Heat and
Mass Transfer

About the Author

D K Dixit was formerly on the faculty of Mechanical Engineering at Visvesvaraya National Institute of Technology, Nagpur, and has more than 40 years of industrial, teaching, and research experience. He has taught almost all subjects, and guided undergraduate and postgraduate mechanical engineering students in projects on heat power engineering. Besides an accomplished teacher and a prolific writer, he is well versed in diversely different fields like journalism, management, astrology, yoga, and spiritualism. Besides engineering, Prof. Dixit has a bachelor's degree in Journalism, a master's degree in Communication and Journalism, a Sahitya Ratna felicitation (Hindi), a Sahityacharya title (Sanskrit), and a few more.

Dr. Dixit has travelled worldwide and visited several countries including the USA, the UK, the Philippines, the Netherlands, Luxemburg, Belgium, Singapore, Hong Kong, Thailand, etc. He has also published and presented more than 150 research papers both nationally and internationally. He has published more than 2000 articles in English, Hindi, and Marathi in almost all leading newspapers and magazines. He has organized many workshops and seminars and was the convener of the All India Seminar on Energy Management organized by the Institution of Engineers (India). Prof. Dixit has been frequently invited by the Akashvani and Doordarshan for topical talks and scientific programmes.

Based on his doctoral research at IIT Bombay, he was awarded the prestigious Best Paper Prize by the Institution of Engineers (India), and he is also the recipient of the K F Antia Award. He also won the Nagpur Times gold medal and Austen Wingate Nazareth Memorial Prize for journalism. Prof. Dixit is a Life Fellow of the Institution of Engineers, a Life Member of the International Solar Energy Society, a Life Member of the Indian Society for Heat and Mass Transfer, and a Life Member of the Indian Society for Technical Education. He has been Visiting Faculty at many institutes like the Department of Business Management, Nagpur University; Bhavan's College of Communication and Management, Department of Mass Communication, Nagpur University; National Power Training Institute, Nagpur, etc. His areas of interest and specialization are Energy Management and Renewable Energy Sources, Refrigeration and Air Conditioning, Thermal Engineering, Heat Transfer, and Marketing Management. He is also the author of a digital book *Heat and Mass Transfer: Problems and Solutions*, published by McGraw Hill Education (India).

Heat and Mass Transfer

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Contents

Preface

Heat Transfer, along with Thermodynamics and Fluid Mechanics, constitutes the *trinity* of the Mechanical Engineering discipline.

It is an important and compulsory subject in the engineering curricula of almost all institutes and universities. The present book is aimed to provide precisely a sound grounding of fundamentals which is the essential prerequisite to master this core course. The target audience is essentially the undergraduate students of Mechanical Engineering, Chemical Engineering, and Aerospace Engineering. The book is also tailored to meet the requirements of candidates aspiring to take IES, GATE, AMIE, and other competitive examinations. Written in a racy and user-friendly style, the book is the quintessence of the author's teaching and research experience of more than thirty-five years.

The USP of the book is clear, coherent, and cogent presentation of the relevant theory followed by a large number of solved examples with different degrees of difficulty. It has a refreshing approach with unique problem-solving methodology coupled with an emphasis on consistency and compatibility of units, so sadly neglected by the student fraternity.

The unsolved problems will be useful in gaining confidence acquired with constant practice. The numerous *objective-type questions, glossary of key terms,* and *points to ponder* along with *review questions* at the end of each chapter will certainly enhance the usefulness of the book.

Salient Features

- Covers the standard topics of heat transfer with an emphasis on physics and real-world everyday applications
- Both theoretical and mathematical derivations covered for all topics
- Balanced physical explanation and mathematical treatment
- Excellent problem-solving approach
- Over 650 illustrations for better conception—excellent-quality figures including 3D views wherever required to aid in better understanding of concepts
- Tutorial approach for solving all examples for creative thinking and development of a deeper understanding
- Excellent pedagogy including
  - *324 Solved Examples*
  - *232 Unsolved Problems*
  - *276 Review Questions*
  - *242 Multiple-Choice Questions*

Chapter Organization

The presentation in this introductory text is logically organized in 15 chapters with a 360 degree approach to provide an overall understanding of the subject and to serve as a springboard for further learning and research. **Chapter 1** introduces the fundamental concepts involving the three cardinal modes of heat transfer, viz., conduction, convection, and radiation, and mass transfer. **Chapter 2** begins with the discussion and derivation of the general three-dimensional conduction equation and subsequently analyzes at length the systems involving steady-state, one-dimensional conduction without heat generation. Practical problems pertaining to conduction with internal heat generation are dealt within **Chapter 3**. Extended surfaces and their applications constitute the subject matter of **Chapter 4**. **Chapters 5 through 9** are devoted to the elegant discussion of convection followed by on-forced convection (both external and internal flow) and free convection, both in depth and detail basics using the latest empirical correlations. **Chapter 10** illustrates and explains boiling and condensation. The vital topic of heat exchangers is described and analyzed fairly exhaustively in **Chapter 11. Chapters 12** and **13** are devoted to the discussion of properties and processes relevant to radiation mode of heat transfer followed by analytical and graphical treatment of radiant heat exchange between bodies. Finally, rudimentary aspects of both diffusion and convective mass transfer are taken up for a short but sharp discussion in **Chapter 14.** Multidimensional conduction is briefly touched upon in **Chapter 15** and is available on the book website.

Online Learning Center

The Online Learning Center can be accessed at http://www.mhhe.com/dixit/hmt1e and contains the following resources.

- *For Instructors*: Lecture PPTs, Solution Manual
- *For Students*: Chapter on *Multi-dimensional Heat Conduction*

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I deeply appreciate the keen interest and involvement of Ms Harsha Singh and Ms Vaishali Thapliyal of McGrawHill Education (India) for ensuring expeditious publication of the book.

I hope that this book will be well received and appreciated by students and teachers alike.

Feedback Request

Despite every possible care, if any errors might have crept in inadvertently, kindly bring them to my notice. Also, please feel free to suggest any changes for further improvement of the text in the next edition.

Dhirendra Kumar Dixit

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1 Basic Concepts of Heat Transfer

1.1 ❑ INTRODUCTION

It is our common observation and experience that when two bodies at different temperatures are brought together, the temperature of the warmer body decreases while the temperature of the colder body increases. When the energy transfer is the result of only the temperature difference without any work interaction, such an energy transfer is referred to as heat transfer. *Heat transfer* is, thus, energy in transit due to temperature difference. Heat transfer can occur within a system, or between two systems. The science of heat transfer identifies the factors which influence the heat interaction between solids and fluids or their combinations. This information is then used to predict the temperature distribution and the rate of heat transfer in engineering systems and devices.

Basically, heat transfer can take place in *three* different modes: *conduction, convection*, and *radiation.* In all cases, heat is transferred from the high-temperature medium to the low-temperature medium. It is very rarely that these modes of heat transfer take place in isolation in a given application. This chapter presents an overview of the study of heat transfer, including multimode heat transfer. The detailed discussion of each mode of heat transfer and the related applications will be dealt with in the following chapters.

1.2 ❑ THE SIGNIFICANCE OF HEAT TRANSFER

The fascinating field of heat transfer embraces almost every sphere of human activity. Heat-transfer phenomena play an important role in several industrial and environmental problems in *aeronautical, chemical, civil, electrical, metallurgical* and, of course, *mechanical engineering.* There is not a single area of application in energy-production-and-conversion systems that does not involve heat-transfer effects. In the generation of electrical power—be it through the combustion of fossil fuels like coal, oil, and gas, the nuclear fission or fusion, the use of geothermal energy sources, the Magneto-HydroDynamic (MHD) processes, or the exploitation of solar or wind energy, etc.—numerous heat-transfer problems are encountered.

The optimal design of components like boilers, turbines, condensers, heat-recovery equipment, radiators, refrigerators, and other heat exchangers is essential not only to determine the technical feasibility but also the economic viability. If the size of heat-transfer equipment is to be kept less to cut down cost, or if the space is at a premium, one needs to *maximize* heat-transfer rates.

In nuclear power plants, accurate determination of heat-transfer rates is essential in view of the problem of efficient removal of heat generated in the reactor core. Adequate cooling of electronic components is a must to preclude the possibility of overheating, and to ensure satisfactory performance. With the current trend of miniaturization in the electronic industry, this problem of faster heat dissipation from the limited surface area available has become rather critical. Heat-transfer analysis is also necessary in the design of electrical machines, transformers, bearings, etc., to avoid conditions likely to cause overheating and damage to the equipment.

To ensure successful operation and to maintain the integrity of materials in high-temperature environment of certain components like gas-turbine blades or the combustion chamber wall, the rapid rate of continuous removal of heat from vulnerable metal parts is necessary. Heat treatment of various metals and ceramics, design of catalytic converters, heat engines, cryogenic storage equipment, cooling towers, desert coolers, refrigeration and air-conditioning systems, jet- and rocket-propulsion systems, solar-energy collection and storage devices, thermal pollution associated with the discharge of a large quantity of waste heat into the environment (*air* and *water*), and dispersion of atmospheric pollutants—all require a thorough understanding of heat-transfer processes and analyses.

The so-called *thermal barrier* in aerodynamics involves exploring means of transferring away from the aircraft huge amounts of heat produced by the dissipative effect of the viscosity of the air. Indeed, since all observable processes in nature are *irreversible,* the attendant *dissipative effects* eventually manifest themselves as heat-transfer processes.

An engineer must, therefore, appreciate the crucial significance of the complex heat-transfer problems. The solution lies in complete understanding of and familiarity with the different modes and laws of heat transfer. One has to bank upon one's ingenuity and experience while making sound assumptions, approximations and idealizations in the course of heat-transfer analysis and in the interpretation of the final results.

1.3 ❑ HEAT TRANSFER AND ITS RELATION TO THERMODYNAMICS

Thermodynamics deals with matter and energy interaction. And energy is central to the very human existence. Energy interaction is basically of two types: *work interaction* and *heat interaction.* Heat interaction involves heat flow or exchange of heat. The flow of heat is all pervasive. In the study of thermodynamics, heat is defined as the energy in transition resulting from temperature difference. Heat interaction, like work interaction, is a *transient* energy-transfer process across the boundary between the system and the surroundings.

It should be remembered that the existence of *temperature difference* is a characteristic feature of the energy form known as *heat.* If there is no temperature difference, there is no heat transfer. Moreover, since the term *heat* is used to describe a transfer process, the heat energy ceases to exist when the process ceases. Thus, heat is *not* a property. Heat transfer being strictly a phenomenon occurring only at boundaries of systems, heat transfer elsewhere in a system is more correctly a re-distribution of internal energy within the system.

Apart from the physical entities of heat and work, the energies in transit, a system can possess external kinetic and potential energies due to motion or position of the working fluid or the system as a whole, as well as internal energy due to kinetic and potential energies of the molecules comprising the system. *The First Law of Thermodynamics* for a *closed* system can be expressed as

$$\boxed{Q = W + \Delta E} \tag{1.1}$$

where $\Delta E \equiv \Delta U + \Delta \mathrm{PE} + \Delta \mathrm{KE}$

Usually, the changes in potential and kinetic energies (ΔPE and ΔKE) are negligible as compared to the change in internal energy (ΔU) in a *closed* system. One can thus simply write

$$\boxed{Q = W + \Delta U} \tag{1.2}$$

For an *open* system in a steady-flow process, the first law can be expressed as

$$\boxed{\dot{Q} - \dot{W} = \dot{m}\Delta\left[\underbrace{h}_{\text{enthalpy}} + \underbrace{V^2/2}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{gz}_{\substack{\text{potential}\\ \text{energy}}}\right]} \tag{1.3}$$

Neglecting changes in potential and kinetic energy,

$$\boxed{\dot{Q} - \dot{W} = \dot{m}\Delta h} \tag{1.4}$$

From a thermodynamic standpoint, the amount of heat transferred (*positive towards the system*) during a process simply equals the work transfer (*positive away from the system*) and the change in internal energy (*positive when the system's energy increases*). It is obvious that thermodynamic analysis considers neither the *mechanism* of heat flow nor the *time* required to transfer heat. Thus, in analyzing a power cycle, we are concerned with gross transfer of heat and work *to* or *from* the system and the resulting thermal efficiency. The time or temperature difference required to bring about the transfer of heat energy, or whether there is a uniform temperature within the system is not considered of any consequence. *But in heat transfer, our focus is on the time rate at which heat is transported or transferred across a specific temperature difference.*

In thermodynamics, we deal with thermodynamic equilibrium including (besides mechanical and chemical equilibria) *thermal equilibrium* between the end states of a process, but in heat transfer, it is the *thermal non-equilibrium* (temperature differential or gradient) that is of interest to us. The knowledge of temperature distribution in a system is, in fact, essential in several heat-transfer studies. The point to point out here is that *heat transfer is inherently a non-equilibrium process.*

The *Second Law of Thermodynamics* enables us to determine the *extent* or *magnitude* of energy conversion from heat into work as also the *direction* of a heat-transfer process (from a hotter to a colder body). Even the limit to the amount of work that can be obtained from a given source of heat cannot be reached in practical engineering processes, thanks to their inherent irreversibilities. These *irreversibilities* may be accounted for in the analysis or calculations but, thermodynamics alone, for want of time scale, cannot permit determination of physical sizes of devices or equipment necessary to accomplish a specified objective.

1.4 ❑ BASIC MODES OF HEAT TRANSFER

It is customary to classify or categorize the various heat-transfer processes into three basic types or modes, namely, *conduction*, *convection*, and *radiation.* These basic heat-transfer mechanisms may occur *separately*, or *simultaneously.* It should be emphasized at this stage that there is hardly any problem of practical significance which does not involve at least two, and sometimes all the three modes occurring at the same time. The three modes, or mechanisms, are inherently different and though they often operate in combination, they may be studied separately. The type of system and temperature levels dictates and decides their relative contributions. Figure 1.1 illustrates the mechanisms of conduction, convection, and radiation.

Now we proceed to take a closer look at each of the three modes of heat transfer in some detail.

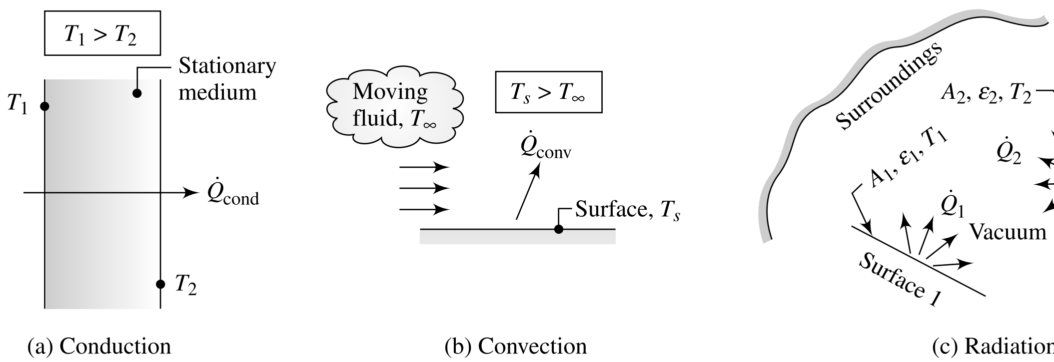


Fig. 1.1 *Schematic diagrams showing the three basic modes of heat transfer*

1.5 ❑ CONDUCTION

Conduction is the transfer of energy from the more energetic particles of a substance to the neighbouring less energetic ones as a result of interactions between the particles. Conduction can take place in any *stationary* medium—*solid*, *liquid*, or *gas*.

When a *temperature gradient* exists as a *driving force* in a *stationary medium (a solid or a fluid),* causing the heat to be transferred across the medium in the direction of the lower temperature in an attempt to establish *thermal equilibrium,* conduction heat transfer occurs. When the *temperature gradient* does not vary with *time,* the process is said to be in a *steady state. Transient*, or *unsteady*, heat transfer involves change in the temperature gradient with time.

Conduction involves the transfer of heat by *direct physical contact* on a molecular scale. It is the only mode by which heat can be transported within opaque solids. Conduction also occurs in liquids and gases but is usually associated with convection, and possibly with radiation too in the case of gases.

The thermal conductivity of a solid is obtained by adding the *lattice* and the *electronic* components. It is noteworthy that the thermal conductivity of *pure metals* is primarily due to the *electronic* component, whereas the thermal conductivity of *non-metals* is essentially due to the *lattice* component. The *lattice* component of thermal conductivity strongly depends upon the way the molecules are arranged. For instance, the thermal conductivity of diamond, which is a highly ordered crystalline solid, is very high compared to the thermal conductivities of even pure metals.

The basic empirical law governing heat conduction established by the French scientist Joseph Fourier in 1822, states that in *steady state, the rate of heat transfer in a given direction per unit area (normal to that flow direction), i.e., the heat flux is proportional to the temperature gradient in that direction.* Figure 1.2 shows the heat transfer by conduction through a plane wall in the *x*-direction.

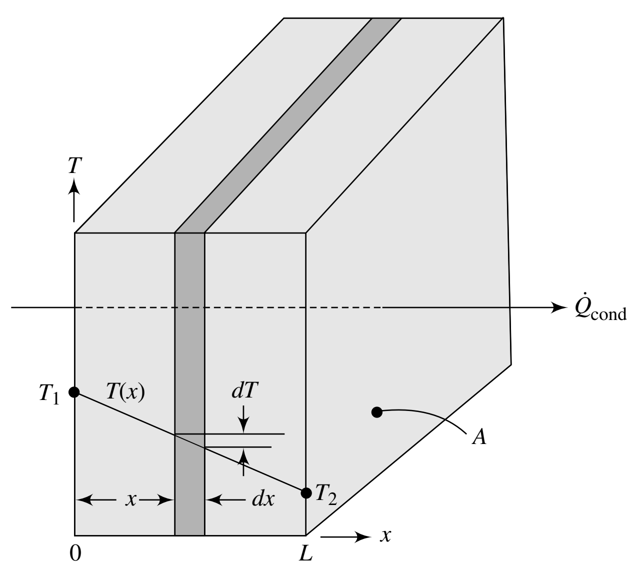


Fig. 1.2 *Conduction heat transfer through a plane wall*

In the x-direction, for example, this *Fourier law* can be mathematically expressed as the rate equation:

$$\boxed{\dot{Q}_x = -kA\frac{dT}{dx}} \quad \text{(W)} \tag{1.5}$$

or

$$q_x = \frac{\dot{Q}_x}{A} = -k\frac{dT}{dx} \tag{1.6}$$

where $\dot{Q}_x$ = the rate of heat flow (W) in the *positive* x-direction

k = the proportionality constant, called the thermal conductivity of the material and is a positive quantity (W/m K or W/m °C)

A = the area of the section perpendicular to the direction of heat flow (m^2)

$\frac{dT}{dx}$ = the temperature gradient at the section, i.e., the rate of change of the temperature T with respect to distance in the direction of heat flow (°C/m or K/m)

The *negative sign* results from the convention of defining a **positive heat flow** in the direction of a *negative temperature gradient.*

In a *unidirectional, steady-state* conduction heat transfer, the heat flux, q, across an infinite slab of thickness, L, with temperature difference between the hot and cold walls, ΔT, can be expressed as

$$\boxed{\frac{\dot{Q}}{A} = q = k\frac{\Delta T}{L}} \quad (\text{W/m}^2) \tag{1.7}$$

1.5.1 • Electrical Analogy

Using the *electrical analogy,* the temperature difference causing heat flow is analogous to the potential difference causing current flow through an electrical circuit. When the potential difference across a resistance becomes zero, the flow of current ceases. Similarly, in thermal equilibrium, when in the equivalent thermal circuit the temperature difference is zero, there is no heat transfer. Furthermore, the higher the electrical resistance, the lower the current flow. By the same token, the greater the equivalent thermal resistance, the lesser would be the heat transfer rate. The reciprocal of thermal resistance is known as *thermal conductance C*, akin to electrical conductance. In conduction heat transfer, thermal resistance, $\boxed{R_{\text{th, cond}} = L/kA.}$

1.5.2 • Thermal Conductivity

The thermal conductivity, *k,* is an extremely important *transport property* of a material or medium. Its value largely determines the suitability of the material for a given application. One should study and remember the order of magnitude of the thermal conductivities of different types of materials. This will help in making suitable assumptions during problem solving. There is a wide difference in the range of thermal conductivities of various engineering materials as illustrated in Table 1.1. It may be stressed here that thermal conductivity of a material depends on the following: (a) the chemical composition of the substance, (b) the phase (i.e., solid, liquid, or gas), (c) the temperature, (d) the pressure, and (e) the direction of heat flow.

In respect of thermal conductivity of materials, the following behavioural characteristics are significant.

- Thermal conductivity of pure metals decreases with temperature.
- Even small amounts of impurities reverse the above process.
- Most liquids have thermal conductivity decreasing with temperature.

Table 1.1 *Average values of thermal conductivities of some materials at 20°C*

Materials	Name of Material	Thermal Conductivity (W/m °C)
Solid metals	Silver (pure) Copper (pure) Aluminium (pure) Brass Carbon steel (0.5% C) Carbon steel (1% C) Stainless steel (18% Cr, 8% Ni)	419 386 204 111 54 43 16.3
Non-metals	Window glass Asbestos cement board Building brick Asbestos Glass wall Plastics Sawdust Wood Wallboard, paper	0.78 0.74 0.69 0.23 0.038 0.58 0.58 0.17 0.076
Liquids	Water Lubricating (Engine) oil R-12 (Freon-12)	0.60 0.145 0.073
Gases	Dry air (1 atm) Saturated steam (1 atm) R-12 (Freon-12)	0.026 0.025 0.009

- Thermal conductivity of gases increases with temperature, but decreases with increasing molecular weight (see Table 1.2).
- Except under very high pressures, thermal conductivity is *not* affected by pressure.

For most materials, the variation of thermal conductivity with temperature is almost linear,

$$\boxed{k = k_0(1 + \beta T)} \tag{1.8}$$

where k_0 is the thermal conductivity at 0°C, and β is a constant whose value depends upon the material. This constant may be *positive* or *negative* depending on whether thermal conductivity *increases* or

Table 1.2 *Thermal conductivities of several gases at atmospheric pressure and 0°C*

S. No.	Gas	Thermal Conductivity (W/m K)	Molecular Weight (kg/kmol)
1.	Hydrogen (H_2)	0.1652	2
2.	Helium (He)	0.1416	4
3.	Methane (CH_4)	0.03042	16
4.	Nitrogen (N_2)	0.02384	28
5.	Air	0.02364	28.97
6.	Argon (Ar)	0.0164	40
7.	Carbon dioxide (CO_2)	0.01456	44

decreases with temperature. The coefficient β is usually *positive* for *non-metals* and *insulating materials* (exception is magnesite bricks) and *negative* for *metallic conductors* (exceptions are aluminium and certain non-ferrous alloys).

The value of the thermal conductivity increases with temperature for gases while it tends to decrease with temperature for most of the liquids, water being a notable exception.

The thermal conductivity of a solid is more than that of a liquid which in turn is greater than that of a gas. A few materials like wood have *directional dependence.* Wood has one value of k along the grain while a different value across it. Such materials are called *anisotropic.* For most solids and liquids, pressure does not affect the value of their thermal conductivities. By and large, most materials are assumed to be *homogeneous* and *isotropic.*

1.6 ❑ CONVECTION

Convection is the mode of energy transfer between the solid surface and the liquid or gas in motion in contact with it. It involves the combined effects of *conduction* and advection (*bulk fluid motion*). The faster the fluid motion, the greater is the convection heat transfer. In the absence of any bulk fluid motion, heat transfer between a solid surface and the adjacent fluid is by pure conduction. The presence of bulk motion of the fluid increases the heat transfer between the solid surface and the fluid, but it also makes the determination of heat-transfer rate a tedious task.

Consider the cooling of a hot plate by blowing of cold air over its top surface. Energy is first transferred to the air layer in the vicinity of the surface of the block by conduction. This energy is then carried away from the surface by convection; that is, by the combined effects of conduction within the air, which is due to random motion of air molecules, and the bulk or macroscopic motion of the air, which removes the heated air near the surface and replaces it by the cooler air.

Convection is called *forced convection* if the fluid is *forced* to flow in a tube (*internal flow*) or over a surface by external means such as a fan, pump or the wind (*external flow*). In contrast, convection is called *free (or natural) convection* if the fluid motion is caused by buoyancy forces induced by density differences due to the variation of temperature in the fluid. For example, in the absence of a fan, heat transfer from the surface of the hot plate will be by natural convection since any motion in the air in this case will be due to the rise of warmer (*and, thus, lighter*) air near the surface and the fall of the colder (*and, thus, heavier*) air to fill its place. Heat transfer between the plate and the surrounding air will be by conduction if the temperature difference between the air and the plate is not large enough to overcome resistance of air to move and, thus, to initiate natural convection currents.

Heat-transfer processes that involve *change of phase* of a fluid are also considered to be convection because of the fluid motion induced during the process such as the rise of the vapour bubbles during *boiling* or the fall of the liquid droplets during *condensation.*

The rate of heat transfer by convection $\dot{Q}_{\text{conv}}$ is determined from *Newton's law of cooling*, expressed as

$$\boxed{\dot{Q}_{\text{conv}} = hA(T_s - T_\infty)} \quad \text{(W)} \tag{1.9}$$

where, the fluid temperature equals the surface temperature of the solid

$\dot{Q}_{\text{conv}}$ = the rate of convective heat transfer (W)

h = the average convective or film heat-transfer coefficient, or unit surface conductance (W/m^{2}K or W/m^2°C)

A = the heat transfer surface area (m^2)

T_s = the surface temperature (°C or K)

T_∞ = the temperature of the undisturbed fluid (usually far away from the surface, i.e., free stream)

It may be noted that the convection heat-transfer coefficient h is not a property of the fluid. It is an experimentally determined parameter whose value depends on all the variables that influence convection such as the *surface geometry*, *the nature of fluid motion, the properties of the fluid, and the bulk fluid velocity*.

It must be kept in mind that the basic energy exchange at the *solid-fluid boundary* is by conduction and this energy is then convected away by the fluid flow. Thus, one can write

$$\boxed{hA(T_s - T_\infty) = -kA\left.\frac{\partial T}{\partial y}\right|_{y=0}} \tag{1.10}$$

where k is the thermal conductivity of the *fluid* and $\left.\frac{\partial T}{\partial y}\right|_{y=0}$ is the temperature gradient in the fluid at the *solid-fluid interface* as shown in Fig. 1.3.

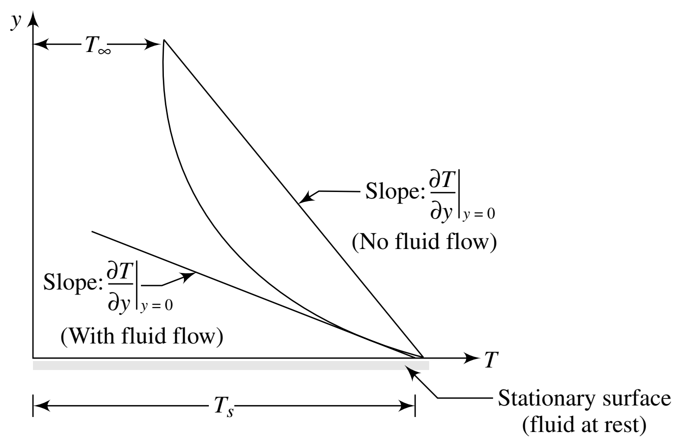


Fig. 1.3 *Slope of temperature profile in fluid at the surface with and without fluid flow.*

Incidentally, h is *not* a property (unlike thermal conductivity) but a function of many parameters encompassing *fluid properties like viscosity, fluid velocity, temperature difference, geometric configuration,* etc. The main problem in the analysis of convective heat transfer is to accurately predict the value of h for design purposes. The typical values of the *convective heat transfer coefficient* are given in Table 1.3 in order to get a feel for figures.

Table 1.3 *Order of magnitude of convective heat-transfer coefficient*

Mechanism and Medium	Connection Coefficient h (W/m^2 °C)
Free (natural) convection: Air Water	5–25 50–300
Forced convection: Air Water	15–250 100–5000
Boiling (water)	2000–50 000
Condensation (steam)	2000–50 000

Using the *electrical analogy,* the convective *thermal resistance,* $\boxed{R_{\text{th, conv}} = 1/hA}$

1.7 ❑ RADIATION

The third fundamental mode of heat transfer, known as radiation, is qualitatively different from the first two. While conduction and convection occur within a solid or fluid material and often simultaneously, radiation does not require any material medium for energy transfer. Radiant exchange between surfaces is, in fact, maximum when no material occupies the intervening space.

The energy we receive from the sun cannot come to us through either conduction or convection except radiation because in the vast region between the sun and the earth, there is no medium. Even if the separating medium like air is present, it remains unaffected by the passage of radiant energy.

Radiant energy is thus transmitted most freely and fast (at the speed of light) in a *vacuum.* It occurs between all material phases. All bodies emit radiant energy continuously by virtue of their temperature in the form of electromagnetic waves. When this energy falls on a second body, it may be partially *absorbed*, *reflected*, or *transmitted.* It is only the absorbed energy that heats the second body. Radiative heat transfer becomes increasingly important as the temperature of the emitting body increases.

All matter at temperatures above absolute zero emit electromagnetic waves of different wavelengths. Visible light together with infra-red and ultraviolet radiation constitutes only a small part of the total electromagnetic spectrum.

The quantity of energy leaving a surface as radiant heat is dependent upon the nature of the surface and its absolute temperature. There will be a continuous energy exchange between two radiating surfaces with a net radiant interchange from the hotter to the colder surface. *Even in the case of thermal equilibrium, the energy exchange does not stop, though the net exchange will be zero.*

The basic law of radiation heat transfer is expressed as the *Stefan–Boltzmann equation:*

$$\boxed{\dot{Q}_{\max} = \sigma A_s T_s^4} \quad \text{(W)} \tag{1.11}$$

$\dot{Q}_{\max}$ = the maximum rate of radiation emitted by a surface (ideal) (W)
A_s = the surface area of a perfectly radiating body (m^2)
T_s = the absolute surface temperature (K)
σ = the constant of proportionality, called the *Stefan–Boltzmann* constant (its numerical value is 5.67×10^{-8} W/m^2 K^4.

It is significant that *radiation heat transfer* is proportional to the *fourth* power of the *absolute* temperature of the surface (K).

At any given temperature, a perfectly radiating body, or the most efficient radiator, called a black body, emits the maximum possible energy at all wavelengths and in all directions.

It should be borne in mind that Eq. (1.15) defines an energy *emission* rather than energy *exchange*.

The radiant heat energy emitted by a real surface is obviously less than that by a black body and is given by

$$\boxed{\dot{Q} = \sigma \varepsilon A_s T^4} \quad \text{(W)} \tag{1.12}$$

where ε is a radiative property of the surface called *emissivity*. Obviously, an ideal emitter has $\varepsilon = 1$. Determination of the net rate at which radiation heat exchange takes place between surfaces is fairly complicated.

In general, the *net radiant energy interchange* between two *real non-black* bodies at different temperatures depends on many factors: their *surface properties*, their *geometry* and their *orientation* with each other.

A fairly common practical situation is the net rate of radiation heat transfer between a relatively small surface of area A_s, emissivity ε and absolute temperature T_s placed in a much larger enclosure (surroundings) at temperature T_{sur} (the emissivity and surface area of the surrounding surfaces do not matter). The intervening medium is usually air. In such a case, we can write

$$\boxed{\dot{Q}_{\text{net, rad}} = \sigma A_s \varepsilon (T_s^4 - T_{\text{sur}}^4)} \quad \text{(W)} \tag{1.13}$$

Often when the temperature difference between a surface and its surroundings is small, a *radiation heat-transfer coefficient, h_r* is defined as follows:

$$\boxed{\dot{Q}_{\text{net}} = h_r A_s (T_s - T_{\text{sur}})} \quad \text{(W)} \tag{1.14}$$

This equation resembles *Newton's law of cooling* in convection heat transfer.
From Eqs (1.13) and (1.14), we have

$$\boxed{h_r = \sigma\varepsilon (T_s^2 + T_{\text{sur}}^2)(T_s + T_{\text{sur}})} \tag{1.15}$$

By *electrical analogy*, radiative thermal resistance, $\boxed{R_{\text{th, rad}} = \dfrac{1}{h_r A_s} = \dfrac{1}{C_{\text{rad}}}}$

1.8 ❑ ENERGY BALANCE FOR A CONTROL VOLUME

In order to solve many problems of heat transfer, the first law of thermodynamics (*the law of energy conservation*) provides a meaningful tool. One has to identify first the control volume, a region of space bounded by a control surface through which matter and energy can pass. A general form of the energy conservation requirement may then be expressed on a rate basis as follows:

The rate at which thermal and mechanical energy enters a control volume, minus the rate at which thermal and mechanical energy leaves the control volume plus the rate at which thermal energy is generated within the control volume, must equal the rate of increase of energy stored within the control volume.

Let us consider applying energy conservation to a control volume shown in Fig. 1.4. We identify the control surface (boundary) by a dashed line. Then we identify the energy terms, the rate at which thermal and mechanical energy enter and leave through the control surface, $\dot{E}_{\text{in}}$ and $\dot{E}_{\text{out}}$. Also, heat (thermal energy) generation within the control volume is included and the rate at which it occurs is denoted by $\dot{E}_{\text{gen}}$. The rate of change of energy stored within the control volume, dE_{st}/dt, is designated as $\dot{E}_{\text{st}}$.

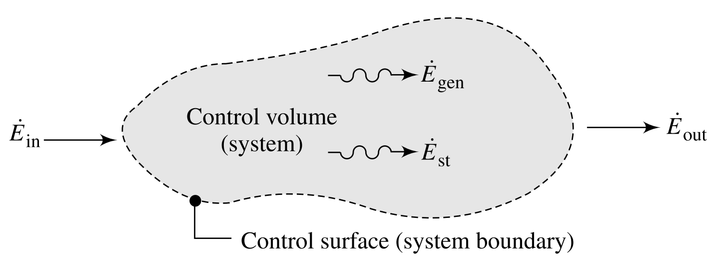


Fig. 1.4 *Control volume energy balance*

$$\boxed{\dot{E}_{\text{in}} + \dot{E}_{\text{out}} + \dot{E}_{\text{gen}} = \frac{dE_{\text{st}}}{dt} \equiv \dot{E}_{\text{st}}} \quad \text{(W)} \tag{1.16}$$

$$\boxed{E_{\text{in}} - E_{\text{out}} + \dot{E}_{\text{gen}} = \Delta E_{\text{st}}} \quad \text{(J)} \tag{1.17}$$

$$\boxed{\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_{\text{gen}} = 0} \quad \text{(steady state)} \quad \text{(W)} \tag{1.18}$$

$$\boxed{\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0} \quad \text{(steady state, no heat generation)} \quad \text{(W)} \tag{1.19}$$

If the inflow and generation of energy exceed the outflow, there will be an increase in the amount of energy stored (*accumulated*) in the control volume; if the converse is true, there will be a *decrease* in energy storage. If the inflow and generation of energy equal the outflow, a steady-state condition must prevail in which there will be no change in the amount of energy stored in the control volume.

It is important to note that the inflow and outflow terms are *surface phenomena*, occurring at the *control surface* and are proportional to the surface area. A common situation involves energy inflow and outflow due to heat transfer by *conduction*, *convection*, and/or *radiation*. The inflow and outflow terms may also include work transfer occurring at the system boundaries. Essentially, $\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}}$ by conduction and/or convection and/or radiation.

The internal *heat generation* means conversion from some other form of energy (*chemical, electrical, electromagnetic,* or *nuclear*) to thermal energy. It is a *volumetric phenomenon*. That is, it occurs within the control volume and is proportional to the magnitude of this volume. For example, in an *exothermic* chemical reaction, heat is evolved (*generated*). If the heat is generated uniformly at the rate of $\overline{q}(\text{W/m}^3)$, $\dot{E}_{\text{gen}} = \overline{q} \times$ Volume.

If heat is absorbed as in an *endothermic* reaction, $\dot{E}_{\text{gen}}$ would be negative.

Energy storage is also a *volumetric phenomenon* as it also depends on the volume. If the density is ρ, volume $\forall$, specific heat C_p increase in temperature ($T_{\text{final}} - T_{\text{initial}}$) or ΔT (as in heating) and the duration (time) Δt, then $\dot{E}_{\text{st}} = \rho \forall C_p \dfrac{\Delta T}{\Delta t}$ or $mC_p dT/dt$. If the final temperature is less (as in cooling) then $\dot{E}_{\text{st}}$ would be negative.

Moreover, latent energy effects involving a phase change are taken care of by the energy-storage term. When phase changes from solid to liquid (*melting*) or from liquid to vapour (*vaporization, evaporation, boiling*), the latent energy increases. Conversely, if the phase change is from vapour to liquid (*condensation*) or from liquid to solid (*solidification, freezing*), the latent energy decreases. Hence, neglecting kinetic- and potential-energy effects, which is almost always the case in heat-transfer analysis, changes in energy storage are contributed by changes in the internal thermal and/or latent energies

$$(\Delta E_{\text{st}} \equiv \Delta U \equiv \Delta U_{\text{sensible}} + \Delta U_{\text{latent}}) \tag{1.20}$$

- **Surface Energy Balance**

Applying energy balance at the system boundary (*control surface*), we have

$$\boxed{\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0 \quad \text{or} \quad \dot{E}_{\text{in}} - \dot{E}_{\text{out}}} \tag{1.21}$$

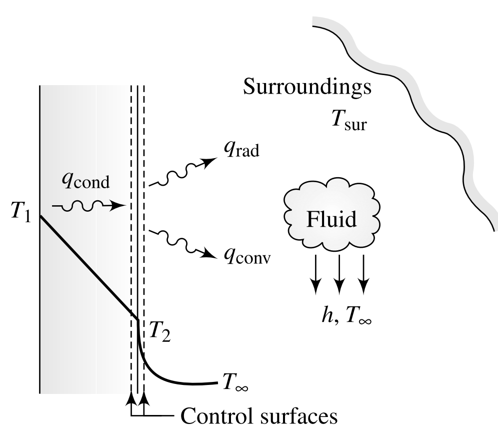


Fig. 1.5 *Control surface energy balance*

This equation is valid for both steady-state and unsteady-state conditions. Moreover, even if there is internal heat generation, it will not affect the control surface energy balance. Referring to Fig. 1.5, one can write

$$\boxed{\dot{Q}_{\text{cond, in}} = \dot{Q}_{\text{conv, out}} + \dot{Q}_{\text{rad, out}}} \tag{1.22}$$

1.9 ❑ COMBINED MODES OF HEAT TRANSFER

So far we have considered the *three* mechanisms of heat transfer separately. In practice, however, heat is usually transferred simultaneously in double mode in gases and liquids. In the case of opaque solids, heat transfer can occur only by conduction.

In the case of evacuated space between two surfaces at different temperatures, heat can be transferred only by radiation. In other cases, with a gas like air and a liquid like water, both single and double modes of heat transfer are possible. Figure 1.6 shows different situations involving single and double modes of heat transfer. Figure 1.7 illustrates simultaneous convection and radiation from a small surface of area A_s and emissivity

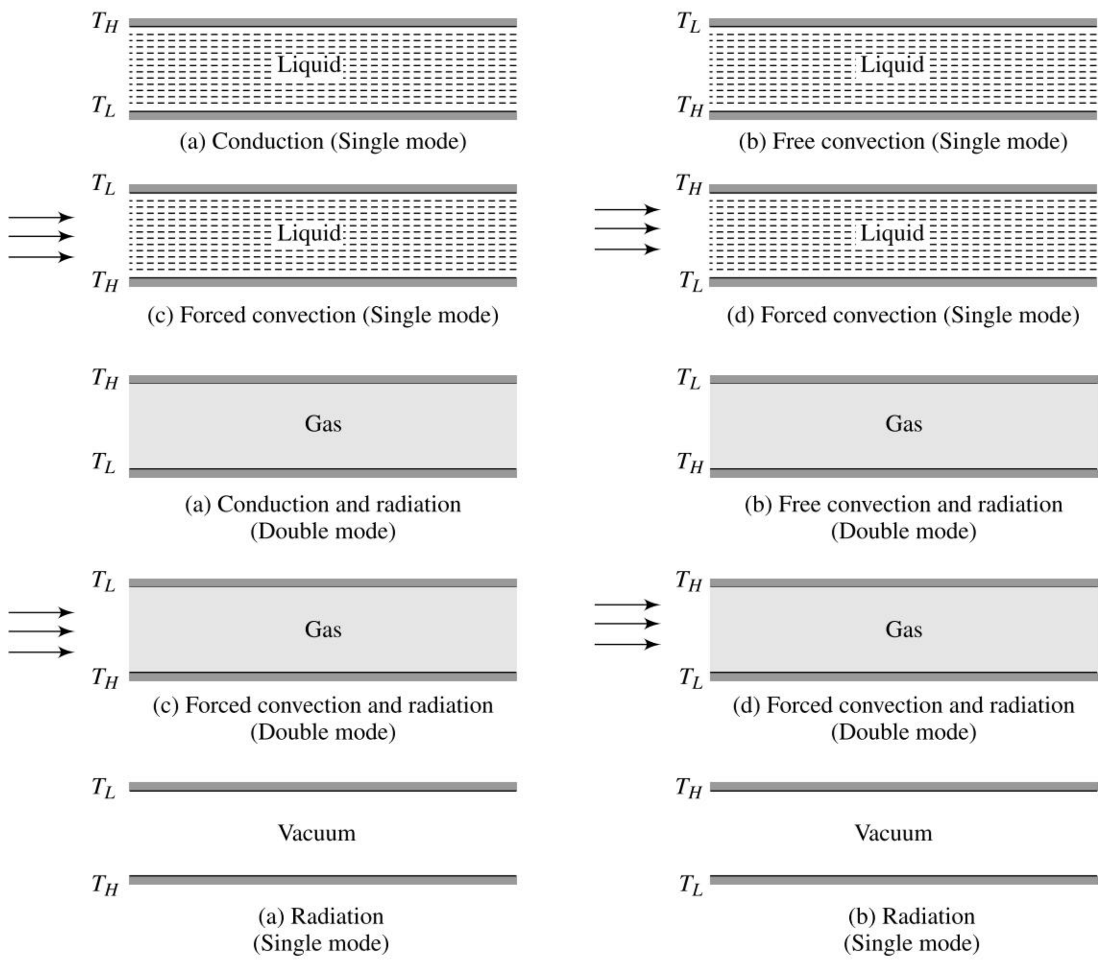


Fig. 1.6 *Single- and double-mode heat transfer in different media*

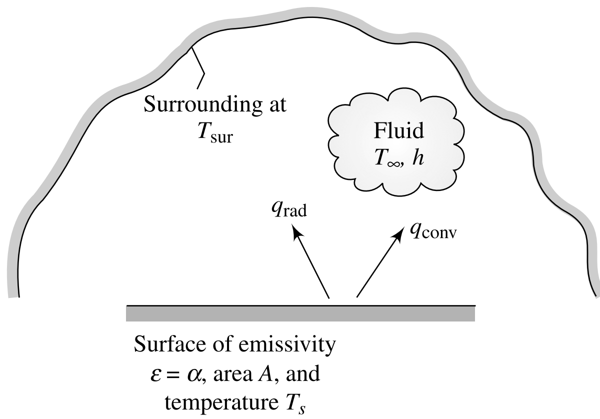


Fig. 1.7 *Combined mode of convection and radiation heat transfer*

ε which is maintained at T_s and exchanges energy by convection with a fluid at T_∞ having heat-transfer coefficient h_c and by radiation with the surroundings at T_{sur}. The heat loss per unit surface area, by the combined mechanism of convection and radiation, is given by

$$\boxed{\frac{\dot{Q}}{A} = q = q_{conv} + q_{rad} = h_c(T_s - T_\infty) + \varepsilon\sigma(T_s^4 - T_{sur}^4)} \tag{1.23}$$

The second term for radiation heat flux in the above equation can be *linearized* if the difference between T_s and T_{sur} is small.

Then
$$q_{rad} = \varepsilon\sigma(T_s^4 - T_{sur}^4) = \varepsilon\sigma(T_s^2 + T_{sur}^2)(T_s + T_{sur})(T_s - T_{sur}) \tag{1.24}$$

$$= 4\varepsilon\sigma\left(\frac{T_s^2 + T_{sur}^2}{2}\right)\left(\frac{T_s + T_{sur}}{2}\right)(T_s - T_{sur}) = 4\varepsilon\sigma T_m^2 T_m (T_s - T_{sur})$$

where
$$T_m \frac{T_s + T_{sur}}{2} = 4\varepsilon\sigma T_m^3 (T_s - T_{sur})$$

$\therefore$
$$\boxed{q_r = h_r(T_s - T_{sur}) \quad \text{where} \quad h_r \equiv 4\varepsilon\sigma T_m^3} \tag{1.25}$$

The orders of magnitude of h_r and h_c are approximately the same in such a case. And if $T_\infty = T_{sur}$, one can write

$$q = q_{conv} + q_{rad} = (h_c + h_r)(T_s - T_\infty) = h(T_s - T_\infty) \tag{1.26}$$

where $h \equiv (h_c + h_r)$, the *combined heat-transfer coefficient.*

1.10 ❑ MASS TRANSFER

Mass transfer is defined as the movement or diffusion of a chemical component (species) in a mixture of liquids or gases with different chemical compositions. Many heat-transfer problems in engineering practice are based on *concentration difference.* In a stationary medium, mass transfer takes place by diffusion from a region of high concentration to a region of low concentration.

The rate equation for diffusion mass transfer is given by *Fick's law* which states that for a binary mixture of species A and B, the diffusion mass flux of the species A is given by

$$\boxed{j_{\text{diff},A} = \frac{\dot{m}_{\text{diff},A}}{A} = -D_{AB}\frac{d\rho_A}{dx}} \quad (\text{kg/m}^2\text{s}) \tag{1.27}$$

where $j_{\text{diff},A} = \dot{m}_{\text{difff},A}/A$, i.e., mass flux of the species A, kg s m^2 in the flow direction x

A = area normal to the direction of flow

ρ_A = concentration of the species A, kg/m^3

$d\rho_A/dx$ = concentration gradient, kg/m^4

D_{AB} = diffusion coefficient or mass diffusivity, m^2/s

Apart from diffusion mass transfer, there is convective mass transfer that involves bulk mass transport. Convective mass transfer occurs on a *macroscopic* scale, while diffusion mass transfer on a *microscopic* level. There are several applications of mass transfer such as *absorption, desorption, distillation, solvent extraction, drying, humidification, sublimation,* etc.

1.11 ❑ PROBLEM-SOLVING METHODOLOGY

A systematic procedure characterized by a specific format clears the cobwebs of confusion and facilitates solution of numerical problems. The methodology employed in this book consists of the following logical steps:

Known Having read the problem carefully and completely, state briefly and concisely the key information given in the problem statement.

Find State in a nutshell the quantities to be determined.

Schematic Draw a neat schematic of the physical system involved. List all the pertinent information (*including units*) on the schematic. Also indicate the relevant heat-transfer processes indicating the directions.

Assumptions List all appropriate simplifying assumptions, approximations and idealizations.

Properties Obtain unknown property values required from property tables (*if not given*) for subsequent calculations. Ensure that properties are evaluated at the correct pressure and temperature. Use linear interpolation, if necessary.

Analysis Put on your thinking cap and examine the system and the processes involved (*open or closed system, steady or unsteady process, etc.*) Apply relevant basic concepts and governing principles (*for example, mass conservation, momentum conservation, energy conservation, force balance, moment balance, etc.*) and introduce rate equations. Simplify them by using the assumptions made. Carry out the analysis as completely as possible before substituting the numerical values for better accuracy in results. Perform the calculations required to get the desired results with due care about the consistency and compatibility of units. Round off the final answer to an appropriate number of significant digits. Do not truncate or round off the intermediate results as far as possible.

Comments Discuss and dissect your results. Verify and justify the validity of the assumptions made. Point out any unusual trend in the results.

Table 1.4 *Summary of heat-transfer processes*

Mode	Mechanism(s)	Rate Equation	Transport Property or Coefficient
Conduction	Diffusion of energy due to random molecular motion	$\dot{Q}_{\text{cond}} = -kA_c \dfrac{dT}{dx}$ (W)	k(W/m K or W/m °C)
Convection	Diffusion of energy due to random molecular motion *plus* energy transfer due to bulk fluid motion (*advection*)	$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty)$ (W)	h(W/m^2 K or W/m^2 °C)
Radiation	Energy transfer by electromagnetic waves	$\dot{Q}_{\text{rad}} = A_s\varepsilon\sigma(T_s^4 - T_{\text{sur}}^4)$ (W) or $\dot{Q}_{\text{rad}} = h_rA_s(T_s - T_{\text{sur}})$ (W)	ε h_r(W/m^2 K or W/m^2 °C)

ILLUSTRATIVE EXAMPLES

(A) Conduction

EXAMPLE 1.1 *An experimental facility is constructed to measure the thermal conductivity of different building materials. The apparatus is designed such that there is one-dimensional, steady-state heat conduction between two isothermal parallel surfaces of the material being tested. A concrete slab measuring 15 cm × 15 cm × 5 cm is placed in the test rig. The two walls, 5 cm apart, are maintained at uniform temperatures of 36°C and 22°C. The heat-transfer rate between the surfaces is 27 kJ/h. Determine the thermal conductivity of the concrete being tested.*

Solution

Known A concrete slab of given dimensions being tested. End surface temperatures and heat rate are specified.

Find k_{concrete} (W/m °C).

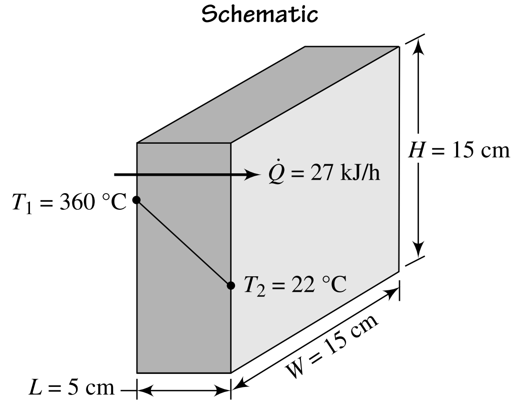
Assumptions (1) Steady state, one-dimensional conduction. (2) Constant thermal properties. (3) End effects are negligible.

Analysis The rate of heat transfer by conduction is

$$\dot{Q} = kA(T_1 - T_2)/L$$

Hence, the thermal conductivity of the concrete being tested is determined to be

Schematic



$$k = \frac{\dot{Q}L}{A(T_1 - T_2)} = \frac{\dot{Q}L}{WH(T_1 - T_2)}$$

$$= \frac{(27\text{ kJ/h})(0.05\text{ m})}{(0.15\text{ m} \times 0.15\text{ m})(36 - 22)°\text{C}}\left|\frac{10^3\text{ J}}{1\text{ kJ}}\right|\left|\frac{1\text{ h}}{3600\text{ s}}\right|\left|\frac{1\text{ W}}{1\text{ J/s}}\right| = \mathbf{1.9\ W/m\ C} \qquad \textbf{(Ans.)}$$

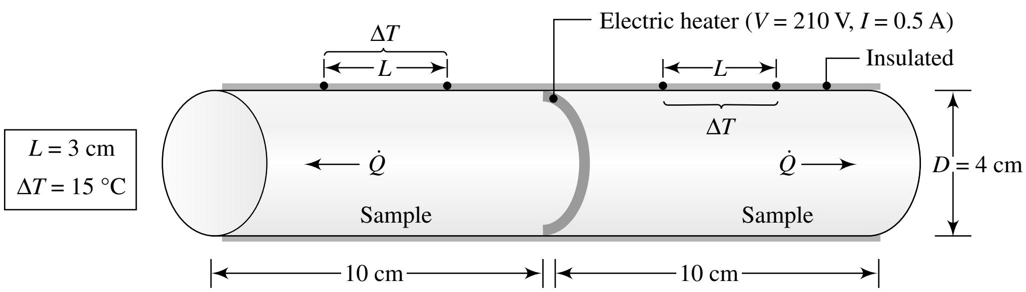
EXAMPLE 1.2 *Two identical cylindrical samples, each 4 cm diameter and 10 cm long, are used in an experiment to measure the thermal conductivity of the sample material. An electric-resistance heater is sandwiched between the two samples for the supply of heat. The curved surface of the samples is effectively insulated. Two thermocouples are placed 3 cm apart in each sample. After initial transients die out, the temperature drop across the distance along each sample is observed to be 15°C. The current and voltage are measured to be 0.5 A and 210 V. Determine the thermal conductivity of the sample material.*

Solution

Known An experiment to measure the thermal conductivity of a sample involves measurement of temperature difference across a certain distance while ensuring one-dimensional heat transfer under steady operating conditions.

Find Thermal conductivity, k (W/m K).

Schematic



Assumptions (1) Steady-state conditions prevail so that temperatures remain constant even with the lapse of time. (2) Thermal symmetry exists. (3) One-dimensional (*axial*) heat conduction since the lateral surface of the cylindrical sample is well insulated.

Analysis The power consumed by the electric heater is converted into heat and is equally divided to supply heat to each identical sample.

Hence, the heat-transfer rate for *each* sample is

$$\dot{Q} = \frac{1}{2}VI = \frac{1}{2}(210\ \text{V})\ (0.5\ \text{A}) = 52.5\ \text{W}$$

For one-dimensional conduction along the axis of the cylindrical sample, one has

$$\dot{Q} = kA(\Delta T/L)$$

where $\Delta T/L$ is the temperature gradient, and A is the area perpendicular to the direction of heat flow, that is, cross-sectional area, $A = \dfrac{\pi D^2}{4}$

With $\dfrac{\Delta T}{L} = \dfrac{15°\text{C}}{3\times10^{-2}\ \text{m}}$ = 500°C/m and $A = \dfrac{\pi}{4}(0.04\ \text{m})^2$ = 1.2566 × 10^{-3} m^2, the thermal conductivity of the sample material is determined to be

$$k = \frac{\dot{Q}}{A(\Delta T/L)} = \frac{52.5\ \text{W}}{(1.2566\times10^{-3}\ \text{m}^2)(500°\text{C/m})}$$

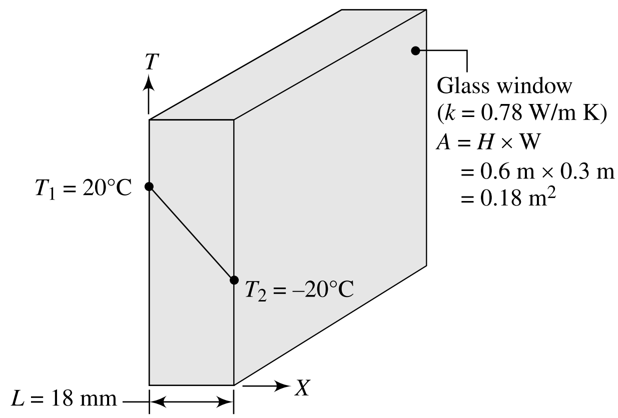
$$= \mathbf{83.5\ W/m\ °C\ (or\ W/m\ K)} \qquad \textbf{(Ans.)}$$

Comment You might have noticed that the area of the sample is the area of cross section, $A_c = \frac{\pi}{4}D^2$ and *not* πDL, which is the surface area, A_s. The length of each sample given (10 cm) was, therefore, *not* used.

It is noteworthy that the additional sample is simply used for providing thermal symmetry so that experimental error can be reduced. Also, it enables us to compare the readings of temperature difference for verification.

EXAMPLE 1.3 *The inside and outside surface temperatures of a glass window are 20 and –20°C respectively. If the glass is 60 cm × 30 cm with 18 mm thickness, determine the heat loss through the glass cover in 3 h. Take the thermal conductivity of the window glass as 0.78 W/m K.*

Schematic



Solution

Known Dimensions and end-surface temperatures of a glass window.

Find Heat loss in 3 h.

Assumptions (1) Steady-state one-dimensional conduction. (2) Constant thermal conductivity.

Analysis Cross-sectional area,

$$A = 0.6 \text{ m} \times 0.3 \text{ m} = 0.18 \text{ m}^2$$

Heat-transfer rate,

$$\dot{Q} = \frac{kA(T_1 - T_2)}{L}$$

$$= \frac{(0.78 \text{ W/m K})(0.18 \text{ m}^2)[20-(-20)]\text{K}}{0.018 \text{ m}}$$

$$= 312 \text{ W}$$

In 3 h, the heat loss is

$$Q = \dot{Q}\cdot\Delta t = 312 \text{ W} \times (3 \times 3600) \text{ s} = \mathbf{3.37 \times 10^6 \text{ W or } 3.37 \text{ MW}} \quad \textbf{(Ans.)}$$

EXAMPLE 1.4 *Determine the rate of heat flow and heat flux through a 0.6 m wide, 0.4 m high and 4 mm thick steel plate, having a thermal conductivity of 45 W/m °C, when the temperature of the surface at* x *= 0 is maintained at a constant temperature of 200°C and its temperature at* x *= 4 mm is 202°C.*

Solution

Known Dimensions, thermal conductivity and end-surface temperatures of a steel plate.

Find Heat rate, $\dot{Q}$(W); Heat flux, q(W/m²).

Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant thermal conductivity.

Analysis From Fourier's rate equation:

Heat rate,

$$\dot{Q} = -kA\frac{dT}{dx} = -kA\frac{T_2 - T_1}{L}$$

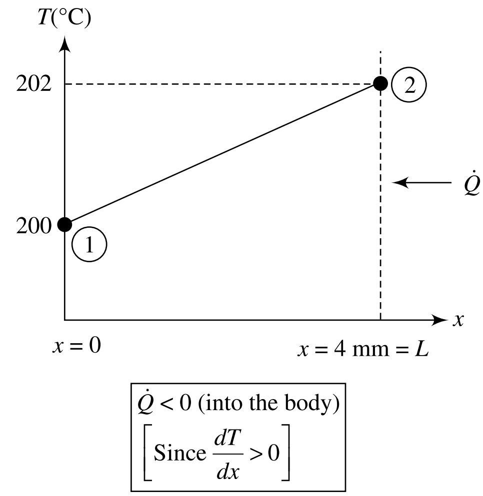
$$= -(45 \text{ W/m °C})\ (0.6 \text{ m} \times 0.4 \text{ m}) \times \frac{(202 - 200)\,°\text{C}}{0.004 \text{ m}}$$

$$= \mathbf{-5400\ W\ or\ -5.4\ kW} \qquad \textbf{(Ans.)}$$

Heat flux,

$$q = \frac{\dot{Q}}{A} = -\frac{5.4 \text{ kW}}{(0.6 \times 0.4)\text{m}^2} = \mathbf{-22.5\ kW/m^2} \qquad \textbf{(Ans.)}$$

Schematic



Comment Since in this case the temperature gradient is *positive* $\left(\frac{dT}{dx} > 0\right)$, the heat rate (and the heat flux) is *negative,* which means that the heat is flowing *inwards.*

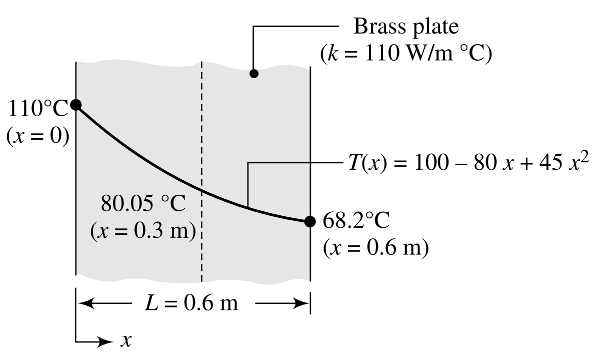
EXAMPLE 1.5 *Determine the heat flux at* x = 0, x = 0.3 m *and* x = 0.6 m *if the temperature distribution across a 0.6 m thick brass plate (*k = 110 W/m °C*) is* T(x) = 100 – 80x + 45x² *where* x *is in metres and* T *in °C. Sketch the temperature distribution.*

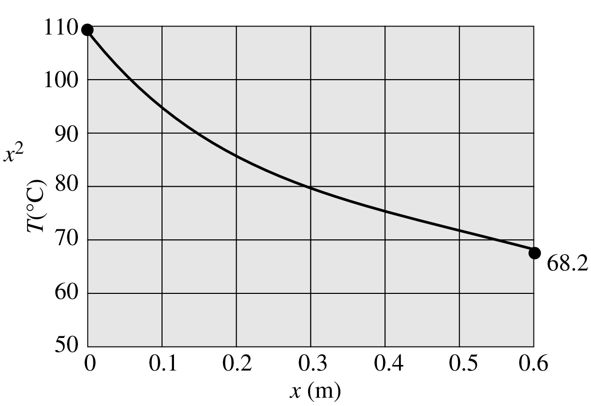
Solution

Known Temperature distribution across a brass plate.

Find Heat flux at x = 0, 0.3 m and 0.6 m.

Schematic





Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant thermal conductivity.

Analysis We have, $T(x) = 100 - 80x + 45x^2$

Differentiating with respect to x, the temperature gradient is $\frac{dT}{dx} = -80 + 90x$

Heat flux at the *three* specified locations is determined from

- $q = -k\left.\dfrac{dT}{dx}\right|_{x=0} = (-110 \text{ W/m°C})(-80\text{°C/m}) = \mathbf{8800\ W/m^2}$ **(Ans.)**
- $q = -k\left.\dfrac{dT}{dx}\right|_{x=0.3\text{m}} = (-110 \text{ W/m°C})(-80 + 90 \times 0.3)\text{°C/m} = \mathbf{5830\ W/m^2}$ **(Ans.)**
- $q = -k\left.\dfrac{dT}{dx}\right|_{x=0.6\text{m}} = (-110 \text{ W/m°C})(-80 + 90 \times 0.6)\text{°C/m} = \mathbf{2860\ W/m^2}$ **(Ans.)**

The temperature distribution is shown in the schematic.

(B) Convection

EXAMPLE 1.6 *Calculate the rate of heat transfer by natural convection from an uninsulated steam pipe (10 cm OD and 15 m long) if the average surface temperature of the pipe is 160°C, the ambient air temperature is 40°C and the average heat-transfer coefficient is 8.4 W/m² °C.*

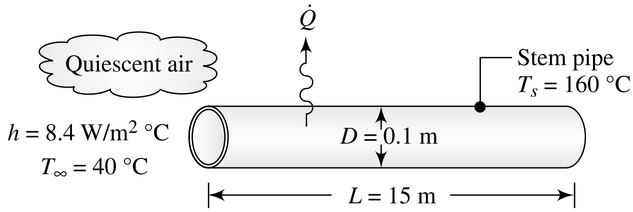
Solution

Known Pipe dimensions and surface temperature. Air temperature and convection coefficient.

Find Heat loss rate, $\dot{Q}$(W).

Assumptions (1) Steady operating conditions. (2) Constant tube surface temperature. (3) Radiation heat loss is not taken into account.

Schematic



Analysis Rate of heat transfer by free convection from the tube surface to the ambient air is given by

$$\dot{Q} = hA_s(T_s - T_\infty) = h(\pi DL)(T_s - T_\infty)$$

$$= (8.4 \text{ W/m}^2\text{ °C})\ (\pi \times 0.1 \text{ m} \times 15 \text{ m})(160 - 40)\text{°C} = \mathbf{4750\ W} \qquad \textbf{(Ans.)}$$

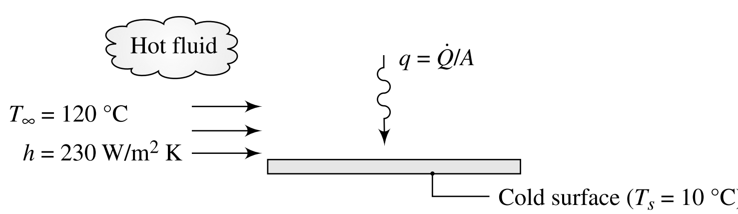
EXAMPLE 1.7 *The forced convection heat-transfer coefficient for a hot fluid flowing over a cold surface is 230 W/m² K. The fluid temperature upstream of the cold surface is 120°C and the surface is held at 10°C. Determine the heat flux from the fluid to the surface.*

Solution

Known A hot fluid is forced to flow over a cold surface.

Find Heat flux, q (W/m²).

Schematic



Assumptions (1) Steady-state conditions. (2) Radiation effects are negligible.

Analysis Heat-transfer rate from the surface is given by, $\dot{Q} = hA(T_s - T_\infty)$

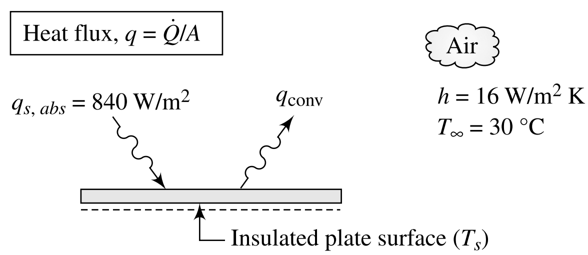
Since the flow of heat is *from the fluid to the surface*, the heat flux from the fluid to the surface is determined to be, $q = \dfrac{\dot{Q}}{A} = h(T_\infty - T_s) = 230\ \text{W/m}^2\,\text{K}\,(120 - 10)\text{K}$

$$= \mathbf{25\ 300\ W/m^2 \quad or \quad 25.3\ kW/m^2} \qquad \textbf{(Ans.)}$$

EXAMPLE 1.8 *A thin metallic plate is insulated at the back surface and is exposed to the sun at the front surface. The front surface absorbs the solar radiation of 840 W/m² and dissipates it essentially by convection to the ambient air at 30°C. If the heat-transfer coefficient between the plate and the air is 16 W/m²K, what is the plate temperature?*

Solution

Schematic



Known A thin plate with one surface exposed to solar radiation loses heat by convection.

Find Plate temperature, T_s (°C).

Assumptions (1) Steady operation is established. (2) Uniform heat-transfer coefficient. (3) Radiation effects are not considered.

Analysis The dominant heat-transfer mode here is convection.

Then, by Newton's law of cooling, $\dot{Q} = hA(T_s - T_\infty)$

It follows that the plate surface temperature is, $T_s = \left(\dfrac{\dot{Q}}{A}\right)\dfrac{1}{h} + T_\infty = \dfrac{q}{h} + T_\infty$

From energy balance: $q = q_{s,\text{abs}} = q_{\text{conv}} = 840\ \text{W/m}^2$

Hence, the temperature of the plate,

$$T_s = \frac{840\ \text{W/m}^2}{16\ \text{W/m}^2\,\text{K}} + 30\,°\text{C} = \mathbf{82.5°C} \qquad \textbf{(Ans.)}$$

(C) Radiation

EXAMPLE 1.9 *A long cylindrical electrically heated rod, 20 mm in diameter, is placed in a large evacuated chamber. The surface emissivity of the rod is 0.85 and the interior walls of the chamber are maintained at 300°C. The rod is held at 30°C in a steady operation. Determine (a) the net rate at which radiation is exchanged between the heating rod and the chamber walls per metre length, and (b) the radiation heat-transfer coefficient.*

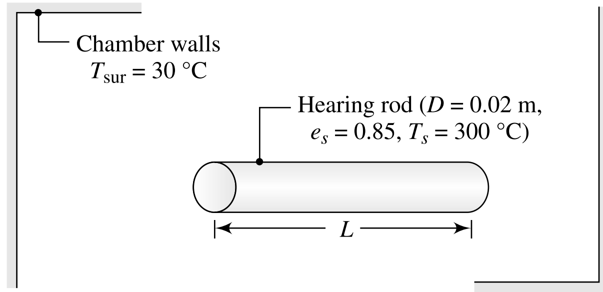
Solution

Known A long rod of specified diameter, emissivity and temperature is installed in a vacuum chamber held at a lower temperature.

Find (a) $\dot{Q}_{\text{net}}/L$ (W/m), (b) h_r(W/m² K).

Assumptions (1) Steady operating conditions. (2) The chamber walls are idealized as a black body. (3) The rod surface is very small compared to the large enclosure (*chamber*).

Schematic



Analysis Under steady-state conditions, the rate of heat loss through the chamber walls must equal the rate at which the heating rod dissipates electrical energy. This also equals the rate of electrical input to the system. As the rod, relatively a small body, is completely enclosed in the large chamber, the entire radiant energy emitted by the rod's surface is intercepted by the chamber walls.

Net rate of radiation heat transfer is

$$\dot{Q}_{net} = \sigma A_s \varepsilon_s [T_s^4 - T_{sur}^4] = \sigma \pi D L \varepsilon_s [T_s^4 - T_{sur}^4]$$

Substituting the numerical values, we have

$$\frac{\dot{Q}_{net}}{L} = (5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4)(\pi \times 0.02 \text{ m})(0.85)[(300 + 273.15)^4 - (30 + 273.15)^4]\text{K}^4$$

$= \mathbf{302.2\ W/m}$ **(Ans.) (a)**

Radiation heat-transfer coefficient,

$$h_r = \frac{\dot{Q}}{(\pi DL)(T_s - T_{sur})} = \frac{302.2 \text{ W/m}}{(\pi \times 0.02)(300 - 30)°\text{C or K}}$$

$= \mathbf{17.8\ W/m^2\ K}$ **(Ans.) (b)**

Comment The only mode of heat transfer in an evacuated space is radiation because both conduction and convection heat transfer require a material medium. In radiation calculations, all temperatures must be *absolute* temperatures. However, the temperature difference, ΔT can be expressed either in °C or K.

(D) Energy Balance and Combined Heat-Transfer Mechanisms

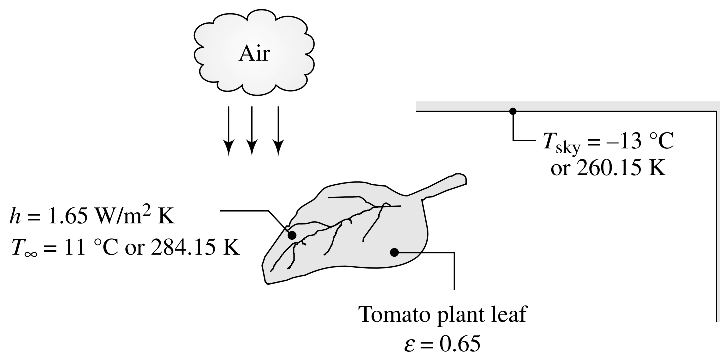
EXAMPLE 1.10 *Tomato plants placed in a garden encounter a clear still night with an effective sky temperature of –13°C. The air temperature is 11°C and the convection heat-transfer coefficient is 1.65 W/m²K. The emissivity of the leaves of the tomato plants is 0.65. Determine the temperature of the tomato plant leaves under equilibrium conditions.*

Solution

Known Tomato plants in a garden are subjected to convective and radiative heat transfer equilibrium conditions.

Find Equilibrium temperature of tomato leaves.

Schematic



Assumptions (1) Steady-state conditions. (2) Quiescent air.

Analysis *Energy balance:* $\begin{pmatrix}\text{Heat transferred by}\\ \text{convection to the leaves}\end{pmatrix} = \begin{pmatrix}\text{Heat transferred to the}\\ \text{night sky by radiation}\end{pmatrix}$

$$\dot{Q}_{conv} = hA(T_\infty - T_s)$$

where T_s is the surface temperature of the leaves

$$\dot{Q}_{rad} = \sigma A\varepsilon(T_s^4 - T_{sky}^4)$$

Equating the two expressions, we get

$$h(T_\infty - T_s) = \sigma\varepsilon(T_s^4 - T_{sky}^4)$$

Substituting the numerical values, we have

$$(1.65\ \text{W/m}^2\ \text{K})(284.15 - T_s)\text{K} = (5.67 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4) \times (0.65)(T_s^4 - 260.15)^4\ \text{K}^4$$

A *trial-and-error* solution is necessary to estimate the tomato plant leaves' surface temperature.

T_s(°C)	T_s(K)	LHS	RHS
–3	270.15	23.1	27.49
–5	268.15	26.4	21.74
–4	**269.15**	**24.75**	**24.60**

At $T_s = -4$°C, both LHS and RHS are almost equal. Hence, the temperature of the tomato leaves is **–4°C**. **(Ans.)**

EXAMPLE 1.11 *A cold spherical drop of water (3 mm diameter) gains heat as it falls through the surrounding air which is at a temperature of 30°C. Simultaneously, the drop also loses mass by evaporation at the rate of 0.63 × 10^{-5} kg/h. If the surface heat-transfer coefficient is 20 W/m^2 K and it is assumed that at any given instant of time, the drop is at a uniform temperature, calculate the rate at which the drop is heating up (in °C/min) when it is at a temperature of 10°C.*

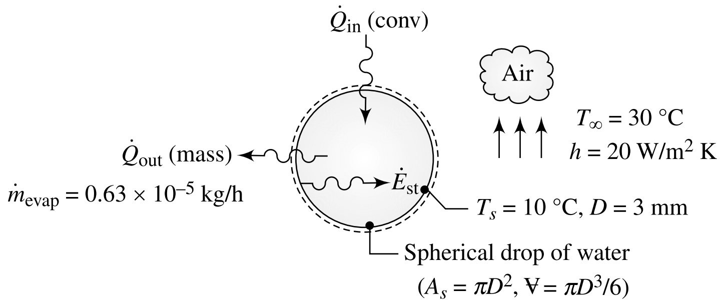
Properties of water: $\rho = 997.7$ *kg/m^3,* $C_p = 4.194$ *kJ/kg K,* $h_{fg} = 2478$ *kJ/kg*

Solution

Known A cold drop of water gains heat by convection and loses mass by evaporation.

Find Rate of temperature rise of drop, dT/dt.

Schematic



Assumptions (1) Steady-state conditions. (2) The drop is at a uniform temperature. (3) Constant properties.

Analysis Applying control volume energy balance,

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \underbrace{\cancelto{0}{\dot{E}_{\text{gen}}}}_{\text{(no heat generation)}} = \dot{E}_{\text{st}}$$

or
$$\underbrace{\dot{Q}_{\text{in,convection}} - \dot{Q}_{\text{out,evaporation}}}_{\dot{Q}_{\text{net,in}}} = \rho \forall C_p \frac{dT}{dt} = \text{energy storage rate}$$

Net rate of heat transfer to the drop,

$$\dot{Q}_{\text{net,in}} = hA_s(T_\infty - T_s) - \dot{m}h_{fg}$$

$$= 20 \text{ W/m}^2\text{ K} \times (\pi \times 0.003^2)/\text{m}^2 \times (30 - 10)\text{K} - \left(\frac{0.63 \times 10^{-5}}{3600}\right) \text{kg/s} \times 2478 \times 10^3 \text{ J/kg}$$

$$= (11.31 \times 10^{-3} \text{ W}) - (4.34 \times 10^{-3} \text{ W}) = 6.97 \times 10^{-3} \text{ W}$$

It follows that $\rho \forall C_p \dfrac{dT}{dt} = \dot{Q}_{\text{net,in}}$

or
$$997.7 \text{ kg/m}^3 \left(\frac{\pi}{6} \times 0.003^3\right) \text{m}^3 \times 4.194 \times 10^3 \text{ J/kg K} \times \left(\frac{dT}{dt}\right) (°\text{C/s}) = 6.97 \times 10^{-3} \text{ W}$$

$$\therefore \quad \frac{dT}{dt} = \mathbf{0.118°C/s \text{ or } 7°C/min}$$ **(Ans.)**

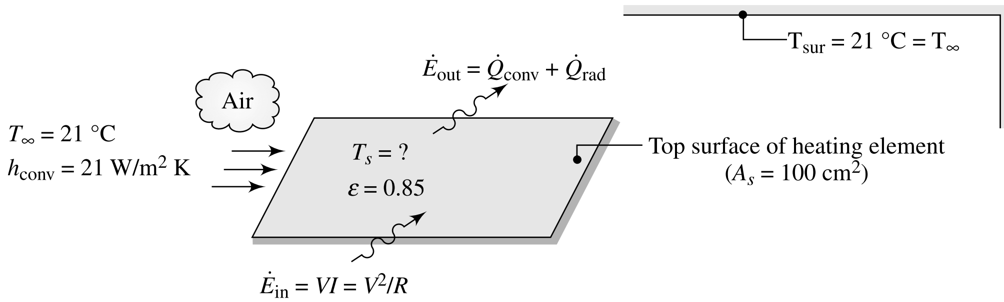
EXAMPLE 1.12 *The top surface area of a heating element is 100 cm². Its resistance is 15 ohms and its emissivity is 0.85. The convective heat-transfer coefficient from the top of the element is 21 W/m² K. If the voltage drop across the element is 30 V, how hot will it become in the steady state? Assume that all the heat is dissipated by convection and radiation from the top of the element and that the room is at 21°C.*

Solution

Known The top surface of a heating element loses heat both by convection and radiation.

Find Steady-state surface temperature, T_s (°C).

Schematic



Assumptions (1) The system is in steady state. (2) Uniform heat-transfer coefficient.

Analysis Under steady operating conditions: $\dot{E}_{in} = \dot{E}_{out}$

Now, $$\dot{E}_{in} = I^2 R = \frac{V^2}{R} = \frac{(30\text{ V})^2}{15\,\Omega} = 60\text{ W}$$

The total heat dissipated, $\dot{E}_{out} = \dot{Q}_{conv} + \dot{Q}_{rad} = h_{tot} A(T_s - T_\infty)$

where h_{tot} is the total heat-transfer coefficient, comprising convective and radiative components. The radiative heat-transfer coefficient is determined from

$$h_{rad} = \varepsilon\sigma(T_s + T_\infty)(T_s^2 + T_\infty^2)$$

However, there is a catch since h_{rad} depends on T_s which is to be determined. To evaluate the radiative heat-transfer coefficient, the surface temperature must be assumed. A reasonable value to start with is 200°C. Later, we can correct it if necessary. Using our assumed value, h_{rad} becomes

$$\begin{aligned} h_{rad} &= \varepsilon\sigma(T_s + T_\infty)(T_s^2 + T_\infty^2) \\ &= 0.85 \times 5.67 \times 10^{-8}\text{ W/m}^2\text{K}^4\,(473.15 + 294.15)\text{K}\,\{(473.15)^2 + (294.15)^2\}\text{K}^2 \\ &= 11.48\text{ W/m}^2\text{ K} \end{aligned}$$

Absolute temperatures must always be used in radiative calculations. Hence, the temperatures have been converted to Kelvin. The total heat-transfer coefficient is

$$h_{tot} = h_{rad} + h_{conv} = 11.49 + 21 = 32.48\text{ W/m}^2\text{ K}$$

Substituting the appropriate values, we have

$$60\text{ W} = (32.48\text{ W/m}^2\text{ K})(100 \times 10^{-4}\text{ m}^2)(T_s - 21)\text{K}$$

Evaluating and solving for T_s yields

$$T_s = 205.74\,°\text{C}$$

Recall that we assumed T_s = 200°C in order to calculate h_{rad}. What would h_{rad} be if we use T_s = 205.74°C? Repeating the calculations gives h_{rad} = 11.77 W/m² K and the total heat-transfer coefficient, h_{tot} = 32.77 W/m² K. The new value of T_s is found to be 204°C. We may continue the iterations if more precision is needed. A table of the assumed and computed values of T_s is given below.

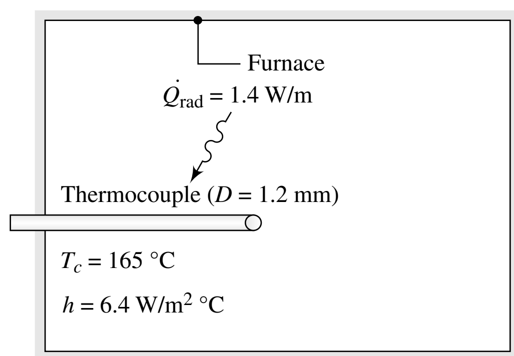
Iteration, #	$h_{rad,}$ W/m² K	Assumed T_s, °C	Calculated T_s, °C
1	11.48	200.0	205.7
2	11.77	205.7	204.1
3	11.69	204.1	204.57
4	11.71	204.6	204.4
5	**11.70**	**204.4**	**204.5**

Hence, the surface temperature, $\boldsymbol{T_s}$ **= 204.5°C** **(Ans.)**

EXAMPLE 1.13 *A thermocouple (1.2 mm OD wire) is used to measure the temperature of a quiescent gas in a furnace. The thermocouple reading is 165°C. It is known, however, that the rate of radiant heat flow per metre length from the hotter furnace walls to the thermocouple wire is 1.4 W/m and the heat-transfer coefficient between the wire and the gas is 6.4 W/m² °C. Estimate the true gas temperature.*

Solution

Schematic



Known A thermocouple reads the gas temperature in a furnace. Convection coefficient and radiant heat transfer are specified.

Find True gas temperature.

Assumptions (1) Uniform thermocouple wire temperature. (2) Thermal equilibrium exists between the thermocouple and the walls.

Analysis Under steady operating conditions: Rate of heat *gain* by the thermocouple by *radiation* = Rate of heat *loss* by the thermocouple by *convection*.

i.e., $$\dot{Q}_{rad} = \dot{Q}_{conv} \quad \text{or} \quad \frac{\dot{Q}_r}{L} L = h(\pi D L)(T_c - T_g)$$

∴ True gas temperature is

$$T_g = T_c - \frac{(\dot{Q}/L)}{h\pi D} = 165°\text{C} - \frac{1.4 \text{ W/m}}{(6.4 \text{ W/m}^2 \text{ °C})(\pi \times 1.2 \times 10^{-3} \text{ m})} = \mathbf{107°C} \qquad \textbf{(Ans.)}$$

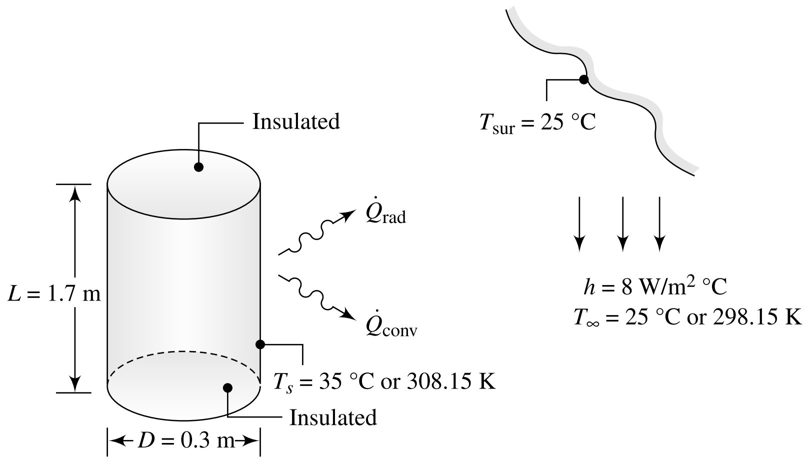
EXAMPLE 1.14 *A person standing in a breezy room at 25°C is modelled as a vertical cylinder of 30 cm diameter and 1.7 m height with both top and bottom surfaces insulated. The lateral (side) surface is at an average temperature of 35°C. The convection heat-transfer coefficient is 8 W/m² °C. The surface emissivity may be assumed to be 0.95. Calculate the total heat-transfer rate.*

Solution

Known The side surface of a person idealized as a vertical cylinder is exposed to convection and radiation.

Find Total heat-transfer rate.

Schematic



Assumptions (1) Steady operating conditions. (2) Diffuse-gray isothermal surface. (3) Constant emissivity and heat-transfer coefficient. (4) Top and bottom surfaces are insulated.

Analysis Total heat-transfer rate from the person,

$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{rad} = hA_s(T_s - T_\infty) + \varepsilon A_s \sigma (T_s^4 - T_{sur}^4)$$

where $A_s = \pi DL = \pi(0.30\text{ m})(1.7\text{ m}) = 1.6\text{ m}^2$

Substituting numerical values,

$$\dot{Q} = [(8\text{ W/m}^2\,°\text{C})(1.6\text{ m}^2)(35 - 25)°\text{C}]$$
$$+ [0.95 \times 1.6\text{ m}^2 \times 5.67 \times 10^{-8}\text{ W/m}^2\text{ K}^4\{308.15^4 - 298.15^4\}\text{K}^4]$$
$$= \mathbf{128\ W + 96.2\ W = 224.2\ W} \qquad \textbf{(Ans.)}$$

EXAMPLE 1.15 *The following data was obtained in an experiment for determining the emissivity of a surface.*

Plate size	*20 × 20 cm*
Heat coil input	*50 W*
Steady state temperature of the surface	*85°C*
Ambient temperature	*30°C*
Heat-transfer coefficient	*16 W/m² K*
Stefan–Boltzmann constant	*5.6688 × 10⁻⁸ W/m²K⁴*

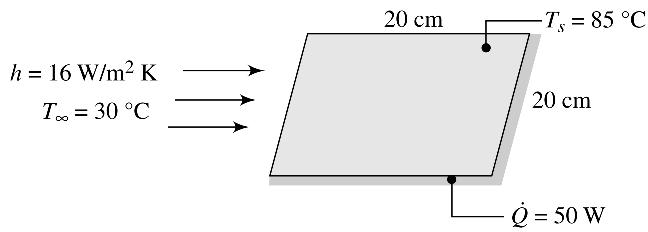
Estimate the emissivity of the surface.

Solution

Known Details associated with experimental arrangement to measure surface emissivity.

Find Surface emissivity, ε.

Schematic



Assumptions (1) Steady-state conditions. (2) Uniform heat-transfer coefficient. (3) Ambient temperature equals surroundings temperature.

Analysis Rate of heat transfer by *convection* and *radiation* is

$$\frac{\dot{Q}}{A} = \frac{50\ \text{W}}{0.2\ \text{m} \times 0.2\ \text{m}} = 1250\ \text{W/m}^2$$

Rate of heat loss due to *convection*

$$\frac{\dot{Q}_{\text{conv}}}{A} = h\,\Delta T = 16\ \text{W/m}^2\ \text{K}\ (85 - 30)\ \text{K} = 880\ \text{W/m}^2$$

Rate of heat loss due to *radiation*

$$\frac{\dot{Q}_{\text{rad}}}{A} = 1250 - 880 = 370\ \text{W/m}^2 \qquad (\textit{by difference})$$

But $$\frac{\dot{Q}_{\text{rad}}}{A} = \varepsilon\sigma(T_s^4 - T_\infty^4)$$

Hence, the emissivity is determined to be

$$\varepsilon = \frac{370\ \text{W/m}^2}{5.6688 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4\ [(273.15 + 85)^4 - (273.15 + 30)^4]\text{K}^4}$$
$$= \mathbf{0.813}$$ **(Ans.)**

(E) Mass Transfer

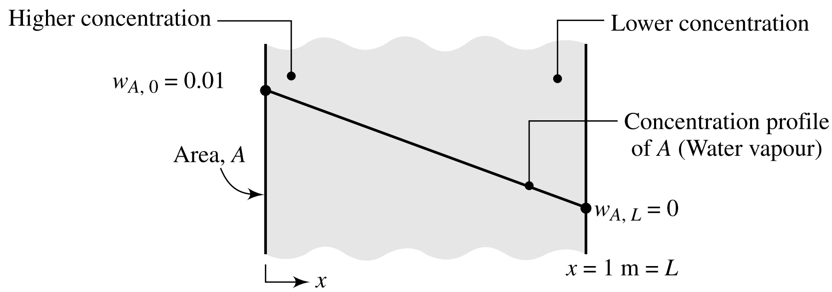
EXAMPLE 1.16 *Air fills a tube that is 1 m in length. There is a small water leak at one end where the water vapour concentration builds to a mass fraction of 0.01. A dissector maintains the concentration at zero on the other side. What is the steady flux of water from one side to the other if the mass diffusivity,* D_{AB} *= 0.000284 m²/s and* ρ_{air} *= 1.18 kg/m³.*

Solution

Known Water-vapour concentration on the two sides of an air-filled tube. Density of air and diffusivity of water vapour in air.

Find Diffusive mass flux.

Schematic



Assumptions (1) Constant mass density. (2) Linear concentration profile.

Analysis Mass transfer by diffusion per unit area per unit time, i.e., diffusive mass flux of the species A is

$$j_{\text{diff},A} = \frac{\dot{m}_{\text{diff,H}_2\text{O vapour}}}{A} = -\rho D_{AB}\frac{d(\rho_A/\rho)}{dx} = -\rho D_{AB}\frac{dw_A}{dx}$$

$$= -\rho D_{AB}\frac{\overset{0}{\cancel{w_{A(x=L)}}} - w_{A(x=0)}}{L-0} = +\rho D_{AB}\frac{w_{A,0}}{L}$$

$$= (1.18\ \text{kg/m}^3)(0.000\ 284\ \text{m}^2/\text{s})\left(\frac{0.01\,\text{kg H}_2\text{O/kg mixture}}{1\ \text{m}}\right)$$

$$= \mathbf{3.35 \times 10^{-6}\ kg/m^2s} \qquad \textbf{(Ans.)}$$

Points to Ponder

- In the radiator, radiation is not the dominant mode of heat transfer.
- In pure metals, the electronic component of thermal conductivity is typically two orders of magnitude greater than the lattice component.
- In the increasing order of thermal conductivity, we have metal alloys, non-metallic crystals and pure metals.
- In conduction and convection problems, we deal with linear temperature differences and any consistent temperature scale (Celsius or Kelvin) may be used.
- Heat transfer takes place in accordance with the second law of thermodynamics.
- Surface energy balance is valid under both steady and unsteady operating conditions.
- The convection heat-transfer coefficient, unlike thermal conductivity, is not a property of the fluid.
- Heat transfer is a non-equilibrium phenomenon.
- The driving force for heat transfer is the temperature gradient.
- The property which is a measure of the ability of a material to conduct heat is called thermal conductivity.
- A material with uniform properties in all directions is known as isotropic.
- The larger the thermal diffusivity, the faster the propagation of heat into the medium.
- Diamond is a better heat conductor than silver.
- Heat transfer is transport of energy due to a temperature difference.
- The three basic modes of heat transfer are conduction, convection and radiation.

- Conduction requires a stationary medium.
- Convection requires a moving fluid.
- Radiation takes place in the absence of any medium.
- The rate of heat conduction is proportional to the area measured normal to the direction of heat flow and to the temperature gradient in the direction of heat flow.
- SI units of thermal conductivity and thermal diffusivity are respectively W/m °C or W/m K and m^2/s.
- Heat-transfer coefficients also depend on the magnitude of surface-to-fluid temperature difference in free convection, unlike forced convection.
- The most important mode of heat transfer at high temperatures is radiation.
- In conduction and convection heat transfer, one deals with linear temperature differences enabling one to express temperatures in °C or K but in radiation heat transfer, the temperature differences are non-linear with temperatures expressed only in kelvin.
- Flow of cigarette smoke in a still room is an example of free convection while the mechanism of heat flow in a car radiator is an example of forced convection.
- If T_m is the mean temperature of the surface and the surroundings in K, ε is the emissivity of the surface, and σ is the Stefan–Boltzmann constant, the radiation heat transfer coefficient h_r is approximately $4\varepsilon\sigma T_m^3$.
- Distant non-black surroundings are effectively black.
- In many cases, convection and radiation heat transfers have roughly equal contributions. The convective and radiative thermal resistances are in parallel if ambient and surroundings temperatures are same.
- The thermal conductivities of most metals decreases with an increase in temperature whereas in the case of non-metals, they increase with an increase in temperature.

GLOSSARY of Key Terms

• **Heat transfer**	Transport of energy due to temperature gradients or differences.
• **Conduction**	Transfer of energy in a substance due to random molecular collision.
• **Convection**	Transfer of energy from one region to another by bulk (macroscopic) fluid motion, added on to the energy transfer by conduction.
• **Forced convection**	Fluid motion caused by an external agency such as a pump or a blower.
• **Free (natural) convection**	Fluid motion occurring due to density variations caused by temperature differences.
• **Radiation**	Transport of energy in the form of electromagnetic waves with or without the material medium.
• **Mass transfer**	The transport of one component in a mixture from a region of higher concentration to one of lower concentration.
• **Thermal conductivity**	A transport property of the material which indicates its ability to conduct heat. It is the conduction heat flux per unit temperature gradient.
• **Heat-transfer coefficient**	Convection heat flux per unit difference between the surface and fluid temperatures.
• **Emissivity**	A radiative property of the surface.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

1.1 Heat transfer takes place according to the following law of thermodynamics:
(a) Zeroth (b) First (c) Second (d) Third

1.2 Consider the following statements:

The Fourier's heat conduction equation, $\dot{Q} = -kA\dfrac{dT}{dx}$ presumes

1. steady-state conditions
2. constant value of thermal conductivity
3. uniform temperatures at the wall surfaces
4. one-dimensional heat flow

Of these statements:
(a) 1, 2 and 3 are correct (b) 1, 2 and 4 are correct
(c) 2, 3 and 4 are correct (d) 1, 3 and 4 are correct.

1.3 For a plate of thickness L, cross-sectional area A_c, and thermal conductivity k, the thermal conductance is given by

(a) $\dfrac{kA_c}{L}$ (b) $\dfrac{L}{kA_c}$ (c) $\dfrac{kL}{A_c}$ (d) $\dfrac{A_c}{kL}$

1.4 Most metals are good conductors of heat because of
(a) energy transport due to molecular vibration
(b) migration of neutrons from hot end to cold end
(c) lattice defects such as dislocations
(d) presence of free electrons and frequent atomic collision

1.5 The ratio of the average thermal conductivities of water and air is of the order of
(a) 5 : 1 (b) 10 : 1 (c) 25 : 1 (d) 50:1

1.6 The steady-state temperature profile in a plane wall with isothermal surfaces is
(a) parabolic (b) linear (c) hyperbolic (d) logarithmic

1.7 In a pulverized-fuel-fired large power boiler, the heat transfer from the burning fuel to the walls of the furnace is
(a) by conduction only (b) by convection only
(c) by conduction and convection (d) predominantly by radiation

1.8 Free convection heat flow depends on all of the following, except
(a) density (b) velocity
(c) coefficient of viscosity (d) gravitational force

1.9 Forced convection in a liquid bath is caused by
(a) density difference brought about by temperature gradients
(b) molecular energy interaction
(c) flow of electrons in a random fashion
(d) intense stirring by an external agency

1.10 Heat transfer in liquids and gases is essentially due to
(a) conduction (b) convection
(c) radiation (d) conduction and radiation

1.11 Match the properties with their respective units:

Property	**Units**
A. Thermal resistance	(P) W/s
B. Thermal conductivity	(Q) K/W
C. Heat-transfer coefficient	(R) N/m^3
D. Heat flow rate	(S) W
	(T) W/m K
	(U) W/m^2 K

Codes:	**A**	**B**	**C**	**D**
(a)	P	R	S	T
(b)	Q	U	R	S
(c)	U	T	Q	P
(d)	Q	T	U	S

1.12 Which of the following correctly represents Newton's law of cooling?

(a) $\frac{\dot{Q}}{A} = -k\frac{dT}{dx}$ (b) $\frac{\dot{Q}}{L} = h\Delta T$ (c) $\frac{\dot{Q}}{L} = k\frac{dT}{dx}$ (d) $\frac{\dot{Q}}{A} = h\Delta T$

1.13 Which mode(s) of heat transfer does (do) not need a material medium?

(a) Conduction (b) Convection (c) Radiation (d) Convection and radiation

1.14 On a summer day, a scooter rider feels more comfortable while on the move than while at a stoplight because

(a) an object in motion captures less solar radiation
(b) air is transparent to radiation and, hence, it is cooler than the body
(c) more heat is lost by convection and radiation while in motion
(d) air has a low specific heat and, hence, it is cooler

1.15 Heat is mainly transferred by conduction, convection and radiation in

(a) insulated pipes carrying hot water (b) refrigerator freezer coil
(c) boiler furnaces (d) condensation of steam in a condenser

1.16 Identify the wrong statement:

(a) A temperature gradient is necessary for heat transfer.
(b) A physical medium is essential for heat flow.
(c) Heat transmission requires flow of heat from higher temperature to lower temperature according to the second law of thermodynamics.
(d) Heat transfer is thermodynamically an irreversible process.

1.17 Which of the following would lead to a reduction in thermal resistance?

1. In conduction, reduction in the thickness of the material and an increase in the thermal conductivity
2. In convection, stirring of the fluid and cleaning the heat surface.
3. In radiation, increasing the temperature and reducing the emissivity.

Select the correct answer using the codes given below:

Codes:

(a) 1, 2, and 3 (b) 1 and 2 (c) 1 and 3 (d) 2 and 3

1.18 A spherical body at a temperature T_1 is surrounded by walls at a temperature T_2. At what rate must the energy be supplied in order to keep the temperature of the body constant if $T_1 > T_2$?

(a) $4\pi r^2 \varepsilon\sigma(T_1^4 - T_2^4)$ (b) $4\pi r^2 \varepsilon\sigma T_1^4$

(c) $k4\pi r^2(T_2 - T_1)$ (d) $4\pi r^2 \varepsilon\sigma(T_2^4 - T_1^4)$

1.19 A cylindrical rod of radius $[10^{-1}/\sqrt{2\pi}]$ cm, 100 cm length and 200 J/s m °C thermal conductivity has a temperature difference of 100°C. How much heat flows axially through the rod in one day? Assume that the heat flow is in steady state.
(a) 7.2×10^3 J (b) 864 J (c) 250 J (d) 1.7×10^{10} J

1.20 A thin 2 m by 2 m flat plate is hanging freely in air at 25°C. Solar radiation is falling on one side of the plate at the rate of 500 W/m^2. The temperature of the plate will remain constant at 30°C, if the convective heat transfer coefficient, h, in W/m^2 K is
(a) 25 (b) 50 (c) 100 (d) 200

1.21 A black surface of 2 m^2 area at 85°C is losing heat by both convection and radiation. The surroundings are at 15°C and the ambient air temperature is 25°C. The convective heat-transfer coefficient is 14.5 W/m^2 K. The total heat loss rate from the surface is
(a) 1004 W (b) 3574 W (c) 2524 W (d) 2824 W

1.22 A closed container of 0.1 m^2 surface area holds one kg of water (C_p = 4.2 kJ/kg °C) at 100°C. The initial rate of cooling in ambient air at 25°C is 3 K/min. The heat-transfer coefficient (W/m^2 °C) is
(a) 26.25 (b) 105 (c) 5.0 (d) 28

1.23 A 20 cm diameter, 1.2 m long cylinder loses heat from its peripheral surface by convection. Surface temperature of the cylinder is constant at 100°C and the fluid temperature is constant at 20°C. The average convection heat-transfer coefficient over the surface of the cylinder is 25 W/m^2 K.
(a) 120 π (W) (b) 240 π (W) (c) 320 π (W) (d) 480 π (W)

1.24 A finned-tube hot-water radiator with a fan blowing air over it is kept in a room during winter. The major part of the heat transfer from the radiator is due to
(a) better conduction (b) convection to the air
(c) radiation to the surroundings (d) combined conduction and radiation

1.25 Air at 20°C blows over a hot plate of 50 × 60 cm made of carbon steel maintained at 220°C. The convective heat-transfer coefficient is 25 W/m^2 K. What will be the heat loss from the plate?
(a) 1500 W (b) 2500 W (c) 3000 W (d) 4000 W

Answers

1.1 (c)	**1.2** (d)	**1.3** (a)	**1.4** (d)	**1.5** (c)	**1.6** (b)
1.7 (d)	**1.8** (b)	**1.9** (d)	**1.10** (b)	**1.11** (d)	**1.12** (d)
1.13 (c)	**1.14** (c)	**1.15** (a)	**1.16** (b)	**1.17** (b)	**1.18** (a)
1.19 (b)	**1.20** (b)	**1.21** (d)	**1.22** (d)	**1.23** (d)	**1.24** (b)
1.25 (c)					

REVIEW QUESTIONS

1.1 Bring out the essential difference between heat transfer and thermodynamics.
1.2 Enumerate some of the important industrial applications of heat transfer.
1.3 Discuss the physical mechanism of heat conduction in solids, liquids and gases.
1.4 Define thermal conductivity of a material and explain its significance.
1.5 What is heat flux? How is it related to the heat-transfer rate?
1.6 Draw the temperature profile for steady-state conduction through a material with constant thermal conductivity.
1.7 Differentiate between thermal conductivity and thermal conductance. State their respective units in SI units.

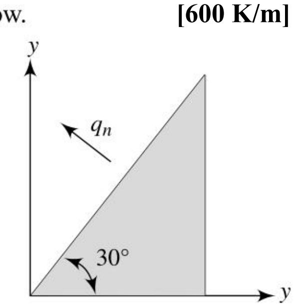
1.8 What is the order of magnitude of thermal conductivity for (a) metals, (b) insulating materials, (c) liquids, and (d) gases?
1.9 Discuss the mechanism of convective heat transmission.
1.10 How does heat convection differ from conduction?
1.11 How does forced convection differ from natural convection?
1.12 Give a few examples of natural convection and forced convection.
1.13 What is the order of magnitude of convective heat-transfer coefficient for (a) free convection in air, (b) forced convection in water, (c) condensation of steam, and (d) boiling of water.
1.14 Comment on the mechanism of heat transfer in a vacuum.
1.15 When would you consider radiation heat exchange to be of significance?
1.16 What is a black body? How do real bodies differ from black bodies?
1.17 Elucidate the concept of driving potential and electrical analogy in heat transfer.
1.18 Write down the expressions for the physical laws that govern each mode of heat transfer, and identify the variables involved in each relation.
1.19 Identify and indicate the important modes of heat transfer in the following cases:
(a) Heat transfer from an automobile radiator
(b) Heat transfer from a room heater
(c) Cooling of an internal combustion engine
(d) Condensation of steam in a condenser
(e) Heat loss from a thermos flask
(f) Heating of water in an immersion water heater
(g) Quenching of a hot steel ingot in water
1.20 Can all three modes of heat transfer occur simultaneously in a medium?
1.21 Can a medium involve (a) conduction and convection, (b) conduction and radiation, or (c) convection and radiation simultaneously?

PRACTICE PROBLEMS

(A) Conduction

1.1 Calculate the heat transfer rate across a plane wall, 15-cm-thick, with a cross-sectional area of 5 m^2, and of thermal conductivity 9.5 W/m K. The steady state end surface temperatures are 120°C and 30°C. Also determine the temperature gradient in the direction of heat flow. **[600 K/m]**

1.2 The heat flux on the 30° diagonal surface of the Bakelite (k = 1.4 W/m °C) wedge shown in the following figure is 2000 W/m^2 in the direction shown. Calculate the heat flux and temperature gradient in the x-and y-directions. **[– 1237 k/m]**



1.3 The inside and outside surface temperatures of a glass window are 20 and –20°C respectively. If the glass is 60 cm × 30 cm with 18-mm-thickness, determine the heat loss through the glass cover in 3 h. Take the thermal conductivity of the window glass as 0.78 W/m K. **[3.37 MW]**

1.4 Heat is generated due to friction at a rate of 250 W/m^2 when a stationary brake shoe made of rigid foamed rubber (k = 0.032 W/m °C) is pressed against a rotating carbon-silicon steel drum (k = 51.9 W/m °C) Ninety percent of this heat enters the drum while the remaining ten percent enters the brake shoe. Calculate the temperature gradients at the contact surface in the steel drum and the brake shoe. **[+4.335 °C/m, –781.25 °C/m]**

1.5 A very long solid cylinder of 40-mm-diameter experiences uniform internal thermal energy generation. The thermal conductivity of the solid material is 15.5 W/m °C. The temperature profile is given by T(r) = 256 – 86 × 10^4 r^2 (°C) where r is expressed in metres. Evaluate (a) the centreline temperature, (b) the wall temperature, (c) the wall heat flux, and (d) the heat rate per metre length.

[(a) 256°C (b) 221.6°C (c) 53.32 kW/m^2 (d) 6.7 kW]

(B) Convection

1.6 A flat plate of dimensions 100 × 50 mm and negligible thickness, maintained at 46°C, loses 1.2 W from both sides by free convection to ambient air at 30°C. Determine the heat transfer coefficient. **[3740 W/m^2]**

1.7 The surface of a small ceramic kiln is at 65°C when the kiln is in operation. The kiln is of cubical shape and is supported in air at 25°C with negligibly small legs, with a heat transfer coefficient of 15 W/m^2 °C (assumed to be an average for all surfaces). If 900 watts are required to keep the kiln in steady-state operation, determine the size of the cubical kiln. What type of heat transfer phenomenon is involved? **[0.5 m or 50 cm]**

1.8 The temperature profile of a flat plate exposed to water at 60°C is given by $T(y) = 50 + 640\,y + 0.1\,y^2$ where T is in °C. Calculate the heat transfer coefficient if the thermal conductivity of the water is 0.65 W/m °C. **[41.6 W/m^2 °C]**

(C) Radiation

1.9 The outside surface of a spacecraft in space has an emissivity of 0.8 and an absorptivity of 0.25 for solar radiation. If solar radiation is incident on the spacecraft at a rate of 1200 W/m^2, calculate the surface temperature of the spacecraft when the radiation emitted equals the solar energy absorbed. **[285.18 K, 12°C]**

(D) Energy Balance and Combined Heat Transfer Mechanisms

1.10 Water at a temperature of 70.6°C is to be evaporated slowly in a vessel. The water is in a low-pressure container which is surrounded by steam. The steam is condensing at 103°C. The overall heat transfer coefficient between the water and the steam is 1200 W/m^2 °C. Determine the surface area of the container which would be required to evaporate water at a rate of 0.01 kg/s. **[0.60 m^2]**

1.11 A metal plate is placed on a driveway and receives 950 W/m^2 of incident radiant energy from the sun. The plate absorbs 80% of the incident solar energy and has an emissivity of 0.05. Consider the lower surface of the plate to be thermally insulated from the driveway. If the air temperature is 20°C and the natural convection heat transfer coefficient between the plate's surface and the surrounding air is 10 W/m^2 K, estimate the temperature of the plate. **[93°C]**

1.12 Electronic power devices are mounted on a heat sink having an exposed surface area of 450 cm^2 and an emissivity of 0.85. When the devices dissipate a total power of 20 W and the air and surroundings are at 27°C, the average sink temperature is 4°C. What average temperature will the heat sink reach when the devices dissipate 30 W for the same environment condition? **[53.7°C]**

2

Steady-State Heat Conduction—One Dimension

2.1 ❑ INTRODUCTION

In the previous chapter, the nature of heat conduction as a mode of heat transfer was discussed. The Fourier's rate equation for one-dimensional heat flow introduced earlier can be expressed in a more general form:

$$\boxed{\dot{Q}_n = -kA\frac{\partial T}{\partial n}} \tag{2.1}$$

where $\dot{Q}_n$ is the rate of conduction heat transfer in the *n*-direction, and $\dfrac{\partial T}{\partial n}$ is the *temperature gradient* in that direction. The *partial derivative* is used here because temperature gradients in other directions may also exist. One-dimensional conduction seldom occurs in practice since a body would have to be either very large so that conduction would then be one-dimensional at its centre, or it would have to be perfectly insulated at its edges.

The *instantaneous heat-transfer rate* expressed in Eq. (2.1) may also be written as

$$\boxed{q_n = \frac{\dot{Q}_n}{A} = -k\frac{\partial T}{\partial n}} \tag{2.2}$$

where q_n is the heat flux in kJ (or J) per unit time and per unit area of cross section in the *n*-direction. It is a *vector quantity* as it has both *magnitude* and *direction.* The maximum heat flux at any *isothermal* surface always occurs perpendicular to that surface.

Heat conduction within a solid can be looked upon as a heat flux which varies with *direction* as well as *position* throughout the material. The temperature within the solid will thus be a function of *spatial* (position) coordinates of the system, e.g. x, y, z in *rectangular coordinates.* Besides, the temperature may also be changing with time, t. Thus, in general, one can write: $T = f(x, y, z, t)$. To determine the *temperature distribution* is one of the main aims of any heat-conduction analysis. This represents how temperature varies with position in the medium. Once this distribution is known, the heat flux at any point in the body or on its surface may be calculated from Fourier's rate equation.

It may be noted that the knowledge of *temperature field or temperature distribution* can also be used to ensure structural integrity by determining thermal stresses, expansions, and deflections. We can also *optimize the insulating material thickness or find out the compatibility of special coatings or adhesives* used with any material, once the temperature distribution is determined.

Consider a solid body having the top view of its geometric shape as shown in Fig. 2.1 which shows a picture of temperature distribution illustrated by joining points of equal temperature to form *isothermal surfaces* shown only in *two dimensions.* The *lines of heat flow* in the direction of *maximum temperature gradient,* that is at right angles to the isothermal surfaces, are also depicted.

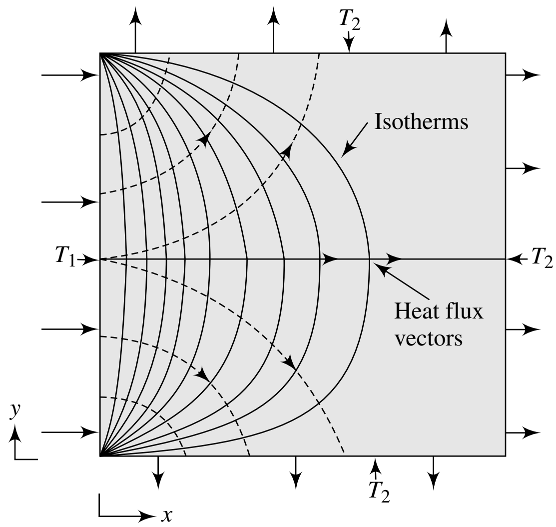


Fig. 2.1 *Lines of constant heat flux and constant temperatures in a solid body*

One side of this rectangle is held at one temperature, T_1, and the remaining three sides are held at the other temperature, T_2 which is less than T_1. The figure indicates lines of constant temperature (*isotherms*). The arrows drawn perpendicular to the isotherms represent the heat fluxes at these locations. The heat flux is greater where the isotherms are more closely spaced. Hence, heat flux is a vector quantity and has both magnitude and direction. For a three-dimensional temperature field, we can write the heat flux vector (q) in terms of its x-y, and z-components, (q_x, q_y, q_z), respectively:

$$\boxed{q = q_x\hat{i} + q_y\hat{j} + q_z\hat{k}} \tag{2.3}$$

where $\hat{i}, \hat{j}$, and $\hat{k}$ are unit direction vectors. The heat fluxes in the *three* directions will be given by

$$\boxed{q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}, \quad q_z = -k\frac{\partial T}{\partial z}} \tag{2.4}$$

For prior determination of the temperature distribution in the field, the differential equation governing the heat conduction will have to be first derived and subsequently solved for prescribed *initial and boundary conditions.* Furthermore, one must also have a basic understanding of the thermal properties of different engineering materials, the most important of which is *thermal conductivity*. The order of magnitude of the thermal conductivities of different types of materials also needs to be studied. In the following sections, we will first discuss in some detail some of the vital thermal properties of substances, particularly, thermal conductivity and derive generalized differential equations of the temperature field in different coordinate systems—*Cartesian* (*rectangular*), *cylindrical*, and *spherical*.

The objective of the conduction heat-transfer analysis may be to determine the *steady-state heat transfer rate* or *temperature distribution* through the solid, or we may want the *transient* (*time-dependent*) *temperature distribution* or *heat-transfer rate* in the solid.

2.2 ❑ GENERAL HEAT CONDUCTION EQUATION IN CARTESIAN COORDINATES

Heat transfer has *direction* as well as *magnitude*. The rate of heat conduction in a specified direction is proportional to the *temperature gradient,* which is the rate of change in temperature with distance in that direction. Heat conduction in a medium, in general, is *three-dimensional* and *time dependent*. The temperature in a medium varies with *position* as well as *time*, that is, $T = T(x, y, z, t)$. Heat conduction in a medium is said to be *steady* when the temperature does not vary with time, and *unsteady* or *transient* when it does. Heat conduction in a medium is said to be *one-dimensional* when conduction is significant in one dimension and negligible in the other two primary dimensions, *two-dimensional* when conduction in the third dimension is negligible, and *three-dimensional* when conduction in all dimensions is significant.

Consider a medium in which temperature gradients are present and the temperature distribution $T(x, y, z)$ is expressed in Cartesian coordinates. We first define an infinitesimally small *(differential)* control volume element $dx\ dy\ dz$, parallel to the coordinates x, y and z with its edges as shown in Fig. 2.2. The material considered is assumed to be *homogeneous* and *isotropic*—homogeneous because its physical properties, viz., density (ρ), specific heat (C_p), and thermal conductivity (k) are the same everywhere in the *temperature field*, and *isotropic* because the thermal conductivity at any point is constant in all directions, i.e., x, y, and z in Cartesian coordinates. The volume of the differential element, $d\forall = dx\ dy\ dz$.

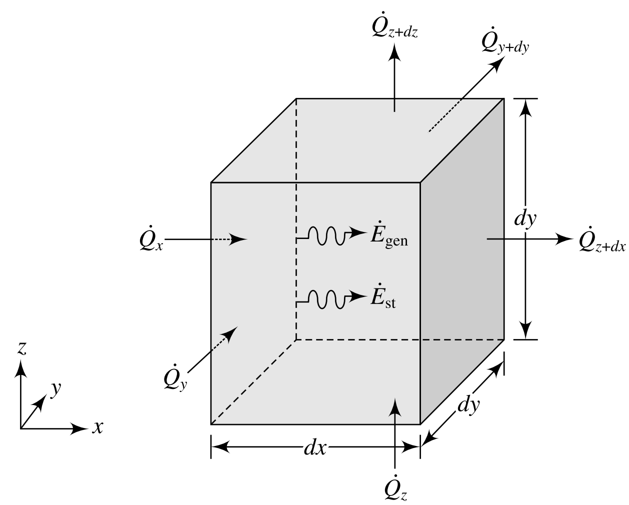


Fig. 2.2 *Three-dimensional differential control volume in a solid for derivation of the heat-conduction equation*

The principle of conservation of energy (the *first law of thermodynamics)* for the control volume gives the energy balance as follows:

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conducted in}\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conducted out}\end{pmatrix} + \begin{pmatrix}\text{Volumetric rate of}\\ \text{thermal energy generation}\\ \text{inside the element}\end{pmatrix} = \begin{pmatrix}\text{Rate of thermal}\\ \text{energy storage}\end{pmatrix}$$

or $$\boxed{(\dot{E}_{\text{in}} - \dot{E}_{\text{out}}) + \dot{E}_{\text{gen}} = \dot{E}_{st}} \tag{2.5}$$

Fourier's law can be used to express the heat flow in all the *three* directions.

In the x-direction Rate of heat flow *entering* the element,

$$\dot{Q}_x = -k(dy\,dz)\frac{\partial T}{\partial x}$$

Rate of heat flow *leaving* the element,

$$\dot{Q}_{x+dx} = \dot{Q}_x + \frac{\partial}{\partial x}(\dot{Q}_x)\,dx$$

NB $f(x + dx) = f(x) + \dfrac{\partial}{\partial x} f(x)\, dx + \ldots$[*Taylor series expansion, neglecting higher order terms*]

Hence, the net rate of heat conduction inside the element

$$\dot{Q}_x - \dot{Q}_{x+dx} = -\frac{\partial}{\partial x}(\dot{Q}_x)\,dx = -\frac{\partial}{\partial x}\left[-k(dy\,dz)\frac{\partial T}{\partial x}\right]dx = k(dx\,dy\,dz)\frac{\partial^2 T}{\partial x^2} \tag{2.6}$$

In the y-direction Rate of heat *inflow*, $\dot{Q}_y = -k(dx\,dz)\dfrac{\partial T}{\partial y}$

Rate of heat *outflow*, $\dot{Q}_{y+dy} = \dot{Q}_y + \dfrac{\partial}{\partial y}(\dot{Q}_y)\,dy$

Hence, net rate of heat conducted is,

$$\dot{Q}_y - \dot{Q}_{y+dy} = -\frac{\partial}{\partial y}(\dot{Q}_y)\,dy = -\frac{\partial}{\partial y}\left[-k(dx\,dy)\frac{\partial T}{\partial y}\right]dy = k(dx\,dy\,dz)\frac{\partial^2 T}{\partial y^2}$$

In the z-direction Rate of heat *inflow*, $\dot{Q}_z = -k(dx\,dy)\dfrac{\partial T}{\partial z}$

Rate of heat *outflow*, $\dot{Q}_{z+dz} = \dot{Q}_z + \dfrac{\partial}{\partial z}(\dot{Q}_z)dz$

Hence, net rate of heat conducted is,

$$\dot{Q}_z - \dot{Q}_{z+dz} = -\frac{\partial}{\partial z}(\dot{Q}_z)\,dz = -\frac{\partial}{\partial z}\left[-k(dx\,dy)\frac{\partial T}{\partial z}\right]dz = k(dx\,dy\,dz)\frac{\partial^2 T}{\partial z^2} \tag{2.7}$$

It follows that

$$\begin{aligned}\dot{E}_{\text{in}} - \dot{E}_{\text{out}} &= (\dot{Q}_x - \dot{Q}_{x+dx}) + (\dot{Q}_y - Q_{y+dy}) + (\dot{Q}_z - Q_{z+dz})\\ &= k\,dx\,dy\,dz\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]\end{aligned} \tag{2.8}$$

If the rate of heat generated *within* the element per unit volume is expressed as $\overline{q}$ then the rate of *thermal energy generation in the volume is given by*

$$\boxed{\dot{E}_{\text{gen}} = \overline{q}\,d\forall = \overline{q}(dx\,dy\,dz)} \tag{2.9}$$

The *rate of change of internal energy* of the material in the element is equal to the product of the mass of the material, its specific heat, and the rate of increase of the element's temperature.

$$\boxed{\dot{E}_{\text{st}} = \rho(dx\,dy\,dz)C_p \frac{\partial T}{\partial t}} \tag{2.10}$$

where t is the time. Incidentally, the subscript for the specific heat in the case of a solid is immaterial ($C_p = C_v = C$). In the case of *steady-state* conduction, this term would vanish and the temperature will be a function of only spatial coordinates (x, y, z).

Substituting the energy quantities expressed in Eqs. (2.9) through (2.11) in Eq. (2.5), we have

$$k(dx\,dy\,dz)\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \overline{q}(dx\,dy\,dz) = \rho(dx\,dy\,dz)\,C_p \frac{\partial T}{\partial t}$$

or
$$k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \overline{q} = \rho C_p \frac{\partial T}{\partial t}$$

or
$$\boxed{\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.11}$$

where the quantity $\alpha = \dfrac{k}{\rho C_p}$, is the *thermal diffusivity* of the material.

• Special Cases of Practical Interest

It may be noted that Eq. (2.9) was derived on the assumption of *constant thermal conductivity.* A more general equation for variable thermal conductivity would be

$$\boxed{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(y\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial z}\right) + \overline{q} = \rho C_p \frac{\partial T}{\partial t}} \tag{2.12}$$

However, for most engineering problems, the assumption of constant thermal conductivity is fairly satisfactory. The generalized three-dimensional differential heat conduction equation with unsteady-state and internal thermal-energy generation in Cartesian coordinates is then given by

$$\boxed{\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.13}$$

Using the *Laplacian* operator ∇^2, we can rewrite the above equation as

$$\boxed{\nabla^2 T + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.14}$$

Note that ∇^2 *is independent of the system of coordinates—rectangular, cylindrical, or spherical.* This equation may be simplified to suit any particular application.

For *steady-state* problems, the temperature does not change at any specified point in the body with time. Hence in such cases, Eq. (2.12) is reduced to

$$\boxed{\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\overline{q}}{k} = 0} \tag{2.15}$$

This is called the *Poisson equation.*

For *steady-state* heat flow in the absence of heat sources, i.e., *without heat generation,* Eq. (2.16) takes the form

$$\boxed{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0} \tag{2.16}$$

This is called the *Laplace equation.*

For *steady-state* conditions *with no thermal energy generation,* Eq. (2.12) becomes

$$\boxed{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.17}$$

This is called the *Fourier equation.*

For *steady-state, one-dimensional heat flow with no heat generation,* Eq. (2.17) is reduced to

$$\boxed{\frac{d^2 T}{dx^2}=0} \tag{2.18}$$

which is an *exact differential.*

2.3 ❑ GENERAL HEAT-CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

Many a time, conduction problems involve solids of cylindrical form, e.g., *solid or* hollow *round bars, tubes, cones*, *etc.* While the general heat-conduction equation derived earlier in Cartesian coordinates is appropriate for heat flows in solids with rectangular boundaries like walls, cubes, etc., it would be more suitable and convenient to use the cylindrical coordinate system for analyzing conduction heat transfer in solids with cylindrical boundaries. We will now derive a general heat-conduction equation in polar or cylindrical coordinates.

Consider the differential element **(**Fig. 2.3**)** having volume $r\ dr\ d\phi\ dz$. The material considered is *homogeneous* and *isotropic.*

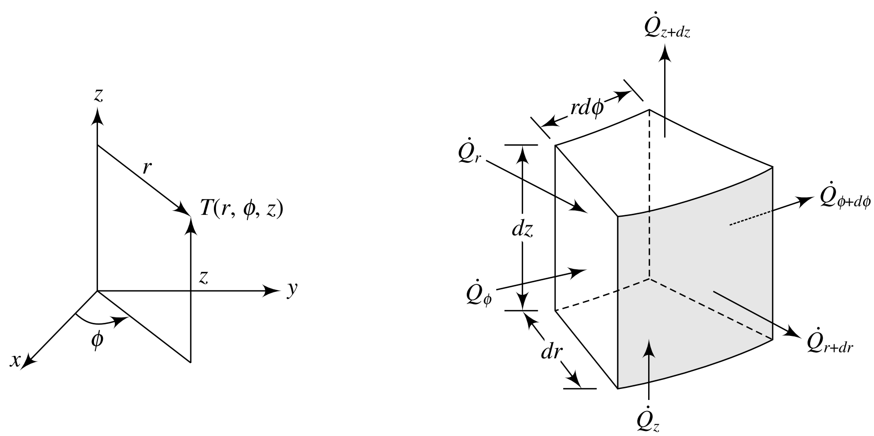


Fig. 2.3 *Cylindrical coordinate system, r, ϕ, z, and a differential control volume*

Energy balance by applying the first law *(conservation of energy)* to this element is

$$\begin{pmatrix}\text{Rate of}\\ \text{heat inflow}\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{outflow}\end{pmatrix} + \begin{pmatrix}\text{Rate of volumetric}\\ \text{thermal energy generation}\end{pmatrix} = \begin{pmatrix}\text{Rate of thermal}\\ \text{energy storage}\end{pmatrix}$$

or

$$\begin{pmatrix}\text{Net rate of heat gain}\\ \text{due to conduction}\end{pmatrix} + \begin{pmatrix}\text{Rate of internal}\\ \text{heat generation}\end{pmatrix} = \begin{pmatrix}\text{Rate of increase}\\ \text{of internal energy}\end{pmatrix}$$

i.e.,

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}}) + \dot{E}_{\text{gen}} = \dot{E}_{\text{st}} \tag{2.19}$$

Radial Direction Heat conducted *in* the radial direction, $\dot{Q}_r = -(rd\phi\, dz)\dfrac{\partial T}{\partial r}$

Heat conducted *out* in the radial direction, $\dot{Q}_{r+dr} = \dot{Q}_r + \dfrac{\partial}{\partial r}(\dot{Q}_r)\, dr$

Net heat gain in the radial direction,

$$\dot{Q}_r - \dot{Q}_{r+dr} = -\frac{\partial}{\partial r}\left[-k(rd\phi dz)\frac{\partial T}{\partial r}\right]dr = k(d\phi\cdot dr\cdot dz)\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = k(d\phi\cdot dr\cdot dz)\left[r\frac{\partial^2 T}{\partial r^2} + \frac{\partial T}{\partial r}\right] \tag{2.20}$$

Circumferential Direction Heat conducted *in* the circumferential direction

$$\dot{Q}_\phi = -k(dr\, dz)\frac{\partial T}{r\partial\phi}$$

Heat conducted *out* in the same direction

$$\dot{Q}_{\phi+d\phi} = \dot{Q}_\phi + \frac{\partial}{r\partial\phi}(\dot{Q}_\phi)\, r\, d\phi$$

Net heat gain in the circumferential direction

$$\dot{Q}_\phi - \dot{Q}_{\phi+d\phi} = -\frac{\partial}{r\partial\phi}[\dot{Q}_\phi]rd\phi = -\frac{\partial}{r\partial\phi}\left[-k(drdz)\frac{\partial T}{r\partial\phi}\right]rd\varphi = k(rd\phi\, dr\, dz)\frac{\partial^2 T}{r^2\partial\phi^2} \tag{2.21}$$

Axial Direction Heat conducted *in* the axial direction,

$$\dot{Q}_z = -k(rd\phi\cdot dr)\frac{\partial T}{\partial z}$$

Heat conducted *out* in the same direction,

$$\dot{Q}_{z+dz} = \dot{Q}_z + \frac{\partial}{\partial z}(\dot{Q}_z)dz$$

Net heat gain in the *axial* direction,

$$\dot{Q}_z - \dot{Q}_{z+dz} = -\frac{\partial}{\partial z}\left[-k(dr\, rd\phi)\frac{\partial T}{\partial z}\right]dz = k(r\, d\phi\, dr\, dz)\frac{\partial^2 T}{\partial z^2} \tag{2.22}$$

Hence,

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = (\dot{Q}_r - \dot{Q}_{r+dr}) + (\dot{Q}_\phi - \dot{Q}_\phi + \dot{Q}_\phi) + (\dot{Q}_z - \dot{Q}_{z+dz})$$

$$= k(rd\phi\cdot dr\cdot dz)\left[\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \left(\frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2}\right) + \frac{\partial^2 T}{\partial z^2}\right] \tag{2.23}$$

Rate of heat generated *within* the element = $\overline{q}\,(r\,d\phi\,dr\,dz)$ (2.24)

Rate at which heat is being stored *within* the element is

$$\boxed{\dot{E}_{\text{gen}} = \rho C_p (r\,d\phi\,dr\,dz)\frac{\partial T}{\partial t}} \tag{2.25}$$

Substitution of equations (2.24) through (2.26) into Eq. (2.20) leads to the general differential equation for conduction heat transfer in three dimensions in cylindrical coordinates.

Thus,

$$k\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \overline{q} = \rho C_p \frac{\partial T}{\partial t}$$

or

$$\boxed{\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.26}$$

Below is a more general equation with variable thermal conductivity

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \overline{q} = \rho C_p \frac{\partial T}{\partial t}$$

or

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.27}$$

This equation can also be simplified to suit any specific problem.

When temperature variation only in the *radial* direction is dominant and important, we have

$$\boxed{\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.28}$$

If *steady-state* condition prevails and there are *heat sources* within the material then the governing differential equation will be

$$\boxed{\frac{1}{r}\left[\frac{d}{dr}\left(r\frac{dT}{dr}\right)\right] + \frac{\overline{q}}{k} = 0} \tag{2.29}$$

For *one-dimensional, steady-state* heat conduction *without heat generation,* Eq. (2.29) becomes

$$\boxed{\frac{d^2 T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0} \tag{2.30}$$

or

$$\frac{1}{r}\left[r\frac{d^2 T}{dr^2} + \frac{dT}{dr}\right] = 0 \quad \text{or} \quad \frac{1}{r}\left[\frac{d}{dr}\left(r\frac{dT}{dr}\right)\right] = 0$$

As $\dfrac{1}{r}$ cannot be zero, $\boxed{\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = 0}$ (2.31)

Note that the above term is an *exact differential* and contains *no partial derivatives.*

As mentioned earlier, ∇^2, the Laplacian operator is not exclusive to rectangular coordinates. The Laplacian can also be expressed in cylindrical coordinates:

$$\boxed{\nabla^2 T \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}} \tag{2.32}$$

The general differential equation in *cylindrical* coordinates can also be obtained from that in the *Cartesian* coordinates by the *coordinate transformation method as explained below.*

- **General Heat-Conduction Equation in Cylindrical Coordinates: Coordinate Transformation Procedure**

The general heat-conduction equation can be transformed from Cartesian coordinates into cylindrical coordinates by using the following fundamental relations between the two coordinate systems (Fig. 2.4).

