other words, there is less basis risk while using derivative contracts on established exchanges to hedge, as compared to the pre-liberalization era. Hence, the need to develop domestic commodity derivatives exchanges is no longer being felt to the same extent.

Financial products like stocks and bonds, however, tend to be country specific and hence attract interest mainly from domestic investors. Similarly, for obvious reasons, currency derivatives are also country specific. Consequently, markets for financial derivatives have become popular in the emerging markets.

A thorough study of the feasibility of derivatives exchanges is a pre-requisite before trading is allowed to commence if such exchanges are to succeed in the developing economies. Tsetsekos and Varangis (2000) list five important lessons based on research and the experiences of various countries.

1. The chances of success of derivatives exchanges are higher if stock index and interest rate derivatives are introduced first.

   Derivatives on agricultural commodities are more difficult to introduce, either because liberalization of agricultural markets in developing countries is more difficult than the liberalization of financial markets or else because the post liberalization correlation between domestic and foreign markets is high enough to permit international exchanges to be used by domestic investors. In general, commodity exchanges have succeeded only in countries with a large domestic market for commodities, like Argentina, Brazil, China, and Malaysia.
2. It is very important to have a proper regulatory and legal framework in place. Uncertainty about the just and fair application of laws and regulations can keep potential investors away. Needless to say, a history of scandals and market failures can spell doom for a developing financial market.
3. Partnerships and joint ventures between new and existing exchanges may be mutually beneficial. Emerging exchanges stand to benefit from the experience and technological knowhow of established exchanges. At the same time, the growth of active trading in emerging markets is in the interest of traders in the developed countries, as the globalization process forges greater and closer links between economies.
4. Policymakers in emerging markets should permit domestic investors greater and relatively unfettered access to foreign markets and foreign investors access to domestic markets. If the barriers to trading abroad are removed, then it will facilitate arbitrage across borders, thereby making domestic exchanges more liquid.
5. Electronic trading, as we have discussed earlier, seems to be the preferred mode of trading in the new exchanges, as opposed to the open outcry system.

Suggestions for Further Reading

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CONCEPT CHECK

State whether the following statements are True or False.

1. A short hedger is long the basis.
2. The higher the correlation between the spot price and the futures price, the lower will be the basis risk.
3. In a contango market, a narrowing basis would lead to a loss for the short hedger.
4. An overpriced contract will benefit a long hedger.
5. The minimum variance hedge ratio will always be less than or equal to 1.0.
6. Hedging effectiveness can be measured by the R^2 of a regression, with $\triangle F$ as the independent variable and $\triangle S$ as the dependent variable.
7. In practice, when you tail your hedge, you would like to do so each time the contract is marked to market.
8. If you do not tail your hedge, then you will always be overhedged.
9. At the time of expiration of the futures contract, the basis will always be zero.
10. The terms widening and narrowing of the basis refer to changes in the absolute value of the basis.
11. Futures and options are interchangeable from the standpoint of speculation.
12. Maximum loss for a speculator who goes long in calls or puts is equal to the premium paid at the outset.
13. Speculation is synonymous with gambling.
14. The estimated hedging effectiveness will always be between 0 and 1.0.

15. The general rule is that the hedger should choose a contract with an expiration month that is as close as possible to the month in which the hedge is being terminated, but after it.
16. Hedging replaces price risk with basis risk.
17. A perfect hedge is feasible only if the contract is delivery settled.
18. A futures hedge may lead to ex-post regret.
19. An options hedge cannot lead to ex-post regret.
20. A selling hedge requires a long position in the futures market.

QUESTIONS AND PROBLEMS

Question-I

What is Basis Risk? What are the factors that give rise to it?

Question-II

Explain why a long hedger would be uncomfortable holding on to a futures contract in its expiration month.

Question-III

What are the factors that you would consider while selecting a futures contract for hedging? Explain each issue in detail.

Question-IV

What does 'Tailing a Hedge' mean? In practice what are the issues to be considered before Tailing?

Question-V

Explain how the effectiveness of a hedge can be computed using empirical data.

Question-VI

'Most derivatives exchanges in emerging markets, tend to introduce contracts on financial products, rather than on agricultural commodities.' Comment.

Question-VII

Consider the following data.

Month	ΔF	ΔS
1	.1525	.2050
2	.1375	.3050
3	.1200	.3000
4	−.0600	.0200
5	−.1100	−.2000
6	−.1500	−.1115
7	.2000	.3555
8	.1500	.4500
9	−.1000	.0400
10	−.1800	−.2015

Steven Alter is currently holding 10,000,000 bushels of wheat which he wants to sell after one month and wants to hedge using futures contracts. Each contract is for 5,000 bushels of wheat.

1. Should Steven go in for a short hedge or a long hedge? Explain.
2. If Steven decides to set up a risk minimizing hedge, how many futures contracts should he use?

   Note: Take all variables to four places of decimal, except for the final answer, which may be rounded off to the nearest whole number.
3. Assume that Steven decides to hedge using a hedge ratio of 1.0, but decides to tail his hedge because of the marking to market feature. Let the daily interest rate be .2%. Assume that he gets into the contract when it has three days left to maturity. The futures prices on these days are as follows.

Day	Futures Price
0	4.20
1	4.00
2	4.10
3	4.00

Describe the tailing procedure that he would adopt. Calculate the profit/loss as of the day of expiration.

Question-VIII

A farmer wants to hedge 72,000 bushels of wheat using futures contracts. Each contract is for 5,000 bushels of wheat. If the date of termination of the hedge is the same as the expiration date of the futures contract, show why the hedge will nevertheless not be perfect in practice.

Question-IX

Michael Smith, a wheat mill owner, wants to procure 2 million bushels of wheat on March 31, 2006. After analyzing past data, he has come to the conclusion, that a hedge ratio of 1.20 is appropriate. Assume that today is May 15, 2005 and that wheat futures contracts are available with expiration dates in July, October, January and April. Although April 2006 contracts are available on May 15, they are not liquid and consequently Michael decides to initiate a position in the July contract and then rollover into October, followed by January, followed by April. The market prices for the various contracts are as follows. Each contract is for 5,000 bushels of wheat.

Date	Contract	Price
May 15, 2005	July, 2005	4.25
June 30, 2005	July, 2005	4.75
June 30, 2005	October 2005	4.40
September 30, 2005	October 2005	4.00
September 30, 2005	January 2006	3.50
December 31, 2005	January 2006	3.95
December 31, 2005	April 2006	3.75
March 31, 2006	April 2006	4.20
March 31, 2006	Spot	4.50

What is the effective price paid by Michael per bushel of wheat?

Question-X

Part (a) Consider a 2-1-1 Crack Spread. That is, assume that every barrel of crude oil (42 gallons) produces 21 gallons each of heating oil and gasoline. Assume that the futures contracts are entered into at t_0, that the crude oil is bought at t_1 and that the finished product is sold at t_2. The prices of gasoline and heating oil are quoted on a per gallon basis, whereas the price of crude oil is quoted on a per barrel basis. Futures contracts on crude oil, as well as on the refined products, are for 1,000 barrels each.

Logically derive the expression for the Gross Refining Margin per barrel.

Part (b) The following futures prices are observed on September 15, 20XX.

Heating Oil: \$3.9725 per gallon
Gasoline: \$3.2125 per gallon
Crude Oil: \$ 128.75 per barrel
Calculate the 2-1-1 Crack Spread per barrel.

4

Orders and Exchanges

4.1 Introduction

An order is an instruction to trade that is given by a party who wishes to take a position in an asset. If he wishes to establish a long position, he will issue a buy order, whereas if he were to desire to go short, he will place a sell order. At the time of placing an order the security in which the investor wishes to take a position must be clearly identified. This is fairly simple in the case of stocks because most companies usually issue only one type of shares. However, in the case of futures contracts, each expiration month for an asset constitutes a different security. In the case of an options contract too the specifications should be precise, for each combination of an expiration month and exercise price, constitutes a different asset.

The quantity in which a long or short position is sought to be taken must be clearly spelt out. Most stocks used to trade in what were known as round lots or board lots, where each lot was usually for 100 shares. However with the advent of *dematerialized* securities or *scripless trading*, this requirement has been dispensed with. The quantity that is specified in an order is known as the *'Order Size'*.

The price at which an investor is willing to transact is obviously a key feature of the order specification. There are two possibilities. There are investors who will accept any price that a market may offer at the point in time at which the order is placed. Such investors will place what are known as *'Market Orders'*. However, there are others who may have a floor or a ceiling in mind. Traders placing buy orders may seek to specify a price ceiling. That is, there is a maximum price that they are prepared to pay. On the other hand, those placing sell orders may wish to specify a price floor. That is, there is a minimum price below which they would not like to transact. Traders who specify a floor or a ceiling for the price place what are known as *'Limit Orders'*. Obviously in the case of market orders, the only parameter that has to be specified is the order size. However, in the case of limit orders in addition to the quantity, the limit price has also to be specified.

Traders are also required to specify the period of time for which they wish their orders to remain valid. Many exchanges allow only *'good today limit orders'* or what are known as *'Day Orders'*. That is, an order is valid only until the close of trading on the day on which it is placed. If it were to fail to get executed on that day, it would automatically stand canceled. Other exchanges may permit orders to be carried forward. Even in such cases the exchange will specify a maximum validity period.

Some traders specify that their orders should be executed on placement or else should be canceled immediately. These are known as *'Fill or Kill'*, *Immediate or Cancel*, or *'Good on Sight'* orders. Others will allow their order to be filled only if the trade results in the fulfillment of the entire quantity that has been specified. That is, they will not accept a partial match. Such orders are known as *'All-or-None'* orders. Such orders must be limit orders and must be filled at one price.

4.1.1 An Illustration

George Elliot a trader in Kansas City has placed a buy order for 200 March 2009 futures contracts on corn. He has specified a price limit of \$ 5.40 per bushel. The order specification states that it is *Good-This-Week*. The asset underlying the order is obviously the March 2009 corn futures contract. The order size is 200 contracts. It is a limit order with a limit price of \$ 5.40 per bushel. Since it is a buy order, the limit price represents an upper limit or price ceiling. The order is valid for the week in which it is placed. Quite obviously the exchange in this case allows orders with a validity of more than one day.

4.2 Market Orders and Limit Orders

Traders who place market orders are content with the best price that the market has to offer. Obviously the trading system should be structured in such a way that an order is executed at the best available price from the standpoint of the trader. What will be the best matching order for a market buy order? It will be the limit sell order with the lowest limit price among all the sell orders that are available when the buy order enters the system. Similarly market sell orders too should be executed at the best available price. In this case, the best matching order will be the limit buy order with the highest specified limit price among all the buy orders that are available when the sell order enters the system.

To ensure that market orders are executed at the best available price from the standpoint of the trader, all limit orders which are unexecuted must be sorted in accordance with certain priority rules. The primary priority rule is the *'Price Priority Rule'*. All limit buy orders that are pending execution are sorted in descending order of price, and all limit sell orders that are awaiting execution are sorted in ascending order of price. Thus an incoming market buy order is assured of execution at the lowest available price on the sell side, whereas an incoming market sell order is guaranteed to be executed at the highest available price on the buy side.

The next issue is what happens when two or more limit orders have the same specified price limit. Here the exchanges invoke the *'Time Priority Rule'*. That is for a given limit price, the orders are arranged in the order in which they are received. That is, an order which came in earlier would be accorded priority over an order that came later. Some exchanges also use order size as a priority rule.

These priority rules are easily enforceable in an electronic or screen-based system, where everything is coded into the software. Later on, we will discuss as to how these rules are enforced in the open-outcry system of trading, where traders crowd around a trading ring or pit.

4.2.1 Limit Order Books

All unexecuted but valid orders need to be stored in the system till they can be matched with an incoming order. The record of such orders is referred to as the *'Limit Order Book'*. Prior to the advent of electronic trading, the record was physically maintained in the form of a book of orders. In modern systems everything is maintained in an electronic form.

Illustration of a Limit Order Book Consider the following Limit Order Book (LOB) at a given point in time. The book represents the situation for futures contracts on a stock, which are scheduled to expire three months later, as of 11:00 A.M on the morning of January 3, 2009.

Table 4.1 Snapshot of the LOB at a Given Point in Time

Buyers			Sellers		
Trader	**Order Size**	**Limit Price**	**Limit Price**	**Order Size**	**Trader**
Alfred	100	99.70	100.00	200	Jack
Betty	200	99.60	100.15	300	Keith
Carol	200	99.50	100.25	500	Larry
David	500	99.45	100.40	700	Mike
Eric	500	99.40	100.50	1,000	Nancy
Fred	1,000	99.25	100.60	1,500	Peter
Harry	1,000	99.15	100.75	1,500	Robby
Irene	2,000	99.00	100.90	2,000	Steve
Tom	3,000	99.00	101.00	2,000	Victor

As can be seen, on the buy side the orders have been arranged in descending order of price, whereas on the sell side they have been arranged in ascending order of price. On the buy side both Irene and Tom have given a limit price of 99.00. Obviously, Irene's order was placed before Tom's.

The best price on the buy side is 99.70. That is, an incoming market sell order with a size of 100 or less, will be filled at 99.70. What if the incoming order has a size of 250? In this case, 100 contracts will be filled at 99.70 and the remaining 150 at 99.60. Similarly, the best price on the sell side is 100.00. The bid-ask spread or the difference between the best ask and the best bid is currently $ 0.30.

4.2.2 Price Control versus Time Control

Traders who place limit orders are able to control the price at which their order will be executed. Take the case of Irene who has placed a buy order at 99.00. She can be assured that she will have to pay a price no higher than the specified limit price, in the event of her order being executed. However, she has no control over the length of time that is taken to execute her order. It is conceivable that it could take an inordinately long time for her order to get executed, and there could be circumstances in which her order fails to get executed.

Let us take the case of Xavier who observes the limit order book depicted in Table 4.1 Assume that he would like to buy 400 contracts and places a limit buy order at 99.20. It will find a place between Fred's order and Harry's order on the buy side. Now assume that a series of market buy orders keep entering the system. The price will gradually get pushed upwards. In such a situation there could be considerable delays in the execution of limit buy orders. In the event of the exchange accepting only day orders, an order like Xavier's may not get executed during the course of the day and consequently will have to be canceled.

4.3 Marketable Limit Orders

Take the case of an investor who is contemplating the placement of a buy order for 200 contracts. If he observes the LOB shown in Table 4.1 he will see that the best available price is 100.00. If he feels that this is less than or equal to what he would like to pay, he would place a market order hoping that it will get executed at 100.00. However, if he feels that this price is too high, he will place a limit buy order at a price of say 99.80. Thus limit buy orders are usually placed at a price that is lower than the best available price in the market, which is nothing but the best price on the sell side. Similarly, limit sell orders will usually be placed at a price that is higher than that of the best bid in the market.

There could be situations however, where a trader prices his limit order very aggressively. How will such aggression manifest itself? On the buy side, the higher the specified limit price, the more aggressive is the order. On the other hand, on the sell side, the lower the limit price the more aggressive is the order.

Marketable limit orders by definition are those limit orders that are likely to be executed upon submission. In the situation depicted in Table 4.1, a limit buy order with a limit price greater than or equal to 100.00 will be classified as a marketable limit order. Similarly a limit sell order with a limit price of 99.70 or less will also be classified as a marketable limit order. Thus the price for a marketable buy order is greater than or equal to the best ask, whereas the price for a marketable sell order is less than or equal to the best bid.

Why would anyone wish to place a marketable limit order when he can place a market order? Assume that an investor called William places a limit buy order with a price of 100.05 for 200 contracts. His anticipation is that it will be matched with the best sell order and that a trade will result. However, what if a market buy order for 200 contracts were to enter the system before the limit order is placed? If so, the best ask after the execution of this order will be 100.15 and not 100.00. Since the bid given by William is lower than 100.15, it will stay in the book in the form of a limit buy order with a limit price of 100.05. However, if William had given a market order and the same sequence of events were to occur, his order would be matched with the sell order at 100.15, and a trade would result at that price.

Marketable limit orders are limit orders which represent a desire for quick execution on the part of the trader. However, in the event of things not working out as planned, the trader still retains an element of control over the price. Market orders as discussed earlier are usually guaranteed to get executed on submission. However, they do not offer any control over the price. Thus, if time is of the essence, the trader will choose a market order, whereas if price is of the essence he will choose a limit order. If speed is important and price is not irrelevant, he may prefer a marketable limit order to a market order. However, it must be remembered that limit orders, whether marketable or otherwise, are subject to execution uncertainty.

4.4 Trade Pricing Rules

The incoming order is referred to as an *'active order'* whereas the orders that are already present in the system are referred to as *'passive orders'*. The trade price is the limit price of the passive order with which the active order is matched.

Consider the LOB depicted in Table 4.1 Assume that William places a market buy order for 200 contracts. It will get matched with Jack's order which is the passive order in this case. Thus the trade will occur at a price of 100.00 which is the limit price corresponding to the passive order.

What would happen if William were to issue a marketable limit order with a limit price of 100.05? Once again the order will be matched with Jack's, and a trade will result at the passive price of 100.00

An interesting situation arises when a market order enters the system and the opposite side of the LOB is empty. If we were to wait until a limit order appears on the other side, the passive price rule cannot be applied since the passive order in this case is a market order. Consequently, in such situations, the market order is usually converted to a limit order with a limit price equal to the last traded price. Subsequently, if it were to be matched with an order on the opposite side, a trade will result at this assigned limit price.

4.4.1 An Illustration

Consider the LOB depicted in Table 4.2. The last trade price was 99.90.

Assume that William gives a market buy order for 200 contracts. Since there is no order on the opposite side, William's order will be converted into a limit buy order with a limit price of 99.90 and will take its place at the top of the buy side of the LOB. Subsequently if a market sell order for 200 contracts were to appear, a trade will result at 99.90.

Table 4.2 Snapshot of the LOB at a Given Point in Time

Buyers			Sellers		
Trader	**Order Size**	**Limit Price**	**Limit Price**	**Order Size**	**Trader**
Alfred	100	99.70			
Betty	200	99.60			
Carol	200	99.50			
David	500	99.45			
Eric	500	99.40			
Fred	1,000	99.25			
Harry	1,000	99.15			
Irene	2,000	99.00			
Tom	3,000	99.00			

4.5 Stop-Loss and Stop-Limit Orders

Stop or stop-loss orders are placed by traders who already have a position in the market. Usually at the time of order placement the trader has no desire to unwind his existing position. However, if market conditions were to suddenly turn adverse the trader may have a threshold price beyond which he is unwilling to tolerate losses. This desire to cap the trader's loss is why such orders are also termed as stop-loss orders.

Let us take the case of William who is long 200 contracts in corn futures and observes the following LOB.

Table 4.3 Snapshot of the LOB at a Given Point in Time

Buyers			Sellers		
Trader	**Order Size**	**Limit Price**	**Limit Price**	**Order Size**	**Trader**
Alfred	100	99.70	100.00	200	Jack
Carol	200	99.50	100.25	500	Larry
David	500	99.45	100.40	700	Mike
Fred	1,000	99.25	100.60	1,500	Peter
Tom	3,000	97.50	101.00	2,000	Victor

The best price available for a trader like William, if he wishes to offset, is 99.70. If he feels that this itself is low enough to stimulate him to cut his losses he may as well place a market sell order and exit the market. The very fact that he is contemplating the placement of a stop sell order signifies that his threshold is lower. Assume that his threshold is 99.45. If so he could place a stop sell order with this threshold price. The order will go into the stop-loss book and will remain dormant until the trigger is hit or breached. If this were to happen, it will get triggered off and will become a market sell order.

Assume that a market sell order for 400 contracts enters the system. The trade price will hit 99.45. Immediately William's order will get triggered off and will get executed at 99.45.

Now consider another case where the incoming market order is for 2,000 contracts. This will cause the trade price to hit 97.50. Once again William's order will get triggered off and will get executed at 97.50. Notice that the eventual price of execution is substantially different from the prescribed trigger price in this case. This is because, once the order is triggered off it becomes a market order and such orders do not afford any price control to the trader.

However, a trader like William can control the final execution price by placing what is known as a Stop-Limit order. Such orders become limit orders once they are triggered off. Consequently two prices need to be specified. One is the trigger price at which the stop sell order is to be activated. The other is a slightly lower price, which will become the limit price of the order if and when it is activated.

Assume that William gives a stop-limit order with a trigger of 99.45 and a limit of 99.20. Now if a market sell order for 800 contracts were to enter, the trade price will hit 99.45 and will cause William's order to get triggered off. It will get executed at 99.25 in this case, which is above the price of 99.20 that has been specified for the limit order. However, if the incoming market sell order were to be for 2,000 contracts, the trade price will hit 97.50. William's order will once again get triggered off. However it will not be executed because his limit price is 99.20. Consequently it will become a limit sell order with a limit price of 99.20. The book will look as follows.

Table 4.4 Snapshot of the LOB at a Given Point in Time

Buyers			Sellers		
Trader	**Order Size**	**Limit Price**	**Limit Price**	**Order Size**	**Trader**
			99.20	200	William
			100.00	200	Jack
			100.25	500	Larry
			100.40	700	Mike
			100.60	1,500	Peter
Tom	2,800	97.50	101.00	2,000	Victor

Stop-loss and stop-limit orders can also be placed by traders who have a short position in the market prior to the placement of the order. Such traders will obviously place buy orders. Their trigger price will be greater than the lowest price on the sell side, which represents the best price that is available when they place such an order. If they were to place a stop-limit order, the limit price specified by them will be slightly higher than their trigger price.

4.6 Equivalence with Options

Limit orders may be perceived as options. A limit buy order is a put option, for it gives other traders the right to sell to the trader who has placed the order. Similarly a limit sell order is a call option. The limit price that is specified in the order is the strike price of the corresponding option.

There are two key differences between conventional options and the options implicit in limit orders. Firstly, limit orders represent options with a zero premium, unlike normal options which require the buyer to pay a premium up front to the writer. Secondly, a conventional option is a contract between the writer and a particular buyer. In the case of limit orders there is, however, no exclusive owner or buyer. Once the order has been placed, anyone is welcome to come and trade by placing a market order or a marketable limit order.

4.7 Validity Conditions

Traders have the option of specifying a validity instruction to indicate as to how long their order should remain valid, in the event of there being a delay in execution. Such instructions may be appended to any kind of order. However, they are used primarily for limit orders and stop orders (with or without a price limit), for such orders have a tendency to experience delays in execution.

4.7.1 Good Till Canceled Orders

Such orders remain valid until they are canceled by the trader. Thus, they would be carried over to the next business day, if they were to remain unexecuted at the close of trading on a particular day. Obviously such orders cannot remain in the system for an indefinite period of time. For this reason the exchange will specify a maximum validity period. The order will be automatically canceled if it were to remain unexecuted at the end of this prescribed time limit.

4.7.2 Good Till Days Orders

In the case of such orders, the trader can specify the validity period. Obviously the time limit specified cannot exceed the validity period for a good-till-canceled order. There are different variations of such orders, such as Good-this-week (GTW) and Good-this-month (GTM) orders.

4.8 Open-Outcry Trading Systems

Open outcry systems, also referred to as oral auctions are a type of continuous bilateral auctions. They were extremely common in the last century, but have of late been replaced to a large extent by electronic trading systems.

In an oral auction there is a central location at the futures exchange, known as a *Pit* where the traders congregate. Traders cry out their bids and offers hoping to entice others who may be willing to take a matching position. While shouting out their requirements, they will concurrently be listening for orders being shouted out by other traders. A trade will result if a buyer were to accept a seller's offer, or if a seller were to accept a buyer's bid. The consummation of a trade is accompanied by a shout of "take it" if the buyer were to accept an offer, or by a shout of "sold" if the seller were to accept a bid.[1]

Since both buyers and sellers will be shouting out their requirements, it is necessary to distinguish between bids and offers. In most oral auctions the convention is that buyers will call out their limit price first followed by the quantity, whereas sellers will call out their required quantity first followed by their limit price. In exchanges like the CME Group, traders have a practice of using a system of hand and finger signals to indicate their limit prices, and the quantities which they wish to trade.

The first rule that a trader is required to follow on an open-outcry exchange, is known as the *'open-outcry'* rule. Put simply, it means that whatever the trader has in mind, needs to be vocally expressed, for after all no one is a mind reader. Once a trader expresses his intention, any other trader standing in the pit may respond. It is a normal practice for traders to take turns in making bids and counter offers, and offers and counter bids, before the two parties are able to agree on a price. When a trader expresses his bid or offer, the first counterparty who accepts it gets to trade with him.

As in the case of electronic systems there are certain priority rules. The principal rule is price priority. That is, a bidder is required to accept the lowest offer, and a seller is required to accept the highest bid. Such a rule is obviously self-enforcing in practice, because an honest bidder will always be on the lookout for the lowest price, while an honest seller will always strive to obtain the highest price.

The price priority rule requires that there should be clarity about the best bid and best offer that are available at a point in time. Obviously inferior bids and offers, that is bids with prices lower than the best bid, and offers with prices higher than the best offer, will only serve to create confusion. Consequently, most open-outcry systems will not allow a trader to bid below the best bid that is currently available, or offer above the best offer that is currently available. However, a trader may at any time improve upon the best bid by bidding higher, or improve upon the best offer by quoting a lower offer. If a trader were to do so, he will obviously attain priority.

Most oral auctions follow what is known as *'floor time preference'*, as a secondary rule. What this means is that time priority is given to the trader who was the first to bid or offer at a particular price by improving upon the previous quote. The implication is that as long as this trader is enjoying time preference, no other trader may add noise by bidding or offering at the same price. Of course,

[1] See Harris (2003).

a competitor is welcome to wrest priority by bidding higher or offering at a lower price. This rule obviously encourages price competition. For, in his quest for priority, an aggressive trader has no choice but to improve the existing quote.

The difference between the time preference rule in an oral auction, and the so called *'strict time preference rule'* that is followed by electronic systems, is that in an oral auction, once a trade is consummated at a particular price, any trader may bid or offer at this price, and all orders at this price have equal priority. As we have already seen, electronic systems rank orders strictly in accordance with their time of arrival.

However, the time preference rule in an oral auction is not self-enforcing. This is because from the standpoint of a potential counterparty, it is immaterial as to whose bid or offer is accepted. The sole criterion is that the trader should get the best available price. Hence, in such auctions a trader who is currently enjoying time preference may have to vocally defend his status. That is, if another party were to bid or offer at the same price, he would have to shout out “That’s my bid" or “That’s my offer" to ensure that he continues to enjoy priority.

Also, in such auctions, bids and offers are valid only for an instant. Hence, if a counterparty is not available, a trader may have to repeatedly shout out his order in order to indicate to potential counterparties that he is very much interested in trading.

The trade pricing rule in such auctions is simple. Once a bid or an offer is accepted by another trader, the trade will take place at the price proposed by the trader whose quote was accepted.

4.9 Merits and Demerits of the Trading Systems

A successful trading system is one that is characterized by high liquidity and low trading costs. Trading related costs include not just direct transactions costs such as commissions, but also indirect costs such as lost revenues due to illiquidity.

Liquidity in an open outcry system is supplied by traders called *locals*. These sell-side traders are always ready to buy and sell on their own account. The problem in practice is that a local is restricted at any point in time to a single pit. In an active exchange futures contracts on different assets trade in different pits. Obviously, a local cannot devote attention to multiple pits at the same time. As a consequence, a local is in practice required to service the traders at his usual pit, even though there may be very little trading activity there.

In an electronic system, however, traders do not face such locational constraints and can effortlessly switch to a different screen displaying the LOB for a different asset, if there is limited trading activity in a particular futures contract. Hence, for assets characterized by relatively lower trading volumes, electronic trading is clearly the preferred mode of trading. Consequently, in emerging markets, where derivatives trading is often less active as compared to established exchanges in developed countries, electronic systems are the chosen modes of trading.

Evidence indicates that even advanced economies are increasingly switching to electronic systems. It must be remembered that exchanges in these countries introduced derivative contracts decades ago. In the earlier years, trading volumes were high for most contracts, because such instruments with their novel features provided opportunities that were simply not available earlier. In most of these cases, trading was active even though the underlying assets were not necessarily very sophisticated. After a point in time, however, the financial instruments on which derivatives have subsequently been introduced tend to be highly specialized. Consequently the number of traders who are attracted to such contracts are relatively less. Thus derivatives on new instruments are characterized by relatively lower trading volumes, which is causing even established exchanges to switch to screen based trading systems. Exchanges like the CME Group, which is a pioneer in derivatives trading, continue to use both types of trading platforms.

The older exchanges are also faced with the specter of declining trading volumes. This is to a large extent due to competition from newly established exchanges which embraced modern trading systems from the very outset. In the face of such rivalry, there is little option for the older exchanges but to embrace electronic systems. Electronic systems also play a key role in cross-border trading. Trading across borders is an accepted fact of the modern economy, characterized by the tenets of LPG - Liberalization, Privatization, and Globalization.

Electronic trading platform are characterized by lower operational costs as compared to open-outcry systems. Screen based trading platforms typically require less labor, skill, and time. Oral auctions are characterized by higher fixed costs due to the need for relatively larger manpower. The overheads such as building and back-office costs also tend to be higher.

The disadvantages not withstanding, open-outcry system continue to have certain merits of their own. As Sarkar and Tozzi (1998) have argued, highly active derivative contracts are better traded on traditional trading platforms, because the traders on such exchanges are more accomplished at executing large and complex orders. Traders in traditional systems also have the advantage that they are well aware of the trading behavior of competitors and counterparties, for the same parties trade with each other virtually every day. Such knowledge is indispensable for predicting the response of counterparties while implementing a trading strategy. In contrast, electronic trading systems are faceless entities, where the counterparties remain unidentified.

Finally, in an oral auction, order revision is relatively simpler because quotes are valid only for an instant after being expressed. However, in an electronic system, modification of an order that has been placed earlier requires the trader to explicitly cancel the prior order. In such a situation, there is always a possibility that the original order will get executed before the change can be effected.

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CONCEPT CHECK

State whether the following statements are True or False.

1. A limit buy order is placed at a price that is below the best price that is available in the market.
2. A stop-loss sell order is placed at a price that is below the best price that is available in the market.
3. A stop-loss order will become a market order if triggered off.
4. The time priority rule is self-enforcing in an open-outcry system.
5. Open-outcry systems generally use a strict time preference rule.
6. A limit order that is executable on submission is called an immediate or cancel order.
7. A trader who has short sold an asset and is worried that the price may abruptly move in an adverse direction, is likely to place a stop-buy or a stop-limit buy order.
8. The limit price for a marketable limit buy order should be greater than or equal to the best available ask.
9. Stop-loss orders give the trader control over the trade price.
10. A ‘good today’ limit order is a day order.
11. All-or-none orders are the same as fill-or-kill orders.
12. The price-priority rule is self-enforcing in an oral auction.
13. A stop-limit order is subject to execution uncertainty.
14. A limit buy order is a put option.
15. Limit orders are European options.
16. The limit price, is the exercise price of the corresponding option.
17. A limit order is an option with a premium of zero.
18. All-or-none orders can be filled only at a single price.
19. An all-or-none order must be a limit order.
20. Order revision is simpler on an open-outcry system as compared to an electronic trading system.

QUESTIONS AND PROBLEMS

Question-I

Market orders are superior to limit orders. Comment

Question-II
Why may a trader prefer a marketable limit order to a market order?
Question-III
What kind of a trader may place a stop-buy order?
Question-IV
What is the difference between a stop buy order and a stop-limit buy order?
Question-V
On an open-outcry system, price priority is self-enforcing but time priority is not. Comment.
Question-VI
What is the difference between a floor time preference rule and a strict time preference rule?
Question-VII
Electronic trading systems are superior to open-outcry systems. Comment.
Question-VIII
Limit orders are like options. Comment. What is the difference between the options implicit in limit orders and conventional options?
Question-IX
What is the difference between All-or-None and Fill-or-Kill orders.
Question-X
In an open-outcry system a trader cannot bid below the best bid or offer above the best ask. However a trader who bids higher than the best available bid or offers at a price lower than the best available ask, automatically acquires priority. Explain.

5

Money Market Futures

5.1 Introduction

Short term debt instruments, that is assets with an original term to maturity of one year or less are referred to as *Money Market Instruments*. In the case of debt securities we must make a distinction between the original term to maturity of a security, and its actual term to maturity. The original term to maturity of an asset is the time to maturity of the instrument at the time of issue. Obviously this cannot change once the security is issued. On the other hand, the actual term to maturity is the current term to maturity of the security. With the passage of time, the actual term to maturity will keep declining. All money market securities are obviously debt securities, for equity securities do not have a maturity date.

The money market is intended for transactions to meet short-term cash needs or what are termed as *current account* transactions. It represents the arena where parties with a temporary cash surplus interact with those faced with the specter of a short-term cash deficit. The nature of transactions range from overnight deals to those with a maturity of as long as one year.

Why do we require a money market? For the majority of businesses as well as government entities, the projected inflows of cash will rarely match with the scheduled outflows. Money markets serve to bridge the gap between receipts and expenditure. Let us first take the case of a government. It will collect revenue primarily in the form of taxes. Taxes are received periodically in lumpy amounts and do not arise uniformly during the course of the financial year. However, cash disbursements have to be made throughout the year on account of expenses such as wages and salaries, office supplies and other expenses. At the time of collection of tax revenues, the government will enter the money market as a lender. At other points in time when there is a deficit, it may borrow in the money market by issuing short term debt securities. Similarly businesses collect sales revenues at points in time that will usually not coincide with budgeted expenditure. Thus the checking account of these firms will fluctuate from large credit balances to low or nil balances. Whenever there is a surplus, such firms will enter the money market to earn some returns on the surplus funds. On the other hand, firms with a temporary deficit will seek to borrow for short periods.

Money is an extremely perishable asset. When idle cash is not gainfully invested, the holder will incur an opportunity cost in the form of interest that is foregone. Such income if not earned, is lost forever. When a large amount of money is involved, the income that is lost from not profitably investing idle funds for even a day can be substantial. Take the case of an institution that has 12MM dollars available for investment overnight. If we assume that the interest rate is 12% per annum and that the year consists of 360 days, which is a common assumption in the case of money market securities, the loss if the funds are kept idle is:

$$12{,}000{,}000 \times 0.12 \times \frac{1}{360} = \$\,4{,}000$$

For a week, this amounts to \$ 28,000 of lost income.

Futures contracts are available on a number of such assets. The CME Group offers contracts on Eurodollars, Treasury Bills, Euroyen, and Federal Funds. We will begin by taking a look at the underlying instruments.

5.2 Eurodollars

The development of the Eurocurrency market was one of the early factors in the growth of international investment. A eurocurrency is a freely traded currency deposited in a bank outside its country of origin. For example, Eurodollar deposits are dollar denominated time deposits held by banks outside the U.S. while Euroyen are yen deposited outside Japan. These deposits are accepted by commercial banks and are not backed by any government guarantees.

5.2.1 LIBOR

LIBOR is an acronym for the *London Inter Bank Offer Rate*. It may be defined as the rate at which a bank with a high credit rating is prepared to lend to a similar bank, and is the main benchmark rate in the London inter bank market. In practice LIBOR is quoted for a number of tenors - 1 month; 2 months; 3 months; 6 months; and 12 months. Thus there are several LIBOR rates that are quoted at any point in time, and any quotation must be prefixed by its term to maturity. Every bank in London quotes its own indicative LIBOR rate for each maturity period, but usually the rates quoted by competing banks are the same with occasional small differences of the magnitude of a few basis points. *A basis point, denoted by b.p, is one hundredth of one percent*. Indicative LIBOR rates for the London market as a whole are produced daily by the British Bankers Association (BBA), for loans with different tenors.

BBA LIBOR BBA LIBOR is the most widely used ‘benchmark’ or reference rate for short-term interest rates. It is compiled by the BBA in conjunction with Reuters and released shortly after 11:00 a.m. London time each day. The BBA maintains a reference panel of at least 8 contributor banks. The objective is to provide a reference panel which reflects the balance of the market—by country and by type of institution. The top quartile and bottom quartile market quotes are disregarded and the middle two quartiles are averaged to arrive at the BBA LIBOR rate. The quotes from all panel banks are published on screen to ensure transparency. BBA LIBOR rates are provided for ten currencies.

Table 5.1 **Currencies for which BBA LIBOR is Reported**

Currency Name	Symbol
Pound Sterling	GBP
US Dollar	USD
Japanese Yen	JPY
Swiss Franc	CHF
Canadian Dollar	CAD
Australian Dollar	AUD
Euro	EUR
Danish Krona	DKK
Swedish Krona	SEK
New Zealand Dollar	NZD

5.2.2 LIBID

LIBID is an acronym for the London Inter Bank Bid Rate. It is the rate that a London bank with a good credit rating will pay on funds deposited with it by another top rated London bank. LIBID just like LIBOR is quoted for a number of tenors. While LIBOR represents the rate that banks seeking to borrow in the inter-bank market might have to pay, LIBID is the rate that banks with surplus funds might have to accept on an inter-bank deposit. LIBID will be lower than LIBOR. When two banks arrange an inter-bank transaction, the rate that is agreed upon will often be somewhere between LIBID and LIBOR, and could possibly be an average of the two. Some banks therefore use LIMEAN, which is an arithmetic average of the LIBID and the LIBOR, as the reference rate for their inter-bank transactions.

5.2.3 Floating Rate Loans and LIBOR

The growth of Eurocurrency deposits has facilitated Eurodollar loans. These are loans made on a floating rate basis, with the base rate being LIBOR. *A floating rate loan is a loan whose interest rate is linked to a base rate, and consequently is reset periodically*. Commercial loans based on LIBOR are priced at LIBOR plus a *Margin or Spread*, which would depend on the credit worthiness of the borrower.

We will illustrate such loans with the help of an example.

Example Consider a conventional fixed rate loan of $ 100,000, with interest payable at the rate of 10% per annum on a semi-annual basis. The interest payable every six months would be \$ 5,000. On the other hand, consider a floating rate loan for the same amount, with the rate being specified as LIBOR + 50 basis points. 50 basis points ≡ 0.50%. Let us assume that the interest rate is reset every six months, and that the LIBOR when the loan is made is 8%. So the interest for the first six months is:

$$0.085 \times \frac{1}{2} \times 100{,}000 = \$\,4{,}250$$

If at the end of the first six months, the LIBOR were to be 8.5%, then the interest for the next six months will be:

$$0.09 \times \frac{1}{2} \times 100{,}000 = \$\,4{,}500$$

As can be seen, the LIBOR that determines the interest rate for a six monthly period, is the rate that prevails at the start of the period. Such a system is called *'determined in advance and paid in arrears'*, since the LIBOR and consequently the applicable rate of interest is determined at the start of the period, but the actual interest is paid at the end of the period. In principle (but rarely if ever observed in practice), a lender may adopt a convention known as *'determined in arrears and paid in arrears'*. As the term suggests, the applicable LIBOR in such cases will be the rate prevailing at the end of the period.

5.3 T-bills

Treasury bills or T-bills are zero coupon securities issued by governments. U.S. Treasury bills are direct obligations of the U.S. government. These securities are important because of their zero default risk, ready marketability, and high liquidity. Regular series bills are issued routinely every week or month by way of competitive auctions. One month (4-week), three month (13-week) and six month (26-week) bills are auctioned every week and one year bills are sold usually once a month. Consequently, at any point in time, T-bills are available with a wide range of maturities and there is a regular supply of new instruments in the market.

The most recently issued securities for a given maturity are called *On-The-Run* securities and tend to be highly liquid. *Off-The-Run* securities, which are instruments issued earlier, tend to be less liquid. For instance on a given day, the most recently issued bill with three months to maturity would be categorized as the on-the-run bill. A bill with an original term to maturity of six months, and with three months remaining to maturity, would be classified as off-the-run. Why is it that on-the-run bills are more liquid? In practice for a short period after the issue, such securities tend to be actively traded. Thereafter, the investors who acquire these instruments mostly tend to hold them to maturity. Consequently, off-the-run issues are less liquid.

T-bills are sold by an auction process. Prices and yields are therefore determined by the market and not by the Treasury. The issue of a new 13-week bill is announced by the Treasury on the Thursday of every week, with bids from investors being due on the following Monday before 1 P.M New York time. T-bills are usually issued on the Thursday following Monday's auction. The Treasury entertains both competitive and non-competitive tenders. Competitive tenders typically are submitted by large investors including banks and securities dealers who bid for several million dollars worth at a time. Such bidding is limited to 35% of the issue amount for each bidder. Non-competitive tenders are submitted by small investors who agree to accept the rate set in the auction. There is a limit of \$ 5 MM for each non-competitive bid. Generally the Treasury fills all non-competitive tenders.

5.3.1 Yield

In the market for fixed income securities, there are different ways of computing the yield on an instrument. It is very important for an investor to be conversant with the various methods used for calculation and the relationship between the corresponding yields, in order for him to understand the various instruments that are available.

Notation We will define the following symbols. The corresponding terms will become clear as we proceed.

- $d \equiv$ quoted yield on a T-bill.
- $D \equiv$ discount from the face value in dollars.
- $V \equiv$ face value of the T-bill.
- $T_m \equiv$ time left to maturity of the T-bill in days.
- $P \equiv$ price of the T-bill.
- $i_{365} \equiv$ Bond equivalent yield of a T-bill with $T_m < 182$ days.
- $y \equiv$ Bond equivalent yield of a T-bill with $T_m > 182$ days.

5.3.2 Discount Rates and T-bill Prices

The quoted yield on a T-bill is a discount rate, which is used to determine the difference between the price of a T-bill and its face value. *For the purpose of calculation, the year is treated as if it has 360 days.*

If d is the quoted yield for a T-bill, with a face value of \$ V and having T_m days to maturity, the dollar discount D is given by

$$D = V \times d \times \frac{T_m}{360} \tag{5.1}$$

Given the discount, the price may be calculated as

$$P = V - D$$

Alternately, the price may be directly computed from the discount rate as follows.

$$P = V - D = V - V \times d \times \frac{T_m}{360}$$

$$= V\left[1 - \frac{d \times T_m}{360}\right] \tag{5.2}$$

We will illustrate these concepts with the help of an example.

Example A T-bill with 90 days to maturity and a face value of \$ 1,000,000, has a quoted yield of 4%. What is the price in dollars?

$$D = 1{,}000{,}000 \times .04 \times \frac{90}{360} = \$\,10{,}000$$

The price is given by

$$P = V - D = 1{,}000{,}000 - 10{,}000 = \$\,990{,}000$$

5.3.3 The Bond Equivalent Yield (BEY)

The objective of calculating the Bond Equivalent Yield, is to facilitate comparisons between the rate of return on securities like T-bills which are sold at a discount from their face value and the yield to maturity on conventional coupon paying bonds.[1] The procedure adopted depends on whether the discount instrument has less than 182 days left to maturity or greater than 182 days left to maturity.

Case A : $T_m < 182$ days *The bond equivalent yield for a T-bill with less than 182 days to maturity, is nothing but the equivalent rate of return on a simple interest basis, assuming that the year has 365 days.*

Remember that in conventional interest calculations, the rate earned is based on the price. That is, the rate of return on an asset that is bought for P and which pays the face value V at maturity, is given by

$$\frac{(V - P)}{P}$$

For a zero coupon security, this will always be greater than

$$\frac{(V - P)}{V} = \frac{D}{V}$$

which is the dollar discount expressed as a fraction of the face value. Thus if a bill is bought at a price corresponding to a yield of $d\%$, and is held to maturity, the rate of return earned by the investor will always be greater than the yield that is quoted in the market.

The BEY in this case is defined as

$$i_{365} = \frac{(V - P)}{P} \times \frac{365}{T_m} \tag{5.3}$$

It can be directly computed given the quoted yield, using the following equation.

$$i_{365} = \frac{(V - P)}{P} \times \frac{365}{T_m} = \left(\frac{V}{P} - 1\right) \times \frac{365}{T_m}$$

[1] The bond equivalent yield is also known as the *Coupon Equivalent Yield*.

$$= \left[\frac{V}{V\left[1 - \dfrac{d \times T_m}{360}\right]} - 1 \right] \times \frac{365}{T_m}$$

$$= \left[\frac{360}{360 - d \times T_m} - 1 \right] \times \frac{365}{T_m}$$

$$= \frac{d \times 365}{360 - d \times T_m} \tag{5.4}$$

We will illustrate this using a suitable example.

Example A T-bill with a face value of $ 1,000,000 and 90 days to maturity, has a quoted yield of 5%. What is the bond equivalent yield?

The price is given by

$$P = V\left[1 - \frac{d \times T_m}{360}\right]$$

$$= 1{,}000{,}000 \times \left[1 - \frac{.05 \times 90}{360}\right] = \$\ 987{,}500$$

The bond equivalent yield is given by

$$i_{365} = \frac{(V - P)}{P} \times \frac{365}{T_m}$$

$$= \frac{(1{,}000{,}000 - 987{,}500)}{987{,}500} \times \frac{365}{90}$$

$$= .0513 \equiv 5.13\%$$

Case B : T_m > 182 days A coupon paying bond with a time to maturity that is greater than 182 days (half a year) will make a coupon payment before maturity. To facilitate a comparison between the BEY for a discount instrument with more than 182 days to maturity and the yield to maturity (YTM) for a conventional bond, the discount security must be treated as if it too pays a semi-annual coupon.

Let us denote the BEY in this case by y. y is given by the following equation.

$$P\left(1 + \frac{y}{2}\right)\left(1 + \frac{y}{2}\left\{\frac{T_m - \dfrac{365}{2}}{\dfrac{365}{2}}\right\}\right) = V$$

The logic is as follows. The future value of P at the end of six months will be

$$P\left(1 + \frac{y}{2}\right)$$

The future value of this expression, as calculated on the date of maturity must equal the face value. The compounding factor for the remaining period, on a simple interest basis, is

$$\left(1 + \frac{y}{2} \times \frac{T_m - \frac{365}{2}}{\frac{365}{2}}\right)$$

Therefore

$$P\left(1+\frac{y}{2}\right)\left(1+\frac{y}{2}\left\{\frac{T_m - \frac{365}{2}}{\frac{365}{2}}\right\}\right) = V \tag{5.5}$$

The expression for y is[2]

$$\frac{\frac{-2T_m}{365} \pm 2\sqrt{\left(\frac{T_m}{365}\right)^2 - \left(\frac{2T_m}{365} - 1\right)\left(1 - \frac{V}{P}\right)}}{\frac{2T_m}{365} - 1} \tag{5.6}$$

As usual, we discard the negative root and retain the positive. We will illustrate the calculation of the BEY for a T-bill with a time to maturity greater than 182 days, with the help of the following example.

Example Consider a T-bill with 240 days to maturity and a face value of \$ 1,000,000, which has a quoted yield of 6%. What is the bond equivalent yield?

The price of the bill is

$$1{,}000{,}000 - 1{,}000{,}000 \times 0.06 \times \frac{240}{360} = \$\ 960{,}000$$

Therefore

$$y = \frac{\frac{-2 \times 240}{365} \pm 2\sqrt{\left(\frac{240}{365}\right)^2 - \left(\frac{2 \times 240}{365} - 1\right)\left(1 - \frac{1000000}{960,000}\right)}}{\frac{2 \times 240}{365} - 1}$$

$$= \frac{-1.315068 + 2\sqrt{(.432351) - (.315068)(-.041667)}}{.315068}$$

$$= .0629 \equiv 6.29\%.$$

5.3.4 Illustration of an Auction

Assume that the bids shown in Table 5.2 have been received for a T-bill auction. The bids have been arranged in ascending order of the yield (discount rate).[3] In addition 2 billion dollars worth of non-competitive bids have been received.

[2] The derivation is given in the appendix.

[3] Had it been a price based auction we would have arranged them in a descending order of price.

Table 5.2 **A Hypothetical Treasury Auction**

Bidder	Bid Rate	Bid Amount in Billions	Aggregate Demand
Alpha	3.50%	5	5
Beta	3.60%	3	8
Charlie	3.70%	2	10
Delta	3.75%	3	13
Gamma	3.75%	2	15
Tango	3.80%	2	17

Assume that the auction is for 15 billion dollars of bills. All non-competitive bids will be accepted. Thus a total of 13 billion dollars worth of securities is available for the competitive bidders. Aggregate demand equals the amount on offer at a discount rate of 3.75%. This is the market clearing yield. Thus everyone who bid this rate or less will be awarded at this yield. However at a yield of 3.75% the amount sought is 5 billion whereas only 3 billion is available. Of the amount demanded, Delta has asked for 3 billion or $\frac{3}{5}$th of 5 whereas Gamma has sought $\frac{2}{5}$th of 5. Therefore, Delta will be awarded $\frac{3}{5}$th of 3 or 1.8 billion. Gamma will obviously receive 1.2 billion. Alpha, Beta, and Charlie will be allotted whatever they have asked for. Tango will get nothing and is said to be *shut-out* of the auction. This type of an auction is known as a single yield or uniform yield auction, since all the winning bidders are allotted securities at the same yield.

The minimum denomination for T-bills is $ 100, and bills are issued in multiples of $ 100 thereafter. The lowest rate at which at least some bills are awarded is called the *stop-out* yield. In this example is it 3.75%. No one bidding more than the stop-out yield will receive any bills in an auction. However, once bills are acquired by successful bidders, many of them will be sold right away in the secondary market, giving the unsuccessful bidders a chance to buy.

5.4 Federal Funds

Federal funds are perhaps the most important of all money market instruments because they are the primary means of making payments in this market. The term refers to money that is available for immediate payment. In earlier years the federal funds market referred to loans made by banks with excess reserves at the Federal Reserve to banks with a deficit in the reserve account. Such funds can be readily transferred in seconds by computer. These days the Fed Funds market is broader in scope. For instance, most banks maintain deposits with large correspondent banks in major financial centers. These deposits may be transferred readily from one party to another. The most important borrowers continue to be commercial banks, which use this instrument as the principal way to adjust their legal reserve account at the Federal Reserve Bank in the district where they are located. Federal funds are same day money. In contrast the normal payments which are made by check take at least 24 hours for the payee to receive the funds. As can be appreciated, such transactions called *clearinghouse funds*, are far too slow for money market participants.

5.4.1 Use of the Fed Funds Market to meet Deposit Reserve Requirements

Banks and other depository institutions must hold in a reserve account assets equal to a fraction of the funds deposited with them. Legal reserve requirements are met by holding vault cash, and reserve balances with the local Federal Reserve Bank. Since these reserves earn little or no income, banks whose reserve balances exceed statutory requirements lend the excess reserves in their possession, to financial institutions with a deficit.

The legal reserve requirement of banks is calculated on a daily average basis over a two week period known as the *reserve computation period*, which stretches from a Tuesday through a Monday two weeks later. The Federal Reserve calculates the daily average of transaction deposits held by each depository institution over this two-week period and then multiplies that average by the required reserve percentage to determine the amount of legal reserves that must be held by each institution. These legal reserves must average the required amount over a two-week period known as the *reserve maintenance period*. This period starts on a Thursday, 30 days after the reserve computation period begins and ends on a Wednesday two weeks later.

Once the borrower and the lender agree on the terms of the loan, the lending institution will contact the district Federal Reserve bank requesting a wire transfer of Federal funds. The FED will then transfer reserves through its wire network, FEDWIRE, to the Federal Reserve bank serving the region where the borrowing institution is located. The transaction will be reversed when the loan is repaid. Most federal funds loans are overnight transactions or continuing contracts that have no specific maturity date, and can be terminated without advance notice by either party. Longer maturity loans are referred to as *term federal funds*.

5.5 Eurodollar Futures

Eurodollar (ED) futures contracts trade on the IMM in Chicago. The underlying interest rate as we have discussed earlier, is the London Inter Bank Offer Rate (LIBOR). Each futures contract is for a time deposit with a principal of 1 MM and three months to maturity.

The quarterly cycle for Eurodollar contracts is March, June, September and December. On the CME, a total of 40 quarterly futures contracts spanning ten years, are listed at any point in time. In addition, the four nearest serial months are also listed. Let us assume that we are standing on June 25, 2008. The four serial months that would be available would be July, August, October and November 2008. The remaining available months would be September 2008, December 2008, March, June, September and December of 2009-2017, March 2018 and June 2018.

Eurodollar futures contracts expire at 11:00 a.m. London time, on the second London bank business day before the third Wednesday of the contract month. The contracts are cash settled to the BBA 3-M LIBOR. The futures price is quoted in terms of an IMM index for Eurodollars and implies an interest rate.

Quoted ED Futures Price = 100.00 − Implicit Interest Rate

The implicit interest rate in this case is, however, an actual or add-on interest rate and not a discount rate as in the case of T-bills. The difference can be illustrated with the help of an example.

5.5.1 Example

A discount rate of 5% per annum, means that for a 90 day loan with a maturity value of 1 MM, you have to pay

$$1{,}000{,}000\left[1 - .05 \times \frac{90}{360}\right] = \$\ 987{,}500$$

The actual rate of return is

$$\left(\frac{1{,}000{,}000 - 987{,}500}{987{,}500}\right) \times \frac{360}{90} \equiv 5.06\%$$

An add-on interest rate of 5% per annum, however, means that if you invest 1 MM for 90 days, you will get

$$1{,}000{,}000\left[1 + .05 \times \frac{90}{360}\right] = \$\ 1{,}012{,}500$$

after 90 days.

5.6 Calculating Profits and Losses on ED Futures

Let us suppose that the futures price is 96. This represents a yearly interest rate of 4% or 1% per quarter. Thus the implied quarterly interest payment on a time deposit of 1 MM is

$$.01 \times 1{,}000{,}000 = \$\ 10{,}000$$

If the futures price were to fall to 95 at the end of the day, then it would represent a quarterly interest payment of $ 12,500 on a deposit of 1 MM.

Consider a person who goes long in an ED futures contract at 96. Such a person is agreeing to lend money at the equivalent interest rate, namely 1% per quarter.[4]

[4]The logic is the same as that underlying futures in the case of a debt security such as a T-bill. In the case of bills, a long position means that you are willing to buy bills at the expiration of the contract. In the case of ED futures, a long position means that you are willing to make a term deposit of three months. In either case you are a lender.

If interest rates rise subsequently to 5% per annum, that is, the futures price goes down to 95 and the contract is marked to market, then the long will lose \$ 2,500. The logic is that when the contract is marked to market, it is as if he is offsetting by going short, which in this case, means that he is agreeing to borrow at 1.25% per quarter. Thus when the interest rates rise the longs will lose. The reverse is true for the shorts, that is, when the interest rates fall, they will lose.

You should by now be able to see the logic behind quoting futures prices in terms of an index, rather than in terms of interest rates. If futures prices were to be quoted in terms of interest rates, then the longs would gain when the futures prices fall and the shorts would gain when the futures prices rise. But in all the other markets, you have been observing that the longs gain when futures prices rise, whereas the shorts gain if futures prices fall. Thus to make money market futures consistent with other futures markets, we do not quote futures prices in terms of interest rates, but do so in terms of an index. When the index rises, the longs will gain and when it falls, the shorts will gain.

Yet another reason for quoting futures prices in terms of an index rather than in terms of interest rates, is to ensure that the bid prices are lower than the ask. Remember, that as an investor, the borrowing rates that you will face will be typically higher than the lending rates confronted by you. Thus the rates underlying a short position will be greater than the rates underlying a long position. To be consistent with the principle that bid prices are always lower than the ask, it is necessary to convert the rates to equivalent index values.

Now consider the case where the ED index changes from F_0 to F_1. The profit for a long is

$$1{,}000{,}000 \times \frac{(100 - F_0)}{100} \times \frac{90}{360} - 1{,}000{,}000 \times \frac{(100 - F_1)}{100} \times \frac{90}{360}$$

$$= 1{,}000{,}000 \times \frac{(F_1 - F_0)}{100} \times \frac{90}{360}$$

The minimum price move or *tick* is .01, which corresponds to a price change of

$$1{,}000{,}000 \times \frac{.01}{100} \times \frac{90}{360} = \$\ 25$$

5.7 Bundles and Packs

A bundle entails the simultaneous purchase or sale of one each of a consecutive series of ED contracts. The first contract in a bundle is generally the first quarterly contract available. For instance, a three-year bundle will consist of the first 12 ED contracts available. On the CME, one, two, three, five, seven and ten year bundles are available. In addition, a five year *forward bundle*, which consists of 20 ED contracts starting from the first quarterly contract of year six, is also listed for trading.

A pack also involves the simultaneous purchase or sale of an equal number of consecutive ED futures. But the number of contracts in a pack is fixed at four. For instance, if today is 1 March, 2008, then a two-year pack will consist of one each of the March, June, September and December contracts of 2009.

5.8 Locking in a Borrowing Rate

We will first illustrate how a borrowing rate can be locked in using ED futures, before deriving the fair price of a contract using cash and carry arbitrage and reverse cash and carry arbitrage arguments.

Assume that today is 15 July, 2009. Ranbaxy is planning to borrow 1 million dollars on 14 September for a period of 90 days. The company is confident that given its rating, it can borrow at LIBOR. It is however worried that interest rates may rise before 14 September, which is the last day of trading for the September futures contract. The current September futures price is 94 and the current LIBOR for a 90 day loan is 5.85%. Since the company is planning to borrow money it requires a short hedge.

Assume that Ranbaxy goes short in one September futures contract. We will consider two different scenarios on 14 September, one where the LIBOR is higher as compared to the rate implicit in the futures price, and the other where it is lower.

5.8.1 Case A: LIBOR = 4%

The interest payable on the loan of 1 MM is

$$0.04 \times 1{,}000{,}000 \times \frac{90}{360} = \$\ 10{,}000$$

Gain/loss from the futures market is

$$1{,}000{,}000 \times \frac{(F_0 - F_1)}{100} \times \frac{90}{360} = 1{,}000{,}000 \times \frac{(94 - 96)}{100} \times \frac{90}{360}$$
$$= (5{,}000)$$

Therefore, the effective interest paid is

$$10{,}000 + 5{,}000 = \$\ 15{,}000$$

5.8.2 Case B: LIBOR = 7%

The interest payable is

$$0.07 \times 1{,}000{,}000 \times \frac{90}{360} = \$\ 17{,}500$$

Profit/loss from the futures position is

$$1{,}000{,}000 \times \frac{(94 - 93)}{100} \times \frac{90}{360} = \$\ 2{,}500$$

Thus, the effective interest paid is

$$17{,}500 - 2{,}500 = \$\ 15{,}000$$

Thus the company can lock in an interest payable of \$ 15,000, irrespective of the prevailing LIBOR on 14 September. This amount corresponds to a rate of

$$\frac{15{,}000}{1{,}000{,}000} \times \frac{360}{90} \equiv 6\%$$

which is nothing but the rate implicit in the initial futures price.

It is important to take cognizance of the following fact. Since the ED futures contract is cash settled, the profit/loss from the futures contract will be paid/received on 14 September, when the contract expires. However, the company is required to pay interest on the loan taken by it, only 90 days thereafter. Thus, if there were to be a profit from the futures position, it can be reinvested. On the contrary, a loss from the futures market would have to be financed. If adjustments were to be made for the interest on such profits/losses, the effective interest paid on the loan would be higher than 6% in case A and lower than 6% in case B. In our study of ED futures contracts, we will ignore such interest on profits/losses.

With this in mind, let us go on to analyze the pricing of ED futures.

5.9 Cash and Carry Arbitrage

Let us assume that an arbitrageur is confronted with the following situation on 15 August, 20XX.

Futures contracts expiring on 18 September are priced at 94. The 90 day ED deposit made on 18 September, will mature on 17 December.

The interest rate for an ED deposit between 15 August and 17 December is 6.75% . The rate for a loan between 15 August and 18 September is 4%.

Consider the following strategy. Borrow one million dollars for 34 days. Simultaneously go short in a futures contract to borrow the maturity amount, for a further period of 90 days. Invest the borrowed funds in a 124 day deposit.[5]

The amount due after 34 days will be $1{,}000{,}000\left(1 + 0.04 \times \frac{34}{360}\right)$. The futures contract will lock in a rate of 6% when this amount is rolled over for another 90 days. Thus, the amount payable after 124 days will be

$$1{,}000{,}000\left(1 + 0.04 \times \frac{34}{360}\right) \times \left(1 + 0.06 \times \frac{90}{360}\right) = \$\ 1{,}018{,}834.44$$

The amount that will be received when the 124 day investment matures is

$$1{,}000{,}000 \times \left(1 + 0.0675 \times \frac{124}{360}\right) = \$\ 1{,}023{,}250$$

Thus there is an arbitrage profit of

$$1{,}023{,}250 - 1{,}018{,}834.44 = \$\ 4{,}415.56$$

[5] 34 is the number of days between 15 August and 18 September, while 124 is the number of days between 15 August and 17 December.

5.10 Reverse Cash and Carry Arbitrage

Let us suppose that all the other variables have the same values as above, except for the futures price, which we will assume is 92.

Consider the following strategy. Borrow one million dollars for 124 days. Invest it for 34 days, and go long in a futures contract to rollover the maturity amount for a further period of 90 days. The amount repayable after 124 days will be:

$$1{,}000{,}000 \times \left(1 + 0.0675 \times \frac{124}{360}\right) = \$\ 1{,}023{,}250$$

The investment in the 34 day deposit will yield $1{,}000{,}000\left(1 + 0.04 \times \frac{34}{360}\right)$. The futures contract will lock in a rate of 8% for this amount. Thus, the amount receivable after 124 days is:

$$1{,}000{,}000\left(1 + 0.04 \times \frac{34}{360}\right) \times \left(1 + 0.08 \times \frac{90}{360}\right) = \$\ 1{,}023{,}853.33$$

Thus there is an arbitrage profit of

$$1{,}023{,}853.33 - 1{,}023{,}250 = \$\ 603.33$$

5.11 The No-Arbitrage Pricing Equation

We will now derive an expression for the futures price, F, which will preclude both cash and carry as well as reverse cash and carry arbitrage, for a given set of interest rates.

We will denote the day on which we are standing as day t. The futures contract is assumed to expire at T. We will denote the Eurodollar rate for a $T - t$ day loan by s_1 and the borrowing / lending rate for a $T + 90 - t$ day loan by s_2.

In order to rule out both forms of arbitrage, we require that,

$$1{,}000{,}000 \times \left[1 + \frac{(100 - F)}{100} \times \frac{90}{360}\right] \times \left[1 + \frac{s_1}{100} \times \frac{T - t}{360}\right]$$

$$= 1{,}000{,}000 \times \left[1 + \frac{s_2}{100} \times \frac{T + 90 - t}{360}\right]$$

$$\Rightarrow \left[1 + \frac{(100 - F)}{100} \times \frac{90}{360}\right]\left[1 + \frac{s_1}{100} \times \frac{T - t}{360}\right] = \left[1 + \frac{s_2}{100} \times \frac{T + 90 - t}{360}\right] \quad (5.7)$$

In our case, $s_1 = 4\%$, and $s_2 = 6.75\%$.

Therefore $$\left[1 + \frac{(100 - F)}{100} \times \frac{90}{360}\right] = 1.0194$$

$$\Rightarrow F = 92.24$$

5.12 Hedging Rates for Periods Not Equal to 90 Days

We have seen as to how ED futures contracts can be used to lock in a borrowing rate or a lending rate for a 90 day loan to be made on the date of expiration of the futures contract. ED futures can also be used to lock in rates for an N day loan if N were to be fairly close to 90. The necessary condition for such a hedge is that the interest rate for the N day loan should move closely with the rate for a 90 day loan. Before giving an illustration, we will first derive the required hedge ratio.

Assume that $d_N = \alpha + d_{90} + \epsilon$, where d_N is the annualized rate for an N day loan and d_{90} is the annualized rate for a 90 day loan. We will assume that $\epsilon = 0$. If so

$$\triangle d_N = \triangle d_{90}$$

Take the case of a party which is raising an N day loan for Q million dollars. The change in the interest payable, due to a rate change is

$$Q \times 1{,}000{,}000 \times \frac{\triangle d_N}{100} \times \frac{N}{360}$$

The profit/loss per contract, from the futures market is

$$1{,}000{,}000 \times \frac{\triangle d_{90}}{100} \times \frac{90}{360}$$

The number of futures contracts, Q_f, ought to be chosen in a way such that

$$Q \times 1{,}000{,}000 \times \frac{\triangle d_N}{100} \times \frac{N}{360} = Q_f \times 1{,}000{,}000 \times \frac{\triangle d_{90}}{100} \times \frac{90}{360} \tag{5.8}$$

Since we have assumed that $\Delta d_N = \Delta d_{90}$, we get the result that

$$Q \times N = Q_f \times 90$$
$$\Rightarrow \frac{Q_f}{Q} = \text{ the hedge ratio } = \frac{N}{90}$$

We will now illustrate as to how this hedge will perform in practice, with the help of the following example.

5.12.1 Example

Once again assume that we are standing on 15 July, 2009, and that Ranbaxy will be borrowing 10 MM USD on 14 September, for a period of 117 days. The current September futures price is 95.75. The firm can borrow at the prevailing LIBOR on 14 September.

Since Ranbaxy is borrowing it would require a short position in ED futures.

$$Q_f = 10 \times \frac{117}{90} = 13$$

Let us examine the performance of this hedge.

Case A : LIBOR = 4%   The actual interest paid by Ranbaxy is

$$0.04 \times 10{,}000{,}000 \times \frac{117}{360} = \$\ 130{,}000$$

The profit/loss from the futures position is

$$13 \times 1{,}000{,}000 \times \frac{(95.75 - 96)}{100} \times \frac{90}{360} = -\$\ 8{,}125$$

Thus, the effective interest paid by the company is $ 138,125.

Case B : LIBOR = 4.5%   The actual interest paid by Ranbaxy is

$$0.045 \times 10{,}000{,}000 \times \frac{117}{360} = \$\ 146{,}250$$

Profit/loss from the futures position is

$$13 \times 1{,}000{,}000 \times \frac{(95.75 - 95.50)}{100} \times \frac{90}{360} = 8{,}125$$

The effective interest paid by the company is $ 138,125

Hence, irrespective of the prevailing LIBOR on 14 September, the company has locked in an interest expense of $ 138,125. This amount corresponds to an interest i such that

$$10{,}000{,}000 \left[1 + i \times \frac{117}{360}\right] = 10{,}138{,}125$$

i is therefore equal to 4.25%, which is nothing but the rate implicit in the initial futures price.

5.13 Creating a Fixed Rate Loan

Assume that YES Bank is able to borrow money at LIBOR for three month periods. Consequently the interest payable on its liability is variable. The bank however has a potential client who wishes to borrow at a fixed rate for a period of one year. Thus the interest receivable from the proposed asset is fixed in nature. The bank would like to use futures contracts to mitigate the risk arising from the fact that it is borrowing at a floating rate, but is lending at a fixed rate. It turns out that ED futures contracts can be used to hedge the funding risk and determine a suitable rate that can be quoted by the bank while negotiating a fixed rate loan. We will illustrate the principle involved with the help of an example.

IT20, a company based in London, wants a loan for 100 million US dollars for a period of one year from 15 September, 20XX, at a fixed interest rate.

Let us assume that the 90 day LIBOR on 15 September, is 3.8% and that December, March, and June contracts, are available at 97.1, and 96.6, and 97 respectively. For ease of exposition, we will assume that the dates on which the bank will rollover its three month borrowings, namely 15 December, 15 March, and 15 June, are the same as the dates on which the futures contracts for those months are scheduled to expire.

Each ED futures contract is on 1 MM USD and the bank is borrowing at a floating rate. Consequently it needs a short position in 100 each of the December, March, and June futures contracts.

For the first quarter, the interest expense for the bank is

$$100{,}000{,}000 \times 0.038 \times \frac{90}{360} = \$\ 950{,}000.$$

There is no uncertainty about this amount since it is based on the current 3M LIBOR. The short position in December contracts, will lock in a rate of 2.9% for a period of 90 days from December to March. This corresponds to an interest expense of

$$100{,}000{,}000 \times 0.029 \times \frac{90}{360} = \$\ 725{,}000.$$

In a similar fashion, the short position in March contracts, will lock in

$$100{,}000{,}000 \times 0.034 \times \frac{90}{360} = \$\ 850{,}000$$

for the 90 day period from March. The June contracts will lock in the following interest amount for the last quarter.

$$100{,}000{,}000 \times 0.03 \times \frac{90}{360} = \$\ 750{,}000$$

Hence the total interest payable by the bank for the twelve month period is:

$$950{,}000 + 725{,}000 + 850{,}000 + 750{,}000 = \$\ 3{,}275{,}000.$$

This corresponds to an annualized interest rate of:

$$\frac{3{,}275{,}000}{100{,}000{,}000} \times 100 = 3.275\%$$

The bank can now quote a fixed rate to the client, based on this effective cost of funding, after factoring in hedging costs and a suitable profit margin.

5.13.1 Stack and Strip Hedges

We have assumed in the above illustration that the bank would use 100 futures contracts for each of the expiration months. Such a hedge, where the same number of contracts are used for each expiration month right from the outset, is called a *Strip Hedge*.

As we have seen in the chapter on hedging, when the maturity of a futures contract is very far away, the contract may not be actively traded. Such illiquidity could be a deterrent for a potential hedger, because he would seek the ability to enter and exit the market at price that is close to the true or fair value of the asset. In this case, when the hedge is initiated in September, it is conceivable that the June futures may be illiquid. In such a situation, rather than starting with an equal number of December, March, and June contracts, the bank may initiate the hedge using an unequal number of December and March contracts, without taking a position in the June contract. Subsequently when the December contracts expire, it will take a position in the June contract. Such a hedge, where the number of contracts for each maturity is not equal at the outset, is known as a *Stack Hedge*. We will illustrate the mechanics of such a hedge with the help of an example.

Example of a Stack Hedge Assume that the June futures contracts are perceived to be illiquid when the hedge is initiated. The bank therefore decides to hedge using 100 December contracts and 200 March contracts. The position in December futures is obviously intended to lock in a rate for that month's borrowing. Out of the 200 March contracts, 100 are intended to lock in a rate for March, while the remaining 100 are meant for hedging the June exposure. The inherent assumption is that on 15 December, the bank will partially offset its March position and go short in 100 June contracts, assuming that they have begun to be actively traded by then.

5.13.2 Performance: Strip versus Stack

The relative performance of a Stack Hedge vis à vis a Strip Hedge, would depend on the movement of interest rates between September and December.

Case A: The Stack Hedge is Equivalent to the Strip Assume that the March futures price moves from 96.6 to 96 between September and December, while the June futures price moves from 97 to 96.4 during the same period.

When the 100 extra March contracts are liquidated in December, the profit from the position will be

$$100{,}000{,}000 \times \left(\frac{96.60 - 96}{100}\right) \times \frac{90}{360} = \$\ 150{,}000$$

The 100 June contracts which the bank will enter into at 96.4 will lock in an interest expense of

$$100{,}000{,}000 \times 0.036 \times \frac{90}{360} = \$\ 900{,}000$$

for the last quarter. The effective interest expense for this quarter is therefore:

$$900{,}000 - 150{,}000 = \$\ 750{,}000$$

which is the same amount that was locked in by the Strip Hedge.

Thus, if the three month ED rate, as contained in the March futures price, changes by the same magnitude and direction as the yield contained in the June futures price, then the Strip and Stack hedges will be equivalent. In interest rate parlance, we would say that there has been a *Parallel Shift* in the *Yield Curve*.

Case B: The Strip Outperforms If the increase in the March yield is less than the increase in the June yield, then the Strip Hedge will outperform the Stack. For instance, assume that the March futures price moves from 96.6 to 96 whereas the June futures price moves from 97 to 96.20.

Since the futures price has declined there will be a profit when the extra March contracts are offset, which as before will be $ 150,000.

However the interest expenses for the last quarter will be higher. This expense will be

$$100,000,000 \times 0.038 \times \frac{90}{360} = \$\ 950,000$$

The effective interest is

$$950,000 - 150,000 = \$\ 800,000$$

which is greater than the amount of 750,000 that was locked in by the Strip Hedge.

The same would be the case if the decrease in the March yield were to be more than the decrease in the June yield. In this case, there will be a loss when the March contracts are offset. However, the interest expense locked in by the June contracts will be lower. For instance assume that the March futures price moves to 96.9, whereas the June futures price moves to 97.1.

The profit from the March position will be

$$100,000,000 \times \left(\frac{96.6 - 96.9}{100}\right) \times \frac{90}{360} = -\$\ 75,000$$

The interest expense for the last quarter, as locked in by the June contracts, will be

$$100,000,000 \times 0.029 \times \frac{90}{360} = \$\ 725,000$$

The effective interest paid is

$$725,000 + 75,000 = \$\ 800,000$$

which is greater than the amount of 750,000 that was locked in by the Strip Hedge. Clearly the gain on account of the decline in yield implicit in the June contracts is not adequate to compensate for the loss incurred when the March contracts are offset.

Case C: The Stack Outperforms If the increase in the March yield is more than the increase in the June yield, then the Stack Hedge will out perform the Strip. For instance, assume that the March futures price moves from 96.60 to 96, while the June futures price moves from 97 to 96.60.

The effective interest rate for the last quarter in this case is:

$$\begin{aligned}100,000,000 \times 0.034 \times \frac{90}{360} - 150,000 &= 850,000 - 150,000 \\ &= \$\ 700,000\end{aligned}$$

The same would be true if the decrease in the March yield were to be less than the decrease in the June yield. In this case, once again there will be a loss when the March contracts are offset. However this will be more than compensated by the decline in the rate for the last quarter. For instance assume that the March futures price moves from 96.60 to 96.90, whereas the June futures price moves from 97 to 97.50.

The loss from the March position will be 75,000. The interest expense for the last quarter will be:

$$100{,}000{,}000 \times 0.025 \times \frac{90}{360} = \$\,625{,}000$$

The effective interest is:

$$625{,}000 + 75{,}000 = \$\,700{,}000$$

which is less than the amount of $ 750,000 that was locked in by the Strip Hedge.

5.14 LIBOR Futures

The underlying asset is a time deposit with a principal of $ 3MM and one month to maturity. The contracts expire at 11:00 a.m. London time on the second bank business day before the third Wednesday of the contract month. At any point in time, contracts for the next 12 consecutive months will be listed. So on June 25, 2008 the available contract months will be July 2008 to June 2009. All contracts are cash settled.

5.15 Euroyen Futures

The underlying asset is a time deposit with a principal of 100MM JPY, and three months to maturity. The contract settles to the Tokyo Inter Bank Offer Rate (TIBOR) and is cash settled. At any point in time 20 contracts will be listed from the March quarterly cycle. So on January 2, 2009, the available months will be March, June, September and December of 2009–2013.

5.15.1 Euroyen LIBOR Futures

The underlying asset is the same as that for Euroyen futures, and the available contract months are identical. The difference is that this contract settles to the London Inter Bank Offer Rate (LIBOR).

5.16 T-bill Futures

The underlying asset in a T-bill futures contract is a 13 week (91 day) T-bill with a face value of $ 1 MM. The contract cash settles into the auction rate on the business day of the weekly 91-day T-bill auction in the week of the third Wednesday of the contract month. At any point in time, four months from the March cycle plus two serial months will be listed. Thus on June 25, 2008 the available months will be July 2008, August 2008, September, 2008, December 2008, March 2009 and June 2009.

The auction is conducted on a Monday for a bill to be issued on a Thursday. Thus the rate that is set at the auction is a forward rate for a 91-day bill. Since there is an auction of a 13-week bill every week, there will be a bill maturing three days after the day of the auction as well as 94 days after the day of the auction. This has ramifications for the pricing of the futures contract as we shall demonstrate.

5.16.1 Creating a Synthetic 91 Day Forward Bill

Assume that we are on day t and that the futures contract expires on day T. There is a bill maturing on $T+3$ which we will refer to as the *short bill*. At the same time, there will be a bill expiring at $T+94$, which we will refer to as the *long bill*. Let s_1 be the rate of return on the short bill and s_2 the rate of return on the long bill.[6] If we denote the 91-day forward rate as $f_{1,2}$, then to preclude arbitrage we require that

$$(1+s_1)(1+f_{1,2}) = (1+s_2)$$
$$\Rightarrow (1+f_{1,2}) = \frac{(1+s_2)}{(1+s_1)} \tag{5.9}$$

If we denote the price of a short bill with a face value of \$ 1 by $P(t, T+3)$ and that of a long bill with a face value of \$ 1 by $P(t, T+94)$, then

$$P(t, T+3) = \frac{1}{(1+s_1)} \text{ and } P(t, T+94) = \frac{1}{(1+s_2)}$$

Thus the price of the implied forward bill $P_t(T+3, T+94)$ is

$$\frac{1}{(1+f_{1,2})} = \frac{P(t, T+94)}{P(t, T+3)} \tag{5.10}$$

To preclude arbitrage the forward bill implicit in the futures contract should track the price of this synthetic forward bill.

5.16.2 The No-arbitrage Condition

The quoted T-bill futures price may be stated as follows:

Quoted Futures Price = 100.00 − Implicit T-bill Yield

The implicit T-bill yield is a discount yield and as in the case of the cash market, is based on a 360 day year.

If we denote the quoted futures price by F, then the actual transactions price or the *Invoice Price* of the futures contract, is given by

$$1{,}000{,}000 - 1{,}000{,}000 \times \frac{(100.00 - F)}{100} \times \frac{91}{360}$$

Let us denote the discount rate for the short bill by d_1 and that for the long bill

[6] These are not annualized rates.

by d_2. To preclude arbitrage, we require that

$$1{,}000{,}000 - 1{,}000{,}000 \times \frac{(100.00 - F)}{100} \times \frac{91}{360} = 1{,}000{,}000 \left[\frac{1 - d_2 \times \dfrac{T + 94 - t}{360}}{1 - d_1 \times \dfrac{T + 3 - t}{360}} \right] \quad (5.11)$$

Illustration Consider a futures contract with 14 days to expiration. Thus the short bill has 17 days to expiration while the long bill has 108 days to maturity. The quoted rate for the short bill is 4% while for the long bill it is 5%. The price of the short bill with a face value of \$ 1 is therefore \$ 0.9981. The price of the long bill is \$ 0.9850. Thus

$$1{,}000{,}000 - 1{,}000{,}000 \times \frac{(100.00 - F)}{100} \times \frac{91}{360} = 1{,}000{,}000 \times \frac{0.9850}{0.9981}$$
$$= 0.9869$$
$$\Rightarrow \quad F = 94.8176$$

5.17 The TED Spread

A spread using Treasury Bill futures and Eurodollar futures is known as a TED Spread. To set up such a spread, the investor needs to hold opposite positions in T-bill and ED futures contracts with the same expiration month.

To go long in the TED spread, the investor needs to take a long position in T-bill futures and a short position in ED futures. Quite obviously, to go short in the spread, he needs to go short in T-bill futures and long in ED futures.

$$\begin{aligned} \text{The TED Spread} &= \text{T-bill Futures Price} - ED \text{ Futures Price} \\ &= (100 - \text{Implied T-bill Rate}) - (100 - \text{Implied ED Rate}) \\ &= \text{Implied ED Rate} - \text{Implied T-bill Rate} \end{aligned} \quad (5.12)$$

Let us take the case of a speculator who expects the differential between the T-bill rate and the ED rate to widen. In such a situation he can operationalize his market view by going long in the TED spread. If he is right, and if the spread does indeed widen, then he will make a profit as we demonstrate in the following example.

5.17.1 Example

Assume that on 15 July 2009, the September 2009 T-bill contract is priced at 96.7, while the ED futures contract is priced at 96.1.

The TED spread is given by the difference in the two futures prices. In this case it is

$$96.7 - 96.1 = 0.60 \equiv 60 \text{ b.p.}$$

We will assume that on 1 September the T-bill futures are priced at 96.8, while the ED futures are priced at 96. The TED spread on this day is

$$96.8 - 96 \equiv 80 \text{ b.p.}$$

The profit from the T-bill position is

$$1{,}000{,}000 \times \frac{(96.80 - 96.7)}{100} \times \frac{90}{360} = \$\ 250$$

The profit from the ED position is

$$1{,}000{,}000 \times \frac{(96.1 - 96)}{100} \times \frac{90}{360} = \$\ 250$$

The total profit is therefore \$ 500.

This profit of \$ 500 represents a widening of 20 b.p. in the TED spread.

$$1{,}000{,}000 \times \frac{\Delta\delta}{100} \times \frac{90}{360} = 500$$

$$\Rightarrow \Delta\delta = 0.20$$

Speculators who expect the TED spread to narrow, can operationalize their view by going short in the TED spread.

5.17.2 Example

Assume that on 1 September, the T-bill futures continues to be priced at 96.80, whereas the ED futures are priced at 96.35. The spread is therefore 45 b.p.

If the trader were to go short in the spread, the profit from the T-bill futures position will be:

$$1{,}000{,}000 \times \frac{(96.7 - 96.8)}{100} \times \frac{90}{360} = -\$\ 250$$

The profit from the ED futures position will be:

$$1{,}000{,}000 \times \frac{(96.35 - 96.10)}{100} \times \frac{90}{360} = \$\ 625$$

The total profit is therefore \$ 375, which represents a narrowing of 15 b.p. in the spread.

5.18 Fed Funds Futures

The underlying asset is Federal Funds with a value of \$ 5 million dollars. Prices are quoted as 100.00 minus the Fed Funds overnight rate. At any point in time contracts for the next 24 consecutive months will be available. The contracts are cash settled.

Suggestions for Further Reading

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Web Sites

- www.cmegroup.com
- www.treasurydirect.gov

Concept Check

State whether the following statements are True or False.

1. A Euroyen deposit refers to Japanese Yen deposits held outside Europe.
2. The rate of return earned on a T-bill, will always be less than the dollar discount expressed as a percentage of the face value.
3. The Bond Equivalent Yield is also known as the Coupon Equivalent Yield.
4. T-bill futures contracts are delivery settled.
5. Eurodollar futures contracts are delivery settled.
6. When interest rates fall, the longs will gain whereas if interest rates rise, the shorts will gain.
7. Borrowers require long hedges while lenders require short hedges.

8. A hedge using contracts of different maturities, where the number of contracts for each maturity is not identical, is called a Stack Hedge.
9. If there is a parallel shift in the Yield Curve, then the performance of the Strip and Stack hedges will be equivalent.
10. The TED Spread can sometimes be negative.
11. On-the-run T-bills will be more liquid than off-the-run T-bills.
12. In a uniform yield auction, bids are arranged in descending order of yield.
13. Euroyen futures are available based on both LIBOR as well as TIBOR.
14. To hedge the rate for a 72 day loan of $ 10 MM using ED futures, we need eight futures contracts.
15. T-bill futures are based on 26-week bills.
16. T-bills with more than 182 days to maturity will pay a coupon.
17. Interest on a floating rate loan is usually determined in arrears and paid in arrears.
18. There is no limit on the amount that a competitive bidder may receive in a T-bill auction.
19. If the investor expects the TED spread to widen, he will go long in T-bill futures and short in ED futures.
20. The number of contracts in an ED pack is fixed at four.

QUESTIONS AND PROBLEMS

Question-I

Short term interest rate futures prices are quoted in terms of index values rather than in terms of rates of interest. Explain.

Question-II

A T-bill with 144 days to maturity and a face value of $ 1,000,000 has a quoted yield of 7.5%.

What will be its price?

Question-III

What is the coupon equivalent yield for the above T-bill?

Question-IV

An investor purchases a T-bill maturing on November 21, 20XX on March 1, 20XX. The bill has a face value of $ 1,000,000 and the purchase price is $ 975,000. If he holds the bill till maturity, then what is the Bond Equivalent Yield?

Question-V

What is a eurocurrency? What are the factors responsible for the growth of the eurodollar market?

Question-VI

A T-bill futures contract is expiring after 30 days. A 33 day T-bill is available at a discount of 4.8%, and a 124 day bill is available at a discount of 5.2%. What should be the no-arbitrage price for the futures contract?

Question-VII

Assume that today is October 1, 2008. ED futures contracts expiring on December 21, are priced at F. A three month ED deposit entered into on December 21, will expire on March 21, 2009. The interest rate for an ED deposit between October 1 and March 21, is 5.5% per annum. The interest rate for an ED deposit between October 1 and December 21 is 3.5% per annum.

What should be the futures price if arbitrage is to be ruled out?

Question-VIII

A bank wishes to hedge the rate on a 117 day loan using ED futures. The relationship between the 117 day rate and the 90 day rate is given by:

$$d_{117} = 0.025 + 1.025 d_{90} + \epsilon$$

There is a futures contract expiring on the day the loan is to be made. The futures price is 95.50.

Show that the effective rate of interest locked in by the bank is the same, irrespective of whether the futures price at expiration is 94 or 96.50.

Question-IX

The current T-bill futures price is 94.3 and the ED futures price is 93.9. Consider a speculator who expects the yield spread to widen.

What strategy will he initiate?

Assume that 45 days later, the T-bill futures price is 94.8 and that the ED futures price is 94.2.

What will be his profit?

Question-X

HCL Infosystems has approached ICICI bank for a six month fixed rate loan. The bank is in a position to borrow money at LIBOR + 75 b.p. for periods of three months at a time. The current 90 day LIBOR is 6.2% and the current 180 day LIBOR is 6.8%. ED futures contract are available that expire after 90 days.

What is the minimum rate that the bank will quote for a six month loan?

Appendix–V

We will derive the expression for y, given that

$$P\left(1+\frac{y}{2}\right)\left(1+\frac{y}{2}\left\{\frac{T_m-\frac{365}{2}}{\frac{365}{2}}\right\}\right)=V$$

From the above expression, when we expand the terms we get,

$$P+\frac{Py}{2}+\frac{Py}{2}\times\frac{2T_m-365}{365}+\frac{Py^2}{4}\times\frac{2T_m-365}{365}=V$$

$$\Rightarrow y^2\times\frac{P}{4}\times\frac{2T_m-365}{365}+y\times\frac{P}{2}\times\left(1+\frac{2T_m-365}{365}\right)+(P-V)=0$$

This is a quadratic equation of the form,

$$ax^2+bx+c=0$$

The roots are given by,

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Substituting the appropriate values for a, b, and c, and simplifying, we get,

$$\frac{\frac{-2T_m}{365}\pm 2\sqrt{\left(\frac{T_m}{365}\right)^2-\left(\frac{2T_m}{365}-1\right)\left(1-\frac{V}{P}\right)}}{\frac{2T_m}{365}-1}$$

6

Bond Market Futures

6.1 Introduction

Debt securities are a type of financial claim that are used by borrowing entities to raise capital. Unlike equity shares which confer ownership rights on parties who subscribe to them, a debt security merely represents the fact that the holder has lent money to the borrower, and is consequently entitled to receive interest at periodic intervals. The principal amount will be repaid at a prespecified maturity date. Such securities usually have a finite life span, unlike equity shares which do not have a stated maturity date. From the standpoint of the borrower, the interest payments on such securities are a contractual obligation. In other words, the borrowers are required to make the promised payments irrespective of the financial performance of the firm in a given time period. In contrast, a firm can always reduce the dividends on equity if it deems it necessary. In fact, there could be periods in which the firm chooses to skip the dividends entirely. Debt holders enjoy priority over shareholders in two respects. First, the interest claims of debt holders have to be satisfied before any dividends can be paid to the shareholders. Second, in the event of bankruptcy or liquidation of the firm, the proceeds from the sale of assets must be used to first settle all outstanding interest and principal that is due to the debt holders. Only the residual amount if any can be distributed among the shareholders.

Debt instruments can be *secured* or *unsecured*. In the case of secured debt, the terms of the contract will specify the assets of the firm that have been pledged as security or collateral. In the event of the company's inability to repay, the bond holders have a right over these assets. In the case of unsecured debt there is no specific pledge of collateral. In the US the word 'Debenture' is specifically used to refer to unsecured debt. Debt instruments can be either *negotiable* or *non-negotiable*. Negotiable instruments are securities which can be endorsed from one party to another, and hence can be bought and sold easily in the financial markets. A non-negotiable instrument is one which cannot be transferred. Equity shares are an obvious example of a negotiable security. However, loans made by commercial banks to business firms, and savings bank accounts and time deposits of individuals, are non-negotiable in nature.

While debt is important for corporations, it is indispensable for central or federal, state, and local (municipalities) governments, for, such entities obviously cannot issue equity shares. Unlike debt issued by entities in the private sector, *US Treasury securities are fully backed by the federal government, and consequently have little or no credit risk associated with them*.[1] Thus, the interest rate on such securities becomes a benchmark for setting the rates of returns on other, more risky, securities. The US Treasury issues three categories of marketable debt instruments. Marketable securities are those that can be readily traded in the financial markets. T-bills are discount or zero coupon securities. That is they are sold at a discount from their face value, and do not pay any interest explcitly. T-notes and T-bonds are issued at face value and are interest bearing instruments which typically make payouts at semi-annual intervals. T-bills are grouped under the category of *Money Market or Short Term Debt* instruments since they have a life of less than one year, whereas T-notes and T-bonds are considered to be *Long Term Debt* instruments, and are consequently classified as *Capital Market* instruments. A T-note is akin to a T-bond, but has a time to maturity between 1-10 years at the time of issue, whereas T-bonds have a life in excess of 10 years.

6.2 Fundamentals of Bond Valuation

The concepts of time value of money, can be used to value any asset, whose value is derived from future cash flows. We will see in this chapter, as to how these principles are applied by investors to establish the value of a bond.

An ordinary bond, also known as a *Plain Vanilla Bond*, is a fairly simple instrument.[2] It will pay interest periodically, typically every six months, and will repay the principal at maturity. *Zero Coupon* or *Deep Discount* securities, like T-bills, pay no interest. They are sold for less than the amount that is payable at maturity. The difference between the terminal payment and the issue price, constitutes the interest.

We will first take a close look at the pricing of plain vanilla bonds and attempt to understand the related principles and conventions. The following notation will be used in the course of our discussion. Additional variables will be defined as we go along.

6.2.1 Notation

- $M \equiv$ Face Value[3]
  The *Face Value* or *Par Value* is the amount that the issuer promises to pay to the lender at maturity. It is also referred to as the *Redemption Value* or *Maturity Value* or *Principal Value*.

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[1]The term *credit risk* refers to the probability of default.
[2]The most basic form of any security is referred to as the *Plain Vanilla* version. More exotic versions are said to have *'Bells and Whistles'* attached.
[3]The symbol $\equiv$ stands for 'is equivalent to'.

- $T \equiv$ Term to Maturity
  A key feature of any bond is its Term to Maturity, which is the time remaining in the life of the bond. This is the period during which the borrower has to meet the conditions of the debt. A bond's term to maturity is the length of time after which the debt shall cease and the issuer will redeem the issue by paying the face value or principal back to the lender.[4] If we denote today as time t and the maturity date as time T, then $T - t$ represents the term to maturity. The words Maturity, Term and Term to Maturity are used interchangeably to refer to the number of years remaining in the life of the bond.
  The maturity of a bond indicates the expected life of the instrument, or the number of periods during which the holder can expect to receive the interest payments.[5] It also represents the number of periods, before the principal will be repaid.
  A plain vanilla bond is a *Term Bond*, that is, it has a single maturity date which is fixed at the outset. But there can arise situations, where a bond issue can be retired early by the issuer, either in full or in part. In the case of callable bonds, which we will discuss later, the issuer can redeem the bonds by paying off the lender before the scheduled maturity date. The presence of *Sinking Fund* provisions can also lead to bonds being redeemed before maturity. In many cases, the sinking fund provisions require the issuer to retire a part of the debt according to a prespecified schedule during the life of the bond.
- $C \equiv$ Coupon
  A bond's coupon is the periodic interest payment made to the owners during the life of the bond. The coupon rate c is the rate of interest that, when multiplied by the par value, provides the dollar value of the annual coupon payment denoted by C. If the annual coupon rate is c, then the semi-annual coupon payment is $C/2$. The word 'coupon' is used because traditionally bonds were issued with a booklet of post-dated coupons. The bondholder was expected to redeem a coupon for cash at the end of every coupon period.
  Example
  Consider a bond with a face value of \$ 1,000, and a coupon of 10% per annum paid semi-annually. Since $C = .10$, the semi-annual interest is given by
  $$\frac{C}{2} = \frac{.10}{2} \times 1{,}000 = \$\ 50$$
- $y \equiv$ Yield to Maturity
  The *yield to maturity* or the *YTM* is the rate of return that the buyer will get, if he buys the bond at the prevailing price and holds it to maturity. It makes an assumption that all intermediate coupon payments are reinvested

[4]Remember, the issuer of the bond is the borrower and the buyer of the bond is the lender.

[5]We use the phrase *Expected Life*, because bonds other than those issued by the government are subject to *Default Risk*, which means that the borrower may cease to honour his obligations before the stated time to maturity. Secondly, in practice many bonds are *Callable*, that is they can be recalled by the issuer well before they are due to mature.

at the YTM itself. Those of you who are familiar with Capital Budgeting, should note that it is similar to the concept of the *Internal Rate of Return* or *IRR* used in project appraisal.

- $N \equiv$ Number of Coupons Left in the Life of the Bond
- $k \equiv$ Time Until the Receipt of the First Coupon Expressed as a Fraction of Six Months
  If we are standing on a coupon date, then $k = 1$ else $k < 1$.
- $P_{i,t} \equiv$ Clean Price of Bond i in the Spot Market at time t.
- $P_d \equiv$ Dirty Price
- $AI_{i,t_1,t_2} \equiv$ Accrued Interest on Bond i from t_1 (the last coupon date) till t_2 (the date of valuation).

6.2.2 Valuation of a Bond as on a Coupon Date

The price of a bond as on a coupon date may be expressed by the following equation.

$$P_d = \sum_{t=1}^{N} \frac{\frac{C}{2}}{\left(1+\frac{y}{2}\right)^t} + \frac{M}{\left(1+\frac{y}{2}\right)^N} \tag{6.1}$$

A cursory glance will reveal that the price of a bond is nothing but the present value of the stream of coupon payments, which is an ordinary annuity, discounted using the YTM, plus the present value of the principal repayable at maturity.

Example Let us assume that today is the 15th of August, 2008. A Treasury Bond is available that matures on 15 August, 2028 and pays a coupon of 9% semi-annually on 15 February and 15 August every year.[6] The face value is \$ 1,000 and the yield to maturity is 8%.

The price of the bond may be calculated as follows.

We know that $M = 1000$, $c = 9\%$, $y = 8\%$ and $N = 40$. Therefore

$$P_d = \sum_{t=1}^{40} \frac{\frac{(.09 \times 1{,}000)}{2}}{\left(1+\frac{.08}{2}\right)^t} + \frac{1000}{\left(1+\frac{.08}{2}\right)^{40}}$$

$$= 45 \times \text{PVIFA}\ (4{,}40) + 1{,}000 \times \text{PVIF}\ (4, 40)$$

$$= 890.6748 + 208.2890 = \$\ 1098.9639$$

where PVIFA is the Present Value Interest Factor Annuity and PVIF is the Present Value Interest Factor.[7]

[6] Unless otherwise stated the coupon rates given represent annual rates of interest.

[7] $\text{PVIFA}\ (r, N) = \frac{\left[1 - \frac{1}{(1+r)^N}\right]}{r}$ and $\text{PVIF}\ (r, N) = \frac{1}{(1+r)^N}$, where r is the discount rate and N is the number of periods.

6.3 Yield to Maturity: A Detailed Exposition

Once we know the market's required rate of return from a bond, we can calculate its price by calculating the present values of the cash flows generated by it. On the other hand if we are given information about the price of a bond, we can find the rate of return that equates the present value of the cash flows to the price. This is nothing but the *Internal Rate of Return (IRR)* of the bond. The IRR of a bond is termed as its *Yield to Maturity (YTM)* or its *Redemption Yield.*

In practice an investor will have a choice between a wide variety of bonds with different coupons and terms to maturity. Besides the credit-worthiness of the issuer will vary from bond to bond. Consequently, although data about bonds is primarily provided in the form of prices, it is the yield measure that facilitates comparison between different instruments.

The YTM is the interest rate that equates the present value of the cash flows from the bond (assuming that the bond is held to maturity), to the price of the bond. It is that value of y, which satisfies the following equation.

$$P = \sum_{t=1}^{N} \left[\frac{\frac{C}{2}}{\left(1 + \frac{y}{2}\right)^t} \right] + \frac{M}{\left(1 + \frac{y}{2}\right)^N}$$

In general, we require a computer program to calculate the YTM. The calculation is however, fairly simple for a coupon bond with two periods left to maturity, in which case we get a quadratic equation, and for zero coupon bonds.

6.3.1 Sources of Returns

The YTM calculation takes into account the coupon payments, as well as any capital gains/losses that accrue to an investor who buys and holds a bond to maturity. Before we analyze the YTM in detail, let us consider the various sources which contribute to the returns received by a bondholder.

A bondholder can expect to receive income from the following sources.

- Coupon payments, which are typically paid every six months.
- A capital gain/loss, that is obtained when the bond matures, or is called before maturity, or is sold before maturity. From the YTM equation, you can see that we are assuming that the bond is held to maturity.
- Reinvestment of coupon payments, from the time each coupon is paid till the time the bond matures, or is sold, or is called. Once again, the YTM calculation assumes that the bond is held till maturity. The reinvestment income is the *Interest on Interest* income.

A satisfactory measure of the yield should take into account all the three sources of income. The YTM does indeed take all sources of income into account. However, it makes two key assumptions.

- The bond is held until maturity.
- The intermediate coupon payments are reinvested at the YTM itself.

The second assumption is in built into the mathematics of the YTM calculation, as we shall shortly demonstrate.[8]

The YTM is called a *Promised Yield*. Why do we use the word *promised*? It is a promise, because, in order to realize it the bondholder has to satisfy the above two conditions. If either of them is violated, he may not get the promised yield. The assumption that the bond is held to maturity is fairly easy to comprehend. Let us now focus on the reinvestment assumption.

Consider a bond which pays coupons on a semiannual basis, i.e. it pays $C/2$ every six months. Let r be the rate at which the investor can reinvest the coupon payments till maturity. r would depend on the prevailing rate of interest when the coupon is received, and need not be equal to y, the yield prevailing at the outset, or c, the coupon rate. For ease of exposition, we will assume that r is constant for the life of the bond.

Thus each coupon payment is reinvested at $\frac{r}{2}$ (for six monthly periods). The coupon stream is obviously an annuity. The final payoff from reinvesting the coupons is therefore given by the future value of this annuity, using a rate of $\frac{r}{2}$ for six monthly periods.

The future value is

$$\frac{\frac{C}{2}}{\frac{r}{2}}\left[\left(1+\frac{r}{2}\right)^N - 1\right]$$

Note, this amount represents the sum of all coupons which are reinvested (which in this case is the principal), plus interest earned on reinvesting the coupons.

The total value of the coupons $= \frac{C}{2} \times N = \frac{NC}{2}$. Thus, interest on interest is equal to

$$\frac{\frac{C}{2}}{\frac{r}{2}}\left[\left(1+\frac{r}{2}\right)^N - 1\right] - \frac{NC}{2}$$

The YTM calculation assumes that $\frac{r}{2} = \frac{y}{2}$.

We will now demonstrate that an investor will actually get a rate of return equal to the YTM, only if he manages to reinvest all the coupons at the YTM.

6.3.2 Example

Consider an IBM bond that has 10 years to maturity. The face value is $ 1,000. It pays a semi-annual coupon at the rate of 10% per annum. The YTM is 12% per annum.

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[8]Remember, the IRR calculation also assumes that intermediate cash flows are reinvested at the IRR itself.

The price can be shown to be \$ 885.295.

We will assume that the semi-annual interest payments can be reinvested at a six monthly rate of 6%, which corresponds to a nominal annual rate of 12%. Let us analyze the sources of income for a bondholder, assuming that he holds the bond till maturity.

1. Total coupon received = $50 \times 20 = \$\ 1000$
2. Interest on interest got by reinvesting the coupons

$$= \frac{50[(1.06)^{20} - 1]}{.06} - 1{,}000$$
$$= 50 \times 36.786 - 1{,}000 = \$\ 839.3$$

   Notice that, if we do not deduct \$ 1,000 $\left(\text{or } \frac{NC}{2}\right)$ in the above equation, we can calculate the income from both the above sources together. We have separated them for ease of exposition.
3. Finally, in the end, the bondholder will get back the face value of \$ 1,000.

Thus, in the end the bondholder will have $1{,}000 + 839.3 + 1{,}000 = \$\,2{,}839.3$.

To get this income he has to pay \$ 885.295 today, which is his investment. So what is the rate of return that he has earned? It is the value of i, that satisfies the following equation.

$$885.295(1 + i)^{20} = 2,839.3$$
$$\Rightarrow (1 + i) = \left[\frac{2839.3}{885.295}\right]^{.05} = 1.0600$$
$$\Rightarrow i = .06 \equiv 6\%$$

Thus the return is 6% on a semi-annual basis, or 12% on a nominal annual basis, which is exactly the same as the YTM. So, how did the bondholder realize the YTM? Only by being able to reinvest the coupons at a nominal annual rate of 12%, compounded on a semiannual basis.

Notice that the reinvestment rate affects only the interest on interest income. The other two sources are unaffected. If $r > y$, the interest on interest would have been higher, and i would have been greater than y. On the contrary, if $r < y$, the interest on interest would have been lower, and i would have been less than y.

Hence, if an investor buys a bond by paying a price which corresponds to a given YTM, he will realize that YTM only if, he holds the bond till maturity, and he is able to reinvest the coupons at the YTM.

The Reinvestment Assumption We can claim that we have received a YTM of $y\%$ per annum, if the compounded semi-annual return on our initial investment is $\frac{y}{2}$.

The initial investment is

$$P = \frac{\frac{C}{2}}{\frac{y}{2}}\left[1 - \frac{1}{\left(1+\frac{y}{2}\right)^N}\right] + \frac{M}{\left(1+\frac{y}{2}\right)^N}$$

The compounded value of the investment is

$$P \times \left(1+\frac{y}{2}\right)^N = \frac{\frac{C}{2}}{\frac{y}{2}}\left[\left(1+\frac{y}{2}\right)^N - 1\right] + M$$

The terminal cash flow from holding the bond, assuming that each coupon is reinvested at $\frac{r}{2}$ per semi-annual period, is

$$= \frac{\frac{C}{2}}{\frac{r}{2}}\left[\left(1+\frac{r}{2}\right)^N - 1\right] + M$$

Equating the two, we get

$$\frac{r}{2} = \frac{y}{2}$$

Thus in order to get an annual YTM of y%, every intermediate cash flow must be reinvested at $\frac{y}{2}$% per six monthly period.