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \tag{2.33}$$

$$z = z$$

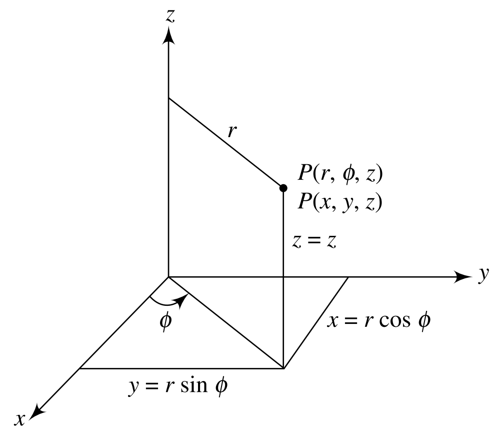


Fig. 2.4 *Relationship between Cartesian and cylindrical coordinates*

Differentiating equations (2.27) and (2.28) alternately with respect to r and ϕ, we get

$$\frac{\partial x}{\partial r} = \cos\phi \tag{2.34}$$

$$\frac{\partial x}{\partial \phi} = -r\sin\phi \tag{2.35}$$

$$\frac{\partial y}{\partial r} = \sin\phi \tag{2.36}$$

$$\frac{\partial y}{\partial \phi} = r\cos\phi \tag{2.37}$$

Both r and ϕ are functions of x and y $\left[r^2 = x^2 + y^2 \text{ and } \phi = \tan^{-1}\frac{y}{x}\right]$

Thus, we can write

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial r} \tag{2.38}$$

$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \phi}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial \phi} \tag{2.39}$$

Substituting the values from Eqs. (2.29), (2.30), (2.31), and (2.32) into Eq. (2.33) and (2.34), one obtains

$$\frac{\partial T}{\partial r} = \cos\phi\frac{\partial T}{\partial x}+\sin\phi\frac{\partial T}{\partial y} \tag{2.40}$$

$$\frac{\partial T}{\partial \phi} = -r\sin\phi\frac{\partial T}{\partial x}+r\cos\phi\frac{\partial T}{\partial y} \tag{2.41}$$

Multiplying both sides of Eq. (2.40) by cos ϕ, one gets

$$\cos\phi\frac{\partial T}{\partial r}=\cos^2\phi\frac{\partial T}{\partial x}+\cos\phi\sin\phi\frac{\partial T}{\partial y} \tag{2.42}$$

Multiplying both sides of Eq. (2.41) by $-\frac{1}{r}\sin\phi$, one has

$$-\frac{1}{r}\sin\phi\frac{\partial T}{\partial\phi}=\left(-\frac{1}{r}\sin\phi\right)(-r\sin\phi)\frac{\partial T}{\partial x}+(r\cos\phi)\left(-\frac{1}{r}\sin\phi\right)\frac{\partial T}{\partial y}$$

or
$$-\frac{1}{r}\sin\phi\frac{\partial T}{\partial\varphi}=\sin^2\phi\frac{\partial T}{\partial x}-\cos\phi\sin\phi\frac{\partial T}{\partial y} \tag{2.43}$$

Summing up equations (2.42) and (2.43), one gets

$$\cos\phi\frac{\partial T}{\partial r}-\frac{\sin\phi}{r}\frac{\partial T}{\partial\phi}=\frac{\partial T}{\partial x}(\cos^2\phi+\sin^2\phi)$$

But $\cos^2\phi+\sin^2\phi=1$

Hence,
$$\frac{\partial T}{\partial x}=\cos\phi\frac{\partial T}{\partial r}-\frac{\sin\phi}{r}\frac{\partial T}{\partial\phi} \tag{2.44}$$

Similarly, multiplying both sides of equations (2.40) and (2.41) by sin ϕ and $\frac{\cos\phi}{r}$ respectively, one finds

$$\sin\phi\frac{\partial T}{\partial r}=\sin\phi\cos\phi\frac{\partial T}{\partial x}+\sin^2\phi\frac{\partial T}{\partial y} \tag{2.45}$$

$$\frac{\cos\phi}{r}\frac{\partial T}{\partial\phi}=-\sin\phi\cos\phi\frac{\partial T}{\partial x}+\cos^2\phi\frac{\partial T}{\partial y} \tag{2.46}$$

Adding equations (2.45) and (2.46), one gets

$$\frac{\partial T}{\partial y}=\sin\phi\frac{\partial T}{\partial r}+\frac{\cos\phi}{r}\frac{\partial T}{\partial\phi}\qquad(\because\ \sin^2\phi+\cos^2\phi=1) \tag{2.47}$$

Now, putting $\frac{\partial T}{\partial x}$ in place of T in Eq. (2.44), one obtains

$$\frac{\partial^2T}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)=\frac{\partial}{\partial x}\left[\cos\phi\frac{\partial T}{\partial r}-\frac{\sin\phi}{r}\frac{\partial T}{\partial\phi}\right]$$

The operator $\frac{\partial}{\partial x}$ can be written as

$$\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}$$

Hence,
$$\frac{\partial^2T}{\partial x^2}=\left(\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial T}{\partial r}-\frac{\sin\phi}{r}\frac{\partial T}{\partial\phi}\right)$$

$$=\cos^2\phi\frac{\partial^2T}{\partial r^2}-\cos\phi\sin\phi\left(-\frac{1}{r^2}\right)\frac{\partial T}{\partial\varphi}-\frac{\cos\phi\sin\phi}{r}\frac{\partial^2T}{\partial r\partial\phi}$$

$$-\frac{\sin\phi}{r}(-\sin\phi)\frac{\partial T}{\partial r}-\frac{\sin\phi}{r\cos\phi}\frac{\partial^2T}{\partial r\partial\phi}+\frac{\sin^2\phi}{r^2}\frac{\partial^2T}{\partial\phi^2}+\frac{\sin\phi}{r}\frac{\partial T}{\partial\phi}\left(\frac{\cos\phi}{r}\right)$$

or $$\frac{\partial^2 T}{\partial x^2} = \cos^2\phi \frac{\partial^2 T}{\partial r^2} + \frac{\sin^2\phi}{r}\frac{\partial T}{\partial r} + \frac{2\cos\phi\sin\phi}{r^2}\frac{\partial T}{\partial \phi} - 2\sin\phi\cos\phi\frac{\partial^2 T}{\partial r\partial\phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2 T}{\partial\phi^2} \tag{2.48}$$

Similarly, putting $\frac{\partial T}{\partial y}$ in place of T in Eq. (2.47), one gets

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left[\sin\phi\frac{\partial T}{\partial r} + \frac{\cos\phi}{r}\frac{\partial T}{\partial\phi}\right]$$

The operator $\frac{\partial}{\partial y}$ can be written as

$$\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}$$

Hence, $$\frac{\partial^2 T}{\partial y^2} = \left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial T}{\partial r} + \frac{\cos\phi}{r}\frac{\partial T}{\partial\phi}\right)$$

$$= \sin^2\phi\frac{\partial^2 T}{\partial r^2} + \frac{\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial r\partial\phi} + \sin\phi\cos\phi\frac{\partial T}{\partial\phi}\left(-\frac{1}{r^2}\right)$$

$$+ \frac{\cos\phi\sin\phi}{r}\frac{\partial^2 T}{\partial r\partial\phi} + \frac{\cos\phi}{r}(\cos\phi)\frac{\partial T}{\partial r} + \frac{\cos^2\phi}{r^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\cos\phi}{r}\frac{\partial T}{\partial\phi}\frac{(-\sin\phi)}{r} \tag{2.49}$$

$$= \sin^2\phi\frac{\partial^2 T}{\partial r^2} + \frac{\cos^2\phi}{r}\frac{\partial T}{\partial r} - \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial T}{\partial\phi} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial r\partial\phi} + \frac{\cos^2\phi}{r^2}\frac{\partial^2 T}{\partial\phi^2} \tag{2.50}$$

By identity, $$\frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial z^2} \tag{2.51}$$

The summation of equations (2.48), (2.49), and (2.50) yields

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial r^2}(\cos^2\phi + \sin^2\phi)$$

$$+ \frac{1}{r}\frac{\partial T}{\partial r}(\sin^2\phi + \cos^2\phi) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2}(\sin^2\phi + \cos^2\phi) + \frac{\partial^2 T}{\partial z^2}$$

As $\sin^2\phi + \cos^2\phi = 1$, one finally gets

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.52}$$

Substituting Eq. (2.51) into Eq. (2.12) gives

$$\boxed{\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.53}$$

One can write

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{r}\left[r\frac{\partial^2 T}{\partial r^2} + \frac{\partial T}{\partial r}\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Therefore, one can also express Eq. (2.52) as

$$\boxed{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.54}$$

2.4 ❑ GENERAL HEAT-CONDUCTION EQUATION IN SPHERICAL COORDINATES

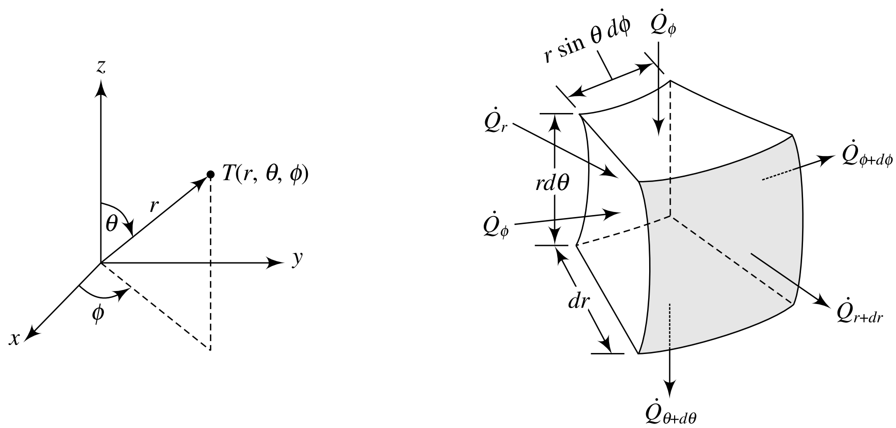
It is often more convenient to use the spherical coordinates for certain shapes of heat conductors.

Let us consider the heat flow through an infinitesimal spherical volume element $d\forall$ of a *homogeneous* and *isotropic* material shown in Fig. 2.5. The three sides of this differential control volume are dr, $r\,d\phi$, and $r \sin\,\theta d\phi$.

Then $d\forall = (dr)\,(r\,d\theta)\,(r \sin\,\theta\,d\phi)$ or $d\forall = r^2 \sin\,\theta\,dr\,d\phi\,d\theta$

where the angle ϕ is measured from the x-axis towards the y-axis, and the angle ϕ is measured from the z-axis towards the x-y plane.

The physical properties, viz., density ρ, specific heat, C_p, and thermal conductivity, k are considered constant.



(2.55)

Fig. 2.5 *Spherical coordinate system r, θ, φ, and a differential control volume*

In the Radial r-direction The rate of heat flow *entering* the control volume, $\dot{Q}_r = -k(r\,d\theta)(r \sin\,\theta\,d\phi)\dfrac{\partial T}{\partial r}$

The rate of heat flow *leaving* the control volume, $\dot{Q}_{r+dr} = \dot{Q}_r + \dfrac{\partial}{\partial r}[\dot{Q}_r]dr$

The *net* rate of heat flow *into* the volume,

$$Q_r - Q_{r+dr} = -\frac{\partial}{\partial r}[Q_r]dr$$

$$= \frac{\partial}{\partial r}\left[kr^2 \sin\theta\,dr\,d\phi\,d\theta\frac{\partial T}{\partial r}\right] = k\,\sin\,\theta\,dr\,d\phi\,d\theta\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right] \tag{2.56}$$

In the Azimuthal φ-direction The rate of heat flow *entering* the volume,

$$\dot{Q}_\phi = -k(r\,d\theta\,dr)\frac{\partial T}{\partial(\phi r \sin\theta)}$$

The rate of heat flow *leaving* the volume,

$$\dot{Q}_{\phi+d\phi} = \dot{Q}_\phi + \frac{\partial[\dot{Q}_\phi]}{\partial(\phi r \sin\theta)}d(\phi r \sin\theta)$$

The *net* rate of heat flow *into* the volume,

$$\dot{Q}_\phi - \dot{Q}_{\phi+d\phi} = \frac{\partial}{\partial(\phi r \sin\theta)}\left[\left\{(k\,r\,d\theta\,dr)\frac{\partial T}{\partial(\phi\, r\,\sin\theta)}\right\}\right] r\sin\theta\,d\phi \tag{2.57}$$

In the Zenith (Polar) θ direction The rate of heat flow *entering* the volume,

$$\dot{Q}_\theta = -k(r\sin\theta\,d\phi\,dr)\frac{\partial T}{\partial(\theta r)}$$

The rate of heat flow *leaving* the volume,

$$\dot{Q}_{\theta+d\theta} = \dot{Q}_\theta + \frac{\partial}{\partial(\theta r)}[\dot{Q}_\theta]d(\theta r)$$

The *net* rate of heat flow *into* the volume,

$$\dot{Q}_\theta - \dot{Q}_{\theta+d\theta} = \frac{\partial}{\partial(\theta r)}\left[k(r\sin\theta d\phi)\,dr\,\frac{\partial T}{\partial(\theta r)}\right] r d\theta \tag{2.58}$$

The net rate of heat added to the volume element is obtained by summing up equations (2.57), (2.58) and (2.59).

The differential operators in equations (2.58) and (2.59) can be evaluated as follows:

$$\frac{\partial}{\partial(\phi r\sin\theta)} = \left(\frac{\partial}{\partial\phi}\right)\frac{\partial\phi}{\partial(\phi r\sin\theta)} = \left(\frac{\partial}{\partial\phi}\right)\frac{1}{\dfrac{\partial}{\partial\phi}(\phi r\sin\theta)} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{2.59}$$

$$\frac{\partial}{\partial(\theta r)} = \left(\frac{\partial}{\partial\theta}\right)\frac{\partial\theta}{\partial(\theta r)} = \left(\frac{\partial}{\partial\theta}\right)\frac{1}{\dfrac{\partial}{\partial\theta}(\theta r)} = \frac{1}{r}\frac{\partial}{\partial\theta} \tag{2.60}$$

By using equations (2.60) and (2.61) and by adding up equations (2.57), (2.58), and (2.59), we obtain

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}}) = (\dot{Q}_r - \dot{Q}_{r+dr}) + (\dot{Q}_\phi - \dot{Q}_{\phi+d\phi}) + (\dot{Q}_\theta - \dot{Q}_{\theta+d\theta}) \tag{2.61}$$

Hence, the *total* rate of heat flow *added* to the differential control volume is

$$\begin{aligned}
\dot{E}_{\text{in}} - \dot{E}_{\text{out}} &= k\sin\theta\,dr\,d\phi\,d\theta\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left[\frac{k\,r\,dr\,d\theta}{r\sin\theta}\frac{\partial T}{\partial\phi}\right] r\sin\theta\,d\phi \\
&\quad + \frac{1}{r}\frac{\partial}{\partial\theta}\left[k(r\sin\theta\,d\phi\,dr)\frac{1}{r}\frac{\partial T}{\partial\theta}\right] r\,d\theta = k\sin\theta\,dr\,d\phi\,d\theta\left[r^2\frac{\partial^2 T}{\partial r^2} + 2r\frac{\partial T}{\partial r}\right] \\
&\quad + \frac{k\,r\,dr\,d\theta(r\sin\theta\,d\phi)}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} + \frac{k}{r^2}d\phi\,dr(r^2 d\theta)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) \\
&= (r^2\sin\theta\,dr\,d\phi\,d\theta)\left\{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right\} + (r^2\sin\theta\,dr\,d\phi\,d\theta)\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \\
&\quad + k(\sin\theta)(d\phi\,dr\,r^2 d\theta)\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) \\
&= [k(r^2\sin\theta\,dr\,d\phi\,d\theta)]\left[\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left(\frac{\partial^2 T}{\partial\phi^2}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)\right]
\end{aligned} \tag{2.62}$$

The rate of internal heat generation is

$$\boxed{\dot{E}_{\text{gen}} = \overline{q}\, d\forall (dr \cdot r\, d\theta \cdot r \sin\theta\, d\phi) = \overline{q}(r^2 \sin\theta\, dr\, d\phi\, d\theta)} \tag{2.63}$$

where $\overline{q}$ is the rate of heat generation per unit volume.
The rate of change of internal energy in the elemental volume

$$\boxed{\dot{E}_{\text{st}} = mC_p \frac{\partial T}{\partial t} = \rho \forall C_p \frac{\partial T}{\partial t} = (r^2 \sin\theta\, dr\, d\theta\, d\phi)\rho C_p \frac{\partial' T}{\partial t}} \tag{2.64}$$

Energy balance on the control volume gives:

$$\underbrace{\begin{pmatrix}\text{Net rate of heat}\\ \text{conducted in}\end{pmatrix}}_{\text{Eq. (2.62)}} + \underbrace{\begin{pmatrix}\text{Rate of internal}\\ \text{heat generated}\end{pmatrix}}_{\text{Eq. (2.63)}} = \underbrace{\begin{pmatrix}\text{Rate of change of}\\ \text{internal energy}\end{pmatrix}}_{\text{Eq. (2.64)}}$$

Hence, $$k\left[\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right)\right] + \overline{q} = \rho C_p \frac{\partial T}{\partial t}$$

or $$\frac{1}{r^2}\left[r^2 \frac{\partial^2 T}{\partial r^2} + 2r\frac{\partial T}{\partial r}\right] + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{\overline{q}}{k} = \frac{\rho C_p}{k}\frac{\partial T}{\partial t}$$

or $$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.65}$$

The spherical coordinate system is related to the Cartesian or rectangular coordinate system by the following relations (Fig. 2.6):

$$\left.\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \end{aligned}\right\}$$
$$z = r \cos\theta$$

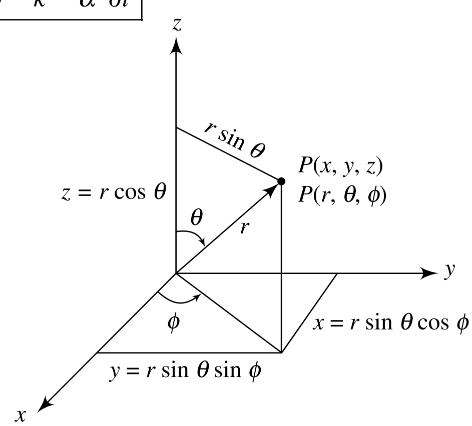


Fig. 2.6 *Relationship between Cartesian and spherical coordinates*

We can transform the equation in Cartesian coordinates into spherical coordinates by making the above substitutions and proceed in a manner similar to that for cylindrical coordinates. The general equation of heat conduction in spherical coordinates will then be given by

$$\boxed{\left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right] + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.66}$$

For *steady-state, one-dimensional* heat conduction with *no heat generation,* the equation in exact differential form is

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dt}\right) = 0} \tag{2.67}$$

It is important to remember that the terms *heat-generation rate* $(\dot{E}_{\text{gen}})$ and heat-storage rate $(\dot{E}_{\text{st}})$ are distinctly different physical processes. While energy generation is due to energy conversion involving *chemical, electrical*, *or nuclear* energy manifested as *thermal* energy. This term is positive *(source)* if the thermal energy is evolved in the material and negative *(sink)* if the thermal energy is consumed or absorbed *(rather than generated)* within the material. The strength of such *sources* or *sinks* is always specified per unit volume. The other term $\dot{E}_{\text{st}}$ is the rate of energy storage in the material as internal energy because of the change of temperature with time. *Under steady-state conditions without heat generation, the heat fluxes as also the temperature gradients will be constant throughout.*

The temperature distribution *within* the solid and the rate of heat transfer *across* the solid boundaries can be determined by integrating the appropriate heat-conduction equation. The *constants of integration* can be evaluated by using appropriate *boundary* and *initial* conditions which will be discussed in the next section.

• A Compact Equation

The *one-dimensional, time-dependent* heat-conduction equation *with thermal energy generation* in the *three* principal coordinate systems can be written in the form of a *single* equation

$$\boxed{\frac{1}{r^n}\frac{\partial}{\partial r}\left[r^n k\frac{\partial T}{\partial r}\right]+\overline{q}=\rho C_p\frac{\partial T}{\partial t}} \tag{2.68}$$

where $n = 0$ for Cartesian coordinates
$n = 1$ for cylindrical coordinates
$n = 2$ for spherical coordinates

In Cartesian coordinates, it is variable with the x-variable.

- *For constant thermal conductivity,* Eq. (2.68) simplifies to

$$\boxed{\frac{1}{r^n}\frac{\partial}{\partial r}\left[r^n\frac{\partial T}{\partial r}\right]+\frac{\overline{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{2.69}$$

- For steady-state heat conduction with internal heat generation, and constants k, Eq. (2.69) becomes

$$\boxed{\frac{1}{r^n}\frac{d}{dr}\left[r^n\frac{dT}{dr}\right]+\frac{\overline{q}}{k}=0} \tag{2.70}$$

For steady-state heat conduction without any energy generation within the medium and constant k, Eq. (2.68) reduces to

$$\boxed{\frac{d}{dr}\left[r^n\frac{dT}{dr}=0\right]} \tag{2.71}$$

2.5 ❑ THERMAL PROPERTIES OF MATERIALS

2.5.1 • Thermal Conductivity

Heat conduction is basically the *transmission of energy by molecular motion.* Metals, particularly pure metals, are generally the best conductors of heat. The transfer of heat by conduction in solids is brought about partly by *free electrons* and partly by *lattice vibrations.* For most metals, the flow of free electrons contributes significantly to the process of heat transmission as mentioned in Chapter 1. When these metals are heated, the increased vibration of the atoms interferes with the motion of the free electrons. *Hence, as the temperature increases, the thermal conductivity of these metals decreases. Aluminium* is an exception. Its thermal conductivity remains fairly constant over a wide temperature range. The other exception is *uranium* whose thermal conductivity *increases* with temperature. For *nickel and platinum,* the thermal conductivity first *decreases* with temperature and then *increases.*

The thermal conductivity of *liquid metals* is usually *lower* than that of solids and *increases* with *increasing* temperature. The thermal conductivity of most *metallic alloys* is less than that of any of the constituents and increases with an increase in temperature with an exception of *aluminium alloys.*

Non-metallic solids are not effective thermal conductors. They are often called *semiconductors or insulators.* As mentioned earlier, *all good electrical conductors are good thermal conductors too.*

All electrical *insulators* ought to dissipate the heat generated by the current in the electrical wires *(ohmic heating)* faster to ensure better current-carrying capacity of the wire. What is more, the higher the voltage, the heavier the insulation and the greater the requirement of an electrical insulator to be an effective thermal conductor. This often poses problems in proper selection of insulating materials for electrical applications.

Amorphous materials like *fire clay* and *ordinary glass* are, sometimes, called *glassy materials.* Their thermal conductivity is usually small but *increases* with *increasing* temperature. *Crystalline* materials like *quartz* have *higher* thermal conductivity which *decreases* with *increasing* temperature.

Non-metallic solids can be porous or non-porous. Glass, plastics, and quartz are non-porous. Materials while brick, cork, wood, leather, and felt are porous. Moisture always causes an increase of density and thermal conductivity. The thermal conductivity of *wood,* an *anisotropic* material, is *larger* in the direction *parallel to the grain* than it is in the direction *across the grain.*

The thermal conductivity of *refractory materials* varies also with apparent density and temperature. It usually increases with density. The value of thermal conductlies for *fire clay* for example, *increases* with *increasing* temperature while that for *magnesite brick*, *decreases* with *increasing* temperature.

The thermal conductivity of *snow* is also proportional to its *density.* The density of *ice* increases with decreasing temperature. Thermal conductivity too varies in the same manner. This information is useful in estimating the rate of ice formation on a lake or elsewhere.

Mercury has the lowest thermal conductivity among *metals*, liquid or solid, while *water* is *the best conducting non-metallic liquid*. The thermal conductivity of water *increases* with *increasing* temperature up to 120°C and then decreases as the temperature continues to rise.

Gases with higher Relative Molecular Mass (RMM) or molecular weight in general have smaller values of k. The thermal conductivity of *gases* increases with *increasing pressure*.

The thermal conductivity of *metals* and *alloys* is usually in the range of 20 to 425 W/m K, that of *non-metallic solids* in the range of 0.02 to 20 W/m K. With the exception of *liquid metals* the thermal conductivity of liquids varies approximately within a narrow range of 0.2 to 2 W/m K. For *gases* and *vapours,* this range is about 0.004 to 0.04 W/m K except *hydrogen* and *helium* whose values of k are in the neighbourhood of 0.2 W/m K.

Silver has a conductivity 50 000 times as great as that of Freon-12 (refrigerant). Generally speaking, a *liquid is a better conductor than a gas* and that *a solid is a better conductor than a liquid.* Take *mercury* for instance. As a solid, the thermal conductivity of mercury (at 193°C) is of the order of 48 W/m K, as a liquid (at 0°C) the conductivity *decreases* to 8 W/m K and as a gas (at 200°C) its thermal conductivity is estimated to be 0.0341 W/m K.

Figure 2.7 shows the variation of thermal conductivity of some metals with temperature, while Fig. 2.8 presents the variation of thermal conductivity of some liquids and gases with temperature.

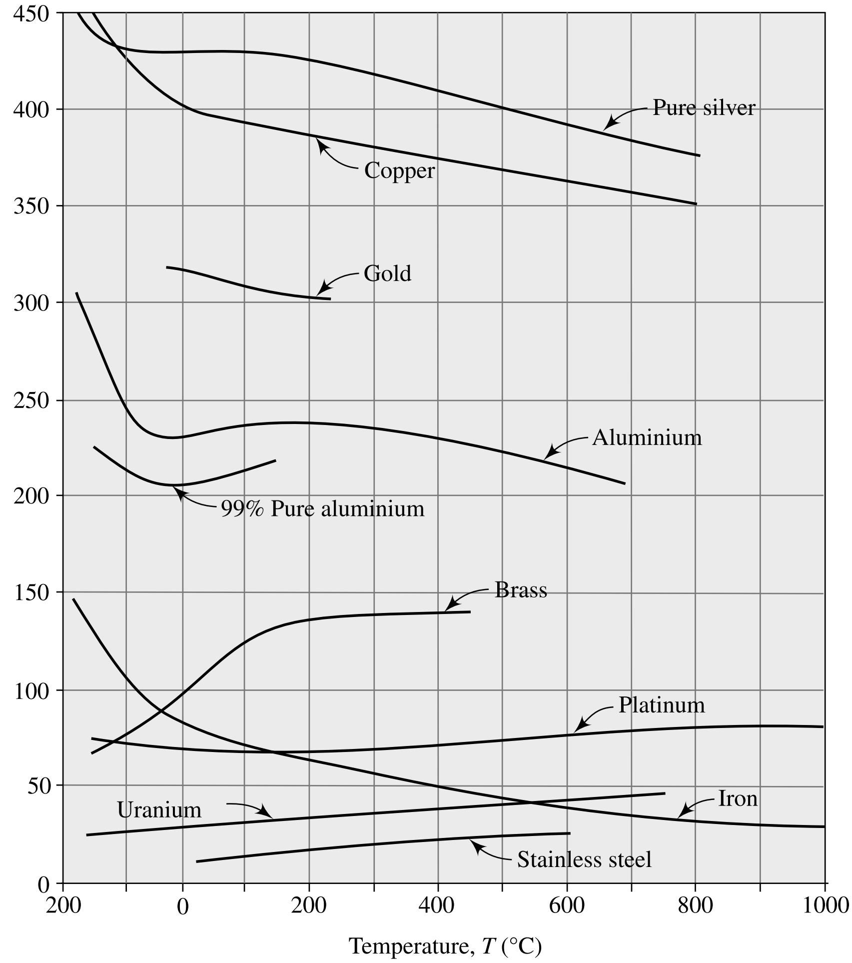


Fig. 2.7 *Variation of thermal conductivity of metallic solids with temperature*

2.5.2 • Thermal Diffusivity

Thermal diffusivity is a significant physical property of the material. Its physical significance is associated with the propagation of heat into the medium in *unsteady-state* heat conduction situations. *The larger the value of* α, *the faster the heat will propagate or diffuse through the material.* This can be appreciated by examining the properties that make up the thermal diffusivity, i.e., k (thermal conductivity) and ρC_p (thermal capacity per unit volume).

Thermal capacity indicates the heat-retention capacity or thermal inertia of a material. Thermal diffusivity can be high because of large value of k, which would imply a rapid heat-transfer rate. A high value of α could also result from a low value of thermal capacity ρC_p, which would indicate that less of the energy moving through the material would be absorbed and used to increase the temperature of the material and consequently, more energy would be available for onward transfer.

Table 2.1 lists the time required for the temperature of a solid initially at 100°C to be reduced to 50°C at a distance of 0.3 m from the boundary surface for materials having different values of α.

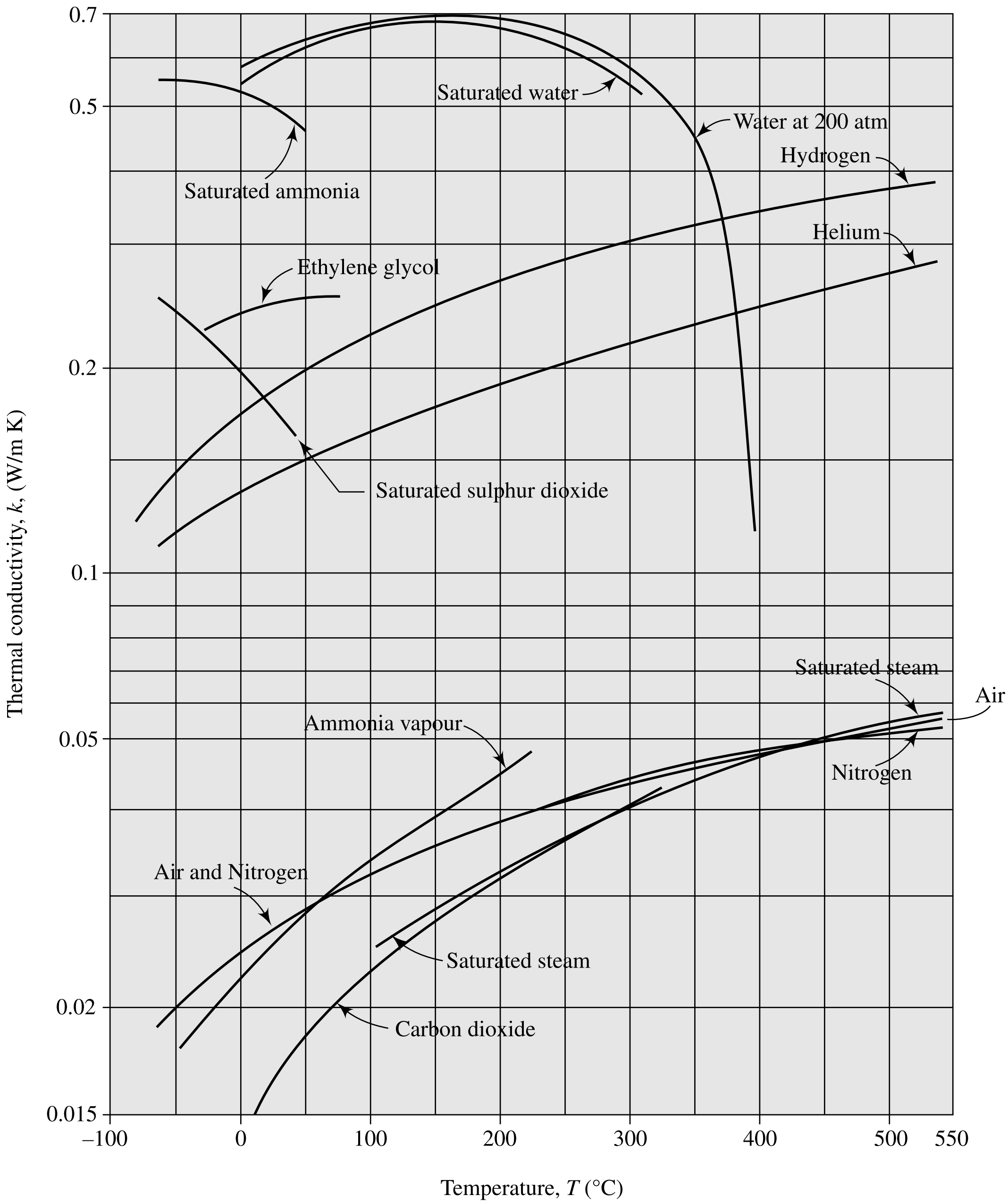


Fig. 2.8 *The temperature dependence of the thermal conductivity of liquids and of gases that are either saturated or at 1 atm pressure.*

Clearly, the larger the thermal diffusivity, the less time is required for heat to penetrate into the solid. In other words, *the thermal diffusivity is a measure of how quickly a material can carry heat away from a heat source.* Since a material does not just transmit heat but must be warmed by it as well, α involves both the conductivity, k, and the volumetric heat capacity, ρC_p.

Table 2.1 *Effect of thermal diffusivity on the rate of heat propagation*

Material	Thermal diffusivity × 10^6 (m^2/s)	Time
Silver	170	9.5 min
Copper	103	16.5 min
Steel	12.9	2.2 h
Glass	0.59	2 days

Metals possess *higher* thermal conductivity, *lower* thermal capacity, and thus *higher* thermal diffusivity. *Liquids* have *lower* thermal conductivity*, higher* thermal capacity and hence *smaller* thermal diffusivity. A large value of α means greater effectiveness of the material in energy transfer by conduction rather than of energy storage. It is instructive to note that the heat diffuses through the *gases* at almost the same rate as it does through the metals, while metals have higher values of k than metals.

Thermal diffusivity is as important in *transient heat conduction* as *thermal conductivity* is in *steady-state heat conduction.* In fact, in a steady state, the term thermal diffusivity has virtually no significance.

2.6 ❑ INITIAL AND BOUNDARY CONDITIONS

We are now in a position to calculate the temperature distribution as well as heat-transfer rate with the help of general heat conduction equation. One can first get $T_{(\text{space, time})}$ and then by differentiating T obtain the heat-transfer rate from the Fourier law. The heat-diffusion equation is a *partial differential equation* not easily amenable to quick solution. After some simplifications, assumptions and approximations, we can obtain one-dimensional, steady-state situations. The heat conduction equation then becomes an *ordinary linear differential equation* the solution of which is not quite difficult.

It is noteworthy that as the heat-diffusion equation is the second-order differential equation in spatial coordinates (x, y, z or r, ϕ, z or r, ϕ, θ), *two* boundary conditions must be expressed for each coordinate required to describe the system. Only one *initial* condition is, however, necessary to be specified because the equation is *first order in time.*

The initial condition specifies the temperature distribution at the origin of the time coordinate (t = 0). In a *steady-state problem, the initial condition is not necessary.*

The most important boundary conditions commonly encountered in practice are

1. Prescribed temperature boundary condition—boundary condition of the *first* kind.
2. Prescribed heat flux boundary condition—boundary condition of the *second* kind.
3. Convection boundary condition—the boundary condition of the *third* kind.
4. Radiation boundary condition—the boundary condition of the *fourth* kind
5. Interface boundary condition

2.6.1 • Boundary Condition of the *First Kind* (or a *Dirichlet Condition*)

This condition occurs when the temperature is prescribed on a boundary surface. One of the more common conditions is a surface with a constant temperature. For example, if the boundary is at $x = L$,

$$\boxed{T(L,t) = T_w \qquad \text{constant surface temperature}} \tag{2.72}$$

The temperature T_w must be a known quantity, and it must be fixed at T_w for all times, t. Physically, this mathematical boundary condition is approximated when a constant-pressure melting ice, boiling liquid or condensing steam with a very large heat-transfer coefficient touches the solid surface of a body. Consider the equation $q = h\Delta T$, where ΔT is the difference between the surface and the fluid. For a finite heat-transfer rate, the temperature difference will approach zero if the heat-transfer coefficient is extremely large, so that the surface can be considered at a constant and uniform temperature. In general, the prescribed surface temperature could be constant, a known function of location, or a known function of time.

2.6.2 • Boundary Condition of the *Second* Kind (*or a Neumann Condition*)

This condition occurs when heat flux is prescribed on the boundary. In general, the prescribed surface heat flux could be constant, a known function of location or a known function of time. For example, if a constant heat flux, q_w, is applied at a boundary at $x = 0$, then

$$\boxed{-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q_w \qquad \text{constant surface heat flux}} \tag{2.73}$$

Fourier's law is used to denote the heat conducted into the solid at the boundary which is balanced by the external heat supplied to the boundary, q_w. This boundary condition is approximated when there is an electrically heated surface with the heat flow rate entering the solid.

There are two *special cases* of this boundary condition:

Adiabatic (Insulated) Boundary For a perfectly insulated (or *adiabatic*) when the heat flux is set equal to zero, we can write

$$\boxed{-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q_w \qquad \text{adiabatic (included) surface}} \tag{2.74}$$

Thermal and Geometric Symmetry Often the known heat flux is zero, such as a plane of symmetry. At the symmetry axis, the temperature gradient in a direction normal to the axis disappears. Thus,

$$\boxed{\frac{\partial T}{\partial x} = 0} \tag{2.75}$$

This considerably simplifies the mathematical formulation of the heat-conduction problem.

In fact, it is more convenient to solve such a problem over half of the region subject to the relevant boundary conditions.

2.6.3 • Boundary Condition of the *Third Kind* (or a *Convection Boundary Condition*)

This condition occurs when is in contact with an adjacent fluid at T_∞ (which could be constant, a known function of location, or a known function of time) surface at a certain temperature, T_w. By performing an energy balance at the boundary, at $x = L$,

$$\boxed{-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T_w - T_\infty] \qquad \text{convection at the surface}} \tag{2.76}$$

2.6.4 • Radiation Boundary Condition

When *radiation* is the only mechanism of heat transfer between the surface and the surroundings, the radiation boundary condition on a surface can be expressed by writing the following energy balance

$$\boxed{-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = \sigma[T_{x=L}^4 - T_{\text{sur}}^4] \qquad \text{radiation at the surface}} \tag{2.77}$$

It is to be emphasized that the rate of heat transfer by radiation introduces *a non-linearity* into the boundary conditions since the radiation heat transfer is proportional to the *fourth power* of the *absolute* temperature. Hence, usually radiation heat exchange at a surface is neglected to avoid the complexities involved due to non-linearity especially when heat transfer at the surface is essentially by convection, and radiation plays a marginal role.

2.6.5 • Interface Boundary Condition

This boundary condition is based on the requirement that

- Two bodies in contact must have the *same temperature* at the area of contact, and
- An interface (*which is a surface*) cannot store any energy.

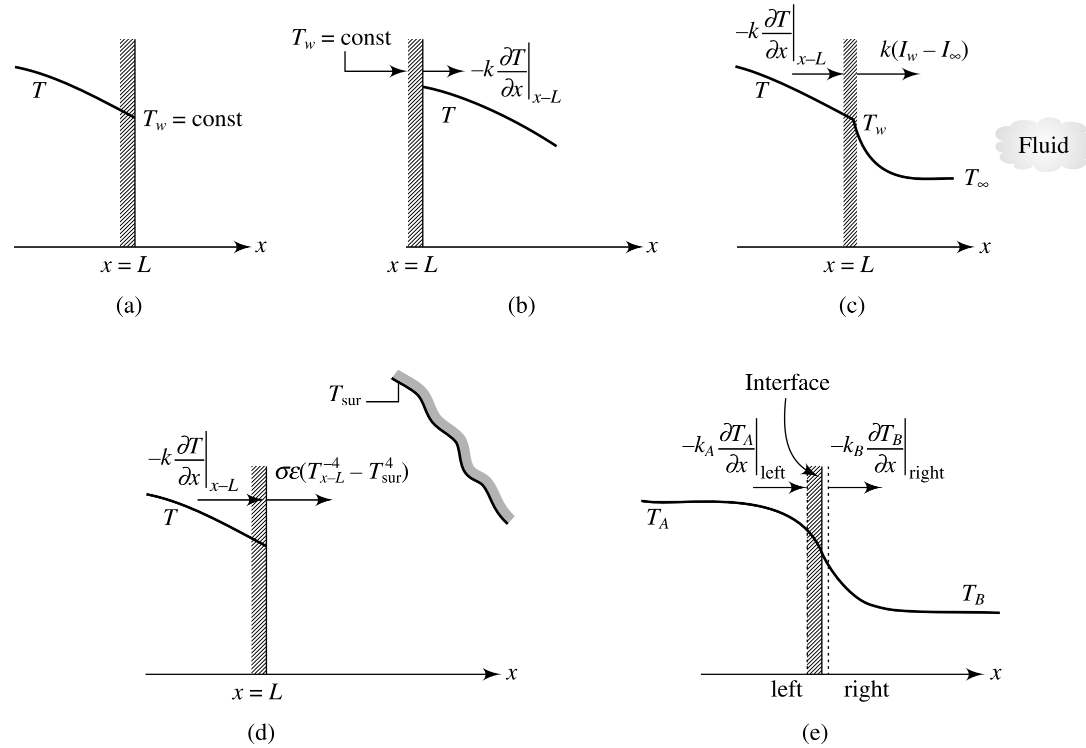


Fig. 2.9 *Types of boundary conditions. (a) First kind, or Dirichlet. (b) Second kind, or Neumann. (c) Third kind, or mixed. (d) Fourth kind, or radiation. (e) Interface.*

Thus, the *heat flux* on the two sides of an interface *must be the same*. The boundary condition at the interface of two bodies A and B in perfect contact can be expressed as (*at the interface*)

$$T_A = T_B \qquad \text{(at the interface)}$$

and

$$\boxed{-k_A \left.\frac{\partial T_A}{\partial x}\right|_{\text{left}} = -k_B \left.\frac{\partial T_B}{\partial x}\right|_{\text{right}}} \tag{2.78}$$

Figure 2.9 illustrates schematically the above five boundary conditions.

2.7 ❑ STEADY-STATE, ONE-DIMENSIONAL CONDUCTION

Let us analyze the steady-state temperature distribution and corresponding heat-transfer rate for the cases where the heat-transfer rate, or the temperature, is a function of only one distance variable such as *large plane walls, long cylinders, short cylinders with the ends insulated, or hollow spheres*. Results will be obtained both for constant thermal conductivity and for the thermal conductivity as a function of temperature.

The general methodology for solving such problems is given below:

1. Draw the neat schematic based on the problem statement.
2. Write the right governing differential equation.
3. State clearly the assumptions made.
4. Explicitly mention the boundary conditions.
5. Carry out the integration between prescribed limits and get the general solution.
6. Evaluate constants of integration.
7. Obtain the temperature distribution by putting back the calculated constants in the general solution.
8. Determine the rate of heat transfer.
9. Play with the solution (*look it over*) and see what it implies.
10. Plot the graph, if necessary, preferably in the dimensionless form.

2.7.1 • Conduction Through a Plane Wall

Consider heat transfer through a plane wall shown schematically in Fig. 2.10. This could represent a wall in a house, a window, or some other large flat plane with specified uniform and constant temperatures on each face. The material properties are also prescribed. We have to determine the temperature distribution in the wall and the heat transfer rate through the wall.

The two extreme surfaces are maintained at constant and uniform temperatures. $T = T_1$ at $x = 0$, and $T = T_2$ at $x = L$.

Assumptions

- One-dimensional conduction, *the plane wall is large with small thickness compared to the dimensions in the y- and z-directions*).
- Steady-state conditions (*the temperature at any point within the wall does not change with time*). The temperatures at different points within the wall will of course be different.
- The area normal to heat flow, A is independent of x.
- No internal heat generation.

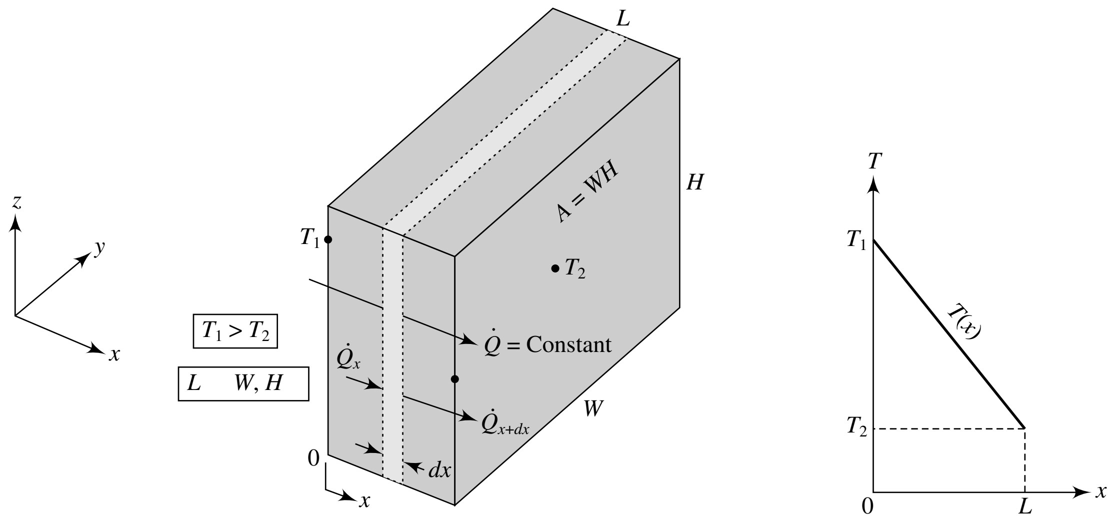


Fig. 2.10 *One-dimensional steady-state heat conduction in a plane wall*

- The material of the wall is homogeneous (*of constant density*) and isotropic (*same thermal conductivity in all directions*).

The appropriate governing differential equation is, $\dfrac{d^2T}{dx^2} = 0$

Integrating once: $\dfrac{dT}{dx} = C_1$

Integrating again: $\boxed{T(x) = C_1 x + C_2}$

This is the general solution for the temperature distribution. The two integration constants C_1 and C_2 can be evaluated from the following two boundary conditions,

BCI: $T = T_1$ at $x = 0$

BCII: $T = T_2$ at $x = L$

It follows that

$$\boxed{T(0) = T_1 = C_2}$$

$$T(L) = T_2 = C_1 L + C_2 = C_1 L + T_1$$

Therefore, $\boxed{C_1 = (T_2 - T_1)/L}$

Substituting values of C_1 and C_2, we get,

$$\boxed{T(x) = \frac{T_2 - T_1}{L} x + T_1} \tag{2.79}$$

The temperature distribution is *linear* and *independent* of thermal conductivity.

In *non-dimensional form*, the temperature profile is given by

$$\boxed{\frac{T(x) - T_1}{T_2 - T_1} = \frac{x}{L}} \tag{2.80}$$

Differentiating ($T(x)$ with respect to x,

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}(1) + 0$$

Applying Fourier's law, the heat flux is given by

$$q = -k\frac{dT}{dx} = -k\frac{T_2 - T_1}{L} = k\frac{T_1 - T_2}{L} (\text{W/m}^2)$$

We note that q is independent of x, i.e., heat flux is the same at every point inside the wall. The heat-transfer rate, $\dot{Q} = qA$. Hence,

$$\boxed{\dot{Q} = \frac{kA(T_1 - T_2)}{L}} \qquad (\text{W}) \tag{2.81}$$

Alternative Approach From Fourier's law, $\dot{Q} = -kA\dfrac{dT}{dx}$

Separating the variables and integrating from $x = 0$ to $x = L$ (with $T = T_1$ to $T = T_2$), we get,

$$\dot{Q}\int_0^L dx = -kA\int_{T_1}^{T_2} dT \qquad (\text{since } \dot{Q}, k, A \text{ are constants})$$

$$\dot{Q}L = -kA(T_2 - T_1) = kA(T_1 - T_2)$$

It follows that

$$\boxed{\dot{Q} = \frac{kA(T_1 - T_2)}{L}} \qquad (\text{W}) \tag{2.82}$$

Integrating between $x = 0$ and $x = x$, $T = T_1$, $T = T(x)$,

$$\dot{Q}\int_0^x dx = -kA\int_{T_1}^{T} dT$$

$$\dot{Q} = \frac{kA(T_1 - T(x))}{x} \qquad (\text{W})$$

$\dot{Q}$ being the same through each layer of the wall, we have

$$\dot{Q} = \frac{kA(T_1 - T_2)}{L} = \frac{kA(T_1 - T(x))}{x}$$

$$\Rightarrow \qquad \boxed{T(x) = T_1 - (T_2 - T_1)\frac{x}{L}} \tag{2.83}$$

The temperature distribution is then given by

$$\boxed{\frac{T(x) - T_1}{T_2 - T_1} = \frac{x}{L}} \tag{2.84}$$

2.7.2 • Concept of Thermal Resistance

In Chapter 1, the concept of electrical analogy was introduced. According to it, for an electrical system, Ohm's law provides an electrical resistance of the form

$$R_e = \frac{E_1 - E_2}{I} = \frac{L}{\sigma A}$$ where σ is the electrical conductivity.

Similarly, for an equivalent thermal circuit,

$$\boxed{R_{\text{t,cond}} = \frac{T_1 - T_2}{\dot{Q}} = \frac{L}{kA}} \tag{2.85}$$

where k is the thermal conductivity of the material. The analogy between the electrical and thermal resistances is thus obvious.

A thermal resistance may also the associated with heat transfer by convection at a surface. From Newton's law of cooling,

$$\dot{Q} = hA(T_s - T_\infty)$$

where T_s and T_∞ are surface and fluid temperature respectively.

The thermal resistance for convection is then

$$\boxed{R_{\text{t,conv}} \equiv \frac{T_s - T_\infty}{\dot{Q}} = \frac{1}{hA}} \quad \text{where } h \text{ is the convection coefficient} \tag{2.86}$$

2.7.3 • Conduction Through a Plane Wall with Convective Surfaces

Often, a plane wall is exposed to the surrounding fluid on both sides and hence the effect of convective resistance should also be considered in the analysis. Let the plane wall of thickness L and thermal conductivity k with its surface temperatures T_1 and T_2 and cross-sectional area A be subjected to convection on the two sides with the fluid temperatures $T_{\infty 1}$ and $T_{\infty 2}$ and the associated heat transfer coefficients h_1 and h_2 as indicated in Fig. 2.11.

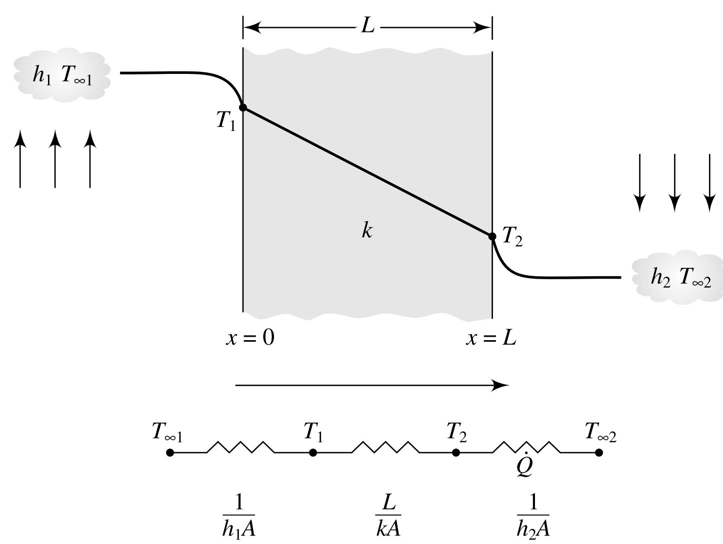


Fig. 2. 11 *A plane wall exposed to convection on both sides*

Figure 2.11 shows the thermal circuit for the plane wall with convective surface conditions. The-heat transfer rate may be determined from separate consideration of each element in the network. Since $\dot{Q}$ is constant throughout the network, it follows that

$$\dot{Q} = \frac{T_{\infty,1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{\infty,2}}{1/h_2 A} \tag{2.87}$$

In terms of the overall temperature difference, $T_{\infty,1} - T_{\infty,2}$, and the total thermal resistance, R_{tot}, the heat-transfer rate may also be expressed as

$$\boxed{\frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}} \tag{2.88}$$

Since the conduction and convection resistances are in series and may be added to find the total resistance which is given by

$$\boxed{R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}} \tag{2.89}$$

2.7.4 • Combined Heat-Transfer Coefficient

There is one more resistance which may be included if a surface is separated from its large surroundings at a temperature T_{sur}. The net radiation heat exchange between the surface and the surroundings may be determined from

$$\boxed{\dot{Q}_{rad} = \sigma A \varepsilon (T_s^4 - T_{sur}^4) = h_r A (T_s - T_{sur})} \tag{2.90}$$

It follows that a thermal resistance for radiation may be defined as

$$\boxed{R_{t,rad} = \frac{T_s - T_{sur}}{\dot{Q}_{rad}} = \frac{1}{h_r A}} \tag{2.91}$$

where h_r is found from surface radiation and convection resistances act in parallel, and if $T_\infty = T_{sur}$, they may be combined to obtain a single, effective (equivalent) surface resistance which can be expressed as

$$\boxed{\frac{1}{R_{eff}} = \left(\frac{1}{h_c}\right)^{-1} + \left(\frac{1}{h_r}\right)^{-1} = (h_c + h_r)A = hA}$$

or $R_{eff} = \dfrac{1}{hA}$ where h_c is convection coefficient and h is the combined convection and radiation heat-transfer coefficient. A typical thermal circuit involving conduction, convection, and radiation thermal resistances is shown in Fig. 2.12. It is noteworthy that circuit representations provide a useful tool for both conceptualizing and quantifying heat-transfer problems.

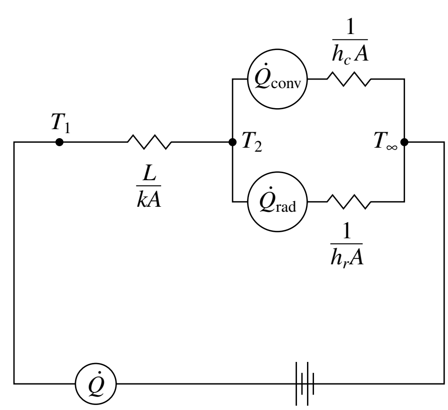


Fig. 2.12 *Thermal circuit for a plane wall exposed to combined convection and radiation*

2.8 ❑ BIOT NUMBER AND ITS SIGNIFICANCE

Consider a plane wall of thickness L, thermal conductivity k, area A, with one face held at temperature T_1 and the other face exposed to the convective environment with convection coefficient h and ambient fluid temperature T_∞.

The equivalent thermal circuit is shown in Fig. 2.13. There are two thermal resistances L/kA due to conduction and $1/hA$ due to convection. The driving potential is the temperature differential $T_1 - T_\infty$. The temperature drop across the wall and the fluid film will be in the ratio of $(L/kA)/(1/hA)$, i.e., $\frac{hL}{k}$ which is known as *Biot number*, Bi and is dimensionless number. Bi can, therefore, be looked upon as the ratio of conduction resistance to convective resistance. Figure 2.13 illustrates graphically the temperature profile for three cases. When $Bi << 1$, the temperature drop across the wall is much smaller compared to that across the film. Heat transfer is then controlled essentially by convective resistance. The temperature drop across the fluid film is however, negligible when $Bi >> 1$ and the exposed surface temperature T_2 closely approaches the fluid temperature. The larger the Biot number, the larger the temperature gradient within the material.

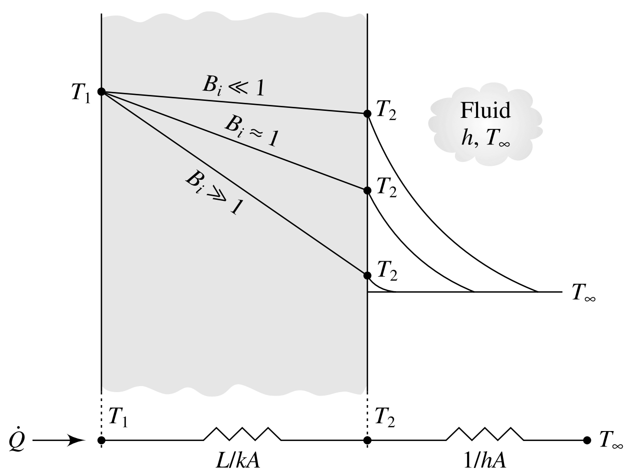


Fig. 2.13 *Effect of Biot number on the temperature profile*

2.9 ❑ CONDUCTION WITH VARIABLE AREA OF CROSS SECTION

Consider the area of cross section in this case is not constant but variable, i.e., A is a function of x as shown in Fig. 2.14 (a truncated cone with its laternal (*curved*) surface insulated)

The left face of the truncated solid is at $x = 0$ and the right face is at $x = L$ as shown. Let the left face be at temperature T_1 and the right face at T_2. Let $T_1 > T_2$.

Consider a differential control volume of thickness dx at any distance x from the left face.

The heat-transfer rate $\dot{Q}$ is the same through all sections and is constant. The area A, however, is a function of x and can be expressed mathematically as $A = A(x)$. It follows that

$$\boxed{\dot{Q} = -k(T)\,A(x)\frac{dT}{dx}}$$

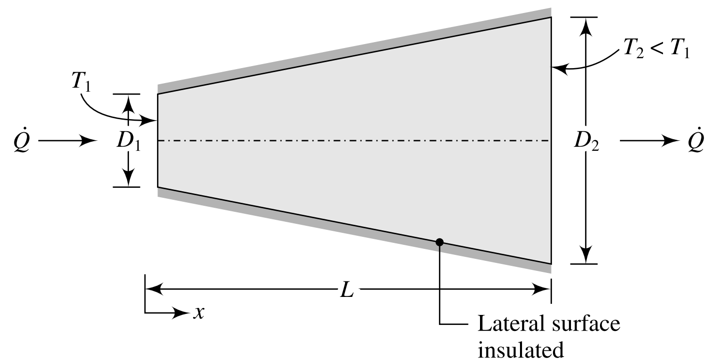


Fig. 2.14 *A truncated cone with insulated curved surface*

for the general case when thermal conductivity, k, is not constant but is a function of temperature, T.

Separating the variables,

$$\boxed{\dot{Q}\,dx = -k(T)A(x)\,dT}$$

Now, integrating between $x = 0$ and $x = L$, with corresponding temperatures at the two end faces, $T = T_1$ and $T = T_2$, we have

$$\boxed{\dot{Q}\int_0^L \frac{dx}{A(x)} = -\int_{T_1}^{T_2} k(T)\,dT} \tag{2.92}$$

If k is a constant and does not vary with temperature, we can write,

$$\dot{Q}\int_0^L \frac{dx}{A(x)} = -k\int_{T_1}^{T_2} dT \tag{2.93}$$

Now, if T_1 and T_2 are known, then $\dot{Q}$ can be calculated. For the temperature distribution in the solid, we integrate between 0 and any x (i.e., temperature varying from T_1 to $T(x)$) and equate this to the already obtained value of $\dot{Q}$.

2.10 ❑ CONDUCTION THROUGH A LONG HOLLOW CYLINDER

Conduction through a *long hollow cylinder* such as a pipe with uniform inside and outside surface temperatures can be looked upon as essentially *one-dimensional* depending only on a single coordinate, the *radial* distance. The simple geometric configuration is of great engineering significance. Consider a very long cylinder with inner and outer surfaces at radii r_1 and r_2, and maintained at uniform temperatures T_1 and T_2, respectively, (Fig. 2.15). The end effects can be

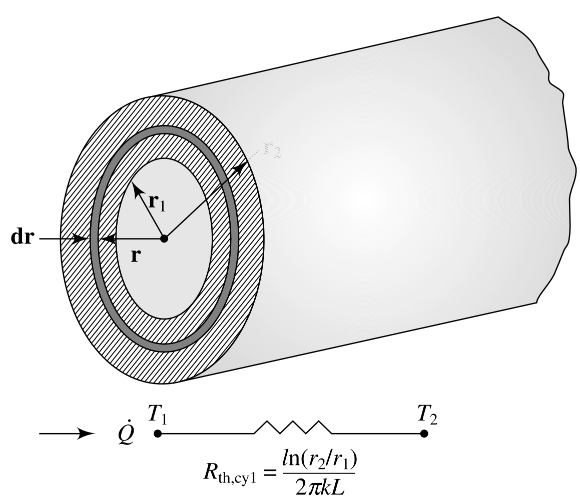


Fig. 2.15 *Long hollow cylinder (cylindrical shell)*

neglected as the pipe is pretty long so that the temperature dependence on *axial* and *circumferential* coordinates, z and ϕ, can be easily ignored. *Steady-state conditions*, *constant thermal conductivity*, and *no internal heat generation* are assumed. The general heat-conduction equation in cylindrical coordinates can then be reduced to the following expression:

$$\boxed{\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0} \tag{2.94}$$

or

$$\boxed{\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \quad \Rightarrow \quad \frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0} \tag{2.95}$$

The boundary conditions are

$$T = T_1 \quad \text{at} \quad r = r_1$$

$$T = T_2 \quad \text{at} \quad r = r_2$$

Integrating Eq. (2.92), we have

$$r\frac{dT}{dr} = C_1 \quad \text{or} \quad \frac{dT}{dr} = \frac{C_1}{r}$$

Integrating again, we get

$$\boxed{T = C_1 \ln r + C_2} \tag{2.96}$$

Substitution of B Cs. in Eq. (2.93), one obtains

$$T_1 = C_1 \ln r_1 + C_2$$

$$T_2 = C_1 \ln r_2 + C_2$$

$$\therefore \qquad T_1 - T_2 = C_1(\ln r_1 - \ln r_2) = C_1 \ln\frac{r_1}{r_2} = -C_1 \ln\frac{r_2}{r_1}$$

or

$$\boxed{C_1 = -\frac{(T_1 - T_2)}{\ln\dfrac{r_2}{r_1}}}$$

Also, $C_2 = T_1 - C_1 \ln r_1$ or $\boxed{C_2 = T_1 + \dfrac{(T_1 - T_2)}{\ln\dfrac{r_2}{r_1}}\ln r_1}$

Putting the values of constants C_1 and C_2 in Eq. (2.96), one gets

$$T = -\left(\frac{T_1 - T_2}{\ln\dfrac{r_2}{r_1}}\right)\ln r + T_1 + \left(\frac{T_1 - T_2}{\ln\dfrac{r_2}{r_1}}\right)\ln r_1$$

$$\therefore \qquad T = T_1 - \frac{(T_1 - T_2)}{\ln(r_1/r_1)}(\ln r - \ln r_1) \tag{2.97}$$

or

$$\boxed{\frac{T - T_1}{T_2 - T_1} = \frac{\ln r/r_1}{\ln r_2/r_1}} \tag{2.98}$$

This is the *logarithmic temperature distribution*.

The net heat flow across any cylindrical surface of radius r will be constant under *steady state conditions* and will be given by the Fourier's *rate* equation: $\dot{Q} = -kA(r)\dfrac{dT}{dr}$

where $A(r)$ = 2 π r L, L being the length of the cylinder

Differentiating Eq. (2.94), we get

$$\frac{dT}{dr} = 0 - \left(\frac{T_1 - T_2}{\ln\dfrac{r_2}{r_1}}\right)\left(\frac{1}{r}\right) = -\frac{1}{r}\left(\frac{T_1 - T_2}{\ln\dfrac{r_2}{r_1}}\right)$$

Steady-state heat flux, $q = -k\dfrac{dT}{dr}$

or
$$q = -\frac{k(T_1 - T_2)\left(-\dfrac{1}{r}\right)}{\ln\left(\dfrac{r_2}{r_1}\right)} = \frac{k}{r}\frac{(T_1 - T_2)}{\ln\left(\dfrac{r_2}{r_1}\right)}$$

Thus, the heat flux *decreases inversely with radius*. This is understandable since the same heat flow passes through each radial surface.

The heat flow, $\dot{Q} = q\,A(r) = (2\pi\, r\, L)q$

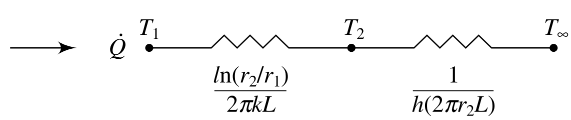
$$\boxed{\dot{Q} = \frac{2\pi\, k\, L(T_1 - T_2)}{\ln\left(\dfrac{r_2}{r_1}\right)}} \tag{2.99}$$

Note that $\dot{Q} \neq f(r)$.

Due to symmetry, any cylindrical surface concentric with the axis of the tube or pipe is an isothermal surface and the direction of heat flow is normal to that surface. The radial heat flow per unit length of the tube will be constant through successive layers and the temperature gradient must decrease with radius.

As we have assumed infinitely long cylinder, it is better to express the heat-transfer rate per unit length as

$$\frac{\dot{Q}}{L} = \frac{2\pi\, k(T_1 - T_2)}{\ln\left(\dfrac{r_2}{r_1}\right)} \tag{2.100}$$



Using electrical analogy, we have the thermal resistance for a cylinder

$$R_{t,\text{cyl}} = \frac{\ln\left(\dfrac{r_2}{r_1}\right)}{2\pi\, k\, L} \tag{2.101}$$

This can be compared with the resistance of a plane wall: $R_{t,\text{wall}} = \dfrac{L}{kA}$

Both resistances are inversely proportional to k, but each reflects a different geometry.

$$\boxed{\frac{T(r) - T_1}{T_2 - T_1} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}} \tag{2.102}$$

It is instructive to note that for a *very thin-walled* cylinder or when $\frac{r_1}{r_2}$ is approximately equal to 1, we have from Eq. (2.102),

$$\ln\frac{r}{r_1} \approx \frac{r}{r_1} - 1 = \frac{r - r_1}{r_1} \quad \text{and} \quad \ln\frac{r_2}{r_1} \approx \frac{r_2 - r_1}{r_1}$$

Eq. (2.102) then becomes

$$\boxed{\frac{T - T_1}{T_2 - T_1} = \frac{r - r_1}{r_2 - r_1}} \tag{2.103}$$

This is a simple *linear profile*, similar to the one we obtained in a plane wall.

Alternative Approach For steady-state, one-dimensional conduction, without internal heat generation, the heat-flow rate is the same and constant at every cross section. We can then directly integrate the *Fourier's rate equation* between the two specified temperatures (at the two known radii).

$$\dot{Q} = -kA\frac{dT}{dr} \quad \text{where} \quad A = 2\pi r L$$

$$\therefore \qquad \dot{Q} = -2\pi k r L\frac{dT}{dr} \quad \text{or} \quad \frac{\dot{Q}}{2\pi k L}\int_{r_1}^{r_2}\frac{dr}{r} = -\int_{T_1}^{T_2} dT$$

$$\text{or} \qquad \frac{\dot{Q}}{2\pi k L}\ln\frac{r_2}{r_1} = T_1 - T_2$$

$\therefore$ heat-transfer rate,

$$\boxed{\dot{Q} = \frac{2\pi k L(T_1 - T_2)}{\ln(r_2/r_1)}} \tag{2.104}$$

Then, at any radius r, the temperature $T(r)$ is calculated by integrating between $r = r_1$ and $r = r$ (with $T = T_1$ and $T = T(r)$).
Replacing r_2 with r and T_2 with $T(r)$, we have

$$\dot{Q} = \frac{2\pi kL(T_1 - T(r))}{\ln(r/r_1)}$$

Equating the two expressions for $\dot{Q}$, we get

$$\frac{2\pi kL(T_1 - T_2)}{\ln(r_1/r_1)} = \frac{2\pi kL(T_1 - T(r))}{\ln(r/r_1)}$$

The temperature profile is then given by

$$\boxed{\frac{T(r) - T_1}{T_2 - T_1} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}} \tag{2.105}$$

- **Log Mean Area**

It is sometimes convenient to express the heat-flow rates through a cylinder and through a plane wall in the same form.

Thus, one can write

$$\boxed{\dot{Q} = \frac{2\pi kL(T_1 - T_2)}{\ln(r_2/r_1)} = \left[\frac{2\pi L}{\ln(r_2/r_1)}\right](\Delta T)(k)} \tag{2.106}$$

If the mean value of the area of the hollow cylinder is designated as A_m,

$$\boxed{\dot{Q}(x) = \frac{A_m}{L}(k)(\Delta T)} \tag{2.107}$$

where L is the thickness of the wall equal to $(r_2 - r_1)$.
Comparing Eqs. (2.106) and (2.107), we have

$$\frac{2\pi L}{\ln(r_2/r_1)} = \frac{A_m}{r_2 - r_1}$$

We can express A_m as

$$\boxed{A_m = \frac{2\pi r_2 L - 2\pi r_1 L}{\ln\dfrac{2\pi r_2 L}{2\pi r_1 L}} = \frac{A_2 - A_1}{\ln\left(\dfrac{A_2}{A_1}\right)}} \tag{2.108}$$

where area of the cylinder: A_1 (*Inside*): $2\pi\, r_1\, L$
A_2 (*Outside*): $2\pi\, r_2\, L$
A_m is called *log mean area.*

If $\dfrac{A_2}{A_1}$ or $\dfrac{r_2}{r_1}$ is less than 2 then the *log mean area* can be replaced by the *arithmetic mean area*, $\bar{A} = \dfrac{A_2 + A_1}{2}$ without any significant sacrifice in accuracy. The *log mean radius* can also be expressed as $r_m = (r_2 - r_1)/\ln(r_2/r_1)$.

2.11 ❑ CONDUCTION THROUGH A HOLLOW SPHERE

A spherical shell or a hollow sphere is one of the most commonly used geometries in industrial applications like *storage tanks*, *nuclear reactors*, *petrochemical plants*, *refineries*, and *cryogenic systems*. Hollow spheres are commonly used in industry for *low-temperature applications* to minimize heat losses because the geometrical configuration of sphere is such that its *surface–volume* ratio is *minimum* and the material requirement to manufacture a sphere is also *minimum* compared to other geometries.

Consider a *hollow sphere* with radii r_1 and r_2, and corresponding *uniform* surface temperatures T_1 and T_2, respectively as illustrated in Fig. 2.16. The constant thermal conductivity of the material of this spherical

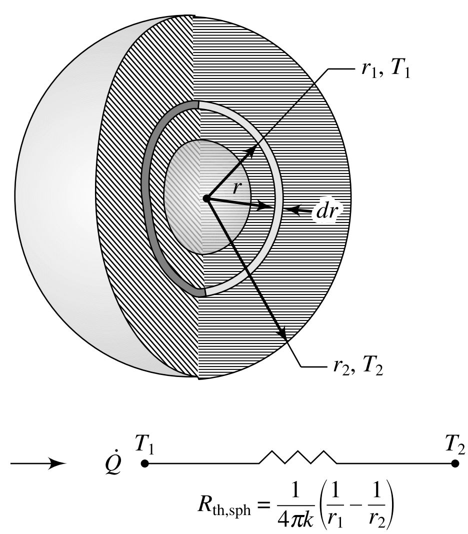


Fig. 2.16 *Heat conduction through a spherical shell*

shell is k. The temperature distribution is *one-dimensional*, i.e., *radial* and *steady state* exists. There is *no internal heat source/sink* either.

The appropriate heat conduction equation in spherical coordinates is

$$\boxed{\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0 \quad \text{or} \quad \frac{d}{dr}\left[r^2 \frac{dT}{dr}\right] = 0} \tag{2.109}$$

Integrating Eq. (2.109), we get

$$r^2 \frac{dT}{dr} = C_1 \quad \text{or} \quad \frac{dT}{dr} = \frac{C_1}{r^2}$$

Integration again yields,

$$T = C_1\left(-\frac{1}{r}\right) + C_2$$

The boundary conditions of the *first kind* are

$$\text{At} \quad r = r_1 \quad T = T_1$$
$$\text{At} \quad r = r_2 \quad T = T_2$$

Using these BCs in Eq. (2.109), we obtain

$$T_1 = -\frac{C_1}{r_1} + C_2 \quad \text{and} \quad T_2 = -\frac{C_1}{r_2} + C_2$$

Subtracting one from the other, we have

$$T_1 - T_2 = -\left[\frac{1}{r_1} - \frac{1}{r_2}\right]C_1$$

$$\therefore \qquad \boxed{C_1 = \frac{-(T_1 - T_2)}{\left[\dfrac{1}{r_1} - \dfrac{1}{r_2}\right]}} \quad \text{and} \quad C_2 = T_1 + \frac{C_1}{r_1}$$

$$\text{or} \qquad \boxed{C_2 = T_1 - \frac{1}{r_1}\frac{T_1 - T_2}{\left[\dfrac{1}{r_1} - \dfrac{1}{r_2}\right]}}$$

Substituting the values of the integration constants C_1 and C_2 in Eq. (2.109), we get

$$T = \frac{1}{r}\left[\frac{T_1 - T_2}{\dfrac{1}{r_1} - \dfrac{1}{r_2}}\right] + T_1 - \frac{1}{r_1}\left[\frac{T_1 - T_2}{\dfrac{1}{r_1} - \dfrac{1}{r_2}}\right]$$

$$\therefore \qquad \boxed{T = T_1 + \frac{T_1 - T_2}{\left[\dfrac{1}{r_1} - \dfrac{1}{r_2}\right]}\left(\frac{1}{r} - \frac{1}{r_1}\right)} \quad \text{or} \quad \frac{T - T_1}{T_2 - T_1} = \frac{\dfrac{1}{r} - \dfrac{1}{r_1}}{\dfrac{1}{r_2} - \dfrac{1}{r_1}} \tag{2.110}$$

To find the steady-state heat-flow rate in the radial direction,

$$\dot{Q}(r) = -kA(r)\frac{dT}{dr} \quad \text{where} \quad A(r) = 4\pi r^2$$

From Eq. (2.110),

$$\frac{dT}{dr} = \frac{C_1}{r^2} \quad \text{or} \quad \frac{dT}{dr} = -\frac{1}{r^2}\left[\frac{T_1 - T_2}{\frac{1}{r_1} - \frac{1}{r_2}}\right]$$

The heat flux, $q = -k\dfrac{dT}{dr} = \dfrac{k}{r^2}\dfrac{(T_1 - T_2)}{\left[\frac{1}{r_1} - \frac{1}{r_2}\right]}$

Thus, the *heat flux is inversely proportional to* r^2, which keeps on decreasing as r increases. But the *heat-flow rate* at any surface is *constant* and is equal to

$$\dot{Q}(r) = k\frac{(4\pi r^2)}{r^2}\left[\frac{T_1 - T_2}{\frac{1}{r_1} - \frac{1}{r_2}}\right]$$

$\therefore$
$$\boxed{\dot{Q} = \frac{4\pi k(T_1 - T_2)}{\frac{1}{r_1} - \frac{1}{r_2}}} \tag{2.111 (a)}$$

or
$$\boxed{\dot{Q} = \frac{4\pi k(T_1 - T_2)r_1 r_2}{r_2 - r_1}} \tag{2.111 (b)}$$

The thermal resistance in this case is

$$\boxed{R_{\text{th}} = \frac{\Delta T}{\dot{Q}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{r_2 - r_1}{4\pi k(r_1 r_2)}} \tag{2.112}$$

Alternative Approach For steady-state, one-dimensional heat conduction, with no internal heat generation, the heat flow rate, $\dot{Q}$ is constant at every cross section. One can directly integrate Fourier's equation between the two prescribed temperatures (and the corresponding, known radii). Then, at any r, the temperature $T(r)$ is calculated by integrating between $r = r_1$ and $r = r$ (with $T = T_1$ and $T = T(r)$).

At any radius r, consider an elemental spherical shell of thickness dr; let the temperature differential across this thin layer be dT. Then, in the steady state, the rate of heat transfer through this layer $\dot{Q}$, can be written from Fourier's law, to be equal to:

or
$$\dot{Q} = -kA(r)\frac{dT}{dr}, \text{ where } A(r) = 4\pi r^2$$

i.e.
$$\dot{Q}\frac{dr}{r^2} = -4\pi k dT$$

Integrating from r_1 to r_2 (with temperature from T_1 to T_2),

$$\dot{Q}\int_{r_1}^{r_2}\frac{dr}{r^2} = -4\pi k\int_{T_1}^{T_2} dT \quad \text{or} \quad \dot{Q}\left[\frac{-1}{r}\right]_{r_1}^{r_2} = 4\pi k(T_1 - T_2)$$

or
$$\dot{Q}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = 4\pi k(T_1 - T_2)$$

Heat-transfer rate,

$$\dot{Q} = \frac{4\pi k(T_1 - T_2)}{\dfrac{1}{r_1} - \dfrac{1}{r_2}} \quad \text{or} \quad \boxed{\dot{Q} = \frac{4\pi k(r_1 r_2)(T_1 - T_2)}{r_2 - r_1}} \qquad \text{(W)} \tag{2.113}$$

At any radius r, let the temperature be $T(r)$.
Replacing r_1 by r and T_2 by $T(r)$, we get

$$\dot{Q} = \frac{4\pi k r_1 r(T_1 - T(r))}{r - r_1}$$

Equating the two expressions for $\dot{Q}$,

$$\frac{4\pi k(r_1 r_2)(T_1 - T_2)}{r_2 - r_1} = \frac{4\pi k r_1 r(T_1 - T(r))}{r - r_1}$$

The temperature profile is then given by

$$\boxed{\frac{T(r) - T_1}{T_2 - T_1} = \frac{r_2}{r} \times \left(\frac{r - r_1}{r_2 - r_1}\right)} \tag{2.114}$$

Geometric Mean Area If one writes the expression for heat-flow rate in the manner of a plane wall then

$$\dot{Q} = \frac{kA_m \Delta T}{L} = kA_m\frac{\Delta T}{(r_2 - r_1)}$$

where, the wall thickness, $L = r_2 - r_1$

Also,
$$\dot{Q} = \frac{4\pi r_1 r_2}{r_2 - r_1} k\Delta T$$

The inner and outer surface areas being A_i and A_o, we can write

$$A_i = 4\pi r_1^2 \quad \text{and} \quad A_o = 4\pi r_2^2$$

$\therefore$
$$A_i A_o = (4\pi)^2 (r_1 r_2)^2 \quad \text{or} \quad \boxed{\sqrt{A_i A_o} = 4\pi r_1 r_2 = A_m} \tag{2.115}$$

$\therefore$ A_m is the *geometric mean* of A_i and A_o.

2.12 ❑ COMPARISON OF TEMPERATURE PROFILES OF PLANE WALL, CYLINDRICAL ANNULUS, AND SPHERICAL SHELL

For a plane wall of thickness L which equals $(r_2 - r_1)$ for a hollow cylinder and a hollow sphere and the same surface temperatures T_1 and T_2, the temperature distribution for the *three* geometries can be expressed as follows:

Plane Wall

$$T(x) = T_1 \frac{x}{L} - (T_1 - T_2)$$

Cylindrical Annulus

$$T(r) = T_1 - \frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_1 - T_2)$$

Spherical Shell

$$T(r) = T_1 - \frac{(1/r) - (1/r_1)}{(1/r_1) - (1/r_2)}(T_1 - T_2)$$

Typically, for r_1 = 5 cm, r_2 = 10 cm, $L = x_2 - x_1$ = 5 cm, T_1 = 300°C and T_2 = 100°C, the three temperature profiles are plotted in Fig. 2.17. Note that for a plane wall, the profile is *linear* while for the two radial systems, it is *non-linear* (*logarithmic* for cylindrical annulus and *hyperbolic* for the spherical shell).

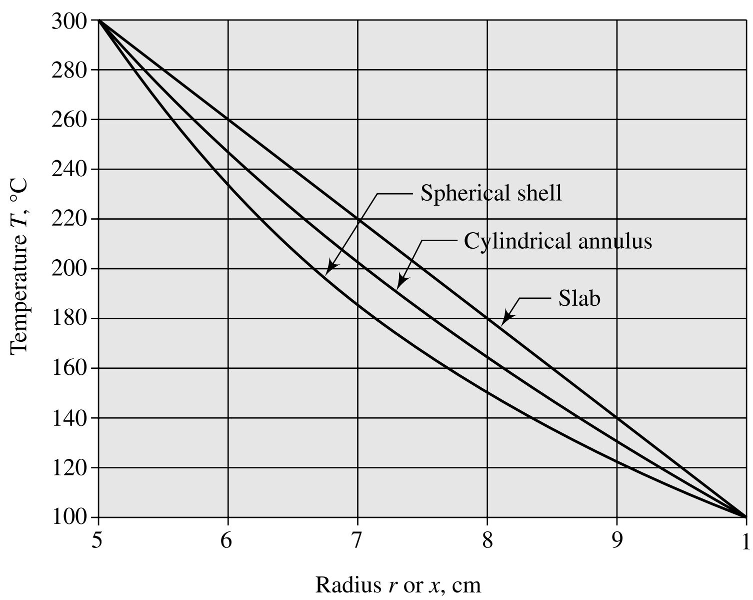


Fig. 2.17 *Comparison of temperature profiles in a slab, a cylindrical annulus, and a spherical shell*

2.13 ❑ THERMAL CONTACT RESISTANCE

We have so far assumed that there is *perfect thermal contact* at the interface, which means that there is no temperature drop at the interface. However, in several cases, particularly when the mating surfaces are rough, this may not be true and there will be a temperature drop at the interface.

When two solid surfaces are pressed together in a composite system, they will not form perfect *thermal* contact, due to *air gaps* resulting from inevitable surface roughness effects. There may be appreciable temperature drop across the interface between materials (Figure 2.18). This temperature change is due to the existence of a finite thermal contact resistance. Heat transfer follows two paths in such an interface. The heat-transfer is due to conduction across the air-filled gaps. The heat conduction path through points of solid-to-solid contact is quite effective but the path through the gap containing air or some other low conductivity gas can be very ineffective. For a unit area of the interface, the contact resistance is defined as

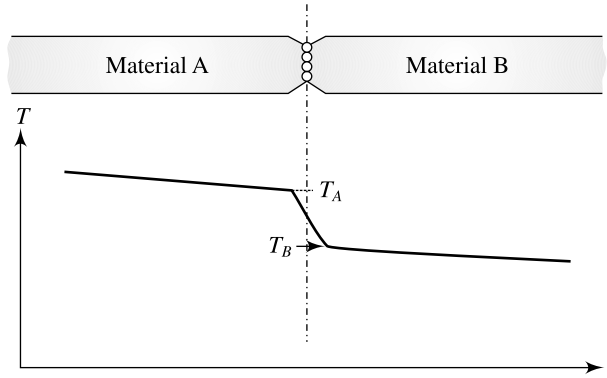


Fig. 2.18 *Temperature drop at the interface due to thermal contact resistance*

$$\boxed{R_{t,c} = \frac{T_A - T_B}{q}}\left(\frac{m^2 K}{W}\right)$$

The contact conductance, h_c is the reciprocal of contact resistance, i.e., $h_c = \dfrac{1}{R_{t,c}} (\mathrm{W/m^2\ °C})$.

2.14 ❑ INSULATION AND R-VALUES

In evaluating the relative performance of insulation, it is common practice in the building industry to use a term, referred to as the *R-value*, which is defined as

$$\boxed{R = \frac{\Delta T}{\dot{Q}/A}[m^2\ K/W] = \frac{L}{k}} \tag{2.116}$$

It may be noted that this is different from the *thermal resistance concept* because here in the denominator *heat flow per unit area*, i.e., heat flux has been used as against the *heat flow rate* while defining the thermal resistance. The *R*-values of different insulating materials, and their permissible temperature ranges provides a useful index for selecting a suitable insulating material for a specific application.

2.15 ❑ COMPOSITE SLABS (MULTILAYERED) WALLS

Let us now analyze those problems where plane walls of different materials are placed in intimate contact so that the heat flows through them in either *series* or *parallel* paths under the two assumptions: *first*, that the contact resistance between the different materials is negligible and *second*, that the heat always flows in one direction. The first one can be a serious limitation if air (*or other gas*) gaps of any appreciable size exist between the different materials. The other limitation is not serious if the thermal conductivities of the various materials are not different, otherwise two-dimensional effects must be considered.

Figure 2.19 shows a composite plane wall comprising three different materials, and having a convective coefficient h_1 on the left side and h_2 on the right side. Figure 2.19 also shows the equivalent thermal circuit from which one can see that the heat-transfer rate is equal to the overall temperature (*potential*) difference divided by the equivalent thermal resistance

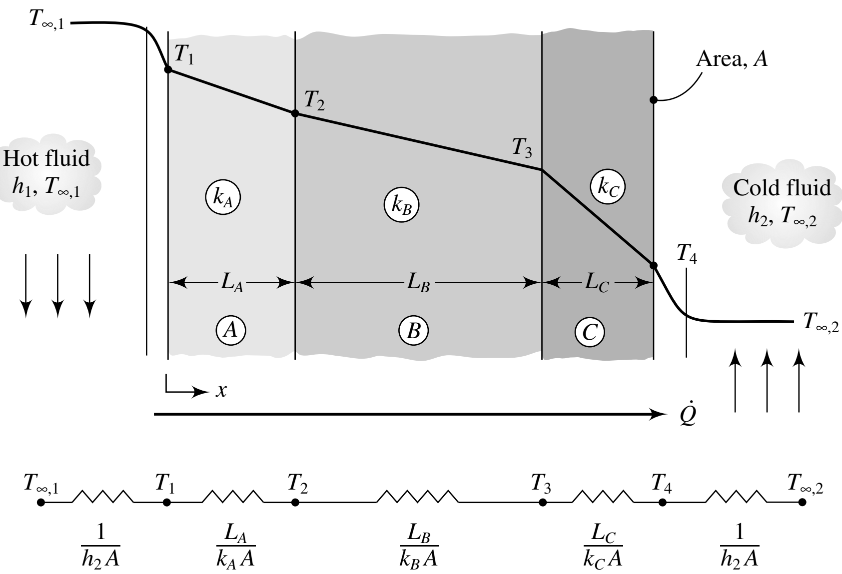


Fig. 2.19 *Composite wall and the equivalent thermal circuit*

$$\boxed{\dot{Q} = \frac{\Delta T_{\text{overall}}}{R_{\text{total}}}}$$

In this case the equivalent thermal resistance is simply the sum total of the individual thermal resistances in series.

$$\boxed{R_{\text{total}} = \left[\frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_2 A}\right]} \tag{2.117}$$

In the steady state, the same heat-flow rate which enters the left side passes through the different materials and leaves the right side at any given time.

Referring to Fig. 2.19 it may be seen that heat flows from the fluid at temperature $T_{\infty 1}$ to the left surface of the slab 1 by convection, then by conduction through slabs A, B, and C then, by convection from the right surface of slab C to the fluid at temperature $T_{\infty 2}$.

Let the area of the slab normal to the heat flow direction be A(m^2). Now, considering each case by turn, we have the following:

- *Convection at the left surface of Slab A:*

$$\dot{Q} = h_1 A(T_{\infty 1} - T_1) \qquad (\textit{Newton's law of cooling})$$

i.e.,
$$T_{\infty 1} - T_1 = \frac{\dot{Q}}{h_1 A}$$

Conduction through Slab A:

$$Q = \frac{k_A A(T_1 - T_2)}{L_A} \qquad (\textit{Fourier's law})$$

i.e., $$T_1 - T_2 = \frac{\dot{Q}L_A}{k_A A}$$

- *Conduction through Slab A:*

$$\dot{Q} = \frac{k_B A(T_2 - T_3)}{L_B} \qquad (\textit{Fourier's law})$$

$$T_2 - T_3 = \frac{\dot{Q}L_B}{k_B A}$$

- *Conduction through Slab C:*

$$\dot{Q} = \frac{k_C A(T_3 - T_4)}{L_C} \qquad (\textit{Fourier's law})$$

i.e. $$T_3 - T_4 = \frac{\dot{Q}L_C}{k_C A}$$

- *Convection at the right surface of Wall 3*

$$\dot{Q} = h_2 A(T_4 - T_{\infty 2}) \qquad (\textit{Newton's law of cooling})$$

or $$T_4 - T_{\infty 2} = \frac{\dot{Q}}{h_2 A}$$

Adding up all the temperature drops, the overall temperature drop is given by

$$\boxed{T_{\infty 1} - T_{\infty 2} = \dot{Q}\left[\frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_2 A}\right]} \tag{2.118}$$

i.e., $T_{\infty 1} - T_{\infty 2} = \dot{Q}\,[R_1 + R_2 + R_3 + R_4 + R_5]$

where R_1 = convective resistance at the left surface of Wall 1,
R_2 = conductive resistance of Wall 1,
R_3 = conductive resistance of Wall 2,
R_4 = conductive resistance of Wall 3, and
R_5 = convective resistance at the right surface of Wall 3.

$$\dot{Q} = \left[\frac{T_{\infty 1} - T_{\infty 2}}{R_1 + R_2 + R_3 + R_4 + R_5}\right] \tag{2.119}$$

From electrical analogy, $(T_{\infty 1} - T_{\infty 2})$ is the total temperature difference, $\dot{Q}$ is the heat-flow rate and the total thermal resistance is the sum of the individual *five* resistances which are connected in series.

A more elaborate arrangement is shown in Fig. 2.20 in which the heat-flow paths are not in a simple series circuit. The expression for heat transfer for this composite wall is the overall temperature potential divided by the total thermal resistance.

Thus, $$\boxed{\dot{Q} = \frac{\Delta T_{\text{overall}}}{R_{\text{total}}}}$$

In this case, the thermal resistances are in a *series-parallel combination*. It follows that

$$R_{\text{total}} = R_1 + R_2 + R_{\text{eq}} + R_5 + R_6$$

where R_{eq} is the equivalent resistance of the two resistances R_3 and R_4 in *parallel*.

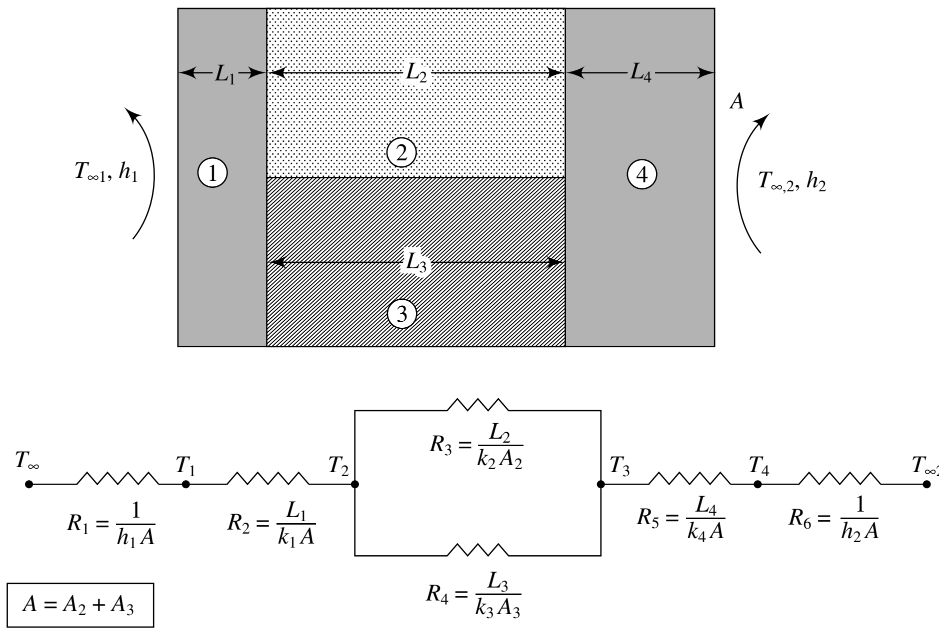


Fig. 2.20 *Composite plane wall with convective surfaces and resistances connected in series and parallel*

And the equivalent thermal resistance becomes $$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_3} + \frac{1}{R_4}} \Rightarrow \boxed{R_{\text{eq}} = \frac{R_3 R_4}{R_3 + R_4}} \qquad (2.120)$$

Then, $$\boxed{R_{\text{total}} = R_1 + R_2 + \frac{R_3 R_4}{R_3 + R_4} + R_5 + R_6} \qquad (2.121)$$

Care must be exercised in evaluating the areas involved: From **Fig. 2.20,** it is obvious that the total area $A = A_1 = A_4$ and $A = A_2 + A_3$.

2.16 ❑ OVERALL HEAT-TRANSFER COEFFICIENT FOR A COMPOSITE WALL

In composite systems, it is often convenient to use the concept of *overall heat-transfer coefficient*, U defined by an expression analogous to Newton's law of cooling. Accordingly,

$$\boxed{\dot{Q} = UA\Delta T_{\text{overall}}} \quad \text{(W)} \qquad (2.122)$$

where $\Delta T_{\text{overall}}$ is the *overall temperature difference*. The overall heat transfer coefficient is related to the total thermal resistance such that,

$$\boxed{UA = \frac{1}{R_{\text{total}}}} \qquad \text{(W/K)} \tag{2.123}$$

For a multilayered wall, one can, therefore, write

$$\dot{Q} = UA(T_{\infty 1} - T_{\infty 2}) = UA\Delta T_{\text{overall}}$$

where, $\dot{Q}$ is the heat transfer rate (W), A is the heat-transfer area perpendicular to the direction of heat transfer.

Also, $$\dot{Q} = \left[\frac{T_{\infty 1} - T_{\infty 2}}{R_1 + R_2 + R_3 + R_4 + R_5}\right]$$

Comparing the two equations,

$$\dot{Q} = UA(T_{\infty 1} - T_{\infty 2}) = \left[\frac{T_a - T_b}{R_1 + R_2 + R_3 + R_4 + R_5}\right] = \frac{T_{\infty 1} - T_{\infty 2}}{\sum R_{\text{th}}}$$

$$UA = \frac{1}{\sum R_{\text{th}}} \quad \Rightarrow \quad U = \frac{1}{A\sum R_{\text{th}}}$$

The overall heat-transfer coefficient is

$$\boxed{U = \left\{\frac{1}{h_1} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_3}{k_3} + \frac{1}{h_2}\right\}^{-1}} \tag{2.124}$$

If thermal contact resistance is included then we have for the composite wall

$$\boxed{U = \frac{1}{R_{t,\text{overall}}\, A[K/W]\,[m^2]} = \left[\frac{1}{h_1} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + R_{t,c} + \frac{L_3}{k_3} + \frac{1}{h_2}\right]^{-1}} \qquad (\text{W/m}^2\text{K}) \tag{2.125}$$

2.17 ❑ COMPOSITE CYLINDER

Consider a multiple-layer cylindrical system comprising various layers of different thermal conductivities. Using the thermal-resistance concept and neglecting interfacial contact resistances, we can write the expression for the heat-transfer rate in the case of a long three-layered cylindrical wall in the radial direction shown in Fig. 2.21.

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{\dfrac{1}{h_1(2\pi r_1 L)} + \dfrac{\ln(r_2/r_1)}{2\pi k_1 L} + \dfrac{\ln(r_3/r_2)}{2\pi k_2 L} + \dfrac{\ln(r_4/r_3)}{2\pi k_3 L} + \dfrac{1}{h_2(2\pi r_4 L)}}$$

This result can be expressed in terms of the *overall heat-transfer coefficient*.

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{\sum R_{\text{th}}} = UA(T_{\infty 1} - T_{\infty 2}) \quad \text{or} \quad UA = \frac{1}{\sum R_{\text{th}}} \tag{2.126}$$

Unlike a composite plane wall, in composite radial systems the area normal to the flow direction (*radial*) is *not* constant and is a function of the radial distance from the centre. It is therefore customary

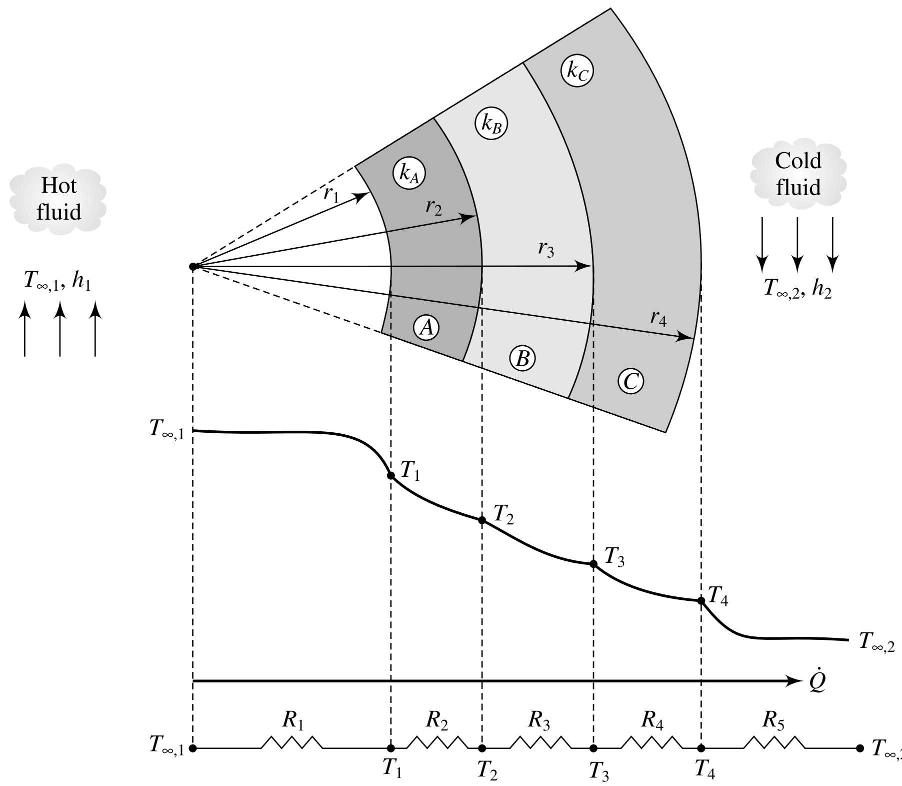


Fig. 2.21 *Temperature distribution in composite cylinder with convective surfaces*

to define the overall heat transfer coefficient either in terms of the inside surface area A_i, i.e., $2\pi\ r_1L$ or in terms of the outside surface area A_o, i.e., 2p r_4L. In fact, one can also *arbitrarily* express U in terms of any of the *intermediate areas* because one might note that

$$U_1A_1 = U_2A_2 = U_3A_3 = U_4A_4 = [\Sigma R_t]^{-1} \tag{2.127}$$

Care should of course be taken to substitute the appropriate U and A values for computation of heat transfer rate. Thus, one may write

$$\dot{Q}_r = U_iA_i\Delta T_{\text{overall}} \quad \text{or} \quad U_oA_o\Delta T_{\text{overall}} \qquad [\text{but } not\ U_iA_o\Delta T_{\text{overall}}]$$

where U_i = overall heat-transfer coefficient based on inside area
U_o = overall heat-transfer coefficient based on outside area
A_i = heat-transfer area on inside
A_o = heat-transfer area on outside

Based on *inner* area or radius, the overall heat-transfer coefficient,

$$U_i = \frac{1}{A_i}\Sigma(R_t)^{-1} \quad \text{or} \quad \frac{1}{U_i} = A_i\sum R_t$$

$$= 2\pi r_1L\left[\frac{1}{(2\pi r_1L)h_1} + \frac{\ln(r_2/r_1)}{2\pi k_1L} + \frac{\ln(r_3/r_2)}{2\pi k_2L} + \frac{\ln(r_4/r_3)}{2\pi k_3L} + \frac{1}{(2\pi r_4L)h_2}\right]$$

Hence,

$$\boxed{U_i = \left[\frac{1}{h_1} + \frac{r_1}{k_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{r_1}{k_2}\ln\left(\frac{r_3}{r_2}\right) + \frac{r_1}{k_3}\ln\left(\frac{r_4}{r_3}\right) + \frac{1}{h_2}\left(\frac{r_1}{r_4}\right)\right]^{-1}} \tag{2.128}$$

Similarly, based on the *outer area* of radius,

$$U_o = \frac{1}{A_o \Sigma R} = \frac{1}{2\pi r_3 L \times \left[\dfrac{1}{2\pi r_1 L h_1} + \dfrac{\ln(r_2/r_1)}{2\pi k_1 L} + \dfrac{\ln(r_3/r_2)}{2\pi k_2 L} + \dfrac{1}{2\pi r_3 L h_2}\right]}$$

or
$$\boxed{U_o = \left[\frac{1}{h_1}\left(\frac{r_4}{r_1}\right) + \frac{r_4}{k_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{r_4}{k_2}\ln\left(\frac{r_3}{r_2}\right) + \frac{r_4}{k_3}\ln\left(\frac{r_4}{r_3}\right) + \left(\frac{1}{h_2}\right)\right]^{-1}} \tag{2.129}$$

Note that the various resistances, namely, the two convective resistances and the two conductive resistances are all in series. By electrical analogy, the total thermal resistance is the sum of the individual resistances. Then $\dot{Q} = \Delta T_{\text{overall}}/R_{\text{total.}}$ Temperatures at the interfaces can be calculated by using the fact that $\dot{Q}$ is the same through each layer and by applying the analogy of Ohm's law for each layer in turn.

2.18 ❑ OVERALL HEAT-TRANSFER COEFFICIENT FOR THE SPHERICAL SYSTEM

As in the case of cylindrical systems, we can also define an overall heat-transfer coefficient for the spherical systems also. Again, in this case too, the area normal to the direction of heat transfer varies with the radius and it is necessary to specify as to on which area the overall heat-transfer coefficient is based. Accordingly, we write $\dot{Q} = U_i A_i (T_{\infty 1} - T_{\infty 2}) = U_o A_o (T_{\infty 1} - T_{\infty 2})$

Therefore, $\dot{Q} = \dfrac{T_{\infty 1} - T_{\infty 2}}{\sum R_t} = U_i A_i (T_{\infty 1} - T_{\infty 2}) = U_o A_o (T_{\infty 1} - T_{\infty 2})$

or
$$U_i A_i = U_o A_o = \frac{1}{\sum R_t}$$

Therefore, $U_i = \dfrac{1}{A_i \sum R_t}$ and $U_o = \dfrac{1}{A_o \sum R_t}$

One can also write

$$U_i = \frac{1}{A_i \sum R_t} = \frac{1}{4\pi r_1^2 \times \left[\dfrac{1}{4\pi h_1 r_1^2} + \dfrac{1}{4\pi h_2 r_3^2} + \dfrac{r_2 - r_1}{4\pi k_1 r_1 r_2} + \dfrac{r_3 - r_2}{4\pi k_2 r_3 r_2}\right]}$$

or
$$\boxed{U_i = \left[\frac{1}{h_1} + \frac{1}{h_2} \times \left(\frac{r_1}{r_3}\right)^2 + \frac{r_1(r_2 - r_1)}{k_1 r_2} + \frac{r_1^2(r_3 - r_2)}{k_2 r_3 r_2}\right]^{-1}} \quad (\text{W/m}^2\text{K}) \tag{2.130}$$

and
$$U_o = \frac{1}{A_o \sum R} = \frac{1}{4\pi r_3^2 \times \left[\frac{1}{4\pi h_1 r_1^2} + \frac{1}{4\pi h_2 r_3^2} + \frac{r_2 - r_1}{4\pi k_1 r_1 r_2} + \frac{r_3 - r_2}{4\pi k_2 r_3 r_2}\right]}$$

or
$$\boxed{U_o = \left[\frac{1}{h_1} \times \left(\frac{r_3}{r_1}\right)^2 + \frac{1}{h_2} + \frac{r_3^2 (r_2 - r_1)}{k_1 r_1 r_2} + \frac{r_3 (r_3 - r_2)}{k_2 r_2}\right]^{-1}} \quad (\text{W/m}^2\,\text{K}) \tag{2.131}$$

2.19 ❑ CRITICAL RADIUS OF INSULATION FOR A CYLINDER

Consider a cylinder (*electric cable*) of radius r_1, the outer surface of which is maintained at a uniform temperature T_1, and is surrounded by a fluid of temperature T_∞ as shown in Fig. 2.22. If T_2 is greater than T_∞ (for instance, when steam flows through a pipe) then heat will be lost to the ambient fluid. The thermal resistance to heat flow will be only through convection (with negligible conduction resistance due to the cylindrical pipe being thin-walled and made up of material assumed to have high thermal conductivity), given by $\dfrac{1}{2\pi r_1 Lh}$.

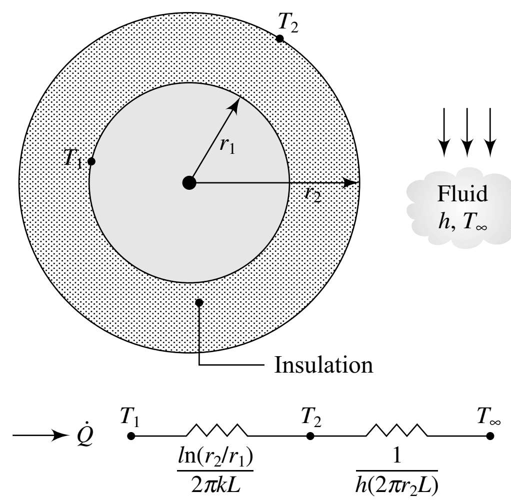


Fig. 2.22 *An insulated cylinder exposed to convection*

In order to reduce the heat losses, one must increase the thermal resistance significantly. This can be accomplished by adding a *conduction* resistance in series, say, by providing insulation to a radius r_2. The thickness of this insulation will be $(r_2 - r_1)$, and the resistance offered by this wrapping material of low thermal conductivity, k will be $\dfrac{\ln(r_2/r_1)}{2\pi kL}$, where L is the length of the insulated cylinder. The lower the thermal conductivity and the larger the thickness (i.e., the more the radius, r_2), the greater will be the thermal resistance. This will necessarily reduce the heat loss or heat dissipation from the cylinder.

But there lies the rub. The convective film resistance is *inversely proportional* to the area (or the radius). The thermal resistance due to convection after putting the insulation will now be $\dfrac{1}{2\pi r_2 Lh}$ in place of $\dfrac{1}{2\pi r_1 Lh}$. The greater the radius (*or the surface area*) depending upon the extent of insulation thickness, the *less* would be the convective thermal resistance offered by the surrounding fluid film. This will defeat the very purpose of providing the additional resistance and may result in increasing the heat loss instead of reducing it.

This apparently paradoxical situation results from two conflicting and mutually contradictory resistances. An increase in the thickness of insulation does *reduce* the heat transfer because of an increase in the *conductive* resistance. The *convective* resistance, on the other hand, tends to *decrease* with increasing radius of insulation, resulting in improved heat transfer or greater heat loss from the cylinder. Clearly, there is a radius of insulation at which the sum total of both *conductive* and *convective* resistances is a *minimum* corresponding to the maximum rate of heat transfer. That radius is called the *critical radius of insulation*.

The heat transfer rate is

$$\dot{Q} = \frac{\Delta T_{\text{overall}}}{R_{\text{total}}} = \frac{T_1 - T_\infty}{\underbrace{\dfrac{1}{2\pi r_2 Lh}}_{R_{t,\text{conv}}} + \underbrace{\dfrac{\ln(r_2/r_1)}{2\pi kL}}_{R_{t,\text{cond}}}} = \frac{2\pi L(T_i - T_\infty)}{\dfrac{1}{hr_2} + \dfrac{1}{k}\ln\left(\dfrac{r_2}{r_1}\right)}$$

As T_1, T_∞ and L are constants, the condition for maximum heat transfer can be obtained by differentiating the denominator representing the total thermal resistance (terms containing the variable r_2) with respect to r_2 and equating the derivative to zero.

Hence,

$$\frac{d}{dr_2}\left[\frac{1}{hr_2} + \frac{1}{k}\ln\left(\frac{r_2}{r_1}\right)\right] = 0 \quad \text{or} \quad \frac{d}{dr_2}\left[\frac{1}{h}(r_2)^{-1} + \frac{1}{k}(\ln r_2 - \ln r_1)\right] = 0$$

or
$$\frac{1}{h}(-1)(r_2)^{-2} + \frac{1}{k}\left(\frac{1}{r_2} - 0\right) = 0 \qquad [\because r_1 \text{ is a constant}]$$

or
$$\frac{1}{hr_2^2} = \frac{1}{kr_2} \quad \text{or} \quad hr_2^2 = kr_2$$

$\therefore$ Critical radius of insulation,

$$\boxed{r_2 = r_{cr} = \frac{k}{h}} \tag{2.132}$$

As r_2 increases, the conductive resistance $R_{t,\text{cond}}$ goes on *increasing* due to greater outer surface area. Initially, the rate of decrease of $R_{t,\text{conv}}$ is faster than the rate of increase of $R_{t,\text{cond}}$. Thus, the net result is an increasing heat transfer. But soon, with the increasing radius of insulation, $R_{t,\text{cond}}$ catches up with $R_{t,\text{conv}}$ and finally becomes predominant resulting in *continual decrease* in heat transfer rate. Below the optimum radius of insulation, i.e., r_{cr} the insulation is not only *not* effective in reducing the heat loss but is counterproductive as it adds to heat losses. Typically, with $k_{\text{insulation}}$ of the order of 0.1 W/m K and h of around 10 W/m K, the value of r_{critical} is about 10 mm.

The critical radius of insulation corresponds to maximum heat dissipation and minimum total resistance. To make sure that the total resistance is minimum, we take the second derivative of the denominator which should be positive. Differentiating Eq. (2.132) again with respect to r_2 and substituting $r_2 = k/h$, we have

$$\left[\frac{d}{dr_2}\left\{-\frac{1}{hr_2^2} + \frac{1}{kr_2}\right\}_{2=r_{cr}}\right] = \left[-\frac{1}{h}(-2)r_2^{-3} + \frac{1}{k}\left(-\frac{1}{r_2^2}\right)_{r_2=r_{cr}}\right] + \left[\frac{2}{hr_2^3} - \frac{1}{kr_2^2}\right]_{r_2=k/h}$$

$$= \frac{2}{h} \times \frac{h^3}{k^3} - \frac{1h^2}{kk^2} = \frac{2h^2}{k^3} - \frac{h^2}{k^3} = +\frac{h^2}{k^3}\,(+\text{ve})$$

Hence, for thin pipes, wrapping insulation is hardly the way to cut down heat losses from the pipe unless one puts a very thick layer of insulation which may be uneconomical. For thick pipes, there is no point of minimum resistance or threshold value, and any addition of insulation is desirable and effective. This shows that R_{tot} is *minimum* and $\dot{Q}$ is *maximum.* Since this result is bound to be always *positive*, the total resistance can never be maximum. Hence there is no optimum thickness of insulation.

Figure 2.23 illustrates graphically the effect of varying thickness of insulation on wrapping (*conduction*) resistance and film (*convection*) resistance as well as the heat-flow rate.

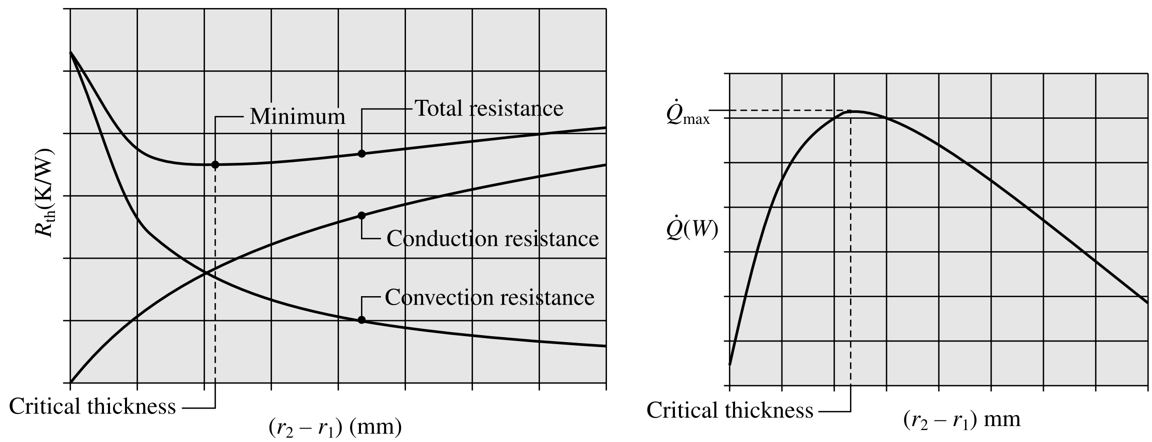


Fig. 2.23 *The effect of variation of thickness of insulation on a cylindrical surface on the thermal resistances and the heat transfer rate*

It is worth noting that insulation is not always needed to bring down the heat losses. In the design of electrical conductors, their current-carrying capacity is limited by the rate of dissipation of ohmic heat, i.e., I^2R. Providing insulation to a current-carrying electrical wire, can not only protect people from dangerous exposure to live wires but also improve the current-carrying capacity of the wire. The insulation thickness being almost always well below the critical value (the wire is thin any way), the rate of ohmic heat dissipation to the atmosphere increases due to decreasing combined thermal resistance. The electrical conductor becomes colder and can carry larger current.

2.20 ❑ CRITICAL THICKNESS OF INSULATION FOR A SPHERE

The case of a sphere is similar to that of a cylinder since here too, as the radius of insulation increases, the surface area increases. So, as the insulation radius is increased, the *conduction resistance of* insulation *increases* and the *convection resistance decreases*.

Let r_1 be the radius of the sphere on which insulation is applied, and let r_2 be the outer radius of insulation (see Fig. 2.24). We would like to investigate the change of R_{tot} as insulation radius r_2 is varied.

We have, for the spherical system:

$$R_{tot} = R_{cond} + R_{conv} = \frac{1}{4\pi k}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] + \frac{1}{h(4\pi r_2^2)}$$

Differentiating R_{tot} with respect to r_2 and equating to zero, we have

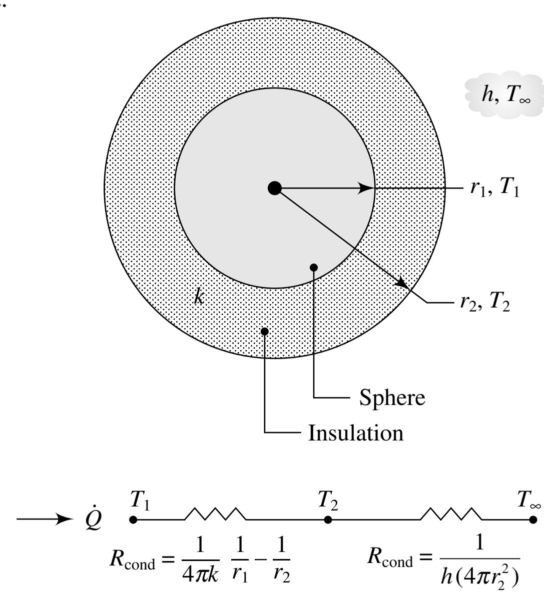


Fig. 2.24 *An insulated sphere exposed to convection*

$$\frac{d}{dr}(R_{\text{tot}}) = \frac{d}{dr_2}\left[\frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{1}{h(4\pi r_2^2)}\right] = 0$$

or
$$q_h = \frac{1}{4\pi k}\left(0 + \frac{1}{r_2^2}\right) - \frac{2}{h 4\pi r_2^3} = 0 \quad \text{or} \quad \frac{1}{4\pi k} - \frac{2}{h 4\pi r_2} = 0$$

∴ Critical radius of insulation,

or
$$\boxed{r_2 = r_{cr} = \frac{2k}{h}}$$

The second derivative, evaluated at $r_2 = r_{cr}$, is

$$\frac{d^2 R_{\text{tot}}}{dr_2^2} = \frac{d}{dr_2}\left[\frac{dR_{\text{tot}}}{dr_2}\right] = \left[-\frac{1}{2\pi k}\frac{1}{r_2^3} + \frac{3}{2\pi h}\frac{1}{r_2^4}\right]_{r_2 = r_{cr}}$$

$$= \frac{1}{(2k/h)^3}\left\{-\frac{1}{2\pi k} + \frac{3}{2\pi h}\frac{1}{2k/h}\right\} = \frac{1}{(2k/h)^3}\frac{1}{2\pi k}\left\{-1 + \frac{3}{2}\right\} > 0 \tag{2.133}$$

Hence, it follows that no optimum R_{tot} exists.

2.21 ❑ EFFECT OF VARIABLE THERMAL CONDUCTIVITY

The assumption of constant thermal conductivity materials may not be valid in some cases when the range of temperature is large. In several cases, particularly for solid substances, the dependence of conductivity on temperature is fairly *linear*, in the limited temperature range.

Let the thermal conductivity, k vary as a function of temperature in a linear fashion.

i.e.,
$$\boxed{k = k_0(1 + \beta T)} \tag{2.134}$$

where k_0 is the thermal conductivity at $T = 0°\text{C}$, T is the temperature in °C at which k is to be calculated and β is a constant which is usually very small.

Typical temperature profiles for a *plane wall* with variable thermal conductivity are illustrated in Fig. 2.25, β is generally *positive* for *insulating materials* and *negative* for *metallic conductors*.

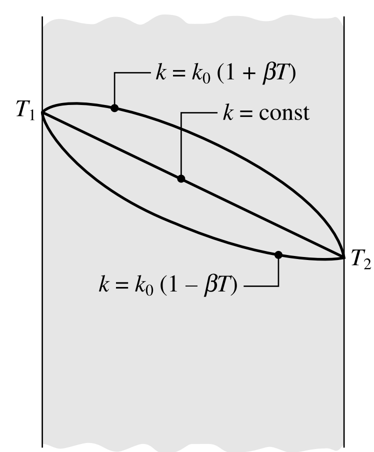


Fig. 2.25 *Variation of thermal conductivity with temperature in a plane wall*

When $\beta = 0$, the thermal conductivity, k equals k_0, a constant. Under steady operating conditions, the heat-transfer rate is constant. Hence, from Fourier's rate equation, the slope or gradient $\frac{dT}{dx}$ of the temperature distribution curve is also a constant, resulting in the temperature profile being a *straight line*.

When β is *positive*, the Fourier's rate equation becomes

$$\dot{Q} = -k_0(1 + \beta T)A\frac{dT}{dx}$$

or
$$\left|\frac{dT}{dx}\right| = \left|\frac{\dot{Q}/A}{k_0(1 + \beta T)}\right| \tag{2.135}$$

As the temperature *decreases* from T_1 to T_2, the term k_o $(1 + \beta T)$ in Eq. (2.22) also *decreases*, and the absolute value of the gradient $\left|\dfrac{dT}{dx}\right|$ *increases*. This indicates that the temperature curve becomes inncreasingly *steeper* resulting in the *convex* profile for $\beta > 0$. Clearly, the curve will have a *decreasing slope* for *negative* values of β resulting in a *concave* profile.

• Heat Transfer Rate With Variable Thermal Conductivity

Plane Wall Consider a plane wall of thickness L. The temperatures at the two boundaries are constant and uniform, i.e., $T = T_1$ at $x = 0$ and $T = T_2$ at $x = L$.

Assumptions

- One-dimensional, steady-state conduction.
- No internal heat generation.
- Thermal conductivity varies linearly with temperature, i.e., $k(T) = k_0\ (1 + \beta T)$.

Heat rate, $\dot{Q} = -k(T)A\dfrac{dT}{dx}$

where $k(T)$ is the *non-uniform* thermal conductivity given by k_0 $(1 + \beta T)$, A is the area normal to the direction of heat flow, and dT/dx is the temperature gradient.

Substituting for $k\ (T)$, separating the variables and integrating from $x = 0$, $T = T_1$ to $x = L$, $T = T_2$, we have:

$$\frac{\dot{Q}}{A}\int_0^L dx = -\int_{T_1}^{T_2} k_0 (1+\beta T)dT$$

or
$$\frac{\dot{Q}L}{A} = +k_0\left[(T_1 - T_2) + k_0\frac{\beta}{2}(T_1^2 - T_2^2)\right] \tag{2.136}$$

or
$$\frac{\dot{Q}L}{A} = +k_0\left[(T_1 - T_2) + \frac{\beta}{2}(T_1 + T_2)\times(T_1 - T_2)\right]$$

or
$$\frac{\dot{Q}L}{A} = (T_1 - T_2)k_0\left[1 + \beta\frac{T_1 + T_2}{2}\right]$$

or
$$\frac{\dot{Q}L}{A} = (T_1 - T_2)\times k_0(1+\beta T_m) = (T_1 - T_2)\times k_m$$

where $k_m = k_0\ (1 + \beta T_m)$ is the *mean thermal conductivity* at the *mean temperature*, $T_m \equiv \dfrac{T_1 + T_2}{2}$

Therefore,
$$\boxed{\dot{Q} = \frac{k_m A(T_1 - T_2)}{L}} \qquad \text{(W)} \tag{2.137}$$

Hollow Cylinder Consider a long, hollow cylinder of length L, the inside radius r_1 and the outside radius r_2. The inner and outer surfaces are held at uniform temperatures of T_1 and T_2, respectively.

From the Fourier's rate equation: $\dot{Q} = -k(T)A(r)\dfrac{dT}{dr}$

where $k(T) = k_0\ (1 + \beta T)$,

$A(r)$ is the area at any radius r, normal to the direction of heat flow $= 2\ \pi\ r\ L$, and dT/dr is the temperature gradient

Substituting for $k(T)$, separating the variables and integrating from $r = r_1$, $T = T_1$ to $r = r_2$, $T = T_2$, we obtain:

$$\dot{Q} = -k_o(1+\beta T)(2\pi rL)\frac{dT}{dr}$$

$$\frac{\dot{Q}}{2\pi L}\int_{r_1}^{r_2}\frac{dr}{r} = -k_0\int_{T_1}^{T_2}(1+\beta T)\,dT$$

or $$\dot{Q}\ln(r_2/r_1) = 2\pi k_0 L\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]$$

or $$\dot{Q}\ln(r_2/r_1) = 2\pi L(T_1 - T_2)k_0\left[1 + \beta\times\frac{(T_1+T_2)}{2}\right] = 2\pi L(T_1 - T_2)k_m$$

or $$\boxed{\dot{Q} = \frac{2\pi k_m L(T_1 - T_2)}{\ln(r_2/r_1)}} \quad \text{(W)} \tag{2.138}$$

where $k_m = k_0(1 + \beta T_m)$ is the mean (*average*) thermal conductivity, and
$T_m = (T_1 + T_2)/2$ is the mean temperature.

Hollow Sphere Consider a hollow sphere with the inside radius r_1 and outside radius r_2. The inner and outer surfaces are at uniform temperatures of T_1 and T_2, respectively ($T_1 > T_2$).

From Fourier's rate equation: $\dot{Q} = -k(T)A(r)\dfrac{dT}{dr}$

where $k(T) = k_0(1 + \beta T)$,
$A(r)$ = area at a radius r, normal to the direction of heat flow = $4\pi r^2$, and
dT/dr is the temperature gradient

Substituting for $k(T)$, separating the variables and integrating from $r = r_1$, $T = T_1$ to $r = r_2$, $T = T_2$

$$\dot{Q}\int_{r_1}^{r_2}\frac{dr}{r^2} = -4\pi k_0\int_{T_1}^{T_2}(1+\beta T)\,dT$$

or $$\dot{Q}\left[\frac{-1}{r}\right]_{r_1}^{r_2} = 4\pi k_0\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]$$

i.e., $$\dot{Q}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 4\pi(T_1 - T_2)k_0\left[1 + \beta\frac{(T_1+T_2)}{2}\right] = 4\pi k_m(T_1 - T_2)$$

where $k_m = k_0(1 + \beta T_m)$ is the mean thermal conductivity, and

and $T_m = (T_1 + T_2)/2$ is the mean temperature

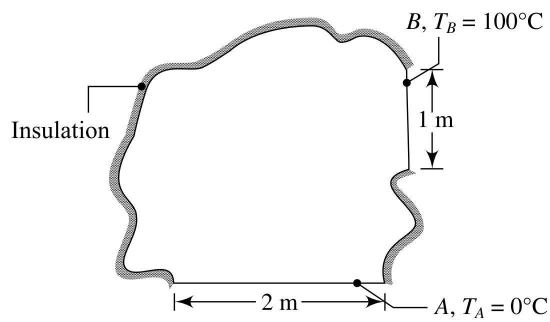
Heat rate,

$$\boxed{\dot{Q} = \frac{4\pi k_m(T_1 - T_2)}{\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)} = \frac{4\pi k_m r_1 r_2(T_1 - T_2)}{r_2 - r_1}} \quad \text{(W)} \tag{2.139}$$

Illustrative Examples

(A*) Differential Equation and Boundary Conditions

EXAMPLE 2.1 *Steady two-dimensional heat conduction takes place in the body shown in the figure. The temperature gradients over surfaces* A *and* B *can be considered to be uniform. The temperature gradient* ∂T/∂y *at the surface* A *is found to be 20 K/m. Surfaces* A *and* B *are maintained at constant temperatures, while the remaining part of the boundary is well insulated. The body has a constant thermal conductivity of 5 W/m K. Determine the values of* ∂T/∂x *and* ∂T/∂y *at the surface* B.

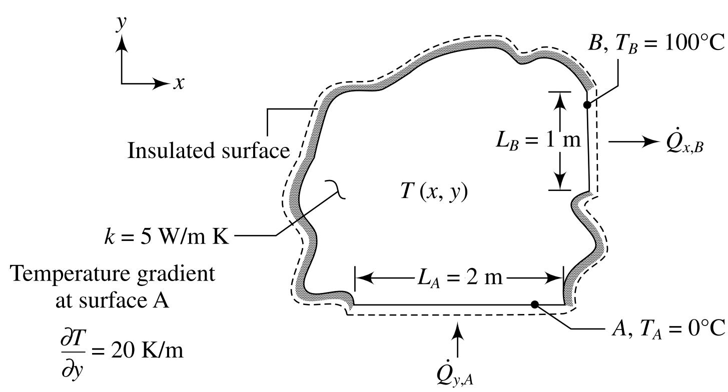


Solution

Known Two-dimensional body with given thermal conductivity. Two isothermal surfaces at specified temperatures. Temperature gradient at one surface is prescribed.

Find Temperature gradients, $\partial T/\partial x$ and $\partial T/\partial y$, at the surface B.

Schematic



Assumptions (1) Two-dimensional conduction. (2) Steady-state conditions. (3) No internal heat generation. (4) Constant properties.

Analysis We know that the heat-flux vector must always be normal to an isothermal surface. Clearly, at the horizontal surface A, the temperature gradient in the x-direction must be zero. That is, $(\partial T/\partial x)_A = 0$.

The heat-flow rate by conduction at the surface A is given by Fourier's law expressed as

$$\dot{Q}_{y,A} = -kL_A\left(\frac{\partial T}{\partial y}\right)_A = -5\ \text{W/m K} \times 2\ \text{m} \times 20\ \text{K/m} = -200\ \text{W/m}$$

Since the heat-flux vector has to be normal to the isothermal surface B, the temperature gradient can be only in the x-direction. On the vertical surface B, it is obvious that $(\partial T/\partial y)_B = 0$.

Using the conservation of energy requirement,

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \underbrace{\cancelto{0}{\dot{E}_{\text{gen}}}}_{\text{(no heat generation)}} = \underbrace{\cancelto{0}{\dot{E}_{\text{st}}}}_{\text{steady state}}$$

It follows that, $\dot{E}_{\text{in}} = \dot{E}_{\text{out}}$ or $\dot{Q}_{y,A} = \dot{Q}_{x,B}$

Using the Fourier's rate equation, $\dot{Q}_{x,B} = -kL_B \left(\dfrac{\partial T}{\partial x}\right)_B$

Since $\dot{Q}_{x,B} = \dot{Q}_{y,A}$, it follows that

$$(\partial T/\partial x)_B = \frac{\dot{Q}_{y,A}}{-k\,L_B} = \frac{-(-200\ \text{W/m})}{5\ \text{W/mK} \times 1\ \text{m}} = 40\ \text{K/m}$$

Thus, at the surface B,

$$\boxed{\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}} = \mathbf{40\ K/m} \quad \textbf{and} \quad \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{y}} = \mathbf{0}} \qquad \textbf{(Ans.)}$$

EXAMPLE 2.2 *The temperature variation within an infinite homogeneous medium at any given instant is given by,* T*(*x, y, z*)* = 2x² – y² – z² – xy + yz

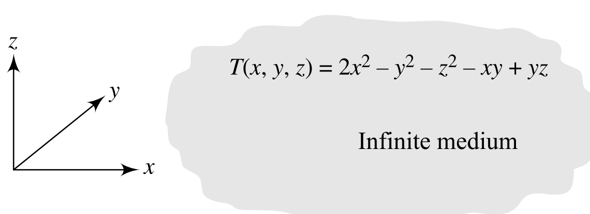
Assuming no internal heat generation and constant properties, identify the regions in the body where transient conditions exist (i.e. temperature changes with time).

Solution

Known Temperature distribution in a large body at any instant.

Find Regions where unsteady state exists.

Schematic



Assumptions (1) Isotropic, homogeneous body. (2) No internal heat generation.

Analysis At any specific time, the temperature distribution throughout a homogeneous medium with constant thermophysical properties in three dimensional Cartesian coordinates is given by

$$T(x, y, z) = 2x^2 - y^2 - z^2 - xy + yz$$

If this satisfies the general differential equation (without heat generation), i.e., Fourier's equation, the conservation of energy in the medium is satisfied. The relevant energy differential equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(A)}$$

Let us calculate the temperature gradients in the three directions:

$$\frac{\partial T}{\partial x} = 4x - y, \quad \frac{\partial T}{\partial y} = -2y - x, \quad \frac{\partial T}{\partial z} = -2z + y$$

Differentiating further,

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = 4, \quad \frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = -2, \quad \frac{\partial^2 T}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) = -2$$

Substituting in Eq. (A), we have

$$4 + (-2) + (-2) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \Rightarrow \quad \frac{\partial T}{\partial t} = 0$$

Hence, at any instant of time, transient conduction does not exist and the temperature does not change with time in any region. **(Ans.)**

(A) Plane Wall with Specified Boundary Temperatures

EXAMPLE 2.3 *In an experiment for determining thermal conductivity of a given metal, a specimen of 2.5 cm diameter and 15 cm long is maintained at 100°C at one end and at 0°C at the other end. If the cylindrical surface is completely insulated and electrical measurements show a heat flow of 5 W, determine the thermal conductivity of the specimen material.*

Solution

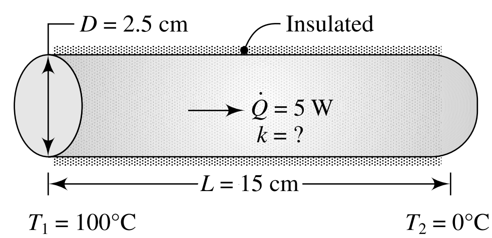
Known Apparatus for measurement of thermal conductivity.

Find Thermal conductivity of specimen material.

Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant thermal conductivity. (3) Curved surface is insulated.

Analysis The cylindrical surface being insulated, heat flow is in an axial direction only. Thus, it is a plane wall problem.

Schematic



Cross sectional area, $A = \frac{\pi}{4}(0.025 \text{ m})^2 = 4.9 \times 10^{-4} \text{ m}^2$

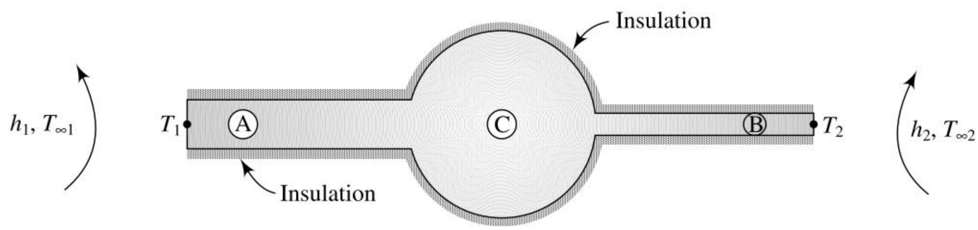
Now $\dot{Q} = kA\left[\frac{T_1 - T_2}{L}\right]$

Thermal conductivity, $k = \frac{\dot{Q}L}{A(T_1 - T_2)} = \frac{(5 \text{ W})(0.15 \text{ m})}{(4.9 \times 10^{-4} \text{ m}^2)(100 - 0)°\text{C}}$

$= \mathbf{15.3 \text{ W/m °C}}$ **(Ans.)**

(B) Plane Wall Bounded by Specified Fluid Temperatures

EXAMPLE 2.4 *Two mild steel (*k = 47 W/m °C*) circular rods* A *and* B *are interconnected via a sphere* C *as shown below:*



The respective cross-sectional areas of the rods A and B are 15 and 7.5 cm². The system is well insulated except for the left-hand face of the rod A and the right-hand face of the rod B. Under steady-state conditions, the following data is available:

$T_{\infty 1} = 80°C$, $T_{\infty 2} = 5°C$, $T_1 = 60°C$, $T_3 = 15°C$, $h_1 = 25$ W/m²°C

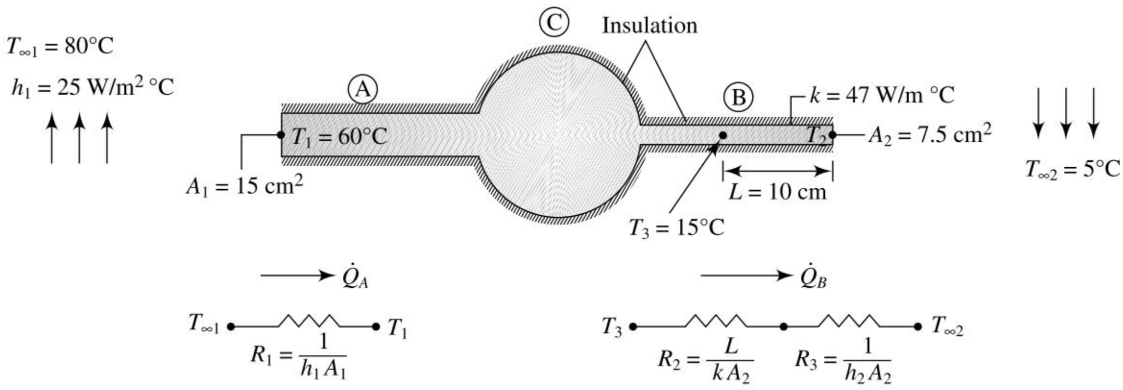
The temperature T_3 is measured at a point which is 10 cm from the right-hand end of the rod B. Determine the heat transfer coefficient, h_2.

Solution

Known Two steel rods connected through a sphere are exposed to convective environment at their ends. The curved surface is insulated.

Find Heat-transfer coefficient, h_2(W/m² °C).

Schematic



Assumptions (1) Steady-state conditions. (2) One-dimensional conduction along the rods. (3) Constant thermal conductivity. (4) Uniform convection coefficients.

Analysis Applying the control volume energy balance to the specified system, we have

$$\dot{E}_{in} - \dot{E}_{out} + \underbrace{\cancelto{0}{\dot{E}_{gen}}}_{\text{no heat generation}} = \underbrace{\cancelto{0}{\dot{E}_{st}}}_{\text{steady state}}$$

Hence, $\dot{E}_{in} = \dot{E}_{out}$

or $$\dot{Q}_A = \dot{Q}_B \quad \Rightarrow \quad \frac{T_{\infty 1} - T_1}{R_1} = \frac{T_3 - T_{\infty 2}}{R_2 + R_3}$$

or $$\frac{R_2 + R_3}{R_1} = \frac{T_3 - T_{\infty 2}}{T_{\infty 1} - T_1} = \frac{(15-5)°\text{C}}{(80-60)°\text{C}} = 0.5$$

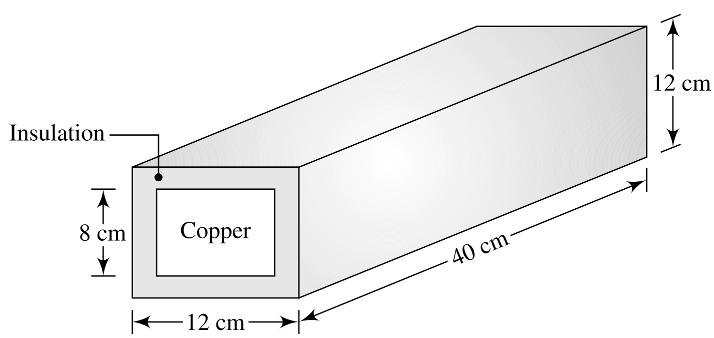
or $$R_3 = 0.5\,R_1 - R_2 \quad \Rightarrow \quad \frac{1}{h_2 A_2} = 0.5 \times \frac{1}{h_1 A_1} - \frac{L}{kA_2}$$

or $$\frac{1}{h_2} = 0.5\frac{(A_2/A_1)}{h_1} - \frac{L}{k}$$

The convection coefficient, h_2 can then be expressed as

$$h_2 = \left[\frac{0.5}{25\ \text{W/m}^2\,°\text{C}}\left(\frac{7.5\ \text{cm}^2}{15\ \text{cm}^2}\right) - \frac{0.10\ \text{m}}{47\ \text{W/m}\,°\text{C}}\right]^{-1} = \mathbf{127\ W/m^2\,°C} \qquad \textbf{(Ans.)}$$

EXAMPLE 2.5 *A copper bus bar of 40 cm length carrying electricity produces 5 W in Joulean heating. The cross section is square (see the figure) and is provided with insulation of thermal conductivity* k = *0.036 W/m °C. All four sides are cooled by air at 24°C with an average convection coefficient* h = *18 W/m²°C. Assuming the copper to be isothermal, estimate the maximum temperature of the insulation.*



Solution

Known A copper bus bar covered with insulation dissipates heat by conduction and convection.

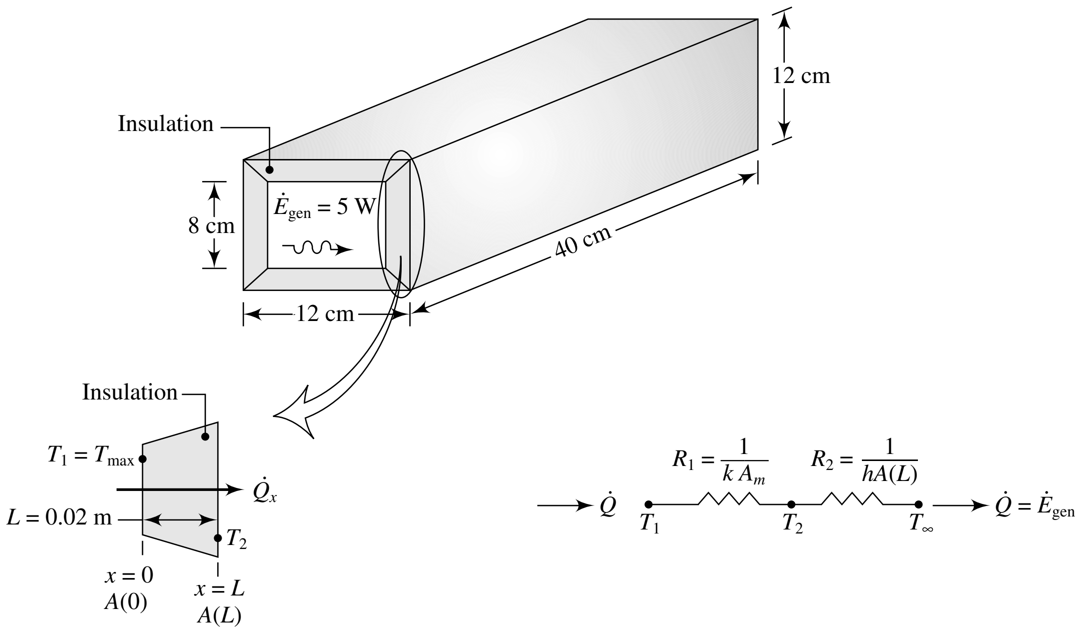
Find Maximum temperature of insulation.

Assumptions (1) One-dimensional conduction. (2) Constant conductivity and uniform convection coefficient. (3) Copper is isothermal. (4) For conduction, average cross-sectional area is considered for analysis. (5) Steady-state conditions.

Analysis The thickness of insulation is much smaller than the dimensions in the other two directions. Assuming one-dimensional conduction in steady state, the heat rate is

$$\dot{Q} = -kA_c(x)\frac{dT}{dx}$$

Schematic



From symmetry, the heat transfer for only *one* side is considered. The heat generated within is dissipated from all *four* sides by conduction through insulation and by convection from the surface to the surroundings.

Hence, $\dot{E}_{gen} = 5W = 4\dot{Q}_x$ (*from symmetry*)

or $$\dot{Q}_x = 1.25\text{ W} = \frac{T_1 - T_2}{R_{cond}} = \frac{T_2 - T_\infty}{R_{conv}} = \frac{T_1 - T_\infty}{R_{cond} + R_{conv}}$$

The maximum temperature of insulation will occur at $x = 0$, i.e., at the copper surface.

Convection resistance, $R_{conv} = \dfrac{1}{hA(L)}$

where $A(L)$ = outer surface area (*one side only*) = 0.4 m × 0.12 m = 0.048 m

$$\therefore \qquad R_{conv} = \frac{1}{18\text{ W/m}^2\text{°C} \times 0.048\text{ m}} = 1.1574\text{°C/W}$$

Outer-surface temperature of insulation is found to be

$$T_2 = T_\infty + \dot{Q}_x R_{conv} = 24\text{°C} + (1.25\text{ W})(1.1574\text{ °C/W}) = 25.45\text{°C}$$

Inner-surface temperature of insulation or its maximum temperature,

$$T_{max} = T_1 = \dot{Q}_x R_{cond} + T_2$$

where $$R_{cond} = \frac{L}{k A_c(x)}$$

We note that the area of cross section is *not* constant.

At $x = 0$, $A_c = A(0) = 0.4 \text{ m} \times 0.08 \text{ m} = 0.032 \text{ m}^2$

At $x = L$, $A_c = A(\text{L}) = 0.4 \text{ m} \times 0.12 \text{ m} = 0.048 \text{ m}^2$.

The average cross sectional area,

$$A_m = \frac{1}{2}[A(0) + A(L)] = \frac{1}{2}[0.032 + 0.048]\text{m}^2 = 0.04 \text{ m}^2 \textit{ (for each of the four sides)}$$

Hence, $$R_{\text{cond}} = \frac{0.02 \text{ m}}{0.036 \text{ W/m °C} \times 0.04 \text{ m}^2} = 13.89\text{°C/W}$$

Maximum temperature,

$$T_{\max} = (1.25 \text{ W})\ (13.89 \text{ °C/W}) + 25.45\text{°C} = \mathbf{42.81°C} \qquad \textbf{(Ans.)}$$

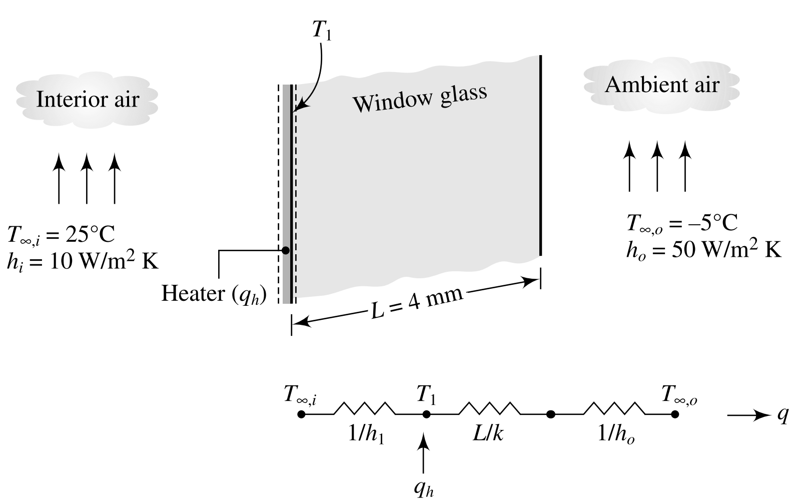
EXAMPLE 2.6 *The rear window of an automobile is defogged by attaching a thin, transparent, film-type heating element to its inner surface. By electrically heating this element, a uniform heat flux may be established at the inner surface. What is the electrical power that must be provided per unit window area to maintain an inner surface temperature of 15°C when the interior air temperature and convective heat transfer coefficient are 25°C and 10 W/m² K, respectively and the exterior (ambient) air temperature and convective heat transfer coefficient are 5°C and 50 W/m² K? The window glass is 4 mm thick and* $k_{\textit{window glass}}$ *= 1.4 W/m K. What would be the inner surface temperature without heater power?*

Solution

Known Desired inner surface temperature of rear window of a car with specified inner and outer air conditions.

Find Heat flux provided by heater to maintain the desired temperature.

Schematic



Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant thermal conductivity.

Analysis *Per unit surface area*, the energy balance at the inner surface gives,

$$\frac{T_{\infty,i}-T_1}{\dfrac{1}{h_i}}+q_h=\frac{T_1-T_{\infty,o}}{\dfrac{L}{k}+\dfrac{1}{h_o}} \quad\Rightarrow\quad q_h=\frac{T_1-T_{\infty,o}}{\dfrac{L}{k}+\dfrac{1}{h_o}}-\frac{T_{\infty,i}-T_1}{\dfrac{1}{h_i}}$$

or
$$\frac{[15-(-5)]^\circ\text{C}}{\dfrac{0.004\text{ m}}{1.4\text{ W/mK}}+\dfrac{1}{50\text{ W/m}^2\text{K}}}-\frac{(25-15)^\circ\text{C}}{\dfrac{1}{10\text{ W/m}^2\text{K}}}=(875-100)\frac{\text{W}}{\text{m}^2}=\mathbf{775\ W/m^2} \quad \textbf{(Ans.)}$$

With $q_h = 0$, $\dfrac{T_{\infty,i}-T_1}{1/h_i}=\dfrac{T_{\infty,i}-T_{\infty,o}}{1/h_i+L/k+1/h_o}$

Inner surface temperature *without* heater would be

$$T_1=T_{\infty,i}-(T_{\infty,i}-T_{\infty,o})\frac{1/h_i}{1/h_i+L/k+1/h_o}$$

$$=25-[25-(-5)]\frac{1/10}{(1/10)+\dfrac{0.004}{1.4}+(1/50)}=\mathbf{0.58^\circ C} \quad \textbf{(Ans.)}$$

(C) Composite Wall with Prescribed Boundary Temperatures

EXAMPLE 2.7 *A furnace wall is made up of refractory brick, fire brick and an outside plaster. There is an air gap with a thermal resistance of 0.15 m² K/W between the refractory brick and the fire brick. The refractory brick, 120 mm thick, has* k_1 *= 1.58 W/m K. The fire brick, 120 mm thick, has* k_2 *= 0.3 W/m K. The outside plaster, 15 mm thick, has* k_3 *= 0.15 W/m K. The two extreme temperatures of this wall are 1000°C and 100°C.*

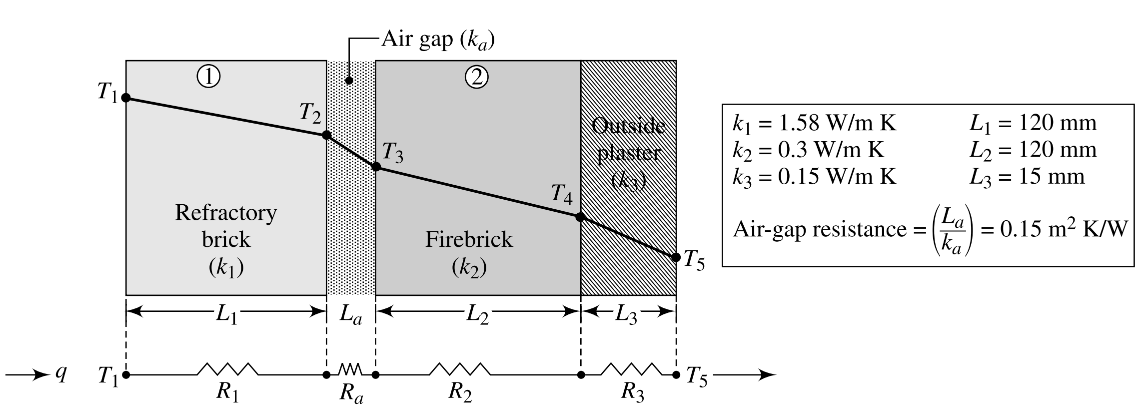
Determine (a) the heat flow rate in kJ/h m², and (b) the interface temperatures.

Solution

Known A composite wall comprising three layers and an air gap with end surfaces at specified temperatures.

Find (a) Heat-flow rate (kJ/h m²), (b) Interface temperatures, T_2, T_3 and T_4 (°C).

Schematic



Assumptions (1) Steady state conduction. (2) Air is stagnant. (3) Constant properties. (4) One-dimensional conduction.

Analysis Total thermal resistance per m² is

$$R_{\text{total}} = \left[\frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_3}{k_3}\right] + \frac{L_a}{k_a} = \left[\frac{0.12}{1.58} + \frac{0.12}{0.3} + \frac{0.015}{0.15}\right] + 0.15 = 0.726 \text{ m}^2 \text{ K/W}$$

Steady-state heat flux is

$$q = \frac{\dot{Q}}{A} = \frac{(1000 - 100)\text{ K}}{0.726 \text{ m}^2 \text{ K/W}}\left[\frac{1 \text{ J/s}}{1 \text{ W}}\right] \times \left[\frac{3600 \text{ s}}{\text{h}}\right]\left[\frac{1 \text{ kJ}}{10^3 \text{ J}}\right] = \mathbf{4463 \text{ kJ/h m}^2} \qquad \textbf{(Ans.)(a)}$$

Interface temperatures: From the thermal network, we have,

$$q = \frac{T_1 - T_2}{L_1/k_1} = \left(4463 \times \frac{1000}{3600}\right)\frac{\text{W}}{\text{m}^2} = \frac{4463}{3.6}\text{W/m}^2$$

$$\therefore \quad T_2 = T_1 - \frac{qL_1}{k_1} = 1000 - \left[\frac{4463 \times 0.12}{3.6 \times 1.58}\right] = \mathbf{906°C} \qquad \textbf{(Ans.)(b)}$$

Also,
$$q = \frac{T_1 - T_3}{\left[\dfrac{L_1}{k_1} + \dfrac{L_a}{k_a}\right]}$$

$$\therefore \quad T_3 = T_1 - q\left[\frac{L_1}{k_1} + \frac{L_a}{k_a}\right] = 1000 - \left[\frac{0.12}{1.58} + 0.15\right]\left(\frac{4463}{3.6}\right) = \mathbf{720°C} \qquad \textbf{(Ans.)(b)}$$

Furthermore,
$$q = \frac{T_1 - T_4}{\left[\dfrac{L_1}{k_1} + \dfrac{L_a}{k_a} + \dfrac{L_2}{k_2}\right]}$$

$$\therefore \quad T_4 = T_1 - \left(\frac{4463}{3.6}\right)\left\{\frac{0.12}{1.58} + 0.15 + \frac{0.12}{0.3}\right\} = \mathbf{224°C} \qquad \textbf{(Ans.)(b)}$$

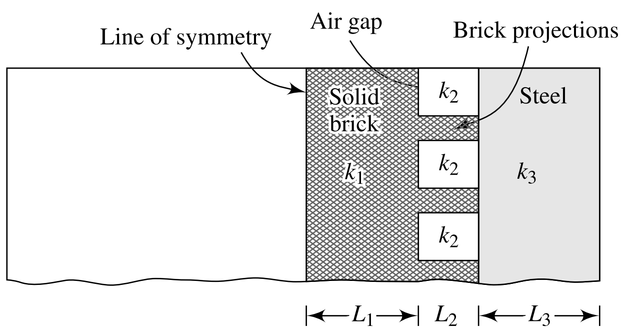
EXAMPLE 2.8 *A layer of 5 cm thick fire brick (*k *= 1.0 W/m K) is placed between two 6 mm thick steel plates (*k *= 52 W/m K). The faces of the bricks adjacent to the plates are rough, having solid to solid contact over only 20% of the total area, with the average height of asperities (projection of rough elements) being 0.8 mm. If the outer-surface temperatures of steel plates are 100°C and 400°C, respectively, find (a) the rate of heat flow per unit area. Assume that the gaps are filled with air (*k *= 0.035 W/m K), and (b) the rate of heat flow per unit area if the faces of the bricks were smooth and have solid to solid contact over the entire area.*

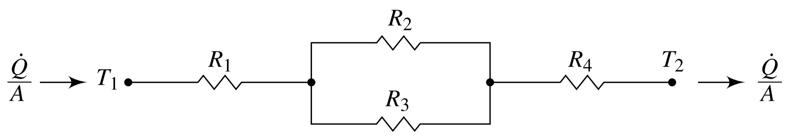
Solution

Known A series–parallel composite wall comprises a brick layer between two steel plates with brick projections and trapped air.

Find (a) Heat-flow rate per unit area, q(W/m²) with (a) 20% solid to solid contact, and (b) solid to solid contact over the whole area.

Schematic





Equivalent thermal circuit

Assumptions (1) Steady operating conditions exist. (2) Constant thermal conductivities. (3) Heat flow through air gap is by conduction. (4) One-dimensional conduction. (5) Brick projections are distributed.

Analysis We consider only half of the system since the composite wall is symmetrical with respect to the centre plane. The equivalent thermal circuit is shown in the schematic. The thermal resistances on the basis of *unit area* are as follows:

Half of the solid brick:

$$R_1 = \frac{L_1}{k_1} = \frac{2.5 \times 10^{-2}\,m}{1.0\text{ W/m K}} = 0.025\text{ K/W} \qquad [L_1 \text{ is half of thickness of the solid brick}]$$

Brick asperities: $R_2 = \dfrac{L_2}{0.2\,k_1} = \dfrac{0.8 \times 10^{-3}\text{ m}}{0.2 \times 1.0\text{ W/m K}} = 0.004\text{ K/W}$

Air gap: $R_3 = \dfrac{L_2}{0.8\,k_2} = \dfrac{0.8 \times 10^{-3}\text{m}}{0.8 \times 0.035\text{ W/m K}} = 0.028\,57\text{ K/W}$

Steel plate: $R_4 = \dfrac{L_3}{k_3} = \dfrac{6 \times 10^{-3}\text{m}}{52\text{ W/m K}} = 0.000\,115\text{ K/W}$

The factors 0.2 and 0.8 in R_2 and R_3 respectively represent the fraction of the total area for the two separate heat flow paths.

Since R_2 (*brick asperities*) and R_3 (*air gap*) are *in parallel,* the equivalent resistance is

$$R_{eq} = \left[\frac{1}{R_2} + \frac{1}{R_3}\right]^{-1} = \left[\frac{1}{0.004} + \frac{1}{0.02857}\right]^{-1} = 0.0035\text{ K/W}$$

The total thermal resistance is

$$R_{tot} = R_1 + R_{eq} + R_4 = (0.025 + 0.0035 + 0.000\ 115)\text{ K/W} = 0.028\ 62\text{ K/W}$$

Heat-flow rate *per unit* area is

$$\frac{\dot{Q}}{A} = \frac{\Delta T}{R_{\text{tot}}} = \frac{(400 - 100)\ \text{K}}{0.028\ 62\ \text{K/W}} = \mathbf{10\ 480\ W/m^2} \qquad \textbf{(Ans.) (a)}$$

If the solid-to-solid contact is 100%,

$$R^*_{\text{tot}} = R^*_1 + R_4 = \frac{(L_1 + L_2)}{k_1} + R_4 = \frac{(0.025 + 0.0008)\,\text{m}}{1.0\ \text{W/m K}} + 0.00\ 015\ \text{K/W}$$

$$= \mathbf{0.0259\ K/W}$$

Hence, the heat flow rate *per unit area* now is

$$\frac{\dot{Q}^*}{A} = \frac{\Delta T}{R^*_{\text{tot}}} = \frac{(400 - 100)\,\text{K}}{0.0259\ \text{K/W}} = \mathbf{11576\ W/m^2} \qquad \textbf{(Ans.) (b)}$$

(D) Composite Wall Bounded by Fixed Fluid Temperatures

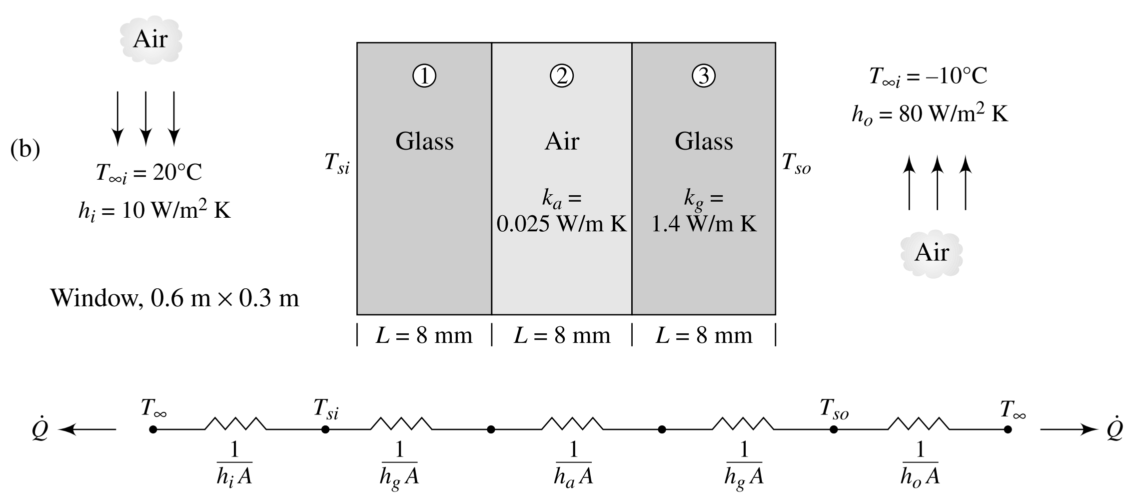
EXAMPLE 2.9 *A thermopane (thermally insulated glass) window that is 0.6 m long by 0.3 m wide comprises two 8 mm thick pieces of glass sandwiching an 8 mm thick stagnant air space. The thermal conductivity of glass is 1.4 W/m K and that of air is 0.025 W/m K. The window separates room air at 20°C from outside ambient air at –10°C. The convection coefficients associated with the inner (room side) and the outer (ambient) surfaces are 10 W/m²K and 80 W/m²K respectively. (a) Determine the heat loss through the window, and the two surface temperatures. (b) What would be the heat loss if the window had a single glass of 8 mm thickness instead of a thermopane? (c) Calculate the heat loss for a triple pane construction in which a third pane and a second air space, each 8 mm thick, are added.*

Solution

Known Dimensions of a thermopane window. Room and ambient air conditions.

Find (a) Heat loss through window, and surface temperatures. (b) Heat loss with a single glass. (c) Heat loss with a triple pane thermopane window.

Schematic



Assumptions (1) Steady state, one dimensional heat flow. (2) Constant properties. (3) Negligible radiation effects.

Analysis (a) From the thermal circuit, the heat loss is

$$\dot{Q} = \frac{\Delta T_{\text{overall}}}{\Sigma R_{\text{th}}} = \frac{T_{\infty,i} - T_{\infty,o}}{\frac{1}{A}\left[\frac{1}{h_i} + \frac{L}{k_g} + \frac{L}{k_a} + \frac{L}{k_g} + \frac{1}{h_o}\right]}$$

$$= \frac{(0.6 \text{ m} \times 0.3 \text{ m})[20 - (-10)]°\text{C or K}}{\left[\frac{1}{10}\frac{\text{m}^2\text{K}}{\text{W}} + \frac{0.008 \text{ m m K}}{1.4 \text{ W}} + \frac{0.008 \text{ m m K}}{0.025\, W} + \frac{0.008 \text{ m m K}}{1.4 \text{ W}} + \frac{1}{80}\frac{\text{m}^2\text{K}}{\text{W}}\right]}$$

$$= \frac{(0.18 \text{ m})^2 (30 \text{ K})}{[0.1 + 0.005714 + 0.32 + 0.005714 + 0.0125] \text{ m}^2\text{K/W}}$$

Heat loss, $\dot{Q}$ = **12.164 W** **(Ans.) (a)**

Now, $T_{\infty,i} - T_{si} = \dot{Q}\frac{1}{h_i A} \Rightarrow T_{si} = 20 - \left\{\frac{12.164}{10 \times 0.18}\right\} = \mathbf{13.24°C}$ **(Ans.) (a)**

Also, $T_{so} - T_{\infty,o} = \dot{Q} \cdot \frac{1}{h_o A} \Rightarrow T_{s,o} = -10 + \left\{\frac{12.164}{80 \times 0.18}\right\} = -\mathbf{9.16°C}$ **(Ans.) (a)**

(b) With single glass:

$$\Sigma R_{\text{th}} = \frac{1}{A}\left[\frac{1}{h_i} + \frac{1}{k_g} + \frac{1}{h_o}\right] = \frac{1}{0.18 \text{ m}^2}\left[\frac{1}{10} + \frac{0.008}{1.4} + \frac{1}{80}\right]\frac{\text{m}^2\text{K}}{\text{W}} = 0.6567 \text{ K/W}$$

∴ Heat loss, $\dot{Q} = \frac{[20 - (-10)] \text{ K}}{(0.6567 \text{ K/W})} = \mathbf{45.68 \text{ W}}$ **(Ans.) (b)**

(c) Triple pane window: $\Sigma R_{\text{th}} = \frac{1}{A}\left[\frac{1}{h_i} + \frac{3 \times L}{k_g} + \frac{2 \times L}{k_a} + \frac{1}{h_o}\right]\frac{\text{K}}{\text{W}}$

$$= \frac{1}{1.8}\left[\frac{1}{10} + \frac{3 \times 0.008}{1.4} + \frac{2 \times 0.008}{0.025} + \frac{1}{80}\right]\frac{\text{K}}{\text{W}} = 4.276 \text{ K/W}$$

∴ Heat loss, $\dot{Q} = \frac{\Delta T}{\Sigma R_{\text{th}}} = \frac{30 \text{ K}}{4.276 \text{ K/W}} = \mathbf{7.02 \text{ W}}$ **(Ans.) (c)**

EXAMPLE 2.10 *A 2 kW heating element of area 0.05 m² is protected on the backside by a 50 mm thick insulating material (*k *= 1.5 W/m K) and on the front side by a 10 mm thick plate (*k *= 55 W/m K). The backside is exposed to cold environment at 5°C with convective heat-transfer coefficient of 10 W/m² K. The front side is exposed to warm room air at 23°C with combined convective cum radiative heat transfer coefficient of 220 W/m²K.*

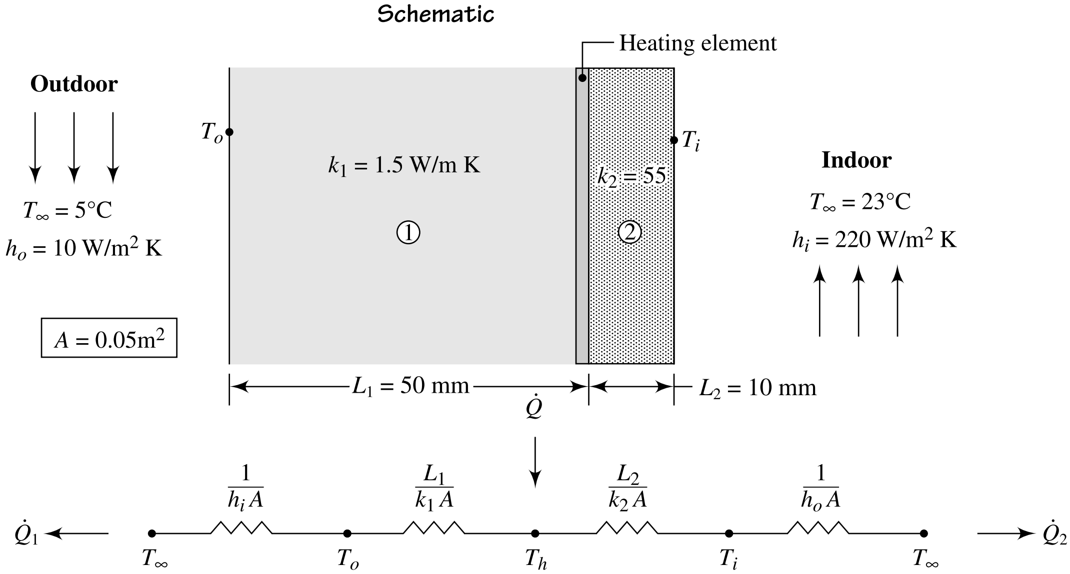
Determine: (a) the heating element temperature (°C), (b) the rate of heat transfer into the room (W), (c) the surface temperatures on the front and backside (°C).

Solution

Known A heating element sandwiched between an insulating layer and a metal plate is exposed to convective atmosphere on both extreme surfaces.

Find (a) Heating element temperature. (b) Heat rate into room. (c) Extreme surface temperatures.

Schematic



Assumptions (1) Steady operating conditions. (2) One-dimensional conduction. (3) Constant properties.

Analysis The heat dissipated by the heating element is partly lost to the surrounding air and the rest enters the room to keep it warm. Thus,

$$\dot{Q} = 2000 \text{ W} = \dot{Q}_1 + \dot{Q}_2 \quad \text{(A)}$$

Now, $$\dot{Q}_1 = \frac{(T_h - T_{\infty o})}{\left\{\frac{L_1}{k_1 A} + \frac{1}{h_i A}\right\}} \quad \text{and} \quad \dot{Q}_2 = \frac{(T_h - T_{\infty i})}{\left\{\frac{L_2}{k_2 A} + \frac{1}{h_o A}\right\}}$$

$\therefore$ $$T_h - 5 = \frac{\dot{Q}_1}{0.05}\left[\frac{0.05}{1.5} + \frac{1}{10}\right] = 2.6667\,\dot{Q}_1 \quad \text{(B)}$$

and $$T_h - 23 = \frac{\dot{Q}_2}{0.05}\left[\frac{0.01}{55} + \frac{1}{220}\right] = 0.09455\,\dot{Q}_2$$

But $\dot{Q}_2 = 2000 - \dot{Q}_1$ from Eq. (A)

Hence, $T_h - 23 = 0.09455\,(2000 - \dot{Q}_1)$ or $T_h - 23 = 189.091 - 0.09455\,\dot{Q}_1$ (C)

From (B) and (C), we have

$$18 = 2.7612\,\dot{Q}_1 - 189.091 \quad \Rightarrow \quad \dot{Q}_1 = \mathbf{75\ W}$$

Heat-flow rate into the room, $\dot{Q}_2 = 2000 - 75 =$ **1925 W** **(Ans.) (b)**

Heating element temperature is obtained from Eq. (B):

$T_h = 5 + (2.6667 \times 75) =$ **205°C** **(Ans.) (a)**

From the equivalent thermal circuit, the extreme surface temperatures are determined to be

$$T_o = T_{\infty o} + \dot{Q}_1 \cdot \frac{1}{h_o A} = 5 + (75)\left(\frac{1}{10 \times 0.05}\right) = \mathbf{155°C} \qquad \textbf{(Ans.) (c)}$$

$$T_i = T_{\infty i} + \dot{Q}_2 \cdot \frac{1}{h_i A} = 23 + (1925)\left(\frac{1}{220 \times 0.05}\right) = \mathbf{198°C} \qquad \textbf{(Ans.) (c)}$$

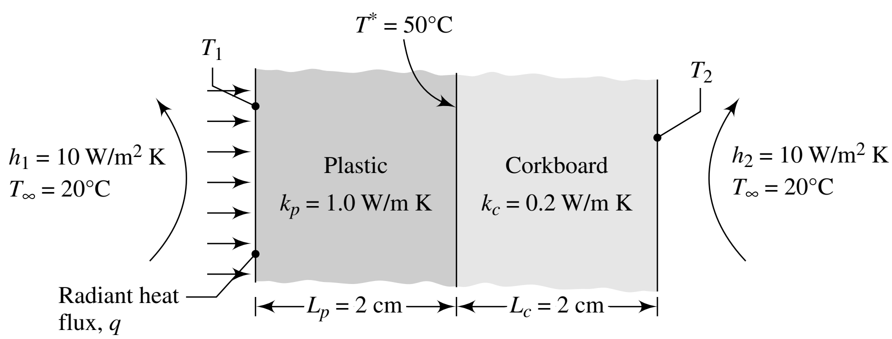
EXAMPLE 2.11 *In a manufacturing process, a large sheet of 2 cm thick plastic is to be glued to a 2 cm thick corkboard. To effect a good bond, the glue should be maintained at 50°C for a long time. This is achieved by heating the outer plastic surface by radiation. The convective heat-transfer coefficient is 10 W/m² K on both sides. The conductivities of plastic and cork are 1 and 0.2 W/m K, respectively and the ambient temperature is 20°C. Find the temperature of outer plastic surface and the radiant heat flux needed. Neglect losses due to radiation.*

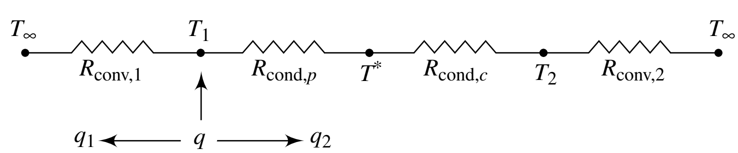
Solution

Known Plastic sheet glued with cork board. Outer plastic surface heated by radiation.

Find Outer plastic surface temperature, T_1 (°C). Radiant heat flux, q (W/m²).

Schematic





Assumptions (1) Steady operating conditions. (2) Constant convection coefficients. (3) Radiation losses neglected. (4) One-dimensional conduction.

Analysis From energy balance: $q_1 = \dfrac{T_1 - T_\infty}{R_{\text{conv},1}}$

$$q_2 = \frac{(T_1 - T^*)}{R_{\text{cond},p}} = (T^* - T_2)/R_{\text{cond},c} = (T_2 - T_\infty)/R_{\text{conv},2}$$

With $R_{\text{cond}, p} = L_p/k_p$, $R_{\text{cond}, c} = L_c/k_c$, $R_{\text{conv}, 1} = 1/h_1$ and $R_{\text{conv}, 2} = 1/h_2$ we have

$$q_1 = h_1(T_1 - T_\infty) \quad \text{and} \quad q_2 = \frac{k_p}{L_p}(T_1 - T^*) = \frac{k_c}{L_c}(T^* - T_2) = h_2(T_2 - T_\infty)$$

To determine T_2, we have

$$\frac{0.2\ \text{W/m K}}{0.02\ \text{m}}(50 - T_2)^\text{o}\text{C} = 10\ \text{W/m}^2\text{K}(T_2 - 20)^\text{o}\text{C}$$

or $$(50 - T_2) = \frac{(0.02)(10)}{(0.2)}(T_2 - 20) = (T_2 - 20) \Rightarrow 2T_2 = 70,$$

Hence, $T_2 = 35°\text{C}$

Heat flux, $q_2 = h_2(T_2 - T_\infty) = 10\ \text{W/m}^2\ \text{K}\ (35 - 20)°\text{C or K} = 150\ \text{W/m}^2$

We note that, $\dfrac{T_1 - T^*}{L_p/k_p} = \dfrac{T^* - T_\infty}{L_c/k_c + 1/h_2}$

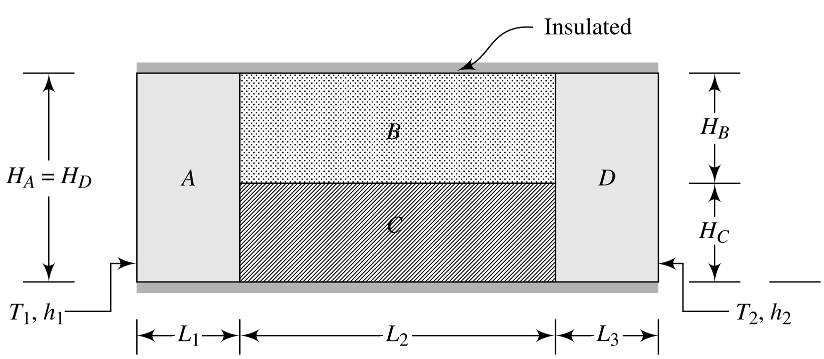
$$\therefore \quad T_1 = T^* + \left\{\frac{L_p/k_p}{L_c/k_c + 1/h_2}\right\}(T^* - T_\infty) = 50 + \left[\frac{0.02/1}{(0.02/02) + (1/10)}\right](50 - 20) = 53°\text{C}$$

Heat flux, $q_1 = h_1\ (T_1 - T_\infty) = 10\ \text{W/m}^2\ \text{K}\ (53 - 20) =$ **330 W/m²** **(Ans.)**

$$\begin{pmatrix}\text{Heat flux received} \\ \text{by radiation } q\end{pmatrix} = \begin{pmatrix}\text{Heat flux lost from both} \\ \text{exposed surfaces i.e. } q_1 + q_2\end{pmatrix}$$

And $q = q_1 + q_2 = 150 + 330 =$ **480 W/m²** **(Ans.)**

EXAMPLE 2.12 *A composite wall, having unit length normal to the plane of paper, is insulated at the top and bottom as shown in the figure. It is comprised of four different materials* A, B, C *and* D.



The dimensions are:

H$_A$ = H$_B$ = *3m,* H$_B$ = H$_C$ = *1.5 m,*

L$_1$ = L$_3$ = *0.05 m,* L$_2$ = *0.1 m*

The thermal conductivities of the materials are:

k$_A$ = k$_D$ = *50 W/m K,* k$_B$ = *10 W/m K,* k$_C$ = *1 W/m K*

The fluid temperature and heat-transfer coefficients (see figure) are

T$_1$ = *200°C,* h$_1$ = *50 W/m K,* T$_2$ = *25°C,* h$_2$ = *10 W/m² K.*

Assuming one dimensional conduction,

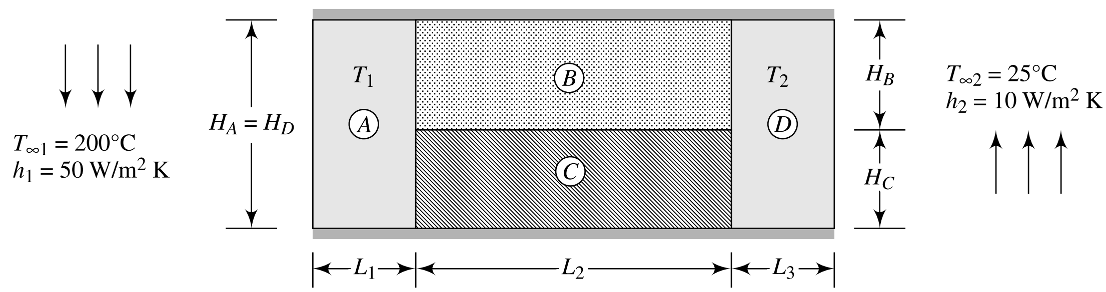
(a) sketch the thermal circuit of the system, (b) determine the rate of heat transfer through the wall, and (c) find the interface temperatures. ***[GATE 2001]***

Solution

Known A composite slab with inner and outer surface conditions, dimensions, and material thermal conductivities.

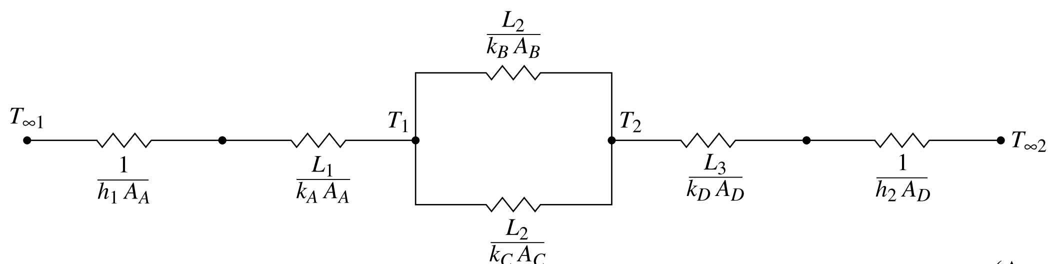
Find (a) Thermal circuit. (b) Heat transfer rate.

Schematic



Assumptions (1) Steady-state conditions. (2) Temperature of the composite varies only with x (surfaces normal to x are isothermal). (3) Negligible contact resistance. (4) Negligible radiation. (5) Constant properties.

Analysis: (a) The appropriate thermal circuit is a *series parallel arrangement* of the form



(Ans.) (a)

For a unit length normal to the paper,

$$A_A = A_D = H \quad \text{and} \quad A_B = A_C = H/2$$

(b) For a unit length normal to the paper, $\dot{Q} = \dfrac{(T_{\infty 1} - T_{\infty 2})}{R_{\text{total}}}$

where
$$R_{\text{total}} = \frac{1}{h_1 H} + \frac{L_1}{k_A H} + \left\{\frac{k_B H}{2L_2} + \frac{k_C H}{2L_2}\right\}^{-1} + \frac{L_3}{k_D H} + \frac{1}{h_2 H}$$

$$= \left[\frac{1}{50 \times 3} + \frac{0.05}{50 \times 3} + \left\{\frac{10 \times 3}{2 \times 0.1} + \frac{1 \times 3}{2 \times 0.1}\right\}^{-1} + \frac{0.05}{50 \times 3} + \frac{1}{10 \times 3}\right] \text{m K/W}$$

$$= 0.0467 \text{ m K/W}$$

Hence, the heat-transfer rate through the wall,

$$\dot{Q} = \frac{(200 - 25)^{\circ}\text{C or K}}{0.0467 \text{ m K/W}} = \mathbf{3745\ W/m} \qquad \textbf{(Ans.) (b)}$$

From the thermal circuit, the interface temperatures are:

$$T_1 = T_{\infty 1} - \dot{Q}\left[\frac{1}{h_1 H} + \frac{L_1}{k_A H}\right] = 200°\text{C - 3745 W/m} \times \left[\frac{1}{50 \times 3} + \frac{0.05}{50 \times 3}\right] \text{mK/W}$$

$$= \mathbf{173.8°C} \qquad \textbf{(Ans.) (c)}$$

Similarly, $$T_2 = T_{\infty 2} + \dot{Q}\left[\frac{1}{h_2 H} + \frac{L_3}{k_D H}\right] = 25^{\circ}\text{C} + 3745 \text{ W/m}\left[\frac{1}{10 \times 3} + \frac{0.05}{50 \times 3}\right] \text{m K/W}$$

$$= \mathbf{151.1°C} \qquad \textbf{(Ans.) (c)}$$

EXAMPLE 2.13 *(a) A furnace wall, 35 cm thick, (*k *= 1.05 W/m °C), has its inner surface maintained at 1300°C while its outer surface is exposed to surrounding air at 40°C with the associated convection coefficient expressed as* h *= 10 (1 + 0.01* ΔT*) (W/m °C) where* ΔT *is the temperature difference between the outer surface and the ambient air, in °C. Calculate the steady-state heat flux through the wall.*

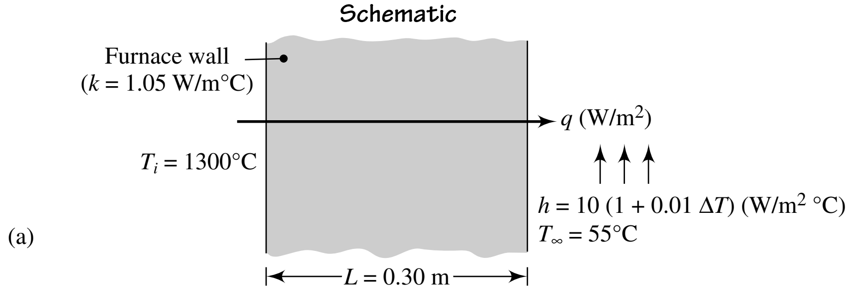
*(b) In order to curtail the heat transfer through the furnace wall by a factor of 7, it is decided to add a layer of red brick (*k *= 0.7 W/m °C) followed by a 30 cm thick layer of silicon brick (*k *= 0.15 W/m °C). Determine the thickness of the red brick layer required to effect this heat transfer.*

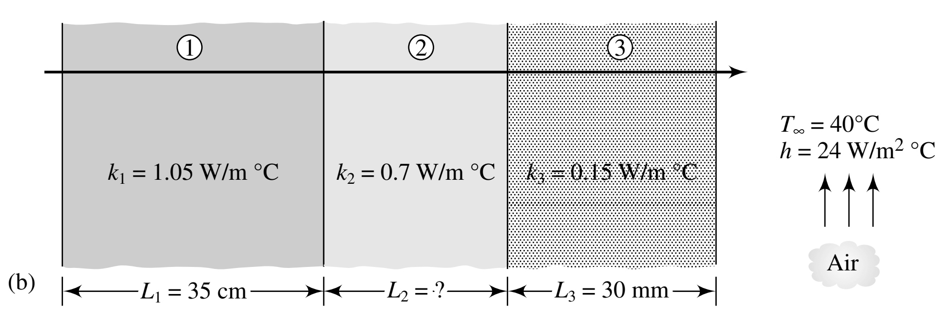
Solution

Known A furnace wall with outer convective surface and a prescribed convection coefficient. Addition of two brick layers to reduce the heat flux to one seventh of the single wall case.

Find (a) Heat flux, q (W/m²); (b) Red brick layer thickness (L_2) (cm).

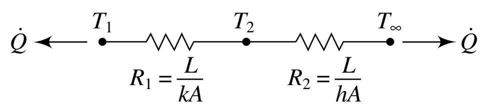
Schematic





Assumptions (1) Steady operating conditions exist. (2) Constant properties. (3) One dimensional heat conduction.

Analysis (a) The equivalent thermal circuit is



Heat transfer rate, $\dot{Q} = \dfrac{T_1 - T_\infty}{\left[\dfrac{L}{kA} + \dfrac{1}{hA}\right]} = \dfrac{T_2 - T_\infty}{1/hA}$

or $$T_2 - T_\infty = (T_1 - T_\infty)\frac{1/hA}{\left[\dfrac{L}{kA} + \dfrac{1}{hA}\right]} = \frac{T_1 - T_\infty}{\left(\dfrac{Lh}{k} + 1\right)} \quad \text{(A)}$$

Since $h = 10\ [1 + 0.01\ \Delta T] = 10 + 0.1\ \Delta T$

$\Delta T = T_2 - T_\infty = (h - 10)\ /\ 0.1 = 10\ h - 100$

Substituting proper values in Eq. (A),

$$10h - 100 = \frac{1300 - 40}{\dfrac{0.35h}{1.05} + 1} \quad \text{or} \quad 10h - 100 = \frac{1260 \times 1.05}{0.35h + 1.05}$$

or $3.5\ h^2 + 10.5\ h - 35\ h - 105 = 1323$

or $3.5\ h^2 - 24.5\ h - 1428 = 0$ or $h^2 - 7h - 408 = 0$

Solving this quadratic equation, we get, $h = 24$ W/m²°C

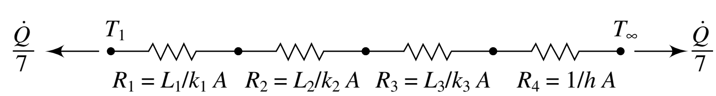
With this value of h, the steady-state heat transfer per unit wall area, that is, heat flux is determined from

$$q = \frac{\dot{Q}}{A} = \frac{T_1 - T_\infty}{(L/k) + (1/h)} = \frac{1300 - 40}{(0.35/1.05) + (1/24)} = \frac{1260}{0.375} = \mathbf{1260\ W/m^2} \quad \textbf{(Ans.) (a)}$$

(b) With two additional brick layers to cut down heat transfer by a factor of 7, one has

$$\frac{\dot{Q}}{7} = \frac{T_1 - T_\infty}{R_1 + R_2 + R_3 + R_4}$$

The equivalent thermal circuit is



or $$\frac{q}{7} = \frac{T_1 - T_\infty}{\left[\underbrace{\dfrac{L_1}{k_1} + \dfrac{1}{h}}_{\mathbf{0.375}} + \dfrac{L_2}{k_2} + \dfrac{L_3}{k_3}\right]}$$

or $$\frac{3360}{7} = \frac{1260}{[0.375 + (L_2/0.7) + (0.3/0.15)]} = \frac{1260}{(L_2/0.7) + 2.375}$$

or $$\frac{L_2}{0.7} = \left(\frac{1260 \times 7}{3360}\right) - 2.375 = 2.625 - 2.375 = 0.25$$

Hence, the thickness of the red brick layer is

$$L_2 = (0.7)\ (0.25)\ \text{m} = \mathbf{0.175\ m\ or\ 17.5\ cm} \qquad \textbf{(Ans.) (b)}$$

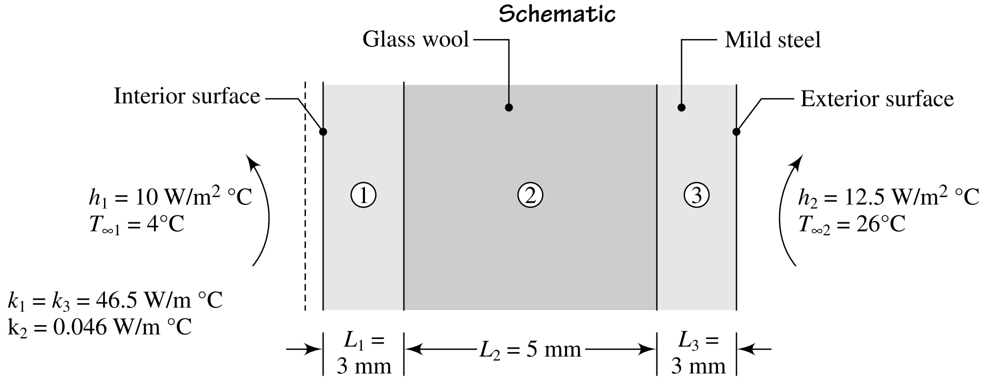
EXAMPLE 2.14 *The inside dimensions of a refrigerator are 50 cm × 50 cm base and 100 cm height. The walls of the refrigerator are constructed of two mild steel sheets 3 mm thick with 5 cm of glass wool insulation between them. If the average heat-transfer coefficients at the inner and outer surfaces are 10 and 12.5 W/m² °C respectively, (a) estimate the steady-state cooling load, i.e., the rate at which the heat must be removed from the interior space to maintain the refrigerated air temperature at 4°C while the surrounding outside air temperature is 26°C. (b) What will be the temperature at the outer surface of the wall? The thermal conductivities of mild steel and glass wool are 46.5 and 0.046 W/m °C, respectively.*

Solution

Known Composite wall of a refrigerator. Convective surface conditions.

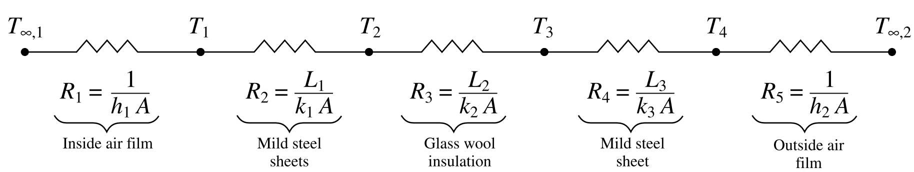
Find (a) Steady-state cooling load, $\dot{Q}$(W). (b) Wall surface temperature, T_4(°C).

Schematic



Assumptions (1) Steady operating conditions. (2) One-dimensional heat conduction. (3) Constant properties. (4) Interfacial contact resistance negligible.

Analysis (a) *The thermal resistance network is shown below:*



Heat-transfer area for the 6 surfaces is

$$A = 2\ [(0.5 \times 0.5) + (0.5 \times 1.0) + (0.5 \times 1.0)]\text{m}^2 = 2.5\ \text{m}^2$$

Thermal resistances connected in series are:

$$R_1 = \frac{1}{h_1 A} = \frac{1}{10 \times 2.5} = 0.004°\text{C/W} \qquad R_2 = \frac{L_1}{k_1 A} = \frac{0.003}{46.5 \times 2.5} = 2.58 \times 10^{-5}°\text{C/W}$$

$$R_3 = \frac{L_2}{k_2 A} = \frac{0.05}{0.046 \times 2.5} = 0.4348°\text{C/W} \quad R_4 = \frac{L_3}{k_3 A} = \frac{0.003}{46.5 \times 2.5} = 2.58 \times 10^{-5}°\text{C/W}$$

$$R_5 = \frac{1}{h_2 A} = \frac{1}{12.5 \times 2.5} = 0.032°\text{C/W}$$

Total thermal resistance is, $R_{total} = \Sigma R = R_1 + R_2 + R_3 + R_4 + R_5 = 0.5068°\text{C/W}$
Rate of heat removal, that is cooling load, is

$$\dot{Q} = \frac{\Delta T_{overall}}{R_{total}} = \frac{(26-4)°\text{C}}{0.5068°\text{C/W}} = \mathbf{43.4\ W} \qquad \textbf{(Ans.) (a)}$$

(b) Temperature at the outer surface of the refrigerator wall is

$$T_4 = T_{\infty_2} - \dot{Q}R_5 = 26°\text{C} - (43.4\text{ W})\,(0.032°\text{C/W}) = \mathbf{24.6°C} \qquad \textbf{(Ans.) (b)}$$

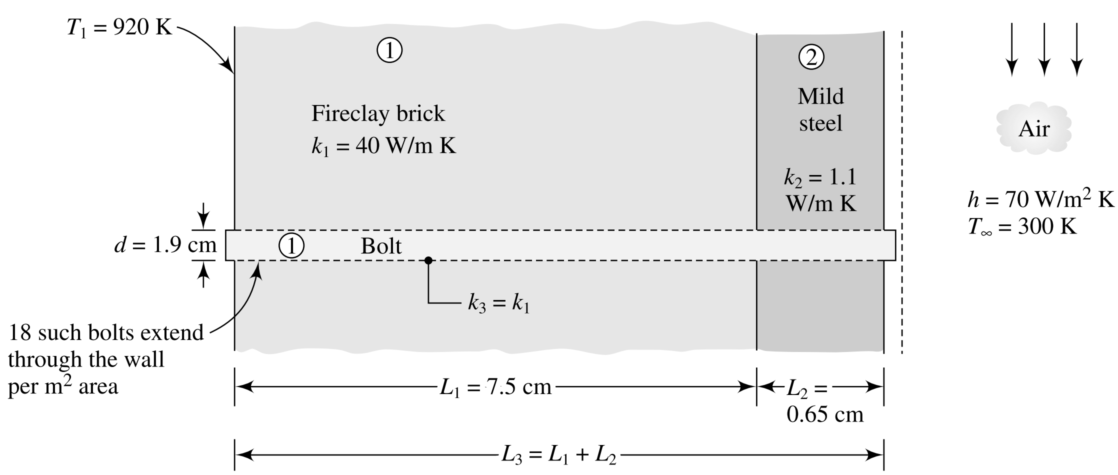
EXAMPLE 2.15 *A furnace wall is constructed with 7.5 cm of fireclay brick (*k *= 1.1 W/m K) next to the fire box and 0.65 cm of mild steel (*k *= 40 W/m K) on the outside. The inside surface of the brick is at 920 K, and the steel is surrounded by air at 300 K with an outside surface coefficient of 70 W/m² K. Find (a) the heat flux through each square metre of furnace wall, (b) the outside surface temperature of the steel, and (c) the percentage increase in the heat flux if, in addition to the conditions specified, eighteen 1.9 cm diameter steel bolts extend through the composite wall per square metre of wall area. How will the outside surface temperature of the steel get affected?*

Solution

Known Composite wall with one boundary surface exposed to convective environment.

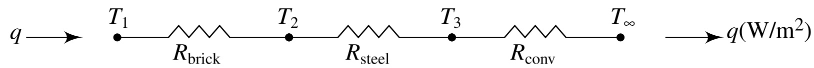
Find (a) Heat flux, q, (b) Outside surface temperature of steel, T_3 (°C), (3) Percent increase in heat flux if 18 bolts pierce through the wall.

Schematic



Assumptions (1) Steady-state one-dimensional conduction. (2) Constant properties. (3) Uniform convection coefficient.

Analysis (a) For the composite slab, the equivalent thermal circuit is



The heat flux may be evaluated, using the following equation: $q = \dfrac{\dot{Q}}{A} = \dfrac{T_1 - T_\infty}{\Sigma R_{\text{th}}}$

The thermal resistances (*per unit area*) are, in turn

$$R_{\text{brick}} = \frac{L_1}{k_1}\bigg|_{\text{brick}} = \frac{0.075\ \text{m}}{(1.10\ \text{W/m K})} = 0.068\,18\ \text{m}^2\ \text{K/W}$$

$$R_{\text{steel}} = \frac{L_2}{k_2}\bigg|_{\text{steel}} = \frac{0.0065\ \text{m}}{(40\ \text{W/m K})} = 1.625 \times 10^{-4}\ \text{m}^2\ \text{K/W}$$

$$R_{\text{conv}} = \frac{1}{h} = \frac{1}{(70\ \text{W/m}^2\text{K})} = 0.0143\ \text{m}^2\ \text{K/W}$$

The combined thermal resistance is

$$\Sigma R_{\text{th}} = [(0.068\ 18) + (1.625 \times 10^{-4}) + (0.0143)] = 0.0826\ \text{m}^2\text{K/W}$$

And, the heat flux is

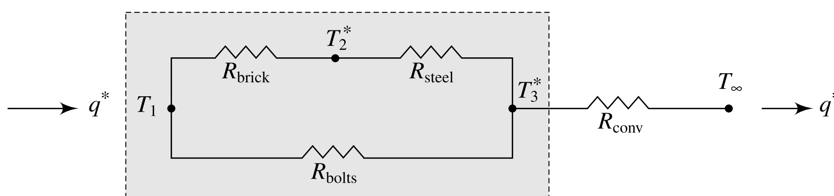
$$q = \frac{\dot{Q}}{A} = \frac{(920 - 300)\ \text{K}}{0.0826\ \text{m}^2\ \text{K/W}} = \mathbf{7506\ W/m^2} \qquad \textbf{(Ans.) (a)}$$

(b) The outside surface temperature of the steel can now be found from

$$q\ R_{\text{conv}} = T_3 - T_\infty$$

$\therefore$ $T_3 = 300\ \text{K} + (7506\ \text{W/m}^2)\ (0.0143\ \text{m}^2\ \text{K/W}) = \mathbf{407.3\ K}$ **(Ans.) (b)**

(c) In the case of additional steel bolts through the wall, there are now two paths whereby heat may flow from the inside of the furnace wall to the outside air. The equivalent thermal circuit in this case is shown below.



R_{bricks}, R_{steel}, and R_{conv} are all known. Thermal resistance for the steel bolts (*per unit area*) is calculated as

$$R_{\text{bolts}} = \frac{L_3}{k_3} = \frac{(0.075 + 0.0065)\ \text{m}}{(40\ \text{W/m K})\left[\left(18\dfrac{\text{bolts}}{m^2}\right)\left(\dfrac{\pi}{4} \times 0.019^2\right)\text{m}^2/\text{bolts}\right]} = 0.399\ \text{m}^2\ \text{K/W}$$

The equivalent resistance of the parallel portion of the circuit is

$$R_{\text{equiv}} = \left[\frac{1}{R_{\text{bolts}}} + \frac{1}{(R_{\text{bricks}} + R_{\text{steel}})}\right]^{-1} = \left[\frac{1}{0.399} + \frac{1}{(0.06818 + 1.625 \times 10^{-4})}\right]^{-1}$$

$$= 0.05835\ \text{m}^2\ \text{K/W}$$

The total thermal resistance for the wall and bolts is

$$R^*_{\text{total}} = R_{\text{equiv}} + R_{\text{conv}} = 0.058\ 35 + 0.0143 = 0.07265\ \text{m}^2\ \text{K/W}$$

The resulting heat flux is

$$q^* = \frac{\dot{Q}}{A} = \frac{\Delta T_{\text{overall}}}{R^*_{\text{total}}} = \frac{(920 - 300)\,\text{K}}{(0.072\ 65)\,\text{m}^2\ \text{K/W}} = \mathbf{8534\ W/m^2} \qquad \textbf{(Ans.) (c)}$$

Percentage *increase* in the heat flux after the bolts are introduced is

$$\left(\frac{q^* - q}{q}\right)(100) = \left(\frac{8534 - 7506}{7506}\right)(100) = \mathbf{13.7\%} \qquad \textbf{(Ans.) (c)}$$

Outside surface temperature of the steel,

$$T_3^* = T_\infty + q^* R_{\text{conv}} = 300°\text{C} + (8534\ \text{W/m}^2)\,(0.0143\ \text{m}^2\ \text{K/W})$$
$$= \mathbf{422°C} \qquad \text{(an increase by 14.7°C)} \qquad \textbf{(Ans.) (c)}$$

(E) Thermal Contact Resistance

EXAMPLE 2.16 *The following table gives details of a composite wall comprising two materials.*

S.No.	Parameter	Material 1	Material 2
1.	Wall thickness (L), mm	10	20
2.	Thermal conductivity (k), W/m °C)	0.1	0.05
3.	Contact resistance between two materials (R_c) = 0.3 m² °C /W		
4.	Convective medium adjoining the material 1: T = 200°C h_1= 10 W/m² °C		
5.	Convective medium adjoining the material 2: $T_{\infty 2}$ = 25°C h_2 = 20 W/m²°C		

Determine: (a) The steady state heat transfer rate per unit area through the wall and (b) The temperature distribution.

Solution

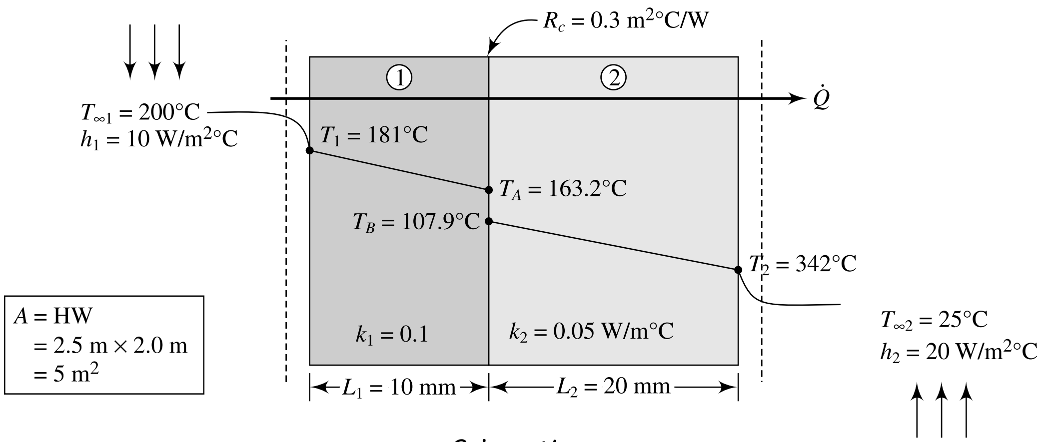
Known Temperatures and convection coefficients associated with fluids at inner and outer surfaces of a composite wall. Contact resistance, dimensions, and thermal conductivities of the two wall materials.

Find (a) Heat-transfer rate, $\dot{Q}$ (W), (b) Temperature variation across the wall.

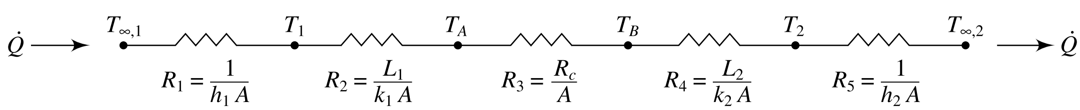
Assumptions (1) Steady-state conditions exist. (2) One-dimensional heat transfer. (3) Constant properties. (4) Negligible radiation.

Analysis (a) There are in all *five* thermal resistances connected in series. Of these, *two* are conduction resistances, *two* convective resistances and *one* contact resistance at the interface of the two walls.

Schematic



Schematic



Total thermal resistance,

$$R_{\text{total}} = \Sigma R = R_1 + R_2 + R_3 + R_4 + R_5 = \frac{1}{h_1 A} + \frac{L_1}{k_1 A} + \frac{R_c}{A} + \frac{L_2}{k_2 A} + \frac{1}{h_2 A}$$

$$= \left[\frac{1}{(10)(5)} + \frac{0.01}{(0.1)(5)} + \frac{0.3}{5} + \frac{0.02}{(0.05)(5)} + \frac{1}{(20)(5)}\right] \text{°C/W}$$

$$= [0.02 + 0.02 + 0.06 + 0.08 + 0.01] \text{ °C/W} = \mathbf{0.19°C\ /W}$$

Rate of heat transfer across the wall,

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} = \frac{(200 - 25)\text{°C}}{0.19\text{°C/W}} = \mathbf{921\ W} \qquad \textbf{(Ans.) (a)}$$

(b) It follows that

$$T_1 = T_{\infty 1} - \dot{Q}R_1 = 200\text{°C} - (921 \text{ W})\ (0.02\text{°C/W}) = \mathbf{181.6°C}$$

$$T_A = T_1 - \dot{Q}R_2 = 181.6 - (921)\ (0.02) = \mathbf{163.2°C}$$

$$T_B = T_A - \dot{Q}R_3 = 163.2 - (921)\ (0.06) = \mathbf{107.9°C}$$

$$T_2 = T_B - \dot{Q}R_4 = 107.9 - (921)\ (0.08) = \mathbf{34.2°C}$$

$$T_{\infty 2} = T_2 - \dot{Q}R_5 = 34.2 - (921)\ (0.01) = \mathbf{25°C}$$

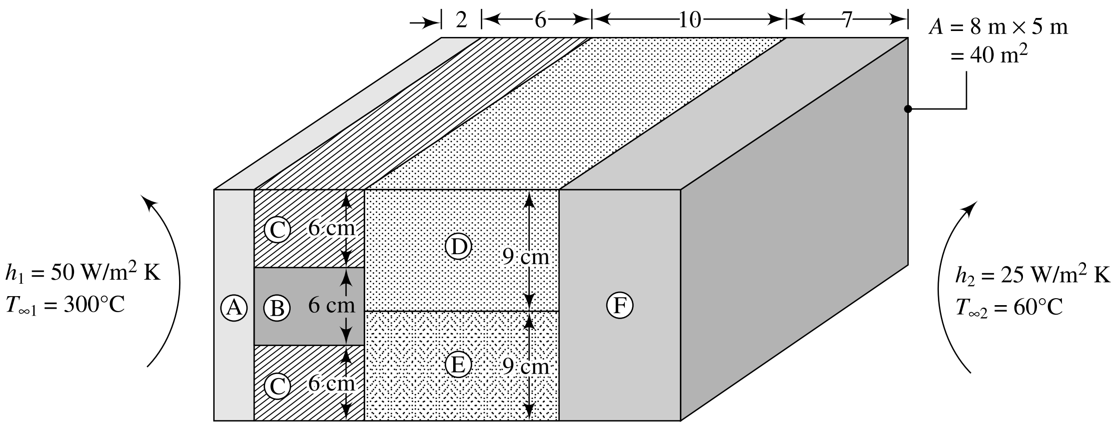
The temperature distribution across the wall is shown in the schematic. (Ans.) (b)

Comment As $\dot{Q}$, k, and A are constant, the temperature gradient across each wall is also constant, implying linear temperature profile. A sudden temperature drop from T_A to T_B is due to contact resistance.

EXAMPLE 2.17 *A multilayered plane wall is 8 m long, 5 m high and 0.25 m thick. A representative cross section of the wall is shown in the figure. The left and right faces of the wall are at 300°C and 60°C respectively. The thermal conductivities of different materials used in the composite system (in W/m K) are* $k_A = k_F = 2$, $k_B = 10$, $k_C = 20$, $k_D = 15$, *and* $k_E = 55$.

The thermal contact resistance at the interfaces D F *and* E F *is* 1.2×10^{-4} m^2 *K/W.*

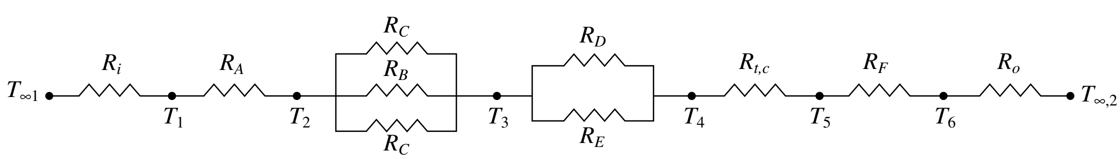
Assuming one-dimensional heat conduction, calculate: (a) the rate of heat flow through the wall, (b) the temperature at the point where the sections B, D, *and* E *meet; and (c) the temperature drop across the section* F, *(d) the temperature drop due to contact resistance.*



Solution

Known A composite slab with a series–parallel combination. Exposed surfaces subjected to convection processes. Dimensions, conductivities, convection coefficients, and fluid temperatures.

Find (a) $\dot{Q}$ (W), (b) T_3(°C), (c) $(T_5 - T_6)$°C, (d) $(T_4 - T_5)$°C.



Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant properties.

Analysis The thermal resistance network for the series–parallel arrangement is shown in the schematic. Let us first determine the individual thermal resistances.

We note that

$$A_A = A_F = A = 5 \times 8 = 40 \text{ m}^2;\ A_B = A_C = \frac{1}{3}A \text{ and } A_D = A_E = \frac{A}{2}$$

Convective resistances: $R_i = \dfrac{1}{h_1 A_1} = \dfrac{1}{(50\ \text{W/m}^2\text{K})(40\ \text{m}^2)} = 0.5 \times 10^{-3}\ \text{K/W}$

$$R_o = \frac{1}{h_2 A_2} = \frac{1}{(25\ \text{W/m}^2\text{K})(40\ \text{m}^2)} = 1.0 \times 10^{3}\ \text{K/W}$$

Contact thermal resistance: $R_{t,c} = \dfrac{R''_{t,c}}{A} = \dfrac{1.2 \times 10^{-4}\ \text{m}^2\ \text{K/W}}{40\ \text{m}^2} = 3 \times 10^{-6}$ K/W

Conduction resistances: $R_A = \dfrac{L_A}{k_A A_A} = \dfrac{0.02\ \text{m}}{2\ \text{W/m}^2\ \text{K} \times 40\ \text{m}^2} = 0.25 \times 10^{-3}\ \text{K/W}$

Three resistances in parallel: $R_C = \dfrac{L_C}{k_C\, A/3} = \dfrac{3 \times 0.06\ \text{m}}{20\ \text{W/m K} \times 40\ \text{m}^2} = 0.225 \times 10^{3}\ \text{K/W}$

$$R_B = \frac{L_B}{k_B A / 3} = \frac{3 \times 0.006\ \text{m}}{10\ \text{W/m} \times 40\ \text{m}^2} = 0.45 \times 10^{3}\ \text{K/W}$$

$$R_{\text{eq}_\text{I}} = \left[\frac{1}{R_C} + \frac{1}{R_B} + \frac{1}{R_C}\right]^{-1} = \left[\frac{1}{0.225 \times 10^{-3}} + \frac{1}{0.45 \times 10^{-3}} + \frac{1}{0.225 \times 10^{-3}}\right]^{-1}$$
$$= 0.09 \times 10^{3}\ \text{K/W}$$

Two resistances in parallel: $R_D = \dfrac{L_D}{k_D A/2} = \dfrac{0.1\ \text{m}}{15\ \text{W/m K} \times 20\ \text{m}^2} = 0.333 \times 10^{-3}\ \text{K/W}$

$$R_E = \frac{L_E}{k_E A/2} = \frac{0.1\ \text{m}}{55\ \text{W/ m K} \times 20\ \text{m}^2} = 0.0909 \times 10^{3}\ \text{K/W}$$

$$R_{\text{eq}_\text{II}} = \left[\frac{1}{R_D} + \frac{1}{R_E}\right]^{-1} = [3000 + 11000]^{-1} = 0.0714 \times 10^{-3}\ \text{K/W}$$

$$R_F = \frac{L_F}{k_F A} = \frac{0.07\ \text{m}}{2\ \text{W/m K} \times 40\ \text{m}^2} = 0.875 \times 10^{-3}\ \text{K/W}$$

Total thermal resistance,

$R_{\text{total}} = R_i + R_A + R_{\text{eq I}} + R_{\text{eq II}} + R_{t,\,c} + R_F + R_o$

$= [(0.5 \times 10^{-3}) + (0.25 \times 10^{-3}) + (0.09 \times 10^{-3}) + (0.0714 \times 10^{-3}) + (3 \times 10^{-6}) + (0.875 \times 10^{-3}) + (1 \times 10^{-3})]$

$= 2.6894 \times 10^{-3}$ K/W

∴ Rate of heat flow through the wall is

$$\dot{Q} = \frac{\Delta T_{\text{overall}}}{R_{\text{total}}} = \frac{(300 - 60)°\text{C}}{2.7894 \times 10^{-3}\ \text{K/W}} \left|\frac{1\ \text{kW}}{10^3 W}\right| = \mathbf{86\ kW} \qquad \textbf{(Ans.) (a)}$$

Temperature at which sections B, D, and E meet is

$T_3 = T_{\infty 1} - \dot{Q}\,[R_1 + R_A + r_{\text{eq I}}]$

$= 300°\text{C} - (86.04 \times 10^3\ \text{W})\,[(0.5 \times 10^{-3}) + (0.25 \times 10^{-3}) + (0.09 \times 10^{-3})]$ K/W

= 227.7°C **(Ans.) (b)**

Temperature drop across the section F,

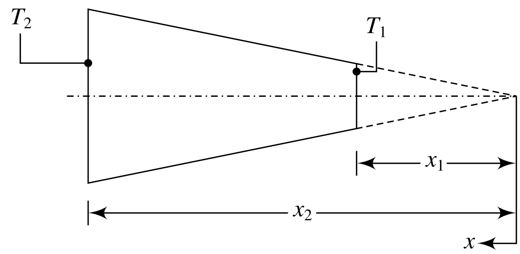
$T_5 - T_6 = \dot{Q} R_F = (86.04 \times 10^3\ \text{W})(0.875 \times 10^{-3}\ \text{K/W}) =$ **75.3°C** **(Ans.) (c)**

Temperature drop due to contact resistance,

$T_4 - T_5 = \dot{Q} R_{tc} = (86.04 \times 10^3\ \text{W})(3 \times 10^{-6}\ \text{K/W}) =$ **0.26°C** **(Ans.) (d)**

(F) Variable Area of Cross Section

EXAMPLE 2.18 *The diagram shows a conical section of circular cross section (k = 3.42 W/m K) with the diameter D = ax, where a = 0.30. The small end is at x_1 = 75 mm and the large end at x_2 = 300 mm. The end temperatures are T_1 = 150°C and T_2 = 350°C, while the lateral surface is well insulated.*



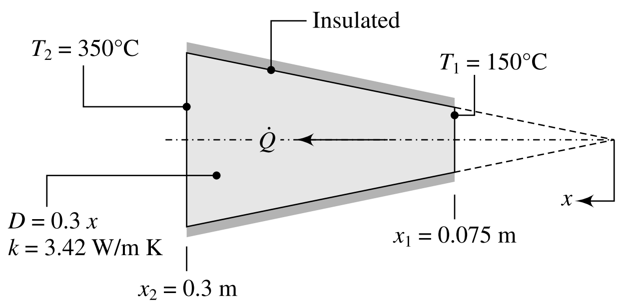
(a) Derive an expression for the temperature distribution assuming steady-state, one-dimensional conduction with no internal heat generation. (b) Compute the heat-transfer rate through the cone. (c) Sketch the temperature distribution and justify the shape of the curve.

Solution

Known Conduction in a conical section with $D = ax$ where a = 0.30 and insulated curved surface.

Find (a) Temperature distribution, $T(x)$. (b) Heat rate, $\dot{Q}$ (W). (c) Temperature profile.

Schematic



Assumptions (1) Steady operating conditions. (2) One-dimensional conduction in x direction. (3) Constant properties.

Analysis (a) Fourier' law of heat conduction is, $\dot{Q} = -k A_c \dfrac{dT}{dx}$

where A_c = cross-sectional area = $\dfrac{\pi D^2}{4} = \dfrac{\pi a^2 x^2}{4}$

$$\dot{Q} = -\frac{\pi k a^2 x^2}{4}\frac{dT}{dx}$$

Separating the variables, we get, $\dfrac{4\dot{Q}dx}{\pi a^2 x^2 k} = -dT$

Integration from x_1 to any x within the cone yields,

$$\frac{4\dot{Q}}{\pi a^2 k}\int_{x_1}^{x}\frac{dx}{x^2} = -\int_{T_1}^{T} dT \quad \text{or} \quad \frac{4\dot{Q}}{\pi a^2 k}\left[-\frac{1}{x}\right]_{x_1}^{x} = -(T - T_1)$$

or $$\frac{4\dot{Q}}{\pi a^2 k}\left(\frac{1}{x_1}-\frac{1}{x}\right)=(T_1-T) \quad \text{or} \quad T=T_1-\frac{4\dot{Q}}{\pi a^2 k}\left(\frac{1}{x_1}-\frac{1}{x}\right) \tag{1}$$

Now, $T(x_2) = T_2$. Substituting this in the above expression, we have

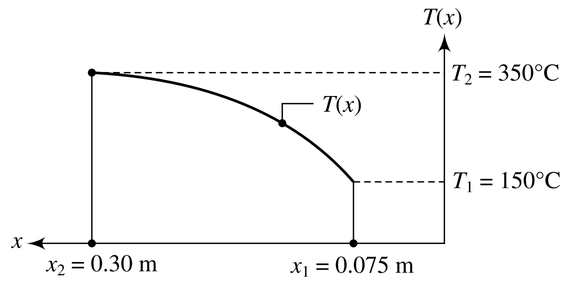
$$T_2=T_1-\frac{4\dot{Q}}{\pi a^2 k}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)$$

Solving for $\dot{Q}$,

$$\boxed{\dot{Q}=\frac{(T_1-T_2)\pi a^2 k}{4[(1/x_1)-(1/x_2)]}} \tag{2}$$

Substituting for $\dot{Q}$ in the expression for T (Eq. 1),

$$T=T_1-\frac{(T_1-T_2)\pi a^2 k}{4\left[\frac{1}{x_1}-\frac{1}{x_2}\right]}\cdot\frac{4}{\pi a^2 k}\left[\frac{1}{x_1}-\frac{1}{x}\right]$$



The temperature distribution is thus given by

$$\boxed{\boldsymbol{T(x)=T_1-(T_1-T_2)\left[\frac{\left(\frac{1}{x_1}\right)-\left(\frac{1}{x}\right)}{\left(\frac{1}{x_1}\right)-\left(\frac{1}{x_2}\right)}\right]}}$$ **(Ans.) (a)**

(b) Substituting the appropriate numerical values, we get

$$\dot{Q}=\frac{(150-350)°\text{C or } K(\pi)\,(0.30\text{ m})^2\,(3.42\text{ W/m K})}{4\left[\frac{1}{0.075\text{ m}}-\frac{1}{0.300\text{ m}}\right]}$$

$= -$ **4.38 W** **(Ans.) (b)**

The temperature distribution can be sketched qualitatively as under.

Comment Since the steady heat rate, $\dot{Q}$ and thermal conductivity, k are constant, the product $\left(A_c\frac{dT}{dx}\right)$ is constant. In the x-direction, the area of cross section A_c is *increasing*. Hence, the slope or temperature gradient dT/dx must *decrease* in the x-direction.

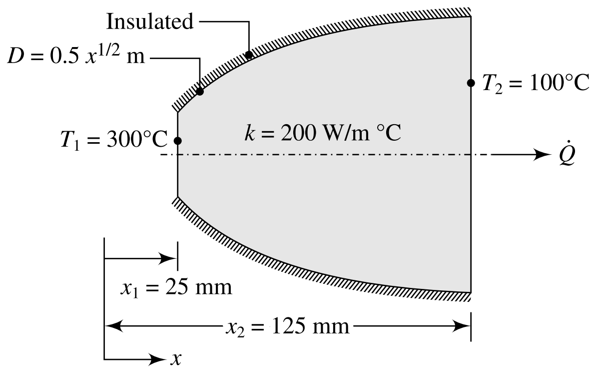
EXAMPLE 2.19 *A conical section of circular cross section (*k *= 200 W/m°C) having diameter* D *=* a$x^{1/2}$ *where* a *= 0.5* $m^{1/2}$ *has its small end located at* x_1 *= 25 mm and the large end at* x_2 *= 125 mm. The end temperatures* T_1 *and* T_2 *are held at* T_1 *= 300°C and* T_2 *= 100°C. The lateral (curved) surface is effectively insulated. Assuming one dimensional conditions, determine (a) the temperature profile, and (b) the heat-transfer rate.*

Solution

Known Geometry and surface conditions of a conical solid of circular cross section.

Find $T(x)$ and $\dot{Q}$.

Schematic



Assumptions (1) Steady, one-dimensional conduction. (2) Constant thermal conductivity.

Analysis (a) Fourier's rate equation

$$\dot{Q} = -kA_c \frac{dT}{dx} = -k\left[\frac{\pi D^2}{4}\right]\frac{dT}{dx} = -\frac{\pi k}{4}(ax^{1/2})^2 \frac{dT}{dx}$$

$$= -\frac{\pi k a^2}{4} x \frac{dT}{dx}$$

Separating the variables and integrating between limits,

$$\frac{4\dot{Q}}{\pi k a^2}\int_{x_1}^{x_2}\frac{dx}{x} = -\int_{T_1}^{T_2} dT \quad \text{or} \quad \frac{4\dot{Q}}{\pi k a^2}\ln\frac{x_2}{x_1} = T_1 - T_2 \qquad \text{(A)}$$

If the limits are identified as $x = x_1$, $T = T_1$ and $x = x$, $T = T$, then

$$\frac{4\dot{Q}}{\pi k a^2}\ln\frac{x}{x_1} = (T_1 - T) \qquad \text{(B)}$$

From equations (A) and (B), $\dfrac{\ln(x/x_1)}{\ln(x_2/x_1)} = \dfrac{T_1 - T}{T_1 - T_2}$

The temperature distribution is then given by

$$\boxed{\boldsymbol{T(x) = T_1 - (T_1 - T_2)\frac{\ln(x/x_1)}{\ln(x_2/x_1)}}} \qquad \textbf{(Ans.) (a)}$$

(b) The heat-transfer rate is given by

$$\dot{Q} = \frac{\pi k a^2 (T_1 - T_2)}{4\ln(x_2/x_1)} = \frac{\pi(200\ \text{W/m°C})(0.5\ \text{m}^{1/2})^2(300-100)\text{°C}}{4\ln(125\ \text{mm}/25\ \text{mm})}$$

$$= \mathbf{4.88 \times 10^3\ W} \quad \text{or} \quad \mathbf{4.88\ kW} \qquad \textbf{(Ans.) (b)}$$

(G) Variable Thermal Conductivity

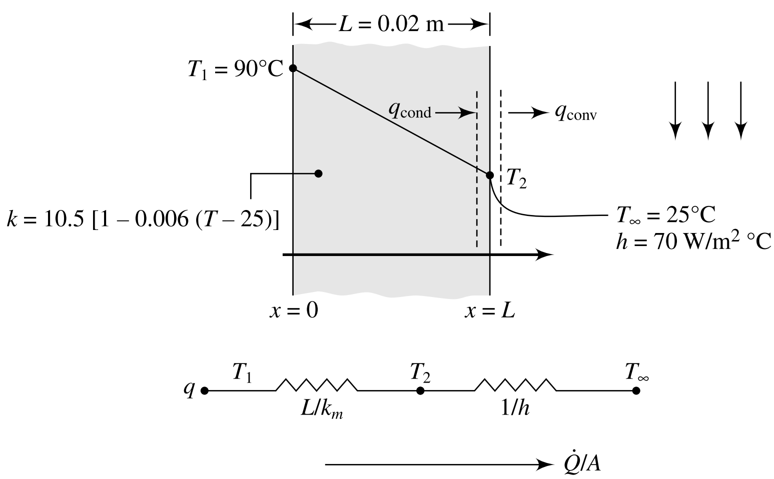
EXAMPLE 2.20 *A 10 cm thick slab has its heated surface maintained at 90°C and the other surface exposed to convective environment at 25°C with a convection coefficient of 70 W/m²°C. The slab material has variable thermal conductivity given by* k (T) *= 10.5 [1 – 0.006 (*T *– 25)] W/m °C where* T *is in °C. This relation is valid for the temperature range 25 °C <* T *< 110°C. Determine (a) the steady state-heat flux (W/m²), (b) the temperature of the cooled surface (°C), and (c) the temperature gradients at the two exposed surfaces of the slab (°C/m).*

Solution

Known A slab made of a variable thermal conductivity material has one isothermal heated surface and the other surface subjected to convective cooling.

Find (a) Heat flux, q (W/m²). (b) Surface temperature, T_2(°C). (c) Temperature gradients, $\left.\dfrac{dT}{dx}\right|_{x=0}, \left.\dfrac{dT}{dx}\right|_{x=L}$

Schematic



Assumptions (1) Steady-state one-dimensional conduction. (2) Variable thermal conductivity. (3) Uniform heat-transfer coefficient.

Analysis The steady-state heat flux or transfer rate per unit area is

$$q = \frac{\dot{Q}}{A} = \frac{\Delta T_{\text{overall}}}{R_{\text{total}}} = \frac{T_1 - T_\infty}{L/k_m + 1/h} = \frac{T_2 - T_\infty}{1/h}$$

i.e., $$\boxed{q = h(T_2 - T_\infty)}$$

Control surface energy balance at the exposed cooled surface ($x = L$).

$$q_{\text{cond}}\big|_{x=L} = q_{\text{conv}} \quad \text{or} \quad \frac{k_m(T_1 - T_2)}{L} = h(T_2 - T_\infty) \tag{A}$$

where k_m is the mean thermal conductivity, expressed as, $k_m = k_0(1 - \beta T_m)$ in which k_0 and β are constants and $T_m = \dfrac{1}{2}(T_1 + T_2)$

In the present case, k_0 = 10.5, β = 0.006, and

$$T_m = \frac{1}{2}[(T_1 - 25) + (T_2 - 25)] = \left(\frac{T_1 + T_2}{2}\right) - 25$$

Hence, from Eq. (A):

$$\frac{k_0(T_1 - T_2)}{L}\left[1 - \beta\left\{\frac{T_1 + T_2}{2} - 25\right\}\right] - h(T_2 - T_\infty) = 0$$

or
$$\frac{k_0 T_1}{L} - \frac{k_0 T_2}{L} - \frac{k_0(T_1^2 - T_2^2)\beta}{2L} + \frac{25\beta k_0(T_1 - T_2)}{L} - hT_2 + hT_\infty = 0$$

or
$$\left(\frac{k_0\beta}{2L}\right)T_2^2 - \left\{\frac{k_0}{L} + \frac{25\beta k_0}{L} + h\right\}T_2 + \left[\frac{k_0 T_1}{L} + \frac{25\beta k_0 T_1}{L} - \frac{\beta k_0 T_1^2}{2L} + hT_\infty\right] = 0$$

With $\dfrac{k_0\beta}{L} = \dfrac{10.5 \times 0.006}{0.1} = 0.63, \dfrac{k_0\beta}{2L} = 0.315, \dfrac{25k_0\beta}{L} = 15.75,$ we have

$$0.315\ T_2^2\ [105 + 15.75 + 70] + [9450 + 1417.5 - 2551.5 + (70 \times 25)] = 0$$

or
$$0.315\ T_2^2 - 190.75\ T_2 = 10066 = 0$$

This is a quadratic equation, the roots of which are

$$T_2 = 547.15°\text{C and } 58.4°\text{C}$$

The first value is absurd because $T_1 > T_2 > T_\infty$. Hence, the temperature of the cooled surface is

T_2 = **58.4°C** **(Ans.) (b)**

Heat flux is determined from

$q = h(T_2 - T_\infty)$ = (70 W/m²°C) (58.4 – 25)°C = **2338 W** **(Ans.) (a)**

The temperature gradients at the two surfaces are obtained from

$$\frac{dT}{dx} = \frac{q}{k_m} = \frac{q}{k_0[1 - \beta\{T - 25\}]}$$

At x = 0 (*the heated surface*),

$$\left|\frac{dT}{dx}\right|_{x=0} = \frac{q}{k_0[1 - \beta(T_1 - 25)]} = \frac{2338\text{ W}}{10.5[1 - 0.006(90 - 25)]} = \mathbf{365°C}$$ **(Ans.) (c)**

and
$$\left|\frac{dT}{dx}\right|_{x=L} = \frac{q}{k_0[1 - \beta(T_2 - 25)]} = \frac{2338\text{ W}}{10.5[1 - 0.006(58.4 - 25)]} = \mathbf{278.5°C/m}$$

Comment The temperature gradient is *not* constant but decreases with an increase in the value of x. The temperature profile is therefore concave upwards. The thermal conductivity decreases with an increase in temperature.

EXAMPLE 2.21 *A 0.5 m long metal piece has a cross section in the form of a sector of a circle of 0.1 m radius and included angle of 90°. The thermal conductivity of the metal piece can be expressed as,* k = *116.3 [1 – 0.0001* T*] (W/m °C) where* T *is in °C.*

Assuming heat transfer only in axial direction, find the heat rate if the two ends of the metal piece are maintained at 120°C and 30°C.

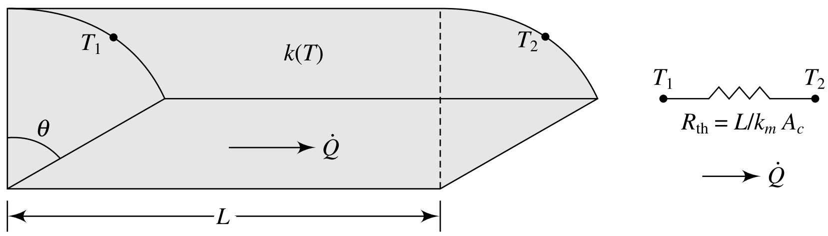
Solution

Known A metal piece with variable thermal conductivity has a cross section in the form of a sector of a circle.

Find Heat rate, $\dot{Q}$ (W) in the axial direction.

Schematic

$r = 0.1$ m
$L = 0.5$ m
$T_1 = 120°C$
$T_2 = 30°C$
$\theta = 90°$ or $\pi/2$ rad



Assumptions (1) One-dimensional (*axial*) heat conduction. (2) Steady-state conditions. (3) Boundary temperatures are uniform.

Analysis Area of cross section,

$$A_c = \left(\frac{\theta}{2\pi}\right)(\pi r^2) = \frac{\theta r^2}{2} = \frac{(\pi/2)(0.1\,\text{m})^2}{2} = 7.854 \times 10^{-3}\,\text{m}^2$$

Mean thermal conductivity, $k_m = k_0[1 + \beta T_m]$

where $T_m = \frac{1}{2}(T_1 + T_2) = \frac{1}{2}(120 + 30)°C = 75°C$

$\therefore$ $k_m = 116.3\ [1 - (0.0001)\ (75)] = 115.43$ W/m °C

Thermal resistance,

$$R_{th} = \frac{L}{k_m A_c} = \frac{0.5\,\text{m}}{115.43\,\text{W/m°C} \times 7.854 \times 10^{-3}\,\text{m}^2} = 0.5515°\text{C/W}$$

Temperature difference,

$$\Delta T = T_1 - T_2 = 120 - 30 = 90°C$$

Hence, the heat rate is determined from

$$\dot{Q} = \frac{\Delta T}{R_{th}} = \frac{90°C}{0.5515°\text{C/W}} = \mathbf{163\ W} \qquad \textbf{(Ans.)}$$

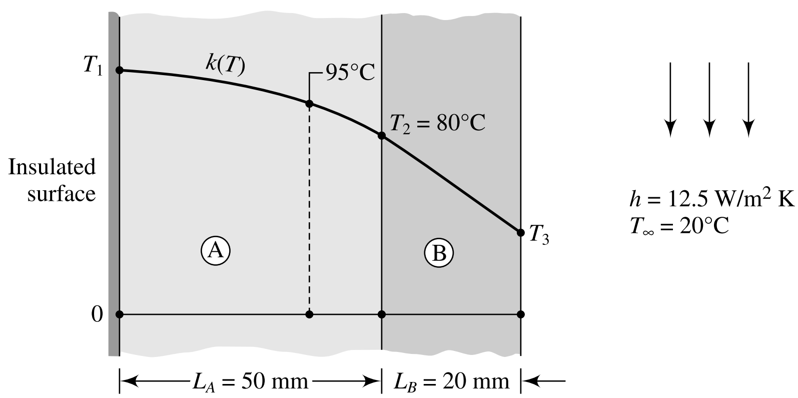
EXAMPLE 2.22 *A composite slab, 70 mm thick, is comprised of two layers of materials* A *and* B. *The layer* A, *50 mm thick, has its thermal conductivity* $k_A = 0.5\ [1 + 0.008\ T]$ *where* T *is in °C. The layer* B *has constant thermal conductivity of 30 W/m K and is 20 mm thick. The exposed surface of layer* A *is effectively insulated and that of the layer* B *is exposed to convective environment at 20°C with a surface heat-transfer coefficient of 12.5 W/m² K. The interface temperature between the two layers is 80°C. Determine (a) the steady-state heat flux, (b) the maximum temperature in the composite slab, and (c) the distance of a plane from the insulated surface where the temperature is 90°C.*

Solution

Known A composite slab made up of two materials, and one of variable thermal conductivity and the other with constant thermal conductivity with one adiabatic surface and the other exposed to convective medium.

Find (a) q(W/m²); (b) T_{max} (°C); (c) x at T = 90°C.

Schematic



Assumptions (1) Steady-state, one-dimensional conduction. (2) Uniform heat-transfer coefficient. (4) No thermal conduct resistance or internal heat generation.

Analysis (a) The steady-state heat-transfer rate through the composite slab in the absence of heat generation is constant throughout. The cross-sectional area being same, the heat flux is also constant.

Using electrical analogy,

$$q = \frac{\dot{Q}}{A} = \frac{T_2 - T_\infty}{R_{\text{cond}} + R_{\text{conv}}} = \frac{T_2 - T_\infty}{\left(\dfrac{L_B}{k_B} + \dfrac{1}{h}\right)} = \frac{(80 - 20)\,\text{K}}{\left(\dfrac{0.02\ \text{m}}{30\ \text{W/mK}}\right) + \left(\dfrac{1}{12.5\ \text{W/m}^2\,\text{K}}\right)}$$

$= \mathbf{743.8\ W/m^2}$ **(Ans.) (a)**

(b) Considering the layer A, one has

$$q = -k(T)\frac{dT}{dx} \quad \text{or} \quad q\,dx = -k_o(1 + \beta T)dT$$

Integrating between the limits: x = 0, $T = T_1$ and $x = L$, $T = T_2$,

$$q\int_0^{L_A} dx = -k_o \int_{T_1}^{T_2} (1 + \beta T)dT \quad \text{or} \quad \frac{qL_A}{k_0} = (T_1 - T_2) + \beta(T_1^2 - T_2^2)/2$$

or
$$\frac{743.8\ \text{W/m}^2 \times 0.05\ \text{m}}{0.5\ \text{W/m}\,°\text{C}} = \left[(T_1 - 80) + \frac{0.008}{2}(T_1^2 - 80^2)\right]°\text{C}$$

or
$$0.004T_1^2 + T_1 - 80 - (0.004 \times 80^2) - \left(\frac{743.8 \times 0.05}{0.5}\right) = 0$$

or $$0.004T_1^2 + T_1 - 179.98 = 0$$

Solving this quadratic equation, T_1 = 121.21°C

Noting that the maximum temperature will occur at the exposed insulated surface (x = 0) where $\frac{dT}{dx} = 0$, T_{max} = $T(0)$ = T_1 = **121.21°C** **(Ans.) (b)**

(c) Since $T(x)$ = 90°C at a distance x from the insulated surface, one gets

$$\frac{qx}{k_0} = (T_1 - T) + \frac{\beta}{2}(T_1^2 - T^2) = (T_1 - T)[1 + \beta T_m] \quad \text{where} \quad T_m = \frac{T_1 + T}{2}$$

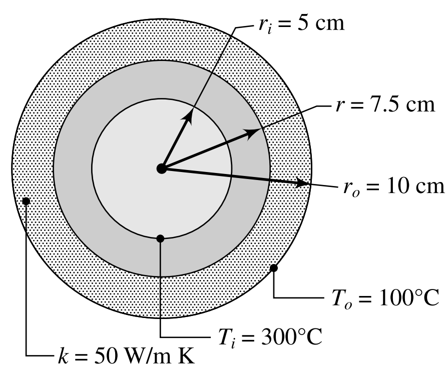
Hence, $$x = \frac{(121.21 - 90)[1 + 0.008 \times (121.21 + 90)/2] \times 0.5}{743.8} = \mathbf{0.0387\ m\ or\ 38.7\ mm}$$ **(Ans.) (c)**

(H) Long Cylinders Bounded by Fixed Surface Temperatures

EXAMPLE 2.23 *A hollow cylinder with 10 cm ID and 20 cm OD has an inner surface temperature of 300°C and an outer surface temperature of 100°C. The thermal conductivity of the cylinder material is 50 W/m K. Calculate the heat flow through the cylinder per linear metre. Also find the temperature half way between the inner and outer surfaces.*

Solution

Schematic



Known Hollow cylinder with specified temperatures, radii, and conductivity of material.

Find Heat rate. Temperature at midradius.

Assumptions (1) Steady-state radial conduction. (2) Constant thermal conductivity.

Analysis Heat transfer per unit length

$$\frac{\dot{Q}}{L} = \frac{2\pi k(T_i - T_o)}{\ln(r_o/r_i)}$$

$$= \frac{2\pi(50\ \text{W/m K})(300 - 100)°\text{C or K}}{\ln\left(\frac{10\ \text{cm}}{5\ \text{cm}}\right)}$$

$$= 90647.2\ \text{W/m} = \mathbf{90.65\ kW/m}$$ **(Ans.)**

The radius *halfway* between inner and outer surfaces,

$$r = \frac{r_i + r_o}{2} = \frac{(5 + 10)\ \text{cm}}{2} = 7.5\ \text{cm}$$

Radial temperature distribution is determined to be

$$\dot{Q} = \frac{2\pi kL(T_i - T_o)}{\ln(r_o/r_i)} = \frac{2\pi kL(T_i - T)}{\ln(r/r_i)} \Rightarrow \frac{T_i - T}{T_i - T_o} = \frac{\ln(r/r_i)}{\ln(r_o/r_i)}$$

or $$\boxed{T = T_i - \frac{(T_i - T_o)\ln(r/r_i)}{\ln(r_o/r_i)}}$$

or $$T = 300°\text{C} - (300 - 100)°\text{C}\frac{\ln(7.5/5)}{\ln(10/5)} = \mathbf{183°C}$$ **(Ans.)**

EXAMPLE 2.24 *Compute the rate of radial heat flow from the blading periphery to the centre for a gas-turbine rotor as shown in the figure. The dimensions are as follows:*

R = *25 cm,* x_0 *= 2 cm,* x_c *= 7 cm*

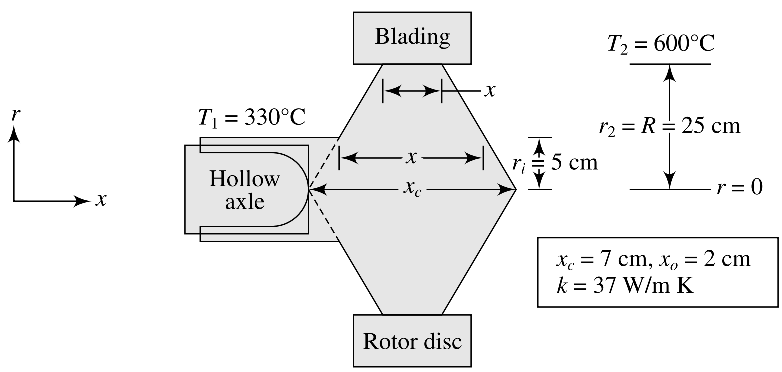
The thermal conductivity of the material is 37 W/m K. The temperatures are 600°C at r_2 = R = *25 cm, and 330°C at* r_1 *= 5 cm.*

Solution

Known Heat flows radially inwards from the periphery of a gas turbine rotor blade to the centre of the rotor disc.

Find Heat flow rate, $\dot{Q}$(W).

Schematic



Assumptions (1) Steady state conditions. (2) Constant properties. (3) One-dimensional heat conduction. (4) Only conduction heat transfer is considered.

Analysis While cooling of a gas-turbine rotor is essentially by gas convection, conduction cooling is also present. Heat flow is in a radial inward fashion from the blading periphery to the centre of the rotor disc. It is then dissipated at the centre along the axle through conduction. Referring to the schematic, let

x_c = Thickness of the rotor disc at the centre (r = 0)

x_0 = Thickness of the rotor disc at the periphery (r = R).

x = Thickness of the rotor disc at a radius r

From the geometry of the rotor disc,

$$\frac{x_c - x}{r} = \frac{x_c - x_0}{R} \quad \text{or} \quad x = x_c - \frac{x_c - x_0}{R} r$$

or $$x = x_c - pr \quad \text{where} \quad p = \frac{x_c - x_0}{R} \tag{A}$$

Rate of radial heat flow is given by

$$\dot{Q} = k(2\pi r x)\frac{dT}{dr} \quad \text{or} \quad \frac{dr}{rx} = \frac{2\pi k}{\dot{Q}} dT$$

From (A): $$\int_{r=r_1}^{r=r_2} \frac{dr}{r(x_c - pr)} = \frac{2\pi k}{\dot{Q}} \int_{T=T_1}^{T=T_2} dT$$

But $$\int_{r_1}^{r_2} \frac{dr}{r(x_c - pr)} = \frac{1}{x_c} \ln \frac{r_2(x_c - pr_1)}{r_1(x_c - pr_2)}$$

Hence, $$\frac{2\pi k}{\dot{Q}}(T_2 - T_1) = \frac{1}{x_c} \ln \frac{r_2}{r_1}\left(\frac{x_c - pr_1}{x_c - pr_2}\right)$$

Heat flow rate from blade periphery to centre is

$$\boxed{\dot{Q} = \frac{2\pi x_c k(T_2 - T_1)}{\ln \dfrac{r_2}{r_1} \dfrac{(x_c - pr_1)}{(x_c - pr_2)}}}$$

With $p = \dfrac{x_c - x_0}{R} = \dfrac{(7-2)\text{ cm}}{25\text{ cm}} = 0.2$, the heat flow rate is

$$\dot{Q} = \frac{2\pi(0.07\text{ m})(37\text{ W/m K})[(600 - 330)°\text{C}]}{\ln\left[\dfrac{0.25\text{ m}(0.07 - 0.2 \times 0.05)\text{ m}}{0.05\text{ m}(0.07 - 0.2 \times 0.25)\text{ m}}\right]} = \mathbf{1622.5\ W} \qquad \text{(Ans.)}$$

Comment The temperature gradient in the radial direction, $\dfrac{dT}{dr}$ is taken as positive because in this case, as r increases, T also increases. Heat flows from the higher temperature to the lower temperature, i.e., from the blade periphery to the centre of the rotor disc.

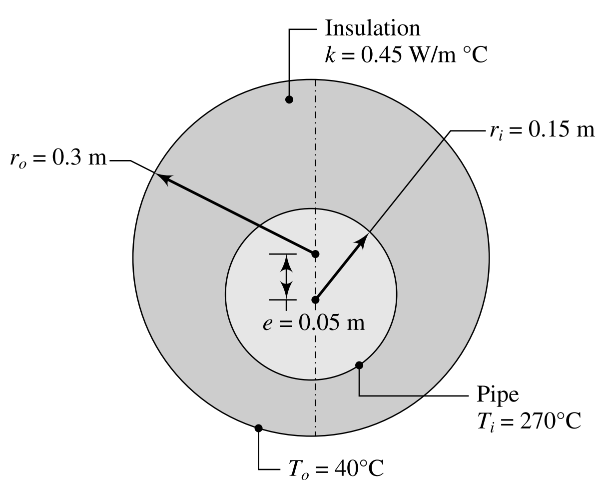
EXAMPLE 2.25 *A pipe having an outer diameter of 300 mm is insulated by a material of thermal conductivity of 0.45 W/m °C. The insulation of outside diameter 600 mm due to restriction of space, is placed with an eccentricity of 50 mm. Determine the heat loss for a length of 1 m if the inner and outer surfaces are at temperatures of 270°C and 40°C, respectively.*

Solution

Known An insulated pipe with specified eccentricity loses heat to the surroundings.

Find Heat loss per m length.

Schematic



Assumptions (1) Steady-state conditions. (2) Constant thermal conductivity. (3) Isothermal surfaces.

Analysis Heat-loss rate is given by

$$\dot{Q} = \frac{2\pi k L(T_i - T_o)}{\ln A/B}$$

where $A = \sqrt{(r_o + r_i)^2 - e^2} + \sqrt{(r_o - r_i)^2 - e^2}$

and $B = \sqrt{(r_o + r_i)^2 - e^2} - \sqrt{(r_o - r_i)^2 - e^2}$

Substituting the numerical values,

$$A = \sqrt{(0.3 + 0.15)^2 - 0.05^2} + \sqrt{(0.3 - 0.15)^2 - 0.05^2}$$

$$= 0.44\ 721 + 0.14142 = 0.58863$$

$$B = 0.44\ 721 - 0.14142 = 0.30579$$

Hence, heat loss per metre length is

$$\dot{Q} = \frac{2\pi(0.45 \text{ W/m °C})(1 \text{ m})(270 - 40)\text{°C}}{\ln \dfrac{0.58863}{0.30579}} = \mathbf{993\ W} \qquad \textbf{(Ans.)}$$

(I) Long Cylinder Bounded by Fixed Fluid Temperatures

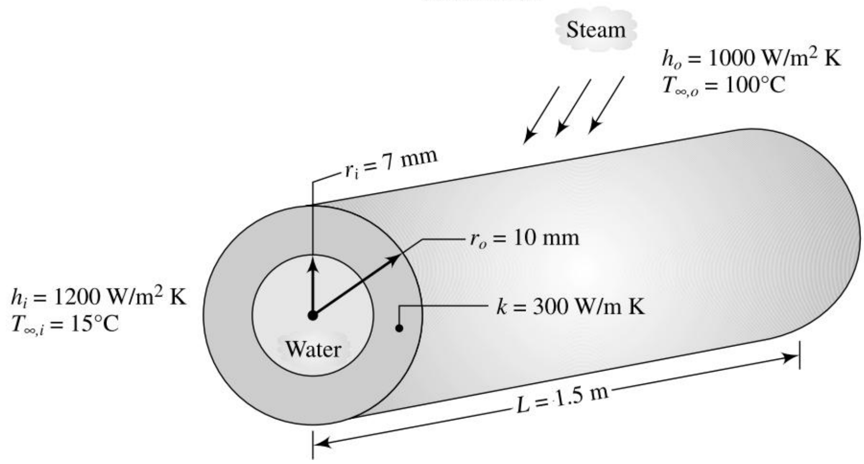
EXAMPLE 2.26 *Determine the heat-transfer rate between two fluids separated by a copper pipe 3 mm thick, 20 mm OD and 1.5 m long, if the inner fluid (water) temperature is 15°C and the outer fluid (steam) temperature is 100°C. The water-side coefficient is 1200 W/m² K, the steam-side coefficient is 10 000 W/m² K, and the thermal conductivity of copper is 300 W/m K. Determine the log mean area and the arithmetic mean area of the conducting surface and show that the arithmetic mean area can be used in place of log mean area.*

Solution

Known A hollow pipe with convective boundaries.

Find Heat-transfer rate; Log mean area, and arithmetic mean area.

Schematic



Assumptions (1) Steady-state, one-dimensional (*radial*) conduction. (2) Constant thermal conductivity. (3) Uniform convection coefficient.

Analysis The log mean area,

$$\text{LMA or } A_m = \frac{A_o - A_i}{\ln\left(\dfrac{A_o}{A_i}\right)} = \frac{2\pi L(r_o - r_i)}{\ln\left(\dfrac{r_o}{r_i}\right)} \quad \text{where } \ln\frac{r_o}{r_i} = \ln\left(\frac{10}{7}\right) = 0.3567$$

$$\therefore \qquad A_m = \frac{(2\pi)(1.5\text{ m})(10.7\times10^{-3}\text{m})}{0.3567} = \mathbf{0.079\ m^2} \qquad \textbf{(Ans.)}$$

The heat-transfer rate,

$$\dot{Q} = \frac{2\pi L(T_i - T_o)}{\dfrac{1}{k}\ln\dfrac{r_o}{r_i} + \dfrac{1}{h_o r_o} + \dfrac{1}{h_i r_i}} = \frac{(2\pi)(1.5\text{ m})(100-15)°\text{C}}{\left\{\dfrac{1}{300}\ln\dfrac{10}{7} + \dfrac{1}{(10000)(0.01)} + \dfrac{1}{(1200)(0.017)}\right\}\dfrac{mK}{W}}$$

$$= 6151.1\text{ W} \quad \text{or} \quad 6.15\text{ kW}$$

The arithmetic mean area,

$$A_{\text{av}} = \frac{A_o + A_i}{2} = 2\pi L\frac{(r_o + r_i)}{2} = \pi(1.5\text{ m})(0.017\text{ m}) = 0.08\text{ m}^2 \approx 0.079\text{ m}^2$$

Hence, the arithmetic mean area can be used instead of log mean area. **(Ans.)**

Comment If $(A_o/A_i) < 2$, i.e., r_o/r_i, LMA or A_m can be approximated as arithmetic mean area. In this case, $\dfrac{A_o}{A_i} = \dfrac{r_o}{r_i} = \dfrac{10\text{ mm}}{7\text{ mm}} = 1.43(<2)$

EXAMPLE 2.27 *A 100 mm diameter pipe carrying a hot chemical at 250°C is covered with two layers of insulation, each 50 mm thick. The length of the pipe is 5 m. The outer-surface temperature of the composite is 35°C. The rate of heat loss through the pipe is 270 W. If the thickness of the outer insulation is increased by 25%, the heat loss is reduced to 260 W. Calculate the thermal conductivities of the two insulating materials.*

Solution

Known A pipe equipped with two insulating layers, prescribed dimensions and extreme surface temperatures. Outer insulation thickness increased by 25%. Heat loss with and without increase in insulation thickness.

Find Thermal conductivities of insulating materials.

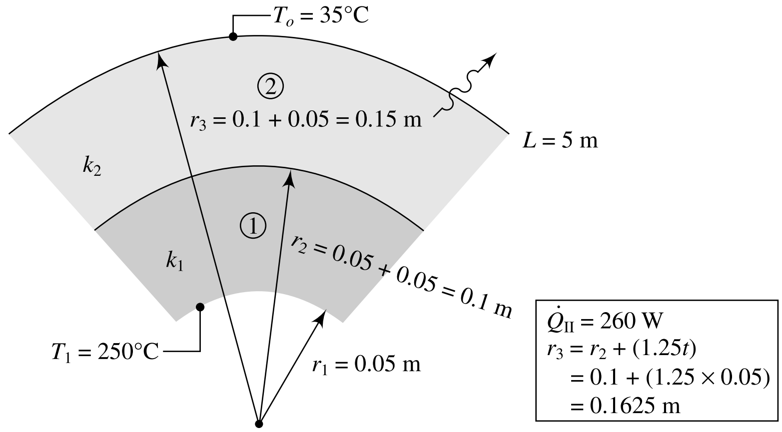
Assumptions (1) Steady-state radial heat conduction. (2) Constant properties. (3) End surface temperatures are same in both cases.

Analysis ***Case I:*** Heat-transfer rate,

$$\dot{Q}_I = \frac{T_i - T_o}{\dfrac{1}{2\pi k_1 L}\ln\dfrac{r_2}{r_1} + \dfrac{1}{2\pi k_2 L}\ln\dfrac{r_3}{r_2}} r$$

$$\frac{1}{k_1}\ln\frac{r_2}{r_1} + \frac{1}{k_2}\ln\frac{r_3}{r_2} = \frac{2\pi(T_i - T_o)}{\dot{Q}_I}$$

Schematic



or $$\frac{1}{k_1}\ln\frac{0.10}{0.05}+\frac{1}{k_2}\ln\frac{0.15}{0.10}=\frac{2\pi(250-35)(5)}{270}$$

or $$\frac{1}{k_1}\ln 2+\frac{1}{\mathrm{k}_2}\ln 1.5=\frac{\pi}{27}\times 215 \qquad \text{(A)}$$

Case II: $$\frac{1}{k_1}\ln\frac{0.10}{0.05}+\frac{1}{k_2}\ln\frac{0.1625}{0.10}=\frac{2\pi(250-35)(5)}{260}$$

or $$\frac{1}{k_1}\ln 2+\frac{1}{k_2}\ln 1.625=\frac{\pi}{26}\times 215 \qquad \text{(B)}$$

Subtracting (A) from (B), one gets

$$215\pi\left[\frac{1}{26}-\frac{1}{27}\right]=\frac{1}{k_2}[\ln 1.625-\ln 1.5] \quad \text{or} \quad 0.96217=\frac{0.08}{k_2}$$

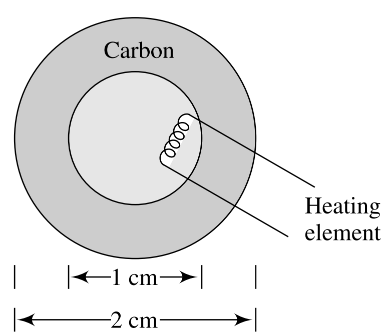
$\therefore$ $$k_2=\frac{0.08}{0.96217}=\mathbf{0.083\ W/m\,K} \qquad \textbf{(Ans.)}$$

From Eq. (A): $$\frac{1}{\mathrm{k}_1}\ln 2+\frac{1}{0.083}\ln 1.5=\frac{\pi}{27}\times 215$$

or $$\frac{1}{k_1}\times 0.693+4.8766=25.016$$

$\therefore$ $$k_1=\frac{0.693}{25.016-4.8766}=\frac{0.693}{20.14}=\mathbf{0.0344\ W/mK}$$ **(Ans.)**

EXAMPLE 2.28 *An electrical heating element is shrunk in a hollow cylinder of amorphous carbon (k = 1.6 W/m K) as shown in the adjoining figure. The outer surface of carbon is in contact with air at 20°C. The convection heat-transfer coefficient is 65 W/m²K. Determine the maximum allowable heat generation rate per metre*



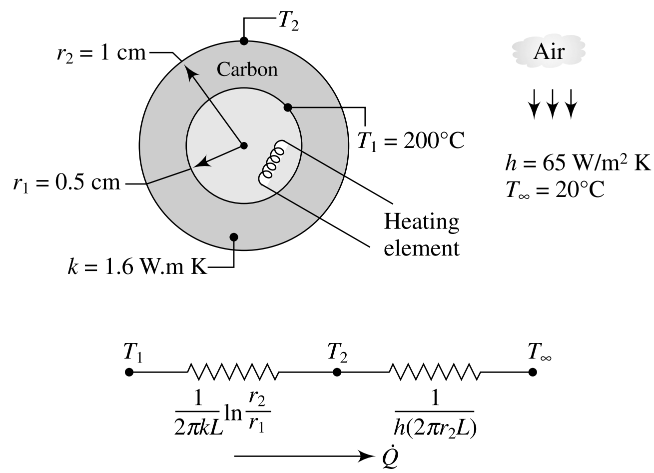
length if the maximum temperature of carbon is not to exceed 200°C. The heating element itself may be assumed to be isothermal.

Solution

Known A hollow cylinder, electrically heated, of prescribed dimensions. Maximum permissible temperature, ambient air temperature and convection coefficient.

Find Maximum permissible electrical heating (W/m).

Schematic



Assumptions (1) Steady-state, one-dimensional conduction. (2) The resistance to heat transfer within the heating element is negligible. (3) Constant thermal conductivity.

Analysis Heat generated by electrical heating is transferred out by conduction to outer surface of the cylinder and dissipated by convection to ambient air. The thermal circuit is shown in the schematic.

Rate of heat transfer, $\dot{Q} = \dfrac{\Delta T_{\text{overall}}}{R_{\text{total}}} = \dfrac{T_1 - T_\infty}{R_{\text{cond}} + R_{\text{conv}}}$

Conduction resistance, per m length,

$$R_{\text{cond}} = \frac{1}{2\pi kL}\ln\frac{r_2}{r_1} = \frac{1}{2\pi(1.6\text{ W/m K})(1\text{ m})}\ln\frac{1\text{ cm}}{0.5\text{ cm}} = 0.06895\text{ K/W}$$

Convection resistance, per m length,

$$R_{\text{conv}} = \frac{1}{h(2\pi r_2 L)} = \frac{1}{(65\text{ W/m}^2\text{K})(2\pi\times 0.01\text{ m}\times 1\text{ m})} = 0.244\,85\text{ K/W}$$

Total thermal resistance,

$$R_{\text{total}} = 0.068\,95 + 0.244\,85 = 0.3138\text{ K/W}$$

Hence, the maximum heat-generation rate is

$$\dot{Q} = \frac{(200 - 20)°\text{C or }K}{0.3138\text{ K/W}} = \mathbf{573.6\ W}$$ **(Ans.)**

(J) Composite Cylinders Bounded by Fixed Boundary Temperatures

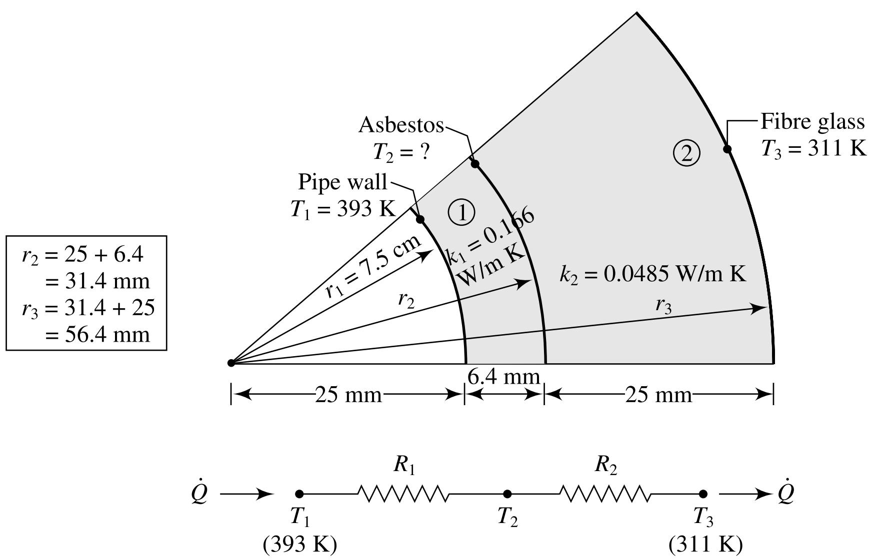
EXAMPLE 2.29 *A steel pipe with 50 mm OD is covered with a 6.4 mm asbestos insulation (*k = *0.166 W/m K) followed by a 25 mm layer of fibre glass insulation (*k = *0.0485 W/m K). The pipe wall temperature is 393 K and the outside insulation temperature is 311 K. Calculate the interface temperature between the asbestos and fibre glass.*

Solution

Known A steel pipe covered with asbestos and fibre glass insulation loses heat by conduction.

Find Interface temperature, T_2(°C).

Schematic



Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant properties. (3) No contact resistance.

Analysis The rate of heat loss is, $\dot{Q} = \dfrac{\Delta T_{\text{overall}}}{\Sigma R_t} = \dfrac{T_1 - T_3}{R_1 + R_2} = \dfrac{T_1 - T_2}{R_1}$

where R_1 and R_2 are the thermal resistances due to asbestos and fibre-glass insulations respectively. T_2 is the interface temperature.

Therefore $$\frac{T_1 - T_3}{T_1 - T_2} = \frac{R_1 + R_2}{R_1} = 1 + \frac{R_2}{R_1} \tag{A}$$

Per unit length: $$R_1 = \frac{1}{2\pi k_1 L}\ln\frac{r_2}{r_1} = \frac{1}{2\pi(0.166\text{ W/m K})\times 1\text{ m}}\ln\frac{31.4\text{ mm}}{25\text{ mm}} = 0.218\,53\text{ K/W}$$

$$R_2 = \frac{1}{2\pi k_2 L}\ln\frac{r_3}{r_2} = \frac{1}{2\pi(0.0485\text{ W/m K})\times(1\text{ m})}\ln\frac{56.4\text{ mm}}{31.4\text{ mm}} = 1.921\,87\text{ K/W}$$

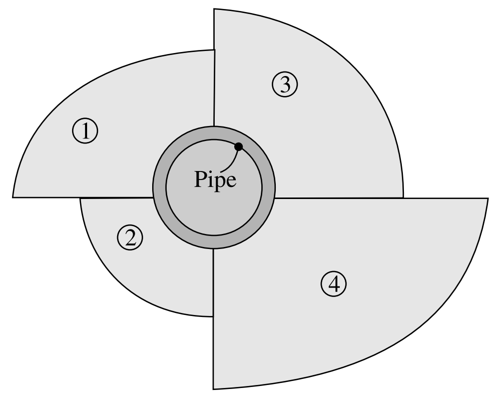
$$\therefore \qquad \frac{R_2}{R_1} = \frac{1.92187\text{ K/W}}{0.21853\text{ K/W}} = 8.7944$$

Substituting in Eq. (A), we have, $\dfrac{393 - 111}{393 - T_2} = 1 + 8.7944 = 9.7944$

Hence, the interface temperature, $T_2 = 393 \text{ K} - \dfrac{(393 - 111)\text{ K}}{9.7944} = \mathbf{384.63\ K}$ **(Ans.)**

EXAMPLE 2.30 *Calculate the net conduction resistance for the arrangement shown in the adjoining figure:*

Given: Pipe material : Stainless steel (k_o = 14.9 W/m K)
Inner diameter of pipe = 20 mm
Outer diameter of pipe = 25 mm
Radius of insulation 1 = 15 mm; k_1 = 0.05 W/m K
Radius of insulation 2 = 18 mm; k_2 = 0.10 W/m K
Radius of insulation 3 = 20 mm; k_3 = 0.15 W/m K
Radius of insulation 4 = 25 mm; k_4 = 0.20 W/m K

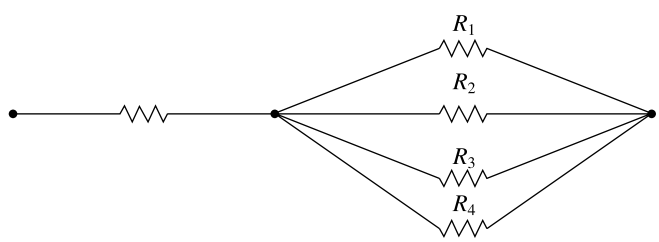


Solution

Known A pipe is provided with four insulations in a prescribed arrangement.

Find Net conduction resistance, R_{tot}(k/W).

Schematic



Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant properties.

Analysis The individual thermal resistances are

Insulation	Radius (mm)	Thermal conductivity (W/m K)	Thermal resistance (K/W)
1	$r_1 = 15$	$k_1 = 0.05$	$R_1 = \dfrac{2}{\pi k_1 L} \ln \dfrac{r_1}{r_o}$
2	$r_2 = 18$	$k_2 = 0.10$	$R_2 = \dfrac{2}{\pi k_2 L} \ln \dfrac{r_2}{r_o}$
3	$r_3 = 20$	$k_3 = 0.15$	$R_3 = \dfrac{2}{\pi k_3 L} \ln \dfrac{r_3}{r_o}$
4	$r_4 = 25$	$k_4 = 0.20$	$R_4 = \dfrac{2}{\pi k_4 L} \ln \dfrac{r_4}{r_o}$

$$R_o = \frac{1}{2\pi kL}\ln\frac{r_o}{r_i}$$

Resistances R_1, R_2, R_3, and R_4 are in parallel. The area in each case is one fourth of the total area.

Net conduction resistance, $R_{\text{tot}} = R_o + \left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right]^{-1}$

$$R_o = \frac{1}{2\pi \times 14.9 \times 1}\ln\frac{12.5}{10} = 2.3835 \times 10^3 \text{ K/W}$$

$$\frac{1}{R_e} = \left[\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right] = \left[\frac{\pi k_1 L}{2\ln r_1/r_o}+\frac{\pi k_2 L}{2\ln r_2/r_o}+\frac{\pi k_3 L}{2\ln r_3/r_o}+\frac{\pi k_4 L}{2\ln r_4/r_o}\right]$$

$$= \frac{\pi}{2}\times 1\left[\frac{0.05}{\ln\frac{15}{2.5}}+\frac{0.10}{\ln\frac{18}{12.5}}+\frac{0.15}{\ln\frac{20}{12.5}}+\frac{0.20}{\ln\frac{25}{12.5}}\right]$$

$$= \frac{\pi}{2}[0.274\,24 + 0.274\,24 + 0.319\,15 + 0.288\,54] = 1.816$$

$\therefore$ $R_{\text{tot}} = R_o + R_e = (2.3835 \times 10^{-3}) + [1.816]^{-1} = \mathbf{0.553\ kW}$ **(Ans.)**

(K) Composite Cylinders Bounded by Fixed, Fluid Temperatures

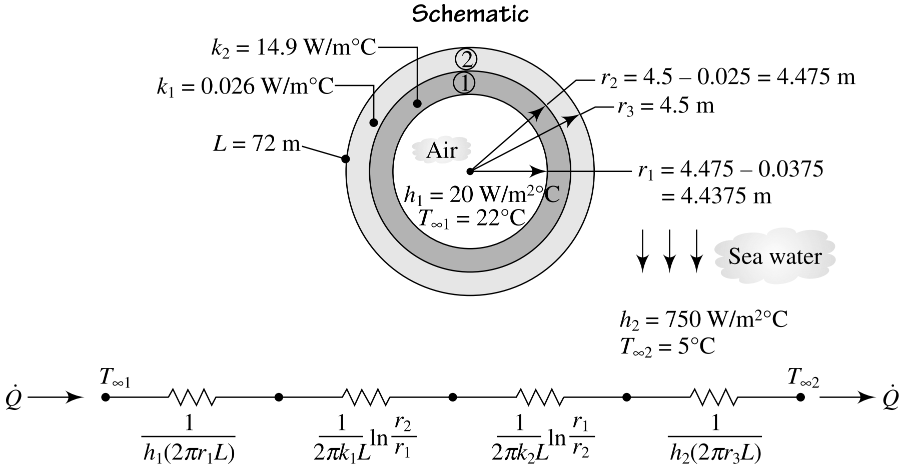
EXAMPLE 2.31 *A submarine has a 25 mm thick stainless steel (k = 14.9 W/m °C) wall insulated on the inside with a 37.5 mm thick layer of PUF (polyurethane foam) (k = 0.026 W/m °C). The convection heat-transfer coefficient on the inside is 20 W/m²°C. At full speed, the outside heat-transfer coefficient is 750 W/m²°C. The submarine can be approximated as a cylindrical system of 9 m diameter and 72 m long. If the sea water is at 5°C, at what rate should the heat be supplied (in kJ/h) to the inside air to maintain it at 22°C. As a first approximation, neglect heat transfer through the ends.*

Solution

Known A composite cylinder with convective boundaries.

Find Rate of heat loss, $\dot{Q}$ (kJ/h).

Schematic



Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant properties and uniform heat-transfer coefficients. (3) Heat transfer through the ends is neglected.

Analysis The rate of heat transfer in the radial direction is, $\dot{Q} = \dfrac{\Delta T_{\text{overall}}}{\Sigma R_{\text{th}}}$

where the overall temperature difference, $\Delta T_{\text{overall}} = 22 - 5 = 17°\text{C}$ and the total thermal resistance, $\Sigma R_{\text{th}} = R_{\text{conv, 1}} + R_{\text{cond, 1}} + R_{\text{cond, 2}} + R_{\text{conv, 2}}$

Let us first calculate the individual thermal resistances.

$$R_{\text{conv},1} = \frac{1}{h_1(2\pi r_1 L)} = \left[20 \frac{W}{\text{m}^2\,°\text{C}} \times 2\pi \times 4.4375 \text{ m} \times 72 \text{ m}\right]^{-1} = 24.907 \times 10^6 \text{ °C/W}$$

$$R_{\text{cond},1} = \frac{1}{2\pi k_1 L}\ln\frac{r_2}{r_1} = \frac{1}{2\pi(0.026 \text{ W/m°C})(72 \text{ m})}\ln\frac{4.475 \text{ m}}{4.4375 \text{ m}} = 715.45 \times 10^{-6} \text{ °C/W}$$

$$R_{\text{cond},2} = \frac{1}{2\pi k_2 L}\ln\frac{r_3}{r_2} = \frac{1}{2\pi(14.9 \text{ W/m°C})(72 \text{ m})}\ln\frac{4.5 \text{ m}}{4.475 \text{ m}} = 0.8265 \times 10^{-6} \text{ °C/W}$$

$$R_{\text{conv},2} = \frac{1}{h_2(2\pi r_3 L)} = [(750 \text{ W/m}^2\,°\text{C})(2\pi \times 4.5 \text{ m} \times 72 \text{ m})]^{-1} = 0.655 \times 10^{-6} \text{ °C/W}$$

$\therefore$ Total thermal resistance,

$$\Sigma R_{\text{th}} = [24.907 + 715.45 + 0.8265 + 0.655]\,(10^{-6}\text{°C/W}) = 741.84 \times 10^{-6} \text{ °C/W}$$

Therefore, the heat-transfer rate from the interior air to the sea water is

$$\dot{Q} = \frac{17°\text{C}}{741.84 \times 10^{-6}\text{°C/W}}\left|\frac{1 \text{ J/s}}{1 \text{ W}}\right|\left|\frac{3600 \text{ s}}{1 \text{ h}}\right|\left|\frac{1 \text{ kJ}}{10^3 \text{ J}}\right| = \frac{17 \times 3.6}{741.84 \times 10^{-6}}\left(\frac{\text{kJ}}{\text{h}}\right) = \mathbf{82.5\ kJ/h} \quad \textbf{(Ans.)}$$

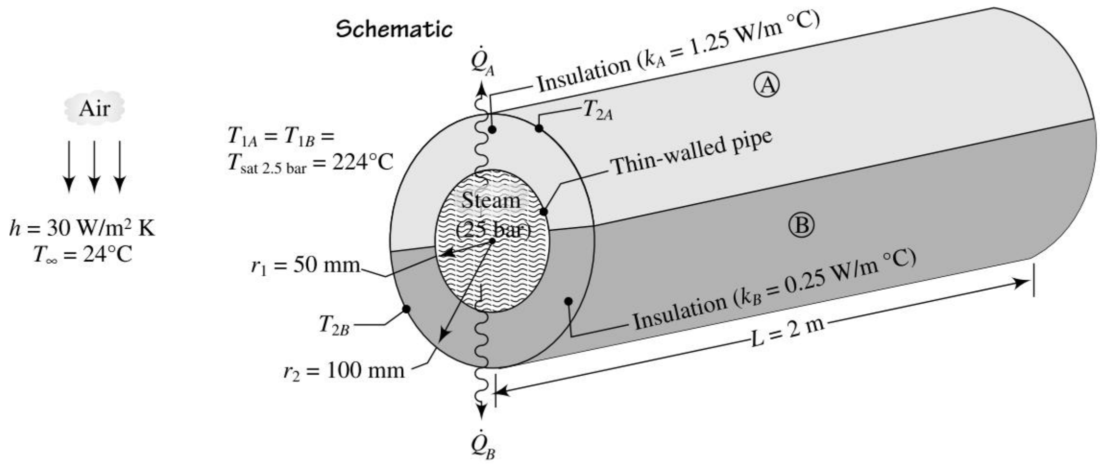
Comment Note the relative magnitudes of different resistances. The conduction resistance through the insulation is the dominant resistance. The contributions of submarine wall resistance and the sea water side convection resistance are comparatively much less significant.

EXAMPLE 2.32 *Steam is flowing through a 2 m long, thin-walled, 100 mm diameter pipe at a pressure of 25 bar. The pipe is equipped with an insulation blanket that is made up of two materials* A *and* B. *The diameter of the insulated pipe is 200 mm with the upper half comprising material* A*(*k_A = *1.25 W/m °C) and the lower half material* B*(*k_B = *0.25 W/m °C). The entire outer surface is exposed to 24°C air with a convection coefficient of 30 W/m²°C. Determine (a) the total heat-dissipation rate from the pipe, and (b) the outer surface temperatures of materials* A *and* B. *(c) Show the equivalent thermal circuit representing the steady state heat transfer situation.*

Solution

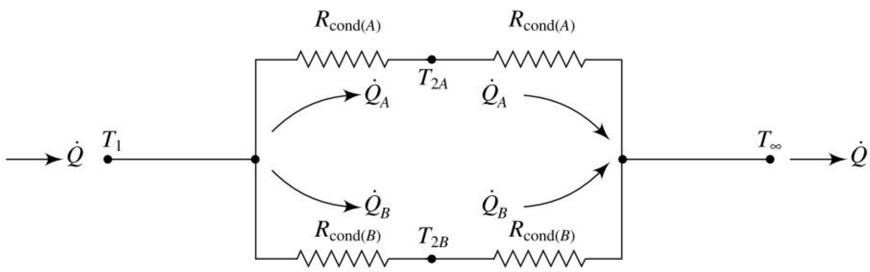
Known Two semicylindrical shells of different materials comprise an insulation blanket over a pipe carrying steam and exposed to ambient air.

Find (a) Heat-loss rate, $\dot{Q}$ (W). (b) Outer surface temperatures of the two insulating materials A and B, T_{2A} and T_{2B} (°C). Thermal circuit representation.



Assumptions (1) Steady operating conditions. (2) One-dimensional (radial) conduction) (3) Constant properties. (4) Negligible contact resistance.

Analysis The equivalent thermal circuit of the specified heat transfer situation is represented below:



Let us first evaluate the various thermal resistances.

$$R_{\text{cond}(A)} = \frac{1}{\pi k_A L}\ln\frac{r_2}{r_1} = \frac{1}{\pi(1.25\text{ W/m°C})(2\text{ m})}\ln\frac{100\text{ mm}}{50\text{ mm}} = 0.08825\text{°C/W}$$

$$R_{\text{cond}(B)} = \frac{1}{\pi k_B L}\ln\frac{r_2}{r_1} = \frac{1}{\pi(0.25\text{ W/m°C})(2\text{ m})}\ln\frac{100\text{ mm}}{50\text{ mm}} = 0.4413\text{°C/W}$$

$$R_{\text{conv}(A)} = \frac{1}{h(\pi r_2 L)} = \frac{1}{(30\text{ W/m}^2\text{ °C})(\pi \times 0.1\text{ m} \times 2\text{ m})} = 0.05305\text{°C/W} = R_{\text{conv}(B)}$$

Note that the above *four* resistances are for *half* cylinder.