6.4 Callable Bonds

Callable bonds give the issuer, the right to retire the debt before the maturity date. This benefits issuers, because if interest rates decline, they can recall the old bonds, and replace them with a new issue carrying a lower coupon. The call provision allows the issuer to alter the maturity of the bond. This works against the investor as it introduces cash flow uncertainty. Thus callable bonds will offer a higher yield than otherwise similar non callable bonds. That is, they will sell for a lower price, compared to non callable bonds with the same face value and coupon. The way to appreciate this is as follows. At the time of issue, a company issuing callable bonds would have to pay a higher coupon, than what it would have to pay were it to issue plain vanilla bonds. After the issue, a callable bond will sell at a lower price, or a higher yield, than a plain vanilla bond of the same credit quality, and an identical coupon.

Certain bonds have a *Call Protection Period*. This is a time period, during which the bonds cannot be recalled. Such bonds are referred to as *deferred callable bonds*. Normally, in order to sweeten the issue, the borrowers will offer a *Call Premium*, which is often equal to one year's coupon. That is, if and when the bond is prematurely called, the holder will be paid the face value plus one year's coupon as a bonus.

Holders of callable bonds face two types of risk. Firstly, they are exposed to reinvestment risk. That is, since the bond will be called back when the rates in the market are falling, they face the possibility of having to reinvest the proceeds at a lower rate of interest. Secondly, the potential for price appreciation is limited in an economic environment where interest rates are falling. This is because the market will expect the bonds to be called back and consequently will not offer the same price as it would for a plain vanilla bond. This feature is referred to as *Price Compression*.

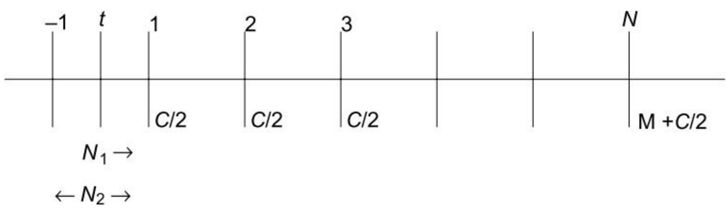
6.5 Valuation of a bond between coupon dates

Valuing a bond in between coupon dates is a more complex exercise. This is because, the next coupon date will obviously be a fraction of a coupon period away. In practice, there is no unique way of computing this fractional period. The logically consistent way of valuing the bond would be to first price it as of the forthcoming coupon date. This price, can then be discounted back for the fraction of the current coupon period that is remaining. We will now illustrate this with the the help of an example.

6.5.1 Illustration

Assume that we are valuing the bond at a time t, such that the time till the next coupon is N_1 days, and that the current coupon period consists of N_2 days. There are N coupons remaining in the life of the bond. The cash flows remaining in the life of the bond can then be depicted as follows:

Figure 6.1 **Cash Flows From a Bond**



In the above timeline:

- -1 is the date of the previous coupon
- t is the valuation date
- 1 is the date of the next coupon

The value of the bond at time 1, which represents the date of the next coupon date, is given by

$$\frac{C}{2} + \frac{C}{2} \times \text{PVIFA}\left(\frac{y}{2}, N-1\right) + M \times \text{PVIF}\left(\frac{y}{2}, N-1\right)$$

Thus, the value of the bond at the current point in time, *time t*, can be expressed as

$$\frac{1}{\left(1+\frac{y}{2}\right)^k}\left[\frac{C}{2} + \frac{C}{2} \times \text{PVIFA}\left(\frac{y}{2}, N-1\right) + M \times \text{PVIF}\left(\frac{y}{2}, N-1\right)\right] \quad (6.2)$$

$k = \frac{N_1}{N_2}$, is the fraction of the current coupon period that is remaining.

6.5.2 Day-Count Conventions

In practice, there is no unique way of computing, k, or the fraction of the current coupon period that is remaining. There are different conventions in different markets, and sometimes for different debt securities in the same market. A day-count convention is a particular method for measuring N_1 and N_2, or in other words to enable us to compute the fractional period k. We will take a detailed look at the *Actual/Actual* method, which is the convention that is used for Treasury notes and bonds in the US.

The Actual-Actual Method Consider a Treasury bond that was issued on 15 February 20XX, and which pays coupons on 15 February and 15 August every year. The bond has thirty years to maturity and has a face value of $ 1,000. The coupon rate is 12% per annum, and coupons are paid on a semi-annual basis. The YTM is 10% per annum. Today, the date of valuation, is 1 June 20XX.

The value of the bond as on the next coupon date will be:

$$60 + 60 \times \text{PVIFA}\ (5, 59) + 1{,}000 \times \text{PVIF}\ (5, 59)$$

The value of the bond today should therefore be:

$$\frac{1}{(1.05)^k}\,[60 + 60 \times \text{PVIFA}\ (5, 59) + 1{,}000 \times \text{PVIF}\ (5, 59)]$$

In the Actual-Actual method, N_1 and N_2, and consequently k, are calculated based on the actual number of days in the corresponding time periods. We have therefore to take cognizance of facts such as whether or not the year in question is a leap year. For the purpose of calculation, we are required to exclude the starting date, and include the ending date, or the other way. In our case, N_1 and N_2 will be computed as shown in Tables 6.1 and 6.2.[9]

So $k = \frac{N_1}{N_2} = \frac{75}{182} = 0.4121$

As per this convention, N_2, or the number of days between two successive coupon dates, will always range between 181 and 184.

[9] Assume that the current year is a leap year.

Table 6.1 **Calculation of N_1**

Month	No. of Days
June	29
July	31
August	15
N_1	75

The value of the bond in this case is therefore

$$\frac{1}{(1.05)^{0.4121}}[60 + 60 \times \text{PVIFA}(5, 59) + 1,000 \times \text{PVIF}(5, 59)] = \$\ 1,223.90$$

Table 6.2 **Calculation of N_2**

Month	No. of Days
February	14
March	31
April	30
May	31
June	30
July	31
August	15
N_2	182

6.5.3 Accrued Interest

Let t_1 and t_2 represent two consecutive coupon payment dates. We will denote the sale date of the bond by t, such that $t_1 < t < t_2$. On day t the seller will be giving away the right to the entire next coupon, which is due on date t_2, although he or she would have held the bond for a part of the current coupon period. This portion of the next coupon that belongs to the seller is known as the the accrued interest. It may be computed as follows.

$$\text{AI} = \frac{C}{2} \times \frac{t - t_1}{t_2 - t_1} = \frac{C}{2} \times \frac{N_2 - N_1}{N_2} = \frac{C}{2} \times \left(1 - \frac{N_1}{N_2}\right) \tag{6.3}$$

The symbols N_1 and N_2 are as defined earlier. The day-count convention that is adopted for calculating the accrued interest for a bond, that is, for computing N_1 and N_2, is the same as the convention that is used to value the bond. Thus for T-bonds and notes, the Actual-Actual method is adopted.

In the case of the US Treasury bond which was sold on 1 June 20XX, the accrued interest is

$$\frac{120}{2} \times \frac{107}{182} = \$\ 35.2747$$

A logical question that a reader may have is as to why is there a need to consider accrued interest. Is it the case that the bond valuation equation fails to take the accrued interest into account? The bond valuation equation values the bond as the present value of all cash flows arising from it, from the point of sale onwards. This is the theoretically correct procedure for valuing any financial security. The reason why we need to take the accrued interest into account is that in practice, the quoted bond prices are reported net of accrued interest. These prices, which do not include the accrued interest are called *Clean* or *Add-Interest* prices. Since the convention is to quote prices net of the accrued interest, the buyer needs to calculate and factor in the accrued interest in order to determine the total amount payable by him. The total price that is payable is called the *Dirty Price*, since it is contaminated with accrued interest.

We will illustrate the rationale behind this market convention with the help of an example. Take the case of the 12% Treasury bond maturing on 15 February 20(XX+30). On 1 June 20XX, the price at a YTM of 10% is \$ 1,223.90.

Let us assume that four days hence, the YTM continues to be 10% per annum. The dirty price of the bond will be

$$\frac{1}{(1.05)^k}[60 + 60 \times \text{PVIFA}\,(5, 59) + 1{,}000 \times \text{PVIF}\,(5, 59)]$$

$$= \frac{1}{(1.05)^{\frac{71}{182}}}[60 + 60 \times \text{PVIFA}\,(5, 59) + 1{,}000 \times \text{PVIF}\,(5, 59)]$$

$$= \frac{1}{(1.05)^{0.3901}}[60 + 60 \times \text{PVIFA}\,(5, 59) + 1{,}000 \times \text{PVIF}\,(5, 59)]$$

$$= 1{,}225.2146$$

The difference in the dirty prices is

$$1{,}225.2146 - 1{,}223.9000 = \$\ 1.3146$$

The accrued interest as computed on the second date is

$$60 \times \frac{111}{182} = 36.5934$$

The change in the accrued interest during this period is therefore

$$36.5934 - 35.2747 = 1.3187$$

which is virtually identical to the change in the dirty price. Thus, the difference in the dirty prices is almost entirely due to the increase in the accrued interest.

From the perspective of a bond analyst, who is tracking the debt market, the issue of concern is the change in the required market yield. Consequently, he ought to be able to focus on yield induced price changes, without being distracted by other factors. If the price were to be quoted including the accrued interest, the effects of yield induced changes cannot be easily discerned. Hence the practice in debt markets is to report prices net of accrued interest. If so, any observed price changes in the very short run, will be entirely yield induced.

6.6 Duration

Bond market watchers have long been aware of the fact that long term bonds are more vulnerable, in terms of price changes, to changes in yield than shorter term bonds. Consider the case of two bonds with a face value of \$ 1,000 each. Let us assume that both pay a coupon of 10% per annum on a semi-annual basis, and that the YTM is 10% per annum in both cases. Bond A has a time to maturity of 10 years, while bond B has a time to maturity of 20 years. Both the bonds will obviously sell at par since the coupon rate is equal to the yield.

Let us now consider an increase of 200 b.p. in the yield, that is, assume that the YTM increases to 12% per annum. The price of bond A will decline to 885.3008, whereas that of bond B will decline to 849.5370. Thus the price of bond A will decline by 11.47%, whereas that of bond B will decline by 15.05%. Therefore, it does indeed seem to be the case that long-term bonds are more impacted more by interest rate changes. What could be the rationale for this fact? We know that the present value of a cash flow is given by

$$\frac{CF}{(1+r)^t}$$

The larger the value of t, the greater will be the impact of a change in the discount rate, for a given cash flow. A 20 year bond has a considerable amount of its cash flows coming in at later points in time as compared to a 10 year bond. Thus, it is the case that its price, which is the sum of the present values of the cash flows arising from it, is more vulnerable to changes in the interest rate.

After answering this question, it was found that there was a related issue which too merited an explanation. Take the case of a 10 year zero coupon bond with a face value of \$ 1,000. When the YTM is 10%, its price is

$$\frac{1{,}000}{(1.05)^{20}} = 376.8895$$

However, when the YTM is 12%, its price is

$$\frac{1{,}000}{(1.06)^{20}} = 311.8047$$

The corresponding decline in price is 17.2689%.

While it is understandable that a long term bond ought to be more vulnerable to interest rate changes, it does not explain why a 10-year zero coupon bond should be more price sensitive than a 10 year bond that pays a coupon of 10%, for after all, both have 10 years to maturity. Further reflection, lead to the following explanation. A coupon paying bond is a series of cash flows arising at six monthly intervals. In other words it is nothing but a portfolio of zero coupon bonds. When we state that a bond has a maturity of N half-years, where N is the number of coupons, we are only taking cognizance of the last of the cash flows. If the bond itself is a portfolio of zero coupon components, then it is obvious that its effective time to maturity ought to be an average of the times to maturity of the component zero coupon bonds. However, a 10-year zero coupon bond will give rise to a single cash flow at maturity. In this case, therefore, its stated time to maturity will be the same as its effective time to maturity. From this perspective, it is not surprising that a ten year zero coupon bond is more price sensitive, for it does have a greater effective time to maturity.

Macaulay came up with the concept of *'Duration'* as a measure of the effective term to maturity of a plain vanilla bond. The duration of a bond is computed by weighting the term to maturity of each component cash flow by the fraction of the total present value of the bond that is contributed by that particular cash flow. We will now give a numerical illustration.

6.6.1 Illustration

Consider a 5 year 10% coupon paying bond with a face value of \$ 1,000. If the YTM is 12% per annum, what should be the duration?

Table 6.3 Computing the Duration

Time = t	Cash Flow = CF_t	Present Value of Cash Flow $= \frac{CF_t}{\left(1+\frac{y}{2}\right)^t}$	Weight of Cash Flow $w_t = \frac{CF_t}{P_0\left(1+\frac{y}{2}\right)^t}$	Weighted Time $w_t \times t$
1	50	47.1698	.05092	.05092
2	50	44.4998	.04804	.09608
3	50	41.9810	.04532	.13596
4	50	39.6047	.04275	.17101
5	50	37.3629	.04033	.20166
6	50	35.2480	.03805	.22829
7	50	33.2529	.03590	.25126
8	50	31.3706	.03386	.27090
9	50	29.5949	.03195	.28752
10	1,050	586.3145	.63290	6.3290
	Total	926.3991	1.00	8.0225

The weighted average time to maturity of the component cash flows is 8.0225 semi-annual periods or 4.0113 years. For a five year zero coupon bond its weighted average time to maturity is the same as its stated time to maturity of 5 years, because it gives rise to a single cash flow. Thus, it is not really surprising that the five year zero coupon bond is more price sensitive than the five year coupon paying bond.

6.6.2 Duration and Price Sensitivity

There is a relationship between the duration of a bond and the rate of change of the percentage change in its price with respect to a yield change. This may be expressed as

$$\frac{\frac{dp}{P}}{dy} = -\frac{D}{(1+\frac{y}{2})} \tag{6.4}$$

where D is the duration of the bond expressed in annual terms.

$$\frac{D}{(1+\frac{y}{2})}$$

is called the modified duration of the bond, D_m. Thus the rate of change of the percentage change in price with respect to the yield is equal to the modified duration of a bond, and not to its time to maturity. Therefore, it is obvious that it is the duration of a bond, and not its stated time to maturity, which accurately captures the relationship between a change in the yield and the corresponding change in its price. Thus the duration of the bond, and not the time to maturity, is a logical measure of interest rate sensitivity.

The correct way to look at duration is as a measure of the interest rate sensitivity of the debt instrument. It cannot always be perceived as a measure of the effective average life of the instrument. As Fabozzi (1993) points out, there are certain instruments like Collateralized Mortgage Obligations (CMOs), where the instrument can have a greater duration than the underlying loan. In fact, some CMOs can have a negative duration!

6.6.3 A Concise Formula

While duration can always be calculated by finding the present value of each cash flow in order to compute the weighted average time to maturity, there is a precise formula for the duration of plain vanilla bonds, which may be stated as follows.

$$D = \frac{1+y}{y} - \frac{(1+y)+N(c-y)}{c[(1+y)^N - 1]+y} \tag{6.5}$$

where c is the semi-annual coupon rate; y is the semi-annual YTM, and N is the number of coupons remaining in the life of the bond.

Illustration Once again, let us consider the 5 year 10% coupon paying bond with a face value of \$ 1,000. The YTM is 12% per annum.

So the duration is

$$\frac{(1.06)}{0.06} - \frac{(1.06)+10(0.05-0.06)}{0.05[(1.06)^{10}-1]+0.06} = 17.6667 - \frac{0.96}{0.0995} = 8.0225$$

6.7 The Cash Market

A Treasury note is a debt instrument issued by the Treasury that has an original term to maturity ranging from one to ten years. In practice, the Treasury issues notes with maturities of 2, 5 and 10 years. 2-year and 5-year notes are auctioned on a monthly basis. The auctions are usually announced on the third or fourth Monday of each month, and are issued on the last day of the month.[10] 10-year note auctions are usually announced on the first Wednesday in February, May, August, and November, and are issued on the 15th of the respective months.[11] The 10-year note is also auctioned as a re-opening of an existing security in March, June, September, and December.

A Treasury bond by definition, is a coupon bearing debt instrument issued by the Treasury, that has an original maturity that is greater than 10 years. In practice, T-bonds are issued for a 30-year period. The bond auction is usually announced on the first Wednesday in February and August, and the issue takes place on the 15th of the month. The 30-year bond is also auctioned as the re-opening of an existing security in May and November.

The minimum amount that can be purchased in note and bond auctions is $ 100. Securities can be acquired in multiples of this amount. As in the case of T-bill auctions, both competitive and non-competitive bids are entertained. The limit for a non-competitive bid is $ 5 MM. A competitive bidder cannot acquire more than 35% of the amount on offer. Auctions are conducted on a uniform yield basis.

6.7.1 Price Quotes

T-Bond and T-Note prices are quoted as a percentage of par plus 32^{nds}. The quotes are for a par value of $ 100.

Bond prices are always quoted per $ 100 of face value. The rationale is the following. Consider two bonds, one with a face value of $ 1,000 and the other with a face value of $ 2,000. If the prices of the two bonds are equal and are quoted as say $ 1,400, the implications would be very different for the two bonds. A quote of $ 1,400 would imply that the $ 1,000 face value bond is trading at a significant premium, whereas the implication would be that the $ 2,000 face value bond is trading at a considerable discount. However, if prices were to be quoted as per $ 100 of face value, a price exceeding $ 100.00 would signify a premium bond, whereas a price of less than $ 100.00 would connote a discount bond.

An actual price quote can best be illustrated with the help of an example.

Example T-Bonds maturing on 15 August, 2021 are currently being quoted at 98-28. What is the corresponding price for bonds with a face value of $ 100,000?

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[10] If the last day of the month is a holiday, then the issue will take place on the first business day of the following month.

[11] If the 15th day of the month were to be a holiday, then the issue will take place on the following business day.

98-28 means that the price of a bond with a face value of \$ 100 is

$$98 + \frac{28}{32} = \$\,98.875$$

Thus the price of bonds with a face value of \$ 100,000 is $98.875 \times 1000 =$ \$ 98, 875.

The quoted prices are always clean prices. In order to calculate the actual price that is payable, the accrued interest has to be calculated and added.

6.8 The Futures Market

6.8.1 2-year T-note Contracts

The underlying asset is a T-note with a face value of \$ 200,000. Multiple grades are allowable for delivery. The deliverable grades must have an original maturity of not more than 5 years and 3 months, and an actual maturity of not less than 1 year and 9 months from the first day of the delivery month, and not more than 2 years from the last day of the delivery month. The actual futures price is subject to a multiplicative adjustment factor called the *conversion factor*.[12] Prices are quoted in terms of points and one quarter of $\frac{1}{32}$ of a point.

A point is \$ 1 for a face value of \$ 100. Thus a quote of 92-165 denotes a price of $92 + \frac{16.5}{32}$ for a bond with a face value of 100. Thus the corresponding price for a note with a face value of \$ 200,000 will be:

$$200{,}000 \times \frac{92.515625}{100} = \$\,185{,}031.25$$

Similarly, a quote of 92-162 denotes a price of $92 + \frac{16.25}{32}$ for a note with a face value of \$ 100, while a quote of 92-167 denotes a price of $92 + \frac{16.75}{32}$ for the same face value. At any point in time, the first five months from the March quarterly cycle will be listed. The last trading day is the last business day of the calendar month. Thus on June 29, 2008 the available contracts will be June 2008, September 2008, December 2008, March 2009, and June 2009.

6.8.2 5-year T-note Contracts

The underlying asset is a T-note with a face value of \$ 100,000. Multiple grades are allowable for delivery. The deliverable grades must have an original maturity of not more than 5 years and 3 months, and an actual maturity of not less than 4

[12]The conversion factor is a multiplicative adjustment factor, and we will shortly describe in detail the procedure for its computation.

years and 2 months from the first day of the delivery month. The actual futures price is subject to a conversion factor. Prices are quoted in terms of points and one quarter of $\frac{1}{32}$ of a point.

Thus a quote of 92-165, denotes a price of

$$100{,}000 \times \frac{92.515625}{100} = \$\ 92{,}515.625$$

for a note with a face value of \$ 100,000. At any point in time, the first five months from the March quarterly cycle will be listed. The last trading day is the last business day of the calendar month. Thus on June 29, 2008 the available contracts will be June 2008, September 2008, December 2008, March 2009, and June 2009.

6.8.3 10-year T-note Contracts

The underlying asset is a T-note with a face value of \$ 100,000. Multiple grades are allowable for delivery. The deliverable grades must have an actual term to maturity of not less than 6 years and 6 months from the first day of the delivery month, and not more than 10 years from that date. The actual futures price is subject to a conversion factor.

Prices are quoted in terms of points and one half of $\frac{1}{32}$ of a point. The last trading day is the seventh business day preceding the last business day of the delivery month. At any point in time, the first five months from the March quarterly cycle will be listed. Thus on June 29, 2008 the available contracts will be September 2008, December 2008, March 2009, June 2009, and September 2009.

6.8.4 30-year T-bond Contracts

The underlying asset is a T-bond with a face value of \$ 100,000. Multiple grades are allowable for delivery. The deliverable grades must, if they are callable in nature, not be callable for at least 15 years from the first day of the delivery month, and if they are not callable in nature, must have a maturity of at least 15 years from the first day of the delivery month. The actual futures price is subject to a conversion factor. Prices are quoted in terms of points and one half of $\frac{1}{32}$ of a point. The last trading day is the seventh business day preceding the last business day of the delivery month. At any point in time, the first five months from the March quarterly cycle will be listed. Thus on June 29, 2008 the available contracts will be September 2008, December 2008, March 2009, June 2009, and September 2009.

All the above contracts are subject to delivery settlement.

6.9 Conversion Factors

As one can see from the contract specifications, a wide variety of notes and bonds with different coupons and maturity dates will be eligible for delivery under any particular futures contract. The choice as to which bond to deliver, will be made by the short and obviously the price received by him will depend on the bond that he chooses to deliver. If the short were to deliver a more valuable bond he should receive more than what he would, were he to deliver a less valuable bond. Thus, in order to facilitate comparisons between bonds, the exchange specifies a conversion factor for each bond that is eligible for delivery. This is nothing but a multiplicative price adjustment system to facilitate comparisons between different bonds that are eligible for delivery.

The conversion factor for a bond is the value of the bond per \$ 1 of face value, as calculated on the first day of the delivery month, using an annual YTM of 6% with semi-annual compounding. For the purpose of calculation, the life of the bond is rounded off down to the nearest three months. If after rounding off, the bond were to have a life that is an integer multiple of semi-annual periods, then the first coupon will be assumed to be paid after six months. However, after rounding off, if the life of the bond were not to be equal to an integer multiple of half-yearly periods, then the first coupon will be assumed to be paid after three months and the accrued interest will be subtracted. We have given detailed examples below to illustrate these principles.

The Invoice Price, which is the price received by the short, is calculated as follows:

$$\begin{aligned}\text{Invoice Price } &= \text{Invoice Principal Amount } + \text{ Accrued Interest}\\ &= CF_i \times F \times 100{,}000 + AI_i \end{aligned} \tag{6.6}$$

where CF_i is the conversion factor of Bond **i**, F is the quoted futures price per dollar of face value[13] and AI_i is the accrued interest.

6.9.1 Example I

Let us assume that we are short in a September futures contract and that today is September 1, 2008. Consider a 5% T-Bond that matures on May 15, 2037. This bond is obviously eligible for delivery under the futures contract.

On September 1, this bond has 28 years and $8\frac{1}{2}$ months to maturity. When we round off down to the nearest three months, we get a figure of 28 years and 6 months.

The first coupon is assumed to be paid after six months. The conversion factor may therefore be calculated as follows.

$$\begin{aligned}\text{CF} &= \frac{\frac{5}{2}\text{PVIFA }(3, 57) + 100\text{ PVIF }(3, 57)}{100} = \frac{67.8773 + 18.5472}{100}\\ &= 0.8642\end{aligned}$$

[13]T-Bond futures prices are quoted in the same way as the cash market prices that is, they are clean prices.

6.9.2 Example II

Instead of the May 2037 bond, consider another bond that is maturing on February 15, 2037, with a coupon of 4.75%. This bond too is suitable for delivery. On September 1, 2008, this bond has 28 years and $5\frac{1}{2}$ months to maturity. The life of the bond when we round off down to the nearest three months is 28 years and 3 months.

In this case, we assume that the first coupon is paid after three months. The CF, can be calculated in three steps as shown below.

1. First find the price of the bond three months from today, using a yield of 6% per annum.

$$\begin{aligned} P &= \frac{4.75}{2} + \frac{4.75}{2}\text{PVIFA}\ (3, 56) + 100\text{PVIF}(3, 56) \\ &= 2.375 + 64.0430 + 19.1036 \\ &= \$\ 85.5216 \end{aligned}$$

2. Discount the price gotten above for another three months.

$$\frac{85.5216}{(1.03)^{\frac{1}{2}}} = \$\ 84.2669$$

3. Subtract the accrued interest for 3 months, from the price obtained in the second step.

$$\text{AI} = \frac{4.75}{2} \times \frac{1}{2} = 1.1875$$

$$\text{CF} = \frac{84.2669 - 1.1875}{100} = 0.8308$$

Why do we adopt two different procedures for calculating the CF? The CF is used to multiply the quoted futures price, which is a clean price. Hence the CF should not include any accrued interest. In the first example, the bond has a life that is an integer multiple of semi-annual periods after rounding off. Consequently, we need not be concerned with accrued interest. However, in the second example, accrued interest for a quarter is present in the value we get in the second step. Hence, in this case, we need to subtract this interest in order to arrive at the conversion factor.

6.10 Calculating the Invoice Price for a T-bond

Assume that on September 15, 2008 the decision to deliver the 5% bond maturing on May 15, 2037, under the September futures contract, is declared by the short. The actual delivery will obviously take place two business days later, that is, on September 17.

The invoice price may be computed as follows. The first step is obviously to calculate the accrued interest. The last coupon would have been paid on May 15, 2008 and the next will be due on November 15, 2008. Between the two coupon dates there are 184 days. Between the last coupon date and the delivery date, there are 125 days. The accrued interest for a T-Bond with a face value of \$ 100,000 is

$$AI = \frac{0.05}{2} \times \frac{125}{184} \times 100{,}000 = \$\ 1{,}698.3696$$

The futures settlement price[14] on September 15 is assumed to be 95-12. This corresponds to a decimal futures price per dollar of face value of

$$\frac{95 + \frac{12}{32}}{100} = .95375$$

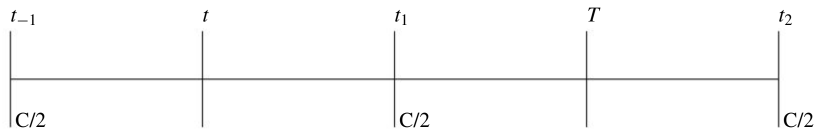
The conversion factor has already been calculated to be 0.8642.

$$\text{Invoice Price} = .95375 \times 0.8642 \times 100000 + 1{,}698.3696 = \$\ 84{,}121.4446$$

6.11 The Cheapest to Deliver Bond

The conversion factor is nothing but an example of a multiplicative adjustment, when more than one grade of the underlying asset is specified for delivery. As we have seen earlier, under such circumstances the short will choose to deliver that bond which is the cheapest to deliver. To illustrate this as well as other concepts which follow, let us depict a hypothetical situation as follows.

Figure 6.2



t represents today. The last coupon of \$C/2 was paid at t_{-1} and the next coupon is due at time t_1. There is a T-Bond futures contract expiring at time T, which will be followed by a coupon at time t_2.

[14]Note that the settlement price is based on the day the intention to deliver is declared, whereas accrued interest is computed as of the Delivery Day. The reason for this is the following. Once the intention to deliver is declared, marking to market will cease and hence the futures price payable by the long will be the settlement price as of that day. However, since the long will receive the bond only on the Delivery Day, accrued interest must be computed until that day.

When the short delivers a particular bond, bond i, at time T, he will receive[15]

$$(F_T \times CF_i + AI_{i,t_1,T}) \times 100{,}000$$

The cost of acquisition of the bond in the spot market at time T is

$$(P_{i,T} + AI_{i,t_1,T}) \times 100{,}000$$

Thus the profit for the short is

$$\begin{aligned}(F_T \times CF_i + AI_{i,t_1,T}) \times 100{,}000 - (P_{i,T} + AI_{i,t_1,T}) \times 100{,}000 \\ = (F_T \times CF_i - P_{i,T}) \times 100{,}000 \qquad (6.7)\end{aligned}$$

If $F_T \times CF_i - P_{i,T}$ is to be maximized or equivalently, $CF_i(F_T - \frac{P_{i,T}}{CF_i})$ is to be maximized, we require that $\frac{P_{i,T}}{CF_i}$ be minimized.

$\frac{P_{i,T}}{CF_i}$ is the *Delivery Adjusted Spot Price* in this case. Thus, as we have seen earlier, the cheapest to deliver bond is the one with the lowest delivery adjusted spot price. The delivery date spot-futures convergence will ensure that $F_T \times CF_i - P_{i,T} = 0$ that is, the quoted futures price at expiration will equal the delivery adjusted spot price of the cheapest to deliver bond.

In practice, T-Bond futures contracts give the short a number of delivery options. Consequently, the quoted futures price at expiration is usually less than the delivery adjusted spot price of the cheapest to deliver bond.

6.11.1 The Cheapest to Deliver Bond Prior to Expiration

Consider the following cash and carry arbitrage strategy. Buy bond i at time t by borrowing at the rate r. Simultaneously go short in CF_i futures contracts expiring at time T, where CF_i is the conversion factor of the bond.

Let $P_{i,t}$ be the clean spot price of the T-Bond at time t and $P_{i,T}$ be the clean spot price at time T. We will denote the corresponding quoted futures prices by F_t and F_T respectively. The payoff at the time of expiration is given by

$$\begin{aligned}(F_T \times CF_i + AI_{i,t_1,T}) \times 100{,}000 + CF_i(F_t - F_T) \times 100{,}000 \\ = (F_t \times CF_i + AI_{i,t_1,T}) \times 100{,}000 \qquad (6.8)\end{aligned}$$

The arbitrageur would receive a coupon of $C/2$ at time t_1. This can be reinvested till time T to yield an amount I_i, which is the future value of the payout per dollar of face value. Hence the implied repo rate is

$$IRR = \frac{F_t \times CF_i + AI_{i,t_1,T} + I_i - (P_{i,t} + AI_{i,t_{-1},t})}{(P_{i,t} + AI_{i,t_{-1},t})} \qquad (6.9)$$

[15]From now on, we will denote the clean price per dollar of face value by P, and the futures price per dollar of face value by F. AI will be used to denote the accrued interest per dollar of face value.

To preclude cash and carry arbitrage, the IRR should be less than the borrowing rate. But in the case of T-Bonds, as, in the case of other futures contracts that allow for multiple deliverable grades, there will be more than one bond that is eligible for delivery, each of which will have its own IRR. If the IRR is greater than the borrowing rate, then arbitrage will be possible. Such arbitrage will continue, until there is no profit to be made from any of the bonds that are eligible for delivery. At this point in time, the cheapest to deliver bond will be the one that maximizes the IRR, which in an arbitrage free setting will just equal the borrowing rate. If we call this bond, bond i then

$$F_t \times CF_i + AI_{i,t_1,T} = (P_{i,t} + AI_{i,t_{-1},t}) \times (1 + r) - I_i$$

$$\Rightarrow F_t = \frac{[(P_{i,t} + AI_{i,t_{-1},t}) \times (1 + r) - I_i - AI_{i,t_1,T}]}{CF_i} \tag{6.10}$$

We will define

$$[(P_{i,t} + AI_{i,t_{-1},t}) \times (1 + r) - I_i - AI_{i,t_1,T}]$$

as the no-arbitrage futures price for bond i less accrued interest, F_i^*, or the *no-arbitrage quoted futures price*. Therefore

$$F_t = \frac{F_i^*}{CF_i} \tag{6.11}$$

Thus prior to expiration, the actual quoted futures price will be equal to the delivery adjusted no-arbitrage quoted futures price for the cheapest to deliver bond. For any other bond j[16]

$$\frac{F_t \times CF_j + AI_{j,t_1^*,T} + I_j - (P_{j,t} + AI_{j,t_{-1}^*,t})}{(P_{j,t} + AI_{j,t_{-1}^*,t})} < r$$

$$\Rightarrow F_t < \frac{[(P_{j,t} + AI_{j,t_{-1}^*,t}) \times (1 + r) - I_j - AI_{j,t_1^*,T}]}{CF_j}$$

$$\Rightarrow F_t = \frac{F_i^*}{CF_i} < \frac{F_j^*}{CF_j} \tag{6.12}$$

6.11.2 Example

Let us illustrate the above concepts in detail using a numerical example. Assume that today is 7 August, 2008. September futures contracts expire on 30 September. There are two bonds that are eligible for delivery. One is a 5% coupon bond maturing on 15 May, 2037 and the other is an 7.5% coupon bond expiring on 15 November, 2024. For ease of exposition, we have chosen bonds which do not pay any coupons between time t, which represents the day on which we are standing and time T, which is the expiration date of the futures contract. Hence we do not have to worry about the future value of payouts I.

[16] $t_{-1}{}^*$ denotes the last coupon date of bond **j**, while $t_1{}^*$ denotes the coupon date corresponding to t_1 for bond **i**.

The conversion factor for the first bond, which we will call bond A, is 0.8642, whereas that for the second[17], which we will call bond B, is 1.1529.

The quoted spot price for bond A is 108-00 and that for bond B is 134-10. These prices correspond to a YTM of 4.5% per annum. The borrowing rate is 9% per annum.

Now if a bond is traded on August 7, the actual settlement will take place on August 8, which is the next business day and is therefore the relevant day for our calculations.

The market price of bond A can be calculated as follows. The number of days from the last coupon date which is May 15, 2008 till August 8, 2008 is 85. The number of days from May 15, 2008 till November 15, 2008, which is the next coupon date, is 184. Hence the accrued interest per \$ 100 of face value is

$$\frac{5}{2} \times \frac{85}{184} = \$\ 1.1549$$

The dirty price per \$100 of face value is therefore

$$108 + 1.1549 = \$\ 109.1549$$

Similarly in the case of bond B, the accrued interest is

$$\frac{7.5}{2} \times \frac{85}{184} = \$\ 1.7323$$

and the total market price is

$$\left(134 + \frac{10}{32}\right) + 1.7323 = \$\ 136.0448$$

The next step is to calculate the delivery adjusted no-arbitrage quoted futures prices for the two bonds.

The number of days between August 8 and September 30 is 53. Hence, for bond A

$$P_d(1 + r) = 109.1549 \times \left[1 + (.09) \times \frac{53}{360}\right] = \$\ 110.6012$$

The accrued interest from the last coupon date till the expiration date of the futures contract is

$$\frac{5}{2} \times \frac{138}{184} = \$\ 1.8750$$

Hence the no-arbitrage futures price of bond A less accrued interest is

$$110.6012 - 1.8750 = 108.7262$$

The delivery adjusted no-arbitrage quoted futures price for bond A is therefore

$$\frac{108.7262}{0.8642} = 125.8113 \equiv 125 * 26$$

[17] Readers should verify these values in order to ensure that they are comfortable with calculating conversion factors

Similarly the no-arbitrage futures price for bond B less accrued interest is

$$136.0448 \times \left[1+(.09)\times\frac{53}{360}\right] - \frac{7.5}{2} \times \frac{138}{184} = 137.8474 - 2.8125 = 135.0349$$

Hence, the delivery adjusted no-arbitrage quoted futures price for bond B is

$$\frac{135.0349}{1.1529} = 117.1263 \equiv 117 * 04$$

Thus bond B is the cheapest to deliver bond as of August 7, 2008.

6.12 Seller's Options

A person who takes a short position in a T-Bond futures contract has a number of options at the time of delivery. These options are referred to as *Quality* options and *Timing* options. A *Quality* option means that the short has the right to select as to which bond he would like to deliver. Whereas a *Timing* option, refers to the fact that the short can choose the time of delivery. Before we go on to analyze the options in detail, let us take a look at the delivery procedure.

6.12.1 The Delivery Process

Delivery in the case of T-Bond and T-Note futures contracts is a three day process. The three day period begins with what is called the *Intention Day*, which is the day on which the short notifies the clearing corporation of his intention to deliver. The Intention Day can be any day from two business days prior to the first business day of the delivery month, till two business days before the last business day of the delivery month. On the next day, which is called the *Notice Day*, the clearing corporation will inform both parties of the other's intention to make or take delivery. The short has to then prepare an invoice for the long which gives details about the security being delivered and the amount of payment for delivery. Finally on the next day, which is the *Delivery Day*, the short has to deliver the bonds to the long in exchange for the amount mentioned in the invoice.

While delivery continues till the end of the delivery month, trading in the futures contracts ceases on the 7th business day prior to the last business day of the delivery month.

6.12.2 The Wildcard Option

Consider the case of a short who decides to deliver under the September 2008 contract on 7 September. The actual delivery will therefore take place on 9 September. The settlement price used to calculate the Invoice Price is the price that is determined at 2 p.m. on the Intention Day[18], which in this case is September 7.

[18]This price will change from day to day during the delivery period, until the last day of trading. For all subsequent deliveries, the settlement price will be the price as of the last trading day.

Now although the settlement price is determined at 2 p.m. on any given Intention Day, the short has until 8 p.m. on that day to notify the exchange as to whether or not he wishes to deliver. Thus the short has an option to lock in the 2 p.m. price by announcing his intention to deliver any time before 8 p.m. This means that if interest rates change after 2 p.m., then the short can profit by delivering a bond which is now cheaper to deliver.

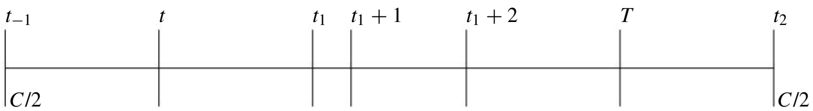
This feature is called the *Wildcard Option*.

In actual practice the short has a series of Wildcard Options. On the first Intention Day[19], he has a Wildcard Option. If interest rates change in a favourable direction between 2 - 8 p.m. on that day, then he can declare his intention to deliver and lock in the price at 2 p.m. However, if there is no change in prices between 2 to 8 p.m., then he can simply wait for the next day hoping that something will happen between 2 to 8 p.m. on that day. This can go on till the last Intention Day which is the third to last business day of the expiration month.

The Wildcard option has a timing component as well as a quality component. We will first illustrate the timing option component of the Wildcard option.

The Timing Option Consider the following time line.

Figure 6.3



t_{-1} denotes the time when the last coupon was paid and t_2 denotes the next coupon date. The contract expires at T.

Let us consider the case of an investor who gets into a cash and carry strategy at time t, by buying 1 unit of bond i and going short in CF_i futures contracts, where CF_i is the conversion factor of the bond that has been bought.

Assume that the investor decides to deliver at time t_1+2 by declaring his intention to deliver at time t_1. We will assume that bond i, the bond in question, is the cheapest to deliver at that point in time. Therefore, the futures settlement price at time t_1, F_{t_1}, will be such that

$$F_{t_1} = \frac{P_{i,t_1}}{CF_i}$$

where P_{i,t_1} is the quoted price of bond i at time t_1.

If there is no timing option, then bond i will indeed be delivered and the proceeds from delivery under the futures contract will be

$$CF_i \times (F_{t_1} \times CF_i + AI_{i,t_{-1},t_1+2}) \times 100{,}000.$$

[19]This is the second to last business day of the month preceding expiration.

The proceeds from the sale of the surplus bonds[20] will be

$$(1 - CF_i)(P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000.$$

The total proceeds are:[21]

$$CF_i \times (F_{t_1} \times CF_i + AI_{i,t_{-1},t_1+2}) \times 100{,}000 + (1 - CF_i)(P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000$$

$$= CF_i \times (P_{i,t_1} + AI_{i,t_{-1},t_1+2}) \times 100{,}000 + (1 - CF_i)(P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000$$

$$= (P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000 \qquad (6.13)$$

Now assume that between 2 p.m. and 8 p.m. on day t_1, the YTM changes. Let the corresponding spot price of bond i be P^*_{i,t_1}. We will assume that bond i continues to be the cheapest to deliver bond, in order for us to be able to focus exclusively on the timing option.

The proceeds from delivery under the futures contract will be the same, namely

$$CF_i \times (P_{i,t_1} + AI_{i,t_{-1},t_1+2}) \times 100{,}000.$$

But the proceeds from the sale of the surplus bonds will be

$$(1 - CF_i)(P^*_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000.$$

The total proceeds will therefore be

$$CF_i \times (P_{i,t_1} + AI_{i,t_{-1},t_1+2}) \times 100{,}000 + (1 - CF_i)(P^*_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000$$

The incremental profit from the yield change is

$$CF_i \times (P_{i,t_1} + AI_{i,t_{-1},t_1+2}) \times 100{,}000 + (1 - CF_i)(P^*_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000$$

$$-(P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000$$

$$= (1 - CF_i)(P^*_{i,t_1} - P_{i,t_1}) \times 100{,}000. \qquad (6.14)$$

If $CF_i > 1$, the incremental profit will be positive if $P^*_{i,t_1} < P_{i,t_1}$, that is, if yields rise. Thus in the case of a bond whose conversion factor is greater than one, the timing option is beneficial if yields rise. Conversely, if the conversion factor were to be less than one, then the yield would have to fall in order to enable the investor to profit from the timing option. We will illustrate this principle with the help of an example.

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[20] If $CF_i > 1$, additional bonds would have to be purchased if the short decides to deliver, which means that there will be an outflow.

[21] $AI_{i,t_{-1},t_1+2} \cong AI_{i,t_{-1},t_1+1}$.

Example Assume that there are two bonds that are eligible for delivery on September 7, 2008. Bond A carries a 5% coupon and matures on 15 May, 2037. Bond B carries a 11% coupon and matures on 15 November, 2028. The conversion factor for bond A is .8642, while that for bond B is 1.5779. If we assume that the YTM for both the bonds is 8%, then the quoted spot price for bond A will be 66-14, and that for bond B will be 129-25. Bond A is cheaper to deliver and the futures price at 2 p.m. will be equal to its delivery adjusted spot price of 76-28.

Let us assume that an investor has initiated a cash and carry strategy on August 7, 2008 with bond A and suddenly announces his intention to deliver on September 7, 2008.

In the absence of a timing option, the investor would have locked in

$$\frac{76+\dfrac{28}{32}}{100}\times .8642\times 100{,}000+\frac{\dfrac{5}{2}}{100}\times\frac{117}{184}\times 100{,}000 = \$\ 68{,}025.0489$$

Let us assume that after 2 p.m., the YTM suddenly falls to 7% . The quoted price of bond A, will now be 75-12. Now, if bond A is delivered, the proceeds will be

$$.8642\times\left(\frac{76+\dfrac{28}{32}}{100}\times .8642+\frac{\dfrac{5}{2}}{100}\times\frac{117}{184}\right)\times 100{,}000$$

$$+(1-.8642)\left(\frac{75+\dfrac{12}{32}}{100}+\frac{\dfrac{5}{2}}{100}\times\frac{116}{184}\right)\times 100{,}000$$

$$= 58{,}787.2473 + 0.1358\times 76{,}951.0870$$

$$= \$\ 69{,}237.2049$$

Thus the timing option has clearly paid off.

The Quality Option Let us examine the wildcard option again. Assume that after 2 p.m. on September 7, the interest rate suddenly rises to 12% . The new spot price of bond A will be 43-23, while that of bond B will be 92-14.

The timing option has no value in this case as one can verify. But let us consider what would happen if we were to deliver bond B, instead of bond A.

If bond A were to be delivered in the absence of the wildcard option, then the payoff would be \$ 68,025.0489. However, if bond A were to be sold at the new price and CF_a units of bond B were to be purchased for delivery under the futures contract, the proceeds would be as follows.

Payoff from delivery under the futures contract

$$= .8642\times\left(\frac{76+\frac{28}{32}}{100}\times 1.5779+\frac{5.5}{100}\times\frac{117}{184}\right)\times 100{,}000$$

$$= \$\ 107{,}850.7298$$

Proceeds from the sale of bond A

$$= \left(\frac{43 + \frac{23}{32}}{100} + \frac{2.5}{100} \times \frac{116}{184} \right) \times 100{,}000$$

$$= \$\ 45{,}294.8370$$

Cost of acquisition of bond B

$$= .8642 \left(\frac{92 + \frac{14}{32}}{100} + \frac{5.5}{100} \times \frac{116}{184} \right) \times 100{,}000$$

$$= \$\ 82{,}881.0071$$

The net proceeds $= 107,850.7298 + 45,294.8370 - 82,881.0071 = \$\ 70,264.5597$, which is greater than the amount of $\$\ 68,025.0489$, that would have been received in the absence of the Wildcard option. The quality option is clearly valuable in this case.

6.12.3 The End of the Month Option

The short can choose to deliver any deliverable bond on any business day of the delivery month. The last trading day for T-bond futures is the eighth to last business day of the delivery month. After that day, all open contracts must be satisfied by delivery. The settlement price on the last trading day is the one used to calculate the Invoice Price on all subsequent days. After the last trading day the seller still has two choices. The first, is the freedom to decide when to deliver. Secondly, he can choose as to which bond to deliver. These quality and timing options are part of the *End of the Month Option*.

Let us assume that interest rates are stable for the days after the last day of trading. Under these conditions, for every additional day the short holds the bond, he earns accrued interest. But he must finance it for every day that he holds it. If the coupon rate of the bond exceeds the financing rate, then he should deliver on the last day. Else he should deliver immediately. This timing option is known as the *Accrued Interest* option.

6.12.4 The Pure Quality Option

We have seen that both the Wildcard as well as the End of the Month options have a quality option component. But even in the absence of these options, the short has a quality option, because he can decide as to which bond to deliver.

Consider the case of the investor who initiates a cash and carry strategy at time t_0 using bond **i**. At time t_1, if bond **i** continues to be the cheapest to deliver bond, then he would deliver it and receive

$$(CF_i \times F_{t_1} + AI_{i,t_{-1},t_1+2}) \times 100{,}000$$

However, if at the time of delivery he were to find that bond **j** is cheaper to deliver, then he would sell off the bond that he is currently holding and acquire the requisite units of bond **j**. The total payoff in this case will be[22]

$$CF_i \times (CF_j \times F_{t_1} + AI_{j,t_{-1}^*,t_1+2}) \times 100,000 + (P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times$$

$$100{,}000 - CF_i(P_{j,t_1} + AI_{j,t_{-1}^*,t_1+1}) \times 100{,}000$$

$$= CF_i(P_{j,t_1} + AI_{j,t_{-1}^*,t_1+2}) \times 100{,}000 + (P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100,000$$

$$-CF_i(P_{j,t_1} + AI_{j,t_{-1}^*,t_1+1}) \times 100{,}000$$

$$= (P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000 \qquad (6.15)$$

The incremental payoff

$$= (P_{i,t_1} + AI_{i,t_{-1},t_1+1}) \times 100{,}000 - (CF_i \times F_{t_1} + AI_{i,t_{-1},t_1+2}) \times 100{,}000$$

$$= (P_{i,t_1} - CF_i \times F_{t_1}) \times 100{,}000 = (P_{i,t_1} - CF_i \times \frac{P_{j,t_1}}{CF_j}) \times 100{,}000$$

$$= CF_i \left[\frac{P_{i,t_1}}{CF_i} - \frac{P_{j,t_1}}{CF_j}\right] \times 100{,}000 \qquad (6.16)$$

Now, $\frac{P_{j,t_1}}{CF_j} < \frac{P_{i,t_1}}{CF_i}$ because by assumption bond **j** is the cheapest to deliver bond. Hence the incremental payoff will be positive. Thus, even in the absence of the Wildcard or the End of the Month options, a cash and carry strategy gives a quality option to the short.

6.13 Hedging

We will first illustrate our arguments using the cheapest to deliver bond and will then discuss the case where other bond portfolios have to be hedged.

6.13.1 Hedging the Cheapest to Deliver Bond

A Naive Approach A naive approach would take the view that for hedging Q bonds, we use Q futures contracts or in other words, that we use a hedge ratio of one. This strategy is not satisfactory, as the following example will illustrate.

Example Let us go back to our earlier example regarding a cash and carry strategy initiated on August 7. We found that bond B which carried a coupon of 7.5% was the cheapest to deliver bond. The quoted spot price was 134.10 and the corresponding futures price was 117.04.

Assume that on September 7, the settlement price is 68.12. The corresponding quoted spot price is 78.27.[23]

[22] We are using t_{-1}^* to denote the time of payment of the last coupon for bond **j**.

[23] We are assuming that bond B continues to be the cheapest to deliver bond.

The bond can be sold in the spot market to yield

$$\frac{78 + \frac{27}{32}}{100} \times 100{,}000 + \frac{3.75}{100} \times \frac{116}{184} \times 100{,}000 = \$\ 81{,}207.8804$$

The profit from the short futures position is

$$\frac{(117 + \frac{04}{32}) - (68 + \frac{12}{32})}{100} \times 100{,}000 = \$\ 48{,}750$$

The total payoff is \$ 129,957.8804.

A perfect hedge would have locked in the original futures price of 117.04 to yield

$$\frac{117 + \frac{04}{32}}{100} \times 1.1529 \times 100{,}000 + \frac{3.75}{100} \times \frac{116}{184} \times 100{,}000 = \$\ 137{,}397.5429$$

In this case we are clearly underhedged.

The Conversion Factor Approach Let us assume that the hedge is initiated at time t_0 and lifted at time t_1, after delivery has commenced.

$$F_{t_0} = \frac{(P_{t_0} + AI_{t_{-1},t_0})(1+r) - AI_{t_{-1},t_1}}{CF_i}$$

$$F_{t_1} = \frac{P_{t_1}}{CF_i}$$

$$\Rightarrow F_{t_1} - F_{t_0} = \frac{P_{t_1} - (P_{t_0} + AI_{t_{-1},t_0})(1+r) + AI_{t_{-1},t_1}}{CF_i}$$

$$\Rightarrow F_{t_1} - F_{t_0} \cong \frac{P_{t_1} - P_{t_0}}{CF_i}$$

$$\Rightarrow \triangle F \cong \frac{\triangle P}{CF_i} \qquad (6.17)$$

if we ignore the cost of carry.

Now a perfect hedge should be such that

$$\triangle P = h^* \triangle F$$

where $h^* = \frac{Q_f}{Q}$ is the optimal hedge ratio.[24] Therefore,

$$h^* = \frac{\triangle P}{\triangle F} = CF_i \qquad (6.18)$$

We will examine the efficiency of this hedge using the same data as for the example on the naive hedging strategy.

Example The proceeds from the spot market when the cheapest to deliver bond is sold is \$ 81,207.8804.

Profit from the futures market = 1.1529 × 48,750 = \$ 56,203.875

The total proceeds = \$ 137,411.7554, which is very close to the value of \$ 137,397.5429, that is implied by the original futures price.

[24] Q is the exposure in the spot market and Q_f is the number of futures contracts.

6.13.2 Hedging a Portfolio Other Than the CTD Bond

The naive hedging strategy has been discredited even for hedging the price of the CTD bond.[25] Hence we will not pursue it further.

Let us now analyze as to how to extend the conversion factor approach to hedge a bond other than the CTD bond.

Our hedge ratio h, should be such that $\triangle P = h\triangle F$. We know that

$$\triangle F \cong \frac{\triangle P_{\text{CTD}}}{CF_{\text{CTD}}}$$

Therefore

$$\triangle P = h\frac{\triangle P_{\text{CTD}}}{CF_{\text{CTD}}} \Rightarrow h = CF_{\text{CTD}}\frac{\triangle P}{\triangle P_{\text{CTD}}}. \tag{6.19}$$

Such hedge ratios are called *Perturbation Hedge Ratios*[26]. To use these ratios, for a given change in yield, we have to calculate the ratio of the change in the price of the bond being hedged to the change in the price of the CTD bond. In practice, the problem is that this ratio will depend on the change in the yield, which cannot be predicted exactly.

Hence for hedging a portfolio of bonds other than the CTD bond, the preferred hedging technique involves the use of the duration of the bond.

Duration Based Hedging For any bond, we know that

$$\frac{dP}{P} = \frac{-D}{\left(1+\frac{y}{2}\right)} \times dy$$

where D is the duration of the bond in annual terms.[27] Therefore

$$\frac{\triangle P}{P} \cong \frac{-D}{\left(1+\frac{y}{2}\right)} \times \triangle y$$

If we denote the bond being hedged as bond h, then the optimal hedge ratio is

$$CF_{\text{CTD}}\frac{\triangle P_h}{\triangle P_{\text{CTD}}} = CF_{\text{CTD}} \times \frac{\dfrac{-D_h}{\left(1+\frac{y_h}{2}\right)} \times P_h \times \triangle y_h}{\dfrac{-D_{\text{CTD}}}{\left(1+\frac{y_{\text{CTD}}}{2}\right)} \times P_{\text{CTD}} \times \triangle y_{\text{CTD}}}$$

$$= CF_{\text{CTD}} \times \frac{D_h \times P_h \times \left(1+\frac{y_{\text{CTD}}}{2}\right) \times \triangle y_h}{D_{\text{CTD}} \times P_{\text{CTD}} \times \left(1+\frac{y_h}{2}\right) \times \triangle y_{\text{CTD}}} \tag{6.20}$$

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[25] CTD stands for Cheapest to Deliver.

[26] See Blake (2000).

[27] We are assuming that the bond pays semi-annual coupons.

If we assume that yield curve movements are parallel, that is $\triangle y_{\text{CTD}} = \triangle y_h$, then the hedge ratio will be[28]

$$CF_{\text{CTD}} \times \frac{D_h \times P_h \times \left(1 + \frac{y_{\text{CTD}}}{2}\right)}{D_{\text{CTD}} \times P_{\text{CTD}} \times \left(1 + \frac{y_h}{2}\right)}$$

which simplifies further to

$$CF_{\text{CTD}} \times \frac{D_h \times P_h}{D_{\text{CTD}} \times P_{\text{CTD}}}$$

if we assume that percentage yield curve movements are parallel.[29] The number of futures contracts required is

$$\frac{\text{Face Value of Spot Exposure}}{\text{Face Value of the Bond Underlying the Futures Contract}} \times h$$

We will now illustrate this hedging technique with the help of an example.

Example Assume that today is 7 August, 2008. September futures contracts expire on 30 September. The cheapest to deliver bond, as we have seen earlier, is a 7.5% coupon bond maturing on 15 November, 2024. Its quoted price is 134.10, which corresponds to a YTM of 4.5% per annum and the conversion factor is 1.1529. The quoted futures price, as calculated before, is 117.04.

Consider a portfolio manager who is holding 10,000 IBM bonds maturing on 7 August, 2033. The face value is \$ 100, the coupon rate is 12% per annum and the YTM is 10% per annum. The manager plans to sell the bonds on 30 September and wants to protect himself against an increase in the yield, using T-bond futures contracts.

The duration of the CTD bond on 7 August is 10.4786 years. The dirty price is 136.0448. The price of the IBM bonds is \$ 118.2559 and the corresponding duration is 9.3570 years.

The hedge ratio is therefore

$$1.1529 \times \frac{(9.3570 \times 118.2559 \times 1.0225)}{(10.4786 \times 136.0448 \times 1.05)} = 0.8714$$

Let us assume that the YTM of the IBM bonds on 30 September is 11% per annum, and that the YTM of the CTD bond is 5.5%. The price of the IBM bond will therefore be 110.1837 and the futures price will be 105.1209. If the yield had remained at 10%, the dirty price of IBM would have been \$ 119.9614.

The loss from the spot market is

$$10{,}000 \times (110.1835 - 119.9614) = \$\ (97{,}779)$$

[28]The per unit face value of the bond being hedged should be taken to be equal to the face value of the bond underlying the futures contract.

[29]This implies that $\frac{\triangle y_{CTD}}{(1+y_{CTD})} = \frac{\triangle y_h}{(1+y_h)}$.

The profit from the futures market is

$$10{,}000 \times 0.8714 \times (117.125 - 105.1209) = \$\ 104{,}604$$

The net profit $= 104{,}604 - 97{,}779 = \$\ 6{,}825$

The tracking error $= \dfrac{6{,}825}{97{,}779} \equiv 6.98\%$

One issue that arises in the context of the duration based hedging approach, is as to whether we should use input values as calculated at the inception of the hedge or as determined at the point of termination. Most authors argue that we should use the expected values of the variables as of the termination date of the hedge. However, in practice, forecasting these variables is not always easy and consequently, the current values are often used as inputs. The two approaches will not yield substantially different results unless the instrument being hedged and the CTD bond are significantly different.[30]

6.13.3 Changing the Duration of a Portfolio of Bonds

Consider a portfolio of bonds which currently has a duration of D_h. Let us assume that we wish to change its duration to D_T, which denotes the target duration, by going long in futures contracts. The question is, what is the appropriate number of futures contracts to use? Let us denote the required hedge ratio by h, the current value per unit of the bond being hedged by P_h and the value of the overall portfolio consisting of the bonds and the futures contracts by V.

We know that,

$$\triangle V = \triangle P + h \triangle F$$

$$\Rightarrow \frac{-D_V}{\left(1 + \frac{y_V}{2}\right)} \times V \times \triangle y_V$$

$$= \frac{-D_h}{\left(1 + \frac{y_h}{2}\right)} \times P_h \times \triangle y_h \; - \; h \times \frac{\dfrac{D_{\text{CTD}}}{\left(1 + \frac{y_{\text{CTD}}}{2}\right)} \times P_{\text{CTD}} \times \triangle y_{\text{CTD}}}{CF_{\text{CTD}}}$$

Let us assume that $\triangle y_v = \triangle y_h = \triangle y_{\text{CTD}}$. Therefore

$$\frac{-D_V}{\left(1 + \frac{y_V}{2}\right)} \times V = \frac{-D_h}{\left(1 + \frac{y_h}{2}\right)} \times P_h \; - \; h \times \frac{\dfrac{D_{\text{CTD}}}{\left(1 + \frac{y_{\text{CTD}}}{2}\right)} \times P_{\text{CTD}}}{CF_{\text{CTD}}}$$

[30] See Daigler (1994).

At the inception of the hedge, $V = P_h$ because the value of the futures contracts is zero. Thus the above expression can be written as

$$\frac{-(D_V - D_h)}{\left(1+\frac{y_h}{2}\right)} \times P_h = -h \times \frac{\dfrac{D_{\text{CTD}}}{\left(1+\frac{y_{\text{CTD}}}{2}\right)} \times P_{\text{CTD}}}{CF_{\text{CTD}}}.$$

If our target duration for the overall portfolio is D_T, then we should choose h such that

$$h = \frac{\dfrac{(D_T - D_h)}{\left(1+\frac{y_h}{2}\right)} \times P_h}{\dfrac{D_{\text{CTD}}}{\left(1+\frac{y_{\text{CTD}}}{2}\right)} \times P_{\text{CTD}}} \times CF_{\text{CTD}}$$

$$\Rightarrow h = \frac{(D_T - D_h) \times P_h \times \left(1+\frac{y_{\text{CTD}}}{2}\right)}{D_{\text{CTD}} \times P_{\text{CTD}} \times \left(1+\frac{y_h}{2}\right)} \times CF_{\text{CTD}} \qquad (6.21)$$

The required number of futures contracts is

$$\frac{\text{Face Value of Spot Exposure}}{\text{Face Value of the Bond Underlying the Futures Contract}} \times h$$

Suggestions for Further Reading

1. Blake, D. *Financial Market Analysis*. John Wiley, 2000.
2. Bodie, Z., Kane A. and Marcus A.J *Investments*. Irwin McGraw-Hill, 1996.
3. Chance, D.M. *An Introduction to Options and Futures*. The Dryden Press, 1991.
4. Edwards, F.R. and C.W. Ma *Futures and Options*. McGraw-Hill, 1992.
5. Kolb, R.W. *Futures, Options, and Swaps*. Blackwell, 2000.
6. Siegel, D.R. and D.F. Siegel *Futures Markets*. The Dryden Press, 1990.

References

1. Daigler, R.T. *Financial Futures and Options Markets: Concepts and Strategies*. Harper Collins, 1994.
2. Fabozzi, F.J. *Bond Markets: Analysis and Strategies*. Prentice-Hall International, 1993.
3. Garbade, K.D. *Fixed Income Analytics*. The MIT Press, 1996.

Concept Check

State whether the following statements are True or False.

1. The Actual-Actual day-count convention is valid for all bond markets.
2. Quoted T-bond prices are inclusive of accrued interest.
3. A coupon bond can be viewed as a portfolio of zero coupon bonds.
4. The duration of a zero coupon bond will always be equal to its time to maturity.
5. The actual quoted futures price prior to the expiration of the contract, will be equal to the delivery adjusted spot price of the cheapest to deliver bond.
6. A quality option means that the short has the right to decide as to which bond he would like to deliver.
7. In the case of a bond with a conversion factor less than 1.0, the timing option is beneficial only if the yield declines.
8. If we wish to increase the duration of a portfolio, then we should go long in futures contracts.
9. T-notes and T-bonds are both capital market instruments.
10. If the YTM is greater than the coupon rate, the bond will sell at a premium.
11. The YTM of a bond is equivalent to its internal rate of return.
12. The YTM calculation assumes that all intermediate coupons are reinvested at the coupon rate.
13. Callable bonds are vulnerable to price compression.
14. Callable bonds are likely to be recalled when interest rates are rising.
15. Compared to a plain vanilla bond with the same coupon, time to maturity, and credit quality, a callable bond will offer a higher yield.
16. In the Actual/Actual method, the number of days between two successive coupon payments will always be between 181 and 184 (both days inclusive).
17. Subjecting deliverable bonds to a conversion factor tantamounts to a multiplicative system of price adjustment.
18. For a plain vanilla bond, the duration will always be less than its stated time to maturity.
19. Due to delivery options, the quoted futures price at expiration, for a T-bond futures contract, will be less than the delivery adjusted spot price of the CTD bond.
20. The end-of-the month option entails a quality as well as a timing option.

QUESTIONS AND PROBLEMS

Question-I

What is Accrued Interest?

Question-II

'Duration can always be perceived as a measure of the interest rate sensitivity of a debt instrument, but not necessarily as the effective time to maturity of the instrument.' Comment.

Question-III

IBM has issued bonds with a face value of $ 1,000, and 10 years to maturity. The bond pays coupons semi-annually at the rate of 8% per annum.

1. What should be the price of the bond if the YTM is 8% per annum?
2. What should be the price of the bond if the YTM is 10% per annum?
3. What should be the price of the bond if the YTM is 6% per annum?

Question-IV

What do you understand by the term 'Wildcard Option'. What are its components and when are they beneficial?

Question-V

Assume that today is 1 September, 2008. There is a T-bond maturing on 15 May, 2028. It has a face value of $ 100,000 and pays an annual coupon of 8% per annum on a semi-annual basis, on 15 May and 15 November every year. The current yield to maturity is 5% per annum.

1. What is the clean price of the bond?
2. What is the conversion factor as calculated today?

Question-VI

Assume that today is 3 January, 2008. The cheapest to deliver bond is a $4\frac{1}{2}\%$ T-bond that matures on 15 November, 2028. The quoted spot price is 89.12. T-bond futures contracts mature on 31 March, 2008. The borrowing/lending rate is 8% per annum.

What is the delivery adjusted no-arbitrage quoted futures price for this bond?

Question-VII

Assume that today is 1 August, 2008. A T-bond expiring on 15 November, 2028 is available. It has a face value of $ 1,000 and pays an 8% coupon on a semi-annual basis, on May 15 and November 15. If the current yield to maturity is 7% per annum, then what is the duration of the bond?

Question-VIII

A T-bond maturing on 15 November, 2028 has a face value of $ 100,000 and pays an 8% coupon on a semi-annual basis. Assume that today is 15 November, 2008 and that the YTM is 12% per annum.

What is the duration of the bond?

What is the modified duration?

Question-IX

Assume that the bond discussed in Question VIII, is being delivered under the December 2008 futures contract. What will be its conversion factor? What will be its conversion factor if it is being delivered under the March 2009 contract?

Question-X

Consider a bond with a face value of \$ 100,000 and a current price of \$ 90,000. The YTM is 12% per annum and the duration is 12.5 years.

The cheapest to deliver bond has a dirty price of \$ 85 per \$ 100 of face value. Its YTM is 10% per annum, and conversion factor is 1.125. It has a duration of 10 years.

Assume that we are holding 10000 bonds. If we want to decrease the duration of our portfolio from 12.5 years to 8 years, how many futures contracts do we require? Should we go long or short?

Appendix–VI

Consider a coupon bond which pays coupons m times per year. It is assumed to have T years to maturity. Let the annual YTM be y, the annual coupon be C, and the face value be M.

$$P = \sum_{t=1}^{mT} \frac{\frac{C}{m}}{\left(1+\frac{y}{m}\right)^t} + \frac{M}{\left(1+\frac{y}{m}\right)^{mT}}$$

Therefore

$$P = \frac{\frac{C}{m}}{(1+\frac{y}{m})} + \frac{\frac{C}{m}}{(1+\frac{y}{m})^2} + \cdots + \frac{\frac{C}{m}}{(1+\frac{y}{m})^{mT}} + \frac{M}{(1+\frac{y}{m})^{mT}}$$

Let us differentiate P with respect to y.

$$\frac{dP}{dy} = -\frac{1}{m} \times \frac{\frac{C}{m}}{(1+\frac{y}{m})^2} - \frac{1}{m} \times 2 \times \frac{\frac{C}{m}}{(1+\frac{y}{m})^3} \cdots$$

$$-\frac{1}{m} \times mT \times \frac{\frac{C}{m}}{(1+\frac{y}{m})^{mT+1}} - \frac{1}{m} \times mT \times \frac{M}{(1+\frac{y}{m})^{mT+1}}$$

$$= -\frac{1}{m} \times \frac{1}{(1+\frac{y}{m})} \left[\frac{1 \times \frac{C}{m}}{(1+\frac{y}{m})} + \frac{2 \times \frac{C}{m}}{(1+\frac{y}{m})^2} + \cdots \right.$$

$$\left. + \frac{mT \times \frac{C}{m}}{(1+\frac{y}{m})^{mT}} + \frac{mT \times M}{(1+\frac{y}{m})^{mT}} \right]$$

Therefore,

$$\frac{dP}{dy} \times \frac{1}{P} =$$

$$-\frac{1}{m} \times \frac{1}{(1+\frac{y}{m})} \left\{ \left[\frac{1 \times \frac{C}{m}}{(1+\frac{y}{m})} + \frac{2 \times \frac{C}{m}}{(1+\frac{y}{m})^2} + \cdots \right. \right.$$

$$\left. \left. + \frac{mT \times \frac{C}{m}}{(1+\frac{y}{m})^{mT}} + \frac{mT \times M}{(1+\frac{y}{m})^{mT}} \right] \frac{1}{P} \right\}$$

The term in curly brackets is nothing but the duration of the bond. Therefore

$$\frac{dP}{dy} \times \frac{1}{P} = -\frac{1}{m} \times \frac{1}{(1+\frac{y}{m})} \times D$$

7

Foreign Exchange Forwards and Futures

7.1 Introduction

The market for the sale and purchase of currencies the world over, is primarily dominated by banks.

We will first study the spot market for foreign exchange and its related conventions in detail, following which we will go on to look at the market for forward contracts. Having covered forward contracts, we will finally focus our attention on the international foreign exchange futures markets.

7.2 Purchase and Sale

An *Exchange Rate* is the price of one country's currency in terms of that of another. In any transaction, there has to be a seller and a purchaser. *In foreign exchange transactions, the words purchase and sale are always used from the dealer's perspective*.

In a purchase transaction the dealer acquires foreign exchange from the customer and pays out the domestic currency. Thus, in the American context, acquisition of foreign currency and payment of dollars by the dealer would constitute a purchase. On the other hand, in a sale transaction, the dealer sells foreign exchange to the customer and acquires the domestic currency in lieu. In an American context therefore, acquisition of dollars and payment of foreign currency by the dealer would constitute a sale.

7.2.1 Example

Consider the case of an Indian exporter who has received US dollars from abroad. When he deposits the draft with his bank and receives rupees in lieu, it represents a purchase transaction. Similarly, if an Indian resident receives a check in riyals from his relative in Saudi Arabia and deposits it in his bank, it would constitute a purchase.

On the other hand, an American importer who is buying machinery from Germany would have to approach his bank for a draft in euros, for which he would have to pay the equivalent amount in dollars. This would represent a sale transaction. Similarly an outward remittance to England by an expatriate manager working in the US would constitute a sale.

7.3 The Spot Market

7.3.1 Quoting Conventions

The exchange rate for a currency can be quoted in two different ways, called Direct Quotes and Indirect Quotes. We will briefly consider the indirect method because it is important that readers understand the principle. However, all our illustrations in this chapter will use the direct method.

In any quotation, a variable number of units of a currency, referred to as the *variable currency*, will be quoted in terms of a fixed number of units of another currency, referred to as the *base currency*. We will use the convention that the symbol for the variable currency will be on the left side of a quote, while the symbol for the base currency will be on the right side of a quote. For instance, if we were to quote the exchange rate as 43.25 INR/USD, the Indian rupee is the variable currency while the US dollar is the base currency.[1]

7.3.2 Direct Quotes

In the direct method, the foreign exchange rate is expressed, as the number of units of the domestic currency per unit of the foreign currency. Thus, the foreign currency is the base currency, while the domestic currency is the variable currency. For example, a rate of 46.80 INR/USD represents a direct quote in India. Thus the direct method expresses the price per unit of the foreign currency in terms of the home currency.

An increase in the quoted value means that the domestic currency has depreciated and that the foreign currency has appreciated. For example, if the rate moves from 46.80 INR/USD to 47.20 INR/USD, then we would say that the rupee has depreciated and that the dollar has appreciated. On the other hand, a decline in the quoted value means that the domestic currency has appreciated and the foreign currency has depreciated. For example, if the rate moves to 46.20 INR/USD, it signifies an appreciation of the rupee relative to the dollar or conversely the depreciation of the dollar relative to the rupee.

An appreciating rupee means that fewer rupees are required to purchase a dollar or in other words that the rupee price of the dollar has gone down. Consequently, it means that imported goods will be cheaper for Indians, whereas exports from India will be costlier for foreigners. On the other hand, a depreciating rupee means

[1]The symbol *INR* stands for Indian Rupees and *USD* for US Dollars. The symbols used for the major currencies are given in Appendix-VII.

that more rupees are required to acquire a dollar, which means that the price of the dollar has gone up. Consequently, it will make imported goods more expensive for Indians, but will ensure that foreigners perceive Indian imports to be more attractive.

Bid-Ask Quotes Let us consider the case of rupee-dollar transactions. From the dealer's perspective, when he is buying dollars, he would like to pay out as little as possible in rupee terms, whereas when he is selling dollars, he would like to charge as much as possible in rupee terms. Thus the buying rate for foreign exchange in the case of the direct quotation method, will be lower than the selling rate. The differential between the buying rate, called the *bid* and the selling rate called the *ask*, is known as the *spread* and constitutes a profit for the dealer. *Thus the maxim in the case of direct quotes is buy low and sell high*. For example, a quote of 46.35/46.65 would imply that the dealer is willing to buy dollars at the rate of rupees 46.35 per dollar, but will charge rupees 46.65 per dollar if he were to sell dollars.

7.3.3 Indirect Quotes

In the indirect method, the exchange rate is expressed as the number of units of the foreign currency per unit of the domestic currency. Thus, the domestic currency is the base currency while the foreign currency is the variable currency. For example, a rate of 2.25 USD per 100 INR represents an indirect quote in India. Thus, the indirect method expresses the price per unit of the domestic currency in terms of the foreign currency.

An increase in the quoted value means that the domestic currency has appreciated and the foreign currency has depreciated. For example, if the rate were to move from 2.25 USD/100 INR to 2.35 USD/100 INR, then we would say that the rupee has appreciated and that the dollar has depreciated. On the other hand, a decline in the quoted value means that the domestic currency has depreciated and the foreign currency has appreciated. For example, if the rate moves to 2.15 USD/100 INR, it signifies a depreciation of the rupee relative to the dollar or conversely an appreciation of the dollar relative to the rupee.

Readers may initially find this confusing, for a larger number means a depreciating rupee if the rates are quoted directly, but an appreciating rupee if they are quoted indirectly. The best way to think of it is in terms of variable and base currencies. If we quote the rate as rupees per dollar, it is a direct quote in India[2]. A larger value would mean that the variable currency, in this case the rupee, has depreciated, whereas the base currency, in this case the dollar, has appreciated. If however, the rate were to be quoted as dollars per INR 100, it will be an indirect quote in India. In this case too, a larger value would mean that the variable currency, namely the dollar, has depreciated, whereas the base currency, the rupee, has appreciated. Similarly, in both the cases, a smaller value would

[2]Remember, a direct quote as perceived by an Indian would be perceived as an indirect quote by an American.

signify an appreciation of the variable currency and a depreciation of the base currency.

Bid-Ask Quotes Once again, let us consider the case of rupee-dollar transactions. From the dealer's perspective, when he is buying dollars he would like to buy as many dollars as he can per rupees 100, whereas when he is selling dollars, he would like to part with as few dollars as possible per rupees 100. Thus the buying rate for foreign exchange in the case of the indirect quotation method, will be higher than the selling rate. *The maxim in the case of indirect quotes is therefore, buy high and sell low*. For example a quote of 2.35/2.25 would imply that the dealer is willing to buy dollars at the rate of 2.35 per 100 rupees, but will give only 2.25 dollars per 100 rupees at the time of sale.

Converting a Direct Quote to an Indirect Quote Consider a direct quote for the US dollar: 42.2500/42.4500 INR/USD

Let us take the bid first. A quote of 42.25 to buy US dollars represents a quote of $\frac{1}{42.2500}$ to sell a rupee or equivalently a rate of $\frac{100}{42.2500} = 2.3669$ USD/100 INR to sell 100 rupees. Similarly a quote of 42.45 to sell dollars represents a quote of $\frac{100}{42.4500} = 2.3557$ USD/100 INR to buy 100 rupees. Consequently a direct quote of 42.2500/42.4500 INR/USD in Mumbai, corresponds to an indirect quote of 2.3669/2.3557 USD/100 INR in Mumbai.

7.3.4 European Terms and American Terms

When an exchange rate is quoted with the US dollar as the base currency, it is referred to as a quote in European terms. For instance a quote of 0.80 EUR/USD would be a quote as per the European convention.

On the other hand, a quote with the US dollar as the variable currency is said to be a quote in American terms. For instance, a quote of 0.65 USD/AUD would be a quote as per the American convention.

7.4 The Forward Market

The forward market allows investors to buy and sell foreign currencies for delivery at a future date. If the forward rate of a currency exceeds the spot rate, then we say that the currency is selling at a *forward premium*. For example consider the following rates.

Spot: 45.8000/46.2000 INR/USD

1 Month Forward: 46.0500/46.5000 INR/USD

In this case, the dollar is at a forward premium.

If however, the forward rate is less than the spot rate, then we say that the foreign currency is at a *forward discount*. For instance, if the 1 month forward rate were to be 45.5000/45.9500, then we would say that the dollar is at a forward discount.

When the spot rate is equal to the forward rate, then we say that the currency is trading *flat*.

If the forward rate is given directly, like it is above, then we call such rates as *outright forward rates*. But in practice, very often, only the difference between the forward rate and the spot rate, called the *forward margin or swap points* is given and we have to construct the forward rates from the spot rates.

Consider the following data.

Spot: 45.2420/45.6150

Forward Points: 45/72

For most currencies, a point is $\frac{1}{10,000}$th of the currency. Thus 45 points would imply 0.0045. For certain currencies, however, a point may connote $\frac{1}{100}$th of a unit.

If we look at the above data, we can see that it has not been specified as to whether the currency is at a forward premium or a discount. So the question to ask ourselves is, do we add the forward margin or do we subtract it?

The thing to remember is that the bid-ask spread will widen as we go forward. That is, the spot market will have the lowest spread. As we move forward in time, the spread will gradually increase.

Thus, if the margin is specified as a/b, where $a < b$, then adding the margin will widen the spread and therefore the forward margin specified indicates that it should be added. Hence, if the margin is specified as a smaller number followed by a larger number, then it signifies a forward premium and must be added. The above quotation therefore implies outright forward rates of 45.2465/45.6222.

However, consider another case where the forward margin is specified as 85/40. In this case, subtracting the margin will widen the spread. Hence, if the margin is specified as a/b, where $a > b$, then the margin should be subtracted. That is, if the margin is specified as a larger number followed by a smaller number, it signifies a discount and must be subtracted. In this case, the implied outright forward rates are 45.2335/45.6110.

It must be remembered that the above logic is valid for Direct Quotes. In the case of Indirect Quotes, if the swap points are specified as a/b where $a < b$, then subtracting the margin will widen the spreads. Thus, a smaller number followed by a larger number, indicates a forward premium, which should be subtracted from the quoted spot rates. Similarly, if the margin is specified as a larger number followed by a smaller number, it signifies that the foreign currency is trading at a discount and hence the swap points have to be added to the quoted spot rates.

7.5 The No-Arbitrage Forward Price

We will now derive the relationship between the spot rate and the forward rate for a given maturity, using arbitrage arguments.

7.5.1 Cash and Carry Arbitrage

The spot rate for Singapore dollars in Mumbai on a given day is INR 25.2025, whereas the rate for a three month forward contract is INR 25.5075. The rate of interest for a three month loan in the Indian money market is 7.5% per annum, whereas the rate for a loan of the same tenor in the Singapore money market is 4.5% per annum.

Take the case of an arbitrageur who decides to undertake the following steps. He borrows INR 25.2025 in Mumbai and buys one Singapore Dollar. This Singapore dollar is immediately invested in the Singapore money market. This will lock in a receivable of

$$\left(1 + \frac{.045}{4}\right) = 1.01125 \text{ SGD}$$

after three months. The arbitrageur can lock in an equivalent amount in Indian rupees, right at the outset, by taking a short position in a forward contract.

At the end of three months, the deposit in Singapore will yield SGD 1.01125, which can be immediately sold under the forward contract to yield INR 25.7945. A part of this, that is

$$25.2025 \times \left(1 + \frac{.075}{4}\right) = \text{INR } 25.6750$$

will be required to pay off the loan that was taken in India. The balance of INR 0.1195, is clearly a pure arbitrage profit.

7.5.2 Reverse Cash and Carry Arbitrage

We will assume that all the other variables have the same values as in the above example, except for the forward rate, which we will assume is INR 25.3075. In such circumstances, a reverse cash and carry arbitrage strategy may be implemented as follows.

The arbitrageur can borrow a Singapore dollar in the Singapore money market, and convert it at the spot rate to INR 25.2025. This can be lent out in India at the rate of 7.5% per annum, which will lock in a receivable of INR 25.6750 after three months. Right at the outset, a long forward position can be used to ensure that the rupees can be used to acquire the following amount in Singapore currency after three months

$$\left(1 + \frac{.045}{4}\right) = 1.01125$$

The cost in rupees will be

$$25.3075 \times 1.01125 = \text{INR } 25.5922$$

At the end of three months, when the deposit in India matures, the arbitrageur can take delivery under the forward contract and pay off his loan in Singapore. The balance of INR 0.0828 is a clear arbitrage profit.

Let us denote the spot rate by S; the forward rate by F; the interest rate in the domestic money market by i_d; and the interest rate in the foreign market by i_f. Note we are using the direct quotation system. The no-arbitrage condition can be expressed as:

$$S(1+i_d) = F(1+i_f)$$
$$\Rightarrow F = S \times \frac{(1+i_d)}{(1+i_f)} \tag{7.1}$$

The kind of arbitrage that we have just illustrated is called *Covered Interest Arbitrage* and the relationship that must exist if such arbitrage is to be precluded, that is:

$$\frac{F}{S} = \frac{(1+i_d)}{(1+i_f)}$$

is called the *interest rate parity* equation.

$$\frac{F}{S} = \frac{(1+i_d)}{(1+i_f)} \Rightarrow \frac{(F-S)}{S} = \frac{(i_d - i_f)}{(1+i_f)} \tag{7.2}$$

In practice, this is often approximated as

$$\frac{(F-S)}{S} = (i_d - i_f) \tag{7.3}$$

because $1 + i_f \cong 1$, if i_f is very small.

In real life there could at times be deviations from the interest rate parity relationship, which an arbitrageur is unable to exploit. One reason is that buying and selling foreign exchange, both in the spot as well as the forward markets, entails the payment of transactions costs. Second, all countries in the world do not permit free flow of funds across their borders. Thus, if such exchange controls were to be in place, it is likely that one will observe deviations from interest rate parity that cannot be arbitraged away. In practice, it has been observed that even a perception that exchange rate controls may be imposed in the future, is adequate to preclude an investor from attempting to execute a covered interest arbitrage transaction. One final factor is that investors have to pay taxes on income and profits, which vary from country to country. Hence the ability to make arbitrage profits on a post tax basis, is likely to depend on the investor's tax status.

7.5.3 Arbitrage in the Presence of Transactions Costs

Let us now consider a situation where both spot and forward markets, as well as the money markets, are subject to bid-ask spreads. We will denote the variables as follows:

$$\text{Spot Quote} \equiv S_b/S_a$$
$$\text{Forward Quote} \equiv F_b/F_a$$
$$\text{Domestic Borrowing/Lending Rates} \equiv r_{db}/r_{dl}$$
$$\text{Foreign Borrowing/Lending Rates} \equiv r_{fb}/r_{fl}$$

Take the case of an investor who is contemplating a cash and carry strategy. He will have to borrow S_a units of the domestic currency to buy 1 unit of the foreign currency, which can be invested at r_{fl} and simultaneously enter into a forward contract at F_b to sell $(1 + r_{fl})$ units of the foreign currency. In order to preclude arbitrage, it must be that

$$(1 + r_{fl}) \times F_b \leq (1 + r_{db}) \times S_a$$
$$\Rightarrow F_b \leq S_a \times \frac{(1 + r_{db})}{(1 + r_{fl})} \quad (7.4)$$

Now let us take the case of reverse cash and carry arbitrage. To implement this strategy, an investor must borrow 1 unit of the foreign currency and convert it to S_b units of the domestic currency, which can be invested at r_{dl}. He must at the same time, go long in a forward contract to acquire $(1 + r_{fb})$ units of the foreign currency at a rate F_a. In order to rule out arbitrage, it must be that

$$S_b \times (1 + r_{dl}) \leq F_a(1 + r_{fb})$$
$$\Rightarrow F_a \geq S_b \times \frac{(1 + r_{dl})}{(1 + r_{fb})} \quad (7.5)$$

We will illustrate these principles with the help of a numerical example.

Example The following market information is available at a given instant.

Spot: 1.5015/1.5125 USD/GBP
Forward: 1.5295/1.5410 USD/GBP
Borrowing/Lending Rates in the US: 5.5%/5.3%
Borrowing/Lending Rates in the UK: 4.5%/4.3%

We will first examine as to whether cash and carry arbitrage is feasible. 1 GBP can be purchased by borrowing 1.5125 USD. This can be invested in the UK to yield 1.043 GBP at maturity. If a forward contract is booked at the outset, then this amount can be sold at 1.5295 USD/GBP. The final proceeds will therefore be

$$1.043 \times 1.5295 = 1.5952685$$

The repayment due on the loan taken in the US is

$$1.055 \times 1.5125 = 1.5956875$$

Consequently, cash and carry arbitrage will not be profitable.

What about reverse cash and carry arbitrage? Consider an investor who borrows 1 GBP in the UK and converts it to 1.5015 USD. This can be invested in the US to yield

$$1.053 \times 1.5015 = 1.5810795 \text{ USD at maturity.}$$

In order to cover the loan repayment in the UK, a forward contract will have to be booked at the outset to acquire 1.045 GBP at 1.5410 USD/GBP. The cost of acquisition will be 1.610345 USD. Hence, reverse cash and carry arbitrage is also not an attractive proposition in this case.

7.6 One Way Arbitrage

In order to carry out covered interest arbitrage using either the cash and carry or the reverse cash and carry strategies, we need to simultaneously transact in four markets, namely, the spot market, the forward market and the money markets for the two currencies. One way arbitrage, entails the use of three out of the four markets. We will first illustrate our arguments without using transactions costs and will then factor in such costs.

Assume that $F < S\frac{(1+r_d)}{(1+r_f)}$. Such a violation of interest rate parity can be exploited with the help of a reverse cash and carry strategy. However, consider the situation from another perspective. An investor who is seeking to acquire the foreign currency immediately has two options. He can either acquire it in the spot market at a rate S or else he can borrow the currency abroad and simultaneously enter into a forward contract to buy $1 + r_f$ units in the future, in order to enable him to repay his loan with interest. The cost of acquisition will be $F \times (1 + r_f)$ in terms of the home currency, which is equivalent to $F \times \frac{(1+r_f)}{(1+r_d)}$ in present value terms. If

$$F \times \frac{(1+r_f)}{(1+r_d)} < S$$

or

$$F < S\frac{(1+r_d)}{(1+r_f)}$$

then such investors would prefer the forward market and would avoid the spot market. Thus, there would be no demand for the foreign currency in the spot market, which clearly represents a disequilibrium situation. Therefore, it must be that

$$F \geq S\frac{(1+r_d)}{(1+r_f)} \tag{7.6}$$

Now assume that

$$F > S\frac{(1+r_d)}{(1+r_f)}$$

If this were to be the case, then a person who needs to acquire the foreign currency at a point of time in the future, would avoid the forward market altogether. This is because if he were to purchase the currency using a forward contract, then he would have to incur an expenditure of F, whereas he has the option of buying $\frac{1}{(1+r_f)}$ units of the foreign currency right now, which can be invested to yield 1 unit at maturity. The current value in terms of the domestic currency is $S \times \frac{1}{(1+r_f)}$, which in future value terms is equivalent to $S \times \frac{(1+r_d)}{(1+r_f)}$. In such a situation, there would be no demand for the foreign currency in the forward market. Therefore, we require that

$$F \leq S\frac{(1+r_d)}{(1+r_f)} \tag{7.7}$$

Hence, in order to ensure demand-supply equilibrium in both the spot as well as the forward markets, it is necessary that

$$F = S\frac{(1+r_d)}{(1+r_f)} \tag{7.8}$$

which is nothing but our interest rate parity condition.

7.6.1 One Way Arbitrage and Transactions Costs

Let us now introduce transactions costs using the same symbols as before.

From the perspective of an individual who needs the foreign currency immediately, one way arbitrage will be ruled out if

$$F_a \geq S_a\frac{(1+r_{dl})}{(1+r_{fb})} \tag{7.9}$$

Let us analyze this relationship. If the investor purchases the foreign currency in the spot market, then he will incur an expenditure of S_a. On the contrary, he can borrow 1 unit of the foreign currency abroad, which will have to be repaid with interest at the rate of r_{fb} and go long in the forward market. The cost of acquisition of the required foreign exchange at a future date will be $F_a(1+r_{fb})$. If the person invests $F_a\frac{(1+r_{fb})}{(1+r_{dl})}$ today, then he will have enough to acquire the foreign currency in the future. Thus, in order to ensure that such investors do not ignore the spot market, it must be that

$$S_a \leq F_a\frac{(1+r_{fb})}{(1+r_{dl})}$$

$$\Rightarrow F_a \geq S_a\frac{(1+r_{dl})}{(1+r_{fb})} \tag{7.10}$$

Now let us take into account the perspective of a person who seeks to acquire the foreign currency at a later date. One option would be to go long in a forward contract and pay F_a at the time of purchase. The other option would be to buy $\frac{1}{(1+r_{fl})}$ units of the foreign currency in the spot market and invest it. The cost of acquisition will be $\frac{S_a}{(1+r_{fl})}$, which can be borrowed at the rate of r_{db}. In order for the investor not to avoid the forward market, it must be that

$$F_a \leq S_a\frac{(1+r_{db})}{(1+r_{fl})} \tag{7.11}$$

Now consider the situation from the standpoint of a foreigner who is seeking to acquire our currency now. One option for him, is to sell a unit of the foreign currency at a rate S_b. Another option for him, is to borrow S_b units of our currency.

He will have to repay $S_b \times (1 + r_{db})$ at a future date, the cost of which can be locked in by going short in a forward contract. The cost in terms of the foreign currency will be

$$\frac{S_b}{F_b} \times (1 + r_{db})$$

the present value of which is

$$\frac{S_b}{F_b} \times \frac{(1 + r_{db})}{(1 + r_{fl})}$$

In order for the spot acquisition to be attractive we require that

$$1 \leq \frac{S_b}{F_b} \times \frac{(1 + r_{db})}{(1 + r_{fl})}$$

$$\Rightarrow F_b \leq S_b \times \frac{(1 + r_{db})}{(1 + r_{fl})} \qquad (7.12)$$

Finally, let us consider the perspective of a person, who is seeking to acquire our currency at a future date. One option is for him to sell 1 unit of the foreign currency at F_b at a future date. Another option is for him to borrow $\frac{1}{(1 + r_{fb})}$ units of the foreign currency, which can be converted to $\frac{S_b}{(1 + r_{fb})}$ of our currency now. This can be invested in our market to yield $\frac{S_b \times (1 + r_{dl})}{(1 + r_{fb})}$. In order for him to be interested in the forward market, it must be the case that

$$F_b \geq S_b \times \frac{(1 + r_{dl})}{(1 + r_{fb})} \qquad (7.13)$$

7.7 The Relationship Between Interest Rate Parity and One Way Arbitrage

As we have seen, in the absence of transactions costs, interest rate parity rules out both covered interest arbitrage, as well as one way arbitrage. However, in the presence of transactions costs, it is possible for interest rate parity to be satisfied in a situation where one way arbitrage is feasible, as also for situations to exist where covered interest arbitrage is profitable while one way arbitrage is not. We will illustrate this with the help of an example.

7.7.1 Example

If covered interest arbitrage is to be precluded, it must be the case that

$$F_a \geq S_b \times \frac{(1 + r_{dl})}{(1 + r_{fb})}$$

and

$$F_b \leq S_a \times \frac{(1 + r_{db})}{(1 + r_{fl})}$$

Consider the following data.

Spot: 1.5100/1.5175

Forward: 1.5225/1.5325

Domestic Interest Rates: 4.90%/4.75%

Foreign Interest Rates: 3.25%/3.10%

$$F_a = 1.5325 > S_b \times \frac{(1 + r_{dl})}{(1 + r_{fb})} = 1.5100 \times \frac{(1.0475)}{(1.0325)} = 1.5319$$

$$F_b = 1.5225 < S_a \times \frac{(1 + r_{db})}{(1 + r_{fl})} = 1.5175 \times \frac{(1.0490)}{(1.0310)} = 1.5440$$

However

$$S_a \times \frac{(1 + r_{dl})}{(1 + r_{fb})} = 1.5175 \times \frac{(1.0475)}{(1.0325)} = 1.5395 > F_a = 1.5325$$

which is a violation of one of the one way arbitrage conditions.

7.8 Option Forwards

Often, while entering into a forward contract, the client may not know the exact date on which he will complete the contract. For instance, an importer may say that he needs foreign exchange some time between one to two months from today. In such a case, he can negotiate a forward contract with an option. The option will allow him to complete the contract on any date during the stated period. The option seller, namely the dealer, will quote rates in such cases, after considering the fact that the contract may be completed on the worst possible day from his point of view. The implications of this depend on whether the dealer is buying or selling and as to whether the currency is at a premium or a discount, as we will illustrate with the help of the following examples.

7.8.1 Example I

Indian Rayon is importing machinery from the US and is required to make the payment between 2 to 3 months from now. However, the company is unable to specify the exact date and wants to enter into a forward contract with the option of taking delivery at any time between 2 and 3 months from today.

Assume that the following rates are prevailing in the interbank market.

Spot: 45.4500/45.8525 INR/USD

1 Month Forward: 45/85

2 Month Forward: 70/110

3 Month Forward: 110/155

The relevant base rate in this case is the selling rate. If the contract is completed at the end of 2 months, then the applicable premium will be 110 points, whereas if it is completed after 3 months, then the relevant premium is 155 points. In this case the dealer will assume that the contract will be completed after 3 months and charge the higher premium. Hence, the quoted forward rate will be 45.8525 + .0155 = 45.8680.

Thus the rule for a sale transaction if the currency is quoting at a premium is, charge the premium for the latest date of delivery.

7.8.2 Example II

Consider the data given above, but assume that the dollar is trading at a forward discount and that the swap points are as follows.

1 Month Forward: 75/35

2 Month Forward: 115/75

3 Month Forward: 140/95

In this case, if the contract is completed after 2 months, then the applicable discount will be 75 points, whereas if it is completed after 3 months, then the relevant discount will be 95 points. In this case the dealer will assume that the transaction will be completed at the end of 2 months and allow the lower of the two discounts. Hence the quoted forward rate will be 45.8525 − .0075 = 45.8450.

Hence, the rule for sale transactions in the case where the currency is trading at a discount is, allow the discount for the earliest date of delivery.

7.8.3 Example III

Parle has exported biscuits to South Africa and will be paid in dollars sometime between 1 to 2 months from today. The current rates in the inter-bank market are as follows:

Spot: 45.3500/45.7320 INR/USD

1 Month Forward: 35/80

2 Month Forward: 65/115

The relevant spot base rate here is the buying rate of Rs. 45.3500. If the contract is completed at the end of 1 month, then the applicable premium is 35 points, whereas, if the party were to deliver the dollars after 2 months, then the premium would be 65 points. Since the bank is buying, it will take the view that the transaction may be completed at the end of 1 month and hence will allow a premium of only 35 points. The quoted forward rate will therefore be, 45.3500 + .0035 = Rs. 45.3535.

Hence, the rule for purchase transactions, in the case where the currency is trading at a premium is, allow the premium for the earliest date of delivery.

Similarly, in the case of purchase transactions where the currency is trading at a discount, the bank will apply the discount for the latest date of delivery.

7.9 Futures Markets

The CME Group trades futures contracts on the following currencies.

Table 7.1 FOREX Futures on the CME: The Underlying Currencies

Australian Dollar	Brazilian Real
British Pound	Canadian Dollar
Chinese Renminbi	Czech Koruna
Euro	Hungarian Forint
Israeli Shekel	Japanese Yen
Korean Won	Mexican Peso
New Zealand Dollar	Norwegian Krone
Polish Zloty	Russian Ruble
South African Rand	Swedish Krona
Swiss Franc	

The contract details are as follows.

Table 7.2 FOREX Futures Contract Details

Currency	Contract Size	Symbol	Point Description	Tick Size
Australian Dollar	100,000	AUD	$ 0.0001	$ 10.00
Brazilian Real	100,000	BRL	$ 0.0001	$ 5.00
British Pound	62,500	GBP	$ 0.0001	$ 6.25
Canadian Dollar	100,000	CAD	$ 0.0001	$ 10.00
Chinese Renminbi	1,000,000	CNY	$ 0.00001	$ 10.00
Czech Koruna	4,000,000	CZK	$ 0.000001	$ 8.00
Euro	125,000	EUR	$ 0.0001	$ 12.50
Hungarian Forint	30,000,000	HUF	$ 0.0000001	$ 6.00
Israeli Shekel	1,000,000	ILS	$ 0.00001	$ 10
Japanese Yen	12,500,000	JPY	$ 0.000001	$ 12.50
Korean Won	125,000,000	KRW	$ 0.0000001	$ 12.50
Mexican Peso	500,000	MXN	$ 0.00001	$ 12.50
New Zealand Dollar	100,000	NZD	$ 0.0001	$ 10.00
Norwegian Krone	2,000,000	NOK	$ 0.00001	$ 20
Polish Zloty	500,000	PLN	$ 0.00001	$ 10.00
Russian Ruble	2,500,000	RUB	$ 0.00001	$ 25
South African Rand	500,000	ZAR	$ 0.00001	$ 12.50
Swedish Krona	2,000,000	SEK	$ 0.00001	$ 20
Swiss Franc	125,000	CHF	$ 0.0001	$ 12.50

The basic unit of a price is a point. The point description denotes the US dollar equivalent of one point, whereas the tick size connotes the minimum observable fluctuation in the value of a contract.

At any point in time six contract months are listed from the March quarterly cycle for all currencies except the Brazilian Real, the Chinese Renminbi, the Korean Won, the Mexican Peso, the Russian Ruble, and the South African Rand.

For instance assuming that today is 26 December 2006, which is the fourth Tuesday of December, the following contracts will be available.
March-2007; June-2007; September-2007, December-2007, March-2008, and June-2008.

Contracts expire on the second business day prior to the third Wednesday of the month. In this case the December-06 contract would have expired on 18 December.

For the Brazilian Real the 12 nearest calendar months are available at any point in time. For the Chinese Renminbi, the Korean Won, the Mexican Peso and the South African Rand, the 13 nearest calendar months, plus the next two months from the March cycle are listed at any point in time. For the Russian Ruble the four nearest months from the March cycle are listed at any point in time.

Thus on 2 January, 2007, the availability of contract months for these currencies would be as follows.

Brazilian Real—January 2007 to December 2007
Chinese Renminbi—January 2007—January 2008, March 2008, June 2008
Korean Won—January 2007—January 2008, March 2008, June 2008
Mexican Peso—January 2007—January 2008, March 2008, June 2008
South African Rand—January 2007—January 2008, March 2008, June 2008
Russian Ruble—March-2007, June-2007, September-2007, and December-2007

The Brazilian Real, the Chinese Renminbi, the Korean Won, and Russian Ruble futures contracts are cash settled. All other contracts are settled by physical delivery.

7.9.1 E-MINI Contracts

The CME offers contracts on certain assets with a smaller contract size. These contracts, called E-MINI contracts, are currently available for the Euro and the Japanese Yen, in the case of foreign currency futures. The contract size is 62,500 Euros for the futures contract on the Euro and 6,250,000 yen for the contract on the Japanese Yen. The tick size is $ 6.25 for both contracts. Contracts are available for the two nearest month from the March cycle. For instance, on 26 December 2006, we would find the March-2007 and June-2007 contracts being traded. All contracts are settled by physical delivery.

7.9.2 Cross Rate Futures

The FOREX futures contracts that we discussed involved the exchange rates of foreign currencies with respect to the US dollar. The CME also trades futures contracts based on the exchange rates of two foreign or non-American currencies with respect to each other. These are known as cross-rate futures contracts.

All contracts, with the exception of the Chinese Renminbi contracts, are settled by physical delivery. At any point in time six contract months are listed from the March quarterly cycle. The Chinese Renminbi contracts are cash settled. At any point in time, the 13 nearest calendar months plus the next two months from the March cycle are listed.

The contract specifications are given in Table 7.3.

Table 7.3 Cross-Rate Contract Details

Cross Rate	Contract Size	Point Description	Tick Size
Australian Dollar/ Canadian Dollar	200,000 AUD	0.0001 CAD/AUD	20.00 CAD
Australian Dollar/ Japanese Yen	200,000 AUD	0.01 JPY/AUD	2,000 JPY
Australian Dollar/ New Zealand Dollar	200,000 AUD	0.0001 NZD/AUD	20.00 NZD
British Pound/ Japanese Yen	125,000 GBP	0.01 JPY/GBP	1,250.00 JPY
British Pound/ Swiss Franc	125,000 GBP	0.0001 CHF/GBP	12.50 CHF
Canadian Dollar/ Japanese Yen	200,000 CAD	0.01 JPY/CAD	2,000 JPY
Chinese Renminbi/ Euro	1,000,000 CNY	0.00001 EUR/CNY	10.00 EUR
Chinese Renminbi/ Japanese Yen	1,000,000 CNY	0.001 JPY/CNY	1,000 JPY
Euro/ Australian Dollar	125,000 EUR	0.0001 AUD/EUR	12.50 AUD
Euro/ British Pound	125,000 EUR	0.0001 GBP/EUR	6.25 GBP
Euro/ Canadian Dollar	125,000 EUR	0.0001 CAD/EUR	12.50 CAD
Euro/ Czech Koruna	4,000,000 CZK	0.000001 EUR/CZK	8.00 EUR
Euro/ Hungarian Forint	30,000,000 HUF	0.0000001 EUR/HUF	6.00 EUR
Euro/ Japanese Yen	125,000 EUR	0.01 JPY/EUR	1,250.00 JPY
Euro/ Norwegian Krone	125,000 EUR	0.001 NOK/EUR	62.50 NOK
Euro/ Polish Zloty	500,000 PLN	0.00001 EUR/PLN	10.00 EUR
Euro/ Swedish Krona	125,000 EUR	0.001 SEK/EUR	62.50 SEK
Euro/ Swiss Franc	125,000 EUR	0.0001 CHF/EUR	12.50 CHF
Swiss Franc/ Japanese Yen	250,000 CHF	0.01 JPY/CHF	1,250.00 JPY

7.10 Hedging an Export Transaction

Merck has exported a consignment of drugs to a company in Sydney and is scheduled to receive 50 MM Australian dollars after one month. The company is worried that the US dollar will appreciate by then and therefore decides to hedge using three month futures contracts. Since the company will be selling Australian dollars, it requires a short hedge.[3]. Since each Australian dollar futures contract is for 100,000 AUD, 500 contracts are required. We will assume that the price of the three month contract is 0.6250 USD/AUD.

Assume that the following prices prevail in the market after one month.

Spot: 0.5985 USD/AUD

2-M futures: 0.6025 USD/AUD

If Merck had not hedged, it would have received:

$$50{,}000{,}000 \times 0.5985 = \text{USD } 29{,}925{,}000$$

However, since it has hedged using the futures contracts, the effective payoff may be calculated as follows. The profit/loss from the futures market is:[4]

$$500 \times 100{,}000 \times (0.6250 - 0.6025) = \text{USD } 1{,}125{,}000$$

Therefore, the total payoff is:

$$29{,}925{,}000 + 1{,}125{,}000 = \text{USD } 31{,}050{,}000$$

The effective exchange rate is:

$$\frac{31{,}050{,}000}{50{,}000{,}000} = 0.6210 \text{ USD/AUD}$$

7.11 Hedging an Import Transaction

United Airlines has ordered spare parts for its aircraft from Airbus in France. The total cost is EUR 50 MM and the payment is due after one month. The company is worried that the euro will appreciate, which means that the cost in dollars will go up. Since it will be buying euros, the company requires a long hedge. Assume that the current futures rate for a three month contract is 1.2325 USD/EUR.

Let us assume that the following rates prevail in the market after one month.

Spot: 1.2250 USD/EUR

2-M Futures: 1.2450 USD/EUR

If the company had not hedged, then it would have had to purchase 50 MM EUR at 1.2250. This would have lead to an outflow of USD 61,250,000. However, since it has hedged, its effective cost can be calculated as follows.

[3]The logic is that if the US dollar appreciates, the US dollar price of Australian dollars will fall and consequently, the short hedger will gain.

[4]In this case it is a profit.