Total heat-dissipation rate is, $\dot{Q} = \dot{Q}_A + \dot{Q}_B$

Heat-transfer rates through *upper half* and the *lower half* of the cylindrical shell are

$$\dot{Q}_A = \frac{T_1 - T_\infty}{R_{\text{cond}(A)} + R_{\text{conv}(A)}} = \frac{(224-24)\text{°C}}{(0.08825 + 0.05305)\text{°C}} = 1415.4\text{ W}$$

and
$$\dot{Q}_B = \frac{T_1 - T_\infty}{R_{\text{cond}(B)} + R_{\text{conv}(B)}} = \frac{(224-24)\text{°C}}{(0.4413 + 0.05305)\text{°C/W}} = 404.6\text{ W}$$

Total heat loss, $\dot{Q}$ = 1415.4 + 404.6 = **1820 W** **(Ans.) (a)**

The outer surface temperatures of the two insulations are determined as follows:

$$T_1 - T_{2A} = \dot{Q}_A \; R_{\text{cond(A)}}$$

or $\quad T_{2A} = 224°\text{C} - (1415.4 \text{ W})\,(0.08825°\text{C/W}) = \mathbf{99.1°C}$ **(Ans.) (b)**

Similarly, $\quad T_1 - T_{2B} = \dot{Q}_B \; R_{\text{cond(B)}}$

or $\quad T_{2B} = 224°\text{C} - (404.6 \text{ W})\,(0.4413°\text{C/W}) = \mathbf{45.5°C}$ **(Ans.) (b)**

Comment The inner-surface temperature of both insulating materials A and B is equal to the saturation temperature at 25 bar which is 224°C (from steam tables). Thin-walled pipe implies negligible pipe wall (conduction) resistance and very high convection coefficients associated with condensing steam implies negligible convection resistance ($T_{\infty 1}$ or $T_{\text{sat}} = T_1$).

This example illustrates the presence of two resistances R_A and R_B connected in parallel in a composite cylinder. Total heat-transfer rate can also be calculated by first determining the equivalent thermal resistance.

$$R_A = 0.08825 + 0.05305 = 0.1413°\text{C/W} \quad \text{and} \quad R_B = 0.4413 + 0.05305 = 0.4943°\text{C/W}$$

$$\therefore \qquad \frac{1}{R_{\text{eq}}} = \frac{1}{R_A} + \frac{1}{R_B} \quad \text{or} \quad R_{\text{eq}} = \left[\frac{1}{0.1413} + \frac{1}{0.4943}\right]^{-1} = 0.1099°\text{C/W}$$

Hence, $\quad \dot{Q} = \dfrac{T_1 - T_\infty}{R_{\text{eq}}} = \dfrac{(224 - 24)°\text{C}}{0.1099°\text{C/W}} = 1820 \text{ W}$

EXAMPLE 2.33 *Steam having a quality of 98% at a pressure of 1.5 bar, is flowing at a velocity of 1 m/s, through a steel pipe (*k *= 43 W/m K) 2.7 cm OD and 2.1 cm ID. The heat-transfer coefficient at the inner surface, where condensation occurs is 567 W/m² K. Scale formation at the inner surface contributes a unit thermal resistance of 0.18 m K/W. Estimate the heat loss per metre length of the pipe if (a) the pipe is bare, (b) the pipe is covered with a 5 cm thick layer of 85% magnesia insulation (*k *= 0.061 W/m K). For both cases, assume that the heat-transfer coefficient at the outer surface is 11 W/m² K, and that the environmental temperature is 20°C.*

Also estimate the change in quality per 3 m length of pipe in both cases.

At P *= 1.5 bar :* T_{sat} *= 111.35°C =* T_i*,* h_{fg} *= 2226.0 kJ/kg,* v_g *= 1.1594 m³/kg*

Solution

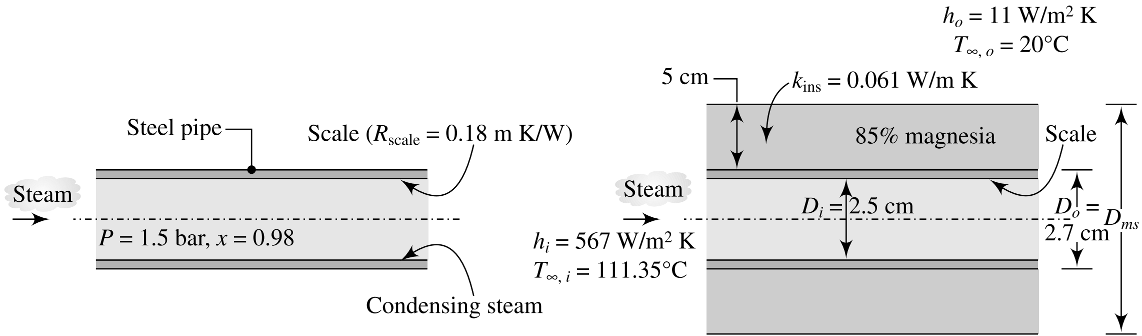
Known A steam pipe with scale formation on inside surface loses heat without and with lagging (*insulation*).

Find (a) Heat loss from bare pipe. (b) Heat loss from insulated pipe. (c) Change in quality of steam per 3 m length for both bare and lagged pipe.

Assumptions (1) Steady-state conditions exist. (2) One-dimensional (radial) heat conduction. (3) Radiation heat transfer from outside surface of the pipe is negligible. (4) Constant properties.

Analysis (a) Heat-transfer rate, $\dfrac{\dot{Q}}{L} = \dfrac{T_i - T_o}{\Sigma R_{\text{th}}}$

Schematic



Case (a) Bare pipe ***Case (b) Lagged pipe***

Total thermal resistance (*without insulation*) per unit length is

$$\Sigma R_{th} = \frac{1}{2\pi r_i h_i} + \frac{\ln\dfrac{r_o}{r_i}}{2\pi k} + \frac{1}{2\pi r_o h_o} + R_{scale}$$

$$= \frac{1}{\pi(0.021\text{ m})(567\text{ W/m}^2\text{K})} + \frac{\ln\left(\dfrac{2.7\text{ cm}}{2.1\text{ cm}}\right)}{2\pi(43)} + \frac{1}{\pi(0.027\text{ m})(11\text{ W/m}^2\text{ K})} + 0.18\text{ m K/W}$$

$$= 0.026733 + 0.00093 + 1.07175 + 0.18 = 1.2794\text{ K/W}$$

Heat loss per metre length is,

$$\frac{\dot{Q}}{L} = \frac{(111.35 - 20)\text{K}}{1.2794\text{ m K/W}} = \mathbf{71.4\ W/m}$$ **(Ans.) (a)**

Total thermal resistance (*with insulation*) is,

$$\Sigma R_{th} = 1.2794 + \frac{\ln\dfrac{D_{ins}}{D_o}}{2\pi k_{ins}} = 1.2794 + \frac{\ln\dfrac{12.7\text{ cm}}{2.7\text{ cm}}}{2\pi(0.061\text{ W/m K})}$$

$$= 1.2794 + 4.0398 = 5.3192\text{ K/W}$$

Heat-loss rate per m length of the pipe *with insulation* is,

$$\frac{\dot{Q}}{L} = \frac{(111.35 - 20)\text{K}}{5.3192\text{ K/W}} = \mathbf{17.2\ W/m}$$ **(Ans.) (b)**

Change in quality for bare pipe: Mass-flow rate of steam,

$$\dot{m}_s = \frac{A_i V_s}{v_s} = \frac{A_i V_s}{x v_g} = \frac{\pi D_i^2 V_s}{4 x v_g}$$ where V_s is the velocity of steam.

$\therefore$ $$\dot{m}_s = \frac{\pi(0.021\text{ m})^2(1\text{ m/s})}{4(0.98)(1.1594\text{ m}^3\text{/kg})} = 0.305 \times 10^{-3}\text{ kg/s} \quad \text{or} \quad 0.305\text{ g/s}$$

Mass of steam condensed in 3 m length of pipe,

$$\dot{m} = \frac{\dot{Q}}{xh_{fg}} = \frac{(71.4 \text{ W})(3 \text{ m})}{(0.98)(2226 \times 10^3 \text{ J/kg})} = 0.0982 \times 10^{-3} \text{ kg/s or } 0.0982 \text{ g/s}$$

Percentage decrease in steam quality

$$= \left(\frac{0.305 - 0.0982}{0.305}\right)(100) = \mathbf{67.8\%} \qquad \textbf{(Ans.) (a)}$$

Change in quality for insulated pipe:

$$\dot{m}_s = \frac{(17.2 \text{ W})(3 \text{ m})}{(0.98)(2226 \times 10^3 J/kg)} = 0.02365 \times 10^{-3} \text{ kg/s} \quad \text{or} \quad 0.02365 \text{ g/s}$$

Percentage decrease in steam quality

$$= \left(\frac{0.305 - 0.02365}{0.305}\right)(100) = \mathbf{92.2\%} \qquad \textbf{(Ans.) (b)}$$

(L) Critical Radius of Insulation (Cylinder)

EXAMPLE 2.34 *A Bakelite coating (k = 1.4 W/m °C) is to be used with a 10 mm diameter circular rod whose surface temperature is maintained at 200°C by passing electrical current through it. The outer surface of the rod is exposed to a convection process characterized by a fluid at a temperature of 30°C and a heat transfer coefficient of 140 W/m²°C.*

Calculate (a) the critical radius of insulation, (b) the heat-loss rate per unit length of the bare rod, (c) the heat-transfer rate per unit length from the coated rod corresponding to critical radius of insulation, and (d) the thickness of insulation required to reduce the heat dissipation rate to 550 W/m. (e) Sketch the relationship between r_o and $\dot{Q}/L$.

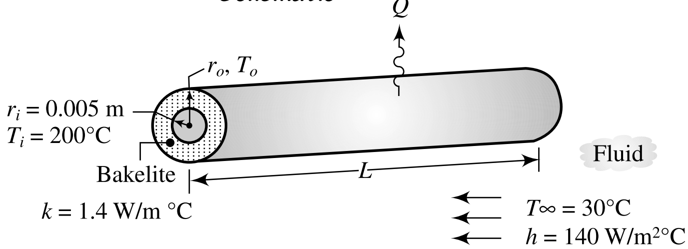
Solution

Known A cylindrical rod with bakelite insulation dissipating heat in a convective environment.

Find (a) r_{cr}(mm), (b) $(\dot{Q}/L)_{\text{bare}}$ (W/m), (c) $(\dot{Q}/L)_{\text{crit ins}}$ (W/m),

(d) $(r_o - r_i)$(mm) for $(\dot{Q}/L)$ = 550 W/m, (e) Graph showing r_o vs $\dot{Q}/L$.

Schematic



$$\xrightarrow{} T_i \;\text{—}\!\text{/\/\/}\!\text{—}\; T_o \;\text{—}\!\text{/\/\/}\!\text{—}\; T_\infty$$

$$\underbrace{\frac{\ln(r_o/r_i)}{2\pi k}}_{R_{\text{cond}}} \qquad \underbrace{\frac{1}{h(2\pi r_o)}}_{R_{\text{conv}}}$$

Assumptions (1) Steady operating conditions. (2) One-dimensional (radial) heat conduction. (3) Contact thermal resistance and radiation effects are negligible. (4) Constant thermal conductivity and uniform heat-transfer coefficient.

Analysis (a) For a circular rod, the critical radius of insulation is

$$r_{cr} = \frac{k}{h} = \frac{1.4\ \text{W/m K}}{140\ \text{W/m}^2\text{K}}\left(\frac{10^3\ \text{mm}}{1\ \text{m}}\right) = \mathbf{10\ mm} \qquad \textbf{(Ans.) (a)}$$

(b) Heat-loss rate from the bare (*uninsulated*) rod per unit length is

$$(\dot{Q}/L)_{\text{bare}} = h(2\pi\, r_i)(T_i - T_\infty) = (140\ \text{W/m}^2{}^\circ\text{C})\ (2\pi \times 0.005\ \text{m})(200 - 30)^\circ\text{C}$$

$$= \mathbf{747.7\ W/m} \qquad \textbf{(Ans.) (b)}$$

(c) Heat-dissipation rate from the insulated (coated) rod with $r_o = r_{cr}$ is the maximum heat transfer rate

Per unit length:

$$(\dot{Q}/L)_{\max} = (\dot{Q}/L)_{\text{crit ins}} = \frac{\Delta T_{\text{overall}}}{R_{\text{cond}} + R_{\text{conv}}} = \frac{T_i - T_\infty}{\dfrac{\ln(r_o/r_i)}{2\pi k} + \dfrac{1}{h(2\pi r_o)}}$$

$$= \frac{2\pi(T_i - T_\infty)}{\dfrac{\ln(r_{cr}/r_i)}{k} + \dfrac{1}{hr_{cr}}} = \frac{2\pi(200-30)^\circ\text{C}}{\dfrac{\ln(10\ \text{mm}/5\ \text{mm})}{1.4\ \text{W/m}^\circ\text{C}} + \dfrac{1}{(140\ \text{W/m}^2\,^\circ\text{C})(10\times 10^{-3}\ \text{m})}}$$

$$= \mathbf{883.2\ W/m} \qquad \textbf{(Ans.) (c)}$$

(d) For $\dfrac{\dot{Q}}{L} = 550\ \text{W/m} = \dfrac{2\pi(T_i - T_\infty)}{\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{hr_o}}$

or
$$\frac{\ln(r_o\ \text{mm}/5\ \text{mm})}{1.4\ \text{W/m}^\circ\text{C}} + \frac{1}{(140\ \text{W/m}^2\,^\circ\text{C})(r_o\times 10^{-3}\ \text{m})} = \frac{2\pi(200-30)^\circ\text{C}}{550\ \text{W/m}}$$

or
$$\underbrace{\frac{\ln(r_o/5)}{1.4} + \frac{1}{0.14\, r_o}}_{\text{LHS}} = \underbrace{1.942}_{\text{RHS}}$$

To solve for r_o, we guess values of r_o to satisfy the above equality.

At r_o = 55 mm, LHS = 1.7128 + 0.1299 = 1.8427

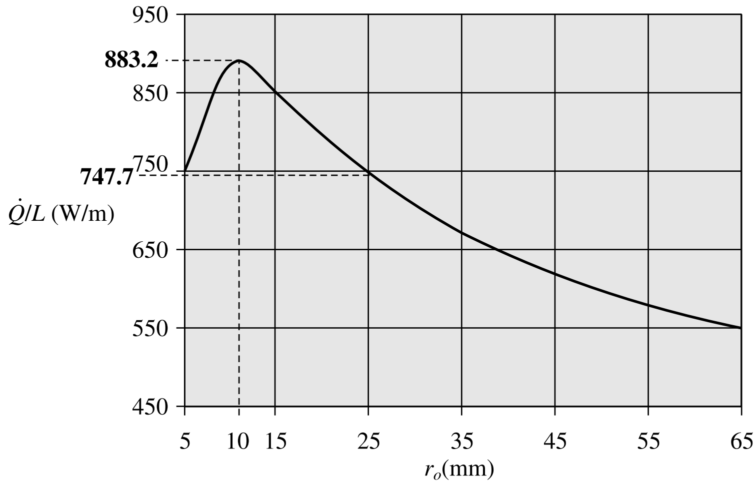
At r_o = 65 mm, LHS = 1.8321 + 0.1099 = 1.942 (= RHS)

Hence, r_o = 65 mm

The required thickness of insulation is

$$r_o - r_i = (65 - 5)\ \text{mm} = \mathbf{60\ mm} \qquad \textbf{(Ans.) (d)}$$

(e) The following graph illustrates the effect of varying the outer radius of insulation (r_o) on the rate of heat loss per metre length $(\dot{Q}/L)$.



Comments It should be noted that for values of r_o other then r_{cr}, the values of $(\dot{Q}/L)$ will be smaller.

Furthermore, the sketch showing $(\dot{Q}/L)$ vs r_o is only qualitative since in reality, the convection coefficient h is a function of both r_o and $(T_o - T_\infty)$, and the thermal conductivity k is a function of temperature.

EXAMPLE 2.35 *A steel pipe of 8 cm outer diameter carries saturated steam at a pressure of 2.5 MPa. A 4 cm thick layer of insulation (*k *= 0.045 W/m °C) is provided. The surface heat-transfer coefficient is 10 W/m °C and the ambient temperature is 12°C. The steam-side convection resistance and the conduction resistance of pipe material (steel) may be neglected. (a) Estimate the heat-loss rate per metre length of pipe. (b) What will be the thickness of the second insulation (*k *= 0.65 W/m °C) to be added to reduce the heat transfer rate by 50 per cent?*

Solution

Known A steam pipe is provided with insulation to reduce heat dissipation.

Find (a) Heat loss per m length, $(\dot{Q}/L)$ (W/m). (b) Thickness of second layer of insulation to reduce heat loss by a factor of 2.

Assumptions (1) Steady-state conditions. (2) One-dimensional (*radial*) heat conduction. (3) Constant properties. (4) Steam-side convection resistance and pipe wall resistance are neglected.

Analysis (a) ***With one insulation:***

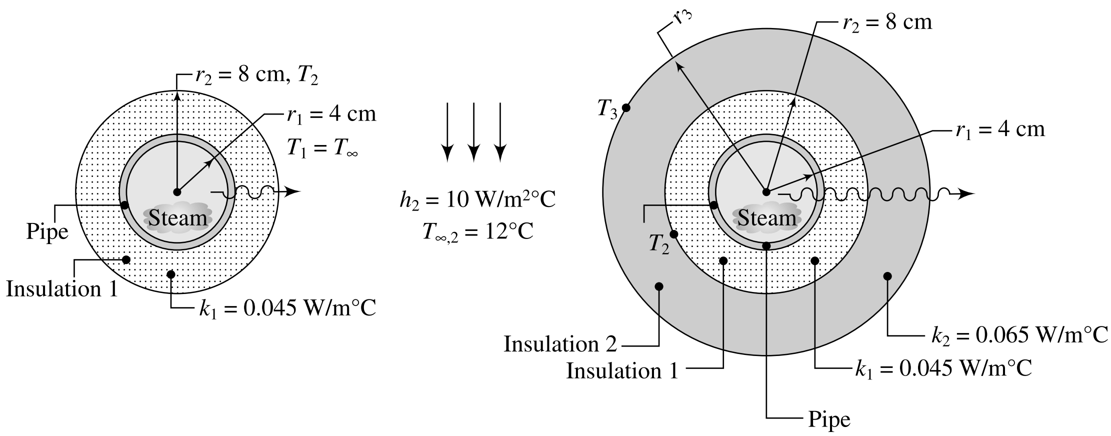
The rate of heat loss per unit pipe length is given by, $\dot{Q} = \dfrac{\Delta T_{\text{overall}}}{R_{\text{total}}} = \dfrac{T_{\infty 1} - \text{T}_{\infty 2}}{R_{\text{cond},1} + \text{R}_{\text{conv}}}$

Note that $R_{\text{conv, 1}}$(*steam side*) and $R_{\text{cond,pipe}}$ are neglected.

Hence, $T_1 = T_{\infty 1} = T_{\text{sat @ 2.5 MPa}} = 224°\text{C}$ (*from Steam Tables*)

$$\therefore \qquad \dot{Q}_a = \frac{T_{\infty 1} - T_{\infty 2}}{\dfrac{1}{2\pi k_1 L}\ln\left(\dfrac{r_2}{r_1}\right) + \dfrac{1}{h_2(2\pi r_2 L)}}$$

Schematic



(a) (b)

or $$\frac{\dot{Q}_a}{L} = \frac{2\pi(T_{\infty 1} - T_{\infty 2})}{\frac{1}{k_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{h_2 r_2}} = \frac{2\pi(224 - 12)°\text{C}}{\frac{1}{0.045\ \text{W/m°C}}\ln\left(\frac{8\ \text{cm}}{4\ \text{cm}}\right) + \frac{1}{(10\ \text{W/m}^2\,°\text{C})(0.08\ \text{m})}}$$

$$= \frac{2\pi \times 212°\text{C}}{16.653\ \text{m}\,^\text{o}\text{C/W}} = \mathbf{80.0\ W/m}$$ **(Ans.) (a)**

(b) ***Two layers of insulation:*** Heat loss is reduced by 50%

$\therefore$ $$\frac{\dot{Q}_b}{L} = 0.5 \times 80.0 = 40.0\ \text{W/m}$$

$$\dot{Q}_b = \frac{T_{\infty_1} - T_{\infty_2}}{\frac{1}{2\pi k_1 L}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{2\pi k_2 L}\ln\left(\frac{r_3}{r_2}\right) + \frac{1}{h_o(2\pi r_3 L)}}$$

or $$\frac{\dot{Q}_b}{L} = \frac{2\pi(T_{\infty_1} - T_{\infty_2})}{\frac{1}{k}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{k_2}\ln\left(\frac{r_3}{r_2}\right) + \frac{1}{h_o r_3}} = 40\ \text{W/m}$$

Hence, $$40\ \text{W/m} = \frac{2\pi(224 - 12)°\text{C}}{\frac{1}{0.015\ \text{W/m}^\text{o}\text{C}}\ln\left(\frac{8\ \text{cm}}{4\ \text{cm}}\right) + \frac{1}{0.065}\ln\left(\frac{r_3(cm)}{8\ cm}\right) + \frac{1}{\left\{\begin{array}{l}10\ \text{W/m}^2\,°\text{C} \\ \times r_3(\text{cm}) \times \left(\frac{1\ \text{m}}{100\ \text{cm}}\right)\end{array}\right\}}}$$

or $$\frac{2\pi \times 212}{40} = \frac{1}{0.045}\ln 2 + \frac{1}{0.065}\ln\frac{r_3}{8} + \frac{100}{10 r_3}$$

or $$\frac{10}{r_3} + \frac{\ln(r_3/8)}{0.065} = 33.3 - 15.4 = 17.9$$

The *trial-and-error* solution is required as tabulated below:

r_3 (cm)	LHS	RHS
20	14.6	17.9
30	20.67	17.9
25	17.93	17.9

Hence, $r_3 = 25$ cm

$\therefore$ Thickness of second insulation = $r_3 - r_2$ = (25 – 8) cm = **17 cm** **(Ans.) (b)**

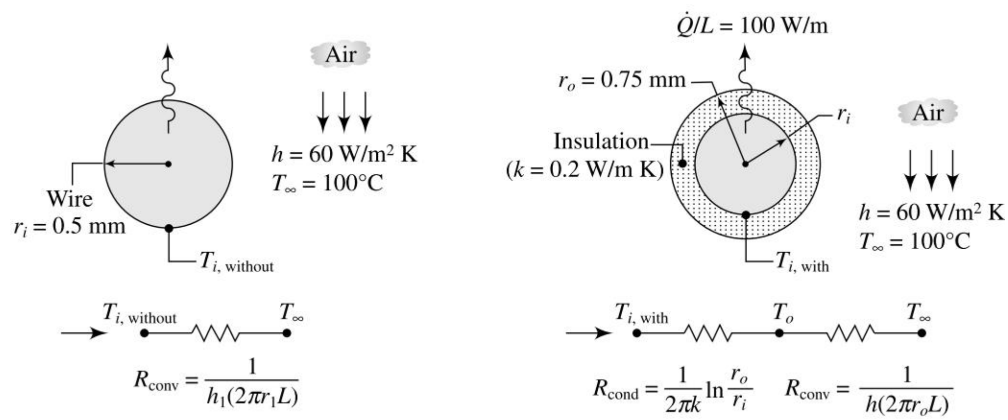
EXAMPLE 2.36 *An electrical wire, 1 mm in diameter, dissipates 100 W/m in an air stream at 100°C. An electrical insulation is added to the wire until the outer diameter becomes 1.5 mm. Calculate the temperature of the wire with and without insulation. Neglect the temperature variations within the wire and assume that the heat transfer coefficient is 60 W/m²K. The thermal conductivity of the insulating material is 0.2 W/m K.*

Solution

Known An electrical wire dissipates heat to the surrounding air. It is provided with insulation for the same heat dissipation rate.

Find Temperature of wire with and without insulation.

Schematic



Assumptions (1) Steady operating conditions prevail. (2) Constant thermal conductivity and uniform heat transfer coefficient. (3) Radial heat conduction. (4) Uniform temperature throughout the wire.

Analysis ***Without insulation:***

Energy balance: $\underbrace{\cancelto{0}{\dot{E}_{\text{in}}}}_{\text{No energy input}} - \dot{E}_{\text{out}} + \dot{E}_{\text{gen}} = \underbrace{\cancelto{0}{\dot{E}_{\text{st}}}}_{\text{Steady state}}$

$\therefore$ $\dot{E}_{\text{out}} = \dot{E}_{\text{gen}}$

Hence, the heat generated inside the electrical wire ($\dot{E}_{\text{gen}}$) is dissipated to the ambient air through convection from the bare wire surface.

$$\therefore \qquad \dot{Q} = \dot{Q}_{\text{conv}} = \frac{T_i - T_\infty}{R_{\text{conv}}} = \frac{T_i - T_\infty}{1/h(2\pi r_i L)}$$

Therefore, the temperature of wire *without* insulation is

$$T_{i,\text{without}} = T_\infty + \frac{\dot{Q}}{L}\left(\frac{1}{h(2\pi r_i)}\right) = 100°\text{C} + (100\ \text{W/m})\left(\frac{1}{60\ \text{W/m}^2\text{K} \times 2\pi \times 0.5 \times 10^{-3}\ \text{m}}\right)$$

$$= 100[1 + 5.305] = \mathbf{630.5°C} \qquad \textbf{(Ans.)}$$

With insulation:

Heat-dissipation rate, $\dot{Q} = \dfrac{\Delta T_{\text{overall}}}{R_{\text{total}}} = \dfrac{T_i - T_\infty}{R_{\text{cond}} + R_{\text{conv}}}$

where $\qquad R_{\text{cond}} = \dfrac{1}{2\pi k L}\ln\dfrac{r_o}{r_i}$ and $R_{\text{conv}} = \dfrac{1}{h(2\pi r_o L)}$

$$\therefore \quad T_i - T_\infty = \frac{\dot{Q}}{L}\left[\frac{1}{2\pi k}\ln\frac{r_o}{r_i} + \frac{1}{h(2\pi r_o)}\right]$$

$$= (100\ \text{W/m})\left[\frac{1}{2\pi(0.2\ \text{W/m K})}\ln\frac{0.75\ \text{mm}}{0.5\ \text{mm}} + \frac{1}{(60\ \text{W/m}^2\ \text{K})(2\pi \times 0.75 \times 10^{-3}\ m)}\right]$$

$$= (100)\,(0.3227 + 3.5368] = 386°\text{C}$$

Hence, temperature of wire *with* insulation is

$$T_{i,\ \text{with}} = 100 + 386 = \mathbf{486°C} \qquad \textbf{(Ans.)}$$

Comments The wire temperature is reduced from **630.5°C** (without insulation) to **486°C** (with insulation) for the *same* heat-dissipation rate since the total thermal resistance is *decreased* after providing insulation. The total (conductive *plus* convective) resistance progressively decreases with *increase* in the insulation till the critical radius of insulation is reached. We note that $r_{o,\ \text{critical}} = \dfrac{0.2\ \text{W/m K}}{60\ \text{W/m}^2\text{K}}\left(\dfrac{10^3\text{mm}}{1\ \text{m}}\right) = 3.33$ mm. In the present case, $r_o = 0.75$ mm which is *less* than $r_{o,\text{critical}}$.

(M) Variable Thermal Conductivity (Cylinder)

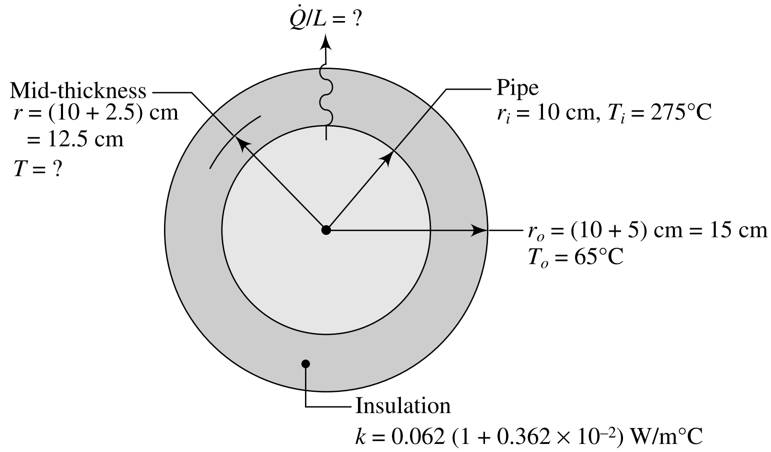
EXAMPLE 2.37 *A 20 cm diameter pipe carrying steam is provided with 5 cm thick insulation whose thermal conductivity varies with temperature as* k(T) *= 0.062 (1 + 0.362 × 10^{-2}T) W/m °C where* T *is in °C. The temperatures at the pipe surface and at the outer surface of the insulation are 275°C and 65°C respectively. Calculate (a) the rate of heat transfer per metre length of the pipe, (b) the temperature at the mid-thickness of insulation, and (c) the temperature gradients at the pipe surface, the mid-thickness of insulation, and the outside surface of the insulation. Sketch the temperature profile.*

Solution

Known A pipe is wrapped with insulation whose thermal conductivity varies with temperature.

Find (a) $\dot{Q}/L$ (W/m), (b) T_m(°C), (c) $\frac{dT}{dr}$ at r = 10 cm, r = 12.5 cm and r = 15 cm.

Schematic



Assumptions (1) Steady-state conditions. (2) One-dimensional radial conduction. (3) Pipe-wall resistance is negligible.

Analysis (a) For a cylindrical layer, the heat-flow rate per unit length is

$$\frac{\dot{Q}}{L} = \frac{2\pi k(T_i - T_o)}{\ln(r_o/r_i)}$$

For linear variation of k (T), k is replaced by mean thermal conductivity determined from

$$k = k(T_m) = 0.062[1 + 0.362 \times 10^{-2}\ T_m]$$

where $$T_m = \frac{1}{2}(T_i + T_o) = \left(\frac{275+65}{2}\right)°\text{C} = 170°\text{C}$$

$\therefore$ $$k_m = 0.062\ (1 + 0.362 \times 10^{-2} \times 170) = 0.1002\ \text{W/m °C}$$

$\therefore$ $$\frac{\dot{Q}}{L} = \frac{2\pi \times 0.1002\ \text{W/m°C} \times (275-65)°\text{C}}{\ln\left(\dfrac{15\ \text{cm}}{10\ \text{cm}}\right)} = \mathbf{326\ W}$$ **(Ans.) (a)**

(b) We note that for variable thermal conductivity, the Fourier's rate equation is

$$\dot{Q} = -k_0(1+bT)(2\pi rL)\frac{dT}{dr} \quad \text{or} \quad \frac{\dot{Q}}{k_0(2\pi L)}\int_{r=r_i}^{r=r}\frac{dr}{r} = -\int_{T=T_i}^{T=T}(1+bT)\,dT$$

or $$\frac{\dot{Q}}{2\pi k_0 L}\ln\frac{r}{r_i} = \left[T + \frac{bT^2}{2}\right]_T^{T_i} = \left[\left(T_i + \frac{b}{2}T_i^2\right) - \left(T + \frac{bT^2}{2}\right)\right]$$

With r = 0.125 m (*at mid-thickness*), k_o = 0.062, L = 1 m,

$\dot{Q}$ = 326 W, r_i = 0.10 m, T_i = 275°C, and b = 0.00362, we have

$$\frac{326}{2\pi \times 0.062 \times 1} \ln \frac{0.125}{0.10} = \left[\left(275 + \frac{0.362 \times 10^{-2}}{2} \times 275^2\right) - \left(T + \frac{bT^2}{2}\right)\right]$$

or $$186.74 = 411.88 - \left(T + \frac{bT^2}{2}\right) \quad \text{or} \quad \frac{bT^2}{2} + T - 225.14 = 0$$

This is a quadratic equation that can be solved:

$$T = \frac{-1 \pm \sqrt{1^2 - 4(b/2)(-225.14)}}{2 \times b/2} = \frac{-1 \pm \sqrt{1 + (2 \times 0.00362 \times 225.14)}}{0.362 \times 10^{-2}}$$

$= \mathbf{171.8°C}$ **(Ans.) (b)**

(c) At $r = r_i = 0.10$ m,

$T_i = 275°C$: $k = 0.062\ [1 + 0.00362 \times 275] = 0.1237$ W/m °C

Temperature gradient,

$$\frac{dT}{dr} = -\frac{\dot{Q}}{k(2\pi r_i L)} = -\frac{326 \text{ W/m}}{0.1237 \text{ W/m°C} \times 2\pi \times 0.10 \text{ m}} = -\mathbf{4194°C/m}$$ **(Ans.) (c)**

At $r = 0.125$ m, $T = 171.8$ °C:

$k = 0.125\ [1 + 0.00362 \times 171.8] = 0.1006$ W/m°C

$$\therefore \quad \frac{dT}{dr} = -\frac{326}{0.1006 \times 2\pi \times 0.125} = -\mathbf{4126°C/m}$$ **(Ans.) (c)**

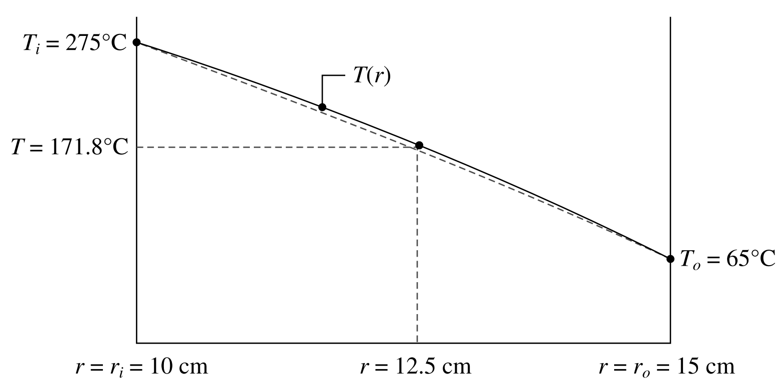
At $r = 0.15$ m, $T = 65°C$:

$k = 0.062\ [1 + 0.00362 \times 65] = 0.0766$ W/m °C

$$\therefore \quad \frac{dT}{dr} = -\frac{326}{0.0766 \times 2\pi \times 0.15} = -\mathbf{4516°C/m}$$ **(Ans.) (c)**

Temperature gradients are *negative* implying that with an increase in radius, the temperature decreases.

The temperature profile is sketched below:



(N) Spherical Shell with Fixed Surface Temperatures

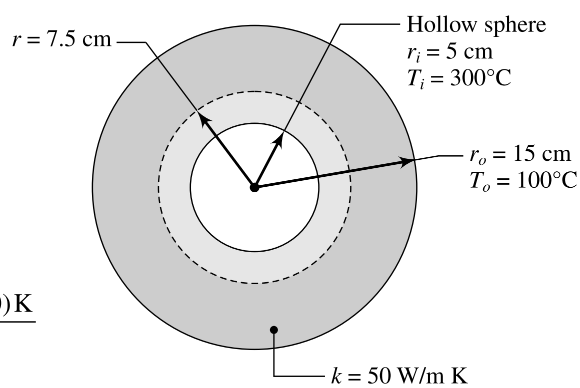
EXAMPLE 2.38 *A hollow sphere (10 cm ID and 30 cm OD) has an inner surface temperature of 300°C and an outer surface temperature of 100°C. The thermal conductivity of the sphere material is 50 W/m K. Calculate the heat flow through the sphere. Determine the temperature one fourth way between the inner and outer surfaces.*

Solution

Known A hollow sphere with prescribed dimensions, and surface temperatures.

Find Heat-transfer rate. Temperature at one fourth of thickness.

Schematic



Analysis Heat flow through the sphere

$$\dot{Q} = \frac{4\pi k(T_i - T_o)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}$$

$$= \frac{(4\pi)(50\ \text{W/m K})(300 - 100)\,\text{K}}{\left(\frac{1}{0.05} - \frac{1}{0.15}\right)\frac{1}{\text{m}}}$$

$= \mathbf{9424.78\ W}$ or $\mathbf{9.425\ kW}$ **(Ans.)**

The temperature distribution is written as

$$T = \frac{r_o}{r}\left[\frac{r - r_i}{r_o - r_i}\right](T_o - T_i) + T_i$$

The value of r at *one-fourth* way of the inner and outer surfaces is

$$r = 5 + \frac{1}{4}(15 - 5) = 7.5\ \text{cm}$$

Temperature at this radius is,

$$\therefore \quad T = \left(\frac{0.15}{0.075}\right)\left(\frac{0.075 - 0.05}{0.15 - 0.05}\right)(100 - 300)°\text{C} + 300°\text{C} = \mathbf{200°C}$$ **(Ans.)**

(O) Spherical Shell Bounded by Known Fluid Temperatures

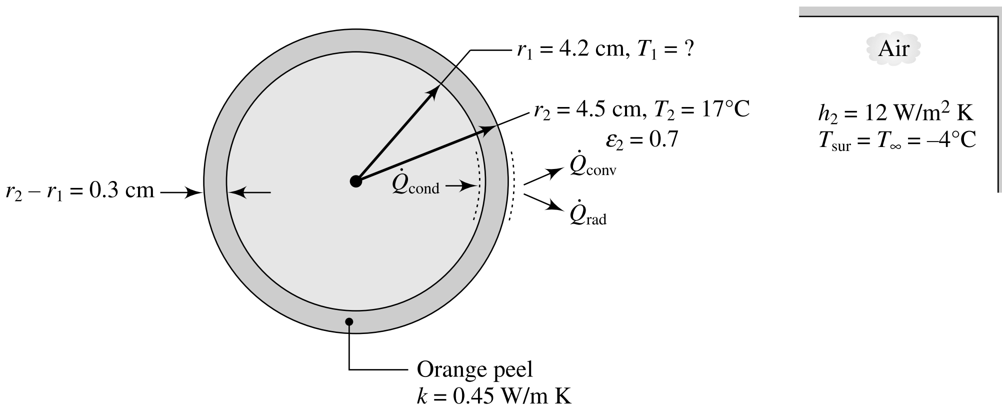
EXAMPLE 2.39 *A 9 cm outer diameter orange is placed in a refrigerator in which the temperature of air is –4°C. The surface temperature of the orange is 17°C and the convection heat-transfer coefficient between the orange surface and the refrigerated air is 12 W/m² K. The orange peel (*k *= 0.45 W/m K) is 3 mm thick and the emissivity of the orange surface may be assumed as 0.7. Calculate the inside surface temperature of the orange peel.*

Solution

Known An orange with specified emissivity is placed in refrigerated air.

Find Inner surface temperature of the orange peel, T_1.

Schematic



Assumptions (1) Steady-state conditions. (2) Constant thermal conductivity. (3) Uniform heat-transfer coefficient.

Analysis Control surface energy balance at the outer surface of the orange yields:

$$\dot{E}_{in} = \dot{E}_{out}$$

i.e.,
$$\begin{pmatrix}\text{Heat transfer by conduction}\\ \text{through orange peel}\end{pmatrix} = \begin{pmatrix}\text{Heat dissipated by convection}\\ \text{to surrounding air}\end{pmatrix} + \begin{pmatrix}\text{Heat lost by radiation}\\ \text{to surroundings}\end{pmatrix}$$

or
$$\dot{Q}_{cond} = \dot{Q}_{conv} + \dot{Q}_{rad}$$

or
$$\frac{4\pi k(r_1 r_2)(T_1 - T_2)}{r_2 - r_1} = h(4\pi r_2^2)(T_2 - T_\infty) + \varepsilon_2 (4\pi r_2^2)\sigma(T_2^4 - T_{sur}^4)$$

or
$$\frac{kr_1(T_1 - T_2)}{r_2(r_2 - r_1)} = h(T_2 - T_\infty) + \varepsilon_2\sigma(T_2^4 - T_\infty^4) \qquad \text{(since } T_{sur} \approx T_\infty\text{)}$$

Inner surface temperature of the orange peel is

$$T_1 = T_2 + \frac{r_2(r_2 - r_1)}{kr_1}[h(T_2 - T_\infty) + \varepsilon_2\,\sigma(T_2^4 - T_\infty^4)]$$

$$= 290.15\text{ K} + \left(\left(\frac{4.5\text{ cm}}{4.2\text{ cm}}\right)\frac{(0.3\times10^{-2}\,m)}{(0.45\text{ W/m°C})}\right)$$

$$\times[12\text{ W/m}^2\text{ K}(290.15 - 269.15)\text{K} + 0.7(5.67\times10^{-8}\text{ W/m}^2\text{ K}^4)$$

$$+ (290.15^4 - 269.15^4)\text{K}^4]$$

$$= \mathbf{292.47\ K} \quad \text{or} \quad \mathbf{19.3°C}$$ **(Ans.)**

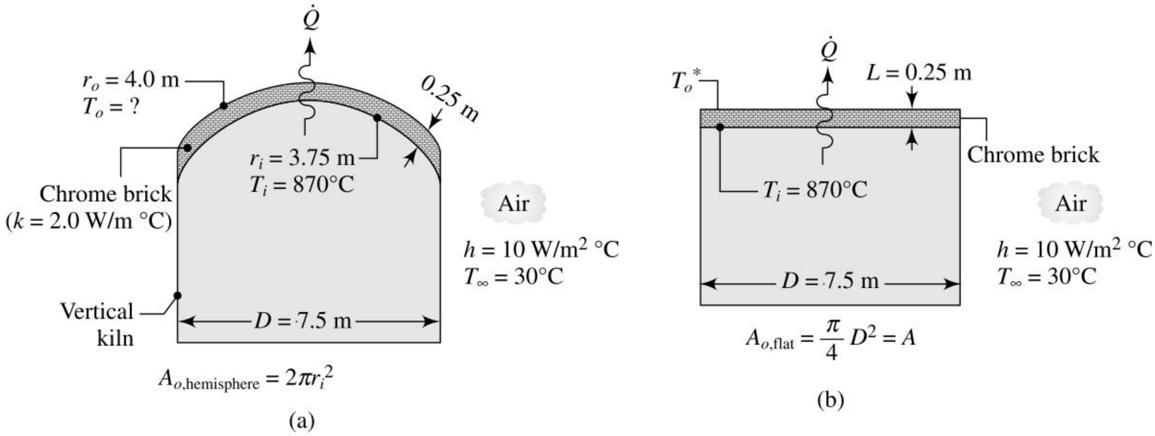
EXAMPLE 2.40 *A hemispherical dome at the top of a 7.5 m vertical kiln is fabricated from 0.25 m thick layer of chrome brick (*k *= 2.0 W /m °C). The inside surface temperature of the dome is 870°C and the ambient air is at 30°C. The surface heat-transfer coefficient is 10 W/m² °C. (a) Calculate the outside surface temperature of the dome and the rate of heat loss from the kiln. (b) Determine the reduction in heat loss if a flat dome 0.25 m thick were to replace this dome of the same material with the kiln operating under the same conditions.*

Solution

Known A hemispherical dome loses heat to surrounding air by conduction and convection. Heat loss if (a) the dome is hemispherical, and (b) the dome is flat.

Find (a) T_o(°C); $\dot{Q}$ (W). (b) Reduction in $\dot{Q}$ if flat dome replaces hemispherical dome.

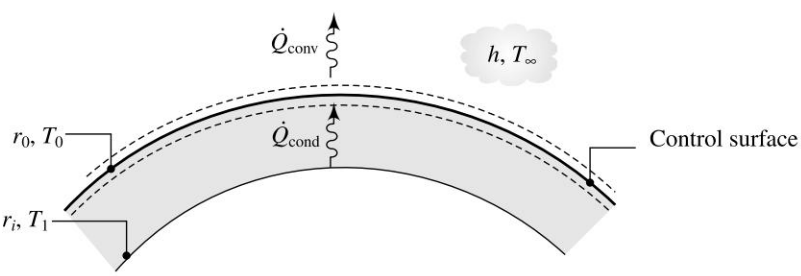
Schematic



Assumptions (1) Steady-state, one-dimensional heat conduction. (2) Constant thermal conductivity and uniform heat transfer coefficient.

Analysis (a) ***Hemispherical dome:*** Control surface energy balance at the outside surface of the dome: $\dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv}}$

$$\frac{2\pi k(T_i - T_o)}{(1/r_i) - (1/r_o)} = h(2\pi r_o^2)(T_o - T_\infty) \quad \text{or} \quad \frac{k(T_i - T_o) r_i r_o}{r_o - r_i} = h r_o^2 (T_o - T_\infty)$$



Substituting the given values, one has

$$= \frac{2.0 \text{ W/m °C}(870 - T_o)\text{°C} \times 3.75 \text{ m} \times 4.0 \text{ m}}{0.25 \text{ m}}$$

$$= (10 \text{ W/m}^2 \text{ °C})(4.0 \text{ m})^2 (T_o - 30)\text{°C}$$

or $\quad 120\ (870 - T_o) = 160\ (T_o - 30) \quad$ or $\quad 280\ T_o = (120 \times 870) + (160 \times 30)$

Hence, the outer surface temperature of the hemispherical dome is

$$T_o = \frac{109200}{280} = \mathbf{390°C}$$ **(Ans.) (a)**

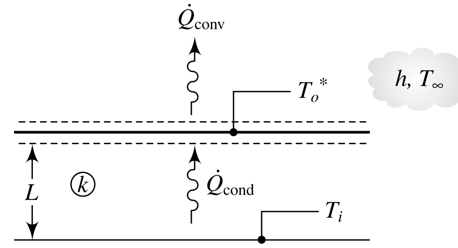
Rate of heat loss,

$$\dot{Q} = h(2\pi r_o^2)(T_o - T_\infty) = 10 \text{ W/m}^2\,°\text{C}\,(2\pi \times 4^2 \text{m}^2)\,(390 - 30)°\text{C}\left|\frac{1 \text{ kW}}{10^3 \text{ W}}\right|$$

$$= \mathbf{361.9\ kW} \qquad \textbf{(Ans.) (a)}$$

(b) ***Flat dome:*** Energy balance at the outer surface gives

$$kA\frac{(T_i - T_o)}{L} = hA(T_o - T_\infty)$$



where L is the thickness of the flat dome. Substituting proper values, one gets

or $$\frac{(2.0 \text{ W/m}\,°\text{C})(870 - T_o)°\text{C}}{0.25 \text{ m}}$$

$$= (10 \text{ W/m}^2\ °\text{C})\ (T_o - 30)°\text{C}$$

or $$8\ (870 - T_o^*) = 10\ (T_o^* - 30)$$

$\therefore$ $$T_o^* = \frac{7260}{18} = 403.3°\text{C}$$

Rate of heat loss,

$$\dot{Q}^* = hA(T_o^* - T_\infty) = h\left(\frac{\pi}{4}D^2\right)(T_o^* - T_\infty) = (10 \text{ W/m}^2\,°\text{C}) = \left(\frac{\pi}{4} \times 7.5^2 \text{ m}^2\right)(403.3 - 30)°\text{C}$$

$$= 164.9 \text{ kW}$$

Percentage reduction in heat loss

$$= \frac{\dot{Q} - \dot{Q}^*}{\dot{Q}} \times 100 = \frac{(361.9 - 164.9)}{361.9} \times 100 = \mathbf{54.4\%} \qquad \textbf{(Ans.) (b)}$$

(P) Composite Sphere with Fixed Boundary Temperatures

EXAMPLE 2.41 *The heat loss from the anterior chamber of the eye through the cornea varies significantly depending upon whether one wears a contact lens. The cornea and the lens constitute one third of the spherical surface area if the eye is considered a spherical system. The inner and outer fluid temperatures are 37 and 20°C, respectively. The inside layer of the cornea is at a radius of 10 mm while the thickness of the cornea and the contact lens are 2.5 and 3.5 mm, respectively. The thermal conductivities of cornea and contact lens are 0.35 and 0.80 W/m K, respectively. The inside and outside unit surface conductances are 10 and 5 W/m² K, respectively. Assume steady-state conditions, and the same outside unit surface conductance with or without contact lens in place.*

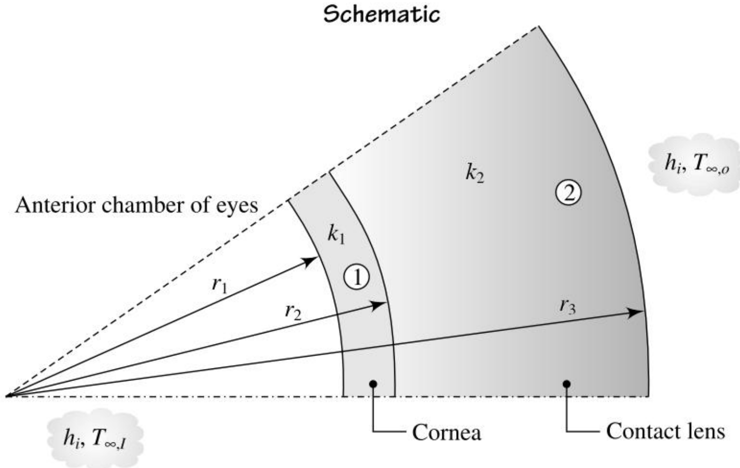
Determine the heat loss from the anterior chamber of the eye with and without the contact lens in place and comment on your results.

Solution

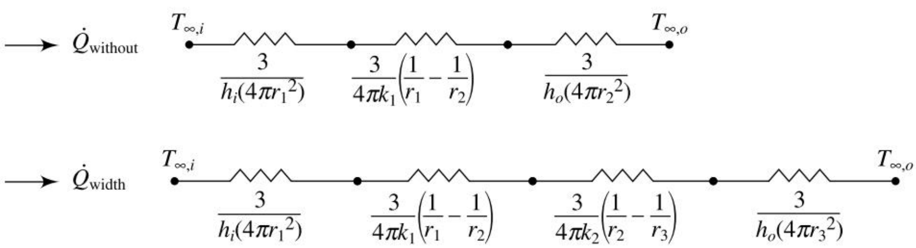
Known An eye with a contact lens is represented as a composite sphere subjected to convection at both inside and outside surfaces.

Find Heat loss from anterior chamber of eye *with* and *without* contact lens.

Schematic



Radius (mm)			Thermal conductivity (W/m K)		Convection coefficient (W/m²K)KKK		Fluid temperature (°C)	
r_1	r_2	r_3	k_1	k_2	h_i	h_o	$T_{\infty,i}$	$T_{\infty,o}$
10	12.5	16.0	0.35	0.80	10	5	37	20



Assumptions (1) Steady-state conditions prevail. (2) Eye is considered as 1/3 sphere. (3) Convection coefficient, h_o is unaffected with or without contact lens. (4) Contact resistance is negligible.

Analysis Note that the eye is represented as 1/3 sphere. Hence, the surface areas are

$$A_i = \frac{4}{3}\pi r_1^2,\ A_o = \frac{4}{3}\pi r_3^2 \text{ (with contact lens) and } A_o = \frac{4}{3}\pi r_2^2 \ (\textit{without contact lens})$$

Now, the rate of heat transfer = $\dot{Q} = \dfrac{T_i - T_o}{\Sigma R_{\text{th}}}$

Total thermal resistance *without* and *with* contact lens in place are now determined:

Without contact lens:

$$\Sigma R_{\text{th}} = \left[\frac{3}{4\pi(0.01)^2(10)} + \frac{3}{4\pi(0.35)}\left[\frac{1}{0.01} - \frac{1}{0.0125}\right] + \frac{3}{4\pi(0.0125)^2(5)}\right]$$

$$= 238.73 + 13.64 + 305.58 = \mathbf{557.95\ K/W}$$

With contact lens:

$$\Sigma R_{\text{th}} = \left[238.73 + 13.64 + \frac{3}{4\pi(0.80)}\left[\frac{1}{0.0125} - \frac{1}{0.016}\right] + \frac{3}{4\pi(0.016)^2(5)}\right]$$

$$= 238.73 + 13.64 + 5.22 + 186.51 = \mathbf{444.10\ K/W}$$

Hence, the heat loss rates from the anterior chamber are

$$\dot{Q}_{\text{without}} = \frac{(37-20)\ \text{K}}{557.95\ \text{K/W}} = \mathbf{0.0305\ W \quad or \quad 30.5\ mW} \qquad \textbf{(Ans.)}$$

$$\dot{Q}_{\text{with}} = \frac{(37-20)\ \text{K}}{444.10\ \text{K/W}} = \mathbf{0.0383\ W \quad or \quad 38.3\ mW} \qquad \textbf{(Ans.)}$$

Comment We find that the heat loss from the anterior chamber increases by $\left(\frac{38.3-30.5}{30.5}\times 100\right)$, i.e., **25.6%** when the contact lens is worn. This implies that the outer radius $r_3 = 0.016$ m is *less* than the critical radius given by $\frac{2k_2}{h_o} = \frac{2\times 0.80}{5} = 0.32$ m.

(Q) Composite Sphere Bounded By Fluid Temperatures

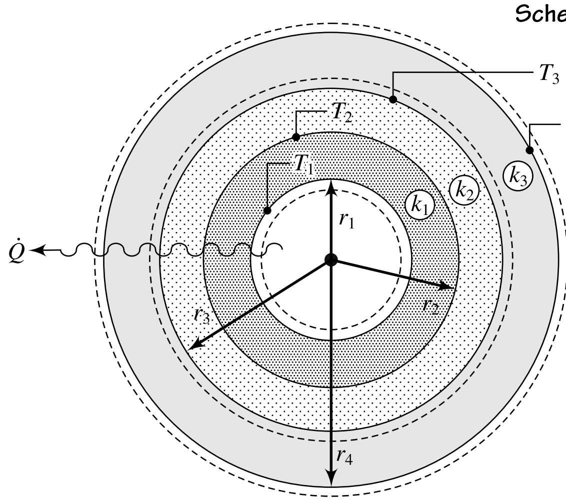
EXAMPLE 2.42 *Hot gas at a constant temperature of 400°C is contained in a spherical shell (2000 mm ID and 50 mm thick) made of steel (k_s = 19 W/m K). It is wrapped with two layers of insulation. The first layer is of mineral wool (k_m = 0.05 W/m K) and is 50 mm thick. The second layer is of asbestos (k_a = 0.2 W/m K) and is also 50 mm thick. Calculate the steady rate at which heat will flow if the outside air is at a temperature of 30°C. Assume that the value of heat transfer coefficient on the inner surface of the steel shell is 30 W/m² K and that on the outer surface of the insulation is 15 W/m² K. In what way, the heat flow will get affected if the sequence of the insulations is changed?*

Solution

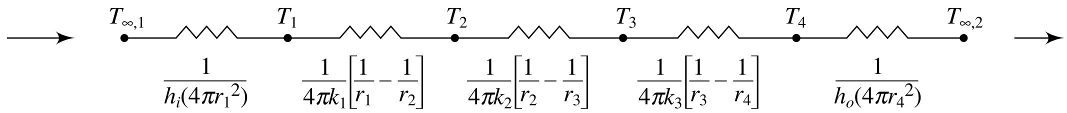
Known Composite spherical shell with convective surfaces.

Find Heat-transfer rate. Effect of alternate placement of insulations.

Schematic



$T_{\infty,1} = 400°\text{C}$, $T_{\infty,2} = 30°\text{C}$
$k_1 = k_s = 19$ W/m K
$k_2 = k_m = 0.05$ W/m K
$k_3 = k_a = 0.20$ W/m K
$\delta_1 = \delta_2 = \delta_3 = 50$ mm
$r_1 = 1$ m
$r_2 = r_1 + \delta_1 = 1.05$ m
$r_3 = r_2 + \delta_2 = 1.10$ m
$r_4 = r_3 + \delta_3 = 1.15$ m
$h_i = 30$ W/m² K
$h_o = 15$ W/m² K



Assumptions (1) Steady-state conditions. (2) One-dimensional heat conduction. (3) No internal heat generation. (4) Constant properties. (5) Uniform heat-transfer coefficients.

Analysis ***Case I***: $\dot{Q}_I = \dfrac{\Delta T_{\text{overall}}}{\sum R_{\text{th}}} = \dfrac{4\pi(T_{\infty,1} - T_{\infty,2})}{\left\{\dfrac{1}{h_i r_1^2} + \dfrac{1}{k_1}\left[\dfrac{1}{r_1} - \dfrac{1}{r_2}\right] + \dfrac{1}{k_2}\left[\dfrac{1}{r_2} - \dfrac{1}{r_3}\right] + \dfrac{1}{k_3}\left[\dfrac{1}{r_3} - \dfrac{1}{r_4}\right] + \dfrac{1}{h_o r_4^2}\right\}}$

$$= \frac{4\pi(400-30)}{\left\{\dfrac{1}{30(1)^2} + \dfrac{1}{19}\left(\dfrac{1}{1} - \dfrac{1}{1.05}\right) + \dfrac{1}{0.05}\left(\dfrac{1}{1.05} - \dfrac{1}{1.1}\right) + \dfrac{1}{0.2}\left(\dfrac{1}{1.1} - \dfrac{1}{1.15}\right) + \dfrac{1}{(15)(1.15)^2}\right\}}$$

$$= \frac{4\pi \times 370}{1.1492} = \mathbf{4045\ W} \qquad \textbf{(Ans.)}$$

***Case II*:** If the sequence of insulations is changed, i.e., the first layer of insulation is of asbestos and the second one is of mineral wool, then

$$\dot{Q}_{II} = \frac{4\pi(400-30)}{\left\{\dfrac{1}{30(1)^2} + \dfrac{1}{19}\left(\dfrac{1}{1} - \dfrac{1}{1.05}\right) + \dfrac{1}{0.2}\left(\dfrac{1}{1.05} - \dfrac{1}{1.1}\right) + \dfrac{1}{0.05}\left(\dfrac{1}{1.1} - \dfrac{1}{1.15}\right) + \dfrac{1}{(15)(1.15)^2}\right\}}$$

$$= \frac{4\pi \times 370}{1.0932} = \mathbf{4253\ W}$$

Thus, by wrapping a better insulating material (*of lower thermal conductivity*) next to the shell, as in the *first* case, the rate of heat loss is *less* than in the *second* case where the inferior insulation (*asbestos*) is next to the shell.
Percentage increase in heat loss by changing the sequence of insulations is

$$\frac{\dot{Q}_{II} - \dot{Q}_I}{\dot{Q}_I} = \frac{(4253-4045)}{4045} \times 100 = \mathbf{5.15\%} \qquad \textbf{(Ans.)}$$

(R) Critical Radius of Insulation (Sphere)

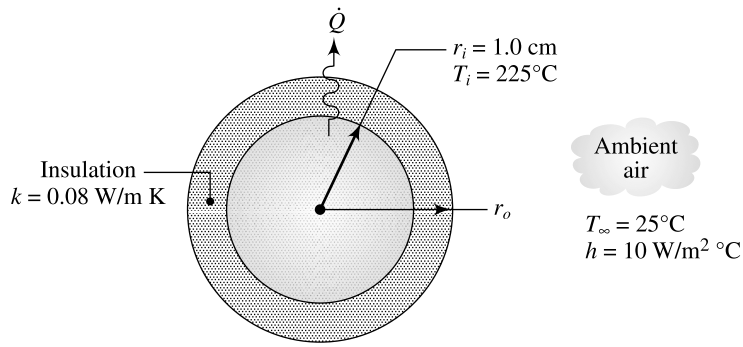
EXAMPLE 2.43 *A sphere of 2 cm outside diameter maintained at a uniform temperature* T_i = *225°C is exposed to an ambient air at* T_∞ = *25°C with a convection heat-transfer coefficient* h *=10 W/m² °C. Calculate the critical thickness of the insulation (*k *= 0.08 W/m°C) required to maximize the rate of heat loss while the sphere is maintained at* T_i = *225 °C.*

Solution

Known Geometry and surface conditions of an insulated sphere

Find Critical thickness of insulation, Percentage increase in heat loss rate.

Schematic



Assumptions (1) Steady-state radial conduction. (2) No internal energy generation. (3) Constant thermal conductivity. (4) Uniform heat transfer coefficient.

Analysis Heat-loss rate *without* insulation is

$$\dot{Q}_{wo} = h(4\pi r_i^2)(T_i - T_\infty)$$

$$= (10\ \text{W/m}^2\,°\text{C})(4\pi \times 0.01^2\ \text{m}^2)(225 - 25)°\text{C}$$

$$= 2.513\ \text{W}$$

Critical radius of insulation for a sphere is

$$r_o = r_{cr} = \frac{2k}{h} = \frac{2 \times 0.08\ \text{W/m}\,°\text{C}}{10\ \text{W/m}^2\,°C} = \mathbf{0.016\ m} \quad \text{or} \quad \mathbf{1.6\ cm}$$

∴ Critical thickness of insulation,

$$r_{cr} - r_i = (1.6 - 1.0)\text{cm} = 0.6\ \text{cm} \quad \text{or} \quad \mathbf{6\ mm}$$ **(Ans.)**

Maximum heat-loss rate corresponds to critical thickness.

Hence, $$\dot{Q}_{\max} = \frac{T_i - T_o}{\dfrac{1}{4\pi k}\left[\dfrac{1}{r_i} - \dfrac{1}{r_o}\right] + \dfrac{1}{(4\pi r_o^2)h}}$$

With $r_o = r_{cr} = 0.016$ m

$$\dot{Q}_{\max} = \frac{4\pi(225 - 25)°\text{C}}{\dfrac{1}{0.08\ \text{W/m}\,°\text{C}}\left[\dfrac{1}{0.01\ \text{m}} - \dfrac{1}{0.016\ \text{m}}\right] + \dfrac{1}{(10\ \text{W/m}^2\,°\text{C}) \times (0.016^2\ \text{m}^2)}} = 2.92\ \text{W}$$

Percentage increase in heat-loss rate

$$= \left(\frac{2.92 - 2.513}{2.513}\right)(100) = \mathbf{16.2\%}$$ **(Ans.)**

(S) Variable Thermal Conductivity (Sphere)

EXAMPLE 2.44 *Estimate the rate of evaporation of liquid oxygen from a spherical container 2 m ID covered with 50 cm of asbestos insulation. The temperatures at the inner and outer surfaces of insulation are –183°C, and 0°C respectively. The boiling point of liquid oxygen is –183°C and its enthalpy of evaporation is 213.54 kJ/kg. The thermal conductivity of insulation is 0.155 and 0.125 W/m K at 0 and –183°C, respectively.*

Assume that the conductivity varies linearly with temperature.

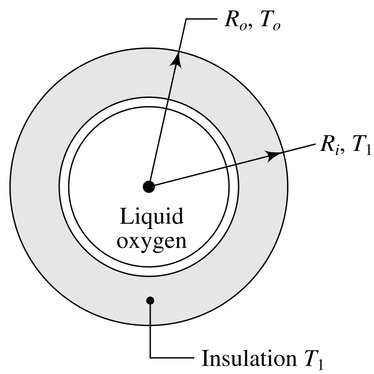
Neglect the thermal resistance of metal container. Derive the equations that you use.

Solution

Known A liquid oxygen spherical container covered with insulation of variable thermal conductivity.

Find Rate of evaporation of liquid oxygen.

Schematic



$R_i = 1$ m
$R_o = (1 + 0.5)$ m $= 1.5$ m
$T_i = -183$°C
$T_o = 0$°C
$h_{fg} = 213.54$ kJ/kg
$k_o = 0.155$ W/m K
$k_i = 0.125$ W/m K

Assumptions (1) Steady-state, one-dimensional (radial) conduction. (2) Thermal conductivity varies linearly with temperature. (3) Thermal resistance of metal container is negligible.

Analysis As the conductivity varies linearly with temperature, we can write

$$\frac{k - k_i}{k_0 - k_i} = \frac{T - T_i}{T_o - T_i}$$

where k, k_i, and k_0 are the thermal conductivities of the sphere material at temperatures T, T_i and T_o, respectively.

Thus, $$k = k_i + (k_0 - k_i)\left[\frac{T - T_i}{T_o - T_i}\right]$$

The Fourier's rate equation for a sphere is

$$\dot{Q} = -k(4\pi r^2)\frac{dT}{dr} \quad \text{or} \quad \dot{Q}\frac{dr}{r^2} = -4\pi k dT = -4\pi\left[k_i + (k_0 - k_i)\left\{\frac{T - T_i}{T_o - T_i}\right\}\right]dT$$

Integrating between the radii R_i and R_o, and the corresponding temperatures T_i and T_o, we have

$$\dot{Q}\int_{R_i}^{R_o}\frac{dr}{r^2} = -4\pi\int_{T_i}^{T_o}\left[k_i + (k_0 - k_i)\left\{\frac{T - T_i}{T_o - T_i}\right\}\right]dT$$

or $$\dot{Q}\left[\frac{1}{R_i}-\frac{1}{R_o}\right] = -4\pi\left[k_i(T_o - T_i) + \left(\frac{k_0 - k_i}{T_o - T_i}\right)\left\{\frac{T_o^2 - T_i^2}{2} - T_i(T_o - T_i)\right\}\right]$$

$$= -4\pi(T_o - T_i)\left[k_i + \left(\frac{k_0 - k_i}{T_o - T_i}\right)\left\{\frac{T_o + T_i}{2} - T_i\right\}\right]$$

$$= 4\pi(T_i - T_o)\left[k_i + \frac{(k_0 - k_i)(T_o - T_i)}{2(T_o - T_i)}\right] = 4\pi(T_i - T_o)\left[\frac{2k_i + k_0 - k_i}{2}\right]$$

$$\therefore \qquad \dot{Q} = \frac{4\pi(T_i - T_o)[(k_i + k_o)/2]}{\left\{\dfrac{1}{R_i} - \dfrac{1}{R_o}\right\}} \quad \Rightarrow \quad \boxed{\dot{Q} = 4\pi(T_i - T_o)\frac{(k_i + k_o)}{2}\left(\frac{R_i R_o}{R_o - R_i}\right)}$$

Substituting the appropriate numerical values, we get

$$\dot{Q} = 4\pi(-183 - 0)\frac{(0.125 + 0.155)}{2}\left(\frac{1 \times 1.5}{1.5 - 1}\right) = -965.85 \text{ W} \quad \text{or} \quad \text{J/s}$$

(*The negative sign indicates heat flow into the spherical container*)

Hence, the rate of evaporation of liquid oxygen,

$$\dot{m} = \frac{\dot{Q}}{h_{fg}} = \frac{(965.85 \text{ J/s})(3600 \text{ s/h})}{(213.54 \text{ kJ/kg})(10^3 \text{ J/kJ})} = \mathbf{16.28 \text{ kg/h}} \qquad \textbf{(Ans.)}$$

Points to Ponder

- Heat transfer is a non-equilibrium phenomenon.
- The heat-flux vector is related to temperature gradient vector according to the relation $q = -k \, \Delta \, T$.
- For steady-state one-dimensional heat conduction, the single governing differential equation is $\dfrac{1}{r^n}\dfrac{d}{dr}\left(r^n \dfrac{dT}{dr}\right) + \dfrac{\overline{q}}{k} = 0$ where $n = 0$, rectangular corrdinates (*replace r with x*)
  $n = 1$, cylindrical coordinates
  $n = 2$, spherical coordinates.
- $\nabla^2 T = 0$ is called the Laplace equation.
- A material with $k_x = k_y = k_z = k$ is known as isotropic.
- Imperfect thermal contact between mating surfaces is accounted for by incorporating contact resistance with units m^2 K/W.
- Critical thickness of insulation is relevant only in radial systems.
- $k_{\text{non-metallic crystals}} > k_{\text{pure metals}} > k_{\text{metal alloys}} > k_{\text{non-metallic solids}}$.
- $k_{\text{solids}} < k_{\text{liquids}} < k_{\text{gases}}$
- Materials that are good electrical conductors are also good thermal conductors.
- Wood has directionally dependent thermal conductivity.
- Thermal diffusivity is a composite parameter defined as $\alpha = k/\rho C_p$.
- The thermal conductivity of air at room temperature is lower than the conductivities of almost all the ordinary insulating materials.

- The thermal conductivity of an alloy of two metals will most likely be less than the thermal conductivities of the two metals.
- Unlike a plane wall, steady-state heat flux is not constant in a long hollow cylinder or a hollow sphere in the radial direction.
- The rate of heat transfer per unit area between two large isothermal plates, a distance apart, will be less when the intervening space is filled with air than when it is filled with an insulating material.
- The R value can be interpreted as the thermal resistance of a 1 m^2 cross section of the material.
- The temperature gradient, dT/dr, decreases with increasing radius in radial systems. For constant k, in a long hollow cylinder. $r\dfrac{dT}{dr}$ = const and $\dot{Q}_r$ is constant, independent of the radial coordinate. For constant k, in a hollow sphere, $r^2\dfrac{dT}{dr}$ is constant. The heat flux q_r is not constant everywhere.
- *Logarithmic mean area* is a term often used in the case of a hollow cylinder and *geometric mean area* in the case of a hollow sphere.
- The temperature profile of a cylindrical annulus is logarithmic while that of a spherical shell, is hyperbolic.
- Heat-transfer coefficient h is not a thermophysical property such as thermal conductivity.
- Biot number is the ratio of conduction resistance to convection resistance.
- Electrical analogy is applicable only when there is no internal heat generation.

GLOSSARY of Key Terms

• **Steady-state conduction**	Temperature is independent of time.
• **One-dimensional conduction**	Temperature is a function of a single space coordinate.
• **Thermal conductivity**	A property of the material that may be considered a heat flux per unit temperature gradient through it. It may vary with temperature for most substances.
• **Thermal resistance**	Temperature difference per unit heat-transfer rate. Analogous to electrical resistance.
• **Thermal contact resistance**	The resistance offered by imperfect thermal contact between adjacent conducting layers (units m^2 K/W).
• **Overall heat-transfer coefficient**	Heat-transfer rate per unit surface area per unit overall temperature difference in the case of multilayered wall/cylinder/sphere. In radial systems, it depends on the area chosen.
• **Heat-conduction differential equation**	The three-dimensional unsteady-state heat conduction in a solid in Cartesian/cylindrical (polar)/spherical coordinates.
• **Isotropic material**	Thermal conductivity at any point is the same for all directions of heat flow.
• **Thermal diffusivity**	An important property which is a measure of the rate at which heat diffuses through a material.

• **Initial condition**	The temperature distribution in the body at a particular instant (t = 0) to begin with. For instance, at t = 0, $T = T_w$ (a constant).
• **Boundary conditions**	Conditions that exist at the surface of the body which may or may not be time dependent.
• **Critical thickness of insulation**	The thickness in the case of an insulated cylinder or sphere (*radial systems*) corresponding to maximum heat-transfer rate. There is no critical thickness in the case of an insulated plane wall since the cross-sectional area remains the same regardless of insulation thickness.
• **Log mean area**	An equivalent area that expresses the heat-transfer rate for a hollow cylinder in the same form as that for a plane wall in terms of inside and outside area of the cylinder.
• **Geometric mean area**	An equivalent area that expresses the heat-flow rate for a hollow sphere in the form of a plane wall in terms of its inner and outer areas.
• **Anisotropic material**	A material whose thermal conducitity parallel to the grain differs from that in the perpendicular direction.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

2.1 The temperature field in a body varies according to the equation $T(x, y) = x^3 + 4xy$. The direction of fastest variation in temperature at the point (1, 0) is given by

(a) $3\hat{i} + 8\hat{i}$ (b) $\hat{i}$ (c) $0.6\hat{i} + 0.8\hat{i}$ (d) $0.5\hat{i} + 0.866\hat{i}$

2.2 One-dimensional unsteady-state heat-transfer equation for a sphere with heat generation at the rate of $\overline{q}$ can be written as

(a) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ (b) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$

(c) $\frac{\partial^2 T}{\partial r^2} = \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ (d) $\frac{\partial^2}{\partial r^2}(rT) + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$

2.3 In case of a one-dimensional heat conduction in a medium with constant properties, T is the temperature at position x, at time t. Then $\frac{\partial T}{\partial t}$ is proportional to

(a) $\frac{T}{x}$ (b) $\frac{\partial T}{\partial x}$ (c) $\frac{\partial^2 T}{\partial x \partial t}$ (d) $\frac{\partial^2 T}{\partial x^2}$

2.4 The temperature distribution across a 50 cm thick plane wall (k = 1.5 W/m K) is given by $T(x) = 40x^2 - 250x + 250$
The outer surface is exposed to a fluid at 60°C. There is no internal energy generation.

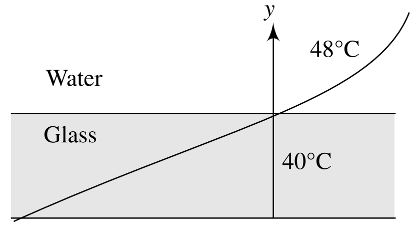
(A) The convection heat-transfer coefficient, h (W/m^2 K) is

(a) 17.5 (b) 21.0 (c) 5.25 (d) 4.2

(B) The rate of change of thermal energy storage per unit area (W/m^2) is

(a) 47.5 (b) 0 (c) 60.0 (d) –60.0

2.5 Heat is being transferred by convection from water at 48°C to a glass plate whose surface that is exposed to the water is at 40°C. The thermal conductivity of water is 0.6 W/m K and the thermal conductivity of glass is 1.2 W/m K. The spatial gradient of temperature in the water at the water glass interface is $\dfrac{dT}{dy} = 1 \times 10^4$ K/m.



(A) The value of the temperature gradient in the glass at the water–glass interface in K/m is

(a) -2×10^4 (b) 0.0

(c) 0.5×10^4 (d) 2×10^4

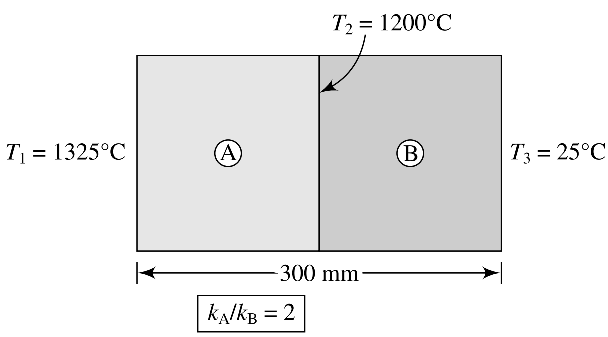
(B) The heat-transfer coefficient h in W/m^2 K is

(a) 0.0 (b) 4.8 (c) 5 (d) 750

2.6. If the ratio of thermal conductivities k_A/k_B is 21.3 and the ratio of densities ρ_A/ρ_B is 2.45, and the ratio of specific heat capacities $C_{p,A}/C_{p,B}$ is 0.27, the ratio of thermal diffusivities α_A/α_B is

(a) 1.45 (b) 62.8 (c) 32.2 (d) 10.7

2.7 A wall as shown below is made up of two layers (A) and (B). The temperatures are also shown in the sketch. The ratio of thermal conductivities of the two layers is. What is the ratio of thicknesses of two layers?

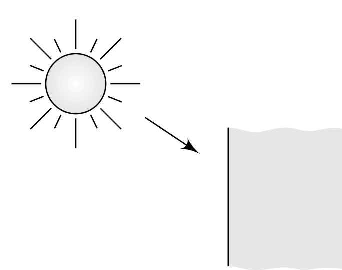


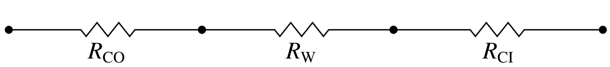
(a) 0.105 (b) 0.213 (c) 0.555 (d) 0.840

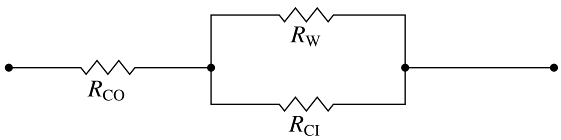
2.8 A composite slab has two layers of different materials having thermal conductivities k_1, and k_2. If each layer has the same thickness then what is the equivalent thermal conductivity of the slab?

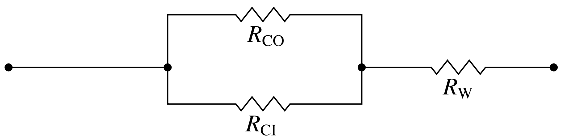
(a) $\dfrac{k_1 k_2}{(k_1 + k_2)}$ (b) $\dfrac{k_1 k_2}{2(k_1 + k_2)}$ (c) $\dfrac{2k_1}{(k_1 + k_2)}$ (d) $\dfrac{2k_1 k_2}{(k_1 + k_2)}$

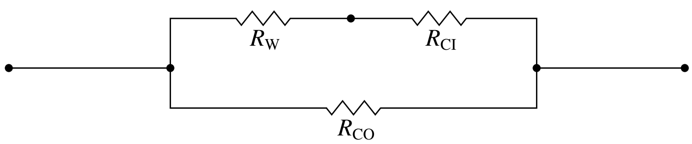
2.9 Solar energy is absorbed by the wall of a building as shown in the figure. Assuming that the ambient temperatures inside and outside are equal and considering steady state, the equivalent circuit will be as shown in the figure (Symbols: $R_{CO} = R_{\text{convection, outside}}$, $R_{CI} = R_{\text{convection, inside}}$, and $R_W = R_{\text{wall}}$)



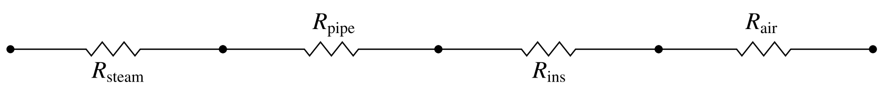
(a) 

(b) 

(c) 

(d) 

2.10 A pipe carrying saturated steam is covered with a layer of insulation and exposed to ambient air. The thermal resistances are as shown in the following figure:



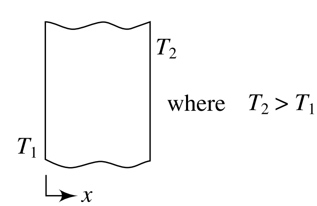
Which one of the following statements is correct in this regard?
(a) R_{steam} and R_{pipe} are negligible as compared to R_{ins} and R_{air}.
(b) R_{pipe} and R_{air} are negligible as compared to R_{ins} and R_{steam}.
(c) R_{steam} and R_{air} are negligible as compared to R_{pipe} and R_{ins}.
(d) No quantitative data is provided; therefore no comparison is possible.

2.11 A furnace has a 20 cm thick wall with thermal conductivity 0.8 W/m K. For the same heat loss from the furnace, what will be the thickness of the wall if the thermal conductivity of the material is 0.16 W/m K?
(a) 4 cm (b) 6.3 cm (c) 10 cm (d) 40 cm

2.12 Which of the following expressions gives the thermal resistance for heat conduction through a hollow sphere of radii r_1 and r_2?
(a) $\dfrac{4\pi k r_1 r_2}{r_2 - r_1}$ (b) $\dfrac{(r_2 - r_1)\ln r_2/r_1}{4\pi k}$ (c) $\dfrac{r_2 - r_1}{4\pi k r_1 r_2}$ (d) $\dfrac{4\pi k(r_2 - r_1)}{r_1 r_2}$

2.13 A furnace wall is 10 cm thick and has a thermal conductivity of 0.1 kW/m K. Inner temperature is maintained at 525°C, while the surrounding temperature outside the furnace is 25°C. If the surface area of the furnace is 20 m^2, the heat flux through the wall is
(a) 50 kW/m^2 (b) 500 kW/m^2 (c) 1000 kW/m^2 (d) 10000 kW/m^2

2.14 Consider one-dimensional steady-state heat conduction, without heat generation, in a plane wall; with boundary conditions as shown in the figure. The conductivity of the wall is given by $k = k_0 + bT$; where k_0 and b are positive constants, and T is the temperature.
As x increases, the temperature gradient (dT/dx) will



(a) remain constant (b) be zero
(c) increase (d) decrease

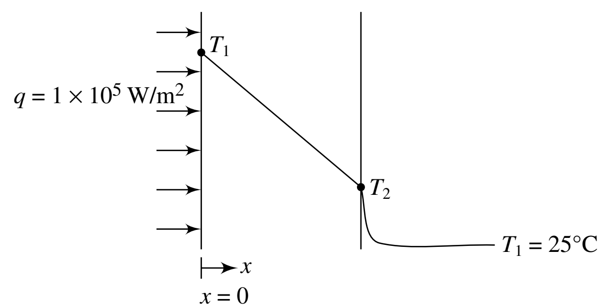
2.15 As the temperature increases, the thermal conductivity of a gas
(a) increases
(b) decreases
(c) remains constant
(d) increases up to a certain temperature and then decreases

2.16 Consider a long cylindrical tube of inner and outer radii, r_i and r_o, respectively, L and thermal conductivity, k. Its inner and outer surfaces are maintained at T_i and T_o, respectively ($T_i > T_o$). Assuming one dimensional steady state heat conduction in the radial direction, the thermal resistance in the wall of the tube is

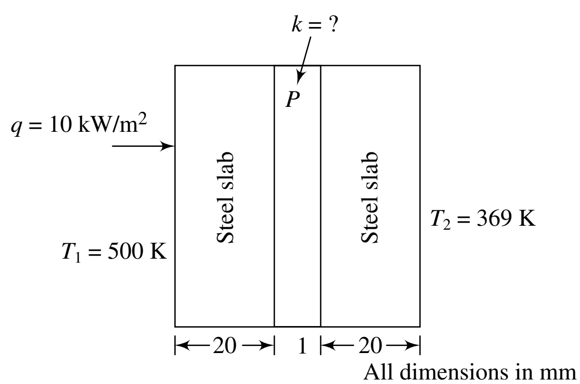
(a) $\dfrac{1}{2\pi kL}\ln(r_i/r_o)$ (b) $\dfrac{1}{2\pi kL}\ln(r_o/r_i)$ (c) $L/\pi r_i\, k$ (d) $\dfrac{1}{4\pi kL}\ln(r_o/r_i)$

● Fill in The Blanks

2.1 Consider one-dimensional steady-state heat conduction across a wall (as shown in figure) of thickness 30 mm and thermal conductivity 15 W/m K. At $x = 0$, a constant heat flux, $q = 1 \times 10^5$ W/m^2 is applied. On the other side of the wall, heat is removed from the wall by convection with a fluid at 25°C and heat-transfer coefficient of 250 W/m^2 K. The temperature (in °C), at $x = 0$ is ______

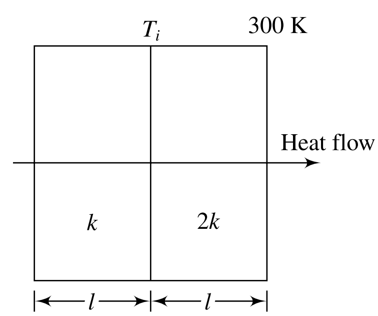


2.2 A material P of 1 mm thickness is sandwiched between two steel slabs, as shown in the figure. A heat flux 10 kW/m^2 is supplied to one of the steel slabs as shown. The boundary temperatures of the slabs are indicated in the figure. Assume thermal conductivity of this steel is 10 W/m K. Considering one-dimensional steady-state heat conduction for the configuration, the thermal conductivity (k, in W/m K) of material P is ______



2.3 An amount of 100 kW of heat is transferred through a wall in steady state. One side of the wall is maintained at 127°C and the other side at 27°C. The entropy generated (in W/K) due to the heat transfer through the wall is ______

2.4 Heat transfer through a composite wall is shown in the figure. Both the sections of the wall have equal thickness (l). The conductivity of one section is k and that of the other is $2k$. The left face of the wall is at 600 K and the right face is at 300 K. The interface temperature T_i (in K) of the composite wall is _____



2.5 A plane wall has a thermal conductivity of 1.15 W/m K. If the inner surface is at 1100°C and the outer surface is at 350°C then the design thickness (in metre) of the wall to maintain a steady heat flux of 2500 W/m^2 should be ______

Answers

Multiple-Choice Questions

2.1	(c)	**2.2**	(b)	**2.3**	(d)	**2.4(A)**	(d)	**(B)**	(c)	**2.5(A)**	(c)
(B)	(d)	**2.6**	(c)	**2.7**	(b)	**2.8**	(d)	**2.9**	(a)	**2.10**	(a)
2.11	(a)	**2.12**	(c)	**2.13**	(b)	**2.14**	(c)	**2.15**	(a)	**2.16**	(b)

Fill in the Blanks

2.1	625	**2.2**	0.1	**2.3**	83.3	**2.4**	400	**2.5**	0.345

REVIEW QUESTIONS

2.1 What is meant by one-dimensional, steady-state heat conduction?

2.2 Write the fundamental relations used for transforming Cartesian coordinates into the cylindrical and spherical coordinates.

2.3 Differentiate between initial and boundary conditions. How many boundary conditions and initial conditions are needed to solve a general heat-conduction equation?

2.4 Discuss different types of boundary conditions applied to heat-conduction problems.

2.5 What is meant by geometric and thermal symmetry?

2.6 State the conditions under which the general heat-conduction equation reduces to the (a) Poisson equation, (b) Fourier, equation, and (c) the Laplace equation.

2.7 Differentiate between isotropic and anisotropic materials.

2.8 How does the thermal conductivity of a pure metal and of a non-metallic solid vary with temperature?

2.9 Is the thermal conductivity of a pure metal always higher than that of its alloys? What is the effect of the amount of each constituent on the thermal conductivity of an alloy?

2.10 Discuss the thermal conductivity of a gas or vapour as a function of temperature and pressure? What is the relation between the thermal conductivity and the molecular weight of a gas?

2.11 What is the approximate range of thermal conductivity for solid metals, non-metallic solids, liquids, and for gases and vapours?

2.12 What is the effect of moisture on the thermal conductivity of building materials?

2.13 How does the thermal conductivity of wood vary with moisture content?

2.14 Explain the physical significance of thermal capacity and thermal diffusivity.

2.15 Explain the concept of *thermal resistance* and the analogy between heat flow and electricity.

2.16 What is meant by *thermal contact resistance*?

2.17 How is the combined (convection plus radiation) heat-transfer coefficient defined? What convenience does it offer in heat-transfer calculations?

2.18 Why are the convection and the radiation resistances at a surface in parallel instead of being in series? Explain how you would determine (a) the single equivalent heat-transfer coefficient, and (b) the equivalent thermal resistance. Assume the ambient fluid and the surroundings to be at the same temperature.

2.19 Show that under steady operating conditions, the radial flow of heat from the surroundings in a thick long cylinder is given by the expression, $\dot{Q} = \dfrac{2\pi kL(T_1 - T_2)}{\ln(r_2/r_1)}$

where T_1 and T_2 are the temperatures inside and outside the thick cylinder which has inner and outer radii r_1 and r_2, length L, and thermal conductivity k.

2.20 What is meant by *overall heat-transfer coefficient*? Write an expression for the overall heat-transfer coefficient, based on the outer area for the heat transfer from a hot fluid flowing in a pipe to a cold fluid to which the pipe is exposed, in terms of convection coefficients and the radii of the pipe.

2.21 Consider one-dimensional heat conduction through a cylindrical rod of diameter D and length L. What is the heat-transfer area of the rod if (a) the lateral surfaces of the rods are insulated and (b) the top and bottom surfaces of the rod are insulated?

2.22 Explain the concept of *log mean area* for a cylindrical shell.

2.23 What is the critical radius of insulation? How is it defined for a cylinder?

2.24 Explain the term *critical insulation thickness*. Derive an expression for the critical radius of insulation for a sphere.

2.25 Obtain an expression for the temperature distribution $T(x)$ in a plane wall of thickness L having uniform wall temperatures T_1 and T_2. The thermal conductivity varies linearly with temperature, $k = k_o\,(1 + bt)$.

2.26 Obtain an expression for the heat-transfer rate per unit length of a hollow cylinder having inner and outer radii r_i and r_o with corresponding surface temperatures T_i and T_o. The thermal conductivity varies linearly with temperature $k = k_o\,(1 + bt)$.

PRACTICE PROBLEMS

(B) Plane Wall Bounded By Specified Fluid Temperature

2.1 It is required to reduce heat loss from a slab by doubling the thickness of brick work. The temperature of inner surface of brick work is 500°C and ambient air is at 30°C. The temperature of outer surface of initial brick work was 200°C. Calculate the percentage reduction in heat loss per m^2 because of doubling the thickness. **[39%]**

2.2 The wind chill, which is experienced on a cold, windy day, is related to increased heat transfer from exposed human skin to the surrounding atmosphere. Consider a layer of fatty tissue that is 3-mm thick and whose interior surface is maintained at a temperature of 36°C. On a calm day the convection heat-transfer coefficient at the outer surface is 25 W/m^2 K but with 30 km/h winds it reaches 65 W/m^2 K. In both cases, the ambient air temperature is –15°C. (a) What is the ratio of the heat loss per unit area from the skin for the calm day to that for the windy day? (b) What will be the skin's outer surface temperature for the calm day and for the windy day? (c) What temperature would the air have to assume on the calm day to produce the same heat loss occurring with the air temperature at –15°C on the windy day? Take k = 0.2 W/m K (for tissue fatty layer).

[(a) 0.552 (b) 22.1°C, 10.8°C (c) 56.3°C]

2.3 Hot gases at 980°C flow past the upper surface of the blade of a gas turbine and the lower surface is cooled by air bled off the compressor. The convective heat-transfer coefficient at the upper and lower surfaces are estimated to be 2830 and 1415 W/m^2 °C respectively. The blade material has a thermal conductivity of 11.6 W/m °C. If the temperature of the blade is limited to 870°C, find the temperature of the cooling air. Consider the blade as a flat plate 0.115 mm thick and presume that steady-state conditions have been reached. **[619.14°C]**

(C) Composite Wall with Prescribed Boundary Temperatures

2.4 An exterior wall of a house may be approximated by a 10-cm layer of common brick [k = 0.7 W/m K] followed by a 4.0-cm layer of gypsum plaster [k = 0.48 W/m K]. What thickness of loosely packed rock-wool insulation [k = 0.065 W/m K] should be added to reduce the heat loss (or gain) through the wall by 80 percent? **[58.8 mm]**

(D) Composite Wall Bounded By Fixed Fluid Temperatures

2.5 A composite insulating wall has three layers of material held together by a 3-cm diameter aluminium rivet per 0.1 m^2 of surface. The layers of material consist of 10-cm-thick brick with hot surface at 200°C, 1-cm-thick timber with cold surface at 10°C. These two layers are interposed by a third layer of insulating material 25-cm-thick. The conductivities of the materials are:

k (*brick*) = 0.93 W/m K k (*insulation*) = 0.12 W/m K

k (*wood*) = 0.175 W/m K k (*aluminium*) = 204 W/m K

Assuming one-dimensional heat flow, calculate the percentage increase in heat transfer rate due to rivet. **[900%]**

2.6 A square plate heater (15 cm × 15 cm) is inserted between two slabs. Slab A is 2-cm-thick (k = 50 W/m K) and slab B is 1-cm-thick (k = 0.2 W/m K). The outside heat transfer coefficients for slabs A and B are 200 and 50 W/m^2 K. The surrounding air temperature is 25°C. If the rating of the heater is 1 kW, determine (a) the maximum temperature in the system, (b) the outer surface temperatures of the two slabs. Draw the equivalent thermal circuit of the system.

[(a) 136.4°C (b) 128.15°C, 56.83°C]

2.7 A steam-to-liquid heat exchanger surface of 3200 cm^2 face area is constructed of 0.7-cm nickel with a 0.2-cm plating of copper on the steam side. The resistivity of a water scale deposit on the steam side is 0.0017 m^2 K/W and the steam and liquid surface conductances are 5465 W/m^2 K and 580 W/m^2 K respectively. The heated steam is at 111°C and the heated liquid is at 75°C. Calculate: (a) Overall steam-to-liquid heat transfer coefficient, (b) Temperature drop across the scale deposit, and (c) Temperature at the copper nickel interface.

Take $\boldsymbol{k}_{\text{(Copper)}}$ = 384 W/m K and $\boldsymbol{k}_{\text{(Nickel)}}$ = 58 W/m K. **[(a) 267.88 W/m^{2}K (b) 16.4°C (c) 92.79°C]**

2.8 A leading manufacturer of household appliances is proposing a self-cleaning oven design that involves use of a composite window separating the oven cavity from the room air. The composite is to consist of two high-temperature plastics (A and B) of thicknesses $L_A = 2L_B$ and thermal conductivities k_A = 0.15 W/m K and k_B = 0.08 W/m K. During the self-cleaning process, the oven wall and air temperatures, T_w and T_a, are 400°C, while the room air temperature T_∞ is 25°C. The inside convection and radiation heat-transfer coefficient h_i and h_r, as well as the outside convection coefficient h_o, are each approximately 25 W/m^{2}K. What is the minimum window thickness, $L = L_A + L_B$, needed to ensure a temperature that is 50°C or less at the outer surface of the window? **[62.7 mm]**

2.9 The inside temperature of a furnace wall, 200-mm-thick, is 1350°C. The mean thermal conductivity of the wall material is 1.35 W/m°C. The heat-transfer coefficient of the outside surface is a function of temperature difference and is given by $h = 7.85 + 0.08\ \Delta T$ where ΔT is the temperature difference between outside wall surface and surroundings. Determine the rate of heat transfer per unit area if the surroundings temperature is 40°C. **[7131.3 W/m^2]**

(E) Thermal Contact Resistance

2.10 Heat losses from the windows are to be reduced by covering them from inside with a polystyrene insulation (k_{ins} = 0.027 W/m K). Consider application of 25-mm-thick insulation panels to 6-mm-thick windows (k_w = 1.4 W/m K). The contact resistance between the glass and the insulation may be approximated as ($R_{t,c}$ = 0.02 m^2 K/W), while the convection coefficient at the outside surface of the window is nominally losing heat (h_o = 20 W/m^2 K). With the insulation, the convection coefficient at the inner surface is h_i = 2 W/m^2 K, and without the insulation it is h_i = 5 W/m^2 K. (a) What is the percentage reduction in heat loss associated with the use of insulation? (b) If the total surface area of the windows is A_s = 12 m^2, what are the heat losses associated with insulated and uninsulated windows for interior and exterior temperatures of $T_{\infty,i}$ = 20°C and $T_{\infty,o}$ = –12°C? (c) If the home is heated by gas furnace operating at an efficiency of η_f = 0.80 and the natural gas is priced at C_g = ₹ 1 per MJ, what is the daily saving associated with covering windows for 12 hours?

[(a) 83.05 % (b) 1510 W (c) ₹ 67.72]

(G) Variable Thermal Conductivity

2.11 A composite slab has two layers of 5 cm and 10 cm thickness. The thermal conductivities of the materials of these layers are temperature dependent and are prescribed by the relations:
k_1 = 0.05 (1 + 0.006 T) W/ m °C, k_2 = 0.04 (1 + 0.007 T) W/ m °C where T = temperature in degree centigrade. The inside and outside surface temperatures of the slab are maintained at 500°C and 200°C. Determine the steady state heat flux through the composite, and the interface temperature.

[424.27°C, 285.7 W/m^2]

2.12 A steam boiler furnace is made of a layer of fireclay 12.5-cm-thick and a layer of red brick 50-cm-thick. If the wall temperature inside the boiler furnace is 1100°C and that on the outside wall is 50°C, determine the amount of heat loss per m^2 of the furnace wall. It is desired to reduce the thickness of the red brick layer in this furnace to half by filling in the space between the two layers by diatomite whose k = [0.113 + 0.00023 T] W/m K. Calculate the thickness of the filling to ensure an identical loss of heat for the same outside and inside temperatures. Assume: k (for fire clay) = 0.533 W/m K, k (for red brick) = 0.7 W/m K

[1106.65 W/m^2, 9.32 cm]

(K) Composite Cylinders Bounded by Fluid Temperatures

2.13 A hot gas at 600°C flows through a long metal pipe of 15-cm outer diameter and 5-mm thick. From the standpoint of safety and reducing the heat loss from the pipe surface, mineral wool insulation (k = 0.02 W/m K) is wrapped around so that the exposed surface of the insulation is at a temperature of 60°C. Calculate the thickness of insulation required to achieve this temperature if the inside and outside surface heat-transfer coefficients are 50 and 20 W/m^2 K and the surrounding air temperature is 28°C. Also find the corresponding heat loss if the pipe length is 3 m. **[1085.7 W]**

(L) Critical Radius of Insulation (Cylinder)

2.14 A heat exchanger shell of 15 cm outside radius is to be insulated with glass wool of thermal conductivity 0.0825 W/m K. The temperature at the surface of the shell is 280°C and it can be assumed to remain constant after the layer of insulation has been applied to the shell. The convective film coefficient between the outside surface of slag wool (insulation material) and the surrounding air is 8 W/m^2 K. It is specified that the temperature at the outer surface of insulation must not exceed 30°C and the loss of heat per metre length of the shell should not be greater than 200 W. Would the slag wool serve the intended purpose of restricting the heat loss. If yes, what should be thickness of the insulating material to suit the prescribed conditions? **[12.6 cm]**

2.15 A copper wire of 0.5 mm radius is insulated uniformly with plastic (k = 0.5 W/m K) sheathing of 1-mm-thick. The wire is exposed to atmosphere at 30°C and the outside surface coefficient is 8 W/m^2 K. Find the maximum safe current carried by the wire, so that no part of the insulated plastic is above 75°C. For copper: Thermal conductivity (k) = 400 W/m K Specific electrical resistance (ρ) = 2×10^{-8} ohm m. Would the capacity of the wire to carry more current will increase or decrease with further addition of insulation? **[11.39 A]**

(M) Variable Thermal Conductivity

2.16 A reinforced concrete smoke stack with an inner diameter of 80 cm and an outer diameter of 130 cm is to be lined with a refractory on the inside. Determine the thickness of the refractory lining and the temperature of the outer surface of the smoke stack if the heat loss from the outer surface of the smoke stack does not exceed 2 kW per metre length of the stack and the temperature of the inner surface of the reinforced concrete smoke stack does not exceed 200°C. The temperature of the inner surface of the lining is 425°C.
k for refractory = (0.84 + 0.0006 T) W/m K, where T is in °C.
k for reinforced concrete = 1.1 W/m K **[20.65 cm, 59.5°C]**

(O) Spherical Shell Bounded by Known Fluid Temperatures

2.17 A spherical, thin-walled, metallic container is used to store liquid nitrogen at –196°C. The container has a diameter of 45 cm and is covered with an insulation 2.5 cm thick. Its outer surface is exposed to ambient air at 30°C. The heat transfer coefficient is 20 W/m^2 K. The latent heat of vaporization and density of liquid nitrogen are 200 kJ/kg and 800 kg/m^3 respectively. Determine: (a) the rate of heat transfer to the liquid nitrogen, and (b) the liquid boil-off rate. Take k (insulation) = 0.0017 W/m K. What is the loss of the cryogenic fluid per day in litres? **[6 litres/day]**

(P) Composite Sphere with Fixed Boundary Temperatures

2.18 A spherical vessel of 50-cm radius contains a liquefied gas at 187°C. It has two jackets of lagging each 10-cm-thick. The thermal conductivities of the inner and outer layers are in the ratio of 2:3. The temperature of the outermost surface is 15°C. If the heat leakage rate into the liquid gas is 225 W, find the thermal conductivities of the lagging material. **[0.0436 W/m K]**

2.19 A cylindrical tank with hemispherical ends is used to store liquid oxygen at –183°C. The diameter of the tank is 1.5 m and the total length is 8 m. The tank is covered with a 10-cm-thick layer of insulation. Determine the thermal conductivity of the insulation so that the boil-off rate does not exceed 10.8 kg/h. The latent heat of vaporization of liquid oxygen is 214 kJ/kg. Assume that the outer surface temperature of the insulation is 27°C and that the thermal resistance of the wall of the tank is negligible. **[0.00752 W/m °C]**

(Q) Composite Sphere Bounded by Known Fluid Temperatures

2.20 A small hemispherical oven is built of inner layer of insulating 12.5-cm-thick, firebrick, and an outer covering of 85% magnesia, 40-mm-thick. The inner surface of the oven is at 800°C and the heat transfer coefficient for the outer surface is 10 W/m^2 °C; the room temperature is 20°C. Calculate (a) the rate of heat loss through the hemisphere, (b) the interface temperatures, and (c) the temperature at the mid-thickness of the layer of firebrick if the inside diameter is 1.2 m. Take the thermal conductivities of firebrick and 85% magnesia as 0.31 and 0.05 W/m °C, respectively.
[(a) 1.93 kW (b) 515.3°C (c) 644.2°C]

3

One-Dimensional Steady-State Heat Conduction with Heat Generation

3.1 ❑ INTRODUCTION

One-dimensional, steady-state heat conduction has been studied in the previous chapter for a few simple geometries in which the heat conducted through the solid came from outside but there was no internal heat generation in the medium. However, there are many cases commonly encountered in practice where there is thermal energy generation *within* the medium resulting in a rise in temperature throughout the medium. One would be interested in finding the temperature distribution within the body as well as the heat-transfer rate at any specified location.

We will consider steady-state situations for some geometries including a plane wall, a long solid cylinder, a cylindrical wall (*hollow cylinder*) a solid sphere, and a spherical wall (*hollow sphere*) with heat sources. In steady state, no thermal energy can be stored in the solid and, hence, the heat generated within the solid must be conducted to the surface and then dissipated to the surroundings by either convection or radiation or both. *The heat source is usually considered uniformly distributed throughout the material.* The source strength or the rate of volumetric internal thermal energy generation is usually expressed as $\overline{q}(W/m^3)$. In most cases, $\overline{q}$ can be considered constant and uniform throughout the solid. However, in some situations, $\overline{q}$ may have a different value at each location in the solid or may vary with temperature. We will also analyze those situations in which the heat sources are non-uniformly distributed.

3.2 ❑ APPLICATIONS

Several problems encountered in heat transfer require an analysis that takes into account the *generation* or *absorption* of heat within a body. Some of the practical applications which involve the conversion of some form of energy into thermal energy in the medium are

- Fission or fusion in nuclear reactors
- Ohmic heating (*resistance heating*) in electrical current-carrying conductors
- Chemical processing industries
- Dielectric heating
- Electronic cooling
- Exothermic chemical reaction in combustion processes

- Curing (drying) and setting of concrete
- Ripening of fruits
- Biological decay processes
- Microwave ovens
- Absorption of solar radiation by water

It is to be noted that chemical reactions can also be *endothermic* (heat absorption) besides being *exothermic* (heat generation).

3.3 ❑ HEAT GENERATION IN A SOLID MEDIUM

There is an increase in the temperature of the medium as a result of absorption of the heat generated within. In steady state, no energy can be stored thermally in the solid, so that all the energy generated must be conducted to the surface and then transferred to the surroundings by either convection or radiation or both (Fig. 3.1). Under steady operating conditions, the temperature of the solid at any location no longer changes and the energy balance can then be expressed as

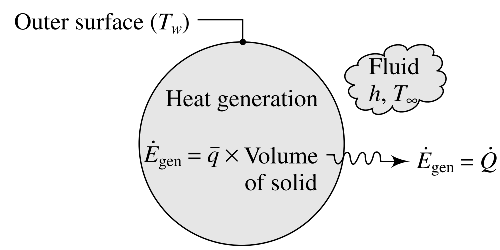


Fig. 3.1 *The heat generated in steady state must leave the solid through the exposed surface to the surrounding fluid.*

$$\begin{pmatrix}\text{Rate of heat transfer } \textit{from} \\ \text{the solid } \textit{to the} \text{ surroundings}\end{pmatrix} = \begin{pmatrix}\text{Rate of heat generation} \\ \textit{within} \text{ the solid}\end{pmatrix}$$

or $$\boxed{\dot{Q} = \dot{E}_{gen} \quad \text{i.e.} \quad \bar{q}\,\forall} \quad \text{(W)} \tag{3.1}$$

We are primarily interested in developing expressions for evaluating the maximum temperature T_{max} that occurs in the medium, and the surface (*wall*) temperature, T_w for common geometries like a *large plane wall*, *long cylinders,* and *spheres*.

3.4 ❑ PLANE WALL WITH UNIFORM INTERNAL HEAT GENERATION

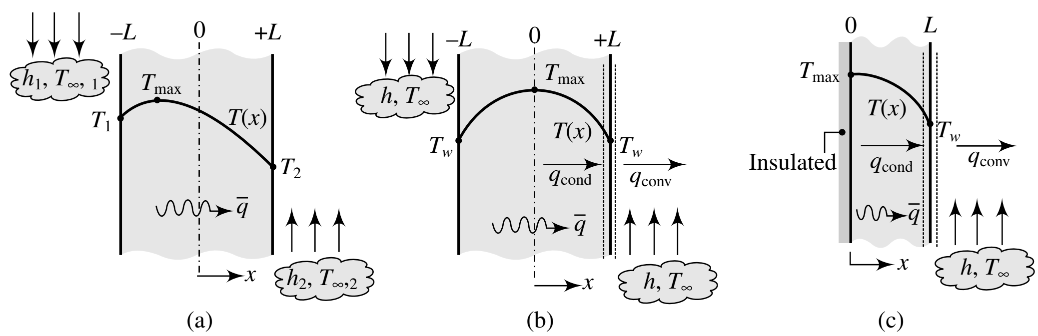


Fig. 3.2 *A plane wall with uniform internal heat generation. (a) Asymmetrical boundary conditions, (b) Symmetrical boundary conditions, (c) Insulated surface at one end*

Consider a large plane wall (*infinite slab*) that experiences heat generation under the following assumptions.

Assumptions

- One-dimensional conduction (*in the x-direction*). Thickness is much smaller than the dimensions in the *y*- and *z*-directions.
- Steady operating conditions since there is no change in temperature at any point within the wall with time although temperatures at different locations within the wall may well be different.
- Uniform thermal energy generation in the wall.
- The material of the wall is homogeneous (same properties with respect to *location*) and isotropic (same properties with respect to *direction*).

The *temperature distribution* within the wall and the *heat-transfer rate* at any point are to be determined. The appropriate differential equation in Cartesian coordinates can be expressed as

$$\underbrace{\frac{\partial^2 T}{\partial x^2}}_{\text{One-dimensional}} + \cancelto{0}{\frac{\partial^2 T}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 T}{\partial z^2}} + \frac{\overline{q}}{k} = \underbrace{\cancelto{0}{\frac{1}{\alpha}\frac{\partial T}{\partial t}}}_{\text{Steady-state}}$$

Heat conduction equation is thus simplified to

$$\boxed{\frac{d^2T}{dx^2} + \frac{\overline{q}}{k} = 0}$$

Integrating this second-order differential equation once, with respect to *x*, we get

$$\frac{dT}{dx} = \frac{-\overline{q}\,x}{k} + C_1$$

Integrating again, the general expression for temperature distribution is given by

$$\boxed{T(x) = \frac{-\overline{q}\,x^2}{2k} + C_1 x + C_2} \tag{3.2}$$

The desired solution for variation of temperature within the medium as a function of *x* can be determined from the appropriate boundary conditions to evaluate the two arbitrary integration constants C_1 and C_2. Specific cases are analyzed as shown in Figures 3.2 (a), (b), and (c).

We note that in cases (a) and (b), the coordinate system is placed at the middle of the plane wall of thickness $2L$ ($x = 0$) and *x* to the right of the centreline is taken to be *positive* while that to the left *negative*. In the case (c), the wall thickness is *L* and *x* is measured from the left end.

Case (a): Plane Wall of thickness 2L with Heat Generation and Different Surface Temperatures (Asymmetrical boundary conditions) Figure 3.2(a) illustrates the geometry we are working with. The asymmetrical boundary conditions (of the *first* kind) are

At $x = -L$, $T = T_1$

At $x = +L$, $T = T_2$

The application of the *first* boundary condition to Eq. (3.2) gives

$$T_1 = -\frac{\overline{q}L^2}{2k} + C_1(-L) + C_2$$

and the application of the *second* boundary condition results in

$$T_2 = -\frac{\overline{q}L^2}{2k} + C_1 L + C_2$$

Subtracting one from the other,

$$T_2 - T_1 = 2C_1 L$$

$$\Rightarrow \qquad \boxed{C_1 = \frac{T_2 - T_1}{2L}} \quad \text{and} \quad \boxed{C_2 = T_1 + \frac{\overline{q}L^2}{2k} + \frac{(T_2 - T_1)}{2} = \frac{\overline{q}L^2}{2k} + \left(\frac{T_1 + T_2}{2}\right)}$$

Substituting for C_1 and C_2 in Eq. (3.2), the temperature distribution is given by

$$T(x) = -\frac{\overline{q}x^2}{2k} + \left(\frac{T_2 - T_1}{2}\right)\frac{x}{L} + \frac{\overline{q}L^2}{2k} + \left(\frac{T_2 + T_1}{2}\right)$$

or

$$\boxed{T(x) = \frac{\overline{q}}{2k}(L^2 - x^2) + \frac{(T_2 - T_1)}{2}\frac{x}{L} + \frac{(T_1 + T_2)}{2}} \tag{3.3}$$

This is a *parabolic temperature distribution*. Note that the variation of temperature within the wall is not symmetrical. Hence, the maximum temperature will not be at the midplane or the line of symmetry ($x = 0$).

To determine the location of maximum temperature, one can differentiate the expression for $T(x)$ with respect to x and equate the resulting derivative to zero.

For finding $x_{\max}$ corresponding to $T_{\max}$, $\dfrac{dT}{dx} = 0$

i.e.,

$$\frac{d}{dx}\left[\frac{\overline{q}}{2k}(L^2 - x^2) + \frac{(T_2 - T_1)}{2}\frac{x}{L} + \left(\frac{T_1 + T_2}{2}\right)\right] = 0$$

or

$$\frac{\overline{q}}{2k}(0 - 2x) + \left(\frac{T_2 - T_1}{2L}\right)(1) + 0 = 0 \quad \text{or} \quad \frac{\overline{q}x}{k} = \frac{T_2 - T_1}{2L}$$

$$\therefore \qquad \boxed{x_{\max} = \frac{k}{\overline{q}}\left(\frac{T_2 - T_1}{2L}\right)} \tag{3.4}$$

By substituting this value of $x_{\max}$ in the expression for $T(x)$, the value of $T_{\max}$ can be easily obtained. Location of $T_{\max}$ is to the *left* of the midplane ($x = 0$), i.e., $x_{\max}$ will be *negative* if $T_1 > T_2$ and *positive* for $T_2 > T_1$, $x_{\max}$ occurring to the *right* of the line of symmetry. And, as the value of $\overline{q}$ increases, the maximum approaches the centreline.

Total heat generated within the wall equals:

$$\boxed{\dot{E}_{\text{gen}} = \overline{q}\underbrace{(A2L)}_{\text{volume}} = \dot{Q}_{\text{left}} + \dot{Q}_{\text{right}}} \tag{3.5}$$

The surface temperatures T_1 and T_2 and fluid temperatures $T_{\infty 1}$ and $T_{\infty 2}$ can be related by surface energy balance:

$$\begin{pmatrix}\text{Heat generated } \textit{within} \\ \text{the medium}\end{pmatrix} = \begin{pmatrix}\text{Heat conducted } \textit{through} \\ \text{the surface}\end{pmatrix} = \begin{pmatrix}\text{Heat carried away } \textit{from} \text{ the surface} \\ \text{to the surrounding fluid by convection}\end{pmatrix}$$

Heat generated between $x = 0$ to $x = x_{max}$ has to go to the *left* and the heat generated in the volume between $x = x_{max}$ and $x = L$ has to go to the *right*, since no heat can cross the plane of maximum temperature (*zero slope*).

$$\dot{Q}_{\text{left}} = \overline{q}\, A\, x_{\max} = +\, kA_c \left.\frac{dT}{dx}\right|_{x=-L} = h_1\, A\, (T_1 - T_{\infty 1})$$

$$\Rightarrow \qquad \boxed{T_1 = T_{\infty 1} + \frac{\overline{q}\, x_{\max}}{h_1}} \tag{3.6a}$$

$$\dot{Q}_{\text{right}} = \overline{q}\, A\, (L - x_{\max}) = -\, kA_c \left.\frac{dT}{dx}\right|_{x=+L} = h_2\, A\, (T_2 - T_{\infty 2})$$

$$\Rightarrow \qquad \boxed{T_2 = T_{\infty 2} + \frac{\overline{q}\, (L - x_{\max})}{h_2}} \tag{3.6b}$$

where h_1 and h_2 are the convection coefficients at the *left* and *right* convective surfaces.

Case (b): Plane Wall of Thickness 2L with Heat Generation and Equal Surface Temperatures (Symmetrical Boundary Conditions)

The general solution is known to be

$$T = -\frac{\overline{q}}{k}\frac{x^2}{2} + C_1 x + C_2$$

The arbitrary constants C_1 and C_2 can be found with reference to the two boundary conditions as follows.

***BC(I)*:** $\qquad \left.\dfrac{dT}{dx}\right|_{x=0} = 0$

Since $\qquad \dfrac{dT}{dx} = -\dfrac{\overline{q}x}{k} + C_1$

At $x = 0$, $\qquad 0 = -\dfrac{\overline{q}(0)}{k} + C_1 \quad \Rightarrow \quad \boxed{C_1 = 0}$

BC (II): $\qquad T(\pm L) = T_w = \dfrac{-\overline{q}L^2}{2k} + C_2 \quad \Rightarrow \quad \boxed{C_2 = T_w + \dfrac{\overline{q}L^2}{2k}}$

Substituting for C_1 and C_2, the temperature profile is obtained as

$$T(x) = \frac{-\overline{q}x^2}{2k} + T_w + \frac{\overline{q}L^2}{2k}$$

or

$$\boxed{T(x) = T_w + \frac{\overline{q}}{2k}(L^2 - x^2)}$$

or

$$T(x) - T_w = \left(\frac{\overline{q}L^2}{2k}\right)\left[1 - \left(\frac{x}{L}\right)^2\right]$$

This can be expressed in neat *dimensionless* form:

$$\boxed{\frac{T - T_w}{\overline{q}L^2 / k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2\right]} \tag{3.7}$$

where L is the *half-thickness* of the wall.

Clearly, the temperature distribution is *parabolic* and *symmetric* about $x = 0$, the plane of symmetry. The maximum temperature naturally occurs at the midplane.

i.e., $T = T_{max}$ at $x = 0$

$$\boxed{T_{max} = T_w + \frac{\overline{q}L^2}{2k}} \tag{3.8}$$

In the *dimensionless* form, the temperature variation can be expressed as

$$\boxed{\frac{T(x) - T_w}{T_{max} - T_w} = \frac{L^2 - x^2}{L^2} = \left[1 - \left(\frac{x}{L}\right)^2\right]} \tag{3.9}$$

- It may be noted that the temperature gradient *dT/dx* (*and, hence, the heat-transfer rate Q*) for a plane wall *with heat generation* depends on the thermal conductivity of the material while it is *independent* of it in the case of a plane wall *without heat generation*.

Convective Boundary Conditions on the Two Faces In many practical cases, the heat generated within the medium is conducted to the surface which may be exposed to the flowing fluid. Heat is thus carried away from the surface to the surrounding fluid characterized by the heat-transfer coefficient h, and the ambient fluid temperature T_∞. Applying the surface energy balance at the two faces ($x = \pm L$), we have

$$-kA\left.\frac{\mathrm{dT}}{\mathrm{dx}}\right|_{x=\pm L} = hA(T_w - T_\infty)$$

or $$-k\left[-\frac{\overline{q}L}{k}\right] = h(T_w - T_\infty) \quad \text{or} \quad T_w - T_\infty = \frac{\overline{q}L}{h}$$

$\Rightarrow$ $$\boxed{T_w = T_\infty + \frac{\overline{q}L}{h}} \tag{3.10}$$

Substituting this value in the expression for temperature distribution, we have

$$T(x) = T_\infty + \frac{\overline{q}L}{h} + \frac{\overline{q}}{2k}(L^2 - x^2)$$

or $$\boxed{T - T_\infty = \frac{\overline{q}}{2k}(L^2 - x^2) + \frac{\overline{q}L}{h}} \tag{3.11}$$

The maximum temperature will obviously occur at the midplane, i.e., at $x = 0$,

$$-kA\left.\frac{dT}{dx}\right|_{x=\pm L} = h(T_w - T_\infty)$$

Hence

$$\boxed{T_{max} - T_\infty = \frac{\overline{q}L^2}{2k} + \frac{\overline{q}L}{h}} \qquad \boxed{T_{max} = T_\infty + \overline{q}L = \left[\frac{L}{2k} + \frac{1}{h}\right]} \tag{3.12}$$

Case (c): Plane Wall of Thickness L with Uniform Heat Generation with One Surface Insulated Consider a plane wall of thickness L, with constant thermal conductivity k, and one of the sides (say, left side) *insulated,* in which heat is generated at a uniform note of $\overline{q}$ per unit volume. The other side of the wall is at a temperature of T_w.

We have seen that in the case of a plane wall of thickness $2L$ with common surface temperature, the maximum temperature occurs at the midplane ($x = 0$). The temperature gradient there is zero and no heat can pass through it. The midplane can, therefore, be looked upon as an adiabatic (*insulated*) surface. This is analogous to a case in which the plane wall is of thickness L (*not 2L*) with one surface insulated (where T_{max} will occur) and the other at the specified temperature, T_w.

Note that in this case, L is *not* the *half-thickness* but the *full thickness* of the wall.

The general solution for temperature distribution is

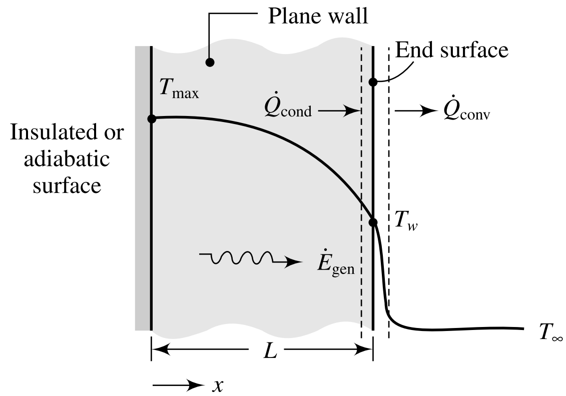


Fig. 3.3 *Temperature profile in a plane wall with internal heat generation with one face insulated and the other exposed to convection*

$$T = \frac{-\bar{q}x^2}{2k} + C_1 x + C_2$$

C_1 and C_2 are obtained by applying the boundary conditions. For the present case:

***BC (I)*:** At $x = 0$, well-insulated surface, i.e., $-kA\ (dT/dx) = 0$, and since k and A are not zero, dT/dx must be zero).

$$\frac{dT}{dx} = \frac{-\bar{q}x}{k} + C_1$$

Then, applying BC (I), we get : $\boxed{C_1 = 0}$

Then, $\dfrac{dT}{dx} = -\dfrac{\bar{q}x}{k}$ and $T(x) = \dfrac{-\bar{q}x^2}{2k} + C_2$

***BC (II)*:** At $x = L$, $T = T_w$, from **BC(II),**

$$\boxed{C_2 = T_w + \frac{\bar{q}L^2}{2k}}$$

Substituting for C_1 and C_2, the temperature variation is given by

$$\boxed{T(x) = T_w + \frac{\bar{q}}{2k}(L^2 - x^2)} \tag{3.13}$$

Maximum Temperature T_{max} must occur on the insulated left surface of the wall since heat being generated in the wall is constrained to flow from left face to right face. Putting $x = 0$ in Eq. (3.13),

$$\boxed{T_{max} = T_w + \frac{\bar{q}L^2}{2k}} \tag{3.14}$$

Substituting for T_w from Eq. (3.3),

$$\boxed{T_{max} = T_\infty + \frac{\bar{q}L}{h} + \frac{\bar{q}L^2}{2k}} \tag{3.15}$$

Equation (3.10) gives T_{max} in terms of the fluid temperature, T_∞.

From Eq. (3.13) and (3.14), we can write

$$\boxed{\frac{T(x)-T_w}{T_{\max}-T_w}=\frac{L^2-x^2}{L^2}=1-\left(\frac{x}{L}\right)^2} \tag{3.16}$$

Heat-transfer rate,

$$\dot{Q}=\dot{E}_{\text{gen}}=\overline{q}\left(\frac{W}{m^2}\right)\times A(m^2)\times L(m)$$

Also, $$\dot{Q}=\dot{Q}_{\text{cond}}=-kA\left.\frac{dT}{dx}\right|_{x=L}=-kA\left(\frac{-\overline{q}L}{k}\right)=\overline{q}AL$$

$\therefore$ $$\dot{Q}_{\text{conv}}=hA(T_w-T_\infty)=\dot{Q}_{\text{cond}}=\dot{E}_{\text{gen}}=\overline{q}\times\text{volume}=\overline{q}AL$$

As $h \to \infty$, $T_{(x=L)}$, i.e., $T_w \approx T_\infty$

This implies that the *surface resistance* between the wall surface and the fluid (R_{conv} = 1/hA) is zero, and the surface temperature equals the ambient temperature. Also, as $h \to 0$, $T(x)$ becomes *infinite*. The physical significance can be better appreciated if one looks at the boundary condition at $x = L$, $\frac{dT}{dx} \to 0$ as $h \to 0$.

At $$x = 0,\ \frac{dT}{dx} = 0.$$

This means that both the end faces are *insulated*. With no scope for escape, the heat generated will continuously increase the temperature with no steady-state solution.

3.5 ❑ LONG CYLINDER (CIRCULAR ROD) WITH INTERNAL HEAT GENERATION AND CONSTANT SURFACE TEMPERATURE

Internal thermal energy generation can occur in many radial geometries, for instance, a *current carrying electrical wire* or a *fuel element in a nuclear reactor*. Consider a long solid cylinder of radius R (radius much smaller compared to its length) in which thermal energy is generated internally per unit volume at a uniform rate of $\overline{q}$. Under steady-state conditions, the heat so generated equals the convective heat-transfer rate from the outer surface of the cylinder to the surrounding fluid. The formulation of the problem can be explained using Fig. 3.3.

The appropriate differential equation is

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dT}{dr}\right]+\frac{\overline{q}}{k}=0 \quad \Rightarrow \quad \frac{d}{dr}\left[r\frac{dT}{dr}\right]=-\frac{\overline{q}r}{k}$$

Integrating once, we get

$$r\frac{dT}{dr}=-\frac{\overline{q}r^2}{2k}+C_1$$

Integrating once again, we have

$$T(r)=-\frac{\overline{q}r^2}{4k}+C_1\,ln\,r+C_2$$

The boundary conditions are

$$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

and $T_{(r=R)} = T_w$

The first boundary condition indicates that the temperature gradient is zero as the centreline of a solid cylinder is a line of symmetry for the temperature distribution. You may recall that we had invoked the same boundary condition at the centre (*midplane*) of a plane wall having symmetrical boundary conditions. Applying the *first* boundary condition,

i.e., $$\left.\frac{dT}{dx}\right|_{r=0} = 0 \quad \text{yields} \quad \boxed{C_1 = 0}$$

Application of the surface boundary condition $T(R) = T_w$, gives

$$\boxed{C_2 = T_w + \frac{\overline{q}R^2}{4k}}$$

The temperature distribution is given by

$$\boxed{T(r) = T_w + \frac{\overline{q}R^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right]} \tag{3.17}$$

At the centreline ($r = 0$), the temperature will be maximum.

$$T(0) = T_{\max} = T_w + \frac{\overline{q}R^2}{4k} \quad \text{or} \quad T_{\max} = T_w + \frac{\overline{q}R^2}{4k}$$

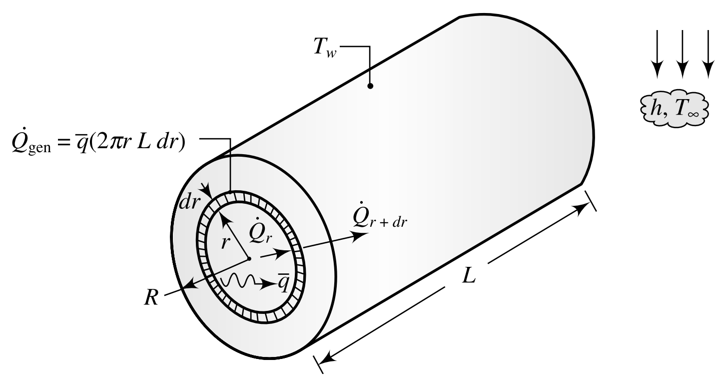


Fig. 3.4 *Solid cylinder with uniform internal heat generation*

In the *non-dimensional* form, the temperature distribution can be expressed as

$$\frac{T(r) - T_w}{\overline{q}R^2/4k} = 1 - \left(\frac{r}{R}\right)^2$$

or $$\boxed{\frac{T(r) - T_w}{T_{\max} - T_w} = 1 - \left(\frac{r}{R}\right)^2} \quad \Rightarrow \quad \boxed{\phi = 1 - \rho^2} \tag{3.18}$$

where ϕ is the *non-dimensional temperature* and ρ is the *non-dimensional radius*.
We note that larger the radius, or smaller the thermal conductivity, larger the centreline temperature. The heat flux is given by

$$q(r) = -k\frac{dT(r)}{dr} = -k\left(-\frac{\overline{q}r}{2k}\right) = \frac{\overline{q}r}{2}$$

At $r = R$, the heat flux at the outer surface,

$$q_{(r=R)} = \frac{\overline{q}R}{2}$$

which is a *positive* quantity. Thus, the heat flow is expectedly in the *positive x-direction*, i.e., *outwards*.

- **Convective Boundary Condition**

In most of the engineering applications, the ambient fluid temperature T_∞ rather than the wall temperature T_w is known. Let us now consider heat conduction in a solid cylinder with uniform volumetric energy-generation rate, $\overline{q}$ subjected to convection at the outer surface with a heat-transfer coefficient h into the surrounding medium at the temperature T_∞.
Applying the energy balance, $\dot{E}_{\text{gen}} = \dot{Q}_{\text{conv}}$

$$\overline{q}\,(\pi R^2 L) = h(2\pi RL)\,(T_w - T_\infty)$$

or $$\overline{q}\,R = 2h(T_w - T_\infty)$$

It follows that

$$\boxed{T_w = T_\infty + \frac{\overline{q}R}{2h}}$$

Substituting T_w in Eq. (3.17), the temperature distribution in the cylinder given by

$$\boxed{T(r) = \frac{\overline{q}R^2}{4k}\left[1-\left(\frac{r}{R}\right)^2\right] + \frac{\overline{q}R}{2h} + T_\infty} \tag{3.19}$$

This expression makes it possible to calculate the temperature at any point on the rod (cylinder) and shows that the temperature curve in a circular rod is parabolic.

At $r = 0$, the maximum centreline temperature is

$$\boxed{T_{\max} = T(0) = \frac{\overline{q}R^2}{4k} + \frac{\overline{q}R}{2h} + T_\infty} \tag{3.20}$$

or $$\boxed{T_{\max} = T_\infty + \frac{\overline{q}R}{2}\left[\frac{R}{2k} + \frac{1}{h}\right]} \tag{3.21}$$

Let us now examine the physical significance of the two extreme cases of solution for $h \to \infty$, and $h \to 0$. For extremely large values of h approaching infinity, the second term in Eq. (3.30) disappears and we get

$$\boxed{T(r) = \frac{\overline{q}R^2}{4k}\left[1-\left(\frac{r}{R}\right)^2\right] + T_\infty} \tag{3.22}$$

which is the same as Eq. (3.17) for constant surface temperature except that T_w is replaced by T_∞ now. This is because as $h \to \infty$, the convective resistance becomes vanishingly small, and the surface temperature, T_w assumes the value of ambient temperature, T_∞.

If $h \to 0$, that is for very small values of h, Eq. (3.18) reveals that the temperature $T(r) \to \infty$, that is, it will become extremely high. From the *second* boundary condition, as $h \to 0$, the temperature gradient, $\left.\frac{dT}{dr}\right|_{r=R} \to 0$. And, negligible temperature gradient corresponds to an almost *insulated* or *adiabatic outer surface* of the cylinder. Naturally, the internal heat generated in the cylinder cannot be convected away resulting in continuous rise in temperature. *No steady-state solution* is possible in this particular situation.

3.6 ❑ CURRENT-CARRYING CONDUCTOR

Cooling of current-carrying conductors improves their current-carrying capacity. Knowledge of temperature distribution is required to make that temperatures leading to *burn-out* of the conductor are not reached. Conductors have to operate safely in *superconducting magnets, transformers, electric motors and electrical machinery*, since sudden failure of a conductor may lead to conditions unsafe to the operator as well as the machine.

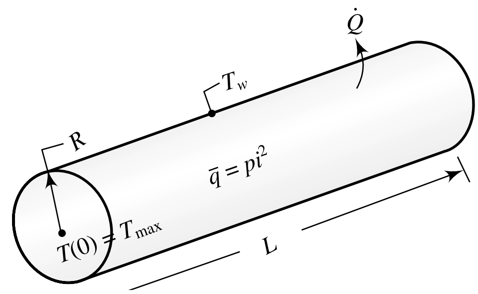


Fig. 3.5 *Current carrying conductor*

In the case of current-carrying conductors, uniform internal heat generation occurs due to ohmic or Joulean heating.

Consider a conductor of cross-sectional area, A_c and length, L. Let the current carried be I in amperes. Let the electrical resistivity of the material be ρ (ohm m).

Then, heat generated per unit volume, $\overline{q} = \dot{E}_{gen}$ per unit volume of the conductor, where $\dot{E}_{gen}$ is the total heat generation rate (W).

$$\dot{Q}_{gen} = I^2 R_e \text{ where } R_e = \text{electrical resistance of the conductor (ohm)}$$

But $$R_e = \frac{\rho L}{A_c}$$

Therefore,

$$\overline{q} = \frac{I^2 R_e}{A_c L} = \frac{I^2(\rho L/A_c)}{A_c L} = (I/A_c)^2 \rho \quad (\text{W/m}^3)$$

or $$\overline{q} = \rho i^2 = \frac{i^2}{k_e}$$

where $i = (I/A_c)$, and is known as the *current density* (A/m^2), and $k_e = \frac{1}{\rho}$ = electrical conductivity, (ohm m)$^{-1}$ or reciprocal of resistivity.

Therefore, the temperature distribution in a current-carrying conductor (of solid, cylindrical geometry) is given by.

$$T(r) = T_w + \frac{\overline{q}}{4k}(R^2 - r^2)$$

Substituting for $\overline{q}$, we get

$$\boxed{T(r) = T_w + \frac{\rho i^2}{4k}(R^2 - r^2)} \tag{3.23}$$

Equation (3.25) gives the temperature variation in the current-carrying conductor, in terms of the surface temperature, T_w. Maximum (centre) temperature is obtained by substituting $r = 0$.

$$\boxed{T_{\max} = T_w + \frac{\rho i^2 R^2}{4k}} \tag{3.24}$$

And, the *non-dimensional* temperature distribution is given by

$$\boxed{\frac{T - T_w}{T_{\max} - T_w} = 1 - \left(\frac{r}{R}\right)^2} \tag{3.25}$$

This indicates a *parabolic temperature profile*.

3.7 ❑ HOLLOW CYLINDER WITH INTERNAL HEAT GENERATION

Radial systems like a hollow cylinder (a long solid tube) have significant practical applications. Many a time, nuclear fuel rods are made of hollow cylinder geometry where the heat generated is carried away by a coolant like *liquid metal* flowing either on the *inside* or *outside* the tubes. Hollow electrical conductors of cylindrical shape are often used for high current-carrying applications, where cooling is effected by a fluid flowing on the inside. Annular reactors, insulated either from inside or outside, are often used in many chemical processes.

3.8 ❑ HOLLOW CYLINDER WITH UNIFORM HEAT GENERATION WITH INSIDE SURFACE INSULATED

The *hollow cylinder* with an *adiabatic inner surface* has application in reactor fuel rod design. Figure 3.6 illustrates the cylinder to be *adiabatic at the inner surface* and on the ends so that heat flow is only in the radial outward direction.

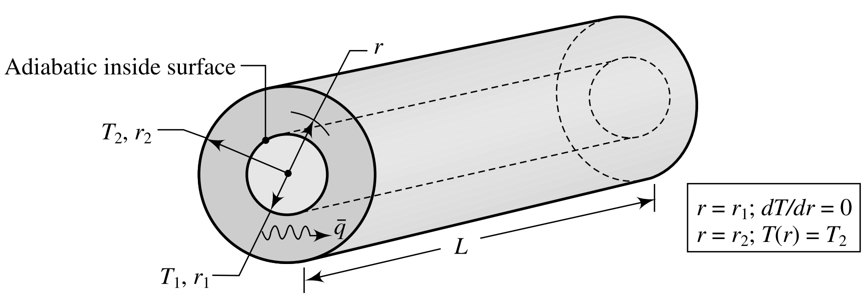


Fig. 3.6 *Hollow cylinder with uniform internal heat generation: adiabatic inner surface and adiabatic ends*

The general solution for temperature distribution is

$$T(r) = -\frac{\bar{q}r^2}{4k} + C_1 \ln r + C_2$$

The first constant C_1 is obtained from the boundary condition at the inner surface,

$$\left(\frac{dT}{dr}\right)_{r=r_1} = 0$$

It follows that

$$\left.\frac{dT}{dr}\right|_{r=r_1} = -\frac{\bar{q}r_1}{2k} + \frac{C_1}{r_1} = 0$$

$$\Rightarrow \qquad \boxed{C_1 = \frac{\bar{q}r_1^2}{2k}}$$

The second constant C_2 can be obtained from the boundary condition at the outer surface: $T(r = r_2) = T_2$

Then

$$\boxed{C_2 = T_2 + \frac{\bar{q}r_2^2}{4k} - \frac{\bar{q}r_1^2}{2k}\ln r_2}$$

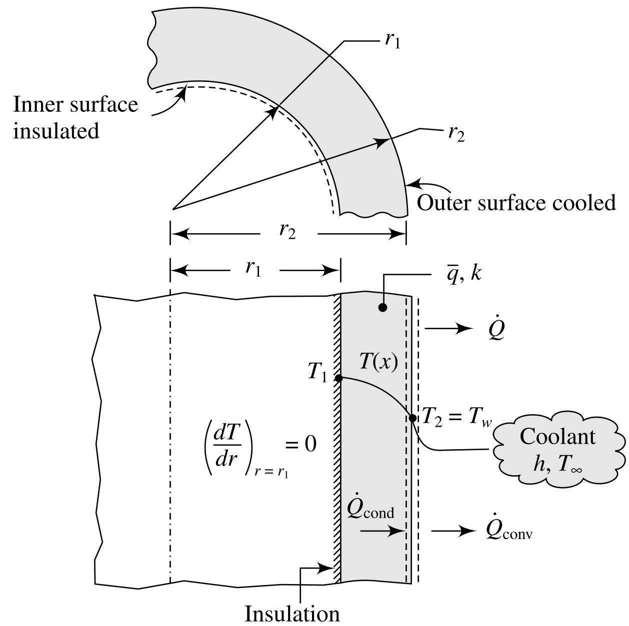


Fig. 3.7 *Heat removal through the outer surface of a cylindrical wall with uniform heat generation*

Substituting for C_1 and C_2, one gets the expression for the temperature field.

$$T(r) = -\frac{\bar{q}r^2}{4k} + \frac{\bar{q}r_1^2}{2k}\ln r + T_2 + \frac{\bar{q}r_2^2}{4k} - \frac{\bar{q}r_1^2}{2k}\ln r_2$$

or

$$\boxed{T(r) = T_2 + \frac{\bar{q}r_1^2}{4k}\left[\left(\frac{r_2}{r_1}\right)^2 - 2\ln\left(\frac{r_2}{r}\right) - \left(\frac{r}{r_1}\right)^2\right]} \qquad (3.26)$$

Also,

$$T(r) = T_2 + \frac{\bar{q}r_2^2}{4k}\left[\left\{1 - \left(\frac{r}{r_2}\right)^2\right\} + 2\left(\frac{r_1}{r_2}\right)^2 \ln\left(\frac{r}{r_2}\right)\right]$$

Maximum temperature will clearly occur at the adiabatic inside surface: $T(r = r_1) = T_1 = T_{max}$

$$\text{since} \left(\frac{dT}{dr}\right)_{r=r_1} = 0 \qquad (\textit{inside surface insulated})$$

The temperature difference across the cylindrical wall is then given by

$$\therefore \qquad \boxed{T_{max} - T_2 = T_1 - T_2 = \frac{\bar{q}r_1^2}{4k}\left[\left(\frac{r_2}{r_1}\right)^2 - 2\ln\left(\frac{r_2}{r_1}\right) - 1\right]} \qquad (3.27)$$

Convective Boundary Condition If the fluid temperature surrounding the outer surface is specified, the entire amount of heat generated must be dissipated by convection to the flowing fluid since the inner surface is adiabatic (*insulated*).

Energy conservation requirement dictates that

$$\dot{Q}_{gen} = \dot{Q}_{conv,out}$$

i.e.,

$$\overline{q}\pi(r_2^2 - r_1^2)L = h(2\pi r_2 L)(T_2 - T_\infty)$$

or

$$\boxed{T_2 = T_\infty + \frac{\overline{q}(r_2^2 - r_1^2)}{2hr_2}} \tag{3.28}$$

Substituting for T_2, one gets

$$\boxed{T(r) = T_\infty + \frac{\overline{q}(r_2^2 - r_1^2)}{2hr_2} + \frac{\overline{q}r_1^2}{4k}\left[\left(\frac{r_2}{r_1}\right)^2 - 2\ln\left(\frac{r_2}{r}\right) - \left(\frac{r}{r_1}\right)^2\right]} \tag{3.29}$$

3.9 ❑ HOLLOW CYLINDER WITH THE OUTSIDE SURFACE INSULATED AND COOLED AT THE INNER SURFACE

Consider steady state, one-dimensional heat-transfer in a hollow cylinder (a long solid tube) of length L, inside radius r_1 and outside radius r_2, with a uniform volumetric heat generation rate $\overline{q}$ and constant thermal conductivity k. The outside surface is effectively insulated (adiabatic) and hence the heat generated in the cylindrical shell will have to flow in the radial inward direction. Let the *inside* and *outside* surface temperatures be maintained at T_1 and T_2, respectively

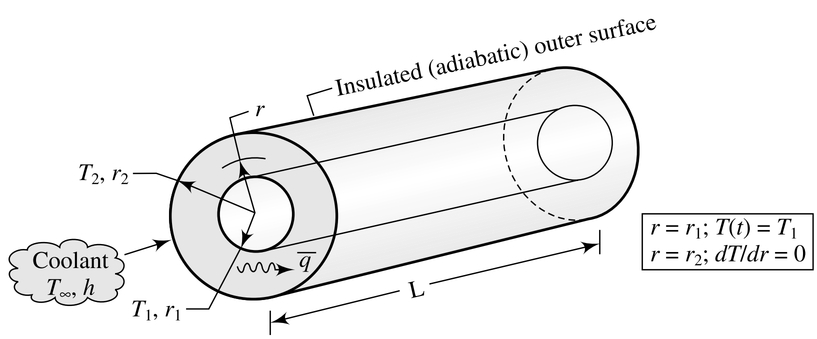


Fig. 3.8 *Hollow cylinder with uniform internal heat generation: adiabatic outer surface and adiabatic ends.*

- **Assumptions**
  - Steady-state conditions
  - One-dimensional (radial) conduction
  - Uniform volumetric heat generation
  - Constant properties
  - Well-insulated outer surface

To determine the temperature distribution we start with the relevant governing differential equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\overline{q}}{k} = 0 \quad \text{or} \quad \frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\overline{q}r}{k}$$

Integrating the above expression,

$$r\frac{dT}{dr} = -\frac{\bar{q}r^2}{2k} + C_1 \quad \text{or} \quad \frac{dT}{dr} = -\frac{\bar{q}r}{2k} + \frac{C_1}{r}$$

Further integration yields,

$$T(r) = -\frac{\bar{q}r^2}{4k} + C_1 \ln r + C_2$$

This is the formal solution for the temperature profile. To obtain the two constants of integration, the two boundary conditions are

$$\left.\frac{dT}{dr}\right|_{r=r_2} = 0 \quad \text{and} \quad T(r_1) = T_1$$

It follows that

$$0 = -\frac{\bar{q}r_2^2}{2k} + C_1 \quad \Rightarrow \quad \boxed{C_1 = \frac{\bar{q}r_2^2}{2k}}$$

and $$T_1 = -\frac{\bar{q}r_1^2}{4k} + C_1 \ln r_1 + C_2$$

$\therefore$ $$\boxed{C_2 = T_1 + \frac{\bar{q}r_1^2}{4k} - \frac{\bar{q}r_2^2}{2k}\ln r_1}$$

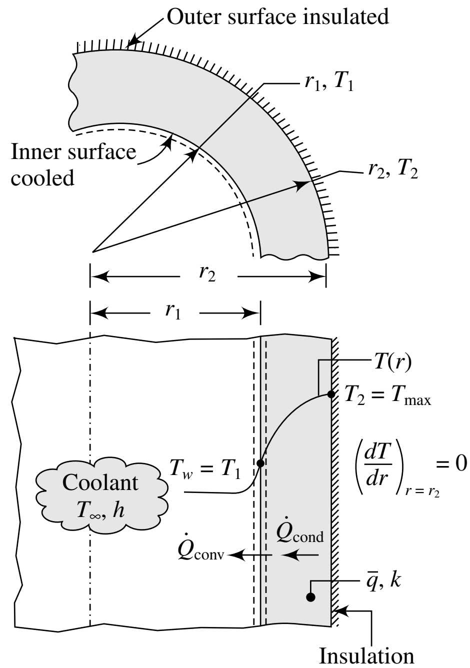


Fig. 3.9 *Removal of heat through the inner surface of a cylindrical wall with uniform heat generation*

Substituting for C_1 and C_2 in the general solution,

$$T(r) = T_1 + \frac{\bar{q}}{4k}(r_1^2 - r^2) + \frac{\bar{q}r_2^2}{2k}\ln r - \frac{\bar{q}r_2^2}{2k}\ln r_1$$

Hence,

$$\boxed{T(r) = T_1 + \frac{\bar{q}}{4k}(r_1^2 - r^2) + \frac{\bar{q}r_2^2}{2k}\ln\frac{r}{r_1}}$$

or $$T(r) = T_1 + \frac{\bar{q}r_2^2}{4k}\left[2\ln\left(\frac{r}{r_1}\right) + \left(\frac{r_1}{r_2}\right)^2 - \left(\frac{r}{r_2}\right)^2\right] \tag{3.30}$$

At $r = r_2$, $T = T_2$ and the total temperature difference across the wall is

$\therefore$ $$\boxed{T_2 - T_1 = \frac{\bar{q}r_2^2}{4k}\left[2\ln\left(\frac{r_2}{r_1}\right) + \left(\frac{r_1}{r_2}\right)^2 - 1\right]} \tag{3.31}$$

Convective Boundary Condition Applying the energy-conservation requirement at the *inner* surface, $\dot{E}_{gen} = \dot{Q}_{conv}$

$$\bar{q}\pi(r_2^2 - r_1^2)L = h(2\pi r_1 L)(T_1 - T_\infty)$$

or $$\boxed{T_1 = T_\infty + \frac{\overline{q}(r_2^2 - r_1^2)}{2hr_1}} \quad \text{and} \quad \boxed{h = \frac{\overline{q}(r_2^2 - r_1^2)}{2r_1(T_1 - T_\infty)}} \tag{3.32}$$

The temperature distribution when heat is removed only through the *inside* surface of the tube to the surrounding fluid is, thus, given by

$$\boxed{T(r) = T_\infty + \frac{\overline{q}(r_2^2 - r_1^2)}{2hr_1} + \frac{\overline{q}r_2^2}{4k}\left[2\ln\left(\frac{r}{r_1}\right) + \left(\frac{r_1}{r_2}\right)^2 - \left(\frac{r}{r_2}\right)^2\right]} \tag{3.33}$$

Since $\dfrac{dT}{dr} = 0$ at $r = r_2$, the maximum temperature will occur at the outside surface and is given by

$$T_{\max} = T_2 = T_\infty + \frac{\overline{q}(r_2^2 - r_1^2)}{2hr_1} + \frac{\overline{q}r_2^2}{4k} + \left[2\ln\left(\frac{r_2}{r_1}\right) + \left(\frac{r_1}{r_2}\right)^2 - 1\right]$$

or $$\boxed{T_{\max} = T_\infty + \frac{\overline{q}r_2^2}{4k}\left[\left\{1 - \left(\frac{r_1}{r_2}\right)^2\right\}\frac{2k}{hr_1} + 2\ln\left(\frac{r_2}{r_1}\right) + \left(\frac{r_1}{r_2}\right)^2 - 1\right]} \tag{3.34}$$

or $$\boxed{T_{\max} = T_\infty + \frac{\overline{q}r_2^2}{4k}\left[\left(\frac{2k}{hr_1} - 1\right)\left\{1 - \left(\frac{r_1}{r_2}\right)^2\right\} + 2\ln\left(\frac{r_2}{r_1}\right)\right]} \tag{3.35}$$

3.10 ❑ HOLLOW CYLINDER WITH BOTH SURFACES HELD AT CONSTANT TEMPERATURES

Consider steady-state, one-dimensional heat-transfer in a hollow cylinder of constant thermal conductivity k, length L, inside radius r_1 and outside radius r_2, with uniform internal heat generation. Let the temperatures of the inside and outside surfaces be T_1 and T_2, respectively. Heat is being dissipated from *both* the surfaces (Fig. 3.10).

The governing differential equation in the cylindrical coordinates is given by

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{\overline{q}}{k} = 0 \quad \text{or} \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\overline{q}}{k}$$

Integrating,

$$r\frac{dT}{dr} = -\frac{\overline{q}r^2}{2k} + C_1 \quad \text{or} \quad \frac{dT}{dr} = -\frac{\overline{q}r}{2k} + \frac{C_1}{r}$$

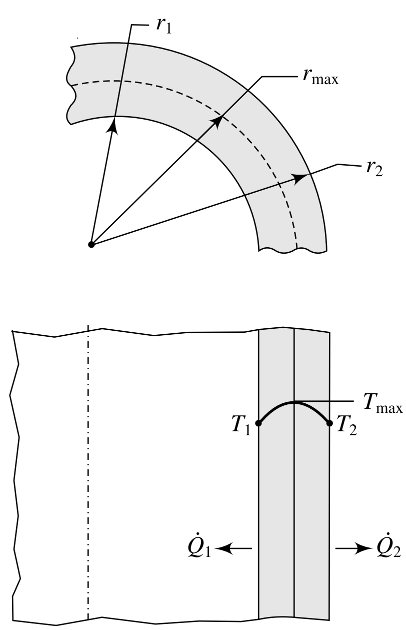


Fig. 3.10 *Heat removal through both surfaces of a cylindrical wall with uniform heat generation*

Integrating again, the following formal solution for temperature distribution is given by.

$$T(r) = \frac{-\bar{q}r^2}{4k} + C_1 \ln r + C_2$$

We now have to determine the constants of integration C_1 and C_2 for the following boundary conditions.

$T = T_1$, at $r = r_1$ (*inside surface*)

$T = T_2$, at $r = r_2$ (*outside surface*)

It follows that

$$\boxed{T_1 = \frac{-\bar{q}r_1^2}{4k} + C_1 \ln r_1 + C_2} \qquad \text{(a)}$$

$$\boxed{T_2 = -\frac{\bar{q}r_2^2}{4k} + C_1 \ln r_2 + C_2} \qquad \text{(b)}$$

Subtracting Eq. (a) from Eq. (b):

$$T_2 - T_1 = -\frac{\bar{q}}{4k}(r_2^2 - r_1^2) + C_1 \ln (r_2/r_1)$$

$$\therefore \qquad \boxed{C_1 = \frac{(T_2 - T_1) + \frac{\bar{q}}{4k}(r_2^2 - r_1^2)}{\ln (r_2/r_1)}}$$

And, from Eq. (a):

$$\boxed{C_2 = T_1 + \frac{\bar{q}r_1^2}{4k} - \frac{(T_2 - T_1) + \frac{\bar{q}}{4k}(r_2^2 - r_1^2)}{\ln (r_2/r_1)} \ln r_1}$$

Substituting for C_1 and C_2 in the general solution:

$$T(r) = -\frac{\bar{q}r^2}{4k} + \frac{(T_2 - T_1) + \frac{\bar{q}}{4k}(r_2^2 - r_1^2)}{\ln (r_2/r_1)} . \ln r + T_1 + \frac{\bar{q}r_1^2}{4k} - \frac{\left(T_2 - T_1\right) + \frac{\bar{q}}{4k}(r_2^2 - r_1^2)}{\ln (r_2/r_1)} \ln r_1$$

The temperature distribution is then given by

$$T(r) - T_1 = \frac{-\bar{q}}{4k}(r^2 - r_1^2) + \left\{(T_2 - T_1) + \frac{\bar{q}}{4k}(r_2^2 - r_1^2)\right\} \frac{\ln (r/r_1)}{\ln (r_2/r_1)}$$

or

$$\frac{T(\mathrm{r}) - T_1}{T_2 - T_1} = \frac{-\bar{q}}{4k}\frac{(r^2 - r_1^2)}{(T_2 - T_1)} + \frac{\bar{q}}{4k}\frac{(r_2^2 - r_1^2)}{(T_2 - T_1)}\frac{\ln (\mathrm{r}/\mathrm{r}_1)}{\ln (r_2/r_1)} + \frac{\ln (\mathrm{r}/\mathrm{r}_1)}{\ln (r_2/r_1)}$$

$$= \frac{\bar{q}(r_2^2 - r_1^2)}{4k(T_2 - T_1)}\left[\frac{\ln\left(r/r_1\right)}{\ln (r_2/r_1)} - \frac{r^2 - r_1^2}{r_2^2 - r_1^2}\right] + \frac{\ln (\mathrm{r}/\mathrm{r}_1)}{\ln (r_2/r_1)}$$

$$\therefore \qquad \boxed{\frac{T(r) - T_1}{T_2 - T_1} = \frac{\bar{q}}{4k}\frac{(r_2^2 - r_1^2)}{(T_2 - T_1)}\left[\frac{\ln (r/r_1)}{\ln (r_2/r_1)} - \frac{(r/r_1)^2 - 1}{(r_2/r_1)^2 - 1}\right] + \frac{\ln (r/r_1)}{\ln (r_2/r_1)}} \qquad (3.36)$$

3.10.1 • Heat Transfer to Both Surfaces

Once the temperature distribution is known, the heat-transfer rate can be easily determined by applying the Fourier's rate equation:

Heat-transfer rate at the *inner* surface, $\dot{Q}\big|_{r=r_1} = -k(2\pi r_1 L)(dT/dr)\big|_{r=r_1}$

Heat-transfer rate at the *outer* surface, $\dot{Q}\big|_{r=r_2} = -k(2\pi r_2 L)(dT/dr)\big|_{r=r_2}$

One must note that the heat transfer to the inner surface will be *negative* since the heat flows from outside to inside, i.e., in the *negative* r-direction.

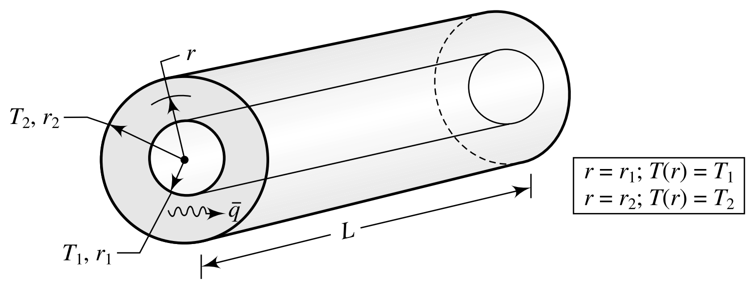


Fig. 3.11 *Hollow cylinder with uniform internal heat generation losing heat from both inner and outer surfaces*

3.10.2 • Maximum Temperature

The location of maximum temperature will be somewhere between r_1 and r_2, since heat is transferred to both inside and outside surfaces. Let that location be at a radius of r_{max}. Then, r_{max} can be found by differentiating $T(r)$ with respect to r and equating the resulting derivative to zero. Then, this value of r_{max} is substituted in the expression for $T(r)$ to obtain T_{max}.

3.10.3 • Convective Boundary Condition

If the wall surface temperatures T_1 and T_2 are not known but the temperatures of the fluids $T_{\infty,1}$ and $T_{\infty,2}$ to which the inner and outer surfaces are exposed respectively and the associated heat-transfer coefficients h_1 and h_2 are specified, then Eq. (3.) must be supplemented with the following equations in order to determine r_{max}.

$$\begin{pmatrix}\text{Heat generated within the wall and conducted}\\ \text{to the inner and outer surfaces}\end{pmatrix} = \begin{pmatrix}\text{Heat lost to the fluids at the two}\\ \text{surfaces by convection}\end{pmatrix}$$

$$\bar{q}\pi(r_{max}^2 - r_1^2)L = h_1(2\pi r_1 L)(T_1 - T_{\infty 1})$$

and
$$\bar{q}\pi(r_2^2 - r_{max}^2)L = h_2(2\pi r_2 L)(T_2 - T_{\infty 2})$$

Since heat is transferred from both inside and outside surfaces, the maximum temperature, T_{max} must naturally occur somewhere in the cylindrical wall. Obviously, the radius at which $T = T_{max}$, i.e., r_{max} will lie between r_1 and r_2. The surface at r_{max} is isothermal and the maximum temperature occurs at r_{max}, i.e., $dT/dr = 0$ at $r = r_{max}$. This means that the surface at r_{max} may be considered as representing an insulated boundary condition $\left(\dfrac{dT}{dr} = 0\right)$.

Thus, the cylindrical annulus (wall) may be considered as being made up of two layers: the inner layer, between $r = r_1$ and $r = r_{max}$, insulated on its *outer surface* and, an outer layer, between $r = r_{max}$ and $r = r_2$, insulated at its *inner surface*.

Then, the maximum temperature for the *inner* and *outer* layers can be determined from the following equations:

For the Inner Layer (Insulated on the Outer Surface) Replacing r_2 by r_{max} and T_2 by T_{max} in Eq. (3.31), we get

$$T_{max} - T_1 = \frac{\bar{q} r_{max}^2}{4k}\left[2\ln\left(\frac{r_{max}}{r_1}\right) + \left(\frac{r_1}{r_{max}}\right)^2 - 1\right] \tag{a}$$

For the Outer Layer (Insulated on the Inner Surface) Replacing r_1 by r_{max} and T_1 by T_{max} in Eq. (3.), we have

$$T_{max} - T_2 = \frac{\bar{q} r_{max}^2}{4k}\left[\left(\frac{r_2}{r_{max}}\right)^2 - 2\ln\left(\frac{r_2}{r_{max}}\right) - 1\right] \tag{b}$$

Subtracting equations (a) from equation (b), we have

$$\begin{aligned}
T_1 - T_2 &= \frac{\bar{q} r_{max}^2}{4k}\left[\left(\frac{r_2}{r_{max}}\right)^2 - 2\ln\left(\frac{r_2}{r_{max}}\right) - 1 - 2\ln\left(\frac{r_{max}}{r_1}\right) - \left(\frac{r_1}{r_{max}}\right)^2 + 1\right] \\
&= \frac{\bar{q} r_{max}^2}{4k}\left[\left(\frac{r_2}{r_{max}}\right)^2 - \left(\frac{r_1}{r_{max}}\right)^2 + 2\ln\left(\frac{r_{max}}{r_2}\right) - 2\ln\left(\frac{r_{max}}{r_1}\right)\right] \\
&= \frac{\bar{q}}{4k}(r_2^2 - r_1^2) + \frac{\bar{q} r_{max}^2}{4k} 2\ln\left(\frac{r_{max}}{r_2}\frac{r_1}{r_{max}}\right) = \frac{\bar{q}}{4k}(r_2^2 - r_1^2) + \frac{\bar{q} r_{max}^2}{4k} 2\ln\left(\frac{r_1}{r_2}\right) \\
&= \frac{\bar{q}}{4k}\left[(r_2^2 - r_1^2) + 2r_{max}^2 \ln\left(\frac{r_1}{r_2}\right)\right]
\end{aligned}$$

or
$$(T_1 - T_2)4k = \bar{q}\left[(r_2^2 - r_1^2) + 2r_{max}^2 \ln\left(\frac{r_1}{r_2}\right)\right]$$

or
$$2\bar{q} r_{max}^2 \ln\left(\frac{r_2}{r_1}\right) = \bar{q}(r_2^2 - r_1^2) - 4k(T_1 - T_2)$$

Solving for r_{max}, it follows that

$$\boxed{r_{max} = \sqrt{\frac{\bar{q}(r_2^2 - r_1^2) - 4k(T_1 - T_2)}{2\bar{q}\ln(r_2/r_1)}}} \tag{3.37}$$

Substituting this value of r_{max} in either of the expressions (a) or (b), one can obtain the maximum temperature in the hollow cylinder.

The temperature distribution in the outer layer is determined from Eq. (3.19).

If the inside and outside surface temperatures, T_1 and T_2 are equal, the location of the maximum temperature is given by

$$\boxed{r_{max} = \sqrt{\frac{r_2^2 - r_1^2}{2\ln(r_2/r_1)}}} \tag{3.38}$$

Note that r_{max} depends *only* on the physical dimensions of the cylindrical wall and *not* on the thermal conditions.

It is worth noting that the location of maximum temperature in the cylindrical wall will not be affected by the value of the uniform volumetric heat generation $\overline{q}$.

3.11 ❑ SOLID SPHERE WITH INTERNAL HEAT GENERATION AND CONSTANT SURFACE TEMPERATURE

The mathematical formulation for steady-state one-dimensional heat conduction in a solid sphere of radius R and constant thermal conductivity k having uniform volumetric heat generation rate $\overline{q}$ in spherical coordinates is

$$\frac{d}{dr}\left[r^2 \frac{dT}{dr}\right] + \frac{\overline{q}r^2}{k} = 0$$

One boundary condition is

$$T = T_w \quad \text{at} \quad r = R \; [\textit{as } T_w \textit{ is specified}]$$

As the entire outer surface of the sphere experiences the same constant temperature T_w and the heat source inside it is also uniformly distributed, the temperature distribution is obviously expected to be symmetrical about the centre of the sphere where the temperature will be maximum and the temperature gradient will be zero.

The second boundary condition then can be written as

$$\frac{dT}{dr} = 0 \quad \text{at} \quad r = 0$$

Separating the variables and integrating once, we have

$$r^2 \frac{dT}{dr} = -\frac{\overline{q}r^3}{3k} + C_1$$

Separating the variables and integrating yet again, we get

$$T(r) = -\frac{\overline{q}r^2}{6k} + \frac{C_2}{r} + C_2$$

Applying the boundary condition at $r = 0$, we obtain

$$0 = -(0) + \frac{C_1}{0}$$

To satisfy the above equation, C_1 has got to be zero. With $\boxed{C_1 = 0,}$ the temperature distribution becomes

$$T(r) = -\frac{\overline{q}r^2}{6k} + C_2$$

The first boundary condition can now be applied to get C_2.

$$\therefore \qquad T_w = -\frac{\overline{q}R^2}{6k} + C_2 \quad \text{or} \quad \boxed{C_2 = T_w + \frac{\overline{q}R^2}{6k}}$$

Substituting this constant of integration in Eq. (3) the resulting temperature distribution is

$$\boxed{T(r) - T_w = \frac{\overline{q}}{6k}(R^2 - r^2) = \frac{\overline{q}R^2}{6k}\left[1 - \left(\frac{r}{R}\right)^2\right]} \tag{3.39}$$

As expected, the maximum temperature in the sphere will be at the centre (r = 0)

$$\boxed{T(0) \quad \text{or} \quad T_{\max} = \frac{\bar{q}R^2}{6k} + T_w} \tag{3.40}$$

It is instructive to note that the temperature distribution is symmetrical about the centre of symmetry. Furthermore, the maximum temperature occurs at a location far away from the outer surface.

3.12 ❑ CONVECTIVE BOUNDARY CONDITION

If instead of the constant surface temperature at the boundary of the solid sphere (at $r = R$), the ambient temperature T_∞ is known (*which is more likely*) then by applying the surface energy balance, we can express T_w in terms of T_∞.

At $r = R$, the heat flux due to *conduction* equals that due to *convection* (the boundary condition of the *second* kind). Thus,

$$-k\left.\frac{dT}{dr}\right|_{r=R} = h(T_{r=R} - T_\infty) \quad \text{or} \quad h(T_w - T_\infty) \tag{3.41}$$

But $$\left.\frac{dT}{dr}\right|_{r=R} = -\frac{\bar{q}R}{3k}$$ From Eq. (3)

Putting this value in Eq.(3), one gets

$$-k\left(-\frac{\bar{q}R}{3k}\right) = h(T_w - T_\infty) \quad \text{or} \quad \boxed{T_w = T_\infty + \frac{\bar{q}R}{3h}} \tag{3.42}$$

Substituting this value of T_w, the temperature distribution in the sphere can now be expressed as

$$\boxed{T(r) - T_\infty = \frac{\bar{q}R^2}{6k}\left[1 - \left(\frac{r}{R}\right)^2\right] + \frac{\bar{q}R}{3h}} \tag{3.43}$$

Maximum temperature (at the centre, i.e., at r = 0) will be

$$T_{\max} = T_\infty + \frac{\bar{q}R^2}{6k} + \frac{\bar{q}R}{3h}$$

or $$\boxed{T_{\max} = T_\infty + \frac{\bar{q}R}{3}\left[\frac{R}{2k} + \frac{1}{h}\right]} \tag{3.44}$$

Similar to the previous cases of the thin plane wall (infinite slab) and long solid cylinder, let us see what happens when the surface heat-transfer coefficient h is either extremely small ($h \to 0$) or inordinately large ($h \to \infty$) in the case of a solid sphere.

As $h \to \infty$, Eq. (3.44) becomes

$$\boxed{T(r) = \frac{\bar{q}R^2}{6k}\left[1 - \left(\frac{r}{R}\right)^2\right] + T_\infty} \tag{3.45}$$

because the second term on the right-hand side is eliminated. The above equation represents the temperature distribution for a sphere with constant surface temperature T_w, with T_w replaced now with T_∞. This is clear when $h \to \infty$, $T_w \to T_\infty$.

As for the second special case of when $h \to 0$, from Eq. (3.44) again

$$\left.\frac{dT}{dr}\right|_{r=R} \to 0 \quad \text{and} \quad T_{r=R} \to T_\infty \tag{3.46}$$

This means that the outer surface (*boundary*) of the sphere becomes *insulated or impervious* to heat transfer. The temperature $T(r)$ cannot attain a steady state value because the heat generated continuously within the sphere results in constant rise in temperature with no scope for escape *(adiabatic wall). Hence, no steady-state solution to the problem exists*.

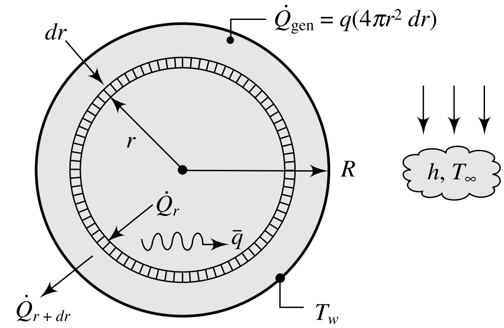


Fig. 3.12 *Solid sphere with uniform internal heat generation*

Heat-transfer by conduction at the outer surface of the sphere is given by the Fourier's rate equation:

$$\dot{Q}_{\text{cond}} = -kA(dT/dr)\big|_{r=R} = -k\,(4\pi R^2)\left(\frac{-\bar{q}R}{3k}\right) = \left(\frac{4}{3}\pi R^3\right)\bar{q}$$
$$= \bar{q}\,(\forall) = \dot{Q}_{\text{gen}}$$

Under steady conditions, the energy balance for the solid sphere can therefore be expressed as

$$\begin{pmatrix}\text{Rate of heat}\\ \text{transfer by}\\ \text{conduction}\end{pmatrix} = \begin{pmatrix}\text{Rate of heat generation}\\ \text{within the sphere from the}\\ \text{solid at the outer surface}\end{pmatrix}$$

3.13 ❑ NON-UNIFORM HEAT GENERATION IN A SOLID SPHERE

Consider a spherical container of radioactive wastes with non-uniform radial distribution of heat dissipation. The outer surface is exposed to convection conditions.

Let us develop an expression for steady-state radial temperature distribution.

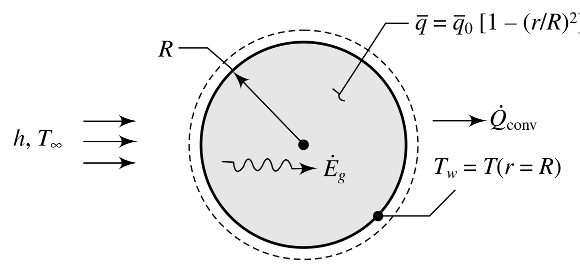


Fig. 3.13 *Solid sphere with non-uniform heat generation*

Assumptions

- Steady-state conditions
- One-dimensional conduction
- Constant properties
- Negligible temperature drop across the container wall

The appropriate form of the heat conduction equation is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{\bar{q}}{k} = -\frac{\bar{q}_0}{k}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{\bar{q}_0}{k}\left[r^2 - \frac{r^4}{R^2}\right]$$

Integrating,

$$r^2\frac{dT}{dr} = -\frac{\dot{q}_0}{k}\left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right) + C_1$$

$$\frac{dT}{dr} = -\frac{\overline{q}_0}{k}\left[\frac{r}{3} - \frac{r^3}{5R^2}\right] + \frac{C_1}{r^2}$$

Further integration yields,

$$\boxed{T(r) = -\frac{\overline{q}_0}{k}\left(\frac{r^2}{6} - \frac{r^4}{20R^2}\right) - \frac{C_1}{r} + C_2} \qquad \text{(A)}$$

From the boundary conditions:

$(dT/dr)|_{r=0} = 0$ (*from symmetry*)

and $-k(dT/dr)|_{r=R} = h[T(R) - T_\infty]$ (*from surface energy balance*)

If follows that $\boxed{C_1 = 0}$

and,
$$\overline{q}_0\left(\frac{R}{3} - \frac{R}{5}\right) = h\left[-\frac{\overline{q}_0}{k}\left(\frac{R^2}{6} - \frac{R^2}{20}\right) + C_2 - T_\infty\right]$$

$\Rightarrow$
$$\boxed{C_2 = \frac{2\overline{q}_0 R}{15h} + \frac{7\overline{q}_0 R^2}{60k} + T_\infty}$$

Substituting for C_1 and C_2 in Eq. (A),

$$T(r) = T_\infty + \frac{2\overline{q}_0 R}{15h} + \frac{7}{60}\frac{\overline{q}_0 R^2}{k} - \frac{1}{6}\frac{\overline{q}_0 r^2}{k} + \frac{1}{20}\overline{q}_0\frac{r^4}{R^2}$$

Simplifying and rearranging, the radial temperature distribution is given by

$$\boxed{T(r) = T_\infty + \frac{2\overline{q}_0 R}{15h} + \frac{\overline{q}_0 R^2}{k}\left[\frac{7}{60} - \frac{1}{6}\left(\frac{r}{R}\right)^2 + \frac{1}{20}\left(\frac{r}{R}\right)^4\right]} \qquad (3.47)$$

Maximum temperature will clearly occur at the centre ($r = 0$)
Then

$$\boxed{T_{\max} = T(0) = \left(T_\infty + \frac{2\overline{q}_0 R}{15h}\right) + \frac{7\overline{q}_0 R^2}{60k}} \qquad (3.48)$$

Surface temperature can be obtained by substituting $r = R$ in the expression for temperature distribution.

Then
$$T(r = R) = T_\infty + \frac{2\overline{q}_0 R}{15h} + \frac{\overline{q}_0 R^2}{k}\left[\frac{7}{60} - \frac{1}{6} + \frac{1}{20}\right]$$

$\therefore$
$$\boxed{T_w = T_\infty + \frac{2\overline{q}_0 R}{15h}} \qquad (3.49)$$

Also
$$\boxed{T_{\max} = T_w + \frac{7}{60}\frac{\overline{q}_0 R^2}{k}} \qquad (3.50)$$

3.14 ❑ HEAT TRANSFER IN A NUCLEAR FUEL ROD (*WITHOUT CLADDING*)

In a nuclear fuel element, the heat generated is *not* uniform throughout the material but varies with the location expressed by the following relation:

$$\bar{q} = q_0\left[1-\left(\frac{r}{R}\right)^2\right]$$

where $\bar{q}$ = volumetric heat generation at the centre (at r = 0), and

R = outer radius of the solid fuel rod.

Our objective is to develop an expression for

- The temperature distribution in the fuel rod,
- The maximum temperature in the rod, and
- The heat transferred.

Assumptions

- Steady-state conditions prevail
- One-dimensional (radial) conduction
- Homogeneous and isotropic material with constant thermal conductivity

Under these assumptions, the appropriate governing differential equation in cylindrical coordinates is

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{\bar{q}}{k}=0 \quad \text{or} \quad \frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\bar{q}r}{k}=0$$

or

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{q_0 r}{k}\left[1-\left(\frac{r}{R}\right)^2\right]=0$$

Integrating with respect to r, one gets

$$r\frac{dT}{dr}+\frac{q_0}{k}\left(\frac{r^2}{2}-\frac{r^4}{4R^2}\right)=C_1 \quad \text{or} \quad \frac{dT}{dr}+\frac{q_0}{k}\left(\frac{r}{2}-\frac{r^3}{4R^2}\right)=\frac{C_1}{r}$$

Integrating again,

$$T(r)+\frac{q_0}{k}\left(\frac{r^2}{4}-\frac{r^4}{16R^2}\right)=C_1\ln r+C_2$$

This is the general solution for temperature profile within the fuel rod. C_1 and C_2 are constants of integration to be obtained from the two boundary conditions, viz.,

- At r = 0, $\dfrac{dT}{dr}=0 \quad \Rightarrow \quad \boxed{C_1=0}$

(*since the temperature is maximum at the centre of the rod*)

- At r = 0, $T = T_{\max} \quad \Rightarrow \quad \boxed{C_2=T_{\max}}$

Thus, the temperature distribution in terms of the centre (maximum) temperature of the fuel rod is given by

$$T(r)+\frac{q_0}{k}\left(\frac{r^2}{4}-\frac{r^4}{16R^2}\right)=T_{\max}$$

$$\therefore \qquad \boxed{T(r)-T_{\max}=\frac{-q_0}{k}\left(\frac{r^2}{4}-\frac{r^4}{16R^2}\right)}$$

Surface temperature of the rod can be obtained by replacing r by R and $T(r)$ by T_w. It follows that

$$T_{max} - T_w = \frac{q_0}{k}\left(\frac{R^2}{4} - \frac{R^4}{16R^2}\right)$$

or
$$\boxed{T_{max} - T_w = \frac{3q_0R^2}{16k}} \tag{3.51}$$

This gives the maximum temperature drop in the fuel rod to ensure that adequate cooling is provided to protect the fuel rod from getting overheated.

Heat Flow from the Surface Knowing the temperature distribution, the heat flow rate at any point can be determined by applying the Fourier's rate equation:
At the surface ($r = R$):

$$\dot{Q} = -kA\left(\frac{dT}{dr}\right)\bigg|_{r=R} = -kA\left[\frac{-q_0}{k}\left(\frac{R}{2} - \frac{R^3}{4R^2}\right)\right]$$

$$\boxed{\dot{Q} = \frac{q_0AR}{4}} \tag{3.52}$$

Convective Boundary Condition

$$\begin{pmatrix}\text{Heat generated}\\ \text{in the fuel rod}\end{pmatrix} = \begin{pmatrix}\text{Heat carried away from the}\\ \text{outside surface by convection}\end{pmatrix}$$

$$\frac{q_0AR}{4} = hA(T_w - T_\infty)$$

or
$$\boxed{T_w = T_\infty + \frac{q_0R}{4h}} \tag{3.53}$$

Substituting this value of surface temperature, we have

$$T_{max} - T_\infty = \frac{q_0R}{4h} + \frac{3q_0R^2}{16k}$$

or
$$\boxed{T_{max} - T_\infty = \frac{q_0R}{4}\left(\frac{1}{h} + \frac{3R}{4k}\right)} \tag{3.54}$$

3.15 ❑ HEAT TRANSFER IN A NUCLEAR FUEL ROD (*WITH CLADDING*)

Usually, the fuel rod used in a nuclear reactor is insulated on the outside with a protective cladding material, to prevent any damage from oxidation of its surface by direct contact with the liquid coolant. Generally, aluminium is used as the cladding material. Our aim is to determine the temperature distribution and the heat-transfer rate in the nuclear *fuel rod with cladding*. It is noteworthy that heat generation occurs only in the fissile material of the fuel rod while the cladding material does not experience any heat generation.

Under steady operating conditions, the heat generated in the fuel rod is conducted through the cladding and subsequently, dissipated to the coolant surrounding the cladding by convection. Any contact resistance between the fuel rod and the cladding is neglected in this analysis.

Let R_F = outer radius of the fissionable fuel rod

k_F = thermal conductivity of the fuel rod

R_C = outer radius of the cladding material

k_C = thermal conductivity of the cladding material

The heat-generation rate in the fuel rod is *not* constant but varies with position according to the following relation:

$$\overline{q} = \overline{q}_0\left[1-\left(\frac{r}{R}\right)^2\right]$$

where $\overline{q}_o$ = volumetric heat generation rate per unit volume at the centre (at r = 0), and

R = outer radius of the solid fuel rod.

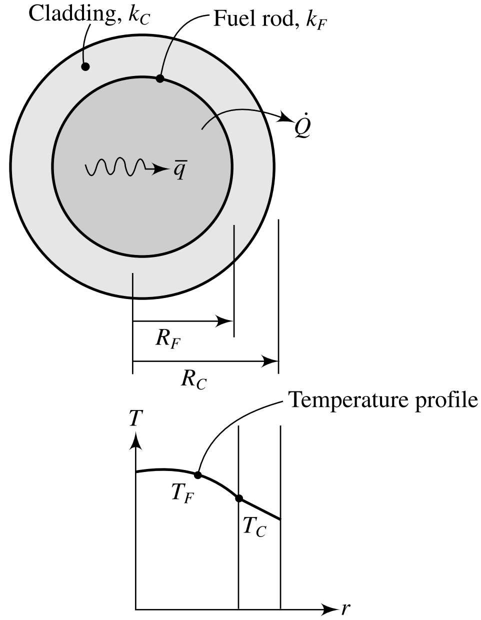


Fig. 3.14 *Cylindrical fuel rod with cladding*

Assumptions

- Steady operating conditions.
- One-dimensional conduction (in the radial direction)
- Homogeneous, isotropic material with constant thermal conductivity
- Non-uniform internal heat generation: $\overline{q} = \overline{q}_0[1-(r/R)^2]$

The governing differential equation in the cylindrical coordinates for one-dimensional (radial), steady-state conduction with internal heat generation and constant thermal conductivity is

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{\overline{q}}{k}=0 \quad \text{or} \quad \frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\overline{q}r}{k}=0$$

From Fourier's rate equation,

Heat flux, $q=-k\dfrac{dT}{dr}$

$\therefore$ Temperature gradient, $\dfrac{dT}{dr}=-\dfrac{q}{k}$

Therefore,

$$\frac{d}{dr}\left(r\frac{-q}{k}\right)+\frac{\overline{q}}{k}=0 \quad \text{or} \quad \frac{d}{dr}(qr)=\overline{q}r$$

Let us denote the fuel and cladding materials by suffices F and C, respectively.

Then, for the *fuel rod*:

$$\frac{d}{dr}(q_F r)=\overline{q}r$$

$$\frac{d}{dr}(q_F r)=\overline{q}_0\left[1-\left(\frac{r}{R_F}\right)^2\right]r$$

Integrating,

$$q_F r = \bar{q}_0 \left(\frac{r^2}{2} - \frac{r^4}{4R_F^2} \right) + C_1$$

or $$\boxed{q_F = \bar{q}_0 \left(\frac{r}{2} - \frac{r^3}{4R_F^2} \right) + \frac{C_1}{r}}$$

For the *cladding*:

$$\frac{d}{dr}(q_C r) = 0 \qquad \text{(\textit{there is no heat generation in the cladding})}$$

Integrating,

$$q_C r = C_2 \quad \text{or} \quad \boxed{q_C = \frac{C_2}{r}}$$

The constants of integration C_1 and C_2 can be determined from the relevant boundary conditions.

BC (I): q_F = finite, at $r = 0$

It follows that $\boxed{C_1 = 0}$

BC (II): $q_F = q_C$, at $r = R_F$, i.e., at the interface

$$q_C = \frac{C_2}{R_F} = q_F = \bar{q}_0 \left(\frac{R_F}{2} - \frac{R_F^3}{4R_F^2} \right) \qquad (\text{at } r = R_F)$$

or $$\frac{C_2}{R_F} = \frac{\bar{q}_0 R_F}{4}$$

∴ $$\boxed{C_2 = \frac{\bar{q}_0 R_F^2}{4}}$$

The heat flux through the fuel rod and cladding may now be re-written as

$$q_F = -k_F \frac{dT_F}{dr} = \bar{q}_0 \left(\frac{r}{2} - \frac{r^3}{4R_F^2} \right) \quad \Rightarrow \quad \frac{dT_F}{dr} = \frac{\bar{q}_0}{k_F} \left[\frac{r^3}{4R_F^2} - \frac{r}{2} \right]$$

and $$q_C = -k_C \frac{dT_C}{dr} = \frac{\bar{q}_0 R_F^2}{4r} \quad \Rightarrow \quad \frac{dT_C}{dr} = -\frac{\bar{q}_0 R_F^2}{4rk_C}$$

To obtain the temperatures T_F and T_C in the fuel rod and cladding, respectively, we can integrate the expressions for temperature gradients, $\frac{dT_F}{dr}$ and $\frac{dT_C}{dr}$

It follows that

$$\int dT_F = \frac{\bar{q}_0}{k_F} \int \left[\frac{r^3}{4R_F^2} - \frac{r}{2} \right] dr \quad \text{or} \quad \boxed{T_F = \frac{\bar{q}_0}{k_F} \left(\frac{r^4}{16R_F^2} - \frac{r^2}{4} \right) + C_3}$$

and $$\int dT_C = -\frac{\bar{q}_0 R_F^2}{4k_C} \int \frac{dr}{r} \quad \text{or} \quad \boxed{T_C = \frac{-q_0 R_F^2}{4k_C} \ln r + C_4}$$

The constants of integration, C_3 and C_4, can be evaluated by applying the following boundary conditions.

BC (III): $T_C = T_w$, at $r = R_C$ (*at the outer surface of cladding*)

BC (IV): $T_C = T_F$ at $r = R_F$ (*at the interface*)

Then, from Eq. and BC (III):

$$\boxed{C_4 = T_w + \frac{\overline{q}_0 R_F^2}{4k_C} \ln R_C}$$

Substituting for C_4, one gets

$$T_C = \frac{-\overline{q}_0 R_F^2}{4k_C} \ln r + \left(T_w + \frac{\overline{q}_0 R_F^2}{4k_C} \ln R_C \right)$$

or

$$\boxed{T_C - T_w = \frac{\overline{q}_0 R_F^2}{4k_C} \ln\left(\frac{R_C}{r}\right)}$$

This gives the temperature drop across the cladding.
And, from Eq. and BC (IV):

$$T_F = \frac{\overline{q}_0}{k_F}\left(\frac{R_F^4}{16R_F^2} - \frac{R_F^2}{4}\right) + C_3 = T_C \quad \text{or} \quad T_F = C_3 - \frac{3\overline{q}_0 R_F^2}{16k_F} = T_C \qquad (at\ r = R_F)$$

$$\therefore \quad C_3 = T_C + \frac{3\overline{q}_0 R_F^2}{16k_F} = \frac{3\overline{q}_0 R_F^2}{16k_F} + T_w + \left\{\frac{\overline{q}_0 R_F^2}{4k_C} \ln\left(\frac{R_C}{R_F}\right)\right\}$$

$$\therefore \quad \boxed{C_3 = T_w + \frac{\overline{q}_0 R_F^2}{4}\left[\frac{3}{4k_F} + \frac{1}{k_C}\ln\left(\frac{R_C}{R_F}\right)\right]}$$

Since
$$T_F = \frac{\overline{q}_0}{k_F}\left(\frac{r^4}{16R_F^2} - \frac{r^2}{4}\right) + C_3$$

The temperature variation in the fuel rod is given by substituting for C_3.

$$\boxed{T_F = \frac{\overline{q}_0}{k_F}\left(\frac{r^4}{16R_F^2} - \frac{r^2}{4}\right) + T_w + \frac{\overline{q}_0 R_F^2}{4}\left[\frac{3}{4k_F} + \frac{1}{k_C}\ln\left(\frac{R_C}{R_F}\right)\right]} \tag{3.55}$$

Maximum temperature in the fuel rod occurs at its centre.
Substituting $r = 0$, we obtain:

$$\boxed{T_{\max} = T_w + \frac{\overline{q}_0 R_F^2}{4}\left[\frac{3}{4k_F} + \frac{1}{k_C}\ln\left(\frac{R_C}{R_F}\right)\right]} \tag{3.56}$$

Illustrative Examples

(A) Plane Wall: Equal Surface Temperatures

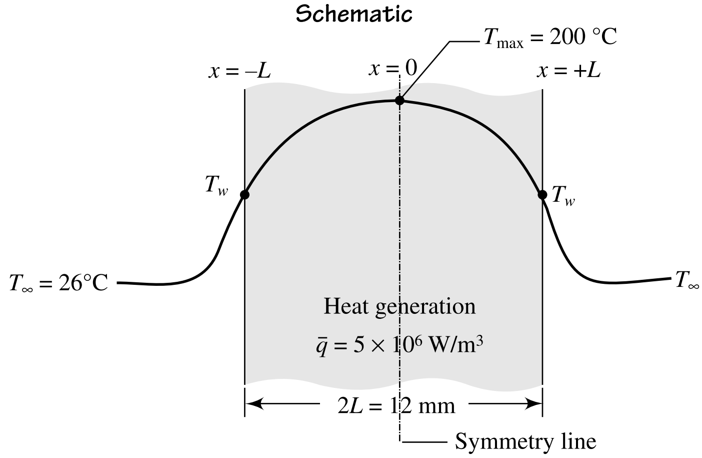
EXAMPLE 3.1 *A fluid of low electrical conductivity at 26°C is heated by a 12 mm thick and 75 mm wide iron plate. The heat is generated uniformly in the plate at a rate of 5 × 10⁶ W/m³ by passing an electrical current through it. Determine the required heat transfer coefficient to maintain the temperature of the plate below 200°C. The thermal conductivity of the plate material is 20 W/m K. Neglect the heat loss from the edges.*

Solution

Known Heat generated in a plate exposed to convection conditions.

Find Heat-transfer coefficient, h (W/m^2 K).

Schematic



Assumptions (1) Steady conditions. (2) Uniform heat generation. (3) Uniform heat-transfer coefficient. (4) Heat loss from the edges is neglected.

Analysis Temperature distribution in an infinite slab of thickness 2 L is given by

or $$(T_{max} - T_w) = \frac{\bar{q}L^2}{2k}$$

Surface temperature,

$$T_w = T_{max} - \frac{\bar{q}L^2}{2k} = 200°\text{C} - \frac{(5\times10^6\ \text{W/m}^3)(6\times10^{-3}\ \text{m})^2}{2\times 20\ \text{W/m K}}$$

$$= 195.5°\text{C}$$

Also, $$T_w - T_\infty = \frac{\bar{q}L}{h}$$

Heat-transfer coefficient,

$$h = \frac{\bar{q}L}{(T_w - T_\infty)} = \frac{(5\times10^6\ \text{W/m}^3)\,(6\times10^{-3}\ \text{m})}{(195.5-26.0)°\text{C or K}} = \mathbf{177\ W/m^2K} \qquad \textbf{(Ans.)}$$

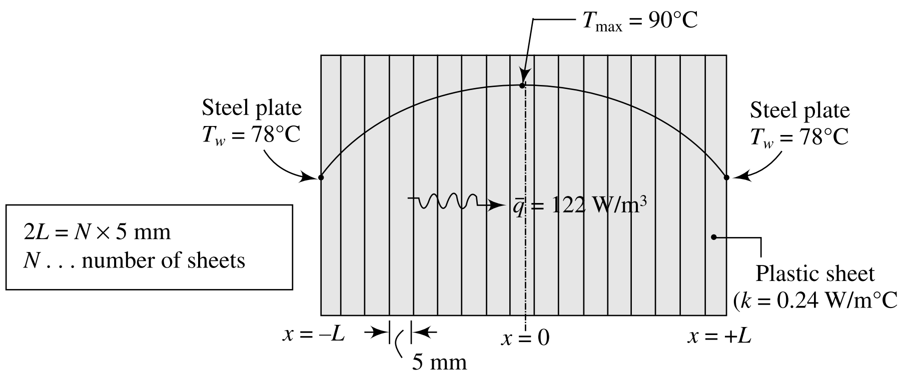
EXAMPLE 3.2 *Laminated sheets, 5 mm thick, are fabricated from single plastic sheets using a suitable adhesive. A number of such sheets are assembled and clamped between steel plates to prevent distortion during the process of hardening. The steel plates are held at 78°C and the maximum allowable temperature in the stack is 90°C. The heat evolved during hardening is equivalent to the volumetric uniform heat generation rate of 122 W/m³. The effective thermal conductivity of the sheet stack may be assumed to be 0.24 W/m°C. Estimate the number of sheets that may be processed at any time under these conditions.*

Solution

Known Laminated plastic sheets stacked together between steel plates. Heat generation during hardening of adhesive.

Find Number of sheets in the stack.

Schematic



Assumptions (1) Steady operating conditions. (2) One-dimensional conduction. (3) Uniform heat generation. (4) Thermal contact resistance due to adhesive is negligible.

Analysis For this geometry,

$$T_{max} - T_w = \frac{\overline{q}L^2}{2k}$$

where L is the half-thickness.

$$\therefore \qquad L = \left[\frac{2k}{q}(T_{max} - T_w)\right]^{1/2} = \left[\frac{2 \times 0.24 \text{ W/m °C}}{122 \text{ W/m}^3}(90 - 78)\text{°C}\right]^{1/2} = 0.2173 \text{ m}$$

Stack thickness, $2L = 2 \times 0.2173 = 0.4346$ m or 434.6 mm

$$\text{Number of sheets } N = \frac{2L\,(\text{mm})}{5 \text{ mm}} = \frac{434.6 \text{ mm}}{5 \text{ mm}} = \mathbf{87}$$ **(Ans.)**

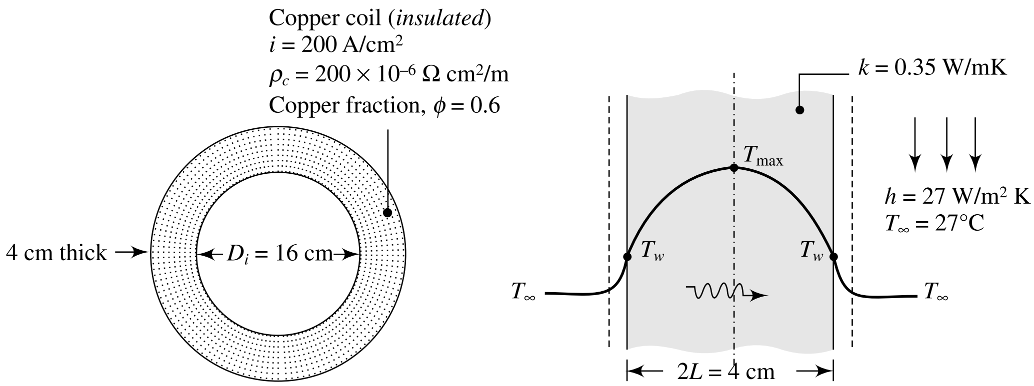
EXAMPLE 3.3 *A cylindrical transformer coil made of insulated copper wire has an inside diameter of 16 cm and an outside diameter of 24 cm. Sixty percent of the total cross section of the coil is copper and the rest is insulation. The current density is 200 A/cm² and the resistivity of copper is 200 × 10⁻⁶ ohm cm² per m. The heat-transfer coefficient on both sides of the coil is 27 W/m²K and the cooling air temperature is 27°C. The effective thermal conductivity of the coil is 0.35 W/m K. Determine the maximum temperature in the coil, assuming the coil to be a plane wall.*

Solution

Known Heat generation in a transformer coil with convective cooling and approximated as a plane wall.

Find Maximum temperature inside the coil.

Schematic



Assumptions (1) Steady operating conditions. (2) One-dimensional (x direction) conduction. (3) Constant properties. (4) Uniform heat-transfer coefficient. (5) Uniform heat generation.

Analysis Considering the transformer coil as a plane wall of thickness $2L$ = 4 cm exposed to convective environment, the maximum temperature in the coil (at the centre, i.e., x = 0) will be

$$T_{\max} = T_\infty + \frac{\overline{q}L}{h} + \frac{\overline{q}L^2}{2k} = T_w + \frac{\overline{q}L^2}{2k}$$

The rate of uniform volumetric thermal energy generation is determined to be

$$\overline{q} = \text{fraction of the coil that is copper (where heat is generated)} \times (\text{current density})^2$$

$$(\text{A/cm}^2)^2 \times \text{copper resistivity (ohm cm}^2\text{/m)} \left|\frac{1\text{ W}}{1A^2\text{ ohm}}\right|\frac{10^4\text{ cm}^2}{1\text{ m}^2}\right| = \phi\rho_e i^2$$

$$= 0.6 \times 200 \times 10^{-6} \times (200)^2 \times 10^4 = 48 \times 10^3 \text{ W/m}^3$$

Substituting the known values, we get

$$T_{\max} = T_\infty + \overline{q}L\left[\frac{1}{h} + \frac{L}{2k}\right]$$

$$= 27°\text{C} + (48 \times 10^3 \text{ W/m}^3)(0.02 \text{ m})\left[\frac{1}{(27\text{ W/m}^2\text{K})} + \frac{0.02\text{ m}}{(2 \times 0.35\text{ W/m K})}\right]$$

$= \mathbf{90.0°C}$ **(Ans.)**

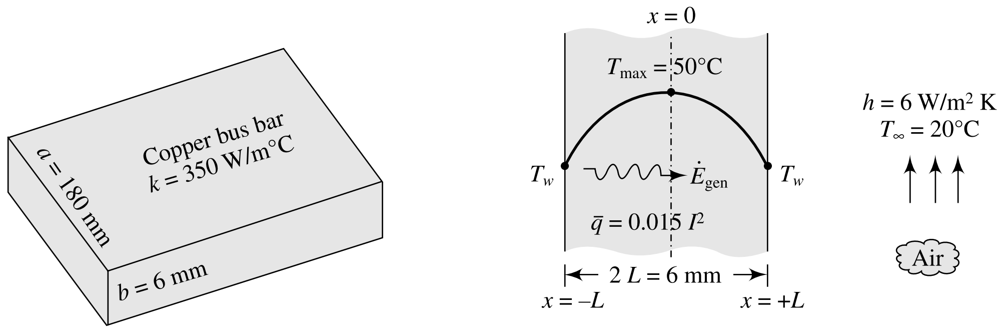
EXAMPLE 3.4 *A copper bus bar (*k *= 350 W/m °C) of rectangular cross section (6 mm × 180 mm) passes an electrical current* I *with the rate of volumetric heat generation,* $\overline{q}$ *= 0.015* I^2*. The bar is exposed to surrounding air at 20°C with an associated convection coefficient of 6 W/m²°C. Calculate the maximum permissible current that can be carried by the bus bar without allowing the temperature to exceed 50°C.*

Solution

Known Heat generation in a rectangular bus bar subjected to convection at the surface.

Find Maximum allowable current, I_{max}(A).

Schematic



Assumptions (1) Steady operating conditions. (2) One-dimensional conduction. (3) Uniform heat-transfer coefficient. (4) Uniform volumetric heat generation. (5) Constant properties.

Analysis Since $b << a$, heat will flow across 6 mm thick plane wall ($2L$ = 6 mm) exposed to convective cooling.

Maximum permissible temperature in the bus bar can be expressed as

$$T_{\max} = T_{(x=0)} = T_w + \frac{\overline{q}L^2}{2k} = T_\infty + \frac{\overline{q}L}{h} + \frac{\overline{q}L^2}{2k}$$

or $$T_{\max} - T_\infty = \overline{q}\left[\frac{1}{h} + \frac{L^2}{2k}\right] \quad \text{or} \quad 50 - 20 = 0.015 I_{\max}^2 \left[\frac{0.003}{6} + \frac{0.003^2}{2\times 350}\right]$$

or $$2000 = I_{\max}^2\,[5\times 10^{-4}]$$

Hence, the maximum allowable current capacity in the bus bar is

$$I_{\max} = \sqrt{\frac{2000}{5\times 10^{-4}}} = \mathbf{2000\ A} \qquad \textbf{(Ans.)}$$

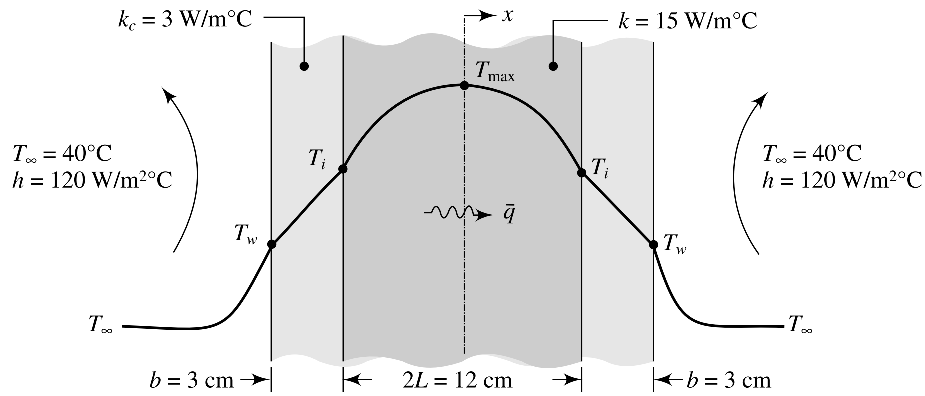
EXAMPLE 3.5 *A composite slab comprising a large slab, 12 cm thick, of thermal conductivity 15 W/m °C with uniform volumetric heat-generation rate of 100 kW/m³ sandwiched between two 30 mm thick slabs of thermal conductivity 3.0 W/m °C (with no heat generation). The two extreme surfaces are exposed to convective environment characterized by a convection coefficient of 120 W/m²°C and the fluid temperature of 40°C. Calculate (a) the interface temperatures of the composite slab, (b) the maximum temperature in the composite slab, and (c) the temperature gradient in each cover slab (without internal heat generation).*

Solution

Known A heat-generating slab covered by two slabs. Convective cooling of exposed surfaces.

Find (a) Interface temperatures, T_i (°C), (b) Maximum temperature, T_{max} (°C). (c) Temperature gradient in the cover slab, $\frac{dT}{dx}$(°C/m).

Schematic



Assumptions (1) Steady-state, one-dimensional conduction. (2) Heat generation in the central slab is uniform. (3) Constant properties. (4) Uniform heat transfer coefficient.

Analysis (a) Energy balance:

$$\underbrace{\cancel{\dot{E}_{in}}^{\,0}}_{\textbf{no heat inflow}} - \dot{E}_{\text{out}} + \dot{E}_{\text{gen}} = \underbrace{\cancel{\dot{E}_{st}}^{\,0}}_{\textbf{steady state}}$$

It follows that, $\dot{E}_{\text{gen}} = \dot{E}_{\text{out}}$

Heat generation rate, $\dot{E}_{\text{gen}} = \dot{E}_{\text{out}} = \overline{q}A(2L)$

By symmetry, the rate of heat transfer towards the left and right will be equal.
Per unit area, the heat-transfer rate passing through each cover slab will be

$$q = \frac{\dot{E}_{\text{out}}}{2A} = \frac{\dot{E}_{\text{gen}}}{2A} = \frac{\overline{q}A(2L)}{2A} = \overline{q}L = (100 \text{ kW/m}^3)(0.06 \text{ m}) = 6 \text{ kW/m}^2$$

We note that, $q = \dfrac{k_c}{b}(T_i - T_w)$

where k_c = thermal conductivity of the cover slab material
b = cover slab thickness
T_i = interface temperature
T_w = outer surface temperature

Therefore,

$$T_i - T_w = \frac{qb}{k_c} = \frac{(6 \times 10^3 \text{ W/m}^2)(0.03 \text{ m})}{3 \text{ W/m°C}} = 60°\text{C}$$

Also, $q = h(T_w - T_\infty)$

so that $$T_w = T_\infty + \frac{q}{h} = 40°\text{C} + \frac{6000 \text{ W/m}^2}{120 \text{ W/m}^2\,°C} = 90°\text{C}$$

Hence, the interface temperatures of the composite slab,

$T(\pm L) = T_i = 60 + 90 =$ **150°C** **(Ans.) (a)**

(b) In the heat-generating slab, the centre temperature (*maximum temperature*) is determined to be

$$T_{\max} = T_i + \frac{\overline{q}L^2}{2k} = 150°\text{C} + \frac{(100 \times 10^3\ \text{W/m}^3)(0.06\ \text{m})^2}{2 \times 15\ \text{W/m °C}} = \mathbf{162°C} \qquad \textbf{(Ans.) (b)}$$

(c) Temperature gradient in each cover slab is

$$\frac{dT}{dx} = \frac{T_w - T_i}{b} = \frac{(90 - 150)°\text{C}}{0.03\ \text{m}} = \mathbf{-2000°C/m} \qquad \textbf{(Ans.) (c)}$$

(B) Plane Wall: One Surface Insulated

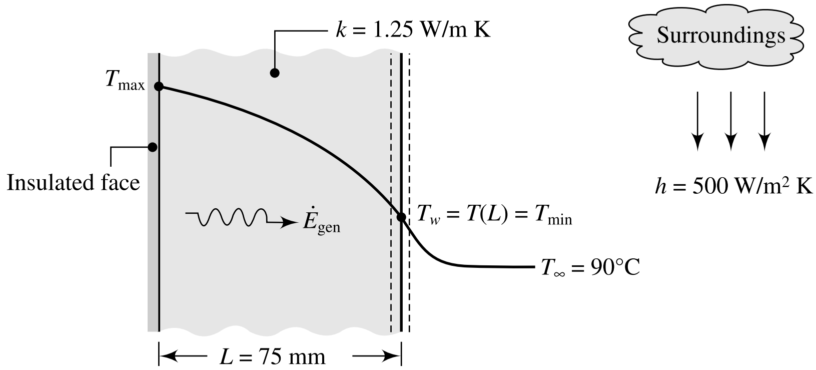
EXAMPLE 3.6 *A plane wall, 75 mm thick, generates heat internally. One side of the wall is insulated, and the other side is exposed to the surroundings at 90°C. The thermal conductivity of the wall is 0.25 W/m K and the convective heat transfer coefficient between the wall and the surroundings is 500 W/m² K. The maximum temperature gradient is limited to –5.4 × 10³ K/m to avoid thermal distortion. (a) Set up the appropriate differential equation and deduce an expression for the variation of temperature in the plate. (b) Compute the rate of volumetric heat generation in the plate. (c) Find the highest and the lowest temperatures in the plate and their locations.*

Solution

Known Heat generation in a plane wall with an insulated surface and the other surface exposed to convective environment.

Find (a) Temperature distribution in the plate. (b) Volumetric heat generation rate, and (c) Maximum and minimum temperatures, and their locations in the plate.

Schematic



Assumptions (1) Steady-state conditions. (2) Constant thermal conductivity. (3) One-dimensional conduction. (4) Uniform heat generation

Analysis (a) The one-dimensional, steady-state heat-conduction equation with internal thermal energy generation is

$$\frac{d^2T}{dx^2} = -\frac{\overline{q}}{k} \tag{A}$$

The boundary conditions are

At $x = 0$: $\frac{dT}{dx} = 0$ (*insulated side*)

At $x = 75$ mm: $-k\frac{dT}{dx} = h(T - T_\infty)$

Integrating Eq. (A) gives

$$\frac{dT}{dx} = -\frac{\overline{q}}{k}x + C_1 \tag{B}$$

Applying the boundary condition at $x = 0$ to Eq. (B),

$$\boxed{C_1 = 0}$$

Integrating Eq. (B), one has

$$T = -\frac{\overline{q}}{k}\frac{x^2}{2} + C_2 \tag{C}$$

Applying the boundary condition at $x = L = 75$ mm:

$$-k\frac{dT}{dx}\bigg|_{x=L} = h[T(L) - T_\infty] \tag{D}$$

Also, from Eq. (B),

$$-k\frac{dT}{dx}\bigg|_{x=L} = \overline{q}L$$

Maximum temperature gradient is at the exposed surface (at $x = L$), and is given by

$$\frac{dT}{dx}\bigg|_{x=L} = -\frac{\overline{q}L}{k} \tag{E}$$

From Eq. (D),

$\therefore$ $h\,[T(L) - T_\infty] = \overline{q}L$ or $T(L) = \frac{\overline{q}L}{h} + T_\infty$

where $T(L) = T_w$, the exposed surface temperature

Substituting this value into Eq. (C), one gets

$$\frac{\overline{q}L}{h} + T_\infty = -\frac{\overline{q}L^2}{2k} + C_2 \quad \Rightarrow \quad \boxed{C_2 = T_\infty + \frac{\overline{q}L}{h} + \frac{\overline{q}L^2}{2k}}$$

The temperature distribution is then obtained by substituting for C_2 in Eq. (C).

$$T(x) = T_\infty + \frac{\overline{q}L}{h} + \frac{\overline{q}}{2k}(L^2 - x^2)$$

or $$\boxed{T(x) = T_w + \frac{\overline{q}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)}$$ **(Ans.) (a)**

(b) From Eq. (E): $\frac{dT}{dx}\bigg|_{x=L} = -\frac{\overline{q}L}{k} = -5.4 \times 10^3$ K/m

Hence, the rate of uniform volumetric heat generation is

$$\bar{q} = -\frac{k}{L}\frac{dT}{dx}\bigg|_{x=L} = -\frac{1.25\ \text{W/m K}}{75\times10^{-3}\ \text{m}}\times(-5.4\times10^{3}\ \text{K/m})$$

$$= \mathbf{90 \times 10^3\ W/m^2} \qquad \textbf{(Ans.) (b)}$$

(c) Maximum (*highest*) wall temperature will obviously occur at $x = 0$, i.e., at the insulated surface, and the minimum (*lowest*) temperature will occur at $x = L$, i.e., at the exposed surface. These values are

$$T_{\min} = T_w = T(L) = T_\infty + \frac{\bar{q}L}{h} = 90°\text{C} + \frac{(90\times10^3\ \text{W/m}^3)\,(75\times10^{-3}\,\text{m})}{500\ \text{W/m}^2\text{K}}$$

$$= \mathbf{103.5°C} \qquad \textbf{(Ans.) (c)}$$

$$T_{\max} = \underbrace{T_\infty + \frac{\bar{q}L}{h}}_{T_w} + \frac{\bar{q}L^2}{2k} = 103.5°\text{C} + \frac{(90\times10^3\ \text{W/m}^3)\times(75\times10^{-3})^2\,\text{m}^2}{2(1.25\ \text{W/m K})}$$

$$= \mathbf{306°C} \qquad \textbf{(Ans.) (c)}$$

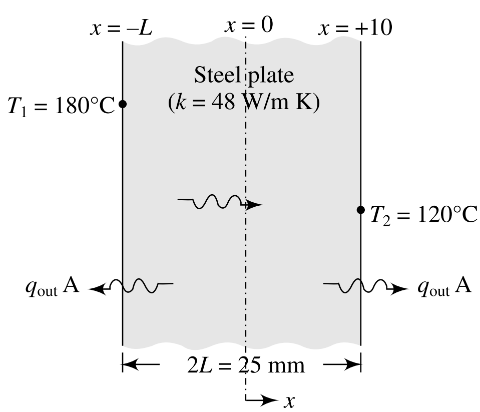
EXAMPLE 3.7 *Consider a large 25 mm thick steel plate (*k *= 48 W/m K) in which heat is generated uniformly at a rate of 30 MW/m³. The two sides of the plate are maintained at 180°C and 120°C respectively. Determine (a) the location where the maximum temperature in the plate will occur, (b) the maximum temperature in the plate and the centreline temperature, and (c) the heat transferred at the two sides of the plate per unit area. Also sketch the temperature distribution.*

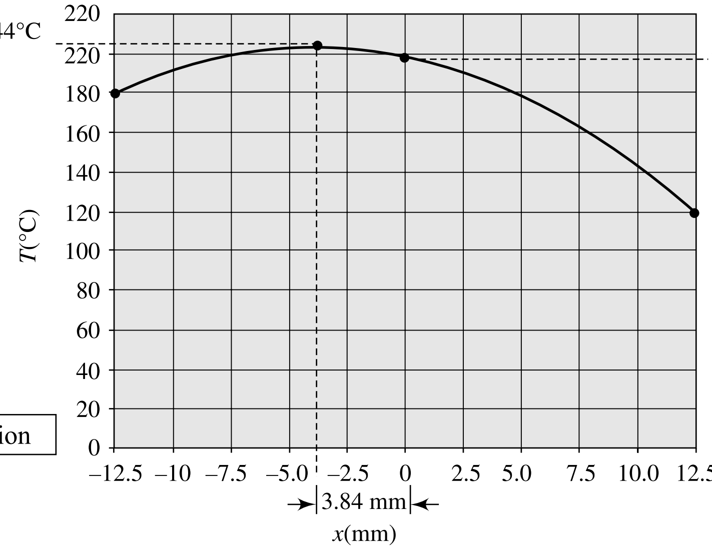
Solution

Known Heat is uniformly generated in a large steel plate whose two sides are maintained at different temperatures.

Find (a) x where $T_{\max}$ will occur, (b) $T_{\max}$, (c) $\dot{Q}$ at $x = -L$ and $x = +L$.

Schematic





Temperature distribution

Assumptions (1) Steady-state, one-dimensional heat conduction. (2) Constant thermal conductivity. (3) Uniform internal thermal energy generation.

Analysis (a) The governing differential equation is

$$\frac{d^2T}{dx^2} = -\frac{\overline{q}}{k}$$

Integrating twice with respect to x, one gets

Temperature gradient, $\dfrac{dT}{dx} = -\dfrac{\overline{q}x}{k} + C_1$ (A)

Temperature at any distance x from the centreline is given by the temperature distribution across the plate and is expressed as

$$T(x) = -\frac{\overline{q}x^2}{2k} + C_1 x + C_2 \tag{B}$$

To evaluate the two integration constants, the following two boundary conditions (BC) can be applied:

BC (I): At $x = -L$, $T = T_1 = 180°C$

BC (II): At $x = +L$, $T = T_2 = 120°C$

Therefore, from Eq. (B), we have

$$T_1 = -\frac{\overline{q}(-L)^2}{2k} + C_1(-L) + C_2 \quad \text{and} \quad T_2 = -\frac{\overline{q}(+L)^2}{2k} + C_1(+L) + C_2$$

Subtracting one from the other,

$$T_1 - T_2 = -2C_1L \qquad \text{or} \qquad \boxed{C_1 = -\left(\frac{T_1 - T_2}{2L}\right)}$$

or $\quad C_1L = T_2 - T_1)/2$

And $$C_2 = T_1 + \frac{\overline{q}L^2}{2k} + C_1 L$$

or $$\boxed{C_2 = \frac{\overline{q}L^2}{2k} + \frac{1}{2}(T_1 + T_2)}$$

Substituting C_1 and C_2 in Eq. (B), we have

$$T(x) = -\frac{\overline{q}x^2}{2k} + \frac{x}{2L}(T_2 - T_1) + \frac{\overline{q}L^2}{2k} + \frac{1}{2}(T_2 + T_1)$$

or $$\boxed{T(x) = \frac{\overline{q}L^2}{2k}\left\{1 - \left(\frac{x}{L}\right)^2\right\} + \frac{x}{2L}(T_2 - T_1) + \frac{1}{2}(T_2 + T_1)} \quad \text{(C)}$$

The temperature gradient,

$$\frac{dT}{dx} = -\frac{\overline{q}x}{k} + \left(\frac{T_2 - T_1}{2L}\right)$$

Maximum temperature will occur where $\dfrac{dT}{dx} = 0$

$$\therefore \quad 0 = \frac{-\overline{q}x}{k} + \frac{(T_2 - T_1)}{2L}$$

Location x from the centreline of the plate is then given by

$$\boxed{x = \frac{k}{\overline{q}}\left(\frac{T_2 - T_1}{2L}\right)} \quad \text{(D)}$$

Location $x = \dfrac{48 \text{ W/mK}}{30 \times 10^6 \text{ W/m}^3} \times \dfrac{(120 - 180)^\circ\text{C or K}}{25 \times 10^{-3} \text{ m}}$

$= \mathbf{-3.84 \times 10^{-3}}$ **m** or **–3.84 mm** **(Ans.) (a)**

Substituting this value of x in Eq. (C) will yield the maximum temperature in the plate. Maximum temperature,

$$T_{\max} = \frac{(30 \times 10^6 \text{ W/m}^3)(12.5 \times 10^{-3} \text{m})^2}{2(48 \text{ W/m K})}\left\{1 - \left(\frac{-3.84 \text{ mm}}{12.5 \text{ mm}}\right)^2\right\}$$
$$+ \left(\frac{-3.84 \text{ mm}}{12.5 \text{ mm}}\right)\left(\frac{120 - 180}{2}\right)^\circ\text{C} + \frac{1}{2}(120 + 180)^\circ\text{C} = \mathbf{203.44^\circ C}$$

(b) Centreline temperature is

$$T(x = 0) = \frac{\overline{q}L^2}{2k} + \frac{1}{2}(T_1 + T_2) = \frac{(30 \times 10^6 \text{ W/m}^3)(12.5 \times 10^{-3} \text{m})^2}{2(48 \text{ W/m K})} + \frac{1}{2}(180 + 120)^\circ\text{C}$$

$=$ **198.83°C** **(Ans.) (b)**

(c) Heat transferred per unit area at $x = +L$ is

$$\left.\frac{\dot{Q}}{A}\right|_{x=L} = -k\left.\frac{dT}{dx}\right|_{x=L} = -k\left[\frac{-\overline{q}x}{k} + \frac{T_2 - T_1}{2L}\right]_{x=L} = \overline{q}L + \frac{(T_1 - T_2)k}{2L}$$

$$= (30 \times 10^6 \text{ W/m}^3)\,(0.0125 \text{ m}) + \frac{(180 - 120)^\circ\text{C}\,(38 \text{ W/mK})}{0.025 \text{ m}}$$

$=$ **490.2 × 103 W/m²** **or** **+490.2 kW/m²** **(Ans.) (c)** i.e., q_{out} (right side) = 490.2 kW/m^2

Heat transferred per unit area at $x = -L$ is

$$\left.\frac{\dot{Q}}{A}\right|_{x=-L} = -k\left.\frac{dT}{dx}\right|_{x=-L} = -k\left[-\frac{\overline{q}(-L)}{k} + \frac{T_2 - T_1}{2L}\right] = -\overline{q}L + \frac{(T_1 - T_2)k}{2L}$$

$$= -(30 \times 10^6 \text{ W/m}^3)(0.0125 \text{ m}) + \frac{(180 - 120)°\text{C}\,(48 \text{ W/m K})}{0.025 \text{ m}}$$

$= \mathbf{-\ 259.8 \times 10^3\ W/m^2}$ or $\mathbf{-259.8\ kW/m^2}$ (*in positive x-direction*)

or q_{out} (left side) = – (–259.8 kW/m^2) = **259.8 kW/m²** **(Ans.) (c)**

Total heat dissipated per unit area,

$$q_{out} = (490.2 + 259.8) \text{ kW/m}^2 = 750 \text{ kW/m}^2$$

Heat generated within the plate per unit area is

$$\frac{\dot{Q}_{gen}}{A} = \frac{\overline{q}A(2L)}{A} = \overline{q}(2L) = (30 \times 10^6 \text{ W/m}^3)\,(25 \times 10^{-3} \text{m})$$

$$= \mathbf{750 \times 10^3\ W/m^2} \quad \text{or} \quad \mathbf{750\ kW/m^2}$$

This satisfies the energy balance. $[\dot{E}_{gen} = \dot{E}_{out}$ as $\dot{E}_{in} = 0$ and $\dot{E}_{st} = 0]$

(C) Plane Wall: Different End Surface Temperatures

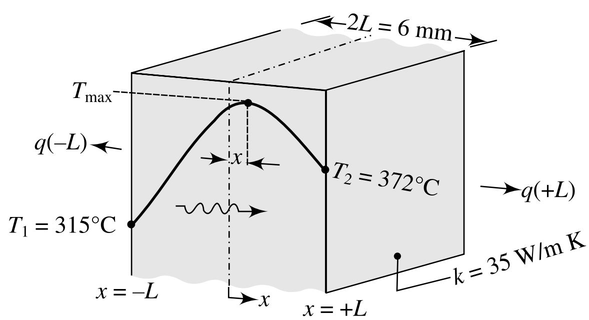
EXAMPLE 3.8 *A reactor core uses enriched uranium plates 6 mm thick (*k *= 35 W/m K) and they are subjected to an internal heat generation rate of 8.3 × 10^8 W/m^3 due to the fission process. One of these plates is near the edge of the core and the coolant maintains one side of the plate at 315°C and the other side at 372°C. Calculate (a) the maximum temperature in the plate under these conditions, and (b) the heat flux at the two surfaces.*

Solution

Known Thermal-energy generation in uranium plates due to nuclear fission under specified operating conditions.

Find (a) Maximum temperature in the plate. (b) Heat flux at the left and right faces.

Schematic



Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant properties. (3) Uniform volumetric thermal-energy generation.

Analysis The temperature distribution in an infinite plate of thermal conductivity (k) with uniform thermal-energy generation per unit volume, $\overline{q}$ and surface temperatures T_1 and T_2 is given by

$$T(x) = \frac{\overline{q}L^2}{2k}\left\{1-\left(\frac{x}{L}\right)^2\right\} + \frac{x}{2L}(T_2 - T_1) + \frac{1}{2}(T_2 + T_1)$$

Maximum temperature will occur at a location x from the centre where $\frac{dT}{dx} = 0$.
Differentiating $T(x)$ with respect to x and equating to zero, we have

$$\frac{dT(x)}{dx} = 0 = \frac{\overline{q}L^2}{2k}\left(0 - \frac{2x}{L^2}\right) + \left(\frac{T_2 - T_1}{2L}\right) + 0$$

or $$\frac{-\overline{q}x}{k} + \frac{T_2 - T_1}{2L} = 0 \quad \text{and} \quad \boxed{x = \frac{k}{\overline{q}}\left(\frac{T_2 - T_1}{2L}\right)}$$

Substituting values,

$$x = \frac{35 \text{ W/m K}}{8.3\times 10^8 \text{ W/m}^3} \times \frac{(372-315)°\text{C or K}}{6\times 10^{-3}\text{ m}}\left|\frac{10^3 \text{ mm}}{1 \text{ m}}\right|$$

$$= \mathbf{0.4\ mm}$$

Hence, the maximum temperature in the plate is

$$T_{\max} = T(x = 0.4 \text{ mm}) = \frac{8.3\times 10^8 \text{ W/m}^3 \times (0.003 \text{ m})^2}{2\times 35 \text{ W/m K}}\left\{1-\left(\frac{0.4 \text{ mm}}{3 \text{ mm}}\right)^2\right\}$$

$$+\left\{(372-315)°\text{C} \times \frac{0.4 \text{ mm}}{6 \text{ mm}}\right\} + \frac{(372+315)°\text{C}}{2}$$

$= \mathbf{452.1°C}$ **(Ans.) (a)**

Heat flux at the left face ($x = -L$) is determined from

$$q(-L) = -k\left.\frac{dT}{dx}\right|_{x=-L} = (-35 \text{ W/m K})\left[\frac{(-8.3\times 10^8 \text{ W/m}^3)(-3\times 10^{-3}\text{ m})}{35 \text{ W/m K}} + \frac{(372-315)^\circ\text{C}}{6\times 10^{-3}\text{ m}}\right]$$

$= \mathbf{2822.5 \times 10^3\ W/m^2}$ **(Ans.) (b)**

Heat flux at the right face ($x = +L$) is determined from

$$q(+L) = -k\left.\frac{dT}{dx}\right|_{x=L} = (-35 \text{ W/m K})\left[\frac{(-8.3\times 10^8 \text{ W/m}^3)(-3\times 10^{-3}\text{ m})}{35 \text{ W/m K}} + \frac{(372-315)^\circ\text{C}}{6\times 10^{-3}\text{ m}}\right]$$

$= \mathbf{2157.5 \times 10^3\ W/m^2}$ **(Ans.) (b)**

(D) Long Cylinder

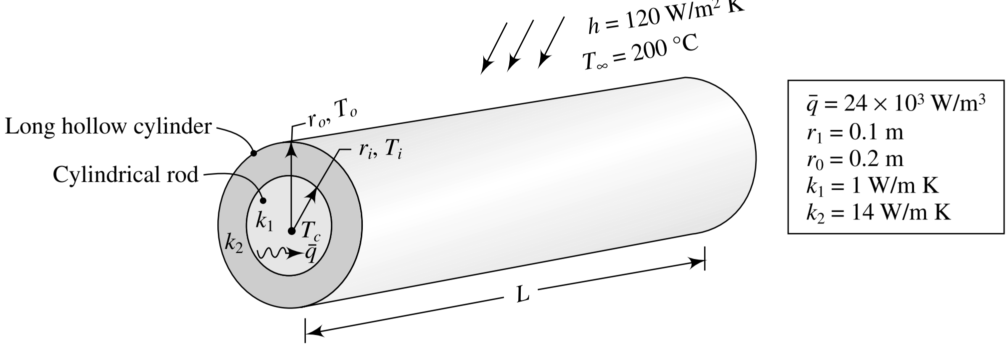
EXAMPLE 3.9 *A long solid cylindrical rod of 10 cm radius is made of a material (*k *= 1 W/m K) generating 24 × 10³ W/m³ uniformly throughout its volume. This rod is tightly encapsulated within a long hollow cylinder (*k *= 14 W/m K), whose inner radius is 10 cm and outer radius is 20 cm. The outer surface is surrounded by a fluid at 200°C and the convective heat transfer coefficient between the surface and the fluid is 120 W/m²K. Find (a) the temperature at the outer surface of the outer cylinder, (b) the temperature at the interface between the two cylinders, and (c) the temperature at the centre of the inner cylinder.*

Solution

Known A long cylindrical rod experiencing uniform internal heat generation is tightly enclosed within a long hollow cylinder exposed to convective environment.

Find (a) T_o (°C), (b) T_i (°C), (c) T_c (°C).

Schematic



Assumptions (1) Steady-state conditions. (2) Uniform volumetric heat generation. (3) Constant properties.

Analysis Consider a rod of length, L [m].

Thermal (conductive) resistance of the hollow cylinder,

$$R_{\text{cond}} = \frac{\ln(r_o/r_i)}{2\pi k_2 L} = \frac{\ln\left(\dfrac{20\text{ cm}}{10\text{ cm}}\right)}{2\pi \times 14\text{ (W/mK)} \times L(m)} = \frac{7.8798 \times 10^{-3}}{L}\text{ K/W}$$

Thermal convective resistance on the outer surface,

$$R_{\text{conv}} = \frac{1}{h(2\pi r_o L)} = \frac{1}{120\text{ W/m}^2\,K \times 2\pi \times 0.2m \times L\,(m)} = \frac{6.6315 \times 10^{-3}}{L}\,\frac{K}{W}$$

Heat-flow rate,

$$\dot{Q} = \dot{Q}_{\text{gen}} = \overline{q}\,\forall = \overline{q}\,(\pi r_i^2 L) = 24 \times 10^3\text{ W/m}^3 \times \pi \times 0.1^2\text{ m}^2 \times L(\text{m})$$

$$= 753.9822\,L \qquad \text{[W]}$$

(a) Temperature at the outer surface of the outer cylinder,

$$T_o = T_\infty + \dot{Q} \cdot R_{\text{conv}} = 200°\text{C} + (753.9822L)\text{ W} \times \left(\frac{6.6315 \times 10^{-3}}{L}\right)\text{ [K/W]}$$

$= \mathbf{205.000°C}$ **(Ans.) (a)**

(b) Temperature at the interface between the two cylinders,

$$T_i = T_o + \dot{Q} \cdot R_{\text{cond}} = 205.000°\text{C} + (753.9822L)\text{ [W]} \times \left(\frac{7.8798 \times 10^{-3}}{L}\right)\text{ [K/W]}$$

$= 205.000 + 5.941 = \mathbf{210.941°C}$ **(Ans.) (b)**

(c) Temperature at the centre of the inner cylinder,

$$T_c = T_i + \frac{\overline{q} r_1^2}{4k_1} = 210.941 + \frac{\overline{q} r_1^2}{4k_1}$$

$$= 210.941°\text{C} + \frac{24 \times 10^3\ \text{W/m}^3 \times 0.1^2\ \text{m}^2}{4 \times 1\ \text{W/m K}} = 210.941 + 60$$

$$= \mathbf{270.941°C} \qquad \textbf{(Ans.) (c)}$$

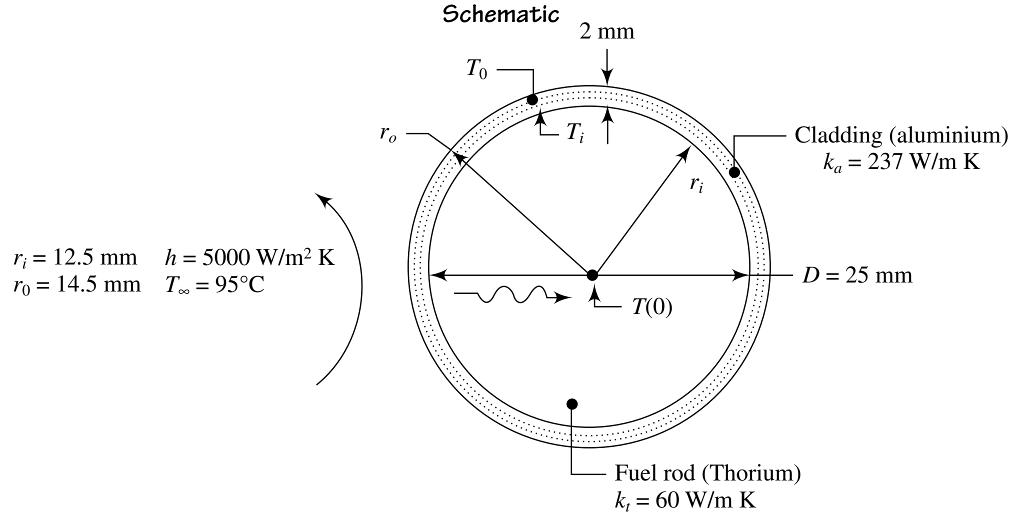
EXAMPLE 3.10 *In a nuclear reactor, the fuel rods consist of a thorium core wrapped in a thin (2 mm thick) aluminium cladding. Consider steady state conditions for which uniform heat generation occurs in the 25 mm diameter fuel rod at a volumetric rate of 600 MW/m³. The outer surface of the fuel rod is exposed to a coolant that is characterized by a temperature of 95°C and a convection coefficient of 5 kW/m² K. The melting point of thorium is 2023 K and that of aluminium is 933 K. The thermal conductivity of thorium is 60 W/m K and that of aluminium is 237 W/m K. Evaluate the safety and feasibility of the fuel element design.*

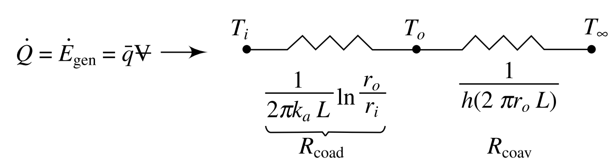
Solution

Known A thorium fuel rod with a thin aluminium cladding experiences heat generation and loses heat to the surrounding fluid.

Find Feasibility of the fuel-rod design.

Schematic





Equivalent thermal circuit

Assumptions (1) Steady-state, one-dimensional conduction with uniform internal heat generation. (2) Constant properties. (3) Uniform heat-transfer coefficient. (4) Negligible contact resistance between thorium and aluminium.

Analysis Maximum temperature of *thorium* will occur at the centre of the fuel rod T (r = 0) and that of *aluminium* will be at the outer surface of the rod $T(r = r_i)$. From the safety standpoint,

these temperatures must be well below the melting points of the respective materials.
Volumetric heat generation,

$$\overline{q} = \frac{\dot{Q}}{\pi r_i^2 L}$$

Heat-transfer rate per unit length,

$$\frac{\dot{Q}}{L} = \pi r_i^2 \overline{q} = \pi(12.5 \times 10^{-3}\,\text{m})^2 \times (600 \times 10^6\ \text{W/m}^3) = 93750\,\pi\ (\text{W/m})$$

Temperature drop = Heat rate × Total thermal resistance

$$\therefore \qquad T_i - T_\infty = \dot{Q}\left[\frac{1}{2\pi k_a L}\ln\frac{r_o}{r_i} + \frac{1}{h(2\pi r_o L)}\right]$$

$$= \frac{\dot{Q}}{2\pi L}\left[\frac{1}{k_a}\ln\frac{r_o}{r_i} + \frac{1}{h r_o}\right] = \frac{93750\pi}{2\pi}\left[\frac{1}{237}\ln\frac{14.5}{12.5} + \frac{1}{5000 \times 0.0145}\right]$$

$$= 46\,875[626.24 \times 10^{-6} + 0.01379] = 675.9°\text{C}$$

$\therefore \qquad T_i = 95 + 675.9 = \mathbf{770.9°C}$

Temperature drop across the aluminium cladding is

$$T_i - T_0 = \dot{Q} \times \frac{1}{2\pi k_a L}\ln\frac{r_o}{r_i} = 46\,875 \times 626.24 \times 10^{-6} = 31.7°\text{C}$$

Also, temperature drop across the convective film is

$$T_o - T_\infty = (T_i - T_\infty) - (T_i - T_o) = 675.9 - 31.7 = 644.2°\text{C}$$

Maximum temperature in the thorium fuel element will occur at the centre,

i.e., $$T_{\max,th} = T(0) = T_i + \frac{\overline{q} r_i^2}{4k_t}$$

$$= 770.9 + \frac{600 \times 10^6 \times 0.0125^2}{4 \times 60} = 1161.5°\text{C} \approx \mathbf{1435\ K}$$

This is well below the melting point of thorium (2023 K). Hence, the design is safe.
The maximum temperature of aluminium will equal

$$T_{\max,\ Al} = T_i = 770.9°\text{C} \approx 1044\ \text{K}$$

This is, however, above the melting point of aluminium (933 K) and therefore the cladding will melt which is not acceptable. The design is *unsafe* from this point of view under the proposed operating conditions. **(Ans.)**

Comment Note that the temperature drop across the *thorium rod* is $(T(0) - T_i)$ = (1161.5 – 770.9)°C = 390.6°C, while that across the *aluminium cladding* is 31.7°C, and across the convective film is a massive 644.2°C.

If there is 'loss of coolant' a remote possibility h would sharply *decrease*, raising T_i further. Using a cladding material with a higher melting point, or *increasing h* or *decreasing q* can get rid of this problem.

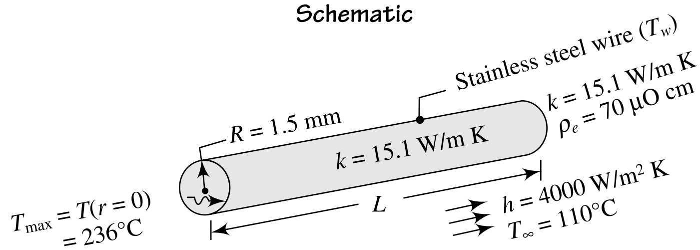
EXAMPLE 3.11 *Determine the current in amperes that is passed through a stainless steel wire (*k *= 15.1 W/m K), 3 mm in diameter. The electrical resistivity of the steel is 70 micro ohm cm and the maximum temperature of the wire is 236°C. The wire is submerged in a liquid at 110°C with a convection heat-transfer coefficient of 4000 W/m² K.*

Solution

Known Current-carrying wire experiencing heat generation dissipates heat to a convective environment.

Find Current, I (A).

Schematic



Assumptions (1) Steady-state, one-dimensional (radial) conduction. (2) Uniform heat generation (3) Constant properties.

Analysis Maximum (*Centre*) temperature of the wire,

$$T_{max} = T(r=0) = T_w + \frac{\overline{q}R^2}{4k}$$

where $$T_w = T_\infty + \frac{\overline{q}R}{2h}$$

It follows that

$$T_{max} - T_\infty = \frac{\overline{q}R}{2}\left[\frac{R}{2k} + \frac{1}{h}\right]$$

i.e., $$236 - 110 = \frac{\overline{q}}{2} \times 1.5 \times 10^{-3}\left[\frac{1.5 \times 10^{-3}}{2 \times 15.1} + \frac{1}{4000}\right] = \overline{q} \times 224.75 \times 10^{-9}$$

$\therefore$ $$\overline{q} = 126/224.75 \times 10^{-9} = 560.6 \times 10^6 \text{ W/m}^3$$

But $$\overline{q} = \frac{\dot{E}_{gen}}{\forall} = \frac{I^2 R_e}{\pi R^2 L} = \frac{I^2 \times \rho_e L}{\pi R^2 \times \pi R^2 L} = \frac{I^2 \rho_e}{(\pi R^2)^2} \qquad \left[\because \; R_e = \frac{\rho_e L}{A_c}\right]$$

$\therefore$ $$I^2 = \frac{560.6 \times 10^6 \times \pi^2 \times (0.0015)^4}{70 \times 10^{-6} \times 10^{-2}} = 40.0 \times 10^3$$

$\therefore$ **Current, I = 200 A** **(Ans.)**

EXAMPLE 3.12

(a) A long stainless steel bar of 20 mm × 20 mm square cross section is perfectly insulated on three sides and is maintained at a temperature of 400°C on the remaining side. Determine the maximum temperature in the bar when it is conducting a current of 1000 amp. The thermal and electrical conductivity of stainless steel may be taken as 16 W/m K and 1.5×10^4 (ohm cm)$^{-1}$ and the heat flow at the ends may be neglected.

(b) Now consider a long stainless steel circular rod of 20 mm diameter carrying 1000 A current with the outer-surface temperature at 400°C and properties same as those for the bar. Calculate the maximum temperature in the rod.

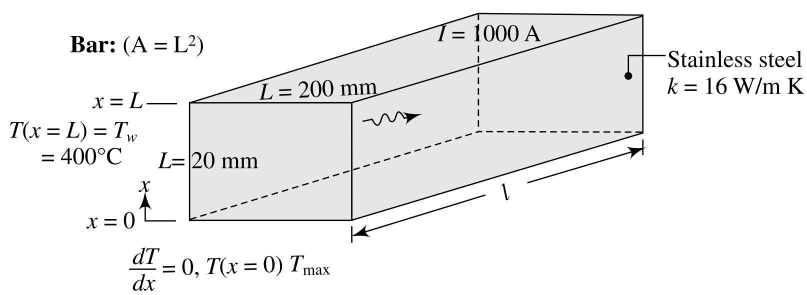
Solution

Known (a) A bar of square cross section insulated on three sides with both ends adiabatic carries current. (b) A rod of 20 mm diameter carries same currents under identical operating conditions.

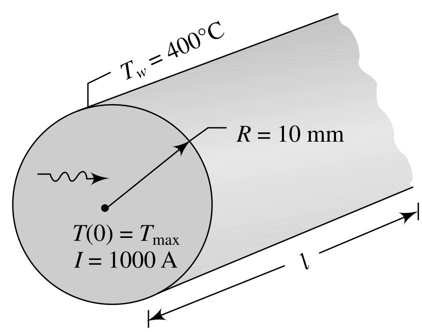
Find Maximum temperature, T_{max} in both cases (a) and (b).

Assumptions (1) Steady operating conditions. (2) One-dimensional (x-direction for the bar) and (r-direction for the rod) heat conduction. (3) Constant properties.

Schematic



(Bottom and side surfaces insulated. Top surface uninsulated. Heat flow from both left and right ends negligible.)



Analysis (a) **Bar of square cross section:**

Let x be measured across the bar from the insulated end where maximum temperature will occur (at $x = 0$, $\frac{dT}{dx} = 0$) towards the uninsulated side ($x = L$) from where heat will flow out and will be equal to the heat generated within.

We note that

$$\bar{q} = \frac{\dot{E}_{\text{gen}}}{\forall} = \frac{I^2 R_e}{Al} = \frac{I^2 \rho_e l}{A^2 l}$$

$$= \left(\frac{I}{A}\right)^2 \rho_e = \left(\frac{I}{A}\right)^2 \frac{1}{k_e} = \left[\frac{1000}{20 \times 20 \times 10^{-6}}\right]^2 \times \frac{1}{1.5 \times 10^4 \times 10^2}$$

$$= 4.167 \times 10^6 \text{ W/m}^3$$

For a plane wall with one surface *insulated*,

$$T_{\max} = T(x=0) = T_w + \frac{\overline{q}L^2}{2k}$$

where L is the *thickness*

Hence,

$$T_{\max} = 400°\text{C} + \frac{(4.167\times10^6\ \text{W/m}^3)(20\times10^{-3}\,\text{m})^2}{2\times16\ \text{W/m K}} = \mathbf{452.1°C} \qquad \textbf{(Ans.) (a)}$$

(b) **Long circular rod:**

For a long cylinder,

$$T_{\max} = T(r=0) = T_w + \frac{\overline{q}R^2}{4k}$$

In this case, $R = 0.01$ m

$$\overline{q} = \left(\frac{I}{A}\right)^2 \rho_e = \left(\frac{I}{\pi R^2}\right)^2 \frac{1}{k_e} = \left(\frac{1000}{\pi\times0.01^2}\right)\times\frac{1}{1.5\times10^4\times10^2}$$

$$= 6.755\times10^6\ \text{W/m}^3$$

Hence, $T_{\max} = 400 + \dfrac{(6.755\times10^6)(0.01)^2}{4\times16}$ = **410.6°C** **(Ans.) (b)**

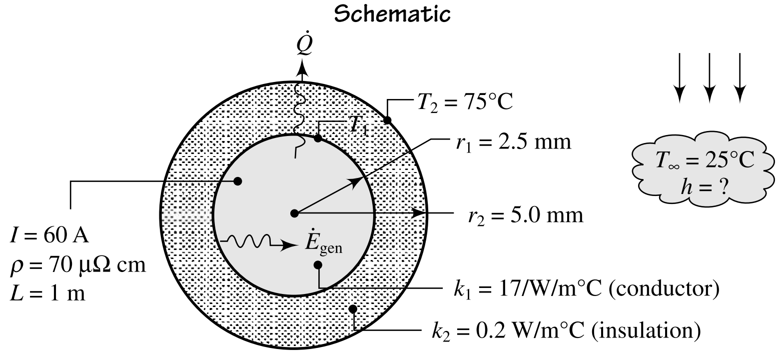
EXAMPLE 3.13 *A current-carrying conductor, 1 m long and 5 mm diameter, is made of stainless steel (k = 17 W/m °C) and is covered with a plastic insulation (k = 0.2 W/m °C), 2.5 mm thick. The outer surface of the insulation held at 75°C is exposed to convective environment at 25°C. The resistivity of the conductor is 70 μΩ cm and the current rating is 60 A. Determine (a) the convection heat-transfer coefficient, (b) the inner surface temperature of the insulation, and (c) the centre temperature of the conductor.*

Solution

Known Heat is generated in a current-carrying wire equipped with insulation and exposed to the convection process.

Find (a) h (W/m² °C); (b) T_1(°C), and (c) T_c or $T_{\max}$ (°C).

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties. (3) One-dimensional conduction.

Analysis Energy balance:

$$\underbrace{\cancel{\dot{E}_{in}}^{\,0}}_{\text{no heat inflow}} - \dot{E}_{out} + \dot{E}_{gen} = \underbrace{\cancel{\dot{E}_{st}}^{\,0}}_{\text{steady state}}$$

$\therefore \quad \dot{E}_{out} = \dot{E}_{gen}$

or $\dot{Q} = \overline{q}\,V$ where $\overline{q}$ is the rate of uniform volumetric heat generation (W/m^3) and V is the volume of the conductor wire $(= \pi r_1^2 L)$

(Rate of heat dissipation) = (Rate of heat generation, $\dot{Q} = I^2 R_e$)

where I is the current in amperes and R_e is the electric resistance of the wire in ohms

With $$R_e = \rho \frac{L}{A_c} = \frac{\rho L}{\pi r_1^2} = \frac{70 \times 10^{-6}\,\text{ohm cm} \times 100\,\text{cm}}{\pi \times (0.25\,\text{cm})^2} = 0.035\,65\,\Omega,$$

the heat-transfer rate is

$$\dot{Q} = I^2 R_e = (60\text{ A})^2\,(0.03565\ \Omega) = 128.34\text{ W}$$

Heat is transferred from the outer surface of the insulated wire to the surrounding environment by convection. That is,

$$\dot{Q} = hA_s\,(T_2 - T_\infty) = h(2\pi\, r_2 L)\,(T_2 - T_\infty)$$

The convective heat-transfer coefficient is

$$h = \frac{\dot{Q}}{(2\pi r_2 L)(T_2 - T_\infty)} = \frac{128.34\text{ W}}{(2\pi \times 0.005\text{ m} \times 1\text{ m})(75 - 25)°\text{C}}$$

$= \mathbf{81.7\ W/m^2\ °C}$ **(Ans.) (a)**

The temperature difference across the insulation is given by

$$T_1 - T_2 = \dot{Q}\,R_{cond}$$

where $$R_{cond} = \frac{1}{2\pi k_2 L}\ln\frac{r_2}{r_1} = \frac{1}{2\pi(0.2\text{ W/m°C})(1\text{ m})}\ln\frac{5\text{ mm}}{2.5\text{ mm}}$$

$$= 0.5516°\text{C/W}$$

Inner surface temperature of insulation or outer surface temperature of the wire is

$$T_1 = T_2 + \dot{Q}\,R_{cond} = 75°\text{C} + (128.34\text{ W})\,(0.5516°\text{C/W})$$

$= \mathbf{145.8°C}$ **(Ans.) (b)**

Heat is generated within the conductor uniformly and the maximum temperature will occur at the centre. The centre temperature is

$$T_c = T(r = 0) = T_1 + \frac{\overline{q} r_1^2}{4k_1} = T_1 + \frac{\dot{Q}}{\pi r_1^2 L} \times \frac{r_1^2}{4k_1} = T_1 + \frac{\dot{Q}}{4\pi L k_1}$$

$$= 146.8°\text{C} + \frac{128.34\text{ W}}{\pi \times 1\text{ m} \times 4 \times 17\text{ W/m°C}} = \mathbf{146.4°C}$$ **(Ans.) (c)**

Comment Note that there is just 0.6°C rise in temperature between the centre and the surface of the current-carrying conductor.

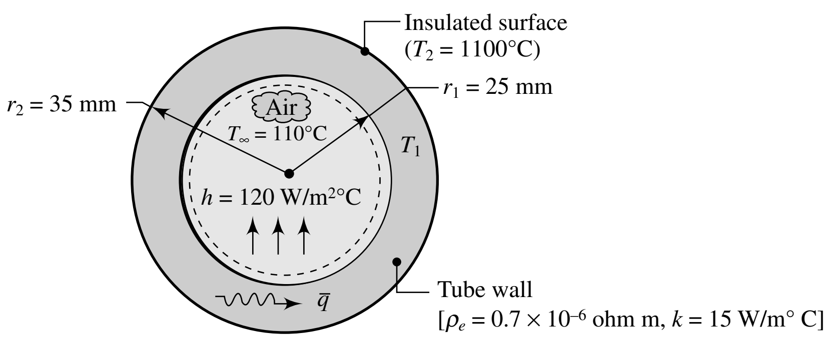
EXAMPLE 3.14 *A stainless steel tube (k = 15 W/m °C) with inside and outside diameters of 50 mm and 70 mm experiences internal heat generation induced by an electric current. The outer surface of the tube is effectively insulated and the heat is dissipated to the air flowing through the tube. The air temperature is 110°C and the heat transfer coefficient is 120 W/m²°C. The maximum allowable operating temperature is 1100°C. The electrical resistivity of the tube wall carrying the current is 0.70 micro-ohm m. Determine the temperature distribution within the tube wall and calculate the maximum permissible electric current.*

Solution

Known Heat generation in a hollow cylinder with outside surface insulated and inside surface exposed to convection.

Find Temperature distribution in the tube wall. Maximum allowable current, I (A).

Schematic



Assumptions (1) Steady-state radial heat conduction. (2) Uniform volumetric heat generation. (3) Constant properties and uniform convection coefficient.

Analysis The appropriate differential equation in cylindrical coordinates for steady-state, one-dimensional conduction with uniform heat generation is

$$\frac{1d}{rdr}\left(r\frac{dT}{dt}\right) = -\frac{\bar{q}}{k}$$

or
$$\frac{d}{dr}\left[r\frac{dT}{dr}\right] = -\frac{\bar{q}r}{k}$$

Integrating with respect to r, one has

$$r\frac{dT}{dr} = -\frac{\bar{q}r^2}{2k} + C_1$$

or Temperature gradient, $\dfrac{dT}{dr} = -\dfrac{\bar{q}r}{2k} + \dfrac{C_1}{r}$

Boundary condition at r = r₂

$$\left.\frac{dT}{dr}\right|_{r_2} = 0 \qquad \textit{(insulated surface)}$$

Hence, $\boxed{C_1 = \dfrac{\bar{q}r_2^2}{2k}}$

***Boundary condition at r = r*₁**

$$\dot{E}_{\text{gen}} = \dot{Q}_{\text{conv, out}} = \dot{Q}_{\text{cond, in}}$$

i.e. $$h[T_1 - T_\infty](2\pi r_1 L) = +k\left.\frac{dT}{dr}\right|_{r_1}(2\pi r_1 L)$$

[*Note that* $\dfrac{dT}{dr}$ *is positive since as r increases, T also increases*]

$$\left.\frac{dT}{dr}\right|_{r_1} = -\frac{\bar{q}r_1}{2k} + \frac{\bar{q}r_2^2}{2kr_1}$$

or $$(T_1 - T_\infty) = -\frac{\bar{q}r_1}{2h} + \frac{\bar{q}r_2^2}{2hr_1} = \frac{\bar{q}(r_2^2 - r_1^2)}{2hr_1} \quad \Rightarrow \quad T_1 = T_\infty + \frac{\bar{q}(r_2^2 - r_1^2)}{2hr_1}$$

We note that

$$\frac{dT}{dr} = -\frac{\bar{q}r}{2k} + \frac{\bar{q}r_2^2}{2kr}$$

Integrating with respect to r yields

$$T(r) = -\frac{\bar{q}r^2}{4k} + \frac{\bar{q}r_2^2}{2k}\ln r + C_2$$

At $r = r_1$,

$$T_1 = C_2 - \frac{\bar{q}r_1^2}{4k} + \frac{\bar{q}r_2^2}{2k}\ln r_1$$

Therefore,

$$C_2 = \frac{\bar{q}r_1^2}{4k} - \frac{\bar{q}r_2^2}{2k}\ln r_1 + T_1 \quad \Rightarrow \quad \boxed{C_2 = T_\infty + \frac{\bar{q}r_1^2}{4k} - \frac{\bar{q}r_2^2}{2k}\ln r_1 + \frac{\bar{q}(r_2^2 - r_1^2)}{2hr_1}}$$

Substituting for C_2 in the expression for $T(r)$, one gets

$$T(r) = T_\infty + \frac{\bar{q}}{4k}(r_1^2 - r^2) + \frac{\bar{q}r_2^2}{2k}(\ln r - \ln r_1) + \frac{\bar{q}(r_2^2 - r_1^2)}{2hr_1}$$

Simplifying,

$$\boxed{\boldsymbol{T(r) = T_\infty + \frac{\bar{q}(r_2^2 - r_1^2)}{2hr_1} + \frac{\bar{q}r_2^2}{4k}\left[2\ln\frac{r}{r_1} + \left(\frac{r_1}{r_2}\right)^2 - \left(\frac{r}{r_2}\right)^2\right]}} \qquad \textbf{(Ans.)}$$

At $r = r_2$, $T = T_2$

Hence $$T_2 - T_\infty = \bar{q}\left[\frac{r_2^2 - r_1^2}{2hr_1} + \frac{r_2^2}{4k}\left\{2\ln\frac{r_2}{r_1} + \left(\frac{r_1}{r_2}\right)^2 - 1\right\}\right]$$

or $$(1100 - 110)°\text{C} = \overline{q}\,(\text{W/m}^3)\left[\frac{(0.035 - 0.025^2)\ \text{m}^2}{2 \times 120\ \text{W/m}^2\,°\text{C} \times 0.025\ \text{m}} + \frac{0.035^2\ \text{m}^2}{4 \times 15\ \text{W/m}\,°\text{C}}\left\{2\ln\left(\frac{35\ \text{mm}}{25\ \text{mm}}\right) + \left(\frac{25\ \text{mm}}{35\ \text{mm}}\right)^2 - 1\right\}\right]$$

or $$990°\text{C} = \overline{q}(\text{W/m}^3)[(1 \times 10^{-4}) + (3.7393 \times 10^{-6})]\frac{\text{m}^3\,°\text{C}}{\text{W}}$$

Therefore,

$$\overline{q} = \frac{990°\text{C}}{1.0374 \times 10^{-4}\ \text{m}^3\,°\text{C/W}} = 9.543 \times 10^6\ \text{W/m}^3$$

Heat-generation rate,

$$\dot{E}_{\text{gen}} = \overline{q}\pi(r_2^2 - r_1^2)L = I^2 R_e$$

where $$R_e = \frac{\rho_e L}{\pi(r_2^2 - r_1^2)} \quad \text{or} \quad I^2 \times \frac{\rho_e L}{\pi(r_2^2 - r_1^2)} = \overline{q}\pi(r_2^2 - r_1^2)L$$

Maximum permissible current,

$$I = \pi(r_2^2 - r_1^2)\sqrt{\frac{\overline{q}}{\rho_e}} = \pi(0.035^2 - 0.025^2)\sqrt{\frac{9.543 \times 10^6}{0.7 \times 10^{-6}}} = \mathbf{6960\ A} \qquad \textbf{(Ans.)}$$

EXAMPLE 3.15 *A long cylindrical rod of 200 mm diameter with thermal conductivity of 0.5 W/m °C experiences uniform volumetric heat generation of 24 kW/m³. The rod is encapsulated by a circular sleeve having an outer diameter of 400 mm and a thermal conductivity of 4 W/m °C. The outer surface of the sleeve is exposed to cross flow of air at 27°C with a convective heat-transfer coefficient of 25 W/²°C.*

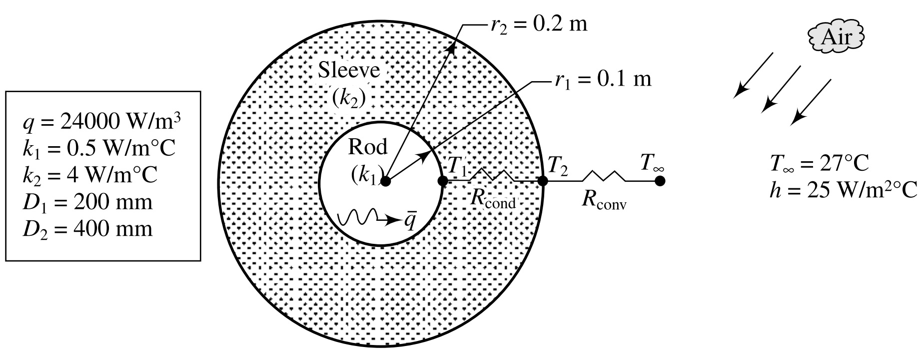
(a) Find the temperature at the interface between the rod and the sleeve on the outer surface. (b) What is the temperature at the centre of the rod?

Solution

Known A long rod experiencing heat generation is encapsulated by a sleeve exposed to convective atmosphere.

Find (a) Interface and outer surface temperatures. (b) Temperature at the centre of the rod.

Schematic



Assumptions (1) Steady-state radial conduction in rod and sleeve. (2) Uniform volumetric heat generation in the rod. (3) Negligible thermal contact resistance between the rod and the sleeve.

Analysis (a) The thermal resistance network shown in the schematic involves two resistances in series which are determined to be

$$R_{\text{cond}} = \frac{1}{2\pi k_2 L}\ln\frac{r_2}{r_1} = \frac{1}{2\pi \times 4\text{ W/m°C} \times 1\text{ m}}\ln\left(\frac{0.2\text{ m}}{0.1\text{ m}}\right)$$

$$= 0.0276\text{°C/W} \qquad (\textit{per m length})$$

$$R_{\text{conv}} = \frac{1}{h(2\pi r_2 L)} = \frac{1}{(25\text{ W/m}^2\text{°C})(2\pi \times 0.2\text{ m} \times 1\text{ m})}$$

$$= 0.03183\text{°C/W} \qquad (\textit{per m length})$$

$$R_{\text{total}} = R_{\text{cond}} + R_{\text{conv}} = 0.05943\text{°C/W} \quad (\textit{per m length})$$

From energy balance: $\dot{Q} = \dot{E}_{\text{gen}} = \overline{q}(\pi r_1^2 L)$

$\therefore$ $\dot{Q} = (24000\text{ W/m}^3)(\pi \times 0.1^2\text{ m}^2 \times 1\text{ m})$

$= 753.98\text{ W}$ (*per m length*)

We note that

$$T_1 - T_\infty = \dot{Q}R_{\text{total}} \quad \Rightarrow \quad T_1 = T_\infty + \dot{Q}R_{\text{total}}$$

Temperature at the interface between rod and sleeve is

$T_1 = 27\text{°C} + (753.98\text{ W})(0.05943\text{ °C/ W}) = \mathbf{71.8°C}$ **(Ans.) (a)**

Outer surface temperature is determined from

$$T_2 - T_\infty = \dot{Q}R_{\text{conv}} \quad \Rightarrow \quad T_2 = T_\infty + \dot{Q}R_{\text{conv}}$$

$\therefore$ $T_2 = 27\text{°C} + (753.98\text{ W})(0.03183\text{°C/W}) = \mathbf{51°C}$ **(Ans.) (a)**

(b) The temperature distribution within the rod with heat generation is

$$T(r) = T_1 + \frac{\overline{q}r_1^2}{4k}\left[1 - \frac{r^2}{r_1^2}\right]$$

Maximum temperature or temperature at the centre of the rod ($r = 0$) is

$T_0 = T_1 + \overline{q}r_1^2/4k = 71.8\text{°C} + (24\ 000\text{ W/m}^2)(0.1\text{ m})^2/4 \times 0.5\text{ W /m °C}$

$= \mathbf{191.8°C}$ **(Ans.) (b)**

EXAMPLE 3.16

(a) A hollow cylindrical copper bar, having internal and external diameters of 13 mm and 50 mm respectively, carries a current density of 5000 A/cm². The thermal conductivity is 381 W/m °C and the electrical resistivity is 2 micro-ohm cm. When the outer surface is maintained at 40°C and no heat is removed through the central hole, determine the position and value of the maximum temperature.

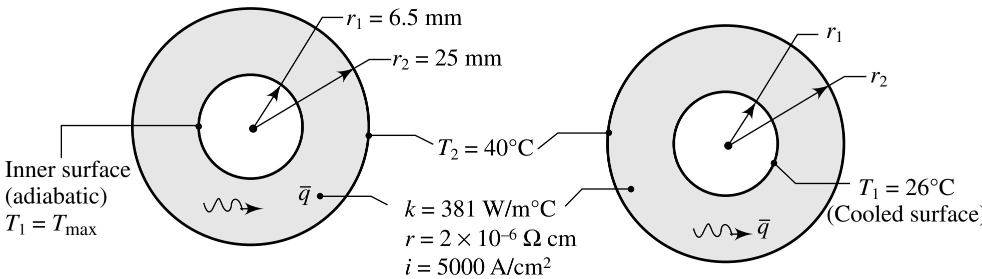
(b) If the inner surface is cooled to 26°C with the outer surface still at 40°C, find the position and value of the maximum temperature. Also calculate the heat removed internally and externally per metre length of the tube.

Solution

Known A hollow cylinder experiences internal heat generation. (a) Inner surface is adiabatic and outer surface temperature is specified. (b) Both inner and outer surface temperatures are prescribed.

Find (a) r_m and T_{max} if inner surface is insulated. (b) r_m and T_{max} if both inner and outer surface temperatures are prescribed. Heat transfer rates, $\dot{Q}\,(r = r_1)$ and $\dot{Q}\,(r = r_2)$.

Schematic



Assumptions (1) Steady operating conditions. (2) Uniform heat generation. (3) Constant properties. (4) One-dimensional (radial) conduction.

Analysis (a) The maximum temperature must occur where the temperature gradient, $\dfrac{dT}{dr} = 0$, i.e., at the *inner surface* which is effectively *insulated* (there is no heat removal, i.e., $\dot{Q}_1 = 0$)

$\therefore \qquad r_m = r_1 = \mathbf{6.5\ mm}$ **(Ans.) (a)**

The rate of uniform volumetric internal heat generation,

$$\bar{q} = \frac{\dot{Q}_{\text{gen}}}{A_c L} = \frac{I^2 R_e}{A_c L}$$

With $\qquad R_e = \rho \dfrac{L}{A_c} \quad \bar{q} = \dfrac{I^2 \rho L}{A_c^2 L} = \rho i^2$

where current intensity, $i = \dfrac{I}{A_c} = 5000$ A/cm²

It follows that

$$\bar{q} = (2 \times 10^{-6}\ \Omega\ \text{cm}) \times 5000^2\ \frac{A^2}{\text{cm}^4} = 50\ \Omega\text{A}^2/\text{cm}^3 \left|\frac{1\ \text{W}}{1\ \Omega\text{A}^2}\right| \left|\frac{10^6\ \text{cm}^3}{1\ \text{m}^3}\right|$$

$$= \mathbf{5 \times 10^7\ W/m^3}$$

For a hollow cylinder, with inner surface insulated:

$$T_1 - T_2 = \frac{\bar{q} r_1^2}{4k}\left[\left(\frac{r_2}{r_1}\right)^2 - 2\ln\left(\frac{r_2}{r_1}\right) - 1\right]$$

The maximum temperature, $T_{max} = T_1$ is obtained by substituting numerical values in the foregoing expression.

$$\therefore \qquad T_{max} = T_1 = 40°C + \frac{(5 \times 10^7 \text{ W/m}^3)(0.0065 \text{ m})^2}{4 \times 381 \text{ W/m°C}}\left[\left(\frac{25}{6.5}\right)^2 - 2\ln\left(\frac{25}{6.5}\right) - 1\right]$$

$= \mathbf{55.4°C}$ **(Ans.) (a)**

(b) For a hollow cylinder with prescribed surface temperatures, (At $r = r_1$, $T = T_1$ and at $r = r_2$, $T = T_2$), the radius at which maximum temperature occurs can be determined from

$$r_m = \sqrt{\frac{\bar{q}(r_2^2 - r_1^2) - 4k(T_1 - T_2)}{2\bar{q}\ln\left(\dfrac{r_2}{r_1}\right)}}$$

$$= \sqrt{\frac{5 \times 10^7 \text{ W/m}^3 \times (25^2 - 6.5^2)\left(10^{-6}\right)m^2 - 4 \times 381 \text{ W/m°C}(26 - 40)°C}{2 \times 5 \times 10^7 \text{ W/m}^3 \times \ln(25/6.5)}}$$

$= \mathbf{0.01936\ m}$ or $\mathbf{19.4\ mm}$ **(Ans.) (b)**

Maximum temperature,

$$T_{max} = T_2 + \frac{\bar{q}r_m^2}{4k}\left[\left(\frac{r_2}{r_m}\right)^2 - 2\ln\left(\frac{r_2}{r_m}\right) - 1\right]$$

$$= 40°C + \frac{(5 \times 10^7 \text{ W/m}^3)(0.0194 \text{ m})^2}{4 \times 381 \text{ W/m °C}}\left[\left(\frac{25}{19.4}\right)^2 - 2\ln\left(\frac{25}{19.4}\right) - 1\right]$$

$= \mathbf{41.9°C}$ **(Ans.) (b)**

Heat removed at the *inner* surface,

$$\dot{Q}_1 = \bar{q}\pi(r_m^2 - r_1^2)L = 5 \times 10^7 \text{ W/m}^3 \times \pi[19.36^2 - 6.5^2](10^{-6}) \text{ m}^2 \times 1 \text{ m}$$

$= \mathbf{52.2 \times 10^3\ W}$ or $\mathbf{52.2\ kW}$ **(Ans.) (b)**

Heat removed at the *outer* surface,

$$\dot{Q}_2 = \bar{q}\pi(r_2^2 - r_m^2)L = 5 \times 10^7 \text{ W/m}^3 \times [25^2 - 19.36^2](10^{-6}) \text{ m}^2 \times 1 \text{ m}$$

$= \mathbf{39.3 \times 10^3\ W}$ or $\mathbf{39.3\ kW}$ **(Ans.) (b)**

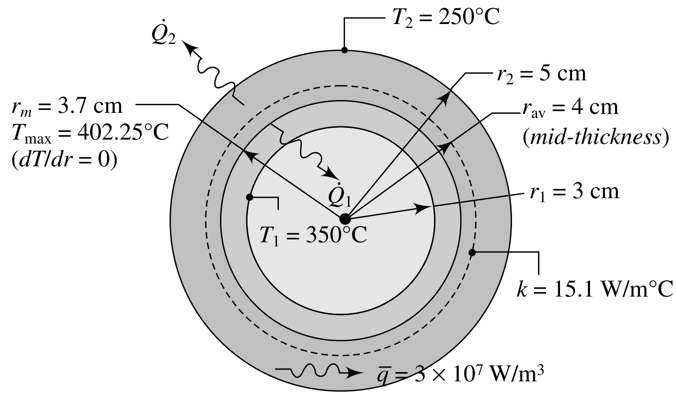
EXAMPLE 3.17 *A long hollow cylindrical shell made of stainless steel (*k *= 15.1 W/m °C) has 6 cm inside diameter and 10 cm outside diameter with uniform internal thermal energy generation at a constant rate of 3×10^7 W/m³. The inner and outer surface temperatures are 350°C and 250°C. Determine (a) the location and value of the maximum temperature, (b) the temperature at the mid thickness of the cylinder, and (c) the rate of heat removal at the inner surface as a percentage of the heat-generation rate.*

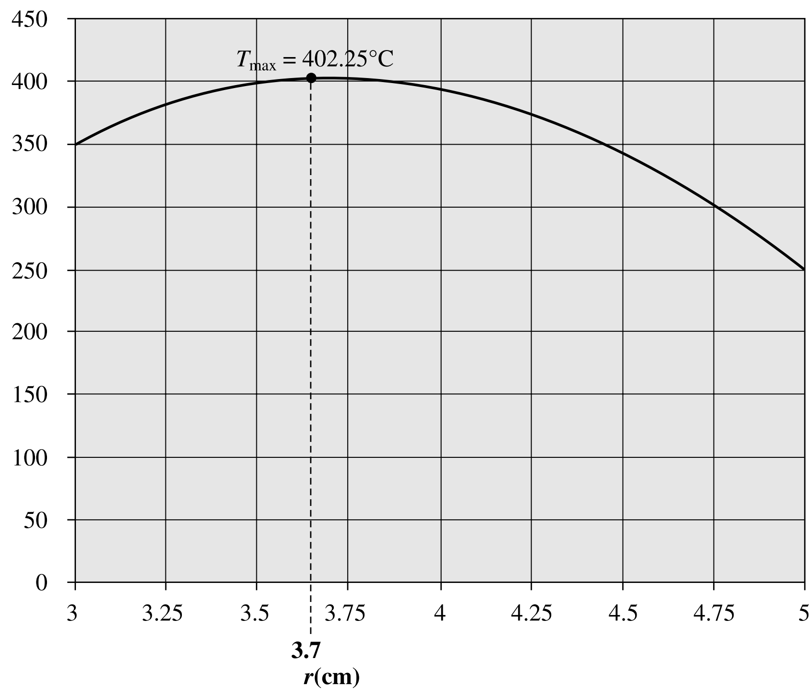
Solution

Known A hollow cylinder with internal heat generation has its surfaces maintained at prescribed temperatures.

Find (a) r_m, T_{max}. (b) T (r = 4 cm). (c) $(\dot{Q}_1/\dot{Q}_{gen})$.

Schematic





Assumptions (1) Steady-state conditions. (2) One-dimensional (r-direction) conduction. (3) Constant properties. (4) Uniform heat generation.

Analysis The relevant differential equation consistent with the foregoing assumptions is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\bar{q}}{k} = 0$$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\bar{q}r}{k}$$

Integrating with respect to r, one gets

$$r\frac{dT}{dr} = -\frac{qr^2}{2k} + C_1 \quad \text{or} \quad \frac{dT}{dr} = -\frac{\overline{q}r}{2k} + \frac{C_1}{r}$$

Integrating again, one obtains

$$T(r) = -\frac{\overline{q}r^2}{4k} + C_1 \ln r + C_2 \tag{A}$$

Boundary conditions:
 I. At $r = r_1$, $T = T_1$
II. At $r = r_2$, $T = T_2$
Substituting these in Eq. (A), one has

$$T_2 = -\frac{\overline{q}r_2^2}{4k} + C_2 \ln r_2 + C_2$$

$$T_1 = -\frac{\overline{q}r_1^2}{4k} + C_1 \ln r_1 + C_2 \tag{B}$$

Subtracting one from the other, one finds

$$T_2 - T_1 = C_1 \ln \frac{r_2}{r_1} - \frac{\overline{q}}{4k}(r_2^2 - r_1^2)$$

$$\therefore \qquad C_1 = \frac{(T_2 - T_1) + (\overline{q}/4k)(r_2^2 - r_1^2)}{\ln(r_2/r_1)}$$

$$= \frac{(250 - 350)°\text{C} + \left(\dfrac{3\times 10^7 \text{ W/m}^3}{4\times 15.1 \text{ W/m°C}}\right)(5^2 - 3^2)(10^{-4})\text{ m}^2}{\ln(5\text{ cm}/3\text{ cm})} = 1359.96$$

From Eq. (B):

$$C_2 = T_2 + \frac{\overline{q}r_2^2}{4k} - C_1 \ln r_2 = 250°\text{C} + \left(\frac{3\times 10^7 \text{ W/m}^3 \times 0.05^2\text{ m}^2}{4\times 15.1 \text{ W/m °C}}\right) - (1359.96)\times \ln 0.05$$

$$= 5565.795$$

Substituting for C_1 and C_2 in Eq. (A), the radial temperature distribution is given by

$$T(r) = -\frac{(3\times 10^7 \text{ W/m}^3)r^2}{4\times 15.1 \text{ W/m°C}} + 1359.96 \ln r + 5565.795$$

or
$$\boxed{T(r) = -496.689\times 10^3 r^2 + 1359.96 \ln r + 5565.795} \tag{C}$$

Temperature at the mid thickness of the cylindrical shell is

T (r = 0.04 m) = **393.55°C** **(Ans.) (b)**

Maximum temperature will occur at a position, r_m where $\dfrac{dT}{dr} = 0$.

$$\frac{dT}{dr} = 0 - 2 \times 496.689 \times 103\ r_m + 1359.96 \times \frac{1}{r_m}$$

Hence, $r_m = \sqrt{\dfrac{1359.96}{2 \times 496689}}\ \text{m} = \mathbf{0.037\ m\ or\ 3.7\ cm}$ **(Ans.) (a)**

Substituting $r_m = 0.037$ m in Eq. (C), one gets

$$T_{\max} = (0.037)^2\ (-\ 496689) + 1359.96\ ln\ (0.037) + 5565.795$$

$= \mathbf{402.25°C}$ **(Ans.) (c)**

Heat removal rate at the inner surface,

$$\dot{Q}_1 = \bar{q}\pi(r_m^2 - r_1^2)L = (3 \times 10^7\ \text{W/m}^3)\ [\pi(0.037^2 - 0.03^2)\, m^2 \times 1\ \text{m}] \left|\frac{1\ \text{kW}}{10^3\ \text{W}}\right|$$

$$= 44.2\ \text{kW}$$

(Total heat-removal rate) = (Rate of heat generation)

$$\dot{Q}_{\text{gen}} = \bar{q}\pi(r_2^2 - r_1^2)\,L = (3 \times 10^7\ \text{W/m}^3)[\pi(0.05^2 - 0.03^2)\,m^2 \times 1\ \text{m}]\left|\frac{1\ \text{kW}}{10^3\ \text{W}}\right|$$

$$= 150.8\ \text{kW}$$

Percentage of heat generated that is removed at the inner surface is

$\left(\dfrac{\dot{Q}_1}{\dot{Q}_{\text{gen}}}\right)(100) = \dfrac{44.2 \times 100}{150.8} = \mathbf{29.3\%}$ **(Ans.) (c)**

(E) Sphere

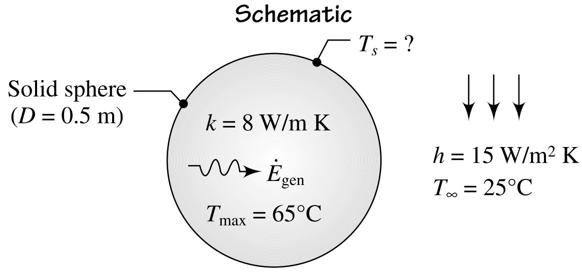
EXAMPLE 3.18 *A solid sphere of 50 cm diameter has a uniformly distributed heat source. The thermal conductivity is 8 W/m K. Calculate the steady-state sphere surface temperature if the maximum temperature in the sphere is 65°C and the ambient temperature is 25°C.*

Solution

Known Uniform volumetric heat generation in a solid sphere.