The number of futures contracts required is:

$$\frac{50{,}000{,}000}{125{,}000} = 400$$

The profit/loss from the futures market is:[5]

$$125{,}000 \times 400 \times (1.2450 - 1.2325) = \text{USD } 625{,}000$$

The effective cost is:

$$61{,}250{,}000 - 625{,}000 = 60{,}625{,}000 \text{ USD}$$

and the effective exchange rate is

$$\frac{60{,}625{,}000}{50{,}000{,}000} = 1.2125 \text{ USD/EUR}$$

7.12 Creating Synthetic Investments

We know that, Spot – Futures = Synthetic T-bill. Foreign exchange futures contracts can be used to make synthetic investments denominated in the domestic currency, with the help of a cash and carry arbitrage strategy. The example that we give below, is yet another manifestation of quasi-arbitrage.

7.12.1 Example

Consider the following information. On July 1, 20XX, a portfolio manager in Sydney has 10 MM AUD to invest till September 21, that is, for a period of 82 days. An investment in domestic T-bills will yield an annualized return of 6.5%. Assume that the following rates are prevalent in the foreign exchange market.

Spot: 1.9750/1.9795 AUD/USD

September 21 Futures: 1.9870/1.9920 AUD/USD

The lending rate in the US is 5% per annum.

It turns out that in such a situation, the manager can earn a higher rate of return without facing exchange risk, with the help of futures contracts. A cash and carry strategy would first entail the conversion of the AUD into an equivalent amount of 5,051,780.75 USD, at the ask rate of 1.9795. This amount can then be invested in the US at 5% per annum. Simultaneously, a short position will have to be taken in the futures contracts. The number of contracts required can be determined as follows. On September 21, the investment in the US will payoff

$$5{,}051{,}780.75\left[1 + .05 \times \frac{82}{360}\right] = 5{,}109{,}314.92 \text{ USD.}$$

[5] In this case, it is once again a profit.

In order to sell this amount on that day, the manager will require 510.9315 futures contracts, assuming that each contract is for 10,000 USD. This will lock in

$$5{,}109{,}314.92 \times 1.9870 = 10{,}152{,}208.75 \text{ AUD}.$$

The rate of return

$$= \frac{(10{,}152{,}208.75 - 10{,}000{,}000)}{10{,}000{,}000} \equiv 1.522\%$$

$$\equiv 6.6823\% \text{ on an annualized basis,}$$

which is greater than the domestic lending rate of 6.5%.

7.13 Borrowing Funds Abroad

Just the way a cash and carry strategy can be employed to make a synthetic investment denominated in the home currency, a synthetic borrowing in the home currency can be undertaken using a reverse cash and carry strategy. The borrower would raise the funds in a foreign capital market and convert it into the domestic currency at the prevailing spot bid rate. He would simultaneously go long in futures contracts to lock in a rate for the foreign currency, in order to take care of his principal and interest payments at the time of maturity. Such quasi-arbitrage would be attractive if the domestic borrowing rate were to be higher than the synthetic borrowing rate.

Suggestions for Further Reading

1. Apte, P.G. *International Financial Management*. Tata McGraw-Hill, 1998.
2. Chance, D.M. *An Introduction to Options and Futures*. The Dryden Press, 1991.
3. Edwards, F.R. and C.W. Ma. *Futures and Options*. McGraw-Hill, 1992.
4. Siegel, D.R. and D.F. Siegel. *Futures Markets*. The Dryden Press, 1990.
5. Steiner, B. *Foreign Exchange and Money Markets*. Butterworth-Heinemann, 2002.

Concept Check

State whether the following statements are True or False.

1. In the case of a direct quotation, an increase in the rate means that the domestic currency has depreciated.
2. An appreciating rupee is likely to lead to an increase in imports into India.

3. In the case of direct quotes, if the forward margin is specified as a/b, where $a > b$, then the swap points should be added.
4. In the case of indirect quotes, if the forward margin is specified as a/b, where $a > b$, it indicates that the foreign currency is at a forward discount.
5. In the case of indirect quotes, if the forward margin is specified as a/b, where $a > b$, then the swap points should be subtracted.
6. If the domestic interest rate is twice the foreign rate, then the forward rate will be twice the spot rate.
7. For a sale transaction in the case of option forwards, if a currency is quoting at a premium, then the premium will be charged for the latest date of delivery.
8. In a market with transactions costs, one way arbitrage may be feasible when covered interest arbitrage is not and vice versa.
9. An American party exporting to the UK, who is going to be paid in pounds, would like to hedge his exposure by going short in futures contracts.
10. In the case of the indirect quotation system, the bid will be higher than the ask.
11. If a quote is given with the US dollar as the base currency, it is said to be a quote in European terms.
12. If the domestic interest rate is equal to the foreign interest rate, the foreign currency will trade flat.
13. In the case of indirect quotes, the maxim is sell high and buy low.
14. A direct quote for US dollars against the Indian rupee, as quoted by a bank in India, will be perceived as an indirect quote by an American investor.
15. The bid-ask spread in the spot market will be lower than the spread in the forward market.
16. In the absence of bid-ask spreads, the conditions required to preclude one-way arbitrage, imply the interest rate parity condition.
17. The slightest hint of exchange control may preclude an investor from taking advantage of a perceived opportunity for covered interest arbitrage.
18. One-way arbitrage requires transactions in three of the following four markets: the spot market, the forward market, and the money markets for the two currencies.
19. In the case of option forwards, the dealer will quote a rate assuming that the deal may be completed on the worst possible day from his point of view.
20. Cross-rate futures contracts on the CME Group are based on the exchange rates of two non-American currencies with respect to each other.

Questions and Problems

Question-I

Explain the difference between direct quotes and indirect quotes.

Question-II

What are European terms and American terms?

Question-III

IDBI Bank Mumbai is quoting the following rates.

Spot: 48.20/48.80 INR/USD

1 Month Forward: 20/10

2 Month Forward: 60/90

Calculate the outright forward rates for 1 Month and 2 Month contracts.

Question-IV

Mitoken Solutions is importing chips from Intel in California and is required to make the payment between 1 to 2 months from now, although it cannot specify the exact date. The company approaches Scotia Bank Bangalore for an option forward contract.

Assume that the rates in the foreign exchange market are as follows.

Spot: 48.15/48.40 INR/USD

1 Month Forward: 40/70

2 Month Forward: 65/95

What rate will the bank quote for the contract?

Question-V

Assume that the spot rates and the exchange margin are the same as in the question above, but that the swap points are as follows.

1 Month Forward: 70/40

2 Month Forward: 95/65

What rate will the bank quote in this case?

Question-VI

The following rates are available on the interbank market in India.

Spot: 47.25/47.75 INR/USD

The borrowing rate in India is 10% per annum and the lending rate is 8%. The borrowing rate in the US is 6% per annum and the lending rate is 4%.

What are the price bands for the bid and ask prices of a six month forward contract, if covered interest arbitrage is to be ruled out?

Question-VII

The following information is currently available. The spot rate is 42.30 INR/USD, and the six month forward rate is 42.95. The riskless rate in India is 12% per annum, whereas in the US, it is 6% per annum.

Is there any potential for making arbitrage profits? If so, how will you go about exploiting the situation?

If you decide to exploit the arbitrage opportunities if any, what are the factors that will force the market back into equilibrium?

Question-VIII

General Motors is importing machinery from Siemens in Germany and is required to pay 100 MM EUR two months from today. Assume that we are standing on April 1, 20XX. Futures contracts expiring on 21 June are available at the following rates.

June Futures: 1.5275/1.5385 USD/EUR

On 1 June, the following rates are observable.

Spot: 1.5875/1.5925

June Futures: 1.5890/1.5945

If the company buys euros in the spot market on 1 June and closes out its futures position, what is the effective rate at which it would have bought the euros?

Question-IX

Techspan, an IT firm based in Nashville, has exported software worth 2 MM EUR to a party in Paris, and is due to be paid after two months. The current rates in the inter-bank market are

Spot: 1.1500-1.1595 USD/EUR

3M Forward: 45/80

Two months later, the rates in the inter-bank market are as follows.

Spot: 1.1825-1.1895 USD/EUR

1M Forward: 30/65

1. What will be the revenue in US dollars if the firm remains unhedged?
2. What will be the revenue in US dollars if the firm enters into a forward contract, and subsequently offsets?
3. Assuming that the firm hedges, what will be the effective exchange rate that is locked in?

Question-X

What are E-mini contracts?

Appendix–VII

Symbols for Major Currencies

Country	Currency	Symbol
Australia	Dollar	AUD
Brazil	Real	BRL
Canada	Dollar	CAD
China	Renmimbi Yuan	CNY
Czech Republic	Koruna	CAK
European Monetary Union	Euro	EUR
Hong Kong	Dollar	HKD
India	Rupee	INR
Israel	Shekel	ILS
Japan	Yen	JPY
Malaysia	Ringgit	MYR
Mexico	Peso	MXN
New Zealand	Dollar	NZD
Norway	Krone	NOK
Russia	Rouble	RUB
Singapore	Dollar	SGD
South Africa	Rand	ZAR
South Korea	Won	KRW
Sweden	Krona	SEK
Switzerland	Franc	CHF
Thailand	Baht	THB
UK	Pound Sterling	GBP
USA	Dollar	USD

8

Stock and Stock Index Futures

8.1 Introduction

Equity shares like debt securities, are a type of financial claims that are issued by companies. However, unlike debt securities, such securities confer ownership rights on the shareholders. A share of stock represents the fundamental unit of ownership of a corporation. Consequently, every firm must have at least one shareholder. On the other hand, a firm need not have a bond holder. There is however no upper limit on the number of shareholders that a firm may have, nor is there any restriction on how many shares a firm can issue to its shareholders. Every shareholder, is a part owner of the company to whose shares he has subscribed, and his stake in the company is equal to the fraction of the total share capital of the firm held by him.

When a firm seeks to reward its shareholders, it will usually pay out a percentage of its profits to them in the form of cash. This income that is received by the shareholders from the firm is called a *Dividend*. Usually, a firm will not pay out the entire profits earned by it in the form of dividends. The normal practice is to plough back a part of it into the firm to meet future cash requirements. Such profits that are reinvested in the firm, are called *Retained Earnings*. These earnings which are retained by the firm, will manifest themselves as an increase in the *Reserves and Surplus* account on the balance sheet.

Shareholders are considered to be residual claimants. This perception is valid in two respects. First, they are eligible for dividends only after all payments due to the other creditors of the firm have been made. The rate of dividends is consequently not fixed and neither is a dividend a contractual obligation. Dividend related decisions are made by the board of directors of a company, and the shareholders cannot make demands on them. However, if it were to be perceived that the directors are not acting in their interests, a group of dissident shareholders with a required majority can outvote the old director(s) and replace them with their nominees. There is another reason as to why shareholders are considered to be residual claimants. If the firm were to declare bankruptcy, then the shareholders would be entitled only to the residual assets, if such a residue were to be available after the claims of all the other creditors have been met.

Unlike debt securities, which usually have a stated maturity date, equity shares never mature. That is, no promoter will ever incorporate a firm with a termination date in mind. Equity shares therefore continue to be in existence as long as the firm itself continues to be in existence. A key feature of a company as opposed to a sole proprietorship or a partnership, is that shareholders have limited liability. That is, no matter how serious the financial difficulties facing a company may be, neither it nor its creditors can make financial demands on the common shareholders. On the other hand, if a partnership were to go bankrupt, its creditors can in most cases pursue the partners personally in order to recover what is owed to them. However, the maximum financial loss that a shareholder may suffer is limited to the investment made by him in the process of acquisition of the shares.

8.2 Dividends

The shareholders of a company cannot demand dividends from it. The decision to pay dividends is entirely at the discretion of the board of directors of the company. Companies usually declare a dividend when they announce their results for a period. In the US, since results are typically declared on a quarterly basis, dividends are also announced every quarter. In the UK most companies pay their annual dividends in two stages. A dividend declaration is a statement of considerable importance. It is an affirmation by the company that its affairs are on track, and that it has adequate resources to reinvest in its operations as well as to reward its shareholders. The dividend announcement will mention a date called the *record date*. Only those shareholders, whose names appear as of the record date on the register of shareholders being maintained by the registrar, will be eligible to receive the forthcoming dividend.

Another important date in the context of dividends is known as the *ex-dividend date*. This date is specified by the exchange on which the shares are traded. If an investor were to acquire the shares on or after the ex-dividend date, then he will not be eligible to receive the forthcoming dividend. Thus, the ex-dividend date will be such that transactions prior to that date will be reflected in the register of shareholders on the record date. However, subsequent transactions will be reflected in the books only after the record date.

There is obviously a relationship between the ex-dividend date and the settlement cycle on the exchange where the shares are being traded. For instance the NYSE follows a T+3 settlement cycle. Thus, if a trade occurs on day T, then delivery of shares to the buyer and payment of funds to the seller will occur on day T+3. Consequently, anyone who purchases shares two days before the record date or later will not be able to ensure that he is the owner of record as of that date. Thus, on the NYSE, the ex-dividend date for an issue will be two business days prior to the record date that has been specified in the dividend announcement.

Prior to the ex-dividend date we say that the shares are being traded cum-dividend. The implication is that potential buyers will receive the forthcoming dividend. However, on the ex-dividend date the shares begin to trade ex-dividend. This signifies that a potential buyer will no longer be eligible to receive the next dividend, and that the forthcoming dividend will therefore be paid to the seller since he is the owner of record. Thus, on the ex-dividend date, the share price ought to decline by the amount of the dividend. For instance, if the cum-dividend price is \$ 75 per share, and the quantum of the dividend is \$ 2.50 per share, then in theory, the share should trade at \$ 72.50 ex-dividend.

8.3 Stock Dividends

A stock dividend is a dividend that is paid to the shareholders in the form of shares. That is the company offers additional shares to the shareholders, without requiring them to pay any money in lieu. The issue of such shares without any monetary consideration entails the transfer of funds from the reserves and surplus account on the balance sheet to the share capital account. Such a fund transfer is known as the *capitalization of reserves*.

Irrespective of whether the funds are retained in the reserves and surplus account or transferred to the share capital account, they continue to belong to the shareholders. Thus, from a theoretical standpoint stock dividends do not create any value for the shareholders. The way to think of it is that it is as if the shareholder has two bank accounts. We are debiting one account and crediting the other. In the process, the shareholder's wealth is not affected. From the company's point of view, the issue of such additional shares does not lead to an increase in its assets. Nor does it have any implications for the earnings capacity of the firm. Consequently, following a stock dividend the share price should theoretically decline.

Assume that a firm has issued 100,000 shares and that the share price prior to the dividend announcement is \$ 60 per share. If the firm announces a 25% stock dividend there will be 125,000 shares outstanding after the dividends are paid. Considering that there is no change in the value of the firm, the ex-dividend price, P, should be such that:

$$60 \times 100{,}000 = P \times 125{,}000$$

$$\Rightarrow P = \$\,48$$

8.4 Splits and Reverse Splits

Unlike stock dividends which entail the capitalization of reserves, stock splits lead to a decrease in the par value of the shares, accompanied by a simultaneous increase in the number of shares outstanding. An n:1 stock split means that n new shares will be issued to the existing shareholders in lieu of one existing share. Take the case of a 5:4 split. What this means is that the holder of four shares will have five shares after the split. This is exactly analogous to a 25% stock dividend. Thus splits and stock dividends are mathematically equivalent. However the mechanics are different. Take the case of a company which has issued 100,000 shares with a par value of \$ 10 each. If it were to announce a 2:1 split, the number of shares issued will increase to 200,000. However the par value will decline to \$ 5. The issued capital will continue to remain at \$ 1,000,000.

Since a split is equivalent to a stock dividend, the share price after a split will behave in the same way as it would after an equivalent stock dividend. Assume that an investor is holding 10,000 shares worth \$ 60 each. The market value of the shares will be \$ 600,000. If the firm were to announce a 6:5 split then he will have 12,000 shares after the split. Since the split is value neutral in theory, the post-split share price ought to be

$$\frac{600{,}000}{12{,}000} = \$\ 50$$

In the case of an n:m reverse split, n will be less than m. Take the case of a company which has issued 100,000 shares with a market price of \$ 48 per share. If the firm were to announce a 4:5 reverse split, the post-reverse split price ought to be

$$\frac{4{,}800{,}000}{80{,}000} = \$\ 60.00$$

8.5 Pre-emptive Rights

The directors of a company must obtain the approval of existing shareholders if they wish to issue shares beyond what has been issued earlier. As per the charter of some corporations, existing shareholders must be given the first right to buy additional shares in proportion to the shares that they already own. In such cases, existing shareholders are said to have a pre-emptive right to acquire new shares as and when they are issued. That is, the existing shareholders are given an opportunity to maintain their proportionate ownership in the company.

Usually such issues are made at a price that is lower than the prevailing market price of the share. In such circumstances, the right acquires a value of its own. The existing shareholders have a choice. That is, they may either exercise their rights and acquire additional shares, or else sell they may sell the rights to another investor.

The value of a right may be determined as follows. Assume that a company has issued 100,000 shares and that it is offering its shareholders the right to purchase one share for every five shares that they are holding. The market price per share is \$ 60 and the new shares are being offered at \$ 48 each. The market value of the firm prior to the rights issue is

$$100{,}000 \times 60 = \$\ 6{,}000{,}000$$

The post issue theoretical value of the firm will be

$$6{,}000{,}000 + 20{,}000 \times 48 = \$\ 6{,}960{,}000$$

The ex-rights price should therefore be:

$$\frac{6{,}960{,}000}{120{,}000} = \$\ 58$$

Since the shareholders are getting a share worth \$ 58 at \$ 48, the value of the right to acquire a share is \$ 10. Considering that the shareholder needs 5 shares to acquire the right to buy one share, the value of a right is \$ 2.

8.6 Adjustment of Stock Futures Contracts for Corporate Actions

In the event of certain corporate actions, the terms of a futures contract will have to be adjusted. The procedures adopted are not uniform and sometimes differ across exchanges.

8.6.1 Dividends

Usually no adjustment is made in the case of cash dividends. However, a dividend may have consequences for a futures contract, if it is perceived to be extraordinary.

Exchanges which prescribe an adjustment procedure for an extraordinary cash dividend adopt the following procedure. Readers should note at the outset that for all corporate actions, changes to the futures settlement price and the contract size, will only be made at the end of the last day on which the security is traded on a cum basis, that is cum-cash dividend or cum-stock dividend. Thus in the case of an extraordinary cash dividend, the previous day's settlement price will be reduced by the amount of the dividend, when the contract is marked to market on the ex-dividend date.

Let us assume that the settlement price on the ex-dividend date is \$ 120, and that the previous settlement price was \$ 126. Each futures contract is for 100 shares, and the dividend is \$ 10 per share. While marking to market on the ex-dividend date the previous day's settlement price will be taken as \$ 116, that is 126 − 10. Hence the profit/loss for a long position in one contract will be

$$(\$\ 120 - \$\ 116) \times 100 = \$\ 400$$

In the case of cash dividends, there will be no adjustment made to the contract size.

8.6.2 Stock Dividends

Let us denote a stock dividend as $A{:}B$. That is, holder of B shares will have $A + B$ shares in his possession after the dividend is declared. For a 20% stock dividend, the ratio will be 1:5. Thus the holder of 5 shares will have 6 shares after the dividend is declared. The adjustment factor in this case is $\frac{A+B}{B}$.

On the ex-dividend day the contract size will be multiplied by the adjustment factor. While marking the contract to market on the ex-dividend date, the settlement price on the previous day will be divided by the adjustment factor.

Let us consider the case of a 1:5 stock dividend. The adjustment factor is 1.20. Thus the contract size will be adjusted from 100 to 120. Assume that the settlement price on the ex-dividend date is \$ 108, while that on the previous day was \$ 126. The adjusted settlement price will be $\frac{126}{1.20} = \$\ 105$. Thus the profit from marking to market for the holder of a long position in one contract will be

$$120 \times (108 - 105) = \$\ 360$$

8.6.3 Stock Splits/Reverse Splits

The adjustment for stock splits and reverse splits is very similar to the procedure that is adopted for stock dividends. Assume that a firm undergoes an $A{:}B$ split or reverse split. That is, the holder of B shares will have A shares after the corporate action. The adjustment factor is therefore $\frac{A}{B}$. For instance, if a stock undergoes a 8:5 split, the adjustment factor will be 1.60.

The terms of the contract will be adjusted as follows. First, the contract size will be multiplied by the factor. Second, on the ex date, the previous day's settlement price will be divided by the factor.

In the case of an 8:5 split, the contract size will be adjusted from 100 to 160. Let us assume that the previous day's settlement price is 112 while the settlement price on the ex-date is 75. Thus the adjusted settlement price will be

$$\frac{112}{1.60} = 70$$

Hence, the profit from marking to market for an investor with a long futures position will be

$$160 \times (75 - 70) = \$\ 800$$

8.6.4 Rights Issues

A company announces a 2:5 rights issue. That is, an existing shareholder is entitled to two shares for every five shares that he is holding. The right can be exercised at a price of \$ 48 per share. At the end of the day prior to the expiry of the right, the share price is \$ 50.80. Obviously, it is profitable to exercise the right. If the rights are exercised the value of the share will drop to

$$\frac{5 \times 50.80 + 2 \times 48}{7} = 50$$

which represents a decline of \$ 0.80 per share. On the next day, the settlement price for the previous day will be reduced by \$ 0.80 while marking to market.[1] No adjustment would be made to the contract size.

8.7 Stock Indices

While tracking the stock market it is impossible for an analyst to draw a meaningful conclusion if he is confronted with a vector of stock prices. He needs

[1] If the right were not to have any value on the previous day, no adjustment would be made.

one number that is a summary measure of the performance of the market as a whole. Such a measure is termed as an index number and is intended to serve as a barometer of the performance of the stock market, or of a particular segment of the stock market. In order for the index to be a good indicator of market movements, the stocks constituting the index ought to be chosen so as to be representative of the market or of the market segment.

8.7.1 Price-Weighted Indices

The first step while computing an index entails the specification of the number of companies that are going to be included, and the identity of these companies. This is the case for all indices and not just those that are price weighted. In a price weighted index, the only factor that is considered is the prices of the component stocks.

To compute the index, the current prices of all the component stocks are added up and divided by a number known as the *Divisor*. On the base date, or the date on which the index is being computed for the first time, the divisor can be set equal to any arbitrary value. Often, it is set equal to the number of stocks constituting the index. Subsequently, if there is a corporate action such a split/reverse split or a stock dividend, the divisor will have to be adjusted.

Let us assume that we are standing at the end of day t, and that the closing price of the i th stock on the day is $P_{i,t}$. If so, the index level I_t, is given by

$$I_t = \sum_{i=1}^{N} \frac{P_{i,t}}{Div_t} \tag{8.1}$$

where Div_t is the applicable value of the divisor for the day, and N is the number of stocks comprising the index.

Numerical Illustration Consider a hypothetical index consisting of four stocks. Assume that we are standing on the base date, and that the starting value of the divisor is 4.0. The closing prices of the component stocks at the end of the day are as shown in Table 8.1.

Table 8.1 Prices of the constituent stocks on the base date

Stock	Price
DuPont	20
PepsiCo	45
IBM	85
Merck	25
Total	175

The end of the day index value will therefore be $\frac{175}{4} = 43.75$.

8.8 Value-Weighted Indices

Unlike a price-weighted index which is based solely on the prices of the component stocks, a value weighted index takes into consideration the market capitalization of the component stocks. In other words, the number of shares issued by a company is also factored in.

Assume that we are standing on day t. Let us denote the base date of the index by b. We will use $P_{i,t}$ and $P_{i,b}$ to denote the market prices of the ith stock on days t and b respectively, and $Q_{i,t}$ and $Q_{i,b}$ to denote the number of shares outstanding on those two days.

On the base date, the index can be assigned any value. We will set it as equal to 100, which is a fairly common practice. The level of the index on day t is then defined as

$$\frac{1}{Div_t}\left(\frac{\sum_{i=1}^{N} P_{i,t}Q_{i,t}}{\sum_{i=1}^{M} P_{i,b}Q_{i,b}}\right) \times 100 \tag{8.2}$$

Div_t represents the value of the divisor on day t. The divisor is assigned a value of 1.0 on the base date. Subsequently it will be adjusted as and when required.

Numerical Illustration We will take the same four stocks as before and use the prices shown in Table 8.1. The difference is that we will now also consider the number of shares issued by each firm.

Table 8.2 Prices, number of shares outstanding, and market capitalization of the components of a value weighted index on the base date

Stock	Price (P)	# of Shares (Q)	Market Capitalization
DuPont	20	900,000,000	18,000,000,000
PepsiCo	45	1,250,000,000	56,250,000,000
IBM	85	1,635,000,000	138,975,000,000
Merck	25	2,220,000,000	55,500,000,000

The total market value is

$$\sum_{i=1}^{4} P_i Q_i = 268{,}725{,}000{,}000$$

We will assign a starting value of 100 for the index. The corresponding value for the divisor is obviously 1.0.

On the following day, we will assume that the prices and number of shares outstanding are as follows.

Table 8.3 **Prices, number of shares outstanding, and market capitalization of the components of a value weighted index on the following day**

Stock	Price (P)	# of Shares (Q)	Market Capitalization
DuPont	22.50	900,000,000	20,250,000,000
PepsiCo	50.00	1,250,000,000	62,500,000,000
IBM	90.00	1,635,000,000	147,150,000,000
Merck	27.50	2,220,000,000	61,050,000,000

The total market value is

$$\sum_{i=1}^{4} P_i Q_i = 290{,}950{,}000{,}000$$

The value of the index on this day is therefore

$$\frac{290{,}950{,}000{,}000}{268{,}725{,}000{,}000} \times 100 = 108.2705$$

Quite obviously, our conclusion would be that the market has moved up.

8.9 Equally-Weighted Indices

An equally weighted index is also a type of barometer for tracking the performance of a portfolio of stocks. Assume that we decide to form such an index using N stocks. Just like a price weighted index, an equally weighted index too is based on the share prices and not on the market capitalization.

The value of the index on day t is defined as

$$I_t = I_{t-1} \times \frac{1}{N} \sum_{i=1}^{N} \frac{P_{i,t}}{P_{i,t-1}} \tag{8.3}$$

The ratio of the prices, $\frac{P_{i,t}}{P_{i,t-1}}$ may be expressed as $(1 + r_{i,t})$ where $r_{i,t}$ is the arithmetic rate of return on the ith stock between day t and day $t-1$. Therefore

$$\frac{1}{N} \sum_{i=1}^{N} \frac{P_{i,t}}{P_{i,t-1}} = \frac{1}{N} \sum_{i=1}^{N} (1 + r_{i,t}) = 1 + \frac{1}{N} \sum_{i=1}^{N} r_{i,t} = 1 + \overline{r}_t \tag{8.4}$$

where $\overline{r}_t$ is the arithmetic average of the returns on all the component stocks between day $t-1$ and day t. Thus

$$I_t = I_{t-1} \times (1 + \overline{r}_t) \tag{8.5}$$

8.10 Tracking Portfolios

Investors can construct portfolios that replicate the behavior of a market index. The method of construction of such a portfolio would depend on the way the underlying index has been defined.

To imitate an equally weighted index, the investor has to put an equal fraction of his wealth in all the assets that constitute the index. That is, if the underlying index consists of four stocks, he has to invest 25% of his wealth in each asset. Since the component securities will have different prices, the number of shares of each security that are required to be held will vary.

In order to track a price weighted index the investor has to hold an identical number of shares of each of the companies that are present in the index.

Forming a portfolio to track a value weighted index is the most complex task. To form such a tracking portfolio, the fraction of the investor's wealth that is invested in each asset should be equal to the ratio of the market capitalization of that stock to the total market capitalization of all the stocks that are present in the index.

8.11 Major Stock Indices

The most famous stock index is undoubtedly the Dow Jones Industrial Average (DJIA), popularly known as the *Dow*. It is a price weighted average of 30 stocks. The list of constituent stocks as of 22 September 2008 is as follows.

Table 8.4 Constituents of the Dow Jones industrial average

3M	Alcoa	American Express
AT&T	Bank of America	Boeing
Caterpillar	Chevron	Citigroup
Coca-Cola	E.I. DuPont de Nemours	Exxon Mobil
General Electric	General Motors	Hewlett-Packard
Home Depot	Intel	IBM
Johnson & Johnson	J.P. Morgan Chase	Kraft Foods
McDonald's	Merck	Microsoft
Pfizer	Procter & Gamble	United Technologies
Verizon Communications	Wal-Mart Stores	Walt Disney

The Nikkei Index, which is a barometer of the Japanese stock market, is also price weighted and includes 225 large Japanese companies.

The Standard & Poor's 500 Index (S&P500) and the Nasdaq 100 index are both value weighted.

8.12 Stock Index Futures

The CME Group trades futures contracts on a number of indices. Details of some of the prominent contracts are summarized in Table 8.5.

Table 8.5 Specifications of futures contracts on major indices

Contract	Index	Multiple	Point Description	Currency Equivalent	Tick Size
E-Mini Dow	DJIA	$ 5	1 Index Unit	$ 5.00	$ 5.00
E-Mini S&P 500	S&P500	$ 50	0.01 Index Units	$ 0.50	$ 12.50
E-Mini NASDAQ 100	NASDAQ-100	$ 20	0.01 Index Units	$ 0.20	$ 5.00
Dow	DJIA	$ 10	1 Index Unit	$ 10	$ 10
Big Dow	DJIA	$ 25	1 Index Unit	$ 25	$ 25
S&P 500	S&P 500	$ 250	0.01 Index Units	$ 2.50	$ 25.00
NASDAQ 100	NASDAQ 100	$ 100	0.01 Index Units	$ 1.00	$ 25.00
Nikkei 225	Nikkei 225	500 JPY	1 Index Unit	500 JPY	2,500 JPY
Nikkei 225	Nikkei 225	$ 5	1 Index Unit	$ 5.00	$ 25.00

The relationship between *'Point Description'*, *'Currency Equivalent'*, and *'Tick Size'* can be illustrated as follows.

Take the case of the E-Mini S&P 500 futures contract. One point is equivalent to 0.01 units in terms of the index. Since each index unit is worth $ 50, each point is worth $ 0.50. The minimum observable change in the futures price is 25 points or $ 12.50 per contract.

In the case of the Nikkei 225 Yen denominated contract, each point is one index unit, which is worth 500 JPY. The minimum observable price change is 5 points or 2,500 JPY.

For contracts on the Dow Jones Industrial average, the size is 10 times the value of the index. There is also a mini-sized Dow futures contract. The contract size in this case is $ 5 times the index level. The exchange has now introduced a Big Dow futures contract. The contract size is $ 25 times the index level.

The expiration dates differ from contract to contract. The E-Mini S&P and the E-Mini NASDAQ 100 contracts normally expire on the third Friday of the contract month. The Nikkei 225 contracts, both dollar denominated as well as yen denominated, expire on the business day preceding the second Friday of the contract month. The S&P 500 and the NASDAQ 100 contracts expire on the business day prior to the third Friday of the expiration month. The Dow contracts, normal, E-Mini, as well as Big, expire on the third Friday of the contract month. All the index futures contracts are cash settled.

The number of contracts that will be available at any point in time is as described below.

Table 8.6 Available expiration months in general

Contract	Available Months
E-Mini Dow	4 Months from the March quarterly cycle
E-Mini S&P 500	5 Months from the March quarterly cycle
E-Mini NASDAQ 100	5 Months from the March quarterly cycle
Dow	4 Months from the March quarterly cycle and 2 additional December months
Big Dow	4 Months from the March quarterly cycle
S&P 500	8 Months from the March quarterly cycle
NASDAQ 100	5 Months from the March cycle
Nikkei 225 (Yen)	5 Months from the March quarterly cycle and 3 Serial months
Nikkei 225 (Dollar)	4 Months from the March quarterly cycle

Assume that today is 30 December 2008, which is the fifth Tuesday of December. The December contracts would have all expired. The following expiration months will be available for the various products.

Table 8.7 Available expiration months on 26 December 2006 available months

E-Mini Dow	March-09; June-09; Sep-09; Dec-09
E-Mini S&P 500	March-09; June-09; Sep-09; Dec-09; March-10
E-Mini NASDAQ 100	March-09; June-09; Sep-09; Dec-09; March-10
Dow	March-09; June-09; Sep-09; Dec-09; Dec-10; and Dec-11
Big Dow	March-09; June-09; Sep-09; Dec-09
S&P 500	March-09; June-09, Sep-09; Dec-09; March-10; June-10; Sep-10; and Dec-10
NASDAQ 100	March-09; June-09; Sep-09; Dec-09; March-10
Nikkei 225 (Yen)	March-09; June-09; Sep-09; Dec-09; March-10; and Jan-09; Feb-09; and April-09
Nikkei 225 (Dollar)	March-09; June-09; Sep-09; and Dec-09

8.13 Pricing of Index Futures

In order to derive the expression for the no-arbitrage futures price, we will start with a numerical illustration. Let us take the case of a price weighted index that is based on five stocks. We will use the price data given below in Table 8.8. We will assume that the divisor is 5.0. Since the price aggregate is 800, the index level is obviously 160.

Table 8.8

Stock	Price
ACC	210
BD	80
CP	210
ECT	70
HUL	230
Total	800

Let us assume that today is 23 June, 20XX and that there is a futures contract based on the above index, that expires on 21 September, 20XX. We will also assume that ACC will pay a dividend of \$ 2.40 on 23 July, that HUL will pay a dividend of \$ 2.40 on 10 August and that CP will pay a dividend of \$ 2.40 on 3 September. The borrowing/lending rate for all investors will be taken to be 7.2% per annum.

Consider the following strategy. Borrow money to buy one share of each of the 5 companies and simultaneously go short in an index futures contract. The cost of the portfolio will be equal to

$$(210 + 80 + 210 + 70 + 230) = 800$$

The borrowed amount will have to be repaid with interest on 21 September. The number of days between 23 June and 21 September is 90. So the amount due on 21 September is

$$800 \times \left(1 + .072 \times \frac{90}{360}\right) = 814.40$$

On 23 July you will get a dividend of \$ 2.40. This can be reinvested till 21 September to yield

$$2.40 \times \left(1 + .072 \times \frac{60}{360}\right) = 2.43$$

Similarly, the future values of the other two dividends, as of 21 September, are 2.42 and 2.41.

To preclude arbitrage opportunities, the futures price, in dollar terms, should be a value F such that,

$$F + 2.43 + 2.42 + 2.41 = 814.40$$

$$\Rightarrow F = \$\ 807.14$$

However, by convention, futures prices are expressed in terms of index units and not in dollar terms. Thus the no-arbitrage futures price should be

$$\frac{807.14}{5} = 161.43$$

8.14 Cash and Carry Arbitrage

Let us assume that F = 163. If so, an investor can borrow $ 800 to buy the five stocks and go short in one futures contract. Each dividend, when received, can be re-invested till the expiration of the futures contract at the prevailing lending rate. At expiration, the futures price will be set equal to the spot index value at that time, since *index futures contracts are always cash settled*. Thus, the arbitrageur should sell the shares at their prevailing market prices.

Let us assume that the spot prices of the shares at the time of expiration of the futures contract are as follows.

Table 8.9

Stock	Price
ACC	225
BD	90
CP	225
ECT	75
HUL	250
Total	865

The index value is $\frac{865}{5}$ = 173. The profit/loss from the futures market is, 163 – 173 = (10) index points, which is equivalent in our case to 10× 5 = $ (50). When the stocks are disposed off by the arbitrageur, there will be a cash inflow of 865. The payoffs from the reinvested dividends is

$$2.43 + 2.42 + 2.41 = \$\ 7.26$$

Thus, the net cash flow at expiration is

$$865 + 7.26 - 814.40 - 50 = \$\ 7.86$$

This value of $ 7.86 is equivalent to $\frac{7.86}{5}$ = 1.57 index units, which is nothing but the difference between the quoted futures price of 163 and the no-arbitrage price of 161.43. It should be noted that the profit will always be equal to the difference between the quoted price and the no-arbitrage price and will be independent of the actual stock prices prevailing at expiration. This is because of the underlying assumption that the arbitrageur is able to sell the shares in the market at the same prices as those used to compute the index value at expiration.

8.15 Reverse Cash and Carry Arbitrage

We will use the same information that we used for demonstrating the cash and carry arbitrage arguments. However, we will assume that the futures price is 159.

An arbitrageur can implement the following strategy. He can short sell all the five stocks which constitute the index and invest the proceeds at the lending rate of 7.2%. Simultaneously, he can take a long position in a futures contract. When the dividends are due, the arbitrageur can borrow the required amounts and pay the person(s) who have lent him the shares. At the time of expiration of the futures contract, the shares present in the index may be acquired at their prevailing spot prices and returned. The arbitrageur will also be required to repay the amounts borrowed for the dividend payments with interest. He will be entitled to the profit/loss from the futures market, and the amount invested by him at the outset, that is the proceeds from the short sale, will now be returned to him with interest.

Therefore, the net cash flow at expiration will be

$$-865 - 7.26 + 814.40 + 5(173 - 159) = \$\ 12.14$$

which is equivalent to 2.43 index units. This, once again, is the difference between the futures price of 159 and the no-arbitrage price of 161.43. Once again, the profit will be independent of the stock prices prevailing at expiration.

8.16 The No-Arbitrage Equation

Let I_t be the spot value of the index on day t. Consider a futures contract maturing on day T, with a current price[2] of F_t. We will denote the borrowing/lending rate by r and the future value of the dividends paid by the component stocks between t and T, as calculated at T, by D_T, where D_T is expressed in terms of index units.[3]

The no-arbitrage equation can therefore be expressed as

$$F_t = I_t\left[1 + r\frac{(T-t)}{360}\right] - D_T \tag{8.6}$$

8.17 Index Arbitrage and Program Trading

If the observed futures price is greater than the price given by the no-arbitrage equation, then arbitrageurs will engage in cash and carry arbitrage as we have

[2]Stock index futures prices are always in index units. They have to be multiplied by the unit size that has been specified by the exchange, in order to calculate the value of the contract in terms of the currency of the country in which they trade. The unit size for the S&P 500 is 250, for the DJIA it is 10 and for the Nikkei it is 5.

[3]Each dividend is assumed to be invested at the rate r between t_i, which is the date on which the ith dividend is paid and T, the expiration date of the contract.

seen. On the other hand, if the observed price is less than the theoretical price, they will resort to reverse cash and carry arbitrage.

Now, unlike in the case of arbitrage using contracts on commodities or even other financial assets, index arbitrage is considerably more complex. Cash and carry index arbitrage requires the arbitrageur to buy all the stocks constituting the index simultaneously and in the same proportions in which they are present in the index, while reverse cash and carry index arbitrage requires that the arbitrageur short sell all the constituent stocks simultaneously, once again in the specified proportions. Obviously, this is the kind of task for which a *computer* is indispensable. The use of a computer program to execute large and complex stock market orders is called *Program Trading*. Consequently, the terms *Index Arbitrage* and *Program Trading* are used synonymously.

8.17.1 Risks Inherent in Index Arbitrage

Index arbitrage has certain underlying risks in practice.

1. Let us assume that you have analyzed the spot and futures prices and have come to the conclusion that there is money to be made through cash and carry arbitrage. To execute it, however, you need to buy the basket of securities constituting the index and sell the appropriate number of futures contracts. Let us assume that you are quickly able to take a short position in the futures market. But, how can you be sure that the prices of some of the stocks will not go up before your purchase transactions are through? If so, you may make a lower profit than anticipated or even make a loss.
2. The second problem with cash and carry arbitrage[4] is that the basket of securities has to be sold on the day of expiration at prices corresponding to those which are used to calculate the settlement price of the futures contract on that day. Why is this so?

   Remember, that since the contract is cash settled, at expiration, the futures price will be set equal to the prevailing spot index value. In order for our earlier observation, that is, that arbitrage profits are independent of the prevailing spot prices at expiration, to be true, it is necessary that we be able to sell the individual stocks at prices which correspond to those used to determine the settlement price of the index.

   In practice, the arbitrageur will issue a *Market on Close Order*, which means that the broker has to sell the shares at the market prices prevailing at the close of trading. In real life, this can be difficult.

   The above risks are called *Execution Risks*.
3. The third problem with index arbitrage, is that stocks may not pay dividends as forecasted by us while identifying arbitrage opportunities. This is called *Dividend Risk*.

[4]Equivalent problems will exist with reverse cash and carry Arbitrage. Readers should be able to work out the logic.

4. And finally, the interest rates that we use to calculate potential arbitrage profits at the outset may not be the same as the rates at which we are eventually able to borrow and lend. This is known as *Interest Rate Risk*.

8.18 Hedging with Index Futures

We will now derive the optimal number of futures contracts for a risk minimizing hedge, for hedging a portfolio P using stock index futures contracts. Let the beta of the portfolio be β_P with respect to the market portfolio. We will assume that the portfolio underlying the futures contract is a suitable proxy for the market portfolio.[5]

8.18.1 Derivation

We will denote today by t and the expiration date of the futures contract by T. The current value of the portfolio which we want to hedge, will be denoted by P_t, the spot value of the index by I_t and the prevailing futures price by F_t.

We will define $\tilde{r}_m$, as the rate of return on the market portfolio where

$$\tilde{r}_m = \frac{(\tilde{I}_T - I_t) + D_T}{I_t} \tag{8.7}$$

The rate of return on the futures contract, $\tilde{r}_F$ can be similarly defined as

$$\tilde{r}_F = \frac{\tilde{F}_T - F_t}{F_t} \tag{8.8}$$

From the no-arbitrage equation, we know that

$$F_t = I_t\left[1 + \frac{r(T-t)}{360}\right] - D_T$$

Therefore

$$\tilde{r}_F = \frac{\tilde{F}_T - I_t\left[1 + \frac{r(T-t)}{360}\right] + D_T}{I_t\left[1 + \frac{r(T-t)}{360}\right] - D_T} \tag{8.9}$$

At expiration

$$\tilde{F}_T = \tilde{I}_T$$

Therefore

$$\tilde{r}_F = \frac{\tilde{I}_T - I_t\left[1 + \frac{r(T-t)}{360}\right] + D_T}{I_t\left[1 + \frac{r(T-t)}{360}\right] - D_T} \tag{8.10}$$

[5] See Stoll and Whaley (1993) for a similar derivation without dividends.

Now, $\tilde{I}_T = I_t(1 + \tilde{r}_m) - D_T$. Therefore

$$\tilde{r}_F = \frac{I_t(1 + \tilde{r}_m) - I_t\left[1 + \frac{r(T-t)}{360}\right]}{I_t\left[1 + \frac{r(T-t)}{360}\right] - D_T}$$

$$= \frac{I_t\tilde{r}_m - I_t\frac{r(T-t)}{360}}{I_t\left[1 + \frac{r(T-t)}{360}\right] - D_T} \tag{8.11}$$

$$\Rightarrow \tilde{r}_m = \tilde{r}_F\left\{1 + \frac{r(T-t)}{360} - \frac{D_T}{I_t}\right\} + \frac{r(T-t)}{360} \tag{8.12}$$

$$\beta_P = \frac{\text{Cov}\,(\tilde{r}_P, \tilde{r}_m)}{\text{Var}\,(\tilde{r}_m)} \tag{8.13}$$

$$= \frac{\text{Cov}\left(\tilde{r}_P, \tilde{r}_F\left\{1 + \frac{r(T-t)}{360} - \frac{D_T}{I_t}\right\} + \frac{r(T-t)}{360}\right)}{\text{Var}\left(\tilde{r}_F\left\{1 + \frac{r(T-t)}{360} - \frac{D_T}{I_t}\right\} + \frac{r(T-t)}{360}\right)}$$

$$= \frac{\text{Cov}\left(\tilde{r}_P, \tilde{r}_F\left\{1 + \frac{r(T-t)}{360} - \frac{D_T}{I_t}\right\}\right)}{\text{Var}\left(\tilde{r}_F\left\{1 + \frac{r(T-t)}{360} - \frac{D_T}{I_t}\right\}\right)}$$

$$= \frac{\text{Cov}\;(\tilde{r}_P, \tilde{r}_F)}{[\text{Var}\,(\tilde{r}_F)] \times \left\{1 + \frac{r(T-t)}{360} - \frac{D_T}{I_t}\right\}} \tag{8.14}$$

The following rules of statistics have been used.

- If X and Y are two random variables and a and b are two constants then

  1. $\text{Var}\,(aX) = a^2\text{Var}\,(X)$
  2. $\text{Var}\,(a) = \text{Var}\,(b) = 0$
  3. $\text{Cov}\,(aX, b) = 0$
  4. $\text{Cov}\,(aX, bY) = ab\,\text{Cov}\,(X, Y)$

$$\tilde{r}_P = \frac{\triangle P}{P_t}$$

$$\tilde{r}_F = \frac{\triangle F}{F_t}$$

Therefore

$$\beta_P = \frac{\text{Cov}\left(\frac{\Delta P}{P_t}, \frac{\Delta F}{F_t}\right)}{\left[\text{Var}\,(\frac{\Delta F}{F_t})\right] \times \left\{1 + \frac{r(T-t)}{360} - \frac{D_T}{I_t}\right\}}$$

$$= \frac{\frac{1}{P_t F_t}\text{Cov}\ (\Delta P, \Delta F)}{\frac{1}{(F_t)^2}[\text{Var}\,(\Delta F)] \times \left\{1 + \frac{r(T-t)}{360} - \frac{D_T}{I_t}\right\}}$$

$$= \frac{\frac{F_t}{P_t}\text{Cov}\,(\Delta P, \Delta F)}{\text{Var}\,(\Delta F)\frac{F_t}{I_t}}$$

$$= \frac{\frac{I_t}{P_t}\text{Cov}\,(\Delta P, \Delta F)}{\text{Var}\,(\Delta F)} \tag{8.15}$$

The number of futures contracts required for a minimum variance hedge is given by

$$h = \frac{\text{Cov}\,(\Delta P, \Delta F)}{\text{Var}\,(\Delta F)}$$

Therefore

$$\beta_P = h\frac{I_t}{P_t}$$

$$\Rightarrow h = \beta_P\frac{P_t}{I_t} \tag{8.16}$$

We will now consider an example using a futures contract on the S&P 500 index. The index is value weighted and the value of the futures contract is \$ 250 per index point.

8.18.2 Example

Let us take the case of a portfolio manager who is handling a portfolio which is currently worth 10 MM dollars. He is worried about the possibility of a market decline, and therefore decides to hedge using index futures. Quite obviously, he needs to go short in index futures.

Assume that the current value of the index is 250. The value of the index in dollar terms is therefore

$$250 \times 250 = \$\ 62{,}500$$

Consider a futures contract with 90 days to expiration, which we will assume represents the hedging horizon of the manager. The riskless rate is assumed to be

10% per annum, and the future value of dividends in index units is assumed to be 10. If so, the no-arbitrage futures price is given by

$$250\left[1 + .10 \times \frac{90}{360}\right] - 10 = 246.25$$

If the beta of the portfolio is assumed to be 1.50, then the required number of contracts for a risk minimizing hedge is

$$1.50 \times \frac{10{,}000{,}000}{62{,}500} = 240 \text{ contracts.}$$

We will examine the performance of the hedge in two different terminal scenarios.

8.18.3 Case A

The index value at expiration is 275. The value of the portfolio may be deduced as follows.

The return on the index is $\dfrac{275 - 250}{250} \equiv 10\%$.

The dividend yield is $\dfrac{10 \times 250}{250 \times 250} \equiv 4\%$.

Hence, the total return on the market = 10% + 4% = 14%.

The riskless rate for 90 days is 2.50%

The portfolio return is therefore

$$2.50 + 1.50(14 - 2.50) = 19.75\%$$

The portfolio value is therefore

$$10(1 + 0.1975) \text{ MM} = \$\ 11.975 \text{ MM}$$

The profit/loss from the futures market is

$$(246.25 - 275) \times 250 \times 240 = \$\ (1{,}725{,}000)$$

Thus, the net value of the investment is: 11.975 – 1.725 = $ 10.25 MM.

The hedged portfolio has earned a 2.50% rate of return. The rationale is as follows. The futures contract has helped remove all the inherent market risk. Therefore, the portfolio has ended up earning the riskless rate of return of 2.5% for 90 days.

8.18.4 Case B

Let the index value at expiration be 220.

The return on the index = $\dfrac{220 - 250}{250} \equiv -12\%$.

The dividend yield is 4%.

Therefore the total return on the market is −8%.

The riskless rate for 90 days is 2.50%.

Hence, the rate of return on the portfolio is

$$2.50 + 1.50(-8 - 2.50) = -13.25\%$$

The portfolio is therefore worth 10 (1 – 0.1325) MM = \$ 8.675 MM

The profit/loss from the futures market is

$$(246.25 - 220) \times 250 \times 240 = \$\ 1{,}575{,}000$$

Thus, the net value of the investment is: 8.675 + 1.575 = \$ 10.25 MM.

Once again, the rate of return on the hedged portfolio is 2.5%.

In practice, however, our hedge may not be perfect. The first reason is that dividends and interest rates may change over the life of the hedge. The second reason is that the return on the index over 90 days may not be perfectly correlated with the return on our portfolio.

8.19 Market Timing with Index Futures

Portfolio managers frequently use index futures contracts to engage in what are known as *Market Timing* strategies. These entail the movement of funds from T-bills to equities and vice versa. If the manager feels that the portfolio is going to do well, he will move funds from T-bills to equity. But, if he anticipates a bear market, then he will do the reverse.

8.19.1 Moving from Equities to T-bills

Take the case of the portfolio manager we considered in the above example on hedging. Let us assume that he is worried about a market decline and wants to shift his funds from equities to T-bills. He can use futures contracts to invest in synthetic T-bills as we have demonstrated above. That is, if he combines a short position in 240 futures contracts with his existing stock position, then it will be tantamount to making a riskless investment, for the overall portfolio value will be independent of the index level at expiration.

8.19.2 Moving from T-bills to Equities

Now let us consider the case of a portfolio manager who is holding T-bills worth 10 MM and wants to shift to an equity portfolio with a beta of .80. We know that

Futures + T-bills = Synthetic Spot.

Thus, the manager needs to combine a long position in the futures contracts with his existing position in T-bills, in order to effectively take an equity position. Using the same logic as before, the number of futures contracts required is

$$0.80 \times \frac{10{,}000{,}000}{62{,}500} = 128 \text{ contracts.}$$

We will examine the performance of the overall portfolio under two different scenarios as before.

Case I: Index = 260 The value of the T-bills = \$ 10,250,000.

The profit/loss from the futures position is

$$128 \times 250(260 - 246.25) = \$\ 440{,}000.$$

Thus the total portfolio value is

$$10{,}250{,}000 + 440{,}000 = \$\ 10{,}690{,}000$$

which implies a rate of return of 6.9% on the investment of 10 MM. A rate of return of 6.9% on the portfolio corresponds to a rate of return of

$$\frac{(6.9 - 2.5)}{.80} + 2.5 = 8\%$$

on the market portfolio. Since the dividend yield is 4%, the implied return on the index is 4%, which is consistent with the observed index level of 260.

Case II: Index = 230 The value of the T-bills = $ 10,250,000.

The profit/loss from the futures position is

$$128 \times 250(230 - 246.25) = \$\ (520{,}000).$$

The total portfolio value is therefore

$$10{,}250{,}000 - 520{,}000 = \$\ 9{,}730{,}000$$

which implies a rate of return of −2.7% for the portfolio, which corresponds to a return of −4% on the market. Since the dividend yield is 4%, the implied return on the index is −8%, which is consistent with an index level of 230.

8.20 Using Index Futures to Change the Beta of a Portfolio

Let us assume that we have a portfolio worth $ P_t, that the current index value is I_t and that the beta of the portfolio is β^*.

Consider taking a long position in N futures contracts. If we denote our overall portfolio as o, the rate of return for this portfolio can be written as

$$r_o = \frac{\triangle P}{P_t} + N\frac{\triangle F}{P_t} \tag{8.17}$$

Therefore

$$E(r_o) = E(r_p) + \frac{N}{P_t}E(\triangle F) = r + \beta^*[E(r_m) - r] + \frac{N}{P_t}E(\triangle F) \tag{8.18}$$

From the systematic risk explanation, we know that

$$[E(I_T) - F_t] = [E(F_T) - F_t] = E(\triangle F) = \beta[E(r_m) - r]I_t$$

$$= [E(r_m) - r]I_t \tag{8.19}$$

because the beta of the futures contract can be taken to be 1.0, since it is a contract on the market portfolio. Hence

$$E(r_o) = r + \beta^*[E(r_m) - r] + \frac{N}{P_t}[E(r_m) - r]I_t \tag{8.20}$$

If we want our overall portfolio to have a beta of β_T, we should choose N such that

$$r + \beta^*[E(r_m) - r] + \frac{N}{P_t}[E(r_m) - r]I_t = r + \beta_T[E(r_m) - r]$$

$$\Rightarrow N = (\beta_T - \beta^*)\frac{P_t}{I_t} \tag{8.21}$$

Thus, if we want to increase the beta, we should go long in the required number of futures contracts, whereas if we want to decrease it, we should go short.

8.21 Stock Picking

Stock picking is a term used for the art of finding stocks that are underpriced or overpriced. Let us depict the rate of return for a stock, stock *i*, as follows.

$$r_i = r + \beta_i(r_m - r) + \epsilon_i + \alpha_i \tag{8.22}$$

ϵ_i is the unsystematic error that is, the return due to unsystematic risk. The expected value of this variable is obviously zero. The term $\beta_i(r_m - r)$ is the excess return due to the systematic or market risk of the stock. α_i is termed as the *abnormal return*, and is related to the mispricing of the stock. If the stock is correctly priced, then α_i will be zero. However, if the stock is underpriced, α_i will be positive, whereas if it is overpriced, α_i will be negative.

Stock pickers are investors who believe that they have an uncanny ability to spot underpriced and overpriced stocks, and seek to take advantage of this perceived skill. However, if an investor were to take a position in the stock because he believes that it is mispriced, without hedging against movements in the market as a whole, then there is a risk that even if the abnormal return were to be realized, the general market movement may be such that he ends up with an overall loss.

8.21.1 Example

A stock picker believes that Colgate Palmolive is underpriced and that he will get a positive abnormal return if he buys it. Let us assume as before that the riskless rate is 2.50% and that the beta is 1.80.

We will assume that the investor is right, and that α_i does turn out to be 0.75%. However, it so happens that there is a sharp decline in the overall market, and $r_m = -7.50\%$. If we assume that $\epsilon_i = 0$ then

$$r_i = 2.50 + 1.8(-7.50 - 2.50) + 0.75 = -14.75\%.$$

Thus although the investor *'backed the right horse'*, he ended up with a negative rate of return. This is because he was exposed to market risk. Such a situation could have been avoided, if he had chosen to hedge using stock index futures.

Let us assume that he invests $ 625,000 in the stock and goes short in S&P index futures when the index level is 250. This corresponds to a futures price of 246.25. The appropriate number of futures contracts is

$$\frac{625,000}{250 \times 250} \times 1.8 = 18$$

Considering that the dividend yield is given to be 4%, a return of −7.50% on the market corresponds to a decline of −11.50% in the index level. Hence the corresponding index value is 221.25.

The rate of return on the stock is −14.75%, which implies that the terminal portfolio value is $625,000(1 - 0.1475) = \$\,532,812.50$

The profit/loss from the futures position is

$$18 \times 250\,(246.25 - 221.25) = \$\,112{,}500$$

The value of the portfolio after factoring in the profit/loss from the futures market is:

$$532{,}812.50 + 112{,}500 = \$\,645{,}312.50$$

The overall rate of return is therefore 3.25%. This value of 3.25% corresponds to the riskless rate of 2.50%, plus the abnormal return of 0.75%.

Thus, if you believe that the stock is underpriced but want to hedge yourself against market risk, you should buy the stock and go short in stock index futures. Similarly, if you believe that the stock is overpriced, short sell the stock and go long in stock index futures.

8.22 Portfolio Insurance

In the section on Market Timing, we saw as to how a portfolio manager could move from an actual spot position worth 10 MM dollars, to a synthetic T-bill position, by going short in futures contracts. In that case, the end result was the creation of a risk free investment, which gave the riskless rate of return.

It is not necessary that a fund manager would always like to convert his entire risky portfolio to a riskless asset. In practice, he may choose to convert a fraction of the portfolio into equivalent synthetic T-bills using futures contracts, while continuing to hold the balance in the form of the risky portfolio. The value of this risky component can at worst go to zero. However, the riskless component of the overall portfolio will continue to earn the riskfree rate of return. Thus, there is a floor on the value of the overall portfolio. The portfolio may end up earning more than the floor, but cannot earn less.

This kind of an asset allocation strategy is referred to as *Portfolio Insurance*. Fund managers constantly monitor the market and sell and buy futures contracts in order to move from equities to synthetic debt and vice versa. The greater the desired level of insurance, the more will be the number of futures contracts sold. This kind of asset management technique is also known as *Dynamic Hedging*.

8.23 Index Futures and Stock Market Volatility

Trading in stock index futures has attracted its share of criticism and controversy. One of the allegations is that trading in such futures contracts has the tendency to increase the volatility of the market as a whole. More specifically, it has been argued that program trading and portfolio insurance make markets more volatile.

Let us take the case of program trading, which is nothing but the initiation of cash and carry and reverse cash and carry arbitrage strategies by traders. What does cash and carry arbitrage entail? It requires the trader to go long in the index and short in futures. On the expiration date of the contract, he will be required to issue a market on close order in order to offload his shares. Off loading of large volumes of shares is likely to have an impact on the market, only if there is a large supply demand imbalance. In this case, *a priori*, we have no reason to presume that such an event will occur. This is because, over the life of the contract, there will be traders entering into both cash and carry as well as reverse cash and carry strategies. Hence we have little theoretical basis to argue that program trading will increase stock market volatility.

The next issue is, what will be the impact of portfolio insurance on stock market volatility? In a bear market, fund managers will seek to insure a larger fraction of their portfolio. This will entail taking short positions in futures contracts. Such widespread selling can depress futures prices, thereby inducing a further drop in stock prices, since the two are inextricably linked by the cost of carry model, thereby exacerbating the problem.

The evidence as to whether trading in index futures contracts leads to greater stock market volatility is not very compelling.[6] But even assuming that index futures are responsible for enhanced volatility, the question remains as to whether volatility per se is bad? Volatility is a manifestation of the arrival of fresh information into the market. As new information is received, buyers and sellers will process it and reassess their perceptions of the values of different assets. The market prices will then change so as to reflect these changed perceptions. It is this process of price adjustment which gives rise to volatility. In free market economics, accurate prices are considered to be absolutely essential for the fair and optimal allocation of resources. From this perspective, volatility may be construed as evidence of a market that is informationally efficient. Hence volatility per se, need not be undesirable. However, it must be pointed out that volatility arising due to factors unrelated to the arrival of market relevant information, is not always desirable.

SUGGESTIONS FOR FURTHER READING

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[6] See Kolb (2000).

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7. *Guide to the Markets—Investor's Business daily*.
8. *Subject: Stocks—Tracking Stock http://invest-faq.com*
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CONCEPT CHECK

State whether the following statements are True or False.

1. If an exchange were to follow a T+5 settlement cycle, the ex-dividend date for a firm will be five days before the record date.
2. Stock splits entail the capitalization of reserves.
3. Stock dividends entail the capitalization of reserves.
4. Stock splits and reverse splits create value for the firm.
5. A loss making company can declare a dividend.
6. Retained earnings manifest themselves as an increase in the reserves and surplus account.
7. Shareholders are residual claimants.
8. Equity shares have a defined maturity date.
9. On the record date, the price of the share should in theory decline by the amount of the dividend.
10. In order to form a portfolio that mimics an equally weighted index, you have to buy an equal number of shares of every stock.
11. The divisor of a price-weighted index must be adjusted if one or more of the component stocks were to declare a stock dividend.

12. On the base date, the divisor of a price-weighted index must always be equal to the number of stocks in the index.
13. Index futures contracts are always cash settled.
14. Index arbitrage is bereft of risk in practice.
15. If we want to increase the beta of our portfolio using futures contracts, we should go long.
16. A person who believes that a stock is underpriced, but wants to hedge against market risk, must buy the stock and go long in index futures.
17. Volatility can be perceived as an indicator of an informationally efficient market.
18. Index futures prices are expressed in terms of index units and not in dollar terms.
19. While hedging a portfolio with index futures, for a given portfolio value and index level, the higher the beta, the greater is the required number of futures contracts.
20. If a stock is fairly priced, the excess return will be zero.

QUESTIONS AND PROBLEMS

Question-I

What is the relationship between the record date and the ex-dividend date? Explain.

Question-II

How will the terms of a futures contract be adjusted for stock splits and stock dividends? Explain with the help of numerical examples.

Question-III

Compare and contrast price-weighted and value-weighted indices.

Question-IV

What are the risks inherent in Program Trading? Explain in detail.

Question-V

What is *Stock Picking*? Why are index futures useful for a stock picker?

Question-VI

What is Portfolio Insurance? Explain the concept of *Dynamic Hedging*.

Question-VII

'Program Trading and Portfolio Insurance serve to make stock markets more volatile.' Comment.

Question-VIII

Consider the following data for a price weighted index.

Quiz—1

Circle or tick the choice that constitutes the best response to the statement.

1. Which of these securities is always traded OTC:
   a. Futures contracts
   b. Forward contracts
   c. Options contracts
   d. None of the above
2. Which of these securities is never traded OTC:
   a. Futures contracts
   b. Forward contracts
   c. Options contracts
   d. Swaps
3. Which of these contracts is never traded OTC:
   a. Futures contracts
   b. European options
   c. American options
   d. Swaps
4. Which of these securities is interchangeable with futures contracts from the standpoint of hedging:
   a. European options
   b. American options
   c. Both (a) and (b)
   d. Neither (a) nor (b)
5. A hedging strategy using the following security may lead to ex-post regret:
   a. Futures contracts
   b. Forward contracts
   c. Options contracts
   d. All of the above
6. A hedging strategy using the following security may lead to ex-post regret:
   a. Futures contracts
   b. European options

c. American options
d. All of the above

7. At the point of expiration of the futures contract, the basis:
   a. May be positive
   b. May be negative
   c. May be zero
   d. Must be zero
8. Which of these is true prior to the expiration of the futures contract:
   a. In a contango market the basis will be negative
   b. In a contango market the basis will be positive
   c. In a contango market the basis may be positive or negative depending on how it is defined
   d. In a contango market the basis will be zero
9. The minimum variance hedge ratio:
   a. Must be greater than one
   b. Must be less than one
   c. Must be equal to one
   d. None of the above
10. When we run a regression to establish the minimum variance hedge ratio, the estimate of the hedging effectiveness is given by:
    a. The slope coefficient
    b. The intercept coefficient
    c. The R^2 of the regression
    d. None of the above
11. The conversion factor for T-bonds and T-notes is:
    a. A multiplicative adjustment factor
    b. An additive adjustment factor
    c. May be (a) or (b)
    d. None of the above
12. The conversion factor for T-bonds and T-notes may:
    a. Be more than 1.0
    b. Be less than 1.0
    c. May be zero
    d. (a) or (b)
13. The conversion factor for T-bonds and T-notes:
    a. Must always be positive
    b. May be positive
    c. May be negative
    d. None of the above
14. If two investors who already have open positions in the futures market enter into a fresh trade with each other:
    a. Open interest will rise
    b. Open interest will fall

c. Open interest will remain unchanged
d. Cannot say

15. When a short delivers the asset under a futures contract, he will receive:
    a. The spot price that was prevailing at the inception of the contract
    b. The futures price that was prevailing at the inception of the contract
    c. The spot price that is prevailing at the time of delivery
    d. (a) and (b)
16. When a short delivers the asset under a futures contract he will receive:
    a. The spot price that was prevailing at the inception of the contract
    b. The spot price that is prevailing at the time of delivery
    c. The futures price that is prevailing at the time of delivery
    d. (b) and (c)
17. If cash and carry arbitrage is possible:
    a. The implied repo rate is greater than the borrowing rate
    b. The futures contract is overpriced
    c. The implied reverse repo rate is less than the lending rate
    d. (a) and (b)
18. If the market is in contango:
    a. The net carry must be positive
    b. The marginal convenience value must be zero
    c. The market must be at full carry
    d. None of the above
19. If the market for a contract on a financial asset is in contango:
    a. The net carry is positive
    b. The future value of payouts is less than the financing cost of the asset
    c. Both (a) and (b)
    d. None of the above
20. Which of these statements is true:
    a. A short hedger is long the basis
    b. A long hedger is short the basis
    c. A short hedger is short the basis
    d. (a) and (b)
21. Which of these statements is true:
    a. A short hedger is short the basis
    b. A long hedger is long the basis
    c. A short hedger is long the basis
    d. (a) and (b)
22. Program trading is a term used for:
    a. Cash and carry index arbitrage
    b. Reverse cash and carry index arbitrage
    c. Both (a) and (b)
    d. Neither (a) nor (b)
23. An investor who anticipates a bull market may take:
    a. A long position in a futures contract

b. A long position in a call option
c. A short position in a put option
d. All of the above

24. If a market is at full carry:
   a. Cash and carry arbitrage is infeasible
   b. Reverse cash and carry arbitrage is infeasible
   c. The asset is pure
   d. All of the above
25. In the case of arbitrage using contracts with multiple deliverable grades:
   a. The implied repo rate may be lower than anticipated
   b. The implied reverse repo rate may be higher than anticipated
   c. The implied reverse repo rate may be lower than anticipated
   d. (a) and (c)
26. The basis will be equal to zero on the expiration date of the futures contract:
   a. Only if the market is in contango
   b. Only if the market is in backwardation
   c. Irrespective of whether the market is in contango or backwardation
   d. None of the above
27. The further away the expiration of the futures contract chosen for hedging, the greater is the basis risk for:
   a. Buying hedges
   b. Selling hedges
   c. Both buying as well as selling hedges
   d. None of the above

Use this information for the next two questions

A bond with a face value of $ 1,000 and 10 years to maturity, paying coupons at the rate of 6% per annum on a semi-annual basis, on April 15 and October 15 every year, has been issued on April 15, 2003. Assume that today is March 8, 2004.

28. Assuming that the bond has been issued by the US Treasury, the accrued interest as of today is:
   a. $ 23.442
   b. $ 23.607
   c. $ 23.769
   d. $ 23.832
29. Assuming that the bond has been issued by the US Treasury, and that the YTM is 6% per annum, the dirty price as of today is:
   a. $ 1,023.442
   b. $ 1,023.607
   c. $ 1,023.769
   d. None of the above
30. A company has 100,000 shares outstanding and announces a 2:5 rights issue. The current stock price is $100 and the proposed issue price is $75. The value of the right to acquire a share is:

a. $ 25.0000
b. $ 92.8571
c. $ 17.8571
d. $ 7.1429

31. A party which has exported goods and expects to be paid in domestic currency will need to take:
    a. A short position in forward contracts
    b. A short position in futures contracts
    c. A long position in futures contracts
    d. None of the above
32. A limit order that can be executed on submission is called:
    a. A market order
    b. A stop-limit order
    c. A marketable limit order
    d. A market if touched order
33. A limit order represents an option. The corresponding limit price is therefore:
    a. The option price
    b. The option premium
    c. The strike price
    d. Both (b) and (c)
34. Assume that an exchange is following a $T + 5$ settlement cycle: The ex-dividend date for a stock will be:
    a. Five days before its record date
    b. Four days before its record date
    c. Five days after its record date
    d. Four days after its record date
35. Which of these corporate actions leads to the capitalization of reserves:
    a. Stock splits
    b. Reverse splits
    c. Stock dividends
    d. Cash dividends
36. Which of these is subject to execution uncertainty:
    a. A limit order
    b. A marketable limit order
    c. A stop-limit order
    d. All of the above
37. In the case of which of these corporate actions will the issued capital remain unchanged:
    a. Stock split
    b. Reverse split
    c. Stock dividend
    d. (a) and (b)

38. If the following type of stock index goes up in value, it means that every component stock has risen in price:
    a. Price weighted index
    b. Value weighted index
    c. Equally weighted index
    d. None of the above
39. If a tracking portfolio has been formed by investing an equal dollar amount in each component stock, then the corresponding index must be:
    a. Price weighted
    b. Value weighted
    c. Equally weighted
    d. Cannot say
40. A currency is said to be trading flat:
    a. If the forward premium is positive
    b. If the forward premium is negative
    c. If the forward premium is zero
    d. None of the above
41. If the domestic interest rate is equal to the foreign interest rate, the foreign currency will trade:
    a. At a premium
    b. At a discount
    c. Flat
    d. Cannot say
42. An Indian exporter who has invoiced in dollars:
    a. Can hedge by going short in a forward contract
    b. Can hedge by buying a put option
    c. Both (a) and (b)
    d. Neither (a) nor (b)
43. If the person who takes a long position in a futures contract is a hedger, the person who takes the opposite position may be:
    a. A hedger
    b. A speculator
    c. An arbitrageur
    d. Any of the above
44. In the case of a direct quote, an increase in the value indicates:
    a. A depreciating foreign currency
    b. An appreciating foreign currency
    c. A depreciating home currency
    d. (b) and (c)
45. In the case of an indirect quote, an increase in the value indicates:
    a. A depreciating foreign currency
    b. An appreciating foreign currency
    c. A depreciating home currency
    d. (b) and (c)

46. The market for which of these contracts is likely to have the highest liquidity:
    a. 1 month forward
    b. 2 months forward
    c. 3 months forward
    d. Cannot say
47. Which of these markets is likely to have the highest bid-ask spread:
    a. 1 month forward
    b. 2 months forward
    c. 3 months forward
    d. Cannot say
48. The swap points have been given as 45/75. It indicates that:
    a. The currency is at a forward premium
    b. The currency is at a forward discount
    c. The currency is trading flat
    d. Depends of whether the quote is direct or indirect
49. An Indian importer who has to pay in dollars can hedge by:
    a. Going short in a forward contract
    b. Buying a put option
    c. Buying a call option
    d. a) and (b)
50. Consider a T-bill with 144 days to maturity and another with 108 days to maturity. The quoted rate for both is 8%. Which of these statements is true:
    a. The 144 day bill will have a lower price
    b. The 144 day bill will have a higher price
    c. The two will have an equal price
    d. Cannot say
51. Consider a T-bill with 144 days to maturity and another with 108 days to maturity. The quoted rate for both is 8%. Which of these statements is true for an investor who buys one of each bill and holds them to maturity:
    a. The return from the 144 days bill be greater than 8%
    b. The return from the 108 day bill will be greater than 8%
    c. Both (a) and (b)
    d. Cannot say
52. A 40% stock dividend is like a:
    a. 5:2 split
    b. 2:5 reverse split
    c. 3:2 split
    d. None of the above
53. The Nikkei stock index is:
    a. Price weighted
    b. Value weighted based on full market capitalization
    c. Value weighted based on free-float market capitalization
    d. Equal weighted

54. The Nikkei index is composed of:
    a. 30 stocks
    b. 50 stocks
    c. 100 stocks
    d. None of the above
55. The FOREX rates in India on a given day are as follows:

    Spot rate: 43.7500 INR/USD

    3M Forward rate: 44.2500 INR/USD

    The interest rate in India is 8% per annum. If arbitrage is to be ruled out, the interest rate per annum in the US should be:
    a. 6.78%
    b. 3.39%
    c. 9.23%
    d. 12.66%
56. Which of these market participants is constantly on the lookout for cost-less, risk-less profits:
    a. Hedgers
    b. Arbitrageurs
    c. Speculators
    d. All of the above
57. Which of these contracts imposes an obligation on the part of both the long as well as the short:
    a. Forward contracts
    b. Futures contracts
    c. Options contracts
    d. (a) and (b)
58. Which of these contracts gives a right to the long and imposes an obligation on the short:
    a. Forward contracts
    b. Futures contracts
    c. Options contracts
    d. (a) and (b)
59. In the case of futures contracts which permit delivery of more than one specified grade, and/or at multiple locations, the right to choose the location and the grade
    a. May in principle be given to either party
    b. Must be given to the short
    c. Is traditionally given to the short
    d. Both (a) and (c)
60. Which of these parties to a futures contract, trades with the clearinghouse:
    a. The long
    b. The short
    c. Both the long as well as the short
    d. Neither the long nor the short

61. In the case of futures contracts:
    a. Delivery commences only after trading has ceased
    b. Delivery may commence before trading ceases
    c. Delivery may continue after trading ceases
    d. (b) and (c)
62. Which of these is a zero sum game:
    a. A futures contract
    b. A forward contract
    c. Both (a) and (b)
    d. Neither (a) nor (b)
63. A VaR number is meaningless unless the following are specified:
    a. The probability level
    b. The holding period
    c. Both (a) and (b)
    d. Neither (a) nor (b)
64. Futures contracts on the following index are cash settled:
    a. The DJIA
    b. The S&P 500
    c. The Nikkei 225
    d. All of the above
65. The CTD grade:
    a. Must have the lowest spot price
    b. Must have the lowest delivery adjusted spot price
    c. Must have the highest delivery adjusted spot price
    d. None of the above
66. Which of these has a linear profit diagram:
    a. A short futures position
    b. A long futures position
    c. Both (a) and (b)
    d. Neither (a) nor (b)
67. Which of these derivatives provides leverage:
    a. Futures contracts
    b. Call options
    c. Put options
    d. All of the above
68. Consider the following quote from Citibank Mumbai:

    Spot: 39.8050–40.2125 INR/USD

    1M Forward: 80/50

    The outright forward rates are:
    a. 39.8000–40.2045 INR/USD
    b. 39.8110–40.2205 INR/USD
    c. 39.8130–40.2175 INR/USD
    d. 39.7970–40.2075 INR/USD

69. Consider the following quote from Citibank Mumbai:
    Spot: 39.8050–40.2125 INR/USD
    1M Forward: 50/80
    The outright forward rates are:
    a. 39.8000–40.2045 INR/USD
    b. 39.8100–40.2205 INR/USD
    c. 39.8130–40.2175 INR/USD
    d. 39.7970–40.2075 INR/USD
70. Profits are finite, while losses are infinite. This statement is true for:
    a. A long spot position
    b. A short spot position
    c. A short futures position
    d. (b) and (c)
71. Stocks and bonds are examples of:
    a. Pure assets
    b. Investment assets
    c. Convenience assets
    d. (a) and (b)
72. Which of these is an example of a crack spread:
    a. 3:2:1
    b. 2:1:1
    c. 5:3:2
    d. All of the above
73. The limit price for a marketable limit buy order must be:
    a. Greater than the best available bid
    b. Greater than or equal to the best available ask
    c. Less than or equal to the best available ask
    d. (a) and (b)
74. Which of these rules is self-enforcing in an oral auction:
    a. The price priority rule
    b. The time priority rule
    c. Both (a) and (b)
    d. Neither (a) nor (b)
75. Which of these will always be less than the prevailing LIBOR:
    a. The prevailing LIBID
    b. The prevailing LIMEAN
    c. Both (a) and (b)
    d. Neither (a) nor (b)
76. ED futures contracts on the CME are based on:
    a. 1 month LIBOR
    b. 3 months LIBOR
    c. 6 months LIBOR
    d. None of the above

77. If there is a parallel shift in the yield curve:
    a. The performance of Stack and Strip hedges will be equivalent
    b. Stack hedges will outperform Strip hedges
    c. Strip hedges will outperform Stack hedges
    d. Cannot say
78. For which of these plain vanilla bonds will the duration be less than the term to maturity:
    a. Par bonds
    b. Premium bonds
    c. Discount bonds
    d. All of the above
79. Which of these securities can give rise to a capital gain:
    a. T-bills
    b. T-notes
    c. T-bonds
    d. All of the above
80. Which of these statements is true about a liquid market:
    a. There will be plenty of buyers and sellers at any point of time
    b. Large trades will have a minimal price impact
    c. The bid-ask spread will be low
    d. All of the above
81. The exact number of days between two successive coupon dates will be:
    a. Between 180 and 183
    b. Between 181 and 183
    c. Between 180 and 184
    d. None of the above
82. Which of these securities is likely to be more liquid:
    a. A 3 month T-bill issued yesterday
    b. A 6 month T-bill issued 3 months ago
    c. Both are likely to be equally liquid
    d. Cannot say
83. If the US dollar is the base currency, then:
    a. It is a direct quote in the US
    b. It is a quote in European terms
    c. It is a quote in American terms
    d. None of the above
84. In the event of the following corporate action, both the settlement price and the contract size must be adjusted for futures contracts:
    a. Stock dividend
    b. Stock split
    c. Reverse split
    d. All of the above
85. In the event of the following corporate action, no adjustment need be made to the contract size:

a. A rights issue
b. A stock split
c. A reverse split
d. All of the above

86. The 99% VaR of a portfolio over a one day horizon is given to be $ 2,500. This means that:
 a. The loss from the portfolio over a one day horizon is likely to be more than or equal to $ 2,500 on 99% of the days.
 b. The loss from the portfolio over a one day horizon is likely to be less than or equal to $ 2,500 on 99% of the days
 c. The maximum possible loss of the portfolio on any day is 0.99 × $ 2,500
 d. The maximum possible loss from the portfolio on any day is $ 2,500

87. To create a synthetic long position in a forward contract, you have to:
 a. Go long in the spot market and short in a T-bill
 b. Go long in the spot market and long in a T-bill
 c. Go short in the spot market and long in a T-bill
 d. Go short in the spot market and short in a T-bill

88. Which of these is true about greater volatility of oil prices:
 a. It gives rise to a greater need for crude oil derivatives
 b. It gives rise to a greater need for derivatives on all commodities
 c. Both (a) and (b)
 d. None of the above

89. Today the world's economies largely follow:
 a. A gold exchange standard
 b. A floating rate standard
 c. A managed floating rate standard
 d. None of the above

90. Which of these is true:
 a. Speculators are calculated risk takers
 b. Speculators are gamblers
 c. Speculators are essential for a market economy
 d. (a) and (c)

91. A speculator who is bearish has a choice between a futures contract at $ 15 and a put option with an exercise price of $ 15 and a premium of $ 1.

 Speculating with futures will be more profitable only if the terminal spot price:
 a. Is more than $ 15
 b. Is less than $ 14
 c. Is less than $ 15
 d. Is less than $ 16

92. Futures contracts are called zero sum games because:
 a. The gain/loss for the long will always be equal to the loss/gain for the short

b. The expected gain/loss for the long as well as the short is zero
c. Neither the long nor the short has to pay a price to enter into the contract
d. None of the above

93. Cash and carry arbitrage is profitable in the case of assets which pay no income, if:
    a. The futures price is greater than the spot price
    b. The difference between the futures price and the spot price is greater than the carrying cost
    c. The implied repo rate exceeds the borrowing rate
    d. (b) and (c)
94. Once a forward contract is entered into:
    a. The forward price will remain fixed while its delivery price will keep increasing
    b. The forward price will remain fixed while its delivery price will keep decreasing
    c. The forward price will keep increasing while its delivery price will remain fixed
    d. None of the above
95. At the end of the day when a futures contract is marked to market, its value will be reset to zero:
    a. Only if it is a financial asset
    b. Only if it is a financial asset or a physical asset held for investment purposes
    c. Only if it is a convenience asset
    d. None of the above
96. If interest rates and futures prices are positively correlated then:
    a. The futures price will be less than the forward price for a comparable contract
    b. The futures price will be more than the forward price for a comparable contract
    c. The long will be financing his losses at higher interest rates
    d. (b) and (c)
97. As per the theory of Normal Backwardation:
    a. Long futures positions can expect a positive payoff
    b. The futures price will be less than the current spot price
    c. Both (a) and (b)
    d. Neither (a) nor (b)
98. Physical assets which are held for investment purposes:
    a. Will always exhibit Contango
    b. Will always exhibit Normal Contango
    c. Will always be delivered at the beginning of the delivery period
    d. (a) and (c)
99. Financial assets are always investment assets. Therefore:
    a. They will always exhibit Contango

b. They will always exhibit Normal Contango
c. They will always exhibit Backwardation
d. None of the above

100. Which of these securities can never be traded in the money market:
a. Equity shares
b. Debt securities
c. Preferred shares
d. (a) and (c)

Company	Price
Ranbaxy	150
Torrent	80
Orchard	120
CIPLA	200
Wockhardt	250

1. If the current divisor is 4.5, what will be the index value?
2. Assume that Torrent announces a 1:5 reverse split. How should the divisor be adjusted?

Question-IX

What is market timing? Explain.

Question-X

Stacy has a hunch that Colgate is overpriced and is likely to yield an abnormal return of .8% over a period of 90 days. The current value of the S&P is 1250 and the beta of Colgate with respect to the index is 1.8. The riskless rate of return is 12% per annum, and the predicted dividend yield on the index over the 90 day period is 5%.

Assume that Stacy decides to short sell $ 25 MM worth of Colgate stock, and that her belief about the abnormal return is translated into reality.

1. If the market goes up by 13% over the 90 day period, what will be the rate of return obtained by Stacy, if she decides not to hedge away the market risk?
2. If she does decide to hedge using index futures, what will be the rate of return obtained by her? Explain your answer.

9

Fundamentals of Options

9.1 Introduction

Thus far, we have covered two types of derivative securities, namely futures and forward contracts, in detail. In a futures contract no money changes hands when the contract is negotiated and nor does the title to the goods. In such a contract, the buyer agrees to pay cash at a future date to the seller, who in turn agrees to transfer the ownership of the asset on that day. The buyer of a futures contract is said to have taken a *Long Position* and is known as the *Long*, while the seller is said to have taken a *Short Position* and is known as the *Short*. The important point to note is that once a futures contract is negotiated, both the long and the short have an obligation at a future date. Since both parties have an obligation, there is always a possibility that the party with a loss at the expiration of the contract may default. Compliance is ensured by requiring both the parties to deposit good faith money or collateral called *Margins* and by adjusting the profits and losses on a daily basis by a procedure known as *Marking to Market*. Forward contracts are similar to futures contracts in the sense that they too impose an obligation on the short to deliver at the time of expiration of the contract, and on the long to take delivery at that time. However, unlike futures contracts which trade on organized exchanges, and consequently have to be designed as per the terms and conditions specified by the exchange, forward contracts are Over-the-Counter contracts which are designed with mutually acceptable terms and conditions by the two parties through bilateral negotiations.

Options contracts which are the focus of this book from now on, are derivative contracts, but by design are different from futures contracts. In an options contract the buyer of the contract, also called the *Holder* or the *Long*, has a right, while the seller, also known as the *Writer* or the *Short*, has an obligation. Thus, in the case of an options contract the long has the freedom to decide as to whether or not he wishes to go through with the transaction, whereas the short, has no choice but to carry out his part of the agreement if and when the holder chooses to exercise his right. Therefore, unlike in the case of futures contracts, both the parties need not deposit collateral. For, if a person has a right, there is no fear of non-compliance since he will exercise his right if it is in his interest and he need not otherwise. Consequently, in the case of options contracts, only the shorts have to deposit margins.

The buyer of an option may either have the right to buy the underlying asset, or the right to sell it, depending on the terms of the agreement. Therefore, there are two types of options contracts, *Calls* and *Puts*. A call option gives the long the right to acquire the underlying asset whereas a put option gives the long the right to sell the underlying asset. An options contract may give the long the right to transact only at a future point in time, or else it may give him the flexibility of exercising his right at any point in time, up to and including the expiration date. Consequently, both calls and puts can be of two types, *European* and *American*. A European option gives the holder the freedom to exercise his right only at the time of expiration of the contract, while an American option may be exercised by the holder at any point in time on or before expiration. It must be noted that the terms European and American have nothing to do with geographical locations and that in practice most exchange traded contracts are American. Given the same features in all other respects, an American option will be more valuable than the corresponding European option. This is because, the holder of an American option has the flexibility to exercise early, whereas the holder of a European option does not have the freedom to exercise prior to expiration.

9.1.1 Capped-style Options

These are options with an established profit cap. The cap price is set equal to the exercise price of the option plus a cap interval in the case of call options, and the exercise price minus a cap interval in the case of put options. Such options are automatically exercised if the price of the underlying asset hits or rises above the cap price in the case of calls, and hits or falls below the cap price in the case of puts.

9.2 Options and Stocks: Similarities and Differences

Exchange traded options contracts have certain features in common with equity shares, although there are certain critical differences between the two. The trading mechanism for exchange traded options is similar to that for equity shares of listed companies. Price discovery takes place through a continuous bilateral auction procedure. However, unlike equity shares which do not have an expiry date, options have a finite life span. Besides, the supply of equity shares is limited to the quantity issued by the firm. On the other hand, the number of options contracts outstanding at any point in time, is dependent entirely on the number of buyers and sellers who have evinced a desire to buy and write options. In certain markets, equity shares are still represented by physical certificates, although with the introduction of dematerialized trading in most markets, scrip less trading is fast becoming the norm. Options however are not backed by certificates of ownership. Evidence of an option position is indicated by a printed statement issued by the brokerage firm of the buyer or the writer. Finally, ownership of a company's stock confers certain privileges on the shareholders, such as the right to participate and vote in annual general meetings, and entitles them to any cash dividends declared by the company. On the other hand, options owners stand to benefit solely from the upward or downward movement of the stock price, depending on whether they own call or put options.

9.3 Common Terms Associated with Options

9.3.1 Exercise Price

This is the price which the holder of a call option has to pay to the writer, per unit of the underlying asset, if he were to decide to exercise his option. In the case of puts, it is the price which the holder of a put option will receive per unit of the underlying asset, were he to exercise. The Exercise Price is also known as the *Strike Price*.

The exercise price enters the picture only if the holder chooses to exercise the option. Since the holder has a right and not an obligation, he may or may not wish to transact, which means that the exercise price may or may not be paid/received.

9.3.2 Expiration Date

This is the point in time after which the contract becomes void. It is the only point in time at which a European option can be exercised and the last point in time at which an American option can be exercised. The Expiration Date is also known as the *Exercise Date*, or *Strike Date* or *Maturity Date*.

9.3.3 Option Premium

This is the price that the holder has to pay to the writer at the outset, in order to acquire the right to exercise. The option premium is a *'sunk cost'*. If the holder were not to exercise prior to expiration, the premium cannot be recovered.

Why does the buyer of an option have to pay a premium at the outset? Options contracts entail the payment of a premium, because the buyer is acquiring a right from the writer, who is taking on an obligation to perform if the buyer were to exercise the right. Rights, it must be understood, are never free and one has to always pay a price to acquire them.

Futures and forward contracts in contrast, do not entail the payment of a premium by the long. Such contracts impose an equivalent obligation on both the long as well as the short. The futures price, which is the price at which the long will acquire the asset at the time of expiration, is set in such a way that from the standpoints of both the long as well as the short, the value of the contract at inception is zero. In other words, the two equivalent and opposite obligations ensure that neither party has to pay the other at the outset.

9.4 Notation

We will use the following symbols to depict the various variables.

- $t \equiv$ today, a point in time before the expiration of the options contract.
- $T \equiv$ the point of expiration of the options contract.
- $S_t \equiv$ the stock price at time t.
- $S_T \equiv$ the stock price at the point of expiration of the options contract.
- $I_t \equiv$ the index value at time t.
- $I_T \equiv$ the index value at the point of expiration of the options contract.
- $X \equiv$ the exercise price of the option.
- $C_t \equiv$ a general symbol for the premium of a call option at time t, when we do not wish to make a distinction between European and American options.
- $P_t \equiv$ a general symbol for the premium of a put option at time $t, whenwedonotwishtomakeadistinctionbetweenEuropeanandAmericanoptions.$
- $C_T \equiv$ a general symbol for the premium of a call option at time T, when we do not wish to make a distinction between European and American options.
- $P_T \equiv$ a general symbol for the premium of a put option at time T, when we do not wish to make a distinction between European and American options.
- $C_{E,t} \equiv$ the premium of a European call option at time t.
- $P_{E,t} \equiv$ the premium of a European put option at time t.
- $C_{A,t} \equiv$ the premium of an American call option at time t.
- $P_{A,t} \equiv$ the premium of an American put option at time t.
- $C_{E,T} \equiv$ the premium of a European call option at time T.
- $P_{E,T} \equiv$ the premium of a European put option at time T.
- $C_{A,T} \equiv$ the premium of an American call option at time T.
- $P_{A,T} \equiv$ the premium of an American put option at time T.
- $r \equiv$ the risk-less rate of interest per annum.

Additional variables will be de ned when required. Some of these variables may be rede ned as we proceed. We will make the modi ed meanings explicit in such cases.

9.5 Exercising Call and Put Options

Exchange traded stock options are American in nature. There are two possible exercise related scenarios. The holder may choose to exercise prior to expiration or at the time of expiration. Index options may be European or American. If they were to be European in nature, they can obviously be exercised only at expiration.

When a call option is exercised before expiration the holder will receive the speci ed number of shares per contract that is being exercised, in return for a payment of $ X per share. On the other hand if a put were to be exercised, the holder will have to deliver the specified number of shares per contract, and will receive $ X per share delivered.

We will now illustrate the cash flows involved when an option is exercised. To keep matters simple we will focus on European options, although the underlying logic is the same in the case of American options. We will use stock options for the purpose of illustration, although, in principle, an options contract can be written on any asset.

9.5.1 Example

Consider a person who buys a call option on IBM expiring in December, with an exercise price of $ 100. Let us assume that the option premium is $ 3.25 per unit of the underlying asset, which in this case is a share of IBM. Option premia are always quoted on a per share basis. The contract size for stock options, or in other words, the number of shares that the option holder can buy or sell per contract, is kept fixed at 100 in the US. So as soon as the deal is struck, the buyer has to pay $ 3.25 × 100 = $ 325 to the writer. In exchange, he gets the right to buy 100 shares of IBM on the expiration date at a price of $ 100 per share.

When will the holder choose to exercise his right? Quite obviously, if the stock price at the time of expiration of the option is greater than $ 100, then it will make sense to exercise the option and buy the shares at $ 100 each. Else, it is best to let the option expire worthless. Readers encountering options for the first time, may feel that it would make sense to exercise only if the stock price at expiration were to be greater than (100 + 3.25). This viewpoint is erroneous. Assume that the terminal stock price is $ 102.00. If the option is exercised, the holder can buy 100 shares for $ 100 each and immediately sell them for $ 102 each. After taking into account the option premium that was paid at the outset, the total profit is[1]

$$\pi = (102 - 100) \times 100 - 325 = \$\,(125)$$

Thus, if the holder were to exercise, he would suffer a loss of $ 125 whereas if he were to allow the contract to expire worthless, he would lose the entire initial premium of $ 325. This argument is an illustration of the maxim that *'sunk costs are irrelevant'* while taking investment decisions.

9.5.2 Example

Let us reconsider the previous example, but assume that the options under consideration are put options. The question is, when will the holder choose to exercise his right? Exercise will be a profitable proposition if the terminal stock price were to be less than $ 100. Else, it would be best to let the option expire worthless. Assume that the terminal stock price is $ 97.50 and that the premium paid for the option is $ 1.25 per share. The profit is

$$\pi = (100.00 - 97.50) \times 100 - 125 = \$\,125$$

[1] We will use the symbol π to denote profits and will indicate losses by putting the numbers in parentheses, after the corresponding currency symbol.

9.6 Cash Settlement versus Delivery Settlement

Index options are *'cash settled'*. That is, if a call holder chooses to exercise an option with an exercise price of X, at a time when the value of the underlying index is I_t, he will be paid an amount equal to $I_t - X$ times a multiplier. The multiplier is generally \$ 100. Similarly if a put holder were to choose to exercise he will be paid an amount equal to $X - I_t$ times a multiplier. When an index option is exercised at expiration, a call holder will receive $I_T - X$ times a multiplier, while a put holder will receive $X - I_T$ times a multiplier.

Cash settlement is the universal practice for index options, for it is a cumbersome process to ensure physical delivery of the stocks constituting an index. The exercise settlement value of an equity index option can be determined in a variety of ways. The two most common are AM and PM settlement.

9.6.1 PM Settlement

In this case the settlement value of the index is based on the last reported prices of the component stocks which constitute the index, at the close of trading on the day of exercise.

9.6.2 AM Settlement

In this case the settlement value is based on the opening prices of the component stocks on the day of exercise.

If one or more of the component securities do not open for trading on the day the settlement value is being determined, then the last reported price(s) are used.

9.6.3 Payoffs and Profits: A Symbolic Representation

The payoff for a call holder at expiration is equal to $S_T - X$, if $S_T > X$, or 0 if $S_T \leq X$. The payoff can therefore be represented as $\text{Max}(0, S_T - X)$.

The profit can be represented as $\text{Max}(0, S_T - X) - C_t$ where C_t is the premium paid for the call when it was acquired.

The payoff for the call writer is $-\text{Max}(0, S_T - X) = \text{Min}(0, X - S_T)$ and the profit is $\text{Min}(0, X - S_T) + C_t$.

In the case of call options therefore, the maximum profit for the holder is unlimited because the stock price at the time of exercise has no theoretical upper limit. The maximum loss is however restricted to the initial premium that is paid. From the standpoint of the call writer, the situation is just the reverse. His maximum loss is unlimited, whereas his maximum profit is the option premium that he receives at the outset.

Similarly, the payoff for a put holder is equal to $X - S_T$, if $S_T < X$, or 0 if $S_T \geq X$. The payoff can therefore be represented as $\text{Max}(0, X - S_T)$.

The profit can be represented as $\text{Max}(0, X - S_T) - P_t$ where P_t is the premium paid for the put when it was acquired.

The payoff for the put writer is $-\text{Max}(0, X - S_T) = \text{Min}(0, S_T - X)$ and the profit is $\text{Min}(0, S_T - X) + P_t$.

For a put holder therefore, the maximum profit is equal to the exercise price less the option premium, because, since stocks have limited liability the price cannot go below zero. His maximum loss is once again the option premium. For the put writer the maximum profit is equal to the premium received, whereas the maximum loss is equal to the exercise price minus the premium. Thus, irrespective of whether it is a call or a put, an option is a *Zero Sum Game*. The holder's profit is equal to the writer's loss and vice versa.

9.7 Exchange Traded Versus OTC Options

Until the introduction of options contracts by the Chicago Board Options Exchange (CBOE) in 1973, options were traded *Over The Counter* (*OTC*). An OTC option is a customized agreement between the long and the short. By customized we mean that the exercise price, the date of expiration, and the contract size are negotiated between the buyer and the seller. On the other hand, options contracts traded on organized exchanges like the CBOE are standardized agreements. By standardized, we mean that the exercise prices and expiration dates for which contracts can be negotiated are specified by the exchange. Individual buyers and sellers are free to strike deals by incorporating any of the allowable exercise prices and expiration dates into the terms of the agreement, but cannot design contracts on their own terms. The contract size or the number of units of the underlying asset per contract, is also fixed by the exchange.

What are the advantages of standardization? Firstly, since only certain exercise prices and expiration dates are permitted, the number of types of contracts available is far less than would be the case, were different individuals free to specify these features. Consequently trading volumes tend to be high and therefore, transactions costs tend to be low. High trading volumes also imply a high degree of liquidity. Secondly, standardized contracts can be offset easily by taking a counterposition on the floor of the exchange. The term *Counterposition* refers to the taking of a short position by a trader who had originally gone long and vice versa. We will illustrate this with the help of an example.

9.7.1 Example

Amy bought an options contract on IBM from Richard a week ago. The contract terms have specified an exercise price of \$ 100 and the contract is scheduled to expire in June. Now assume that Amy wants to get out of her position. All she has to do, is to find a person on the floor of the exchange who would like to go long in a contract on IBM expiring in June, with an exercise price of \$ 100. This person need not be Richard, the individual with whom she initially traded. Notice that to offset a position in a call option, the trader needs to take an opposite position in a call option with the same exercise price and maturity date. The same holds true for a put option.

One reason why this kind of offsetting is feasible, is because the contract is standardized. In the case of customized agreements, there is an infinite number of exercise prices and expiration dates that can be specified, as a consequence of which the odds of finding a third party who is willing to transact as per the terms of the original contract, are considerably reduced. A customized options contract, is a private contract between two parties, with terms and features that are unique to it. Consequently, such contracts can be abrogated by the long or the short, only by seeking out the other party.

The other advantage of exchange traded options, is that credit risk is minimized. Every exchange has an associated clearinghouse which enters the picture once the trade is executed.[2] Once the deal is struck, the clearinghouse positions itself between the two parties, and becomes the effective buyer for every seller and the effective seller for every buyer. In the case of futures contracts it guarantees the performance of both the parties whereas, in the case of options contracts, it need guarantee the performance of only the writer as explained earlier. The fact that the clearinghouse becomes the effective counterparty for both the parties after the trade is consummated, is another reason why taking an offsetting position is easy. For, the link between the original long and the short is broken, and consequently both parties need only deal with the clearinghouse subsequently.

The clearinghouse consists of members called *Clearing Firms*, who are essentially brokerage houses. All orders must eventually be routed through a clearing member. Thus, if your broker is not a clearing member, he must route your order through another member who happens to be a member of the clearinghouse. All writers have to maintain margins or collateral with their brokers. If the broker is not a clearing member, then he has to maintain a margin account with a clearing member. All clearing members have to maintain margins with the clearinghouse.

9.8 Moneyness of the Option

9.8.1 Call Options

- If $S_t > X$ the call option is said to be *In-the-money*.
  The term arises from the fact that the option holder has the right to buy the underlying asset at a price which is lower than what he would have to pay, were he to acquire it in the spot market.
- If $S_t = X$ the call option is said to be *At-the-money*.
- If $S_t < X$ the call option is said to be *Out-of-the-money*.
- If $S_t \approx X$ the call option is said to be *Near-the-money*.

[2]The major clearinghouse in the US is the Options Clearing Corporation (OCC). The OCC is a registered clearing corporation with the Securities Exchange Commission (SEC). The company has a AAA credit rating assigned by the Standard & Poor's Corporation (S&P).

9.8.2 Example

Consider a stock that is currently trading at \$ 100. A call option with an exercise price of \$ 100 will be said to be at-the-money. A call with a lower exercise price. say \$ 95, will be said to be in-the-money, whereas one with a higher exercise price, say \$ 105, will be said to be out-of-the-money. Obviously, a call option will be exercised only if it happens to be in-the-money.

9.8.3 Put Options

- If $S_t > X$ the put option is said to be *Out-of-the-money*.
- If $S_t = X$ the put option is said to be *At-the-money*.
- If $S_t < X$ the put option is said to be *In-the-money*.
  The term arises from the fact that the option holder has the right to sell the underlying asset at a price which is greater than what he would receive, were he to sell it in the spot market.
- If $S_t \approx X$ the put option is said to be *Near-the-money*.

9.8.4 Example

Consider a stock that is currently trading at \$ 100. A put option with an exercise price of \$ 100 will be said to be at-the-money. A put with a lower exercise price, say \$ 95, will be said to be out-of-the-money, whereas one with a higher exercise price, say \$ 105, will be said to be in-the-money. Obviously, a put option too will be exercised only if it happens to be in-the-money.

9.8.5 Exercise by Exception

The OCC has devised a procedure to automatically exercise expiring options. This procedure is known as *'exercise by exception'* or *'ex-by-ex'*. The OCC has established certain in-the-money thresholds for expiring options. Every contract, whose payoff if exercised will be greater than or equal to its in-the-money threshold, will be automatically exercised unless the clearing firm through whom the contract was written, explicitly instructs OCC to the contrary. However, a contract whose payoff if exercised is less than the threshold, will not be exercised unless the brokerage firm specifically instructs the OCC to do so.

9.9 Contract Specification

Since exchange traded options are standardized agreements, the exchange will have to specify the following features before trading can commence.

9.9.1 The Underlying Asset

The assets on which contracts can be traded have to be clearly specified. In the case of stock options, this entails specifying the list of companies on whose shares contracts can be written.

9.9.2 Contract Size

The number of units of the underlying asset that are deliverable per contract has to be clearly specified. In the US all stock options contracts are for 100 shares of stock.

9.9.3 Expiration Dates

Options are referred to by the month in which they expire. For instance, when we refer to a January call, we mean a call option expiring in January.

In the US stock option contracts expire on the Saturday following the third Friday of the expiration month. Expressions such as *'Saturday following the third Friday of the month'* are fairly common in derivatives markets. The phrase refers to the fourth Saturday of the month, in case the first day of the month happens to be a Saturday, or else to the third Saturday of the month. The last day of trading is the third Friday. If the third Friday of the month is an exchange holiday, the last trading day is the Thursday immediately preceding the holiday.

9.9.4 Exercise Prices

The exchange also has to specify the exercise prices at which contracts can be written. There will always be an At-the-Money (ATM) or Near-the-Money contract. There will also be a number of In-the-Money (ITM) and Out-of-the-Money (OTM) contracts that are eligible for trading at any point in time, depending on the rules of the exchange on which the contracts are being traded.

9.10 Choosing Expiration Months

Stock options are designed to be on *Quarterly Cycles*. A particular company could be assigned to either a January, February or a March cycle. The January cycle comprises of the following months: January, April, July and October. A February cycle comprises of February, May, August and November. And the March cycle comprises of March, June, September and December. At any point in time, the available expiration months are the *current month*, the *following month*, and the *next two* months of the cycle to which the company is assigned. We will illustrate this with the help of an example.

9.10.1 Example

Let us assume that today is 1 September, 2008. Consider a company that is assigned to a February cycle. Options contracts will be available with the following expiration months: September, October, November and February. Of these, September is the current month, October is the following month, and November 2008 and February 2009 are the next two months after October 2008 from the February cycle.

When the September contracts expire, options with the following expiration months will be available: October, November, February and May.

When the October options expire, the following contracts will be available: November, December, February and May.

After the expiry of the November options, the following months will be available: December, January, February and May.

And finally, after the December contracts expire, January, February, May and August will be available.

In addition to these contracts, the CBOE and the AMEX offer options contracts on indices as well as individual stocks, with up to *three years* to expiration. Such options are called *Long Term Equity Anticipation Securities* or *LEAPS*. LEAPS expire in January each year, on the Saturday following the third Friday of the expiration month.

9.11 Specification of Allowable Exercise Prices

As mentioned earlier, at any point in time, there will always be an At-the-Money or Near-the-Money contract that is available for trading. This holds true for both calls as well as puts. In addition there will be other contracts which are in-the-money or out-of-the-money to varying degrees.

When new contracts, that is, contracts with a new expiration date, are introduced, two strike prices that are closest to the stock price at that point in time will be chosen. If the stock price moves outside the price band formed by these two exercise prices, then contracts with a fresh exercise price will be introduced for trading. While fixing the exercise prices, the CBOE usually follows the rule that the strike prices be in intervals of \$ 2.50 if the stock price is less than \$ 25, in intervals of \$ 5 if the stock price is between \$ 25 and \$ 200, and that they be spaced \$ 10 apart if the stock price exceeds \$ 200. These rules can best be illustrated with the help of an example.

9.11.1 Example

Let us assume that a company XYZ is assigned to the February cycle and that the May contracts are just being introduced. We will assume that the prevailing stock price is \$ 87.