Find Surface temperature.

Schematic



Assumptions (1) Steady-state conditions. (2) One-dimensional (*radial*) conduction. (3) Constant properties. (4) Constant heat-generation rate.

Analysis The temperature distribution in a sphere is expressed as

$$T(r) = T_\infty + \frac{\bar{q} r_o}{3h} + \frac{\bar{q}(r_o^2 - r^2)}{6k}$$

Maximum temperature will occur at the centre ($r = 0$).

$$T_{\max} = T_\infty + \frac{\bar{q} r_o}{3h} + \frac{\bar{q} r_o^2}{6k} = T_s + \frac{\bar{q} r_o^2}{6k}$$

Substituting the values, we get

$$65°\text{C} = 25°\text{C} + \bar{q}\,(\text{W/m}^3)\left[\frac{0.25\text{ m}}{3 \times 15\text{ W/m}^2\text{ K}} + \frac{(0.25\text{ m})^2}{6 \times 8\text{ W/m K}}\right]$$

or $$\bar{q} = \frac{40}{0.00686} = 5833\text{ W/m}^3$$

Surface temperature of the sphere is

$$T_s = T_\infty + \frac{\bar{q} r_o}{3h} = 25°\text{C} + \frac{(5833\text{ W/m}^3)(0.25\text{ m})}{3 \times 15\text{ W/m}^2\text{ K}}$$

$= \mathbf{57.4°C}$ **(Ans.)**

EXAMPLE 3.19 *A solid sphere of 10 cm radius generates heat according to the law $\bar{q} = 5000\,(1 + 2r)$ [W/m³].*

Obtain the temperature distribution equation if the solid exchanges heat with the surrounding fluid at 100°C. The thermal conductivity of the material is 0.5 W/m K and the unit thermal conductance is 20 W/m² K. Also find the maximum temperature and the temperature at the surface. Sketch the temperature distribution.

Solution

Known Heat generation in a solid sphere exposed to convective environment.

Find $T_{\max}$ and T_w(°C).

Assumptions (1) Steady-state, one-dimensional heat conduction. (2) Constant thermal conductivity.

Analysis The governing differential equation is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = -\frac{\bar{q}}{k} = \frac{-5000(1+2r)}{k} \quad \text{or}$$

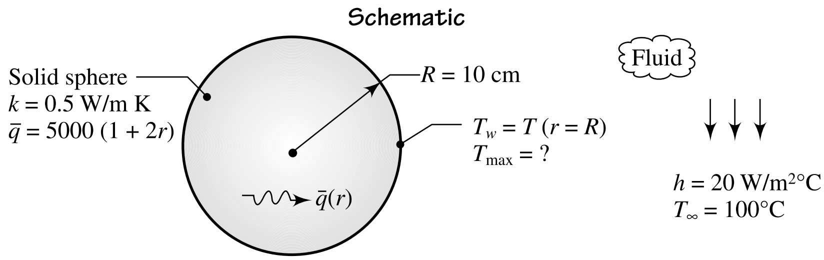
$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = \frac{-5000}{k}(r^2 + 2r^3)$$

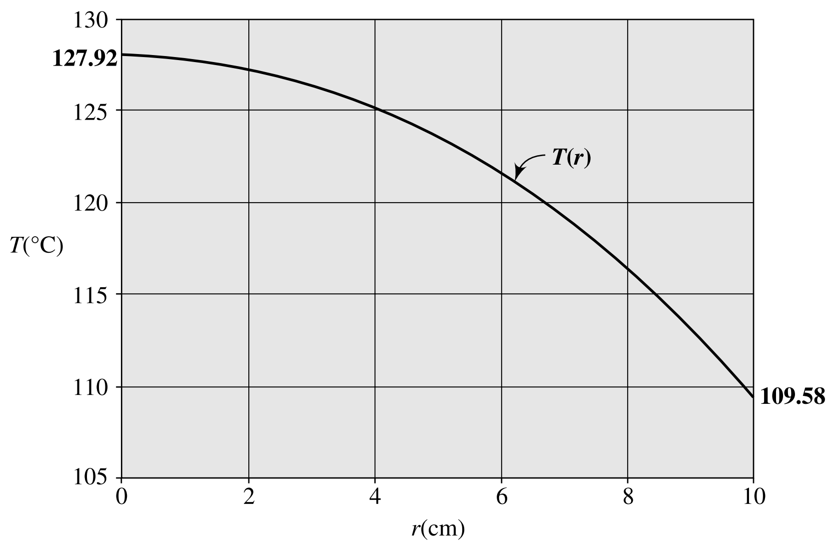
Integrating with respect to r,

$$r^2 \frac{dT}{dr} = \frac{-5000}{k}\left[\frac{r^3}{3} + \frac{2r^4}{4}\right] + C_1 \quad \text{or} \quad \frac{dT}{dr} = -\frac{5000}{k}\left[\frac{r}{3} + \frac{r^2}{2}\right] + \frac{C_1}{r^2} \tag{A}$$

Further integration yields

$$T(r) = -\frac{5000}{k}\left[\frac{r^2}{6} + \frac{r^3}{6}\right] - \frac{C_1}{r} + C_2 \tag{B}$$





Boundary conditions required equal the number of constants of integration.

BC(I): At $r = 0$, $\dfrac{dT}{dr} = 0$ *(by symmetry)*

From Eq. (A): $\boxed{C_1 = 0}$

Then, the radial temperature gradient is

$$\frac{dT}{dr} = -\frac{5000}{k}\left[\frac{r}{3} + \frac{r^2}{2}\right] \tag{C}$$

and the temperature distribution is given by

$$T = -\frac{5000}{6k}[r^2 + r^3] + C_2 \tag{D}$$

BC (II): At the outer surface of the sphere at $r = R$,

$q_{conv} = q_{cond}$

$$h(T_{r=R} - T_\infty) = -k\left.\frac{dT}{dr}\right|_{r=R}$$

With equations (C) and (D), we have

$$h\left[-\frac{5000}{6k}\{R^2+R^3\}+C_2-T_\infty\right]=-k\left[\frac{-5000}{k}\left\{\frac{R}{3}+\frac{R^2}{2}\right\}\right]$$

or $$-\frac{5000}{6k}(R^2+R^3)-T_\infty+C_2=\frac{5000}{h}\left(\frac{R}{3}+\frac{R^2}{2}\right)$$

$\therefore$ $$C_2=T_\infty+5000\left[\frac{R}{3h}+\frac{R^2}{2h}+\frac{R^2}{6k}+\frac{R^3}{6k}\right]$$

$$=100+5000\left[\frac{0.1}{3\times 20}+\frac{0.01}{2\times 20}+\frac{0.01}{6\times 0.5}+\frac{0.001}{6\times 0.5}\right]$$

$\therefore$ $$\boxed{C_2=127.92}$$

Substituting for C_2 in Eq. (D), we have

$$\boxed{\boldsymbol{T(r)=127.92-\frac{5000}{3}(r^2+r^3)}}$$ **(Ans.)**

Maximum temperature (at r = 0) is

T_{max} = **127.92°C** **(Ans.)**

Surface temperature (r = R = 0.1 m) is

$$T_w = 127.92 - \frac{5000}{3}(0.1^2 + 0.1^3)$$

$= \mathbf{109.58°C}$ **(Ans.)**

The temperature profile is shown in the schematic.

(F) Nuclear

EXAMPLE 3.20 *Consider a nuclear fuel element* (k_f *= 60 W/m °C) covered with a steel cladding* (k_s *=15 W/m °C). One surface is insulated and the other exposed to a coolant fluid characterized by temperature* T_∞ *= 200°C and a convection coefficient of* h *= 10 000 W/m² °C. Heat is generated in the nuclear fuel at a constant rate of* $\overline{q}$ *= 2 × 10⁷ W/m³. The thickness of cladding on either side is* b *= 4 mm. Develop an expression for temperature distribution* T(x) *in the nuclear fuel element of thickness* 2L *= 40 mm. Sketch the temperature profile for the composite system. Also find the maximum and minimum temperatures in the fuel element and their corresponding positions. What is the temperature of the exposed surface of the cladding?*

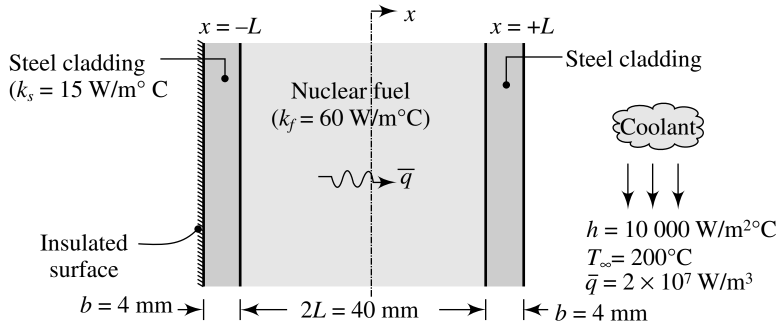
Solution:

Known Nuclear fuel element with steel cladding. Heat generation in fuel. One extreme surface adiabatic and the other subjected to convective cooling.

Find Temperature distribution, $T(x)$. T_{max} and T_{min} in the fuel element and their respective locations.

Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant properties and uniform heat generation. (3) Negligible thermal contact resistance.

Schematic



Analysis The governing differential equation for the nuclear fuel element is

$$\frac{d^2T}{dx^2} = -\frac{\overline{q}}{k_f}$$

Integrating twice successively,

$$\frac{dT}{dx} = -\frac{\overline{q}x}{k_f} + C_1 \quad \text{and} \quad T(x) = -\frac{\overline{q}x^2}{2k_f} + C_1 x + C_2$$

Heat flux at $x = -(L + b) = 0$ because the surface is insulated. As there is no heat generation in the cladding, the heat flux at $x = -L$ will also be zero, i.e, $\frac{dT}{dx} = 0$ between $x = -(L + b)$ and $x = -L$. As a result, $T_{\max}$ will occur in the steel cladding on the left side throughout its thickness, b.

At $\qquad x = -L, \ \frac{dT}{dx} = 0$

$$\therefore \qquad \boxed{C_1 = \frac{-\overline{q}L}{k_f}} \quad \text{and} \quad T(x) = -\frac{\overline{q}x^2}{2k_f} - \frac{\overline{q}Lx}{k_f} + C_2 \tag{A}$$

Heat generated in the fuel, $\dot{E}_{\text{gen}}$ has no scope for escape to the left side.

Hence, $\dot{E}_{\text{gen}} = \dot{Q}_{\text{out,conv}} = \dot{Q}_{\text{cond}}$

$$\dot{E}_{\text{gen}} = \overline{q}(2L \cdot A) = \dot{Q}_{\text{cond}} = \frac{k_s}{b}(T_1 - T_2)A = \dot{Q}_{\text{conv}} = hA(T_2 - T_\infty)$$

Therefore, $T_1 = \frac{\overline{q}(2Lb)}{k_s} + T_2$

where $T_2 = \frac{\overline{q}(2L)}{h} + T_\infty$

$$\text{Hence,} \quad T_1 = T_\infty + \frac{\overline{q}(2Lb)}{k_s} + \frac{\overline{q}(2L)}{h} = T(L) \tag{B}$$

From Eq. (A):

$$T(L) = \frac{-\overline{q}L^2}{2k_f} + \frac{(-\overline{q}L)L}{k_f} + C_2 = -\frac{3}{2}\frac{\overline{q}L^2}{k_f} + C_2 \tag{C}$$

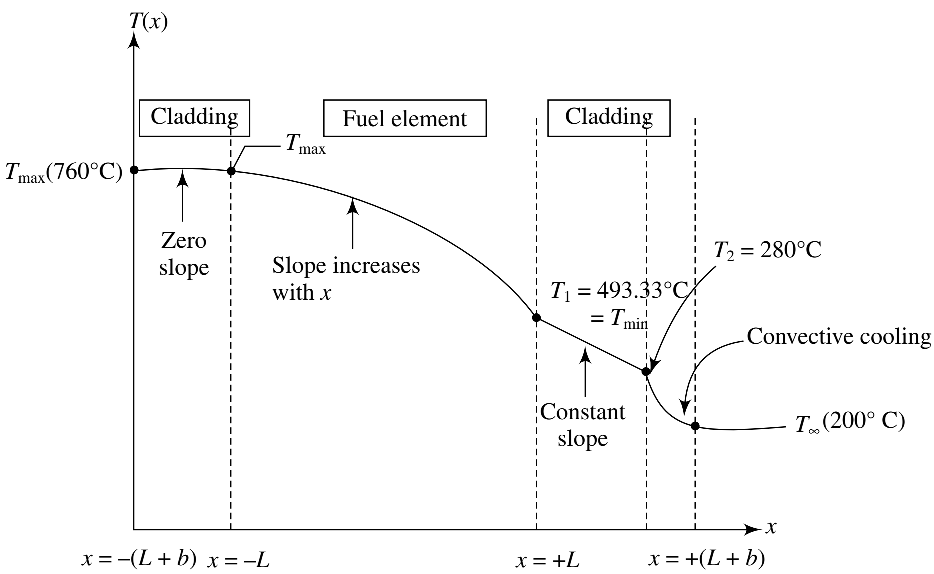
Equating (B) and (C),

$$\boxed{C_2 = T_\infty + \overline{q}L\left[\frac{2b}{k_s} + \frac{2}{h} + \frac{3L}{2k_f}\right]}$$

The temperature distribution for ($-L \le x \le +L$) is

$$\boxed{\boldsymbol{T(x) = T_\infty - \frac{\overline{q}x^2}{2k_f} - \frac{\overline{q}Lx}{k_f} + \overline{q}L\left[\frac{2b}{k_s} + \frac{2}{h} + \frac{3}{2}\frac{L}{k_f}\right]}}$$ **(Ans.)**

The temperature profile for the whole system is sketched below:



Substituting numerical values, the maximum temperature in the fuel element is

$$T_{\max} = T(x = -L)$$

$$= -\frac{\overline{q}L^2}{2k_f} + \frac{\overline{q}L^2}{k_f} + \overline{q}L\left[\frac{2b}{k_s} + \frac{2}{h} + \frac{3}{2}\frac{L}{k_f}\right] + T_\infty$$

$$= \left[\frac{1}{2} \times 2 \times 10^7 \text{ W/m}^3 \times \frac{(0.02 \text{ m})^2}{60 \text{ W/m °C}}\right] + (2 \times 10^7 \text{ W/m}^3 \times 0.02 \text{ m})$$

$$\times \left[\frac{2 \times 0.004 \text{ m}}{15 \text{ W/m °C}} + \frac{2}{10000 \text{ W/m}^2 \text{ °C}} + \frac{3}{2} \times \frac{0.02 \text{ m}}{60 \text{ W/m °C}}\right] + 200°\text{C}$$

$= 66.67 + 693.33 =$ **760°C** **(Ans.)**

Minimum temperature in the fuel element is

$$T_{\min} = T(x = +L) = T_1$$

$$= -\frac{3}{2}\times\frac{2\times10^7\times0.02^2}{60}+(2\times10^7\times0.02)\left[\frac{0.008}{15}+\frac{2}{10000}+\frac{3}{2}\times\frac{0.02}{60}\right]+200$$

$$= -200 + 693.33 = \mathbf{493.33°C} \qquad \textbf{(Ans.)}$$

Outer surface temperature of cladding *(convective side)* can be found from

$$q_{\text{cond}} = q_{\text{conv}} \Rightarrow \frac{k_s}{b}(T_1 - T_2) = h(T_2 - T_\infty) = \bar{q}\,(2L)$$

$$\therefore \quad T_2 = \frac{\bar{q}(2L)}{h} + T_\infty = \frac{2\times10^7\ \text{W/m}^3\times0.04\ \text{m}}{10000\ \text{W/m}^2\,°\text{C}} + 200°\text{C}$$

$$= \mathbf{280°C} \qquad \textbf{(Ans.)}$$

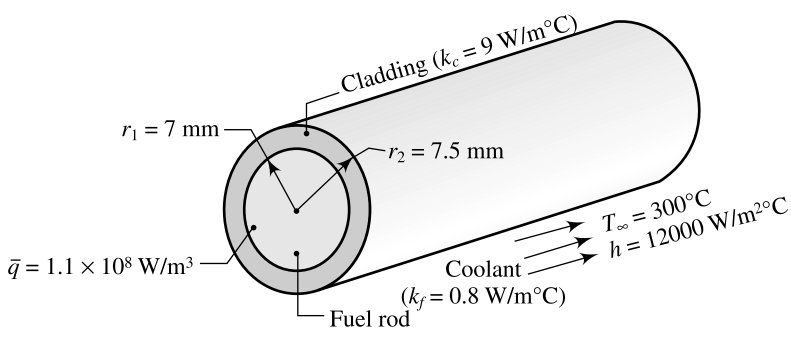
EXAMPLE 3.21 *(a) A long solid cylindrical nuclear fuel element of outer radius* r_1 *and thermal conductivity* k_f *experiences uniform internal heat generation at a constant volumetric rate* $\bar{q}$. *It is encased with an annular layer of cladding material of thermal conductivity* k_c *and outer radius* r_2 *to prevent oxidation of fuel-rod surface due to direct contact with the coolant. The outer surface of cladding is subjected to convective conditions characterized by coolant temperature* T_∞ *and the heat-transfer coefficient* h. *Derive from first principles equations for temperature distribution in the fuel element,* $T_f(r)$ *and the cladding material,* $T_c(r)$. *(b) Calculate the maximum temperature in the fuel element for* h = *12000 W/^{2}°C,* T_∞ *= 300°C and* $\bar{q}$ *= 1.1 × 10^8 W/m^2. For fuel rod:* k_f *= 0.8 W/m °C,* r_1 *= 7 mm. For cladding:* k_c *= 9 W /m °C,* r_2 *= 7.5 mm.*

Solution

Known Uniform heat generation in a cylindrical nuclear fuel rod encapsulated in a cladding.

Find (a) Expressions for temperature distribution $T_f(r)$ and $T_c(r)$. (b) Maximum temperature in the fuel rod for the prescribed conditions.

Schematic



Assumptions (1) Steady-state, one-dimensional conduction with uniform heat generation. (2) Constant properties. (3) Negligible thermal contact resistance.

Analysis (a) The relevant differential equation for the fuel is

$$\frac{1}{r}\frac{d}{dr}\left(k_f r\frac{dT_f}{dr}\right) = \bar{q} \qquad [0 \le r \le r_1]$$

Integrating, $r\dfrac{dT_f}{dr} = -\dfrac{\overline{q}r^2}{2k_f} + C_1$

Temperature gradient, $\dfrac{dT_f}{dr} = -\dfrac{\overline{q}r}{2k_f} + \dfrac{C_1}{r}$ (A)

Integrating, $T_f(r) = -\dfrac{\overline{q}r^2}{4k_f} + C_1 \ln r + C_2$ (B)

The pertinent differential equation for the cladding is

$$\frac{1}{r}\frac{d}{dr}\left(k_c r \frac{dT_c}{dr}\right) = 0 \qquad [r_1 \le r \le r_2]$$

Integrating, $\dfrac{dT_c}{dr} = \dfrac{C_3}{k_c r}$ (C)

Integrating again, $T_c(r) = \dfrac{C_3}{k_c}\ln r + C_4$ (D)

Since there are four constants of integration, four boundary conditions are required.

BC (I): $\left(\dfrac{dT_f}{dr}\right)_{r=0} = 0$ (E)

BC (II): $T_f(r_1) = T_c(r_1)$ (F)

BC (III): $-k_f \left.\dfrac{dT_f}{dr}\right|_{r=r_1} = -k_c \left.\dfrac{dT_c}{dr}\right|_{r=r_1}$ (G)

BC (IV): $-k_c \left.\dfrac{dT_c}{dr}\right|_{r=r_2} = h[T_c(r_2) - T_\infty]$ (H)

Applying Eq. (E) to Eq. (A), we find that

$$\boxed{C_1 = 0}$$

It follows that,

$$T_f(r) = -\frac{\overline{q}r^2}{4k_f} + C_2 \qquad \text{(I)}$$

Applying Eq. (F) to equations (I) and (D), we get

$$-\frac{\overline{q}r_1^2}{4k_f} + C_2 = \frac{C_3}{k_c}\ln r_1 + C_4 \qquad \text{(J)}$$

From Eq. (G), we obtain with equations (A) and (C)

$$-k_f\left[-\frac{\overline{q}r_1}{2k_f}\right] = -k_c\frac{C_3}{k_c r_1} \quad \text{or} \quad \boxed{C_3 = -\frac{\overline{q}r_1^2}{2}} \qquad \text{(K)}$$

Equation (H) with equations (C), (K) and (D) yields

$$\boxed{C_4 = \frac{\bar{q}r_1^2}{2hr_2} + \frac{\bar{q}r_1^2}{2k_c}\ln r_2 + T_\infty}$$

or $$\frac{-\bar{q}r_1^2}{4k_f} + C_2 = -\frac{\bar{q}r_1^2}{2k_c}\ln r_1 + \frac{\bar{q}r_1^2}{2hr_2} + \frac{\bar{q}r_1^2}{2k_c}\ln r_2 + T_\infty \quad \text{(L)}$$

Substituting equations (K) and (L) into Eq. (J), we have

$$\boxed{C_2 = T_\infty + \frac{\bar{q}r_1^2}{4k_f} + \frac{\bar{q}r_1^2}{2k_c}\ln\frac{r_2}{r_1} + \frac{\bar{q}r_1^2}{2hr_2}} \quad \text{(M)}$$

Substituting Eq. (M) into Eq. (I), we have the temperature distribution in the fuel rod.

$$\boxed{\boldsymbol{T_f(r) = T_\infty + \frac{\bar{q}(r_1^2 - r^2)}{4k_f} + \frac{\bar{q}r_1^2}{2k_c}\ln\frac{r_2}{r_1} + \frac{\bar{q}r_1^2}{2hr_2}}} \quad \textbf{(Ans.) (a)}$$

Finally, substitution of equations (K) and (L) into Eq. (D) gives the temperature distribution in the cladding.

$$\boxed{\boldsymbol{T_c(r) = T_\infty + \frac{qr_1^2}{2hr_2} + \frac{qr_1^2}{2k_c}\ln\frac{r_2}{r}}} \quad \textbf{(Ans.) (a)}$$

(b) The maximum fuel temperature will occur at $r = 0$ where $\frac{dT_f}{dr} = 0$. Substituting numerical values in the expression for $T_f(r)$, we obtain

$$T_{\max} = T_f(r = 0) = 300°\text{C}$$
$$+ \frac{(1.1\times10^8\,\text{W/m}^3)^2\,(7\times10^{-3}\text{m})^2}{2\times9\ \text{W/m°C}}\ln\frac{7.5\ \text{mm}}{7\ \text{mm}} + \frac{(1.1\times10^8\text{W/m}^3)(7\times10^{-3}\ \text{m})^2}{2(12\,000\ \text{W/m}^{2}\text{°C})(7.5\times10^{-3}\ \text{m})}$$
$$= \mathbf{2035°C} \quad \textbf{(Ans.) (b)}$$

Comment The temperature drop in the fuel element,

$$[T_f(r = 0) - T_f(r = r_1)] = \frac{\bar{q}r_1^2}{4k_f} = \frac{1.1\times10^8\times0.007^2}{4\times0.8} \approx \mathbf{1684°C}$$

which is too high to justify the assumption of constant properties, i.e., constant k_f.

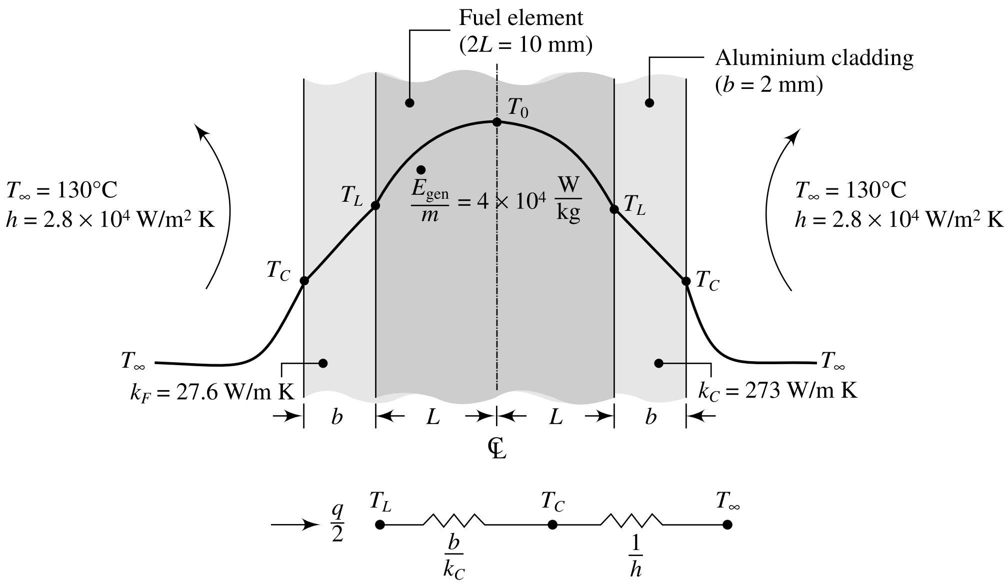
EXAMPLE 3.22 *A flat-plate fuel element made of uranium (*k *= 27.6 W/m K, ρ = 19070 kg/m³) for a nuclear reactor is 10 mm thick, and the rate of internal heat generation is 4 × 10⁴ W/kg. The fuel element is clad with 2 mm thick aluminium sheet (*k *= 237 W/m K) on each face. The heat generated within the nuclear fuel is removed by a coolant fluid at 130°C adjoining the cladding surface and characterized by the convection coefficient of 2.8 × 10⁴ W/m² K. Determine (a) the temperature at the free surface of the aluminium, (b) the temperature at the uranium aluminium interface, and (c) the temperature at the centre of the fuel element.*

Solution

Known A flat-plate fuel element with aluminium cladding on both sides exposed to convective environment.

Find (a) T_L(°C), (b) T_C(°C), and (c) T_0(°C).

Schematic



Assumptions (1) The fuel element can be considered an infinite slab. (2) Uniform heat-transfer coefficient. (3) Constant thermal conductivities. (4) One-dimensional heat conduction. (5) Steady state conditions.

Analysis Heat generated within the fuel element,

$$\dot{E}_{\text{gen}} = qA = \overline{q}A(2L) = \left(\frac{\dot{E}_{\text{gen}}}{m}\right)(\rho \cancel{V}) = \left(\frac{\dot{E}_{\text{gen}}}{m}\right)(\rho A(2L))$$

Heat flux, $q = (\dot{E}_{\text{gen}}/\dot{m})(\rho \times 2L)$

$$= \left(4 \times 10^4 \frac{\text{W}}{\text{kg}}\right)\left(19\,070 \frac{\text{kg}}{\text{m}^3} \times 10 \times 10^{-3}\text{ m}\right) = 7.628 \times 10^6 \text{ W/m}^2$$

Half of this will flow to the *either* side due to symmetry,

i.e., $\dfrac{q}{2} = 3.814 \times 10^6 \text{ W/m}^2$

Rate of volumetric heat generation,

$$\overline{q} = (\dot{E}_{\text{gen}}/\dot{m})\,\rho = (4 \times 10^4 \text{ W/kg})(19070 \text{ kg/m}^3) = 7.628 \times 10^8 \text{ W/m}^3$$

At the free surface of the cladding:

Control surface energy balance: $\dot{E}_{\text{in}} = \dot{E}_{\text{out}}$

i.e., $\dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv}}$ or $k_C A \dfrac{(T_L - T_C)}{b} = hA(T_C - T_\infty) = \dfrac{qA}{2}$

It follows that, $T_C - T_\infty = \dfrac{q/2}{h}$

Temperature at the *free surface* of the cladding is

$$T_C = T_\infty + \frac{q/2}{h} = 130°\text{C} + \frac{3.814 \times 10^6 \text{ W/m}^2}{2.8 \times 10^4 \text{ W/m}^2\text{K}}$$

$= \mathbf{266.2°C}$ **(Ans.) (a)**

Also, $$q = \frac{k_C}{b}(T_L - T_C)$$

Hence, the temperature at the uranium-aluminium *interface* is

$$T_L = T_C + \frac{(q/2)b}{k_C} = 266.2°\text{C} + \frac{(3.814 \times 10^6 \text{ W/m}^2)(2 \times 10^{-3} \text{ m})}{237 \text{ W/m K}}$$

$= \mathbf{298.4°C}$ **(Ans.) (b)**

Temperature at the *centre* of the fuel element is

$$T_0 = T_L + \frac{\overline{q}L^2}{2k_F} = 298.4°\text{C} + \frac{(7.628 \times 10^8 \text{ W/m}^3)(5 \times 10^3 \text{ m})^2}{2 \times 27.6 \text{ W/m K}}$$

$= \mathbf{643.9°C}$ **(Ans.) (c)**

Comment The temperature profile and the thermal circuit are shown in the schematic.

Points to Ponder

- Heat generation is basically the conversion of some form of energy into thermal energy in the medium.
- Internal heat generation results in the temperature rise throughout the medium.
- Transformation of electrical energy into thermal energy (*when the electrical current passes through a wire*), exothermic (*heat evolving*) chemical reactions involving change of chemical energy into thermal energy, and conversion of nuclear energy into thermal energy in nuclear fuel rods are some common examples of heat conduction with heat sources.
- In steady state, heat generated in a solid equals heat lost to the surrounding fluid.
- The usual assumptions in analyzing problems involving heat sources are the following:
  steady-state, one-dimensional, uniform volumetric heat generation with constant thermal conductivity and uniform heat -transfer coefficient between the exposed surface and the ambient fluid.
- Since the heat-transfer rate is not constant when there is internal heat generation, electrical analogy and equivalent thermal circuit are not applicable in such situations.
- The internally distributed heat source in a nuclear reactor is *not* uniform.

GLOSSARY of Key Terms

• **Heat generation**	A process occurring within the medium in which thermal energy is liberated due to conversion from some other energy form.
• **Volumetric generation rate**	The rate of generation of heat per unit volume of the material.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

3.1 A plane wall of thickness $2L$ has a uniform volumetric heat generation $\overline{q}$ (W/m^3). It is exposed to local ambient temperature T_∞ at both the ends ($x = \pm L$). The surface temperature T_s of the wall under steady-state conditions (where h and k have their usual meanings) is given by

(a) $T_s = T_\infty + \dfrac{\overline{q}L}{h}$ (b) $T_s = T_\infty + \dfrac{\overline{q}L^2}{2k}$ (c) $T_s = T_\infty + \dfrac{\overline{q}L^2}{h}$ (d) $T_s = T_\infty + \dfrac{\overline{q}L^3}{2k}$

3.2 How can the temperature drop in a plane wall with uniformly distributed heat generation can be reduced?

(a) By reducing thermal conductivity of wall material
(b) By reducing wall thickness
(c) By reducing convection coefficient at the surface
(d) By reducing heat generation rate.

3.3 A plane wall, of thickness L, has uniform internal heat generation. If the left hand face is insulated, the maximum temperature in the wall will occur at x equal to

(a) L (b) $L/2$ (c) $L/4$ (d) 0

3.4 A plane wall of thickness $2L$ and thermal conductivity k has uniform heat generation rate $\overline{q}$. The left and right surface temperatures are T_1 and T_2. The location x (measured from the midplane) at which the temperature in the wall would be maximum is given by

(a) $\dfrac{\overline{q}(T_1 - T_2)}{kL}-$ (b) $\dfrac{\overline{q}}{k}\left(\dfrac{T_2 - T_1}{2L}\right)$ (c) $\dfrac{\overline{q}(T_1 - T_2)}{kL^2}$ (d) $\dfrac{\overline{q}(T_2 - T_1)}{k^2 L}$

3.5 Which one of the following is the proper expression for steady-state, one-dimensional, constant thermal conductivity, heat-conduction equation with internal heat generation?

(a) $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) + \dfrac{\overline{q}}{k} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$ (b) $\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = 0$

(c) $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) + \overline{q} = \rho C_p \dfrac{\partial T}{\partial t}$ (d) $\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = -\dfrac{\overline{q}}{k}$

3.6 In a long cylindrical rod of radius R and a surface heat flux of q_0, the uniform internal heat generation rate is

(a) $\dfrac{2q_0}{R}$ (b) $2q_0$ (c) $\dfrac{q_0}{R}$ (d) $\dfrac{q_0}{R^2}$

3.7 The temperature distribution for a long solid cylinder of radius R with uniform internal heat generation is given by

[T_0.... centreline temperature, T_w .... cylinder wall temperature]

(a) $\dfrac{T - T_w}{T_0 - T_w} = 1 - \left(\dfrac{r}{R}\right)^2$ (b) $\dfrac{T - T_w}{T_0 - T_w} = \ln\dfrac{r}{R}$

(c) $\dfrac{T - T_w}{T_0 - T_w} = 1 - \left(\dfrac{r}{R}\right)$ (d) $\dfrac{T - T_w}{T_0 - T_w} = \left\{\dfrac{1}{r} - \dfrac{1}{R}\right\}$

3.8 For a long cylinder with uniformly distributed heat sources, the temperature gradient (dT/dr) at half the radius will be

(a) one-fourth of that at the wall (b) one-half of that at the wall
(c) twice that at the wall (d) four times that at the wall

3.9 Heat is uniformly generated in a copper cable with current density I (A/m²) and electrical resistivity ρ (ohm m). The rate of volumetric heat generation $\overline{q}$ (W/m³) is

(a) ρ/i^2 (b) ρi^2 (c) $\rho^2 i$ (d) ρi

3.10 A hollow cylinder of thermal conductivity k with insulated *inner* surface and adiabatic ends having uniform heat generation $\overline{q}$ (W/m³) has temperatures T_1 and T_2 at radii r_1 and r_2. The temperature distribution is given by

$$T(r) = T_2 + \frac{\overline{q}}{4k}(r_2^2 - r^2) + \frac{\overline{q}}{2k} r_1^2 \ln\left(\frac{r}{r_2}\right)$$

With $\overline{q}$ = 6 MW/m³, r_1 = 6 cm, r_2 = 9 cm and k = 30 W/m K, the maximum temperature difference is

(a) 53°C (b) 46.1°C (c) 79°C (d) 256.7°C

3.11 A hollow cylinder of thermal conductivity k with insulated *outer* surface and adiabatic ends having uniform heat generation $\overline{q}$ (W/m³) has temperatures T_1 and T_2 at radii r_1 and r_2. The temperature distribution is given by

$$T(r) = T_1 + \frac{\overline{q}}{4k}(r_1^2 - r^2) + \frac{\overline{q}}{2k} r_2^2 \ln\frac{r}{r_1}$$

With $\overline{q}$ = 6 MW/m³, r_1 = 6 cm, r_2 = 9 cm and k = 30 W/m K, the maximum temperature difference is

(a) 53.5°C (b) 103.4°C (c) 283.2°C (d) 76.0°C

3.12 A long solid tube with uniform heat generation $\overline{q}$ (W/m³) is insulated at the outer radius r_2 and cooled at the inner radius r_1 with coolant temperature T_∞.
With $\overline{q}$ = 5 MW/m³, r_1 = 2 cm, r_2 = 2.5 cm, k = 15 W/m K, T_∞ = 75°C and T_1 = 300°C, the convection coefficient h(W/m² K) is

(a) 180 (b) 40 (c) 110 (d) 125

3.13 Current of 200 A passes through a 3 mm diameter wire, 1 m long, with a resistivity of 70 μΩ cm. The rate of volumetric heat generation in W/m³ is

(a) 5.6×10^8 (b) 0.099 (c) 1.6×10^9 (d) 1.4×10^4

3.14 One-dimensional, steady-state heat-transfer equation for a sphere with heat generation at the rate of $\overline{q}$ can be written as

(a) $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) + \left(\dfrac{\overline{q}}{k}\right) = \left(\dfrac{1}{a}\right)\dfrac{\partial T}{\partial t}$ (b) $\left(\dfrac{\partial^2 T}{\partial r^2}\right) + \dfrac{\overline{q}}{k} = \dfrac{1}{a}\dfrac{\partial T}{\partial t}$

(c) $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial T}{\partial r}\right) + \dfrac{\overline{q}}{k} = \dfrac{1}{a}\dfrac{\partial T}{\partial t}$ (d) $\dfrac{\partial^2}{\partial r^2}(rT) + \dfrac{\overline{q}}{k} = \dfrac{1}{a}\dfrac{\partial T}{\partial t}$

Answers

Multiple-Choice Questions

3.1 (a) **3.2** (b) **3.3** (d) **3.4** (b) **3.5** (d) **3.6** (a)
3.7 (a) **3.8** (b) **3.9** (b) **3.10** (c) **3.11** (b) **3.12** (d)
3.13 (a) **3.14** (c)

REVIEW QUESTIONS

3.1 Give some examples of heat sources present in materials.

3.2 Sketch the temperature distribution for conduction in a plane wall with uniform volumetric heat generation with the following boundary conditions: (a) asymmetrical, (b) symmetrical, and (c) insulated surface on one side.

3.3 Show that a plane wall having uniform heat generation with same convective conditions at both exposed surfaces is similar to a plane wall half as thick with the same uniform heat generation but with one adiabatic surface, and the other, a convective surface.

3.4 Why is it *not* possible for a material experiencing uniform volumetric heating to be represented by a thermal resistance in an equivalent thermal circuit?

3.5 If both the surfaces of a plane wall with uniform volumetric heating are held at the same constant temperature, which point of the plate will have maximum temperature?

3.6 Where will the maximum temperature occur in a long solid cylinder that experiences uniform heat generation and whose outer surface is maintained at constant temperature?

3.7 Write down the expression to describe the temperature distribution in a sphere, with uniform heat generation, the outer surface of which is held at a constant temperature.

3.8 Is it possible to attain steady operating conditions in *a long solid cylinder or a sphere* with heat generation and its outer surface well insulated?

3.9 The difference between the centre (*maximum*) temperature and surface temperature of a copper cable in an electrical transmission line is very small. What is it due to?

3.10 Consider a plane slab, a long cylinder, and a solid sphere with the same uniform volumetric heat-generation rate, made of the same material and having the same characteristic dimension (*radius* in *cylinder* and *sphere*, and *half-thickness* in a *slab*). In which case would the difference between the centre temperature and the wall temperature be minimum?

PRACTICE PROBLEMS

(A) Plane Wall with Equal Surface Temperatures

3.1 A small dam, idealized as a large slab 1.2-m-thick, has both surfaces maintained at 20°C. The hydration of the concrete results in the equivalent of a distributed source of constant strength of 112 W/m^3. The thermal conductivity of the wet concrete may be taken as 8.4×10^{-3} W/m K. Determine the maximum temperature to which the concrete will be subjected, assuming steady state conditions.

[2420°C]

3.2 A plane wall of thickness 2L has an internal heat generation that varies according to $\overline{q} = \overline{q}_0 \cos ax$, where $\overline{q}_0$ is the heat generated per unit volume at the centre of the wall ($x = 0$) and a is a constant. If both sides of the wall are maintained at a constant temperature of T_w, derive an expression for the total heat loss from the wall per unit surface area.

$$\left[\boldsymbol{q_{\text{loss}} = \frac{2\overline{q}_0}{a} \sin aL} \right]$$

(B) Plane Wall with One Surface Insulated

3.3 The inside surface of a large flat plate of thickness L at $x = 0$ is insulated, the outside surface at $x = L$ is maintained at a uniform temperature T_2, and the heat generation term is in the form, $\overline{q}(x) = \overline{q}_0 e^{\gamma x} (\text{W/m}^3)$ where $\overline{q}_0$ and γ are constants and x is measured from the insulated inside surface. Develop (a) an expression for the temperature distribution in the plate, and (b) an expression for the temperature at the insulated surface (i.e., $x = 0$) of the plate. (c) Also deduce an expression for the heat flux at the outer surface, $x = L$.

$$\left[\begin{array}{l} \textbf{(a) } T(x) = T_2 + \dfrac{\bar{q}_0}{k\gamma}(L - x) + \dfrac{\bar{q}}{k\gamma^2}(e^{-\gamma L} - e^{-\gamma x}) \\ \textbf{(b) } T_1 = T_{\max} = T(0) = T_2 + \dfrac{\bar{q}_0 L}{k\gamma} + \dfrac{\bar{q}\,(e^{-\gamma L} - 1)}{k\gamma^2} \\ \textbf{(c) } q = (L) = \dfrac{\bar{q}_0}{k\gamma}(1 - e^{-\gamma L}) \end{array}\right]$$

(C) Plane Wall: Different End Surface Temperatures

3.4 A plane wall, 10-cm-thick, has one surface maintained at 0°C and the other at 100°C. The thermal conductivity of the wall material is 0.1 W/m °C and the rate of uniform volumetric heat generation is 2.2 k W/m^3. Determine (a) the temperature at the midplane of the wall, (b) the location and magnitude of maximum temperature in the wall, and (c) the magnitude and direction of heat flux at each surface.

[(a) 77.5 °C (b) 100.23°C (c) 10 W/m^2]

(D) Long Cylinder

3.5 A thin hollow tube, 6-mm outside diameter, and 4-mm inside diameter carries a current of 1000 A. Water at 27°C is circulated inside the tube for cooling it. The electrical resistivity of the material is 0.1 Ω-m^2/m, the thermal conductivity of the tube material is 18 W/m K, and the water-side heat transfer coefficient is estimated as 22 k W/m^2 K. Calculate the inner surface temperature of the tube if its outer surface is insulated. **[50.0°C]**

3.6 In a graphite moderated gas cooled reactor of the type used for commercial power production, the uranium rods of 6-cm diameter are generating heat at the rate of 1.35×10^7 W/m^3. The rods are jacketed by an annulus in which the coolant gas is flowing. For an average gas temperature of 175°C and a surface heat transfer coefficient of 500 W/m^{2}K, estimate the maximum temperature of the uranium rod. Thermal conductivity of uranium is 25 W/m K. **[701.5°C]**

3.7 Consider the rise in temperature in a body muscle fibre induced by heat generated from exercising in a gym. Modelling the muscle as a cylinder of 20-mm-diameter with a volumetric heat generation at the constant rate of 6 kW/m^3, determine the maximum temperature in the muscle if the thermal conductivity of muscle fibre is 0.4 W/m °C and the muscle surface temperature is 37°C. **[37.4°C]**

3.8 An electrical transmission wire made of a 2.5-cm-diameter annealed copper wire carries 200 amp and has a resistance of 0.4×10^{-4} ohm per cm length. If the surface temperature is 200°C and the ambient air temperature is 15°C, determine (a) the convective heat transfer coefficient between the wire surface and the ambient air, and (b) the maximum temperature in the wire. Assume $k = 60$ W/m K.

[11.0 W/m^2 K, 200.22°C]

3.9 A power transmission line of copper, 1 cm diameter has a resistance of 0.005 Ω/m. It carries a current of 200 A. Determine the centreline and surface temperature of the wire on a breezeless day. Assume : $h = 40$ W/m^2 K, $k_{\text{copper}} = 375$ W/m K, $T_{\text{ambient}} = 30$°C. **[189.16 °C, 189.20°C]**

3.10 Heat is generated uniformly at a constant rate of 3 MW/m^3 in a long resistance wire of 0.6-cm-diameter and thermal conductivity 15 W/m K. The wire is embedded in a 0.5-cm-thick layer of ceramic whose thermal conductivity is 1.2 W/m K. The outer surface of the ceramic layer loses heat by convection to the ambient air at 20 °C with an average combined heat transfer coefficient of 15 W/m^2 K. Assuming one-dimensional heat transfer determine (a) the temperatures at the centre of the resistance wire and the wire-ceramic layer interface under steady operating conditions.

[143.5°C, 144.4°C]

3.11 Consider a tube that is 3-m long, has a 12.5-mm inner diameter and a 15.5-mm outer diameter, and is well insulated. The tube's thermal conductivity is 14.3 W/m K. The fluid (C_p = 2.4 kJ/kg K) enters the tube at 24°C at a flow rate of 0.7 kg/s. Electric current heats the tube; the current is 475 A, and the voltage drop across the length of the tube is 5.6V. At a distance of 2.5 m from the inlet, the measured temperature on the outside surface of the tube is 41.5°C. Determine the heat transfer coefficient at that location. **[1500 W/m²K]**

(E) Sphere

3.12 Develop an expression for the steady state temperature distribution $T(r)$ in a solid sphere of radius $r = r_o$ in which heat is generated at a rate of $\overline{q}(r) = \overline{q}_o\left(1 - \frac{r}{r_o}\right)\mathrm{W/m^3}$ where $\overline{q}_o$ is a constant and the boundary surface at $r = r_o$ is maintained at a uniform temperature T_o. Find the position and value of the maximum temperature. $\left[\boldsymbol{T_{\max} = T_o + \frac{\overline{q}_o r_o^2}{12\,k}}\right]$

3.13 850 W/m³ of heat is generated within a 15-cm-diameter nickel-steel sphere for which k = 10 W/m K. The ambient temperature is 25°C and the free convection heat transfer coefficient is 10 W/m² K around the outside of the sphere. Find the maximum temperature in the sphere at steady state. **[27.2°C]**

(F) Nuclear Systems

3.14 The shielding for a nuclear reactor can be approximated as a plane wall of thickness L. Heat is generated per unit volume at a rate $\overline{q}(x)$ within the shielding according to the relation $\overline{q}(x) = \overline{q}_0\,\alpha\, e^{-\alpha x}$ where $\overline{q}_0$ is the incident radiation flux and α is the absorption coefficient of the shielding material. (a) Obtain an expression for steady-state temperature distribution if the inner (x = 0) and outer ($x = L$) surfaces of the shielding are kept at T_1 and T_2 respectively. (b) Determine the location in the shield at which maximum temperature will occur.

$$\left[\begin{array}{l}\textbf{(a) } \boldsymbol{T(x) = T_1 + \frac{\overline{q}_0}{k\alpha}[1 - e^{-\alpha x}] + \frac{x}{L}\left[(T_2 - T_1) - \frac{\overline{q}_0}{k\alpha}(1 - e^{-\alpha L})\right]} \\ \textbf{(b) } \boldsymbol{x = -\frac{1}{\alpha}\ln\left[\frac{k(T_1 - T_2)}{q_0 L} + \frac{1 - e^{-\gamma L}}{\gamma L}\right]}\end{array}\right]$$

3.15 A nuclear fuel rod assembly, consisting of an outer cladding and the inner nuclear material, has an outside diameter of 75 mm. The outer cladding is 10-mm thick and is made of a material with a thermal conductivity of 3.2 W/m K. The nuclear reaction generates 90 000 W/m³ uniformly in the inner nuclear material. The outside of the assembly is surrounded by water at 300°C, and the convection coefficient is 100 W/m² K. Determine: (a) the temperature at the assembly surface, and (b) the temperature at the interface between the inner nuclear material and the outer cladding.

[(a) 309.1°C (b) 312.4°C]

4 Heat Transfer from Extended Surfaces

4.1 ❑ INTRODUCTION

There are many applications where our primary aim is to increase the rate of heat transfer between the surface of a solid and the surrounding fluid. How can we go about it?

Consider a solid surface dissipating heat to the surrounding fluid by convection. Then, the basic equation governing convective heat-transfer rate $\dot{Q}$, is given by *Newton's law of cooling*:

$$\boxed{\dot{Q} = hA_s(T_s - T_\infty) = \frac{\Delta T}{R_{\text{conv}}}} \tag{4.1}$$

where $\Delta T \equiv (T_s - T_\infty)$ and $R_{\text{conv}} = 1/hA_s$

h = heat-transfer coefficient between the surface and the ambient fluid

A_s = exposed heat-transfer area of the solid surface

T_s = surface temperature of the solid

T_∞ = temperature of the surrounding fluid

In order to increase the heat-transfer rate from the parent (*primary*) surface, there are *three* options:

- *Increase* the driving potential, i.e., *temperature difference* between the surface and the adjacent fluid $(T_s - T_\infty)$. However, this may not always be realistic because large changes in these temperatures may not directly be possible.
- *Increase* the *heat-transfer coefficient h*. However, this may not always be possible since the fluids are fixed in most industrial applications. Using an external fan or pump to increase the fluid velocity and flow rate can increase the value of h. There are practical limits to the maximum convection coefficient that can be achieved. The additional cost may not always be justifiable.
- *Increase* the *surface area* A_s. Yet another method of increasing the rate of heat transfer would be to increase the effective area of the heat-transfer surface, that is, the surface area in contact with the fluid.

Surface area can be increased by adding an *extended surface* (*or fin*) projecting from the *base surface*. This seems to be the satisfactory solution in several situations. In the film *Jurassic Park,* you might have noticed dinosaurs with projected parts on their bodies. Dinosaurs, being huge animals (*now defunct*), required these projections (*cooling fins*) to cool the warm blood pumped by their heart.

Generally, fins are provided on that side of the surface where the heat-transfer coefficient is less and the thermal resistance is more. Heat-transfer coefficients are usually less for gases as compared to liquids. Therefore, one can observe that fins are attached on the outside of the tubes in a car radiator, where cooling liquid flows inside the tubes and air flows on the outside across the fins, and when heat is to be dissipated from a space vehicle, where in the absence of convection, finned surfaces which radiate thermal energy are employed.

In this chapter, we will analyze three types of extended surface configurations–straight fins, annular fins, and triangular fins. Performance parameters like fin effectiveness and fin efficiency will be discussed. Selection and design considerations will also be outlined. Error in temperature measurement will also be touched upon.

4.2 ❑ APPLICATIONS OF FINS

There are several engineering applications where fins are employed to dissipate large quantities of heat from relatively smaller areas particularly when the surface-film heat-transfer coefficient is rather modest. Some typical application areas of fins are:

- Automobile radiators
- Air cooling of cylinder heads of internal combustion engines (e.g., scooters, motor cycles, aircraft engines), air compressors, etc.
- Economisers of steam power plants.
- Fins on solid-state devices.
- Cooling of electric motors, transformers, etc.
- Fin cooling of electronic components, chips, integrated circuit boards, etc.
- Cooling fins for radiation source in an optical instrument.
- Finned evaporators and condensers in refrigerators and air conditioners.
- Heat exchangers used in industrial and commercial installations.

4.3 ❑ TYPES OF FINS

There are several types of fins used in engineering practice. The fins may be of uniform area (*longitudinal fins*) or of non-uniform area (*radial fins*).

Fins are available in different geometries and configurations. They may be of uniform or variable cross section. Fins are essentially classified as a *straight fin, an annular fin,* or a *spine*. The cross-sectional shape of the fin in a plane normal to the surface is called the *profile*. Fins can have *rectangular*, *parabolic, trapezoidal,* or *truncated conical* profiles. The *straight fin* is an extended surface added to a plane wall rectangular in shape and generally of uniform cross section. The *spine fin,* or *pin fin,* is simply a short thin rod protruding from the surface. It may be of *cylindrical* or *conical* shape. An *annular fin* is the one attached *circumferentially* to a cylindrical surface to increase its surface area. Cross-sectional areas of annular fins vary with the radius. In contrast, rectangular cylindrical spines have constant cross-sectional areas. *Triangular* or *parabolic* fins are used when one optimizes the fins from the standpoint of weight or volume.

Fins can be of rectangular cross-section along the length of the tube, called *longitudinal* fins, or *concentric annular discs* around a tube, referred to as *circumferential* fins. Figure 4.1 illustrates different types of fins commonly encountered in practice.

4.4 ❑ ONE-DIMENSIONAL APPROXIMATION OF A FIN

Consider a thin rectangular fin of rectangular cross section as shown in Fig. 4.2. The problem of heat conduction in a fin losing heat to the surrounding fluid by convection is, strictly speaking, *not* a one-dimensional problem.

The Biot number, *Bi,* compares *conduction resistance* to *convection resistance*. When the internal resistance (*conduction*) is small compared to the external resistance (*convection*), the temperature variations

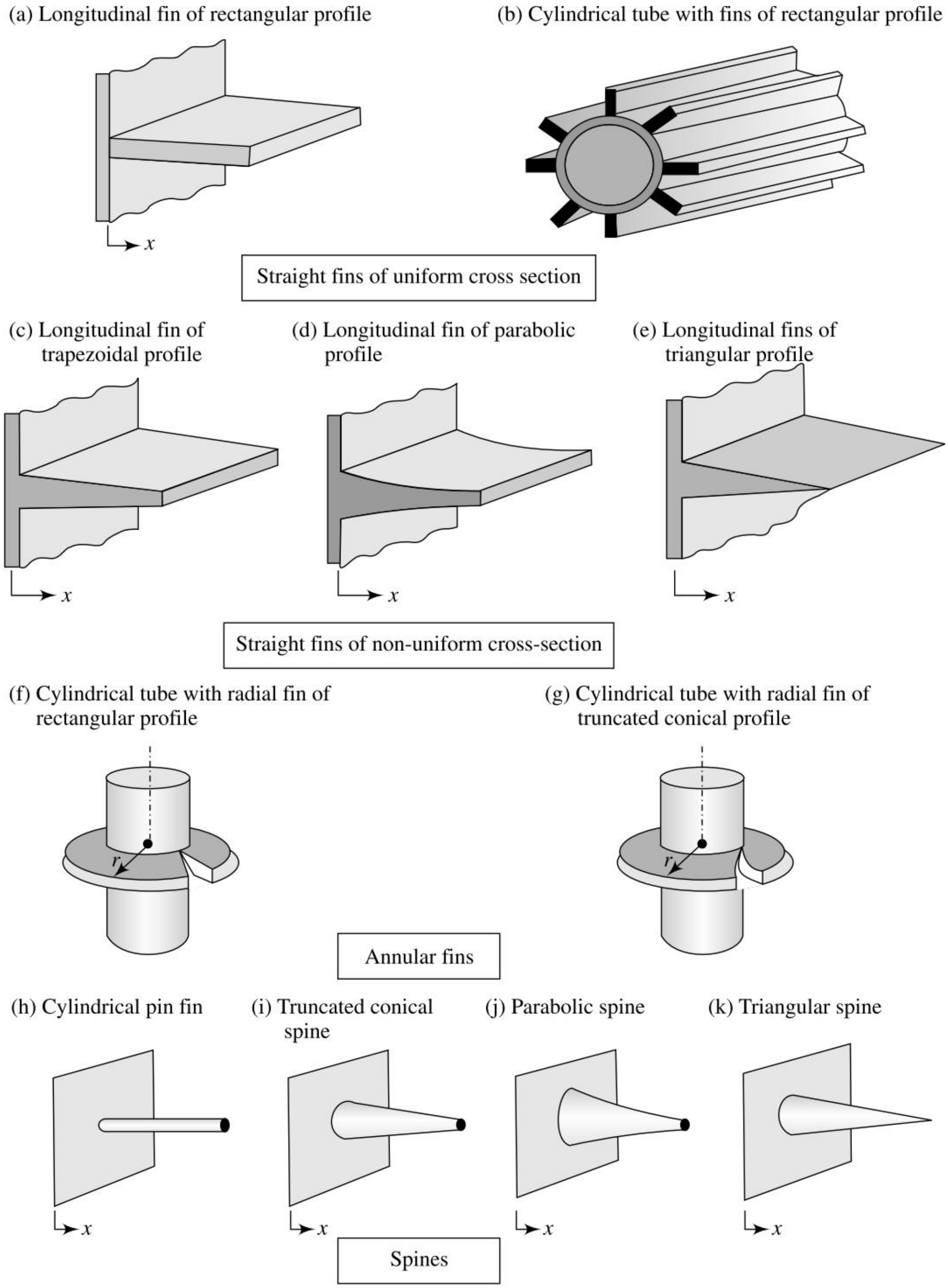


Fig. 4.1 *Schematic diagrams of various types of fins*

in the solid are negligible. If we compare the fin conduction resistance in the *transverse* direction to the convection resistance, we can ignore temperature variations in the transverse direction if

$$\boxed{Bi = \frac{hL_c}{k} \leq 0.1} \tag{4.2}$$

where L_c is the characteristic length (*dimension*) of the fin, which is the plate's *half thickness*, $\frac{t}{2}$ for *rectangular* fins and the radius R for *cylindrical* fins.

In a majority of the cases, the length (*the dimension measured perpendicular to the prime surface to which fins are attached*) of a *straight rectangular fin* from the heated surface is large compared to its maximum thickness, or in the case of a pin fin, the length of the cylindrical rod of *circular* cross section is very long compared to its diameter. Under such circumstances, the temperature can be assumed to be uniform over any cross section taken normal to the axis. Thus, the temperature of the fin can be assumed to depend only on *axial coordinates.* Similarly, a thin *annular* fin can be expected to have only *radial* temperature distribution.

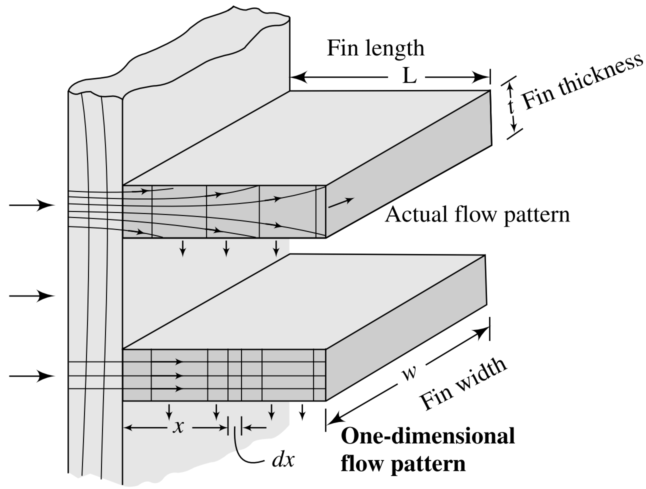


Fig. 4.2 *Heat flow through a cooling fin—actual and one-dimensional flow pattern*

Figure 4.2 shows both the *actual* flow pattern as well as the *one-dimensional* flow pattern in the case of straight fin of uniform thickness. In the latter case, the fin is considered thin so that the heat flow lines are *parallel* and only in the *axial* direction.

4.5 ❑ HEAT TRANSFER FROM A FIN OF UNIFORM CROSS SECTION

Consider two configurations of fins of uniform cross section to be treated in an identical manner—a straight fin of length or height L, width w and thickness t, and a spine (pin fin) of length L and diameter D as shown in Fig. 4.3. Let the temperature of the wall surface and the root of the fin be T_b, the ambient temperature, T_∞, uniform cross-sectional area A_c, constant perimeter of the cross section P, and the convective heat-transfer coefficient between the fin surface and the surrounding fluid, h.

Our aim is to determine the *temperature distribution* from the base to the tip of the fin and consequently, the *heat-transfer rate* from the fin.

For the heat-transfer analysis of a fin, the following assumptions are made:

- Steady operating conditions exist.
- There is no internal thermal energy generation.
- The heat conduction is one-dimensional and the temperature at any cross section of the fin is uniform over the entre cross section of the fin, i.e., $T = T(x)$ only. Temperature varies only along the length of the fin, *not* across the fin.
- The fin shape is constant over the whole length of the fin. The perimeter P and cross-sectional area A_c are *not* functions of the distance x.
- No *contact resistance* exists at the fin base. The fin material is integral with the prime surface.

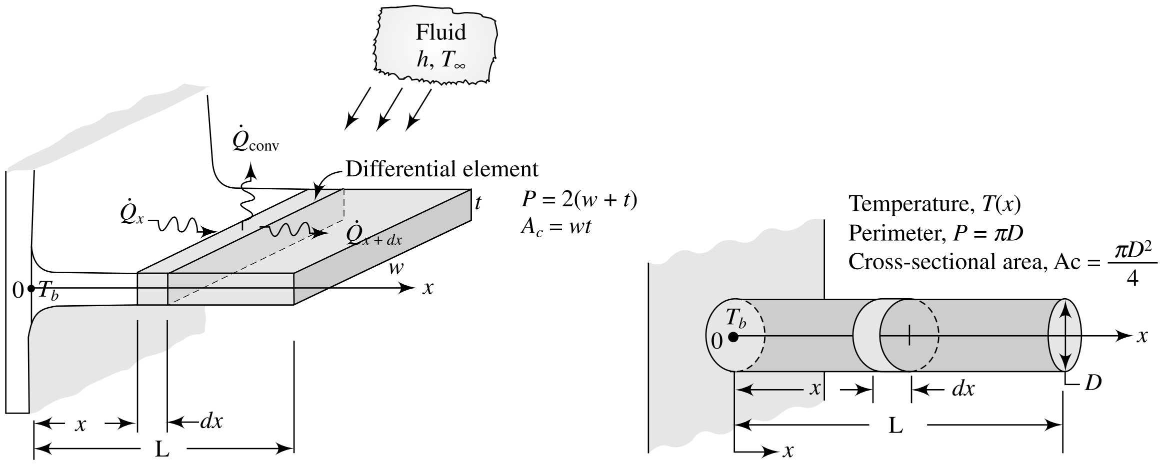


Fig. 4.3 *Analysis of a fin with uniform area of cross section*

- The material of the fin is *homogeneous* and *isotropic*, having *constant thermal conductivity*, k, The thermal conductivity k of the fin material does not vary with temperature.
- Heat loss by radiation and from the side edges of the fin is negligible.
- The heat-transfer coefficient h is uniform over the entire fin surface and does not vary with either the temperature or the location.

Consider an infinitesimal element of the fin dx in length, at a distance x from the root (*base*) of the fin or the wall. The steady-state energy balance for this element is:

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conducted } \textit{into} \text{ the}\\ \text{element at } x.\end{pmatrix} = \begin{pmatrix}\text{Rate of heat conducted}\\ \textit{out} \text{ of the element}\\ \text{at } (x+dx).\end{pmatrix} + \begin{pmatrix}\text{Rate of heat convected } \textit{out} \text{ (heat loss) from}\\ \text{the lateral surface of the element}\\ \text{exposed to the ambient fluid.}\end{pmatrix}$$

i.e., $$\dot{Q}_x = \dot{Q}_{x+dx} + \dot{Q}_{conv}$$

or $$\dot{Q}_x = \left[\dot{Q}_x + \frac{d}{dx}(\dot{Q}_x)dx\right] + \dot{Q}_{conv}$$

or $$-kA_c\frac{dT}{dx} = -kA_c\frac{dT}{dx} + \frac{d}{dx}\left(-kA_c\frac{dT}{dx}\right)dx + h(Pdx)(T - T_\infty)$$

or $$-kA_c\frac{d^2T}{dx^2} + hP(T - T_\infty) = 0 \quad \text{or} \quad \frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) = 0$$

or $$\boxed{\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0} \tag{4.3}$$

where $$m^2 \equiv \frac{hP}{kA_c}$$

Equation (4.3) is the *ordinary, second-order, linear, non-homogeneous* differential equation with constant coefficients. The equation describes the temperature as a function of m and x. The quantity m is a function of the properties of the fin material and the ambient fluid.

Since T_∞ is a constant ambient temperature, we can replace $(T - T_\infty)$ by θ, the local excess temperature, so that,

$$\frac{d^2\theta}{dx^2} \equiv \frac{d^2(T - T_\infty)}{dx^2} \equiv \frac{d^2T}{dx^2} \qquad (\because T_\infty = \text{constant})$$

Equation (4.3) can then be written as

$$\boxed{\frac{d^2\theta}{dx^2} - m^2\theta = 0} \tag{4.4}$$

The general solution of *Eq.* (*4.4*) is

$$\boxed{\theta = C_1 e^{mx} + C_2 e^{-mx}} \tag{4.5}$$

where $m = \sqrt{\dfrac{hP}{kA_c}}$, and C_1 and C_2 are constants of integration to be determined from the boundary conditions for the problem.

It may be noted that m is not a dimensionless constant. It has the unit m^{-1} or reciprocal of length. The parameter mL is of course *dimensionless* and its physical significance is that it is the ratio of *surface conductance* to the *internal conductance*.

$$mL = \sqrt{\frac{hPL}{kA_c/L}}$$

In other words, one can also express mL as

$$\left[\frac{L/kA_c}{1/h(PL)}\right]^{1/2} = \sqrt{\frac{R_{\text{cond}}}{R_{\text{conv}}}}$$

i.e., the ratio of *conductive thermal resistance* to the *convective (surface) thermal resistance*. Note that A_c is the area of cross section while PL is A_s, the surface area.

If m is less, i.e., convective resistance is predominant due to small surface area or lower value of h, then equipping the base surface with fins will *reduce* this surface resistance and *increase* heat transfer in spite of marginal addition of conductive resistance.

In the analysis and design of a finned surface, the heat dissipated by a single fin of a given geometry is determined from the *temperature gradient* and the *cross-sectional area* available for heat transfer rate *at the base of the fin.*

The total number of fins necessary to dissipate a specified total heat-transfer rate is determined from

$$\boxed{\text{Number of fins, } N = \frac{\dot{Q}_{\text{total}}}{\dot{Q}_{\text{single fin}}}} \tag{4.6}$$

4.5.1 • Specific Cases and Boundary Conditions

Customarily, there are *two* boundary conditions specified at the *two* ends of the fin. The *first* one which is at one end (*the base or root*) of the fin is of course *common* to all the cases. This common Boundary

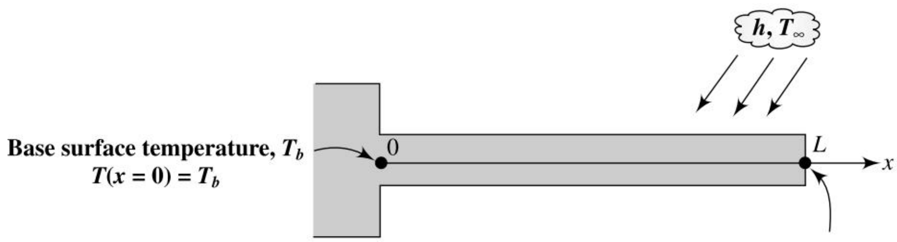
Condition (BC) is that the temperature of the primary surface or wall which is also the base temperature of the fin is known. That is,

BC I: At $x = 0$, $T = T_b$ or $\theta(0) = \theta_b \equiv T_b - T_\infty$

where θ_b is the base excess temperature

The *second* boundary condition depends on the specific case depending on the physical situation. *Four* possible sets of boundary conditions could be imposed.

To illustrate the physical significance of different conditions at the *fin tip* ($x = L$), the solution of the fin problem for each of these cases will now be presented in terms of the *temperature distribution* and hence the *heat-flow rate* through the fin. The schematic representation of the *four* sets of boundary conditions is shown in Fig. 4.4.



(a) Very long fin [$T(L \to \infty) = T_\infty$]

(b) Negligible heat loss from the fin tip (*adiabatic or insulated tip*) $\left[\left.\dfrac{dT}{dx}\right|_{x=L} = 0\right]$

(c) Convection from the tip $k\left.\dfrac{dT}{dx}\right|_{x=L} = h\,(T(x = L) - T_\infty]$

(d) Specified sin tip temperature $T(x = L) = T_L$

Fig. 4.4 *Boundary conditions at the fin base and the fin tip.*

Case I: Very Long Fin ($L \to \infty$) Let us consider a very long cylindrical fin of uniform cross section ($A_c = A_b$, i.e., area at the base). One can assume that the base is hotter than the fluid, although the same processes are applicable when the fluid is hotter than the base. Heat conducts from the base into the fin and is dissipated by convection at the outer surface of the fin. The fin is usually made of a high thermal conductivity material to facilitate the flow of heat from the base to the tip. With heat being convected away from the surface, the temperature of the fin decreases from base to tip, as shown in Figure 4.5.

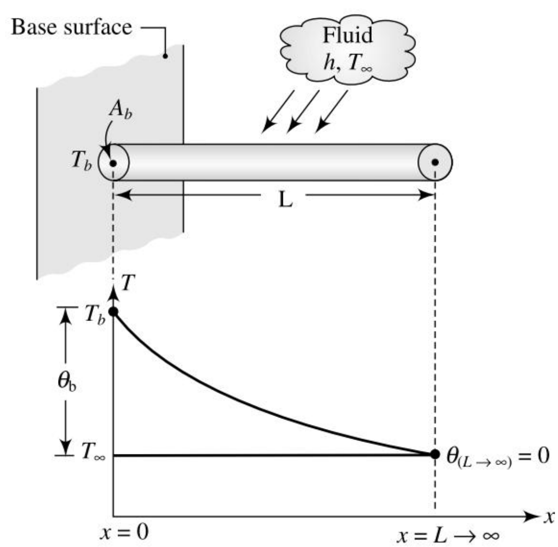


Fig. 4.5 *Temperature variation in a very long fin*

4.5.2 • Temperature Distribution

For an infinitely long fin, it is reasonable to assume that the temperature at the tip of the fin approaches the ambient fluid temperature T_∞. That is,

BC II: As $x = L \to \infty$, $\theta(x) = 0$

Applying the *first* boundary condition at the fin base to Eq. (4.5), we have

$$\boxed{\theta_b = C_1 + C_2} \tag{4.7}$$

Applying the *second* boundary condition at the fin tip, we get

$$0 = C_1 e^{m\infty} + C_2 e^{-m\infty} = C_1 e^{\infty} + C_2 e^{-\infty}$$

or $$0 = C_1(\infty) + C_2(0)$$

This equality is valid only when

$$\boxed{C_1 = 0}$$

Thus,

$$\boxed{C_2 = \theta_b} \qquad \text{from Eq. (4.7)}$$

Substituting the values of C_1 and C_2 in Eq. (4.4), the temperature distribution becomes

$$\boxed{\theta = \theta_b e^{-mx}} \quad \text{or} \quad \boxed{\frac{\theta}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx}} \tag{4.8}$$

This equation gives the *dimensionless* temperature distribution along the length of an infinite (*very long*) fin of uniform cross section. This is shown graphically in Fig. 4.6. Temperature drops exponentially as the distance from the base increases and with increasing value of the fin parameter m, this drop is sharper.

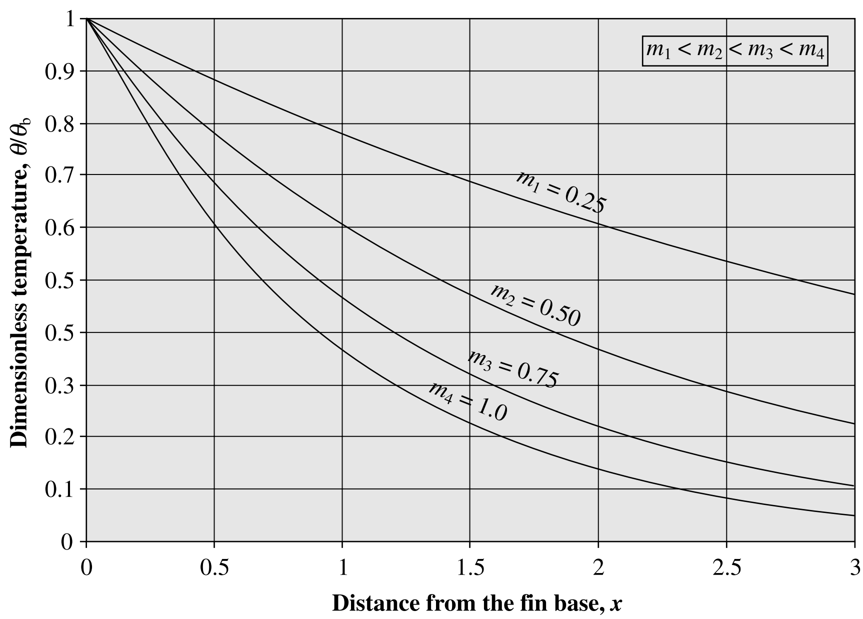


Fig. 4.6 *Temperature profile in a very long fin along its length*

4.5.3 • Heat-Transfer Rate

The heat-flow rate through the fin can be determined by *two* different methods. From Fig. 4.7, the following is noteworthy:

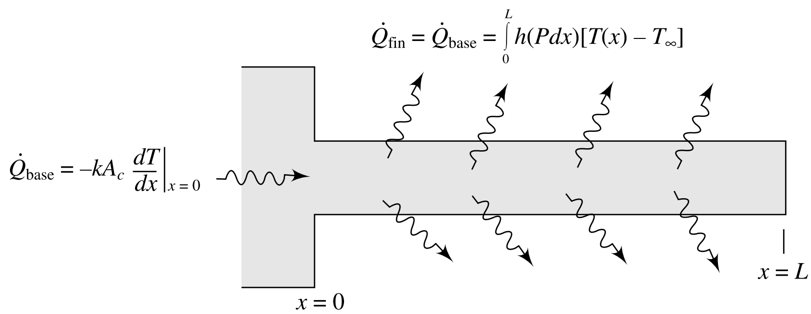


Fig. 4.7 *Under steady operating conditions, heat loss by convection from the fin surface to the surrounding fluid equals heat conduction to the fin at its base.*

$$\begin{pmatrix}\text{Heat conducted } \textit{across} \\ \text{the base of the fin.}\end{pmatrix} = \begin{pmatrix}\text{Heat transferred by convection } \textit{from} \\ \text{the surface of the fin } \textit{to} \text{ the surrounding fluid.}\end{pmatrix}$$

Thus, $\dot{Q}_{\text{base}} = \dot{Q}_{\text{fin}}$

From Fourier's law of heat conduction,

$$\dot{Q}_{\text{base}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} \tag{4.9}$$

Differentiating Eq. (4.8) and substituting the result for $x = 0$ in Eq. (4.9), we have

$$\frac{dT}{dx} = \frac{d\theta}{dx} = \theta_b(-m)e^{-mx}$$

$$\therefore \quad \dot{Q}_{\text{base}} = -kA_{\text{c}}[-m\theta_b e^{-mx}]_{x=0} = mkA_c\theta_b$$

But $\quad m = \sqrt{\dfrac{hP}{kA_c}}$ and one can also write

$$\dot{Q}_{\text{base}} = \sqrt{hP}\,\frac{kA_c}{\sqrt{kA_c}}\theta_b = \sqrt{hPkA_c}\;\theta_b$$

As mentioned earlier, the heat that enters the base in steady state is dissipated to the environment all along the exposed surface of the fin so that we could also calculate the heat-transfer rate by integrating the expression for the surface convection heat transfer over the entire length of the fin. It follows that,

$$\dot{Q}_{\text{fin}} = \int_0^\infty hP[T(x) - T_\infty]dx = \int_0^\infty h\theta(Pdx) = hP\theta_b \int_0^\infty e^{-mx}dx$$

$$= \frac{-hP\theta_b}{m}[e^{-mx}]_0^\infty = -\frac{hP\theta_b}{m}(\cancelto{0}{e^{-m(\infty)}} - e^{-m(0)}) = \frac{hP\theta_{\text{b}}}{m} \quad (\text{since } e^{-0} = 1)$$

But we recognize that $m^2 = hP/kA_c$, so that we have

$$\dot{Q}_{\text{fin}} = \frac{m^2 kA_c\theta_b}{m} = mkA_c\theta_b$$

Hence, the fin heat-transfer rate is

$$\boxed{\dot{Q}_{\text{fin}} = mkA_c\theta_b \quad \text{or} \quad \sqrt{hPkA_c}\ \theta_b = M \cdot \theta_b}$$

where $$M \equiv \sqrt{hPkA_c} = mkA_c \tag{4.10}$$

Case II: Fin with Negligible Heat Loss at the Tip or an Insulated End $\left[\left.\dfrac{dT}{dx}\right|_{x=L} = 0\right]$

Usually, the heat-transfer area at the fin tip is small compared with the lateral surface area of the fin for heat transfer. In such situations, the heat loss from the fin tip is negligible *(as if the fin were insulated)* compared to that from the lateral surfaces, as shown in Fig. 4.8. The *second* boundary condition which characterizes this situation requires that the temperature gradient be zero at the tip (*end*) of the fin. That is,

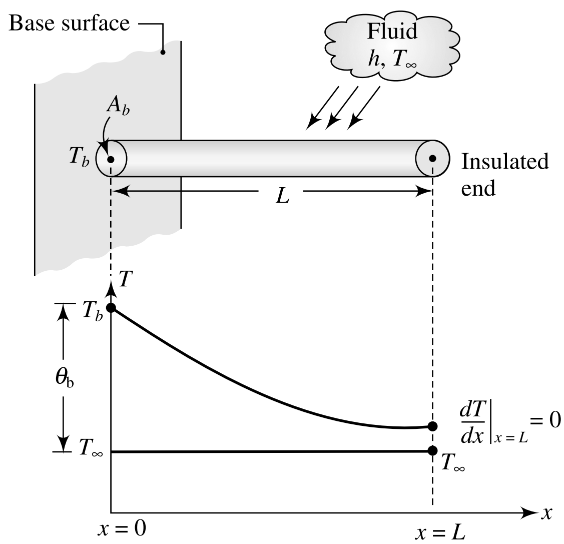


Fig. 4.8 *Temperature variation in a fin with an insulated tip*

BC II: At $x = L, \dfrac{dT}{dx} = \dfrac{d\theta}{dx} = 0$

This assumption is justified in those cases where h is small at the end of the fin, and k of the fin material is large so that the ratio $(h/k) \to 0$, i.e., heat loss from the fin tip is negligible.

The *first* boundary condition is the same (as in *Case I*) which requires that

$$\boxed{\theta_b = C_1 + C_2}$$

Differentiating Eq. (4.4) with respect to x, one gets

$$\frac{d\theta}{dx} = mC_1e^{mx} - mC_2e^{-mx}$$

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0 = m[C_1e^{mL} - C_2e^{-mL}]$$

$\therefore$ $$\boxed{C_1 = C_2e^{-2mL}}$$

From Eq. (4.7), we now have

$$\theta_b = C_2\,e^{-2mL} + C_2 = C_2$$

$\therefore$ $$\boxed{C_2 = \frac{\theta_b}{1 + e^{-2mL}}}$$

and $$C_1 = \theta_b - \frac{\theta_b}{1 + e^{-2mL}} = \theta_b\left[1 - \frac{1}{1 + e^{-2mL}}\right] = \frac{(e^{-2mL})\theta_b}{(1 + e^{-2mL})}\frac{(e^{2mL})}{(e^{2mL})} = \frac{\theta_b}{e^{2mL} + 1}$$

The temperature distribution along the fin is then obtained by putting the values of C_1 and C_2 in Eq. (4.4):

$$\theta = \frac{\theta_b}{1 + e^{2mL}} e^{mx} + \frac{\theta_b}{1 + e^{-2mL}} e^{-mx}$$

or

$$\frac{\theta}{\theta_b} = \frac{e^{mx}}{1 + e^{2mL}} + \frac{e^{-mx}}{1 + e^{-2mL}}$$

The equation can be expressed in a more compact form in terms of hyperbolic functions if we multiply the numerator and denominator of the *first term* on the right-hand side by $\overline{e}^{mL}$ and the numerator and denominator of the *second term* by e^{+mL}.

Then

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{e^{-mL} e^{mx}}{e^{-mL} + e^{mL}} + \frac{e^{+mL} e^{-mx}}{e^{+mL} + e^{-mL}}$$

$$= \frac{e^{-m(L-x)} + e^{+m(L-x)}}{e^{mL} + e^{-mL}} = \frac{[e^{m(L-x)} + e^{-m(L-x)}]/2}{[e^{+mL} + e^{-mL}]/2}$$

Noting that the hyperbolic cosine is defined as, $\cos x \; x = \dfrac{e^x + e^{-x}}{2}$, the temperature profile can be expressed as:

$$\boxed{\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cos m(L - x)}{\cos mL}} \tag{4.11}$$

The temperature gradient at the root of the fin, i.e.,

$$\left.\frac{dT}{dx}\right|_{x=0} \quad \text{or} \quad \left.\frac{d\theta}{dx}\right|_{x=0}$$

is given by differentiating BC II with respect to x and then putting $x = 0$.

Thus,

$$\left.\frac{dT}{dx}\right|_{x=0} = \left.\frac{-m \sin m(L - x)}{\cos mL}\right|_{x=0} = -\theta_b\, m \tan mL$$

Hence, the heat dissipated by the fin

$$\dot{Q}_{\text{fin}} = -kA_c(-\theta_b\, m \tan mL) = -mkA_c\theta_b \tan mL$$

or

$$\boxed{\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,\theta_b \tan mL = M \tan mL} \tag{4.12}$$

where $\quad M \equiv mkA_c = \sqrt{hPkA_c}$

Case III: Fin with Convection at the Tip A more realistic physical boundary condition at the other extremity of the fin ($x = L$) includes convective heat transfer from the fin tip to the surrounding fluid (Fig. 4.9).

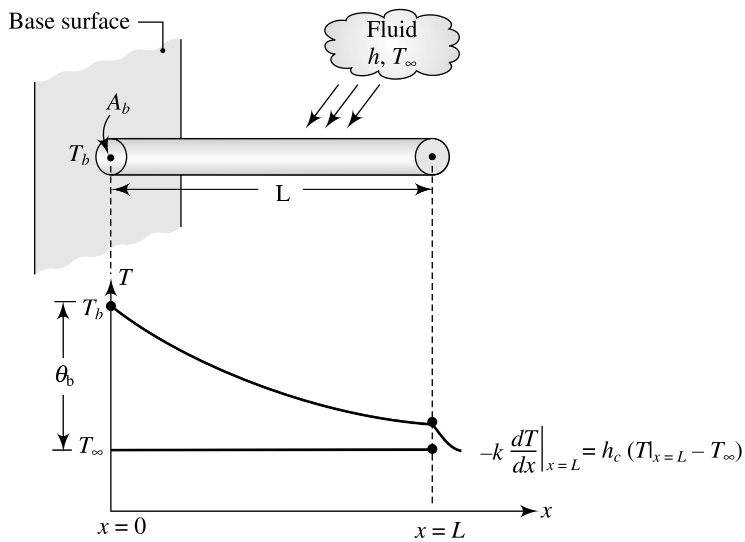


Fig. 4.9 *Temperature variation in a fin with a convective tip*

The result obtained after applying the *first* boundary condition [$T(0) = T_b$] is already given by

$$\boxed{\theta_b = C_1 + C_2}$$

BC II: The *second* boundary condition requires that,

$$\boxed{-k\left.\frac{d\theta}{dx}\right|_{x=L} = h_L\theta\big|_{x=L}} \tag{4.13}$$

where h_L is the heat-transfer coefficient between the fin tip and the ambient fluid.

Now, $\theta_{x=L} = C_1e^{mL} + C_2e^{-mL}$

and $\left.\frac{d\theta}{dx}\right|_{x=L} = m[C_1e^{mL} - C_2e^{-mL}]$

Substituting these values in Eq. (4.11), and assuming $h_L = h$, we have

$$-mk[C_1e^{mL} - C_2e^{-mL}] = h[C_1e^{mL} + C_2e^{-mL}]$$

or $$C_2e^{-mL} - C_1e^{mL} = \frac{h}{mk}[C_1e^{mL} + C_2e^{-mL}]$$

or $$C_1[1 + (h/mk)]e^{mL} = C_2[1 - (h/mk)]e^{-mL}$$

But $C_1 = \theta_b - C_2$ [from *Eq.(4.7)*]

$\therefore$ $$(\theta_b - C_2)\{1 + (h/mk)\}e^{mL} = C_2\{1 - (h/mk)\}e^{-mL}$$

or $$C_2[\{1 + (h/mk)\}e^{mL} + \{1 - (h/mk)\}e^{-mL}] = \theta_b\{1 + (h/mk)\}e^{mL}$$

$\therefore$ $$\boxed{C_2 = \frac{\theta_b\{1 + (h/mk)\}e^{mL}}{(e^{mL} + e^{-mL}) + (h/mk)(e^{mL} - e^{-mL})}}$$

$$\therefore \qquad C_1 = \theta_b - C_2 = \theta_b\left[\frac{(e^{mL}+e^{-mL}) + (h/mk)(e^{mL}-e^{-mL}) - \{1+(h/mk)\}e^{mL}}{(e^{mL}+e^{-mL}) + (h/mk)(e^{mL}+e^{-mL})}\right]$$

$$= \theta_b\left[\frac{\{1+(h/mk)\}e^{mL} + \{1-(h/mk)\}e^{-mL} - \{1+(h/mk)\}e^{mL}}{(e^{mL}+e^{-mL})1 + (h/mk)(e^{mL}-e^{-mL})}\right]$$

$$\therefore \qquad \boxed{C_1 = \frac{\theta_b(1-(h/mk))e^{-mL}}{(e^{mL}+e^{-mL}) + (h/mk)(e^{mL}-e^{-mL})}}$$

Substituting the values of the constants of integration C_1 and C_2 into Eq. (4.4), the temperature profile is given by

$$\frac{\theta}{\theta_b} = \frac{e^{mx}e^{-mL}(1-(h/mk)) + e^{mL}e^{-mx}(1+(h/mk))}{(e^{mL}+e^{-mL}) + (e^{mL}+e^{-mL})}$$

$$= \frac{[e^{m(L-x)} + e^{-m(L-x)}] + (h/mk)[e^{m(L-x)} - e^{-m(L-x)}]}{(e^{mL}+e^{-mL}) + (h/mk)(e^{mL}-e^{-mL})}$$

Noting that

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2}, \text{ we finally get}$$

$$\boxed{\frac{\theta}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}} \tag{4.14}$$

The heat-flow rate through the fin is

$$\dot{Q}_{\text{fin}} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0}$$

Differentiating Eq. (4.11) with respect to x, we have

$$\left.\frac{1}{\theta_b}\frac{d\theta}{dx}\right|_{x=0} = \left.\frac{(-m)\sinh m(L-x) + (h/mk)(-m)\cosh m(L-x)}{\cosh mL + (h/mk)\sinh mL}\right|_{x=0}$$

$$= \frac{-m[\sinh mL + (h/mk)\cosh mL]}{\cosh mL + (h/mk)\sinh mL}$$

$$\therefore \qquad \boxed{\dot{Q}_{\text{fin}} = mkA_c\theta_b\left[\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}\right]}$$

$$\text{or} \qquad \boxed{\dot{Q}_{\text{fin}} = M\left[\frac{\tanh mL + (h/mk)}{1 + (h/mk)\tanh mL}\right]} \quad \text{where } M \equiv mkA_c\theta_b = \sqrt{hPkA_c}\,\theta_b \tag{4.15}$$

If we consider a fin of length L, and then apply the relevant boundary conditions associated with a *very long fin*, the *adiabatic tip fin*, and the *finite convective tip fin*, the following observation about the fin-tip temperature for these *three* cases can be made:

The very long fin would have the *lowest tip temperature*, the *adiabatic tip fin* would have the *highest tip temperature*, and the *finite fin with convective tip* would have a tip temperature *somewhere between that of the infinitely long fin and the adiabatic tip fin* (Fig. 4.10).

$$\boxed{T(L)_{\text{very long}} < T(L)_{\text{convective tip}} < T(L)_{\text{adiabatic tip}}} \tag{4.16}$$

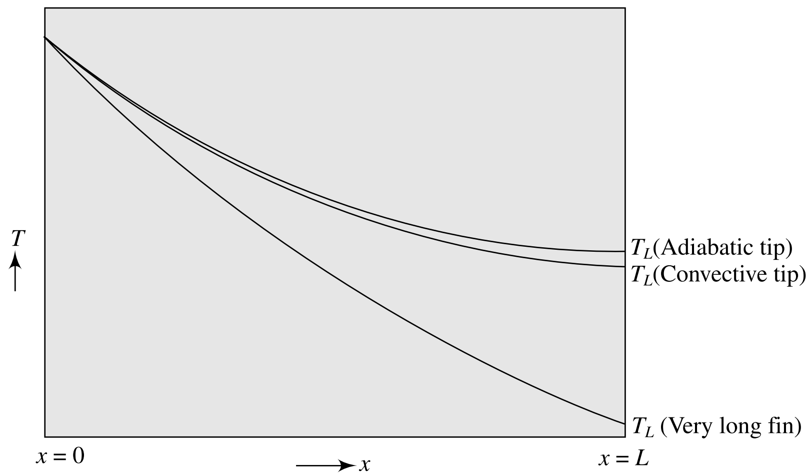


Fig. 4.10 *Comparison of fin-tip temperatures for three different boundary conditions*

4.5.4 • Corrected Length

The expressions for the fin with convection from its tip are rather tedious compared to those of the insulated tip fin. One can obtain a reasonable approximation to the convective tip fin if in the insulated-tip fin expressions, one uses a *corrected* length, L_c, to account for the additional area of the fin tip (Fig. 4.11**).** Corrected fin length L_c is defined such that the heat-transfer rate from a fin of length L_c with adiabatic tip equals the heat-transfer rate from the actual fin of length L with convection at the fin tip, and is given by

$$\boxed{L_c = L + A_c/P} \tag{4.17}$$

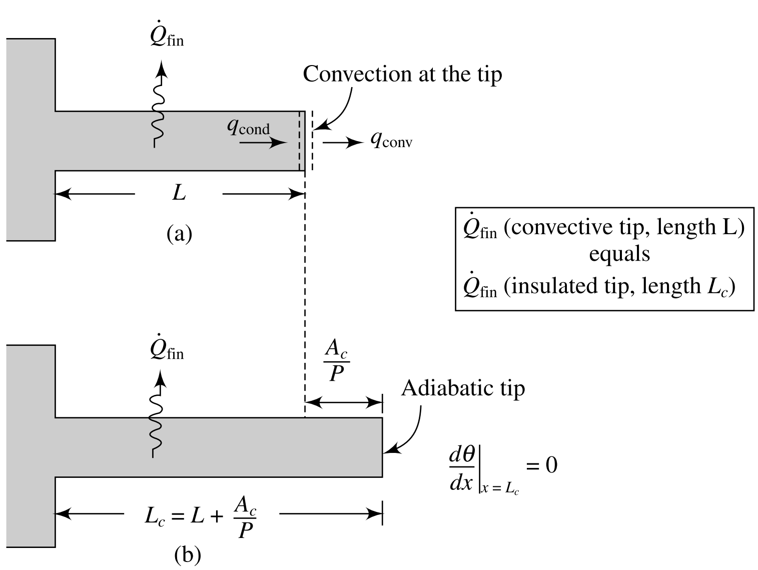


Fig. 4.11 *(a) Actual fin of length L with convection at the tip (b) Equivalent fin of corrected length L_c with an adiabatic tip.*

if the convective coefficients on surface and end, are equal. For the pin fin, $L_c = L + D/4$, and for a rectangular fin, $L_c = L + t/2$, where t is the fin thickness. One would get essentially the same performance for the adiabatic tip fin of length L_c with the uninsulated fin tip of length L.

One must note that the error associated with the adiabatic tip approximation is negligible if

$$\boxed{(ht/k) \text{ or } (hD/2k) \leq 0.0625}$$

Case IV: Fin of Finite Length with Specified Temperature at its End ***($T_{(x\,=\,L)} = T_L$)*** Consider a thin fin of length L with its two ends attached to two parallel walls, maintained at temperatures T_b and T_L, as shown in Fig. 4.12. The fin loses heat by convection to the ambient fluid at T_∞. Let us analyse such a fin with a specified tip temperature, T_L, and base temperature $T_o = T_b$.

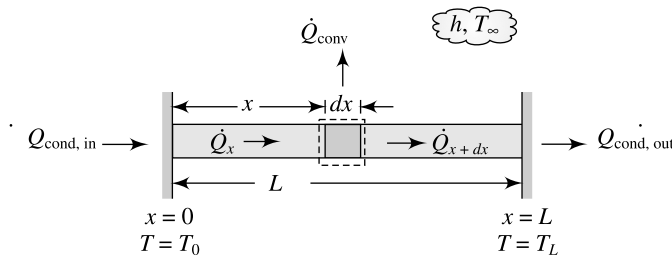


Fig. 4.12 *A fin with fixed temperature at the tip*

The general solution of temperature distribution is

$$\boxed{\theta \equiv T - T_\infty = C_1 e^{mx} + C_2 e^{-mx}} \tag{A}$$

Boundary conditions are

BC I: $\theta(x = 0) = \theta_b$

and BC II: $\theta(x = L) = \theta_L$

From boundary condition I:

$$\theta_b = C_1 + C_2 \quad \text{or} \quad \theta_b e^{mL} = C_1 e^{mL} + C_2 e^{mL} \tag{a}$$

From boundary condition II:

$$\theta_L = C_1 e^{mL} + C_2 e^{-mL} \tag{b}$$

Subtracting (b) from (a), we have

$$\theta_b e^{mL} - \theta_L = C_2 (e^{mL} - e^{-mL})$$

or

$$C_2 = \frac{\theta_b e^{mL} - \theta_L}{e^{mL} - e^{-mL}} \quad \text{and} \quad C_1 = \theta_b - C_2 = \theta_b - \frac{\theta_b e^{mL} - \theta_L}{e^{mL} - e^{-mL}}$$

$$= \frac{\theta_b e^{mL} - \theta_b e^{-mL} - \theta_b e^{mL} + \theta_L}{e^{mL} - e^{-mL}} = \frac{\theta_L - \theta_b e^{-mL}}{e^{mL} - e^{-mL}}$$

Substituting for C_1 and C_2 in Eq. (A), one gets

$$\theta = \left(\frac{\theta_L - \theta_b e^{-mL}}{e^{mL} - e^{-mL}}\right) e^{mx} + \left(\frac{\theta_b\, e^{mL} - \theta_L}{e^{mL} + e^{-mL}}\right) e^{-mx}$$

$$= \frac{\theta_L e^{mx} - \theta_b e^{-mL} e^{mx} + \theta_b e^{mL} e^{-mx} - \theta_L e^{-mx}}{e^{mL} - e^{-mL}}$$

$$= \frac{\theta_L(e^{mx} - e^{-mx}) + \theta_b\{e^{m(L-x)} - e^{-m(L-x)}\}}{e^{mL} - e^{-mL}}$$

$$= \theta_L \frac{e^{mx} - e^{-mx}}{e^{mL} - e^{-mL}} + \theta_b \frac{e^{m(L-x)} - e^{-m(L-x)}}{e^{mL} - e^{-mL}}$$

We note that $\sinh mx = \dfrac{e^{mx} + e^{-mx}}{2}$ and $\sinh m(L-x) = \dfrac{e^{m(L-x)} + e^{-m(L-x)}}{2}$

Hence, $$\boxed{\boldsymbol{\theta = \theta_L \frac{\sinh mx}{\sinh mL} + \theta_b \frac{\sinh m(L-x)}{\sinh mL}}}$$

or $$\boxed{\boldsymbol{\frac{\theta}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}}} \tag{4.18}$$

Differentiating with respect to x, one has

$$\frac{1}{\theta_b}\frac{d\theta}{dx} = \frac{(\theta_L/\theta_b)m\cosh mx + (m)(-1)\cosh m(L-x)}{\sinh mL}$$

or $$\left.\frac{d\theta}{dx}\right|_{x=0} = \frac{\theta_b[m(\theta_L/\theta_b) - m\cosh mL]}{\sinh mL}$$

Heat flow rate at $x = 0$ is

$$\dot{Q}_b = \dot{Q}_{\text{fin}} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} = -kA_c \frac{m\theta_b}{\sinh mL}[(\theta_L/\theta_b) - \cosh mL]$$

or $$\boxed{\dot{Q}_{\text{fin}} = mkA_c\,\theta_b \frac{[\cosh mL - (\theta_L/\theta_b)]}{\sinh mL}} \tag{4.19}$$

Also, at $x = L$, the temperature gradient is

$$\left(\frac{d\theta}{dx}\right)_{x=L} = \frac{m\theta_b}{\sinh mL}\{(\theta_L/\theta_b)\cosh mL - 1\}$$

Hence, the rate of heat flow from the other end ($x = L$) is

$$\dot{Q}_L = -kA_c \left.\frac{d\theta}{dx}\right|_{x=L} = -kA_c \frac{\theta_b m}{\sinh mL}\{(\theta_L/\theta_b)\cosh mL - 1\}$$

or $$\boxed{\boldsymbol{\dot{Q}_L = mkA_c\,\theta_b \frac{[1 - (\theta_L/\theta_b)\cosh mL]}{\sinh mL}}}$$

Net heat lost by the fin is

$$\dot{Q}_0 - \dot{Q}_L = \dot{Q}_{\text{cond,in}} - \dot{Q}_{\text{cond,out}} = \dot{Q}_{\text{conv}}$$

$$= \frac{mkA_c\theta_b}{\sinh mL}[\cosh mL - (\theta_L/\theta_b) - 1 + (\theta_L/\theta_b)\cosh mL]$$

$$= \frac{mkA_c\theta_b}{\sinh mL}\left(1 + \frac{\theta_L}{\theta_b}\right) \times [\cosh mL - 1]$$

or $$\boxed{\dot{Q}_{conv} = mkA_c(\theta_b + \theta_L)[(\cosh mL - 1)/\sinh mL]} \tag{4.20}$$

4.5.5 • Minimum Temperature in the Fin

To find the location and magnitude of minimum temperature that will occur in the fin, let us differentiate the expression for θ with respect to x and equate the resulting derivative to zero.

$$\frac{d\theta}{dx} = \frac{\theta_b}{\sinh mL}\frac{d}{dx}\left[\left(\frac{\theta_L}{\theta_b}\right)\sinh mx + \sinh m(L-x)\right] = 0$$

or $$\frac{\theta_L}{\theta_b} \cosh mx\ (m) + \cosh m\ (L - x)\ (m)\ (-1) = 0$$

or $$\cosh mx = \frac{\theta_b}{\theta_L} \cosh m\ (L - x)$$

or $$\cosh mx = A \cosh m\ (L - x) \qquad \text{where } A \equiv \frac{\theta_b}{\theta_L}$$

As $$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\frac{e^{mx} + e^{-mx}}{2} = A\left[\frac{e^{(mL-mx)} + e^{(-mL+mx)}}{2}\right]$$

or $$e^{mx} + e^{-mx} = A[e^{mL}e^{-mx} + e^{-mL}e^{mx}]$$

or $$e^{mx}[1 - Ae^{-mL}] = e^{-mx}[Ae^{mL} - 1]$$

or $$e^{2mx} = \left(\frac{Ae^{mL} - 1}{1 - Ae^{-mL}}\right) = \frac{e^{mL}(Ae^{mL} - 1)}{(e^{mL} - A)}$$

Taking log on both sides, we have

$$2mx = \ln\left[\frac{e^{mL}}{e^{mL} - A} \times (Ae^{mL} - 1)\right]$$

Hence, minimum temperature will occur at

$$\boxed{x_{min} = \frac{1}{2\text{ m}}\ln\left[\frac{e^{mL}(Ae^{mL} - 1)}{e^{mL} - A}\right]} \qquad \text{where } A = \theta_b/\theta_L \tag{4.21}$$

Substituting for x_{min} in the equation of $\theta(x)$, we can obtain the *minimum* temperature.

Case IV Special Case: When both Ends of the Fin are at the Same Temperature Obviously, *the minimum* temperature will occur at the centre, i.e., at $x = L/2$ (*due to symmetry*).

Then, substituting $\theta_b = \theta_L$ and $x = L/2$ in the expression for $\theta(x)$, we get the *minimum* temperature. We note that

$$\theta(x) = \frac{\theta_b \sinh m(L - x) + \theta_b \sinh mx}{\sinh mL}$$

Therefore,

$$\theta_{min} = \frac{\theta_b \sinh\left[m\left(L - \frac{L}{2}\right)\right] + \theta_b \sinh\left(m\frac{L}{2}\right)}{\sinh mL}$$

or
$$\boxed{\theta_{\min} = \frac{2\theta_b \sinh\left(\dfrac{mL}{2}\right)}{\sinh mL}} \tag{4.22a}$$

Since
$$\sinh mL = 2\sinh\left(\frac{mL}{2}\right)\cosh\left(\frac{mL}{2}\right),$$
$$\theta_{\min} = \frac{2\theta_b \sinh(mL/2)}{2\sinh(mL/2)\cosh(mL/2)}$$

or
$$\boxed{\theta_{\min} = \theta_b/\cosh(mL/2)} \tag{4.22b}$$

where
$$\theta_{\min} = T_{\min} - T_\infty$$

Table 4.1 *Temperature distribution and rate of heat transfer for fins of uniform cross section*

Case	Tip condition ($x = L$)	Temperature distribution, θ/θ_b	Fin heat transfer rate, $\dot{Q}_{\text{fin}}$
I.	Infinite (Very long) fin: ($L \to \infty$) $\theta(L) = 0$	e^{-mx}	M
II.	Adiabatic fin tip: $\left.\dfrac{d\theta}{dx}\right\|_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M \tanh mL$
III.	Convection from top: $h\theta(L) = -k\left.\dfrac{d\theta}{dx}\right\|_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	$M\left\{\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}\right\}$ or $M\left\{\dfrac{\tanh mL + (h/mk)}{(1 + (h/mk)\tanh mL}\right\}$
IV.	Specified tip temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ Location of minimum temperature, $T_{\min}$: $e^{2mx_{\min}} = \dfrac{e^{mL} - A}{A - e^{-mL}}$ where $A \equiv \dfrac{\theta_L}{\theta_b}$ If $e^{2mx_{\min}}$ is *negative*, $T_{\min}$ *does not exist*.	$\dfrac{M}{\sinh mL}[\cosh mL - \theta_L/\theta_b]$
	Special case: Same fin-end temperatures $\theta_L = \theta_b$	$\dfrac{\sinh mx + \sinh m(L-x)}{\sinh mL}$ $T_{\min}$ at $x = L/2$, i.e., midpoint. $\theta_{\min} = \dfrac{2\theta_b \sinh(mL/2)}{\sinh mL} = \dfrac{\theta_b}{\cosh(mL/2)}$	$\dfrac{M[\cosh mL - 1]}{\sinh mL}$

$\theta \equiv T - T_\infty$	$m^2 \equiv \dfrac{hP}{kA_c}$
$\theta_b \equiv \theta(0) = T_b - T_\infty$	$M \equiv mkA_c\theta_b = \sqrt{hPkA_c}\ \theta_b$

4.6 ❑ CONDITION FOR VERY LONG FIN

Consider the ratio of the heat-transfer rate for a fin with an *adiabatic tip* $(\dot{Q}_{\text{finite}})$ to the heat-transfer rate for an *infinitely long fin*. $(\dot{Q}_\infty)$ that is, $\dot{Q}_{\text{finite}}/\dot{Q}_\infty = \tanh mL$. Figure 4.13 shows the comparative magnitude of the heat-transfer rates as the value of mL is increased. When the ratio approaches 1 the fin can be considered *infinitely long*. Notice that after a value of $mL \approx 2.65$, very little increase in heat transfer is obtained by increasing the length of the fin. Hence, the additional (*minor*) increase in heat transfer can hardly be justified for the extra cost of the longer fin.

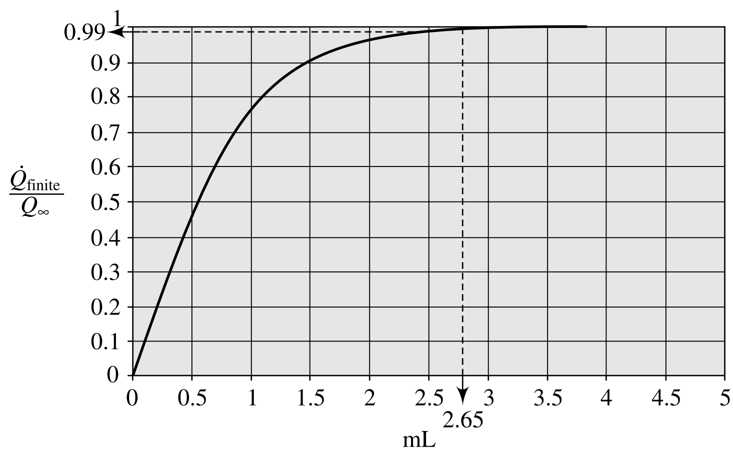


Fig. 4.13 *Comparison of heat transfer from a finite fin versus an infinite fin*

Let us define the following *three* dimensionless parameters:

- Dimensionless axial position, $\xi = \dfrac{x}{L}$
- Dimensionless temperature, $\dfrac{\theta}{\theta_b} = \dfrac{T - T_\infty}{T_b - T_\infty} = \dfrac{\cosh mL(1-\xi)}{\cosh mL}$
- Dimensionless heat-transfer rate, $\dfrac{\dot{Q}}{mkA_c\theta_b} = \tanh mL$

The variation of dimensionless temperature with dimensionless axial position and mL as a parameter is shown in Fig. 4.14 for a straight, one-dimensional fin with adiabatic tip.

Also shown in Fig. 4.15 is the dimensionless tip temperature and dimensionless heat transfer varying with mL. The dimensionless temperature drops to about 0.014 when mL reaches a value of 5. We also need to understand that no noticeable improvement in heat flow will result by increasing mL beyond 3. Additional length (m excess of L = 3/m) would merely increase the cost without being much effective (Fig. 4.13).

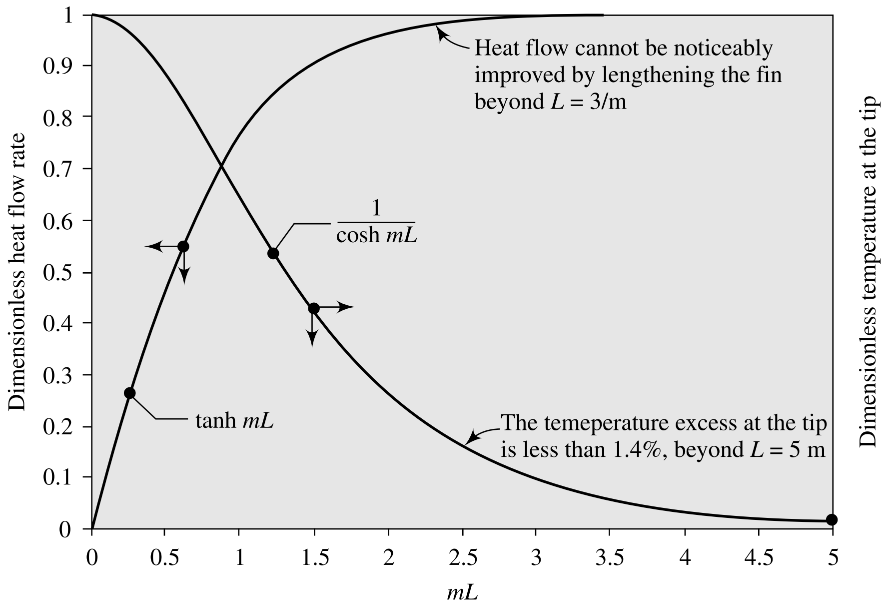


Fig. 4.14 *Heat-flow rate and the tip temperature in a straight one-dimensional fin with the insulated tip*

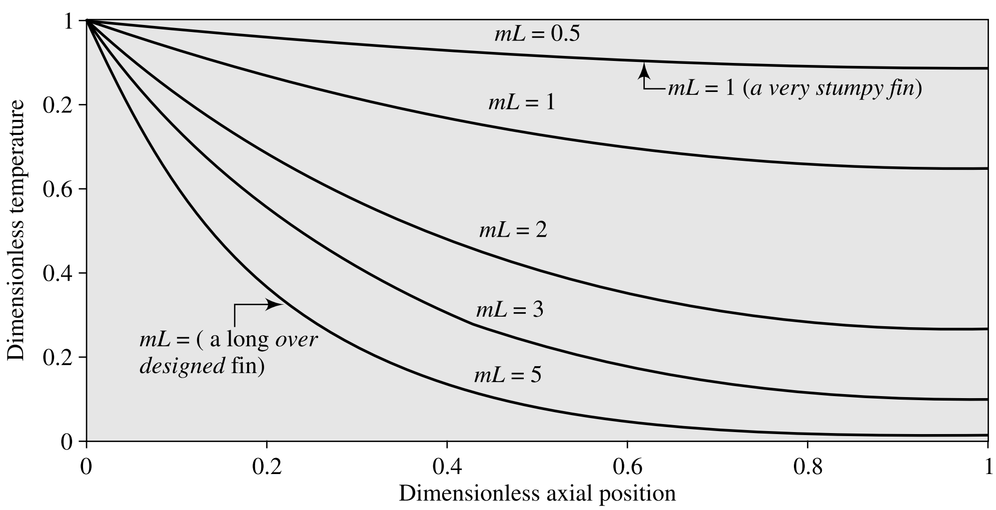


Fig. 4.15 *The temperature distribution in a straight one-dimensional fin with the insulated tip.*

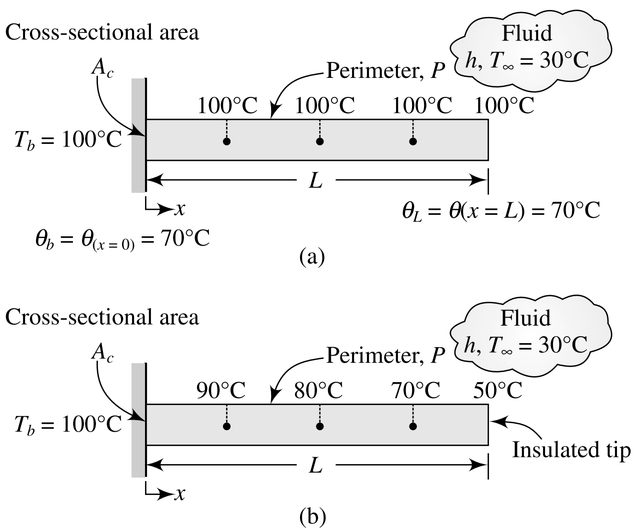
4.7 ❑ FIN-PERFORMANCE PARAMETERS

Fin Efficiency The area equipped with fins is called the *secondary area*, while the bare or unfinned area is known as the *primary area*. While the heat is being convectively transferred from the fin to the surroundings fluid, the temperature of the fin at its base is not constant throughout its length (or height). In fact, the temperature continuously decreases along the length of the fin because of conduction heat transfer. The tip of the fin is ultimately at a temperature lower than that at its root or base. The average surface temperature is thus *decreased*.

It is noteworthy that by artificially providing additional effective heat flow area in the form of fins, the effective *convective thermal resistance* $(1/hA_s)$ is significantly *reduced*, thereby *increasing* the rate of heat exchange between the surface and the fluid. However, to offset this advantage, there is an *increase* in the *conductive thermal resistance* due to the additional metal of the fin which tends to *lower* the heat transmission rate. The convective heat-transfer rate is proportional to the temperature difference $(T_s - T_\infty)$. As T_s decreases along the fin length from the root to the tip, the net rate of heat removal is adversely affected.

An *ideal* fin would be one having *infinite thermal conductivity* so that there is no temperature gradient. (Recall that heat flux is proportional to the product of thermal conductivity and temperature gradient.) In that event, the fin will be at a uniform temperature (the base temperature) right from the root to the tip, with no conductive resistance whatsoever. If the surface is properly proportioned, one can minimize the conductive resistance to extract the maximum benefit from the fins. It should also be borne in mind that if h is already large (*low convective resistance*) then fins may not be a sensible or attractive proposition.

Hence, the *fin efficiency* is defined as the amount of heat actually transferred by a prescribed fin to the ideal amount of heat that would be transferred if the entire fin were at its base temperature (Fig. 4.16).



$$\boxed{\eta_{\text{fin}} = \frac{\dot{Q}_{\text{actual fin}}}{\dot{Q}_{\text{ideal fin}}} = \frac{\tanh mL}{mL}}$$

Fig. 4.16 *(a) Ideal fin (of infinite thermal conductivity and zero heat conduction resistance)*
(b) Actual fin with insulated tip

$$\boxed{\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{max}}}} \tag{4.23}$$

where

$\dot{Q}_{\text{fin}}$ = actual amount of heat transferred from the fin, and

$\dot{Q}_{\text{max}}$ = maximum (*or ideal*) amount of heat that would be transferred form the fin, if the entire fin surface were at the temperature of the base (*primary*) surface.

It is good to remember that the fin efficiency will be *low for long fins, thin fins, or fins of low thermal conductivity* material. Also, the efficiency *decreases* as the heat-transfer coefficient *increases*. For *natural convection* such as in air-cooled heat-transfer equipment, fins can be made *fairly large* and of *low conductivity* materials, such as *steel* instead of *copper* or *aluminium*. On the other hand, for applications where large heat-transfer coefficients are involved, such as in condensation or boiling, fins are not advisable.

(a) For an Infinitely Long Fin For an infinitely long fin, actual heat transferred is given by

$$\boxed{\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,\theta_b = \sqrt{hPkA_c}\,(T_b - T_\infty)}$$

To calculate $\dot{Q}_{\max}$, if the entire fin surface were at a temperature of T_b, the convective heat transfer from the surface would be

$$\boxed{\dot{Q}_{\max} = h(A_{\text{fin}})(T_b - T_\infty) = h(PL)(T_b - T_\infty)} \tag{4.24}$$

where P is the perimeter of the fin and (PL) is the surface area A_s or A_{fin} of the fin.

Then $$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\max}} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)}{hPL(T_b - T_\infty)} = \frac{1}{\sqrt{\dfrac{hP}{kA_c}}\,L} = \frac{1}{mL} \quad \text{where} \quad m = \sqrt{\frac{hP}{kA_c}}$$

Thus, $$\boxed{\eta_{\text{fin}} = \frac{1}{mL}} \tag{4.25}$$

(b) For a Fin with Insulated End For the case of a fin with an *insulated* end, the actual heat transferred $\dot{Q}_{\text{fin}}$ is

$$\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,(T_b - T_\infty)\tanh mL$$

and, the fin efficiency is expressed as

$$\eta_{\text{fin}} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)\tanh mL}{hPL(T_b - T_\infty)} = \frac{\tanh mL}{\sqrt{\dfrac{hP}{kA_c}}\,L}$$

or $$\boxed{\eta_{\text{fin}} = \frac{\tanh mL}{mL}} \tag{4.26}$$

Figure 4.17 shows a plot of fin efficiency with mL for a straight fin with an adiabatic tip. With increasing L, the fin efficiency decreases continuously as shown, so that the most efficient fin is the fin of *zero* length.

We note that $\eta_{\text{fin}} = \dfrac{\tanh mL}{mL}$

At $L = 0$, $\eta_{\text{fin}} = \dfrac{\tanh 0}{0} = \dfrac{0}{0}$, i.e., indeterminate

η_{fin} can be evaluated at $L = 0$ by using L' Hospital's rule.

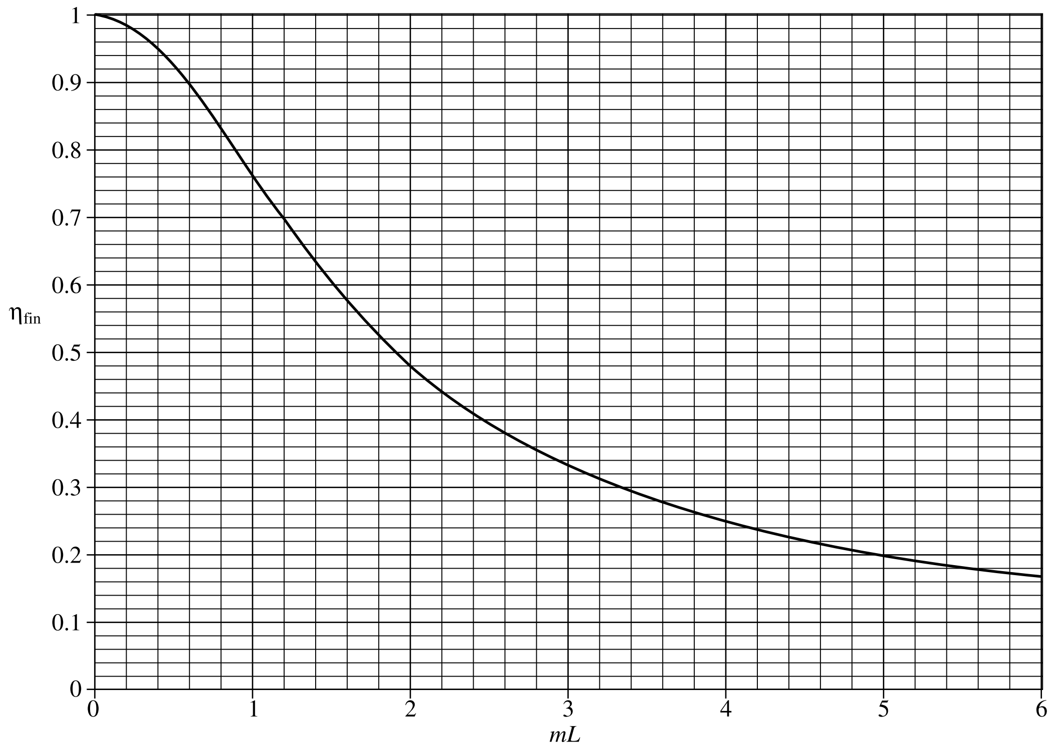


Fig. 4.17 *Fin efficiency versus mL for a straight fin of uniform thickness with negligible heat loss form the fin tip*

With $L \to 0$, $mL \to 0$

$$\therefore \qquad \eta_{\text{fin}} = \lim_{mL \to 0} \frac{\dfrac{d}{d(mL)}(\tanh mL)}{\dfrac{d}{d(mL)}(mL)} = \lim_{mL \to 0} \frac{\text{sech}^2(mL)}{1} = \lim_{mL \to 0} [1 + \tanh^2 mL]$$

$$= 1 + \tanh^2 0 = 1.$$

Thus, $\boxed{\eta_{\text{fin}} = 1}$ for a fin with insulated tip and of zero length

(c) For a Fin with an Active Tip If there is heat loss from the fin end, $\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,\theta_b \dfrac{\tanh mL + (h/mk)}{1 + (h/mk)\tanh mL}$

and the fin efficiency is given by $$\boxed{\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\max}} = \left(\frac{1}{mL}\right)\frac{\tanh mL + (h/mk)}{1 + (h/mk)\tanh mL}} \qquad (4.27)$$

Using the corrected length approximation,

$$\boxed{\dot{Q}_{\text{fin}} = mkA_c\theta_b \tanh mL_c}$$

and the corresponding fin efficiency as

$$\boxed{\eta_{\text{fin}} = \frac{\tanh mL_c}{mL_c}} \qquad (4.28)$$

We must recognize that fin efficiency is useful only for comparison of performance of different fins. There is, however, no simple single criterion for comparison. Generally, the comparison is based on the *volume of material involved*, since volume is related to cost.

• Fin Effectiveness (ε_{fin})

Consider a fin of uniform cross-sectional area A_c, projecting from a parent or base surface. The aim of the fin is to increase the heat transfer. In the absence of fin, heat would be transferred from the base area A_b, by convection. By attaching the fin, the area for convection increases, i.e., convective resistance (= $1/h_{PL}$ or $1/hPL$, decreases. However, conduction resistance due to the solid fin (= L/kA_c) is now introduced and the total heat transfer would depend upon the combined thermal resistance. As one goes on increasing the length of a fin the convection resistance will go on *decreasing* but conduction resistance will go on *increasing*. A fin may not, therefore, necessarily result in effectively increasing the heat transfer. Therefore, how effective the fin is in enhancing the heat transfer is characterized by a parameter called *fin effectiveness* (Fig. 4.18).

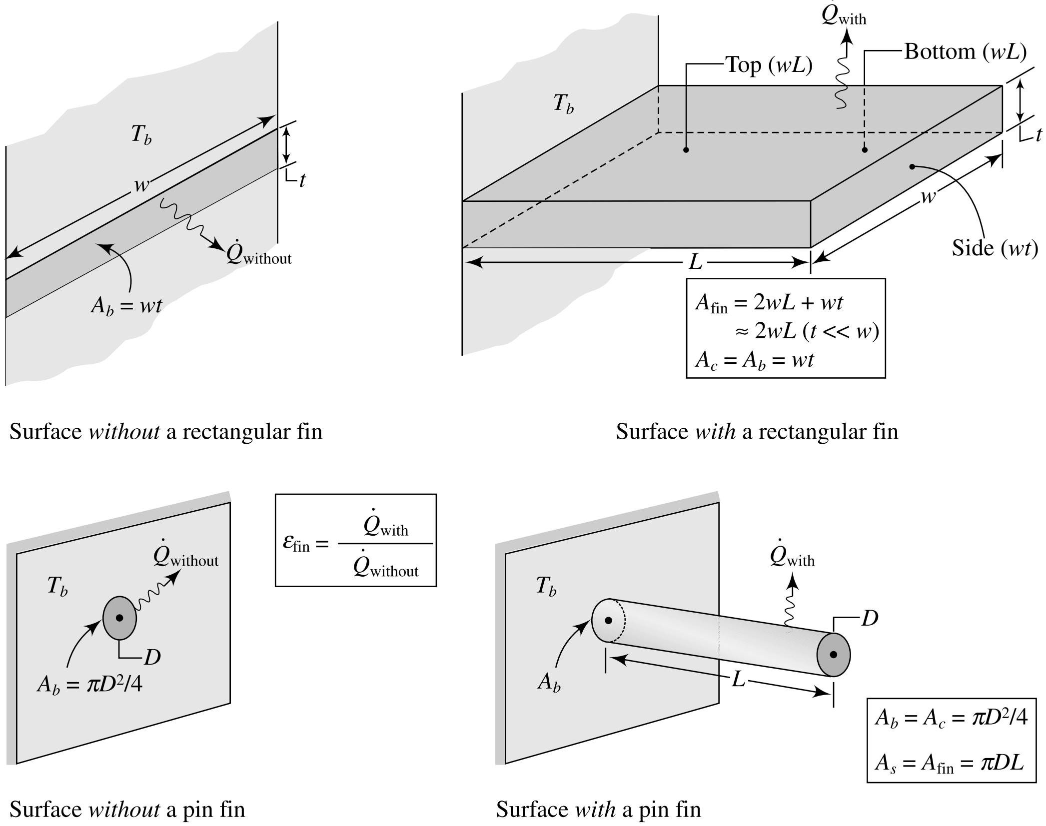


Fig. 4.18 *Fins increase heat transfer by increasing the surface area*

The *fin effectiveness,* ε_{fin}, is defined as the ratio of the heat transfer from a fin with *base area* A_b to the heat-transfer rate from the *same base area* that would exist *without* the fin:

$$\boxed{\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{without fin}}} = \frac{\eta_{\text{fin}} h A_{\text{fin}} (T_b - T_\infty)}{h A_b (T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\eta_{\text{fin}} \quad \text{or} \quad \frac{A_s}{A_c}\eta_{\text{fin}}} \tag{4.29}$$

where $A_{\text{fin}} = A_s = PL$ and $A_b = A_c$

If $\varepsilon_{\text{fin}} = 1$, the addition of the fin is *not* advantageous and, the added material is wasted. If $\varepsilon_{\text{fin}} < 1$, the fin *insulates* the surface. Hence, the value of ε_{fin} should be as *large* as possible, taking into account practical considerations, but certainly not less than about 2. A fin is often specified as the result of optimization, taking into account *cost*, *weight*, *manufacturability*, *pressure drop*, and so on.

While it is possible to have very high values of fin effectiveness, there are practical limitations. Consider a very long fin, whose efficiency is given by $\eta_{\text{fin}} = 1/mL$. If we substitute this into the expression for fin effectiveness and simplify, then

$$\varepsilon_{\text{fin}} = \left(\frac{1}{mL}\right)\left(\frac{PL}{A_c}\right) = \sqrt{\frac{kA_c}{hP}\frac{P^2}{A_c^2}} = \sqrt{\frac{kP}{hA_c}}$$

or

$$\boxed{\varepsilon_{\text{fin}} = \left(\frac{kP}{hA_c}\right)^{1/2}} \tag{4.30}$$

To obtain a large value of fin effectiveness, one would like to use the fin when the convective heat-transfer coefficient, h, is low. For example, consider the car radiator. Fins are used on the air side (*low* heat-transfer coefficient) while water (*high* heat-transfer coefficient) flows inside the unfinned tubes.

- The fin thermal conductivity, k, should be *high* (aluminum and copper are, therefore, often used, though steel may be used in some applications. Aluminium alloys are also the preferred choice due to their *lower cost* and *weight*.
- The ratio of fin perimeter to cross-sectional area (P/A_c) should be *large*. This suggests that slender or thin fins should be used. Very thin fins with very close spacing are used in the radiator. This combination is typical because it ensures a *large surface area* without obstructing the flow to such an extent that the heat transfer coefficient is reduced to an unacceptable level.

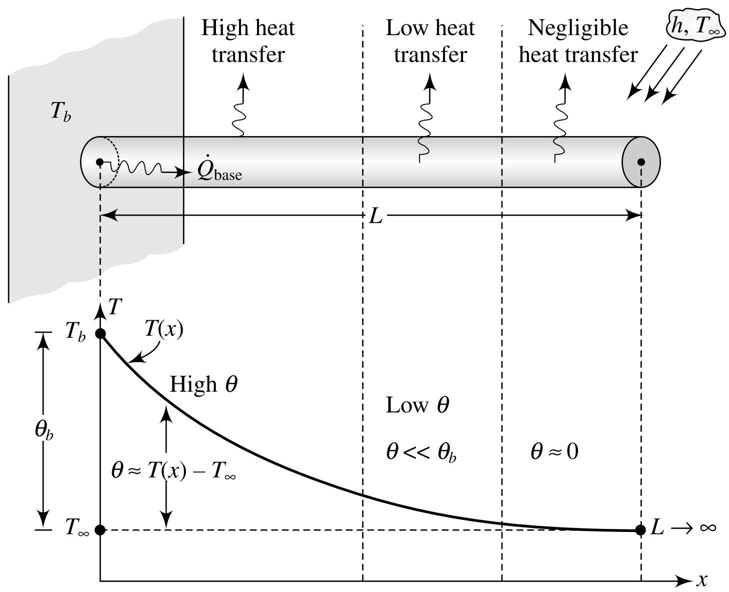


Fig. 4.19 *Due to the gradual temperature drop along the fin, the heat transfer progressively decreases till it is vanishingly small near the tip of a long fin.*

- We have already defined a dimensionless parameter, Biot number $(Bi) \equiv \frac{hL_c}{k}$ where L_c, the characteristic length equals (A_c/P). It follows that $\varepsilon_{\text{fin}} = \left(\frac{k}{hL_c}\right)^{1/2} = (Bi)^{-1}$. The effectiveness is, therefore, inversely proportional to the square root of the Biot number. *An effective fin is one with a small Biot number.*

One of the considerations that should weigh with us in the design of fins is the determination of the proper length of the fin once the fin material and the fin cross section are fixed. We note that the longer the fin, the larger the surface area and, thus, the higher the rate of heat transfer. Therefore, for maximum heat transfer, the fin should be infinitely long. However, the temperature drops along the fin exponentially and reaches the ambient temperature at some length (Fig. 4.19). The part of the fin beyond this length does not contribute to heat transfer since it is at the temperature of the surroundings. Designing such an *extra long* fin is, thus, ruled out since it results in *waste of material*, *excessive weight*, and *increased size* with *increased* cost. (*Such a long fin can, as a matter of fact, adversely affect the performance since it will suppress fluid motion and thus reduce the convection heat transfer coefficient*).

4.8 ❑ GENERALIZED EQUATION FOR FINS

In case the profile of a fin is *non-uniform*, and both the area and the perimeter at any section are functions of x then a generalized equation is useful in solving such problems. Consider an extended surface of unspecified configuration protruding from some primary surface as shown in Fig. 4.20. Let the area of cross section A_c and the perimeter P be $A_c = A_c(x)$ and $P = P(x)$. An appropriate general fin equation can be derived by making the following simplifying assumptions:

- Steady-state conditions prevail.
- One-dimensional heat conduction in the longitudinal direction, i.e., $T = T(x)$, since the fin is thin.
- The fin is homogeneous and radiation heat transfer, if any, is accounted for by radiation heat-transfer coefficient.
- The heat-transfer coefficient is constant and uniform along the length of the fin.
- The surrounding fluid temperature is constant.
- There is no internal heat generation in the extended surface.
- Constant thermal conductivity of the fin material.

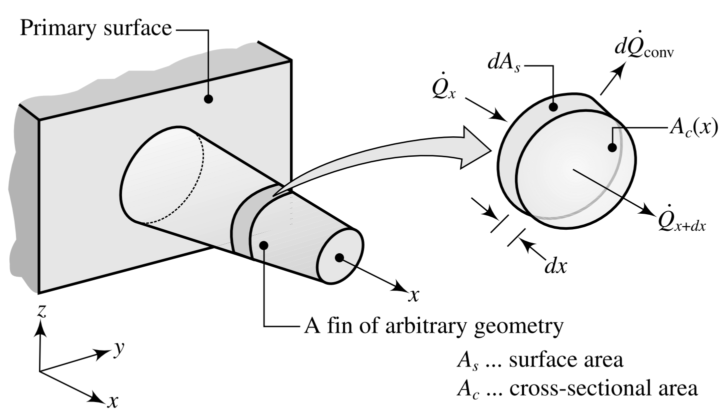


Fig. 4.20 *A generalized extended surface (fin): Energy balance*

Performing an energy balance on an infinitesimal element of thickness dx at a distance x, one has

$$\underbrace{\dot{Q}_x}_{\substack{\text{Heat conducted } into \\ \text{the element}}} = \underbrace{\dot{Q}_{x+dx}}_{\substack{\text{Heat conducted } out \\ of \text{ the element}}} + \underbrace{\dot{Q}_{\text{conv}}}_{\substack{\text{Heat } loss \text{ by convection} \\ \text{to the surrounding fluid}}}$$

where $$\dot{Q}_x = -kA_c \frac{dT(x)}{dx} \qquad \text{(from Fourier's law)}$$

Using a Taylor's series expansion, $f(x+dx) = f(x) + \dfrac{d}{dx}[f(x)]dx$

Hence, $$\dot{Q}_{x+dx} = \left\{-kA_c \frac{dT(x)}{dx} + \frac{d}{dx}\left[-kA_c \frac{dT(x)}{dx}\right]dx\right\}$$

$$\dot{Q}_{\text{conv}} = hdA_s(x)[T(x) - T_\infty]$$

where dA_s is the surface area of the element equal to Pdx.
It follows that

$$\dot{Q}_x - \dot{Q}_{x+dx} = \dot{Q}_{\text{conv}}$$

or $$-kA_c \frac{dT(x)}{dx} - \left\{-kA_c \frac{dT(x)}{dx} + \frac{d}{dx}\left[-kA_c \frac{dT(x)}{dx}\right]dx\right\} = hPdx(T(x) - T_\infty)$$

Rearranging, one gets

$$\frac{d}{dx}\left[kA_c \frac{dT(x)}{dx}\right] = hP[T(x) - T_\infty]$$

Since A_c and P are both functions of x for a fin of arbitrary cross section, the above equation on differentiation with respect to x becomes

$$kA_c(x)\frac{d^2T(x)}{dx^2} + \frac{dA_c}{dx}k\frac{dT(x)}{dx} = hP[(T(x) - T_\infty)]$$

Dividing throughout by kA_c and putting $(T(x) - T_\infty) \equiv \theta(x)$, the governing differential equation is

$$\boxed{\frac{d^2\theta(x)}{dx^2} + \underbrace{\frac{dA_c}{A_c\,dx}}_{\text{Varies with } x}\frac{d\theta(x)}{dx} - \underbrace{\frac{hP}{kA_c}}_{\text{Varies with } x}\theta(x) = 0} \tag{4.31}$$

This is the most general from of equation of *steady, one-dimensional* heat conduction in fins of any profile or cross section. From this governing equation, corresponding to a specified fin configuration and using the appropriate boundary conditions, the temperature distribution and hence the fin heat transfer rate can be evaluated.

It is worth noting that for fins of uniform cross section or thickness, $\dfrac{dA_c}{dx} = 0$ since A_c would be constant. In such a situation, the differential equation will be reduced to

$$\boxed{\frac{d^2\theta}{dx^2} - \left(\frac{hP}{kA_c}\right)\theta = 0} \tag{4.32}$$

4.9 ❑ CIRCUMFERENTIAL (*ANNULAR*) FINS OF RECTANGULAR CROSS SECTION

Analysis of fins of varying cross-sectional area becomes more involved than for the simple pin fin. In that case, the second term of Eq. (4.31) must be retained. The solution is now not in the form of simple *exponential* or *hyperbolic* functions. Of great engineering significance is an annular fin of constant thickness attached circumferentially to a circular cylinder. Such fins find their application in liquid-to-gas heat exchangers and the cylinders of air-cooled engines. Consider the *annular* (*radial flat*) fin shown in Fig. 4.21.