At the outset, both call and put option contracts with exercise prices of \$ 85 and \$ 90 will be permitted for trading.[3] As long as the stock price remains between \$ 85 and \$ 90, only contracts with these two exercise prices will be allowed for trading. If the stock price were to go above \$ 90, new call and put contracts with an exercise price of \$ 95 will be introduced. Similarly, if the stock price were to

[3]The prevailing stock price of \$ 87 is between \$ 25 and \$ 200, and so the exercise prices chosen will be \$ 5 apart.

fall below \$ 85, new contracts with an exercise price of \$ 80 will be permitted for trading.

At any given point in time, contracts with many different exercise prices will be trading for each of the expiration months. The number of different exercise prices that are observable at a given point in time, would depend on the movement in the price of the underlying stock from the inception of trading in contracts for that particular expiration month. Of course, not all contracts will be equally active.

All contracts on a given stock which are of the same type, that is calls or puts, are said to constitute an *Option Class*. Thus all the calls that are available on XYZ stock at a given point in time, irrespective of the exercise price or the expiration date, would be said to constitute an Option Class. Similarly, so will all the puts.

All the contracts in a given class, that is, the Call Class or the Put Class, which have the same exercise price and the same expiration date, are said to constitute an *Option Series*. Thus all call option contracts on XYZ stock with $X = \$ 75$ and a time to expiration of three months would be said to belong to the same series.

9.12 FLEX Options

Despite their disadvantages, customized options contracts nevertheless have their appeal for certain categories of investors. Institutional investors in particular, often desire the freedom to design options contracts to meet their specific requirements, for from their standpoint, exchange traded contracts are inadequate. Traditionally, such investors have had to approach other institutional investors, like commercial banks, who are seeking to write options with similar features. Of late, the exchanges are making an effort to capture a slice of the growing OTC market. To do this, they have created products known as FLEX Options for stocks and stock indices, where FLEX stands for FLexible EXchange.

While such options allow investors to specify contract terms as per their requirements, they also allow them to take advantage of exchange traded products. Such options are cleared by the OCC, thereby virtually eliminating counterparty risk. Price discovery takes place in a competitive market and listings are totally transparent. Finally, traders enjoy the freedom to offset or alter their positions.

FLEX options allow investors to choose most of the contract terms. The terms that can be customized are:

- Expiration Dates: The time to maturity of an option has to be a minimum of one day and cannot be more than five years from the trade date. The expiration date must be a business day and cannot be the third Friday of the expiration month, or two business days on either side of the third Friday.
  For instance if the third Friday of September is the 19th, then the expiration date cannot be the 17th, 18th, 19th, 22nd or 23rd of September. If the third Friday were to be a holiday then the preceding business day cannot be an expiration day, nor can two business days on either side of that day. For instance, if the 19th of September (Friday) is a holiday, then the 18th,

assuming that it is not a holiday, cannot be an expiration date. In this case, the 16th, 17th, 22nd, and 23rd also cannot be expiration days.

- Exercise Style: Options may be European style or American style
- Exercise Prices and Option Premiums: Exercise prices and premiums may be stated in dollar terms or as a percentage of the price of the underlying stock. Exercise prices must be rounded to the nearest $\frac{1}{8}$th of a dollar, while premiums must be rounded to the nearest cent.
- Size: One contract is for 100 shares of the underlying stock. A minimum of 250 contracts or $ 1 MM in terms of value of the underlying stock, whichever is less, is required to open a new series. For subsequent transactions in an existing series, a minimum of 100 contracts is required. A closing transaction may be for the lesser of 25 contracts or the remaining underlying size. The size may be expressed either in terms of the number of contracts or in terms of dollars of underlying value. If the size were to be expressed in dollar terms, the exchange will use the current stock price to determine if the minimum value size requirement is satisfied.
- Mode of settlement: All contracts are settled by physical delivery. The settlement price is the closing stock price on the day of exercise or expiration as the case may be.

9.12.1 Open Outcry Trading

In order to trade in these contracts, investors have to submit a request for a quote (RFQ). This will contain the details of the contract sought by them, namely whether it is a call or a put, the exercise price, the time to maturity, and whether they want a European or an American style contract. The RFQ is then acted upon by market makers on the exchange, who submit quotes for the option premium.

Each RFQ is assigned a Request Response Time (RRT). This is the period of time during which exchange members may submit bids and offers on behalf of their customers. Quotes may be modified at any time during the RRT. At the end of the RRT, the best bid and offer (BBO) are announced by the member who submitted the RFQ. He may accept a part or all of the BBO, improve it, or reject it in its entirety. The member submitting the RFQ is under no obligation to accept the BBO. If the member declines the BBO, any other member may accept the best bid or offer up to the size currently available. On the other hand, if the submitting member accepts the BBO, but there is excess size available at the BBO, then other members may trade the balance available. Once it is determined that there is no further interest in the bids and offers submitted in response to an RFQ, the market for that RFQ is deemed to be closed, and the quotes are no longer considered to be valid.

9.12.2 CFLEX

CFLEX is an internet based electronic system for trading index and equity FLEX options that was launched by the CBOE in 2007.

9.12.3 Index FLEX Options

Options are available on any of the indices listed on the CBOE, including:

- S&P 100
- S&P 500
- NASDAQ 100
- DJIA

Customization Features Investors may select the terms underlying the options contract. These include:

- Expiration Date: The expiration date may be as far as 10 years from the date of creation of the option. It must be a business day and cannot be the third Friday of the expiration month (or the first preceding business day if the third Friday is a holiday). Also, it cannot fall within two days prior to or subsequent to the third Friday, or the preceding business day if the third Friday were to be a holiday.
- Exercise Style: Options may be European style or American style.
- Exercise Price: The strike price may be specified as an index level, as a percentage of or a numerical deviation from the closing or intra-day value of an index, or in the form of any other readily understandable method. It must be rounded to the nearest 100th of an index point.
- Contract Size: The size may be specified either in the form of a number of contracts, or in terms of the notional value of the underlying index. For transactions that lead to the creation of a new series, a minimum of $ 10 MM of notional value is required. For an existing series. a minimum of $ 1 MM of notional value is required. For closing transactions the remaining balance is acceptable even if it were to be less than $ 1 MM in terms of the notional value. In those cases where the size is specified in terms of the notional value, the number of contracts is determined by dividing the dollar amount by the value of the index multiplied by $ 100.
- Settlement Value: Settlement may be based either on the opening or on the closing settlement value. Exercise will result in the receipt/payment of cash, on the following business day. For American style options, settlement value determination only applies to options that are slated to expire. In the event of early exercise, settlement is against the closing value of the index on the day of exercise.
  - Opening Settlement Value: It is calculated using the opening price in the primary market for each component stock on the day of expiration. If a security does not open for trading on that day, the last reported sales price for the security is used.

    The method employed is however different for options on the NASDAQ 100 index. The settlement value is calculated as a volume-weighted average of prices reported in the first five minutes of trading for each of the component stocks.

– Closing Settlement Value: It is calculated using the closing price in the primary market for each component security on the day of expiration. If a security does not open for trading, then its last reported sales price is used.

9.13 Assignment of Contracts

When an option holder decides to exercise his option, it is not necessary that the writer with whom he traded initially will still be in the market, for he could have offset his position. In practice when a holder decides to exercise an option, he will have to inform his clearing firm, which will then place an exercise order with the clearinghouse. The clearinghouse will then randomly select a clearing firm through which an investor has written an option with the same features. The clearing firm that is selected, has to then choose a particular writer to deliver in the case of calls, or to accept delivery in the case of puts. The writer who is chosen is said to be *assigned*.

In order to ensure that the option is exercised on a particular day, the holder must inform his broker before the cutoff time specified by the concerned brokerage firm for accepting exercise instructions. The cutoff times for receiving such instructions can vary from broker to broker, and in the case of a particular brokerage firm, could depend on the class of options that are sought to be exercised.

9.14 Contract Value Margining

In an option contract, there is no risk that the buyer will not perform. Consider a call holder. If $S_t > X$ he may consider exercising, whereas if $S_t < X$, he will not, and in any case, he need not. Similarly, a put buyer will consider exercising if $S_t < X$, else he will let the option expire worthless. So there is no apprehension regarding non performance on the part of the holder.

Then why do we need to worry about margins? Consider the case of a call writer. If $S_t > X$, and the holder decides to exercise, what if the writer refuses to deliver the shares. Similarly, if $S_t < X$, we will have a problem if the put writer refuses to accept the shares. So in the case of options, it is the *Writer* who has to maintain a margin account.

We will now look at the traditional margin requirements on the CBOE for naked options. The CBOE has switched over to a portfolio based margining system in April 2007, for certain products and strategies. An *Uncovered* or *Naked* options position, refers to a transaction in which a person writes a call or a put on a stock, without combining it with an offsetting position in the underlying asset.[4]

[4] Many derivatives exchanges follow a portfolio based margining system called SPAN (Standard Portfolio Analysis of Risk). We will describe the mechanics of SPAN in detail in a subsequent chapter.

9.14.1 Margins for Naked Options

The initial margin required for a naked call is equal to

$$\text{Max}[C_t + .1S_t, C_t + .2S_t - \text{Max}(0, X - S_t)]$$

In words, the initial margin is the greater of the following amounts.

- Call premium + 10% of the stock's value, that is, $C_t + .1S_t$
 and
- Call premium + 20% of the stock's value, less the amount by which the option is out-of-the-money, that is, $C_t + .2S_t - \text{Max}(0, X - S_t)$

In the above expressions, C_t is the call premium collected at the time of sale; S_t is the corresponding value of the stock; and X is the exercise price. Max $(0, X - S_t)$ is the amount by which the call is out-of-the-money.

After the day on which the contract is entered into, margins must be continued to be maintained at adequate levels. The level of collateral required on subsequent days is called the *Maintenance Margin*. The formula for calculating the maintenance margin is the same as that used for computing the initial margin, except that, instead of using the original premium paid, we use the premium prevailing at the end of the day in question. Similarly, we would use the closing stock price for the day on which we are calculating the maintenance margin.

These points will become clear when we go through the following example.

Example Ryan has written a naked call option contract on IBM with an exercise price of \$ 100. The premium per share is \$ 4.50, and the prevailing stock price is \$ 95.

The value of the option is \$ 4.50. Ten percent of the prevailing stock price is $0.1 \times 95 =$ \$ 9.50. Twenty percent of the prevailing stock price is obviously \$ 19.00. The amount by which the call is out-of-the-money is given by Max[0, 5] = \$ 5.00. The initial margin is given by

$$100 \times \text{Max}[4.50 + 9.50, 4.50 + 19.00 - 5] = \$\ 1{,}850$$

Now suppose the next day, the call price is \$ 9.00, and the share price is \$ 102, we will compute the maintenance margin as follows.

$$100 \times \text{Max}[9.00 + .1 \times 102, 9.00 + .2 \times 102 - \text{Max}\{0, (100 - 102)\}]$$
$$= \text{Max}(1{,}920; 2{,}940) = \$2{,}940$$

So, an additional \$ 1,090 has to be deposited per contract.

For put options, the initial margin is given by the greater of these two amounts.

- Put premium + 10% of the exercise price, that is, $P_t + .1X$
 and
- Put premium + 20% of the stock's value, less the amount by which the option is out-of-the-money, that is, $P_t + .2S_t - \text{Max}(0, S_t - X)$

Thus the corresponding formula for the initial margin is

$$\text{Max}[P_t + .1X, P_t + .2S_t - \text{Max}(0, S_t - X)]$$

Example Rhonda has written a put options contract on IBM with an exercise price of \$ 100. The prevailing stock price is \$ 115, and the premium for the options is \$ 1.50 per share. The initial margin is given by

$$\begin{aligned}100 \times \text{Max}[1.50 + .1 \times 100, 1.5 + .2 \times 115 - 15]\\ = 100 \times \text{Max}[11.5, 9.5]\\ = 100 \times 11.5 = \$\ 1{,}150\end{aligned}$$

9.15 Adjusting for Stock Splits and Stock Dividends

Exchange traded options have to be adjusted for splits and stock dividends. Let us suppose that you own a share which is currently priced at \$ S_t. Consider an n for m split. The new theoretical price will be $\frac{mS_t}{n}$. For instance, if there is a 5:2 split, the new price will be $\frac{2S_t}{5}$.

In the event of a split, the terms of the option contract will be adjusted as follows. For an n:m split, the exercise price of the contract will be changed from X to $\frac{mX}{n}$. And the number of shares per contract will be changed to $\frac{n}{m} \times$ number of shares per original contract.

9.15.1 Example

Consider a person who owns a call which entitles him to buy 100 shares at \$ 100 per share. Now, let us say the company announces a 5:2 split. The terms of the contract will be adjusted as follows. Each contract will now allow him to buy $\frac{5}{2} \times 100 = 250$ shares at an exercise price = $\frac{2 \times 100}{5}$, that is, at \$ 40.

Similar adjustments will also take place for stock dividends. Consider a 40% stock dividend. It means that you get .4 extra shares per share that you hold, or two extra shares for every five existing shares. Thus a 40% stock dividend is like a 7:5 split. When the contract is adjusted, the stock dividend will be treated like a 7:5 split.

9.16 The Put-Call Ratio

The put-call ratio, is the ratio of the trading volume of put options to the trading volume of call options. Changes in the ratio are interpreted as an indication of the prevailing market sentiment. An increase in the ratio, which means that the put volume is relatively higher than that of calls, is interpreted as a signal of a bear market. On the other hand, a decrease in the ratio, which means that the call volume is relatively higher than that of puts, is interpreted as a signal of a bull market.

Suggestions for Further Reading

1. Hull J.C. *Options, Futures, and Other Derivatives.* Prentice-Hall, 2006.
2. *Options: Essential Concepts & Trading Strategies*. McGraw-Hill, 1999.

References

1. *Long-Term Equity Anticipation Securities.* The Options Clearing Corporation, 2004.
2. *Chicago Board Options Exchange: Margin Manual.* The CBOE, 2000.
3. *Understanding Stock Options.* The Options Clearing Corporation, 1994.
4. *Understanding Index Options.* The CBOE, 2001.
5. www.cboe.com

Concept Check

State whether the following statements are True or False.

1. Capped style options must be calls by design.
2. The supply of options is subject to a limit related to the number of shares issued by the firm on whose stock the contracts are written.
3. Call option holders are entitled to cash dividends declared during the life of the option by the firm on whose stock the contracts are written.
4. Put option holders enjoy voting rights.
5. The option premium is a sunk cost.
6. A call option should be exercised only if the stock price at the time of exercise is greater than the sum of the exercise price and the option premium.
7. Index options are always cash settled.
8. Exchange traded stock options in the US are American style.
9. Exchange traded index options in the US may be European or American in nature.
10. Call holders face the prospect of infinite profits and finite losses.
11. Put writers face the prospect of finite profits and infinite losses.
12. Both calls and puts are zero sum games.
13. European options are zero sum games while American options are not.

14. If a call option with a given exercise price is in the money then a put option on the same asset and with the same exercise price must be out of the money.
15. There can be only two option classes on a given stock but there can be many series.
16. FLEX options are cleared by the clearinghouse.
17. FLEX options are always European in nature.
18. In the case of options contracts only the writers need to post margins.
19. Holders of put options must post margins.
20. Writers of both calls and puts have a contingent obligation.

QUESTIONS AND PROBLEMS

Question-I

Futures contracts require both parties to deposit margins, whereas in the case of options contracts only the shorts need do so. Discuss.

Question-II

Options and stocks, although similar in certain respects, differ in other respects.

Discuss.

Question-III

Sunk costs are irrelevant while taking investment decisions. Discuss using both call and put options as examples.

Question-IV

Assume that today is the 5th of September 2008. What will be the available expiration months for options on a stock which is assigned to a January cycle? What if the stock were to be assigned to a February or a March cycle?

Question-V

What are FLEX options? What advantages do they offer to the investor with respect to OTC options? What about with respect to exchange traded options?

Question-VI

Consider a call options contract on IBM. The current stock price is $ 100, the exercise price is $ 100, and the premium is $ 7.95 per share. Each contract is for 100 shares. What will be the applicable margin?

Question-VII

How will your answer to the above question change, if the exercise price were to be $ 112.50 and the premium, $ 1.05 per share?

Question-VIII

Options on IBM are available with an exercise price of $ 50 and a contract size of 100. The stock undergoes a 5:4 split during the life of the contract. How will the terms of the contract be adjusted?

Question-IX

What is exercise by exception? Discuss.

Question-X

Discuss the concepts of AM and PM settlement for index options.

10

Arbitrage Restrictions

10.1 The Absence of Arbitrage

We will now prove certain results pertaining to option prices. All option prices must satisfy these properties, or else there will be arbitrage opportunities. What do we mean by an arbitrage opportunity? *Arbitrage refers to the ability of a trader to make costless and riskless profits*. The phrase of significance here is *costless and riskless*. A costly but risky investment should yield a risk adjusted expected rate of return. A strategy that entails an investment, but is devoid of risk, should yield the riskless rate of return. The ability of an individual to earn a return without an investment of his own, in an environment devoid of risk, is referred to as arbitrage. In other words, a strategy that yields a cash inflow at certain points in time, and a zero cash flow at other points in time, can be termed as an arbitrage strategy, because it leads to non-negative returns for the investor without requiring an investment (which would have manifested itself as a cash outflow) at any point in time. As should be obvious, an arbitrage opportunity will be exploited by anyone who perceives it, to the maximum possible extent till it is eliminated.

The following conditions must hold if arbitrage is to be ruled out. They, are however independent of the way the option prices are determined. In other words, these conditions are not specific to a particular option valuation model.

10.2 Assumptions

In order to derive the arbitrage-free properties of option prices, we will make the following assumptions.

1. Traders do not have to pay commissions or incur any form of transactions costs.
2. There are no bid-ask spreads.
3. All traders are price takers.
4. There are no taxes.
5. Traders receive the full premium on options written by them.

6. If a share is sold short, the trader is entitled to receive the full proceeds from the sale immediately.
7. Investors can trade in the stock and options markets instantaneously. If an option written by an investor is exercised, he will get immediate notice of the assignment.
8. Dividends are received on the ex-dividend date. The price decline on the ex-dividend date is exactly equal to the quantum of the dividend.
9. Investors are non-satiated. That is, they are constantly seeking opportunities to increase their wealth.
10. Arbitrage opportunities are exploited as soon as they appear, and till they cease to exist.

10.3 Non Negative Option Premia

An option cannot have a negative price. The existence of a negative price would mean that the writer is prepared to pay the holder to buy the option. If so, the holder can acquire the option, pocket the payment, and simply forget about the contract. The reason as to why the holder need not worry about further adverse consequences, or in other words, subsequent negative cash flows, is because the option is a right and not an obligation. Obviously, the presence of negative option premia would be a classic example of an arbitrage opportunity.

10.4 Properties of American Options

10.4.1 American Calls

We will use the symbol $C_{A,t}$ to denote the price of an American call option at time t. $C_{A,t}$, should be greater than or equal to $\text{Max}[(S_t - X), 0]$.

Proof If $S_t - X$ is less than zero, we can say that $C_{A,t} \geq 0$, since an option cannot have a negative value. However, if $S_t - X > 0$, $C_{A,t}$ must be worth at least $S_t - X$. The proof can be demonstrated as follows. Consider a case where $S_t - X > 0$ and $C_{A,t} < S_t - X$. If so, you can buy the option by paying $C_{A,t}$ and immediately exercise it. The profit of $S_t - X - C_{A,t}$ that you will get is clearly an arbitrage profit. Therefore, to preclude arbitrage, we require that

$$C_{A,t} \geq \text{Max}[(S_t - X), 0] \tag{10.1}$$

Example Let $S_t = \$\,105$ and $X = \$\,100$. Assume that $C_{A,t} = \$\,4.50$. An arbitrageur can buy the call and immediately exercise it. This will yield an arbitrage profit of:

$$(105 - 100) - 4.50 = \$\,0.50$$

per share.

10.4.2 American Puts

We will use the symbol $P_{A,t}$ to denote the premium of an American put option at time t. The put price, $P_{A,t}$, should be greater than or equal to $\text{Max}[(X - S_t), 0]$.

Proof If $X - S_t$ is less than zero, we can say that $P_{A,t} \geq 0$, since an option cannot have a negative value. However, if $X - S_t > 0$, $P_{A,t}$ must be worth at least $X - S_t$. The proof can be demonstrated as follows. Consider a case where $X - S_t > 0$ and $P_{A,t} < X - S_t$. If so, you can buy the option by paying $P_{A,t}$ and immediately exercise it. The profit of $X - S_t - P_{A,t}$ that you will get is clearly an arbitrage profit. Therefore, to preclude arbitrage, we require that

$$P_{A,t} \geq \text{Max}[(X - S_t), 0] \tag{10.2}$$

Example Let $S_t = \$\ 100$ and $X = \$\ 105$. Assume that $P_{A,t} = \$\ 4.50$. An arbitrageur can buy the put and immediately exercise it. This will yield an arbitrage profit of:

$$(105 - 100) - 4.50 = \$\ 0.50$$

per share.

10.5 Put-Call Parity for European Options

The Put-Call Parity theorem states that

$$C_{E,t} - P_{E,t} = S_t - \frac{X}{(1+r)^{T-t}} \tag{10.3}$$

where $C_{E,t}$ represents the price of a European call with an exercise price of X and time to expiration equal to $T - t$ periods, and $P_{E,t}$ represents the price of a European put with the same exercise price and expiration date.

10.5.1 Proof

Consider the following strategy and the corresponding cash flows.

Let us analyze Table 10.1 carefully, for you will repeatedly come across such strategies in the course of your study of options. The first column in the table indicates the transactions that form components of the overall strategy. The second column indicates the initial cash flow associated with each transaction. All inflows will be positive and outflows negative. When an asset is bought there will obviously be an outflow, whereas when an asset is sold, there will be an inflow. Similarly when funds are borrowed there will be an inflow, whereas when funds are lent there will be an outflow.

The third and fourth columns represent the situation at the time of expiration of the option. Since the key variable of interest is the stock price at expiration and its level with respect to the exercise price, there are two possible situations which can arise. That is, either the stock price can be greater than the exercise price or it can be less than it. In either case the stock can be sold for its prevailing price, S_T. If the call is in the money it will be exercised by the party who has acquired it

Table 10.1 Illustration of Put-Call Parity: Strategy 1

Action	Initial Cash Flow	Terminal Cash Flow	
		If $S_T > X$	If $S_T < X$
Buy the Stock	$-S_t$	S_T	S_T
Sell the Call	$C_{E,t}$	$-(S_T - X)$	0
Buy the Put	$-P_{E,t}$	0	$X - S_T$
Borrow the P.V. of X	$\frac{X}{(1+r)^{T-t}}$	$-X$	$-X$
Total	$-S_t + C_{E,t} - P_{E,t} + \frac{X}{(1+r)^{T-t}}$	0	0

P.V. stands for 'Present value'.

from the arbitrageur. His inflow upon exercise will be $S_T - X$, so obviously the arbitrageur's cash flow will be $-(S_T - X)$. If the put is exercised when it is in the money, the arbitrageur will receive $X - S_T$. Finally, since the present value of X has been borrowed, an amount equal to X, which represents the principal plus interest, must be repaid at expiration.

Notice that in Table 10.1, the overall cash flow at expiration is zero, irrespective of whether the stock price at that point in time is above the exercise price or below it. Consequently, to rule out arbitrage, the initial cash flow must be non-positive, because a positive initial cash flow followed by no further outflows is a clear manifestation of arbitrage. Therefore, to rule out arbitrage, it must be the case that

$$-S_t + C_{E,t} - P_{E,t} + \frac{X}{(1+r)^{T-t}} \leq 0$$

However, we can demonstrate that it is possible to make an arbitrage profit if

$$-S_t + C_{E,t} - P_{E,t} + \frac{X}{(1+r)^{T-t}} < 0$$

This is because in such a situation, we can simply reverse the above strategy, thereby making an arbitrage profit, as shown in Table 10.2.

The initial cash flow in this case will be positive if

$$-S_t + C_{E,t} - P_{E,t} + \frac{X}{(1+r)^{T-t}} < 0$$

whereas the terminal cash flows in either scenario will continue to be zero. Therefore, to preclude arbitrage, it must be the case that

$$-S_t + C_{E,t} - P_{E,t} + \frac{X}{(1+r)^{T-t}} = 0$$

Table 10.2 Illustration of Put-Call Parity: Strategy 2

Action	Initial Cash Flow	Terminal Cash Flow	
		If $S_T > X$	If $S_T < X$
Short Sell the Stock	S_t	$-S_T$	$-S_T$
Buy the Call	$-C_{E,t}$	$(S_T - X)$	0
Sell the Put	$P_{E,t}$	0	$-(X - S_T)$
Lend the P.V. of X	$-\dfrac{X}{(1+r)^{T-t}}$	X	X
Total	$S_t - C_{E,t} + P_{E,t} - \dfrac{X}{(1+r)^{T-t}}$	0	0

$$\Rightarrow C_{E,t} - P_{E,t} = S_t - \frac{X}{(1+r)^{T-t}}$$

This relationship is known as the Put-Call parity theorem. It should be noted that it applies only to European options on stocks which do not pay a dividend during the life of the option.

We will now illustrate the potential to make arbitrage profits when put-call parity is violated.

10.5.2 Example

Consider a stock which is selling for \$ 100. Call and put options with six months to expiration are selling for \$ 8 and \$ 3 respectively. The exercise price of the options is \$ 100, and the riskless rate is 10% per annum.

$$C_{E,t} - P_{E,t} = 8 - 3 = 5$$

$$S_t - \frac{X}{(1+r)^{(T-t)}} = 100 - \frac{100}{(1.10)^{0.50}} = 4.65$$

Thus, clearly there is a violation of put-call parity. Consider the following arbitrage strategy. Buy the stock; sell the call; buy the put; and borrow the present value of the exercise price. The initial inflow is:

$$-100 + 8 - 3 + \frac{100}{(1.10)^{0.50}} = 0.35$$

Subsequent cash flows, it can be verified will be zero, irrespective of the stock price at expiration. Thus, the initial inflow is clearly an arbitrage profit.

10.5.3 Example

Consider a stock which is selling for \$ 100. Call and put options with six months to expiration are selling for \$ 7 and \$ 3 respectively. The exercise price of the options is \$ 100, and the riskless rate is 10% per annum.

$$C_{E,t} - P_{E,t} = 7 - 3 = 4$$

$$S_t - \frac{X}{(1+r)^{(T-t)}} = 100 - \frac{100}{(1.10)^{0.50}} = 4.65$$

Thus, clearly there is a violation of put-call parity. Consider the following arbitrage strategy. Short sell the stock; buy the call; sell the put; and lend the present value of the exercise price. The initial inflow is:

$$100 - 7 + 3 - \frac{100}{(1.10)^{0.50}} = 0.65$$

Subsequent cash flows, it can be verified will be zero, irrespective of the stock price at expiration. Thus, the initial inflow is clearly an arbitrage profit.

10.5.4 European Options on a Dividend Paying Stock

The put-call parity relationship for European options on a stock which pays a known dividend D at time t^*, where t^* is a point in time before the expiration of the option such that $t < t^* < T$, may be stated as:

$$C_{E,t} - P_{E,t} = S_t - \frac{D}{(1+r)^{t^*-t}} - \frac{X}{(1+r)^{T-t}} \tag{10.4}$$

Proof Consider the strategy and the corresponding cash flows depicted in Table 10.3.

Table 10.3 Illustration of Put-Call Parity for a Dividend Paying Stock: Strategy 1

Action	Initial Cash Flow	Intermediate Cash Flow	Terminal Cash Flow	
			If $S_T > X$	If $S_T < X$
Buy the Stock	$-S_t$	D	S_T	S_T
Sell the Call	$C_{E,t}$		$-(S_T - X)$	0
Buy the Put	$-P_{E,t}$		0	$X - S_T$
Borrow the P.V. of D	$\frac{D}{(1+r)^{t^*-t}}$	$-D$		
Borrow the P.V. of X	$\frac{X}{(1+r)^{T-t}}$		$-X$	$-X$
Total	$-S_t + C_{E,t} - P_{E,t} + \frac{D}{(1+r)^{t^*-t}} + \frac{X}{(1+r)^{T-t}}$	0	0	0

As in the case of European options on a non-dividend paying stock, the terminal payoff from the strategy is zero, irrespective of the stock price at expiration. The only difference here is that the stock will pay a dividend of \$ D at time t^*. Since the investor has borrowed the present value of the dividend at the outset, the net cash flow at t^* will be zero. Since the subsequent cash flows are guaranteed to be zero, that initial cash flow must be non positive. That is:

$$-S_t + C_{E,t} - P_{E,t} + \frac{D}{(1+r)^{t^*-t}} + \frac{X}{(1+r)^{T-t}} \leq 0$$

$$\Rightarrow C_{E,t} - P_{E,t} \leq S_t - \frac{D}{(1+r)^{t^*-t}} - \frac{X}{(1+r)^{T-t}}$$

However, if

$$C_{E,t} - P_{E,t} < S_t - \frac{D}{(1+r)^{t^*-t}} - \frac{X}{(1+r)^{T-t}}$$

we can as before reverse the above strategy and make arbitrage profits. Consider Table 10.4.

Table 10.4 Illustration of Put-Call Parity for a Dividend Paying Stock: Strategy 2

Action	Initial Cash Flow	Intermediate Cash Flow	Terminal Cash Flow	
			If $S_T > X$	If $S_T < X$
Short sell the Stock	S_t	$-D$	$-S_T$	$-S_T$
Buy the Call	$-C_{E,t}$		$(S_T - X)$	0
Sell the Put	$P_{E,t}$		0	$-(X - S_T)$
Lend the P.V. of D	$-\frac{D}{(1+r)^{t^*-t}}$	D		
Lend the P.V. of X	$-\frac{X}{(1+r)^{T-t}}$		X	X
Total	$S_t - C_{E,t} + P_{E,t} - \frac{D}{(1+r)^{t^*-t}} - \frac{X}{(1+r)^{T-t}}$	0	0	0

At t^*, the arbitrageur has to pay an amount equal to the dividend to the investor who has lent him the share to facilitate the short sale. However, since he has lent an amount equal to the present value of the dividend at the outset, he will have an inflow of \$ D at t^*, which will be just adequate to pay the dividend. Once again, since all the subsequent cash flows are zero, the initial cash flow must be non-positive. That is:

$$S_t - C_{E,t} + P_{E,t} - \frac{D}{(1+r)^{t^*-t}} - \frac{X}{(1+r)^{T-t}} \leq 0$$

$$\Rightarrow C_{E,t} - P_{E,t} \geq S_t - \frac{D}{(1+r)^{t^*-t}} - \frac{X}{(1+r)^{T-t}}$$

In order for both the conditions to hold simultaneously, it must be the case that:

$$C_{E,t} - P_{E,t} = S_t - \frac{D}{(1+r)^{t^*-t}} - \frac{X}{(1+r)^{T-t}}$$

This is the put-call parity condition for European options on a stock that pays a known dividend during the life of the option.

10.6 Intrinsic Value and Time Value

The Intrinsic Value of an option is equal to the amount by which it is in the money, if it is in the money, else it is equal to zero. Therefore, the intrinsic value of a call option is

$$\text{I.V.} = \text{Max}[(S_t - X), 0] \tag{10.5}$$

and that of a put option is

$$\text{I.V.} = \text{Max}[(X - S_t), 0] \tag{10.6}$$

The difference between an option's price and its intrinsic value, is called the Time Value or the Speculative Value of the option.

10.6.1 Example

Assume that the price of a stock is \$ 100 and that the exercise price of a call option is \$ 97.50. If the call premium is \$ 3.50, then

$$100.00 - 97.50 = 2.50$$

is the intrinsic value of the call option, and

$$3.50 - 2.50 = 1.00$$

is its time value.

10.6.2 Example

Assume that the price of a stock is \$ 100 and that the exercise price of a put option is \$ 97.50. The put premium is \$ 1.25. In this case, the intrinsic value is zero because the put is out of the money. Therefore, the entire premium of \$ 1.25 is the time value of the option.

We have already shown that

$$C_{A,t} \geq \text{Max}[(S_t - X), 0] \text{ and } P_{A,t} \geq \text{Max}[(X - S_t), 0]$$

Therefore, both call as well as put options that are American in nature, must have a non-negative time value.

What about European options? From Put-Call parity, we know that

$$C_{E,t} - P_{E,t} = S_t - \frac{X}{(1+r)^{T-t}}$$

$$\Rightarrow C_{E,t} = P_{E,t} + S_t - \frac{X}{(1+r)^{T-t}}$$

$$\Rightarrow C_{E,t} = P_{E,t} + (S_t - X) + X - \frac{X}{(1+r)^{T-t}}$$

If $(S_t - X) > 0$, then $C_{E,t} \geq (S_t - X)$ because $P_{E,t} \geq 0$, since an option cannot have a negative premium and $X - \frac{X}{(1+r)^{T-t}}$ is also guaranteed to be non-negative. Hence, a European call on a non-dividend paying stock will have a non-negative time value if the option is in the money.

What if the option is out of the money? In that case, the entire premium will comprise of time value, which is guaranteed to be non-negative, since an option cannot have a negative premium. *Therefore, a European call on a non-dividend paying stock will always have a non-negative time value.*

What about European puts? From Put-Call parity, we know that

$$P_{E,t} = \frac{X}{(1+r)^{T-t}} - S_t + C_{E,t}$$

$$\Rightarrow P_{E,t} = X - S_t + \frac{X}{(1+r)^{T-t}} - X + C_{E,t}$$

If the option is out of the money, the time value cannot be negative since the premium cannot be negative. What if the option is in the money? If so, $(X - S_t)$ which is the intrinsic value will be positive. $\frac{X}{(1+r)^{T-t}} - X$ will always be less than or equal to zero. So whether or not the time value is negative, would depend on the value of $C_{E,t}$ compared to the value of $\frac{X}{(1+r)^{T-t}} - X$. Obviously, the lower the value of $C_{E,t}$, the lower will be the time value. For a given value of the exercise price, the lower the value of the stock price, the lower will the value of $C_{E,t}$. Consequently, for the European put to have a negative time value, the corresponding call must be deep out of the money, which means that the put itself must be deep in the money. *Thus, deep in the money European puts may have a negative time value.*

10.7 Option Premia at Expiration

American options and European calls on non dividend paying stocks have a non-negative time value, as we have seen earlier, while European puts on non dividend paying stocks may have either a positive or a negative time value depending on the extent to which the option is in the money. At the time of expiration, the time value of an option must be zero, that is, the option premium must be equal to the intrinsic value.

10.7.1 Proof

Consider a call option that is in the money at expiration. Assume that the time value is positive, that is, $C_{A,T}$ or $C_{E,T}$ is greater than $S_T - X$. If so, an arbitrageur will immediately sell the call. If it is exercised, the net cash flow for him is $C_T - (S_T - X)$, which by assumption is positive. The same holds true if the option is out of the money and the time value is positive, since in this case the writer need not worry about exercise. On the other hand, if the time value were to be negative, that is, $C_{A,T}$ or $C_{E,T}$ is less than $(S_T - X)$, an arbitrageur will buy the option and immediately exercise it. The net cash flow will be $S_T - X - C_T$ which is guaranteed to be positive. Thus, to preclude arbitrage, an option must sell for its intrinsic value at expiration. The same rationale holds for put options, whether European or American.

10.8 Upper Bounds for Call Options

At any point in time, the price of a call option, whether European or American, must be less than the prevailing stock price. The rationale is simple. An option gives the holder the right to acquire a stock by paying the exercise price. If the price of the call were to exceed that of the stock, a rational investor who finds the stock to be attractive, would rather buy it directly, rather than pay a higher price for the option, which would entail a further payment of the exercise price were it to be exercised subsequently.

Here is a formal proof. In proofs of this nature we will assume the converse of the proposition that we wish to prove, and take all the variables to one side so that the expression is greater than zero. In order to set up the corresponding arbitrage strategy, all variables which have a positive sign must be sold, while those with a negative sign must be bought. When it comes to borrowing and lending an amount, if the cash flow has a positive sign, it indicates that the amount must be borrowed, whereas if it has a negative sign it signifies that the amount has to be lent. If we can demonstrate that all cash flows at subsequent points in time are non-negative, then a positive cash flow at inception will tantamount to arbitrage. However, if the cash flow in even one of the subsequent scenarios were to be negative, then we will be unable to make such an assertion about the non-existence of an arbitrage opportunity.

In this case for instance, we wish to show that $C_t \leq S_t$. Assume that C_t is $> S_t$. Therefore $C_t - S_t > 0$. Now consider the following strategy and the corresponding cash flows.

10.8.1 Proof

Notice that we have incorporated an additional column in Table 10.5. This corresponds to a time t^*, where $t < t^* < T$. This is to take into account the possibility of early exercise of the option. The need for this column arises, because the proof is intended to be common for European as well as American options, and while dealing with American options we must always factor in the possibility of early exercise. Since the option that has been sold is a call, the counterparty may exercise it prematurely only if $S_{t^*} > X$.

Table 10.5 **Upper Bound for Call Options**

Action	Initial Cash Flow	Intermediate Cash Flow	Terminal Cash Flow	
		If $S_{t^*} > X$	If $S_T > X$	If $S_T < X$
Sell the Call	C_t	$-(S_{t^*} - X)$	$-(S_T - X)$	0
Buy the Stock	$-S_t$	S_{t^*}	S_T	S_T
Total	$C_t - S_t$	$X > 0$	$X > 0$	$S_T > 0$

In Table 10.5, the initial cash flow is positive by assumption. Since the subsequent cash flows are guaranteed to be positive, it is a clear manifestation of an arbitrage opportunity. Consequently, to rule out arbitrage, we require that

$$C_t - S_t \leq 0 \Rightarrow C_t \leq S_t \tag{10.7}$$

10.9 Lower Bounds for Call Options on Non-dividend Paying Stocks

It can be demonstrated that $C_t \geq \text{Max}\left[0, S_t - \dfrac{X}{(1+r)^{(T-t)}}\right]$. If the expression $S_t - \dfrac{X}{(1+r)^{(T-t)}}$ were to be less than zero, then all that we can assert is that $C_t \geq 0$, because we know that an option cannot have a negative premium. However, if the expression were to be positive, then the option premium must be greater than or equal to it to preclude arbitrage.

10.9.1 Proof

Assume that $C_t < S_t - \dfrac{X}{(1+r)^{(T-t)}} > 0$ or that $S_t - \dfrac{X}{(1+r)^{(T-t)}} - C_t > 0$. Consider Table 10.6.

In this table, the cash flow at inception is positive by assumption. The cash flows at expiration are non-negative. However, the intermediate cash flow will be negative if the call is exercised. But, the important thing to note is that the intermediate stage will lead to a negative cash flow only if the arbitrageur exercises the option at that point in time. Since the call is under the control of the arbitrageur, he is under no compulsion to exercise, and consequently need not worry about the spectre of a negative cash flow. Thus, the above strategy is guaranteed to yield non-negative cash flows subsequently, and hence the initial cash flow must be less than or equal to zero. It, therefore, must be the case that

$$C_t \geq S_t - \frac{X}{(1+r)^{T-t}}$$

Table 10.6 **Lower Bound for Call Options**

Action	Initial Cash Flow	Intermediate Cash Flow	Terminal Cash Flow	
		If $S_{t^*} > X$	If $S_T > X$	If $S_T < X$
Short Sell the Stock	S_t	$-S_{t^*}$	$-S_T$	$-S_T$
Buy the Call	$-C_t$	$(S_{t^*} - X)$	$(S_T - X)$	0
Lend the P.V. of X	$-\dfrac{X}{(1+r)^{T-t}}$	$\dfrac{X}{(1+r)^{T-t^*}}$	X	X
Total	$S_t - C_t - \dfrac{X}{(1+r)^{T-t}}$	$\dfrac{X}{(1+r)^{T-t^*}} - X < 0$	0	$X - S_T > 0$

Combining the two conditions, we can therefore state that

$$C_t \geq \text{Max}\left[0, S_t - \frac{X}{(1+r)^{T-t}}\right] \tag{10.8}$$

We know that an American call option must trade for at least its intrinsic value. That is

$$C_{A,t} \geq \text{Max}[0, S_t - X]$$

We have just demonstrated that

$$C_{A,t} \geq \text{Max}\left[0, S_t - \frac{X}{(1+r)^{T-t}}\right]$$

because the proof that has just been presented is valid for both European as well as American calls.

The second condition gives a tighter lower bound for American calls on non-dividend paying stocks. For if $C_{A,t} \geq S_t - \dfrac{X}{(1+r)^{T-t}}$, then it automatically implies that $C_{A,t} \geq S_t - X$. Based on this observation, we can now draw some important conclusions about American calls on non-dividend paying stocks.

1. As in the money, American call will always have a positive time value except at expiration. That is because, if $S_t > X$, then

$$C_{A,t} \geq S_t - \frac{X}{(1+r)^{T-t}} \geq S_t - X$$

2. An American call on a non-dividend paying stock will never be exercised early. If the holder chooses to exercise an in the money call, he will get the intrinsic value of the option. However, if he chooses to offset by selling an identical call, he will get the prevailing call premium which, is guaranteed

to exceed the intrinsic value. Thus, an American call on a non-dividend stock would rather be offset than exercised prior to expiration.

3. An American call on a non-dividend paying stock will have the same premium as a European call on the same stock, with the same exercise price and expiration date. This is because since the American call is never going to be exercised early, the early exercise option inherent in it has no value for the holder. Consequently, the call will trade for the same premium as a European call with identical features.

10.10 Upper Bounds for Puts

Let us first consider European puts. It can be shown that $P_{E,t} \leq \frac{X}{(1+r)^{T-t}}$. To prove this, consider the following strategy.

10.10.1 Proof

Table 10.7 Upper Bound for European Puts

Action	Initial Cash Flow	Terminal Cash Flow	
		If $S_T > X$	If $S_T < X$
Sell the Put	$P_{E,t}$	0	$-(X - S_T)$
Lend the P.V. of X	$-\frac{X}{(1+r)^{T-t}}$	X	X
Total	$P_{E,t} - \frac{X}{(1+r)^{T-t}}$	$X > 0$	$S_T > 0$

In order to preclude arbitrage the initial cash flow must not be positive. Therefore

$$P_{E,t} - \frac{X}{(1+r)^{T-t}} \leq 0$$

$$\Rightarrow P_{E,t} \leq \frac{X}{(1+r)^{T-t}} \tag{10.9}$$

The above proof is not however valid for American puts. For such options, we have to factor in the possibility of early exercise, and when we do so, the above strategy does not give us an explicit condition to rule out arbitrage as can be seen from Table 10.8.

While the terminal cash flows are guaranteed to be non-negative, the sign of the intermediate cash flow is ambiguous. If it were guaranteed to be positive,

Table 10.8 **Upper Bound for American Puts: An Invalid Proof**

Action	Initial Cash Flow	Intermediate Cash Flow	Terminal Cash Flow	
		If $S_{t^*} < X$	If $S_T > X$	If $S_T < X$
Sell the Put	$P_{A,t}$	$-(X - S_{t^*})$	0	$-(X - S_T)$
Lend the P.V. of X	$-\dfrac{X}{(1+r)^{T-t}}$	$\dfrac{X}{(1+r)^{T-t^*}}$	X	X
Total	$P_{A,t} - \dfrac{X}{(1+r)^{T-t}}$	$S_{t^*} + \dfrac{X}{(1+r)^{T-t^*}} - X$	$X > 0$	$S_T > 0$

we could categorically state that $P_{A,t} > \dfrac{X}{(1+r)^{T-t}}$ implies arbitrage. However, because of the ambiguity, we cannot make this assertion. At the same time we cannot rule out the possibility of early exercise, because we have sold the put option. In strategies that involve the acquisition of an option by the arbitrageur, a cash outflow at an intermediate stage can never arise because the option is under his control. However, the same cannot be said for strategies which entail the sale of an option by the arbitrageur.

So what is the applicable upper bound for an American put? We can demonstrate that $P_{A,t} < X$. Consider the following strategy.

10.10.2 Proof

Table 10.9 **Upper Bound for American Puts: A Valid Proof**

Action	Initial Cash Flow	Intermediate Cash Flow	Terminal Cash Flow	
		If $S_{t^*} < X$	If $S_T > X$	If $S_T < X$
Sell the Put	$P_{A,t}$	$-(X - S_{t^*})$	0	$-(X - S_T)$
Lend X	$-X$	$X(1+r)^{(t^*-t)}$	$X(1+r)^{(T-t)}$	$X(1+r)^{(T-t)}$
Total	$P_{A,t} - X$	$S_{t^*} + X(1+r)^{(t^*-t)} - X > 0$	$X(1+r)^{(T-t)} > 0$	$S_T + X(1+r)^{(T-t)} - X > 0$

Since the subsequent cash flows are guaranteed to be non-negative, the initial cash flow must be non-positive to rule out arbitrage. Thus, we require that $P_{A,t} \leq X$.

10.11 Lower Bounds for Puts

For American puts on non-dividend paying stocks, we have already demonstrated that $P_{A,t} \geq \text{Max}[0, X - S_t]$. In the case of European puts, the no-arbitrage condition is:

$$P_{E,t} \geq \text{Max}\left[0, \frac{X}{(1+r)^{(T-t)}} - S_t\right] \quad (10.10)$$

This condition can be interpreted as follows. If $\frac{X}{(1+r)^{T-t}} - S_t < 0$, then all that we can assert is that $P_{E,t} > 0$, because an option cannot have a negative premium. However, if $\frac{X}{(1+r)^{T-t}} - S_t > 0$ then we can show that $P_{E,t} \geq \frac{X}{(1+r)^{T-t}} - S_t$. To prove it, consider the following strategy.

10.11.1 Proof

Table 10.10 Lower Bound for European Puts

Action	Initial Cash Flow	Terminal Cash Flow	
		If $S_T > X$	If $S_T < X$
Buy the Put	$-P_{E,t}$	0	$(X - S_T)$
Buy the Stock	$-S_t$	S_T	S_T
Borrow the P.V. of X	$\frac{X}{(1+r)^{T-t}}$	$-X$	$-X$
Total	$\frac{X}{(1+r)^{T-t}} - P_{E,t} - S_t$	$S_T - X > 0$	0

To rule out arbitrage we require that

$$\frac{X}{(1+r)^{T-t}} - P_{E,t} - S_t \leq 0$$

$$\Rightarrow \frac{X}{(1+r)^{T-t}} - S_t \leq P_{E,t}$$

Although it must be the case that $P_{E,t}$ is greater than $\frac{X}{(1+r)^{T-t}} - S_t$, it is not necessary that $P_{E,t} > X - S_t$. Thus, a European option on a non-dividend paying stock can sell for less than its intrinsic value, or in other words it can have a negative time value. This of course is a possibility that we have already proved earlier.

What about American options on a non-dividend paying stock? We have already demonstrated that $P_{A,t} \geq \text{Max}[0, X - S_t]$ which obviously implies that $P_{A,t} \geq \text{Max}\left[0, \dfrac{X}{(1+r)^{T-t}} - S_t\right]$. Thus we get a tighter lower bound for American put options.

10.12 Early Exercise of American Options

We have already shown that an American call on a non-dividend paying stock will never be exercised early. That is, a holder would rather offset than exercise. Let us now consider it from the perspective of an investor who intends to exercise because he desires to acquire the stock.

Consider two points in time $t1$ and $t2$ where $t1 < t2$. Exercise at $t1$ would be profitable only if $S_{t1} > X$. Take the case of an investor who has a call option plus cash equal to \$ X at time $t1$. Assume that he exercises and acquires the stock. Now consider the situation at $t2$. Since the stock has been acquired at $t1$, its value would be S_{t2}.

However, if the investor had not exercised at $t1$, he would have cash equal to \$ $X(1+r)^{(t2-t1)}$ with him. At this point in time, there are two possibilities, that is, the call may or may not be in the money. If the call is in the money, an amount equal to \$ X can be used to acquire the stock at $t2$, which would obviously be worth S_{t2}, leaving a remainder of \$ $X(1+r)^{(t2-t1)} - X$ in cash. Clearly, the investor would be better of if he had waited till $t2$ to exercise. If he chooses not to exercise at $t2$, because the call is out of the money, he would have an amount of \$ $X(1+r)^{(t2-t1)}$ with him in cash plus a call option worth $C_{A,t2}$. By assumption in this case, $S_{t2} < X$, and $X(1+r)^{(t2-t1)} + C_{A,t2} > S_{t2}$. Thus irrespective of his decision at $t2$, it would have been better for him to wait.

As per this logic it is always better to wait than to exercise prior to expiration. Thus, it is never optimal to prematurely exercise an American call on a non-dividend paying stock.

What about American puts on such stocks? An investor would exercise at $t1$ if $S_{t1} < X$. Take the case of an investor who has a share of stock plus a put option at time $t1$. If he exercises, he will have a cash inflow equal to \$ X. This can be invested till $t2$ to yield \$ $X(1+r)^{(t2-t1)}$.

However, if the investor chooses not to exercise at $t1$, he would have the stock with him at $t2$, which would be worth S_{t2}. At this point in time there are two possibilities, that is, the put may or may not be in the money. If the investor exercises at $t2$, it would be because the put is in the money. In such a scenario he would have a cash inflow of X at $t2$. Clearly he would have been better off by exercising at $t1$. However, if the put is out of the money at $t2$ and he consequently chooses not to exercise, he would have a stock worth S_{t2} plus an option worth $P_{A,t2}$ with him. This may or may not be worth more than \$ $X(1+r)^{(t2-t1)}$. If it is worth more, then he stands to gain from delaying the exercise. Thus, in this case it not necessary that the investor would always be better off waiting. Clearly there could be circumstances where an investor may deem it appropriate to exercise prior to expiration.

Take a situation where the investor is convinced that the option is guaranteed to finish in the money, and that all investors believe the same. Assume that the put is selling for its intrinsic value of $X - S_t$. Waiting to exercise is clearly inferior if the put is guaranteed to be in the money. This is because, immediate exercise would lead to an inflow of X, which can be invested to earn the riskless rate of return. Waiting, on the contrary, would delay the receipt of the cash flow, and consequently would result in a loss of interest. At the same time the put cannot be offset for a price of $X - S_t$, because no investor would want to buy and hold it. Consequently, the holder is better off exercising under such circumstances.

10.12.1 The Case of Dividends

Now consider two points in time $t1$ and $t2$, such that the stock trades cum-dividend at $t1$ and ex-dividend at $t2$. In this case $t2$ is an instant after $t1$. Take the case of an investor who has a call option plus cash equal to \$ X at time $t1$. Assume that he exercises and acquires the stock. He would have a stock worth \$ S_{t2} plus a cash amount of \$ D with him. If he does not exercise at $t1$, he will have \$ X with him in cash plus the option. Once the stock goes ex-dividend, the call will become an option on a non-dividend paying stock and there would be no question of early exercise.

We also know that $C_{A,t2} \geq S_{t2} - \dfrac{X}{(1+r)^{(T-t2)}}$. Assume that

$$C_{A,t2} = S_{t2} - \frac{X}{(1+r)^{(T-t2)}} + \tilde{V}$$

where $\tilde{V}$ is the amount by which the premium exceeds the lower bound for the option.

Thus, if

$$S_{t2} + D > X + S_{t2} - \frac{X}{(1+r)^{(T-t2)}} + E[\tilde{V}]$$

$$\Rightarrow D > X - \frac{X}{(1+r)^{(T-t2)}} + E[\tilde{V}] \qquad (10.11)$$

the option will be exercised early. For deep in the money calls with short times to expiration, $E[\tilde{V}]$ will be approximately zero. In such cases, the condition for early exercise is

$$D > X - \frac{X}{(1+r)^{(T-t2)}} \qquad (10.12)$$

Example Assume that the current stock price is \$ 100. Consider a call option with an exercise price of \$ 90 and one month to expiration. Let the riskless rate of interest be 10% per annum. The stock is about to go ex-dividend and when it does the price will fall by the amount of the dividend. The size of the dividend is \$ 4. Assume that, once the stock goes ex-dividend, the option is likely to trade at its lower bound.

If the investor exercises before the stock goes ex-dividend he will receive \$ 100 − \$ 90 = \$ 10. However if he waits he will have a call with a value of

$$96 - \frac{90}{(1.10)^{0.0833}} = \$\,6.71$$

Clearly early exercise is optimal in this case.

10.13 The Put-Call Parity Equivalent for American Options

In the case of American options, the put-call parity condition consists of two inequalities. In the case of options on a stock which is not scheduled to pay dividends during the life of the option, the condition may be stated as

$$S_t - X \leq C_{A,t} - P_{A,t} \leq S_t - \frac{X}{(1+r)^{T-t}} \tag{10.13}$$

10.13.1 Proof

Let us first consider the R.H.S. To prove it, consider the following strategy and the corresponding cash flows (Table 10.11).

Table 10.11 Illustration of Put-Call Parity for American Options on a Non-dividend Paying Stock: Strategy 1

Action	Initial Cash Flow	Terminal Cash Flow	
		If $S_T > X$	If $S_T < X$
Buy the Stock	$-S_t$	S_T	S_T
Sell the Call	$C_{A,t}$	$-(S_T - X)$	0
Buy the Put	$-P_{A,t}$	0	$X - S_T$
Borrow the P.V. of X	$\frac{X}{(1+r)^{T-t}}$	$-X$	$-X$
Total	$-S_t + C_{A,t} - P_{A,t} + \frac{X}{(1+r)^{T-t}}$	0	0

Since the terminal payoffs are guaranteed to be zero, the initial cash flow must be non-positive. Therefore it must be the case that

$$-S_t + C_{A,t} - P_{A,t} + \frac{X}{(1+r)^{T-t}} \leq 0$$

$$\Rightarrow S_t - \frac{X}{(1+r)^{T-t}} \geq C_{A,t} - P_{A,t}$$

One question that the reader may have is, why has the possibility of early exercise of the options not been considered. The reason is the following. The arbitrageur has sold the call. Since it is a call on a non-dividend paying stock, it will never be optimal for the counter-party to exercise early. In the case of the put, the arbitrageur has bought it. Since an American put on a non-dividend paying stock may be exercised early, it may be optimal for the arbitrageur to exercise early. However, since the put is in his control, he will exercise early only if the overall payoff is guaranteed to be positive. Consequently there is no possibility of a negative cash flow at an intermediate stage. Thus, if the cash flow at inception is non-positive, then there can be no arbitrage profits.

Let us now focus on the L.H.S. Consider the following strategy and the corresponding cash flows as depicted in Table 10.12.

Table 10.12 **Illustration of Put-Call Parity for American Options on a Non-dividend Paying Stock: Strategy 2**

Action	Initial Cash Flow	Intermediate Cash Flow	Terminal Cash Flow	
		If $S_{t^*} < X$	If $S_T > X$	If $S_T < X$
Short sell the Stock	S_t	$-S_{t^*}$	$-S_T$	$-S_T$
Buy the Call	$-C_{A,t}$	T.V.	$S_T - X$	0
Sell the Put	$P_{A,t}$	$-(X - S_{t^*})$	0	$-(X - S_T)$
Lend X	$-X$	$X(1+r)^{(t^*-t)}$	$X(1+r)^{(T-t)}$	$X(1+r)^{(T-t)}$
Total	$S_t - C_{A,t} + P_{A,t} - X$	$X(1+r)^{(t^*-t)} - X + T.V. > 0$	$X(1+r)^{(T-t)} - X > 0$	$X(1+r)^{(T-t)} - X > 0$

We will now analyze the entries in the above table. The overall payoff at expiration is self-explanatory, and is guaranteed to be positive. At the intermediate stage, the call will never be exercised, because it is on a non-dividend paying stock. However, the put that has been sold by the arbitrageur may be exercised early by the counterparty. The put will be exercised only if it is in the money. If the put is in the money, the call will obviously be out of the money. If so, it will have no intrinsic value, and its entire premium will consist of the time value which we have denoted by T.V. Thus if the position is unwound at an intermediate stage, the call can be sold for its time value. This explains the entries corresponding to the payoffs if the put is exercised prior to expiration. Since all subsequent cash flows are guaranteed to be positive, the initial cash flow must be non-positive.

Suggestions for Further Reading

1. Dubofsky D.A. *Options and Financial Futures: Valuation and Uses.* McGraw-Hill, 1992.
2. Figlewski S., Silber W.L., and M.G. Subrahmanyam *Financial Options: From Theory to Practice.* Irwin, 1990.

Concept Check

State whether the following statements are True or False.

1. The arbitrage-free restrictions that we have derived are specific to an option pricing model.
2. American options can never have a negative premium whereas deep out-of-the money European puts may.
3. For at-the-money European options on a non-dividend paying stock, the call premium will always be greater than the put premium.
4. The intrinsic value of out-of-the money calls and puts may be negative.
5. Out of the money options will have only time value.
6. American calls and puts will always have a non-negative time value.
7. An in-the-money American call will always have a positive time value prior to expiration.
8. An American call on a non-dividend paying stock will never be exercised early.
9. The premium of an American call on a non-dividend paying stock will be equal to that of a European call on the same stock, with the same exercise price and expiration date.
10. An American put on a non-dividend paying stock may be exercised early.
11. A deep in-the-money European call may have a negative time value.
12. A deep in-the-money European put will always have a negative time value.
13. The time value of American options at expiration must be zero.
14. The time value of European options at expiration must be zero.
15. Out-of-the-money European as well as American options will have a premium of zero at expiration.
16. If an option were to have a positive time value, the holder would rather offset than exercise.
17. A call option on a stock will always have a premium that is less than or equal to the prevailing stock price.

18. European calls and puts have an identical upper bound.
19. European puts have a tighter upper bound than American puts.
20. European puts have a tighter lower bound than American puts.

Questions and Problems

Question-I
Options, both European and American, can never have a negative premium. Discuss.
Question-II
American calls and puts can never sell for less than their intrinsic value. Discuss.
Question-III
An American call on a non-dividend paying stock will never be exercised early. Discuss.
Question-IV
A deep-in-the-money European put may have a negative time value. Discuss.
Question-V
At expiration, all options must sell for their intrinsic value. Discuss.
Question-VI
American puts have a tighter lower bound than European puts. Discuss.
Question-VII
A stock is scheduled to pay a dividend of \$ D_1 at time t_1, and \$ D_2 at time t_2. $t < t_1 < t_2 < T$ where t is the current point in time, and T is the time of expiration of the option. Consider European calls and puts with an exercise price of \$ X. Derive the put-call parity relationship.
Question-VIII
Assume that the options in Question-VII are American. Derive the put-call parity relationship.
Question-IX
Consider a European call option with an exercise price of X, and a time to expiration of $T - t$. Assume that the stock will pay a dividend of \$ D at time t^*, where $t < t^* < T$. The current stock price is S_t and the option premium is $C_{E,t}$. Derive the lower bound for the option.
Question-X
European calls on XYZ are selling at \$ 5.75 per share. The stock price is \$ 100 and the exercise price is \$ 102. The time to expiration of the options is six months, and the risk less rate is 8% per annum. Consider European puts on XYZ with the same exercise price and time to expiration.

What is the intrinsic value of the put options? What is their time value?

11

Option Strategies and Profit Diagrams

11.1 Introduction

In this chapter, we will analyze various trading strategies that can be set up using call options, put options, and combinations of the two. For each strategy we will compute the payoff and profit for various scenarios at the expiration date of the options. The corresponding breakeven price(s) will be computed, and the profit profile will be graphically illustrated.

11.1.1 Notation

We will use the following symbols to denote the corresponding variables.

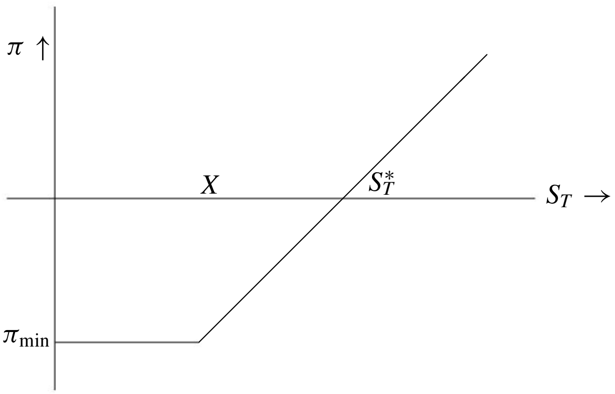
- ${S_T}^* \equiv$ breakeven price
- ${S_T}^{**} \equiv$ second breakeven price if there is more than one.
- $C_t \equiv$ call option premium at time 't' for an option expiring at time 'T'.
- $P_t \equiv$ put option premium at time 't' for an option expiring at time 'T'.
- $r \equiv$ riskless rate of interest per annum.
- $X \equiv$ exercise price of the option.
- $\pi_{\min} \equiv$ maximum loss.
- $\pi_{\max} \equiv$ maximum profit.

In all the graphs, the terminal stock price S_T will be plotted along the X axis, and the profit from the strategy, π will be depicted along the Y axis.

11.2 Long Call

Buying a call option is a popular strategy for bullish investors. Consider a call option with an exercise price of X, and a premium of C_t. If the stock price at expiration, $S_T \leq X$, the option will expire worthless and the investor will lose the entire premium. So for the price range, $0 \leq S_T \leq X$, $\pi = -C_t$, where π denotes the profit from the strategy. As the terminal stock price rises above X, for every dollar increase in the price, the investor's profit will go up by one dollar. Therefore, for the price range $S_T > X$, $\pi = S_T - X - C_t$. The breakeven price $S_T{}^*$ is equal to $X + C_t$. The maximum loss, $\pi_{\min}$, is obviously the premium paid at the outset. The maximum profit on the other hand is unlimited because the stock price has no upper bound.

Figure 11.1



11.2.1 Example

Consider a stock whose current price is \$ 100. The riskless rate of interest is 10% per annum, and the volatility is 30% per annum. The premium for an option with an exercise price of \$ 100, and a time to expiration of six months, is \$ 10.91.[1]

If the stock price at expiration is less than \$ 100, the option will expire out of the money. The loss will be equal to the premium paid for the option which is \$ 10.91. As the stock price at expiration rises above \$ 100, for every dollar increase in the stock price, the profit from the call increases by one dollar. The breakeven stock price is $X + C_t = 100 + 10.91 = \$\ 110.91$. The maximum profit is obviously unlimited.

Let us list the values of the call for stock price values ranging from \$ 90 to \$ 110 (in intervals of \$ 2.50), for times to expiration ranging from 0.5 years to 0.1 years (in intervals of 0.1 years). The last column in Table 11.1 gives the percentage loss over the life of the option, if the stock price were to remain constant, and the options are acquired with 0.5 years to maturity, and held till expiration. For instance, if the option is acquired when the stock price is \$ 90, and there are 0.5 years to expiration, a premium of \$ 5.52 would have to be paid. If held till expiration, the option will expire out of the money if the stock price remains constant. Thus the loss for an investor is 100% of the premium paid. On

[1] We have used the Black–Scholes formula to compute the option prices.

the contrary, if the option is acquired when the initial stock price is \$ 110 and there are 0.50 years to maturity, a premium of \$ 18.05 would have to be paid. At expiration the option will expire in the money, and the payoff will be equal to its intrinsic value of \$ 10. The loss for the investor is

$$\frac{10 - 18.05}{18.05} \equiv -44.60\%$$

Table 11.1 Call Premia for Different Stock Prices and Expiration Times

Stock Price	Time in Years						% age Loss in 0.5 Years
	0.50	0.40	0.30	0.20	0.10	0.00	
			Call Premia				
90.00	5.52	4.40	3.23	2.00	0.74	0.00	–100%
92.50	6.68	5.48	4.20	2.80	1.26	0.00	–100%
95.00	7.97	6.69	5.32	3.79	2.01	0.00	–100%
97.50	9.38	8.05	6.60	4.97	3.01	0.00	–100%
100.00	10.91	9.53	8.04	6.34	4.28	0.00	–100%
102.50	12.55	11.15	9.62	7.90	5.81	2.50	–80.08%
105.00	14.29	12.87	11.34	9.62	7.57	5.00	–65.01%
107.50	16.12	14.71	13.18	11.49	9.54	7.50	–53.47%
110.00	18.05	16.64	15.13	13.48	11.66	10.00	–44.60%

The breakeven stock prices corresponding to options with different values for the initial stock price, as a function of the times to expiration are as depicted in Table 11.2. Let us analyze a few of the entries in the first row of Table 11.1. If the initial stock price is \$ 90, the premium for an option with 0.50 years to expiration is \$ 5.52. After 0.10 years, that is with 0.40 years to expiration, the stock price would have to be \$ 92.60 if the option is to have a premium of \$ 5.52. Thus the breakeven stock price, corresponding to the original option is \$ 92.60 when there are 0.40 years left to expiration. Similarly, after 0.20 years, that is when there are 0.30 years left to expiration, the stock price will have to be at \$ 95.43 if an option with 0.30 years to maturity is to have a price of \$ 5.52. Consequently the breakeven stock price corresponding to the original option is \$ 95.43 when there are 0.30 years to expiration. The remaining entries in the table can be similarly interpreted.

As can be seen from Table 11.2, irrespective of the stock price at which the position is initiated, the breakeven price is higher than the stock price at inception. Thus, the strategy is truly bullish in nature, in the sense that the investor can make money, only if the price of the stock were to rise.

The price which the stock must attain in order for the position to breakeven depends on whether the position is held to expiration, or is offset prior to that. For instance assume that the call is purchased when the stock price is \$ 100 and the

Table 11.2 Breakeven Stock Prices Corresponding to Options with Different Values for the Initial Stock Price as a Function of the Time to Expiration

Initial Stock Price	Time in Years				
	0.40	0.30	0.20	0.10	0.00
	Breakeven Stock Prices				
90.00	92.60	95.43	98.54	102.06	105.52
92.50	94.98	97.66	100.57	103.78	106.68
95.00	97.38	99.90	102.62	105.53	107.97
97.50	99.76	102.15	104.67	107.32	109.38
100.00	102.16	104.39	106.76	109.13	110.91
102.50	104.55	106.67	108.86	111.01	112.55
105.00	106.95	108.95	110.98	112.92	114.29
107.50	109.35	111.23	113.11	114.88	116.12
110.00	111.77	113.54	115.29	116.90	118.05

time to expiration is 0.50 years. The breakeven stock price if the position is held to expiration is \$ 110.91. However if we sell the call when the time to expiration is 0.40 years, the breakeven stock price is \$ 102.16. Thus, the corresponding price at expiration is the highest value that the stock must attain if the position is to breakeven.

The more out of the money the option, at the time of inception of the strategy, the more bullish is the strategy. For instance, the option premium when the stock price is \$ 90, and the time to expiration is 0.50 years, is \$ 5.52. The breakeven stock price for this option at expiration is \$ 105.52, which is \$ 15.52 above the prevailing stock price. However, if the stock price at the outset is \$ 105, the breakeven stock price at expiration is \$ 114.29, which is only \$ 9.29 above the prevailing stock price.

A more bullish strategy is obviously riskier and should yield better returns if the investor's price expectations are met. Take the case of the call when the stock price is \$ 95 and the time to expiration is 0.5 years. If the price of the call were to double when the time to expiration is 0.4 years, the stock price would have to rise to \$ 109.10. On the other hand, a call acquired when the stock price is \$ 105 and the time to expiration is 0.5 years, would require the stock price to rise to \$ 123.80, if it were to double in value over the next 0.10 years.

Out of the money calls also experience a greater premium decay with the passage of time. Consider a call with X = \$ 100 when the prevailing stock price is \$ 90. The premium when there are six months left to expiration is \$ 5.52. If the stock price were to remain constant, the premium when there are 0.1 years left to expiration is \$ 0.74. The percentage decline is

$$\frac{0.74 - 5.52}{5.52} \equiv -87\%$$

Now take the case of a call with $X = \$\,100$, when the stock price is \$ 105. The premium decay when the time to expiration declines from 0.50 years to 0.10 years is

$$\frac{7.57 - 14.29}{14.29} \equiv -47\%$$

11.2.2 Speculating with Calls

Andrew Smith is bullish about IBM and wants to speculate using call options. Options with an exercise price of \$ 100 and three months to expiration are available at a premium of \$ 8. The current stock price is \$ 100. If we assume that Andrew goes long in one contract, his total investment is

$$8 \times 100 = \$\,800$$

Assume that at expiration, the price of IBM is \$ 125. Since the option is in the money, Andrew will exercise. His profit will be

$$(125 - 100) \times 100 - 800 = \$\,1{,}700$$

11.2.3 The 90/10 Strategy

This strategy entails an investment of 10% of the funds at an investor's disposal in call options, and the balance 90% in a money market instrument such as a T-bill.

Assume that Andrew has \$ 8,000 available with him for investment. He decides to buy call options with an exercise price of \$ 100, and with three months to expiration, by paying a premium of \$ 800, which is 10% of his wealth. He then invests the balance of \$ 7,200 is T-bills, which we will assume yield 8% per annum.

The return from the T-bills over three months is

$$7{,}200 \times 0.08 \times 0.25 = \$\,144$$

The effective cost of the options contract is therefore

$$800 - 144 = \$\,656$$

If the stock price at expiration is less than \$ 100, Andrew will incur a loss of \$ 656. This is his maximum possible loss. If the terminal stock price is between \$ 100 and \$ 106.56, he will incur a loss which is given by

$$-656 \leq \pi \leq 0$$

Thus, the breakeven stock price is \$ 106.56. For values of the terminal stock price in excess of \$ 106.56, Andrew will make a profit. His maximum profit is unbounded.

11.2.4 Buying Calls to Lock in a Share Price

IBM is currently trading at \$ 100 per share. Andrew would like to acquire 100 shares but does not have the required funds. He is however confident that he will have adequate funds in three months time. Three month options on IBM are available at a premium of \$ 8 per share. Assume that Andrew goes long in one contract with an exercise price of 100.

If the stock price after three months is \$ 125, the options can be exercised and 100 shares of IBM can be acquired at a cost of

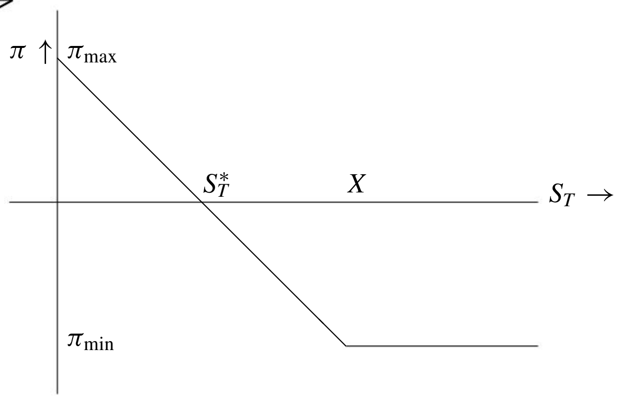
$$100 \times 100 + 800 = \$\ 10{,}800$$

or \$ 108 per share. The cost of the shares would have been \$ 125 each, if Andrew had not taken a position in the options.

11.3 Long Put

Buying a put option is a strategy for bearish investors. Consider a put option with an exercise price of X, and a premium of P_t. If the stock price at expiration, $S_T \geq X$, the option will expire worthless and the investor will lose the entire premium. So for the price range $S_T \geq X, \pi = -P_t$. As the terminal stock price declines below X, for every dollar decline in the price, the investor's profit will go up by one dollar. Therefore, for the price range $0 \leq S_T < X, \pi = X - S_T - P_t$. The breakeven price $S_T{}^*$ is equal to $X - P_t$. The maximum loss, $\pi_{\min}$ is obviously equal to the premium paid at the outset, which is P_t. The maximum profit, $\pi_{\max}$, is equal to $X - P_t$. This is because due to the limited liability feature of stocks, the lowest possible share price is zero.

Figure 11.2



11.3.1 Example

Consider a stock whose current price is \$ 100. The riskless rate of interest is 10% per annum, and the volatility is 30% per annum. The premium for a put option with an exercise price of \$ 100, and a time to expiration of six months, is \$ 6.03.

If the stock price at expiration is greater than \$ 100, the option will expire out of the money. The loss will be equal to the premium paid for the option which is

\$ 6.03. As the stock price at expiration falls below \$ 100, then for every dollar decrease in the stock price, the profit from the put increases by one dollar. The breakeven stock price is

$$X - P_t = 100 - 6.03 = \$\,93.97$$

The maximum profit is equal to \$ 93.97, which corresponds to a terminal stock price of zero.

Let us list the values of the put for stock price values ranging from \$ 90 to \$ 110 (in intervals of \$ 2.50), for times to expiration ranging from 0.5 years to 0.1 years (in intervals of 0.1 years). The last column in Table 11.3 gives the percentage loss over the life of the option, if the stock price were to remain constant, and the options are acquired with 0.50 years to expiration, and held until maturity.

Table 11.3 Put Premia for Different Stock Prices and Expiration Times

Stock Price	Time in Years						% age Loss in 0.5 Years
	0.50	0.40	0.30	0.20	0.10	0.00	
			Put Premia				
90.00	10.64	10.48	10.28	10.02	9.74	10.00	−6.02%
92.50	9.30	9.06	8.74	8.32	7.77	7.50	−24.00%
95.00	8.09	7.77	7.36	6.81	6.01	5.00	−38.20%
97.50	7.00	6.63	6.14	5.49	4.52	2.50	−64.29%
100.00	6.03	5.61	5.08	4.36	3.29	0.00	−100%
102.50	5.17	4.72	4.16	3.42	2.32	0.00	−100%
105.00	4.41	3.95	3.38	2.64	1.58	0.00	−100%
107.50	3.75	3.29	2.72	2.01	1.04	0.00	−100%
110.00	3.17	2.72	2.17	1.50	0.67	0.00	−100%

The more out of the money the option, at the time of inception of the strategy, the more bearish is the strategy. For instance, if the put is acquired when the prevailing stock price is \$ 90, the premium is \$ 10.64. The corresponding breakeven point is 100 − 10.64 = \$ 89.36. Thus the stock would have to decline by \$ 0.64 in order for the position to breakeven. On the other hand, if one were to acquire the put when the prevailing stock price is \$ 107.50, the premium will be only \$ 3.75, and the corresponding breakeven price is 96.25. In this case the stock would have to decline by \$ 11.25 in order for the position to breakeven.

11.3.2 Speculating with Puts

Mike Faye is bearish about IBM and is confident that the stock will decline in value over the next three months. One alternative is for him to short sell shares of IBM. But this strategy can give rise to unlimited losses. This is because, he will have to acquire the stock eventually to cover his short position, and the stock price has no upper bound. Besides, to establish a short position, he needs to set up a margin account, and margin calls may compel him to cover the position prematurely despite the fact that the position may still have profit potential. A long put on the other hand, allows him to speculate while restricting his maximum potential loss to the premium paid for the options.

Assume that the current stock price is \$ 100, and that put options with an exercise price of \$ 100 and three months to expiration, are available at a premium of \$ 4.75 per share. If the stock price at expiration were to be \$ 80 per share, Mike would exercise the options. His profit will be

$$(100 - 80) \times 100 - 4.75 \times 100 = \$\ 1{,}525$$

However, if the terminal stock price were to be \$ 120, Mike would refrain from exercising the options, and his loss would be the premium paid at the outset, which is \$ 475.

11.3.3 Buying Calls to Hedge a Short Sale

This strategy entails the purchase of one call option for every share that is sold short.

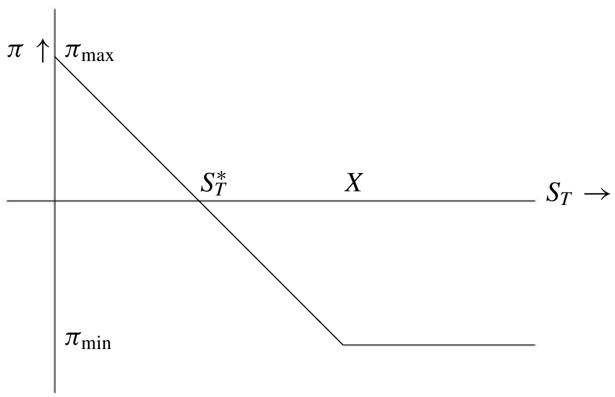
Profit from the short position in the stock $= S_t - S_T$. Profit from the call $=$ $\text{Max}[0, S_T - X] - C_t$. The total profit is given by $\pi = S_t - S_T + \text{Max}[0, S_T - X] - C_t$.

If $S_T \leq X, \pi = S_t - S_T - C_t$. For every dollar that the stock price declines below X, the investor gains a dollar. The breakeven point S_T^* is $= S_t - C_t$. The maximum profit π_{max} is $= S_t - C_t$.

If $S_T > X, \pi = S_t - X - C_t$, which is the maximum loss.

The profit diagram may be depicted in Fig. 11.3.

Figure 11.3



The payoff looks like the one for a long put. This is not surprising, because from put-call parity,

$$P_t - \frac{X}{(1+r)^{T-t}} = C_t - S_t$$

Thus buying a call, and shorting a share, is equivalent to buying a put and borrowing.

Example Andrew has short sold 100 shares of IBM at a price of \$ 100. Quite obviously he is expecting the shares to decline in value. However, there is always a possibility that the stock may rise in value, and lead to a loss when Andrew eventually covers his short position. One way to cap the loss is to invest in call options.

Assume that call options with an exercise price of \$ 100 and three months to expiration are available at a premium of \$ 8, and that Andrew buys one contract. If the share price after three months is \$ 120, it would cost Andrew \$ 10,000 to acquire 100 shares and cover his position. After factoring in the cost of the options, his total cost will be \$ 10,800, which amounts to \$ 108 per share. The loss from the strategy is

$$10{,}000 - 10{,}800 = \$\ (800)$$

If the calls had not been acquired the loss would have been

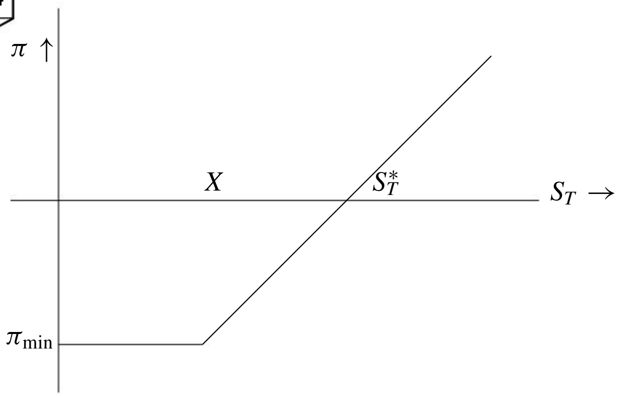
$$(100 - 120) \times 100 = \$\ (2{,}000)$$

11.3.4 The Protective Put

The strategy entails buying the stock and then buying a put option on it, to gain downside protection. $\pi = S_T - S_t - P_t + \text{Max}[0, X - S_T]$. If $S_T \geq X, \pi = S_T - S_t - P_t$, while if S_T is $< X$, $\pi = X - S_t - P_t$. The maximum profit is unlimited. The maximum loss $\pi_{\max}$ is $X - S_t - P_t$. The breakeven price $= S_t + P_t$.

The profit diagram may be depicted as follows.

Figure 11.4



For obvious reasons, the profit diagram resembles that of a long call.

In this case the put acts like an insurance policy. By buying the option with an exercise price of X, the investor ensures that he will not have to sell the stock at a lower price. The higher the exercise price, the greater will be the floor for the stock price. However, the higher the exercise price, the greater will be the option premium, and consequently the higher will be the breakeven point. Thus, an in the money put offers greater downside protection, but requires a larger increase in the stock price for the strategy to turn profitable.

Example Larry King owns 100 shares of IBM that are currently trading at $ 100. Put options with six months to expiration are available. An option with an exercise price of $ 90 is available at a premium of $ 2.65, while an option with an exercise price of $ 110 is available for $ 11.16.

If Larry acquires a contract with X = $ 90, his maximum loss will be

$$90 - 100 - 2.65 = \$\ (12.65)$$

and the breakeven stock price will be $ 102.65. On the contrary if Larry were to acquire a contract with X = $ 110, his maximum loss will be

$$110 - 100 - 11.16 = \$\ (1.16)$$

However the breakeven in this case will be $ 111.16.

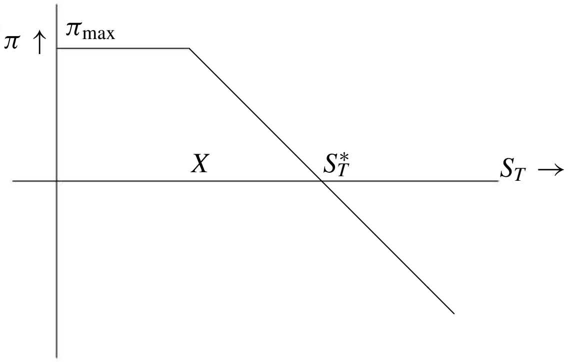
Thus the greater downside protection comes with a cost. And the cost is that the investor will have to give up more of the upside gains, if the stock were to go up in value.

11.4 Writing a Naked Call

This strategy requires the investor to sell call options without taking a position in the underlying stock. Consider a call with an exercise price of X and a premium of C_t. If the stock price at expiration, $S_T \leq X$, the call will expire worthless and the profit for the call writer will be the premium received at the outset. So if $0 \leq S_T \leq X, \pi = C_t$. As the terminal stock price rises above X, for every dollar increase in the stock price, the profit will decline by a dollar. Therefore, for the price range $S_T > X, \pi = -(S_T - X) + C_t$. The breakeven price is, S_T^* equal to $X + C_t$. The maximum profit is the premium received at the outset. The maximum loss is unbounded.

The profit diagram may be depicted as shown in Fig. 11.5.

Figure 11.5



11.4.1 Example

Consider a stock which is currently trading at \$ 100. The riskless rate of interest is 10% per annum, and the volatility is 30% per annum. The premium for a call option with an exercise price of \$ 100, and three months to expiration, is \$ 7.22.

If the stock price at expiration is less than the exercise price of \$ 100, the option will expire out of the money, and the profit for the writer is the premium received at the outset which is $7.22 \times 100 =$ \$ 722 per contract. This represents the maximum profit for the writer. As the stock price rises above \$ 100, for every dollar increase in the price, the profit declines by one dollar. The breakeven stock price is $X + C_t = 100 + 7.22 =$ \$ 107.22. The maximum loss is unlimited.

11.4.2 In the Money versus Out of the Money Calls

Consider the following data for call options with various exercise prices. The current stock price is \$ 100.

Table 11.4 Call Options with Varying Exercise Prices

Exercise Price	Premium	Breakeven Price
\$ 90	\$ 17.03	\$ 107.03
\$ 100	\$ 10.91	\$ 110.91
\$ 110	\$ 6.52	\$ 116.52

The at-the-money call yields the maximum profit if the terminal stock price, S_T is equal to \$ 100. The option also yields a positive profit in the range, $100 \leq S_T \leq 110.91$, since the breakeven price is \$ 110.91. The out of the money option has the lowest premium, which is \$ 6.52 in this case. The profit for the call writer is less, but he gets greater protection against a rising stock price, since the terminal stock price must exceed \$ 110 in order for exercise to be a worthwhile proposition for the buyer. The in-the-money option has the highest premium. If the stock price remains at \$ 100, the profit from the position is

$$-(100 - 90) + 17.03 = \$\ 7.03$$

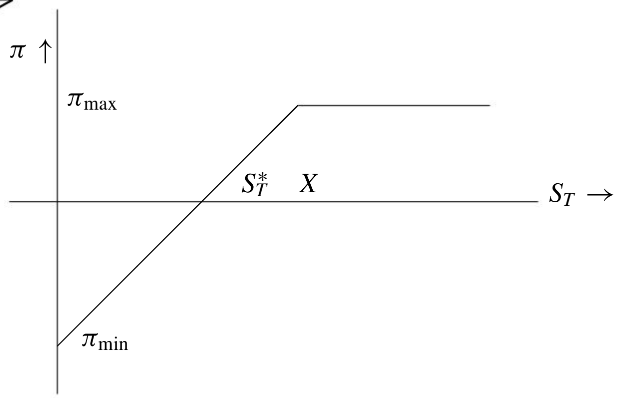
In order for the writer to realize the maximum profit from this option, the stock price must decline from its current level of \$ 100 to \$ 90 or below. Thus, the sale of an in-the-money option represents a more bearish strategy than the sale of an at-the-money or out of the money option.

11.5 Writing a Put

Consider a put option with an exercise price of X, and a premium of P_t. If the stock price at expiration, $S_T \geq X$, the option will expire worthless, and the profit for the writer will be P_t. So in the price range, $S_T \geq X$, $\pi_{\max} = P_t$. As the terminal stock price declines below X, the writer's profit will decline dollar for dollar. Thus, for the price range, $0 \leq S_T \leq X$, $\pi = -(X - S_T) + P_t$. The breakeven price S_T^* is equal to $X - P_t$. The maximum profit is obviously the premium that is received at the outset. The magnitude of the maximum loss is $X - P_t$, because the stock price has a lower bound of zero.

The profit diagram may be depicted as shown in Fig. 11.6.

Figure 11.6



11.5.1 Example

Consider a stock which is currently trading at \$ 100. The riskless rate of interest is 10% per annum, and the volatility is 30% per annum. The premium for a put option with an exercise price of \$ 100, and three months to expiration, is \$ 4.75.

If the stock price at expiration is greater than the exercise price of \$ 100, the option will expire out of the money, and the profit for the writer is the premium received at the outset which is $4.75 \times 100 = \$ 475$ per contract. This represents the maximum profit for the writer. As the stock price declines below \$ 100, for every dollar decrease in the price, the profit declines by one dollar. The breakeven stock price is $X - P_t = 100 - 4.75 = \$ 95.25$. The maximum loss is \$ 95.25, which will arise if the stock price were to attain a value of zero.

11.5.2 Short Put Combined with Short Stock

This strategy entails the sale of a put option for every share that is sold short. The profit from the short stock is $S_t - S_T$, while that from the put is $P_t - \text{Max}[0, X - S_T]$. Thus the total profit from the strategy is

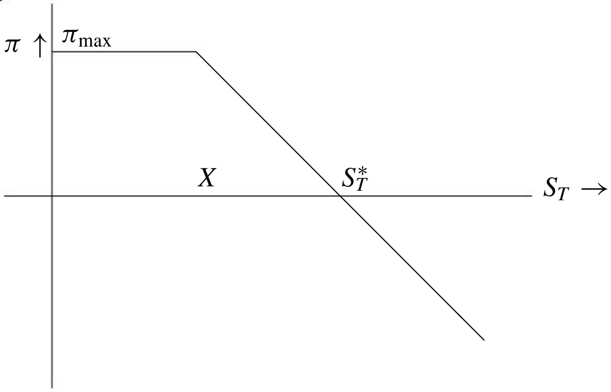
$$\pi = P_t - \text{Max}[0, X - S_T] + S_t - S_T \qquad (11.1)$$

If $S_T \leq X, \pi = S_t + P_t - X$, whereas if $S_T > X, \pi = P_t + S_t - S_T$

The maximum loss is unlimited because the stock has no upper bound. The breakeven price is, S_T^*, is $P_t + S_t$. The maximum profit is equal to $S_t + P_t - X$.

The profit diagram may be depicted as shown in Fig. 11.7.

Figure 11.7



Example Bob Harris has sold short 100 shares of IBM at a price of \$ 100. Put options with three months to expiration, and an exercise price of \$ 100 are available at a premium of \$ 10.16 per share. Let us consider the terminal profit from the investment if Bob writes one put contract.

If the terminal stock price exceeds \$ 100, the puts will not be exercised. Bob's profit from the short stock position will be $(100 - S_T) \times 100$. After factoring in the premium for the options, the overall profit will be

$$\pi = (100 - S_T) \times 100 + 10.16 \times 100$$

The breakeven stock price is $100 + 10.16 = \$\ 110.16$. The maximum loss is unbounded. If the terminal stock price, S_T, is less than \$ 100, the puts will be exercised. The profit from the strategy will be

$$\pi = (10.16 + 100 - 100) \times 100 = \$\ 1{,}016$$

This represents the maximum profit.

11.6 Writing a Covered Call

This strategy entails the sale of a call option for every share that is owned by the investor. There are two possibilities. The calls can be sold at the time of acquisition of the shares. Or else, the purchase of the shares can precede the sale of the options. The first strategy is called *'Buy-writing'*, while the second is termed as *'Over-writing'*.

The profit from the long stock is $S_T - S_t$, while that from the call is $C_t - \text{Max}[0, S_T - X]$. Thus the total profit from the strategy is

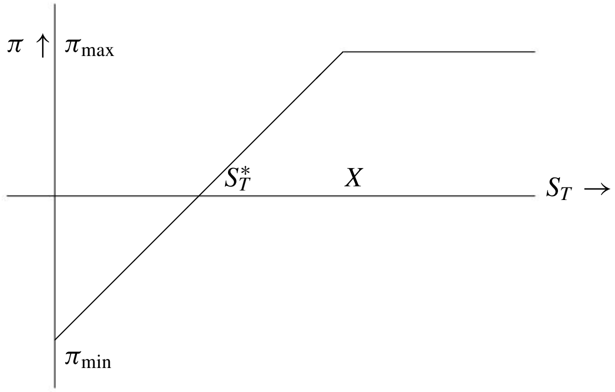
$$S_T - S_t + C_t - \text{Max}[0, S_T - X] \quad (11.2)$$

$$\text{If } S_T \geq X, \pi = X - S_t + C_t, \text{ whereas if } S_T < X, \pi = S_T - S_t + C_t$$

As the terminal stock price goes below X, the profit declines dollar for dollar. The maximum loss occurs when the terminal stock price is zero, and is equal to $C_t - S_t$. The breakeven price is $S_t - C_t$. The maximum profit is equal to $X - S_t + C_t$.

The profit diagram may be depicted as shown in Fig. 11.8.

Figure 11.8



Notice that the profit diagram looks like the one for a put writer.

11.6.1 Example

Maureen Smith owns 100 shares of IBM which she acquired at a price of \$ 100. She has simultaneously sold a call options contract with an exercise price of \$ 100, and three months to expiration, for a premium of \$ 7.22 per share. The profit from the stock position is $S_T - 100$, while that from the call is 7.22 – Max[0, $S_T - 100$]. If the terminal stock price is below \$ 100, the counterparty will not exercise the calls. The profit for Maureen in this scenario will therefore be

$$\pi = (S_T - 100) \times 100 + 7.22 \times 100$$

The range for the profit on a per share basis is

$$-92.78 \leq \pi \leq 7.22$$

If S_T were to be greater than \$ 100, the share will be called away, and the profit for Maureen will be capped at \$ 7.22 per share. If Maureen had not sold the calls, the profit would have been $S_T - 100$. If the stock price were to exceed \$ 107.22 or $X + C_t$, she will regret the fact that she had sold the calls. Thus $X + C_t$ is called the *point of regret*.

Consider the maximum profit for the strategy which is $X - S_t + C_t$. However high the stock price may rise, this represents the maximum profit from the strategy. Thus, in exchange for the option premium, the call writer accepts an upper limit on the profit. The option premium however serves to reduce the magnitude of the loss if the stock price were to decline, as compared to a standalone long stock position.

The profit from the position for various values of the terminal stock price is as depicted in Table 11.5.

Table 11.5 **Profit from the Covered Call for Various Values of the Terminal Stock Price**

Stock Price	Payoff from Call	Profit/Loss from Call	Profit/Loss from Stock	Total Profit/Loss
80	0	722	(2,000)	(1,278)
85	0	722	(1,500)	(778)
90	0	722	(1,000)	(278)
95	0	722	(500)	222
100	0	722	0	722
101	(100)	622	100	722
105	(500)	222	500	722
107.22	(722)	0	722	722
110	(1,000)	(278)	1,000	722
115	(1,500)	(778)	1,500	722
120	(2,000)	(1,278)	2,000	722

As can be seen, below a stock price of $ 100, the loss from the stock is reduced by $ 722 due to the receipt of the premium from the call. As the call goes into the money, the stock is called away and the total profit is $ 722, which is the premium received from the call. The point of regret is $ 107.22. Beyond this stock price the investor will regret the fact that he wrote the call, since he will be unable to realize the profit from the stock.

11.6.2 In-the-money, At-the-Money, and Out-of-the-Money

Assume that the initial stock price is $ 100. Consider three call options with three months to expiration, and exercise prices of $ 90, $ 100, and $ 110 respectively. The corresponding premia are $ 13.71, $ 7.22, and $ 3.22. Let us consider covered call strategies with each of these contracts.

An at the money option would be favoured by a neutral investor, because the maximum return is obtained if the stock price were to remain at its initial level. A strategy with an out of the money option, is bullish by design, because the maximum profit is realized if the stock price rises above its current level to reach a value equal to the exercise price.

In the above illustration, the stock price must rise from a current level of $ 100 to reach the exercise price of $ 110, in order for the investor to obtain the maximum profit. The total profit for 100 shares is $ 1,322, if the stock price

Table 11.6 Option Premia for Varying Degrees of Moneyness

Exercise Price	Maximum Profit
$ 90	$ 3.71
$ 100	$ 7.22
$ 110	$ 13.22

rises to $ 110. Out of this $ 1,000 arises from the stock component, and $ 322 from the option component. While the maximum profit from the out of the money call is the greatest, there is a trade off if the stock price were to decline. This is because, the out of the money call has the smallest premium, and consequently provides the least protection if the share were to fall in value. In our illustration, the at-the-money call provides a protection of $ 7.22 per share in a declining market, whereas the out of the money call gives a protection of only $ 3.22 per share.

11.7 Spreads

A spread is a strategy that involves taking a position in two or more options of the same type. That is all the options must be calls, or all of them should be puts.

11.7.1 Vertical Spreads

To create a vertical spread the investor must buy an option with one exercise price, and sell another with a different exercise price. Both the options must have the same time to expiration. Vertical spreads are also known as *Strike* or *Money* or *Price* spreads.

11.7.2 Horizontal Spreads

To create this kind of a spread, the investor must buy an option with a given exercise price and expiration month, and sell another option with the same exercise price, but with a different expiration month. Horizontal spreads are also known as *Time* or *Calendar* spreads.

11.7.3 Diagonal Spreads

These are strategies where you buy an option with a given exercise price and time to maturity, and sell another option with a different exercise price and time to maturity.

We will focus only on vertical spreads.

11.8 Bull Spreads

11.8.1 With Calls

To create a bull spread with calls, the investor has to buy a call option with an exercise price = X_1, and sell a call on the same asset with an exercise price = X_2, where $X_1 < X_2$. Let $C_{t,1}$ be the premium for the first option, and $C_{t,2}$, the premium for the second. Since $X_1 < X_2$, $C_{t,1}$ must be $> C_{t,2}$.

The initial cash flow $= -C_{t,1} + C_{t,2}$ which is < 0. Thus a bull spread with calls involves a net investment.

Let us consider the payoff from a bull spread at expiration.

Table 11.7 Payoffs from a Bull Spread with Calls

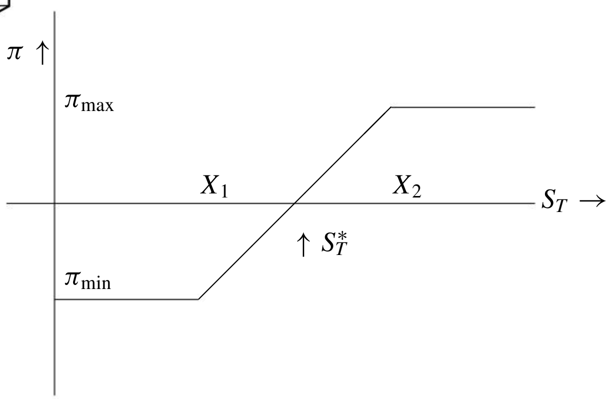
Terminal Price Range	Payoff from Long Call	Payoff from Short Call	Total Payoff
$S_T < X_1$	0	0	0
$X_1 < S_T < X_2$	$S_T - X_1$	0	$S_T - X_1$
$S_T > X_2$	$S_T - X_1$	$-(S_T - X_2)$	$X_2 - X_1$

To calculate the profit in each scenario, we must subtract the initial investment from the payoff. The maximum payoff is obviously $X_2 - X_1$. The minimum payoff = 0. Thus, the maximum profit is $X_2 - X_1 - C_{t,1} + C_{t,2}$, and the maximum loss is $-C_{t,1} + C_{t,2}$. A bull spread, therefore, limits the upside potential while putting a limit on the downside risk. The breakeven stock price is given by,

$$\pi = S_T^* - X_1 - C_{t,1} + C_{t,2} = 0$$
$$\Rightarrow S_T^* = X_1 + C_{t,1} - C_{t,2} \tag{11.3}$$

The profit diagram may be depicted as shown in Fig. 11.9.

Figure 11.9



Example Amy Hepburn has bought a call option with an exercise price of $ 90 at a premium of $ 13.71, and sold a call option with the same maturity, but with a higher exercise price of $ 110, for a premium of $ 3.22. The position obviously represents a bull spread. The initial investment is $ 13.71 - $ 3.22 = $ 10.49. This is also the maximum loss from the strategy. The breakeven point is given by

$$S_T^* - 90 - 10.49 = 0 \Rightarrow S_T^* = 100.49$$

The maximum payoff from the spread is the difference between the two exercise prices, which is $ 20 in this case. The maximum profit is $ 20 – $ 10.49 = $ 9.51. The maximum profit is obtained for all values of the terminal stock price equal to or greater than $ 110.

Table 11.8 **Profit from the Bull Spread for Various Values of the Terminal Stock Price**

Stock Price	Payoff from Long Call	Payoff from Short Call	Payoff from the Spread	Total Profit/Loss
80	0	0	0	(1,049)
85	0	0	0	(1,049)
90	0	0	0	(1,049)
95	500	0	500	(549)
100	1,000	0	1,000	(49)
100.49	1,049	0	1,049	0
105	1,500	0	1,500	451
110	2,000	0	2,000	951
115	2,500	(500)	2,000	951
120	3,000	(1,000)	2,000	951

The maximum loss is $ 1,049 which is the net premium paid for the spread. $ 100.49 is the breakeven point. The maximum profit from the spread is $ 951, which is equal to the difference between the two exercise prices, less the net premium paid. That is $ 951 = 100 × (110 – 90 – 10.49).

Bull Spread or Long Call A long call position limits the investor's loss to the premium that is paid at the outset, without imposing an upper bound on the profit from the strategy. The question naturally arises as to what makes an investor like Amy prefer a bull spread to a long call, considering that the spread can yield only a limited profit.

Let us take Amy's example. If she had bought a call option with X = $ 90, she would have made an investment of $ 13.71 per share. The bull spread however limits her investment to $ 10.49 per share, because there is an inflow from the option that is sold. An investor like Amy may be bullish, but may not consider the additional investment to be warranted, considering her expectations from the stock.

11.8.2 With Puts

To create a bull spread with puts, the investor has to buy a put option with an exercise price equal to X_1, and sell a put on the same asset with an exercise price equal to X_2, where $X_1 < X_2$. Let $P_{t,1}$ be the premium for the first option, and $P_{t,2}$ the premium for the second. Since $X_1 < X_2$, $P_{t,1}$ must be less than $P_{t,2}$.

The initial cash flow = $-P_{t,1} + P_{t,2}$ which is greater than zero. Thus a bull spread with puts leads to an inflow at inception.

Let us analyze the payoff from the spread at expiration.

Table 11.9 **Payoffs from a Bull Spread with Puts**

Terminal Price Range	Payoff from Long Put	Payoff from Short Put	Total Payoff
$S_T < X_1$	$X_1 - S_T$	$-(X_2 - S_T)$	$X_1 - X_2$
$X_1 < S_T < X_2$	0	$-(X_2 - S_T)$	$S_T - X_2$
$S_T > X_2$	0	0	0

To calculate the profit in each scenario, we must add the initial inflow to the payoff. The minimum payoff is obviously $X_1 - X_2$. The maximum payoff is zero. Thus the maximum loss is $X_1 - X_2 - P_{t,1} + P_{t,2}$, and the maximum profit is $-P_{t,1} + P_{t,2}$. As is to be expected, the upside potential is limited, as is the downside risk. The breakeven stock price is given by

$$S_T^* - X_2 - P_{t,1} + P_{t,2} = 0$$
$$\Rightarrow S_T^* = X_2 + P_{t,1} - P_{t,2} \quad (11.4)$$

Example Stacey Smith has bought a put option with an exercise price of \$ 90 at a premium of \$ 1.49, and sold a put option with the same maturity, but with a higher exercise price of \$ 110, for a premium of \$ 10.51. The initial inflow is \$ 10.51 – \$ 1.49 = \$ 9.02. This represents the maximum profit from the strategy. The breakeven point is given by

$$S_T^* - 110 - 1.49 + 10.51 = 0 \Rightarrow S_T^* = \$\ 100.98$$

The minimum payoff from the strategy is 90-110 =-\$ 20. The maximum loss is therefore $-20 + 9.02 = -\$\ 10.98$.

11.9 Bear Spreads

11.9.1 With Calls

To create a bear spread with calls, the investor has to sell a call option with an exercise price of X_1, and buy a call on the same asset with an exercise price of X_2, where $X_1 < X_2$. Let the respective prices of the options be $C_{t,1}$ and $C_{t,2}$. We know that $C_{t,1} > C_{t,2}$.

The initial cash flow $= -C_{t,2} + C_{t,1}$, which is > 0. Thus a bear spread with calls leads to an inflow at inception.

Let us consider the payoff from the spread at expiration.

Table 11.10 **Payoffs from a Bear Spread with Calls**

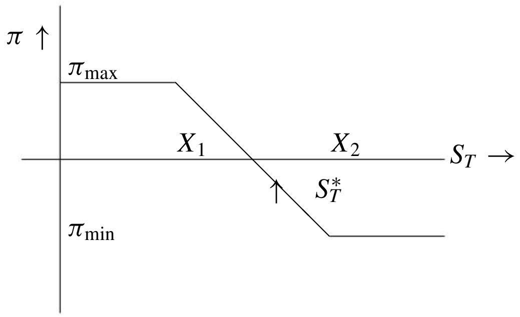
Terminal Price Range	Payoff from Long Call	Payoff from Short Call	Total Payoff
$S_T < X_1$	0	0	0
$X_1 < S_T < X_2$	0	$-(S_T - X_1)$	$X_1 - S_T$
$S_T > X_2$	$S_T - X_2$	$-(S_T - X_1)$	$X_1 - X_2$

The maximum payoff is obviously zero. The minimum payoff is $X_1 - X_2$. Thus, the maximum profit is $C_{t,1} - C_{t,2}$, while the maximum loss is $X_1 - X_2 + C_{t,1} - C_{t,2}$. Thus, just like the bull spread, the bear spread limits the upside potential while putting a limit on the downside risk. The breakeven stock price is given by

$$\pi = X_1 - S_T^* + C_{t,1} - C_{t,2} = 0 \Rightarrow S_T^* = X_1 + C_{t,1} - C_{t,2} \qquad (11.5)$$

The profit diagram may be depicted as follows.

Figure 11.10



Example Kevin Long has sold a call option with an exercise price of \$ 90 at a premium of \$ 13.71, and bought a call with the same maturity, but with an exercise price of \$ 110, for a premium of \$ 3.22. The initial cash inflow is \$ 13.71 – \$ 3.22 = \$ 10.49. This is also the maximum profit from the strategy. The breakeven point is given by

$$90 - S_T^* + 10.49 = 0 \Rightarrow S_T^* = \$\ 100.49$$

The minimum payoff from the strategy is the difference between the two exercise prices which is –\$ 20 in this case. The maximum loss is therefore \$ 10.49 – \$ 20 = –\$ 9.51. The maximum loss occurs for all values of the stock price equal to or greater than \$ 110.

11.9.2 With Puts

To create a bear spread with puts, the investor has to sell a put option with an exercise price of X_1 and buy a put on the same asset with an exercise price of X_2, where $X_1 < X_2$. Let $P_{t,1}$ be the price of the first option, and $P_{t,2}$ the price of the second. Since $X_1 < X_2$, $P_{t,1} < P_{t,2}$.

The initial cash flow = $-P_{t,2} + P_{t,1}$, which is less than zero. Thus the strategy entails a cash outflow at inception.

Let us analyze the payoff from the spread at expiration (Table 11.11).

Table 11.11 **Payoffs from a Bear Spread with Puts**

Terminal Price Range	Payoff from Long Put	Payoff from Short Put	Total Payoff
$S_T < X_1$	$X_2 - S_T$	$-(X_1 - S_T)$	$X_2 - X_1$
$X_1 < S_T < X_2$	$(X_2 - S_T)$	0	$X_2 - S_T$
$S_T > X_2$	0	0	0

The minimum payoff is obviously zero, while the maximum payoff is $X_2 - X_1$. Thus the maximum loss is $P_{t,1} - P_{t,2}$, while the maximum profit is $X_2 - X_1 + P_{t,1} - P_{t,2}$. The breakeven stock price is given by

$$X_2 - S_T^* - P_{t,2} + P_{t,1} = 0$$

$$\Rightarrow S_T^* = X_2 - P_{t,2} + P_{t,1} \tag{11.6}$$

Example Shirlene Kennedy has sold a put option with an exercise price of \$ 90 at a premium of \$ 1.49, and bought a put option with an exercise price of \$ 110, for a premium of \$ 10.51. The initial investment is \$ 10.51 – \$ 1.49 = \$ 9.02. This also represents the maximum loss from this strategy. The breakeven point is given by

$$110 - S_T^* - 9.02 = 0 \Rightarrow S_T^* = \$\ 100.98$$

The maximum payoff from the strategy is \$ 20. Consequently the maximum profit is 20 – 9.02 = \$ 10.98.

11.10 The Convexity Property

Before we go on to analyze the next strategy, we need to establish an important result that must be satisfied by both call and put options.

Consider three exercise prices X_1, X_2, and X_3 such that $X_1 < X_2 < X_3$. Let $X_2 = wX_1 + (1 - w)X_3$. That is, X_2 is a weighted average of the other two exercise prices. If $C_{t,1}$, $C_{t,2}$, and $C_{t,3}$ represent the prices of the corresponding calls, we can show that

$$C_{t,2} \leq wC_{t,1} + (1 - w)C_{t,3} \tag{11.7}$$

11.10.1 Proof

Assume that $C_{t,2} > wC_{t,1} + (1-w)C_{t,3}$

$$\Rightarrow C_{t,2} - wC_{t,1} - (1-w)C_{t,3} > 0$$

Consider the following strategy. Sell a call with $X = X_2$; buy w calls with $X = X_1$; and buy $(1-w)$ calls with $X = X_3$. The initial cash flow is $C_{t,2} - wC_{t,1} - (1-w)C_{t,3}$ which by assumption is positive.

Let us now consider the payoffs from the portfolio at expiration (Table 11.12).

Table 11.12 Payoffs from Portfolio at Expiration

Terminal Price Range	**Payoff from Calls with** $X = X_1$	**Payoff from Calls with** $X = X_3$	**Payoff from Call with** $X = X_2$	**Total Payoff**
$S_T < X_1$	0	0	0	0
$X_1 < S_T < X_2$	$w(S_T - X_1)$	0	0	$w(S_T - X_1)$
$X_2 < S_T < X_3$	$w(S_T - X_1)$	0	$-(S_T - X_2)$	$(w-1)(S_T - X_1)$ $+(X_2 - X_1)$
$S_T > X_3$	$w(S_T - X_1)$	$(1-w)(S_T - X_3)$	$-(S_T - X_2)$	0

Consider the total payoff for the price range $X_2 < S_T < X_3$, which is

$$(w-1)(S_T - X_1) + (X_2 - X_1)$$

$$(w-1) = \frac{X_1 - X_2}{X_3 - X_1}$$

$$\Rightarrow (w-1)(S_T - X_1) + (X_2 - X_1) = \left[\frac{X_1 - X_2}{X_3 - X_1}\right](S_T - X_1) + (X_2 - X_1)$$

$$= (X_2 - X_1)\left[1 - \frac{S_T - X_1}{X_3 - X_1}\right] \geq 0$$

Since the terminal cash flow in every scenario is non-negative, our assumption of a positive cash flow at inception is an indication of an arbitrage profit. Thus, to preclude arbitrage, we require that

$$C_{t,2} - wC_{t,1} - (1-w)C_{t,3} \leq 0$$

$$\Rightarrow C_{t,2} \leq wC_{t,1} + (1-w)C_{t,3}$$

A similar condition must hold for put options.

11.11 Butterfly Spread

11.11.1 With Calls

This strategy requires the investor to take a position in four options, with three different exercise prices. A long butterfly spread with calls requires the investor to buy an in the money as well as an out of the money call, and sell two calls which are at the money. Let us denote the exercise prices by X_1, X_2, and X_3, where $X_1 < X_2 < X_3$. The exercise prices are usually chosen such that $X_2 = \frac{(X_1 + X_3)}{2}$. If we denote the corresponding option prices by $C_{t,1}$, $C_{t,2}$, and $C_{t,3}$, then from the convexity property

$$C_{t,2} < \frac{(C_{t,1} + C_{t,3})}{2} \Rightarrow 2C_{t,2} < C_{t,1} + C_{t,3}$$

Thus a long butterfly entails a net investment equal to $C_{t,1} + C_{t,3} - 2C_{t,2}$.

Let us consider the payoffs from the strategy at expiration (Table 11.13).

Table 11.13 **Payoffs from a Long Butterfly Spread**

Terminal Price Range	Payoff from Call with $X = X_1$	Payoff from Call with $X = X_3$	Payoff from Calls with $X = X_2$	Total Payoff
$S_T < X_1$	0	0	0	0
$X_1 < S_T < X_2$	$S_T - X_1$	0	0	$S_T - X_1$
$X_2 < S_T < X_3$	$S_T - X_1$	0	$-2(S_T - X_2)$	$X_3 - S_T$
$S_T > X_3$	$S_T - X_1$	$S_T - X_3$	$-2(S_T - X_2)$	0

The minimum payoff is zero, which is realized if the terminal stock price is either below the lowest of the three exercise prices, or above the highest. In these scenarios the profit is equal to the initial investment, that is

$$\pi_{\min} = 2C_{t,2} - C_{t,1} - C_{t,3} \tag{11.8}$$

This amount represents the maximum loss from the strategy.

If $X_1 < S_T < X_2, \pi = S_T - X_1 + 2C_{t,2} - C_{t,1} - C_{t,3}$

If $X_2 < S_T < X_3, \pi = X_3 - S_T + 2C_{t,2} - C_{t,1} - C_{t,3}$

The maximum profit is realized when $S_T = X_2$, and is equal to

$$X_2 - X_1 + 2C_{t,2} - C_{t,1} - C_{t,3} \quad \text{or} \quad X_3 - X_2 + 2C_{t,2} - C_{t,1} - C_{t,3}$$

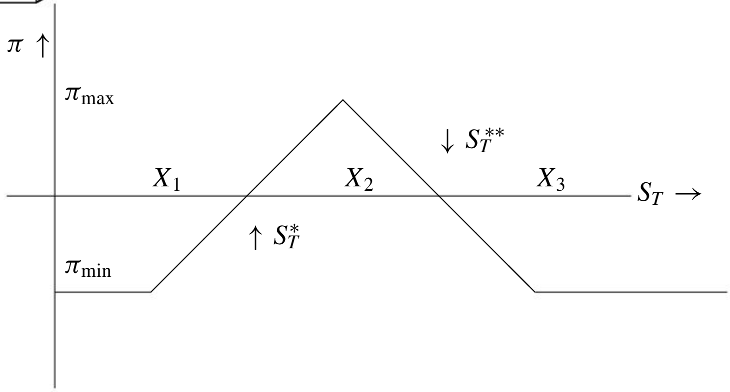
There are two breakeven prices S_T^* and S_T^{**}

$$S_T^* = X_1 - 2C_{t,2} + C_{t,1} + C_{t,3} \tag{11.9}$$

$$S_T^{**} = X_3 + 2C_{t,2} - C_{t,1} - C_{t,3} \tag{11.10}$$

The profit diagram may be depicted in Fig. 11.11.

Figure 11.11



Example Caroline Jones has decided to take a long position in a butterfly spread. She wants to sell two options on IBM with an exercise price of \$ 100, and buy two options on the same stock with exercise prices of \$ 90 and \$ 110 respectively. All the options have three months to expiration. The premia of the options are listed in Table 11.14.

Table 11.14 **Premia for Call Options with Varying Exercise Prices**

Exercise Price	Call Premium
\$ 90	\$ 13.71
\$ 100	\$7.22
\$ 110	\$ 3.20

The initial investment is $13.71 + 3.20 - 2 \times 7.22 = \$\ 2.47$. This is also the maximum possible loss from the strategy. The maximum profit is obtained at a terminal stock price of \$ 100, and is given by

$$100 - 90 + 2 \times 7.22 - 13.71 - 3.2 = \$\ 7.53$$

There are two breakeven prices

$$S_T^* = 90 + 2.47 = \$\ 92.47 \text{ and } S_T^{**} = 110 - 2.47 = \$\ 107.53$$

The maximum loss is equal to the net premium paid, which is $100 \times 2.47 =$ \$ 247. There are two breakeven prices, \$ 92.47 and \$ 107.53.

11.11.2 With Puts

A long butterfly spread can also be set up using put options. The investor should buy puts with exercise prices of X_1 and X_3, and sell two puts with an exercise price of X_2, where $X_1 < X_2 < X_3$ and $X_2 = \dfrac{(X_1 + X_3)}{2}$. If we denote

Table 11.15 Profit from the Butterfly Spread for Various Values of the Terminal Stock Price

Stock Price	Payoff from Long Call with X = 90	Payoff from Long Call with X = 110	Payoff from Short Calls with X = 100	Total Profit/Loss
80	0	0	0	(247)
85	0	0	0	(247)
90	0	0	0	(247)
92.47	247	0	0	0
95	500	0	0	253
100	1,000	0	0	753
105	1,500	0	(1,000)	253
107.53	1,753	0	(1,506)	0
110	2,000	0	(2,000)	(247)
115	2,500	500	(3,000)	(247)
120	3,000	1,000	(4,000)	(247)

the corresponding option prices by $P_{t,1}$, $P_{t,2}$, and $P_{t,3}$, then from the convexity property

$$2P_{t,2} < P_{t,1} + P_{t,3}$$

Thus the strategy entails a net initial investment. The magnitude of this investment represents the maximum loss from the position, which arises if the terminal stock price is either below the lowest exercise price, or above the highest exercise price.

The maximum profit = $X_2 - X_1 + 2P_{t,2} - P_{t,1} - P_{t,3}$ or $X_3 - X_2 + 2P_{t,2} - P_{t,1} - P_{t,3}$. The breakeven stock prices are

$$S_T^* = X_1 - 2P_{t,2} + P_{t,1} + P_{t,3}$$

and

$$S_T^{**} = X_3 + 2P_{t,2} - P_{t,1} - P_{t,3} \quad (11.11)$$

11.12 The Condor

This strategy also entails the assumption of positions in four options. Consider four options with exercise prices X_1, X_2, X_3, and X_4, such that $X_1 < X_2 < X_3 < X_4$. Let $X_2 = \frac{(X_1 + X_3)}{2}$ and $X_3 = \frac{(X_2 + X_4)}{2}$. To set up a long condor, the investor has to buy two call options with exercise prices of X_1 and X_4 respectively, and sell two call options with exercise prices of X_2 and X_3 respectively.

The initial cash flow is $C_{t,2} + C_{t,3} - C_{t,1} - C_{t,4}$. From the convexity property of options

$$C_{t,2} < \frac{(C_{t,1} + C_{t,3})}{2}$$

and
$$C_{t,3} < \frac{(C_{t,2} + C_{t,4})}{2}$$
$$\Rightarrow 2C_{t,2} + 2C_{t,3} < C_{t,1} + C_{t,3} + C_{t,2} + C_{t,4}$$
$$\Rightarrow C_{t,2} + C_{t,3} - C_{t,1} - C_{t,4} < 0 \qquad (11.12)$$

Thus the long condor requires a net initial investment. Let us consider the payoffs from the strategy at expiration.

Table 11.16 **Payoffs from a Long Condor**

Terminal Price Range	Payoff from Call with $X = X_1$	Payoff from Call with $X = X_4$	Payoff from Call with $X = X_2$	Payoff from Call with $X = X_3$	Total Payoff
$S_T < X_1$	0	0	0	0	0
$X_1 < S_T < X_2$	$S_T - X_1$	0	0	0	$S_T - X_1$
$X_2 < S_T < X_3$	$S_T - X_1$	0	$-(S_T - X_2)$	0	$X_2 - X_1$
$X_3 < S_T < X_4$	$S_T - X_1$	0	$-(S_T - X_2)$	$-(S_T - X_3)$	$X_4 - S_T$
$S_T > X_4$	$S_T - X_1$	$S_T - X_4$	$-(S_T - X_2)$	$-(S_T - X_3)$	0

The minimum payoff from the position is zero, which occurs if $S_T < X_1$ or if $S_T > X_4$. Consequently the maximum loss is equal to the initial investment, which is $C_{t,2} + C_{t,3} - C_{t,1} - C_{t,4}$. The maximum payoff is $X_2 - X_1$, and thus the maximum profit is

$$X_2 - X_1 + C_{t,2} + C_{t,3} - C_{t,1} - C_{t,4} \qquad (11.13)$$

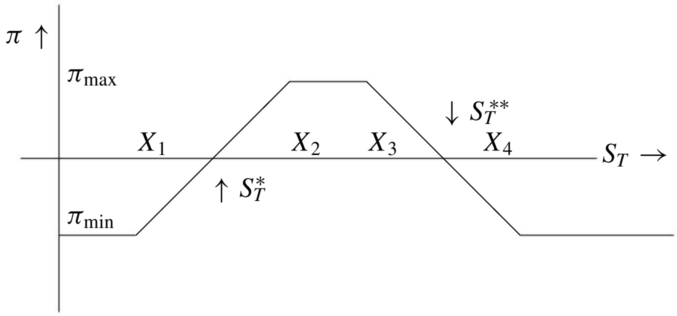
There are two breakeven points, given by

$$S_T{}^* = X_1 - C_{t,2} - C_{t,3} + C_{t,1} + C_{t,4} \qquad (11.14)$$

and
$$S_T^{**} = X_4 + C_{t,2} + C_{t,3} - C_{t,1} - C_{t,4} \qquad (11.15)$$

The profit diagram may be depicted in Fig. 11.12.

Figure 11.12



Example Andy Hunt has taken a long position in a condor by buying two call options on IBM with exercise prices of \$95 and \$ 110 respectively, and selling two call options with exercise prices of \$ 100 and \$ 105 respectively. All the options have three months to expiration. The premia of the options are listed in Table 11.17.

Table 11.17 Premia for Call Options with Varying Exercise Prices

Exercise Price	Call Premium
\$ 95	\$ 10.16
\$ 100	\$7.22
\$ 105	\$ 4.92
\$ 110	\$ 3.20

The initial investment is $10.16 + 3.20 - 7.22 - 4.92 = \$\ 1.22$. This is also the maximum possible loss from the strategy. The maximum profit is obtained in the price range $100 \leq S_T \leq 105$, and is equal to $105 - 100 - 1.22 = \$\ 3.78$. There are two breakeven prices:

$$S_T^* = 95 + 1.22 = \$\ 96.22 \text{ and } S_T^{**} = 110 - 1.22 = \$\ 108.78$$

11.12.1 With Puts

A long condor can also be set up using put options. Consider four options with exercise prices $X_1, X_2, X_3,$ and X_4, such that $X_1 < X_2 < X_3 < X_4$. Let $X_2 = \frac{(X_1 + X_3)}{2}$ and $X_3 = \frac{(X_2 + X_4)}{2}$. To set up a long condor, the investor has to buy two put options with exercise prices of X_1 and X_4 respectively, and sell two put options with exercise prices of X_2 and X_3 respectively.

11.13 Combinations

Combinations are strategies that involve taking positions in both calls and puts on the same stock.

11.13.1 The Straddle

A long straddle requires the investor to buy a call as well as a put option on a stock, with the same exercise price and expiration date. The position will yield a profit if the stock goes up or down substantially. The long call will pay off if the stock goes up in value, while the long put will pay off if the stock declines in value. The position is suitable for an investor who is anticipating a large price move, but is unsure about its direction.

The initial investment, which is the sum of the premia, is $C_t + P_t$.

Let us consider the payoff Table 11.18.

Table 11.18 **Payoffs from a Long Straddle**

Terminal Price Range	Payoff from Call	Payoff from Put	Total Payoff
$S_T < X$	0	$X - S_T$	$X - S_T$
$S_T > X$	$S_T - X$	0	$S_T - X$

If $S_T > X, \pi = S_T - X - P_t - C_t$. Thus the maximum profit is unlimited in this region. If $S_T < X, \pi = X - S_T - P_t - C_t$. The maximum profit in this region is $X - P_t - C_t$, which arises if the stock price declines to zero. Above X, the profit increases dollar for dollar with the stock price. Below X, as S_T goes from X to zero, the profit increases dollar for dollar. The maximum loss occurs at $S_T = X$, and is equal to $-(P_t + C_t)$. There are two breakeven points:

$$X - S_T^* - P_t - C_t = 0$$

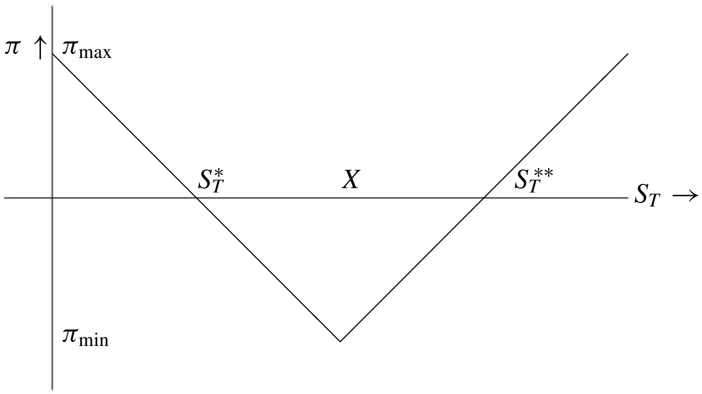
$$\Rightarrow S_T^* = X - P_t - C_t, \text{ and} \tag{11.16}$$

$$S_T^{**} - X - P_t - C_t = 0$$

$$\Rightarrow S_T^{**} = X + P_t + C_t \tag{11.17}$$

The profit diagram may be depicted as follows.

Figure 11.13



Example Mathew Henderson has bought a call option and a put option on IBM. Both options have an exercise price of \$ 100, and have three months to expiration. The premium for the call is \$ 7.22, while that for the put is \$ 4.75. Thus the initial investment is $7.22 + 4.75 = \$ 11.97$. This amount represents the maximum potential loss from the strategy. If the stock price rises above the exercise price, the profit increases dollar for dollar. The profit for this price range is given by

$$S_T - 100 - 11.97 = S_T - 111.97$$

The maximum profit is obviously unbounded.

Table 11.19 Profit from the Straddle for Various Values of the Terminal Stock Price

Stock Price	Payoff from Long Call	Payoff from Long Put	Total Payoff	Total Profit/Loss
80	0	2,000	2,000	803
85	0	1,500	1,500	303
88.03	0	1,197	1,197	0
90	0	1,000	1,000	(197)
95	0	500	500	(697)
100	0	0	0	(1,197)
105	500	0	500	(697)
110	1,000	0	1,000	(197)
111.97	1,197	0	1,197	0
115	1,500	0	1,500	303
120	2,000	0	2,000	803

If the terminal stock price declines below the exercise price, the profit again increases dollar for dollar. The profit in this price range is given by

$$100 - S_T - 11.97 = 88.03 - S_T$$

The maximum profit is obviously \$ 88.03. There are two breakeven points, $S_T^* = \$\ 111.97$, and $S_T^{**} = \$\ 88.03$.

The maximum loss occurs at a stock price of \$ 100 and is equal to the premium paid for the two options, that is $100 \times (7.22 + 4.75)$. There are two breakeven prices, \$ 88.03 and \$ 111.97.

11.14 The Strangle

This strategy, once again, requires the investor to buy a call and a put on the same stock. However, although the two options must have the same time to expiration, their exercise prices should be different. Let the call have an exercise price $= X_1$, and the put, an exercise price $= X_2$. There are two possibilities, $X_1 > X_2$, and $X_1 < X_2$. Consequently we have two types of strangles, called out-of-the money strangles and in-the-money strangles respectively. The initial cash flow, which is the sum of the premia, is given by $-(P_t + C_t)$.

11.14.1 An Out-of-the-Money Strangle

Let us consider the payoff Table 11.20 from an out-of-the-money strangle.

Table 11.20 **Payoffs from an Out-of-the-Money Long Strangle**

Terminal Price Range	Payoff from Call	Payoff from put	Total Payoff
$S_T < X_2$	0	$X_2 - S_T$	$X_2 - S_T$
$X_2 < S_T < X_1$	0	0	0
$S_T > X_1$	$S_T - X_1$	0	$S_T - X_1$

If $S_T < X_2, \pi = X_2 - S_T - P_t - C_t$. The maximum profit in this region $= X_2 - P_t - C_t$. If $X_2 < S_T < X_1, \pi = -(P_t + C_t)$. If $S_T > X_1, \pi = S_T - X_1 - P_t - C_t$. The maximum profit in this region is unbounded. The maximum loss is $(P_t + C_t)$, which occurs in the region $X_2 < S_T < X_1$. There are two breakeven prices:

$$X_2 - S_T^* - P_t - C_t = 0$$

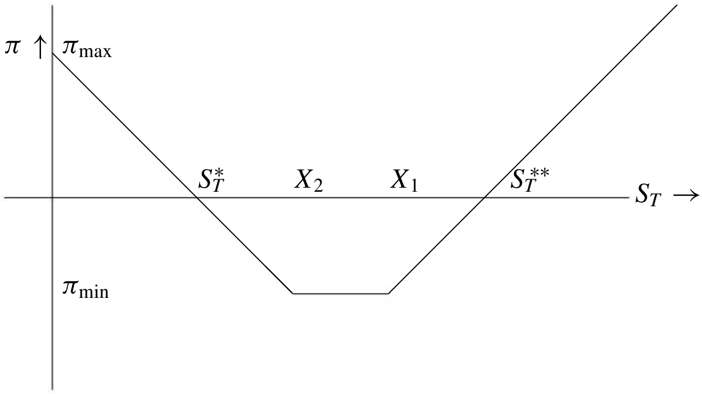
$$\Rightarrow S_T^* = X_2 - P_t - C_t \tag{11.18}$$

$$S_T^{**} - X_1 - P_t - C_t = 0$$

$$\Rightarrow S_T^{**} = X_1 + P_t + C_t \tag{11.19}$$

The profit diagram may be depicted in Fig. 11.14.

Figure 11.14



11.14.2 Example

Ruth Kelly has bought a put option on IBM with an exercise price of \$ 95, and a call option on the same stock with an exercise price of \$ 105. The stock is currently priced at \$ 100, and both options have three months to expiration. The put premium is \$ 2.81 and the call premium is \$ 4.92. The initial investment is $2.81 + 4.92 =$ \$ 7.73, which is also the maximum potential loss. The position leads to a loss equal in magnitude to this amount, if the stock price at expiration is between \$ 95 and \$ 105. As the stock price declines below \$ 95, the profit increases dollar for dollar. The profit in this price range is given by $95 - S_T - 7.73 = \$\,87.27 - S_T$. The maximum profit in this price range is \$ 87.27 which corresponds to a stock price of zero. The breakeven stock price is \$ 87.27.

As the stock price increases above \$ 105, the profit once again increases dollar for dollar. The profit in this price range is given by $S_T - 105 - 7.73 = S_T - 112.73$. The maximum profit is obviously unbounded. The breakeven stock price is \$ 112.73.

11.14.3 An In-the-Money Strangle

Let us consider the payoff table from an in-the-money strangle.

Table 11.21 **Payoffs from an In-of-the-Money Long Strangle**

Terminal Price Range	Payoff from Call	Payoff from Put	Total Payoff
$S_T < X_1$	0	$X_2 - S_T$	$X_2 - S_T$
$X_1 < S_T < X_2$	$S_T - X_1$	$X_2 - S_T$	$X_2 - X_1$
$S_T > X_2$	$S_T - X_1$	0	$S_T - X_1$

If $S_T < X_1, \pi = X_2 - S_T - P_t - C_t$. The maximum profit in this region $= X_2 - P_t - C_t$. If $X_1 < S_T < X_2, \pi = X_2 - X_1 - (P_t + C_t)$. If $S_T > X_2$, $\pi = S_T - X_1 - P_t - C_t$. The maximum profit in this region is unbounded. The maximum loss is $(X_2 - X_1) - (P_t + C_t)$, which occurs in the region $X_1 < S_T < X_2$. There are two breakeven prices:

$$X_2 - S_T^* - P_t - C_t = 0$$
$$\Rightarrow S_T^* = X_2 - P_t - C_t \tag{11.20}$$
$$S_T^{**} - X_1 - P_t - C_t = 0$$
$$\Rightarrow S_T^{**} = X_1 + P_t + C_t \tag{11.21}$$

11.14.4 Example

Anne Smith has bought a call option on IBM with an exercise price of \$ 95, and a put option with an exercise price of \$ 105. The stock is currently priced at \$ 100, and both the stocks have three months to expiration. The call premium is \$ 10.16

and the put premium is \$ 7.33. The initial investment is 10.16 + 7.33 = \$ 17.49. The maximum possible loss is

$$105 - 95 - 17.49 = -\$\,7.49$$

As the stock price declines below \$ 95, the profit increases dollar for dollar. The profit in this price range is given by $105 - S_T - 17.49 = \$\,87.51 - S_T$. The maximum profit in this price range is \$ 87.51 which corresponds to a stock price of zero. The breakeven stock price is \$ 87.51.

As the stock price increases above \$ 105, the profit once again increases dollar for dollar. The profit in this price range is given by $S_T - 95 - 17.49 = S_T - 112.49$. The maximum profit is obviously unbounded. The breakeven stock price is \$ 112.49.

The profit diagram is identical to that for an out-of-the-money strangle.

11.15 A Strap

To go long in a strap, the investor needs to acquire calls and puts with the same exercise price and expiration date. The difference between a straddle and a strap is that, in the case of a strap, for every put that the investor buys, he needs to buy two calls. The initial investment is $2C_t + P_t$.

Let us consider the payoff Table 11.22.

Table 11.22 Payoffs from a Long Strap

Terminal Price Range	Payoff from Calls	Payoff from Put	Total Payoff
$S_T < X$	0	$X - S_T$	$X - S_T$
$S_T > X$	$2(S_T - X)$	0	$2(S_T - X)$

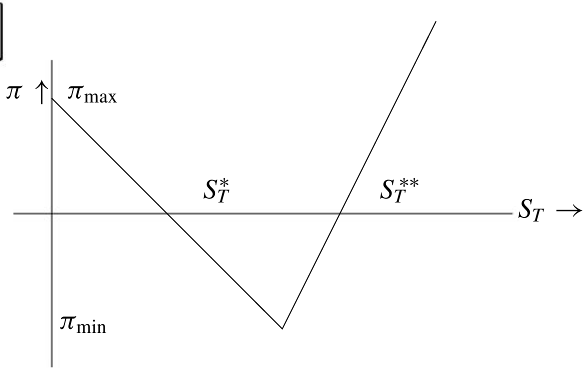
If $S_T < X, \pi = X - S_T - 2C_t - P_t$. The maximum profit in this region $= X - 2C_t - P_t$. If $S_T > X, \pi = 2S_T - 2X - 2C_t - P_t$. The maximum profit in this region is unlimited. The maximum loss occurs when $S_T = X$, and is equal to $-2C_t - P_t$, which is nothing but the initial investment. There are two breakeven points:

$$S_T^* = X - 2C_t - P_t$$

$$S_T^{**} = X + C_t + \frac{P_t}{2} \tag{11.22}$$

Consider the profit diagram shown in Fig. 11.15.

Figure 11.15



As compared to a straddle, the upside breakeven stock price is easier to reach, and the downside breakeven stock price is harder to reach. Thus by increasing the bet on a bull market, the payoff in a bull market is higher, but the payoff in a bear market is lower. Investors who use straps consider an increase in price to be more likely than a decrease.

11.15.1 Example

Mitch Andrews has bought two call options and a put option on IBM. Both options have an exercise price of \$ 100, and have three months to expiration. The premium for the call is \$ 7.22, while that for the put is \$ 4.75. Thus the initial investment is $2 \times 7.22 + 4.75 = \$\ 19.19$. This amounts to the maximum potential loss from the strategy. If the stock price rises above the exercise price, the profit increases dollar for dollar. The profit for this price range is given by

$$2S_T - 200 - 19.19 = 2S_T - 219.19$$

The maximum profit is obviously unbounded.

If the terminal stock price declines below the exercise price, the profit again increases dollar for dollar. The profit in this price range is given by

$$100 - S_T - 19.19 = 80.81 - S_T$$

The maximum profit is obviously \$ 80.81. There are two breakeven points, $S_T^* = \$\ 109.595$, and $S_T^{**} = \$\ 80.81$.

11.16 A Strip

A strip requires the investor to buy two puts and a call, with the same exercise price and time to expiration. Thus it tantamounts to a bigger bet on a bear market. The initial investment is $C_t + 2P_t$.

Let us consider the payoff Table 11.23.

If $S_T < X, \pi = 2X - 2S_T - 2P_t - C_t$. The maximum profit in this region $= 2X - 2P_t - C_t$. If $S_T > X, \pi = S_T - X - 2P_t - C_t$. The maximum profit in

Table 11.23 **Payoffs from a Long Strip**

Terminal Price Range	Payoff from Call	Payoff from Puts	Total Payoff
$S_T < X$	0	$2(X - S_T)$	$2(X - S_T)$
$S_T > X$	$S_T - X$	0	$S_T - X$

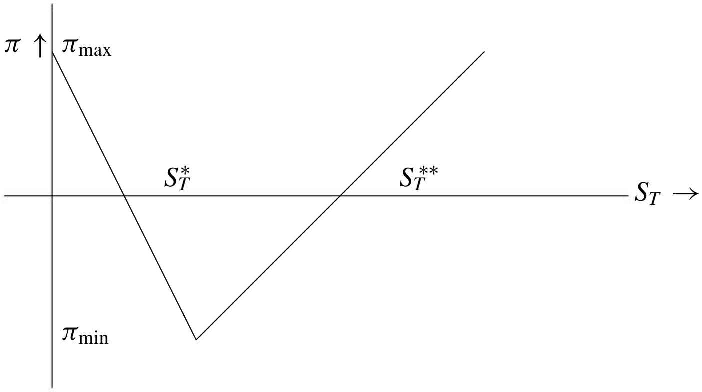
this region is unlimited. The maximum loss occurs when $S_T = X$, and is equal to $-2P_t - C_t$, which is nothing but the initial investment. There are two breakeven prices.

$$S_T^* = X - P_t - \frac{C_t}{2} \tag{11.23}$$

$$S_T^{**} = X + 2P_t + C_t$$

The profit diagram is shown in Fig. 11.16.

Figure 11.16



As compared to a straddle, the upside breakeven stock price is harder to reach, and the downside breakeven stock price is easier to reach. Thus by increasing the bet on a bear market, the payoff in a bear market is higher, but the payoff in a bull market is lower. Investors who use straps consider a decrease in price to be more likely than an increase.

11.16.1 Example

Alice Keaton has bought two put options and a call option on IBM. Both options have an exercise price of \$ 100, and have three months to expiration. The premium for the call is \$ 7.22, while that for the put is \$ 4.75. Thus the initial investment is $7.22 + 2 \times 4.75 =$ \$ 16.72. This amounts to the maximum potential loss from the strategy. If the stock price rises above the exercise price, the profit increases dollar for dollar. The profit for this price range is given by

$$S_T - 100 - 16.72 = S_T - 116.72$$

The maximum profit is obviously unbounded.

If the terminal stock price declines below the exercise price, the profit again increases dollar for dollar. The profit in this price range is given by

$$200 - 2S_T - 16.72 = 183.28 - 2S_T$$

The maximum profit is obviously \$ 183.28. There are two breakeven points, $S_T^* = \$\ 116.72$, and $S_T^{**} = \$\ 91.64$.

11.17 Box Spreads

A box spread is a combination of a bull spread with calls, and a bear spread with puts. Consider two exercise prices X_1 and X_2, such that $X_1 < X_2$. The strategy requires the investor to buy a call with an exercise price of X_1, and sell a call with an exercise price of X_2. It requires him to simultaneously sell a put with $X = X_1$, and buy a put with $X = X_2$. The initial cash flow is:

$$-C_{t,1} + C_{t,2} + P_{t,1} - P_{t,2} < 0 \qquad (11.24)$$

because $C_{t,1} > C_{t,2}$ and $P_{t,1} < P_{t,2}$.

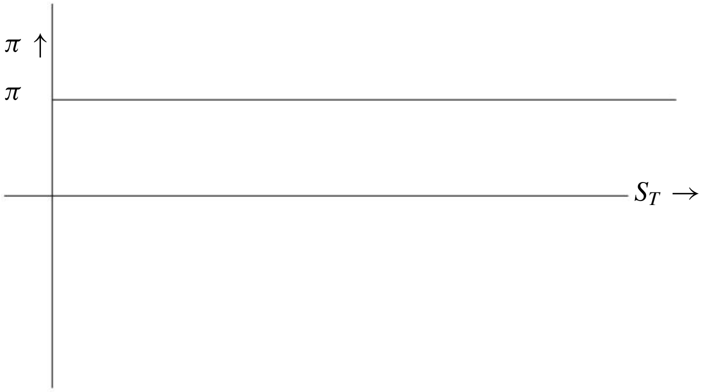
Let us consider the payoff from the position at expiration.

Table 11.24 Payoffs from a Box Spread

Terminal Price Range	Payoff from Call with $X = X_1$	Payoff from Call with $X = X_2$	Payoff from Put with $X = X_1$	Payoff from Put with $X = X_2$	Total Payoff
$S_T < X_1$	0	0	$-(X_1 - S_T)$	$X_2 - S_T$	$X_2 - X_1$
$X_1 < S_T < X_2$	$S_T - X_1$	0	0	$X_2 - S_T$	$X_2 - X_1$
$S_T > X_2$	$S_T - X_1$	$-(S_T - X_2)$	0	0	$X_2 - X_1$

As can be seen the payoff from the spread is independent of the terminal stock price. Thus the spread represents a riskless investment, and if it is correctly priced, the present value of the payoff at expiration discounted at the riskless rate, should equal the initial investment. If the present value exceeds the initial investment, the box spread will lead to an arbitrage profit. On the other hand, if the present value were to be less than the initial investment, an arbitrageur can make a riskless gain by reversing the positions, that is, by executing a short box spread.

Figure 11.17



Example Ralph Fleming has bought a call option with X = \$ 100, and a put option with X = \$ 110. He has simultaneously sold a call option with X = \$ 110, and a put option with X = \$ 100. The initial investment is

$$-7.22 + 3.22 - 10.51 + 4.75 = -\$\ 9.76$$

The terminal payoff is 110 − 100 = \$ 10. The rate of return is given by

$$10 = 9.76e^{.25r} \Rightarrow r = 4[\ln(10) - \ln(9.76)] = .10 \equiv 10\%$$

The rate of return of 10% is the value that we had used to obtain the option prices using the Black–Scholes formula. Since the options are fairly priced, it is but obvious that the box spread will yield the riskless rate that was assumed.

SUGGESTIONS FOR FURTHER READING

1. Chance D.M. *An Introduction to Derivatives & Risk Management.* Thomson; South-Western, 2004.
2. Dubofsky D.A. *Options and Financial Futures: Valuation and Uses.* McGraw-Hill, 1992.
3. Figlewski S., Silber W.L., and M.G. Subrahmanyam, *Financial Options: From Theory to Practice.* Irwin, 1990.
4. *Options: Essential Concepts & Trading Strategies*. McGraw-Hill, 1999.
5. *Understanding Stock Options.* The Options Clearing Corporation, 1994.

Concept Check

State whether the following statements are True or False.

1. A long call is a bullish strategy.
2. A long put is a bearish strategy.
3. The sale of an in-the-money call is more bearish than the sale of an out-of-the-money call.
4. Both buy-writing and overwriting are covered call strategies.
5. Buywriting may be perceived as a risky strategy whereas overwriting cannot.
6. In the case of vertical spreads the options have the same time to expiration.
7. In the case of horizontal spreads the options have the same time to expiration.
8. Bull spreads lead to finite profits as well as losses.
9. Butterfly spreads lead to finite profits as well as losses.
10. Both call as well as put options display the convexity property.
11. Bull spreads, bear spreads, and butterfly spreads may be set up with calls or with puts.
12. The straddle is suitable for investors who anticipate a large price move, but are unsure of its direction.
13. A strap is more bullish than a straddle.
14. The strip is more bearish than a straddle.
15. In the case of straps as well as strips, all the options must have the same exercise price and expiration date.
16. In the case of a strangle the call must have the same exercise price as the put.
17. In the case of a strangle, the call must have a lower exercise price than the put.
18. Both straddles and strangles entail finite losses.
19. Both straps and strips entail finite losses.
20. A box spread is a risk-less strategy.

Questions and Problems

Question-I

Paul Anka has bought 100 shares of EXXON which are currently priced at $ 85 each. He has also sold a call options contract with an exercise price of $ 90 for a premium of $ 2.75 per share.

What is the maximum payoff from this strategy? What is the point of regret?

Question-II

Discuss the following strategies: Vertical Spreads, Horizontal Spreads, and Diagonal Spreads.

Question-III

Demonstrate the convexity property for put options.

Question-IV

Set up the payoff table for a condor with put options. What is the initial cash flow? What are the breakeven points?

Question-V

Discuss the difference between an in-the-money strangle and an out-of-the-money strangle.

Question-VI

Discuss the difference between a strip and a strap.

Question-VII

Andy Harvey has bought call and put options on IBM with an exercise price of \$ 100 and six months to expiration. The call premium is \$ 8 per share while the put premium is \$ 3.75 per share.

Set up the payoff table and calculate the breakeven prices. Plot the profit diagram.

Question-VIII

Assume that Andy buys three call options for every put that he buys. Set up the payoff table and calculate the breakeven prices.

Question-IX

Assume that Andy buys three put options for every call that he buys. Set up the payoff table and calculate the breakeven prices.

Question-X

Call options with X = \$ 100 are available for a premium of \$ 10.75, while put options with the same exercise price are available at a premium of \$ 6.00.

Call options with an exercise price of \$ 107.50 are available for a premium of \$ 7.50 while put options with the same exercise price are available for a premium of \$ 9.75.

Set up a box spread. Calculate the risk-less rate of interest using continuous compounding, and assuming that the options have six months to expiration.

12

Valuation of Options

A futures contract, which entails an obligation on the part of both the long as well as the short, is relatively easy to price as compared to an option. The value of a futures contract in an arbitrage free setting can be derived by ruling out the profitability of *Cash and Carry* and *Reverse Cash and Carry* strategies. Let us recapitulate these strategies.

12.1 Cash and Carry and Reverse Cash and Carry Arbitrage Strategies

Consider a futures contract on an asset which has a current spot price of S_t. Let the riskless rate of interest be r, for the period during which the arbitrage strategy is in place. That is, r is not an annualized rate. If the futures price is greater than $S_t(1 + r)$, then an arbitrageur can profit by engaging in a cash and carry strategy. This would entail borrowing \$ S_t and buying the stock in the spot market and simultaneously going short in the futures market to sell the asset at a future date for a price of F_t. Since by assumption, $F_t > S_t(1 + r)$, where $S_t(1 + r)$ represents the amount that has to be repaid by the arbitrageur at maturity, this strategy would clearly lead to an arbitrage profit.

On the other hand, if $F_t < S_t(1 + r)$, then a *reverse cash and carry* arbitrage strategy would yield a profit. This would entail short selling the stock and lending out the proceeds and simultaneously going long in a futures contract to reacquire the asset subsequently at a price F_t, in order to cover the short position. Since by assumption $S_t(1 + r)$, which is what the arbitrageur will get when the amount lent by him is repaid, is greater than F_t, this strategy would give rise to an arbitrage profit.

Consequently, in order to preclude arbitrage, it must be the case that

$$F_t = S_t(1 + r)$$

In our analysis, we have assumed that there is no risk of default. In practice, the integrity of both the parties is ensured by imposing margins and marking the contract to market on a daily basis. Notice that nowhere in the above argument, did we have to make an assumption regarding the process of evolution of the stock price across time. The only two variables of interest were the current stock price and the riskless rate of interest.

In the case of an option however, the holder or the long has a right and not an obligation. The attractiveness of the option would therefore depend on his perception of the probability of his being able to exercise having entered into the contract, and the corresponding payoff. From the writer's standpoint, what is of consequence is the possibility of the holder not exercising the option and therefore of his being able to retain the premium.

To see why a typical cash and carry strategy does not yield an explicit price for an option, consider the case of an arbitrageur who borrows and buys a stock worth S_t, and a European put option worth $P_{E,t}$. At expiration there are two possibilities, namely $S_T > X$, that is the put is out of the money, or else $S_T < X$, that is the put is in the money.

The terminal payoff for the arbitrageur after repaying the loan with interest is

$$S_T - (S_t + P_{E,t})(1 + r)$$

if the put is out of the money, and

$$X - (S_t + P_{E,t})(1 + r)$$

if the put is in the money. To rule out arbitrage, we require that

$$X - (S_t + P_{E,t})(1 + r) \leq 0$$

$$\Rightarrow \frac{X}{(1 + r)} - S_t \leq P_{E,t} \tag{12.1}$$

which is a condition that is required as the lower bound for a European put. However, this strategy does not yield a precise expression for the put premium, in terms of the dependent variables.

Hence in the case of an options contract, valuation entails the postulation of a process for the evolution of the stock price through time. Corresponding to every hypothesis that we make about the price process, we will get an option price. In some cases we will be able to derive precise formulae for the price or closed-form solutions, while in other cases we will have to make do with numerical approximations.

12.2 Variables of Interest

12.2.1 The Current Stock Price

The prevailing stock price would obviously be a major factor in determining the value of an option. Everything else remaining constant, the higher the current stock price, the greater will be the value of a call option and the lower will be the value of a put option.

12.2.2 The Exercise Price

As one would expect, the higher the exercise price, for a given set of values for the other variables, the lower will be the value of a call option and the higher will be the value of a put option.

12.2.3 Dividends

A dividend payout will lead to a drop in the stock price as the stock goes ex-dividend. Consequently, dividends which are paid out during the life of the option will lead to a reduction in call values and an increase in put values. *Exchange traded options are usually not payout protected from the standpoint of cash dividends*. That is, the terms of the original option contract will not be modified were the underlying stock to pay a dividend subsequently. The terms of the agreement will however be changed in the event of stock splits or stock dividends.

12.2.4 Volatility

Modern finance theory is based on the assumption that all investors are risk averse. Consequently an increase in the volatility, as measured by the variance of the rate of return of the stock, will be perceived negatively and will therefore lead to a higher risk premium being demanded.

In the case of an options contract however, the holder is protected on one side, since his maximum loss is limited to the initial premium paid by him. Therefore, an increase in the volatility will be perceived positively, although it signals a greater probability of higher stock prices as well as lower stock prices. Hence an increase in the volatility of the rate of return on the underlying asset will lead to an increase in the value of both call as well as put options.

12.2.5 Time to Maturity

American options and European calls on non dividend paying stocks have a non-negative time value while European puts on non dividend paying stocks may have either a positive or a negative time value depending on the extent to which the option is in the money.

At the time of expiration, the time value of an option as we have seen earlier must be zero, that is, the option premium must be equal to the intrinsic value. Therefore, everything else remaining constant, the value of an option will generally decline with the passage of time. We use the word *'generally'*, because certain European puts which are deep in the money may have a negative time value which will approach zero at expiration. Hence, options are also known as *Wasting Assets*, because they experience a decline in value with the passage of time.

12.2.6 Riskless Rate of Interest

Let us take the case of an investor who is contemplating the purchase of a stock. One alternative would be to buy a call instead, which can be subsequently exercised if he so desires. We know that the price of a call must be less than the prevailing stock price.

So if a person who has an amount equal to the value of the stock with him, were to decide to buy the call instead of the stock, then he can invest the difference at the riskless rate of interest. Everything else being the same, the higher the rate of interest, the more attractive will be the strategy of buying the call and investing the surplus. Therefore, it follows that the higher the interest rate, the more will be the price of the call.

What about puts? Consider the case of a person who already owns the stock and is contemplating selling it. One alternative would be to buy a put, which would ensure that he will receive a minimum price of \$ X for it subsequently. The higher the interest rate the more attractive will be the alternative of selling immediately in the spot market, rather than buying a put. Consequently, everything else being the same, the higher the interest rate, the less attractive will be the put option. Hence, put prices will decline as interest rates rise.

12.3 The Binomial Model

We will now go on to study theoretical pricing models for call and put options. The first model, which we will study is called the Binomial Option Pricing Model (*BOPM*). Subsequently we will study the famous *Black–Scholes* model (*BSOPM*).

The binomial model assumes that given the current stock price, during the next period the price can either change by $X\%$ or by $Y\%$, where X and Y are specified. Since the stock price can take on only two possible values at the end of the period, the model is called *Binomial*. A realistic portrayal of the behaviour of a stock would require the specification of Y as a negative number.

12.3.1 Assumptions

The assumptions underlying the model are the following.

1. There are no frictions in the market such as transactions costs or taxes.
2. There are no margin requirements.
3. The investor is entitled to use the full proceeds from a short sale, if an asset is sold short.
4. Securities are infinitely divisible, that is investors can trade in fractions of securities.
5. Every investor is a price taker. There is a single riskless rate of return in the economy and everyone can borrow or lend unlimited amounts at this rate.
6. Investors are never satiated from the standpoint of wealth acquisition. Consequently, arbitrage opportunities if any will be fully exploited till they cease to exist.

12.4 The One Period Model

In the one period case, we consider only two points in time. These are, the present time $(T-1)$, and the expiration time of the option T.

Let the current stock price be S_t. The stock price at the end of the period can take on only the following values.

$$S_T = S_t\left(1+\frac{X}{100}\right) = uS_t \tag{12.2}$$

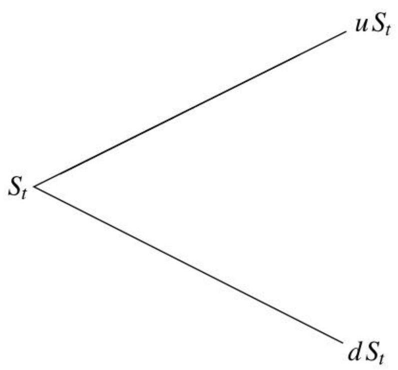
where $u = \left(1+\frac{X}{100}\right)$, and stands for the *Up State*,

or
$$S_T = S_t\left(1+\frac{Y}{100}\right) = dS_t \tag{12.3}$$

where $d = \left(1+\frac{Y}{100}\right)$, and stands for the *Down State*. Y, in this case, is a negative number.

The price tree may be depicted in Fig. 12.1.

Figure 12.1 Stock Price Tree for the One Period Case



Now, the call value at expiration will be $\text{Max}[0, uS_t - E]$, if the up state is reached, or $\text{Max}[0, dS_t - E]$, if the down state is reached. We are going to use the symbol 'E' to denote the exercise price, since the symbol 'X' has already been used to denote the extent of an up movement.

Let $C_u = \text{Max}[0, uS_t - E]$

and $C_d = \text{Max}[0, dS_t - E]$

C_u is the call value at expiration if the up state is reached, and C_d is the call value at expiration if the down state is reached.

Our objective is to find the value of the call today, that is, C_t.

Consider the following strategy. Let us buy α shares of stock and write one call option. The current value of this portfolio will be $\alpha S_t - C_t$.

If the up state is reached, the portfolio will have a value = $\alpha u S_t - C_u$, while if the downstate is reached, it will have a value = $\alpha d S_t - C_d$.

Now let us make this portfolio riskless by choosing α, such that

$$\alpha u S_t - C_u = \alpha d S_t - C_d$$

$$\Rightarrow \alpha S_t(u-d) = C_u - C_d$$

$$\Rightarrow \alpha = \frac{C_u - C_d}{S_t(u-d)} \tag{12.4}$$

α is known as the hedge ratio.

This portfolio is riskless because the payoff is independent of the state of nature at time T. Since our portfolio by design is riskless, it must earn the riskless rate of return to preclude arbitrage. Therefore,

$$\alpha u S_t - C_u = \alpha d S_t - C_d = (\alpha S_t - C_t)r$$

where $r = 1$ + riskless rate of interest. The symbol r is often used to denote the riskless rate of interest per annum. In the binomial model, r represents the riskless rate per period, and is defined as one plus the periodic interest rate.

We will typically assume that the parameters u and d, as well as the riskless rate of return r are constant. However, while this is done in order to simplify the calculations, this is not a necessary condition. The model merely requires that these variables be deterministic, that is they are known with certainty to all investors.

Thus,

$$\alpha u S_t - C_u = (\alpha S_t - C_t)r$$

$$\Rightarrow \left(\frac{C_u - C_d}{S_t(u-d)}\right) \times u S_t - C_u = \left(\frac{C_u - C_d}{S_t(u-d)}\right) \times r S_t - C_t r$$

$$\Rightarrow \left(\frac{C_u - C_d}{u-d}\right) \times (u-r) - C_u = -C_t r$$

$$\Rightarrow C_t = \left(\frac{C_u\left(\frac{r-d}{u-d}\right) + C_d\left(\frac{u-r}{u-d}\right)}{r}\right) \tag{12.5}$$

Let $\frac{r-d}{u-d} = p$. Therefore, $\frac{u-r}{u-d} = 1-p$

$$\Rightarrow C_t = \left(\frac{pC_u + (1-p)C_d}{r}\right) \tag{12.6}$$

This is the one period binomial call option pricing formula.[1]

[1] We will interpret p later. Notice that it seems to be some kind of probability.

12.4.1 Example

Let $S_t = 100$, and $X = 100$. $u = 1.2$, $d = .80$, $r = 1.05$

$$C_u = \text{Max}[0, 1.2 \times 100 - 100] = 20$$

$$C_d = \text{Max}[0, 0.8 \times 100 - 100] = 0$$

$$p = \frac{r-d}{u-d} = \frac{1.05 - .80}{1.2 - .80} = .625$$

$$1 - p = .375$$

$$\text{The hedge ratio} = \frac{C_u - C_d}{S_t(u-d)} = \frac{20 - 0}{100(1.2 - .80)} = .5$$

That is, we have to buy .5 shares for every call that we write, in order to form a riskless portfolio.

$$C_t = \frac{.625 \times 20 + .375 \times 0}{1.05} = 11.9048$$

It can be shown that $u > r > d$, that is, p and $(1 - p)$ should both be positive. Otherwise, one can make arbitrage profits. The proof follows.

12.4.2 Proof

Case A Suppose $u > d > r$. Consider the following strategy.

At time $T - 1$, borrow \$ S_t for one period at the riskless rate, and buy a share of stock. At time T, you will get uS_t if the up state occurs. After repaying rS_t, you will get a net profit of $(u - r)S_t$ which is > 0. Similarly, if the down state occurs, you will get $(d - r)S_t$ which is > 0. Hence to preclude arbitrage opportunities, we require that r be $> d$.

Case B Suppose $r > u > d$. Consider the following strategy.

Short sell the stock at $T - 1$, and deposit S_t at the riskless rate. At time T, you will get rS_t. If the up state occurs, you will have to buy back the stock at uS_t. This will lead to a profit of $(r - u)S_t$, which is > 0. Similarly, in the down state, you will get $(r - d)S_t$, which is > 0. Hence, we require that u be $> r$.

Combining the two conditions, we get the requirement that $u > r > d$.

12.5 Pseudo Probabilities and Risk Neutrality

p and $(1 - p)$ are referred to as the pseudo probabilities of the call having a value of C_u and C_d respectively.

Let the actual probability of the stock being in the up state be q, and the probability of it being in the down state be $1 - q$. As you can see, q does not appear anywhere in the pricing equation. Thus different people may disagree on the probabilities of the stock being in the up and down states, and yet agree on the call price.

You will also notice that risk preferences of individual investors do not appear in the pricing equation.[2] So people with different levels of risk aversion, can agree on the option price.

So what is p? Now, if risk aversion levels are irrelevant for pricing, everyone including risk neutral investors, will agree on the call price calculated above. Consider a risk neutral investor. He will accept any risky investment which offers an expected rate of return equal to the riskless rate.

For the sake of argument, let us consider a world full of risk neutral investors. Let the probability of an up move be q, and that of a down move be $(1 - q)$. The expected rate of return on the stock in this scenario, will be equal to the riskless rate. Therefore,

$$\frac{quS_t + (1-q)dS_t - S_t}{S_t} = \frac{rS_t - S_t}{S_t}$$

$$\Rightarrow q = \frac{r-d}{u-d} \tag{12.7}$$

which is the same as our p. Thus p is the probability of an up move, in a world full of risk neutral investors.

How would a risk neutral person value the call option? He would calculate the expected value of the terminal payoff, and then discount it at the riskless rate. So, for such an investor

$$C_t = \frac{qC_u + (1-q)C_d}{r} = \frac{pC_u + (1-p)C_d}{r}$$

So to value an option using the binomial model, we can act as if everyone were risk neutral, and use p as the probability of an up move.[3]

12.6 A Replicating Portfolio

As we have seen, a portfolio of α shares of stock and a short position in a call option yields the riskless rate of return. Thus the portfolio is equivalent to an investment of \$ B in a riskless asset. Therefore

$$\alpha S_t - C_t = B$$

$$\Rightarrow \alpha S_t - B = C_t \tag{12.8}$$

Thus an investment in a call option is equivalent to buying α shares of stock and borrowing an amount of \$ B at the riskless rate.

$$\alpha = \frac{C_u - C_d}{S_t(u-d)}$$

[2]We have made no reference to utility functions.

[3]We are not saying that everyone is risk neutral. What we are saying is that we can pretend as if everyone is risk neutral. It is in this sense, that p is a pseudo probability.

This will be non-negative since $C_u \geq C_d$ and $u > d$.

$$B = \alpha S_t - C_t = \frac{C_u - C_d}{u - d} - \frac{pC_u + (1-p)C_d}{r} \tag{12.9}$$

$$= \frac{rC_u - rC_d - [p(u-d)C_u + (1-p)(u-d)C_d]}{r(u-d)}$$

$$= \frac{[(rC_u - rC_d) - (r-d)C_u - (u-r)C_d]}{r(u-d)}$$

$$= \frac{dC_u - uC_d}{r(u-d)} \tag{12.10}$$

In our example, $\alpha = 0.50$. $B = \dfrac{(0.8 \times 20 - 1.2 \times 0)}{(1.05)(1.2 - 0.8)} = \$\,38.10$. The value of the replicating portfolio is

$$0.5 \times 100 - 38.10 = \$\,11.90$$

which is the value that we obtained for the call. If the up state is reached, the payoff from the portfolio will be

$$0.5 \times 120 - 38.10 \times 1.05 = \$\,20$$

If the down state is reached the payoff will be

$$0.5 \times 80 - 38.10 \times 1.05 = 0$$

In both cases the payoff from the replicating portfolio is identical to that of the call.

12.7 The Two Period Case

Thus far, we have assumed that the option has only one period left to expiration. In general, the stock price will move many times between the date of valuation and the expiration date. In the multi period case also, the same arguments will hold.

Let the stock price two periods before the expiration date be S_t. The stock price tree, may then be depicted as shown in Fig. 12.2.

At $T-1$, there is only one period left to expiration, and so, we an apply the one period model to get C_u and C_d at $T-1$.

$$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{r}$$

$$C_d = \frac{pC_{ud} + (1-p)C_{dd}}{r}$$

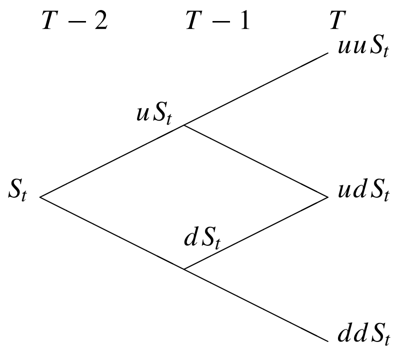
We know C_{uu}, C_{ud}, and C_{dd} because they represent terminal values of the option.

C_{uu} is the option price at T, if there are two upticks in the stock price.

C_{ud} is the option price at T, if there is an uptick followed by a downtick or vice versa.

C_{dd} is the option price at T, if there are two downticks in the stock price.

Figure 12.2 **Stock Price Tree for the Two Period Case**



Once we find C_u and C_d, we can work backwards to find C_t.

$$C_t = \frac{pC_u + (1-p)C_d}{r}$$

$$= \frac{p^2C_{uu} + 2p(1-p)C_{ud} + (1-p)^2C_{dd}}{r^2} \tag{12.11}$$

Using the same iterative process, we can work backwards, moving one period at a time, to solve the 'N' period case.

12.7.1 Example

Let us use the same data as in the earlier example.

$S_t = X = 100$, $u = 1.2$, $d = 0.8$, $r = 1.05$.

The price tree is depicted in Fig. 12.3.

$$p = .625,\ (1-p) = 0.375$$

$$C_{uu} = \text{Max}[0, 144 - 100] = 44$$

$$C_{ud} = \text{Max}[0, 96 - 100] = 0$$

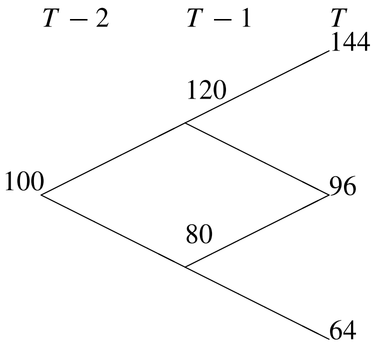
$$C_{dd} = \text{Max}[0, 64 - 100] = 0$$

$$C_u = \frac{.625 \times 44 + .375 \times 0}{1.05} = 26.1905$$

$$C_d = \frac{.625 \times 0 + .375 \times 0}{1.05} = 0$$

$$C_t = \frac{.625 \times 26.1905 + .375 \times 0}{1.05} = 15.5896$$

Figure 12.3 **Stock Price Tree for the Two Period Example**



12.7.2 The Self-Financing Property

As we have seen a call option can be replicated by buying α shares of stock and borrowing \$ B at the riskless rate. The hedge ratio at time $T-2$ is given by

$$\alpha_{T-2} = \frac{26.1905 - 0}{100 \times (1.2 - 0.80)} = 0.6548$$

$$B = \frac{0.80 \times 26.1905 - 1.20 \times 0}{(1.05)(1.2 - 0.80)} = 49.89$$

Thus we need to invest \$ 65.48 in the stock and borrow \$ 49.89 at the riskless rate of interest.

This portfolio has to be rebalanced at time $T-1$, irrespective of whether the upstate or the downstate is reached. If the upstate is reached, the new hedge ratio is given by

$$\alpha_{u,T-1} = \frac{44 - 0}{120 \times (1.20 - 0.80)} = 0.9167$$

The new value of B is given by

$$\frac{0.80 \times 44 - 1.20 \times 0}{(1.05)(1.20 - 0.80)} = 83.81$$

The additional investment required in the stock is

$$(0.9167 - 0.6548) \times 120 = \$\ 31.43$$

The amount borrowed in the previous period was \$ 49.89. This would have grown to

$$49.89 \times 1.05 = \$\ 52.38$$

The additional amount that has to be borrowed is

$$83.81 - 52.38 = \$\,31.43$$

which is exactly equal to the additional investment in the stock.

If the down state is reached, the new hedge ratio is given by

$$\alpha_{d,T-1} = \frac{0 - 0}{80 \times (1.20 - 0.80)} = 0$$

The new value of B is given by

$$\frac{0.80 \times 0 - 1.20 \times 0}{(1.05)(1.20 - 0.80)} = 0$$

Thus we need to sell off 0.6548 units of the stock. This will lead to an inflow of

$$0.6548 \times 80 = 52.38$$

This will be exactly adequate to pay off the amount borrowed in the previous period with interest. Thus if we start with a replicating portfolio, subsequent changes required in the composition of the portfolio are self-financing. That is the outflow/inflow due to investment/divestment in the stock is exactly equal to the amount that has to be borrowed/repaid at the riskless rate. We will demonstrate later, that the self-financing property of the replicating portfolio breaks down in cases where it is optimal to exercise an option before expiration.

12.8 The Multi Period Case

If there are N periods left to expiration

$$C_t = \frac{1}{r^N}\sum_{k=0}^{N}\left[\frac{N!}{(N-k)!k!}p^k(1-p)^{N-k}\text{Max}(0, u^k d^{N-k} S_0 - X)\right] \qquad (12.12)$$

12.9 Binomial Pricing for European Puts

Consider the one period case. The put value at expiration is Max$[0, E - uS_t]$ if the up state is reached, or is Max$[0, E - dS_t]$ if the down state is reached. Therefore,

$$P_u = \text{Max}[0, E - uS_t]$$

and

$$P_d = \text{Max}[0, E - dS_t]$$

Using arguments similar to what we used for the call, it can be shown that

$$P_t = \frac{pP_u + (1-p)P_d}{r} \qquad (12.13)$$

where, p and $(1 - p)$ are as defined before.

12.9.1 Example

We will use the same data that we used for the call. That is, $S_t = X = 100$, $u = 1.2$, $d = .8$, $r = 1.05$.

$$P_u = \text{Max}[0, 100 - 120] = 0$$

$$P_d = \text{Max}[0, 100 - 80] = 20$$

$$P_t = \frac{.625 \times 0 + .375 \times 20}{1.05} = 7.1429$$

12.10 Replicating Portfolio

Consider a portfolio consisting of α shares of stock and \$ B in the riskless asset. The payoff from the portfolio will be:

$$\alpha u S_t + Br \text{ if the upstate is reached}$$

and $$\alpha d S_t + Br \text{ if the downstate is reached.}$$

Equating the payoffs from the portfolio to the payoff from the option, we get

$$\alpha u S_t + Br = P_u \quad \text{and} \quad \alpha d S_t + Br = P_d$$

$$\Rightarrow \alpha(u - d)S_t = P_u - P_d$$

$$\Rightarrow \alpha = \frac{P_u - P_d}{S_t(u - d)} \leq 0 \qquad (12.14)$$

since $P_u \leq P_d$.

$$Br = P_u - \alpha u S_t = P_u - u\frac{(P_u - P_d)}{(u - d)}$$

$$= \frac{P_u(u - d) - u(P_u - P_d)}{(u - d)}$$

$$\Rightarrow B = \frac{uP_d - dP_u}{r(u - d)} \qquad (12.15)$$

Thus, the replicating portfolio entails the short selling of α shares of stock and an investment of \$ B in the riskless asset.

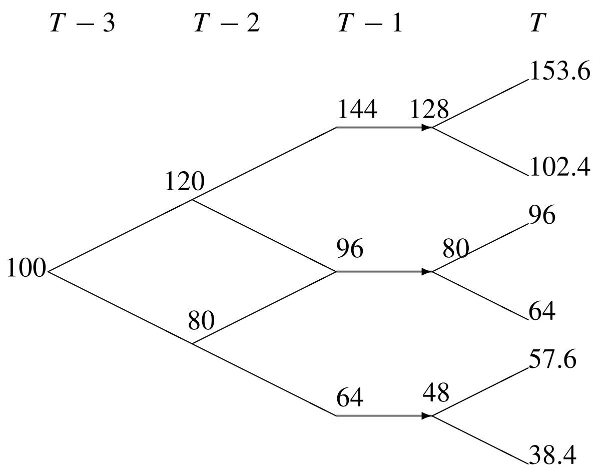
12.11 Valuing European Calls on Dividend Paying Stocks

Consider the data given in the earlier examples. Assume that the option has three periods left to maturity. When there is one period left to go for expiration, the stock pays a dividend of \$ 16. That is, if the option expires at T, then, at $T - 1$ the stock trades ex-dividend. In other words, an instant before it trades at $T - 1$, a dividend is paid.[4]

[4] When valuing options on dividend paying stocks, it is critical to know as to when exactly the dividend will be paid.

Let us first model the stock price tree (Fig. 12.4).

Figure 12.4 **Stock Price Tree Corresponding to a Discrete Dividend**



Notice that because of the dividend, you get additional branches at time T.

$$C_{uu} = \frac{.625 \times 53.6 + .375 \times 2.4}{1.05} = 32.7619$$

C_{uu} is the value of the call at $T - 1$, if there are two upticks, that is, the node corresponding to an ex-dividend price of 128.

$$C_{ud} = \frac{.625 \times 0 + .375 \times 0}{1.05} = 0$$

$$C_{dd} = \frac{.625 \times 0 + .375 \times 0}{1.05} = 0$$

Using C_{uu}, C_{ud}, and C_{dd} we can work backwards to calculate, C_u and C_d.

$$C_u = \frac{.625 \times 32.7619 + .375 \times 0}{1.05} = 19.5011$$

$$C_d = \frac{.625 \times 0 + .375 \times 0}{1.05} = 0$$

Therefore,

$$C_t = \frac{.625 \times 19.5011 + .375 \times 0}{1.05} = 11.6078$$

12.12 Valuing American Calls on Dividend Paying Stocks

Consider the stock price tree given in Fig. 12.4. The only difference while valuing American calls is that at each node of the stock price tree, we have to check as to whether the option will be exercised early (killed), or kept alive.

If there are two upticks, $C_{uu} = 32.7619$. If the option is exercised, the payoff is $(128 - 100) = 28$, which is less. So the option will not be exercised early.

If there is an uptick and a downtick, $C_{ud} = 0$. The option is out of the money, so there is no question of early exercise.

Finally, in the case of two downticks, $C_{dd} = 0$. The option is out of the money. So there is no need to consider the possibility of early exercise.

$C_u = 19.5011$. If exercised early, the payoff = 20. So the option will be exercised at this stage.

$C_d = 0$. The option is out of the money. So there is no possibility of early exercise.

The difference between European options, and American options, is that at each node, you have to take the greater of the model value and the payoff if exercised early, in order to work backwards.

Thus, at $T - 3$ C_t is calculated as follows.

$$\frac{.625 \times 20 + .375 \times 0}{1.05} = 11.9048$$

As you can see, the American call is more valuable than the corresponding European call. But this is not all. You have to compare 11.9048 with the intrinsic value at $T - 3$ which in this case is 0. If the intrinsic value is greater, then the call must sell for the intrinsic value. In this case however, the call will sell for the model price, which is 11.9048.

12.13 Rationale for Early Exercise

If the value of the option at a particular node as determined from the model is lower than its intrinsic value then a rational investor will choose to exercise early. If he does not wish to continue to hold the option, then obviously early exercise will occur. Offsetting the option is not a worthwhile proposition, because a potential buyer will not pay more than the model value since he can always replicate the option at that value. Consequently the holder stands to benefit more by exercising. Even if the holder were to desire to continue holding the option, it would be more beneficial for him to exercise the option and replicate it at a cost which is lower than the intrinsic value.

12.14 The Breakdown of the Self-Financing Property

At time $T-3$

$$\alpha = \frac{C_u - C_d}{S_{T-3}(u-d)} = \frac{20-0}{100 \times (1.20-0.80)} = 0.50$$

$$B = \frac{0.80 \times 20 - 1.20 \times 0}{1.05 \times (1.20-0.80)} = \$\,38.10$$

If the upstate is reached, the new hedge ratio is given by

$$\alpha_{u,T-2} = \frac{32.7619-0}{120 \times (1.20-0.80)} = 0.6825$$

$$B = \frac{0.80 \times 32.7619 - 0}{1.05 \times (1.20-0.80)} = \$\,62.40$$

The additional investment in the stock = (0.6825 − 0.50) × 120 = \$ 21.90. The additional amount to be borrowed = 62.40 − 38.10 × 1.05 = \$ 22.40. Clearly the replicating portfolio is not self-financing at a node where early exercise of the option is optimal.

12.15 European versus American Puts

Consider the data used in the earlier examples. We will assume that the stock pays no dividends.

Let us look at a put option with three periods to expiration. The stock price tree may be depicted as shown in Fig. 12.5.

12.15.1 Valuing a European Put

To value the option, we will proceed backwards as follows.

$$P_{uu} = \frac{.625 \times 0 + .375 \times 0}{1.05} = 0$$

$$P_{ud} = \frac{.625 \times 0 + .375 \times 23.20}{1.05} = 8.2857$$

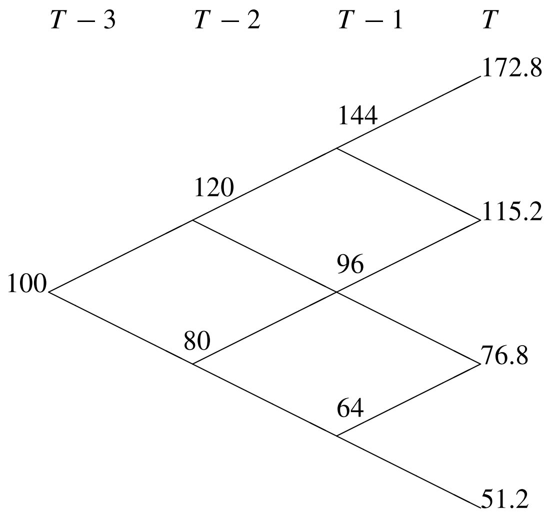
$$P_{dd} = \frac{.625 \times 23.2 + .375 \times 48.8}{1.05} = 31.2381$$

$$P_u = \frac{.625 \times 0 + .375 \times 8.2857}{1.05} = 2.9592$$

$$P_d = \frac{.625 \times 8.2857 + .375 \times 31.2381}{1.05} = 16.0884$$

$$P_t = \frac{.625 \times 2.9592 + .375 \times 16.0884}{1.05} = 7.5073$$

Figure 12.5 **Stock Price Tree for the Three Period Case**



12.15.2 Valuing an American Put

For American puts, we have to, at each node, compare the value obtained from the model with the intrinsic value. If the intrinsic value is greater, we have to use it for subsequent calculations.

At uuS_t, the value according to our previous calculations is 0. The intrinsic value is also 0.

At udS_t, the model price is 8.2857, and the intrinsic value is 4, which is less.

Thus P_u will have the same value for an American put.

But at ddS_t the model price is 31.2381, while the intrinsic value is 36. So we will calculate P_d as follows.

$$P_d = \frac{.625 \times 8.2857 + .375 \times 36}{1.05} = 17.7891$$

The intrinsic value at the node corresponding to a stock price of dS_t is 20, which is greater than 17.7891.

So we will calculate P_t as follows.

$$P_t = \frac{.625 \times 2.9592 + .375 \times 20}{1.05} = 8.9043$$

This is not the final step however. You must compare 8.9043 with the intrinsic value at $T - 3$ which in this case is 0. Thus the option will sell for 8.9043.

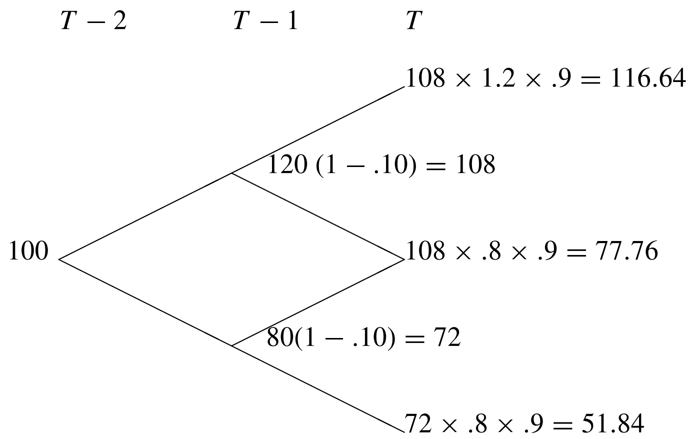
As you can see, the American put is priced higher than the European put. This is because early exercise is optimal at an intermediate stage.

12.16 Another Approach to Dividends

Consider a case where the stock pays a constant dividend yield of 10% per period. Thus each time the stock price moves to a new value, it immediately declines by 10% as it goes ex-dividend.

Let us consider a two period call option on this stock. Except for the dividend yield, other variables have the same value as before. Consider the stock price tree.

Figure 12.6 Stock Price Tree in the Case of a Constant Dividend Yield



$C_{uu} = 16.64$, $C_{ud} = 0$, and $C_{dd} = 0$

$$C_u = \frac{.625 \times 16.64 + .375 \times 0}{1.05} = 9.9048$$

The intrinsic value = 8.

$$C_d = \frac{.625 \times 0 + .375 \times 0}{1.05} = 0$$

The intrinsic value = 0.

$$C_t = \frac{.625 \times 9.9048 + .375 \times 0}{1.05} = 5.8957$$

12.17 The Black–Scholes Model

A variable is said to follow a stochastic process, if its value changes over time in an unpredictable way. As you may have guessed, a stock price is an obvious candidate. Stochastic processes may be either of the discrete time variety, or of the continuous time type. A discrete time process is used to represent variables which can take on values only at certain fixed points in time. On the other hand, variables which are described using continuous time processes, can take on values at any point in time. A stochastic process may be of the continuous state space variety, or of a discrete state space type. If the state space is discrete, the variable can take on only certain discrete values, whereas if the state space in continuous, it can take on any value in a specified range.

To describe stock prices, Black–Scholes assumed a continuous time, continuous state space process. The process assumed by them may be described as follows.

$$\frac{dS}{S} = \mu dt + \sigma dZ \tag{12.16}$$

Thus the rate of return on the stock as per this model, consists of a deterministic component as well as a random component. μdt is the deterministic component. μ is a measure of the average return per unit time, and is called the '*drift*' of the process. The second term σdZ captures the random component of the return. σ is referred to as the volatility of the rate of return, and is measured as a percentage per annum.

dZ is referred to as a Wiener process. It may be expressed as

$$dZ = \epsilon\sqrt{dt} \tag{12.17}$$

where ϵ is a number drawn from a standard normal distribution, that is, a normal distribution with a mean of zero and a variance of one.

It has the following properties.

1. The expectation of dZ is zero.

$$dZ = \epsilon\sqrt{dt} \Rightarrow E[dZ] = E[\epsilon\sqrt{dt}] = \sqrt{dt}E[\epsilon] = 0 \tag{12.18}$$

2. The variance of dZ is dt.

$$\begin{aligned}\text{Var}(dZ) &= E[\epsilon\sqrt{dt}]^2 - \{E[\epsilon\sqrt{dt}]\}^2 \\ &= dtE[\epsilon]^2 = dt\end{aligned} \tag{12.19}$$

3. The values of $\triangle Z$ for any two non-overlapping short intervals of time $\triangle t$ are independent.

   Let the change in Z over a period of time T, be $Z(T) - Z(0)$. T can be divided into N non-overlapping intervals of $\triangle t$. Therefore

$$Z(T) - Z(0) = [Z(T) - Z(T-1)] + [Z(T-1) - Z(T-2)] + ...$$

$$= \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t} \tag{12.20}$$

$$E[Z(T) - Z(0)] = \sqrt{\Delta t} \sum_{i-1}^{N} E(\epsilon_i) = 0$$

$$\Rightarrow E[Z(T)] = Z(0) \tag{12.21}$$

$$\text{Var}[Z(T) - Z(0)] = \sum_{i=1}^{N} \text{Var}[\epsilon_i \sqrt{\Delta t}]$$

$$= \Delta t \sum_{i=1}^{N} 1.0 = N\Delta t = T \tag{12.22}$$

12.17.1 An Ito Process

Consider the process

$$dS = \mu S dt + \sigma S dZ \tag{12.23}$$

This is a process of the form

$$dS = a(S, t)\, dt + b(S, t) dZ \tag{12.24}$$

That is, the drift and volatility of the process are a function of the variable itself and time. Such a process is known as an Ito process.

Ito's Lemma Assume that a variable X follows an Ito process. That is

$$dX = a(X, t) + b(X, t)\, dZ \tag{12.25}$$

Define G as a function of X and time. That is, $G = G(X, t)$. According to Ito's lemma

$$dG = \left(\frac{\partial G}{\partial X} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} b^2 \right) dt + \frac{\partial G}{\partial X} b dZ \tag{12.26}$$

In other words, G also follows an Ito process with a drift of

$$\left(\frac{\partial G}{\partial X} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} b^2 \right)$$

and a variance rate of $\left(\frac{\partial G}{\partial X} b \right)^2$.

12.17.2 The Lognormal Property of Stock Prices

Let the stock price S follow an Ito process, such that

$$dS = \mu S dt + \sigma S dZ$$

Consider a function G, where $G = \ln S$. Therefore

$$\frac{\partial G}{\partial S} = \frac{1}{S}; \frac{\partial G}{\partial t} = 0; \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$$

Thus, according to Ito's lemma

$$dG = \left(\frac{1}{S}\mu S + 0 - \frac{1}{2} \times \frac{1}{S^2}\sigma^2 S^2\right) dt + \frac{1}{S}\sigma S dZ$$

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dZ \qquad (12.27)$$

Over a discrete time interval from t to T, we can state that

$$\ln S_T - \ln S_t \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$$

$$\Rightarrow \ln S_T \sim N\left[\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right] \qquad (12.28)$$

Thus the logarithm of S follows a normal distribution which implies that S itself follows a lognormal distribution. The process $dS = \mu S dt + \sigma S dZ$ is referred to as a lognormal random walk.

12.18 Estimating Volatility from Historical Data

To estimate the volatility of returns using past data, we have to observe the stock prices at fixed intervals in time. Consider the case where we have $n+1$ observations of the stock price S. That is S can take on the values S_0 to S_n. Define $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$. That is, u_i is the continuously compounded rate of return.[5] An unbiased estimate of the variance of the returns is given by

$$\hat{s}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(u_i - \overline{u})^2$$

$$\Rightarrow \hat{s}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}{u_i}^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2 \qquad (12.29)$$

We know that $\ln S_T - \ln S_t \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$

[5] It is not an annualized rate.

Therefore, our variance estimate $\hat{s}^2$ is an estimate of $\sigma^2(T-t)$. Thus $\frac{\hat{s}^2}{(T-t)}$ is an estimate of σ^2, or in other words, $\frac{s}{\sqrt{(T-t)}}$ is an estimate of σ. One of the key issues in estimating σ, is choosing the appropriate value for n. The more data we use, the more accurate our estimate is likely to be. However, in practice, it has been found that σ changes over time, and hence very old data cannot be used for predicting the future. In real life, people often use daily closing prices over the most recent 3 to 6 months.[6]

In the definition of u given above, we have excluded dividends. However, if day j is an ex-dividend day, we have to define u_j as,

$$u_j = \ln\left(\frac{S_j + D}{S_{j-1}}\right) \tag{12.30}$$

where D is the quantum of the dividend. For all the other days, we will use

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

12.18.1 Illustration

Consider the vector of daily stock prices given in Table 12.1. We have converted the prices to price relatives and then taken the natural logarithm of the price relatives to compute the rates of return. Finally we have computed the standard deviation of the vector of returns.

$$\hat{s}^2 = 0.00050881 \Rightarrow \hat{s} = 0.02255683$$

If we assume that there are 252 trading days in a year, the time period in years is $\frac{1}{252}$. Thus

$$\sigma = \hat{s}\sqrt{252} = 0.358079 \equiv 35.8079\% \text{ per annum}$$

12.18.2 Parkinson's Method

Michael Parkinson developed a volatility measure based on the high and low of stock prices in a particular period. Let $S_{H,t}$ be the stock's highest price on day t, and let $S_{L,t}$ be its lowest price on the same day. The return is defined as

$$r_t{}^{HL} = \ln\left(\frac{S_{H,t}}{S_{L,t}}\right) \tag{12.31}$$

The annual volatility is then computed as:

$$\sigma_{HL} = \sqrt{\left[\frac{252}{N}\sum_{t=1}^{N}\frac{1}{4\ln(2)}(r_t{}^{HL})^2\right]} \tag{12.32}$$

[6]See Hull (2005).

Table 12.1 Computing the Standard Deviation of Returns

Day	Price	$\frac{S_t}{S_{t-1}}$	$\ln\left(\frac{S_t}{S_{t-1}}\right) = r_t$
1	102.26		
2	108.95	1.065398273	0.027511988
3	102.12	0.937330735	−0.028107142
4	106.76	1.045470920	0.019311957
5	101.13	0.947185443	−0.023564985
6	103.00	1.018561154	0.007987109
7	107.22	1.040898150	0.017408237
8	109.58	1.022091327	0.009489703
9	101.03	0.921966616	−0.035284804
10	109.25	1.081311631	0.033950874
11	100.48	0.919746755	−0.036331736
12	104.50	1.039975114	0.017022947
13	102.84	0.984092982	−0.006963865
14	107.93	1.049526378	0.020993358
15	106.89	0.990350224	−0.004211196
16	108.46	1.014760684	0.006363632
17	103.29	0.952300156	−0.021226144
18	101.93	0.986826706	−0.005759106
19	108.68	1.066218662	0.02784628
20	107.31	0.987365741	−0.005521946
21	102.19	0.952325088	−0.021214774
22	109.50	1.071572170	0.030021426
23	100.58	0.918497328	−0.036922103
24	108.66	1.080299943	0.033544353
25	108.66	1.000054576	2.37013E-05
26	106.90	0.983789929	−0.007097628
27	107.92	1.009547820	0.004126895
28	101.95	0.944641661	−0.024732905
29	107.65	1.055971567	0.023652225
30	107.82	1.001507445	0.000654182
31	101.59	0.942217809	−0.025848691

Parkinson showed that his measure contains more information about the underlying return generating process, than the standard volatility measure that relies solely on closing prices.

12.18.3 The Garman–Klass Method

This method of volatility computation takes into account Open, High, Low, and Closing prices for a stock.

Notation

- $N \equiv$ number of historical prices used for the estimate

- $O_t \equiv$ opening price on day t
- $H_t \equiv$ the highest observed price on day t
- $L_t \equiv$ the lowest observed price on day t
- $C_t \equiv$ the closing price on day t

The annual volatility is computed as:

$$\sigma = \sqrt{\frac{252}{N}\sum_{t=1}^{N}\left[\frac{1}{2}\times\left[\ln\left(\frac{H_t}{L_t}\right)\right]^2 - [2\ln(2)-1]\left[\ln\left(\frac{C_t}{O_t}\right)\right]^2\right]} \tag{12.33}$$

The Garman and Klass estimator assumes Brownian motion with a drift of zero, and no opening jumps. That is, it assumes that the opening price is equal to the closing price for the previous period. It is 7.4 times more efficient than the standard estimator using close-to-close returns.[7]

12.18.4 The Yang Zhang Modification

Yang and Zhang extended the Garman Klass estimator by allowing for the possibility of opening jumps. It is 8 times more efficient than the standard estimator.[8] The annual volatility is computed as:

$$\sigma = \sqrt{\frac{252}{N}\sum_{t=1}^{N}\left[\left[\ln\left(\frac{O_t}{C_{t-1}}\right)\right]^2 + \frac{1}{2}\times\left[\ln\left(\frac{H_t}{L_t}\right)\right]^2 - [2\ln(2)-1]\left[\ln\left(\frac{C_t}{O_t}\right)\right]^2\right]} \tag{12.34}$$

12.19 The Distribution of Discretely Compounded versus Continuously Compounded Rates of Return

Consider the process

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

Thus the rate of return in a discrete time interval is given by

$$\frac{\Delta S}{S} \sim N\left[\mu(T-t), \sigma^2(T-t)\right]$$

However we know that

$$d\ln S \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$$

If we define

$$S_T = S_t e^{i(T-t)}, \text{ then}$$

$$\ln S_T = \ln S_t + i(T-t)$$

[7] www.sitmo.com
[8] www.sitmo.com

$$\Rightarrow \ln \frac{S_T}{S_t} = i(T - t)$$

i is the annual continuously compounded rate of return.

So is the expected rate of return per annum μ or is it $\mu - \frac{\sigma^2}{2}$. The answer is that it depends on how we define the rate of return. If the return is continuously compounded then the expected return per annum is $\mu - \frac{\sigma^2}{2}$. However if the return is computed in a discrete fashion as $\frac{(S_T - S_0)}{S_0}$, then the mean rate of return per annum is μ.

12.20 The Black–Scholes Differential Equation

Black and Scholes derived a partial differential equation which must be satisfied by the price of any derivative security whose value depends on that of a non dividend paying stock. The assumptions underlying the Black–Scholes model are as follows.

1. There are no frictions in the market such as transactions costs or taxes.
2. There are no margin requirements.
3. The investor is entitled to use the full proceeds from a short sale, if an asset is sold short.
4. Securities are infinitely divisible, that is investors can trade in fractions of securities.
5. Every investor is a price taker. There is a single riskless rate of return in the economy and everyone can borrow or lend unlimited amounts at this rate. This rate is a constant and is the same for all maturities.
6. Investors are never satiated from the standpoint of wealth acquisition. Consequently, arbitrage opportunities if any will be fully exploited till they cease to exist.
7. No dividends are paid on the underlying asset during the life of the derivative security.
8. All securities are traded continuously.
9. The stock price can be described by the Ito process with μ and σ being constant.

12.20.1 Derivation

Assume that the stock price follows an Ito process. Therefore,

$$dS = \mu S dt + \sigma S dZ$$

Let F be the price of a derivative security, which is a function of S and t. By Ito's Lemma,

$$dF = \left(\frac{\partial F}{\partial S}\mu S + \frac{\partial F}{\partial t} + \frac{1}{2} \times \frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial F}{\partial S}\sigma S dZ \qquad (12.35)$$

Now, the Wiener process underlying S and F is the same. That is, $dZ = \epsilon\sqrt{dt}$ is the same for both the equations. Black and Scholes designed a portfolio using the stock and the derivative asset in a manner which eliminated the Wiener process.

Consider a portfolio which is long in $\frac{\partial F}{\partial S}$ units of the stock, and short in one unit of the derivative security. Let π be the value of this portfolio. Therefore

$$\pi = \frac{\partial F}{\partial S}S - F \qquad (12.36)$$

The change in the value of this portfolio during a time interval of Δt, is

$$\Delta\pi = \frac{\partial F}{\partial S}\Delta S - \Delta F = \left(-\frac{\partial F}{\partial t} - \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)\Delta t \qquad (12.36)$$

Notice that this equation does not involve ΔZ. Hence it must be riskless during the time interval Δt. If it is riskless, it must earn the riskless rate of return during this period to preclude arbitrage.[9]

Therefore

$$\Delta\pi = r\pi\Delta t \qquad (12.38)$$

where r is the riskless rate of interest. Hence,

$$\left(-\frac{\partial F}{\partial t} - \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(-F + \frac{\partial F}{\partial S}S\right)\Delta t$$

$$\Rightarrow \frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 - rF = 0 \qquad (12.39)$$

This the Black–Scholes partial differential equation. This equation has many different solutions which correspond to the various derivative securities which can be defined with the stock price as the underlying variable. The solution obtained will depend on the *boundary conditions* which are specified. Boundary conditions are the values of the derivative security at the boundaries of the possible values that S and t can take.

Boundary Conditions for European Calls and Puts One of the boundary conditions for a European call option is $C(S, T) = \text{Max}(0, S_T - X)$. The analogous condition for a European put is $P(S, T) = \text{Max}(0, X - S_T)$. These conditions arise from the fact that an option must be worth its intrinsic value at expiration. If the stock price goes to zero, then the change in price dS is also zero. Consequently if the stock were to ever attain a value of zero then it would stay there. The call option would therefore become worthless in such a situation, irrespective of the time left to maturity. Thus $C(0, t) = 0$ for calls. On the other

[9]It must be remembered that our portfolio is not permanently riskless. As S and t change, $\frac{\partial F}{\partial S}$ will also change. Hence to keep the portfolio riskless, it must be continuously rebalanced.

hand, if the stock price were to attain a value of zero, a European put is guaranteed to be exercised at expiration. Hence its present value must be the discounted value of the exercise price. That is $P(0, t) = Xe^{-r(T-t)}$. Finally as the stock price tends to infinity it is increasingly likely that the call will be exercised and the magnitude of the exercise price becomes more and more insignificant. Thus the call will tend to behave like the stock in such circumstances and we can state that $C(S, t) \rightarrow S$ as $S \rightarrow \infty$. Similarly if the stock price tends towards infinity it is increasingly unlikely that a European put will be exercised. Consequently $P(S, t) \rightarrow 0$ as $S \rightarrow \infty$.

12.21 Risk Neutral Valuation

When we studied the binomial model, we dealt with pseudo probabilities and risk neutral valuation. Similar principles are applicable here.

Let us consider the partial differential equation used by Black and Scholes. Notice that it does not contain any variables which pertain to the risk preferences of individuals. For instance, μ, the expected rate of return, is conspicuous by its absence. If μ were to be present, the equation would not be independent of risk preferences.[10]

Considering this fact, we can use the following argument. If risk aversion measures are absent, then we can assume any kind of risk preferences while valuing the option, since it will not affect the solution. To keep matters simple, therefore, we can assume that everyone is risk neutral.

Now, in a risk neutral world, the expected rate of return on all securities will be the riskless rate r.[11] And, the value of any cash flow can be gotten by discounting its expected value using the riskless rate.

This assumption of risk neutrality considerably simplifies option pricing. We can now assume that the expected rate of return on the stock is r, and can then calculate the expected value of the option at time T. This expected value can then be discounted back using r, to get the option price.

Remember once again that we are not saying everyone is risk neutral. By assuming that everyone is risk neutral, we are only trying to simplify the analysis. The final solution is valid for all investors, irrespective of their risk preferences.

12.22 The Black–Scholes Formula

Black and Scholes obtained exact formulae for valuing call and put options on non-dividend paying stocks, by assuming that stock prices follow a lognormal process. The formulae obtained by them are

$$C_{E,t} = S_t N(d_1) - Xe^{-r(T-t)} N(d_2) \tag{12.40}$$

[10]The more risk averse the investor, the greater will be the value of μ.

[11]This is because risk neutral investors do not require a premium to induce them to take risks.

and

$$P_{E,t} = Xe^{-r(T-t)}N(-d_2) - S_t N(-d_1) \tag{12.41}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \tag{12.42}$$

and

$$d_2 = d_1 - \sigma\sqrt{(T-t)} \tag{12.43}$$

$N(X)$ is the cumulative probability distribution function for a standard normal variable, and σ is the standard deviation of the rate of return on the stock.

12.22.1 Example

Consider a stock which is currently selling for \$ 100. Call and put options are available with X = 100, and time to expiration = 6 months.

The riskless rate of interest = 10% per annum, and the volatility is 30% per annum. So, $S_t = X = 100$; $T - t = .5$ years; r = 10% ≡ .10; $\sigma = 30\% \equiv .30$

Let us consider the call first.

$$d_1 = \frac{\ln\left(\frac{100}{100}\right) + \left(.10 + \frac{(.30)^2}{2}\right).5}{.3\sqrt{.5}} = \frac{.0725}{.2121} = .3418$$

$$d_2 = .3418 - .2121 = .1297$$

$N(d_1)$ and $N(d_2)$ have to be calculated using interpolation. $N(.34) = .6331. N(.35) = .6368$. So $N(.3418) = .6338$

$N(.12) = .5478$. $N(.13) = .5517$. So $N(.1297) = .5516$.

$$C_{E,t} = 100 \times .6338 - 100e^{-.10\times.5} \times .5516 = 10.9102$$

$$P_{E,t} = 100e^{-.10\times.5}N(-.1297) - 100N(-.3418)$$

Now, $N(-X) = 1 - N(X)$. So, $N(-.1297) = 1 - .5516 = .4484$, and $N(-.3418) = .3662$. Thus, $P_{E,t} = 6.0331$.

12.23 Put-Call Parity

We will now show that the Black–Scholes formula satisfies put-call parity.

12.23.1 Proof

$$C_{E,t} = S_t N(d_1) - Xe^{-r(T-t)}N(d_2)$$

$$= S_t[1 - N(-d_1)] - Xe^{-r(T-t)}[1 - N(-d_2)]$$
$$= Xe^{-r(T-t)}N(-d_2) - S_tN(-d_1) + S_t - Xe^{-r(T-t)}$$
$$= P_t + S_t - Xe^{-r(T-t)}$$

12.24 Interpretation of $N(d_1)$ and $N(d_2)$

The Black–Scholes formula for call options states that

$$C_{E,t} = S_tN(d_1) - Xe^{-r(T-t)}N(d_2)$$

The option premium is invariant to the risk preferences of investors and is consequently valid in a world characterized by risk neutrality. A risk neutral investor would value the option as the discounted value of its expected payoff. Consequently, since

$$C_{E,t} = e^{-r(T-t)}[S_te^{r(T-t)}N(d_1) - XN(d_2)]$$

it is obvious that

$$[S_te^{r(T-t)}N(d_1) - XN(d_2)]$$

is the expected payoff from the option from the perspective of a risk neutral investor.

$S_te^{r(T-t)}N(d_1)$ is the expected value of a variable in a risk neutral world, that is equal to S_T if the option is exercised, and is equal to zero otherwise. $N(d_2)$ is the probability that the option will be exercised in a risk neutral world. If the option is exercised there will be an outflow of X else the outflow will be zero. Consequently, $XN(d_2)$ is the expected outflow on account of the exercise price.

The formula for puts states that

$$P_{E,t} = Xe^{-r(T-t)}N(-d_2) - S_tN(-d_1)$$
$$\Rightarrow = e^{-r(T-t)}[XN(-d_2) - S_te^{r(T-t)}N(-d_1)]$$

If $N(d_2)$ is the probability that a call option with an exercise price of X is exercised in a risk neutral world then $1 - N(d_2)$ or $N(-d_2)$ is the probability that a put with the same exercise price will be exercised. Thus $XN(-d_2)$ is the expected inflow on account of the exercise price. $S_te^{r(T-t)}N(-d_1)$ is the expected value of a variable that will be equal to S_T if the option is exercised and is equal to zero otherwise.

12.25 Implied Volatility

Of all the variables required to estimate the option premia using the Black–Scholes formula, the most difficult to obtain is an estimate of the volatility. The stock price and the riskless rate can be deduced from market data, and the exercise price and time to maturity are known from the terms of the option contract. One way of estimating the volatility is using a vector of historical stock price data, as we have seen above. Another way is to estimate the volatility implicit in the option prices that are observed in the market.

Let C_i, i ranging from 1 to N be a vector of option prices that are obtained from the market. Estimate the vector of option prices for an assumed value of σ using the Black–Scholes formula, for a given set of other variables. Let us denote the model values by M_i, i ranging from 1 to N. The error for the ith observation is given by $e_i = C_i - M_i$. The optimal estimate of σ is the value that minimizes the sum of squared errors for the series of data. That is, the optimal estimate $\hat{\sigma}$ is the one that minimizes

$$\sum_{i=1}^{N}[C_i - M_i]^2$$

The volatility that is obtained in this manner is referred to as the implied volatility.

If the Black–Scholes model is strictly valid, then all options on a given stock, which have the same time to expiration, should give an identical implied volatility. However, very often it is observed in practice that at-the-money options give the lowest implied volatility, while deep-in-the-money and deep-out-of-the-money options give the highest volatility.

When the implied volatility is plotted against the exercise price, for options with the same time to expiration, we get a u-shaped pattern that is referred to as the *Volatility Smile*. The very fact that we observe such patterns tells us that the Black–Scholes model does not fully capture all the facets of option pricing. Options with higher implied volatilities are obviously more expensive than those with lower implied volatilities, a phenomenon that the Black–Scholes formula is unable to explain.

12.26 European Options on Dividend Paying Stocks

For the ensuing analysis, we will assume that we can predict with certainty the dividends that will be paid during the life of the option.[12]

We will assume that the stock price at a given point in time is the sum of two components. There is a component that grows at the riskless rate of return, which will be used to pay the dividends during the life of the option. In addition, there is a risky component.

The value of the riskless component is the present value of all dividends during the life of the option. The difference between the total stock price S_t and the present value of the riskless component, is the value of the risky component S_t^*. The Black–Scholes formula can then be applied using S_t^* instead of S_t.

12.26.1 Example

Consider the same data we just used. Assume that a dividend of \$ 5 is paid after 3 months.

[12]This may not be unreasonable for short term options.

$$S_t^* = 100 - 5e^{-.10\times.25} = 95.1235.$$

$$d_1 = \frac{\ln\left(\frac{95.1235}{100}\right) + \left(.10 + \frac{(.30)^2}{2}\right).5}{.3\sqrt{.5}} = .1061$$

$$d_2 = .1061 - .2121 = -.1060$$

$$N(.1061) = .5422; N(-.1060) = .4578$$

$$C_{E,t} = 95.1235 \times .5422 - 100e^{-.10\times.5} \times .4578 = 8.0287$$

$$P_{E,t} = 100e^{-.10\times.5} \times .5422 - 95.1235 \times .4578 = 8.0281$$

As you can see, due to the dividend, the call price has come down, while the put price has gone up.

12.27 Using the Binomial Model in Practice

While illustrating the binomial model, we chose the parameters u and d in a rather arbitrary fashion. In practice, the parameters u, d, and p, are chosen in such a way that the moments of the discrete time stock price process being modeled correspond to the moments of the lognormal distribution of stock prices, which is the assumption underlying the Black–Scholes model. We will now illustrate two approaches to estimating the parameters in practice.

12.27.1 Method-A

Given the current price S_t, the price next period, S_T, can be either uS_t where u represents the up state, or dS_t where d represents the down state. The expected value of S_T given S_t is

$$puS_t + (1-p)dS_t \tag{12.44}$$

In a risk neutral world we know that for a lognormal distribution

$$E(S_T) = S_t e^{r(T-t)} \tag{12.45}$$

If we equate the two, we get

$$pu + (1-p)d = e^{r(T-t)}$$

$$\Rightarrow p = \frac{e^{r(T-t)} - d}{u-d} \tag{12.46}$$

If we denote $T-t$ as $\triangle t$, we can state the risk neutral probability of an up move as $p = \frac{e^{r\triangle t} - d}{u-d}$. The variance of S_T from the binomial process is

$$pu^2 S_t{}^2 + (1-p)d^2 S_t{}^2 - [puS_t + (1-p)dS_t]^2$$
$$= [pu^2 + (1-p)d^2 - e^{2r\triangle t}]S_t{}^2 \tag{12.47}$$

The variance of S_T from the lognormal distribution is

$$e^{2r\triangle t}[e^{\sigma^2\triangle t} - 1]S_t{}^2$$

If we equate the two, we get

$$pu^2 + (1-p)d^2 = e^{(2r+\sigma^2)\Delta t} \tag{12.48}$$

We now have two equations in three unknowns. The third condition that is normally imposed is $u = \frac{1}{d}$. The system of equations can now be solved to yield the following values for the three parameters.[13]

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{12.49}$$

$$u = A + \sqrt{A^2 - 1} \text{ and } d = A - \sqrt{A^2 - 1}$$

where

$$A = \frac{1}{2}\left[e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\right] \tag{12.50}$$

12.27.2 Method-B

Another approach is to specify the following values for u and d.[14]

$$u = e^{\sigma\sqrt{\Delta t}} \text{ and } d = e^{-\sigma\sqrt{\Delta t}}$$

Obviously d is the reciprocal of u. The variance of the binomial process is given by

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}$$

Using the result that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$ and discarding terms with Δt^2 and higher powers of Δt, it can be shown that the variance of the binomial process tends towards $\sigma^2(T-t)$ as $T - t = \Delta t \to 0$.

Example Take the case of a stock which is currently priced at \$ 100. Consider a one year call option with an exercise price of \$ 100 and one year to expiration. Let the riskless rate be 10% per annum, and the volatility be 30% per annum. If we model the stock price process as a two period model, we will get the following parameters.

$$u = e^{0.30\sqrt{0.5}} = 1.2363; d = 0.8089; p = \frac{e^{0.10\times 0.5} - 0.8089}{1.2363 - 0.8089} = 0.5671$$

The variance of the binomial process is

$$e^{r\Delta t}(u+d) - ud - e^{2r\Delta t} = e^{0.10\times 0.5}(1.2363 + 0.8089) - 1.0 - e^{2\times 0.10\times 0.5}$$

$$= 0.044889$$

The variance of the continuous time process is

$$0.3 \times 0.3 \times 0.5 = 0.045$$

The approximation is excellent.

[13]See Wilmott et al. (1999).

[14]See Hull (2005).

12.28 An Introduction to the Greeks

We have discussed the variables which influence the option premium, and their relationship with the price of an option. The rate of change of the option price with respect to these variables is denoted by various Greek symbols, and consequently this area of option theory is often referred to as 'The Greeks'.

12.28.1 Delta

Delta represents the rate of change of the option premium with respect to the price of the underlying asset, keeping all the other variables constant. In other words, delta is the partial derivative of the option price with respect to the price of the underlying asset. The delta of a call option will always lie between zero and one. For deep out of the money options, delta will be close to zero, whereas for options that are very deep in the money, the delta will be close to one. It must be remembered that delta represents the change in the option price for an infinitesimal change in the asset price. Consequently, as the asset price changes, so will the delta. Thus, while we may interpret a delta value of say 0.45 to mean that for a one dollar move in the asset price, the option premium will move by 45 cents, it must be remembered that is accurate only for an infinitesimal change in the asset price.

Even if the asset price were to remain constant, the delta of an option will change with the sheer passage of time. As the call option approaches maturity, the delta will tend towards one if the option were to be in the money, whereas it will tend towards zero, if the option were to be out of the money.

For put options, quite obviously delta will be between 0 and -1.