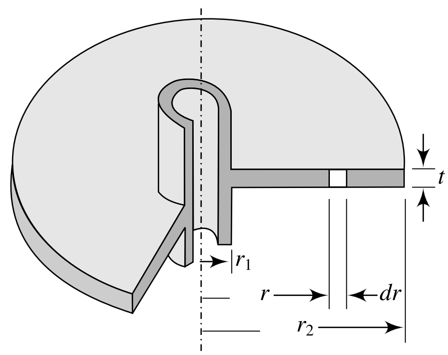


Fig. 4.21 *A circumferential (annular) fin of uniform thickness*

Although the fin thickness is uniform (*t is independent of r*), the cross-sectional area, $A_c = 2\pi rt$, increases linearly with r. Replacing x by r in *Eq. (4.31)* and expressing the surface area as $A_s = 2\pi(r^2 - r_1^2)$, the general form of the fin equation now becomes.

$$\boxed{\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kt}(T - T_\infty) = 0}$$

With $m^2 = (2h/kt)$ and $\theta \equiv T - T_\infty$,

$$\boxed{\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0}$$

This expression is a *modified Bessel equation* of *zero order*, and its general solution is of the form

$$\boxed{\theta(r) = C_1 I_0(mr) + C_2 K_0(mr)}$$

where I_0 and K_0 are *modified, zero-order Bessel functions* of the *first* and *second* kinds, respectively. The temperature at the base of the fin is specified, $\theta(r_1) = \theta_b$, and the fin tip is assumed adiabatic, i.e., $d\theta/dr|_{r_2} = 0$.

Temperature Distribution: The constants C_1 and C_2 may be determined by the boundary conditions to yield a temperature distribution of the form.

$$\boxed{\frac{\theta}{\theta_b} = \frac{I_0(mr)K_1(mr_2) + K_0(mr)I_1(mr_2)}{I_0(mr_1)K_1(mr_2) + K_0(mr_1)I_1(mr_2)}} \tag{4.33}$$

where $I_1(mr) = \frac{d}{d(mr)}[I_0(mr)]$ and $K_1(mr) = -\frac{d}{d(mr)}[K_0(mr)]$ are *modified, first order Bessel functions* of the *first* and *second* kinds, respectively. Properties of Bessel functions are tabulated in Table 4.2.

Table 4.2 *Modified Bessel functions of the first and second kinds*

x	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
0	1.0000	0	∞	∞
0.1	1.0025	0.0501	2.4271	9.8538
0.2	1.0100	0.1005	1.7527	4.7760
0.3	1.0226	0.1517	1.3725	3.0560
0.4	1.0404	0.2040	1.1145	2.1844
0.5	1.0635	0.2579	0.9244	1.6564
0.6	1.0920	0.3137	0.7775	1.3028
0.7	1.1263	0.3719	0.6605	1.0503
0.8	1.1665	0.4329	0.5653	0.8618
0.9	1.2130	0.4971	0.4867	0.7165
1.0	1.2661	0.5652	0.4210	0.6019
1.1	1.3262	0.6375	0.3656	0.5098
1.2	1.3937	0.7147	0.3185	0.4346
1.3	1.4693	0.7973	0.2782	0.3725
1.4	1.5534	0.8861	0.2437	0.3208
1.5	1.6467	0.9817	0.2138	0.2774
1.6	1.7500	1.0848	0.1880	0.2406
1.7	1.8640	1.1963	0.1655	0.2094
1.8	1.9896	1.3172	0.1459	0.1826
1.9	2.1277	1.4482	0.1288	0.1597
2.0	2.2796	1.5906	0.1139	0.1399
2.1	2.4463	1.7455	0.1008	0.1227
2.2	2.6291	1.9141	0.0893	0.1079
2.3	2.8296	2.0978	0.0791	0.0950
2.4	3.0493	2.2981	0.0702	0.0837
2.5	3.2898	2.5167	0.0623	0.0739
2.6	3.5533	2.7554	0.0554	0.0653
2.7	3.8417	3.0161	0.0493	0.0577
2.8	4.1573	3.3011	0.0438	0.0511
2.9	4.5027	3.6126	0.0390	0.0453
3.0	4.8808	3.9534	0.0347	0.0402
3.1	5.2945	4.3262	0.0310	0.0356
3.2	5.7472	4.7343	0.0276	0.0316
3.3	6.2426	5.1810	0.0246	0.0281
3.4	6.7848	5.6701	0.0220	0.0250

contd.

contd.

3.5	7.3782	6.2058	0.0196	0.0222
3.6	8.0277	6.7927	0.0175	0.0198
3.7	8.7386	7.4357	0.0156	0.0176
3.8	9.5169	8.1404	0.0140	0.0157
3.9	10.3690	8.9128	0.0125	0.0140
4.0	11.3019	9.7595	0.0112	0.0125
4.1	12.3236	10.6877	0.0100	0.0111
4.2	13.4425	11.7056	0.0089	0.0099
4.3	14.6680	12.8219	0.0080	0.0089
4.4	16.0104	14.0462	0.0071	0.0079
4.5	17.4812	15.3892	0.0064	0.0071
4.6	19.0926	16.8626	0.0057	0.0063
4.7	20.8585	18.4791	0.0051	0.0057
4.8	22.7937	20.2528	0.0046	0.0051
4.9	24.9148	22.1993	0.0041	0.0045
5.0	27.2399	24.3356	0.0037	0.0040
5.1	29.7889	26.6804	0.0033	0.0036
5.2	32.5836	29.2543	0.0030	0.0032
5.3	35.6481	32.0799	0.0027	0.0029
5.4	39.0088	35.1827	0.0024	0.0026
5.5	42.6946	38.5882	0.0021	0.0023
5.6	46.7376	42.3283	0.0019	0.0021
5.7	51.1725	46.4355	0.0017	0.0019
5.8	56.0381	50.9462	0.0015	0.0017
5.9	61.3766	55.9003	0.0014	0.0015
6.0	67.2344	61.3419	0.0012	0.0013
6.1	73.6628	67.3194	0.0011	0.0012
6.2	80.7179	73.8859	0.0010	0.0011
6.3	88.4616	81.1000	0.0009	0.0010
6.4	96.9616	89.0261	0.0008	0.0009
6.5	106.2929	97.7350	0.0007	0.0008

contd.

contd.

6.6	116.5373	107.3047	0.0007	0.0007
6.7	127.7853	117.8208	0.0006	0.0006
6.8	140.1362	129.3776	0.0005	0.0006
6.9	153.6990	142.0790	0.0005	0.0005
7.0	168.5939	156.0391	0.0004	0.0005
7.1	184.9529	171.3834	0.0004	0.0004
7.2	202.9213	188.2503	0.0003	0.0004
7.3	222.6588	206.7917	0.0003	0.0003
7.4	244.3410	227.1750	0.0003	0.0003
7.5	268.1613	249.5844	0.0002	0.0003
7.6	294.3322	274.2225	0.0002	0.0002
7.7	323.0875	301.3124	0.0002	0.0002
7.8	354.6845	331.0995	0.0002	0.0002
7.9	389.4063	363.8539	0.0002	0.0002
8.0	427.5641	399.8731	0.0001	0.0002
8.1	469.5006	439.4843	0.0001	0.0001
8.2	515.5927	483.0477	0.0001	0.0001
8.3	566.2551	530.9598	0.0001	0.0001
8.4	621.9441	583.6570	0.0001	0.0001
8.5	683.1619	641.6199	0.0001	0.0001
8.6	750.4612	705.3773	0.0001	0.0001
8.7	824.4499	775.5115	0.0001	0.0001
8.8	905.7973	852.6635	0.0001	0.0001
8.9	995.2399	937.5389	0.0001	0.0001
9.0	1093.6	1030.9	0.00005	0.00005
9.2	1320.7	1246.7	0.00004	0.00004
9.4	1595.3	1507.9	0.00003	0.00035
9.6	1927.5	1824.1	0.00003	0.00003
9.8	2329.4	2207.1	0.00002	0.00002
10.0	2815.2	2670.7	0.00002	0.00002

Heat Loss and Fin Efficiency The fin heat-transfer rate can be expressed as

$$\dot{Q}_{\text{fin}} = \dot{Q}_{\text{base}} = -kA_c \left.\frac{dT}{dr}\right|_{r=r_1} = -k(2\pi r_1 t)\left.\frac{d\theta}{dr}\right|_{r=r_1}$$

It follows that

$$\boxed{\dot{Q}_{\text{fin}} = 2\pi k r_1 t \theta_b m \frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2) + I_0(mr_1)K_1(mr_2)}} \tag{4.34}$$

The fin efficiency is given by

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\max}} \quad \text{where} \quad \dot{Q}_{\max} = h[2\pi(r_2^2 - r_1^2)](T_b - T_\infty)$$

$$\boxed{\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{h[2\pi(r_2^2 - r_1^2)]\theta_b} = \frac{2r_1}{m(r_2^2 - r_1^2)} \frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2) + I_0(mr_1)K_1(mr_2)}} \tag{4.35}$$

This result may be applied to an active (*convective*) tip, if the tip radius r_2 is replaced by a *corrected* radius of the form $\boxed{r_{2c} = r_2 + (t/2)}$.

The fin resistance can then be calculated as

$$\boxed{R_{\text{fin}} = \frac{1}{hA_{\text{fin}}\eta_{\text{fin}}}} \tag{4.36}$$

4.10 ❑ GRAPHICAL PROCEDURE TO FIND FIN EFFICIENCY

If the width of a straight *rectangular* fin is much larger than its thickness, $w >> t$, the perimeter may be approximated as $P = 2w$

$$\boxed{mL_c = \left(\frac{hP}{kA_c}\right)^{1/2} L_c = \left(\frac{2h}{kt}\right)^{1/2} L_c}$$

Multiplying the numerator and denominator by $L_c^{1/2}$, we have the correct fin profile area is defined as $A_p = L_c t$.

It follows that

$$\boxed{mL_c = \left(\frac{2h}{kA_p}\right)^{1/2} L_c^{3/2}} \tag{4.37}$$

Hence, the efficiency of a rectangular fin with tip convection may be represented as a function of $L_c^{3/2}(h/kA_p)^{1/2}$ which is $\sqrt{2} \cdot mL_c$.

In plotting various fin efficiencies, the profile area, A_p is generally used.

The stepwise procedure for calculating the heat flow rates from real fins is summarized below:

Step 1*: *Calculate the parameters required to obtain the fin efficiency

- For a straight *rectangular* fin:

  $L_c = L + (t/2) \qquad A_p = L_c t$

- For a straight *triangular* fin:

  $L_c = L \qquad A_p = L_c(t/2)$

- For a *circumferential* fin of *rectangular* cross section with an inner radius, r_1, and the outer radius, r_2:

  $L = r_2 - r_1 \qquad L_c = L + (t/2)$

  $r_{2c} = L_c + r_1 \qquad A_p = (r_{2c} - r_1)t = L_c t$

Step 2: Calculate the parameter ξ defined as

$$\xi = L_c^{3/2}\left(\frac{h}{kA_p}\right)^{1/2}$$

Step 3: Using the appropriate chart, determine η_{fin}.
Step 4: Calculate the maximum heat-transfer rate for the fin using its corrected fin length and assuming its entire surface to be at the base temperature.

(a) For a straight rectangular fin:

$$\dot{Q}_{\max} = hPL_c(T_b - T_\infty)$$

(b) For a straight triangular fin:

$$\dot{Q}_{\max} = hPL_c(T_b - T_\infty)$$

(c) For circumferential fins of rectangular cross section:

$$\dot{Q}_{\max} = h[2\pi(r_{2c}^2 - r_1^2)](T_b - T_\infty)$$

where P is the perimeter of the fin

Step 5*: *Find the actual heat-transfer rate by multiplying $\dot{Q}_{\max}$ by η_{fin}, i.e., $\boxed{\dot{Q}_{\text{fin}} = \eta_{\text{fin}} \cdot \dot{Q}_{\max}}$

Note that as $r_{2c}/r_1 \to 1.0$, the efficiency of circumferential fin approaches that of the straight fin of rectangular profile.

Figure 4.23 shows fin efficiency of straight fins of rectangular, triangular and parabolic profile plotted against mL and ξ respectively. (Figure 4.22 is the graphical representation of fin efficiency of circumferential fin.)

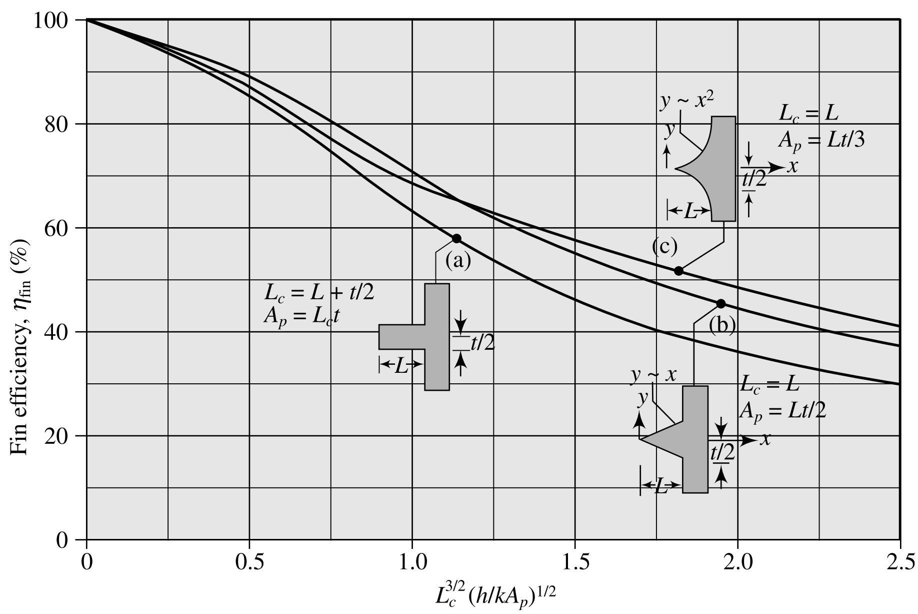


Fig. 4.22 *Efficiency of straight fins (rectangular, triangular and parabolic profiles)*

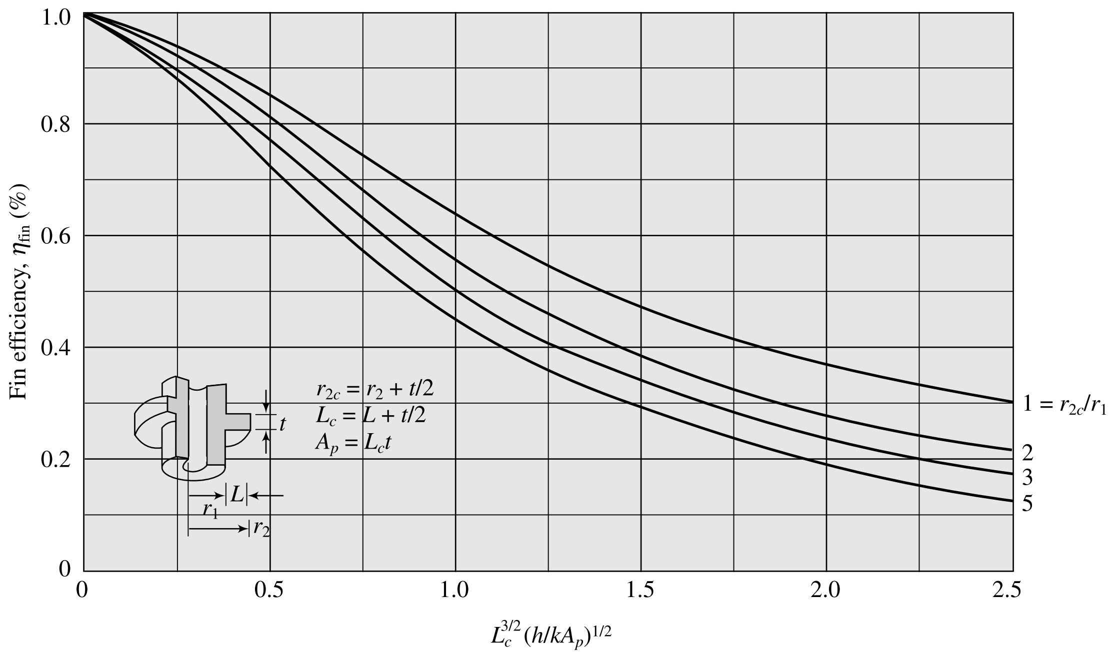
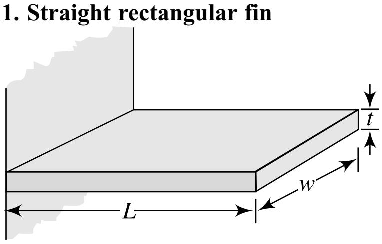


Fig. 4.23 *Efficiency of circumferential (annular) fins of rectangular profile*

Table 4.3 summarizes expressions for the efficiency and surface area of some common fin geometries. Although results for the fins of uniform thickness or diameter were obtained by assuming an *adiabatic tip*, the effects of convection may be treated by using a *corrected length* or *radius*. The *triangular* and *parabolic* fins are of *non-uniform* thickness, which reduces to zero at the fin tip.

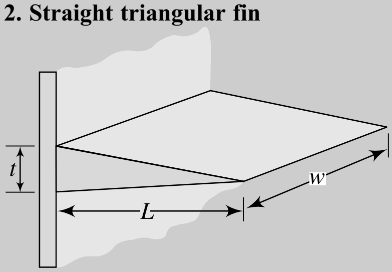
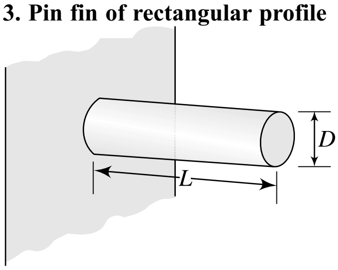
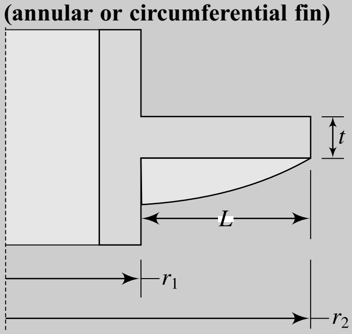
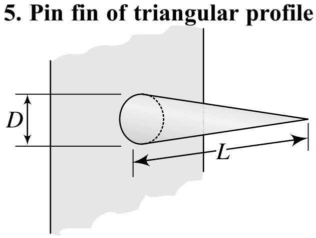
Table 4.3 *Fin efficiency for common fin configurations*

[(A_c = area of cross section, A_{fin} = total fin surface area, A_p = profile area, L_c = corrected length, P = perimeter of fin section, h = heat-transfer coefficient, fin parameter, $m = (hP/kA_c)^{1/2}$]

Description	Parameters	Fin efficiency
1. Straight rectangular fin 	$A_{fin} = 2wL_c$ $L_c = L + (t/2)$ $m = \sqrt{\dfrac{2h}{kt}}$ $A_p = L_c t$	$\eta_{fin} = \dfrac{\tanh mL_c}{mL_c}$

contd.

contd.

2. Straight triangular fin 	$A_{\text{fin}} = 2w[L^2 + (t/2)^2]^{1/2}$ $m = \sqrt{\dfrac{2h}{kt}}$ $A_p = L_t/2$ $L_c = L$	$\eta_{\text{fin}} = \dfrac{1}{mL}\dfrac{I_1(2mL)}{I_0(2mL)}$
3. Pin fin of rectangular profile 	$A_{\text{fin}} = \pi D L_c$ $L_c = L + (D/4)$ $m = \sqrt{\dfrac{4h}{kD}}$	$\eta_{\text{fin}} = \dfrac{\tanh mL_c}{mL_c}$
4. Circular fin of rectangular profile (annular or circumferential fin) 	$A_{\text{fin}} = 2\pi(r_{2c}^2 - r_1^2)$ $r_{2c} = r_2 + (t/2) = r_1 + L_c$ $m = \sqrt{\dfrac{2h}{kt}}$ $L_c = L + (t/2)$ $A_p = (r_{2c} - r_1)t = L_c\, t$	$\eta_{\text{fin}} = C_2\left[\dfrac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}\right]$ $C_2 = \dfrac{(2r_1/m)}{(r_{2c}^2 - r_1^2)}$
5. Pin fin of triangular profile 	$A_{\text{fin}} = \dfrac{\pi D}{2}[L^2 + (D/2)^2]^{1/2}$ $m = \sqrt{\dfrac{4h}{kD}}$	$\eta_{\text{fin}} = \dfrac{2}{mL}\dfrac{I_2(2mL)}{I_1(2mL)}$

4.11 ❑ DESIGN ASPECTS OF FINS

The main considerations in the optimum design of fins are

- Heat-dissipation rate
- Mass, particularly in the case of aircraft and automobiles

- Geometrical configuration
- Manufacturing difficulties
- Pressure drop (*resistance to fluid flow*)
- Manufacturing cost

We have so far discussed only the thermal performance of fins. For optimum design, the fins must be *light in mass*, offer *minimum resistance to fluid flow, should be easy to manufacture* and *should have least fabrication cost.*

The fin may be integral to the surface (*e.g., the cooling fins on an air-cooled two-wheeler are cast as part of the cylinder head*) or the fins may be manufactured separately. The fins can then be brazed or press-fitted onto the tube. Both *heat transfer* and *fluid flow* considerations are important in the selection of *fins*. *The choice of the number of fins, spacing, length, thickness, shape, and material are equally significant*. Manufacturing, maintenance, and operating costs are also key factors in fin design.

A straight *triangular* fin is quite appealing because, for equivalent heat transfer, it requires much less volume (*fin material*) compared to a *rectangular* profile. It is worth noting that heat dissipation per unit volume, $(\dot{Q}/\forall)_{\text{fin}}$, is the largest for a *parabolic profile*. But, since $(\dot{Q}/\forall)_{\text{fin}}$, for the *parabolic* profile is only *marginally* more than that for a *triangular profile*, its use can hardly be justified in view of its *larger manufacturing costs*. The circumferential (*annular*) *fin of rectangular profile is* commonly used to increase the heat transfer to or from the circular tubes.

When selecting fins for a given application, *the available space, weight, and cost* must all be considered. In addition, *the thermal properties of the fluid flowing over the fins* must be considered along with the *pump work* necessary to *pump the fluid* across the *fins* if they are used in a *forced convection* system.

If fins are machined as an integral part of the prime surface from which they extend, the two materials (wall and fin) are assumed to be in perfect contact and there is *no contact resistance* at their base. However, in practice, usually fins are manufactured separately and are attached to the wall by a *metallurgical or adhesive joint*. The other option is *a press fit*, for which the fins are forced into slots machined on the wall material. This involves *a thermal contact resistance* $R_{t,c}$, which may reduce the overall thermal performance.

Design of the dimensions of a fin of given *mass* or *profile area* to get the maximum heat transfer rate is described below.

Consider a straight rectangular fin of length L, uniform thickness t, and width w. Then the mass of a fin is

$$\boxed{m = \rho\, A_p\, w}$$

where ρ is the density of the fin material, and A_p is the profile area, i.e., the area taken in a plane that is parallel to the fin length, and normal to the width, w. For a fixed amount of material, $A_p = Lt$ is constant. Our objective is to determine the *condition for optimum length* and *thickness* of a fin for *maximum heat dissipation*.

Let the face width of the fin be w, the length L, and the thickness, t. In most practical applications, the width is large as compared to the thickness, so that the perimeter P can be expressed as

$$\boxed{P = 2(w + t) \approx 2w}$$

The cross-sectional area of the fin is given by

$$A_c = wt$$

and
$$m = \sqrt{\frac{hP}{kA_c}} \approx \sqrt{\frac{h \times 2w}{kwt}} \approx \sqrt{\frac{2h}{kt}}$$

The expression for the heat-flow rate in a fin with an adiabatic tip and of specified profile area is given by

$$\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,\theta_b \tanh mL = \sqrt{2whkwt}\,\theta_b \tanh(\sqrt{2h/kt}\,L)$$

$$= w\sqrt{2hkt}\cdot\theta_b \tanh\left(\sqrt{2h/kt}\cdot\frac{Lt}{t}\right)$$

or
$$\frac{\dot{Q}_{\text{fin}}}{w} = \theta_b\sqrt{2hkt}\tanh\left(A_p\sqrt{\frac{2h}{kt^3}}\right)$$

where $A_p = Lt$

Rate of heat transfer *per unit width,*

$$\dot{Q}_{\text{fin}}/w = \theta_b\sqrt{2hkt}\,\tanh(A_P\sqrt{2h/kt^3})$$

For maximum heat-dissipation rate, we differentiate the heat flow rate, $\dot{Q}_{\text{fin}}$ with respect to thickness t and equate the resulting derivative to zero. Then

$$\frac{d\dot{Q}_{\text{fin}}}{dt} = \theta_b\sqrt{2kh}\left[\frac{1}{2\sqrt{t}}\tanh\left(A_p\sqrt{\frac{2h}{k}}t^{-3/2}\right) + t^{1/2}\,\text{sech}^2[A_p\sqrt{2h/k}\,t^{-3/2}]\right]\frac{d}{dt}[A_p\sqrt{2h/k}\,t^{-3/2}] = 0$$

Therefore,

$$\frac{1}{2\sqrt{t}}\tanh\left[A_p\sqrt{\frac{2h}{k}}t^{-3/2}\right] + t^{1/2}\,\text{sech}^2\left[A_p\sqrt{\frac{2h}{k}}t^{-3/2}\right]\left[A_p\sqrt{\frac{2h}{k}}(-3/2)t^{-5/2}\right] = 0$$

or
$$\frac{1}{2\sqrt{t}}\tanh\left[A_p\sqrt{\frac{2h}{k}}t^{-3/2}\right] - \frac{3}{2}\frac{1}{\sqrt{t}}A_p\sqrt{\frac{2h}{k}}t^{-3/2}\,\text{sech}^2\left[A_p\sqrt{\frac{2h}{k}}t^{-3/2}\right] = 0$$

or
$$\tanh\left[A_p\sqrt{\frac{2h}{k}}t^{-3/2}\right] = 3A_p\sqrt{\frac{2h}{k}}t^{-3/2}\,\text{sech}^2\left[A_p\sqrt{\frac{2h}{k}}\,t^{-3/2}\right]$$

where
$$\lambda \equiv A_p\sqrt{\frac{2h}{k}}\,t^{-3/2} \quad\Rightarrow\quad \boxed{\tanh\lambda = 3\lambda/\cosh^2\lambda}$$

This transcendental equation can be solved graphically as shown in Fig. 4.24 and its solution is found to be

$$\lambda_{\text{opt}} = A\sqrt{\frac{2h}{k}}t^{-3/2} = \sqrt{\frac{2h}{kt}}\frac{A_p}{t} = 1.4192$$

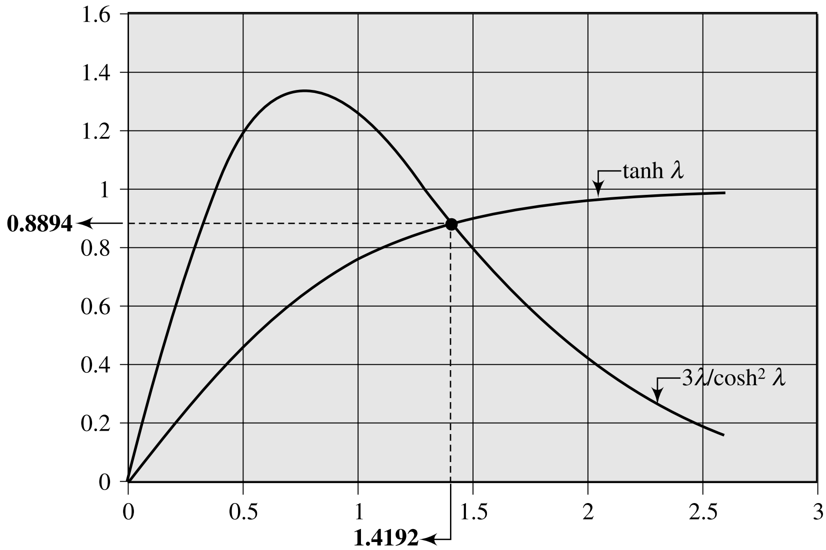
With
$$A_p = Lt, \qquad L\sqrt{\frac{2h}{kt}} = 1.4192$$

or
$$L\sqrt{\frac{2h(t/2)}{kt(t/2)}} = 1.4192 \quad\text{or}\quad \frac{2L}{t}\sqrt{\frac{h(t/2)}{k}} = 1.4192$$

Since

$$\frac{h(t/2)}{k} \equiv Bi$$

(*where Bi is the Biot number with its characteristic length equal to half thickness*).



$$\lambda = A_p \sqrt{\frac{2h}{k}}\, t^{-3/2}$$

Fig. 4.24 *Solution of transcendental equation*

$$\frac{2L}{t}\sqrt{Bi} = 1.4192$$

$$\boxed{\frac{L}{t} = \frac{1.4192}{2\sqrt{Bi}} = \frac{0.7096}{\sqrt{Bi}}} \tag{4.38}$$

where (L/t) is known as the *optimum* ratio of *fin height to thickness* of the fin

It follows that the maximum heat-dissipation rate per unit width is

$$\frac{\dot{Q}_{\text{fin}}}{w} = \theta_b \sqrt{2hkt}\tanh\lambda = \sqrt{2hkt}\cdot\theta_b \tanh 1.4192 = 0.8894\,\theta_b\sqrt{2hkt}$$

$$= 0.8894\,\theta_b\sqrt{2hkt}$$

The optimum fin thickness in terms of the fin heat-loss rate is given by

$$t = \frac{(\dot{Q}_{\text{fin}}/w\theta_b)^2}{(0.8894)^2\,2hk} = \frac{0.6321}{hk}\left(\frac{\dot{Q}_{\text{fin}}}{w\theta_b}\right)^2$$

Optimum length of fin can then be expressed as

$$L = 1.4192\sqrt{\frac{k}{2h}}\sqrt{t} = \frac{1.4192}{\sqrt{2}}\frac{\sqrt{k}}{\sqrt{h}}\frac{\sqrt{0.6321}}{\sqrt{h}\sqrt{k}}\left(\frac{\dot{Q}_{\text{fin}}}{w\theta_b}\right)$$

$$\therefore \qquad \boxed{L = \frac{0.7979}{h}\left(\frac{\dot{Q}_{\text{fin}}/w}{\theta_b}\right)} \tag{4.39}$$

4.12 ❑ CONDITIONS WHEN FINS DO NOT HELP

It is worth noting that the installation of fins on a heat-transfer surface will not necessarily increase the heat-transfer rate. Situations may exist when providing fins may in fact decrease the rate of heat transfer for specified values of h, k, A_c, m, and θ_b. Heat transfer rate from the base of the fin is expressed as

$$\dot{Q}_{\text{fin}} = mkA_c\theta_b\left[\frac{(h/mk) + \tanh mL}{1 + (h/mk)\tanh mL}\right]$$

Differentiating this equation with respect to L and equating the resulting derivative to zero, one gets

$$\frac{d}{dL}\left[\left(\frac{h}{mk} + \tanh mL\right)\Big/\left(1 + \frac{h}{mk}\tanh mL\right)\right] = 0$$

or $$\{1 + (h/mk)\tanh mL\}(0 + \text{sech}^2\, mL\, m) - \left\{(h/mk + \tanh mL)\left(0 + \frac{h}{mk}\, m\, \text{sech}^2\, mL\right)\right\} = 0$$

or $$\left(1 + \frac{h}{mk}\tanh mL\right)\frac{m}{\cosh^2 mL} - \left(\frac{h}{mk} + \tanh mL\right)\frac{h/k}{\cosh^2 mL} = 0$$

or $$\frac{m}{\cosh^2 mL} + \frac{h}{k}\frac{\tanh mL}{\cosh^2 mL} - \frac{h^2}{m^2 k^2}\frac{m}{\cosh^2 mL} - \frac{h}{k}\frac{\tanh mL}{\cosh^2 mL} = 0$$

or $$\frac{1}{\cosh^2 mL}\left(m - \frac{h^2}{mk^2}\right) = 0 \quad \text{or} \quad m - \frac{h^2}{k^2 m} = 0$$

i.e, $$m^2 = \left(\frac{h}{k}\right)^2 \quad \text{or} \quad \frac{hP}{kA_c} = \frac{h^2}{k^2} \quad \text{or} \quad hA_c = kP$$

Defining (A_c/P) as a characteristic linear dimension L_c, one can write

$$\frac{h(A_c/P)}{k} = 1 \quad \text{or} \quad \frac{hL_c}{k} = Bi$$ where Bi is the Biot number.

Biot number can also be defined as

$$Bi = \frac{hL_c}{k} = \frac{L_c/k}{1/h} = \frac{\text{Internal resistance (conductive) of the fin material}}{\text{External resistance (convective) of the fluid at the fin surface}}$$

For a rectangular fin of length L, width w, and thickness, t, the characteristic dimension,

$$L_c \equiv \frac{A_c}{P} = \frac{wt}{2(w+L)} \approx \frac{t}{2}$$, i.e., *half- thickness*.

Introducing Biot number, Bi, in the equation for heat transfer, one gets,

$$\dot{Q}_{\text{fin}} = \frac{h}{mk} \times mkA_c\theta_b\,[1 + \tanh mL/(h/mk)]/[1 + (h/mk)\tanh mL]$$

where $$\sqrt{Bi} = \sqrt{\frac{hL_c}{k}} = \sqrt{\frac{h}{k}\frac{A_c}{P}\frac{m}{m}} = \sqrt{\frac{hA_c}{kP}} \times \sqrt{\frac{hP}{kA_c}} \times \frac{1}{m} = \frac{h}{mk}$$

$$\boxed{\dot{Q}_{\text{fin}} = hA_c\theta_b \frac{\left(1 + \dfrac{\tanh mL}{\sqrt{Bi}}\right)}{(1 + \sqrt{Bi}\tanh mL)}} \tag{4.40}$$

From *Eq*. (*4.40*), *three cases* are classified for heat transfer from extended surfaces. These are:

- When ***Bi* = 1**, i.e., $\dfrac{h}{mk} = 1$

  In this case, the *internal conduction resistance* is equal to the *external convection resistance* and $\dot{Q}_{\text{fin}} = hA_c\theta_b$, which is the heat loss from the primary surface with no extended surfaces i.e. $\dot{Q}_{\text{no fin}}$. An extended surface in this case will *not* increase the heat transfer, no matter what the fin length L is

- When ***Bi* > 1**, i.e., $\dfrac{h}{mk} > 1$

  In this *case*, the *internal resistance is greater is* than the *external resistance* and $\dot{Q}_{\text{fin}} < hA_c\theta_b$. Adding fins will, therefore, *reduce* the heat transfer rate. This will happen when the value of h is *very high*, for example, in condensers and evaporators that involve change of phase.

- When ***Bi* < 1**, i.e., $\dfrac{h}{mk} < 1$

  In this case, the *external resistance* is *greater* than the *internal resistance*, and $\dot{Q}_{\text{fin}} > hA_c\theta_b$. This means that attaching fins will certainly *increase* the heat transfer rate. For fins to be effective, fins of high thermal conductivity, for instance, aluminium or copper should be provided.

  For *gases* compared to *liquids,* the value of h is less. Hence, fins are very effective with *gases*, less effective with *liquids* and are disadvantageous with *two-phase fluids*. All the three cases discussed above are shown in Fig 4.25.

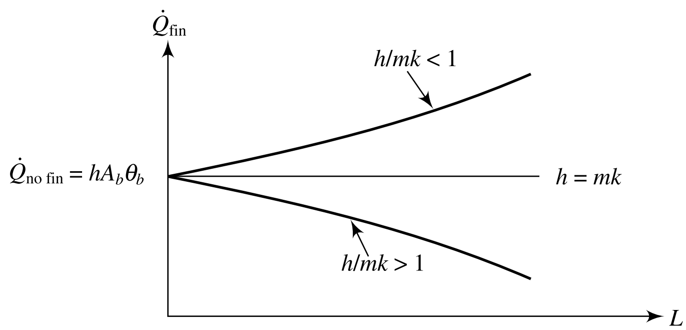


Fig. 4.25 *Heat transfer from a fin against length for different values of h/mk.*

4.13 ❑ THERMAL RESISTANCE OF FIN

The heat leaving the surface equipped with fins is transferred to the ambient fluid in *two* ways. *One* is the heat that goes from the *unfinned* or *bare* surface, *directly* to the fluid. The *second* is the heat transferred to the fins and then from the fins to the surrounding fluid. The two processes occur *in parallel*. If one considers the heat flow to be one-dimensional—an approximation which is usually acceptable for most of the situations-the thermal circuit is as shown in Fig. 4.26. The resistance of the bare (*unfinned*) surface is

$$\boxed{R_{\text{unfin}} = \frac{1}{hA_{\text{unfin}}}}$$

We must recognize that the thermal resistance is the ratio of the temperature difference across the resistance and the heat-transfer rate through it.

■ The expression for the resistance offered by the fins is

$$\boxed{R_{\text{fin}} = \frac{1}{N\sqrt{hPkA_c}\tanh mL} = \frac{1}{NmkA_c \tanh mL}} \tag{4.41}$$

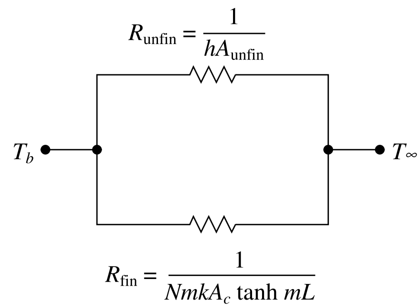


Fig. 4.26 *Thermal circuit for a number of fins attached to a surface*

where N is the total number of fins attached to the surface.

If the tip of the fin has a convective boundary condition and losing heat, the fin length should be increased by adding to it (A_c/P), thus $L_c = L + A_c/P$ total effective. The total effective fin resistance of a fin array can be calculated from

$$\boxed{R_{\text{tot}} = \frac{\theta_b}{\dot{Q}_{\text{tot}}} = \frac{1}{\eta_o hA_{\text{tot}}}} \tag{4.42}$$

Figure 4.26 illustrates the thermal circuits corresponding to the parallel paths and their representation in terms of an effective resistance.

4.14 ❑ HEAT TRANSFER FROM A FIN ARRAY

Very rarely a single fin is used. Generally, a parent (*primary or base*) surface is covered by an array of fins. Figure 4.27 shows fin arrays for *straight* fins and *circular* fins. In such a case, we can use either of the following concepts: (a) *overall surface efficiency, or* (b) *effectiveness of a fin array*.

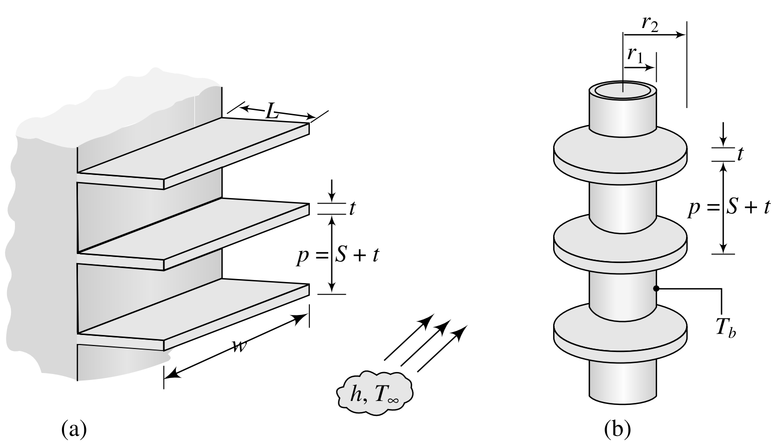


Fig. 4.27 *Fin arrays for (a) straight rectangular fins, and (b) annular fins*

(a) Overall Surface Efficiency (or Area-weighted Fin Efficiency) The fin efficiency discussed earlier is concerned with performance of a *single* fin. However, in many applications an array of fins attached to the primary surface is commonly employed. In such cases, it is useful to define an *overall surface efficiency* or *area-weighted fin efficiency* which gives a measure of the performance of the total exposed surface of the array comprising both the *finned* and *unfinned* surfaces. Let

A_{fin} = exposed surface area of all fins only

A_{tot} = total exposed surface area, including the finned and unfinned surface

η_{fin} = efficiency of each individual fin in the fin array

The overall surface efficiency η_o, is defined as *the ratio of the actual heat transferred by the array to that it would transfer if its entire surface were maintained at the base temperature (i.e., maximum possible heat transfer rate).*

A_{unfin} = exposed unfinned base (*prime*) area in the fin array, i.e., $A_{\text{unfin}} = A_{\text{tot}} - A_{\text{fin}}$.
The total heat-transfer rate from the fin array may then be expressed as

$$\begin{aligned}\dot{Q}_{\text{tot}} &= Q_{\text{base}} + \dot{Q}_{\text{fins}} = A_b h\theta_b + A_{\text{fin}} h\eta_{\text{fin}}\theta_b = (A_{\text{tot}} - A_{\text{fin}})h\theta_b + A_{\text{fin}} h\eta_{\text{fin}}\theta_b \\ &= A_{\text{tot}} h\theta_b + A_{\text{fin}}(\eta_{\text{fin}} - 1)h\theta_b = hA_{\text{tot}}\theta_b\left[1 - \frac{A_{\text{fin}}}{A_{\text{tot}}}(1 - \eta_{\text{fin}})\right]\end{aligned}$$

where θ_b is the temperature excess of the base temperature with respect to the ambient, and h is the same uniform heat-transfer coefficient for heat transfer by convection from all—both finned and exposed prime (unfinned) surfaces. The maximum possible heat-transfer rate is

$$\boxed{\dot{Q}_{\max} = hA_{\text{tot}}\theta_b} \tag{4.43}$$

It follows that

$$\eta_o = \frac{\dot{Q}_{\text{tot}}}{\dot{Q}_{\max}} = \frac{hA_{\text{tot}}\theta_b\left[1 - \dfrac{A_{\text{fin}}}{A_{\text{tot}}}(1 - \eta_{\text{fin}})\right]}{hA_{\text{tot}}\theta_b}$$

Hence, the overall surface efficiency,

$$\boxed{\eta_o = 1 - \frac{A_{\text{fin}}}{A_{\text{tot}}}(1 - \eta_{\text{fin}})} \tag{4.44}$$

with overall effective thermal resistance, $R_{t,o} = \theta_b/\dot{Q}_{\text{tot}} = \dfrac{1}{\eta_o \cdot hA_{\text{tot}}}$

Sometimes, the term *finning factor*, f_F is used to denote the ratio of the finned surface area to the area of the non-finned surface.

Clearly, for a plane wall (*flat surface*) without fins, the *finning factor* will be unity.

$$\boxed{f_F = \frac{(A_{\text{fin}} + A_{\text{unfin}})}{A_{\text{no fin}}}} \tag{4.45}$$

4.15 ❑ THERMOMETRIC ERROR

The theory of finned surfaces may be extended to estimate the error in the value of the temperature measured by a thermometer placed in a thermometer well. A thermometer well is a small tube welded radially in a large pipe or duct for the measurement of temperature of the fluid, (*liquid or gas*) flowing through it. The thermometer well is filled with some other liquid and a thermometer is inserted into it for temperature measurement.

The temperature recorded by the thermometer will be that of the bottom of the well. The surface of the thermometer well is uniformly exposed to fluid temperature and its root being fixed with the pipe wall will have the temperature equal to the pipe wall temperature. Since the wall temperature of the pipe is *less* than the fluid temperature being exposed to atmosphere, the heat will flow from the bottom of the well to the pipe wall through the well wall by conduction. Thus, the bottom of the well will have a temperature *lower* than that of the flowing fluid. The thermometer will indicate a temperature somewhere between the true temperature of the fluid and the pipe wall temperature, thus involving an error. This error may be corrected by using the principles of analysis of finned surface.

Figure 4.28 shows a thermometer pocket with various temperatures mentioned in it. As an approximation it is assumed that there is no heat flow between top of the well and the fluid flowing in the pipe.

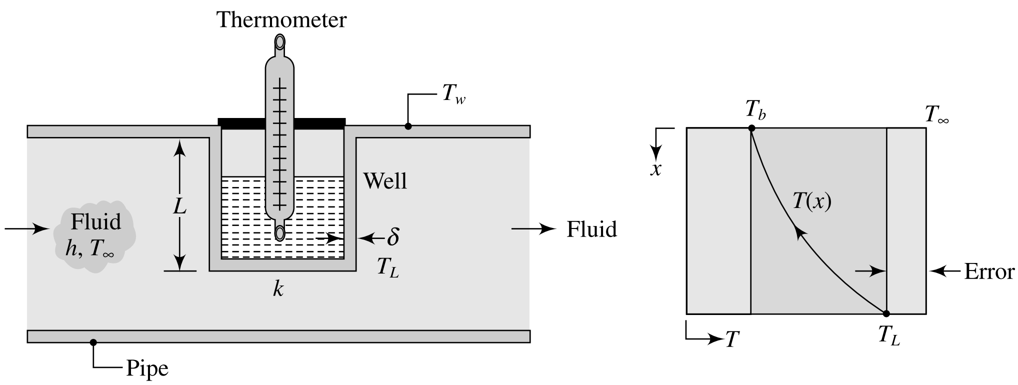


Fig. 4.28 *Error in temperature measurement*

The temperature distribution at a distance x, measured from the pipe wall, along the thermometer well is given by,

$$\frac{\theta}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

At $x = L$, we have,

$$\frac{T_L - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - L)}{\cosh mL}$$

$$\frac{T_L - T_\infty}{T_b - T_\infty} = \frac{1}{\cosh mL}$$

The error in temperature measurement is

(True fluid temperature, T_∞ – Measured fluid temperature, T_L).

Also the thermometric error is defined as

$$\boxed{\frac{T_\infty - T_L}{T_\infty - T_b} = \frac{1}{\cosh mL}} \tag{4.46}$$

where T_L = temperature recorded by the thermometer at the bottom of the well.
We note that perimeter of well, $P = \pi(d + 2\delta) \approx \pi d$
Cross-sectional area, $A_c = \pi d\delta$

$$\therefore \qquad \frac{P}{A_c} = \frac{\pi d}{\pi d\delta} = \frac{1}{\delta} \quad \text{and} \quad m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h}{k\delta}}$$

Note that the temperature measured by the thermometer is not affected by the diameter of the well.

To reduce the error in temperature measurement, a good thermometer well should be designed by

- Lagging (*insulating*) the pipe to increase the pipe wall temperature (T_w).
- Increasing the product (mL).

To achieve this

(a) L can be increased by making the thermometer well oblique (inclined) or letting it project beyond the pipe or tube axis.

(b) m ($\equiv h/k\ \delta$) can be increased by using thinner pipe (less thickness δ) or using a material of low thermal conductivity (k) for the pocket.

Incidentally, using the material of too low a thermal conductivity may increase the radial resistance to heat flow to such an extent that the temperature distribution is far from one-dimensional. Hence, for the sound design of a thermometer pocket or well, the measures like *thinner well wall* and *longer well* are to be preferred compared with relatively *low thermal conductivity*.

Illustrative Examples

(A) Very Long Fin

EXAMPLE 4.1 *Consider two very long cylinder rods of the same diameter but of different materials. One end of the each rod is attached to a base surface maintained at 100°C, while the surfaces of rods are exposed to ambient air at 20°C. By traversing the length of each rod with a thermocouple, it was observed that the temperatures of the rods were equal to the positions* x_A *= 0.15 m and* x_B *= 0.075 m, where* x *is measured from the base surface. If the thermal conductivity of rod A is known to be* k_A *= 72 W/m K, determine the value of* k_B *for the rod* B.

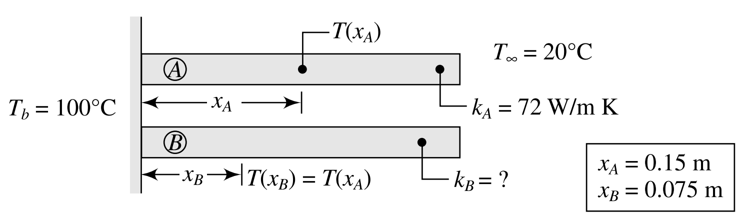
[AMIE W 2012]

Solution

Known Two long rods of different materials, but same diameters with same base and ambient temperatures.

Find Thermal conductivity of the rod B.

Schematic



Assumptions Infinite fins

Analysis Excess temperature, $\theta(x) = T(x) - T_\infty$

$$\frac{T_{(x_A)} - T_\infty}{T_b - T_\infty} = e^{-m_A x_A} \quad \text{and} \quad \frac{T_{(x_B)} - T_\infty}{T_b - T_\infty} = e^{-m_B x_B}$$

Since $\quad e^{-m_A x_A} = e^{-m_B x_B}$

or $\quad m_A x_A = m_B x_B$

or $$\frac{m_A}{m_B} = \frac{x_B}{x_A} = \frac{0.075}{0.15} = 0.5$$

or $$\sqrt{\frac{h(\pi D)}{k_A(\pi D^2/4)}} \times \sqrt{\frac{k_B(\pi D^2/4)}{h(\pi D)}} = 0.5$$

or $$\sqrt{\frac{k_B}{k_A}} = 0.5 \quad \Rightarrow \quad \frac{k_B}{k_A} = 0.25$$

$\therefore \quad k_B = (0.25)(72) = \mathbf{18\ W/mK}$ **(Ans.)**

(B) Fins of Finite Length

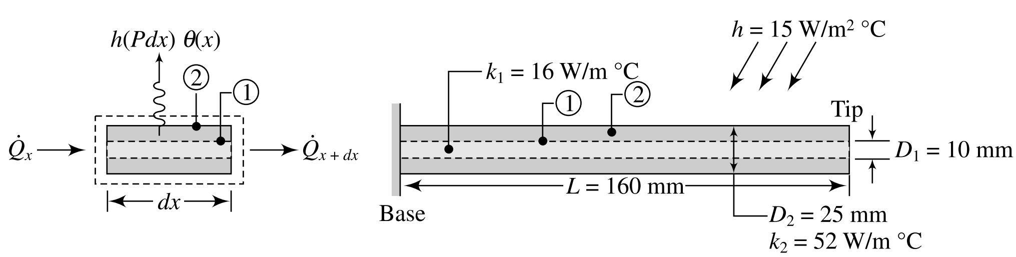
EXAMPLE 4.2 *A composite fin is made from two materials. The inner material of 10 mm diameter has a thermal conductivity of 16 W/m °C while the outer material of 25 mm outside diameter has a thermal conductivity of 52 W/m °C. The convection coefficient is 15 W/m²°C and the fin length is 160 mm. Determine the fin efficiency assuming adiabatic fin tip.*

Solution

Known A composite fin constructed from two different materials and of different diameters is exposed to convective environment.

Find Fin efficiency.

Schematic



Assumptions (1) Steady operating conditions exist. (2) Constant properties and uniform convection coefficient. (3) No internal heat generation. (4) Fin tip is insulated. (5) One-dimensional conduction.

Analysis The general differential equation for this case is obtained from energy balance:

$$\dot{E}_{in} - \dot{E}_{out} + \cancelto{0}{\dot{E}_{gen}} = \cancelto{0}{\dot{E}_{st}}$$

An energy balance on a thin element of thickness dx gives

$$\dot{Q}_x - \dot{Q}_{x+dx} - \dot{Q}_{\text{conv}} = 0 \quad \text{or} \quad -\frac{d}{dx}(\dot{Q}_x)dx - h(Pdx)(T(x) - T_\infty) = 0$$

or
$$-\frac{d}{dx}\left[\left(-k_1A_1\frac{dT}{dx}\right) + \left(-k_2A_2\frac{dT}{dx}\right)\right] - hP\theta(x) = 0$$

or
$$(+k_1A_1 + k_2A_2)\frac{d^2T}{dx^2} - hP\theta = 0$$

where
$$\theta(x) = \theta \equiv T - T_\infty \quad \text{and} \quad \frac{d^2\theta}{dx^2} \equiv \frac{d^2T}{dx^2}$$

$\therefore$
$$(k_1A_1 + k_2A_2)\frac{d^2\theta}{dx^2} - hP\theta = 0 \quad \text{or} \quad \frac{d^2\theta}{dx^2} - \left(\frac{hP}{k_1A_1 + k_2A_2}\right)\theta = 0$$

or
$$\ddot{\theta} - m^2\theta = 0$$

where
$$m = \sqrt{\frac{hP}{k_1A_1 + k_2A_2}} = \sqrt{\frac{h\pi D_2}{k_1\left(\frac{\pi}{4}D_1^2\right) + k_2\frac{\pi}{4}(D_2^2 - D_1^2)}} = \sqrt{\frac{4hD_2}{k_1D_1^2 + k_2(D_2^2 - D_1^2)}}$$

Substituting the appropriate numerical values,

$$m = \left[\frac{4 \times 15\ \text{W/m}^2\,°\text{C} \times 0.025\ \text{m}}{(16\ \text{W/m}\,°\text{C})(0.010\ \text{m})^2 + (52\ \text{W/m}\,°\text{C})(0.025^2 - 0.01^2)\text{m}^2}\right]^{1/2}$$

$= 7.2044\ \text{m}^{-1}$ and $mL = (7.2044\ \text{m}^{-1})\ (0.16\ \text{m}) = 1.1527$

$\tanh mL = 0.8184$

Fin efficiency of the composite fin is

$$\eta_f = \frac{\tanh mL}{mL} = \frac{0.8186}{1.1527} = \mathbf{0.71\ or\ 71\%} \qquad \textbf{(Ans.)}$$

EXAMPLE 4.3 *The handle of a ladle used for pouring molten lead at 328°C is 30 cm long. Originally the handle was made of 1.3 cm by 2.0 cm mild steel bar stock. To reduce the grip temperature, it is proposed to form a hollow handle of 1.5 mm thick mild steel tubing to the same rectangular shape. The average heat-transfer coefficient over the handle surface is 17 W/m² °C, when the ambient air temperature is 28°C. The thermal conductivity of mild steel is 43 W/m °C. Determine the reduction in the temperature of the grip, stating the assumptions made.*

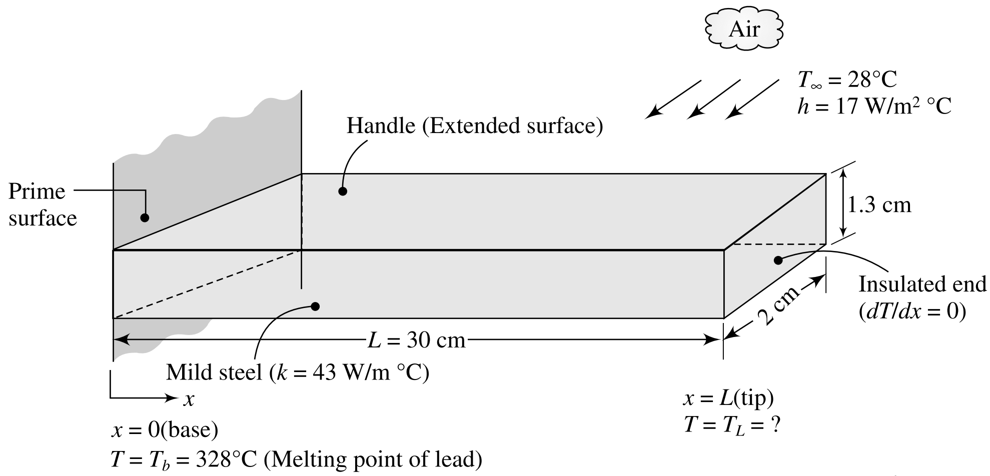
Solution

Known A handle for pouring molten lead is made of mild steel with rectangular cross section. It is to be made hollow to reduce the temperature at the grip.

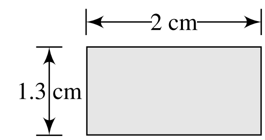
Find Reduction in the grip temperature.

Assumptions (1) One-dimensional steady-state conduction. (2) The tip is insulated. (3) Constant properties and uniform heat-transfer coefficient. (4) Heat transfer from the inner surface of the hollow shape is neglected.

Schematic



Analysis ***Rectangular cross section (solid shape):***



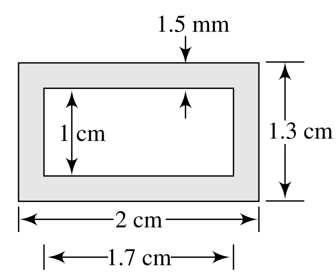
Cross-sectional area, A_c = 2 cm × 1.3 cm = 2.6 cm^2
Perimeter, P = 2 (2 + 1.3) cm = 6.6 cm

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{(17\text{ W/m}^2\,°\text{C})(6.6\text{ cm})}{(43\text{ W/m}\,°\text{C})(2.6\text{ cm}^2)}\left|\frac{100\text{ cm}}{1\text{ m}}\right|} = 10.02\text{ m}^{-1}$$

and $\quad mL = (10.02\text{ m}^{-1})(0.30\text{ m}) = 3.0$

$\quad \cosh mL = 10.12$

Hollow shape:



$P^* = P$ = 6.6 cm (*same as before*)

$$A_c^* = [(2 \times 1.3) - (1.7 \times 1)]\text{cm}^2 = 0.90\text{ cm}^2$$

$$\therefore \quad m^* = \sqrt{\frac{hP^*}{kA_c^*}} = \sqrt{\frac{17 \times 6.6 \times 100}{43 \times 0.90}} = 17.03\text{ m}^{-1}$$

and $\quad m^*L = (17.03)\ (0.3) = 5.11$

$\quad \cosh m^*L = 82.68$

Temperature distribution along the length of a fin with insulated tip is given by

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

At the grip of the handle,

$T(x) = T_L$ and $\cosh m\ (L - x) = \cosh 0 = 1$

$$\text{Hence,} \quad T_L - T_\infty = (T_b - T_\infty) \cdot \frac{1}{\cosh mL} \tag{A}$$

Similarly, for the proposed hollow handle:

$$T_L^* - T_\infty = (T_b - T_\infty) \cdot \frac{1}{\cosh m^* L} \tag{B}$$

Subtracting (B) from (A), we have

$$T_L - T_L^* = (T_b - T_\infty)\left[\frac{1}{\cosh mL} - \frac{1}{\cosh m^* L}\right] = (328 - 28)°\text{C}\left[\frac{1}{10.12} - \frac{1}{82.68}\right]$$

$$= \mathbf{26.0°C} \qquad \textbf{(Ans.)}$$

EXAMPLE 4.4 *In a chemical process, the heat transfer from a surface to distilled water is increased by a number of thin fins, each 2 mm thick and 50 mm long. The metal fins are coated with a 0.1 mm thick layer of plastic to prevent ionization of the water and the ends of the fins are fitted against an insulated wall. The temperature at the base of the fins is 80°C, the mean water temperature is 20°C and the heat-transfer coefficient between the water and the plastic coating is 200 W/m² °C. Determine (a) the temperature at the tip of the fins, (b) the actual heat transferred to the complete fin per unit width, (c) the fin efficiency, and (d) the fin effectiveness. Take thermal conductivity for aluminium as 237 W/m °C and that for plastic as 0.50 W/m °C. Sketch the temperature profile.*

Solution

Known Dimensions and base temperature of plastic-coated aluminium fins with their base attached to a plane wall and insulated tip. Surrounding water conditions.

Find (a) Temperature at the tip of the fins, $T_{(x = L)}$°C (b) Rate of heat transfer from a single fin, $\dot{Q}(W)$ (c) Fin efficiency, η_f(%) (d) Fin effectiveness, ε_f.

Assumptions (1) Steady operating conditions exist. (2) One-dimensional conduction without heat generation. (3) Adiabatic fin tip. (4) Uniform convection coefficient. (5) Constant properties.

Analysis In this problem there is the layer of plastic to be taken into account and the total thermal resistance to heat transfer is not only $1/h$ but also the thermal resistance through the plastic, given by t_p/k (*where t_p is the coat thickness*).

The overall heat-transfer coefficient, U can then be obtained by the following expression.

$$\frac{1}{U} = \frac{1}{h} + \frac{t_p}{k}$$

$$\therefore \qquad U = \left[\left(\frac{1}{200\ \text{W/m}^2\ °\text{C}}\right) + \left(\frac{0.1 \times 10^{-3}\ \text{m}}{0.5\ \text{W/m}\ °\text{C}}\right)\right]^{-1} = \mathbf{192.3\ W/m^2\ °C}$$

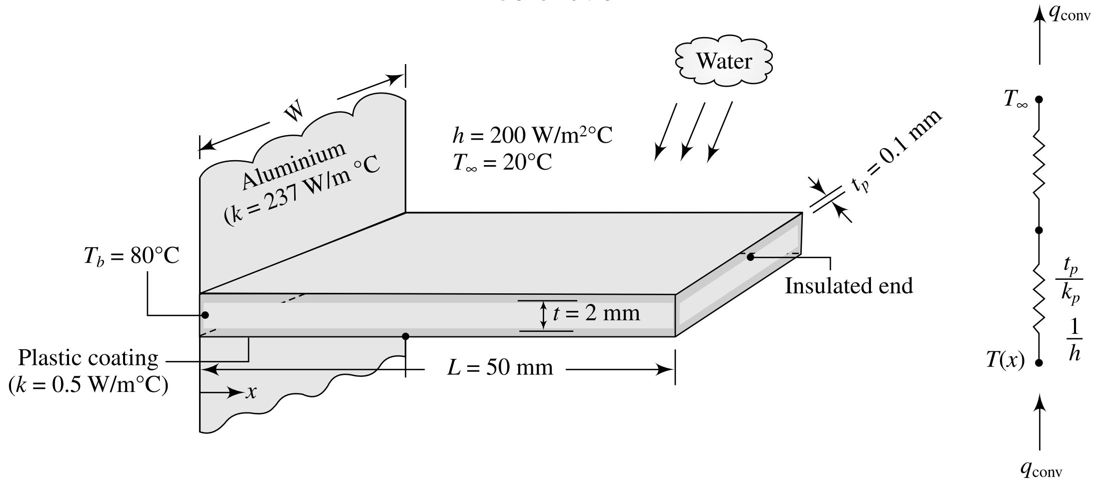
Temperature distribution along the length of a fin, with negligible heat transfer from the adiabatic tip of the fin, is given by

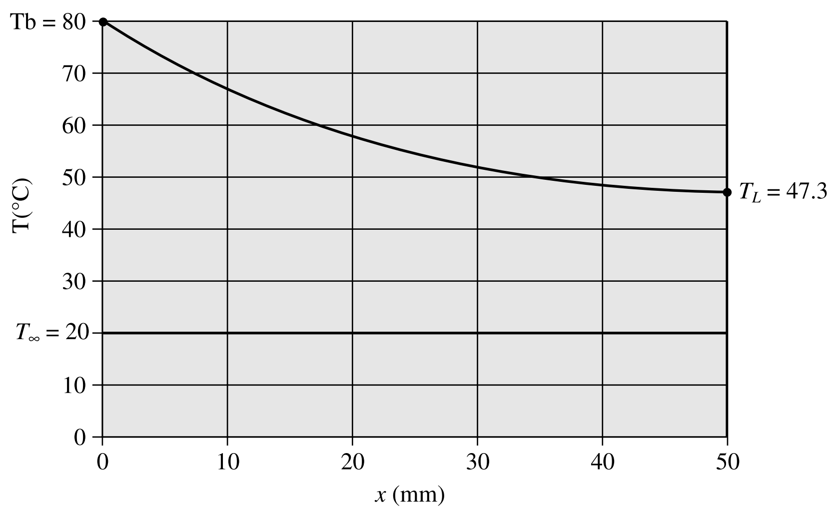
$$\frac{\theta}{\theta_b} \equiv \frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

The temperature at the tip of the plastic coated fins is obtained at $x = L$

$$\therefore \qquad T_{(x=L)} = T_\infty + \frac{(T_b - T_\infty)}{\cosh mL} \cdot 1$$

Schematic





where $$m = \sqrt{\frac{U2(W+t)}{kWt}} \cong \sqrt{\frac{2U}{kt}}$$

(*It is worth noting that h is replaced by U in the expression for m because of the plastic-coated film's additional resistance to heat transfer*).

Now, $$m = \sqrt{\frac{2(192.3\ \text{W/m}^2\ {}^\circ\text{C})}{(237\ \text{W/m}^\circ\text{C})(2\times10^{-3}\ \text{m})}} = 28.485\ \text{m}^{-1}$$

$\therefore$ $$mL = (28.485\ \text{m}^{-1})(0.05\ \text{m}) = 1.424$$

$$T_{(x=L)} = 20^\circ\text{C} + \frac{(80-20)^\circ\text{C}}{\cosh 1.424} = \mathbf{47.3^\circ C}$$ **(Ans.) (a)**

Heat-transfer rate from fin,

$$\dot{Q}_f = mkA_c\theta_b \tanh mL$$

For a rectangular fin, $A_c = Wt$ where W is width and t is thickness.
For unit width,

$$A_c = 1 \times 2 \times 10^{-3} \text{ m}^2$$

$$\therefore \quad \dot{Q}_{\text{fin}} = (28.485 \text{ m}^{-1})(237 \text{ W/m °C})(1 \times 2 \times 10^{-3} \text{ m}^2)\ (80 - 20)\text{°C} \tanh 1.424$$

$$= \mathbf{721.4\ W} \qquad \textbf{(Ans.) (b)}$$

Fin efficiency is expressed as

$$\eta_f = \frac{\tanh mL}{mL}$$

$$\therefore \quad \eta_f = \frac{\tanh 1.424}{1.424} = \mathbf{0.625} \text{ or } \mathbf{62.5\%} \qquad \textbf{(Ans.) (c)}$$

Heat-transfer rate without fin,

$$\dot{Q}_{\text{no fin}} = hA_c\theta_b = h(Wt)\theta_b = (192.3 \text{ W/m}^2\text{°C})\ (1 \text{ m} \times 0.002 \text{ m})\ (60\text{°C}) = \mathbf{23\ W}$$

$\therefore$ Fin effectiveness,

$$\varepsilon_f = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{721.4 \text{ W}}{23 \text{ W}} = \mathbf{31.3} \qquad \textbf{(Ans.) (d)}$$

Addition of fins in this case results in **31.3** times more heat dissipation.
The temperature profile is sketched in the schematic.

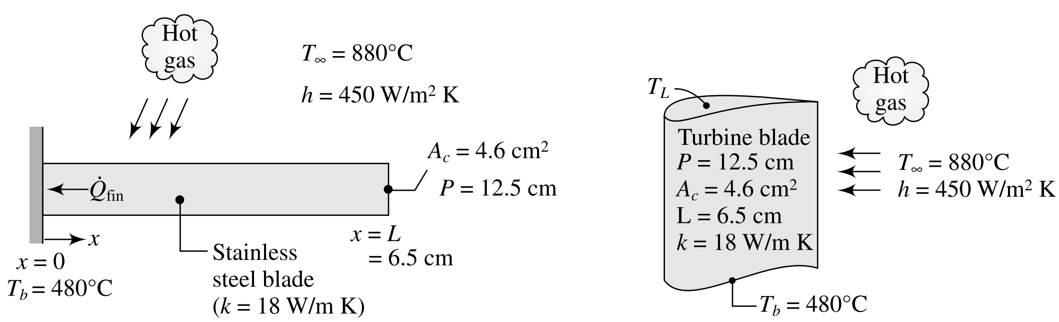
EXAMPLE 4.5 *A 6.5 cm long turbine blade, with a cross-sectional area of 4.6 cm^2 and a perimeter of 12.5 cm, is made of stainless steel (k = 18 W/m K). The temperature at the root is 480°C. The blade is exposed to a hot gas from the combustion chamber at 880°C and the convection heat-transfer coefficient is 450 W/m^2 K. Determine (a) the temperature distribution, (b) the rate of heat flow at the root of the blade, and (c) the temperature at the tip. Assume that the blade tip is insulated.*

Solution

Known A stainless-steel turbine blade is exposed to a hot gas.

Find (a) Temperature distribution, $T(x)$ (b) Heat-flow rate, $\dot{Q}_0(W)$ (c) Temperature at the tip, T_L(°C)

Schematic



Assumptions (1) Steady operating conditions. (2) One-dimensional conduction. (3) Constant thermal conductivity (4) Uniform surface heat-transfer coefficient (5) Adiabatic fin tip.

Analysis Fin parameter,

$$m = \sqrt{\frac{hP}{kA}} = \left[\frac{(450\ \text{W/m}^2\ \text{K})(125 \times 10^{-2}\ \text{m})}{(18\ \text{W/m}^2\ \text{K})(4.6 \times 10^{-4}\ \text{m}^2)}\right]^{1/2} = 82.42\ \text{m}^{-1}$$

The temperature distribution in the case of a fin with its tip insulated is given by

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

$$\therefore \qquad T - T_\infty = (480 - 880)\left\{\frac{\cosh 82.42(0.065 - x)}{\cosh(82.42)(0.065)}\right\} = \frac{-400 \cosh 82.42(0.065 - x)}{106}$$

or $\boxed{\boldsymbol{T - T_\infty = -\,3.77 \cosh 82.42\,(0.065 - x)}}$ **(Ans.) (a)**

The rate of heat flow at the root of the blade,

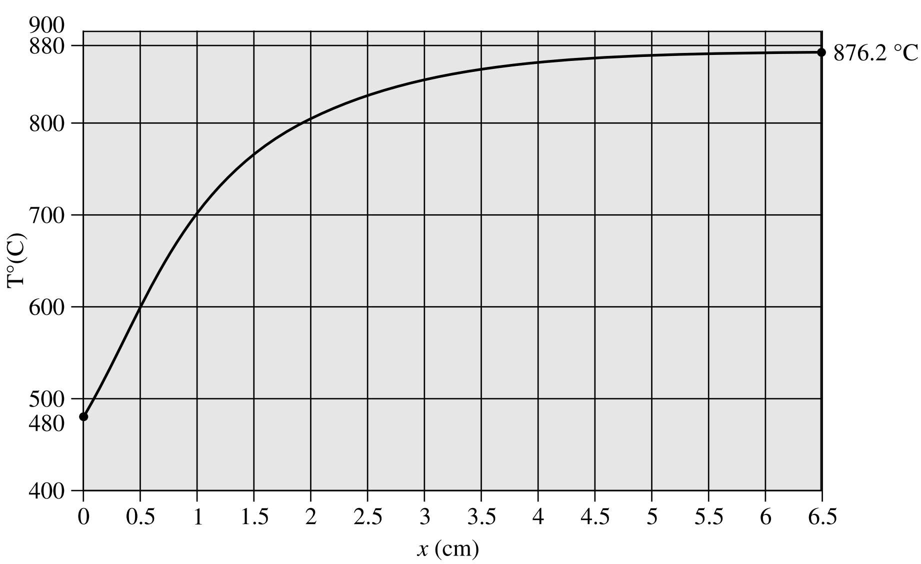
$$\dot{Q}_0 = \dot{Q}_{\text{fin}} = \sqrt{hPkA}\ \theta_b \tanh mL$$

$$= \sqrt{(450\ \text{W/m}^2\ \text{K})(0.125\ \text{m})(18\ \text{W/mK})(4.6 \times 10^{-4}\ \text{m}^2)}\ (-400°\text{C}) \tanh(82.42 \times 0.065)$$

$= \mathbf{-\ 273\ W}$ **(Ans.) (b)**

[*Negative sign implies that heat is transferred from the gas to the turbine blade.*]

The temperature profile is sketched below.



Temperature at the tip of the blade, $T_L = T_\infty + (T_b - T_\infty)/\cosh mL$

$\boxed{\mathbf{= 880 + (480 - 880)/\cosh\,(82.42 \times 0.065)}}$ **= 876.2°C** **(Ans.) (c)**

EXAMPLE 4.6 *The cylinder of an engine is 1 m long and has an outside diameter of 6 cm. The outside surface temperature of the cylinder is 200°C when the ambient temperature is 30°C. The film coefficient of heat transfer is 25 W/m^2 K. The cylinder is provided with 12 longitudinal straight fins of 0.1 cm thickness and 3 cm length. The thermal conductivity of cylinder and fin material is 75 W/m K. Assuming that the fins have insulated tip, determine (a) the percentage increase in heat dissipation due to addition of fins, (b) the temperature at the centre of the fin, (c) the fin efficiency and fin effectiveness, and (d) the overall fin effectiveness.*

Solution

Known Longitudinal fins are attached to a long cylinder to enhance heat dissipation.

Find (a) Percentage increase in heat transfer due to fins (b) Fin temperature at the centre (c) Efficiency and effectiveness of fin (d) Overall fin effectiveness.

Assumptions (1) Steady operating conditions exist. (2) Constant thermal conductivity (*same for cylinder and fins*). (3) Fin tip is insulated. (4) Uniform heat-transfer coefficient.

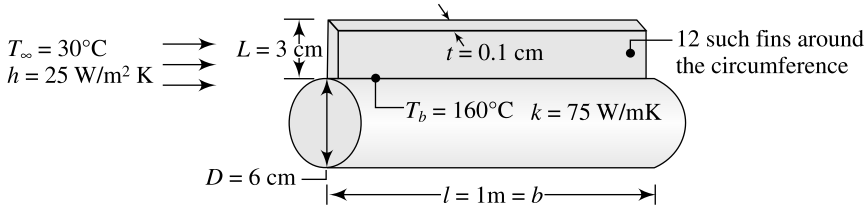
Analysis Fin heat-transfer rate,

$$\dot{Q}_{\text{fin}} = NM \tanh mL$$

where N = number of fins = 12

$$M = mk\,A_c(T_b - T_\infty)$$

Schematic



Fin parameter,

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h2(b+t)}{kbt}} = \left\{\frac{25\text{ W/m}^2\text{ K} \times 2(1\text{ m} + 0.1\times 10^{-2}\text{ m})}{75\text{ W/m K} \times 1\text{ m} \times 0.1 \times 10^{-2}\text{ m}}\right\}^{1/2} = 25.83\text{ m}^{-1}$$

$$mL = (25.83\text{ m}^{-1})\,(0.03\text{ m}) = 0.775$$

$$\tanh mL = 0.6498$$

$M = mk\,A_c(T_b - T_\infty) = (25.83\text{ m}^{-1})(75\text{ W/m K})\,(1\text{ m} \times 0.001\text{ m})\,(200 - 30)°\text{C}$ or K

$= 329.37$ W

$$\therefore \quad \dot{Q}_{\text{fin}} = 12 \times 329.37\text{ W} \times 0.6498 = \mathbf{2568\ W}$$

Heat dissipation from the unfinned surface,

$$\dot{Q}_{\text{unfin}} = hA_{\text{unfin}}(T_b - T_\infty)$$

where $A_{\text{unfin}} = A_{\text{cylinder}} - NA_{c,\text{fin}} = \pi DL - Nbt = (\pi D - Nt)b$ since $b = l$

$$= \{(\pi \times 0.06\text{ m}) - (12 \times 0.001\text{ m})\}\,(1\text{ m}) = 0.1765\text{ m}^2$$

Hence, $\dot{Q}_{\text{unfin}} = (25\ \text{W/m}^2\ \text{K})(0.1765\ \text{m}^2)$ (200 – 30)°C or K = **750 W**

Total heat dissipation from the finned cylinder is

$$\dot{Q}_{\text{total, fin}} = \dot{Q}_{\text{fin}} + \dot{Q}_{\text{unfin}} = (2568 + 750)\ \text{W} = \mathbf{3318\ W}$$

Heat loss from the cylinder *without* fins is

$$\dot{Q}_{\text{no fin}} = hA_{\text{cyl}}(T_b - T_\infty) = h(\pi Dl)(T_b - T_\infty)$$

$$= (25\ \text{W/m}^2\ \text{K})\ (\pi \times 0.06\ \text{m} \times 1\ \text{m})(200 - 30)°\text{C or } K = \mathbf{801.0\ W}$$

Percentage increase in heat transfer due to addition of fins

$$= \left(\frac{\dot{Q}_{\text{total, fin}} - \dot{Q}_{\text{no fin}}}{\dot{Q}_{\text{no fin}}}\right)(100) = \left(\frac{3318 - 801}{801}\right)(100)$$

$= \mathbf{314\%}$ **(Ans.) (a)**

Temperature distribution along the height of the fin is given by

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

At the centre of the fin, $x = L/2$

$$\therefore \qquad T_{(x=L/2)} = T_\infty + \frac{(T_b - T_\infty)}{\cosh mL}\cosh(mL/2)$$

$$= 30°\text{C} + (200 - 30)°\text{C} \times \frac{\cosh(0.775/2)}{\cosh 0.775} = \mathbf{169°C}$$ **(Ans.) (b)**

Fin efficiency,

$$\eta_f = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{mkA_c\theta_b \tanh mL}{h(PL)\theta_b} = \frac{mkA_c \tanh mL}{m^2kA_cL} = \frac{\tanh mL}{mL}$$

$$= \frac{\tanh 0.775}{0.775} = \mathbf{0.8385} \text{ or } \mathbf{83.85\%}$$ **(Ans.) (c)**

Fin effectiveness, $\varepsilon_f = \dfrac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}}$

where $\dot{Q}_{\text{no fin}} = h\ A_b(T_b - T_\infty)$

$$= (25\ \text{W/m}^2\text{K})(1\ \text{m} \times 0.001\ \text{m})(12) \times (200 - 30)°\text{C or } K = 51.0\ \text{W}$$

$$\therefore \qquad \varepsilon_f = \frac{2568\ \text{W}}{51\ \text{W}} = \mathbf{50.35}$$ **(Ans.) (c)**

Overall fin effectiveness of the finned surface is given by

$$\varepsilon_{\text{total, fin}} = \frac{h(A_{\text{unfin}} + \eta_f A_{\text{fin}})(T_b - T_\infty)}{hA_{\text{no fin, total}}(T_b - T_\infty)} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}}$$

$$= \frac{[A_{\text{unfin}} + \eta_f A_{\text{fin}}]}{A_{\text{no fin, total}}} = \frac{[\pi Dl - Nbt] + \eta_f[2Lb + bt]N}{\pi Dl}$$

$$= \frac{\pi Dl - \{1 - \eta_f\}Nbt + \eta_f\, 2LbN}{\pi Dl}$$

$$= \{(\pi \times 0.06\text{ m} \times 1\text{ m}) - (1 - 0.8385)(12 \times 1\text{ m} \times 0.001\text{ m})\} + \frac{\{(0.8385) \times 2 \times 0.03\text{ m} \times 1\text{ m} \times 12\}}{\pi \times 0.06\text{ m} \times 1\text{ m}}$$

$= 0.79/0.1885 =$ **4.19** **(Ans.) (d)**

Note that the breadth (width) of the sin b is the length of the cylinder, l.

EXAMPLE 4.7 *An array of eight aluminium alloy fins, each 3 mm wide, 0.4 mm thick, and 40 mm long, is used to cool a transistor. When the base is at 342 K and the ambient air is at 300 K, calculate (a) the fin efficiency, and (b) the power the fins would dissipate if the combined convection and radiation heat-transfer coefficient is estimated to be 8 W/m² K. The alloy has a thermal conductivity of 177 W/m K.*

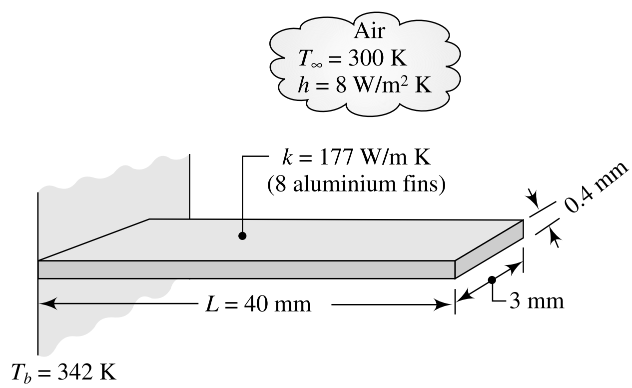
Solution

Known Aluminium fins to cool a transistor

Find (a) Fin efficiency (b) Power dissipated by 8 fins

Assumptions (1) Heat-transfer coefficient is constant along the fin. (2) Heat loss from the fin tip is negligible.

Schematic



Analysis For one fin,

$$A_c = (0.003)(0.0004) = 1.2 \times 10^{-6}\text{ m}^2$$

$$P = 2(0.003 + 0.0004) = 6.8 \times 10^{-3}\text{ m}$$

$$m = \sqrt{\frac{hP}{kA_c}} = \left[\frac{(8.0\text{ W/m}^2\text{ K})(6.8 \times 10^{-3}\text{ m})}{(177\text{ W/mK})(1.2 \times 10^{-6}\text{ m})}\right]^{1/2}$$

$$= 16.0\text{ m}^{-1}$$

$$mL = (16.0\text{ m}^{-1})(0.040\text{ m}) = 0.64$$

Fin efficiency,

$$\eta_{\text{fin}} = \frac{\tanh mL}{mL} = \frac{\tanh(0.64)}{0.64} = \mathbf{0.883} \quad \textbf{or} \quad \mathbf{88.3\%} \qquad \textbf{(Ans.) (a)}$$

Power dissipated,

$$\wp = \dot{Q}_{\text{fin, total}} = NmkA_c\,\theta_b \tanh mL$$

$$= 8 \times 16 \times 177 \times 1.2 \times 10^{-6} \times (342 - 300) \times \tanh 0.64 = \mathbf{0.645\ W} \qquad \textbf{(Ans.) (b)}$$

EXAMPLE 4.8 *A rectangular fin of 30 cm length, 30 cm width, and 2 mm thickness is attached to a surface at 300°C. The fin is made of aluminium (*k *= 204 W/m K) and is exposed to air at 30°C. The fin end is uninsulated and can lose heat through its end also. The convection heat-transfer coefficient between the fin surface and air is 15 W/m² K.*

Determine (a) the temperature of the fin at 30 cm from the base, (b) the rate of heat transfer from the fin, and (c) the fin efficiency.

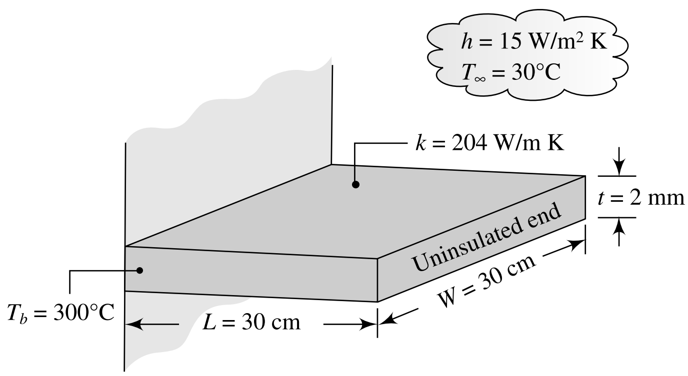
[IES 2012]

Solution

Known A straight rectangular aluminium fin with heat loss at the end.

Find (a) Fin tip temperature, $T(x = L)$ (b) Fin heat-transfer rate, $\dot{Q}_{\text{fin}}$ (c) Fin efficiency, η_{fin}.

Schematic



Assumption Heat-transfer coefficient is constant along the fin.

Analysis Fin parameter,

$$m = \sqrt{\frac{h2[W+t]}{kWt}} = \sqrt{\frac{15 \times 2 \times (0.3 + 0.002)}{204 \times 0.3 \times 0.002}} = 8.6\ \text{m}^{-1}$$

$$mL = 8.6 \times 0.3 = 2.581$$

$$\cosh mL = 6.6433$$

$$\sinh mL = 6.5676$$

$$\frac{h}{mk} = \frac{15}{8.6 \times 204} = 0.00855$$

(a) At $x = L = 0.30$ m,

$$\frac{\theta(x=L)}{\theta_b} = \frac{1}{\cosh mL + (h/mk)\sinh mL}$$

Temperature of the fin at 30 cm from the base is

$$\therefore \qquad T_{(x=L)} = T_\infty + \frac{T_b - T_\infty}{\cosh mL + (h/mk)\sinh mL}$$

$$= 30 + \frac{(300-30)}{6.6433 + (0.00855)(6.5676)} = \mathbf{70.3°C} \qquad \textbf{(Ans.) (a)}$$

(b) Rate of heat transfer from the fin is

$$\dot{Q}_{\text{fin}} = mkA_c\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

$$= 8.6 \times 204 \times 0.3 \times 0.002 \times (300-30) \times \frac{6.5676 + (0.00855 \times 6.6433)}{6.6433 + (0.00855 \times 6.5676)}$$

$$= \mathbf{281.0\ W} \qquad \textbf{(Ans.) (b)}$$

(c) Fin efficiency,

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin,max}}} = \frac{\dot{Q}_{\text{fin}}}{h(PL)\theta_b}$$

$$= \frac{281}{15 \times [2(0.3+0.002)](0.3) \times 270} = \frac{281}{733.86} \times 100 = \mathbf{38.3\%} \qquad \textbf{(Ans.) (c)}$$

(C) Number of Fins on Plane Wall

EXAMPLE 4.9 *A metal tank containing cooling coil is sought to be equipped with straight rectangular fins (*k *= 275 W/m °C) attached to its wall that is maintained at 105°C. The fins proposed to be used are 4 mm thick and are spaced 10 cm between centres. Heat is dissipated from the tank's surface to the ambient air at 25°C with a convection heat transfer coefficient on both finned and bare surfaces as 30 W/m^2 °C. For the finned configuration the prime surface temperature is expected to fall to 100°C. Determine the length (height) of the fins required for 50 % increase in the heat-transfer rate as a result of addition of fins.*

Solution

Known Dimensions and base temperature of straight rectangular fins. Ambient air conditions.

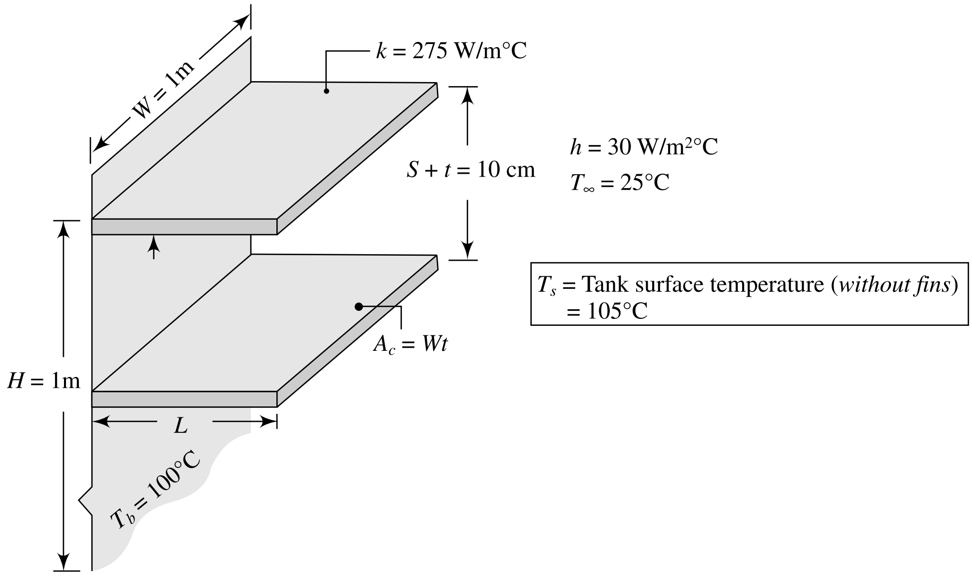
Find Length of fins required, L (cm).

Assumptions (1) Steady-state conditions. (2) Constant thermal conductivity and uniform convection coefficient. (3) Tank surface temperature is reduced by 5°C after fins are provided.

Analysis Let the tank have dimensions W (*width*) = 1 m and H (*height*) = 1 m. The area without fins, $A_{\text{no fin}} = WH = 1\ \text{m}^2$

As the spacing (*centre to centre*) between fins, $S + t = 10$ cm, the number of fins per metre length (*height*) of the tank wall, $N = \dfrac{H}{S+t} = \dfrac{1\ \text{m}}{0.1\ \text{m}} = 10$.

Schematic



Before fins are attached:

Heat dissipation rate, *without fins* is

$$\dot{Q}_{wo} = hA_{\text{no fin}}(T_s - T_\infty) = (30 \text{ W/m}^2\text{°C})\ (1 \text{ m}^2)\ (105 - 25)\text{°C} = \mathbf{2400\ W}$$

After fins are added:

Heat-transfer rate *with* fins is

$$\dot{Q}_w = 1.5\dot{Q}_{wo} = (1.5)\ (2400) \text{ W} = \mathbf{3600\ W}$$

The total heat loss per m^2 surface area comprise two components: Heat transfer from the unfinned surface, $\dot{Q}_{\text{unfin}}$ and that from the finned surface, $\dot{Q}_{\text{fin}}$

Area of unfinned surface,

$$A_{\text{unfin}} = WH - NW\,t = W\,[H - Nt] = 1 \text{ m } [1 \text{ m} - (10)\ (0.004 \text{ m})] = 0.96 \text{ m}^2$$

$$\therefore \qquad \dot{Q}_{\text{unfin}} = hA_{\text{unfin}}(T_b - T_\infty) = (30 \text{ W/m}^2\ \text{°C})\ (0.96 \text{ m}^2)\ (100 - 25)\text{°C} = \mathbf{2160\ W}$$

Therefore, heat-dissipation rate from the finned surface,

$$\dot{Q}_{\text{fin}} = \dot{Q}_w - \dot{Q}_{\text{unfin}} = (3600 - 2160) \text{ W} = 1440 \text{ W}$$

$$\dot{Q}_{\text{fin}} = NmkA_c\,\theta_b \tanh mL_c$$

where L_c = corrected length = $L + (t/2)$

$$NA_c = NWt = 10 \times 1 \text{ m} \times 0.004 \text{ m} = 0.04 \text{ m}^2$$

$$k = 275 \text{ W/m °C}$$

Also, $$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h2(W+t)}{kWt}} = \sqrt{\frac{2 \times 30 \text{ W/m}^2\ \text{°C} \times (1 + 0.004)\text{m}}{275 \text{ W/m°C} \times 1 \text{ m} \times 0.004 \text{ m}}} = 7.4 \text{ m}^{-1}$$

and, $$\theta_b = T_b - T_\infty = 100 - 25 = 75\text{°C}$$

$$\therefore \qquad \tanh mL_c = \frac{\dot{Q}_{\text{fin}}}{NmkA_c(T_b - T_\infty)} = \frac{1440 \text{ W}}{(0.04 \text{ m}^2)(7.4 \text{ m}^{-1})(275 \text{ W/m°C})(75\text{°C})}$$

$$= 0.236$$

$$\therefore \qquad mL_c = \tanh^{-1}(0.236) = 0.2405$$

Corrected length of fin to account for heat loss from the fin tip,

$$L_c = L + (t/2) = \frac{0.2405}{7.4 \text{ m}^{-1}} = 0.0325 \text{ m}$$

Fin length required, $L = \{0.0325 - (0.004/2)\}$ m = 0.0305 m = **3.05 cm** **(Ans.)**

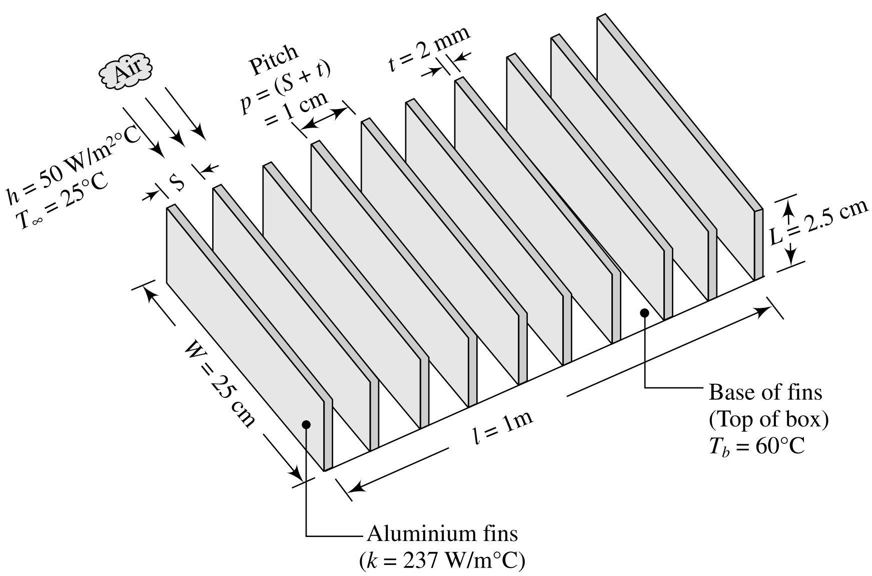
EXAMPLE 4.10 *An aluminium box encasing electronic equipment dissipates the heat generated within to the surrounding air at 25°C with a heat-transfer coefficient of 50 W/m² °C. The box surface temperature is not to exceed 60°C. Hence to aid heat removal, vertical rectangular aluminium fins (*k *= 237 W/m °C) are attached to the top of the box. If 10 fins, spaced 1 cm centre to centre apart are employed which are 2.5 cm long, 2 mm thick, and 25 cm wide, determine (a) the rate of heat dissipation, (b) fin efficiency, (c) fin effectiveness, and (d) overall surface efficiency.*

Solution

Known Vertical rectangular fins attached to a plane wall aid heat dissipation under specified conditions.

Find (a) Heat-dissipation rate (b) Fin efficiency (c) Fin effectiveness (d) Overall efficiency

Schematic



Assumptions (1) Steady operating conditions exist. (2) Constant thermal conductivity. (3) Uniform heat-transfer coefficient.

Analysis Fin parameter,

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{2h}{kt}} = \sqrt{\frac{2 \times 50 \text{ W/m}^2 \text{ °C}}{237 \text{ W/m °C} \times 0.002 \text{ m}}} = 14.525 \text{ m}^{-1}$$

$L_c = L + (t/2) = 2.5$ cm + (0.2 cm/2) = 2.6 cm = 0.026 m

$\therefore$ $mL_c = (14.525 \text{ m}^{-1})(0.026 \text{ m}) = 0.3776$

tanh $mL_c = 0.3607$

Fin efficiency, $\eta_f = \frac{\tanh mL_c}{mL_c} = \frac{0.3607}{0.3776}$ = **0.955** **(Ans.) (b)**

Fin heat-transfer rate,

$$\dot{Q}_{\text{fin}} = NmkA_c\theta_b \tanh mL_c$$

where $A_c = Wt = (0.25 \text{ m})(0.002 \text{ m}) = 0.0005 \text{ m}^2$

$\theta_b = T_b - T_\infty = 60 - 25 = 35°\text{C}$

Substituting the known values,

$$\dot{Q}_{\text{fin}} = (10)(14.525 \text{ m}^{-1})(237 \text{ W/m °C})(0.0005 \text{ m}^2)(35°\text{C}) \times (0.3607)$$
$$= \mathbf{217.27 \text{ W}}$$

As there are 10 fins, 1 cm apart,

$H = N(S + t) = 10 \times 1$ cm = 10 cm

Total area without fins, $A_{\text{no fin}} = WH = 25 \text{ cm} \times 10 \text{ cm} = 0.025 \text{ m}^2$

Heat transferred from the unfinned portion is

$$\dot{Q}_{\text{unfin}} = hA_{\text{unfin}}\theta_b$$

where $A_{\text{unfin}} = WH - NW\,t = W(H - Nt) = 0.25 \text{ m}(0.1 \text{ m} - 10 \times 0.002 \text{ m}) = 0.02 \text{ m}^2$

$\therefore$ $\dot{Q}_{\text{unfin}} = (50 \text{ W/m}^2 \text{ °C})(0.02 \text{ m}^2)(35°\text{C}) = \mathbf{35 \text{ W}}$

Total heat dissipation rate *with* fins

$\dot{Q}_w = \dot{Q}_{\text{fin}} + \dot{Q}_{\text{unfin}}$ = (217.27 + 35.0) W = **252.3 W** **(Ans.) (a)**

Heat-transfer rate *without* fins,

$\dot{Q}_{wo} = hA_{\text{no fin}}\theta_b$ = (50 W/m^2 °C) (0.025 m^2) (35°C) = **43.75 W**

$\therefore$ Effectiveness of fin, $\varepsilon_f = \frac{\dot{Q}_w}{\dot{Q}_{wo}} = \frac{252.3 \text{ W}}{43.75 \text{ W}}$ = **5.8** **(Ans.) (c)**

Overall surface efficiency,

$$\eta_o = 1 - \frac{A_{\text{fin}}}{A_{\text{total}}}(1 - \eta_f)$$

where $A_{\text{fin}} = 2\,LW + Wt + 2\,Lt$

$= 2(0.025 \text{ m} \times 0.25 \text{ m}) + (0.25 \text{ m} \times 0.002 \text{ m}) + 2(0.025 \text{ m} \times 0.002 \text{ m})$

$= 0.013 \text{ m}^2$ (*for a single fin*) and for 10 fins, A_{fin} = 0.13 m^2

$$A_{\text{total}} = A_{\text{fin}} + A_{\text{unfin}} = 0.13\ \text{m}^2 + 0.02\ \text{m}^2 = 0.15\ \text{m}^2$$

$$\therefore \qquad \eta_o = 1 - \frac{0.13\ \text{m}^2}{0.15\ \text{m}^2}\ (1 - 0.955) = \mathbf{0.961\ or\ 96.1\%} \qquad \textbf{(Ans.) (d)}$$

(D) Bar With Two Ends At Specified Temperatures

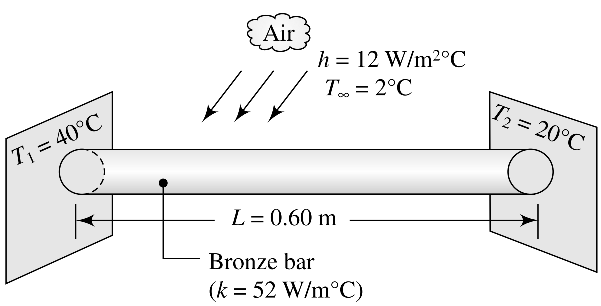
EXAMPLE 4.11 *A round bronze bar (k = 52 W/m °C), 0.6 m long and 2 cm diameter extends between two walls, one at 40°C and the other at 20°C. Ambient air at 2°C with a convection heat transfer coefficient of 12 W/m^2 °C on the surface of the bar. Determine (a) the location, and (b) the magnitude of the minimum temperature in the bar. (c) What will be the total heat-transfer rate from the fin?*

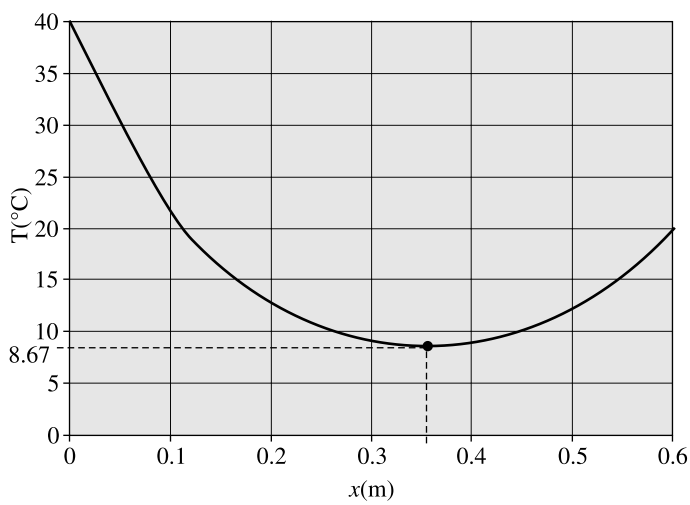
Solution

Known A bronze bar spans two walls held at different temperatures with ambient air surrounding it.

Find (a) Location and magnitude of minimum bar temperature (b) Fin heat-transfer rate

Schematic





Assumptions (1) Steady-state, one-dimensional conduction. (2) Constant thermal conductivity. (3) Heat-transfer coefficient is uniform on all fin surfaces.

Analysis The present case is one of a known tip temperature, $T_2 = T_L = 20°C$ while the base temperature of the fin, $T_1 = T_b = 40°C$. The temperature distribution is given by

$$\frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL} \tag{A}$$

Now, $$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h(\pi D)}{k\pi D^2/4}}$$

$$= \sqrt{\frac{4h}{kD}} = \left(\frac{4 \times 12 \text{ W/m}^2\,°\text{C}}{52 \text{ W/m}\,°\text{C} \times 0.02 \text{ m}}\right)^{1/2} = 6.794 \text{ m}^{-1}$$

$\therefore$ $mL = (6.794)\ (0.6) = 4.076$

$\sinh mL = 29.45,\ e^{mL} = 58.91,\ e^{-mL} = 0.017$

$$\frac{\theta_L}{\theta_b} = \frac{18°\text{C}}{38°\text{C}} = 0.4737$$

Location for minimum temperature is given by

$\therefore$ $\boldsymbol{x_{\min} = 0.357}$ **m** **(Ans.) (a)**

Substituting this value in Eq. (A), we get

$$\frac{\theta}{\theta_b} = \frac{(0.4737)\sinh(6.794 \times 0.357) + \sinh\{(6.794 \times (0.6 - 0.357)\}}{29.45}$$

$$= \frac{(0.4737 \times 5.61) + 2.51}{29.45} = 0.1755$$

$\therefore$ $\theta = (0.1755)\ (38) = 6.67°C = T - T_\infty$

and, the minimum bar temperature,

$T_{\min} = (2 + 6.67)°C \approx$ **8.67°C** **(Ans.) (b)**

The temperature profile is shown below.

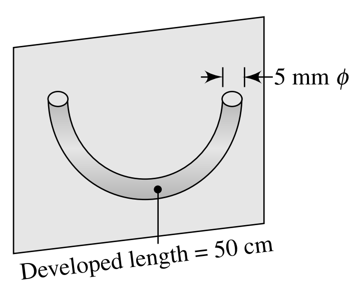
Total fin heat-transfer rate,

$$\dot{Q}_{\text{fin}} = mkA_c\,(\theta_b + \theta_L)(\cosh mL - 1)/\sinh mL$$

$$= (6.794 \text{ m}^{-1})\ (52 \text{ W/m °C}) \left(\frac{\pi}{4} \times 0.02^2 \text{ m}^2\right)\ (38 + 18)°\text{C} \times \left(\frac{\cosh 4.076 - 1}{\sinh 4.076}\right)$$

$=$ **6.0 W** **(Ans.) (c)**

EXAMPLE 4.12 *The two ends of a 5 mm copper U-shaped rod (see schematic) are rigidly fixed to a vertical wall maintained at a temperature of 90°C. The developed length of the rod is 50 cm and the ambient air temperature is 35°C. The combined convective and radiative heat-transfer coefficient is 35 W/m² K. (a) Compute the temperature at the centre point of the rod. (b) Also find the heat transfer from the rod.*

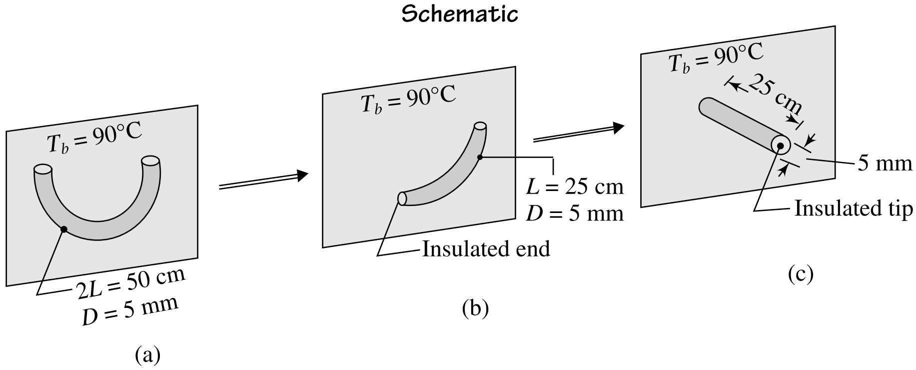


Solution

Known Two ends of a U-shaped rod rigidly fixed to a vertical wall dissipates heat to the surrounding air.

Find (a) Temperature at the centre point of the rod (b) Heat-transfer rate.

Schematic



By symmetry, Figure (a) can be transformed into Figure (b), and the equivalent Figure (c) is obtained by neglecting the effect of curvature of the rod.

Assumptions (1) Steady state prevails. (2) Across a section of the rod, the temperature is uniform. (3) The U-shaped rod can be approximated as a straight rod of the same length.

Analysis The temperature distribution for a fin with insulated tip and uniform cross section is given by

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh m(L - x)}{\cosh mL}$$

Fin parameter,

$$m = \sqrt{\frac{hP}{kA_c}} = \left[\frac{(35 \text{ W/m}^2\text{ K})(\pi \times 5 \times 10^{-3}\text{ m})}{(400 \text{ W/mK})\left(\dfrac{\pi}{4}\right)(5 \times 10^{-3}\text{ m})^2}\right]^{1/2} = 8.366 \text{ m}^{-1}$$

$$mL = (8.366 \text{ m}^{-1})(0.25 \text{ m}) = 2.09$$

At the fin tip,

$$\frac{T_L - T_\infty}{T_b - T_\infty} = \frac{1}{\cosh mL} = \frac{1}{\cosh 2.09} = 0.243$$

$\therefore$ $\quad T_L - T_\infty = 0.243\ (T_b - T_\infty)$

Hence, $\quad T_L = 35°\text{C} + 0.243\ (90 - 35)°\text{C} = \mathbf{48.4°C}$ **(Ans.)**

Fin heat-transfer rate, for one half of the U-shaped rod is

$$\dot{Q}_{\text{fin}} = mkA_c\theta_b \tanh mL$$

$$= (8.366 \text{ m}^{-1})\ (400 \text{ W/m K}) \left(\frac{\pi}{4} \times 0.005 \text{ m}\right)^2\ (90 - 35)°\text{C} \tanh (2.09)$$

$$= 3.614 \tanh (2.09) = (3.614)\ (0.97) = \mathbf{3.5\ W}$$

∴ Total heat-transfer rate,

$$\dot{Q} = 2 \times 3.5\ \text{W} = \mathbf{7.0\ W} \qquad \textbf{(Ans.)}$$

EXAMPLE 4.13 *A cylindrical pin fin of 1 cm diameter and 5 cm length is attached to a wall at 230°C. The thermal conductivity of the fin material is 204 W/m K. The fin is exposed to an environment at 30°C with a surface heat transfer coefficient of 200 W/m² K. Calculate the fin-tip temperature and the heat dissipation rate from the fin for the following four boundary conditions: (a) very long fin, (b) adiabatic fin tip, (c) convective fin tip, and (d) the fin tip is maintained at 180°C.*

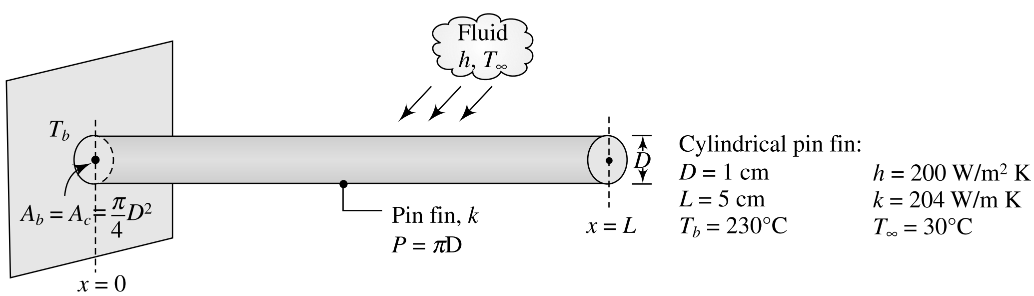
Plot the temperature distribution along the length of the fin.

Solution

Known A pin fin dissipates heat to the surroundings under different boundary conditions at the tip.

Find Fin-tip temperature, $T(x = L)$, heat-flow rate, $\dot{Q}_{\text{fin}}(W)$, and temperature profile for the given four boundary conditions.

Schematic



Assumption (1) Steady operating conditions. (2) One-dimensional conduction. (3) Constant thermal conductivity and uniform heat-transfer coefficient.

Analysis Fin parameter,

$$m = \left(\frac{hP}{kA_c}\right)^{1/2} = \sqrt{\frac{4h}{kD}} = \sqrt{\frac{4 \times 200\ \text{W/m}^2\text{K}}{204\ \text{W/mK} \times 0.01\ \text{m}}} = 19.8\ \text{m}^{-1}$$

$$mL = 19.8 \times 0.05 = 0.99$$

$$\frac{h}{mk} = \frac{200\ \text{W/m}^2\ \text{K}}{19.8\ \text{m}^{-1} \times 204\ \text{W/mK}} = 0.0495$$

■ **Fin-tip temperature:**

(a) $T_{(x=L)} = T_\infty + (T_b - T_\infty)\exp(-mL)$

$$= 30 + (230 - 30)\exp(-0.99) = \mathbf{104.3°C}$$

(b) $T_{(x=L)} = T_\infty + (T_b - T_\infty)/\cosh mL$

$$= 30 + 200/\cosh(0.99) = \mathbf{160.6°C}$$

(c) $T_{(x=L)} = T_\infty + (T_b - T_\infty)/[\cosh mL + (h/mk)\sinh mL]$

$$= 30 + 200 \times \left[\frac{1}{\cosh 0.99 + \left(\dfrac{200}{19.8 \times 204}\right)\sinh 0.99}\right] = \mathbf{155.9°C}$$

Corrected length approximation:

Since L_c = 52.5 mm,

$$= T_\infty + (T_b - T_\infty)/\cosh mL_c = 30 + 200/\cosh(19.8 \times 0.0525)$$

= 155.7°C which is almost same as with the convective tip.

(d) $T_{(x=L_c)} = \mathbf{180°C}$ (given)

$$\theta_L = 180 - 30 = 150°\text{C}$$
$$\theta_b = 230 - 30 = 200°\text{C}$$
$$(\theta_L/\theta_b) = 0.75$$

Location of minimum temperature,

$$x_{\min} = \frac{1}{2\text{ m}}\ln\left[\frac{e^{mL} - (\theta_L/\theta_b)}{(\theta_L/\theta_b) - e^{-mL}}\right] = 0.0413\text{ m} = \mathbf{41.3\text{ mm}}$$

■ **Fin heat-transfer rate:**

(a) $\dot{Q}_{\text{fin}} = mkA_c\theta_b = M$

$$= 19.8\text{ m}^{-1} \times 204\text{ W/mK} \times \frac{\pi}{4} \times 0.01^2\text{ m}^2 \times 200\text{ K}$$

= 63.457 W ≈ **63.5 W**

(b) $\dot{Q}_{\text{fin}} = M\tanh mL = 63.5\tanh 0.99$ = 48.06 W ≈ **48.1 W**

(c) $\dot{Q}_{\text{fin}} = M\dfrac{\tanh mL + (h/mk)}{1 + (h/mk)\tanh mL} = 63.5\text{ W} \times \dfrac{\tanh 0.99 + (0.04905)}{1 + (0.04905)(\tanh 0.99)}$

= 0.7777 M **= 49.4 W**

Corrected length approximation

Since L_c = 52.5 mm,

tanh mL_c = tanh 1.0397 = 0.77775

∴ $\dot{Q}_{\text{fin}} = mkA_c\theta_b\tanh mL_c = M\tanh mL_c$

$$= 63.5\text{ W} \times 0.77775 = 49.35\text{ W} \approx \mathbf{49.4\text{ W}}$$

which is same as with the convective tip.

(d) $\dot{Q}_{\text{fin}} = mkA_c(\theta_b + \theta_L)\dfrac{(\cosh mL - 1)}{\sinh mL}$

$$= 19.8 \text{ m}^{-1} \times 204 \text{ W/mK} \times \frac{\pi}{4}(0.01)^2 \text{ m}^2 \times (200 + 150)\text{ K} \times \frac{(\cosh 0.99 - 1)}{\sinh 0.99}$$

$$= 50.88 \text{ W} \approx \mathbf{50.9 \text{ W}}$$

■ **Temperature variation along the fin length**

(a) $\dfrac{\theta}{\theta_b} = \dfrac{T - T_\infty}{T_b - T_\infty} = \exp(-mx)$

$\therefore \quad T(x) = T_\infty + e^{-mx}(T_b - T_\infty) = 30 + e^{-19.8x}(230 - 30)$

$$\boxed{T(x) = 30 + 200\exp(-19.8x)}$$

(b) $\dfrac{\theta}{\theta_b} = \dfrac{\cosh m(L - x)}{\cosh mL}$

$$\theta = \frac{\cosh 19.8(0.05 - x)}{\cosh 0.99} \times 200$$

$$\boxed{T(x) = 30 + 130.6 \cosh 19.8(0.05 - x)}$$

(c) $\dfrac{\theta}{\theta_b} = \dfrac{\cosh m(L - x) + (h/mk)\sinh m(L - x)}{\cosh mL + (h/mk)\sinh mL}$

$$\theta = [\cosh 19.8(0.05 - x) + 0.0495 \sinh 19.8(0.05 - x)] \times 200/[\cosh 0.99 + (0.0495) \times \sinh 0.99]$$

$$\boxed{T(x) = 30 + 125.88[\cosh 19.8(0.05 - x) + 0.0495 \sinh 19.8(0.05 - x)]}$$

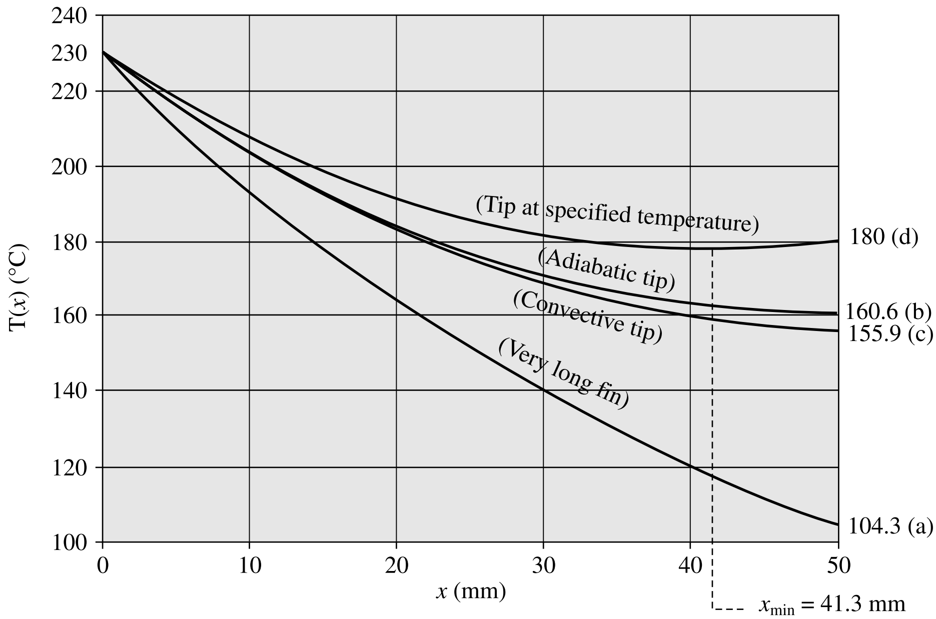
(d) $\dfrac{\theta}{\theta_b} = \dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L - x)}{\sinh mL}$

$$\theta = [0.75 \sinh 19.8x + \sinh 19.8(0.05 - x)] \times 200/\sinh 0.99$$

$$\boxed{T(x) = 30 + 172.4[0.75 \sinh 19.8x + \sinh 19.8(0.05 - x)]}$$

Based on the above *four* expressions for $T(x)$, the results are tabulated and graphically shown below:

x (mm)	$T(x)$ [Case (a)]	$T(x)$ [Case (b)]	$T(x)$ [Case (c)]	$T(x)$ [Case (d)]
0	230	230	230	230
10	194.0	203.8	202.9	207.1
20	164.6	184.3	182.7	191.1
30	140.4	171.0	168.4	181.5
40	120.6	163.2	159.6	177.8
50	104.3	160.6	155.9	180.0



Temperature variation along the fin length for four different cases

Comment The infinite (very long) boundary condition can fairly accurately predict fin temperatures if mL is at least equal to 2.65. In this case, mL is only 0.99. Clearly, the very long fin approximation results in significant underestimates of temperature at all locations. For case (a) to be valid, $L_\infty \approx 2.65/m \approx (2.65/19.8)(1000) = \mathbf{134\ mm}$

(E) Circumferential (*Annular*) Fins

EXAMPLE 4.14 *Circular aluminium disk fins of constant rectangular profile are attached to a tube having 2.5 cm OD with a pitch of 8 mm. Fins are 1 mm thick with 15 mm height and a thermal conductivity of 200 W/m °C. The tube wall is maintained at a temperature of 180°C and the fins dissipate heat by convection into ambient air at 40°C with a heat-transfer coefficient of 80 W/m²°C. Determine (a) the fin efficiency, (b) the area weighted (total) fin efficiency, (c) the net heat loss per metre length of tube, (d) the heat loss per metre length of tube if no fins were provided, and (e) the overall effectiveness of the finned tube.*

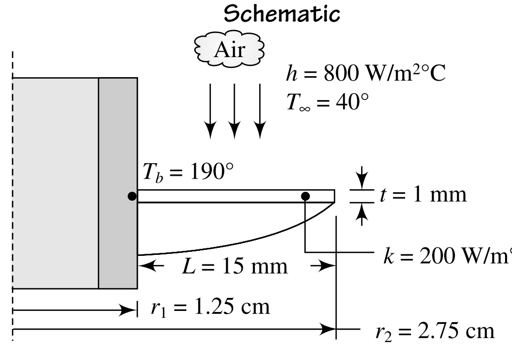
Solution

Known Circumferential fins provided on a tube dissipate heat under specified conditions.

Find (a) η_{fin}, (b) η_o, (c) $\dot{Q}_{\text{total,fin}}$ (W), (d) $\dot{Q}_{\text{no fin}}$. (e) $\varepsilon_{\text{fin, overall}}$

Assumptions (1) Steady operating conditions. (2) Constant conductivity and uniform heat-transfer coefficient. (3) One dimensional representation. (4) Radiation effects negligible.

Schematic



Analysis Fin parameter = $L_c^{3/2}(h/kA_p)^{1/2}$

where $L_c = L + \dfrac{t}{2}$ = 15 mm + $\dfrac{1}{2}$ mm = 15.5 mm

$A_p = L_c t$ = 15.5 mm × 1 mm = 15.5 mm^2

h = 80 W/m^2 °C, k = 200 W/m °C

$$\therefore \quad L_c^{3/2}\left(\frac{h}{kA_p}\right)^{1/2}$$

$$= (0.0155\ \text{m})^{3/2}\left[\frac{80\ \text{W/m}^2\,°\text{C} \times 10^6\ \text{mm}^2/1\ \text{m}^2}{200\ \text{W/m}\,°\text{C} \times 15.5\ \text{mm}^2}\right]^{1/2} = 0.31$$

$r_{2c} = r_2 + (t/2)$ = 27.5 mm + 0.5 mm = 28 mm

r_1 = 12.5 mm

$$\therefore \quad \frac{r_{2c}}{r_1} = \frac{28}{12.5} = 2.24$$

Fin efficiency,

$$\eta_{\text{fin}} = C_2 \frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$

$$C_2 = \frac{(2r_1/m)}{(r_{2c}^2 - r_1^2)}$$

where $$m = \left(\frac{hP}{kA_c}\right)^{1/2} = \left(\frac{2h}{kt}\right)^{1/2} = \sqrt{\frac{2(80\ \text{W/m}^2\,°\text{C})}{200\ \text{W/m}\,°\text{C} \times 0.001\ \text{m}}} = 28.284\ \text{m}^{-1}$$

mr_1 = (28.284 m^{-1})(0.0125 m) = 0.3536

mr_{2c} = (28.284 m^{-1}) (0.028 m) = 0.792

From the modified Bessel Functions table,

$K_0(mr_1)$ = 1.2342 $K_1(mr_1)$ = 2.588

$I_0(mr_1)$ = 1.0321 $I_1(mr_1)$ = 0.1797

$I_1(mr_{2c})$ = 0.428 02 $K_1(mr_{2c})$ = 0.876 88

and $$C_2 = \frac{(2 \times 0.0125\ \text{m}/28.284\ \text{m}^{-1})}{(0.028^2 - 0.0125^2)\text{m}^2} = 1.408$$

Fin efficiency,

$$\eta_{\text{fin}} = (1.408)\left\{\frac{(2.5888)(0.428\ 02) - (0.1797)(0.876\ 88)}{(1.0321)(0.876\ 88) + (1.2342)(0.428\ 02)}\right\}$$

= 1.408 × 0.6631 = **0.934** **(Ans.) (a)**

$$A_{\text{fin}} = 2\pi(r_{2c}^2 - r_1^2) = 2\pi[0.028^2 - 0.0125^2]\text{m}^2 = 3.944 \times 10^{-3}\ \text{m}^2$$

$$\dot{Q}_{\text{fin}} = \eta_f \dot{Q}_{\text{fin,max}} = \eta_f h A_f (T_b - T_\infty)$$

$$= (0.934)\ (80\ \text{W/m}^2\ °\text{C})\ (3.944 \times 10^{-3}\ \text{m}^2)\ (190–40)°\text{C} = \mathbf{44.2\ W}$$

$$\dot{Q}_{\text{unfin}} = hA_{\text{unfin}}(T_b - T_\infty)$$

$$A_{\text{unfin}} = 2\pi r_1(p - t) = 2\pi \times 0.0125\ \text{m} \times (7-1)(10^{-3})\text{m} = 0.55 \times 10^{-3}\ \text{m}^2$$

where $\dot{Q}_{\text{unfin}} = (80\ \text{W/m}^2\ °\text{C})(0.55 \times 10^{-3}\ \text{m}^2)\ (190 - 40)°\text{C} = \mathbf{6.6\ W}$

$\therefore$ $\dot{Q}_{\text{total,fin}} = N(\dot{Q}_{\text{fin}} + \dot{Q}_{\text{unfin}})$

where N = number of fins or interfin spacings

$$= \frac{1\ \text{m}}{8 \times 10^{-3}\ \text{m}} = 125$$

$\therefore$ $\dot{Q}_{\text{total,fin}}$ = 125 (44.2 + 6.6) W

= **6350 W or 6.35 kW per m tube length** **(Ans.) (c)**

Area weighted (total) efficiency,

$$\eta_o = 1 - \frac{A_{\text{fin}}}{A}(1 - \eta_{\text{fin}}) = 1 - \frac{3.944 \times 10^{-3}\ \text{m}^2}{(3.944 + 0.55) \times 10^{-3}\ \text{m}^2}(1 - 0.934)$$

= **0.942 or 94.2%** **(Ans.) (b)**

Heat transfer per m tube length without fin is

$$\dot{Q}_{\text{no fin}} = hA_{\text{no fin}}(T_b - T_\infty) = h(2\pi r_1 \times 1)(T_b - T_\infty)$$

$$= (80\ \text{W/m}^2°\text{C})\ (2\ \pi \times 0.0125\ \text{m} \times 1\ \text{m})\ (190 - 40)°\text{C}$$

= **942.5 W or 0.94 kW** per m length of tube **(Ans.) (d)**

The overall effectiveness of the finned tube is

$$\varepsilon_{\text{fin,overall}} = \frac{\dot{Q}_{\text{total,fin}}}{\dot{Q}_{\text{no fin}}} = \frac{6350\ \text{W/m}}{942.5\ \text{W/m}} = \mathbf{6.74}$$ **(Ans.) (e)**

EXAMPLE 4.15 *Steam at 1 atm condenses inside a copper pipe. Circular circumferential fins are fixed on the outside, which is exposed to air.*

Pipe: *ID = 30 cm,* *OD = 36 cm*
k_{pipe} *= 380 W/m K*

Fins: *Thickness = 1 mm,* *Pitch = 20 mm, Tip diameter = 100 mm*
k_{fin} *= 45 W/m K* h_{air} *= 25 W/m² K*

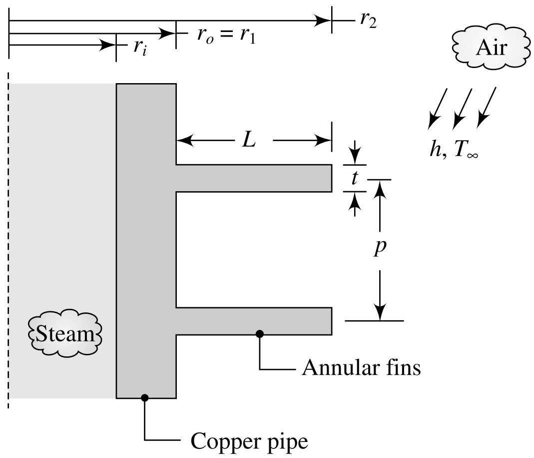
Determine (a) fin efficiency, and (b) overall heat-transfer coefficient based on the tube inner area.
[IIT, Bombay]

Solution

Known Circular fins are attached to a copper steam pipe on its outside which is exposed to air.

Find (a) Fin efficiency, η_{fin}, (b) Overall heat-transfer coefficient based on *inside* area, U_i

Schematic



- r_i = 15 mm = 0.015 m
- $r_o = r_1$ = 18 mm = 0.018 m
- r_2 = 50 mm = 0.05 m
- k_{fin} = 45 W/m K
- $h_{\text{air}} = h_o$ = 25 W/m^2 K
- t = 1 mm = 0.001 m
- p = 20 mm = 0.02 m

Assumptions (1) Steady operating conditions. (2) Constant thermal conductivity. (3) Uniform heat-transfer coefficients. (4) Negligible radiation effects and contact thermal resistance.

Analysis The overall heat-transfer coefficient based on inner area is given by

$$\frac{1}{U_i} = \frac{1}{h_i} + \frac{r_i}{k_{\text{pipe}}} \ln\frac{r_o}{r_i} + \frac{A_i}{(A_{o,\text{unfin}} + A_{o,\text{fin}} \cdot \eta_{\text{fin}})} \cdot \frac{1}{h_o}$$

$$= \frac{1}{h_i} + \frac{r_i}{k_{\text{pipe}}} \ln\frac{r_o}{r_i} + \frac{1}{\left(\dfrac{A_{o,\text{unfin}}}{A_i} + \dfrac{A_{o,\text{fin}}}{A_i}\eta_{\text{fin}}\right) h_o} \tag{A}$$

Condensation of steam on inside is associated with a very high heat-transfer coefficient. Hence, the first term, $\frac{1}{h_i}$, may be neglected in comparison with other terms.

To determine the fin efficiency, we use the graph in which the efficiency of circular fins of length L and constant thickness t is plotted. The parameters that govern the problem are

Radius ratio, $\dfrac{r_{2c}}{r_1} = \dfrac{r_2 + \frac{1}{2}t}{r_1} = \dfrac{\left(50 + \frac{1}{2} \times 1\right)\text{mm}}{18 \text{ mm}} = 2.80$

Non-dimensional fin parameter, $\xi = L_c^{3/2}\left(\dfrac{h}{kA_p}\right)^{1/2}$

where $L_c = L + \frac{1}{2}t = r_2 - r_1 + \frac{1}{2}t$ = (50 – 18 + 0.5) mm = 32.5 mm or 0.0325 m

Profile area,

$$A_p = L_c t = 0.0325 \text{ m} \times 0.001 \text{ m} = 32.5 \times 10^{-6} \text{ m}^2$$

$$\therefore \qquad \xi = (0.0325)^{3/2}\left[\frac{25 \times 10^6}{45 \times 32.5}\right]^{1/2} = 0.766$$

With $\dfrac{r_{2c}}{r_1} = 2.80$ and $\xi = 0.766$, from the graph:

Fin efficiency, $\eta_{\text{fin}} = 0.62$ **or 62%** **(Ans.) (a)**

$$\frac{A_{o,\text{unfin}}}{A_i} = \frac{2\pi(p-t)r_o}{2\pi r_i p} = \frac{(20-1)\times 18}{15\times 20} = 1.140$$

Now,
$$\frac{A_{o,\text{fin}}}{A_i} = \frac{2\pi(r_{2c}^2 - r_1^2)}{2\pi r_i p} = \frac{(50.5^2 - 18^2)}{15\times 20} = 7.421$$

Substituting the known values in Eq. (A), we have

$$\frac{1}{U_i} = \cancel{\frac{1}{h_i}}^{\,0} + \frac{0.015}{380}\ln\frac{18}{15} + \frac{1}{(1.140 + 7.421\times 0.62)\times 25} = 6.98\times 10^{-3}\,\frac{\text{m}^2\text{K}}{\text{W}}$$

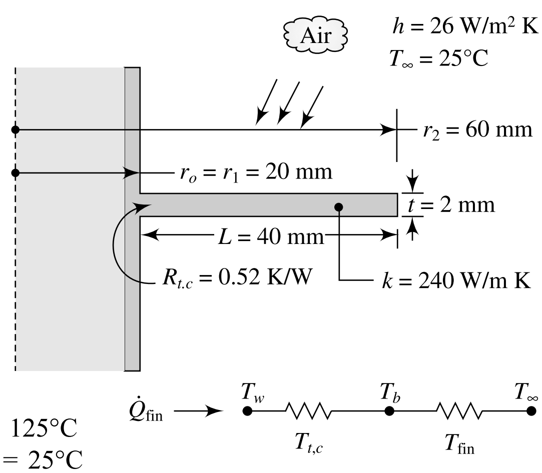
$\therefore$ overall heat-transfer coefficient based on inner area,

U_i **= 143.3 W/m² K** **(Ans.) (b)**

EXAMPLE 4.16 *Circumferential fins of rectangular cross section (2 mm thick and 40 mm long) are force fitted on a tube, 40 mm OD. The fins and the tube are made of aluminium (k = 240 W/m K). The thermal contact resistance between each fin and the tube is known to be 0.52 K/W. Calculate the rate of heat transfer from each fin if the tube wall is at 125°C, the ambient air is at 25°C and the convective heat-transfer coefficient is 26 W/m² K. Also find the heat-transfer rate without the base contact resistance.*

Solution

Schematic



Known Circumferential aluminium fins of constant thickness are attached to a tube.

Find Fin heat-transfer rate, $\dot{Q}_{\text{fin}}(W)$.

Assumptions (1) Steady operating conditions. (2) Thermal conductivity is constant. (3) Heat-transfer coefficient is uniform over the entire fin surface. (4) Radiation effects are negligible.

Analysis Heat-dissipation rate from each fin,

$$\dot{Q}_{\text{fin}} = \frac{T_w - T_\infty}{R_{t,c} + R_{\text{fin}}}$$

where T_w = tube wall temperature = 125°C

T_∞ = ambient air temperature = 25°C

$R_{t,c}$ = thermal contact resistance = 0.52 K/W (*given*)

R_{fin} = thermal resistance of the radial fin of uniform thickness

$$= \frac{1}{h(\eta_{\text{fin}}A_{\text{fin}})} = \frac{1}{h\eta_{\text{fin}}\{2\pi(r_{2c}^2 - r_1^2)}$$

Radius ratio,

$$\frac{r_{2c}}{r_1} = \frac{r_2 + (t/2)}{r_1} = \frac{[60 + (2/2)]\text{mm}}{20\text{ mm}} = \frac{0.061\text{ m}}{0.02\text{ m}} = \mathbf{3.05}$$

Non-dimensional fin parameter, $\xi = L_c^{3/2}\sqrt{\dfrac{h}{kA_p}}$

$$L_c = L + (t/2) = [40 + (2/2)]\ \text{mm} = 41\ \text{mm or } 0.041\ \text{m}$$

$$h = 26\ \text{W/m}^2\text{K}, \quad k = 240\ \text{W/mK}$$

where $\quad A_p = Lt = (0.04\ \text{m})(0.002\ \text{m}) = 8 \times 10^{-5}\ \text{m}^2$

$$\therefore \quad \xi = (0.041)^{3/2}\left[\frac{26}{240 \times 8 \times 10^{-5}}\right]^{1/2} = \mathbf{0.3055}$$

From the graph for circumferential fin attached to a circular tube with $\dfrac{r_{2c}}{r_1} = 3.05$ and ξ = 0.3055, the fin efficiency is, η_{fin} = **0.91**

$$\therefore \quad R_{\text{fin}} = [26\ \text{W/m}^2\ \text{K} \times 0.91 \times 2\pi(0.061^2 - 0.02^2)\text{m}^2]^{-1} = \mathbf{2.0255\ K/W}$$

Fin heat-transfer rate,

$$\dot{Q}_{\text{fin}} = (125 - 25)\text{K}/(2.0255 + 0.52)\text{K/W} = \mathbf{39.3\ W} \qquad \textbf{(Ans.)}$$

Without the contact resistance, $T_w = T_b$ = 125°C, and

$$\dot{Q}_{\text{fin}} = \frac{\theta_b}{R_{\text{fin}}} = \frac{(125 - 25)\text{K}}{2.0255\ \text{K/W}} = \mathbf{49.4\ W} \qquad \textbf{(Ans.)}$$

(F) Triangular Fins

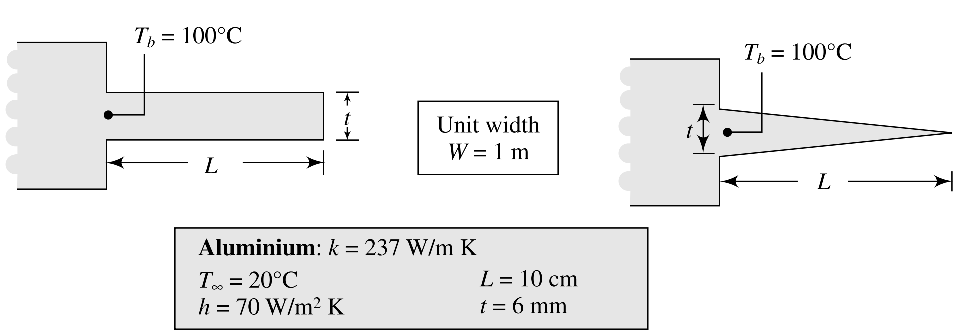
EXAMPLE 4.17 *A straight, uniform area fin and a triangular fin, both 10 cm long with 6 mm thickness are to be compared. The base and the surrounding temperature in both cases are 100°C and 20°C respectively. The material of the fins is aluminium with thermal conductivity 237 W/m K and density of 2700 kg/m^3 respectively. The surface heat-transfer coefficient is 70 W/m^2 K in both cases. Determine the fin efficiency, fin effectiveness, and the rate of heat loss per unit mass for the two configurations.*

Solution

Known Dimensions, base temperature and surrounding conditions associated with uniform area and triangular aluminium fins.