12.28.2 Gamma

The rate of change of delta with respect to the asset price is called the Gamma of the option. Gamma will always be positive and tends to be at its peak when the option is near the money.

12.28.3 Vega

Vega is the derivative of the option price with respect to the volatility or the standard deviation of the rate of return of the underlying asset. The vega will be positive for both call and put options, as should be obvious from the arguments presented earlier.

12.28.4 Theta

As we have discussed earlier, options are wasting assets. That is, with everything else remaining constant, their values decline with the passage of time. Theta is a measure of the time decay of the option premium. It is expressed as the negative of the rate of change of the option premium with respect to the time to maturity. For European and American calls, theta will always be negative, indicating that option premia decline with time. The same is true for American puts. For European puts theta is usually negative. However, for certain deep in the money puts, theta can be positive.

12.28.5 Rho

Rho represents the rate of change of the option premium with respect to the riskless rate of interest. For call options rho will be positive, whereas for put options it will be negative.

12.28.6 The Black–Scholes Formula and the Greeks

Delta The delta for a European call option is equal to $N(d_1)$ and that for a European put option is equal to $-N(-d_1)$. As we would expect the delta for a call is positive, while that for a put is negative.

Gamma The gamma for both European call and put options is equal to $\frac{n(d_1)}{S_t\sigma\sqrt{T-t}}$. Thus gamma is positive for both calls as well as puts.

Vega The vega for European call and put options is equal to $S_t n(d_1)\sqrt{T-t}$. As we would expect, vega is positive for both calls as well as puts.

Theta Theta is defined as the negative of the rate of change of the option premium with respect to the time to maturity. For European call options

$$\text{Theta} = -rXe^{-r(T-t)}N(d_2) - S_t n(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

while for European puts it is

$$rXe^{-r(T-t)}N(-d_2) - S_t n(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

For call options theta is negative, which is consistent with the fact that ceteris paribus, call premiums decline with the passage of time. For puts theta may be either positive or negative. As we have seen earlier, certain deep in the money puts can have a negative time value. In such cases the option premium will actually increase with the passage of time, if all other variables were to remain constant.

Rho For a European call option

$$\text{Rho} = X(T-t)e^{-r(T-t)}N(d_2)$$

while for European put options

$$\text{Rho} = -X(T-t)e^{-r(T-t)}N(-d_2)$$

As we would expect, for call options, rho is positive, while for put options it is negative.

Sensitivity with respect to the Exercise Price

$$\frac{\partial C}{\partial X} = -e^{-r(T-t)}N(d_2)$$

$$\frac{\partial P}{\partial X} = e^{-r(T-t)}N(-d_2)$$

Table 12.2 Call and Put Option Deltas for Various Stock Prices

Stock Price	Call Delta	Put Delta
5	0	−1.0
10	0	−1.0
15	0	−1.0
20	2.17×10^{-14}	−1.0
25	2.96×10^{-10}	−1.0
30	4.82×10^{-08}	−1.0
35	2.04×10^{-06}	−0.999998
40	3.48×10^{-05}	−0.999965
45	0.00031	−0.99969
50	0.001718	−0.998282
55	0.006635	−0.993365
60	0.019410	−0.980599
65	0.045613	−0.954387
70	0.090186	−0.909814
75	0.155201	−0.844799
80	0.238808	−0.761192
85	0.335654	−0.664346
90	0.438448	−0.561552
95	0.539816	−0.460184
100	0.633737	−0.366263
105	0.71626	−0.28374
110	0.785547	−0.214453
115	0.841493	−0.158507
120	0.885171	−0.114829
125	0.918292	−0.081708
130	0.942782	−0.057218
135	0.960496	−0.039504
140	0.973067	−0.026933
145	0.98184	−0.01816
150	0.987875	−0.012125

Thus the call option premium declines with an increase in the exercise price, keeping all the other variables constant, while the put option premium increases with an increase in the exercise price.

12.28.7 Computing the Delta for Varying Degrees of Moneyness

Consider the following data. The exercise price is equal to $ 100. The riskless rate is 10% per annum and the volatility is 30% per annum. The options have six months to maturity. Table 12.2 above lists the values of delta for both European calls and puts for stock prices ranging from $ 5 to $ 150 in increments of $ 5.

The relationship between delta and the underlying stock price is graphically depicted for European call and put options in Figs 12.7 and 12.8 respectively.

Figure 12.7 Delta for a Call Option

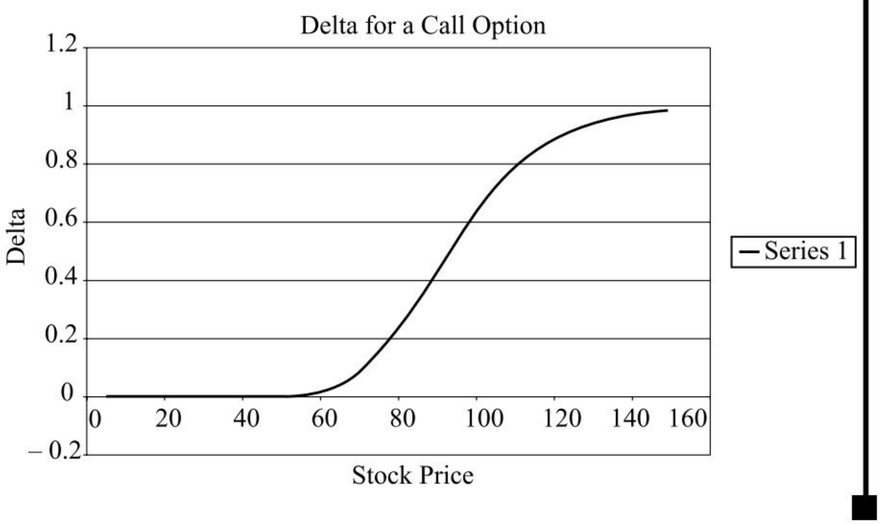
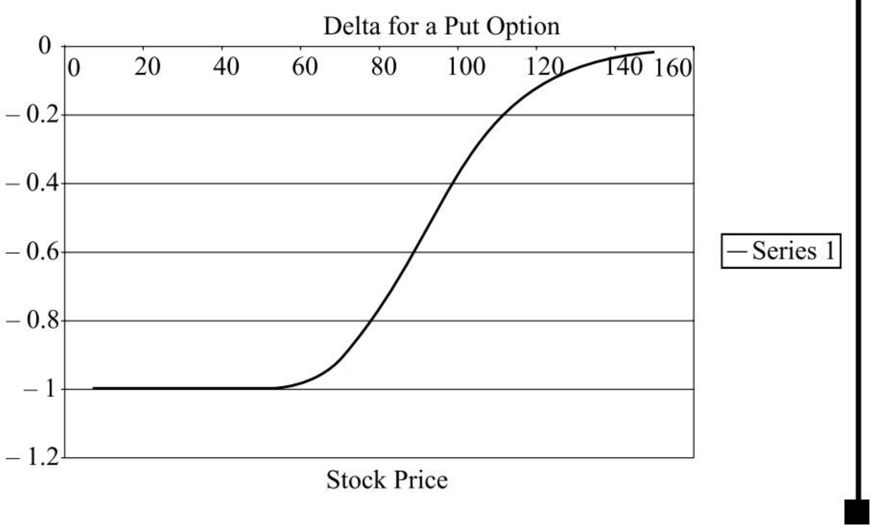


Figure 12.8 Delta for a Put Option



12.28.8 Creating a Delta Neutral Position

Consider a portfolio of N_1 shares and N_2 call options. The delta for a call is $\frac{\partial C}{\partial S}$, while the delta for a stock is one. If we denote the value of the portfolio as π

$$\frac{\partial \pi}{\partial S} = N_1 + N_2 \frac{\partial C}{\partial S} \tag{12.51}$$

If the portfolio is to be delta neutral, N_1 and N_2 must be chosen in such a way that

$$\frac{\partial \pi}{\partial S} = 0$$

$$\Rightarrow N_1 = -\frac{\partial C}{\partial S} N_2 \tag{12.52}$$

Consider an investor who has sold 50 call options contracts, each of which is for 100 shares. Let the delta of the option be 0.6337. In order to make the position delta neutral, he needs to go long in

$$0.6337 \times 50 \times 100 = 3{,}168.50 \text{ shares}$$

If the stock price were to go up by \$ 1, he would gain \$ 3,168.50 from the shares. However, a dollar's increase in the stock price would lead to an increase of 0.6337 in the option premium. Thus the short position in the option would lead to a loss of \$ 3,168.50.

It must be remembered that delta is a function of the stock price, and hence will change as the stock price changes. Consequently, in order to maintain the neutrality of the position, it must be periodically rebalanced.

The delta of a put option is negative. Hence to hedge a short position in puts, the investor needs to go short in the required number of shares.

SUGGESTIONS FOR FURTHER READING

1. Chance D.M. *An Introduction to Derivatives and Risk Management.* Thomson; South-Western, 2004.
2. Dewynne J., Howison S., and P. Wilmott *The Mathematics of Financial Derivatives: A Student Introduction.* Cambridge University Press, 1999.
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4. Kolb R.W. *Futures, Options, and Swaps.* Blackwell, 1999.
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- www.ivolatility.com
- www.optiontradingpedia.com
- www.sitmo.com

Concept Check

State whether the following statements are True or False.

1. Exchange traded options are usually dividend protected.
2. The binomial model requires the riskless rate of interest to be a constant.
3. The self-financing property breaks down in those cases where it is optimal to exercise an option prior to expiration.
4. Black–Scholes assumed that stock prices follow an Ito process.
5. An Ito process must have a constant drift and variance rate.
6. The Black–Scholes p.d.e. is valid only for European calls and puts on non-dividend paying stocks.
7. The Black–Scholes formula is applicable only for a world full of risk-neutral investors.
8. In a lognormal random walk, if the stock price were to attain a value of zero, it would stay there.
9. In a risk neutral world, the expected rate of return on all securities will be the riskless rate.
10. In a risk-neutral world, the probability that a European call will be exercised is $N(d_2)$.
11. The delta of a call will range between zero and one.
12. The delta of a put will range from minus one to zero.
13. Delta is a constant for European calls and puts.
14. Gamma is positive for both calls and puts.
15. Gamma tends to be at its peak for near-the-money options.
16. Vega is positive for both calls and puts.
17. Theta may be positive for deep-out-of-the-money European calls.
18. Theta may be positive for deep-in-the-money European puts.
19. As per the Black–Scholes model, gamma is identical for both calls and puts on the same stock, with the same exercise price and time to expiration.
20. As per the Black–Scholes formula, vega is identical for both calls and puts on the same stock, with the same exercise price and time to expiration.

Questions and Problems

Question-I

A stock is currently priced at \$ 100. European call and put options are available with an exercise price of \$ 100. Given a value for the stock price, it may either go up by 25% at the end of the next period, or else it may go down by 25%. The riskless rate of interest is 10% per period. The stock is not scheduled to pay any dividends during the life of the option.

1. Compute the value of a call as well as a put using the two period Binomial model.
2. Compute the value of an American put option using the two period Binomial model.
3. Form a replicating portfolio for the European call option and show that it is self-financing.
4. Form a replicating portfolio for the American put option. Show that the self-financing property breaks down when the option is exercised early.

Question-II

A stock is currently priced at $ 100. It pays a constant dividend yield of 10% per period. Given a value for the stock price, it may either go up by 25% at the end of the next period, or else it may go down by 25%. The riskless rate of interest is 15% per period. Take the case of a call option with an exercise price of $ 100.

1. Compute the value of the option using the two period Binomial model assuming that it is European in nature.
2. Compute the value of the option using the two period Binomial model assuming that it is American in nature.

Question-III

IBM shares are currently priced at $ 100. European call and put options with an exercise price of $ 90 are available with nine months to maturity. The riskless rate of interest is 10% per annum and the volatility is 25% per annum.

1. Compute the value of the call using the Black–Scholes model.
2. Compute the value of the put using the Black–Scholes model.
3. Calculate delta, gamma, vega, theta, and rho for the call as well as the put options.

Question-IV

Consider the data given in the previous question. Assume that the stock will pay a cash dividend of $ 5 after three months, and once again after six months.

Recompute the value of the call option using the Black–Scholes model.

Question-V

GM shares are currently priced at $ 60. Call options with an exercise price of $ 60 and six months to maturity are available. The riskless rate is 10% per annum and the volatility is 20% per annum.

Compute the value of a European call option using the two period Binomial model.

Question-VI

A stock is currently priced at $ 40. European call and put options with an exercise price of $ 50, and six months to expiration, are available. The riskless rate of interest is 8% per annum, and the volatility is 25% per annum.

1. What is the probability that a call option will be exercised in a risk-neutral world?

2. What is the expected value of a variable that is equal to S_T if the call is exercised and zero otherwise?
3. What is the probability that a put option will be exercised in a risk-neutral world?
4. What is the expected value of a variable that is equal to S_T if the put is exercised and zero otherwise?

Question-VII

What exactly is the concept of risk-neutral valuation? Discuss.

Question-VIII

The current stock price is \$ 80. European call and put options with an exercise price of \$ 80 and six months to expiration are available. The risk-less rate is 10% per annum and the volatility is 20% per annum.

An investor wants to create a delta neutral position using call and put options. If he goes long in 100 call contracts, where each contract is for 100 shares, what position should he take in the put options?

Question-IX

Consider the following data vector.

Day	Closing Price
1	44.64
2	46.43
3	45.18
4	46.58
5	45.23
6	45.29
7	45.45
8	44.69
9	46.80
10	46.14
11	45.61
12	46.97
13	45.96
14	45.78
15	45.44
16	44.66
17	46.84
18	46.46
19	45.85
20	45.35
21	45.73
22	45.07
23	45.89
24	44.93
25	46.36

Calculate the annual volatility assuming that the year has 252 days.

Question-X

Consider the following data.

Day	Low Price	High Price
1	42.43	46.09
2	42.30	46.37
3	42.10	45.57
4	44.30	47.40
5	42.80	47.96
6	42.24	47.80
7	42.17	47.01
8	42.97	47.95
9	42.97	46.66
10	43.50	46.45
11	42.34	46.51
12	44.00	45.59
13	43.13	46.04
14	43.24	47.70
15	42.73	46.82
16	44.17	47.67
17	42.38	47.23
18	43.26	47.63
19	43.32	45.96
20	42.58	46.56
21	43.63	45.71
22	43.86	47.02
23	42.17	47.20
24	43.50	45.64
25	43.23	46.18

Calculate the annual volatility using Parkinson's formula, assuming that the year has 252 days.

Appendix–XII

Derivation of Delta for the Black–Scholes Model

The Black–Scholes formula states that

$$C_{E,t} = S_t N(d_1) - Xe^{-r(T-t)}N(d_2)$$

where
$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

and $$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

$$N(d_1) = \int_{-\infty}^{d_1} n(d_1)$$

and $$N(d_2) = \int_{-\infty}^{d_2} n(d_2)$$

where $$n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$\frac{\partial C_{E,t}}{\partial S_t} = N(d_1) + S_t\frac{\partial N(d_1)}{\partial S_t} - Xe^{-r(T-t)}\frac{\partial N(d_2)}{\partial S_t}$$

$$\frac{\partial N(d_1)}{\partial S_t} = \frac{\partial N(d_1)}{\partial d_1} \times \frac{\partial d_1}{\partial S_t}$$

$$\frac{\partial N(d_2)}{\partial S_t} = \frac{\partial N(d_2)}{\partial d_2} \times \frac{\partial d_2}{\partial S_t}$$

$$\frac{\partial N(d_1)}{\partial d_1} = n(d_1)$$

and $$\frac{\partial N(d_2)}{\partial d_2} = n(d_2)$$

$$\frac{\partial d_1}{\partial S_t} = \frac{1}{\sigma\sqrt{T-t}} \times \frac{1}{X} \times \frac{X}{S_t} = \frac{1}{\sigma S_t\sqrt{T-t}}$$

$$\frac{\partial d_2}{\partial S_t} = \frac{\partial d_1}{\partial S_t} = \frac{1}{\sigma S_t\sqrt{T-t}}$$

$$\frac{\partial C_{E,t}}{\partial S_t} = N(d_1) + S_t n(d_1)\frac{1}{\sigma S_t\sqrt{T-t}} - Xe^{-r(T-t)}n(d_2)\frac{1}{\sigma S_t\sqrt{T-t}}$$

$$= N(d_1) + \frac{n(d_1)}{\sigma\sqrt{T-t}} - Xe^{-r(T-t)}\frac{n(d_2)}{\sigma S_t\sqrt{T-t}}$$

$$n(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{{d_1}^2}{2}}$$

$$n(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{{d_2}^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}}e^{\left[\frac{-{d_1}^2}{2} - \frac{\sigma^2(T-t)}{2} + \frac{2d_1\sigma\sqrt{T-t}}{2}\right]}$$

$$\frac{\partial C_{E,t}}{\partial S_t} = N(d_1) + \frac{n(d_1)}{\sigma\sqrt{T-t}} - Xe^{-r(T-t)}\frac{n(d_2)}{\sigma S_t\sqrt{T-t}}$$

$$= N(d_1) + \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sigma\sqrt{T-t}}\left[e^{-\frac{d_1{}^2}{2}} - \frac{Xe^{-r(T-t)}}{S_t}e^{-\frac{d_2{}^2}{2}}\right]$$

$$= N(d_1) + \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sigma\sqrt{T-t}}\left[e^{-\frac{d_1{}^2}{2}} - \frac{X}{S_t}e^{-\left[\frac{d_1{}^2}{2} - \ln\left(\frac{S_t}{X}\right)\right]}\right]$$

$$= N(d_1) + \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sigma\sqrt{T-t}}\left[e^{-\frac{d_1{}^2}{2}} - e^{-\frac{d_1{}^2}{2}}\right] = N(d_1)$$

13

Options on Stock Indexes, Foreign Currencies, Futures Contracts, and Volatility Indexes

13.1 The Merton Model

Before we go on to analyze options on stock indexes, foreign currencies, and futures contracts, let us first derive an equivalent of the Black–Scholes formula for a stock that pays a continuous dividend yield. This model was derived by Merton and has implications for option pricing models for other financial assets such as stock indexes and foreign currencies.

Take the case of a stock that evolves from a current price of S_t to a value of S_T by time T. If this stock were to pay a continuous dividend yield at the rate of δ, the dividend can be construed as a leakage of value from it. Thus, if it were to pay such a dividend it would evolve to a value of $S_T e^{-\delta(T-t)}$ by time T. This price movement is identical to what a non-dividend paying stock which is currently price at $S_t e^{-\delta(T-t)}$ would experience. Thus the Black–Scholes formula can be applied to the stock paying a continuous yield of δ, if we replace its price S_t with $S_t e^{-\delta(T-t)}$.

We know that for a non-dividend paying stock

$$C_{E,t} = S_t N(d_1) - X e^{-r(T-t)} N(d_2) \tag{13.1}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \tag{13.2}$$

and

$$d_2 = d_1 - \sigma\sqrt{(T-t)} \tag{13.3}$$

If we substitute S_t with $S_t e^{-\delta(T-t)}$ we get

$$C_{E,t} = S_t e^{-\delta(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2) \tag{13.4}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \tag{13.5}$$

and

$$d_2 = d_1 - \sigma\sqrt{(T-t)} \tag{13.6}$$

The corresponding formula for European puts on a stock that pays a continuous dividend yield is

$$P_{E,t} = X e^{-r(T-t)} N(-d_2) - S_t e^{-\delta(T-t)} N(-d_1) \tag{13.7}$$

where d_1 and d_2 are as defined for the call options.

13.1.1 The Underlying Rationale

We defined the Ito process for the stock price of a non-dividend paying stock as

$$dS = \mu S dt + \sigma S dZ \tag{13.8}$$

The equivalent process for a stock that pays a continuous dividend yield is

$$dS = (\mu - \delta) S dt + \sigma S dZ \tag{13.9}$$

Consider a derivative security F that is a function of S and t.

$$dF = \left[\frac{\partial F}{\partial S}(\mu - \delta)S + \frac{\partial F}{\partial t} + \frac{1}{2} \times \frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right]dt + \frac{\partial F}{\partial S}\sigma S dZ \tag{13.10}$$

Let π be the value of a portfolio that is long in $\frac{\partial F}{\partial S}$ units of the stock and short in one unit of the derivative.

$$\pi = \frac{\partial F}{\partial S} S - F \tag{13.11}$$

The change in π over a small interval of time Δt is given by

$$\Delta\pi = \frac{\partial F}{\partial S}[\Delta S + \delta S \Delta t] - \Delta F$$

$$= \left[-\frac{\partial F}{\partial t} - \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 + \frac{\partial F}{\partial S}\delta S\right]\Delta t \tag{13.12}$$

Since this portfolio by construction is instantaneously riskless

$$\Delta\pi = r\pi\Delta t$$

where r is the riskless rate of interest. Hence

$$\left[-\frac{\partial F}{\partial t} - \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 + \frac{\partial F}{\partial S}\delta S\right]\Delta t = r\left(-F + \frac{\partial F}{\partial S}S\right)\Delta t$$

$$\Rightarrow \frac{\partial F}{\partial t} + (r-\delta)S\frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 - rF = 0 \quad (13.13)$$

The Merton model is the solution to this equation by applying the relevant boundary conditions. For a European call option, the conditions are:

- $C(S, T) = \text{Max}[0, S_T - X]$
- $C(0, t) = 0$
- $C(S, t) \sim Se^{-\delta(T-t)}$ as S $\rightarrow \infty$

The first two conditions are identical to those for a non-dividend paying stock. The third condition is however different, and reflects the fact that there is leakage of value from the stock price.

Example Consider a stock that is currently priced at \$ 100. Call and put options with an exercise price of \$ 100 and six months to maturity are available. The riskless rate of interest is 10% per annum and the volatility of stock returns is 30% per annum. The stock pays a continuous dividend yield of 5% per annum.

$$d_1 = \frac{\ln\left(\frac{100}{100}\right) + \left(0.10 - 0.05 + \frac{0.30 \times 0.30}{2}\right)0.50}{0.30\sqrt{0.50}} = 0.2240$$

$$d_2 = 0.2240 - .2121 = 0.0119$$

$$N(d_1) = 0.5886 \text{ and } N(d_2) = 0.5047$$

$$C_{E,t} = 100e^{-.05\times 0.5} \times 0.5886 - 100e^{-.10\times 0.5} \times 0.5047 = \$\ 9.3982$$

$$P_{E,t} = 100e^{-.10\times 0.5} \times 0.4953 - 100e^{-.05\times 0.5} \times 0.4114 = \$\ 6.9902$$

Applying the Binomial Model for a Given Value of Sigma The Binomial model can be used to value options on a stock paying a continuous dividend yield. The parameters are typically defined as follows.

$$u = e^{\sigma\sqrt{\Delta t}} \quad (13.14)$$

$$d = e^{-\sigma\sqrt{\Delta t}} \quad (13.15)$$

The risk neutral probabilities can be derived as follows. Consider a portfolio consisting of α units of the stock and \$ B of riskless debt. Assume that the stock pays a dividend at a rate of δ% per annum and that the dividends are re-invested in the stock. Let us choose α and B such that:

$$\alpha u S_t e^{\delta\Delta t} + Be^{r\Delta t} = C_u \quad (13.16)$$

and

$$\alpha d S_t e^{\delta\Delta t} + Be^{r\Delta t} = C_d \quad (13.17)$$

$$\text{So } \alpha = \frac{C_u - C_d}{S_t(u-d)} e^{-\delta \Delta t} \tag{13.18}$$

and

$$B = [C_u - \alpha u S_t e^{\delta \Delta t}] e^{-r\Delta t} \tag{13.19}$$

$$= [C_u - \frac{u(C_u - C_d)}{u-d}] e^{-r\Delta t} (13.20) \tag{13.20}$$

$$\alpha S_t + B = C_t.$$

Therefore

$$\frac{C_u - C_d}{(u-d)} e^{-\delta \Delta t} + [C_u - \frac{u(C_u - C_d)}{u-d}] e^{-r\Delta t} = C_t$$

$$\Rightarrow \frac{C_u - C_d}{(u-d)} e^{(r-\delta)\Delta t} + [C_u - \frac{u(C_u - C_d)}{u-d}] = C_t e^{r\Delta t}$$

$$\Rightarrow C_u \left[\frac{e^{(r-\delta)\Delta t} - d}{u-d}\right] + C_d \left[\frac{u - e^{(r-\delta)\Delta t}}{u-d}\right] = C_t e^{r\Delta t}$$

$$\Rightarrow C_t = \left\{ C_u \left[\frac{e^{(r-\delta)\Delta t} - d}{u-d}\right] + C_d \left[\frac{u - e^{(r-\delta)\Delta t}}{u-d}\right] \right\} e^{-r\Delta t} \tag{13.21}$$

This is of the form $C_t = [pC_u + (1-p)C_d]e^{-r\Delta t}$ where $p = \left[\frac{e^{(r-\delta)\Delta t} - d}{u-d}\right]$.

13.1.2 Arbitrage Restrictions

13.2 Lower Bound for European Call Options

It can be demonstrated that $C_{E,t} \geq \text{Max}\left[0, S_t e^{-\delta(T-t)} - X e^{-r(T-t)}\right]$. If the expression $S_t e^{-\delta(T-t)} - X e^{-r(T-t)}$ were to be less than zero, then all that we can assert is that $C_{E,t} \geq 0$, because we know that an option cannot have a negative premium. However, if the expression were to be positive, then the option premium must be greater than or equal to it to preclude arbitrage.

13.2.1 Proof

Assume that

$C_{E,t} < S_t e^{-\delta(T-t)} - X e^{-r(T-t)} > 0$ or that $S_t e^{-\delta(T-t)} - X e^{-r(T-t)} - C_t > 0$. Consider Table 13.1.

The initial strategy entails the short sale of $e^{-\delta(T-t)}$ units of the stock. If the lender had not parted with the stock he could have reinvested the dividends in the stock itself, and consequently would have had one unit of the stock at the time of expiration of the option. Consequently the short seller must return one share of the stock to cover his short position, which explains the outflow of S_T at time T.

In Table 13.1, the cash flow at inception is positive by assumption. The cash flows at expiration are non-negative. Thus, this table reflects an arbitrage

Table 13.1 Lower Bound for a European Call Option

Action	Initial Cash Flow	Terminal Cash Flow	
		If $S_T > X$	If $S_T < X$
Short sell $e^{-\delta(T-t)}$ units of the Stock	$S_t e^{-\delta(T-t)}$	$-S_T$	$-S_T$
Buy the Call	$-C_{E,t}$	$(S_T - X)$	0
Lend the P.V. of X	$-Xe^{-r(T-t)}$	X	X
Total	$S_t e^{-\delta(T-t)} - Xe^{-r(T-t)} - C_{E,t}$	0	$X - S_T > 0$

opportunity. To preclude it, it therefore must be the case that

$$S_t e^{-\delta(T-t)} - Xe^{-r(T-t)} - C_{E,t} \leq 0$$

$$\Rightarrow C_{E,t} \geq S_t e^{-\delta(T-t)} - Xe^{-r(T-t)} \quad (13.22)$$

13.3 Lower Bound for European Put Options

In the case of European puts, the no-arbitrage condition is

$$P_{E,t} \geq \text{Max}[0, Xe^{-r(T-t)} - S_t e^{-\delta(T-t)}]$$

This condition can be interpreted as follows. If $Xe^{-r(T-t)} - S_t e^{-\delta(T-t)} < 0$, then all that we can assert is that $P_{E,t} > 0$, because an option cannot have a negative premium. However, if $Xe^{-r(T-t)} - S_t e^{-\delta(T-t)} > 0$ then we can show that $P_{E,t} \geq Xe^{-r(T-t)} - S_t e^{-\delta(T-t)}$. To prove it, consider the following strategy.

Table 13.2 Lower Bound for a European Put Option

Action	Initial Cash Flow	Terminal Cash Flow	
		If $S_T > X$	If $S_T < X$
Buy the Put	$-P_{E,t}$	0	$(X - S_T)$
Buy the $e^{-\delta(T-t)}$ units of the Stock	$-S_t e^{-\delta(T-t)}$	S_T	S_T
Borrow the P.V. of X	$Xe^{-r(T-t)}$	$-X$	$-X$
Total	$Xe^{-r(T-t)} - P_{E,t} - S_t e^{-\delta(T-t)}$	$S_T - X > 0$	0

To rule out arbitrage we require that

$$Xe^{-r(T-t)} - P_{E,t} - S_t e^{-\delta(T-t)} \leq 0$$

$$\Rightarrow P_{E,t} \geq Xe^{-r(T-t)} - S_t e^{-\delta(T-t)} \quad (13.23)$$

13.3.1 Put Call Parity for European Options

The put-call parity condition for European options on a stock paying a continuous dividend yield is

$$C_{E,t} - P_{E,t} = S_t e^{-\delta(T-t)} - Xe^{-r(T-t)}$$

The proof is given in Table 13.3.

Table 13.3 **Put Call Parity for European Options**

Action	Initial Cash Flow	Terminal Cash Flow If $S_T > X$	Terminal Cash Flow If $S_T < X$
Sell the Call	$C_{E,t}$	$-(S_T - X)$	0
Buy the Put	$-P_{E,t}$	0	$(X - S_T)$
Buy $e^{-\delta(T-t)}$ units of the Stock	$-S_t e^{-\delta(T-t)}$	S_T	S_T
Borrow the P.V. of X	$Xe^{-r(T-t)}$	$-X$	$-X$
Total	$C_{E,t} + Xe^{-r(T-t)} - P_{E,t} - S_t e^{-\delta(T-t)}$	0	0

The initial cash flow has to be less than or equal to zero to preclude arbitrage. However, if it were to be less, then we could reverse the above strategy and make arbitrage profits. Consequently to rule out any form of arbitrage, the initial cash flow must be zero. This implies that

$$C_{E,t} - P_{E,t} = S_t e^{-\delta(T-t)} - Xe^{-r(T-t)} \qquad (13.24)$$

13.3.2 The Greeks for European Options on a Stock Paying a Continuous Dividend Yield

13.3.3 Delta

For a European call: Delta = $e^{-\delta(T-t)}N(d_1)$
For a European put: Delta = $-e^{-\delta(T-t)}N(-d_1)$

13.3.4 Gamma

Gamma for both calls and puts is given by $\dfrac{e^{-\delta(T-t)}n(d_1)}{S_t\sigma\sqrt{T-t}}$

13.3.5 Vega

Vega for both calls and puts is given by $e^{-\delta(T-t)}S_t n(d_1)\sqrt{T-t}$

13.3.6 Theta

For a European call:

$$\text{Theta} = \delta e^{-\delta(T-t)}S_t N(d_1) - rXe^{-r(T-t)}N(d_2) - \frac{e^{-\delta(T-t)}S_t\sigma n(d_1)}{2\sqrt{T-t}}$$

For a European put:

$$\text{Theta} = -\delta e^{-\delta(T-t)} S_t N(-d_1) + rXe^{-r(T-t)} N(-d_2) - \frac{e^{-\delta(T-t)} S_t \sigma n(d_1)}{2\sqrt{T-t}}$$

13.3.7 Rho

For a European call: Rho = $e^{-r(T-t)} X(T-t)N(d_2)$
For a European put: Rho = -$e^{-r(T-t)} X(T-t)N(-d_2)$

13.4 Index Options

Options on prominent stock indices are traded in venues across the globe. In the US the CBOE offers options contracts on a number of indices. These include:

- The Dow Jones Industrial Average
- The S&P 100 Index
- The S&P 500 Index
- The NASDAQ 100 Index

Both European as well as American style options are offered on the S&P 100 index, whereas on the other indices only European options are available. All options expire on the Saturday following the third Friday of the expiration month. The contracts are cash settled and the multiplier is \$ 100. Thus if a call option is exercised, the holder will receive \$ 100 × $(I_T - X)$, whereas if a put is exercised he will receive \$ 100 × $(X - I_T)$. I_T is the index level at the time of exercise/expiration.[1]

13.4.1 Example

European call options on the S&P 100 index are available with an exercise price of 700, and six months to expiration. The riskless rate is 6% per annum and the volatility is 20% per annum. The dividend yield is 4% per annum and the current level of the index is 700.

The option price can be calculated using the Merton model.

$$d_1 = \frac{\left(0.06 - 0.04 + \frac{0.2 \times 0.2}{2}\right) 0.5}{0.2 \times \sqrt{0.5}} = 0.1414$$

$$d_2 = 0.1414 - 0.2 \times \sqrt{0.5} = 0$$

$$N(d_1) = 0.5562 \text{ and } N(d_2) = 0.5000$$

$$C_{E,t} = 700 \times e^{-0.04 \times 0.5} \times 0.5562 - 700 \times e^{-0.06 \times 0.5} \times 0.5000$$

$$= 381.6306 - 339.6559 = 41.9747$$

The premium per contract is $100 \times 41.9747 = \$\,4{,}197.47$.

[1]For options on the Dow Jones index, the index level is taken to be one hundredth of the value of the index.

13.5 Foreign Currency Options

Foreign currency options are traded on a number of exchanges. The Philadelphia Stock Exchange (PHLX) is the world's leading platform for exchange traded currency options. The exchange currently lists options contracts on six foreign currencies. These currencies and the contract sizes for the respective options are given in Table 13.4. The options are European style. At any point in time, contracts are available for the four quarterly months, March, June. September and December, and for the two nearest calendar months. Contracts expire on the Saturday following the third Friday of the expiration month. Exercise prices are expressed in terms of US cents per unit of foreign currency. For instance a call option on British pounds with an exercise price of \$ 195 would give the option buyer the right to buy pounds at \$ 1.95 per pound. Premiums for options are quoted in US cents per unit of the underlying currency. For instance a premium of 2.78 for a given option on British pounds is \$ 0.0278 per pound. Since each option contract is for 10,000 GBP, the total premium per contract is $0.0278 \times 10{,}000 = \$\,278.00$.

Table 13.4 Currencies and Contract Sizes

Currency	Contract Size
Australian Dollar	10,000 AUD
British Pound	10,000 GBP
Canadian Dollar	10,000 CAD
Euro	10,000 EUR
Japanese Yen	1,050,000 JPY
Swiss Franc	10,000 CHF

All the above contracts are cash settled.

13.5.1 Arbitrage Restrictions

A foreign currency is like a stock paying a continuous dividend yield. The leakage of value in this case, takes place at a rate of r_f, which is the riskless rate of interest in the foreign currency. So the lower bounds for European call and put options may be expressed as

$$C_{E,t} \geq \text{Max}[0, S_t e^{-r_f(T-t)} - X e^{-r(T-t)}] \quad (13.25)$$

and

$$P_{E,t} \geq \text{Max}[0, X e^{-r(T-t)} - S_t e^{-r_f(T-t)}] \quad (13.26)$$

The put-call parity relationship for European options is given by

$$C_{E,t} - P_{E,t} = S_t e^{-r_f(T-t)} - X e^{-r(T-t)} \quad (13.27)$$

13.6 The Garman Kohlhagen Model

This model is an extension of the Black–Scholes model. According to it

$$C_E = S_t e^{-r_f(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2) \tag{13.28}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - r_f + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{13.29}$$

and $$d_2 = d_1 - \sigma\sqrt{T-t}$$

13.6.1 Example

Consider European calls on British pounds.

Let $X = 200$, and the spot rate be 185. The riskless rate of interest in the US is 4.5%, while the rate in the UK is 6%. The time to expiration is six months, and the volatility of the spot rate is 30%.

$$d_1 = \frac{\ln\left(\frac{185}{200}\right) + \left(0.045 - 0.06 + \frac{(0.30)^2}{2}\right)0.5}{0.30\sqrt{0.5}} = -.2970$$

$$d_2 = -.5091 N(d_1) = 0.3832$$

and $$N(d_2) = 0.3053 C_E = 185e^{-0.06\times0.5} \times 0.3832$$
$$- 200e^{-0.045\times0.5} \times 0.3053 = 9.0953$$

13.6.2 The Binomial Model with Given Parameters

Let us take the case of call options on British pounds. Let S_t be the spot exchange rate quoted as USD/GBP. Consider a portfolio consisting of α USD of riskless debt in the foreign currency, and B USD of riskless debt in the domestic currency.

Let r = 1 + domestic riskless rate, and r_f = 1 + foreign riskless rate.

In a given period, the spot rate can either go up to uS_t, or go down to dS_t.

If the spot rate were to increase, the portfolio will be worth

$$\frac{\alpha}{S_t} r_f u S_t + Br \tag{13.30}$$

while, if it were to decline, it will be worth

$$\frac{\alpha}{S_t} r_f d S_t + Br \tag{13.31}$$

Let us choose α and B such that

$$\alpha u r_f + Br = C_u \tag{13.32}$$

and

$$\alpha d r_f + Br = C_d \tag{13.33}$$

Therefore

$$\alpha = \frac{C_u - C_d}{r_f(u-d)} \tag{13.34}$$

and

$$B = \frac{1}{r}\left[\frac{uC_d - dC_u}{u-d}\right] \tag{13.35}$$

Thus

$$C_t = \alpha + B = \frac{pC_u + (1-p)C_d}{r} \tag{13.36}$$

where

$$p = \frac{\frac{r}{r_f} - d}{u-d} \tag{13.37}$$

and

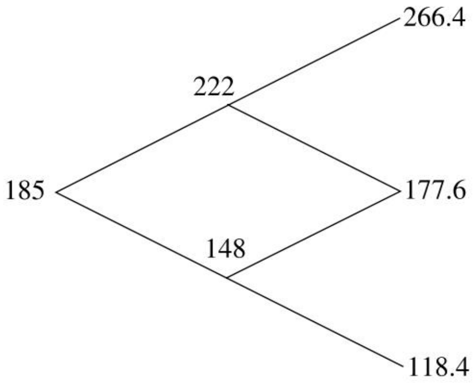
$$(1-p) = \frac{u - \frac{r}{r_f}}{u-d} \tag{13.38}$$

13.6.3 Example

We will now compute the price of a call option on British pounds using a two period Binomial model. Let $u = 1.2$, $d = .8$, $S_t = 185$, $r = 1.045$, $r_f = 1.06$, and $X = 200$.

The spot exchange rate tree, may be depicted as in Fig. 13.1.

Figure 13.1



$C_{uu,T} = 66.40, C_{ud,T} = 0, C_{dd,T} = 0$

$$p = \frac{\frac{1.045}{1.06} - 0.8}{0.4} = 0.4646 \text{ and } (1 - p) = 0.5354$$

$$C_{u,T-1} = \frac{0.4646 \times 66.4 + 0.5354 \times 0}{1.045} = 29.5210$$

The intrinsic value at this node is 22.

$$C_{d,(T-1)} = \frac{0.4646 \times 0 + 0.5354 \times 0}{1.045} = 0$$

The intrinsic value at this node is zero.

$$C_t = \frac{0.4646 \times 29.5210 + 0.5354 \times 0}{1.045} = 13.1248$$

The intrinsic value at this node is zero. Thus the call will be worth 13.1248 cents.

Applying the Binomial Model for Given Value of Sigma The formula for the option price using continuously compounded rates of interest r and r_f and the volatility of the exchange rate σ is similar to what we derived for a stock paying a continuous dividend yield. The only difference is that we replace δ with the riskless rate in the foreign currency r_f.

13.6.4 The Greeks

The greeks for foreign currency options are similar to those for a stock paying a continuous dividend yield. We have to once again replace δ with the foreign riskless rate r_f. In the case of Rho however we get two expressions, one for the partial derivative with respect to the domestic interest rate, ρ_r, and the other for the partial derivative with respect to the foreign riskless rate, ρ_{r_f}.

For a European call:

$$\rho_r = e^{-r(T-t)} X(T - t) N(d_2) \text{ and } \rho_{r_f} = -e^{-r_f(T-t)} S_t(T - t) N(d_1) \quad (13.39)$$

For a European put:

$$\rho_r = -e^{-r(T-t)} X(T - t) N(-d_2) \text{ and } \rho_{r_f} = e^{-r_f(T-t)} S_t(T - t) N(-d_1) \quad (13.40)$$

13.7 Futures Options

A futures option is an option on a futures contract. Such options are available on a number of successful futures contracts in many of the leading exchanges. Examples include options on index futures, interest rate futures, FOREX futures, and commodity futures.

13.7.1 Call Options

A call futures option gives the buyer the right to assume *a long position in a futures contract*. If and when the call holder decides to exercise, a long position will be established for him in the futures contract, and the contract will be immediately marked to market. Once the long position is entered into, the investor must either deposit adequate funds to meet the initial margin requirement of the futures contract, or else he must offset the futures position.

When the call holder exercises, a short position in the futures contract is established for the writer of the call. The writer's position will also be marked to market, and he must also deposit adequate margin for the futures position, or else, liquidate it.

Example Consider a person who buys a call futures option on crude oil. Each contract is for 1,000 barrels of oil. We will assume that the current futures price of oil is \$ 75.00 per barrel, and that the exercise price of the call is \$ 72.50 per barrel.

If the call holder exercises, he will get a long position in the futures contract. At the same time, due to marking to market, he will receive an inflow of $(75.00 - 72.50) \times 1,000 = \$\ 2,500$. How can we explain this inflow of \$ 2,500? The futures option contract gives the holder the right to go long in a futures contract at a price of \$ 72.50. If he exercises, and a long position is established for him, it will be at the prevailing futures price of \$ 75.00. To ensure that he effectively gets a long position at \$ 72.50, the difference between \$ 75.00 and \$ 72.50 must be paid to him. Notice, that if F_t is the current futures price, and X the exercise price, the call holder will exercise only if $F_t > X$. When he exercises, he will get an inflow equal to \$ $(F_t - X)$, which is the intrinsic value of the option.

13.7.2 Put Options

A put futures option gives the holder the right to go short in the futures contract at the exercise price. If the holder exercises, a long position in the futures contract will be established for the writer. Both the holder's and the writer's positions will be marked to market when the option is exercised, and they will both have to post the required margin for the futures contract, or else, liquidate their positions.

Example Consider a person who owns a put futures option on crude oil. The exercise price is \$ 75.00 per barrel, and the current futures price of oil is \$ 72.00 per barrel.

If the holder chooses to exercise, he will get a short position in the futures contract. At the same time, due to marking to market, he will receive a cash inflow of $(75.00 - 72.00) \times 1,000 = \$\ 3,000$. The rationale for this inflow should be obvious from the logic presented for call futures options. Notice, that if F_t is the current futures price, and X the exercise price, the call holder will exercise only if $F_t < X$. When he exercises, he will get an inflow equal to \$ $(X - F_t)$, which is the intrinsic value of the option.

13.8 Arbitrage Restrictions

13.8.1 Lower Bounds for European Call Futures Options

It can be demonstrated that $C_{E,t} \geq \text{Max}[0, (F_t - X)e^{-r(T-t)}]$. As before we know that $C_{E,t} \geq 0$ and consequently that explains the first part of the expression. However, if $(F_t - X)e^{-r(T-t)} \geq 0$ then we can demonstrate that the option premium must be greater in order to preclude arbitrage.

Proof Assume that $C_{E,t} < (F_t - X)e^{-r(T-t)} > 0$ or that $(F_t - X)e^{-r(T-t)} - C_{E,t} > 0$. Consider the following Table 13.5.

Table 13.5 Lower Bound for European Calls

Action	Initial Cash Flow	Terminal Cash Flow	
		$F_T > X$	$F_T \leq X$
Buy a call	$-C_{E,t}$	$F_T - X$	0
Sell a futures contract	0	$F_t - F_T$	$F_t - F_T$
Borrow P.V. of $(F_t - X)$	$(F_t - X)e^{-r(T-t)}$	$-(F_t - X)$	$-(F_t - X)$
Total	$-C_{E,t} + (F_t - X)e^{-r(T-t)}$	0	$X - F_T$

In this table, the cash flow at inception is positive by assumption and the subsequent cash flows are non-negative. Consequently, this table reflects an arbitrage opportunity. Therefore, in order to preclude arbitrage it must be the case that

$$-C_{E,t} + (F_t - X)e^{-r(T-t)} \leq 0$$
$$\Rightarrow C_{E,t} \geq (F_t - X)e^{-r(T-t)} \tag{13.41}$$

We also know that $C_{E,t}$ must be greater than zero. Therefore, the lower bound is

$$C_{E.t} \geq \text{Max}[0, (F_t - X)e^{-r(T-t)}] \tag{13.42}$$

The lower bound for a European call on the underlying asset, with the same expiration date is

$$C_{E,t} \geq \text{Max}[0, S_t - Xe^{-r(T-t)}] \tag{13.43}$$

For an asset which does not make any payouts, $F_t = S_t e^{r(T-t)}$. Therefore, the lower bound for a European futures call option may be written as

$$C_{E,t} \geq \text{Max}[0, S_t - Xe^{-r(T-t)}] \tag{13.44}$$

Thus, the two lower bounds are equivalent. Hence, if the futures option, and the futures contract expire at the same time, a European call on the futures contract is equivalent to a European call on the spot commodity. The logic is that, at expiration, $S_T = F_T$. Hence $\text{Max}[0, S_T - X] = \text{Max}[0, F_T - X]$. If the payoffs from the two options are the same, the options must be equivalent.

What about American call futures options? An American option must always be worth at least its intrinsic value. So for an American futures option,

$$C_A \geq \text{Max}[0, F_t - X] \tag{13.45}$$

This is a tighter lower bound because $(F_t - X) > (F_t - X)e^{-r(T-t)}$. Hence an American option will be priced higher than a corresponding European option, and may be exercised early. Remember that for a stock which makes no payouts, we showed that American call options will never be exercised early. American calls on futures contracts are clearly different.

13.8.2 Lower Bound for European Put Futures Options

In the case of European puts, the no-arbitrage condition is

$$P_{E,t} \geq \text{Max}[0, (X - F_t)e^{-r(T-t)}] \tag{13.46}$$

This condition has the usual interpretation. If $(X - F_t)e^{-r(T-t)} < 0$, then all that we can assert is that $P_{E,t} > 0$. However, if the expression is positive then we can demonstrate that $P_{E,t} \geq (X - F_t)e^{-r(T-t)}$. To prove it, consider the following strategy.

Table 13.6 **Lower Bound for European Puts**

Action	Initial Cash Flow	Terminal Cash Flow	
		$F_T \geq X$	$F_T < X$
Buy a put	$-P_{E,t}$	0	$X - F_T$
Buy a futures contract	0	$F_T - F_t$	$F_T - F_t$
Lend PV of $(F_t - X)$	$-(F_t - X)e^{-r(T-t)}$	$(F_t - X)$	$(F_t - X)$
Total	$-P_{E,t} - (F_t - X)e^{-r(T-t)}$	$F_T - X$	0

Proof To preclude arbitrage, we require that

$$-P_{E,t} - (F_t - X)e^{-r(T-t)} \leq 0$$

$$\Rightarrow P_{E,t} \geq (X - F_t)e^{-r(T-t)}$$

Thus, we can assert that

$$P_{E,t} \geq \text{Max}[0, (X - F_t)e^{-r(T-t)}]$$

For an asset which does not make any payouts, $F_t = S_t e^{r(T-t)}$. Therefore, the lower bound for a European futures put option may be written as

$$P_{E,t} \geq \text{Max}[0, Xe^{-r(T-t)} - S_t] \tag{13.47}$$

Thus the lower bound is equivalent to what we derived for a put option on the underlying asset. Hence, if the futures option, and the futures contract expire at the same time, a European put on the futures contract is equivalent to a European put on the spot commodity. The intrinsic value of an American put futures option is Max(0, $X - F_t$), which provides a tighter lower bound. Hence American put futures options may be exercised early.

13.8.3 Put-Call Parity

The put-call parity condition for European options on a futures contract is:

$$C_{E,t} - P_{E,t} = (F_t - X)e^{-r(T-t)}$$

The proof is given in Table 13.7.

Table 13.7 Put Call Parity

Action	Initial Cash Flow	Terminal Cash Flow	
		$F_T \geq X$	$F_T < X$
Sell a call	$C_{E,t}$	$-(F_T - X)$	0
Buy a put	$-P_{E,t}$	0	$(X - F_T)$
Buy futures	0	$F_T - F_t$	$F_T - F_t$
Lend P.V. of $(F_t - X)$	$-(F_t - X)e^{-r(T-t)}$	$(F_t - X)$	$(F_t - X)$
Total	$C_{E,t} - P_{E,t} - (F_t - X)e^{-r(T-t)}$	0	0

So to preclude arbitrage, we require that

$$C_{E,t} = P_{E,t} + (F_t - X)e^{-r(T-t)} \tag{13.48}$$

13.9 The Black Model

This model is a variation of the Black–Scholes model and is applicable for pricing European options on futures contracts. Assuming that the options and the underlying futures expire at the same time, Black showed that

$$C_{E,t} = e^{-r(T-t)}[F_t N(d_1) - XN(d_2)] \tag{13.49}$$

and

$$P_E = e^{-r(T-t)}[XN(-d_2) - F_t N(-d_1)] \tag{13.50}$$

where

$$d_1 = \frac{\ln\left(\frac{F}{X}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T-t}} \tag{13.51}$$

and $d_2 = d_1 - \sigma\sqrt{T-t}$

13.9.1 Example

The price of a six months futures contract on crude oil is \$ 75. Options are available with an exercise price of \$ 70 and six months to expiration. The riskless rate is 8% per annum and the volatility is 30% per annum.

$$d_1 = \frac{\ln\left(\frac{75}{70}\right) + \left(\frac{0.30 \times 0.30}{2}\right)0.5}{0.30\sqrt{0.5}}$$

$$= \frac{0.0915}{0.2121} = 0.4314$$

$$d_2 = 0.4314 - .2121 = 0.2193$$

$$N(0.4314) = 0.6669; N(0.2193) = 0.5868$$

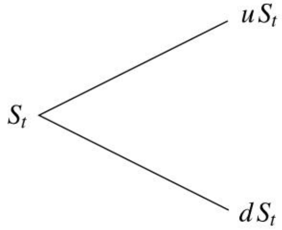
$$C_{E,t} = e^{-0.08\times.5}[75 \times 0.6669 - 70 \times 0.5868] = 8.5909$$

$$P_{E,t} = e^{-0.08\times.5}[70 \times 0.4132 - 75 \times 0.3331] = 3.7870$$

13.9.2 The Binomial Model

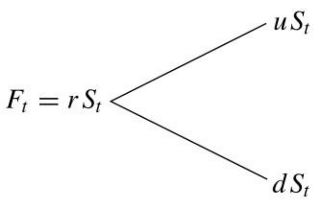
We will now see as to how the binomial model can be used to price futures options. To start with we will focus on the one period model. Let the stock price follow a binomial model as described in Fig. 13.2.

Figure 13.2



From the cost of carry model, the futures price at any node is given by $F = Sr^{(T-t)}$, where r is 1 + riskless rate and $T - t$ is the number of periods left till the maturity of the futures contract. Hence, the futures price can also be modeled using a binomial process as shown in Fig. 13.3.

Figure 13.3



Let us form a hedge portfolio consisting of a long position in α futures contracts, and \$ B in riskless debt. The initial investment = \$ B, because it costs nothing to get into the futures contract. If this portfolio is to replicate the call, we require that

$$\alpha\left(\frac{u}{r}F_t - F_t\right) + Br = C_u \tag{13.52}$$

and

$$\alpha\left(\frac{d}{r}F_t - F_t\right) + Br = C_d \tag{13.53}$$

Therefore

$$\alpha = \frac{C_u - C_d}{F_t\left(\frac{u}{r} - \frac{d}{r}\right)} \tag{13.54}$$

Let $u^* = \frac{u}{r}$ and $d^* = \frac{d}{r}$.

Therefore

$$\alpha = \frac{C_u - C_d}{F_t(u^* - d^*)} \tag{13.55}$$

Substituting for α we get

$$B = \frac{1}{r}\left[\frac{1 - d^*}{u^* - d^*}C_u + \frac{u^* - 1}{u^* - d^*}C_d\right] \tag{13.56}$$

$$p = \frac{1 - d^*}{u^* - d^*} = \frac{1 - \frac{d}{r}}{\frac{u}{r} - \frac{d}{r}} = \frac{r - d}{u - d} \tag{13.57}$$

Thus p and $(1 - p)$ can be calculated using either u^* and d^* or u and d. Therefore

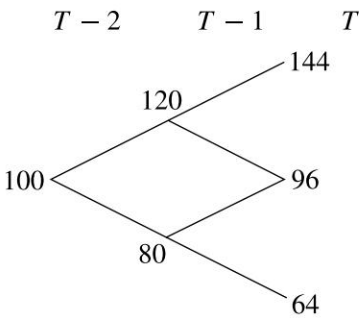
$$B = \frac{pC_u + (1 - p)C_d}{r} \tag{13.58}$$

Since the payoffs from the portfolio are the same as that from the call, $C_t = B$

Example A stock is currently valued at \$ 100. Every period the stock price may go up by 20% or go down by 20%. The riskless rate of interest is 5% per period. What is the value of a call futures option with two periods to expiration?

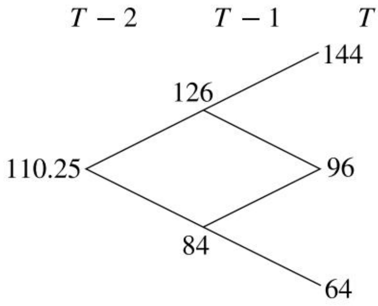
The stock price tree may be depicted as shown in Fig. 13.4.

Figure 13.4



The futures price tree can be depicted as shown in Fig. 13.5 for a contract with two periods to expiration. As you can see, at expiration, the futures price is equal to the spot price.

Figure 13.5



If we assume that the futures contract and the option on the futures contract expire at the same time, then

$$C_{uu,T} = 44,\ C_{ud,T} = 0,\ \text{and}\ C_{dd,T} = 0;\ p = 0.625\ \text{and}\ 1-p = 0.375$$

$$C_{u,T-1} = \frac{0.625 \times 44 + 0.375 \times 0}{1.05} = 26.1905$$

If we are dealing with American options, we have to check and see as to whether this is less than the intrinsic value. The I.V. at this node is 26, which is less. So the option will not be exercised early.

$$C_{d,T-1} = \frac{0.625 \times 0 + 0.375 \times 0}{1.05} = 0$$

The intrinsic value at this node is zero since the option is out of the money. Therefore

$$C_t = \frac{0.625 \times 26.1905 + 0.375 \times 0}{1.05} = 15.5896$$

Once again, we have to check for early exercise. The intrinsic value at $T-2$ is 10.25, so the option will not be exercised early. Thus the value of the American call futures option = \$ 15.5896.

13.9.3 Applying the Binomial Model for a Given Value of Sigma

The binomial model can be used to value futures options using the following parameters.

$$u = e^{\sigma\sqrt{\Delta t}} \tag{13.59}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{13.60}$$

$$p = \frac{1-d}{u-d} \tag{13.61}$$

13.10 Options on Futures versus Options on the Underlying

Futures options are often more actively traded than options on the underlying asset. One of the main reasons is that the underlying asset in the case of a futures option, namely the futures contract, is more liquid and consequently its prices are more reliable. In practice futures prices can be easily ascertained from the futures exchange, while the price of the underlying commodity may not be so transparent. This is particularly true for underlying assets like Treasury bonds which are largely traded over-the-counter.[2]

The other reason for the popularity of futures options is that the holder is not required to take possession of the underlying asset upon exercise, which is normally the case for options on the underlying asset. Since the exercise of a futures option leads to the establishment of a futures position for the holder, he has the option to offset and exit the market without taking delivery.

We have demonstrated earlier that a European futures option, whether a call or a put, must be worth the same as a European option on the underlying asset with the same time to maturity. However, if the European futures option were to expire before the expiration date of the underlying futures contract, then this would no longer be the case. In such a situation, the futures call will be worth more than a call on the underlying asset if the market is in contango, whereas it will be worth less than a call on the underlying asset if the market is in backwardation. For puts however, the case is the opposite. Puts on futures will be worth less than puts on the underlying if the futures market is in contango, whereas they will be worth more than puts on the underlying if the market is in backwardation. The same is the case for American futures options, irrespective of whether the futures contract expires at the same time as the option or after it.[3]

13.11 Portfolio Insurance

The term *portfolio insurance* refers to a strategy for protecting the principal value of a portfolio against a decline in the price. The strategy entails the acquisition of a put option, such that the floor value of the portfolio at the time of expiration of the put is the principal value of the portfolio.

[2]See Hull (2006).
[3]See Hull (2006).

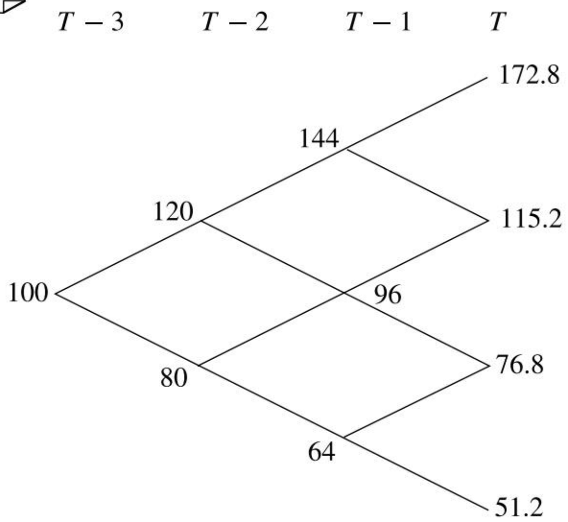
This kind of a protective put strategy may be accomplished by acquiring an exchange traded put option, or by replicating a put option using positions in the underlying asset and the riskless asset, as we have demonstrated in our discussion of the Binomial model in the previous chapter. The first type of insurance may be termed as *static portfolio insurance* while the second may be considered as *dynamic portfolio insurance*.[4]

13.11.1 Dynamic Option Replication

We will assume in this illustration that we have more capital at the outset than our target insurance floor. More specifically, we are going to assume we have enough capital to buy one unit of the underlying asset plus one put option on it.

Consider an asset that is currently valued at \$ 100. We will assume that every period the stock price may go up by 20% or go down by 20%. The riskless rate of interest is 5% per period. The evolution of the underlying asset over three periods may be depicted as in Fig. 13.6.

Figure 13.6



Let us now consider a European put with an exercise price of \$ 100. The value of the put at each node can be computed using the binomial model. The risk neutral probabilities are:

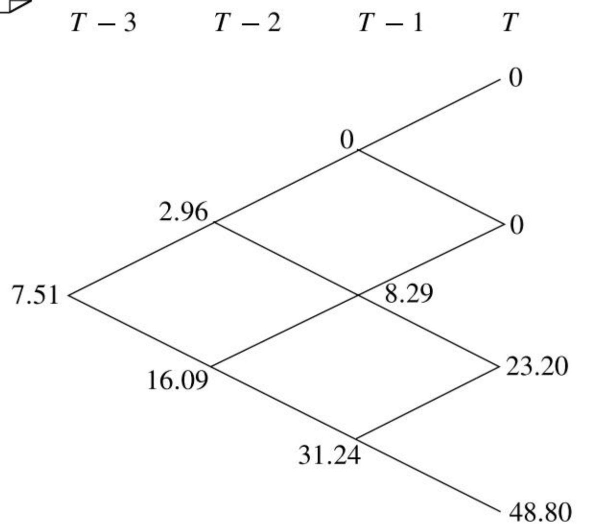
$$p = \frac{1.05 - 0.80}{1.20 - 0.80} = 0.625 \text{ and } 1 - p = 0.375$$

$$P_{uuu} = 0; P_{uud} = 0; P_{udd} = 23.20; P_{ddd} = 48.80$$

[4]See Stoll and Whaley (1993).

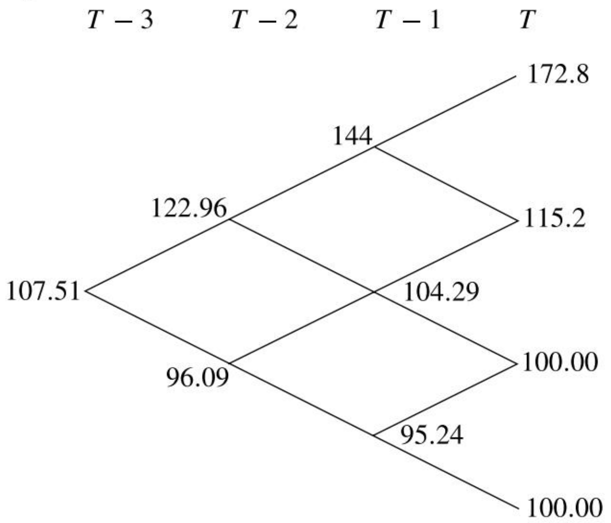
P_{uu} is obviously zero. It can be shown that $P_{ud} = 8.29$, $P_{dd} = 31.24$, $P_u = 2.96$, $P_d = 16.09$,and $P_t = 7.51$. The evolution of the put price can consequently be depicted as shown in Fig. 13.7.

Figure 13.7



So the insured portfolio consisting of one unit of the underlying asset and a put option follows the price process depicted in Fig. 13.8.

Figure 13.8



As we have already seen from the study of the binomial model, a put option can be replicated by a combination of positions in the underlying asset and the riskless asset. We will denote the respective positions by α and B. For instance

$$\alpha_{T-3} = \frac{P_u - P_d}{S_{T-3} \times (u-d)} = \frac{2.96 - 16.09}{100 \times (1.20 - 0.80)} = -0.32825$$

$$B_{T-3} = \frac{uP_d - dP_u}{r \times (u-d)} = \frac{1.20 \times 16.09 - 0.8 \times 2.96}{1.05 \times 0.40} = \$\,40.3333$$

The composition of the replicating portfolios at the other nodes can be derived in a similar fashion. The insured portfolio consists of one unit of the underlying asset and a put option. Consequently, to replicate the insured portfolio, we need to start with a portfolio consisting of $1.0 - 0.32825 = 0.67175$ units of the underlying asset and \$ 40.3333 of riskless debt. The composition of such a replicating portfolio at each node is depicted in Fig. 13.9.

Figure 13.9

$T-3$	$T-2$	$T-1$
		$\alpha = 1.0000$ $B = \$\,0.0000$
	$\alpha = 0.8273$ $B = \$\,23.6857$	
$\alpha = 0.67175$ $B = \$\,40.3333$		$\alpha = 0.3958$ $B = \$\,66.2857$
	$\alpha = 0.2828$ $B = \$\,73.4667$	
		$\alpha = 0.0000$ $B = \$\,95.2381$

As we have demonstrated in the previous chapter, this dynamic replication process is self-financing at each node.

13.11.2 The Cost of Insurance

One way of looking at the cost of insurance is by considering the price of the implied put option in the dynamic replication strategy. In our example we had to take out \$ 7.51 out of our initial capital of \$ 107.51 to insure the portfolio at a floor value of \$ 100.

Another way of looking at the cost of insurance is by considering the return foregone on the upside. In our case, by creating a put worth \$ 7.51, the manager is able to devote only $\frac{100}{107.51} \equiv 93\%$ of the total capital to the underlying asset. On the other hand, if he had not insured he would have been able to devote the entire \$ 107.51 to the underlying asset. The consequence is that, in the two states where the insurance is not required, namely $S_T = 172.80$ and $S_T = 115.20$, the insured portfolio is able to capture only 93% of the value that it could have captured had the insurance not been availed off.

13.11.3 Portfolio Insurance and the Black–Scholes Model

From put-call parity we know that:

$$C_{E,t} - P_{E,t} = S_t - \frac{X}{(1+r)^{T-t}}$$

$$\Rightarrow S_t + P_{E,t} = C_{E,t} + \frac{X}{(1+r)^{T-t}}$$

$$= S_t N(d_1) - \frac{X}{(1+r)^{T-t}} N(d_2) + \frac{X}{(1+r)^{T-t}}$$

$$= S_t N(d_1) + \frac{X}{(1+r)^{T-t}} N(-d_2) \qquad (13.62)$$

Thus in the Black–Scholes world, an insured portfolio can be created by investing in $N(d_1)$ shares and investing an amount of $N(-d_2)$ times the present value of the exercise price in the riskless asset. Thus, in this scenario, $N(d_1)$ or the delta of the call option plays the role of the hedge ratio for the insured portfolio.

From put-call parity

$$\Rightarrow S_t + P_{E,t} = C_{E,t} + \frac{X}{(1+r)^{T-t}} \qquad (13.63)$$

Thus a portfolio consisting of a call option with an exercise price of \$ 100 plus an investment equal to the present value of the exercise price in the riskless asset, will behave in the same fashion as our insured portfolio consisting of positions in a share plus a put option. Thus this combination offers an alternative way of creating our insured portfolio.

13.12 Options on Volatility

The CBOE offers options on volatility computed using data for S&P500 index options. These trade under the symbol *VIX* where VIX is an acronym for volatility index. It is an estimate of expected volatility computed using the bid and ask quotes for options on the S&P500 index. For the purpose of computation, data is used for the nearby and second nearby options contracts with at least eight days left to expiration. The data spans a range of exercise prices. The computation is independent of any option pricing model.

The significance of VIX may be explained as follows. Option values are influenced by a number of variables, of which volatility is one factor. VIX isolates the expected volatility from the other factors, and allows investors to directly take a position on the volatility.

The contract multiplier is $ 100, and the expiration date is the Wednesday thirty days prior to the third Friday of the calendar month immediately following the expiration month. The exercise style is European. Generally up to three near-term contract months plus up to three additional months from the February cycle are listed.

13.13 SPAN

SPAN is an acronym for *Standard Portfolio Analysis of Risk*. As the name suggests it is a portfolio based margining system, and was developed by the Chicago Mercantile Exchange (CME) in 1988. SPAN attempts to identify the overall risk of a portfolio of derivatives on a given underlying asset.

SPAN is extremely simple to implement in practice. The complex aspects such as option price calculations are performed by the exchanges and clearinghouses that use SPAN. The output of these calculations is known as a *Risk Array*. The risk array and other required inputs for margin computation are then packaged in a file called the SPAN *risk parameter file*. This process is referred to as the *SPAN front end*. The clearing firms and other end users of SPAN use the data contained in these risk parameter files in conjunction with their portfolios containing positions in various derivative securities on a given underlying asset, to determine the required margin. This entails only simple arithmetic calculations, and is referred to as the *SPAN back end*. Thus the end users are not required to concern themselves with the complex aspects of portfolio margin computation such as option valuation. This simplicity is largely responsible for the wide acceptance of SPAN. One important aspect of SPAN is that since the valuation and revaluation of the underlying derivatives is undertaken by the exchanges and clearinghouses, they are in effect able to ensure that the risk parameters being used are the same for all end users and reflect their margin policies.

The objective of SPAN is to identify the overall risk of a portfolio of derivative securities. Thus it simultaneously evaluates all derivatives with a common underlying asset, be they options, futures, or futures options. These derivatives which are defined on a common underlying security are said in the parlance of SPAN to belong to a single *combined commodity*. SPAN also takes cognizance of both inter-commodity and inter-month risk relationships. These concepts will become clear as we proceed.

At the core of the SPAN system is what is called the risk array. The risk array for a derivative security represents the change in its value over a specified period of time called the *look-ahead time*, under various scenarios that represent changing market conditions. In practice the look-ahead period is usually taken to be one trading day. The various market scenarios that are considered are called *risk scenarios*. Each scenario is defined in terms of how much the price of the underlying asset is expected to change from its current value over the look-ahead period, as well as how much the volatility of the rate of return on the underlying asset is expected to change in the same period. The change in value for a derivative security for each risk scenario is termed as the risk array value for that particular risk scenario. The vector of risk array values for a derivative under the full set of 'what-if scenarios' (risk scenarios) is termed as the risk array for that security.

Risk array values are calculated for a long position in one unit of the derivative security. Since SPAN is more concerned with potential losses rather than potential gains, losses are represented as positive values, whereas gains are depicted as negative values.

SPAN evaluates the change in value for a security over sixteen risk scenarios. Every scenario is specified in terms of two parameters called the *price scan range* and the *volatility scan range*. The price scan range is a specified value for the potential change in the price of the underlying asset over the look-ahead period, while the volatility scan range is a specified value for the potential change in the volatility of the rate of return on the underlying asset over the same period. Let us denote the price scan range as $\triangle P$ and the volatility scan range as $\Delta\sigma$. The sixteen scenarios are defined in Table 13.8.

Table 13.8 **Definition of Risk Scenarios**

Scenario No.	Change in Price	Change in Volatility
1	0	$+\Delta\sigma$
2	0	$-\Delta\sigma$
3	$+\frac{\triangle P}{3}$	$+\Delta\sigma$
4	$+\frac{\triangle P}{3}$	$-\Delta\sigma$
5	$-\frac{\triangle P}{3}$	$+\Delta\sigma$
6	$-\frac{\triangle P}{3}$	$-\Delta\sigma$
7	$+\frac{2\triangle P}{3}$	$+\Delta\sigma$
8	$+\frac{2\triangle P}{3}$	$-\Delta\sigma$
9	$-\frac{2\triangle P}{3}$	$+\Delta\sigma$
10	$-\frac{2\triangle P}{3}$	$-\Delta\sigma$
11	$+\triangle P$	$+\Delta\sigma$
12	$+\triangle P$	$-\Delta\sigma$
13	$-\triangle P$	$+\Delta\sigma$
14	$-\triangle P$	$-\Delta\sigma$
15	$+2\triangle P$	0
16	$-2\triangle P$	0

SPAN endeavours to determine the largest loss that a portfolio may be reasonably expected to suffer over the look-ahead period. The exchanges and clearinghouses that use SPAN have to define what they consider to be a reasonable loss over this period. For instance, an exchange may define the loss as a value that is unlikely to be exceeded over 99% of the trading days. It will then set the price scan range and the volatility scan range accordingly.

13.13.1 Composite Delta

Delta which is the rate of change of the value of the derivative with respect to the price of the underlying asset is required by SPAN to form spreads. Delta for a long futures position is always 1.0, whereas for options it will range from -1.0 to $+1.0$. For options, as we have seen earlier, delta is not a constant and is a function of the price of the underlying asset. SPAN calculates one delta value for the entire contract, called the *Composite Delta*. It is derived as a weighted average of the deltas associated with each of the price scan points. The weight associated with a price scan point is based on the probability of the price movement. Thus, more likely price changes receive higher weights.[5] The composite delta may be perceived as the best estimate of what the delta for the contract will be at the end of the look-ahead period. The value that is computed in this fashion, is a constituent of the risk parameter file that an exchange/clearinghouse transmits to its members.

13.13.2 Illustration

We will illustrate the mechanics of SPAN using futures and futures options on crude oil. First let us take the case of futures contracts. Let the current futures price be \$ 75. We will set the price scan range, $\triangle P$, as \$ 15, and the volatility scan range as 2.5%. The volatility is of no consequence while valuing futures contracts, but will obviously have a role to play in the valuation of futures options. Each crude oil contract is for 1,000 barrels of oil. The gains and losses under the various risk scenarios are depicted in Table 13.9.

Let us analyze Table 13.9. The gain/loss for a long futures position over a one day horizon is $F_{t+1} - F_t$. However since SPAN considers the profits to be negative numbers and losses to be positive, we define the gain/loss per barrel as $-[F_{t+1} - F_t]$. The gain/loss per contract is obviously $-[F_{t+1} - F_t] \times 1,000$. This explains the first 14 entries in the above table. The last two scenarios depict what are called extreme value scenarios. These are specified to capture the risk associated with deep out of the money short option positions. The price change corresponding to these scenarios is a multiple of the price scan range. Most exchanges take it to be twice the price scan range. That is, the price change is considered to be $\pm 2\triangle P$, keeping the volatility constant. However, only 35% of the change in value is considered for these extremes by most exchanges. Since the

[5] See www.cme.com

Table 13.9 Risk Array for a Futures Contract

Scenario No.	Change in Price	Change in Volatility	Gain/Loss per Barrel	Gain/Loss per Contract
1	0	+2.5%	0	0
2	0	−2.5%	0	0
3	+$ 5	+2.5%	−$ 5	−$ 5,000
4	+$ 5	−2.5%	−$ 5	−$ 5,000
5	−$ 5	+2.5%	+$ 5	+$ 5,000
6	−$ 5	−2.5%	+$ 5	+$ 5,000
7	+$ 10	+2.5%	−$ 10	−$ 10,000
8	+$ 10	−2.5%	−$ 10	−$ 10,000
9	−$ 10	+2.5%	+$ 10	+$ 10,000
10	−$ 10	−2.5%	+$ 10	+ $ 10,000
11	+$ 15	+2.5%	−$ 15	−$ 15,000
12	+$ 15	−2.5%	−$ 15	−$ 15,000
13	−$ 15	+2.5%	+$ 15	+$ 15,000
14	−$ 15	−2.5%	+$ 15	+$ 15,000
15	+$ 30	0	−$ 10.5	−$ 10,500
16	−$ 30	0	+$ 10.5	+$ 10,500

total change in value is ±30, 000 the gain/loss is taken as −$ 10,500 in scenario 15 and +$ 10,500 in scenario 16.

Now we will apply the principles of SPAN to a long position in a call futures option. Assume that the current futures price is $ 75, and that the volatility of the rate of return is 30%. We will consider a call option with an exercise price of $ 75 and 61 days to maturity. The risk-less rate is assumed to be 8% per annum. Using the Black model, the theoretical options price is

$$C_{E,t} = e^{-r(T-t)}[F_t N(d_1) - XN(d_2)]$$

We will take the time to maturity as $\frac{61}{365} = 0.1671$. The option premium is $ 3.6183 per barrel or $ 3,618.30 per contract.

The risk array is illustrated in Table 13.10.

Let us analyze the first entry in Table 13.10. The change in value for a long position in a call option is $[C_{t+1} - C_t]$ where C_t is the theoretical value of the option at the time of margin computation and C_{t+1} is the theoretical value at the end of the look-ahead period for a specified set of parameters. Remember that in this case, the option which has 61 days to expiration at the time of margin computation will have only 60 days left at the end of the look-ahead period. The first row in Table 13.10 is for a scenario where the futures price is assumed to remain unchanged while the volatility is expected to increase by 2.50% from 30% to 32.50%. The value of the call in the this scenario is $ 3.8885. Thus the change in value is

$$3.8885 - 3.6183 = \$\ 0.2702$$

Table 13.10 Risk Array for a Call Futures Option

Scenario No.	Change in Price	Change in Volatility	Gain/Loss per Barrel	Gain/Loss per Contract
1	0	+2.5%	−$ 0.2702	−$ 270.19
2	0	-2.5%	$ 0.3274	$ 327.36
3	+$ 5	+2.5%	−$ 3.3387	−$ 3,338.74
4	+$ 5	−2.5%	−$ 2.8031	−$ 2,803.10
5	−$ 5	+2.5%	+$ 1.8240	+$ 1,824.01
6	−$ 5	−2.5%	+$ 2.3150	+$ 2,315.03
7	+$ 10	+2.5%	−$7.2008	−$ 7,200.78
8	+$ 10	−2.5%	−$ 6.8270	−$ 6,827.01
9	−$ 10	+2.5%	+$ 2.9786	+$ 2,978.60
10	−$ 10	−2.5%	+$ 3.2563	+ $ 3,256.34
11	+$ 15	+2.5%	−$ 11.5913	−$ 11,591.28
12	+$ 15	−2.5%	−$ 11.3788	−$ 11,378.82
13	−$ 15	+2.5%	+$ 3.4564	+$ 3,456.42
14	−$ 15	−2.5%	+$ 3.5559	+$ 3,555.91
15	+$ 30	0	−$ 9.1033	−$ 9,103.30
16	−$ 30	0	+$ 1.2664	+$ 1,266.40

Since SPAN considers gains to be negative, the gain/loss is defined as $-[C_{t+1} - C_t]$. The gain/loss per contract is

$$-0.2702 \times 1{,}000 = -\$\,270.20$$

Table 13.11 Portfolio Gain/Loss

Scenario	Gain/Loss from Futures Position	Gain/Loss from Options Position	Portfolio Gain/Loss
1	0	$ 1,080.75	$ 1,080.75
2	0	−$ 1,309.45	−$ 1,309.45
3	−$ 5,000	$ 13,354.97	$ 8,354.97
4	−$ 5,000	$ 11,212.41	$ 6,212.41
5	$ 5,000	−$ 7,296.04	−$ 2,296.04
6	$ 5,000	−$ 9,260.13	−$ 4,260.13
7	−$ 10,000	$ 28,803.10	$ 18,803.10
8	−$ 10,000	$ 27,308.03	$ 17,308.03
9	$ 10,000	−$ 11,914.40	−$ 1,914.40
10	$ 10,000	−$ 13,025.35	−$ 3,025.35
11	−$ 15,000	$ 46,365.11	$ 31,365.11
12	−$ 15,000	$ 45,515.30	$ 30,515.30
13	$ 15,000	−$ 13,825.66	$ 1,174.34
14	$ 15,000	−$ 14,223.62	$ 776.38
15	−$ 10,500	$ 36,413.20	$ 25,913.20
16	$ 10,500	−$ 5,065.60	$ 5,434.40

The other entries can be interpreted in a similar fashion. Once again, for scenarios 15 and 16, only 35% of the gain/loss has been taken into account.

Let us compute the gain/loss under each scenario for a portfolio consisting of one futures contract and a short position in four call futures option contracts.

13.13.3 Scanning Risk Charge

The scanning risk charge for a portfolio is the largest loss that it is likely to suffer under the 16 scenarios being considered. In our illustration (see Table 13.11) the largest loss is $ 31,365.11 which corresponds to scenario 11.

13.13.4 Inter-Commodity Spread Credits

A derivatives exchange may list contracts on two different products whose prices are correlated. Consequently, positions in the two contracts on opposite sides of the market may be risk reducing because losses from one instrument may be offset against gains from the other. In such cases, the exchange may set an inter-commodity spread to take cognizance of the fact that the risk of the combined position is lower than the sum of the risks for the two positions taken in isolation. An inter-commodity spread credit, if applicable, reduces the required margin for the combined position. Here is an illustration.[6]

Effect of Inter-commodity Spread

Margins as of April 12, 2001

Margin for a short position in one S&P500 September futures contract	$ 17,250
Margin for a long position in one NASDAQ 100 June futures contract	$ 27,000
Margin if the two positions are treated separately	$ 44,250
Inter-commodity Spread Credit	($ 26,993)
Margin if the position is treated as a portfolio	$ 17,257

13.13.5 Intra-Commodity Spread Risk Charge

SPAN assumes that the price movements in the underlying instrument correlate perfectly across contract months. For instance, it will assume that the price scan range for a three month futures contract is the same as that for a six month futures contract. However, in practice, the correlation will be less than perfect for price movements of different contract months. In order to take this factor into account, SPAN allows the exchanges to levy an intra-commodity spread charge. In the absence of such a charge, a position that is long in three month futures and short in six month futures, would have a margin requirement of zero. The

[6]See www.cftc.gov.

intra-commodity charge is also referred to as a *calendar spread charge*, and is added to the scanning risk associated with each futures and options contract. For each contract, SPAN identifies the associated delta, and then forms spreads using these deltas across contract months. The calendar spread charge is assessed for each spread that is formed in this fashion.

13.13.6 Short Option Minimum Charge

Short positions in options contracts which are extremely deep out of the money, may appear to have little risk across the entire scanning range. However, if the underlying market conditions were to change significantly, such options may move into the money, thereby generating large losses for the writer. To cover the risk associated with such options, SPAN levies a minimum margin for each short option position in the portfolio. This serves as a minimum charge towards margin requirements for each short position in an options contract. The short option minimum charge is usually applied to the greater of the short calls or short puts in the product, for it is unlikely that both short calls as well as short puts will lose due to a large one way price movement in the underlying asset. However, clearing members are given the option to calculate the short option minimum for their customers based on the total of short calls and puts.

13.13.7 Spot Month Add-on Charge

Most exchanges and clearinghouses that use SPAN, levy a charge to recognize the additional risk inherent in portfolios which have positions in futures contracts that are in their delivery month. This charge is usually only for physically deliverable products and applies only to open positions held in their delivery month.

13.13.8 Computing the Final Risk Based Margin

The procedure for determining the total margin that has to be posted as per SPAN may be defined as follows.

1. The scanning risk charge for the entire portfolio of derivatives has to be first computed.
2. Intra-commodity spread risk charges and spot month add-on charges, if applicable, have to be added to the scanning risk charge.
3. Any inter-commodity spread credits should be subtracted from the figure arrived at in step (2).
4. The short option minimum charge has to be computed.
5. The margin as computed in step (3) has to be compared with the short option minimum charge that is computed in step (4). The larger of the two amounts represents the risk margin requirement as per SPAN.

Suggestions for Further Reading

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References

- www.cboe.com
- www.cftc.gov
- www.cme.com

Concept Check

State whether the following statements are True or False.

1. Gamma is identical for European calls and puts, as per the Merton model, on a stock that gives a continuous dividend yield.
2. Vega is identical for European calls and puts, as per the Merton model, for stocks that give a continuous dividend yield.
3. Vega is positive for European calls and puts on a stock that gives a continuous dividend yield.
4. Theta may be positive for European calls on a stock that gives a continuous dividend yield.
5. European calls on a stock that gives a continuous dividend yield may not always be wasting assets.
6. European puts on a stock that gives a continuous dividend yield are always wasting assets.
7. A call futures option gives the holder the right to assume a long position in a futures contract.
8. If a call futures option is exercised it will always lead to an inflow for the option holder.

9. If a call futures option is exercised, both the holder and the writer must post margins or else offset their respective positions.
10. If a put futures option is exercised it will lead to an inflow for the option writer.
11. A put futures option gives the writer the right to assume a short position in a futures contract.
12. A European call option on a futures contract will be worth the same as a European call option on the underlying asset if the futures contract and the options expire at the same time.
13. American calls on a futures contract may be exercised early even if the underlying asset does not make any payouts.
14. Futures options are usually more liquid than option on the underlying assets.
15. Portfolio insurance can be created using put options but not with call options.
16. The SPAN back-end requires only simple arithmetic calculations.
17. SPAN treats futures, options, and futures options on a given underlying asset, as a single combined entity.
18. Inter-commodity spreads serve to increase the applicable margin for the overall position.
19. The short option minimum charge is usually applied to the sum of short calls and short puts.
20. The spot-month add-on charge is applicable to futures contracts that are in their delivery month.

Questions and Problems

Question-I

A stock is currently priced at \$ 50. Call and put options with an exercise price of \$ 45 and nine months to maturity are available. The risk-less interest rate is 8% per annum, and the volatility of the rate of return is 25% per annum. The stock gives a continuous dividend yield of 10% per annum.

1. Compute the value of European calls and puts using the Merton model.
2. Compute the value of delta, gamma, vega, theta, and rho, for both calls and puts.

Question-II

Call and put options on Euros are available. The spot rate is 135 and the exercise price of the options is 150. Every period the spot rate may go up by 20% or go down by 20%. The risk-less rate of interest is 8% per period in the US, and is 6% per period in Europe.

Assuming that the options are European in nature, value both calls and puts expiring after three periods using the Binomial model.

Question-III

Futures options are often more actively traded than options on the underlying assets. Discuss.

Question-IV

European futures options, both calls and puts, must be worth the same as options on the underlying asset, if both the options as well as the futures contracts expire at the same time. Discuss.

Question-V

What is portfolio insurance, and how can it be achieved using put options? Discuss the difference between static and dynamic insurance.

Question-VI

What is SPAN? What the components of the margin as calculated using SPAN? What is the concept of a short option minimum charge in SPAN?

Question-VII

The spot price of an asset is \$ 100. The volatility of the rate of return is 25% per annum, and the risk-less rate of interest is 10% per annum. Put options are available with an exercise price of \$ 105 and 91 days to maturity. Each contract is for 5,000 units of the underlying asset.

An investor has a short position in 10 option contracts and 4 futures contracts. The price scan range is \$ 12 and the volatility scan range is 2.5%.

Compute the scanning risk charge using SPAN. For the extreme scenarios you need consider only 35% of the gain/loss.

Question-VIII

The volatility of the rate of return is 25% per annum and the risk-less rate is 10% per annum. The stock pays a continuous dividend yield at the rate of 7.5% per annum. Each period is equal to 0.25 years. The current spot price is \$ 100 and the exercise price is also \$ 100.

Value European call options with three periods to expiration, using the Binomial model.

Question-IX

The price of a 6 months futures contract on gold is \$ 850 per ounce. Call and put options with 6 months to expiration are available with an exercise price of \$ 870. The risk-less rate is 10% per annum and the volatility is 40% per annum. Each future contract is for 100 ounces.

Compute the prices of European calls and puts using the Black model.

Question-X

A stock is currently priced at \$ 80. Every period the price may go up by 25% or go down by 20%. The risk-less rate of interest is 10% per period.

Futures contracts are available with three period to expiration, and an exercise price of \$ 80. Compute the values of European call and put options with three period to expiration, on the futures contract.

14

Exotic Options

14.1 Introduction

Thus far we have dealt with standard European and American options, which are what a market professional would term as *plain vanilla products*. Most exchange traded products fall within this category. However, it is common for investment banks to offer options with non-standard features to their clients in the OTC markets. Such options have come to be known as *exotic options*. We will discuss some of the relatively more common exotic products. The list is by no means exhaustive, and options with more and more novel features are constantly being devised. From the standpoint of the investment banks, such derivatives not only pose a constant intellectual challenge to their so called *rocket scientists*, they also serve as an additional source of revenue in markets which are characterized by a constant shrinking of spreads due to enhanced competition.

14.2 Digital or Binary Options

There are two types of digital options, also known as binary options, which may be termed as *cash-or-nothing* and *asset-or-nothing* options. Let us first consider cash-or-nothing options.

14.2.1 Cash-or-nothing

Consider a cash-or-nothing call option with an exercise price of X, and a time to expiration of $T - t$. Such an option will result in the holder receiving a payoff of \$ X from the writer, if $S_T > X$, that is if the option finishes in the money. Otherwise the option will simply expire worthless.

Similarly, in the case of a cash-or-nothing put option, the writer will pay \$ X to the holder if $S_T < X$. Obviously, if the option were to expire out of the money, it would simply expire worthless.

14.2.2 Asset-or-nothing

An asset-or-nothing call option will result in one unit of the underlying asset being delivered to the holder by the writer if the terminal stock price is greater than the exercise price. Otherwise the option will expire worthless.

On the other hand, an asset-or-nothing put option will result in a unit of the underlying asset being delivered to the buyer by the writer if the terminal stock price is less than the exercise price. Else, the option will expire worthless.

The difference between asset-or-nothing and plain vanilla European options is that irrespective of whether it is a call or a put, an asset-or-nothing option does not require the holder to pay anything to the writer if and when he receives the underlying asset.

14.2.3 Equivalence with Plain Vanilla Options

Consider a long position in an asset or nothing call option with an exercise price of X and time to expiration $T - t$, and a short position in a cash or nothing call option with the same exercise price and time to expiration. If the terminal stock price were to be higher than the exercise price, such a position would result in the holder receiving the underlying asset from the writer in return for the payment of an amount equal to the exercise price. Thus, this portfolio is nothing but a standard European call option.

Similarly a European put option is equivalent to a long position in a cash-or-nothing put option plus a short position in an asset-or-nothing put option.

From the Black–Scholes formula

$$C_{E,t} = SN(d_1) - Xe^{-r(T-t)}N(d_2) \qquad (14.1)$$

The first term, $SN(d_1)$ is the value of an asset-or-nothing call option with an exercise price of X and time to expiration equal to $T - t$. The second term, $Xe^{-r(T-t)}N(d_2)$, is the value of a cash-or-nothing call option with the same exercise price and expiration date. The minus sign before this term denotes a short position in such an option.

A European put option can be similarly analyzed in terms of its components. From the Black–Scholes formula

$$P_{E,t} = Xe^{-r(T-t)}N(-d_2) - SN(-d_1) \qquad (14.2)$$

$Xe^{-r(T-t)}N(-d_2)$ is the value of a cash-or-nothing put option while $SN(-d_1)$ is the value of an asset-or-nothing put option.

14.3 Asian Options

An Asian option is a derivative in which the payoff is not based on the price of the asset at expiration but rather on the average price of the asset over the life of the option. Consequently such options are also known as *average price options*. Most Asian options can be exercised only at expiration. However, in certain cases early exercise may be permissible.

Let us denote the average asset price over the life of the option as $\overline{S}$. The payoff from an average price option, which cannot be exercised prior to maturity, may be defined as Max[0, $\overline{S} - X$], in the case of call options, and as Max[0, $X - \overline{S}$], in the case of put options. The strike price X in the case of such options is pre-defined like in the case of European and American options.

A variation of these options is known as the *average strike option*. In the case of such options the final payoff is based on the difference between the asset price at expiration and the average price over the life of the option. Thus, the exercise price in the case of such options is the average price of the underlying asset over the life of the option. The payoff from an average strike option, which cannot be exercised prior to maturity, may be defined as Max$[0, S_T - \overline{S}]$, in the case of call options, and as Max$[0, \overline{S} - S_T]$, in the case of put options.

The method of computing the average price needs to be specified in advance. There are two possibilities, arithmetic averaging and geometric averaging. If we have a vector of N stock prices the arithmetic mean may be defined as

$$\frac{1}{N} \times \sum_{i=1}^{N} S_i \tag{14.3}$$

and the geometric average as

$$\left[\prod_{i=1}^{N} S_i\right]^{\frac{1}{N}} \tag{14.4}$$

The advantage of using the geometric average, is that if the price of the underlying security is assumed to be lognormally distributed, then the average of the prices is also lognormal. However, such an assertion cannot be made in the case of an arithmetic average of the prices. In practice Asian options are usually defined in terms of the arithmetic average. For a given series of prices, the geometric average will always be lower than the arithmetic average, except in the trivial case where all the observations are equal.

In the case of a standard Asian option, all the observations are weighted equally while computing the average. However, options with unequal weights for the observations can be defined. The period during which the average is to computed and the monitoring frequency have also to be specified. For instance, in the case of an option with one year to maturity, it is conceivable that the average price as per the terms of the option is computed using the observations for the first three or six months only. The monitoring frequency is also important. For instance, should the prices used for computing the average be based on the daily closing values or should they be sampled with a different frequency.

14.3.1 Average Price Options

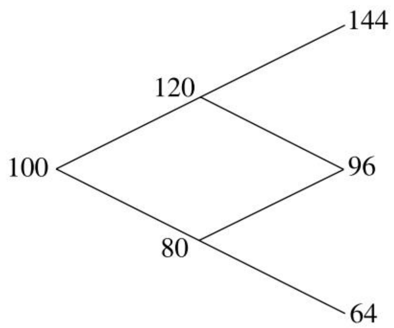
We will use the two period Binomial model to compute the price of an average price call option. The illustration will first assume that the option has been defined in terms of the arithmetic average of prices. Subsequently, we will recompute the price using the geometric average of prices.

Consider the following data. The current stock price is \$ 100. Every period the price may go up by 20% or may decline by 20%. An average price Asian option with two periods to expiration, and an exercise price of \$ 100 is available. The riskless rate is 5% per period.

$$u = 1.2, d = .8, r = 1.05$$
$$\Rightarrow p = 0.625 \text{ and } (1 - p) = 0.375$$

The stock price tree for the two period case may be modeled in Fig. 14.1.

Figure 14.1



As we can see, there are four paths to expiration. The average price attained along each path, and the probability of the asset price following that particular path, is given in Table 14.1. Both the arithmetic as well as the geometric average prices are given. We will illustrate the computation for the first path. Readers should be able to derive the remaining values.

Path 1: 100 → 120 → 144

$$\text{Arithmetic average} = \frac{100 + 120 + 144}{3} = 121.33$$

$$\text{Geometric average} = (100 \times 120 \times 144)^{\frac{1}{3}} = 120$$

Table 14.1 **Average Price for Each Path**

Path No.	Probability	Arithmetic Average	Geometric Average
1	0.390625	121.33	120.00
2	0.234375	105.33	104.83
3	0.234375	92.00	91.58
4	0.140625	81.33	80.00

If the stock follows path 1 or path 2, the call option will be in the money, else it will be out of the money. Similarly the put will be in the money if the stock follows paths 3 or 4, else it will be out of the money.

We can compute the price of the option as the discounted value of the expected payoff at expiration, using the risk-neutral probabilities.

The Case of the Arithmetic Average

$$C_t = \frac{1}{(1.05)^2} \times [.390625 \times (121.33 - 100) + .234375 \times (105.33 - 100) + .234375 \times 0 + .140625 \times 0] = 8.6905$$

$$P_t = \frac{1}{(1.05)^2} \times [.390625 \times 0 + .234375 \times 0 + .234375 \times (100 - 92) + .140625 \times (100 - 81.33)] = 4.0821$$

The Case of the Geometric Average

$$C_t = \frac{1}{(1.05)^2} \times [.390625 \times (120 - 100) + .234375 \times (104.83 - 100) + .234375 \times 0 + .140625 \times 0] = 8.1130$$

$$P_t = \frac{1}{(1.05)^2} \times [.390625 \times 0 + .234375 \times 0 + .234375 \times (100 - 91.58) + .140625 \times (100 - 80)] = 4.3410$$

Since the geometric average price is lower than the arithmetic average price, for a given value of the exercise price the call premium will be lower when the option is defined is terms of the geometric average while the put premium will be higher.

14.3.2 Average Strike Options

An average strike call will be in the money if the stock follows paths 1 or 3, else it will be out of the money. As before the price of the option can be calculated as the present value of the expected payoff at expiration. Thus, if the option is defined in terms of an arithmetic average

$$C_t = \frac{1}{(1.05)^2} \times [.390625 \times (144 - 121.33) + 0.234375 \times (96 - 92)] = 8.8825$$

However, if the option were to be defined in terms of the geometric average, then

$$C_t = \frac{1}{(1.05)^2} \times [.390625 \times (144 - 120) + 0.234375 \times (96 - 91.58)] = 9.4430$$

An average strike put will be in the money if the stock follows paths 2 or 4, else it will be out of the money. If the option is defined in terms of the arithmetic average

$$P_t = \frac{1}{(1.05)^2} \times [.234375 \times (105.33 - 96) + 0.140625 \times (81.33 - 64)] = 4.1939$$

However, if the option were to be defined in terms of a geometric average

$$P_t = \frac{1}{(1.05)^2} \times [.234375 \times (104.83 - 96) + 0.140625 \times (80 - 64)] = 3.9179$$

In the case of average strike options, the call premium is higher if the geometric average of the prices is taken, while the put premium is lower. This is because in the case of such options the average price is used in lieu of the exercise price, and the lower the exercise price the higher will be the value of a call and the lower will be the value of a put.

14.4 Lookback Options

Lookback call options allow the holder to acquire an asset for its lowest price during the life of the option, while lookback put options allow the holder to sell the asset for its highest price during the life of the option. That is, for a call option, the strike price at expiration is set equal to the minimum observed price of the asset during the life of the option, while for a put, the exercise price is set equal to the maximum asset price observed during the life of the option. Thus, for lookback options, the exercise price depends on the path taken by the asset price. Consequently, just like Asian options, lookbacks too are path dependent.

Let us denote the minimum and maximum prices attained during the life of the option by $S_{\min}$ and $S_{\max}$ respectively. The payoff from a lookback call at expiration is $\text{Max}[0, S_T - S_{\min}]$, while that from a put is $\text{Max}[0, S_{\max} - S_T]$. Hence, both call and put options are guaranteed to expire in the money or at the money.

Such options are also termed as *no-regrets* options because the holder will not have to face the ex-post regret of having sold or exercised the option at the wrong time. These options are also referred to as *hindsight options*.[1]

We will once again use the risk neutral valuation technique to price such options. A risk neutral investor will calculate the expected payoff, and then discount it back at the riskless rate for the number of periods remaining, which we will denote by $T - t$. Therefore

$$C_t = \frac{1}{(r)^{T-t}} E[\text{Max}(0, S_T - S_{\min})] \tag{14.5}$$

and

$$P_t = \frac{1}{(r)^{T-t}} E[\text{Max}(0, S_{\max} - S_T)] \tag{14.6}$$

Now, by definition

$$S_T - S_{\min} \geq 0 \text{ and } S_{\max} - S_T \geq 0$$

Therefore

$$C_t = \frac{1}{(r)^{T-t}} E[S_T - S_{\min}] = \frac{1}{(r)^{T-t}} E(S_T) - \frac{1}{(r)^{T-t}} E(S_{\min}) \tag{14.7}$$

and
$$P_t = \frac{1}{(r)^{T-t}} E(S_{\max}) - \frac{1}{(r)^{T-t}} E(S_T) \tag{14.8}$$

[1] See Chance (1998).

For a risk neutral investor, the expected asset price at expiration is equal to the current stock price compounded at the riskless rate of interest. That is, $E(S_T) = S_t(r)^{T-t}$. Therefore

$$C_t = S_t - \frac{1}{(r)^{T-t}} E(S_{\min}) \tag{14.9}$$

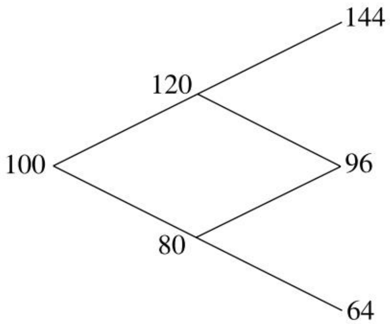
and

$$P_t = \frac{1}{(r)^{T-t}} E(S_{\max}) - S_t \tag{14.10}$$

14.4.1 Computing the Value of a Call

We will use the same data as before. The stock price tree is as depicted in Fig. 14.2.

Figure 14.2



The minimum and maximum prices for each path, and the corresponding probability of the path are shown in Table 14.2.

Table 14.2 Minimum and Maximum Prices for Each Path

Path No.	Probability	Minimum Price	Maximum Price
1	0.390625	100	144
2	0.234375	96	120
3	0.234375	80	100
4	0.140625	64	100

$$E(S_{\min}) = 0.390625 \times 100 + 0.234375 \times 96 + 0.234375 \times 80 + 0.140625 \times 64 = 89.3125$$

Therefore, the call premium is given by

$$C_t = 100 - \frac{1}{(1.05)^2} \times 89.3125 = \$\ 18.99$$

14.4.2 The Value of a Put

$$E(S_{max}) = 0.390625 \times 144 + 0.234375 \times 120 + 0.234375 \times 100 + 0.140625 \times 100 = 121.875$$

Therefore, the put premium is given by

$$P_t = \frac{1}{(r)^T} E(S_{max}) - S_t$$

$$P_t = \frac{1}{(1.05)^2} \times 121.875 - 100 = \$\ 10.54$$

14.4.3 Modified Lookback Options

In the case of a modified lookback option, the exercise price is specified at the outset. However, the payoff is based on the maximum price observed during the life of the option, if it is a call, and the minimum price observed during the life of the option if it is a put. We will illustrate such options with the help of an example.

Example Consider a modified lookback call with an exercise price of \$ 100. We will continue to use the same stock price process that we have considered earlier. The payoff from a call option is

$$\text{Max}[0, S_{max} - X]$$

while that from a put is

$$\text{Max}[0, X - S_{min}]$$

The payoff from call and put options if the stock price follows a given path is given below in Table 14.3.

Table 14.3 **Payoffs from Call and Put Options**

Path	Payoff from a Call	Payoff from a Put
1	44	0
2	20	4
3	0	20
4	0	36

The call option price is given by

$$C_t = \frac{1}{(1.05)^2}[44 \times 0.390625 + 20 \times 0.234375] = \$\ 19.8413$$

The put option price is given by

$$P_t = \frac{1}{(1.05)^2}[4 \times 0.234375 + 20 \times 0.234375 + 36 \times 0.140625] = \$\ 9.6939$$

14.4.4 Lookbacks versus European Options

If the stock price were to follow the process described above, it can be shown that a European call with an exercise price of \$ 100 will be priced at \$ 15.5896, while a European put with an exercise price of \$ 100 will be priced at \$ 6.2925. From these prices we can observe that standard lookback calls and puts are more expensive than standard European options. We will justify this by comparing the payoffs of the lookback options, both calls and puts, with the payoffs from European call and put options with an exercise price of \$ 100.

Table 14.4 Comparison of Payoffs from Lookback Options and European Options

Path	Payoff from Lookback Call	Payoff from European Call	Payoff from Lookback Put	Payoff from European Put
1	44	44	0	0
2	0	0	24	4
3	16	0	4	4
4	0	0	36	36

As can be seen from Table 14.4, the payoff from a European call is less than or equal to that from a lookback call irrespective of the path taken by the stock price. The same is true for European puts. Consequently, both lookback calls and puts are valued higher than the corresponding European options.

In the case of modified lookback options, if the asset price at the outset is greater than or equal to the exercise price of the option, then a call option is guaranteed to expire in the money. Similarly, if the asset price at the outset were to be less than the exercise price of the option, a put option is guaranteed to expire in the money. If this condition is not satisfied, there is no guarantee that a modified lookback will finish in the money. Consequently, in such situations, modified lookbacks will be priced lower than standard lookbacks which are guaranteed to finish in or at the money.

However, modified lookbacks will always be valued higher than European options with the same exercise prices. Consider the case of modified lookback call options. The payoff is $\text{Max}[0, S_{\text{max}} - X]$. Except for a situation where $S_{\text{max}} = S_T$, where S_T is the asset price at expiration, the payoff from the modified lookback call will be greater than that of a European call. Consequently, the modified lookback will be priced higher. A similar line of argument applies to modified lookback put options. We will compare the payoffs from modified lookback calls and puts with those from the corresponding European options in order to substantiate our argument. As can be seen from Table 14.5, the payoff from both modified lookback calls as well as puts is greater than or equal to the payoff from the corresponding European options irrespective of the path taken by the stock price.

Table 14.5 Comparison of Payoffs from Lookback Options and European Options

Path	Payoff from Modified Lookback Call	Payoff from European Call	Payoff from Modified Lookback Put	Payoff from European Put
1	44	44	0	0
2	20	0	4	4
3	0	0	20	4
4	0	0	36	36

14.5 Compound Options

A compound option is an option on an option. There are four types of compound options. Let us take the case of calls first. We can have a call that gives the holder the right to buy a call if exercised or a call that gives the right to buy a put if exercised. Similarly we can have a put that gives the holder the right to sell a call if exercised or a put that gives the right to sell a put if exercised.

Take the case of a two period compound call on a call. If this call is bought at $T-2$, the investor would have acquired the right to acquire a one period call at $T-1$. Assume that the stock price tree is the same as what we have considered earlier, and that the option gives the right to acquire a call with an exercise price of \$ 100 at $T-1$, by paying a premium of \$ 20.

Let us value the one period call at $T-1$.

$$C_u = \frac{1}{(1.05)}[.625 \times (144 - 100) + .375 \times 0] = 26.1905$$

and $$C_d = \frac{1}{(1.05)}[.625 \times 0 + .375 \times 0] = 0$$

Thus if the up state is reached at $T-1$, the compound call is worth exercising because the holder gets a one period call worth \$ 26.1905 for \$ 20. However, if the down state is reached it is better to let the compound option expire worthless. Hence, the value of the compound option at $T-2$ is

$$\frac{1}{(1.05)}[.625 \times (26.1905 - 20) + .375 \times 0] = \$\ 3.6848$$

14.6 Barrier Options

A barrier option is an option that gets activated or deactivated if the underlying asset price crosses a price barrier. There are two kinds of barrier options, *knock-in* options and *knock-out* options. In the case of knock-out options, an existing option will be deactivated if the barrier is breached. If the option is deactivated the holder may be entitled to a positive *rebate* in certain cases. If the barrier is not breached, then the option will be equivalent to a standard European option with the same exercise price, and will have an identical payoff at expiration. In the case of knock-in options, the option itself will come into existence if the barrier is breached. Once it is activated, it will subsequently behave like a European option with the same exercise price. However if it is not activated the holder may be given a positive rebate in certain cases.

Since the barrier can be either above or below the current stock price, knock-in options can be of two types, *up-and-in* or *down-and-in*.The word *up* denotes that the barrier is above the current stock price, while the word *down* implies that the barrier is below the current stock price. Similarly, knock-out options may be *up-and-out* or *down-and-out*. Of course every type of barrier options may be either a call or a put. Thus in all there can be eight types of barrier options.

The characteristics of the four kinds of barrier call options is summarized in the table below. The same logic is applicable to put options.

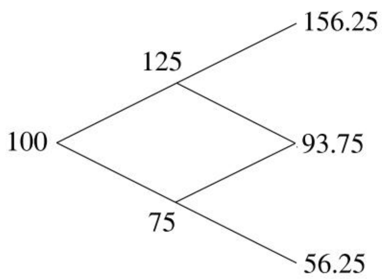
Table 14.6 **Features of Barrier Calls**

Nature of Option	Location of barrier B Relative to the Initial Stock Price S_t	Impact of Crossing/ Not Crossing the Barrier On the Option Payoff: If Crossed	If Not Crossed
Down and Out	Below	Rebate R$\geq$ 0	Standard Call
Down and in	Below	Standard Call	Rebate $R \geq 0$
Up and out	Above	Rebate $R \geq 0$	Standard Call
Up and in	Above	Standard Call	Rebate $R \geq 0$

14.6.1 Example of an Up and Out Call

Assume that the current stock price is \$ 100. Every period the stock may go up by 25% or go down by 25%. The riskless rate of interest is 10% per period. Consider a call option with an exercise price of \$ 80 and a barrier of \$ 110. We will assume that if the option is deactivated, then the holder will get a rebate of \$ 10. The stock price tree may be depicted in Fig. 14.3.

Figure 14.3



$$p = \frac{1.10 - 0.75}{1.25 - 0.75} = 0.70; 1 - p = 0.30$$

We will inspect each possible path that the stock price can take.

Path 1: 100 → 125 → 156.25. The option will get deactivated when the price hits 125 at $T - 1$. So the holder will get a rebate of $ 10 at expiration.

Path 2: 100 → 125 → 93.75. Once again the option will get deactivated. The holder will receive $ 10 at expiration.

Path 3: 100 → 75 → 93.75. The option will stay alive, and will payoff (93.75 – 80) = $ 13.75 at expiration.

Path 4: 100 → 75 → 56.25. The option will stay alive, but will finish out-of-the-money at expiration.

Using the risk-neutral valuation technique,

$$C_t = \frac{1}{(1.10)^2}[(0.70)^2 \times 10 + 0.70 \times 0.30 \times 10 + 0.30 \times 0.70 \times 13.75 + (0.30)^2 \times 0] = 8.1715$$

14.6.2 Example of an Up and In Call

We will use the same data that we used in the previous example.

Path 1: 100 → 125 → 156.25. The option will get activated when the price hits 125 at $T - 1$. The call will finish in the money at expiration, and the holder will consequently get (156.25-80) = $ 76.25.

Path 2: 100 → 125 → 93.75. Once again the option will get activated, and will finish in the money. The holder will receive (93.75-80) = $ 13.75.

Path 3: 100 → 75 → 93.75. The option will not come into existence. The holder will get a rebate of $ 10.

Path 4: 100 → 75 → 56.25. The option will once again not come into existence. Consequently the holder will receive a rebate of $ 10.

$$C_t = \frac{1}{(1.10)^2}[(0.70)^2 \times 76.25 + 0.70 \times 0.30 \times 13.75 + 0.30 \times 0.70 \times 10 + (0.30)^2 \times 10] = 35.7438$$

14.6.3 Equivalence With European Options

It can be shown that an up and out call plus an up and in call is equivalent to a European option with the same exercise price, if the rebate for the barrier options is set at zero. We can see this by comparing the payoffs at expiration from a portfolio consisting of the two barrier options.

As can be seen from the last column of the above table, the payoff from the portfolio is identical to that of a European call. This is because while an up and out call has value only if the barrier is not breached an up and in call has value only if the barrier is breached. Since there are only two possibilities, that is, either

Table 14.7 Payoffs from a Portfolio of Barrier Options

Path	Payoff from an Up and Out Call	Payoff from an Up and In Call	Payoff from the Portfolio
Path 1	0	76.25	76.25
Path 2	0	13.75	13.75
Path 3	13.75	0	13.75
Path 4	0	0	0

the barrier is breached or it is not, it is but logical that a portfolio of the two barrier options should behave like a European option.

Similarly it can be shown that a portfolio consisting of a down and out call plus a down and in call is equivalent to a European call. The same holds true for put options.

Suggestions for Further Reading

1. Arditti, F.D. *Derivatives*. Harvard Business School Press (1996).
2. *Exotic Options*. Euromoney Publications.

Concept Check

State whether the following statements are True or False.

1. The average price in the case of Asian options may be an arithmetic average or a geometric average.
2. The geometric average will always be less than or equal to the arithmetic average.
3. If prices are lognormally distributed the arithmetic average will also be lognormal.
4. In the case of Asian options all the observations must be weighted equally.
5. In the case of average strike options the call premium will be higher if a geometric average is used.
6. Lookback options are path dependent.
7. Average strike options are path dependent while average price options are not.
8. Lookback calls and puts are guaranteed to expire at or in the money.
9. Lookback options are also known as no-regrets options.
10. A European put option is equivalent to a long position in a cash-or-nothing put plus a long position in an asset-or-nothing put.

11. In the case of average price options the call premium will be higher if a geometric average is used.
12. Lookback calls and puts are always more expensive than European options on the same asset with the same time to maturity.
13. Modified lookback options are always guaranteed to finish at or in the money.
14. Modified lookback options will always be valued higher than European options with the same exercise prices.
15. An up and out call plus a down and in call is equivalent to a European option with the same exercise price if the rebate is set at zero.
16. An up and out call plus an up and in call is equivalent to a European option with the same exercise price if the rebate is set at zero.
17. A down and out put and a down and in put is equivalent to a European option with the same exercise price if the rebate is set at zero.
18. Asset-or-nothing options, whether calls or puts, do not require the holder to pay anything to the writer if and when the options are exercised.
19. In the case of lookback call options, the exercise price is set equal to the minimum observed price of the underlying asset during the life of the option.
20. Barrier options are path dependent.

QUESTIONS AND PROBLEMS

Question-I

A combination of cash-or-nothing and asset-or-nothing options is equivalent to a European option. Discuss with respect to both calls and put.

Question-II

A stock is currently priced at $ 100. Binary options are available with an exercise price of $ 100 and six months to expiration. The volatility of the underlying asset is 25% per annum, and the riskless rate is 10% per annum.

Compute the values of cash-or-nothing calls and puts, as well as asset-or-nothing calls and puts.

Question-III

Consider the following vector of stock prices.

Serial No.	Price	Serial No.	Price
1	105.25	7	102.30
2	104.00	8	99.95
3	102.35	9	101.60
4	106.85	10	104.40
5	108.10	11	107.15
6	104.50	12	109.95

Compute the arithmetic as well as the geometric average.

Question-IV

A stock is currently priced at $ 80. Everey period the price may go up by 25% or go down by 25%. Thr riskless rate is 10% period.

Compute the values of average price calls and puts with an exercise price of $ 75. Use arithmetic averaging.

Question-V

A stock is currently price at $ 75. Every period the price may go up by 20% or down by 20%. The riskless rate is 10% per period.

Compute the values of average strike calls and puts with three periods to expiration. Use arithmetic averaging.

Question-VI

Consider the data given in Question-V. Using the same, compute the value of look-back calls and puts with three periods to expiration.

Question-VII

A stock is currently priced at $ 100. Every period the price may go up by 25% or down by 25%. The riskless rate is 10% per period.

Compute the values of modified look-back options, both calls and puts, with three periods to expiration, and an exercise price of $ 100.

Question-VIII

Using the data in Question-VII demonstrate that the payoffs at expiration for the modified look-back options are greater than those for conventional European options with the same exercise price.

Question-IX

The current stock price is $ 80. Every period the price may go up by 25% or down by 25%. The riskless rate is 10% per period. Consider call options with an exercise price of $ 80 and a barrier of $ 100. If the barrier is not crossed, the holder will get a nil rebate.

Compute the value of an up-and-out call and an up-and-in call with three periods to expiration. Demonstrate the relationship of the calculated prices with that of a three period European call with the same exercise price.

Question-X

A stock is priced at $ 100. Every period the price may go up by 20% or down by 20%. The riskless rate is 8% per period. Consider put options with an exercise price of $ 110 and a barrier of $ 80. If the barrier is not crossed, the holder will get a rebate of $ 10.

Compute the value of a down-and-in put as well as that of a down-and-out put.

15

The Term Structure of Interest Rates and the Valuation of Interest Rate Options

15.1 Introduction

At any point in time, an investor who is contemplating an investment in fixed income securities, will typically have access to a large number of bonds with different yields and varying times to maturity. It is common for investors and traders to frequently examine the relationship between the yields on bonds belonging to a particular risk class. A plot of the yields of bonds that differ only with respect to their time to maturity, versus their respective times to maturity is called a *Yield Curve*. The curve is an important indicator of the state of the bond market, and provides valuable information.

While constructing the yield curve it is very important that the data pertain to bonds of the same risk class, and having comparable degrees of liquidity. For example, a curve may be constructed for government securities or AAA rated corporate bonds, but not for a mixture of both. The primary yield curve in any domestic capital market is the government bond yield curve, for these instruments are free of default risk. In the US debt market for instance, the primary yield curve is the US. Treasury yield curve.

15.2 Analyzing the Yield Curve

The yield curve is an indication of where the bond market is trading currently. It also has implications for the level of trading for the future or at least for what the market thinks will be happening in the future.

The yield curve sets the yield for all debt market instruments. First it fixes the price of money over the maturity structure. In practice the yields of government bonds from the shortest maturity instrument to the longest set the benchmark for yields on all other debt instruments. This would mean that if a 5 year government security is trading at a yield of 5%, all other 5 year bonds. irrespective of the issuer, will be trading at yields over 5%. The excess over the yield on the corresponding government security is called the *spread*. Thus issuers of debt in the market use the yield curve to price all debt securities.[1]

The yield curve also acts as an indicator of future yield levels. It assumes certain shapes in response to market expectations of future interest rates. Market participants therefore analyze the current shape of the curve in order to determine the direction of future interest rates. The curve is scrutinized in order to divine information about future rates by bond traders and fund managers.

Central banks and government treasury departments also analyze the curve for its information content. This information is then used to set rates for the economy as a whole.

Portfolio managers use the curve to assess the relative values of investments across the maturity spectrum. The curve indicates the returns that are available at different points of time and is therefore important for fixed income fund managers who can use this information to assist them in assessing the points along the curve that offer the best returns. The curve can also be analyzed to determine which bonds are relatively cheap or costly.[2]

15.3 Spot Rates

The spot rate of interest for a particular time period is the discount rate that is applicable for a zero coupon instrument maturing at the end of the period.

For instance assume that the price of a six month zero coupon bond with a face value of \$ 1,000 is \$ 961.54. If we consider six months to be equivalent to one period, then the one period spot rate is given by

$$961.54 = \frac{1,000}{(1+s_1)} \Rightarrow s_1 = 0.04 \equiv 4\%$$

Similarly if a one year or a two period zero coupon bond has a price of \$ 873.44, then the two period spot rate is given by

$$873.44 = \frac{1,000}{(1+s_2)^2} \Rightarrow s_2 = 0.07 \equiv 7\%$$

[1] See Choudhury (2004).
[2] See Choudhury (2004)

15.4 Relationship Between Spot Rates and the YTM

A plain vanilla bond consists of a series of cash flows arising at six monthly intervals. Thus such a bond is equivalent to a portfolio of zero coupon bonds, where each cash flow represents the face value of a zero coupon bond maturing at that particular instant.

The correct way to price a bond is by discounting each cash flow at the spot rate for the corresponding period.

Let us take the case of a bond with a face value of \$ 1,000 and one year to maturity. Assume that it pays a coupon of 7% per annum on a semi-annual basis. Using the spot rates derived in the previous section, we can calculate the price of the bond to be

$$P = \frac{35}{(1.04)} + \frac{1,035}{(1.07)^2} = \$\ 937.66$$

The yield to maturity of this bond is given by

$$937.66 = \frac{35}{\left(1 + \frac{y}{2}\right)} + \frac{1,035}{\left(1 + \frac{y}{2}\right)^2}$$

$$\Rightarrow \frac{y}{2} = 0.069454 \equiv 6.9454\%$$

The yield to maturity is therefore a complex average of the spot rates. The problem with the yield to maturity is that it is a function of the coupon rate, for bonds with identical terms to maturity but with different coupons.

For instance let us take a 12% coupon bond with a face value of \$ 1,000 and one year to maturity. Its price is given by

$$P = \frac{60}{(1.04)} + \frac{1,060}{(1.07)^2} = \$\ 983.54$$

The yield to maturity of this bond is given by

$$983.54 = \frac{60}{\left(1 + \frac{y}{2}\right)} + \frac{1,060}{\left(1 + \frac{y}{2}\right)^2}$$

$$\Rightarrow \frac{y}{2} = 0.069092 \equiv 6.9092\%$$

Why is there a difference in the yields to maturity of the two bonds? After all they both have one year to maturity.

Let us take the 7% bond first. It has

$$\frac{\frac{35}{(1.04)}}{937.66} = 0.035891 \equiv 3.5891\%$$

of its value tied up in one period money and the balance 96.4109% tied up in two period money.

However, in the case of the 12% bond

$$\frac{\frac{60}{(1.04)}}{983.54} = 0.058658 \equiv 5.8658\%$$

of its value is tied up in one period money whereas the balance 94.1342% is tied up in two period money.

The one period spot rate is less than the two period spot rate, which implies that one period money is cheaper than two period money. Since the second bond has a greater percentage of its value tied up in one period money, its yield to maturity is less.

This is a manifestation of the *coupon effect*. In other words, the yield to maturity is a complex weighted average of spot rates where the weights depend on the pattern of cash flows receivable from a particular security. Consequently it tends to vary with the coupon rate on the bond, when comparisons are sought to be made among bonds with the same time to maturity. Thus a given measure of the YTM is specific to a particular security and cannot be used to estimate the fair value of another security.[3]

15.5 Yield Curve versus The Term Structure

Technically speaking, the term 'Yield Curve' is a graph depicting the relationship between the yield to maturity, which is plotted along the Y-axis, and the time to maturity, which is plotted along the X-axis. To reiterate, for the purpose of constructing the yield curve it is imperative that the bonds being compared belong to the same credit risk class. This is the most commonly used version of the yield curve for the simple reason that the YTM is the most commonly used measure of the yield from a bond.

The expression 'Term Structure of Interest Rates' on the other hand, refers to a graph depicting the relationship between spot rates of interest as shown along the Y-axis, and the corresponding time to maturity, which is plotted along the X-axis. Once again, to facilitate meaningful inferences, the data used to construct the graph should be applicable to bonds of the same risk class. The 'Term Structure of Interest Rates' is also referred to as the 'Zero Coupon Yield Curve' for obvious reasons. The zero coupon yield curve is considered to be the true term structure of interest rates because there is no reinvestment risk. The curve can be obtained from coupon bonds using the bootstrapping procedure as demonstrated below.

The YTM yield curve does not distinguish between different payment patterns that may result from bonds with different coupons, that is, the fact that low coupon bonds pay a higher portion of their cash flows in present value terms at a later

[3] See Babbel and Merill (1996).

date as compared to higher coupon bonds of the same maturity. To compensate for this bond analysts sometimes construct a coupon yield curve which plots the YTM against the term to maturity for a group of bonds with the same coupon.

The yield curve will be equivalent to the term structure if the term structure is flat, or in other words the spot rates are the same for all maturities. This is because, when the term structure is flat, the YTM which is a complex average of spot rates, will be equal to the observed spot rate.

15.6 Bootstrapping

In practice, we are unlikely to have data for the prices of zero coupon bonds maturing at regularly spaced intervals of time. Bootstrapping is a technique for determining the term structure of interest rates, given price data for a series of coupon paying bonds, and it works as follows.

Assume that we have the following data for four bonds, each of which matures at the end of the stated period of time. For ease of exposition we will also assume that the bonds pay coupons on an annual basis.

Table 15.1 Inputs for Determining the Zero Coupon Yield Curve

Time to Maturity	Price in dollars	Coupon
1 year	1,000	6%
2 years	975	8%
3 years	950	9%
4 years	925	10%

The one year spot rate is obviously 6%. Using this information, the two year spot rate can be determined as follows:

$$975 = \frac{80}{(1.06)} + \frac{1080}{(1+s_2)^2}$$
$$\Rightarrow s_2 = 9.57\%$$

Similarly the three year spot rate can be determined as follows:

$$950 = \frac{90}{(1.06)} + \frac{90}{(1.0957)^2} + \frac{1090}{(1+s_3)^3}$$
$$\Rightarrow s_3 = 11.32\%$$

And finally, using the same logic

$$925 = \frac{100}{(1.06)} + \frac{100}{(1.0957)^2} + \frac{100}{(1.1132)^3} + \frac{1,100}{(1+s_4)^4}$$
$$\Rightarrow s_4 = 12.99\%$$

15.7 Practical Difficulties with Bootstrapping

In the above example we used the prices of four bonds, maturing after 1, 2, 3, and 4 years respectively. In some cases there may be several bonds of the same risk class, maturing at a given point in time. Usually each will have its own coupon. While estimating spot rates for a given maturity, obviously the coupon rates of the bonds being used is a factor. One of the other major issues while estimating the term structure using bootstrapping, is that a bond may not exist, or else it may not actively trade, for a particular maturity. And it is not necessary that we will always have access to a set of bonds whose maturity dates are conveniently spaced exactly one period apart.

Finally, all traded bonds may not be plain vanilla in nature. The US Treasury has issued bonds which can be recalled after a point in time. This too has implications for the bootstrapping procedure.

15.8 Coupon Yield Curves and Par Bond Yield Curves

One of the problems with bootstrapping is that in practice we receive data in the form of prices of bonds with different coupons. One way of getting over this estimation related problem is by using data for bonds all of which have the same coupon. The 'Yield Curve' that is so obtained is called the 'Coupon Yield Curve'. If we construct such a curve we can see that in general higher coupon bonds trade at a discount (have higher yields) relative to low coupon bonds. This is due to reinvestment risk.

The 'Par Bond Yield Curve' on the other hand, is an estimate of the yield curve obtained from data for bonds which have different coupons, but all of which trade at par. In this case, the coupon for each of these bonds is nothing but its Yield to Maturity. The method of bootstrapping can then be applied to such a data set in order to derive the vector of spot rates as before. Here is a numerical illustration of this procedure.

Table 15.2 The Par-Bond Approach to Bootstrapping

Time to Maturity	Price in Dollars	YTM = Coupon
1 year	1,000	6%
2 years	1,000	8%
3 years	1,000	9%
4 years	1,000	10%

The one year spot rate is obviously 6%. Using this information, the two year spot rate can be determined as follows.

$$1{,}000 = \frac{80}{(1.06)} + \frac{1{,}080}{(1+s_2)^2}$$
$$\Rightarrow s_2 = 8.08\%$$

Similarly the three year spot rate can be determined as follows.

$$1{,}000 = \frac{90}{(1.06)} + \frac{90}{(1.0808)^2} + \frac{1{,}090}{(1+s_3)^3}$$
$$\Rightarrow s_3 = 9.16\%$$

And finally, using the same logic

$$1{,}000 = \frac{100}{(1.06)} + \frac{100}{(1.0808)^2} + \frac{100}{(1.0916)^3} + \frac{1,100}{(1+s_4)^4}$$
$$\Rightarrow s_4 = 10.30\%$$

The par bond yield curve is not commonly encountered in secondary market trading. However it is often constructed and used by people in Corporate Finance departments and others who are involved with issues in the primary market. Investment bankers use par bond yield curves to determine the required coupon for a new bond that is to be issued at par. This is because new issues are typically issued at par and consequently the banker needs to know the coupon that needs to be offered in order to ensure that the bonds can be issued at par. In practice the market uses data from non-par plain vanilla bonds to first derive the zero coupon yield curve. This information is then used to deduce the hypothetical par yields that would be observed if traded par bonds were to be available.[4]

15.9 Deducing a Par Bond Yield Curve

The par bond yield curve can be derived using a vector of spot rates. We have already obtained the following information while analyzing the data given in Table 15.1.

The yield for a one year par bond is obviously 6%.

The yield or equivalently the coupon for the two year par bond can be deduced as follows.

$$1{,}000 = \frac{C}{(1.06)} + \frac{1{,}000 + C}{(1.0957)^2}$$
$$\Rightarrow C = \$\ 94.0441 \Rightarrow c = 9.4044\%$$

[4]Notice that even though the bonds are priced at par, the vector of spot rates that is deduced, is different from the vector of observed yields to maturity.

Table 15.3 **Using Spot Rates to Infer a Par Bond Yield Curve**

Time to Maturity	Spot Rate
1 year	6%
2 years	9.57%
3 years	11.32%
4 years	12.99%

Similarly

$$1{,}000 = \frac{C}{(1.06)} + \frac{C}{(1.0957)^2} + \frac{1{,}000 + C}{(1.1132)^3}$$
$$\Rightarrow C = \$\ 109.9837 \Rightarrow c = 10.9984\%$$

and

$$1{,}000 = \frac{C}{(1.06)} + \frac{C}{(1.0957)^2} + \frac{C}{(1.1132)^3} + \frac{1{,}000 + C}{(1.1299)^4}$$
$$\Rightarrow C = \$\ 124.0742 \Rightarrow c = 12.4074\%$$

Thus a two year bond of this risk class ought to be issued with a coupon of 9.4044% if it is to be sold at par. Similarly three year and four year bonds should carry coupons of 10.9984% and 12.4074% respectively.

15.10 Implied Forward Rates

Consider an investor who is contemplating a two period loan. He will be indifferent between a two-period spot rate of s_2 and a one period spot rate of s_1 with a forward contract to rollover his one period loan at maturity for one more period at a rate f_1^1, provided

$$(1 + s_2)^2 = (1 + s_1)(1 + f_1^1) \tag{15.1}$$

f_1^1 is the one period forward rate or the rate for a one period loan to be made one period later. It is known as the implied forward rate that is contained in the term structure. In general if we have an n period spot rate and an m period spot rate, where $m > n$, then

$$(1 + s_m)^m = (1 + s_n)^n (1 + f_n^{m-n})^{m-n} \tag{15.2}$$

where f_n^{m-n} is the $m - n$ period implied forward rate for a loan to be made after n periods.

Forward rates are believed to convey information about the expected interest rate structure in the future. In fact, one school of thought called the *unbiased expectations hypothesis* believes that such rates are nothing but current expectations of future interest rates.

15.10.1 Illustration

The one year spot rate is 8%; the two year spot rate is 10%, and the three year spot rate is 11.25%. Using this data what information can we deduce about implied forward rates?

$$(1 + s_2)^2 = (1 + s_1)(1 + f_1^1) \Rightarrow f_1^1 = 0.1204 = 12.04\%$$

Similarly

$$(1 + s_3)^3 = (1 + s_1)(1 + f_1^2)^2 \Rightarrow f_1^2 = 0.1291 \equiv 12.91\%$$
$$(1 + s_3)^3 = (1 + s_1)(1 + f_1^1)(1 + f_2^1) \Rightarrow f_2^1 = 0.1379 \equiv 13.79\%$$

15.11 Fitting the Yield Curve

When plotting a yield curve we fit a series of discrete points of yield against maturity. Similarly for the term structure of interest rates we plot spot rates for a fixed time period against the time period. The yield curve itself is however a smooth curve drawn through these discrete points. We require a method that allows us to fit the curve as accurately as possible. This aspect of yield curve analysis is known as *yield curve modeling* or *estimating the term structure*.

The yield curve is derived from coupon bond prices and yields. In attempting to model the curve from bond yield data we need to take cognizance of two fundamental issues. First, there is the problem of gaps in the maturity spectrum as in reality there will not be bonds maturing at regular intervals along the term structure. Second the term structure is defined in terms of spot rates or zero-coupon interest rates. But in many markets there is no zero coupon bond market.

We will study four methods for modeling the yield curve.[5]

15.11.1 Interpolation

The simplest method that can be used to fit the yield curve is *linear interpolation*. For example assume that we are given the following data.

$$s_5 = 8\% \text{ and } s_{10} = 9\%$$

We can then calculate the eight-year spot rate as:

$$s_8 = 8.00 + \frac{(8-5)}{(10-5)} \times (9.00 - 8.00) = 8.60\%$$

15.11.2 Polynomial Models

A polynomial of degree k may be expressed as

$$y_i = \alpha + \beta_1 t + \beta_2 t^2 + ... + \beta_k t^k + u_i \tag{15.3}$$

[5] See Choudhury (2004).

where y_i is the YTM of bond **i**; **t** is its term to maturity; and u_i is the residual error. To determine the coefficients of the polynomial we minimize the sum of the squared residual errors given by

$$\sum_{i=1}^{N} {u_i}^2$$

where N is the number of bonds used.

15.11.3 Regression Models

This is a variation of the polynomial approach. The method uses bond prices as the dependent variable and the coupons and face values of the bonds as the independent variables. The standard format is given by

$$P_i = \beta_1 C_{1i} + \beta_2 C_{2i} + \ldots\ldots + \beta_N (C_{Ni} + M) + u_i \tag{15.4}$$

In the above expression, P_i is the dirty price of bond **i**; C_{ji} is the coupon of bond **i** in period **j**; and u_i is the residual error. The spot rates can be derived from the estimated relationships using the expression

$$\beta_n = \frac{1}{(1 + s_n)^n} \tag{15.5}$$

15.12 The Nelson–Siegel Model

Before we describe the Nelson–Siegel technique, we need to familiarize ourselves with bond pricing in a continuous time framework.

Take a zero coupon bond which pays \$ 1 after n periods. In a discrete time setting, we would express the bond price as:

$$P = \frac{1}{(1 + s_n)^n} \tag{15.6}$$

where s_n is the n-period spot rate at time 0. If we were to compound interest m times per period where $m > 1$, then we would express the bond price as

$$P = \frac{1}{\left(1 + \dfrac{s_n}{m}\right)^{mn}} \tag{15.7}$$

For instance if we were to compound four times every period them m would be equal to 4. In the limit as $m \to \infty$ we get the case of continuous compounding. In the limit we can express the price of the bond as

$$P = e^{-s_n \times n} \tag{15.8}$$

We know that in the discrete time framework the n-period spot rate can be expressed as

$$(1 + s_n)^n = (1 + s_1)(1 + f_1^1)(1 + f_2^1)\ldots\ldots\ldots\ldots\ldots(1 + f_{n-1}^1) \tag{15.9}$$

In the case of continuous compounding the equivalent representation is

$$s_n \times n = \int_0^n f_s ds \tag{15.10}$$

where f_s is the instantaneous forward rate at time s as perceived at time zero.

Nelson–Siegel proposed the following representation for the instantaneous forward rate.

$$f_s = \beta_0 + \beta_1 e^{-\frac{s}{\theta}} + \beta_2 \times \frac{s}{\theta} e^{-\frac{s}{\theta}} \tag{15.11}$$

Integrating this function we get the following expression for the n-period spot rate.

$$s_n = \beta_0 + \beta_1 \times \left[\frac{1 - e^{\frac{-n}{\theta}}}{\frac{n}{\theta}} \right] + \beta_2 \times \left[\frac{1 - e^{\frac{-n}{\theta}}}{\frac{n}{\theta}} - e^{\frac{-n}{\theta}} \right] \tag{15.12}$$

The parameters $\beta_0, \beta_1, \beta_2,$ and θ have to be empirically estimated.

The Nelson–Siegel method for estimating the term structure has a number of advantages. Firstly, its functional form can handle a variety of the shapes of the term structure that are observed in the market. Secondly, the model avoids the need to introduce other assumptions for interpolation between intermediate points. For instance, the bootstrapping approach will give us a vector of spot rates spaced six months apart. To value a bond whose life is not an integer multiple of semi-annual periods, we would obviously need to interpolate. On the contrary, using the Nelson–Siegel approach we can derive the spot rate at any point in time and not just at certain discrete points.

15.13 Theories of the Term Structure

From observing yield curves in different markets at various points in time, an individual who studies the bond market will notice that the yield curve tends to adopt one of the three basic shapes.

- **Upward Sloping**: In the case of a yield curve with a positive slope, also termed as a rising yield curve, short term yields will be lower than long term yields.
- **Downward Sloping**: In the case of a yield curve with a negative slope, also termed as an inverted yield curve, long term rates will be lower than short term rates.
- **Humped**: A humped yield curve is characterized by lower rates at the short end of the spectrum. The curve then rises, reaching a peak at the middle of the maturity spectrum, and then gradually slopes downward at longer maturities.

A great deal of effort is expended by analysts and economists in analyzing and interpreting the yield curve, for there is often substantial information that is associated with the curve at any point in time. Various theories have been advanced that purport to explain the observed shapes of the curve. However, no theory by itself is able to explain all aspects of the curves that are observed in practice. So often, analysts seek to explain specific shapes of the curve using a combination of the accepted theories.

15.13.1 The Pure or Unbiased Expectations Hypothesis

The unbiased expectations hypothesis states that current implied forward rates are unbiased estimators of future spot rates. As per this theory long-term rates are geometric averages of expected future short-term rates. Thus a positively sloped yield curve would be consistent with the argument that the market expects spot interest rates to rise. If rates are expected to rise then investors in long term bonds will be perturbed for they face the specter of a capital loss. This is because rising interest rates will lead to declining bond prices, and long term bonds are more sensitive to rising interest rates than short term bonds. In such a situation investors will start selling long dated securities and buying short dated securities. This will lead to an increase in yields on long term bonds and a decline in yields on short term bonds. The overall result will be an upward sloping yield curve. On the contrary, an inverted yield curve would indicate that the market expects future spot rates to fall.

The hypothesis can be used to explain any shape of the yield curve. For instance a humped yield curve would be consistent with the explanation that investors expect short-term rates to rise and long-term rates to fall.

Expectations or views on the future direction of the market are a function mainly of the expected rate of inflation. If the market expects inflationary pressures in the future, the yield curve will be positively sloped, whereas if inflation is expected to decline then the slope of the yield curve will be negative.

15.13.2 The Liquidity Preference Theory (LPT)

Intuitively, most of us would feel that longer maturity instruments are more risky than those with a shorter maturity. An investor lending money for five years will usually demand a higher rate of interest than if he were to lend money to the same customer for a year. This is because the borrower may not be able to repay the loan over a longer term period. For this reason, long dated yields should be higher than short dated yields.

Take the case where the market expects inflation to remain fairly stable over time. The expectations hypothesis would postulate that this scenario would be characterized by a flat yield curve. However the liquidity preference hypothesis would predict a positively sloping yield curve. The argument would be as follows. Generally a borrower would like to borrow over as long a term as possible while a lender would like to lend over as short a term as possible. Thus lenders have to be suitably rewarded if they are to be induced to lend for longer periods of time. This compensation may be considered as a premium for the loss of liquidity from the standpoint of the lender. The premium may be expected to increase the further the investor lends across the term structure, so that the longest dated instruments will, all else being equal, have the highest yield.

As per this hypothesis, the yield curve should almost always be upward sloping, reflecting the bond holders' preference for liquidity. However, inverted yield curves can still be explained by this theory by postulating that interest rates are likely to decline in the future, as a consequence of which, despite the liquidity premium, long term rates are lower than short-term rates.

15.13.3 The Expectations Hypothesis versus the LPT: A Mathematical Analysis

As per the expectations hypothesis forward rates are unbiased expectations of future spot rates. Thus

$$f_n^{m-n} = E_0[\,_nS_{m-n}] \tag{15.13}$$

In other words the $(m-n)$ period forward rate, n periods from now, is the current expectation of the $(m-n)$ period spot rate that is expected to prevail n periods from now. The expectations hypothesis can explain any shape of the term structure. For instance, an expectation that future short term interest rates will be above the current level would lead to an upward sloping term structure.

Illustration of an Upward Sloping Term Structure Assume that $s_1 = 5.50\%$; $E[\,_1s_1] = 6.0\%$; $E[\,_2s_1] = 7.5\%$, and that $E[\,_3s_1] = 8.5\%$. If so then

$$s_2 = [(1.055)(1.06)]^{\frac{1}{2}} - 1 = 5.75\%$$

$$s_3 = [(1.055)(1.06)(1.075)]^{\frac{1}{3}} - 1 = 6.33\%$$

$$s_4 = [(1.055)(1.06)(1.075)(1.085)]^{\frac{1}{4}} - 1 = 6.87\%$$

According to the expectations hypothesis, investors care only about expected returns and not about risk. Let us take the case of an investor who chooses to invest for two periods. He can buy a two period bond yielding a rate of s_2. Or else he can buy a one period bond that yields s_1 and then roll over into another one period bond at maturity. As per the hypothesis the investor will be indifferent between the two strategies if the expected returns in both cases are equal.

In other words the hypothesis predicts that the market will be in equilibrium if

$$(1+s_2)^2 = E[(1+s_1)(1+\,_1s_1)] \tag{15.14}$$

But we know that if arbitrage is to be ruled out, then

$$(1+s_2)^2 = (1+s_1)(1+f_1^1) \tag{15.15}$$

Thus as per the expectations hypothesis $f_1^1 = E(\,_1s_1)$

Now let us focus on the liquidity preference theory. Take the case of an investor who has a one period investment horizon. He can buy a one period bond and lock in a rate of s_1. Or else he can buy a two period bond and sell it after one year. In the second case, the rate of return will be uncertain at the outset, for it will depend on the one period rate that will prevail one period from now.

Consider a two period zero coupon bond with a face value of \$ 1,000. Its current price will be

$$\frac{1{,}000}{(1+s_2)^2} = \frac{1{,}000}{(1+s_1)(1+f_1^1)} \tag{15.16}$$

The expected price of the bond after one period is

$$E\left[\frac{1,000}{(1+\ {}_1s_1)}\right] \geq \frac{1,000}{1+E(\ {}_1s_1)} \tag{15.17}$$

How do we know that the expectation of the price after one year will be greater than or equal to the face value discounted by the expected one period spot rate one period from now. This conclusion arises from a result called Jensen's Inequality. It states that for a convex function, *the expectation of the function is greater than or equal to the function of the expectation*. That is

$$E[f(X)] \geq f[E(X)] \tag{15.18}$$

In our case, $\dfrac{1{,}000}{1+\ {}_1s_1}$ is a convex function because the second derivative is positive.

The expected rate of return from the two period bond over the first year will be

$$\frac{E\left[\dfrac{1{,}000}{(1+\ {}_1s_1)}\right] - \dfrac{1{,}000}{(1+s_1)(1+f_1^1)}}{\dfrac{1{,}000}{(1+s_1)(1+f_1^1)}} \geq \frac{\dfrac{1{,}000}{1+E(\ {}_1s_1)} - \dfrac{1{,}000}{(1+s_1)(1+f_1^1)}}{\dfrac{1{,}000}{(1+s_1)(1+f_1^1)}}$$

$$\Rightarrow \frac{E\left[\dfrac{1{,}000}{(1+\ {}_1s_1)}\right] - \dfrac{1{,}000}{(1+s_1)(1+f_1^1)}}{\dfrac{1{,}000}{(1+s_1)(1+f_1^1)}} \geq \frac{(1+s_1)(1+f_1^1)}{1+E(\ {}_1s_1)} - 1$$

Obviously, the expected one period return from the two period bond will be greater than s_1 only if $f_1^1 > E(\ {}_1s_1)$.

Now an investor with a one period investment horizon will obviously choose to hold a two period bond only if its expected return is greater than the assured return on a one period bond. This is because if he chooses to hold the two period bond, he will have to sell it after one period at a price that is unknown at the outset. From the above analysis this would imply that the forward rate must be higher than the expected one period spot rate. Thus if investors are risk averse, which is the normal assumption made in Finance theory, the forward rate will exceed the expected spot rate by an amount equal to the risk premium or what may be termed as the *liquidity* premium.

We know that

$$(1 + s_2)(1 + s_2) = (1 + s_1)(1 + f_1^1)$$

As per the LPT

$$(1 + s_1)(1 + f_1^1) > (1 + s_1)[1 + E({}_1s_1)]$$

Therefore

$$(1 + s_2)(1 + s_2) > (1 + s_1)[1 + E({}_1s_1)]$$

Consider a downward sloping yield curve. This would imply that $s_1 > s_2$. Therefore it must be the case that $E({}_1s_1)$ is substantially less than s_1. In other words the market expects spot rates to decline substantially. For instance if $s_1 = 7\%$ and $s_2 = 6\%$ then $f_1^1 = 5.01\%$ As per the expectations hypothesis $E({}_1s_1) = 5.01\%$. However, as per the LPT, $E({}_1s_1) < 5.01\%$. If we assume that the liquidity premium is 0.50%, then $E({}_1s_1) = 4.51\%$.

Now let us take the case of a flat term structure. As per the expectations hypothesis

$$s_1 = s_2 = f_1^1 = E({}_1s_1)$$

However, according to the LPT $E({}_1s_1) < s_1 = s_2$. Thus while the expectations hypothesis would imply that the market expects spot rates to remain unchanged, the prediction according to the liquidity preference theory is that the market expects spot rates to decline. For instance if $s_1 = s_2 = 7\%$ then according to the expectations hypothesis $E({}_1s_1) = 7\%$. However, according to the LPT, $E({}_1s_1) = 7 - 0.50 = 6.50\%$.

Finally let us take the case of an upward sloping yield curve. If $s_1 < s_2$ then for a slightly upward sloping yield curve the LPT would be consistent with the expectation that rates are going to marginally decline. However, if the curve were to be steeply upward sloping then the LPT would be consistent with the expectation that short term rates are going to rise. For instance, assume that $s_1 = 7\%$ and that $s_2 = 7.1\%$. If so

$$f_1^1 = 7.2\% \Rightarrow E({}_1s_1) = 7.20 - 0.50 = 6.70\%$$

However, if $s_2 = 7.3\%$ then

$$f_1^1 = 7.6\% \Rightarrow E({}_1s_1) = 7.60 - 0.50 = 7.10\%$$

In both these cases however, the expectations hypothesis would predict that spot rates are likely to rise. In the first scenario, as per the expectations hypothesis, $E({}_1s_1) = 7.20\%$, whereas in the second case $E({}_1s_1) = 7.60\%$.

15.13.4 The Money Substitute Hypothesis

According to this hypothesis short-term bonds are substitutes for holding cash. Investors hold only short dated bonds because they are viewed as having low or negligible risk. As a result the yields on short dated bonds are depressed due to increased demand, and consequently long term yields are greater than short term yields. Borrowers on the other hand prefer to issue debt for long maturities and on as few occasions as possible to minimize costs. Thus the yields on long term securities are driven upward due to increased supply and lower liquidity.[6]

15.13.5 The Market Segmentation Hypothesis

This theory states that the capital market is made up of a wide variety of issuers, each with different requirements. Certain classes of investors will prefer short dated bonds while others will prefer long dated bonds. The theory argues that activity is concentrated in certain specific areas of the market, and that there are no interrelationships between these segments of the market. The relative amount of funds invested in each market segment causes differentials in supply and demand which leads to humps in the yield curve.

Thus, as per this theory the observed shape of the yield curve is determined by the supply and demand for specific maturity investments, and the dynamics in a particular market segment has no relevance for any other part of the curve. For example, banks concentrate a large part of their activity at the short-end of the curve as a part of daily cash management known as asset-liability management, and for regulatory purposes. On the other hand, fund managers such as pension funds and insurance companies are active at the long end of the market.[7] Few institutions however have a preference for medium dated bonds. This behavior leads to high prices and low yields at both the short and long ends of the maturity spectrum and to high yields in the middle of the term structure.

15.13.6 The Preferred Habitat Theory

This is a slightly modified version of the segmentation hypothesis. This suggests that different market participants have an interest in specified areas of the yield curve but can be persuaded to hold bonds from other parts of the maturity spectrum if they are provided with sufficient incentives. Hence, banks which typically operate at the short-end of the spectrum, may at times hold long dated bonds once the prices of these bonds fall to a certain level, thereby ensuring that the returns from holding such bonds is commensurate with the attendant risk. Similar considerations may persuade long-term investors to hold short term debt. So the incentive for an investor to shift out of his preferred habitat is the inducement offered by way of a higher rate of interest.

15.13.7 Short Rates

A short rate of interest is a future spot rate of interest that might arise over time. Usually it is denoted as a single period rate of interest, and is defined as the rate for the shortest period of time considered by a model.

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[6] See Choudhry (2004).
[7] See Choudhry (2004).

At any point in time, if we consider a series of spot rates corresponding to different lengths of time, we will be able to derive a unique vector of forward rates. However, for any future point in time, there are an infinite number of short rates that may arise. Each possible value has a varying probability of occurrence. At the initial point in time, the one period spot rate will be equal to the one period forward rate, which will be equal to the one period short rate. However, if we look at a time horizon beyond the first period, these interest rate measures will in general not be equal to each other.

15.14 Issues in the Valuation of Interest Rate Derivatives

A Black–Scholes approach or a standard binomial model, which work well in the case of stock options, are inappropriate for interest rate dependent financial claims and their derivatives. For instance, let us take the case of a bond maturing after N years. Although the price before maturity is expected to evolve in a random fashion, as the time to expiration approaches zero, the price of the underlying security should tend towards its par value. Consequently, option pricing models based on lognormal or binomial outcomes for the price of the underlying asset are unsuitable for valuing a derivative like an option on an interest rate dependent asset such as a bond.

There is no standard model for pricing options on debt securities. However, almost all of the models that have been postulated are based on the following three-step approach.[8]

- The random character of interest rate movements is first modeled.
- The interest rate process is used to infer the distribution of prices of the underlying debt securities.
- The distribution of prices of the underlying asset is used to value the option.

Thus, the difference between competing models lies in the way they model the interest rate process.

15.15 Equilibrium Models of the Term Structure

Such models derive a stochastic process for the evolution of the short rate by making a set of assumptions about economic variables. In a one-factor model, the stochastic process for the short rate is the only source of uncertainty. In general,

[8] See Rendleman (2002).

short rates are assumed to follow an Ito process of the form

$$dr = \mu(r)\,dt + \sigma(r)\,dZ \tag{15.19}$$

That is, the drift μ, and the variance rate σ^2 are assumed to be functions of the short rate, but are considered to be independent of time.

There is, however, a critical problem with this approach to the pricing of interest rate derivatives. In practice, it is usually the case that the prices of the underlying debt securities, which are inferred from the postulated interest rate process, are not equal to their observed market prices. It is obvious that if we are unable to correctly price the underlying security, it will be impossible for us to have faith in the derivative prices that are obtained from the same.

15.15.1 The Cox, Ingersoll, and Ross (CIR Model)

The CIR model may be specified as

$$dr = \alpha(\beta - r)\,dt + \sigma\sqrt{r}dZ \tag{15.20}$$

This model incorporates mean reversion. Mean reversion may be described as follows. When the short rate (r), is higher than the long-run rate (β), it will display a tendency to decline and will gradually tend towards β. On the other hand, if r is lower than beta, it will gradually rise and will once again tend towards β. In other words, when the short rate is high, the process will have a negative drift while when the short rate is low, it will be characterized by a positive drift. One of the appealing features of the CIR model is that it does not allow interest rates to become negative.

The long-run value for the short rate should not be misconstrued as a long-term rate of interest.[9] While the long-term interest rate in an economy is a variable that is expected to evolve stochastically, the long-run rate is a parameter to which the short rate may be expected to converge over a period of time. The parameter α determines the speed of adjustment of the short rate. If α is high the adjustment will be rapid. However if α is low, then the movement of the short rate to its long-run value will be more gradual.

15.15.2 The Rendleman and Bartter Model

Rendleman and Bartter derived the following process for the short rate

$$dr = \mu r dt + \sigma r dZ \tag{15.21}$$

This is analogous to the process used by Black and Scholes to define the evolution of the stock price over time, and implies that the short rate follows geometric Brownian motion. One of the shortcomings of this model is that it does not incorporate mean reversion.

[9] See Ho and Lee (2005).

15.15.3 The Vasicek Model

Vasicek derived the following process for the short rate

$$dr = \alpha(\beta - r)\,dt + \sigma dZ \qquad (15.22)$$

where α, β, and σ are constants. This process is referred to as the Ornstein–Uhlenbeck process, and it too incorporates mean reversion. That is, as per the model, the short rate displays a tendency to revert to its long-term mean of β. Vasicek obtained the prices of bonds of different maturities using a constant market price of risk.[10] The model has a drawback in the sense that it allows the short rate to become negative.

15.16 Arbitrage-Free Term Structure Models

This class of option pricing models takes the observed term structure of interest rates as an input, unlike the earlier class of models which generate the term structure as an output. The stochastic process for the short rate that is implied by the observed term structure is deduced and from it the values of the underlying debt securities are inferred. Finally, the price of the derivative security is obtained. The appealing feature of this approach is that the prices of the debt securities that are derived in this fashion are entirely consistent with the observed term structure.

We will now look at alternative models of the term structure.

15.16.1 The Ho and Lee Model

As per this model, the evolution of the short rate may be specified as

$$dr_t = \mu_t dt + \sigma dZ \qquad (15.23)$$

15.16.2 The Hull and White Model

This model specifies the evolution of the short rate as

$$dr_t = [\mu_t - \alpha r_t]dt + \sigma dZ \qquad (15.24)$$

15.16.3 The Kalotay–Williams–Fabozzi Model

As per this model, the evolution of the short rate may be described as

$$d\ln r_t = \mu_t dt + \sigma dZ \qquad (15.25)$$

15.16.4 The Black–Derman Toy Model

The model may be stated as

$$d\ln r_t = \mu_t dt + \sigma_t dZ \qquad (15.26)$$

[10]We will shortly have more to say about the market price of risk.

15.16.5 The Black–Karasinski Model

As per this model

$$d \ln r_t = [\mu_t - \alpha ln r_t]dt + \sigma_t dZ \tag{15.27}$$

15.17 The Fundamental Bond Pricing Equation in Continuous Time

If $P(t, T)$ is the price of a zero coupon bond maturing at time T, then the instantaneous return may be defined as

$$\frac{dP}{P} = \mu(t, T)\,dt + \sigma(t, T)\,dZ \tag{15.28}$$

where Z is a Wiener process.

Consider a portfolio of two bonds of different maturities, T_1 and T_2. Assume that we invest a proportion w of our wealth in the first bond and $1 - w$ in the second bond. If we denote the portfolio by V, then the rate of return on this portfolio may be expressed as

$$\frac{dV}{V} = [w\mu(t, T_1) + (1 - w)\mu(t, T_2)]dt + [w\sigma(t, T_1) + (1 - w)\sigma(t, T_2)]dZ \tag{15.29}$$

Let us choose w in a way such that

$$w\sigma(t, T_1) + (1 - w)\sigma(t, T_2) = 0$$

$$\Rightarrow w = \frac{\sigma(t, T_2)}{\sigma(t, T_2) - \sigma(t, T_1)}$$

In such a case the portfolio return will be riskless and can be expressed as

$$[w\mu(t, T_1) + (1 - w)\mu(t, T_2)]\,dt = \left[\frac{\mu(t, T_1)\sigma(t, T_2)}{\sigma(t, T_2) - \sigma(t, T_1)} - \frac{\mu(t, T_2)\sigma(t, T_1)}{\sigma(t, T_2) - \sigma(t, T_1)}\right]dt$$

Since the portfolio is risk-less by construction, the rate of return must be equal to the riskless rate. Thus

$$\left[\frac{\mu(t, T_1)\sigma(t, T_2)}{\sigma(t, T_2) - \sigma(t, T_1)} - \frac{\mu(t, T_2)\sigma(t, T_1)}{\sigma(t, T_2) - \sigma(t, T_1)}\right]dt = r(t)dt$$

$$\Rightarrow \mu(t, T_1)\sigma(t, T_2) - \mu(t, T_2)\sigma(t, T_1) = r(t)[\sigma(t, T_2) - \sigma(t, T_1)]$$

$$\Rightarrow [\mu(t, T_1) - r(t)]\sigma(t, T_2) = [\mu(t, T_2) - r(t)]\sigma(t, T_1)$$

$$\Rightarrow \frac{[\mu(t, T_1) - r(t)]}{\sigma(t, T_1)} = \frac{[\mu(t, T_2) - r(t)]}{\sigma(t, T_2)} = \lambda(r, t) \tag{15.30}$$

$\lambda(r, t)$ is referred to as the market price of risk. Since μ is equal to $r + \lambda\sigma$, we can state that

$$\frac{dP}{P} = [r(t) + \sigma(t, T)\lambda(t, T)]dt + \sigma(t, T)\,dZ$$

A general form of a single factor model for the short rate is:

$$dr = \alpha(r, t)dt + \rho(r, t)dZ$$

Since $P = P(r, t)$ from Ito's lemma

$$dP = \left[\frac{\partial P}{\partial t} + \alpha \times \frac{\partial P}{\partial r} + \frac{1}{2} \times \rho^2 \times \frac{\partial^2 P}{\partial r^2}\right]dt + \rho \times \frac{\partial P}{\partial r}dZ$$

If we compare it with the earlier representation of dP, which is

$$dP = [r(t) + \sigma(t, T)\lambda(t, T)]P(t, T)\,dt + \sigma(t, T)P(t, T)\,dZ$$

we can state that

$$\left[\frac{\partial P}{\partial t} + \alpha \times \frac{\partial P}{\partial r} + \frac{1}{2} \times \rho^2 \times \frac{\partial^2 P}{\partial r^2}\right] = [r(t) + \sigma(t, T)\lambda(t, T)]P(t, T)$$

and

$$\rho \times \frac{\partial P}{\partial r} = \sigma(t, T)P(t, T)$$

Thus

$$\frac{\partial P}{\partial t} + \alpha \times \frac{\partial P}{\partial r} + \frac{1}{2} \times \rho^2 \times \frac{\partial^2 P}{\partial r^2} = r(t)P(t, T) + \rho \times \lambda \times \frac{\partial P}{\partial r}$$

$$\Rightarrow \frac{\partial P}{\partial t} + [\alpha - \rho \times \lambda] \times \frac{\partial P}{\partial r} + \frac{1}{2} \times \rho^2 \times \frac{\partial^2 P}{\partial r^2} - rP(t, T) = 0 \qquad (15.31)$$

This is the fundamental bond pricing equation.

15.18 The Binomial Tree Approach to the Term Structure

We will now illustrate no-arbitrage models of the term structure using a discrete time framework. Before we proceed we need to define certain variables.

We will denote the length of a time period in the binomial tree as $\triangle t$, where time is measured in years. Interest rates will consequently be specified as annual rates. s_n is the n-period spot rate at time 0, quoted as a percentage per annum, but compounded at an interval equal to $\triangle t$. Thus if a dollar is invested at time 0 for n periods, the future value will be

$$\text{F.V.} = (1 + s_n \triangle t)^n$$

We will define r_n as the short rate that will prevail n periods later. r_0 is obviously equal to the one period spot rate s_1. The time step in the binomial tree will be taken as six months for the purpose of illustration. Consequently, $\triangle t = 0.50$.

Consider a bond that pays \$ 1 after two periods. Its present value will be $\frac{1}{\left(1+\frac{s_2}{2}\right)^2}$. The short rate after one period will be either r_u or r_d. Thus the value of the bond after one period will be either

$$P_u = \frac{1}{\left(1+\frac{r_u}{2}\right)} \quad \text{or} \quad P_d = \frac{1}{\left(1+\frac{r_d}{2}\right)}$$

The present value of the one period bond will therefore be $\frac{P_u}{\left(1+\frac{s_1}{2}\right)}$ or $\frac{P_d}{\left(1+\frac{s_1}{2}\right)}$.

Thus

$$\frac{1}{\left(1+\frac{s_2}{2}\right)^2} = \frac{q \times P_u}{\left(1+\frac{s_1}{2}\right)} + \frac{(1-q) \times P_d}{\left(1+\frac{s_1}{2}\right)}$$

$$\Rightarrow \frac{1}{\left(1+\frac{s_2}{2}\right)^2} = \frac{q}{\left(1+\frac{s_1}{2}\right)\left(1+\frac{r_u}{2}\right)} + \frac{1-q}{\left(1+\frac{s_1}{2}\right)\left(1+\frac{r_d}{2}\right)}$$

$$\Rightarrow \frac{\left(1+\frac{r_u}{2}\right)\left(1+\frac{r_d}{2}\right)\left(1+\frac{s_1}{2}\right)}{\left(1+\frac{s_2}{2}\right)^2} = q\left(1+\frac{r_d}{2}\right) + (1-q)\left(1+\frac{r_u}{2}\right)$$

$$\Rightarrow \frac{\left(1+\frac{r_u}{2}\right)\left(1+\frac{r_d}{2}\right)\left(1+\frac{s_1}{2}\right)}{\left(1+\frac{s_2}{2}\right)^2} - q\left(1+\frac{r_d}{2}\right) - (1-q)\left(1+\frac{r_u}{2}\right) = 0 \tag{15.32}$$

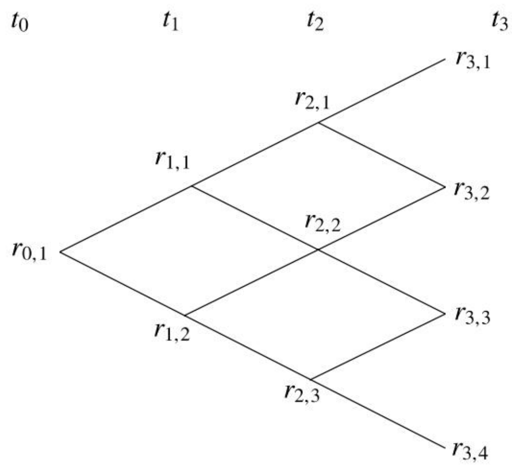
This is the fundamental equation that is used in no-arbitrage models to determine the short rates. To apply these models in practice one has to model r_u and r_d. The models differ in their approach to modeling these parameters.

15.18.1 Some Insights into the Ho-Lee Model

Let us depict a four-period interest rate tree. Each node will be denoted as $r_{k,j}$ where k represents the point in time that we are at and j represents the state. The tree may be modeled as shown in Fig. 15.1.

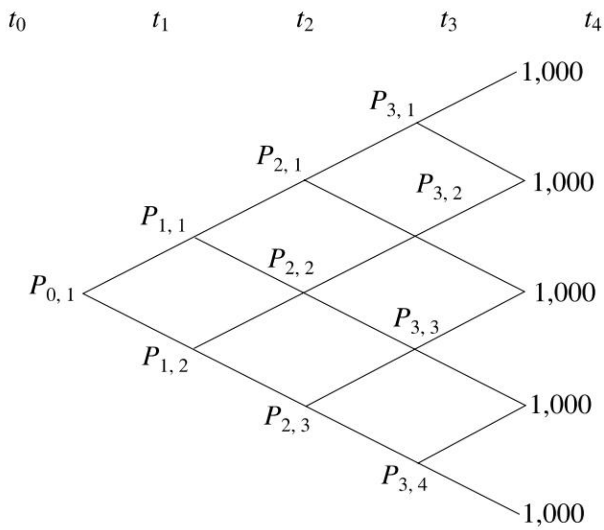
Thus an up move from node $r_{k,j}$ is a move to node $r_{k+1,j}$ while a down move, is a move to node $r_{k+1,j+1}$. Notice that the above tree is recombining, that is, r_{ud} is the same as r_{du}. The recombination condition not only reduces the number of nodes at each point in time, and consequently makes computation relatively easier, it also has implications for σ as well shall shortly demonstrate.

Figure 15.1 **Interest Rate Tree**



Interest rates span time whereas prices do not. Consequently the above interest rate tree can be used to price bonds with up to four periods to maturity. That is, if we have a bond that pays \$ 1,000 at time t_4, the price tree corresponding to the above interest tree may be depicted in Fig. 15.2.

Figure 15.2 **Price Tree**



Consider time point t_k where $t_k = k\Delta t$ and let r_k be the short rate at this point in time. For the process

$$dr = \mu_t dt + \sigma_t dZ$$

we can approximate dr as $r_{k+1,j+1} - r_{k,j}$ and can write

$$r_{k+1,j} = r_{k,j} + \mu_k \Delta t + \sigma_k \sqrt{\Delta t}$$

and
$$r_{k+1,j+1} = r_{k,j} + \mu_k \Delta t - \sigma_k \sqrt{\Delta t}$$

Thus

$$r_{1,1} = r_{0,1} + \mu_0 \Delta t + \sigma_0 \sqrt{\Delta t}$$

and
$$r_{1,2} = r_{0,1} + \mu_0 \Delta t - \sigma_0 \sqrt{\Delta t}$$

$$\Rightarrow r_{1,1} - r_{1,2} = 2\sigma_0 \sqrt{\Delta t}$$

Hence the spread between the upstate and the down state is a function of the volatility parameter. Similarly

$$r_{2,1} = r_{1,1} + \mu_1 \Delta t + \sigma_1 \sqrt{\Delta t}$$

$$r_{2,2} = r_{1,1} + \mu_1 \Delta t - \sigma_1 \sqrt{\Delta t} = r_{1,2} + \mu_1 \Delta t + \sigma_1 \sqrt{\Delta t}$$

$$r_{2,3} = r_{1,2} + \mu_1 \Delta t - \sigma_1 \sqrt{\Delta t}$$

Notice that $r_{2,2}$ can be expressed as a down-move from $r_{1,1}$ or as an up-move from $r_{1,2}$ because of the recombination requirement. Thus

$$r_{1,1} + \mu_1 \Delta t - \sigma_1 \sqrt{\Delta t} = r_{1,2} + \mu_1 \Delta t + \sigma_1 \sqrt{\Delta t}$$

$$\Rightarrow 2\sigma_1 \sqrt{\Delta t} = r_{1,1} - r_{1,2}$$

$$\Rightarrow \sigma_1 = \frac{r_{1,1} - r_{1,2}}{2\sqrt{\Delta t}} = \frac{2\sigma_0 \sqrt{\Delta t}}{2\sqrt{\Delta t}} = \sigma_0$$

Therefore if the interest rate tree is to recombine, then it must be the case that σ is a constant for all points in time. In other words, $\sigma_k = \sigma \forall k$.

15.19 Calibrating the Ho and Lee Model

We will now illustrate as to how the Ho and Lee model can be fitted to the observed term structure using the binomial tree approach. The parameters of the model will be chosen in a way such that the branches of the tree recombine. This, as you have already seen, implies that the standard deviation, σ, is constant for all points in time. In practice the value of σ is chosen arbitrarily. It is also assumed that given a state of nature, there is a 50% probability of an up move, and therefore obviously a 50% probability of a down move.

The movements in the short rate may be expressed as follows.

$$r_{t+1} = r_t + \mu_{t+1} + \epsilon_{t+1}$$

where $\epsilon_{t+1} = \sigma$ with a probability of 0.50 and is equal to $-\sigma$ with a probability of 0.50. Thus

$$r_{k+1,j} = r_{k,j} + \mu_{t+1} + \sigma$$

and

$$r_{k+1,j+1} = r_{k,j} + \mu_{t+1} - \sigma$$

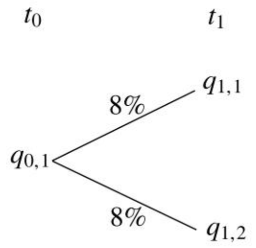
The values of μ will be determined so as to generate the term structure that is observed in practice. Before we proceed to illustrate the calibration technique, let us first introduce the concept of an Arrow–Debreu security, a concept which we will use to subsequently value interest rate dependent securities.

15.19.1 Arrow–Debreu Securities

Such securities, also known as *'Pure Securities'* will pay a dollar if a particular state of nature were to occur. However in all other states of nature, the payoff will be zero. Obviously each node of a binomial tree represents a particular state of nature. The value at time t_0, of a pure security that pays off \$ 1 in state (k, j), will be denoted by $q_{k,j}$.

Let us first consider a one period binomial tree. We will assume that the one period interest rate is 8% per annum. Each period represents a time interval of six months.

Figure 15.3 **One Period Interest Rate and State Price Tree**



Thus the price of a security that pays off \$ 1 in the upstate is given by

$$q_{1,1} = 0.50 \times \frac{1}{(1 + 0.08 \times 0.5)} = \frac{1}{1.04} = 0.4808$$

$q_{1,2}$ or the price of a security that pays one dollar in the down state is obviously the same. The values of q that are calculated in this fashion are referred to as *'State Prices'*.

A riskless security in such an environment is obviously one that pays off a dollar irrespective of the state of nature after one period. Its price is obviously given by

$$q_{1,1} + q_{1,2} = 0.4808 + 0.4808 + 0.9616$$

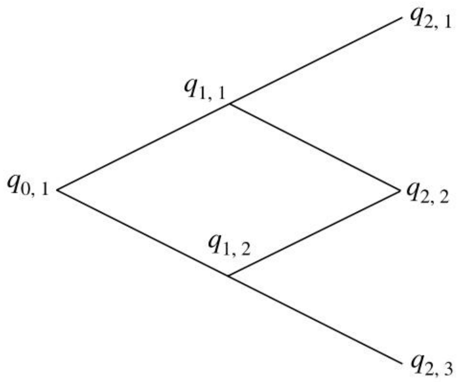
The riskless rate of interest is therefore

$$\frac{1 - 0.9616}{0.9616} \equiv 4\%$$

which is what we would expect.

Now let us extend the model by a period. Assume that if the up state were to be reached after a period, the rate of interest will be 8.75% per annum , whereas if the down state were to be reached it will 7.25% per annum.

Figure 15.4 **Two Period Interest Rate and State Price Tree**



At the end of two periods there will obviously be three states of nature. We can calculate the prices of the Arrow–Debreu securities as follows.

$$q_{2,1} = \frac{0.5 \times 0.5}{(1.04) \times \left(1 + \dfrac{0.0875}{2}\right)}$$

$$= q_{1,1} \times \frac{0.5}{1.04375} = 0.4808 \times \frac{0.5}{1.04375} = 0.2303$$

Let us analyze this expression. The value of an Arrow–Debreu security that pays off \$ 1 at state (2,1) is $\frac{0.5}{1.04375}$ at state (1,1). We know that the value of a pure security that pays off \$ 1 in state (1,1) is \$ 0.4808 as calculated at state (0,1). Consequently the value of a pure security that pays off \$ 1 in state (2,1), as calculated at state (0,1), is

$$0.4808 \times \frac{0.5}{1.04375} = 0.2303$$

Now let us consider a pure security that will pay off \$ 1 if state (2, 2) were to occur. This state can be attained via two paths. Hence, using the same logic

$$q_{2,2} = q_{1,1} \times \frac{0.5}{\left(1 + \dfrac{0.0875}{2}\right)} + q_{1,2} \times \frac{0.5}{\left(1 + \dfrac{0.0725}{2}\right)}$$

$$= 0.4808 \times \frac{0.5}{1.04375} + 0.4808 \times \frac{0.5}{1.03625} = 0.4623$$

Similarly

$$q_{2,3} = q_{1,2} \times \frac{0.5}{1.03625} = 0.2320$$

Thus this approach enables us to price pure securities at various states of nature by working our way forward through the tree.

Now consider the term structure depicted in Table 15.4.

Table 15.4 Vector of Spot Rates

Period	Spot Rate
1	6.00%
2	5.80%
3	6.05%
4	5.90%

Assume that $\sigma = 1\%$. $r_{0,1}$ is obviously 6%. $q_{0,1} = 1$

We know that

$$r_{1,1} = r_{0,1} + \mu_1 + 1 = 7\% + \mu_1$$

and

$$r_{1,2} = r_{0,1} + \mu_1 - 1 = 5\% + \mu_1$$

$$q_{1,1} = q_{1,2} = \frac{0.50}{\left(1 + \frac{0.06}{2}\right)} = 0.4854$$

The prices of the pure securities for the next period may be computed as follows.

$$q_{2,1} = q_{1,1} \times \frac{0.50}{\left(1 + \frac{0.07 + \mu_1}{2}\right)} = \frac{0.4854}{2.07 + \mu_1}$$

$$q_{2,2} = q_{1,1} \times \frac{0.50}{\left(1 + \frac{0.07 + \mu_1}{2}\right)} + q_{1,2} \times \frac{0.50}{\left(1 + \frac{0.05 + \mu_1}{2}\right)}$$

$$= \frac{0.4854}{2.07 + \mu_1} + \frac{0.4854}{2.05 + \mu_1}$$

$$q_{2,3} = q_{1,2} \times \frac{0.50}{\left(1 + \frac{0.05 + \mu_1}{2}\right)} = \frac{0.4854}{2.05 + \mu_1}$$

Obviously

$$q_{2,1} + q_{2,2} + q_{2,3} = \frac{1}{(1 + 0.5 \times s_2)^2} = \frac{1}{(1.029)^2} = 0.9444$$

Thus

$$\frac{2}{2.07+\mu_1}+\frac{2}{2.05+\mu_1}=\frac{0.9444}{0.4854}=1.9456$$

This can be expressed as

$$1.9456\mu^2+4.0159\mu+0.0161=0$$
$$\Rightarrow \mu_1=-0.4017\%$$

Thus $r_{1,1}=.07-.004017=.065983\equiv 6.5983\%$ and $r_{1,2}=.05-.004017=.045983\equiv 4.5983\%$. The prices of the pure securities are: $q_{2,1}=0.2349$, $q_{2,2}=0.4722$, and $q_{2,3}=0.2372$.

Now let us proceed to the next point in time.

$$r_{2,1}=r_{1,1}+\mu_2+1=6.5983+1+\mu_2=7.5983+\mu_2$$
$$r_{2,2}=r_{1,1}+\mu_2-1=6.5983-1+\mu_2=5.5983+\mu_2$$
$$r_{2,3}=r_{1,2}+\mu_2-1=4.5983-1+\mu_2=3.5983+\mu_2$$

The prices of pure securities for the next point in time may be expressed as follows.

$$q_{3,1}=q_{2,1}\times\frac{0.50}{\left(1+\dfrac{0.075983+\mu_2}{2}\right)}=\frac{0.2349}{2.075983+\mu_2}$$

$$q_{3,2}=q_{2,1}\times\frac{0.50}{\left(1+\dfrac{0.075983+\mu_2}{2}\right)}+q_{2,2}\times\frac{0.50}{\left(1+\dfrac{0.055983+\mu_2}{2}\right)}$$
$$=\frac{0.2349}{2.075983+\mu_2}+\frac{0.4722}{2.055983+\mu_2}$$

$$q_{3,3}=q_{2,2}\times\frac{0.50}{\left(1+\dfrac{0.055983+\mu_2}{2}\right)}+q_{2,3}\times\frac{0.50}{\left(1+\dfrac{0.035983+\mu_2}{2}\right)}$$
$$=\frac{0.4722}{2.055983+\mu_2}+\frac{0.2372}{2.035983+\mu_2}$$

$$q_{3,4}=q_{2,3}\times\frac{0.50}{\left(1+\dfrac{0.035983+\mu_2}{2}\right)}=\frac{0.2372}{2.035983+\mu_2}$$

Obviously

$$q_{3,1}+q_{3,2}+q_{3,3}+q_{3,4}=\frac{1}{(1+0.5\times s_3)^3}=\frac{1}{(1.03025)^3}=0.9145$$

Thus

$$2\times\frac{0.2349}{2.075983+\mu_2}+2\times\frac{0.4722}{2.055983+\mu_2}+2\times\frac{0.2372}{2.035983+\mu_2}=0.9145$$

$$\Rightarrow \mu_2 = 0.009364 = 0.9364\%$$

Therefore

$$r_{2,1} = 7.5983 + 0.9364 = 8.5347\%$$

$$r_{2,2} = 5.5983 + 0.9364 = 6.5347\%$$

$$r_{2,3} = 3.5983 + 0.9364 = 4.5347\%$$

The prices of the pure securities are

$$q_{3,1} = 0.1126; q_{3,2} = 0.3413; q_{3,3} = 0.3446; q_{3,4} = 0.1160$$

Finally let us proceed to the penultimate point in time.

$$r_{3,1} = r_{2,1} + \mu_3 + 1 = 8.5347 + 1 + \mu_3 = 9.5347 + \mu_3$$

$$r_{3,2} = r_{2,1} + \mu_3 - 1 = 8.5347 - 1 + \mu_3 = 7.5347 + \mu_3$$

$$r_{3,3} = r_{2,2} + \mu_3 - 1 = 6.5347 - 1 + \mu_3 = 5.5347 + \mu_3$$

$$r_{3,4} = r_{2,3} + \mu_3 - 1 = 4.5347 - 1 + \mu_3 = 3.5347 + \mu_3$$

The prices of the pure securities at this point in time may be expressed as

$$q_{4,1} = q_{3,1} \times \frac{0.50}{\left(1 + \frac{0.095347 + \mu_3}{2}\right)} = \frac{0.1126}{2.095347 + \mu_3}$$

$$q_{4,2} = q_{3,1} \times \frac{0.50}{\left(1 + \frac{0.095347 + \mu_3}{2}\right)} + q_{3,2} \times \frac{0.50}{\left(1 + \frac{0.075347 + \mu_3}{2}\right)}$$

$$= \frac{0.1126}{2.095347 + \mu_3} + \frac{0.3413}{2.075347 + \mu_3}$$

$$q_{4,3} = q_{3,2} \times \frac{0.50}{\left(1 + \frac{0.075347 + \mu_3}{2}\right)} + q_{3,3} \times \frac{0.50}{\left(1 + \frac{0.055347 + \mu_3}{2}\right)}$$

$$= \frac{0.3413}{2.075347 + \mu_3} + \frac{0.3446}{2.055347 + \mu_3}$$

$$q_{4,4} = q_{3,3} \times \frac{0.50}{\left(1 + \frac{0.055347 + \mu_3}{2}\right)} + q_{3,4} \times \frac{0.50}{\left(1 + \frac{0.035347 + \mu_3}{2}\right)}$$

$$= \frac{0.3446}{2.055347 + \mu_3} + \frac{0.1160}{2.035347 + \mu_3}$$

$$q_{4,5} = q_{3,4} \times \frac{0.50}{\left(1 + \frac{0.035347 + \mu_3}{2}\right)} = \frac{0.1160}{2.035347 + \mu_3}$$

Obviously

$$q_{4,1} + q_{4,2} + q_{4,3} + q_{4,4} + q_{4,5} = \frac{1}{(1 + 0.5 \times s_4)^4} = \frac{1}{(1.0295)^4} = 0.8902$$

Thus

$$2 \times \frac{0.1126}{2.095347 + \mu_3} + 2 \times \frac{0.3413}{2.075347 + \mu_3} + 2 \times \frac{0.3446}{2.055347 + \mu_3} + 2 \times \frac{0.1160}{2.035347 + \mu_3} = 0.8902$$

$$\Rightarrow \mu_3 = -0.010481 = -1.0481\%$$

Therefore

$$r_{3,1} = 9.5347 - 1.0481 = 8.4866\%$$

$$r_{3,2} = 7.5347 - 1.0481 = 6.4866\%$$

$$r_{3,3} = 5.5347 - 1.0481 = 4.4866\%$$

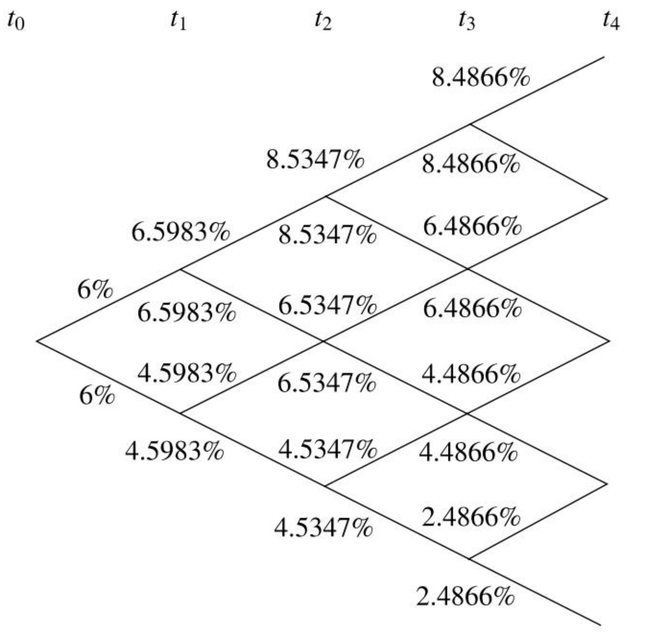
$$r_{3,4} = 3.5347 - 1.0481 = 2.4866\%$$

The prices of the pure securities are

$$q_{4,1} = 0.0540; q_{4,2} = 0.2193; q_{4,3} = 0.3338; q_{4,4} = 0.2258; q_{4,5} = 0.0573$$

The no-arbitrage interest rate tree derived by us may be depicted in Fig. 15.5.

Figure 15.5 **No Arbitrage Interest Rate Tree**



The state price tree may be depicted as shown in Fig. 15.6.

Figure 15.6 **State Price Tree**

t_0	t_1	t_2	t_3	t_4
				0.0540
			0.1126	
		0.2349		0.2193
	0.4854		0.3413	
1.0		0.4722		0.3338
	0.4854		0.3446	
		0.2372		0.2258
			0.1160	
				0.0573

15.19.2 Valuation of a Plain Vanilla Bond

Consider a four period bond with a face value of \$ 1,000, which pays a coupon of \$ 40 every period. Using the state price tree, we can compute the value of this bond as follows.

$$1{,}040 \times (0.3338 + 0.2193 + 0.2258 + 0.0540 + 0.0573)$$
$$+40 \times (0.3413 + 0.3446 + 0.1126 + 0.1160)$$
$$+40 \times (0.2349 + 0.4722 + 0.2372) + 40 \times (0.4854 + 0.4854)$$
$$= 1{,}040 \times 0.8902 + 40 \times 0.9145 + 40 \times 0.9443 + 40 \times 0.9708 = \$\,1{,}038.9920$$

15.19.3 Valuation of a Zero Coupon Bond

Consider a zero coupon bond that pays \$ 1,000 at time t_4. Its value at various nodes of the tree can be computed by working backwards.

At t_3, its value at node (3, 1) is given by

$$\frac{1{,}000}{\left(1 + \dfrac{0.084866}{2}\right)} = \$\,959.2943$$

At node (3, 2) the value is given by

$$\frac{1{,}000}{\left(1 + \dfrac{0.064866}{2}\right)} = \$\,968.5859$$

Similarly at node (3, 3), the value is

$$\frac{1,000}{\left(1+\frac{0.044866}{2}\right)} = \$\,978.0592$$

Finally at node (3, 4), the value is

$$\frac{1,000}{\left(1+\frac{0.024866}{2}\right)} = \$\,987.7197$$

Working backwards we can calculate the prices at time t_2 as follows. At node (2, 1) the value is

$$0.5 \times \frac{959.2943}{\left(1+\frac{0.085347}{2}\right)} + 0.5 \times \frac{968.5859}{\left(1+\frac{0.085347}{2}\right)} = 924.4889$$

Similarly at node (2, 2), the value is

$$0.5 \times \frac{968.5859}{\left(1+\frac{0.065347}{2}\right)} + 0.5 \times \frac{978.0592}{\left(1+\frac{0.065347}{2}\right)} = 942.5269$$

Finally at node (2, 3), the value is

$$0.5 \times \frac{978.0592}{\left(1+\frac{0.045347}{2}\right)} + 0.5 \times \frac{987.7197}{\left(1+\frac{0.045347}{2}\right)} = 961.0980$$

Using the same logic, the value at (1, 1) is

$$0.5 \times \frac{924.4889}{\left(1+\frac{0.065983}{2}\right)} + 0.5 \times \frac{942.5269}{\left(1+\frac{0.065983}{2}\right)} = 903.6937$$

while the value at (1, 2) is

$$0.5 \times \frac{942.5269}{\left(1+\frac{0.045983}{2}\right)} + 0.5 \times \frac{961.0980}{\left(1+\frac{0.045983}{2}\right)} = 930.4207$$

At the starting node, the value is therefore

$$0.5 \times \frac{903.6937}{\left(1+\frac{0.06}{2}\right)} + 0.5 \times \frac{930.4207}{\left(1+\frac{0.06}{2}\right)} = 890.3468$$

The bond prices may be depicted in a tree format as shown in Fig. 15.7.

Figure 15.7 **Evolution of the Price of a Zero Coupon Bond**

t_0	t_1	t_2	t_3
			959.2943
		924.4889	
	903.6937		968.5859
890.3468		942.5269	
	930.4207		978.0592
		961.0980	
			987.7197

15.19.4 Valuing a European Call

Consider a two period European call on the zero coupon bond with an exercise price of \$ 940. The payoff will be zero in node (2, 1); \$ 2.5269 at node (2, 2) and \$ 21.0980 at node (2, 3). The value of the option at t_0 is

$$0.4722 \times 2.5269 + 0.2372 \times 21.098 = \$\ 6.1976$$

15.19.5 Valuing an American Put

Valuing an American option is more complex because we have to check for the possibility of early exercise at every node. Consider a two period put on the zero coupon bond with an exercise price of \$ 950. The payoffs from the option at time t_2 are \$ 25.5111 at node (2, 1); \$ 7.4731 at node (2, 2) and 0 at node (2, 3).

The model value at (1, 1) is

$$.5 \times \frac{25.5111}{\left(1 + \frac{0.065983}{2}\right)} + .5 \times \frac{7.4731}{\left(1 + \frac{0.065983}{2}\right)} = 15.9654$$

The payoff from early exercise is \$ 46.3063. Thus the option will be exercised early.

The model value at node (1, 2) is

$$.5 \times \frac{7.4731}{\left(1 + \frac{0.045983}{2}\right)} + .5 \times \frac{0}{\left(1 + \frac{0.045983}{2}\right)} = 3.6526$$

The payoff from early exercise is \$ 19.5793. Thus the option will be exercised early.

The model value at (0, 1) is

$$.5 \times \frac{46.3063}{\left(1 + \frac{0.06}{2}\right)} + .5 \times \frac{19.5793}{\left(1 + \frac{0.06}{2}\right)} = 31.9833$$

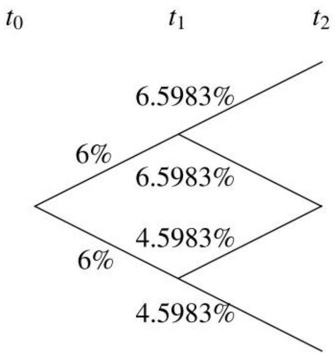
The payoff from early exercise is \$ 59.6532. So the option will be exercised right at the outset, that is at time t_0.

15.19.6 Caps, Floors, and Collars

Before we go on to analyze caps and floors, let us first define *Caplets* and *Floorlets*.

A caplet is a call option on an interest rate, while a floorlet is a put option on an interest rate. To evaluate these options we will first reproduce a segment of the no-arbitrage interest rate tree that we derived earlier.

Figure 15.8 **Segment of the No-arbitrage Interest Rate Tree**



Consider a one period caplet with an exercise price of 5%. Let the underlying principal be \$ 1,000,000. Each time period is assumed to be of six months duration, and we will take it as 0.5 years in order to avoid issues regarding day-count conventions.[11]

At time t_1 if the node (1,1) were to be attained, the call will be in the money. The payoff will be

$$1{,}000{,}000 \times (0.065983 - 0.05) \times 0.5 = \$\,7,991.50$$

In the case of interest rate options, the payoff will occur not when the option expires, but at a point in time when the next interest payment is due. In the above case, the rate as determined at t_1 is applicable for computing the interest due at t_2. Consequently the caplet will payoff at t_2.

If however we were to attain node (1, 2) at time t_1, the caplet will expire out of the money and the payoff will be zero.

Thus the value of the caplet at time t_0 may be computed as

$$0.5 \times \frac{7991.50}{(1.03)\left(1 + \dfrac{0.065983}{2}\right)} = \$\,3,755.47$$

[11] In practice the length of the time period will have to be calculated as per the day-count convention that is applicable in the market in question.

A floorlet is a put option on an interest rate. Let us consider a floorlet with the same exercise price and underlying principal as the caplet. The payoff will be zero in state (1, 1). However there will be a positive payoff at state (1, 2), that is given by

$$1{,}000{,}000 \times [0.05 - 0.045983] \times 0.5 = \$\ 2{,}008.50$$

This payoff will obviously occur at time t_2. Consequently the value of the floorlet at t_0 is

$$0.5 \times \frac{2{,}008.5}{(1.03)\left(1 + \dfrac{0.045983}{2}\right)} = \$\ 953.09$$

A cap is a portfolio of caplets, that can be acquired by a borrower who has availed of a floating rate loan, in order to protect himself against an increase in interest rates. Similarly, a floor is a portfolio of floorlets, that can be acquired by a lender, who has made a loan on a floating rate basis, to protect himself against a decline in interest rates.

Using the interest rate tree that we have derived (Fig. 15.5) let us price a three period cap. It consists of three caplets, expiring after one, two and three periods respectively. The value of the two period caplet may be calculated as follows.

The payoff if state (2, 1) is attained is

$$1{,}000{,}000 \times [0.085347 - 0.05] \times 0.5 = \$\ 17{,}673.50$$

The payoff at state (2, 2) is given by

$$1{,}000{,}000 \times [0.065347 - 0.05] \times 0.5 = \$\ 7{,}673.50$$

The value at time t_0 of the first payoff is

$$0.5 \times 0.5 \times \frac{1}{1.03} \times \frac{1}{1.032992} \times \frac{17{,}673.50}{\left(1 + \dfrac{0.085347}{2}\right)} = \$\ 3{,}982.7226$$

Since there are two paths which lead to state (2, 2), the value of the payoff if this state were to occur is given by

$$0.5 \times 0.5 \times \frac{1}{1.03} \times \frac{1}{1.032992} \times \frac{7{,}673.50}{\left(1 + \dfrac{0.065347}{2}\right)}$$

$$+ 0.5 \times 0.5 \times \frac{1}{1.03} \times \frac{1}{1.022992} \times \frac{7{,}673.50}{\left(1 + \dfrac{0.065347}{2}\right)}$$

$$= \$\ 1{,}745.9680 + \$\ 1{,}763.0353 = \$\ 3{,}509.0033$$

Thus the value of the caplet = 3,982.7226 + 3,509.0033 = \$ 7,491.7259

Similarly, the value of a three period caplet is given by

$$0.5 \times 0.5 \times 0.5 \times \frac{1}{1.03} \times \frac{1}{1.032992} \times \frac{1}{1.042674} \times \frac{1}{1.042433}$$
$$\times\ 1{,}000{,}000 \times [0.084866 - 0.05] \times 0.5$$
$$+0.5 \times 0.5 \times 0.5 \times \frac{1}{1.03} \times \frac{1}{1.032992} \times \frac{1}{1.042674} \times \frac{1}{1.032433}$$
$$\times 1{,}000{,}000 \times [0.064866 - 0.05] \times 0.5$$
$$+0.5 \times 0.5 \times 0.5 \times \frac{1}{1.03} \times \frac{1}{1.032992} \times \frac{1}{1.032674} \times \frac{1}{1.032433}$$
$$\times 1{,}000{,}000 \times [0.064866 - 0.05] \times 0.5$$
$$+0.5 \times 0.5 \times 0.5 \times \frac{1}{1.03} \times \frac{1}{1.022992} \times \frac{1}{1.032674} \times \frac{1}{1.032433}$$
$$\times 1{,}000{,}000 \times [0.064866 - 0.05] \times 0.5 = \$\,4{,}341.6315$$

Thus the value of a three period cap is

$$\$\,3{,}755.47 + \$\,7{,}491.7259 + \$\,4{,}341.6315 = \$\,15{,}588.8274$$

The value of a three period floor can be computed in a similar fashion.

An interest rate collar is a combination of a cap and a floor. It requires the investor to take a long position in a cap and a short position in a floor.

SUGGESTIONS FOR FURTHER READING

1. Buetow G.W and J. Sochacki *Term-Structure Models Using Binomial Trees.* AIMR Research Foundation, 2001.
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5. Figlewski S., W.L. Silber, and M.G. Subrahmanyam *Financial Options: From Theory to Practice.* Irwin, 1990.
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6. www.1728.com.

Concept Check

State whether the following statements are True or False.

1. The term structure of interest rates is also referred to as the zero coupon yield curve.
2. While constructing the yield curve care must be taken to ensure that all the bonds under consideration belong to the same risk class.
3. The yield curve will be the same as the term structure, if the term structure is flat.
4. If we have a series of bonds of different maturities, all of which are priced at par, then the deduced term structure will be flat.
5. The par bond yield curve is used by investment bankers to determine the required coupon rate for a new issue.
6. The unbiased expectations theory is incapable of explaining an inverted yield curve.
7. Lenders like to lend short term whereas borrowers like to borrow long term.
8. As per the liquidity preference theory, the liquidity premium must always be positive.
9. For a convex function, the expectation of the function is greater than or equal to the function of the expectation.
10. The one period spot rate will always be equal to the one period short rate.
11. A process that assumes that bond prices are lognormally distributed is suitable for valuing options on bonds.
12. In a one factor model, the stochastic process for the short rate is the only source of uncertainty.
13. The CIR model for the evolution of interest rates exhibits mean reversion.
14. In a model with mean reversion, the long-run value for the short rate is the same as the long term rate of interest.
15. While determining the evolution of rates as per the Ho and Lee model, the assumption that the tree is recombining is tantamount to assuming that the standard deviation is constant across time.
16. An Arrow–Debreu security will pay a dollar if a particular state of nature were to occur, but will not pay anything in any other state.

17. A caplet is a call option on an interest rate.
18. Caplets and floorlets will pay off at the time of expiration of the option, assuming of course that the option is in the money.
19. A collar is a combination of a long position in a cap and a long position in a floor.
20. A floor can be used by a lender to protect himself against a decline in interest rates.

QUESTIONS AND PROBLEMS

Question-I

What is bootstrapping? What are the practical difficulties in implementing this technique?

Question-II

Discuss the various theories of the term structure?

Question-III

The one year spot rate is 6% per annum and the two year spot rate is 8% per annum.

Consider a bond with a face value of \$ 1,000, with two years to maturity, and paying a coupon of 7% per annum. What is the YTM? If the bond were to offer a coupon of 8% per annum, what will be the YTM?

Why is the YTM different in the two cases?

Question-IV

Consider the following data. Assume that all the bonds have a face value of \$ 1,000, and pay interest on an annual basis.

Time to Maturity	Price	Coupon
1 year	\$ 950	6%
2 years	\$ 1,000	8%
3 years	\$ 1,050	10%
4 years	\$ 1,100	12%

Compute the following: $s_1; s_2; s_3;$ and s_4.

Compute the following: $f_1^1; f_2^1; f_1^2; f_1^3; f_2^2;$ and f_3^1.

Question-V

Consider the following data. Assume that all bonds have a face value of \$ 1,000 and pay interest on an annual basis.

Time to Maturity	Spot Rate
1 year	6%
2 years	7%
3 years	8%
4 years	10%

What should be the coupons of 2, 3, and 4 year par bonds?

Question-VI

What is the difference between equilibrium models of the term structure and no-arbitrage models?

Question-VII

The one period spot rate is 8% per annum while the two period rate is 7.85% per annum. Each time period represents six months. Caliberate the Ho and Lee model using this information, assuming that $\sigma = 0.50$.

Compute the following state prices: $q_{1,1}; q_{1,2}; q_{2,1}; q_{2,2};$ and $q_{2,3}$.

Question-VIII

Consider a two period plain vanilla bond with a face value of $ 1,000. Assume that it pays a coupon of $ 35 every period.

What should be its value if the applicable state prices are as derived in Question-VII?

Question-IX

Consider a two period zero coupon bond. What will be its value at the following nodes: (0, 1); (1, 1) and (1, 2). Use the interest rate tree derived in Question-VII.

Question-X

What are caps, floors and collars?

16

Fundamentals of Swaps

16.1 Introduction

A swap is an exchange between two parties, of two payment streams that are different from each other. In the case of an interest rate swap, the two payments are denominated in the same currency, but are computed using different interest rates. On the other hand, in the case of a currency swap, the two counterparties are required to exchange streams of payments denominated in two different currencies. As we have explained earlier, in the case of an interest rate swap, both cash flow streams may be based on floating rates of interest, or else one stream may be based on a fixed rate while the other can be based on a floating rate. A fixed-rate to fixed-rate interest rate swap is infeasible for it would tantamount to arbitrage for the party who is making payments based on a lower rate of interest. That is, if such a swap were to exist, the party which is employing a lower rate to compute its payments, will always be receiving more than what it would have to pay to the counterparty. However, in the case of a currency swap, since the cash flow streams are denominated in two different currencies, all the three possibilities exist. That is, the swap may be fixed-fixed, fixed-floating or floating-floating.

16.2 Interest Rate Swaps

16.2.1 Contract Terms

The following terms must be explicitly stated while designing an interest rate swap contract.

- The identities of the two counterparties.
- The maturity of the swap. This is the date on which the last exchange of cash flows will take place between the two parties.
- The interest rate used by the first party to calculate its payments. It may be fixed or floating.
- The interest rate used by the second party to calculate its payments. It must be floating if the first party is making payments based on a fixed rate of interest.

- The day-count convention for the computation of interest.
- The frequency of payment.
- The notional principal. The principal is said to be notional because there is no need for the two parties to exchange the specified amount, considering the fact that all payments are denominated in the same currency. Nevertheless a principal amount is required to be specified in order to facilitate the computation of interest.

Illustration Consider a three year fixed-floating swap between HSBC and Barclays Bank. HSBC will be making payments based on an annual rate of 8%. Barclays will compute its liability based on the six month LIBOR. We will assume that every month consists of 30 days and that the year as a whole consists of 360 days. The exchange of payments will take place on a semi-annual basis. The notional principal amount is $ 5,000,000.

Assume that the six month LIBOR as observed at semi-annual intervals over the next three years is as follows.

Table 16.1 Observed Values of LIBOR

Time	LIBOR
Start	7.00%
After 6M	7.75%
After 12M	8.50%
After 18M	8.75%
After 24M	8.25%
After 30M	8.75%
After 36M	7.25%

Using this data, let us compute the payments to be made by the two parties every six months.

Table 16.2 Cash Flows in an Interest Rate Swap

Time	Payment by HSBC	LIBOR	Payment by Barclays	Net Payment
Start	–	7.00%	–	–
6M	$ 200,000	7.75%	$ 175,000	$ 25,000
12M	$ 200,000	8.50%	$ 193,750	$ 6,250
18M	$ 200,000	8.75%	$ 212,500	$ (12,500)
24M	$ 200,000	8.25%	$ 218,750	$ (18,750)
30M	$ 200,000	8.75%	$ 206,250	$ (6,250)
36 M	$ 200,000	7.25%	$ 218,750	$ (18,750)
Total	$ 1,200,000		$ 1,225,000	$ (25,000)

Let us analyze the cash flows to be exchanged after six months. HSBC has to pay \$ 200,000. This may be calculated as follows:

$$0.5 \times 0.08 \times 5{,}000{,}000 = \$\ 200{,}000$$

Barclays has to pay \$ 175,000. This may be calculated as follows.

$$0.5 \times 0.07 \times 5{,}000{,}000 = \$\ 175{,}000$$

Notice the following. The fixed rate payment is constant because of two reasons. First the interest rate used by HSBC to compute its obligation is constant. Second we have assumed a 30/360 day count convention, as a consequence of which every semi-annual period is exactly one-half of a year. If we had assumed an ACT/360 convention, the payments to be made by HSBC would have varied across periods, despite the fact that it is paying a fixed rate.

The payments that are made by Barclays are based on the LIBOR that is observed at the start of the period concerned. The interest itself is however payable at the end of the period. For instance, the payment made by Barclays at the end of the first period is based on a interest rate of 7.00%, which corresponds to the LIBOR at the start of the period. This system of computation is known as determined in advance and paid in arrears.

Also, since the payments to be made by the two parties are denominated in the same currency, they can be netted, and the party which owes the larger amount can simply pay the difference to the other. For instance, after the first six months, HSBC owes \$ 200,000 and in return is owed \$ 175,000. Consequently all that it has to do is to make a payment of \$ 25,000 to Barclays. This is shown in the last column of Table 16.2. Positive entries connote that HSBC has to pay the counterparty, whereas negative amounts indicate that the counterparty has to pay HSBC. The advantage of netting is that it reduces delivery risk. For instance, if Barclays were to pay the gross amount owed by it to HSBC, it is exposed to the risk that the payment of \$ 200,000 owed by HSBC may never arrive. On the contrary, because of netting the risk for Barclays in this case is that the sum of \$ 25,000 will not be received if HSBC were to renege.

16.2.2 Risk

An interest swap exposes both the parties to interest rate risk. In the case of HSBC, in the above illustration, the risk is that LIBOR may decline during the life of the swap. If so, the payments due to it may stand reduced, while the payments to be made by it are invariant to interest rate changes. On the other hand, the risk for Barclays is that LIBOR may increase over the life of the swap. If so, the quantum of payments to be made by it will increase. Obviously, the payments due to be received by it are invariant to rate changes.

16.2.3 Motivation for the Swap

Swaps may be used for speculation, hedging, or for arbitrage.

Speculation Assume that HSBC expects interest rates to rise, whereas Barclays expects that rates will fall. In such a situation the swap that we have just illustrated can be used by both parties to speculate. If HSBC's view point were to be correct, and rates were indeed to rise, then the payments received by it, which are calculated on a floating rate basis, would be higher than the payments made by it, which are computed on a fixed rate basis. On the contrary, if Barclays were to read the movement of rates correctly, it would profit from the swap, for the payments made by it would be smaller in magnitude than what it receives from HSBC.

Hedging Consider a situation where HSBC has taken a loan from Alpha Bank on a floating rate basis. Now after the loan has been made, it is worried that interest rates may rise. If so it can hedge by entering into a swap where it pays a fixed rate and receives payments based on a floating rate. If it is correct and rates do rise, then although it will have to pay more to the party from which it has borrowed, the extra funding cost will be mitigated by the profit from the swap.

On the other hand, Barclays too may use the swap to hedge. For instance, Barclays may have made a floating rate loan to Google. If interest rates were to fall, the payments from the borrower would stand reduced. However, in such circumstances, the bank would make a profit from the swap.

16.3 Terminology

- **Coupon Swap**: The type of interest rate swap that we illustrated, where one party makes payments based on a fixed rate while the other party makes payments based on a floating rate, is referred to as a coupon swap.
- **Basis Swap**: An interest rate swap where both the parties make payments based on floating rates is known as a basis swap. For instance, one party may commit to making payments on the basis of LIBOR, while the other may agree to make payments based on the T-bill rate.
- **Payer and Receiver**: In a coupon swap, the party which agrees to make payments based on a fixed rate, is referred to as the 'payer'. The counterparty, which is committed to making payments on a floating rate basis, is referred to as the 'receiver'.

  Quite obviously these terms cannot be used in the case of basis swaps, since both the cash flow streams are determined based on floating rates. Consequently, in order to be explicit and avoid ambiguities, it is a good practice to describe for each of the two parties, the rate on the basis of which it is scheduled to make payments, as also the rate on the basis of which it is scheduled to receive payments.

  In the case of coupon swaps, some markets refer to the fixed rate payer as the *'buyer'* and the fixed rate receiver as the *'receiver'*.
- **Swap Rate**: The fixed rate of interest that has been agreed upon in a coupon swap is referred to as the swap rate. If the swap rate is quoted as a percentage,

it is referred to as an *all-in price*. However in certain interbank markets the fixed rate is not quoted as a percentage. Instead, what is quoted is the difference, in basis points, between the agreed upon fixed rate and a benchmark interest rate. The benchmark that is chosen to compute this differential, is usually the government security whose remaining term to maturity is closest to the life of the swap in question. For instance, in the case of the HSBC-Barclays swap that we studied, the fixed rate was 8% per annum. If the swap rate were to be quoted as an all-in price, it would obviously be reported as such. However, in the second convention the rate would be quoted as follows. Assume that a three year T-note has a yield to maturity of 7.55%. The swap price will be quoted as 8% minus 7.55% or as 45 basis points.

16.4 Futures and Options versus Swaps

Swaps are exclusively traded OTC, unlike futures contracts that are traded on an organized exchange. Options of course may be traded OTC or on an exchange. Exchange traded contracts are accompanied by the intervention of a clearinghouse and the related requirement of margins. This ensures that default risk is considerably minimized. However, in the case of OTC instruments like swaps, default risk is definitely an issue. In this respect they are similar to forward contracts.

In practice, futures and exchange listed options contracts are usually available only for short to medium periods of time. However, swaps by virtue of being OTC instruments can be negotiated for any tenor and for any currency (in the case of currency swaps), provided of course that a suitable counterparty can be found.

16.5 Illustrative Swap Rates

The following quotes (Table 16.3) were available for Euro denominated interest rate swaps on 22 December, 2008. The corresponding floating rate is the six month Euribor, except in the case of the one year swap in which case the quote is against the three month Euribor.

16.5.1 Interpretation

Why do we have two swap rates for each maturity? It must be understood that these rates are quoted by professional swap dealers and represent a *'paying'* rate and a *'receiving'* rate. The rationale is that the dealer should make a net profit if he were to undertake a fixed-floating swap with one party and offset it by doing a floating-fixed swap with another party. Thus the lower rate or the 'bid' represents the rate that the dealer is willing to pay if he were to do a swap wherein he pays fixed and receives floating. Obviously, the higher rate or the 'ask' is the rate that he would like to receive in a swap where he receives fixed and pays floating.

Table 16.3 **Quotes Rates for Euro Denominated**

Interest Rate Swaps		
Maturity	**Bid**	**Ask**
1 year	2.74	2.79
2 years	2.74	2.79
3 years	2.89	2.95
5 years	3.19	3.24
10 years	3.69	3.74
20 years	3.82	3.87
25 years	3.64	3.69

Source: www.swap-rates.com

When the swap market was at its nascent stage the normal practice was for investment banks to play the role of an intermediary. These banks would arrange the transaction by bringing together two counterparties, and in return would be paid an arrangement fee. Over a period of time the role of an intermediary evolved from that of an agent who facilitated a swap to that of a principal. One of the main reasons for this was that parties to swaps did not want their identities to be revealed to the counterparty. Second, as we have seen swaps expose both counterparties to default risk. For this reason, parties to a swap were more comfortable dealing with a bank, whose creditworthiness was easier to appraise.

Initially, the banks would do what are called *'reversals'*. That is, they would agree to be the principal to a swap only if there was an equal and opposite swap that was immediately available. For instance assume that Google approaches Citibank for a swap where it pays fixed and receives floating. Citibank would agree only if it could simultaneously find another party with whom it could enter into an offsetting swap wherein the bank pays fixed and receives floating. A dealer who carries two offsetting swaps in his books, is said to be running a *'matched book'*.

It must be understood that a dealer who maintains a matched book is exposed to default risk from both the parties with whom it has entered into swaps. These days market makers have become less fastidious about maintaining a matched book. That is, they are usually willing to accept a temporary exposure to an unmatched or naked position.

16.6 Valuing an Interest Rate Swap

In our illustration of a swap between HSBC and Barclays Bank, we arbitrarily assumed that the swap rate was 8%. We will now demonstrate as to how this rate will be determined in practice.

Assume that instead of entering into a swap, HSBC had issued a three year fixed rate note with a principal of $ 5 MM on which it has to make semi-annual interest payments at the rate of 8% per annum. This money could have been used to acquire a three year floating rate note with the same principal, and which pays coupons semi-annually based on the LIBOR observed at the start of the six-monthly period.

If we look at the cash flows of these two transactions, the result is equivalent to that on a three year fixed-floating swap. At the outset there will be an inflow of \$ 5 MM when the fixed rate note is issued. But this amount will be just adequate to purchase the floating rate loan. Thus the net cash flow is zero. Similarly, at the point of termination, that is after three years, HSBC will receive \$ 5MM when the floating rate note matures. However, this will be just adequate to retire the fixed rate note. The net result is that there is no exchange of principal, either at the outset or at the end, which is consistent with what we have seen for interest rate swaps.

Every six months, the floating rate note will pay a coupon based on the LIBOR at the start of the period. HSBC will receive this amount and would be required to pay interest at the rate of 8% per annum to service the fixed rate note that it has issued. Consequently the cash flows every six months will be identical to that of the swap.

Thus a long position in a floating rate note coupled with a short position in a fixed rate note is equivalent to a swap which requires fixed rate payments in return for payments based on a floating rate. We will demonstrate as to how the fixed rate of a coupon swap can be determined.

Assume that the current term structure is as shown in Table 16.4.

Table 16.4 **Observed Term Structure**

Time to Maturity	Rate
6M	8.25%
12M	8.40%
18M	8.50%
24M	8.20 %
30M	7.80%
36M	7.50%

Since today by assumption is the start of the next six monthly period, the price of the three year floating rate note will be equal to its face value of \$ 5 MM. For, on a coupon reset date, the price of a floating rate bond will revert back to its face value. The question is, what should be the coupon rate for the fixed rate note, so that it too has a current price of \$ 5 MM.

Let us first determine the discount factors corresponding to the observed LIBOR rates. The discount factor for a given maturity is the present value of a dollar to be received at the end of the stated period. The convention in the LIBOR market is that if the number of days for which the rate is quoted is N, then the corresponding discount factor is given by $\dfrac{1}{\left(1+i\times\dfrac{N}{360}\right)}$. For instance, the discount factor for an investment of 18 months will be $\dfrac{1}{\left(1+i\times\dfrac{540}{360}\right)}$ where i is obviously the quoted 18 M LIBOR. Thus the vector of discount factors for our example is

Table 16.5 Discount Factors

Time to Maturity	Discount Factor
6 M	0.9604
12 M	0.9225
18 M	0.8869
24 M	0.8591
30 M	0.8368
36 M	0.8163

If we denote the semi-annual coupon by $\frac{C}{2}$, it must be that

$$\frac{C}{2}\times 0.9604+\frac{C}{2}\times 0.9225+\frac{C}{2}\times 0.8869$$
$$+\frac{C}{2}\times 0.8591+\frac{C}{2}\times 0.8368+\left[\frac{C}{2}+5{,}000{,}000\right]\times 0.8163$$
$$=5{,}000{,}000$$
$$\Rightarrow 5.2282\times\frac{C}{2}=918{,}500$$
$$\Rightarrow \frac{C}{2}=173{,}892.465\Rightarrow C=347{,}784.93$$

Thus the swap rate is

$$\frac{347{,}784.93}{5{,}000{,}000}\times 100=6.96\%$$

16.6.1 Valuing a Swap at an Intermediate Stage

Assume that three months have elapsed since the above swap was initiated. Consider the following term structure (Table 16.6).

The value of the fixed rate note may be computed as follows.

$$\frac{C}{2}\times[0.9802+0.9421+0.9065+0.8758+0.8491+0.8271]$$

Table 16.6 **Observed Term Structure after Three Months**

Time to Maturity	Rate	Discount Factor
3 M	8.10%	0.9802
9 M	8.20%	0.9421
15 M	8.25%	0.9065
21 M	8.10 %	0.8758
27 M	7.90%	0.8491
33 M	7.60%	0.8271

$$+ 5{,}000{,}000 \times 0.8271 =$$

$$173{,}892.465 \times 5.3808 + 5{,}000{,}000 \times 0.8271 = 935{,}680.58$$

$$+ 4{,}135{,}500 = \$\,5{,}071{,}180.58$$

The value of the floating rate bond may be computed as follows. Three months hence it will pay a coupon based on the original six month rate which is 8.25%. The quantum of this coupon is

$$0.5 \times 0.0825 \times 5{,}000{,}000 = \$\,206{,}250$$

Once this coupon is paid, the value of the bond will revert back to its face value of $ 5,000,000. Consequently, its value today is

$$5{,}206{,}250 \times 0.9802 = \$\,5{,}103{,}166.25$$

From the standpoint of the fixed rate payer, the swap is tantamount to a long position in a floating rate note that is combined with a short position in a fixed rate note. Thus the value of the swap is

$$5{,}103{,}166.25 - 5{,}071{,}180.58 = \$\,31{,}985.67$$

For the counterparty the value will obviously be −$ 31,985.67.

16.7 Terminating a Swap

Let us suppose that after three months have elapsed, the fixed rate payer in the above swap decides that it no longer wants to be a party to the swap. It could get out of the current situation in a variety of ways. One way is by way of a reversal. That is, it can enter into a swap with 33 months to maturity, wherein it is required to pay floating and receive fixed. The index for the floating rate payments must obviously be the same. In this case, there are two swaps which are in existence. So the party is exposed to credit risk in both swaps.

The second way to exit the swap would be by selling it to a new party. In this case, HSBC would receive $ 31,985.67 from the acquirer since the swap has a positive value. If the value were to be negative, the acquirer would have to be paid to take over the swap. In this case, the original counterparty to the swap, that is Barclays, would have to agree to the deal.

Finally, Barclays itself may buy out of the swap by paying HSBC the current value. This is known as *buy-back* or *close-out*.

16.8 Equivalence with FRAs

We have already established that a position in an interest rate swap is equivalent to a combination of a position in a fixed rate note and a position in a floating rate note. We will now show that a swap is equivalent to a series of *Forward Rate Agreements* also known as *FRAs*.

16.8.1 FRA

A forward rate agreement is nothing but a forward contract on an interest rate. Consider the case of an investor who has taken a long position in a FRA with two periods to maturity and a notional principal of \$ 5,000,000. As per the agreement the fixed rate of interest is 8%, while the floating rate is the 6M LIBOR. Assume that each period exactly corresponds to six months. If the interest rate at the end of two periods is 9%, the investor will receive

$$5{,}000{,}000 \times (0.09 - 0.08) \times 0.5 = \$\,25{,}000$$

On the contrary, if the rate after two periods were to be 7%, then the investor will experience an outflow of

$$5{,}000{,}000 \times (0.07 - 0.08) \times 0.5 = -\$\,25{,}000$$

In practice, these cash flows have to be discounted at the time of pay off. This is because, while in normal circumstances the payoff based on LIBOR will be determined in advance and paid in arrears, in the case of the FRA the payment occurs as soon as the LIBOR for the period is determined. Hence the above payoffs have to be discounted using the relevant rate. Thus if the 6M LIBOR were to be 9%, the payoff from the FRA will be

$$\frac{25{,}000}{\left(1 + \frac{0.09}{2}\right)} = \$\,23{,}923.44$$

In the second case, where the LIBOR happens to be 7%, the outflow will be

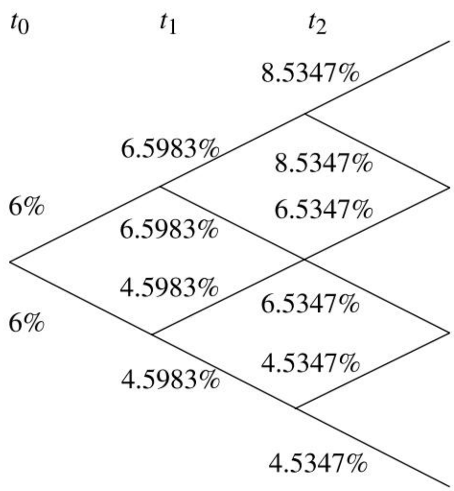
$$\frac{25{,}000}{\left(1 + \frac{0.07}{2}\right)} = \$\,24{,}154.59$$

16.8.2 Determining the Fixed Rate

In the above illustration we arbitrarily assumed a value of 8% for the fixed rate of interest. In practice it will depend on how the term structure is expected to evolve.

Let us reproduce a segment of the no-arbitrage interest rate tree that we derived in the previous chapter using the Ho and Lee model.

Figure 16.1 **No Arbitrage Interest Rate Tree**



The payoff at node (2, 1) is

$$\frac{5{,}000{,}000 \times 0.5 \times (0.085347 - X)}{\left(1 + \frac{0.085347}{2}\right)}$$

The payoff at node (2, 2) is

$$\frac{5{,}000{,}000 \times 0.5 \times (0.065347 - X)}{\left(1 + \frac{0.065347}{2}\right)}$$

And at node (2, 3) it is

$$\frac{5{,}000{,}000 \times 0.5 \times (0.045347 - X)}{\left(1 + \frac{0.045347}{2}\right)}$$

The unknown fixed rate should be set in such a way that the risk-neutral expected payoff is zero. That is

$$0.5 \times 0.5 \times \frac{5{,}000{,}000 \times 0.5 \times (0.085347 - X)}{\left(1 + \frac{0.085347}{2}\right)} + 2 \times 0.5$$

$$\times 0.5 \times \frac{5{,}000{,}000 \times 0.5 \times (0.065347 - X)}{\left(1 + \frac{0.065347}{2}\right)}$$

$$+0.5 \times 0.5 \times \frac{5{,}000{,}000 \times 0.5 \times (0.045347 - X)}{\left(1 + \frac{0.045347}{2}\right)} = 0$$

$$\Rightarrow X = 0.065251 \equiv 6.5251\%$$

The equivalence of swaps and FRAs may be demonstrated as follows. Consider the three year swap that HSBC did with Barclays. The very first payment due from Barclays is known at the very outset because the applicable LIBOR is determined at the start of the period. Hence the first transaction may be viewed as a spot transaction in which HSBC agrees to give Barclays $ 25,000 six months hence. The remaining five transactions are FRAs in which HSBC agrees to make payments based on a rate of 8% per annum, and receive payments based on the LIBOR that prevails at the start of the corresponding period.

16.9 Currency Swaps

A currency swap may be defined as a contract under which two parties commit themselves to exchange over a stated time period two streams of payments in two different currencies, each calculated using a different interest rate. They also agree to exchange, at the end of the stated time period, the corresponding principal amounts, at an exchange rate that is agreed upon at the outset.[1]

16.9.1 Illustration

HSBC and JP Morgan Chase have entered into a contract wherein over a period of two years, the first party, namely HSBC, will pay interest in US dollars at a rate of 3% on a principal amount of $ 5 MM, while the counterparty, namely JP Morgan Chase, will over the same period of time pay interest in British pounds at a rate of 4% on a principal amount of £3.125 MM. The cash flows will be exchanged every six months over the period of two years. The interest rates to be used by the two parties will be fixed at the outset. As we explained earlier, since the payments are denominated in two different currencies, both parties may use fixed rates; or else one party may use a fixed rate while the other may use a variable rate such as LIBOR; or else both may opt to use floating rates.

In this case the two banks also commit themselves to exchange, at the end of two years, the principal amounts of $ 5 MM and £3.125 MM on which the respective interest payments are being calculated. That is, HSBC will pay $ 5 MM to JP Morgan Chase and in return will receive a payment of £3.125 MM. The implicit exchange rate of 1.6000 USD/GBP is agreed upon right at the outset.

By definition, a currency swap requires an exchange of principal at the time of maturity. However, it is possible to structure the swap in such a way that there is an exchange of principal at inception as well. Such swaps are also referred to as *cash swaps*.

[1] See Geroulanos (1998).

Let us assume that the swap between HSBC and JP Morgan Chase is a cash swap. Such a swap would normally entail the following transactions.

- HSBC would borrow £3.125 MM in London.
- JP Morgan Chase would borrow $ 5 MM in New York.
- The two parties would then swap the principal amounts. The equivalent principal amount in the other currency is normally computed using the prevailing spot rate in the foreign exchange market. However this need not always be the case.
- At the end of two years, the principal amounts will be swapped back. The re-exchange is usually at the exchange rate that was used at inception. Thus HSBC would pay $ 5 MM to JP Morgan Chase in return for a payment in sterling of £3.125. Such a swap where the same amount of principal is exchanged is known as a *par swap*.[2]
- HSBC would use the pounds received by it to pay off its original loan.
- JP Morgan would use the dollars received by it to pay off its loan in New York.

Notice that during the life of the loan, HSBC would be receiving interest in pounds from JP Morgan, which will be used to service the loan that it has taken in London in British pounds. During the same period, JP Morgan Chase would be receiving interest in dollars from HSBC, which will be used to service the dollar denominated loan that it has taken in New York. Thus the swap transaction enables each party to service the debt of the counterparty.

16.9.2 Cross-Currency Swaps

The term *currency swap* strictly speaking should only be used for transactions between currencies that are on a fixed-rate to fixed-rate basis. Fixed-floating and floating-floating swaps involving two currencies are referred to technically as *cross-currency* swaps. A cross-currency coupon swap is on a fixed-floating basis, whereas a cross-currency basis swap is on a floating-floating basis.

16.10 Inherent Risks

A currency swap, as we would expect, exposes both the parties to currency risk. Let us take the case where HSBC borrows and makes a payment in pounds to JP Morgan Chase in return for a payment in dollars. The exchange rate used was the prevailing spot rate of 1.600 USD/GBP.

At maturity HSBC would pay back $ 5 MM USD to JP Morgan Chase and would receive £3.125 MM from it. Now assume that in the intervening three years, the dollar has appreciated to 1.5000 USD/GBP. JP Morgan Chase would be benefited by the fact that the terminal exchange of principal is based on the

[2] See Geroulanos (1998).

original exchange rate. For, if £3.125 MM were to be converted at the prevailing rate, it would receive only $ 4.6875 MM USD. Thus the counterparty which is slated to receive the currency that has appreciated during the life of the swap, stands to gain, while the other party stands to lose. In our illustration, JP Morgan Chase avoids a loss of $ 0.3125 MM USD while HSBC foregoes an opportunity to save an equal amount.

16.11 Valuation

Let us consider the swap between HSBC and JP Morgan Chase. Assume that it is on a fixed-rate to fixed rate basis. The swap can be analyzed as follows. It is as if HSBC has issued a bond with a face value of $ 5 MM, converted the amount to £3.125 MM at the prevailing spot rate and invested it in a bond denominated in pounds. From the perspective of the counterparty the transaction may be viewed as follows. JP Morgan Chase has issued a bond with a face value of £3.125 MM, converted it to dollars at the prevailing spot rate and invested it in a bond denominated in US dollars. Thus a currency swap between two parties is equivalent to a transaction in which each party issues a bond in one currency to the other, and uses the proceeds to acquire a bond issued by the counterparty. Consequently the fixed rate that is applicable in either currency is nothing but the coupon rate that is associated with a par bond in that currency, as we have seen in the case of a fixed-floating interest rate swaps. In this case, however we need to know the term structure for both currencies.

Consider the following data given in Table 16.7.

Table 16.7 LIBOR Rates for USD and GBP

Maturity	USD LIBOR	Discount Factor	GBP LIBOR	Discount Factor
6 M	2.50%	0.9877	3.30%	0.9838
12 M	2.75%	0.9732	3.50%	0.9662
18 M	2.60%	0.9625	3.40%	0.9515
24 M	2.85%	0.9461	3.75%	0.9302

The value of the fixed rate for USD may be determined as follows. Consider a bond with a face value of $ 1. Thus the coupon for a par bond denominated in USD is given by

$$\frac{C}{2} \times .9877 + \frac{C}{2} \times 0.9732 + \frac{C}{2} \times 0.9625 + \left(1 + \frac{C}{2}\right) \times 0.9461 = 1$$

$$\Rightarrow \frac{C}{2} = 0.01393 \Rightarrow C = 2.7856\%$$

Similarly

$$\frac{C}{2} \times .9838 + \frac{C}{2} \times 0.9662 + \frac{C}{2} \times 0.9515 + \left(1 + \frac{C}{2}\right) \times 0.9302 = 1$$

$$\Rightarrow \frac{C}{2} = 0.01822 \Rightarrow C = 3.6433\%$$

for the British pound denominated par bond.

If the swap had been such that the rate in dollars was fixed while that in pounds was floating, the applicable rate would be 2.7856% for the dollar denominated payments and LIBOR for the pound denominated payments. On the other hand, if the rate were to be fixed for the payments in pounds and floating for the dollar denominated payments, the applicable rate would be LIBOR for the payment in dollars and 3.6433% for the payments in pounds. Finally if payments in both currencies were to be on a floating rate basis, it would be LIBOR for LIBOR.

16.12 Swaptions

A swaption is nothing but an option on a swap. Such an option requires the buyer to pay an up front premium, for the right to enter into a coupon swap. There are two possibilities. The holder of the option may either enter in into a coupon swap as a fixed rate payer or as a fixed rate receiver. Consequently there are two types of swaptions. In the case of a *payer swaption* the option holder can exercise in order to enter into the swap as a fixed rate payer. On the other hand, a *receiver swaption* gives the holder the right to enter into a swap as a fixed rate receiver.

The exercise price specified in the swaption is an interest rate. The underlying asset is a swap with a specified term to maturity. A payer swaption will be exercised only if the prevailing rate for a swap with the specified maturity is higher than the exercise price of the swaption. Quite obviously, a receiver swaption will be exercised only if the prevailing swap rate is lower than the exercise price. Swaptions may be European or American in nature.

Suggestions for Further Reading

1. Chance, D.M. *An Introduction to Derivatives and Risk Management*. South Western, 2004.
2. Geroulanos P. *An Introduction to Swaps*. The Securities Institute, 1998.
3. Hull, J. *Options, Futures, and Other Derivatives*. Prentice-Hall, 2006.

Concept Check

1. A fixed-fixed interest rate swap will be tantamount to arbitrage.
2. The principal involved need not be exchanged in the case of interest rate swaps.
3. Netting for payments is feasible in the case of both interest rate as well as currency swaps.
4. Netting eliminates delivery risk.
5. An interest rate swap exposes only the floating rate payer to interest rate risk.
6. Swaps can be used for hedging but not for speculation.
7. The party who pays at a fixed rate in an interest rate swap is referred to as the 'buyer'.
8. Swaps can be traded on an organized exchange.
9. Swaps can be negotiated in principle for any tenor and any currency.
10. When a dealer quotes two swap rates, the bid will be higher than the ask.
11. The bid quoted by a dealer represents the fixed rate that he is prepared to pay in a coupon swap.
12. A dealer who carries two offsetting swaps in his books is said to be running a 'Matched Book'.
13. A currency swap must be accompanied by an exchange of principal at inception.
14. Currency swaps that entail an exchange of principal at inception are referred to as par swaps.
15. A cross-currency swap must be a coupon swap.
16. A payer swaption gives the holder the right to enter into a coupon swap as a fixed rate payer.
17. A payer swaption will be exercised only if the prevailing swap rate is lower than the exercise price of the swaption.
18. Swaptions must be European by design.
19. At the time of inception, the value of a swap must be zero to both parties.
20. In a currency swap the exchange of principal must be at the prevailing spot foreign exchange rate.

Questions and Problems

Question-I

A fixed-floating interest rate swap may be used for hedging, speculation, or arbitrage. Illustrate with suitable examples.

Question-II

Compare and contrast swaps with exchange traded products such as futures and options.

Question-III

What are the different ways in which a swap can be terminated? Discuss.

Question-IV

What are swaptions? Illustrate the payoffs at expiration of the various types of swaptions with suitable examples.

Question-V

What are the risks inherent in a currency swap for the two couterparties? Discuss from the perspective of currency swaps as well as cross-currency swaps.

Question-VI

What is a forward rate agreement. Consider a FRA with two periods to maturity and a notional principal of 50MM USD. The fixed rate is 6% per annum, while the floating rate is 6M LIBOR. Assume that each period is exactly equal to 0.5 years.

If the LIBOR after two periods is 7.5% per annum what will be the payoff for an investor with a long position? What if the LIBOR after two periods is 5% per annum.

Question-VII

A fixed-floating interest rate swap is equivalent to simultaneous positions in fixed and floating rate notes. Discuss.

Question-VIII

Bank Afrique and Bank Europeana have entered into an interest rate swap. Bank Afrique will make payments every six months based on a fixed rate of 6% per annum. Bank Europeana, on the other hand, will make payments on the basis of the 6M Euribor prevailing at the start of the period. Payments will be made on 15 May and 15 November every year based on an ACT/360 day-count convention. The swap has a maturity of two years and the notional principal is 25MM euros. The deal was negotiated on 15 May 2008 and the first payments are scheduled to be exchanged on 15 November 2008. The Euribor rates as observed at semi-annual intervals are assumed to be as follows.

Date	Euribor
15 May 2008	5.60%
15 November 2008	5.40%
15 May 2009	5.00%
15 November 2009	5.80%
15 May 2010	6.20%

Show the payments to be made by each bank, as well as the net payments.

Question-IX

Consider the following term structure for LIBOR.

Term to Maturity	Interest Rate
6 M	5.80%
12 M	5.40%
18 M	5.20%
24 M	5.50%

Consider a swap with a notional principal of 100MM USD. Every six months Bank Alpha will make payments based on a fixed rate of *'k%'*, while Bank Beta will base its payments on LIBOR. Determine the value of *k*, that will ensure that the swap has nil value to both parties at inception.

Question-X

Assume that the term structure after three months is as follows.

Term to Maturity	Interest Rate
3 M	5.40%
9 M	5.20%
15 M	5.00%
21 M	5.20%

What will be the value of the swap for Bank Alpha, if we assume that it has entered into the swap as described in Question-IX.

Quiz–2

Circle or tick the choice that constitutes the best response to the statement

1. Asian options:
   a. Can be exercised only at expiration
   b. May be exercised prior to expiration
   c. Must be exercised prior to expiration
   d. Depends on whether it is an average price or an average strike option
2. The exercise price is pre-defined for:
   a. Average price Asian options
   b. European options
   c. American options
   d. All of the above
3. If the asset price is lognormally distributed then the average price is also lognormal if:
   a. The average is arithmetic
   b. The average is geometric
   c. Both (a) and (b)
   d. None of the above
4. The geometric average is always:
   a. Less than the arithmetic average
   b. Greater than the arithmetic average
   c. Less than or equal to the arithmetic average
   d. Greater than or equal to the arithmetic average
5. In the case of average price Asian options, for a given value of the exercise price:
   a. The call premium will be lower if geometric averaging is used
   b. The put premium will be higher if geometric averaging is used
   c. Both (a) and (b)
   d. None of the above
6. In the case of average strike Asian options:
   a. The call premium will be lower if geometric averaging is used
   b. The put premium will be higher if geometric averaging is used

c. Both (a) and (b)
d. None of the above

7. In the case of average strike Asian options:
   a. The call premium will be higher if geometric averaging is used
   b. The put premium will be lower if geometric averaging is used
   c. Both (a) and (b)
   d. None of the above
8. In the case of average strike Asian options:
   a. The call premium will be higher if geometric averaging is used
   b. The put premium will be higher if geometric averaging is used
   c. Both (a) and (b)
   d. None of the above
9. A European call option may be expressed as:
   a. A long position in an asset or nothing call with a long position in a cash or nothing call
   b. A long position in an asset or nothing call with a short position in a cash or nothing call
   c. A short position in an asset or nothing call with a long position in a cash or nothing call
   d. None of the above
10. Which of these options is path dependent:
    a. An American option
    b. A European option
    c. (a) and (b)
    d. None of the above
11. Which of these options is path dependent:
    a. An American option
    b. An Asian option
    c. A lookback option
    d. All of the above
12. Which of these is guaranteed to finish at or in the money:
    a. A lookback call
    b. A lookback put
    c. Both (a) and (b)
    d. None of the above
13. Which of these is guaranteed to finish at or in the money:
    a. An Asian call
    b. An Asian put
    c. Both (a) and (b)
    d. None of the above
14. A standard lookback will be valued higher than a corresponding European option if:
    a. It is a call
    b. It is a put

c. Both (a) and (b)
d. None of the above

15. Assume that all the options under consideration have the same exercise price and that the rebate on the barrier options is zero. A European call will be equal to:
   a. An up and out call plus an up and in call
   b. An up and out call plus a down and in call
   c. An up and out call plus an up and in put
   d. None of the above
16. Assume that all the options under consideration have the same exercise price and that the rebate on the barrier options is zero. A European call will be equal to:
   a. A down and out call plus a down and in call
   b. A down and out call plus an up and in call
   c. A down and out call plus a down and in put
   d. None of the above
17. Assume that all the options under consideration have the same exercise price and that the rebate on the barrier options is zero. A European put will be equal to:
   a. An up and out put plus an up and in put
   b. A down and out put plus a down and in put
   c. Both (a) and (b)
   d. None of the above
18. The expected stock return is $\mu - \frac{\sigma^2}{2}$ if:
   a. Returns are calculated using discrete compounding
   b. Return are calculated using continuous compounding
   c. Both (a) and (b)
   d. Neither (a) nor (b)
19. $S_t e^{r(T-t)} N(d_1)$ is the expected value of a variable that:
   a. Takes on a value of S_T in a risk-neutral world at the time of expiration of the option
   b. Takes on a value of S_T in a risk neutral world at the time of expiration of the option if the call option is in the money
   c. Takes on a value of zero in a risk neutral world at the time of expiration of the option if the call option is out of the money
   d. Both (b) and (c)
20. The expected value of a variable that takes on a value of S_T at the time of expiration of the call option if the option is in the money and a value of zero if the option is out of the money is given by:
   a. $S_t N(d_1)$
   b. $S_t e^{r(T-t)} N(d_1)$
   c. $N(d_1)$
   d. None of the above

21. $SN(d_1)$ is the value of:
    a. An asset or nothing call with an exercise price of X and a time to expiration of $T - t$
    b. An asset or nothing put with an exercise price of X and a time to maturity of $T - t$
    c. A cash or nothing call with an exercise price of X and a time to expiration of $T - t$
    d. None of the above
22. $Xe^{-r(T-t)}N(d_2)$ is:
    a. The present value of the expected outflow on account of the exercise price for a European call option in a risk neutral world
    b. The premium of a cash or nothing call option with the same exercise price and time to expiration
    c. Both (a) and (b)
    d. None of the above
23. Which of these investors has to post margins:
    a. Buyers of calls
    b. Writers of calls
    c. Buyers of puts
    d. (b) and (c)
24. Consider European call options on a stock with an exercise price of X and a premium of C. The call should be exercised if the terminal stock price S_T is such that:
    a. $S_T > X$
    b. $S_T > X + C$
    c. $S_T > X - C$
    d. None of the above
25. To offset a long position in a call option with X = \$100, and 3 months to expiration, the investor needs to:
    a. Go long in a put option with X = 100 and 3 months to expiration
    b. Go short in a put option with X = 100 and 3 months to expiration
    c. Go short in a call option with X = 100 and 3 months to expiration
    d. Depends on whether the options are European or American
26. Which of these is a Zero Sum game:
    a. A European call option
    b. A European put option
    c. An American call option
    d. All of the above
27. A call with an exercise price of X_1 and a put with an exercise price of X_2 are available on the same stock. If both of then are in the money, which of these statements must be true:
    a. $X_1 < X_2$
    b. $X_1 > X_2$

c. $X_1 = X_2$
d. Cannot say

28. Which of these statements is false:
    a. A call option will be exercised only if it is in the money
    b. A put option will be exercised only if it is in the money
    c. A put option will be exercised only if it is out of the money
    d. (a) and (c) are both false
29. FLEX and E-FLEX options:
    a. May be calls or puts
    b. May be European or American
    c. Both (a) and (b)
    d. Neither (a) nor (b)
30. Which of these can have a negative premium:
    a. A deep out of the money European put
    b. A deep out of the money American put
    c. Both (a) and (b)
    d. None of the above
31. Consider European calls and puts on XYZ, a non-dividend paying stock, both with X = \$100 and 1 month to expiration. If the current stock price is \$100, which of these statements is true:
    a. The two options will have an equal premium
    b. The call will have a higher premium
    c. The put will have a higher premium
    d. Depends on the riskless rate of interest
32. Which of these can have a negative intrinsic value:
    a. European puts
    b. American calls
    c. American puts
    d. None of the above
33. Which of these can have a negative intrinsic value:
    a. An at-the-money European put
    b. An in-the-money European put
    c. An out-of-the money European put
    d. None of the above
34. Which of these can have a negative time value:
    a. European calls
    b. European puts
    c. American calls
    d. American puts
35. At expiration the time value must be zero for:
    a. European calls
    b. American calls
    c. European puts
    d. All of the above

36. A call option will have a premium that cannot exceed the stock price if:
    a. The option is European
    b. The option is American
    c. Both (a) and (b)
    d. None of the above
37. An American call on a non-dividend paying stock:
    a. Will be exercised early
    b. May be exercised early
    c. Will never be exercised early
    d. Cannot say
38. An American call on a dividend paying stock:
    a. Will be exercised early
    b. Will never be exercised early
    c. May be exercised just before the stock goes ex-dividend
    d. May be exercised just after the stock goes ex-dividend
39. Volatility will have a positive impact on the option price if:
    a. The option is European
    b. The option is American
    c. Both (a) and (b)
    d. None of the above
40. Volatility will have a positive impact on the option price in the case of:
    a. European calls
    b. European puts
    c. American calls
    d. All of the above
41. Which of these is a Wasting Asset:
    a. An American call
    b. An American put
    c. A European call
    d. All of the above
42. Which of these may not be a Wasting asset:
    a. Am American call
    b. An American put
    c. A European put
    d. All of the above
43. Which of these securities offers leverage:
    a. Futures contracts
    b. European options
    c. American options
    d. All of the above
44. Consider the Binomial model. The riskless rate per period is 10%, the magnitude of the up move is 20% and that of a down move is 20%. The pseudo probability of an up move is:

a. 75%
b. 25%
c. 1.0
d. None of the above

45. Consider the Binomial model. Which of these represents an arbitrage opportunity:
    a. $u = 1.2$; $d = 0.80$; $r = 0.75$
    b. $u = 1.2$; $d = 0.80$; $r = 1.25$
    c. Both (a) and (b)
    d. Neither (a) nor (b)
46. Consider a covered call strategy. The current stock price is $100, and the exercise price is $95. If the premium is $8, the point of regret is:
    a. 108
    b. 93
    c. 92
    d. 87
47. Which of these is a covered call strategy:
    a. Overwriting
    b. Buy writing
    c. Both (a) and (b)
    d. None of the above
48. Which of these is perceived as a risky strategy:
    a. Overwriting
    b. Buy writing
    c. Both (a) and (b)
    d. None of the above
49. A bull spread is:
    a. A vertical spread
    b. A horizontal spread
    c. A diagonal spread
    d. None of the above
50. Which of these can be set up using both calls and puts:
    a. Bull spreads
    b. Bear spreads
    c. Butterfly spreads
    d. All of the above
51. Which of these strategies involves two breakeven prices:
    a. Bull spreads
    b. Bear spreads
    c. Butterfly spreads
    d. All of the above
52. An Ito process is a:
    a. Continuous time process
    b. Continuous state space process

c. Both (a) and (b)
d. None of the above

53. The Black–Scholes p.d.e. is valid for:
    a. European options on a non-dividend paying stock
    b. American options on a non-dividend paying stock
    c. Futures contracts on a non-dividend paying stock
    d. Any derivative on a non-dividend paying stock
54. Black and Scholes assumed that:
    a. There is a single riskless rate in the economy
    b. That this rate is the same for all maturities
    c. That this rate is constant
    d. All of the above
55. Which of these is a valid boundary condition for a call on a non-dividend paying stock:
    a. $C(0, t) = 0$
    b. $C(S, t) \rightarrow S$ as $S \rightarrow \infty$
    c. $C(S, T) = \text{Max}[0, S_T - X]$
    d. All of the above
56. The probability that a European call will be exercised in a risk-neutral world is:
    a. d_1
    b. $N(d_1)$
    c. $-N(d_1)$
    d. None of the above
57. The probability that a European put will be exercised in a risk-neutral world is:
    a. d_2
    b. $N(d_2)$
    c. $-N(d_2)$
    d. $N(-d_2)$
58. The delta of a call will be between:
    a. 0 and 1
    b. -1 and 0
    c. 0 and 0.50
    d. -0.5 and 0
59. As per the Black–Scholes formula, the delta of a call is:
    a. $N(d_1)$
    b. $N(-d_1)$
    c. $N(d_2)$
    d. $N(-d_2)$
60. Which of these is true about the delta of a call:
    a. For deep out of the money options it is close to zero

b. For deep in the money options it is close to 1
c. As the option approaches maturity, delta will tend towards 1 for in the money calls
d. All of the above

61. Which of these is true about gamma:
  a. It is identical for both calls and puts in a Black–Scholes world:
  b. It is positive for both calls and puts
  c. It will be at its peak when the option is near the money
  d. All of the above

62. Which of these is true about Vega:
  a. It is positive for both calls and puts
  b. It is identical for calls and puts in a Black–Scholes world
  c. Both (a) and (b)
  d. Neither (a) nor (b)

63. An interest rate swap may be used for:
  a. Hedging
  b. Speculation
  c. Arbitrage
  d. All of the above

64. A currency swap that requires the exchange of principal at inception is known as:
  a. A cash swap
  b. A par swap
  c. A coupon swap
  d. A basis swap

65. A cross-currency swap may be on a:
  a. Fixed-fixed basis
  b. Fixed-floating basis
  c. Floating-floating basis
  d. (b) or (c)

66. For a convex function:
  a. The expectation of the function is greater than or equal to the function of the expectation
  b. The expectation of the function is less than or equal to the function of the expectation
  c. The expectation of the function is less than the function of the expectation
  d. The expectation of the function is equal to the function of the expectation

67. A flat term structure is consistent with the likelihood of declining spot rates if:
  a. The unbiased expectations hypothesis is valid
  b. The liquidity premium hypothesis is valid
  c. Both (a) and (b)

d. None of the above

68. Which of these is true about exchange traded options:
    a. Price discovery takes place through a bilateral auction process
    b. They have a finite life span
    c. There is no limit on the number of contracts that can be traded
    d. All of the above
69. Which of these models for interest rates exhibits mean reversion:
    a. The CIR model
    b. The Vasicek model
    c. The Ornstein-Uhlenbeck process
    d. All of the above
70. An interest rate collar requires:
    a. A long position in a cap
    b. A long position in a floor
    c. Both (a) and (b)
    d. None of the above
71. An interest rate collar requires:
    a. A long position in a cap
    b. A short position in a floor
    c. Both (a) and (b)
    d. None of the above
72. A foreign currency is like:
    a. A non-dividend paying stock
    b. A stock paying a constant dividend
    c. A stock paying a continuous dividend yield
    d. None of the above
73. Which of these will lead to an inflow for the holder if exercised:
    a. A futures call option
    b. A futures put option
    c. Both (a) and (b)
    d. None of the above
74. An American call on a futures contract:
    a. Will never be exercised early
    b. May be exercised early
    c. Will be equal in value to a European call with the same terms
    d. None of the above
75. Which of these options may be exercised early:
    a. American call futures options
    b. American put futures options
    c. Both (a) and (b)
    d. Neither (a) nor (b)
76. A portfolio insurance strategy may be implemented using:
    a. Futures contracts
    b. Put options

c. Call options
d. All of the above

77. The convexity property is valid for:
a. Call options
b. Put options
c. Both (a) and (b)
d. Neither (a) nor (b)

78. Which of these strategies requires an initial investment:
a. A long butterfly spread
b. A long condor
c. A long straddle
d. All of the above

79. Which of these requires an initial investment:
a. A long position in an in the money strangle
b. A long position in an out of the money strangle
c. Both (a) and (b)
d. Neither (a) nor (b)

80. Which of these strategies entails finite losses and infinite profits:
a. A long straddle
b. A long in the money strangle
c. A long out of the money strangle
d. All of the above

81. Which of these strategies requires positions in two or more options of the same type:
a. Vertical spreads
b. Horizontal spreads
c. Diagonal spreads
d. All of the above

82. Which of these is a practical difficulty with the bootstrapping procedure:
a. There may be several bonds of the same risk class with the same maturity date
b. A bond may not exist for a given maturity
c. A bond may not be plain vanilla in nature
d. All of the above

83. A yield curve may be:
a. Downward sloping
b. Humped
c. U-shaped
d. Any of the above shapes

84. Which of these rates will always be equal:
a. The one period spot rate
b. The one period forward rate
c. The one period short rate
d. All of the above

85. Which of these options can be priced without postulating a process for the evolution of the stock price:
    a. European calls
    b. American calls
    c. American puts
    d. None of the above
86. Which of these statements is true about a Wiener process:
    a. It has a constant mean
    b. It has a constant variance
    c. Both (a) and (b)
    d. Neither (a) nor (b)
87. Which of these phenomena is observed in practice about the implied volatility derived using the Black–Scholes formula:
    a. At the money options give the lowest implied volatility
    b. Deep-in-the-money options give high implied volatilities
    c. Deep-out-of-the-money options give high implied volatilities
    d. All of the above
88. Consider at-the-money European calls and puts on a dividend paying stock:
    a. The call will be priced higher
    b. The call may be priced higher
    c. The call and put will be equally priced
    d. Depends on the option pricing model
89. Consider at-the-money European calls and puts on a dividend paying stock:
    a. The call may be priced higher
    b. The call may be priced lower
    c. The call and the put may be equally priced
    d. Any of the above
90. Which of these statements is true about American calls on non-dividend paying stocks:
    a. They will always have a positive time value except at expiration
    b. They will always have a positive intrinsic value prior to expiration
    c. They will always have a non-negative premium prior to expiration
    d. (a) and (c)
91. The Binomial model requires that:
    a. The risk-neutral probabilities be constant
    b. The riskless rate be constant
    c. Both (a) and (b)
    d. None of the above
92. Which of these statements is true about capped-style options:
    a. The cap price is above the exercise price for calls
    b. The cap price is below the exercise price for puts
    c. Both (a) and (b)
    d. None of the above

93. Consider a call options contract for 100 shares of the underlying stock. Assume that a 20% stock dividend is declared on the underlying stock. Which of these statements is true:
    a. The exercise price will be adjusted downwards
    b. The exercise price will be adjusted upwards
    c. There will be no adjustment to the exercise price
    d. The adjustment would depend on whether the underlying stock pays a cash dividend
94. Consider a put options contract for 100 shares of the underlying stock. Assume that a 20% stock dividend is declared on the underlying stock. Which of these statements is true:
    a. The exercise price will be adjusted downwards
    b. The exercise price will be adjusted upwards
    c. There will be no adjustment to the exercise price
    d. The adjustment would depend on whether the underlying stock pays a cash dividend
95. Consider an options contract for 100 shares of the underlying stock. Assume that a 20% stock dividend is declared on the underlying stock. Which of these statements is true:
    a. The number of shares deliverable will be adjusted upwards
    b. The number of shares deliverable will be adjusted downwards
    c. There will be no adjustment to the number of deliverable shares
    d. Depends on whether it is a call or a put option
96. Which of these is true about FLEX options:
    a. They have features of exchange traded products
    b. They have features of OTC products
    c. Both (a) and (b)
    d. None of the above
97. The Black–Scholes formula assumes that:
    a. The riskless rate of interest is a constant
    b. The volatility of the rate of return on the underlying stock is a constant
    c. The stock does not pay any dividends during the life of the option
    d. All of the above
98. Which of these models assumes that the underlying riskless rate of interest is a constant:
    a. The Black–Scholes model
    b. The Merton model
    c. The Garman–Kohlhagen model
    d. All of the above
99. Which of these models assumes that the underlying asset has a constant volatility:
    a. The Black–Scholes model
    b. The Merton model

c. The Garman–Kohlhagen model
d. All of the above

100. Consider at the money European call and put futures options. Which of these statements is true:
   a. The call will be priced higher
   b. The put will be priced higher
   c. The two will be equally priced
   d. Depends on the volatility

17

Financial Derivatives: The Indian Market

17.1 Revival of Derivatives Trading in India

Financial sector reforms have been an integral part of the overall liberalization strategy for the Indian economy, which commenced in 1991. Before the introduction of relatively sophisticated instruments like derivatives, there was an urgent need to reform and revitalize the cash market for securities. Consequently, in the initial years, the focus was on the development of a modern, efficient and more transparent cash market. The most important developments in this regard may be summarized as follows.

1. A new *Electronic* exchange—The National Stock Exchange, was set up, backed by India's leading financial institutions. It commenced trading in equities in the November of 1994. The NSE introduced screen based trading and linked up investors across the country using VSAT technology. This exchange provided healthy competition to the Mumbai Stock Exchange (BSE) which was earlier India's premier stock trading platform. The consequences were soon apparent, for the BSE too chose to abandon the traditional open-outcry system of trading and switched to a screen-based system.
2. The National Securities Clearing Corporation (NSCCL) was set up to clear all trades.
3. The National Securities Depository (NSDL) was set up to facilitate trading of securities in the dematerialized mode. Subsequently a second depository the Central Depository Services (India) Limited (CDSL) was set up.

As liberalization progressed, and the command economy was dismantled, it became obvious that market participants in India would face greater price risk. Liberalization brought in foreign capital, in the form of Foreign Direct Investment (FDI) as well as capital market investments by Foreign Institutional Investors (FIIs). However, capital does not flow across borders in isolation. Risk is an integral accompaniment. The next step, consequent to the streamlining of the cash market, was therefore the creation of a market for exchange traded derivatives. It was felt that derivatives were integral to the philosophy of economic reforms, because they give firms the freedom to decide as to what risks they are comfortable bearing and what they would rather hedge away.

However, there were legal barriers to the introduction of derivatives trading, that were a part of India's socialist legacy. The Securities Contracts (Regulation) Act, SC(R)A, prohibited trading in derivatives. Using the powers bestowed by this Act, the Government of India had prohibited all forward trading in securities in 1969. The first step was therefore to repeal some of the provisions of this Act. It must be pointed out that although forward trading in securities was proscribed, parties with legitimate foreign exchange transactions were permitted to protect themselves against foreign currency risks by entering into forward contracts with commercial banks. Indian regulations also permitted the trading of futures contracts in certain commodities, although the volumes of trading in practice were insignificant.

The Securities Exchange Board of India (SEBI) set up a committee headed by Dr. L.C. Gupta in 1996, to develop an appropriate regulatory framework to facilitate derivatives trading. The committee after extensive deliberations, submitted its report in March 1998. It recommended that derivatives be declared as securities so that the existing regulatory framework for the trading of securities could be extended to cover derivatives as well. Among its other recommendations, it suggested an automated screen based trading system for derivative securities. It mooted a separate clearing corporation and recommended strict eligibility criteria for brokers seeking to trade in the derivatives segment of an exchange.

Subsequently, the Securities Laws (Amendment) Bill was introduced in the Parliament, and was converted into an Act in December, 1999. The Government of India lifted the prohibition on forward trading in securities in March 2000. The SC(R)A post-amendment included derivatives as part of a broader definition of securities. However, the amended regulations clearly stated that trading in derivatives contracts would be legal and valid only if such trading were to take place on a recognized stock exchange. Thus trading of derivative contracts over the counter, was precluded.

SEBI then gave approval to the BSE and the NSE to commence trading in approved derivative contracts, which included only index futures to begin with. The chronology of the introduction of equity derivatives is as follows.

- On 9 June, 2000 the BSE launched futures contracts on the Sensex.
- The NSE began trading futures contracts on the S&P CNX Nifty index from 12 June, 2000.
- European options on the Sensex began trading from 1 June, 2001.
- The NSE commenced trading in Nifty options (also European) on 4 June, 2001.
- American options on individual stocks began trading on the NSE from 2 July, 2001.
- The BSE introduced similar contracts on 9 July, 2001.

- On 9 November, 2001 the NSE launched single stock futures.
- Single stock futures were introduced by the BSE on 9 November, 2002.

SEBI specified that stock futures and options were to be settled in cash. This means that if a call holder were to exercise the option, rather than paying the exercise price and receiving physical delivery of a share, he will receive the difference between the stock price at the time of exercise and the exercise price. Similarly, a put holder will receive the difference between the exercise price and the stock price, if he chooses to exercise. In the case of stock futures, all contracts are marked to market on the last day and all positions are declared closed. Index options, just like index futures, are cash settled, as is the global practice.

In 1998 SEBI constituted a committee under the Chairmanship of J.R.Varma, a Professor at IIM Ahmedabad, to formulate a risk management system for the proposed derivatives market. This committee designed a suitable margining system. It recommended the use of the Value at Risk methodology for computing margins, and suggested a procedure for computing the VaR. Considering the fact that even the best of statistical methods for computing potential losses cannot prevent the possibility of default in 100% of the cases, the committee recommended additional protection for preventing default, by requiring clearing members of the exchange to have a minimum liquid net worth.

17.2 Equity Derivatives on the NSE

The NSE currently offers futures and options on seven indices. These are:

Table 17.1 NSE's Index Based Derivative Products

The S&P CNX NIFTY Index
The CNX NIFTY Junior Index
The CNX IT Index
The CNX 100 Index
The CNX Bank Index
The NIFTY Midcap 50 Index
The S&P CNX DEFTY Index

17.2.1 The Underlying Indices: Key Features

S&P CNX NIFTY This is a well diversified index of 50 stocks, which together represent 22 sectors of the Indian economy. The index is owned and managed by India Index Services and Products Ltd. (IISL). IISL is a joint venture between the National Stock Exchange and CRISIL, which is India's leading credit rating agency. It has a marketing and licensing agreement with Standard & Poor's (S&P).

As on 30 January 2009, these 50 stocks represented about 63.98% of the total market capitalization of all the stocks traded on the exchange. The index is computed on a value weighted basis, that is it is based on the market capitalization of the constituent stocks. The base period for the index is the close of 3 November 1995. The base value of the index is 1,000 and the base market capitalization is Rs 2,060,000,000,000.

CNX NIFTY Junior This index too is based on 50 stocks. The securities selected for this index represent the next rung, in terms of liquidity, after the 50 stocks that constitute the NIFTY. Taken together, the 100 stocks which comprise these two indices make up the 100 most liquid stocks in India. As of 30 January 2009 these 50 stocks represented about 9.62% of the total market capitalization of all exchange traded stocks. The index is value weighted. The base date is 3 November 1996 and the base value is 1,000. The base period market capitalization was Rs 430,000,000,000.

CNX IT Index India is a world leader in the field of information technology and the IT industry is the key driver of the modern Indian economy. IISL has therefore developed an index to gauge the performance of this sector. The constituent companies, have more than 50% of their turnover from IT related activities like:

- IT Infrastructure
- IT Education and Software Training
- Telecommunication Services and Networking Infrastructure
- Software Development
- Hardware manufacturing, vending, support and maintenance

The stocks in the index together represented about 91.92% of the total market capitalization of the IT sector as on 30 January 2009, and about 6.92 of the total market capitalization of exchange traded stocks. The base date is December 1995 and the base value was 1,000. It was subsequently revised to 100 from 28 May 2004.

The CNX 100 Index This is an index of 100 stocks that represent 35 sectors of the Indian economy. The stocks in the index represent about 73.60% of the market capitalization of all exchange traded stocks as of 30 January 2009. The index is value weighted and has a base date of 1 January 2003. The base value is 1,000. The 100 stocks in the index represent the 50 stocks that constitute the NIFTY as well as the 50 stocks in the NIFTY Junior index. Thus any changes in either of these two indices will automatically be reflected in the CNX 100 index.

The CNX Bank Index The banking sector is a critical component of the liberalized Indian economy. To evaluate the performance of this sector, IISL developed the CNX Bank index. The index is composed of 12 stocks that comprise the most liquid and well capitalized Indian banks. These 12 stocks constituted about 86.06% of the total market capitalization of the banking sector as of 30 January 2009, and about 8.63% of the total market capitalization of all exchange traded stocks as on that day. The index is value weighted. The base date is 1 January, 2000 and the base value is 1,000.

The NIFTY Midcap 50 Index This is a value weighted index, designed to capture the performance of the mid-cap segment of the market. The base date is 1 January, 2004 and the base value is 1,000. The 50 stocks in the index represented about 3.78% of total market capitalization as of 30 January, 2009.

The S&P CNX DEFTY Index The DEFTY index allows investors to measure their returns on Indian equities in terms of US dollars. The index is nothing but a USD version of the S&P CNX NIFTY.

The index is computed using the values of the NIFTY and the INR-USD exchange rate. The value of the index on a given day, is given by

$$\text{S\&P CNX DEFTY}_t = \frac{\text{NIFTY}_t \times S_b}{S_t}$$

where S_b is the exchange rate on the base date and S_t is the exchange rate on day t. During the day the exchange rate is determined on a real time basis. The closing value however is computed by taking an average of spot rates in the last 30 minutes of trading. The base date for the index is 3 November 1995, and the base value is 1,000. The NIFTY also had a value of 1,000 on the same base date. The spot exchange rate on that day was 34.65 INR/USD.

In addition the exchange offers mini index futures as well as options, and long term index options. These are based on the S&P CNX NIFTY Index. The key advantage of long-term options is that investors can take a long term view on the underlying without having to resort to the strategy of using a combination of shorter term contracts.

The NSE offers currently offers Individual Stock Futures (ISFs) and stock options on 250 securities. Index options, mini index options, and long term index options are European. Stock options are American. All derivative contracts expire on the last Thursday of the expiration month. If the last Thursday were to be a market holiday, then the contracts will expire on the previous trading day. The exchange opens at 09:55 a.m. and closes at 15:30 p.m.

The tick size for equity derivative contracts is Rs. 0.05 or five paise.

At any point in time, with the exception of long term index options, three contract months are available for all futures and options contracts. These are the near month, the next month, and the far month. For instance assume that we are on 15 March 2009. The last Thursday of the month is 26 March 2009. So on the 15th, we will find that the following contracts are available for each of the underlying products: March 2009, April 2009, and May 2009. On 26 March 2009 the March contracts will expire. Consequently on 27 March 2009 the June 2009 contracts will be introduced.

For long term index options the number of contracts available is more. At any point in time, in addition to the near month, the next month, and the next to next month, the next three quarterly expiration months from the March cycle, and the following five half-yearly expiration months from the June–December cycle are available. This may be illustrated as follows. Assume that today is 15 March 2009. March 2009, April 2009, and May 2009 contracts will obviously be available. June 2009, September 2009, and December 2009 contracts represent the next three quarterly expiration months from the March cycle and will consequently be available. June 2010, December 2010, June 2011, December 2011, and June 2012 are the following five half-yearly expiration months from the June–December cycle. Hence these contracts will also be available for trading.

The lot sizes for the index derivative contracts are as given in Table 17.2. For stock futures and options the lot size varies from stock to stock. The lot size for mini index futures and options is 20.

Table 17.2 **Lot Sizes for Index Derivatives on the NSE**

Underlying	Symbol	Lot Size
S&P CNX NIFTY	NIFTY	50
CNX NIFTY Junior	JUNIOR	100
CNX IT	CNXIT	100
CNX 100	CNX100	100
Bank NIFTY	BANKNIFTY	50
NIFTY Midcap 50	NFTYMCAP50	300
S&P CNX Defty	DEFTY	150

17.2.2 Option Contracts on the NSE and Exercise Prices

When a new contract month is being introduced for stock options, the number of exercise prices that will be permissible at inception is seven. Of these, one will be at the money, three in the money, and three out of the money. The interval between strike prices, is a function of the stock price, and is determined as per the following schedule.

Table 17.3 **Strike Price Intervals**

Stock Price Range	Strike Price Interval
Less than or equal to Rs. 50	Rs. 2.50
> Rs. 50 to less than or equal to Rs. 250	Rs. 5.00
> Rs. 250 to less than or equal to Rs. 500	Rs. 10.00
> Rs. 500 to less than or equal to Rs. 1000	Rs. 20.00
> Rs. 1000 to less than or equal to Rs. 2500	Rs. 30.00
> Rs. 2500	Rs. 50.00

The procedure for determining the exercise prices may be illustrated with the help of an example.

Example Assume that Reliance is trading at 1,550 and that June 20XX contracts are being introduced. The relevant strike price interval is Rs. 30. Since the current strike price is 1,550 the nearest multiple of 30 is Rs. 1,560. Thus this price would be taken as the strike price of the in-the-money contract.[1] With respect to this strike price three in-the-money and three-out-of-the-money contracts will be introduced. The in-the-money strike prices for calls will be Rs. 1,470, Rs. 1,500, and Rs. 1,530. The out-of-the-money strike prices for calls will be Rs. 1,590, Rs. 1,620, and Rs. 1,650. For puts, quite obviously, the first three would constitute out-of-the-money strike prices, while the last three would represent in-the-money exercise prices. On the first day, a trader will have a choice of only these seven strike prices.

Now assume that at the end of the first day, the stock price is Rs. 1,500. The at-the-money strike price for the next day is obviously Rs. 1,500, since it is a multiple of Rs. 30. With respect to this strike price, five out-of-the-money calls and five in-the-money puts are already available. We are obviously referring to the exercise prices of Rs. 1,530, Rs. 1,560, Rs 1,590, Rs. 1,620 and Rs. 1,650. Since at any point in time the exchange assures that a minimum of three in-the-money and three-out-of-the-money contracts will be available, there is no need for it to introduce new exercise prices in excess of Rs. 1,500. However there is only one in-the-money call, or out-of-the-money put, with an exercise price of Rs. 1,470. Consequently on the second day, two additional exercise prices, that is, Rs. 1,410 and Rs. 1,440 will also be introduced for trading. Thus, the number of exercise prices that are available on a given day, is a function of how volatile the stock price has been, since contracts were introduced for that particular month.

For options on the S&P CNX Nifty, the number of exercise prices at the time of contract introduction are given in Table 17.4. The relevant strike price intervals are as given in Table 17.5.

Table 17.4 **Index Levels and No. of Strikes for NIFTY Options**

Index Level	First, Second and Third Month Expiries	Three Quarterly Expiries	Five Half-Yearly Expiries
up to 2,000	4-1-4	6-1-6	4-1-4
> 2,001 up to 4,000	4-1-4	6-1-6	4-1-4
> 4,001 up to 6,000	5-1-5	8-1-8	5-1-5
> 6,000	5-1-5	8-1-8	5-1-5

Explanatory Illustration Assume that today is 27 February 2009, the last Friday of the month. May 2009 contracts are being introduced. Consider options on the NIFTY. Assume that the index is at 3100. This level is between 2,001 and 4,000. Since May is the third month, nine exercise prices will be introduced. Of these, one will be at-the-money, four out-of-the-money, and four in-the-money (4-1-4). The relevant strike price interval is 50. Since 3,100 is a multiple of 50, the strike price for at-the-money calls and puts will be 3,100. 2,900, 2,950, 3,000,

[1] In this case it is more appropriate to term it as the near-the-money contract.

Table 17.5 **Index Levels and Strike Interval for NIFTY Options**

Index Level	First, Second and Third Month Expiries	Three Quarterly Expiries	Five Half-Yearly Expiries
up to 2,000	25	25	50
> 2,001 up to 4,000	50	50	100
> 4,001 up to 6,000	50	50	100
> 6,000	50	50	100

and 3,050 will constitute the exercise prices of the four in-the-money calls and the four out-of-the-money puts. 3,150, 3,200, 3,250 and 3,300 will be the exercise prices of the four out-of-the-money calls and the four in-the-money puts.

The number of exercise prices that are available for an index options contract at the time of introduction, as well as the interval between strike prices for the following indices, is given in Table 17.6.

- NIFTY Junior
- CNX IT
- CNX 100
- Bank NIFTY
- NIFTY Midcap
- CNX Defty

Table 17.6 **Index Levels, Strike Interval, and No. of Strikes**

Index Level	Strike Interval	# of Strikes
up to 2,000	25	4-1-4
> 2,001 up to 4,000	50	4-1-4
> 4,001 up to 6,000	50	5-1-5
> 6,000	50	6-1-6

17.2.3 Equity Derivatives on the BSE

The BSE offers derivatives on the following indices. The lot sizes are also given in Table 17.7.

Key information for these indices, all of which are value weighted, is given in the Table 17.8.

Mini futures and options contracts are available on the Sensex with a lot size of five. The exchange also offers futures and options contracts on the stocks of 109 companies.

Table 17.7 **Underlying Indices and the Contract Lot Sizes**

Index	Lot Size
BSE 30 SENSEX	15
BSE TECk	124
BSE BANKEX	50
BSE Oil & Gas	38
BSE Metal	60
BSE FMCG	175

Table 17.8 **Index Related Information**

Index	Base Year	Base Value	Date of Launch
Sensex	1978-79	100	01-01-1986
BSE TECk	2 April 2001	1,000	11-07-2001
BSE BANKEX	1 January 2002	1,000	23-06-2003
BSE Oil & Gas	1 February 1999	1,000	23-08-2004
BSE Metal	1 February 1999	1,000	23-08-2004
BSE FMCG	1 February 1999	1,000	9-08-1999

17.2.4 Weekly Options

In response to the market demand for options of shorter maturity, the BSE launched weekly options on 13 September 2004. These are available on the Sensex, regular as well as mini, and on three stocks, Reliance, State Bank of India, and Tata Steel. The lot sizes are given in Table 17.9. These options are introduced on the Monday of every week and have a maturity of two weeks. Contracts are scheduled on expire on Friday of the expiring week.

Table 17.9 **Lot Sizes for Weekly Options**

Underlying Asset	Lot Size
Sensex	15
Mini Sensex	5
RIL	300
SBI	264
TISCO	1,528

For all options contracts on the BSE, weekly, regular and long-dated, a minimum of three strike prices will be available at any point in time. That is there will be a near-the-money contract, an in-the-money contract, and an out-of-the-money contract.

The BSE too functions from 09:55 a.m. to 15:30 p.m. Equity derivatives expire on the last Thursday of the expiration month, and on the previous business day if the last Thursday were to be a holiday. All contracts are cash settled. At any point in time, contracts for the current month, the next month, and the far month will be available. The number of long-dated options contracts that are available at any point in time, is the same as for the NSE.

17.3 Currency Futures

Both the BSE and the NSE have recently introduced currency futures. As of now the contracts are available only on the US dollar.

The contract size is 1,000 USD, and the tick size is 0.25 paise or Rs. 0.0025. Contracts are traded from, Monday to Friday and the trading hours are 9.00 a.m. to 5.00 p.m. At any point in time, contracts are available for the next 12 months. Thus on 4 April 2009, contracts are available for April–December 2009, and January–March 2010.

The last trading day is two working days prior to the last business day of the expiration month. Trading ceases at 12.00 p.m. The final settlement day is the last working day of the expiry month excluding Saturdays. The contracts are cash settled. The settlement price on all days is a weighted average price based on the last half-hour of trading. The final settlement price is the reference rate provided by the Reserve Bank of India.

17.4 Currency Forwards

In India, in order for an agency to deal in foreign exchange buying and selling transactions, it is necessary for it to acquire the prior approval of the Reserve Bank of India (RBI). Agencies which have secured permission from the RBI are called *Authorized Dealers* (*ADs*).

We have already explained the quoting conventions for spot as well as forward transactions. In India the indirect quotation system was prevalent until 2 August 1993. Subsequently the market switched over to the direct method. Before we go on to discuss modification of forward contracts, it is necessary to introduce the concepts of *merchant rates* and *exchange margins*.

17.4.1 Merchant Rates and Exchange Margins

Consider a purchase transaction. When the AD buys foreign currency from a customer, he has to sell it in the inter-bank market. The rate at which the AD can sell is obviously the inter-bank buying rate. Thus the inter-bank buying rate is the *base rate* in the case of purchase transactions, to which the AD has to factor in a profit margin before quoting a rate to the customer. The rates quoted by banks to their customers are called *merchant rates*.

Similarly, in the case of a sale transaction, when the AD sells foreign currency to the customer, he has to first acquire it from the inter-bank market. This acquisition will obviously take place at the inter-bank selling rate. Thus the inter-bank selling rate is the *base rate* for sale transactions, to which the AD will factor in a profit margin before arriving at the merchant rate.

Now in order to make a profit on a purchase transaction, the base rate must be higher than the merchant rate. Thus, in this case the profit margin, known in market parlance as the *exchange margin*, must be subtracted from the base rate. On the other hand, if the AD is to make a profit on a sale transaction, then the base rate must be lower than the merchant rate. Hence, in this case, the exchange margin must be added to the base rate, in order to arrive at the merchant rate. We will illustrate these principles with the help of the following numerical examples.

Example I An exporter has received a draft for USD 1MM from a party in New York and has presented it to his bank, HDFC Bank. The current rates in the inter-bank market are as follows:

Spot: 45.8250/46.2315 INR/USD

Assuming that the bank levies an exchange margin of .05%, what is the applicable exchange rate?

The base rate in this case is the inter-bank buying rate, which is Rs. 45.8250. Since the bank is quoting a buying rate, it will subtract the exchange margin. Hence the rate quoted by the bank will be

$$45.8250 \times (1 - .0005) = \text{Rs. } 45.8021.$$

Hence, the equivalent amount payable by the bank in INR is Rs. 45,802,100.

Example II An importer needs to pay 100,000 SGD to a party in Singapore after 3 months and requests her bank, ICICI Bank, for a forward contract. The prevailing rates in the inter-bank market are as follows:

Spot: 24.6135/25.2180 INR/SGD

1 Month Forward: 95/50

2 Months Forward: 110/65

3 Months Forward: 140/90

Assuming that the bank levies an exchange margin of .075%, what is the applicable forward rate?

The base spot rate in this case is the spot selling rate, which is 25.2180. Since the Singapore Dollar is being quoted at a discount, the forward margin must be subtracted. Hence, the corresponding three months forward base rate is

$$25.2180 - .0090 = \text{Rs. } 25.2090$$

Since the bank is quoting a selling rate, it will add the exchange margin. Hence the rate quoted by the bank will be

$$25.2090(1 + .00075) = \text{Rs. } 25.2279$$

Hence the equivalent amount payable by the importer in INR is Rs. 2,522,790.

17.5 Modification of Forward Contracts

A forward contract is an agreement between the buyer and the seller to exchange currencies at a future date, at a rate that is fixed at the outset. But having entered into a forward contract, a bank's client may be unable to carry out his obligations as per the original agreement, often for reasons beyond his control. Consequently, there could be subsequent variations to the terms of the original agreement. In practice the following situations can arise.

- The client may ask for early delivery or delivery before the due date.
- He may ask for late delivery or delivery after the due date. The request for late delivery may be made
  1. Before the original due date or
  2. On the original due date or
  3. After the original due date
- The client may seek to cancel the contract. Once again, the request for cancellation may be made
  1. Before the original due date or
  2. On the original due date or
  3. After the original due date

We will now examine each of these situations with the help of suitable examples.[2] All the illustrations will be based on transactions between a single client, WIPRO and one commercial bank, IDBI Bank, in order to enable us to focus on the methodology, without being distracted by data that is constantly changing.

17.5.1 Early Delivery

We will illustrate the procedure adopted by the bank in the case of early delivery involving a forward purchase contract, as well as in the case of a forward sale contract.

Forward Sale Contract Consider a case where WIPRO has booked a contract with IDBI Bank, Bangalore for buying 1 MM USD on 30 November, 2008. Let us assume that the contract was booked on 1 September, 2008 and that the bank has agreed to sell the currency at 45.7500 INR/USD. However, today, that is 30 October, 2008, the company suddenly makes a request for immediate delivery of foreign exchange, because the shipment from abroad has arrived earlier than anticipated. What will IDBI Bank do in practice?

Now, on 1 September, at the time of booking the contract, IDBI would have entered into a forward contract on the inter-bank market to buy 1MM USD on 30 November. This contract will have to be immediately offset by the bank, by entering into a one month forward contract to sell 1 MM USD on 30 November.[3]

[2]These examples illustrate conventions followed in India, which are not necessarily universal.
[3]There is one month left till the expiration of the original forward contract entered into by the bank.

Having done that, the bank will have to purchase 1MM USD in the spot market for immediate delivery to WIPRO.

These two transactions that the bank enters into on 30 October are called swap transactions. In this case, the swap entails a spot purchase and a forward sale. If the rate at which the bank buys in the spot market is higher than the rate at which it sells forward, then it will incur a swap loss, which is recoverable from the client. However, if the spot buying rate is lower than the forward selling rate, there will be a swap gain, which will have to be passed on to the client.

Let us now analyze the situation using numbers. Assume that the following terms are available in the inter-bank market on 30 October.

Spot: 46.2055/46.5175

1 Month Forward: 35/75

The corresponding outright forward rates are therefore 46.2090/46.5250.

The bank will therefore buy 1 MM USD at 46.5175 and sell 1 month forward at Rs. 46.2090. Thus there is a swap loss of

$$(46.2090 - 46.5175) \times 1{,}000{,}000 = \text{Rs. }(308{,}500).$$

This amount is recoverable from the client.

The bank will get an inflow of INR 45,750,000 when it sells the dollars to WIPRO as per the terms of the original agreement. However, it will incur an expenditure of INR 46,517,500 at the time of purchase in the spot market. Thus, there will be a cash outlay of

$$45{,}750{,}000 - 46{,}517{,}500 = \text{Rs. }(767{,}500).$$

The bank will recover interest on the cash outlay from the client, from the date of early delivery till the original due date. If however, there had been a net inflow, the bank could have, if it had desired, paid interest to the client. Interest on cash outlays is recoverable at a rate that is not lower than the bank's *Prime Lending Rate* (*PLR*), whereas interest on cash inflows may be paid at the rate applicable for term deposits of the same period.

If we assume that the interest rate is 15% per annum, then the interest payable by WIPRO is

$$767{,}500 \times .15 \times \frac{1}{12} = \text{Rs. } 9593.75$$

Finally, the bank will levy a processing fee, which we will assume is, Rs. 100 for the transaction. Hence, the total amount payable by the client for early delivery is

$$308{,}500 + 9593.75 + 100 = \text{Rs. } 318{,}193.75.$$

Let us now examine the rationale behind the method. Assume that the bank had originally entered into a forward contract to buy forward on 30 November, at a price F_0. If the client had not requested for early delivery, the bank would have received an inflow of 45.75 per dollar on that day and would have incurred an outflow of F_0. Thus its net cash inflow would have been

$$(45.75 - F_0) \times 1{,}000{,}000$$

The present value of this cash flow on 30 October is

$$\frac{(45.75 - F_0) \times 1{,}000{,}000}{(1+r)}$$

where $r = \frac{.15}{12}$. However, due to early delivery, the bank will now receive 45.75 on 30 October and will incur an expenditure of 46.5175 per dollar acquired in the spot market. It will offset the forward contract at 46.2090, which will consequently have a value of $\frac{46.2090 - F_0}{(1+r)}$. The amount payable by the client for early delivery, which we will denote by X, should be such that

$$(45.75 - 46.5175) \times 1{,}000{,}000 + \frac{46.2090 - F_0}{(1+r)} \times 1{,}000{,}000 + X$$

$$= \frac{(45.75 - F_0) \times 1{,}000{,}000}{(1+r)}$$

$$\Rightarrow X = \frac{(46.5175 - 46.2090) + r(46.5175 - 45.75)}{1+r} \times 1{,}000{,}000.$$

$(46.5175 - 46.2090) \times 1{,}000{,}000$, is the swap loss, and $(46.5175 - 45.75) \times 1{,}000{,}000$, is the cash outlay. Instead of charging the present value of the swap loss and the interest on the cash outlay, the bank is charging the undiscounted value.

Forward Purchase Contract Let us assume that WIPRO had booked a contract with IDBI Bank for selling 1MM USD on 30 November, 2008 at 45.7500 INR/USD. What should the bank do if the company suddenly decides to deliver on 30 October, due to early receipt of the payment from abroad?

At the time of booking the contract on 1 September, IDBI would have entered into a forward contract to sell 1MM USD on 30 November on the inter-bank market. It will now have to offset it by entering into a 1 month forward contract to buy 1 MM USD. Simultaneously, the bank will have to sell 1MM USD in the spot market. Thus, in this case, the swap involves a spot sale and a forward purchase.

In this particular case, the bank can sell in the spot market at Rs. 46.2055. It will buy 1 month forward at 46.5250. Thus, there is a swap loss of

$$(46.2055 - 46.5250) \times 1{,}000{,}000 = \text{Rs. } (319{,}500).$$

The bank will pay 45,750,000 when it buys the dollars from WIPRO, but will get 46,205,500 when it sells the dollars in the spot market. Thus, there is an inflow of

$$46{,}205{,}500 - 45{,}750{,}000 = \text{ Rs. } 455{,}500.$$

Let us assume that the bank pays 12% interest on this. Therefore, the amount payable to the client is

$$455{,}500 \times .12 \times \frac{1}{12} = \text{ Rs. } 4{,}555.$$

The bank will as usual, levy a flat fee of Rs. 100. Hence, the total amount payable by the client is

$$319{,}500 - 4{,}555 + 100 = \text{Rs. } 315{,}045.$$

Extension on Due Date Assume that on 1 September, 2008, WIPRO had entered into a forward contract with IDBI Bank to buy 1MM USD on 30 October, at 45.7500INR/USD. On 30 October the company requests that the contract be extended by 1 month until 30 November. What should the bank do?

On 1 September, the bank would have entered into a forward contract on the inter-bank market to buy 1MM USD on 30 October. This will have to be sold in the market at the prevailing rate of 46.2055. The difference, which amounts to

$$(46.2055 - 45.7500) \times 1{,}000{,}000 = \text{Rs. } 455{,}500$$

is in favour of the client. Thus, this amount, less a flat transactions fee of Rs. 100, will be payable to the customer. Simultaneously, a fresh forward contract will be booked at the prevailing rate of Rs. 46.5250.[4]

Early Cancellation Consider the case where WIPRO had entered into a forward contract to buy 1MM USD on 30 November at 45.7500 INR/USD. Assume that on 30 October, the company requests the bank to cancel the contract. What should the bank do?

Now, since the client has originally gone long in a forward contract expiring on 30 November, it can cancel by going short in a contract expiring on 30 November, which is now 1 month away. The outright forward rates for the 1 month contract on 30 October in the inter-bank market, as calculated earlier are, 46.2090/46.5250. If we assume that the exchange margin on both purchase and sale transactions is .05%, then the corresponding merchant rates are 46.1859/46.5483. Thus, the applicable rate for the offsetting position is 46.1859. The exchange difference is

$$(46.1859 - 45.7500) \times 1{,}000{,}000 = \text{Rs. } 435{,}900$$

which in this case is in the client's favour and is therefore payable to it. After subtracting the transaction fee of Rs. 100, the bank will pay Rs. 435,800.

Cancellation on the Due Date Assume that WIPRO had originally entered in to a contract to buy 1MM USD on 30 October and decides to cancel the contract on 30 October. What should IDBI bank do?

The bank will treat the transaction as if it has sold 1MM at the originally agreed upon rate of Rs. 45.7500 and bought back 1MM USD at the current merchant rate applicable for purchase transactions. If we assume that the exchange margin is .05%, the corresponding merchant rate for purchase transactions is $46.2055 \times (1 - .0005) = 46.1824$. The exchange difference is

$$(46.1824 - 45.7500) \times 1{,}000{,}000 = \text{Rs. } 432{,}400$$

[4] The Foreign Exchange Dealers' Association of India (FEDAI), has specified that exchange margins need not be factored in to the calculations, when a contract is effectively cancelled and rebooked at the same time.

which is in the customer's favour. Hence the bank will pay 432,400 – 100 = Rs. 432,300 to the client.

Extension Before the Due Date Let us assume that on 1 September, 2008, WIPRO had booked a contract with IDBI bank to buy 1 MM USD on 30 November at 46.8500 INR/USD. On 30 October, the company suddenly seeks permission to extend the contract until 30 December, 2001. What will the bank do?

The bank will first cancel the original contract. The procedure is the same as that described above, for cancelling contracts prior to expiration. The exchange difference will be

$$(46.2090 - 46.8500) \times 1{,}000{,}000 = \text{Rs. } (641{,}000)$$

which in this case is payable by the client. Thus the total amount payable by the client will be Rs. 641,100. A fresh two month forward contract will then be booked at the prevailing inter-bank rate.

Cancellation after the Due Date If a forward contract remains due without any instructions from the client, then the bank is required to automatically cancel it on the 15th day from the date of maturity. However, before the 15th day, the customer may request that the contract be cancelled. Whether at the customer's request or on its own, the bank will adopt the following procedure for cancellation.

Let us assume that WIPRO's 3 month contract to buy 1MM USD has expired on 30 November. On 7 December, the company requests the bank to cancel the forward contract. What will the bank do?

We will assume that the delivery price as per the original contract is 46.8500. On 1 September when the contract was entered into, the bank would have covered itself by booking a forward contract to buy 1MM USD, on the inter-bank market, on 30 November. Let us assume that the applicable rate was 46.7750.

On 30 November, when it finds that the client is not going through with the contract, the bank will have to sell 1MM USD in the spot market. Consider the following rates for 30 November, on the inter-bank market.

Spot: 46.2500/46.7500

1 Month Forward: 45/85

Hence the bank will have to sell at 46.2500. Now, since the client has not cancelled the original contract, the bank has an outstanding position on its books, or in other words it still has a commitment to sell to the client. Hence, to cover itself, the bank may buy an option forward at the prevailing rate for 1 month contracts, which is 46.7585.

On 7 December, when the client requests that the contract be cancelled, the bank will compute the charges payable as follows. We will assume that the following rates prevail on 7 December.

Spot: 46.6500/47.1500

1 Month Forward: 75/115

If the bank charges an exchange margin of .05%, then the contract will be cancelled at $46.6500 \times (1 - .0005) = 46.6267$. The exchange difference is

$$(46.6267 - 46.8500) \times 1{,}000{,}000 = \text{Rs. } (223{,}300)$$

which is payable by the client. Had the exchange difference been in favour of the client, the bank would not have been required to pay, since the contract was overdue.

On 30 November, the bank had to do a swap by selling spot at 46.2500 and buying forward at 46.7585. Hence there is a swap loss of

$$(46.2500 - 46.7585) \times 1{,}000{,}000 = \text{Rs. } (508{,}500)$$

which is payable by the client. Once again, had there been a swap gain, the client would not have been entitled to it.

On 30 November, the bank had to buy 1MM USD at 46.7750 and sell it at 46.2500. Thus, there is a cash outlay of

$$(46.2500 - 46.7750) \times 1{,}000{,}000 = \text{Rs. } (525{,}000).$$

The bank will charge interest on this amount from 30 November till 7 December, at a rate that we will assume is 15% per annum. Thus the interest payable by the client is

$$525{,}000 \times .15 \times \frac{1}{52} = \text{Rs. } 1514.4231.$$

Finally, as usual, the client has to pay a flat fee of Rs. 100. Hence, the total amount payable by the client is

$$223{,}300 + 508{,}500 + 1514.4231 + 100 = \text{Rs. } 733{,}414.4231.$$

Let us examine the rationale behind this. We will ignore exchange margins for the purpose of our illustration.

Assume that WIPRO had originally contracted to buy from the bank on 30 November at a rate E. The bank would have covered itself by buying forward at a rate, which we will denote by F^*. If the company had taken delivery as per schedule, then the bank would have had an inflow of $(E - F^*)$ on 30 November.

Now, if the company fails to take delivery on 30 November and instead cancels the contract on 7 December, then the sequence of cash flows from the standpoint of the bank can be computed as follows. Let the rates on 30 November be:

Spot: $S_{b,0}/S_{a,0}$

1 Month Forward: $F_{b,0}/F_{a,0}$

Assume that the spot rates on 7 December are $S_{b,1}/S_{a,1}$. On 30 November, the bank will take delivery under the original forward contract at F^* and will sell spot at $S_{b,0}$. Thus its net inflow $= S_{b,0} - F^*$. On 7 December, it will collect the cancellation charges from the client, which we will denote by X. It will also take delivery under the forward contract that it had booked on 30 November, when it realized that the client was not taking delivery and will sell spot. Thus, its net inflow on 7 December will be

$$X + (S_{b,1} - F_{a,0})$$

The cancellation amount payable by the client should be such that

$$(E - F^*) = (S_{b,0} - F^*) + \frac{X}{1+r} + \frac{(S_{b,1} - F_{a,0})}{1+r}$$

$$\Rightarrow X = (E - S_{b,1}) + (F_{a,0} - S_{b,0}) + r(E - S_{b,0})$$

The first term on the RHS represents the exchange difference collected by the bank. The second term, is the swap loss that is recovered by the bank. Strictly speaking, the bank should charge interest on the difference between E and $S_{b,0}$, which in our example is (46.85 − 46.25). However, in practice, it charges interest on the difference between F^* and $S_{b,0}$, which in our example is (46.7750 − 46.25).

Execution after the Due Date Let us suppose that on 7 December, WIPRO makes a request for the contract to be executed. What will the bank do?

The bank will first cancel the existing contract and recover Rs. 733,414.4231 from WIPRO. The contract will then be executed at the merchant selling rate on 7 December, which is

$$47.15 \times (1.0005) = 47.1736$$

assuming an exchange margin of .05%.

Extension after the Due Date Finally, let us consider the case where WIPRO makes a request to the bank on 7 December to extend the contract till 7 January. If so, the bank will proceed as follows.

It will first cancel the original contract and collect Rs. 733,414.4231 from WIPRO. It will then sell 1MM USD one month forward at

$$47.1615 \times (1.0005) = 47.1851.$$

17.6 Clearing and Settlement on the NSE

We have four categories of exchange members in India. These are:

- Trading Members (TMs)
- Trading Members-Clearing Members (TM-CMs)
- Self Clearing Members (SCMs)
- Professional Clearing Members (PCMs)

Trading Members or TMs are the equivalent of non-clearing FCMs. They can open accounts on behalf of their clients, but in turn, must clear through a clearing member. Trading Members Clearing Members or TM-CMs, are the equivalent of clearing FCMs. They can accept and clear trades on behalf of their clients, and can also clear trades routed through them by other TMs. Professional Clearing Members or PCMs are FCMs, whose only function is to clear and settle transactions routed through them by other TMs. Finally, there is a category of brokers called Self Clearing Members (SCMs), who can clear trades on behalf of their clients, but are not authorized to have trades routed through them by other trading members for the purpose of clearing and settlement.

Professional Clearing Members are required to have a minimum net worth of Rs 300 lakh or 3 crore, as are Trading Members-Clearing Members. Trading Members and Self Clearing Members are required to have a minimum net worth of Rs 100 lakh or 1 crore.

The net worth of a member is calculated as the sum of his capital and free reserves less certain non-allowable assets.

Every clearing member is also required to maintain a base minimum capital of Rs 50 lakh with the National Securities Clearing Corporation Limited (NSCCL). Of this, Rs 25 lakh must be in the form of cash. The balance Rs 25 lakh may be in or more of the following forms:

- Cash
- Fixed Deposit Receipts issued by approved banks and deposited with approved custodians or the NSCCL
- Bank Guarantee in favour of the NSCCL from approved banks
- Approved securities in demat form deposited with approved custodians

In addition, for every TM whose trades a clearing member undertakes to clear and settle, an amount of Rs 10 lakh has to be maintained as Base Minimum Capital. Of this, Rs 2 lakh should be in the form of cash and the balance Rs 8 lakh in one or more of the forms described above.

All open futures positions are marked to market at the end of the day. Any losses incurred by a party must be paid in from 11:30 a.m. onwards on the following business day. Pay-outs to parties who have made a profit will be made from 12.00 p.m. onwards on the following business day.

The profits/losses due to marking to market are computed by taking the difference between:

1. The trade price and the settlement price for contracts executed during the day which have not been offset
2. The previous day's settlement price and the current settlement price for contracts which have been brought forward and which have not been offset
3. The previous day's settlement price and the trade price for contracts which have been brought forward and offset during the day
4. The buy price and the sell price for contracts which have been executed during the day and which have been offset

Here are two detailed illustrations.

17.6.1 Illustration 1

Sandeep Srivastava had a long position in 250 futures contracts as of the end of yesterday. Yesterday's settlement price was Rs. 122. Each contract is for 50 units of the underlying asset. Today Sandeep went long in an additional 125 contracts at a trade price of Rs 125.75 of these contracts were subsequently offset at a price of Rs. 127.50. Today's settlement price is Rs. 130.

The profit/loss due to marking to market may be computed as follows.

Profit/loss on account of contracts carried over from the previous day and which have not been offset is:

$$250 \times 50 \times (130 - 122) = \text{Rs. } 100{,}000$$

Profit/loss on account of contracts entered into during the day, and which have been offset is:

$$75 \times 50 \times (127.50 - 125) = \text{Rs. } 9{,}375$$

Profit/loss on account of contracts entered into during the day, and which have not been offset:

$$50 \times 50 \times (130 - 125) = \text{Rs. } 12{,}500$$

Total inflow = 100,000 + 9,375 + 12,500 = Rs. 121,875

17.6.2 Illustration 2

Shefali Talwar had a long position in 200 futures contracts as of yesterday. Yesterday' settlement price was Rs. 114. Each contract is for 50 units of the underlying asset. Today she went long in an additional 100 contracts at a trade price of Rs 118. 125 contracts were subsequently offset at a price of Rs 122.50. Today's settlement price is Rs. 126.

The profit/loss due to marking to market may be computed as follows.

Profit/loss on account of contracts carried over from the previous day and which have not been offset is:

$$175 \times 50 \times (126 - 114) = \text{Rs. } 105{,}000$$

Profit/loss on account of contracts entered into during the day, and which have been offset is:

$$100 \times 50 \times (122.50 - 118) = \text{Rs. } 22{,}500$$

Profit/loss on account of contracts carried over from the previous day, and which have been offset:

$$25 \times 50 \times (122.50 - 114) = \text{Rs. } 10{,}625$$

Total inflow = 105,000 + 22,000 + 10,625 = Rs. 138,125

In order to calculate the settlement price for the purpose of marking to market the weighted average of futures prices observed during the last half hour of trading is taken. There could, however, be contracts which are illiquid or in other words trade infrequently. For such contracts a theoretical settlement price is computed according to the formula $F = Se^{rT}$, where S is the value of the underlying asset in the spot market, T is the time remaining to expiration, and r is the current value of MIBOR (the Mumbai Inter Bank Offer Rate).

On the expiration date, the open positions will be marked to market for the last time. On this day, the settlement price is set equal to the weighted average of the spot prices observed during the last half hour of trading. This procedure is legitimate since we know that spot and futures prices must converge at the time of expiration.

It must be noted that the concept of a threshold level up to which mark to market losses can be tolerated, or in other words the concept of a Maintenance Margin, is not prevalent in India. Consequently, all losses, however small, must be settled on the business day following the trade date.

17.6.3 Settlement of Options

Stock options have three types of settlements:

- Daily Premium Settlement
- Interim Exercise Settlement
- Final Exercise Settlement

For index options, since they are European in nature, there is no possibility of interim exercise.

Premium Settlement Long positions in options entail the payment of a premium while short positions will lead to inflows of premia. The premium payable amount and the premium receivable amount are netted across all options contracts for each clearing member at the client level, in order to determine the net amount payable/receivable by each client. The clearing member is responsible for collecting and settling premium amounts from their clients as well as other trading members who have chosen to clear through them.

Pay-in is on the business day following the trade day, at or after 11.30 a.m. Pay-out is at or after 12.00 p.m. on the same day.

Interim Exercise Settlement If and when an in-the-money stock option contract is exercised prior to expiration, settlement is effected at the close of trading on the day of exercise. The settlement value is the difference between the strike price of the option and the settlement price.

To ensure that the option is exercised on the same day, the buyer must direct his trading member to exercise before the cut-off time for accepting exercise instructions for that day. Usually the system will accept such orders till the close of trading. However, different trading members may have different cut-off times for accepting such instructions from their clients, and the timings may vary across contracts. Exercise notices given on a particular day, are processed by the NSCCL after the close of trading on that day, irrespective of the time of submission of the instruction.

The settlement price is the weighted average price of the underlying security in the last half-hour of trading on that day. Pay-in is at or after 11.30 a.m. on the following business day, and pay-out is at or after 12.00 p.m. on that day.

Final Exercise Settlement Settlement is automatically effected for in-the-money contracts which are open at the close of trading on the expiration day. That is, buyers are not required to give an exercise notice. Long positions are automatically assigned to short positions belonging to the same series, on a random basis. The settlement price is the weighted average spot price in the last half-hour of trading on the expiration day of the option. Pay-in is at or after 11.30 a.m. on the following business day, and pay-out is at or after 12.00 p.m. on that day.

17.6.4 Assignment

Exercise notices are assigned in standardized market lots to short positions in the same option series, that is, contracts with the same underlying asset, expiry date, and strike price. Assignments are done at the client level on a random basis. NSCCL determines those positions which are eligible to be assigned and allocates the exercised contracts to one or more short positions. Contracts are assigned at the end of the day on which the exercise instructions are received by the NSCCL. However, a client with a short position may not receive notification to that effect until the following day.

17.6.5 Clearing Banks

Since profits and losses are to be credited or debited on a daily basis, the clearing members must obviously have some kind of an approved bank account which is accessible to the clearing corporation. The National Securities Clearing Corporation Limited (NSCCL) offers settlement of funds through the following 13 clearing banks.

1. Axis Bank
2. Bank of India
3. Canara Bank
4. Citibank N.A.
5. HDFC Bank
6. HSBC
7. ICICI Bank
8. IDBI Bank
9. IndusInd Bank
10. Kotak Mahindra Bank
11. Standard Chartered Bank
12. State Bank of India
13. Union Bank of India

Every clearing member must open a clearing account with one of these banks for the purpose of settlement of transactions. These accounts are to be used exclusively for clearing operations, that is, for settling of funds and obligations to the NSCCL including payments of margins and penalty charges. In the case of clearing members who are also members of the Capital Market Segment of the NSE, a separate clearing account has to be maintained for derivative transactions.

All clearing members are required to authorize their respective clearing banks to access their accounts for debiting and crediting their accounts, and for reporting of balances and other information as may be required by the NSCCL. The clearing banks are required to debit/credit the clearing account of the members as per the instructions received from the NSCCL.

17.7 Margining

NSCCL has a sophisticated risk management system for the derivatives market. It facilitates online position monitoring and margining, on an intra-day basis. The system is known as *Parallel Risk Management System* or PRISM. NSCCL uses SPAN for the purpose of computing initial margins. The volatility of the underlying asset, which is required as an input for SPAN, is determined using an exponential moving average method.

17.7.1 The Exponential Moving Average Method

In order to calculate the 99% VaR, an estimate of the standard deviation of returns or the volatility has to be first computed. According to the exponential moving average technique, the volatility at the end of day t is given by

$$\sigma_t{}^2 = \lambda\sigma_{t-1}{}^2 + (1-\lambda)r_t{}^2 \tag{17.1}$$

The Varma committee recommended a value of 0.94 for λ.

The return on day t, r_t, is defined as

$$r_t = \ln\left(\frac{I_t}{I_{t-1}}\right) \tag{17.2}$$

where I_t is the index futures price at the end of day t.

17.7.2 Client Level Margining

The National Stock Exchange follows a system of Gross or Client Level margining, as opposed to the Net or Broker Level margining systems that are used by many international exchanges.

As far as a clearing member is concerned, he is first required to classify a trade as either *Proprietary* (that is, on his own account), or as *Client* (that is, induced by a client). For proprietary trades, the open position is calculated as the total long position less the total short position. From the stand point of the clients, while the long and short positions for a particular client can be netted out in order to arrive at a net position, netting across clients is not permissible.

Similarly as far as the trades of a trading member who is clearing through this clearing member are concerned, the trading member's proprietary trades will be netted, and the net position for each client of the trading member will be taken into account, once again without netting across the clients of the trading member.

We will illustrate these principles with the help of an example.

Shadab Khan, a TM-CM, has placed the following orders on his own behalf, as well as on behalf of two of his clients, Suhasini Singh and Harita Reddy. In addition, Abraham Jacob, a TM, is also clearing his trades through Shadab.

Shadab's proprietary trades are follows.

Table 17.10 **Shadab's Proprietary Orders**

Buy Orders		Sell Orders	
Price	Quantity	Price	Quantity
1020	600	1035	200

So from the standpoint of proprietary trades, Shadab has a net long position of 600 – 200 = 400 contracts.

Suhasini's and Harita's trades can be summarized as shown in Table 17.11.

Table 17.11 **Shadab's Client Orders**

	Buy Orders		Sell Orders	
Client	Price	Quantity	Price	Quantity
Suhasini	1030	400	1015	200
Harita	1010	400	1025	800

Suhasini has a net long position of 400 – 200 = 200 contracts.

Harita has a net short position of 800 – 400 = 400 contracts.

Notice that we cannot net between Suhasini and Harita since we are following a Client Level system and not a Broker Level system.

Now assume that Abraham Jacob's proprietary trades are as follows.

Table 17.12 **Abraham's Proprietary Orders**

Buy Orders		Sell Orders	
Price	Quantity	Price	Quantity
1020	600	1035	1200

So the net proprietary position for Abraham is short 600 contracts.

Also assume that two clients, Preeti Raghu and Sangeeta Shetty trade through Abraham. Their positions are as follows (Table 17.13).

So Preeti has a net short position of 600 contracts, while Sangeeta has a net long position of 600 contracts.

Table 17.13 **Abraham's Client Orders**

	Buy Orders		Sell Orders	
Client	Price	Quantity	Price	Quantity
Preeti	1030	1400	1015	2000
Sangeeta	1010	2400	1025	1800

For the purpose of computing the initial margin, Shadab's total position will be determined as follows.

Long positions = 400 (his own) + 200 (Suhasini's) + 600 (Sangeeta's) = 1200 contracts.

Short positions = 400 (Harita's) + 600 (Abraham's proprietary trades) + 600 (Preeti's) = 1600 contracts.

Thus Shadab will be margined for 1200 long positions and 1600 short positions.

Initial margins have to be paid up front but need not be in cash. They could be in the form of cash, bank guarantees, fixed deposit receipts, or approved securities.

Initial margins are computed on a real time basis, that is, after every trade. During the day, if the required margin is such that it causes a member's liquid net worth to dip below Rs 50 lakh at any point in time, it is considered a violation

At the end of each trading day, the required initial margin is once again computed. If the required amount exceeds what is currently being maintained by the member, then the shortfall is collected on the next business day.

17.7.3 Types of Margins

Initial Margins These are based on a 99% value at risk over a one-day horizon. In the case of futures contracts, where it may not be possible to collect mark to market dues before the commencement of trading on the following day, the initial margin may be computed over a two-day horizon.

Premium Margins In addition to the initial margin, premium margin is levied at the client level and is payable by buyers of options till the premium settlement is completed.

Assignment Margins This is levied in addition to the initial and premium margins. Every CM has to pay the margin on the positions that have been assigned to him as a part of interim and final exercise settlement. The margin is applied on the net exercise settlement value payable by a CM towards interim and final exercise settlement.

17.7.4 Liquid Net Worth

The purpose of collecting initial margins is to prevent default. However the VaR technique that is used for computational purposes is merely a statistical tool. There is always a possibility of a price move, however remote, that will lead to a loss in excess of the Initial Margin that is collected from an investor.

It is for this reason that the Varma Committee mandated that a member ought to have a minimum liquid net worth at all times. This will serve as additional collateral in the event of losses exceeding the Initial Margin.

The Varma Committee defined liquid net worth as

- the total liquid assets deposited with the exchange/clearing corporation towards Initial Margin and capital adequacy

LESS

- the Initial Margin applicable to the total gross open positions at any given point in time of all trades cleared through the member

The Committee recommended that the member's liquid net worth ought to satisfy the following two conditions on a real-time basis:

- The liquid net worth of a member shall not be less than Rs 50 lakh at any point in time

AND

- the mark to market value of gross open positions at any point in time, of all trades cleared through the clearing member, shall not exceed $33\frac{1}{3}$ times the member's liquid net worth, for index futures, and 20 times the member's liquid net worth for stock futures.

For the purpose of computation, the committee specified that liquid assets shall be taken to include cash, fixed deposit receipts, bank guarantees, T-bills, government securities, and other approved dematerialized equity securities, pledged in favour of the exchange/clearing corporation.

Furthermore, 50% of the total liquid assets, as stipulated by the Committee, ought to be in the form of cash equivalents, where the term 'cash equivalents', includes cash, fixed deposit receipts, bank guarantees, T-bills, and government securities.

We will illustrate the applicability of the liquid net worth criterion with the help of an illustration, using stock index futures contracts.

Kamal Lohia, a clearing member, has deposited Rs 50 lakh in the form of cash and cash equivalents with the clearing corporation. In addition he has deposited dematerialized approved equity securities worth Rs 60 lakh.

So the total value of the liquid assets is Rs 1.1 crore. But remember, there is a stipulation that at least 50% of the amount shall be in the form of cash equivalents. So Lohia will be given credit for only Rs 1 crore, that is Rs 50 lakh $\times 2$.

Assume that Lohia has a proprietary net long position of 200 contracts, and that he has one client Punit, who has a net short position of 200 contracts. The current futures price is 1000 and each contract is for 200 units of the underlying asset. We will take the initial margin percentage for both long and short positions to be 5%.

So the amount of initial margin required to be maintained by Lohia is

$$200 \times 200 \times 1,000 \times 0.05 + 200 \times 200 \times 1,000 \times 0.05 = \text{Rs. } 40 \text{ lakh}$$

Thus his current liquid net worth is Rs 100 lakh - Rs 40 lakh = Rs 60 lakh, which is greater than the minimum requirement of Rs 50 lakh.

There is however an additional requirement that the mark to market value of the open positions shall not exceed $33\frac{1}{3}$ times the net worth.

The mark to market value of the open positions is

$$200 \times 200 \times 1,000 + 200 \times 200 \times 1,000 = \text{Rs. 8 crore}$$

$33\frac{1}{3}$ times the liquid net worth of Rs 60 lakh is Rs 20 crore, which is clearly more than the figure of Rs 8 crore. So Lohia is not violating the capital adequacy norms.

Stock Futures and Options The exposure limit for derivatives on individual securities is the higher of 5% or 1.5 standard deviations of the notional value of gross open positions for stock futures, and gross short option positions for stock options. This amount is adjusted from the liquid net worth on a real time basis.

For futures contracts the notional value is the contract value at the last traded price. For options contracts, it is the value of the underlying shares based on the last available price.

17.8 Risk Management on the BSE

The BSE also uses an online position monitoring and margining system. The exchange uses SPAN for margin computation.

For index futures, the price scan range is taken to be three standard deviations, where the standard deviation used is the daily volatility of the underlying Index or Index Futures, whichever is higher. Contracts are marked to market on a daily basis. The margin is paid-in/paid-out on the next business day. Final settlement is also done in cash on the business day following the last trading day.

For index options the price scan range is taken as 3σ, where σ is the daily volatility of the underlying index. The volatility scan range is taken as 4%. The Black–Scholes model is used for computing option prices.

For stock futures, the price scan range is taken as 3.5σ where σ is the daily volatility of the underlying stock. Contracts are marked-to-market on a daily basis. The margin is paid-in/paid-out on the next business day. For payment of mark-to-market margin to BSE, the requirements are netted out at the Member level. Final settlement is also done in cash on the business day following the last trading day.

For stock options, the price scan range is taken as 3.5σ, where σ is the daily volatility of the respective stock. The volatility scan range is taken as 10%. The Black–Scholes model is used to value the options.

17.8.1 Second Line of Defense

In terms of the liquid net worth requirements, the conditions for stock and index derivatives are the same as those applicable for trading on the NSE.

Suggestions for Further Reading

1. Apte, P.G. *International Financial Management*. McGraw-Hill, 2008.
2. Jeevanandam, C. *A Brief Course on Foreign Exchange Arithmetic*. Sultan Chand & Sons, 2008.

Web Sites

1. www.bseindia.com
2. www.nse-india.com

Concept Check

1. India has two securities depositories.
2. Index options that are traded on Indian exchanges are American in nature.
3. Stock options that are traded on Indian exchanges are American in nature.
4. Stock futures and options that are traded on Indian exchanges are currently cash settled.
5. The indices underlying index derivatives on the NSE are value weighted.
6. All exchange traded derivatives in India, except for weekly options on the BSE, expire on the last Thursday of the expiration month.
7. Weekly options on the BSE have a maturity of one week.
8. Currently currency futures on Indian exchanges are available only on the U.S. dollar.
9. At any point in time, currency futures are available for 3 delivery months.
10. The tick size for equity derivatives in India is 5 paise.
11. The tick size for currency derivatives in India is 2.5 paise.
12. In the case of purchase transactions for a bank, the inter-bank buying rate is the base rate.
13. To make a profit in a purchase transaction, the base rate must be lower than the merchant rate.
14. In the event of modification of a forward contract, interest on cash outlays is recoverable by the bank at a rate that is not lower than its PLR.
15. In the case of early delivery under a forward sale contract, the swap entails a spot purchase and a forward sale.

16. There are three categories of clearing members in India.
17. The settlement price for futures contracts is the weighted average of prices observed during the last half-hour of trading.
18. Clearing members who are also a part of the Capital Market segment of the NSE, must maintain a separate clearing account for derivative transactions.
19. Both BSE and NSE use SPAN for margining derivative contracts.
20. A member should have a minimum liquid net worth of Rs. 50 lakh at any point in time.

QUESTIONS AND PROBLEMS

Question-I

Write a brief note on the recommendations of the L.C. Gupta Committee.

Question-II

Write a brief note on the recommendations of the J.R. Varma Committee.

Question-III

Discuss the differences between TM-CMs, PCMs, and SCMs.

Question-IV

An exporter had entered into a forward contract on 1 September, 2000 to sell 1,00,000 USD on 1 December, 2000, at a rate of 44.65. However, he subsequently has got paid early and so delivers the dollars on 1 October. The inter-bank rates on 1 October are as follows.

Spot: 44.10/44.45 INR/USD

2 Month Forward Margin: 25/10

The bank charges a flat fee of Rs. 100 on the transaction and an annual interest rate of 20% on cash outlays. Discuss the transactions involved in this case and calculate the amount payable/receivable by the party.

Question-V

IDBI Bank Mumbai is quoting the following rates.

Spot: 48.20/48.80 INR/USD

1 Month Forward: 20/10

2 Month Forward: 60/90

The exchange margin on both purchase and sale transactions is .05%.

Calculate the merchant buying and selling rates for 1 Month and 2 Month forward contracts.

Appendix–XVII-A

Constituents of the S&P CNX NIFTY
As on 30 March 2009

ABB	ACC	Ambuja Cements	Axis Bank
BHEL	BPCL	Bharti Airtel	Cairn India
CIPLA	DLF	GAIL	Grasim
HCL	HDFC Bank	Hero Honda	HINDALCO
Hindustan Unilever	HDFC	ITC	ICICI Bank
Idea Cellular	INFOSYS	L&T	Mahindra & Mahindra
Maruti Suzuki	NTPC	NALCO	ONGC
Power Grid Corporation	Punjab National Bank	Ranbaxy	Reliance Capital
Reliance Communications	Reliance Industries	Reliance Infrastructure	Reliance Petroleum
Reliance Power	Siemens	SBI	SAIL
Sterlite Industries	Sun Pharmaceuticals	Suzlon	Tata Communications
TCS	Tata Motors	Tata Power	Tata Steel
UNITECH	WIPRO		

Appendix–XVII-B

Constituents of the CNX NIFTY Junior
As on 30 March 2009

Adani Enterprises	Aditya Birla Nuvo	Andhra Bank	Apollo Tyres
Ashok Leyland	Asian Paints	Bank of Baroda	Bank of India
Bharat Electronics	Bharat Forge	BIOCON	Canara Bank
Chennai Petroleum	Container Corporation of India	Corporation Bank	Cummins
Dr. Reddy's Laboratories	GMR Infrastructure	Glaxosmithkline	Glenmark Pharmaceuticals
Housing Development and Infrastructure	IDBI Bank	IFCI	Indian Hotels
Indian Overseas Bank	IDFC	JSW Steel	Jaiprakash Associates
Jindal Steel & Power	Kotak Mahindra Bank	LIC Housing Finance	LUPIN
MRPL	Moser Baer	MphasiS	Mundra Port and SEZ
Oracle	Patni Computers	Power Finance Corporation	Raymond
Reliance Natural Resources	Sesa Goa	Syndicate Bank Teleservices	Tata
Tech Mahindra	UltraTech Cement	Union Bank	United Spirits
Vijaya Bank	Wockhardt		

Appendix–XVII-C

Constituents of the NIFTY MIDCAP 50 As on 30 March 2009			
Allahabad Bank Ashok Leyland Birla Corporation	Alstom Projects Aurobindo Pharma Bombay Dyeing	Amtek Auto Ltd. BEML CESC	Andhra Bank Bajaj Hindusthan Chennai Petroleum Corporation
Corporation Bank	Divi's Laboratories	Edelweiss Capital	GVK Power and Infrastructure
Great Eastern Shipping	Hindustan Construction	Hotel Leelaventure	IVRCL Infrastructures and Projects
India Cements	Indian Hotels	Kesoram Industries	Lanco Infratech
Lupin	Maharashtra Seamless	Mahindra Lifespace Developers	Moser Baer
MphasiS	Nagarjuna Construction	Patel Engineering	Peninsula Land
Petronet LNG	Praj Industries	Punj Lloyd	Reliance Natural Resources
Rolta	Shipping Corporation of India	Sintex Industries	Sterling Biotech
Syndicate Bank Tata Teleservices	TVS Motor Titan	Tata Chemicals Vijaya Bank	Tata Tea Voltas
Welspun Gujarat Stahl Rohren	Wockhardt		

Appendix–XVII-D

Constituents of the CNX IT Index
As on 30 March 2009

CMC	Core Projects and Technologies	Educomp Solutions	Financial Technologies
First Source Solutions	GTL	HCL Infosystems	HCL Technologies
Hexaware Technologies	Infosys Technologies	MindTree	Moser Baer
MphasiS	Oracle	Patni Computers	Polaris
Rolta	TCS	Tech Mahindra	WIPRO

Appendix–XVII-E

Constituents of the CNX Bank Index

As on 30 March 2009

Axis Bank	Bank of Baroda	Bank of India	Canara Bank
HDFC Bank	ICICI Bank	IDBI Bank	Kotak Mahindra Bank
Oriental Bank of Commerce	Punjab National Bank	SBI	Union Bank of India

Appendix–XVII-F

Constituents of the SENSEX
As on 31 March 2009

ACC	BHEL	Bharti Airtel	DLF
Grasim Industries	HDFC	HDFC Bank	Hindalco
Hindustan Unilever	ICICI Bank	Infosys	ITC
Jaiprakash Associates	L&T	Mahindra & Mahindra	Maruti Suzuki
NTPC	ONGC	Ranbaxy	Reliance Communications
Reliance Industries	Reliance Infrastructure	SBI	Sterlite
Sun Pharmaceuticals	TCS	Tata Motors	Tata Power
Tata Steel	Wipro		

Appendix–XVII-G

Constituents of the BSE TECK Index
As on 31 March 2009

Adlabs Films	Aptech	Balaji Telefilms	Bharti Airtel
Deccan Chronicle	Dish TV	Financial Technologies	GTL
HCL Technologies	Himachal Futuristic	HT Media	IBN18 Broadcast
Idea Cellular	Infosys	IOL Netcom	Jagran Prakashan
MTNL	Moser Baer	MphasiS	NDTV
NIIT	Oracle	Patni Computers	Reliance Communications
Rolta	Sun TV	Tanla Solutions	Tata Communications
TCS	Tata Teleservices	Tech Mahindra	Television Eighteen
UTV Software Communications	Wipro	Wire and Wireless	Zee Entertainment
Zee News			

Appendix–XVII-H

Constituents of the BSE BANKEX
As on 31 March 2009

Allahabad Bank	AXIS Bank	Bank of Baroda	Bank of India
Canara Bank	Federal Bank	HDFC Bank	ICICI Bank
IDBI Bank	Indian Overseas Bank	Indusind Bank	Karnataka Bank
Kotak Mahindra Bank	Oriental Bank of Commerce	Punjab National Bank	SBI
Union Bank	Yes Bank		

Appendix–XVII-I

Constituents of the BSE Oil & Gas Index As on 31 March 2009	
Cairn India	Aban Offshore
BPCL	Essar Oil
Gail	HPCL
Indian Oil	ONGC
Reliance Industries	Reliance Natural Resources
Reliance Petroleum	

Appendix–XVII-J

Constituents of the BSE Metal Index
As on 4 April 2009

Jai Corp	Gujarat NRE Coke
Hindalco Industries	Hindustan Zinc
Ispat Industries	Jindal Saw
Jindal Steel & Powers	JSW Steel
National Aluminium Co.	NMDC
Sesa Goa	Steel Authority of India
Sterlite Industries	Tata Steel
Welspun Gujarat Stahl Rohren	

Appendix–XVII-K

Constituents of the BSE FMCG Index As on 4 April 2009	
Ruchi Soya Industries	Britannia Industries
Colgate Palmolive	Dabur India
Godrej Consumer Products	Hindustan Unilever
ITC	Marico
Nestle India	Tata Tea
United Breweries	United Spirits

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