Find Efficiency, effectiveness, and heat loss per metre width associated with each fin.

Schematic



Assumptions (1) Steady operating conditions. (2) One-dimensional conduction. (3) Constant properties. (4) Uniform convection coefficient. (5) Negligible radiation and contact resistance.

Analysis ***Case I:* Uniform area fin**

Fin parameter, $m = \sqrt{\dfrac{hP}{kA_c}} = \sqrt{\dfrac{h \cdot 2(W+t)}{k(Wt)}}$

As $t << W$, $W + t \approx W$

$$m = \sqrt{\frac{2h}{kt}} = \sqrt{\frac{2 \times 70 \text{ W/m}^2\text{ K}}{(237 \text{ W/mK})(0.006 \text{ m})}} = 9.922 \text{ m}^{-1}$$

$$L_c = L + t/2 = 0.1 \text{ m} + (0.006 / 2) \text{ m} = 0.103 \text{ m}$$

$\therefore$ $mL_c = (9.922 \text{ m}^{-1})\,(0.103 \text{ m}) = 1.022$

$\tanh mL_c = 0.7707$

Fin efficiency, $\eta_{\text{fin}} = \dfrac{\tanh mL_c}{mL_c} = \dfrac{0.7707}{1.022} = \mathbf{0.754}$ **(or 75.4 %)**

Heat-loss rate per unit width,

$$\dot{Q}_{\text{fin}} = mkA_c(T_b - T_\infty)\tanh mL_c$$

$$= (9.922 \text{ m}^{-1})\,(237 \text{ W/m K})\,(1 \text{ m} \times 0.006 \text{ m})\,(100 - 20)°\text{C} \times 0.7707$$

$$= \mathbf{870\ W}$$

Fin effectiveness,

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_{c,b}(T_b - T_\infty)} = \frac{870 \text{ W}}{(70 \text{ W/m}^2\text{ K})(1 \text{ m} \times 0.006 \text{ m})(100 - 20)°\text{C}}$$

$$= 870 \text{ W}/33.6 \text{ W} = \mathbf{25.9}$$

Fin volume per unit width,

$$\forall = tL = (0.006 \text{ m})\,(0.1 \text{ m}) = 0.0006 \text{ m}^3$$

Mass of fin per unit width,

$$m = \rho\forall = (2700 \text{ kg/m})\,(0.0006 \text{ m}^3) = 1.62 \text{ kg}$$

Heat loss per unit mass,

$$\dot{Q}_f/m = \frac{870 \text{ W}}{1.62 \text{ kg}} = \mathbf{537\ W/kg}$$

Case II: Triangular fin

$$mL = 9.922 \text{ m}^{-1} \times 0.1 \text{ m} = 0.9922$$

$$2mL = 1.9844$$

$$I_1(1.9844) = 1.5684$$

$$I_o(1.9844) = 2.2559$$

$\therefore$ $\eta_{\text{fin}} = \dfrac{1}{mL}\dfrac{I_1(2\,mL)}{I_o(2\,mL)} = \dfrac{1}{0.9922} \times \dfrac{1.5684}{2.2559} = 0.7$ or **70%**

Heat loss per unit width,

$$\dot{Q}_{\text{fin}} = 2\eta_{\text{fin}} h \left\{ L^2 + \left(\frac{t}{2}\right)^2 \right\}^{1/2} (T_b - T_\infty)$$

$$= 2 \times 0.70 \times 70 \text{ W/m}^2 \text{ K } \{(0.1^2 \text{ m}^2 + 0.003^2 \text{ m}^2)\}^{1/2} (100 - 20)°\text{C}$$

$$= \mathbf{784\ W}$$

Fin effectiveness,

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_f}{hA_{c,b}(T_b - T_\infty)} = \frac{784 \text{ W}}{33.6 \text{ W}} = \mathbf{23.3}$$

Fin volume per unit width,

$$\forall = \frac{1}{2} tL \times 1 = 0.006 \text{ m} \times 0.1 \text{ m} \times 1 \text{ m} / 2 = 0.0003 \text{ m}^3$$

Mass of fin per unit width,

$$m = \rho \forall = (2700 \text{ kg/m}^3)(0.0003 \text{ m}^3) = 0.81 \text{ kg}$$

∴ Heat loss per unit mass,

$$\dot{Q}_{\text{fin}}/m = \frac{784 \text{ W}}{0.81 \text{ kg}} = \mathbf{968\ W/kg}$$

The results are tabulated below for the sake of comparison:

Case #	Fin efficiency	Fin effectiveness	Heat loss per unit mass	Heat loss per unit width
I. Straight uniform area rectangular fin	75.4%	25.9	537 W/kg	870 W
II. Straight triangular fin	70 %	23.3	968 W/kg	784 W

(Ans.)

Comment Although the rate of heat loss is slightly less in the triangular fin, it is obvious that *per unit mass basis,* it dissipates (968 / 537 = 1.8) times more heat compared to uniform area fin. However, it may be noted that a fin of triangular profile is relatively structurally weak near the tip.

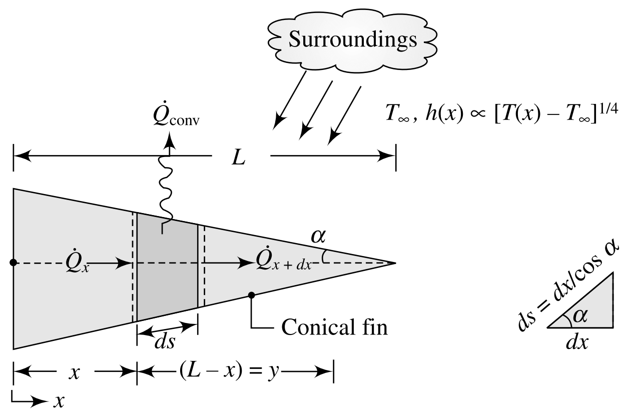
EXAMPLE 4.18 *Derive the differential equation for determining the steady-state temperature distribution in a conical fin with a semi vertex angle α and height* L*, and the surroundings at* T_∞*. Assume that the local heat-transfer coefficient h is directly proportional to the one fourth power of the temperature difference between the fin and the surroundings* $\theta (\equiv T - T_\infty)$*. One-dimensional heat conduction (along the fin length) may be assumed.*

Solution

Known A conical fin having semi-angle α and length L is exposed to convective environment with an ambient at T_∞ and variable heat-transfer coefficient.

Find Differential equation for steady state, one-dimensional temperature variation.

Schematic



Assumptions (1) Steady-state conditions exist. (2) One-dimensional conduction. (3) Heat-transfer coefficient is proportional to one fourth power of the excess temperature. (4) Constant thermal conductivity.

Analysis Consider a volume element of the conical fin at location x having a length dx. An energy balance on this volume element under steady operating conditions can be expressed as

$$\begin{pmatrix}\text{Rate of heat conduction}\\ \textit{into} \text{ the element at } x\end{pmatrix} = \begin{pmatrix}\text{Rate of heat conduction}\\ \textit{from} \text{ the element at } x + dx\end{pmatrix} + \begin{pmatrix}\text{Rate of heat convection}\\ \textit{from} \text{ the element}\end{pmatrix}$$

or $$\dot{Q}_x = \dot{Q}_{x+dx} + \dot{Q}_{\text{conv}}$$

where $$\dot{Q}_x = -kA_c\frac{dT}{dx} \quad \text{and} \quad \dot{Q}_{x+dx} = \dot{Q}_x + \frac{d}{dx}(\dot{Q}_x)dx$$

Hence, $$-\frac{d}{dx}(\dot{Q}_x)\,dx = h(x)(Pds)(T - T_\infty) \quad \text{or} \quad -\frac{d}{dx}\left[-kA_c(x)\frac{dT}{dx}\right]dx = h\,(x)\,Pds\,(T - T_\infty)$$

We note that

$$A_c(x) = \pi r^2 = \pi[(L - x)\tan\alpha]^2$$

$$P = 2\pi r = 2\pi[(L - x)\tan\alpha]$$

$h = C\,(T - T_\infty)^{1/4}$ where C is a constant.

Therefore, Eq. (A) can be expressed as

or $$k\frac{d}{dx}\left[\{\pi(L - x)^2\tan^2\alpha\}\frac{dT}{dx}\right]dx = 2\pi[(L - x)\tan\alpha]C(T - T_\infty)^{5/4}\frac{dx}{\cos\alpha}$$

or $$k\tan\alpha\left\{(L - x)^2\frac{d^2T}{dx^2} + \frac{dT}{dx}(-2(L - x))\right\} = \frac{2C}{\cos\alpha}(L - x)(T - T_\infty)^{5/4}$$

or $$(L - x)^2\frac{d^2T}{dx^2} - 2(L - x)\frac{dT}{dx} - \frac{2C}{k\tan\alpha\cos\alpha}(L - x)(T - T_f)^{5/4} = 0$$

Substituting

$(L - x) = y, - dx = dy$ and $(T - T_\infty) = \theta$ and $dT = d\theta$, we have

$$y^2 \frac{d^2\theta}{dx^2} + 2y\frac{d\theta}{dy} - \frac{2C}{k \sin\alpha} y\theta^{5/4} = 0$$

or
$$y \frac{d^2\theta}{dx^2} + 2\frac{d\theta}{dy} - \frac{2C}{k \sin\alpha} \theta^{5/4} = 0$$

or
$$\boxed{(L-x)\frac{d^2T}{dx^2} - 2\frac{dT}{dx} - \frac{2C}{k \sin\alpha}(T - T_\infty)^{5/4} = 0} \qquad \textbf{(Ans.)}$$

(G) Error in Temperature Measurement

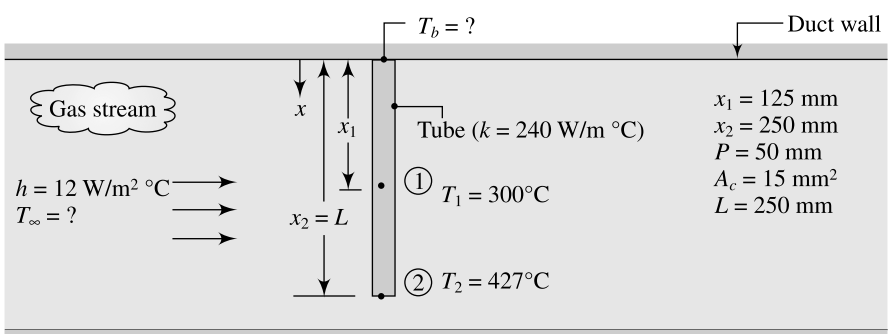
EXAMPLE 4.19 *A gas stream flows through a long duct. In order to estimate the gas temperature, two thermocouples are attached to a tube that is mounted normal to the duct wall. The tube is 250 mm long with a perimeter of 50 mm and an area of cross section of 15 mm^2. The location of the thermocouples measured from the duct wall is 125 mm and 250 mm with the corresponding temperatures measured being 390°C and 427°C. The thermal conductivity of the tube material is 240 W/m °C and the combined convection and radiation heat-transfer coefficient between the tube surface and the gas stream is 12 W/m^2°C. Neglecting heat loss from the exposed tube tip, determine the gas stream temperature and the base (tube wall) temperature.*

Solution

Known Thermocouples measure temperatures at two locations along a tube projected from the wall of a duct carrying a hot gas stream.

Find Gas stream temperature, T_∞ and the duct wall (base) temperature, T_b.

Schematic



Assumptions (1) Steady operating conditions exist. (2) Adiabatic fin tip. (3) Uniform heat-transfer coefficient. (4) Constant thermal conductivity. (5) Negligible contact resistance.

Analysis Temperature distribution along the length of a fin *with insulated end* is given by

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

The tube protruding from the duct wall can be treated as a fin with adiabatic end. Then

$$\frac{T(x_1) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x_1)}{\cosh mL} \tag{A}$$

and $$\frac{T(x_2) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x_2)}{\cosh mL} \tag{B}$$

Fin parameter, $m = \sqrt{\frac{hP}{kA_c}} = \left[\frac{12 \text{ W/m}^2\,°\text{C} \times 50 \text{ mm}}{240 \text{ W/m}\,°\text{C} \times 15 \text{ mm}^2} \times \frac{10^3 \text{ mm}}{1 \text{ m}}\right]^{1/2} = 12.91 \text{ m}^{-1}$

and $mL = (12.91 \text{ m}^{-1})\ (0.25 \text{ m}) = 3.2275$

$\therefore$ $\cosh mL = 12.628$

$x_1 = 125$ mm and $x_2 = L = 250$ mm

$\cosh m\ (L - x_1) = \cosh \{(12.91)\ (0.25 - 0.125)\} = \cosh (1.61375) = 2.6104$

Equations (A) and (B) can now be written as

$$\frac{390 - T_\infty}{T_b - T_\infty} = \frac{2.6104}{12.628} = 0.2067$$

$$\frac{427 - T_\infty}{T_b - T_\infty} = \frac{1}{12.628} = 0.0792$$

Dividing one by the other, we have

$$\frac{390 - T_\infty}{427 - T_\infty} = \frac{0.2067}{0.0792} = 2.61$$

or $(427)\ (2.61) - 2.61\ T_\infty = 390 - T_\infty$

$1.61\ T_\infty = (427 \times 2.61) - 390$

$\therefore$ Gas temperature, T_∞ = **450°C** **(Ans.)**

From Eq. (A):

$$\frac{390 - 450}{T_b - 450} = 0.2067$$

Duct-wall (base) temperature,

$$T_b = 450 - \frac{(450 - 390)}{0.2067} = \mathbf{159.7°C}$$ **(Ans.)**

EXAMPLE 4.20 *Superheated steam at a mean temperature* T_∞ *= 200°C flows through a pipe of* D *= 10 cm diameter. A brass pocket dips radially into the pipe with its closed end on the centre line, and the root of the pocket is at* T_b *= 140°C. The diameter of the pocket is* d *= 1.25 cm and its wall thickness is* δ *= 1 mm. The thermal conductivity* k *for brass is 112 W/m K and it is estimated that the combined convection and radiation heat-transfer coefficient* h *for the pocket surface is 401 W/m*^{2} *K. Predict the thermometer reading* T_L.

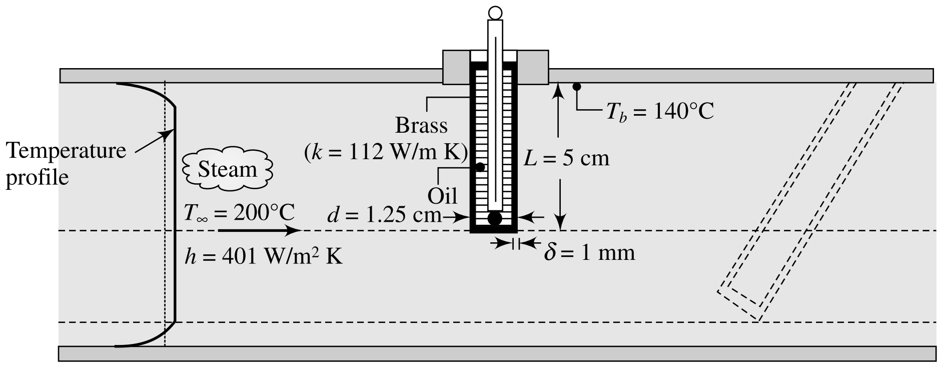
Assume that the thermometer reads the temperature of the bottom plate of the pocket, and that conduction in the axial direction through the thermometer and the surrounding oil is negligible compared with the conduction along the pocket wall. How can the error in temperature measurement be reduced?

Solution

Known Thermometer pocket is provided in a pipe carrying superheated steam for temperature measurement.

Find Thermometer reading, T_L (°C).

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties and uniform convection coefficient. (3) One-dimensional conduction. (4) Adiabatic thermometer tip.

Analysis The temperature at the end of the pocket is given by putting $x = L$ in the expression for temperature distribution with insulated tip.

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

It follows that

$$T_L - T_\infty = (T_b - T_\infty)\frac{\cosh m(L - L)}{\cosh mL} = \frac{1}{\cosh mL}(140 - 200)°C = \frac{-60°C}{\cosh mL}$$

The effective cross-sectional area to be used is approximately equal to $\pi d\delta$, and the circumference of the rod is πd. Hence,

$$m = \left(\frac{h\pi d}{k\pi d\delta}\right)^{1/2} = \left(\frac{h}{k\delta}\right)^{1/2} = \left(\frac{401\ \text{W/m}^2\ \text{K}}{112\ \text{W/mK} \times 0.001\ \text{m}}\right)^{1/2} = 59.84\ \text{m}^{-1}$$

$$\cosh mL = \cosh(59.84\ \text{m}^{-1} \times 0.05\ \text{m}) = 9.99$$

$$\therefore \qquad T_L - T_\infty = \frac{-60}{9.99} = -6\ \text{K} \quad \text{or} \quad -6°C$$

Hence, the thermometer reading is

$$\boldsymbol{T_L = 200 - 6 = 194°C} \qquad \textbf{(Ans.)}$$

Comment The error in temperature measurement can be reduced by lagging the pipe to increase the wall (*base*) temperature T_b. The other option is to increase, the product (mL).

Fin parameter, $m\left(\sqrt{\dfrac{h}{k\delta}}\right)$ can be increased by using a thinner tube, or by using a metal of lower thermal conductivity. However if the thermal conductivity is reduced drastically, the radial resistance to heat flow may increase to such an extent that the temperature distribution will no longer be *one-dimensional*.

Length, L may be increased by slanting the pocket and letting it project beyond the axis of the pipe as shown on the right by dotted line in the schematic.

Points To Ponder

- Fins are used to improve heat transfer by increasing the effective surface area.
- The temperature distribution in a fin is assumed to be one dimensional, i.e., along the length of the fin.
- For thin fins, the temperature drop across the fin is very small compared to the temperature drop along the fin. One-dimensional approximation for temperature distribution is valid in such cases.
- If the Biot number Bi, based on the cross-sectional dimensions of the fin, i.e., $\frac{hR}{k}$ for a pin fin and $\frac{ht}{2k}$ for a straight rectangular fin, is *less* than 0.1, the temperature variation in the direction of its thickness is negligible.
- For $mL > 2.65$, the fin may be considered of infinite length.
- There are 4 options for the fin tip boundary condition: (1) Very long fin ($T \to T_\infty$), (2) Adiabatic tip $\left(\left.\frac{dT}{dx}\right|_{x=L} = 0\right)$, (3) Convective heat loss $\left[-k\left.\frac{dT}{dx}\right|_{x=L} = h(T_L - T_\infty)\right]$, (4) Specified tip temperature ($T_{(x\,=\,L)} = T_L$).
- A small value of mL corresponds to relatively short, thick fins with large thermal conductivity while larger values of mL mean thin, long fins with lower thermal conductivity.
- Knowledge of fin efficiency can be used to calculate the fin thermal resistance $R_{\text{fin}} = 1/h\eta_{\text{fin}}A_{\text{fin}}$.
- An actual fin with active tip (*heat loss from the fin end*) is equivalent to a longer, hypothetical fin with an insulated tip. The corrected fin length, $L_c = L + (D/4)$ for a pin fin and $L_c = L + (t/2)$ for a rectangular fin.
- The corrected length approximation is justified if (ht/k) or $(hD/2k) \le 0.0625$.
- The use of fins is justified if the fin effectiveness exceeds 5.
- The main considerations in the selection of fins for a given application are the available space, weight and cost besides the fluid properties and the pump work if forced convection is involved.

GLOSSARY of Key Terms

● **Fin**	An extended surface used to effectively increase the heat transfer from a surface to the surrounding fluid.
● **Fin effectiveness**	The ratio of heat transfer with fin to that which would be obtained without fin.
● **Fin efficiency**	A ratio of the heat-transfer rate from the fin surface to the heat-transfer rate from an identical fin of infinite thermal conductivity.
● **Fin resistance**	The ratio of the base-temperature difference to the total heat transfer rate.

- **Overall surface efficiency** The ratio of total heat-transfer rate from the *array* of the fins and the unfinned surface to the maximum possible heat transfer rate if the whole surface (*finned and unfinned*) were maintained at the base temperature.

OBJECTIVE-TYPE QUESTIONS

- **Multiple-Choice Questions**

4.1 Consider the following statements pertaining to heat transfer through fins:
(1) Fins are equally effective irrespective of whether they are on the hot side or cold side of the fluid.
(2) The temperature along the fin is variable and, hence the rate of heat transfer varies along the elements of the fin.
(3) The fins may be made of materials that have a higher thermal conductivity than the material of the wall.
(4) Fins must be arranged at right angles to the direction of flow of the working fluid.
Of these statements,
(a) 1 and 2 are correct. (b) 1 and 3 are correct.
(c) 2 and 4 are correct. (d) 2 and 3 are correct.

4.2 Addition of fins to the surface increases the heat transfer if $\sqrt{\dfrac{hA}{kP}}$ is
(a) equal to one (b) greater than one
(c) less than one (d) greater than one but less than two

4.3 The parameter $m = \sqrt{hP/kA_c}$ has been stated to increase in a long fin. If all other parameters are maintained constant then
(a) The temperature profile will remain the same.
(b) The temperature drop along the length will be at a lower rate.
(c) The temperature drop along the length will be steeper.
(d) The parameter m influences the heat flow.

4.4 In a particular heat-transferring situation, a cast-iron fin is to be replaced by a copper fin of identical configuration. If all other parameters are maintained constant, such a replacement will
(a) increase the heat-flow rate
(b) decrease the heat-flow rate
(c) heat-flow rate is influenced only by the base temperature
(d) will affect only the temperature distribution

4.5 Consider the following statements pertaining to large heat-transfer rate using fins:
(1) Fins should be used on the side where the heat-transfer coefficient is small.
(2) Long and thick fins should be used.
(3) Long and thin fins should be used.
(4) The thermal conductivity of the fin material should be large.
Which of the above statements are correct?
(a) 1, 2, and 3 (b) 1, 2, and 4 (c) 2, 3, and 4 (d) 1, 3, and 4

4.6 The effectiveness, ε_f, for a very long fin:
(i) Increases with increasing length of the fin.

(ii) Is equal to the efficiency when the surface area of the fin is equal to its cross-sectional area.
(iii) Should always be greater than 1.
Of these statements,
(a) only (i) is correct (b) only (i) and (ii) are correct
(c) only (i) and (iii) are correct (d) all are correct

4.7 The fins attached to a surface are determined to have an effectiveness of 0.9. The rate of heat transfer from the surface as a result of the addition of these fins
(a) remains the same (b) becomes negligible
(c) decreases (d) increases

4.8 The temperature distribution in a stainless-steel fin (thermal conductivity 0.17 W/cm °C) of constant cross-sectional area of 2 cm^2 and length of 1 cm exposed to an ambient of 40°C (with a surface heat-transfer coefficient of 0.0025 W/cm^2 °C) is given by
$(T - T_\infty) = 3x^2 - 5x + 6$ where T is in °C and x is in cm. If the base temperature is 100°C then the heat dissipated by the fin surface will be:
(a) 6.8 W (b) 3.4 W (c) 1.7 W (d) 0.17 W

4.9 The values of the Biot number for three identical cylindrical fins made of copper, stainless steel, and Teflon exposed to the same operating conditions are 3.2×10^{-5}, 8×10^{-4}, and 0.0343 respectively. The three fins in the decreasing order of their fin tip temperature are:
(a) Teflon, Stainless steel, Copper (b) Stainless steel, Teflon, Copper
(c) Copper, Teflon, Stainless steel (d) Copper, Stainless steel, Teflon

4.10 A 25 cm long, 1 cm diameter copper fin (k = 400 W/m °C) is attached to a prime surface at 150°C while the surrounding medium is at 25°C. For a heat-transfer coefficient of 30 W/m^2 °C the rate of heat loss and the fin efficiency, assuming the fin to be very long are:
(a) 21.5 W, 0.73 (b) 12.8 W, 0.325 (c) 15.7 W, 0.80 (d) 5.7 W, 0.933

4.11 A finned surface consists of root or base area of 1 m^2 and fin-surface area of 2 m^2. The average heat-transfer coefficient for the finned surface is 20 W/m^2 K. Effectiveness of fins provided is 0.75. If the finned surface with root or base temperature of 50°C is transferring heat to a fluid at 30°C, the rate of heat transfer is:
(a) 300 W (b) 800 W (c) 1000 W (d) 1200 W

4.12 The efficiency of a 5 mm diameter and 50 mm long cylindrical pin fin with adiabatic tip is 65%. The fin effectiveness is:
(a) 39 (b) 26 (c) 13 (d) 0.3

4.13 A fin length L protrudes from a surface held at temperature T_b greater than the ambient temperature T_∞. The heat dissipation from the free end of the fin is assumed negligible. The temperature gradient at the fin tip $(dT/dx)_{x=L}$ is
(a) Zero (b) $\dfrac{T_L - T_\infty}{T_b - T_\infty}$ (c) $h(T_b - T_L)$ (d) $\dfrac{T_b - T_L}{L}$

4.14 From a metallic wall at 100°C, a metallic rod protrudes to the ambient air. The temperature at the tip will be minimum when the rod is made of:
(a) aluminium (b) steel (c) copper (d) silver

4.15 The efficiency of a pin fin with insulated tip is
(a) $\dfrac{\tanh mL}{(hA/kP)^{0.5}}$ (b) $\dfrac{\tanh mL}{mL}$ (c) $\dfrac{mL}{\tanh mL}$ (d) $\dfrac{(hA/kP)^{1/2}}{\tanh mL}$

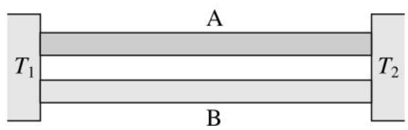
4.16 A thermocouple in a thermowell measures the temperature of hot gas flowing through the pipe. For the most accurate measurement of temperature, the thermowell should be made of:
(a) Steel (b) Brass (c) Copper (d) Aluminium

4.17 Usually, fins are provided to increase the rate of heat transfer. But fins also act as insulation. Which one of the following non-dimensional numbers decides this factor?
(a) Eckert number (b) Biot number (c) Fourier number (d) Peclet number

4.18 A thin pin fin (k = 100 W/m °C) of 6 mm diameter and 100 mm length exposed to 20°C air (h = 24 W/m^2 °C) is held between two walls with base temperatures of 120°C and 20°C. What will be the fin heat-transfer rate?
(a) 2.0 W (b) 20.0 W (c) 200 W (d) 2.4 kW

4.19 A finned metallic mount is used to protect computer memory chips from overheating. If the chip temperature should not be more than 60°C and the ambient air temperature is 20°C, the rate of heat dissipation for the specified (UA) product of a typical finned mount of 0.15 W/K is:
(a) 3 W (b) 4 W (c) 5 W (d) 6 W

4.20 Two rods of metals A and B are positioned as shown below:



The thermal conductivities, specifies heats, densities and areas of cross-section of the rods are (k_A, k_B), (C_A, C_B) (ρ_A, ρ_B) and (A_A, A_B) respectively. Heat will then flow at the same rate in both rods if

(a) $\dfrac{k_A}{k_B} = \dfrac{A_B}{A_A}$ (b) $\dfrac{k_A}{k_B} = \dfrac{A_A}{A_B}$ (c) $\dfrac{k_A}{k_A} = \sqrt{A_B/A_A}$ (d) $\dfrac{k_A}{k_B} = \left(\dfrac{A_B}{A_A}\right)^2$

4.21 Extended surfaces are used to increase the rate of heat transfer. When the convective heat transfer coefficient, $h = mk$, an addition of extended surface will:
(a) Increase the rate of heat transfer.
(b) Decrease the rate of heat transfer.
(c) Not increase the rate of heat transfer.
(d) Increase the rate of heat transfer when the length of the fin is very large.

4.22 The ratio of heat-transfer area of the circumferential fins attached to a tube to the total heat-transfer area composed of the fin surface and the unfinned portion is 0.8. The individual fin efficiency is 90%. The area-weighted fin efficiency is
(a) 92% (b) 78.5% (c) 88.2% (d) 98%

4.23 Straight rectangular fins are mounted on a flat plate of 1 m^2 area and 6 mm thickness (k = 60 W/m K). The fin resistance is 0.025 K/W and the bare surface resistance is 0.15 K/W. The base surface temperature is 125°C and the ambient air temperature is 25°C. The rate of heat loss from the plate is
(a) 2275 W (b) 1600 W (c) 998 W (d) 4645 W

4.24 Which one of the following statements is correct?
(a) Fins should be attached on the side where heat-transfer coefficients are high.
(b) Effectiveness of fins depends on thermal conductivity only.
(c) Fins must have small thickness for better heat dissipation.
(d) In boiling heat-transfer appliances, fins will be very effective.

4.25 A fin will be necessary and effective only when
(a) k is small and h is large (b) k is large and h is also large
(c) k is small and h is also small (d) k is large and h is small

Answers

Multiple-Choice Questions

4.1 (d)	**4.2** (c)	**4.3** (c)	**4.4** (a)	**4.5** (d)	**4.6** (d)
4.7 (c)	**4.8** (c)	**4.9** (d)	**4.10** (a)	**4.11** (a)	**4.12** (b)
4.13 (a)	**4.14** (b)	**4.15** (b)	**4.16** (a)	**4.17** (b)	**4.18** (a)
4.19 (d)	**4.20** (a)	**4.21** (c)	**4.22** (a)	**4.23** (d)	**4.24** (c)
4.25 (d)					

REVIEW QUESTIONS

4.1 What accounts for the extensive use of fins in several heat-transfer applications? Name some application areas of fins.

4.2 Explain with neat sketches the different types of fins used in practice.

4.3 In the analysis of extended-surface heat transfer, under what conditions will the assumption of one-dimensional conduction be a good approximation?

4.4 When can a fin be considered of infinite length to yield a fairly accurate estimate heat loss from it?

4.5 Derive the equation for temperature distribution in a fin with an insulated tip. Then obtain an expression for heat dissipation by integrating the convective losses along its surface.

4.6 To compensate for the fact that there is a convective heat loss from the tip of a real fin, the corrected length L_c is used. Give the expression for L_c and the condition for its applicability.

4.7 Consider *three* boundary conditions: (a) very long fin, (b) fin with insulated tip, and (c) fin with convective tip. In which case will the fin-tip surface temperature be lowest and in which case, the highest?

4.8 Consider three pin fins made of (a) aluminium, (b) mild steel, and (c) glass under identical conditions. In which case will the fin-tip temperature be the lowest and in which case, the highest?

4.9 Define fin efficiency and state its importance in the design of fins. How does the *fin effectiveness* differ from the *fin efficiency*? What is the range of their possible values?

4.10 The fins attached to a surface are found to have an effectiveness of 0.8. Does it imply that the rate of heat transfer from the surface has decreased after providing fins?

4.11 A plate fin of 8 mm thickness is split into two 4 mm thick fins. Will it increase the fin heat-transfer rate?

4.12 In a steam condenser using water as a cooling liquid, fins are never used. Comment on this statement.

4.13 Show that the fin efficiency of a straight fin with an insulated end at the base (*with zero length*) is unity (i.e., 100%).

4.14 Heat is being transferred across a metal wall from a hot gas to a cold liquid. The gas-side heat-transfer coefficient is 50 W/m^2 K and the liquid-side coefficient is 500 W/m^2 K. It is desired to increase the heat-transfer rate. However, it is not possible to alter the values of the heat-transfer coefficients on either side. It is, therefore, proposed to provide fins. On which side should the fins be provided and why?

4.15 Heat is transferred from hot water flowing through a pipe to the air flowing across the pipe. To increase the heat-transfer rate, should fins be provided on the internal or external surface of the pipe?

4.16 Is it better to use a large number of closely spaced thin fins or a small number of thick fins?

4.17 Will an increase in the (a) height (or length), (b) thickness (or diameter), (c) thermal conductivity, and (d) heat-transfer coefficient of a fin increase or decrease the *fin effectiveness* and *fin efficiency*?

4.18 Can fins prove not only not effective but also counterproductive? Discuss.

4.19 The heat-transfer surface area of a fin is equal to the sum of all surfaces of the fin exposed to the surrounding fluid, including the surface area of the fin tip. Under what conditions can we ignore heat transfer from the fin tip?

4.20 Explain the importance of insulated tip solution for the fins used in practice.

4.21 A fin can be manufactured as an integral part of the prime surface or it can be separately brazed or adhered to the base surface. Which alternative would you prefer and why?

4.22 What is *fin resistance* and what are its units?

4.23 What is the difference between the *overall effectiveness* of a finned surface and the efficiency of a single fin?

4.24 What considerations should weigh with you in the design and selection of the fins? Discuss the factors considered in the optimum design of fins.

4.25 Define (a) *the finning factor*, and (b) the *area-weighted fin efficiency*.

4.26 A thin rod of length L has its two ends connected to two reservoirs at temperatures T_1 and T_2 respectively. The rod has the perimeter P and cross-sectional area A. The rod loses heat to the environment at the temperature T_∞. The convection coefficient of heat transfer between the rod and the environment is h. Assuming the thermal conductivity of the material of the rod to be constant k, derive an expression for (a) the temperature distribution in the rod, and (b) the heat transfer from the rod to the environment.

4.27 Derive an expression for the overall heat-transfer coefficient across a plane wall (thickness b, thermal conductivity k) having rectangular fins on both sides. Given that over an area A of the wall, the bare areas on the two sides not covered by the fins are A_{b_1} and A_{b_2}, the surface areas of the fins are A_{f_1} and A_{f_2}, the fin efficiencies are η_{f_1} and η_{f_2}, and the heat-transfer coefficients are h_1 and h_2.

4.28 Discuss the estimation of error in temperature measurement with the help of a thermometer well. Why in many applications is a thermometer well (pocket) fixed inclined to the wall?

4.29 Define thermometric error and mention the steps necessary to reduce the error in the measurement of temperature of a fluid flowing through a duct by means of a thermometer well.

PRACTICE PROBLEMS

(A) Very Long Fin

4.1 Estimate the energy input required to solder together two very long pieces of bare copper wire 1.6-mm diameter with a solder that melts at 196°C. The wires are positioned horizontally in air at 25 °C and the heat transfer coefficient on the wire surface is 17 W/m^2 K. The thermal conductivity of the wire alloy is 337 W/m K. **[1.30 W]**

4.2 The end of a very long cylindrical stainless steel rod is attached to a heated wall and its surface is in contact with a cold fluid. (a) If the rod diameter were doubled, by what percentage would the rate of heat removal increase? (b) If the rod were made of aluminium, by what percentage would the heat transfer rate change from that of the stainless steel? k_{steel} = 16.17 W/m K, $k_{\text{aluminium}}$ = 204.7 W/m K **[(a) 300% (b) 1166%]**

(B) Fins of Finite Length

4.3 Consider an array of 8 pin fins used to cool a transistor. They are black anodized aluminium fins 3-mm wide, 0.4-mm thick, and 5-cm long. When the bases of the fins are at 60°C and the ambient air is at 35°C, how much power do they dissipate? Take h = 8 W/m^2 K and k = 200 W/m K. **[0.46 W]**

4.4 A saucepan of 20-mm diameter is exposed to 99°C temperature during a cooking process. The temperature of air in the kitchen is 25°C and the convection coefficient between the handle surface and the ambient air is 8 W/m^2 K. The thermal conductivity of the handle material is 17 W/m K. The temperature in the last 10-cm of the handle used for hand grip should not exceed 35°C. Assuming heat transfer from the fin tip to be negligible, estimate the length of the saucepan handle. **[32 cm]**

4.5 An electronic semi-conductor device generates 16×10^{-2} kJ/h of heat of heat. To keep the surface temperature at an upper safe limit of 75 °C, it is desired that the generated heat should be dissipated to the surrounding environment which is at 30 °C. This can be achieved by attaching aluminium fins 0.5-mm-square protruding 10 mm from the surface. Estimate the number of fins. Thermal conductivity of the fin material is 700 kJ/m h °C. The heat transfer coefficient is 45 kJ/m^2 h °C. Neglect the heat loss from the tip of the fin. **[4]**

(C) Number of Fins on Plane Wall

4.6 Circular cross-section studs of radius 12 mm, length 90 mm, and thermal conductivity 25 W/m K are attached to a flat surface with their axes perpendicular to the surface on a square pitch of 3.0 cm. The primary surface is at 200°C. A fluid at 25°C is forced across the surface such that the average heat transfer coefficient is 135 W/m^2 K. Determine the rate of heat dissipation per unit area of the studded surface. Assume that the heat transfer coefficient is the same for the primary surface and for the rod surfaces. **[77.3 kW /m^2]**

4.7 In order to reduce the thermal resistance at the surface of a vertical plane wall (50 × 50 cm), 100 pin fins (1 cm diameter, 10 cm long) are attached. If the pin fins are made of copper having a thermal conductivity of 300 W/m K and the value of the surface heat transfer coefficient is 15 W/m^2 K, calculate the decrease in the thermal resistance. Also calculate the consequent increase in the heat transfer rate from the wall if it is maintained at a temperature of 200 °C and the surroundings are at 30 °C. Assume heat transfer from the tip is negligible. **[731.5 W]**

(D) Bar with Two Ends At Specified Temperatures

4.8 (a) A straight fin of uniform cross-section A_c, length L, perimeter P, thermal conductivity k is maintained at a temperature, above the ambient fluid, of θ_o at the end where $x = 0$ and at θ_L at the end where $x = L$. The convective heat transfer coefficient at the exposed surface is h. Derive the following expressions for the heat flow rate at the two ends (*positive from x = 0 to x = L*):

$$\dot{Q}_o = mkA_c \frac{\theta_o \cosh mL - \theta_L}{\sinh mL} \quad \dot{Q}_L = mkA_c \frac{\theta_o - \theta_L \cosh mL}{\sinh mL}$$

What will be the net heat transfer from the fin?

(b) A fin of diameter 12.5 mm, and a length of 30 cm has its left end maintained at 225°C, the right end at 75°C, and the ambient fluid is at 25°C. For a conductivity of 45 W/m °C and a convection coefficient of 25 W/m^2 °C, determine the rate of heat flow out of each source and the total heat loss from the fin. **[(a) 14.6 W, –3.15 W (b) 17.75 W]**

4.9 A carbon steel rod (k = 56.7 W/m °C), 5-cm diameter, is installed as a structural support between two surfaces that are at a temperature of 200 °C. The length of the rod exposed to 25 °C air is 1.2 m long. The unit surface conductance (convection heat transfer coefficient) is 30 W/m^2 °C. Determine the total rate of heat transfer from the bar to the surrounding air. Sketch the temperature profile and find the location and magnitude of minimum temperature. **[253.3 W, 0.6 m, 32.056°C]**

4.10 The top of a 30-cm I-beam made of steel is maintained at a temperature of 350°C, while the bottom is at 150°C. The thickness of the web is 1.25 cm. The thermal conductivity of steel is 52 W/m K and the heat transfer coefficient when air at 350°C is blowing along the side of the beam is 25 W/m^2°C. Determine the temperature distribution along the web from top to bottom and plot the results.

x **(cm)**	***T*****(°C)**
5	**350.0**
10	**337.0**
15	**321.2**
20	**300.0**
25	**268.8**
30	**221.8**
35	**150.0**

4.11 Two vessels are kept at 132°C and 49°C respectively, in a room at an air temperature of 19°C. The vessels are connected by 4 non-insulated bolts, each 1.27 cm in diameter, and 30.48 cm long. The thermal conductivity of the nickel steel is 41.5 W/m °C and the surface heat transfer coefficient between the rod and the ambient air is 5.68 W/m^2 °C. Calculate how much heat energy per hour is conducted away from the warmer vessel and how much is supplied to the colder vessel by the connecting rods? **[54.0 kJ/h, 0.0 kJ/h]**

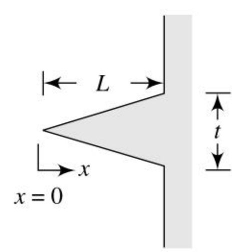
(E) Circumferential (*Annular*) Fins

4.12 Short integral radial fins of 0.3-mm thickness and 7-mm length with a spacing of 3-mm are arranged on a brass tube (k = 110 W/m °C) with inner and outer radii of 3-mm and 3.5-mm. Hot water at 90°C flows inside the tube with a convection coefficient of 340 W/m^2 °C. The fins and the external surface of the tube are exposed to ambient air at 30°C with a convection coefficient of 40 W/m^2 °C. Calculate the rate of heat transfer per metre tube length. **[212 W]**

4.13 To increase the heat dissipation from a 3-cm-*OD* tube, circumferential aluminium fins of rectangular profile (k = 205 W/m °C) are soldered to the outer surface. The fins are 2-mm thick and have an outer diameter of 6-cm. The tube surface temperature is 200°C and the ambient air temperature is 35°C. The average heat transfer coefficient is 100 W/m^2 °C. Calculate, the rate of heat loss per fin, and the fin efficiency. **[76.3 W, 73.3 W]**

(F) Triangular Fins

4.14 A triangular fin made up of stainless steel (k = 18.9 W/m K) is attached to a surface maintained at 440°C. The fin thickness is 6 mm, length 25 mm and is exposed to a convective environment at 90°C with associated heat transfer coefficient of 30 W/m^2 K. Calculate the heat loss from the fin. **[455 W]**



4.15 Derive a differential equation for the temperature distribution in a straight triangular fin of length L and base width t. For convenience take the coordinate axis as shown in the sketch below and assume one-dimensional heat flow. Solve the equations assuming you know the base temperature, the environmental temperature, and the heat transfer coefficient, and present your results in dimensionless coordinates

$$\left[\textbf{(a)}\; x\frac{d^2T}{dx^2} + \frac{dT}{dx} - \frac{2hL\sqrt{4L^2+t^2}}{kt^2}(T - T_\infty) = 0 \quad \textbf{(b)} \quad = \frac{I_0(2\sqrt{K\xi})}{I_0\,(2\sqrt{K})} \right]$$

(G) Error In Temperature Measurement

4.16 A mercury thermometer is placed in a well filled with oil is required to measure the temperature of compressed air flowing in a pipe. The well is 140-mm long and is made of steel (k = 58 W/m K), 1.0-mm thick. The temperature recorded by the well is 100°C while the pipe wall temperature is 50°C. The air film conductance outside the wall may be assumed to be 29 W/m^2 K. Estimate the true temperature of air. **[104.78°C]**

4.17 A thermometric pocket filled with oil is used for measurement of temperature of air flowing inside a pipe. The thickness of well is 2.0 mm and it is 150-mm-long. The temperature recorded by mercury thermometer dipped in well is 80°C, whereas the pipe wall temperature is 40°C. Estimate the true temperature of air and the percentage error in the measurement of temperature. Take air film conductance outside wall = 30 W/m^2 K and conductivity of well material = 60 W/m K. **[89°C, 10.2%]**

4.18 Air temperature in the receiver of a compressor unit is measured with a mercury thermometer placed in a steel thermometer well filled with oil. The thermometer well is 140-mm-long and the thickness of the wall of the well is 1 mm. The thermometer shows the temperature at the end of the well which is lower than the air temperature owing to the transfer of heat along the well. Find the error in the temperature measurement if the thermometer indicates a temperature of 100°C. The temperature at the base of the well is 50°C. Take the thermal conductivity of the metal as 50 W/m K and the compressed air-to-well heat transfer coefficient as 25 W/m^2 K. **[+ 4.8°C]**

5 Unsteady-State Heat Conduction

5.1 ❑ INTRODUCTION

In the earlier chapters, we discussed steady-state heat conduction in solids in which temperature within the body varied with *position* but not with *time*. In this chapter, we will focus our attention on *time-dependent* heat-transfer problems which are typically encountered when the system boundary conditions are changed. In such cases of *unsteady* or *transient* conduction, temperature variation is a function of both *time* and *location*. When the surface temperature of a body is altered, the temperature at each point in the body will also start changing. It takes quite some time before the *steady state* temperature distribution is attained and the temperature no longer changes with the lapse of time. All the process equipment used in engineering practice, such as *boilers, heat exchangers, regenerators*, etc., have to pass through an unsteady state. Eventually, steady state is reached after sufficient time has passed.

Solutions to unsteady-state problems are often difficult because of the additional independent variable of time, i.e., $T = f(x, y, z, t)$. In the Cartesian coordinate system, the transient heat-conduction equation with constant thermal conductivity is

$$\boxed{\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\overline{q}}{k} = \frac{\rho C_p}{k}\frac{\partial T}{\partial t}} \tag{5.1}$$

In one-dimensional transient problems, without heat generation, $T = f(x, t)$ and the governing differential equation is given by

$$\boxed{\frac{\partial^2 T}{dx^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \text{ where } \alpha = \frac{k}{\rho C_p} \tag{5.2}$$

The solution of this equation requires *three* conditions to be satisfied: *Two* boundary conditions (*thermal boundary conditions at the surfaces of the body*) and *One* initial condition (*initial temperature distribution within the body*).

Many methods are available for determining the temperature distribution in a solid during a *transient* process.

- ***Lumped-Capacity Method*** Simpler approach using the *lumped-capacity model* in which *internal temperature gradients* are ignored and temperature is a function of *time alone.*

- ***Analytical Methods*** When internal temperature gradients cannot be neglected, analytical solutions are preferred. These techniques are, however, quite complex and involved, especially pertaining to *two-and three-dimensional* transient conduction problems.

- ***Graphical Techniques*** The transient temperature charts for simple geometries like a *plane wall, a long cylinder, a sphere and a semi-infinite medium* are available. The product solution for multidimensional configurations can also be obtained.
- ***Numerical Techniques*** These include finite difference method and finite element method. Realistic boundary conditions can be handled easily using numerical methods.

5.2 ❑ TYPES AND APPLICATIONS OF TRANSIENT CONDUCTION

Transient conduction may be broadly classified into two categories: (a) *periodic,* and (b) *non-periodic*

Periodic In *periodic* heat-flow systems, the temperature within the system or body goes through *periodic* changes which are *cyclic* in nature and may be *regular* or *irregular*. *Periodic* temperature changes at a surface occur in many applications, for example, (1) Annual or diurnal (daily) temperature variation of buildings. (2) The ground exposed to solar radiation. (3) Heat flow through the cylinder wall of an internal combustion engine. (4) Cyclic temperature fluctuation in the wall of a furnace whose heating element is turned on and off at regular intervals.

Non-Periodic In the *non-periodic* heat-flow systems, *the temperature at any location within the system varies with time in a non-linear* fashion. Such effects are manifest in several engineering applications such as (1) Heat treatment of metals, (2) Electric irons, (3) Boiler tubes, (4) Cooling and freezing of perishable foodstuff, (5) Rocket nozzles, (6) Space re-entry vehicles, (7) Nuclear reactor components, and (8) Other industrial heating or cooling processes.

5.3 ❑ LUMPED-CAPACITY ANALYSIS

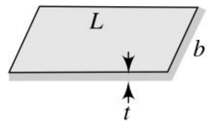
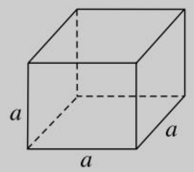
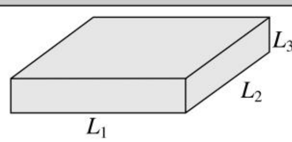
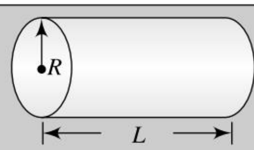
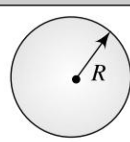
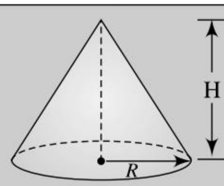
In many cases, the temperature gradients within the solid are rather negligibly small (*i.e., the internal resistance to conduction is negligible*). Heat is transferred by *conduction* from within the body to the surface and then, from the surface to the medium by *convection*. When the body is very small (of slender shape) or when the thermal conductivity of the material of the body is very large, temperature gradients within the body will be negligibly small. In such a case, the temperature within the body is only a function of time and is independent of spatial coordinates. The entire body acts as a lump and the temperatures at all points within the body decrease (*or increase if the object is being heated*) uniformly. Heat-transfer process from the body, in this case, is controlled by the convection resistance at the surface rather than by the conduction resistance in the solid. Such an analysis, where the internal resistance of the body for heat conduction is negligible is known as *lumped-capacity analysis.*

The analysis of *lumped systems* in which the temperature varies with time only and remains uniform (*spatially isothermal*) throughout at any time provides great convenience and simplification in several cases of transient heat conduction problems with little sacrifice in accuracy. Hence, it would be appropriate to establish the suitable criterion for the applicability of this analysis.

To fix the criterion for which lumped-system analysis is applicable, let us define Biot number, in general, as follows: $Bi = \frac{h L_c}{k}$ where, h is the heat-transfer coefficient between the solid surface and the surroundings, k is the thermal conductivity of the solid, and L_c is a characteristic length defined as the ratio of the volume of the body to its surface area, $\forall / A_s$.

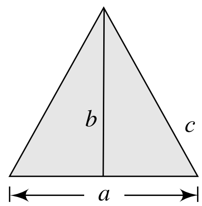
Table 5.1 gives the characteristic length for some well-known geometries.

Table 5.1 *Characteristic length for different geometries in lumped capacity formulation*

Geometry	Schematic	Characteristic length, $L_c = \dfrac{\forall}{A_s}$
Large plate of thickness, t		$\forall = Lbt$, $A_s = 2\,Lb$ L_c = Half-thickness, $t/2$
Cube of side, a		$\forall = a^3$ $A_s = 6a^2$ $L_c = a/6$
Rectangular parallelepiped		$L_c = \dfrac{\forall}{A_s} = \dfrac{L_1L_2L_3}{2[L_1L_2 + L_2L_3 + L_1L_3]}$
Cylinder		$\forall = \pi R^2 L$ $A_s = 2\pi R^2 + 2\pi RL$ $L_c = \dfrac{RL}{2(R+L)}$
Long cylinder ($R << L$)		$\forall = \pi R^2 L$ $A_s = 2\pi RL$ $L_c = R/2$
Thin shell		$\forall = (\pi D t)$, $A_s = 2\pi DL$ L_c = Half-thickness, $t/2$
Sphere		$\forall = \dfrac{4}{3}\pi R^3$, $A_s = 4\pi R^2$ $L_c = R/3$
Right circular cone		$\forall = \dfrac{1}{3}\pi R^2 H$ $A_s = \dfrac{1}{2}(2\pi R)\sqrt{R^2 + H^2}$ $L_c = \dfrac{RH}{3\sqrt{R^2 + H^2}}$

contd.

contd.

Triangle	 $c = \sqrt{\left(\frac{a}{2}\right)^2 + b^2}$	$V = \left(\frac{1}{2}ab\right)L$ $A_s = \left(\frac{1}{2}ab\right) \times 2 + (cL) \times 2$ $L_c = V/A_s$

With this definition of Bi and L_c, for solids such as a *plane wall* (*infinite slab*), a *long cylinder* and a *sphere*, it is found that the transient temperature distribution within the solid at any instant is uniform, with the error being less than about 5%, if the following criterion is satisfied:

$$\boxed{Bi = \frac{h L_c}{k} < 0.1}$$

In other words, if the conduction resistance of the body is less than 10% of the convective resistance at its surface, the temperature distribution within the body will be uniform within an error of 5%, during transient conditions.

Biot number is a measure of the temperature drop in the solid relative to the temperature drop in the convective layer. It is also interpreted as the ratio of conduction resistance in the solid to the convection resistance at its surface. This is precisely the criterion we are looking for. It suggests that one can assume a uniform temperature distribution within the solid if $Bi << 1$.

5.4 ❑ CRITERION FOR NEGLECTING INTERNAL TEMPERATURE GRADIENTS

The governing differential equation for one-dimensional transient heat conduction without internal heat generation is given by

$$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{5.3}$$

where $T = f(x, t)$ and $T = T_i$, at $t = 0$

Let, $\theta(x, t) = \theta = (T - T_\infty)$

Then, $$\boxed{\alpha \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}} \tag{5.4}$$

Assuming the product solution as, $\theta(x, t) = \theta = F(x)\, G(t)$ (5.5)

Substituting in Eq. (5.3), we get

$$\alpha F''(x) G(t) = F(x) G'(t) \text{ or } \frac{F''(x)}{F(x)} = \frac{1}{\alpha}\frac{G'(t)}{G(t)}$$

Introducing a separation constant λ as

$$\frac{F''(x)}{F(x)} = \frac{1}{\alpha}\frac{G'(t)}{G(t)} = \pm\lambda^2 \qquad \text{(a separation constant)}$$

The function $G(t)$ must decay exponentially with time, therefore, for $\lambda^2 < 0$ (i.e., $-\lambda^2$), the solutions are

$$F(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$$

$$G(t) = C_3 \exp(-\alpha\lambda^2 t)$$

Substituting, these solutions in Eq. (5.5), we have

$$\theta = \{C_1 \cos(\lambda x) + C_2 \sin(\lambda x)\}[C_3 \exp(-\alpha\,\lambda^2 t)]$$

Introducing, new constants A and B as

$A = C_1C_3$ and $B = C_2C_3$, we have

$$\theta = \{A \cos(\lambda x) + B \sin(\lambda x)\} \exp(-\alpha\lambda^2 t) \tag{5.6}$$

Three unknowns, A, B, and λ are determined from the following boundary conditions:

- At $x = 0, \dfrac{d\theta}{dx} = 0$
- At the surfaces of the wall, i.e., at $x = L$, $-k\dfrac{d\theta}{dx} = h(\theta_{x=L})$

Differentiating Eq. (5.6) with respect to x, we get

$$\frac{d\theta}{dx} = \{-A \sin(\lambda x) + B \cos(\lambda x)\}(\lambda)\exp(-\alpha\lambda^2 t)$$

Applying the first boundary condition, at $x = 0$, we have

$$\frac{d\theta}{dx} = 0 = \{-A \sin \lambda(0) + B \cos \lambda(0) + B \cos \lambda(0)\}\ (\lambda) \exp(-a\ \lambda^2 t)$$

$\therefore \qquad B = 0$

Equation (5.6) reduces to

$$\theta = \{A \cos(\lambda(x)\} \exp(-\alpha\ \lambda^2 t)$$

Applying the second boundary condition at $x = L$, one has

$$-k\{[-A \sin(\lambda L)](\lambda)\exp(-\alpha\lambda^2 t) = h\{A \cos(\lambda L) \exp(-\alpha\,\lambda^2 t)\}$$

or
$$\cot(\lambda L) = \frac{\lambda k}{h} = \frac{\lambda L}{(hL/k)} = \frac{\lambda L}{Bi} \Rightarrow \boxed{\cot \lambda L = \frac{\lambda L}{Bi}} \tag{5.7}$$

which is a *transcendental equation* with an infinite number of roots or *eigenvalues*. For a given value of the Biot number, the eigenvalues (n = 1, 2, 3…) can be calculated.

The solution is in the following form of infinite series:

$$\theta = \sum_{n=1}^{\infty} \{A_n \cos \lambda_n x\} \exp(-\alpha\,\lambda_n^2 t)$$

At the initial condition,

$$t = 0,\ \theta_i = \sum_{n=1}^{\infty} \{A_n \cos(\lambda_n x)\}$$

For $B = 1$, at $x = L$, i.e., at the boundary surface,

$$\theta_s = A_1 \cos(\lambda_1 L)$$

At the centre plane, i.e., at $x = 0$, $\theta_0 = A_1$.

Hence, the dimensionless temperature distribution becomes, $\dfrac{\theta_s}{\theta_0} = \cos(\lambda_1 L)$

For internal temperature gradients within 5%, i.e., the temperatures within the solid varying less than 5 per cent (*negligible*), one gets

$$\frac{\theta_s}{\theta_0} \geq (1 - 0.05) \quad \text{or} \quad \frac{\theta_s}{\theta_0} \geq 0.95$$

or $\qquad \cos(\lambda_1 L) \geq 0.95 \quad \text{or} \quad \lambda_1 L \geq 0.3175$ radians

Substituting in Eq (5.7), we have

$$\cot(0.3175) = \frac{0.3175}{Bi} \quad \text{or} \quad \boxed{Bi = 0.1} \tag{5.8}$$

Thus, when the Biot number is less than 0.1, the internal temperature gradients within the solid can be ignored. The temperature inside the solid is then essentially the same at all positions and changes only with time.

5.5 ❑ TRANSIENT CONDUCTION ANALYSIS WITH NEGLIGIBLE INTERNAL TEMPERATURE GRADIENTS

Consider an arbitrarily shaped body that is initially at a uniform high temperature T_i, as shown in Figure 5.1.

The body is suddenly immersed in an ambient cold fluid at temperature T_∞.

All the mass within the solid body is *lumped* together and assumed to be at the same temperature. The temperature of the body varies with *time* but *not* with *location*. Hence, at any moment, the body is *spatially isothermal*.

With this approximation, we can now apply the energy balance. In the rate form,

$$\cancelto{0}{\dot{E}_{\text{in}}} - \dot{E}_{\text{out}} + \cancelto{0}{\dot{E}_{\text{gen}}} = \dot{E}_{\text{st}} \quad \Rightarrow \quad \dot{E}_{\text{out}} = -\dot{E}_{\text{st}}$$

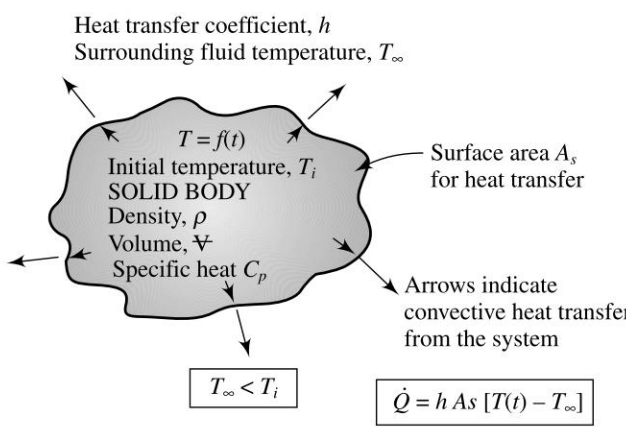


Fig. 5.1 *In a lumped system, the body temperature depends only on time since internal temperature gradients are neglected.*

We assume the entire mass of the body (*including its surface*) is at a uniform temperature $T(t)$ or T at any instant. Convective heat transfer is $\dot{E}_{\text{out}} = hA_s(T - T_\infty)$ and the rate of decrease of internal energy is $-\dot{E}_{\text{st}} = -mC_p \dfrac{dT}{dt}$

It follows that

$$\boxed{hA_s(T - T_\infty) = -mC_p \frac{dT}{dt}} \tag{5.9}$$

where T_∞ is the temperature of the fluid.

To solve this differential equation, we separate the variables so that

$$\frac{dT}{(T - T_\infty)} = -\frac{hA_s}{mC_p}dt$$

Integrating from an initial temperature, T_i, at $t = 0$ to a final state $T(t)$, at time t, we have

$$\int_{T_i}^{T(t)} \frac{dT}{T - T_\infty} = -\frac{hA_s}{mC_p}\int_0^t dt \quad \text{or} \quad \ln\left[\frac{T(t) - T_\infty}{T_i - T_\infty}\right] = -\frac{hA_s}{mC_p}(t - 0) = -\frac{hA_s}{mC_p}t$$

or
$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{hA_s}{mC_p}t\right)$$

Noting that $m = \rho \forall$ and $L_c = \forall / A_s$, we get

$$\boxed{\frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\left[-\frac{hA_s}{\rho \forall C_p}t\right] = \exp\left[\frac{-h}{\rho C_p L_c}t\right]} \tag{5.10}$$

Let,
$$a \equiv \frac{hA_s}{\rho \forall C_p} \quad \text{or} \quad \frac{h}{\rho C_p L_c}$$

Then,
$$\boxed{\frac{\theta(t)}{\theta_i} = \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-at}} \tag{5.11}$$

Figure 5.2 illustrates this exponential behaviour when the ratio $\theta(t)/\theta_i$ is plotted against time t for different values of a.

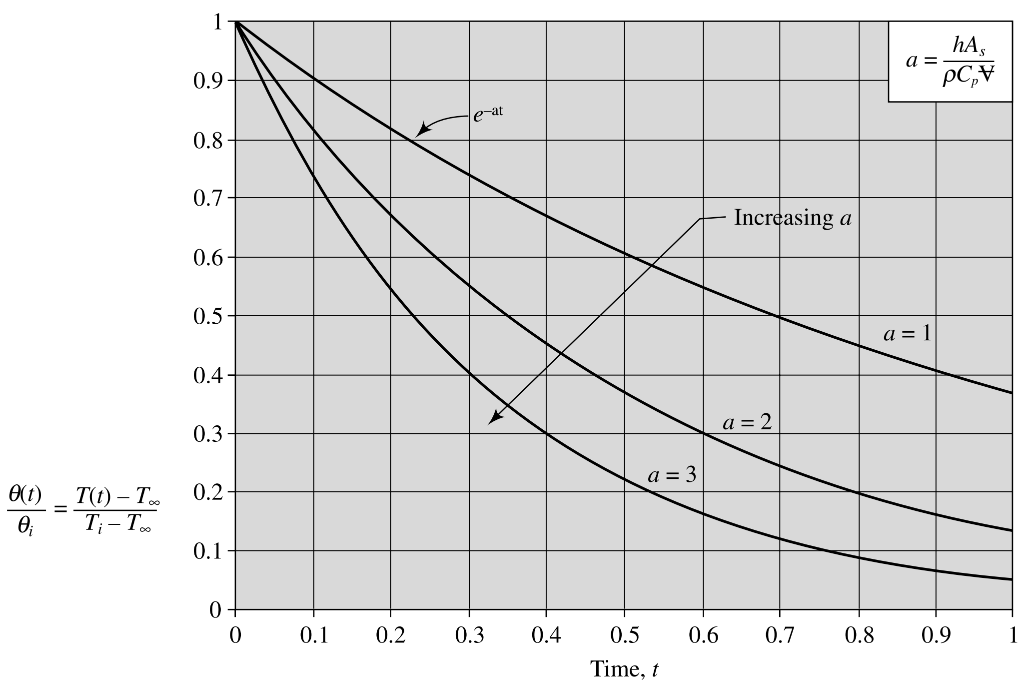


Fig. 5.2 *Dimensionless temperature $\theta(t)/\theta_i$ exponentially decaying as a function of time.*

Note that both sides of Eq. (5.11) are *non-dimensional*. This equation is applicable to either a hot solid immersed in a cold fluid or to a cold solid immersed in a hot fluid. Both $\theta(t)$ and θ_i are *positive* if the body is *cooled* and *negative* if the body is *heated*. The ratio $\theta(t)/\theta_i$ is however always a positive dimensionless number in the range of 0 to 1.

Solving for the time, t, to reach a specified temperature $T(t)$, we have from Eq. (5.11),

$$\boxed{t = -\frac{\rho C_p L_c}{h} \ln\left[\frac{T(t) - T_\infty}{T_i - T_\infty}\right] \qquad Bi \lesssim 0.1} \tag{5.12}$$

Furthermore, Eq. (5.12) can be rearranged to give the body temperature after an elapsed time, t, as

$$\boxed{T(\mathrm{t}) = (T_i - T_\infty)\exp\left[\frac{-h}{\rho C_p L_c} t\right] + T_\infty \quad Bi \lesssim 0.1} \tag{5.13}$$

• Thermal Time Constant and Fourier Number

The quantity $(\rho C_p L_c/h)$ or $1/a$ is called the thermal *time constant* of the system, denoted by τ and it controls the transient behaviour of the body, Eq. (5.11) can now be expressed as

$$\boxed{\frac{\theta(t)}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-t/\tau)} \tag{5.14}$$

As shown in Fig. 5.3, in both *heating and cooling*, we find that initially the temperature changes steeply and then later approaches the fluid temperature asymptotically. After about *five* time constants, the body temperature will have essentially reached a steady-state value. If we multiply the numerator and denominator of the exponent in Eq. (5.13) by $(L_c k)$ and re-arrange the variables, $\dot{m}$ we get

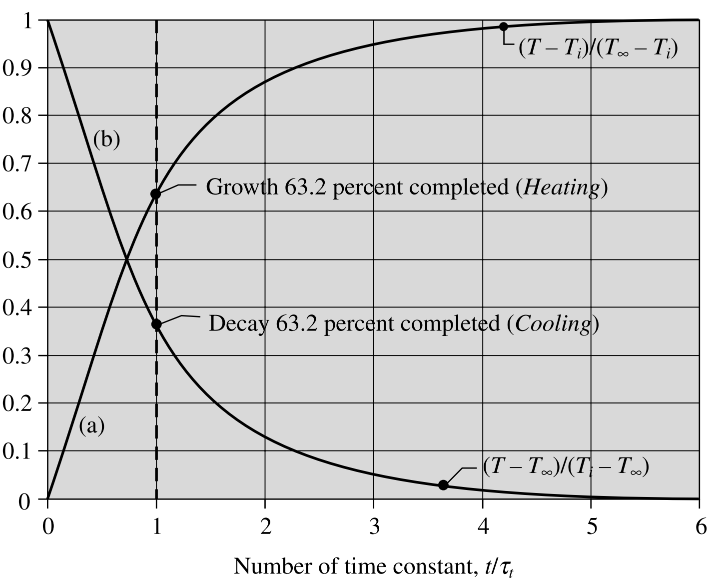


Fig. 5.3 *(a) Exponential heating and (b) exponential cooling as a function of number of time constants*

$$\frac{ht}{\rho C_p L_c}\left[\frac{L_c k}{L_c k}\right]=\underbrace{\left[\frac{hL_c}{k}\right]}_{Bi}\underbrace{\left[\frac{k}{\rho C_p}\frac{t}{L_c^2}\right]}_{Fo} \tag{5.15}$$

The *first factor* on the right-hand side is of course the non-dimensional *Biot number, Bi* = hL_c/k. Physically, it represents a ratio of *internal* (*conduction*) and *external* (*convection*) thermal resistances. The *second factor* is also non-dimensional and is called the *Fourier number,* $Fo = \alpha t/L_c^2$ where α is the *thermal diffusivity* of the material. Physically, the *Fourier number* represents a ratio of the rate at which heat is *conducted* across a body to the rate at which heat is *stored* within the body.

We can now rewrite Eq. (5.12) as

$$\frac{T(t)-T_\infty}{T_i-T_\infty}=\exp(-BiFo)$$

or

$$\boxed{\ln\left\{\frac{T(t)-T_\infty}{T_i-T_\infty}\right\}=-BiFo} \tag{5.16}$$

Thus, the dimensionless temperature can be expressed in terms of the product of two dimensionless numbers, *Bi* and *Fo*.

Equation (5.15) is represented graphically in Figure 5.4.

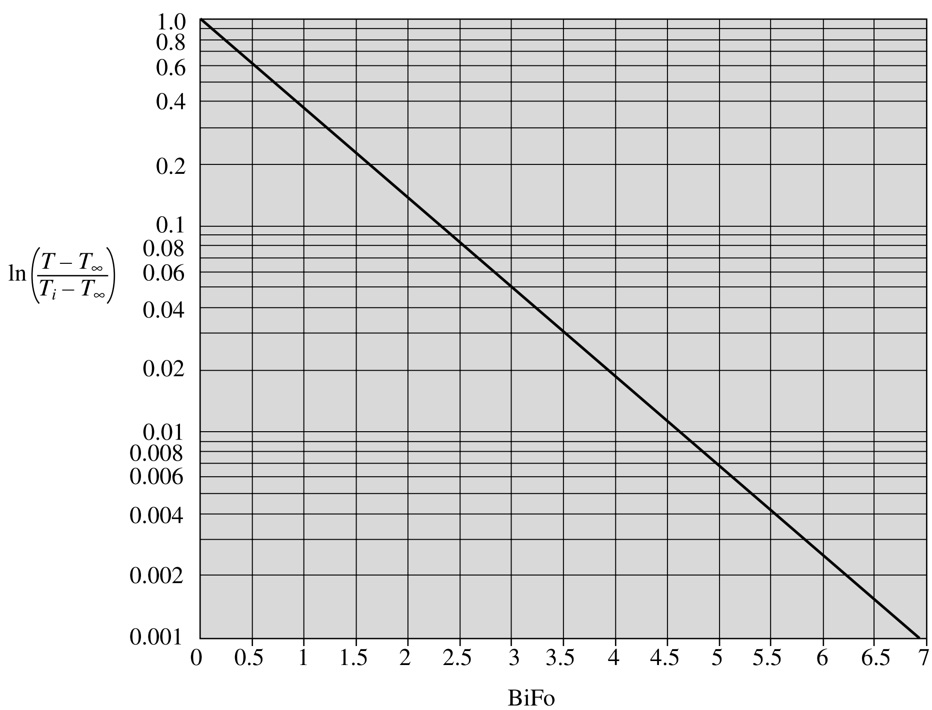


Fig. 5.4 *Temperature variation in a lumped-parameter system without internal heat generation.*

5.6 ❑ INSTANTANEOUS HEAT-TRANSFER RATE AND TOTAL (*CUMULATIVE*) AMOUNT OF HEAT TRANSFER

Instantaneous heat-transfer rate at any time t is given by

$$\dot{Q} = hA_s[T(t) - T_\infty]$$

But, $$\frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\left[-\frac{hA_s t}{\rho ¥ C_p}\right]$$

Therefore, $$\dot{Q} = hA_s(T_i - T_\infty)\exp\left[-\frac{hA_s}{\rho C_p ¥}t\right]$$

or $$\boxed{\dot{Q} = hA_s(T_i - T_\infty)\exp(-t/\tau)} \tag{5.17}$$

or $$\boxed{\dot{Q} = hA(T_i - T_\infty)\exp[-(Bi)(Fo)]} \tag{5.18}$$

In some situations, the change in internal energy ΔU of the system is important. This simply equals the quantity of heat transferred Q during a time interval (0 to t).

$$Q = \int_0^t \dot{Q}dt = hA_s\int_0^t [T(t) - T_\infty]dt$$

But, $$T(t) - T_\infty = (T_i - T_\infty)\exp-\left[-\frac{hA_s}{\rho C_p ¥}t\right] \text{ and } \boxed{\tau = \frac{\rho C_p ¥}{hA_s}}$$

Simplification and integration gives

$$\therefore \quad Q = hA_s(T_i - T_\infty)\int_0^t e^{-t/\tau}dt = hA_s(T_i - T_\infty)(-\tau)[e^{-t/\tau}]_0^t$$

$$= -hA_s\tau(T_i - T_\infty)[e^{-t/\tau} - e^0] = hA_s\tau(T_i - T_\infty)[1 - \exp(-t/\tau)]$$

or $$\boxed{Q = \rho C_p ¥ (T_i - T_\infty)[1 - \exp(-t/\tau)]} \tag{5.19}$$

or $$\boxed{Q_{max} = \rho C_p ¥[1 - \exp(-\infty/\tau)](T_i - T_\infty)} \tag{5.20}$$

If time t is allowed to go to infinity (∞), we obtain a situation corresponding to steady state. Then the cumulative heat transfer or total change in the internal energy of the lumped parameter system that the body would experience in going from the initial temperature T_i to the temperature T_∞ of the environment in which it is placed will be maximum and is given by substituting, $t = \infty$ in Eq. (5.20).

$$Q_{max} = \rho C_p ¥[1 - \exp(-\infty/\tau)](T_i - T_\infty) \qquad [\text{since } e^{-\infty/\tau} = 0]$$

With $\rho ¥ = m$, we have

$$\boxed{Q_{max} = mC_p(T_i - T_\infty)} \quad \text{(J)} \tag{5.21}$$

5.7 ❑ ELECTRICAL ANALOGY: THE *RC* CIRCUIT

The transient behaviour of a lumped-capacity solid is *analogous* to the voltage decay which occurs when a capacitor is discharged through a resistor in an electrical *RC* circuit. The process can thus be represented by the *equivalent thermal circuit* shown in Figure 5.5. With the switch is closed, the capacitor is initially charged to the temperature T_i. When the switch is suddenly opened, the capacitor discharges, through the thermal resistor, the energy that is stored in the solid. As a result, the temperature of the solid *drops* or *decays* with time.

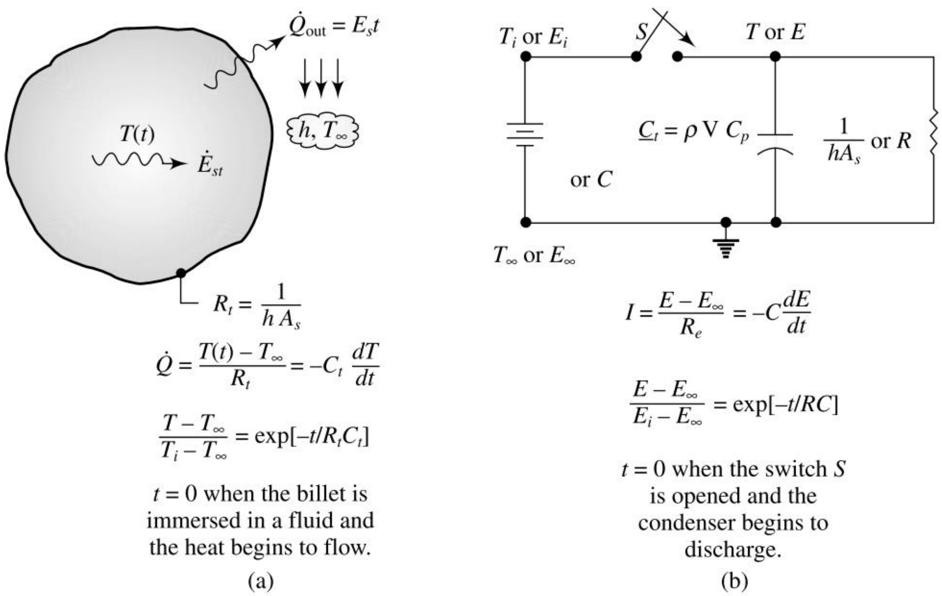


Fig. 5.5 *Equivalence of electrical and thermal circuits in lumped system analysis*

The thermal time constant is analogous to electrical time constant, *RC* and can be expressed as

$$\boxed{\tau = \left(\frac{1}{hA_s}\right)(\rho C_p \forall) = R_t C_t} \quad \text{(s)} \tag{5.22}$$

where R_t represents the *thermal resistance* of the solid and C_t is the *lumped thermal capacity (capacitance)* of the solid. Any increase either in R_t or C_t will increase the *time constant* which will cause a solid to respond slowly to changes in the thermal environment.

5.8 ❑ TEMPERATURE RESPONSE OF A THERMOCOUPLE

One of the applications of the lumped-capacity model is in the case of measurement of temperature by a thermometer or a thermocouple. It is very important that the thermocouple indicates the source temperature as quickly as possible. If the thermocouple is measuring unsteady temperatures then, it should follow the temperature changes at a rate faster than the rate of temperature change. The temperature response of a thermocouple (or thermometer) is defined as the time required for it to attain the source temperature after being exposed to it.

Let us recall that

$$\boxed{\frac{\theta}{\theta_i} = \frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\left(\frac{-hA_s t}{\rho C_p \forall}\right)} \qquad \text{(if } Bi < 0.1\text{)}$$

The parameter group $(\rho C_p \forall)/(hA_s)$ has dimensions of time and can be interpreted as *time constant*, τ. At $t = \tau$, i.e., at a time interval of one time constant, we have

$$\boxed{\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-1} = 0.368} \tag{5.23}$$

or

$$\boxed{\frac{T - T_i}{T_\infty - T_i} = 1 - \frac{T - T_\infty}{T_i - T_\infty} = 1 - 0.368 = 0.632}$$

The body has an excess temperature at $t = \tau$ which is only $1/e$ of what it was at $t = 0$. This can be used as a measure of response time of the body to the changes in temperature of its environment. From Eq. (5.24), it is clear that after an interval of time equal to one time constant of the given temperature measurement device, the temperature difference between the body (thermocouple) and the source would be 36.8% of the initial temperature difference, i.e. the temperature difference would be reduced by 63.2% (see Figure 5.3).

The time required by a thermocouple to attain 63.2% of the value of initial temperature difference is called *thermal time constant* and is indicative of its *sensitivity*.

The ratio t/τ can be seen as

$$\boxed{\frac{t}{\tau} = \frac{hA_s t}{\rho C_p \forall} = \frac{\text{Capacity for convection at the surface}}{\text{Thermal capacity of the body}}} \tag{5.24}$$

For satisfactory response, the response time should obviously be as small as possible. Usually, it is recommended that reading of the thermocouple should be taken after a time equal to about four time constants (4τ) has elapsed. The time constant of a thermocouple is usually maintained between 0.04 to 2.5 s.

For rapid response, the term $(hAt)/(\rho C_p \forall)$ or t/τ should be *large* enough so that the exponential term will reach zero faster. The exponential decay of (*excess*) temperature can be hastened and the sensitivity of the instrument increased by:

- *Increasing* $(A_s/\forall)$ or $(\pi D^2/(\pi D^3/6)$, i.e., $(6/D)$, i.e., *decrease* the wire diameter.
- *Decreasing* density and specific heat.
- *Increasing* the value of the heat-transfer coefficient h.

5.9 ❑ MIXED BOUNDARY CONDITION

Consider a slab of thickness L, initially at a uniform temperature T_i. Heat is supplied to the slab from one of its boundary surfaces at a constant and uniform heat flux q (W/m^2). Heat is dissipated by convection from the other boundary surface into a surrounding medium at a uniform temperature T_∞ with a heat-transfer coefficient h. Figure 5.6 shows the geometry and the boundary conditions for the problem.

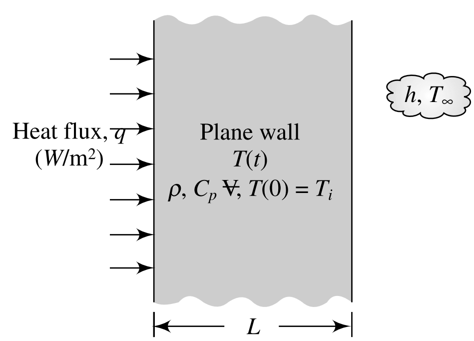


Fig. 5.6 *Transient heat conduction in a plane wall with mixed boundary conditions*

Applying energy balance, we have

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \cancelto{0}{\dot{E}_{\text{gen}}} = \dot{E}_{\text{st}}$$

$$qA - hA(T(t) - T_\infty) = \frac{\rho \forall C_p dT(t)}{dt}$$

or
$$\frac{dT(t)}{dt} + \frac{hA(T(t) - T_\infty)}{\rho \forall C_p} - \frac{qA}{\rho \forall C_p} = 0 \tag{5.25}$$

Substituting $\theta \equiv T(t) - T_\infty$

i.e.,
$$\frac{d\theta}{dt} = \frac{dT(t)}{dt}, \; a \equiv \frac{hA}{\rho \forall C_p}$$

and
$$b \equiv \frac{qA}{\rho \forall C_p}$$

Equation (5.26) then becomes $\dfrac{d\theta}{dt} + a\theta - b = 0$

Now, let $\theta' \equiv \theta - \dfrac{b}{a}$, i.e., $a\theta' = a\theta - b$

Then,
$$\frac{d\theta'}{dt} = \frac{d\theta}{dt}$$

and
$$\boxed{\frac{d\theta'}{dt} + a\theta' = 0} \tag{5.26}$$

Separating the variables and integrating from $t = 0$ to $t = t$, (and $\theta' = \theta'$, to $\theta' = \theta'$), we get

$$\boxed{\frac{\theta'}{\theta'_i} = \exp(-at)} \tag{5.27}$$

Substituting now for θ' and θ,

$$\boxed{\frac{(T(t) - T_\infty) - (b/a)}{(T_i - T_\infty) - (b/a)} = \exp(-at)} \tag{5.28}$$

Dividing the numerator and denominator of the left-hand side by $(T_i - T_\infty)$, we have

$$\frac{\dfrac{T(t) - T_\infty}{T_i - T_\infty} - \dfrac{(b/a)}{T_i - T_\infty}}{1 - \dfrac{(b/a)}{T_i - T_\infty}} = \exp(-at)$$

or
$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \frac{(b/a)}{T_i - T_\infty} + \exp(-at) \times \left[1 - \frac{(b/a)}{T_i - T_\infty}\right]$$

or
$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{(b/a)}{T_i - T_\infty}[1 - \exp(-at)] \tag{5.29}$$

and, also from Eq. (5.28),

$$\boxed{t = \frac{-1}{a} \ln \left[\frac{T(t) - T_\infty - (b/a)}{T_i - T_\infty - (b/a)} \right]} \tag{5.30}$$

Note that for $t = \infty$, Eq. (5.30) reduces to

$$\boxed{T(t) = T_\infty + \frac{b}{a} = T_\infty + \frac{q}{h}} \tag{5.31}$$

Equation (5.30) gives the *steady-state temperature* in the slab.

5.10 ❑ TRANSIENT HEAT CONDUCTION IN LARGE PLANE WALLS, LONG CYLINDERS, AND SPHERES WITH CONVECTIVE BOUNDARY CONDITIONS ($Bi > 0.1$)

There are many situations when the temperature gradients in the solid are *not* negligible and the lumped-capacity analysis is no longer applicable. The analysis of heat-conduction problems in such cases involves temperature distribution within the solid as a function of both *position* and *time*. To determine *heat-flow rate* and *temperature distribution in* the case of simple and common geometries like a *plane wall* (*slab*), a long *cylinder*, and a *sphere* for *Bi greater than 0.1*, there are *two* options-*analytical* and *chart solutions*.

5.10.1 • Analytical Solution of One-dimensional Transient Conduction Problem

The analysis of transient temperature distributions in bodies in which the internal temperature gradients cannot be neglected becomes much more complex because now the temperature is a function of both space and time.

Let us now consider the general case where the conduction and convection resistances are of comparable magnitude with $Bi > 0.1$.

Consider a *plane wall* of thickness $2L$, a *long cylinder* of radius r_o, and a *sphere* of radius r_o, initially at a *uniform temperature* T_i, as shown in Figure 5.7. At time $t = 0$, each geometry is placed in an ambient fluid which is at a constant temperature T_∞ and placed in it for $t > 0$. Heat transfer by convection takes place between the body and its ambient with a *uniform* and *constant* heat transfer coefficient h. In all *three* cases, there is a geometric and thermal symmetry. The plane wall is symmetrical about its *midplane* ($x = 0$), the cylinder is symmetrical about its *centreline* ($r = 0$), and the sphere is symmetrical about its *centre* ($r = 0$). *Radiation* heat transfer between the body and its surroundings is neglected or its effect is included in the convection heat-transfer coefficient h.

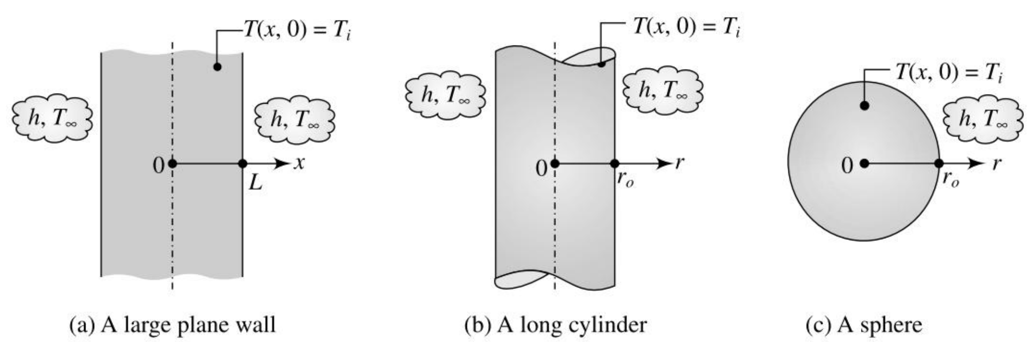


Fig. 5.7 *Schematic of a convectively cooled slab (wall), cylinder, and sphere*

5.10.2 • Plane Wall-Analytical Solution

Consider a large plane wall of thickness $2L$ in the x-direction which extends infinitely in the y- and z-directions. Figure 5.8 shows a plane wall with convective boundary conditions.

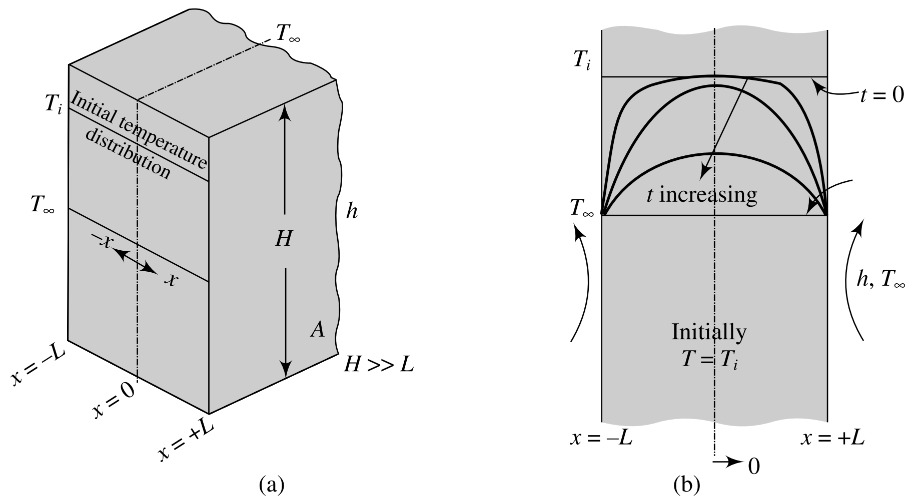


Fig. 5.8 *(a) Schematic and (b) Transient temperature profiles in a large slab exposed to convection from its surface for $T_i > T_\infty$.*

The wall has uniform and constant convective heat transfer coefficient on both surfaces (at $x = L$ and at $x = -L$) and the temperature inside the slab is uniform at all locations at T_i before the plate is brought into convective contact with the surrounding fluid maintained at constant temperature, T_∞.

- *Initial condition*: ($t \le 0$)

  $T = T_i \qquad -L \le x \le L$ (5.32)

- *Boundary condition*: ($t > 0$)

  $$-kA\frac{\partial T}{\partial x} = hA(T_s - T_\infty) \text{ at } x = L$$

Such conditions result in thermal and geometric symmetry about the centreline of the wall. The temperature profile has a horizontal slope at the centreline (i.e., $dT/dx = 0$). Since $dT/dx = 0$ the heat-transfer rate is zero at $x = 0$. Temperature-wise, we will get identical result for a wall with *thickness* L but *insulated* on one face with a convective heat transfer on the other face.

We start with the general three-dimensional heat conduction equation in Cartesian coordinates and proceed as follows:

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\bar{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{5.33}$$

Since, there is no internal heat generation, $\bar{q}/k = 0$. Also, the temperature does not vary in the y or z directions, giving $\dfrac{\partial^2 T}{\partial y^2} = \dfrac{\partial^2 T}{\partial z^2} = 0$

The appropriate form of the differential equation becomes

$$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{5.34}$$

Due to symmetry, we can only consider the region $0 \le x \le L$ (*For* $-L \le x \le 0$, *the transient temperature response is identical*).
It is convenient to transform the governing differential equation in the dimensionless form.
Let us define a dimensionless space variable $X = x/L$ and dimensionless temperature.

$$\theta(x, t) = [T(x, t) - T_\infty]/[T_i - T_\infty]$$

We note that, $\dfrac{\partial \theta}{\partial X} = \dfrac{\partial \theta}{\partial (x/L)} = \dfrac{L}{T_i - T_\infty}\dfrac{\partial T}{\partial x}$

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{T_i - T_\infty}\frac{\partial^2 T}{\partial x^2} \quad \text{and} \quad \frac{\partial \theta}{\partial t} = \frac{1}{T_i - T_\infty}\frac{\partial T}{\partial t}$$

Substituting into Eqs. (5.31) and (5.33) and rearranging, one gets

or $$\frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{\alpha}\frac{\partial \theta}{\partial t} \quad \text{and} \quad \frac{\partial \theta(1, t)}{\partial X} = \frac{hL}{k}\theta(1, t)$$

Therefore, the proper form of the dimensionless time is $Fo = \alpha t/L^2$, which is called the *Fourier number Fo*, and we know that $Bi = \dfrac{hL}{k}$ is the *Biot number*. The mathematical formulation of the one-dimensional transient heat conduction problem in a plane wall can then be expressed in the non-dimensionalized form as

Dimensionless differential equation: $$\boxed{\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial Fo}} \tag{5.35}$$

Dimensionless boundary conditions: $$\boxed{\frac{\partial \theta(0, Fo)}{\partial X} = 0 \quad \text{and} \quad \frac{\partial \theta(1, Fo)}{\partial X} = -Bi\,\theta(1, Fo)} \tag{5.36}$$

Dimensionless initial condition: $$\boxed{\theta(X, 0) = 1} \tag{5.37}$$

where

$\boxed{\theta(X, Fo) = \dfrac{T(x,t) - T_\infty}{T_i - T_\infty}}$ *Dimensionless temperature*

$\boxed{X = \dfrac{x}{L}}$ *Dimensionless coordinate* (*distance from the centre*)

$\boxed{Bi = \dfrac{hL}{k}}$ *Dimensionless heat-transfer coefficient (Biot number)*

$\boxed{Fo = \dfrac{\alpha t}{L^2}}$ *Dimensionless time (Fourier number)*

Non-dimensionalisation reduces the number of independent variables and parameters from 8 to 3, i.e., from x, L, t, k, α, h, T_i, and T_∞ to just X, Bi, and Fo. The functional relationship is now given by

$$\boxed{\theta = f(X, Bi, Fo)} \tag{5.38}$$

The temperature-time history thus depends on three dimensionless parameters: the relative distance X, Biot number Bi and Fourier number Fo. Non-dimensionalisation of the results with the above-mentioned dimensionless numbers enables us to present the results practically over a wide range of operating parameters, either in *tabular* or *graphical* forms.

The solution to this problem is, at best, long and tedious. It has been worked out by a number of people and put in the form of charts. The solution takes the form of an infinite series.

$$\boxed{\frac{T(x,t)-T_\infty}{T_i-T_\infty} = 4\sum_{n=1}^{\infty}\left(\frac{\sin\lambda_n}{2\lambda_n+\sin 2\lambda_n}\right)\exp[-\lambda_n^2\,Fo]\cos\left(\lambda_n\frac{x}{L}\right)} \tag{5.39}$$

where the eigenvalues (*positive roots*), viz., λ_1, λ_2 ... λ_n are obtained from the solution to the characteristic transcendental equation given below and shown graphically in Figure 5.9.

$$\boxed{\cot\lambda_n = \lambda_n\left(\frac{1}{Bi}\right)} \quad \text{or} \quad \boxed{\lambda_n\tan\lambda_n = Bi} \tag{5.40}$$

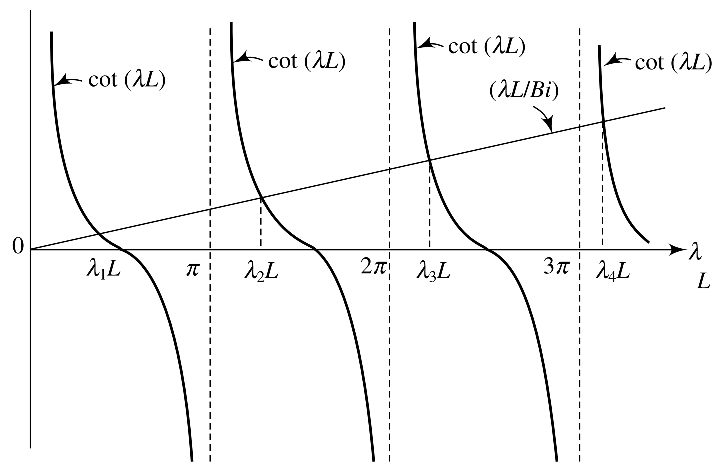


Fig. 5.9 *Graphical solution of transcendental equation*

For most practical purposes, truncation of a series solution after a few terms gives fairly accurate results.

The strategy for obtaining solutions to other geometries like a *long cylinder* or a *sphere* is similar. We start with the appropriate differential equation for *one-dimensional*, *time-dependent* conduction in *cylindrical* or *spherical* coordinates. Boundary conditions will be the same except that x is replaced by r_o and L is replaced by r_o. Again, results are non-dimensionalised with the dimensionless parameters mentioned above.

Characteristic length in *Biot number* is taken as *half-thickness* L for a plane wall, radius r_o for a long cylinder and sphere instead of being calculated as V/A_s, as in the lumped system analysis.

For all these *three* geometries, the solution involves infinite series, which converges rapidly for long times. For Fo > 0.2, only the *first term* of the series needs to be retained and other terms can be neglected, involving an error less than 2 percent. The single-term approximation solution, for all these *three* cases in terms of dimensionless temperature, Fourier number, and dimensionless distance from the centre is presented below

Plane wall: $$\boxed{\theta(x,t)_{\text{wall}} = \frac{T(x,t)-T_\infty}{T_i - T_\infty} = A_1\, e^{-\lambda_1^2 Fo}\cos(\lambda_1 x/L)} \quad (Fo > 0.2) \tag{5.41}$$

Long cylinder: $$\boxed{\theta(r,t)_{\text{cyl}} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = A_1\, e^{-\lambda_1^2 Fo}\, J_0(\lambda_1 r/r_o)} \quad (Fo > 0.2) \tag{5.42}$$

Sphere: $$\boxed{\theta(r,t)_{\text{sph}} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = A_1\, e^{-\lambda_1^2 Fo}\, \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}} \quad (Fo > 0.2) \tag{5.43}$$

In the above equations, the constants A_1 and λ_1 are functions of *Biot number* only. A_1 and λ_1 are calculated from the following relations:

Plane wall: $$\boxed{\begin{aligned} &\lambda_1 \tan\lambda_1 = Bi \\ &A_1 = \frac{4\sin\lambda_1}{2\lambda_1 + \sin 2\lambda_1}\end{aligned}} \tag{5.44}$$

Long cylinder: $$\boxed{\begin{aligned} &\lambda_1 \frac{J_1(\lambda_1)}{J_0(\lambda_1)} = Bi \\ &A_1 = \frac{2J_1(\lambda_1)}{\lambda_1[(J_0^2(\lambda_1)) + (J_1^2(\lambda_1))]}\end{aligned}} \tag{5.45}$$

Sphere: $$\boxed{\begin{aligned} &1 - \lambda_1 \cot\lambda_1 = Bi \\ &A_1 = \frac{4[\sin\lambda_1 - \lambda_1\cos\lambda_1]}{2\lambda_1 - \sin 2\lambda_1}\end{aligned}} \tag{5.46}$$

Table 5.2 *Coefficients in the one-term approximation solution for a plane wall, a long cylinder, and a sphere*

Plane Wall			Long Cylinder			Sphere		
$Bi = hL/k$	λ_1 (radian)	A_1	$Bi = hr_o/k$	λ_1 (radian)	A_1	$Bi = hr_o/k$	λ_1 (radian)	A_1
0	0	1.000	0.01	0.1412	1.0025	0.01	0.1730	1.0030
0.001	0.0316	1.0002	0.02	0.1995	1.0050	0.02	0.2445	1.0060
0.002	0.0447	1.0003	0.03	0.2439	1.0075	0.03	0.2989	1.0090
0.004	0.0632	1.0007	0.04	0.2814	1.0099	0.04	0.3450	1.0120
0.006	0.0774	1.0010	0.05	0.3142	1.0124	0.05	0.3852	1.0149
0.008	0.0893	1.0013	0.06	0.3438	1.0148	0.06	0.4217	1.0179
0.01	0.0998	1.0017	0.07	0.3708	0.0173	0.07	0.4550	1.0209
0.02	0.1410	1.0033	0.08	0.3960	1.0197	0.08	0.4860	1.0239
0.03	0.1732	1.0049	0.09	0.4195	1.0222	0.09	0.5150	1.0268
0.04	0.1987	1.0066	0.10	0.4417	1.0246	0.10	0.5423	1.0298
0.05	0.2217	1.0082	0.15	0.5376	1.0365	0.15	0.6608	1.0445
0.06	0.2425	1.0098	0.20	0.6170	1.04483	0.20	0.7593	1.0592
0.07	0.2615	1.0114	0.25	0.6856	1.0598	0.25	0.8448	1.0737
0.08	0.2791	1.0130	0.30	0.7465	1.0712	0.30	0.9208	1.0880

contd.

contd.

0.09	0.2956	1.0145	0.40	0.8516	1.0932	0.40	1.0528	1.1164
0.10	0.3111	1.0160	0.50	0.9408	1.1143	0.50	1.1656	1.1441
0.15	0.3779	1.0237	0.60	1.0185	1.1346	0.60	1.2644	1.1713
0.20	0.4328	1.0311	0.70	1.0873	1.1539	0.70	1.3525	1.1978
0.25	0.4801	1.0382	0.80	1.1490	1.1725	0.80	1.4320	1.2236
0.30	0.5218	1.0450	0.90	1.2048	1.1902	0.90	1.5044	1.2488
0.40	0.5932	1.0580	1.00	1.2558	1.2071	1.00	1.5708	1.2732
0.50	0.6533	0.0701	2.00	1.5995	1.3384	2.00	2.0288	1.4793
0.60	0.7051	1.0814	3.00	1.7887	1.4191	3.00	2.2889	1.6227
0.70	0.7506	1.0919	4.00	1.9081	1.4698	4.00	2.4556	1.7201
0.80	0.7910	1.1016	5.00	1.9898	1.5029	5.00	2.5704	1.7870
0.90	0.8274	1.1107	6.00	2.0490	1.5253	6.00	2.6537	1.8338
1.00	0.8603	1.1191	7.00	2.0937	1.5411	7.00	2.7165	1.8674
1.50	0.9882	1.1537	8.00	2.1286	1.5526	8.00	2.7654	1.8921
2.00	1.0769	1.179	9.00	2.1566	1.5611	9.00	2.8044	1.9106
2.5	1.1422	1.1966	10.0	2.1795	1.5677	10.0	2.8363	1.9249
3.00	1.1925	1.2102	20.0	2.2881	1.5919	20.0	2.9857	1.9781
4.00	1.2646	1.2287	30.0	2.3261	1.5973	30.0	3.0372	1.9898
5.00	1.3138	1.2402	40.0	2.3455	1.5993	40.0	3.0632	1.9942
6.00	1.3496	1.2479	50.0	2.3572	1.6002	50.0	3.0788	1.9962
7.00	1.3766	1.2532	100.0	2.3809	1.6015	100.0	3.1102	1.9990
8.00	1.3978	1.2532	∞	2.4050	1.6018	∞	3.1415	2.0000
9.00	1.4149	1.2570	—	—	—	—	—	—
10.0	1.4289	1.2598	—	—	—	—	—	—
12.0	1.4505	1.2650	—	—	—	—	—	—
14.0	1.4664	1.2669	—	—	—	—	—	—
16.0	1.4786	1.2683	—	—	—	—	—	—
18.0	1.4883	1.2692	—	—	—	—	—	—
20.0	1.4961	1.2699	—	—	—	—	—	—
30.0	1.5202	1.2717	—	—	—	—	—	—
40.0	1.5325	1.2723	—	—	—	—	—	—
50.0	1.5400	1.2727	—	—	—	—	—	—
60.0	1.5451	1.2728	—	—	—	—	—	—
70.0	1.5487	1.2729	—	—	—	—	—	—
80.0	1.5514	1.2730	—	—	—	—	—	—
90.0	1.5535	1.2731	—	—	—	—	—	—
100.0	1.5552	1.2731	—	—	—	—	—	—
∞	$\pi/2$	$4/\pi$	—	—	—	—	—	—

Table 5.3 *Bessel functions of the first kind of the zeroth order and the first order*

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	1.5	0.5118	0.5579
0.1	0.9975	0.04999	1.6	0.4554	0.5699
0.2	0.9900	0.0995	1.7	0.3980	0.5778
0.3	0.9776	0.1483	1.8	0.3400	0.5815
0.4	0.9604	0.1960	1.9	0.2818	0.5812
0.5	0.9385	0.2423	2.0	0.2239	0.5767
0.6	0.9120	0.2867	2.1	0.1666	0.5683
0.7	0.8812	0.3290	2.2	0.1104	0.5560
0.8	0.8463	0.3688	2.3	0.0555	0.5399
0.9	0.8075	0.4059	2.4	0.0025	0.5202
1.0	0.7652	0.4400	2.6	–0.0968	–0.4708
1.1	0.7196	0.4709	2.8	–0.1850	–0.4097
1.2	0.6711	0.4983	3.0	–0.2601	–0.3391
1.3	0.6201	0.5220	3.2	–0.3202	–0.2613
1.4	0.5669	0.5419	—	—	—

We are usually interested in the centre temperature of the body. Recognizing that cos 0° = 1, $J_0(0) = 1$, and the limit of (sin x/x) is also 1, the dimensionless temperature at the centre of the body is given by:

***Centre of plane wall:* (*x* = *0*)**
$$\boxed{\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{\lambda_1^2 Fo}} \tag{5.47}$$

***Centre of long cylinder:* (*r* = *0*)**
$$\boxed{\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{\lambda_1^2 Fo}} \tag{5.48}$$

***Centre of sphere:* (*r* = *0*)**
$$\boxed{\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}} \tag{5.49}$$

The first step in the solution is to calculate the *Bi*ot number. Once the Biot number is known, constants A_1 and λ_1 are found. One can then use the relations given above to find the temperature at any specific location.

In addition to the temperature distribution, it is often useful to know the amount of heat lost (*or gained*) by the body, Q, during the time interval $t = 0$ to $t = t$, i.e., from the beginning to the specified time. Again, we non-dimensionalize Q by dividing it by Q_{max}, the maximum possible heat transfer. Obviously, the maximum amount of heat is transferred when the body reaches equilibrium with the ambient fluid. Hence,

$$\boxed{Q_{max} = \rho \forall C_p (T_i - T_\infty) = m C_p (T_i - T_\infty)} \tag{5.50}$$

where ρ is the density, $\forall$ is the volume, $(\rho\forall)$ is the mass, C_p is the specific heat of the body.

If Q_{max} is *positive*, the body is *losing* energy; and if it is *negative*, the body is *gaining* energy.

Using the *one-term approximation* solution, (Q/Q_{max}) for the *three* geometrical configurations is calculated from the following relations:

Plane wall:
$$\boxed{\frac{Q}{Q_{max}} = 1 - \theta_0 \frac{\sin \lambda_1}{\lambda_1}} \tag{5.51}$$

Long cylinder:
$$\boxed{\frac{Q}{Q_{max}} = 1 - 2\theta_0 \frac{J_1(\lambda_1)}{\lambda_1}} \tag{5.52}$$

Sphere:
$$\boxed{\frac{Q}{Q_{max}} = 1 - 3\theta_0 \left(\frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^3}\right)} \tag{5.53}$$

5.11 ❑ TRANSIENT-TEMPERATURE CHARTS

Analytical results of even one-dimensional transient conduction are quite complex and inconvenient to use. Approximate one-term approximation solutions of adequate accuracy for some common geometries have already been explained.

These results have been presented conveniently in graphical form as transient temperature charts for rapid engineering calculations for a wide range of parameter values. The *graphical plots or charts* offer computational convenience and being in terms of dimensionless parameters enjoy universal application.

These charts are available for (1) An infinite slab (*a large plane wall*), (2) An infinite cylinder (*a long cylinder*), (3) A sphere, and (4) A semi-infinite solid.

The *assumptions* made in the use of these charts are: (1) One-dimensional conduction without internal heat generation, and (2) Initially the body is at a uniform temperature throughout. (3) All surfaces of the body are exposed to same constant and uniform convection coefficient, h and ambient fluid temperature T_∞. (4) Constant thermal properties.

Some salient features of these transient temperature charts are the following:

- For one-dimensional transient conduction, charts giving temperature variation as a function of time are available. The parameters are usually represented in dimensionless form as follows:
  $$Bi = \frac{hL_c}{k},\ Fo = \frac{\alpha t}{L_c^2},\ \theta \equiv \frac{T - T_\infty}{T_i - T_\infty}$$
- The *initial conditions* for all the *three* chart solutions require that the solid be at a uniform temperature T_i initially and that at time $t = 0$ the entire surface of the body is in contact with the fluid at a temperature of T_∞.
- *One boundary condition* requires that the temperature gradient at the *midplane of the slab*, the *axis of the cylinder*, and *the centre of the sphere* be equal to zero. Physically, it amounts to *insulated* or *no-heat-flow condition* at these *central* positions.
- *The other boundary condition* requires that the conduction heat flux at the solid–fluid interface is *equal* to the convective heat flux. That is,

$$\boxed{-k \left.\frac{\partial T}{\partial n}\right|_s = h(T_s - T_\infty)} \tag{5.54}$$

where h is the *uniform and constant heat transfer coefficient*. The subscript s refers to conditions at the surface and n to the coordinate direction normal to the surface.

In the limiting case when $Bi \to \infty$, $h \to \infty$, convective thermal resistance $(1/hA) \to 0$ and then the prescribed surface temperature, T_s equals the ambient temperature T_∞.

- The *first* chart in each of these figures gives the non-dimensionalized centre temperature θ_0, i.e., at $x = 0$ for the slab of thickness $2L$, and at $r = 0$ for the cylinder and sphere of outer radius r_o, at a given time t.
- The *second* chart, called *position correction chart*, enables us to calculate the temperature at any other position at the same time t.
- The *third* chart gives the dimensionless heat loss $Q/Q_{\max}$, so that the heat loss (or *gain*) can be evaluated.

The general step-wise procedure for using these charts to solve a numerical problem is as follows:

Step 1: To begin with, calculate the Biot number, Bi from the given data, with the usual definition of Bi, i.e., $Bi = (hL_c)/k$, where L_c is the characteristic dimension, defined as $L_c = (\forall / A_s)$, [$L_c = L$, the *half-thickness*, for a plane wall, $L_c = r_o/2$ for a long cylinder, and $L_c = r_o/3$ for a sphere.] If $Bi < 0.1$, use the lumped-heat-capacity model. Otherwise, go in for the one-term approximation or the chart solution.

Step 2: For $Bi > 0.1$, one needs to use the one-term approximation or the chart solution. In this case, calculate the Biot number again with $Bi = (hL/k)$ for a *plane wall* where L is *half-thickness*, and $Bi = (hr_o/k)$ for a *cylinder or sphere*, where r_o is the outer radius. Also, calculate the Fourier number, $Fo = \alpha t/L^2$ for the *plane wall*, and $Fo = \alpha t/r_o^2$ for a *cylinder or sphere*.

Step 3: To calculate the centre temperature, use the appropriate first chart depending on the geometry being considered. Enter the chart on the x-axis with the calculated value of Fourier number, Fo, and draw a vertical line to intersect the Bi line. From the point of intersection, move horizontally to the left to the y-axis to read the value of $(T_0 - T_\infty)/(T_i - T_\infty)$. Here, T_0 is the centre temperature, which can now be calculated since T_i and T_∞ are known.

Step 4: To calculate the temperature at any other position, use the appropriate *second* (position correction) chart. Enter the chart with Bi, on the x-axis, move vertically up to intersect the (x/L) or (r/r_o) curve, as the case may be, and from the point of intersection, move to the left to read on the y-axis, the value of $(T - T_\infty)/(T_0 - T_\infty)$. The desired temperature T at the specified location can then be evaluated from:

$$\boxed{\underbrace{\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)}_{I\ chart}\underbrace{\left(\frac{T - T_\infty}{T_0 - T_\infty}\right)}_{II\ chart} = \frac{T - T_\infty}{T_i - T_\infty}} \tag{5.55}$$

Step 5: To calculate the amount of heat transferred Q, during a specified time interval t from the beginning (i.e., $t = 0$), use the appropriate *third* (Gröber) chart, depending upon the geometry. Enter the x-axis with the value of $(Bi^2\, Fo)$ and move vertically up to intersect the curve representing the given value of Bi, and move to the left to read on the y-axis, the value of $Q/Q_{\max}$. Calculate $Q_{\max}$ from $Q_{\max} = \rho \forall C_p (T_i - T_\infty)$.

Then, $$\boxed{Q = (Q/Q_{\max})\, Q_{\max}} \tag{5.56}$$

The following points may be noted regarding these charts:

- The transient-temperature charts are valid for Fourier number $Fo > 0.2$.
- Specifically remember that while calculating *Bi*ot number, characteristic length (L_c) used is L, the *half-thickness* for a *plane wall*, and outer radius, r_o for the *cylinder* and *the sphere* (L_c is *not* equal to $(\forall/A_s)$).
- In these charts, $Bi = \infty$ corresponds to $h \to \infty$, which means that at $t = 0$, the surface of the body is suddenly brought to a temperature of T_∞ and maintained at $T_\infty = T_s$ thereafter at all times.
- From the *position correction charts* at $Bi < 0.1$, the temperature within the body can be taken as uniform, without introducing an error of more than 5%. This was precisely the condition for application of *lumped-capacitance analysis.*
- Let us face the fact that it is difficult to read the transient temperature charts accurately. Hence, the use of *one-term approximation* with tabulated values of A_1 and λ_1 seems to be a better option. The chart solution, however, is far more valuable from the point of view of convenience though at the cost of some accuracy.
- Generally, two types of transient problems can be solved by using the charts. In one, the time is known while the local temperature is to be calculated. Such types of problems can be relatively easily solved in a straightforward manner. However, in the other type, where time is unknown while the local temperature is given, this may involve a *trial and error* procedure.

5.11.1 • Heisler and Gröber Charts

Heisler had presented the charts in 1947 in a different format. Heisler presented as a *semi-log plot* with the temperature ratio on the *log scale* and Fourier number on a *linear scale*. Unfortunately, these charts are very inconvenient to read in the range of Fourier number below 1 where a lot of action takes place. The use of reciprocal of the *Bi*ot number (1/*Bi*) as a parameter used in the *Heisler* charts is also tedious and confusing.

The transient temperature charts given in this book are more user-friendly and convenient to use.

Apart from the temperature distribution, we need to know the heat loss (*or gain*) during a given time interval. *Gröber* charts come in handy for this purpose. The heat lost or gained during time t may be determined through the use of *Gröber* chart in which dimensionless heat transfer (Q/Q_{max}) is plotted against ($h^2 \times t/k^2$ or $Bi^2\,Fo$) for several values of the Biot number with the *Bi*ot number as a parameter, where Q represents the amount of heat lost or gained during time, t.

5.11.2 • Charts for Plane Wall (Infinite Slab)

The *transient*, *non-dimensional temperature distribution* depends upon the dimensionless distance, (x/L), the *Biot* and *Fourier* numbers. Both *Bi* and *Fo* are significant measures as to how a system will respond to changes in temperature. The charts are presented in Figure 5.10 to 5.12. Figure 5.10 may be used to evaluate the *midplane* or centreline temperature of the slab *at any time* during the transient process. Here the dimensionless temperature at the *midplane*, is defined as:

$$\boxed{\theta(0,\mathrm{t}) = \frac{T(0,\mathrm{t}) - T_\infty}{T_i - T_\infty}} \tag{5.57}$$

It is plotted as a function of dimensionless time, i.e., Fourier number Fo, with the Biot number (*Bi*) as the constant parameter. The curve for $Bi = \infty$ corresponds to the case $h \to \infty$, or negligible surface resistance

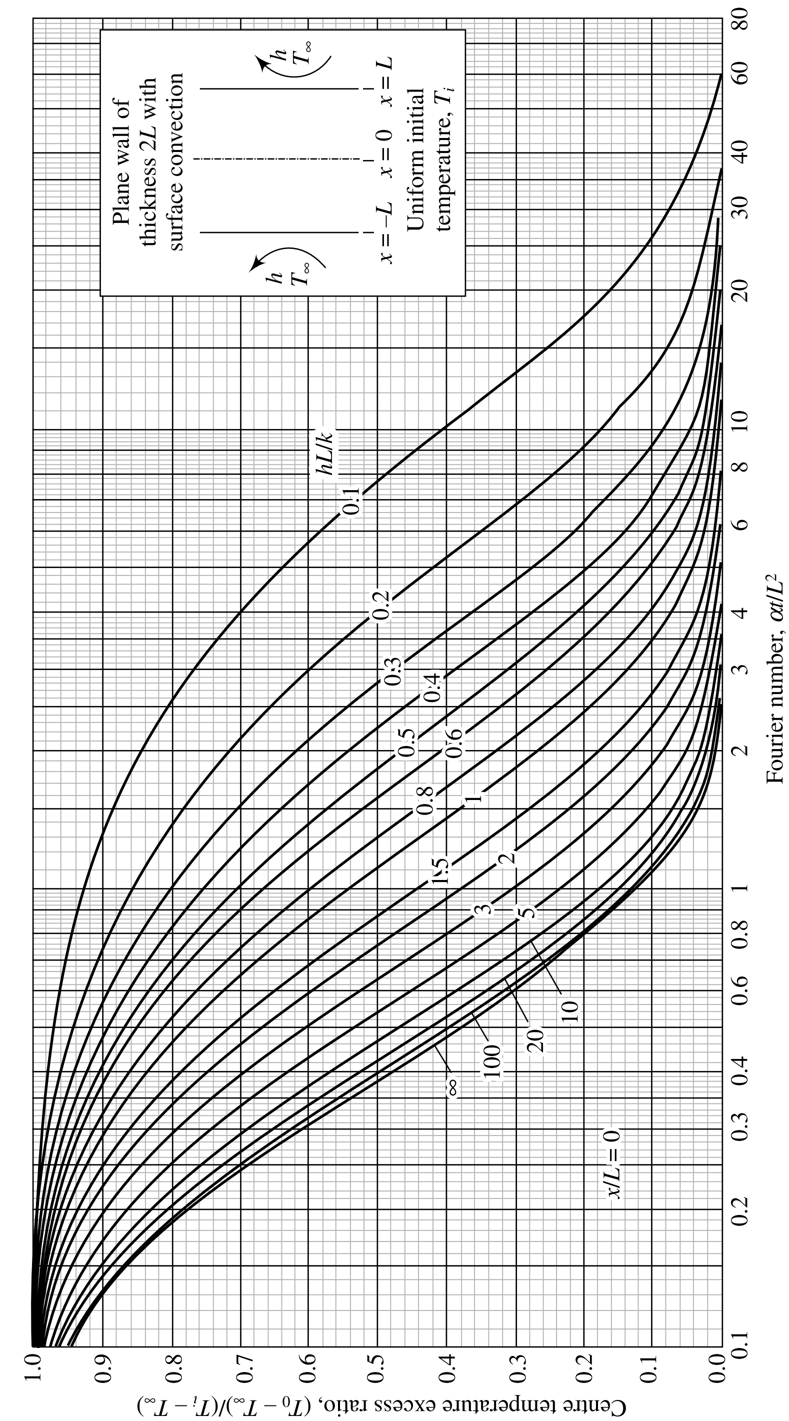


Fig. 5.10 *Dimensionless midplane temperature for a plane wall*

when the surfaces of the slab are maintained at the ambient temperature T_∞. For a case with very low *Bi* (of negligible internal thermal resistance and fairly uniform temperature distribution within the solid) the temperature in the slab is dependent *not* on *x* but *only t* (*lumped-capacitance model*, discussed earlier).

Figure 5.10 is first used to determine the *centreline* temperature T_0 at time *t*. From this knowledge of $T(0, t)$, Fig. 5.11 is then used to evaluate $T(x, t)$. The time dependence of any temperature off the centreline corresponds to the time dependence of the centreline temperature. Hence, in Fig. 5.11, the Fourier number is *absent*. The desired temperature at any distance from the midplane, *x* can be calculated from

$$\boxed{\left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\text{wall}} = \underbrace{\left(\frac{T_0-T_\infty}{T_i-T_\infty}\right)}_{\text{Fig. 6.10}}\underbrace{\left(\frac{T(x,t)-T_\infty}{T_0-T_\infty}\right)}_{\text{Fig. 6.11}}} \tag{5.58}$$

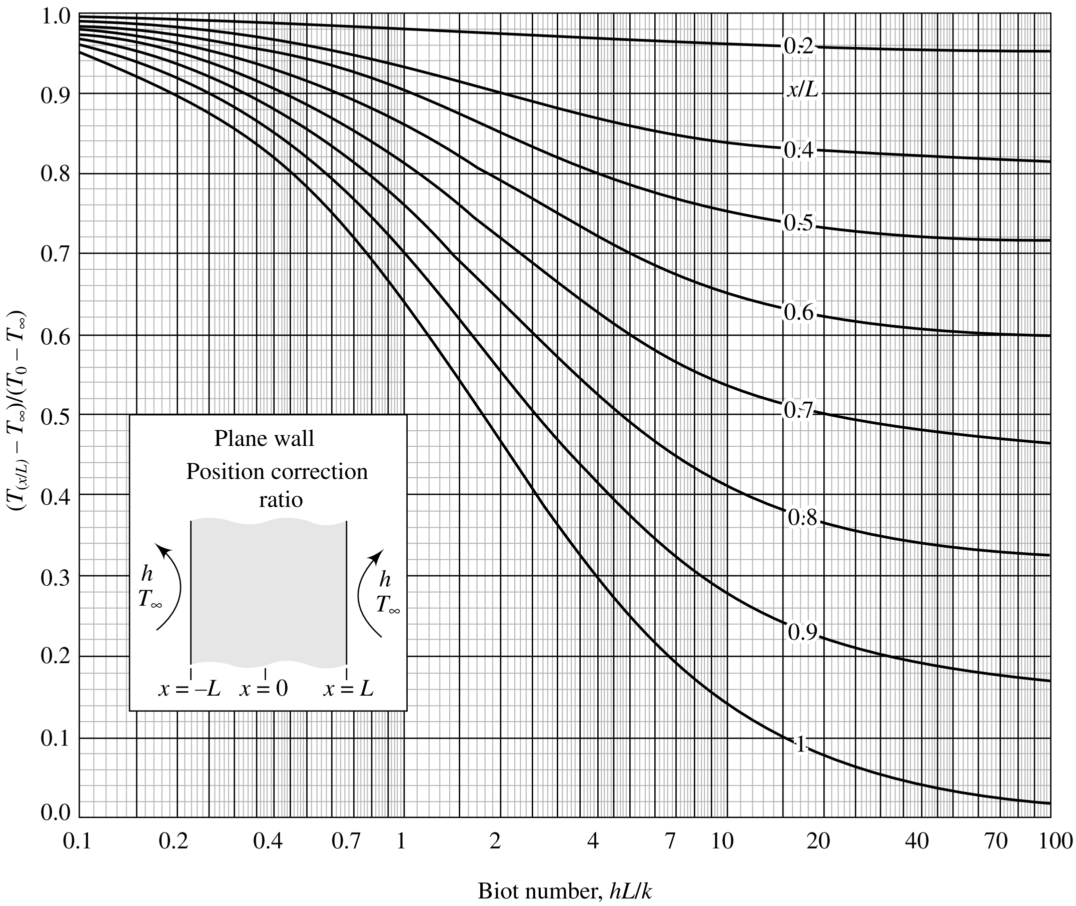


Fig. 5.11 *Position correction ratio for a plane wall*

Note that *x* is measured from the adiabatic surface of the plane of symmetry, i.e., midplane (or centreline) ($x = 0$) to either face (or extremity) of the slab irrespective of the direction. Thus, the temperature at the two locations on the slab or wall of thickness $2L$ on either side of the midplane will be the same. This, of course, implies that both *h* and T_∞ are the same for both the left and right faces of the slab.

If one face of the slab is *insulated*, the temperature distribution can be obtained by letting $x = 0$ at the insulated surface and $x = L$ at the surface exposed to the convective environment. This is possible because the chart solution corresponds to the case where at $x = 0$, $dT/dx = 0$, and at $x = L$, there is convection heat transfer to the environment. The only difference is that L is now the entire wall thickness of the insulated wall.

The heat loss or gain during time t may be determined through the use of Fig. 5.12. The quantity of heat is equal to the change in the internal energy during time t,

$$\boxed{Q = \rho C_p \forall (T - T_\infty)} \tag{5.59}$$

Defining the one-dimensional energy transfer or internal energy as Q/Q_{max} or U/U_i where $Q_{max} = \rho C_p \forall (T - T_\infty)$. where Q_{max} represents the initial energy transfer or internal energy of the slab relative to the ambient temperature T_∞. In Fig. 5.12, Q/Q_{max} is plotted exclusively against $Bi^2\ Fo$ or ($h^2\alpha t/k^2$) with the Biot number, Bi, as a parameter.

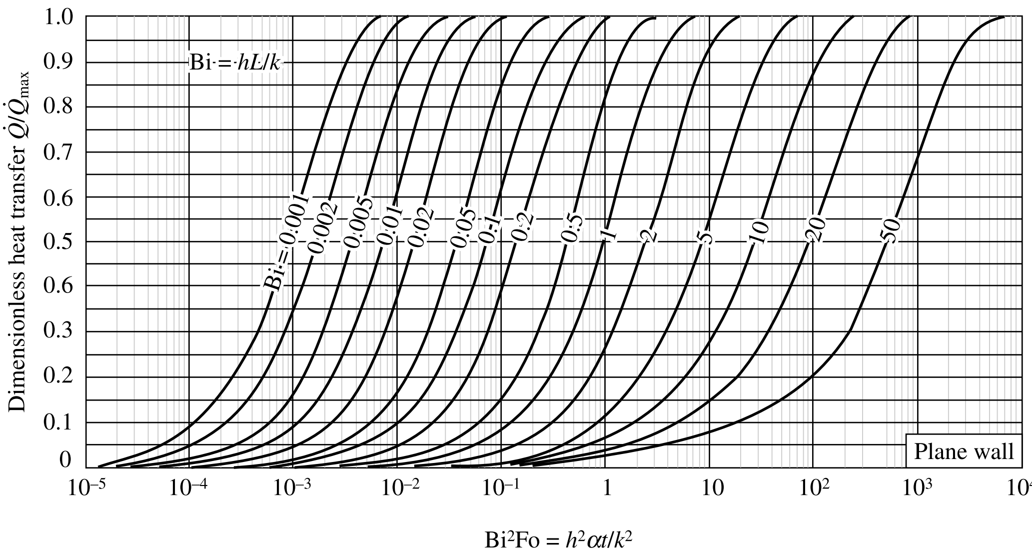


Fig. 5.12 *Dimensionless heat transfer for a plane wall.*

5.11.3 • Charts for Infinitely Long Cylinder

For an infinite (*long*) cylinder of radius r_o which is at an initial uniform temperature T_i and undergoes change in the convective environment at a constant temperature T_∞, the heat-transfer coefficient h on the cylindrical (*curved*) surface is constant when the temperature distribution and the heat transfer can be obtained in a manner similar to that for the infinite slab (*plane wall*).

Figure 5.13 gives the axis (*centreline*) temperature for an infinite (*very long*) cylinder of radius r_o. The dimensionless axis (*centreline*) temperature at $r = 0$ defined as

$$\boxed{\theta(0, t) = \left[\frac{T(0, t) - T_\infty}{T_i - T_\infty}\right]} \tag{5.60}$$

is plotted against dimensionless time or Fo for different values of the parameter Bi. Figure 5.14 relates the temperature at various radial locations within the cylinder to the centreline temperature. Hence, to obtain

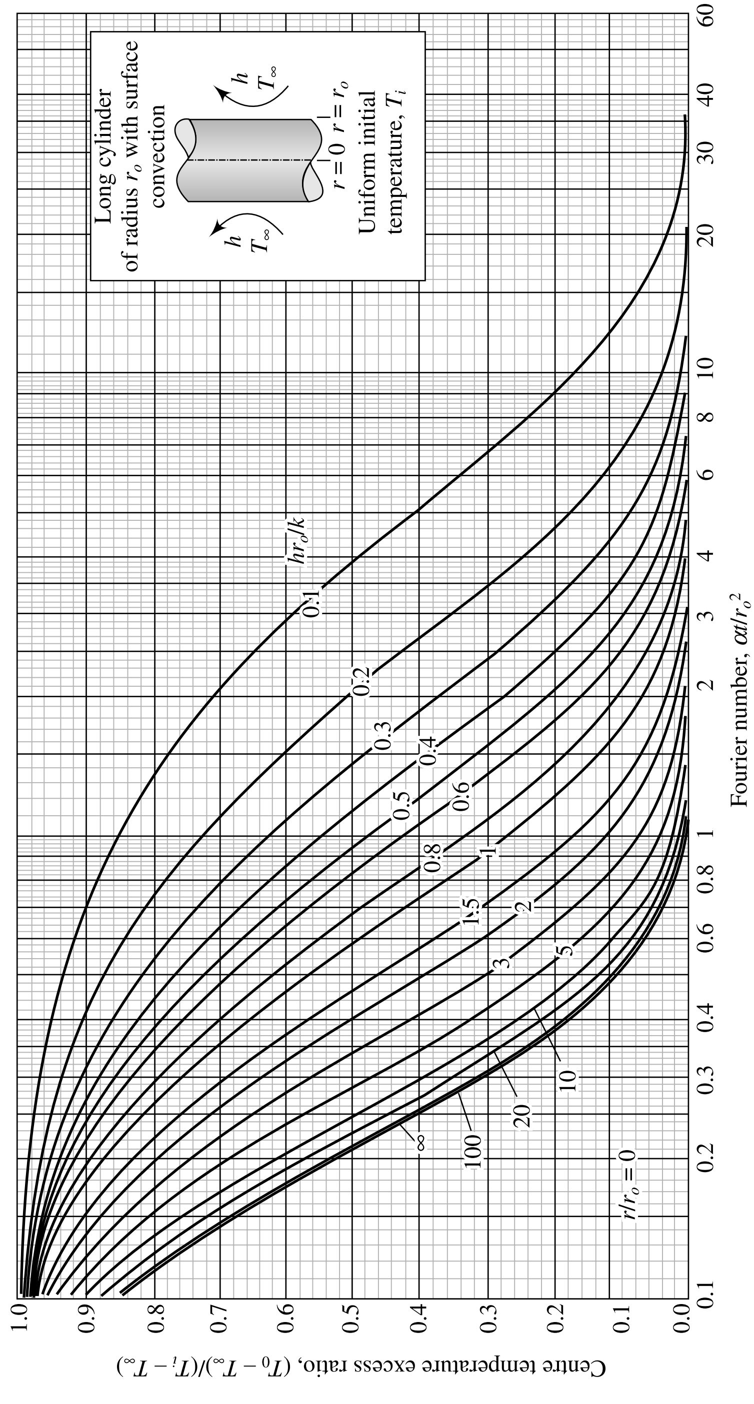


Fig. 5.13 *Dimensionless centreline temperature for a long solid cylinder*

the temperature at any radial location r, at some time t, Fig. 5.13 must be used in conjunction with Fig. 5.14. The dimensionless radial coordinate is defined as r/r_o (*like* x/L in an infinite slab). Note that the Biot number and Fourier number are defined in terms of r_o in contrast with the *lumped-capacity* method where the characteristic dimension is defined as $\frac{V}{A} = \frac{\pi R_o^2 L}{2\pi r_o L}$ or $r_o/2$. Thus, Bi and Fo are defined here as $\frac{hr_o}{k}$ and $\frac{\alpha t}{r_o^2}$, respectively.

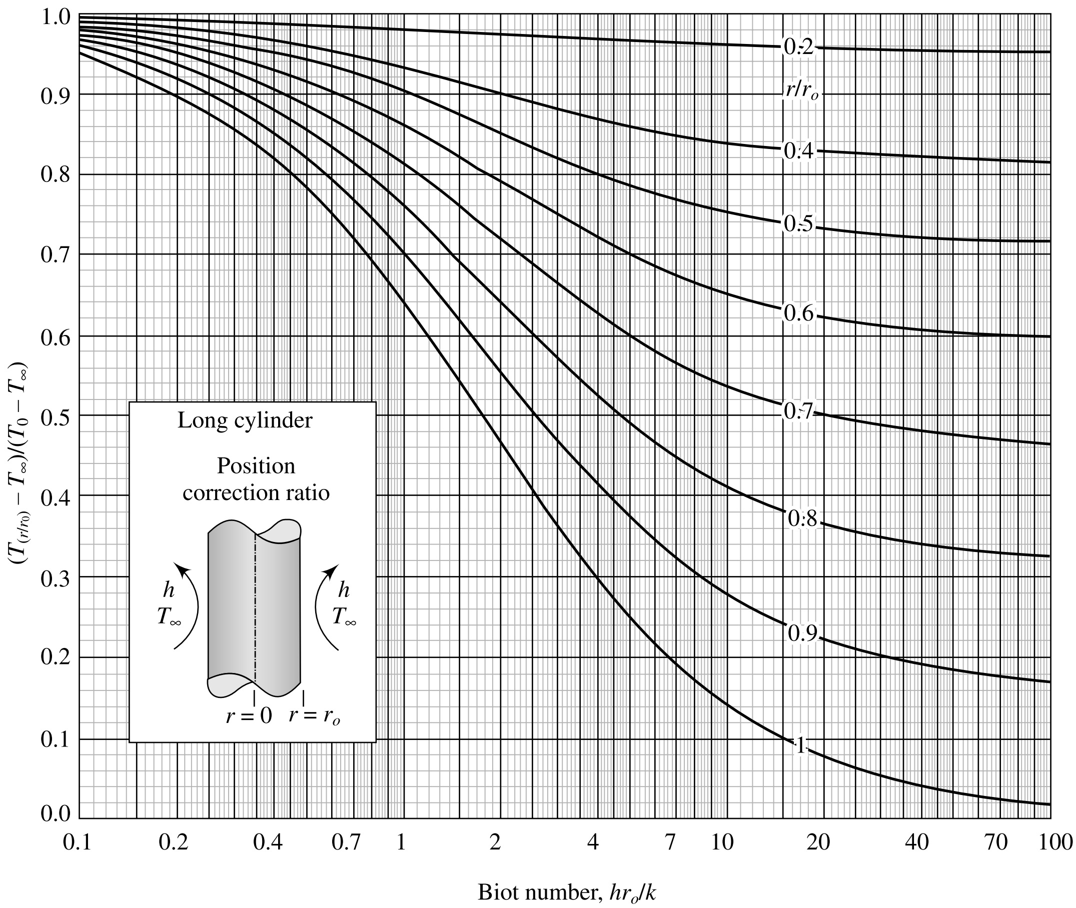


Fig. 5.14 *Position correction ratio for a long solid cylinder*

To determine the local temperature at any radius r at any time t, we have

$$\boxed{\left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\text{cylinder}} = \underbrace{\left[\frac{T(0,t)-T_\infty}{T_i-T_\infty}\right]}_{\text{Fig. 6.13}}\underbrace{\left[\frac{T(r,t)-T_\infty}{T(0,t)-T_\infty}\right]}_{\text{Fig. 6.14}}} \tag{5.61}$$

Once the temperature distribution is known, the *instantaneous* heat transfer rate *to* or *from* the solid surface can be evaluated from the Fourier law and the heat loss or change in internal energy over a time interval t can be computed by integrating the *instantaneous* heat-transfer rates. If we denote the internal energy relative to the ambient fluid at time interval t by $Q(t)$ and the initial internal energy relative to the fluid by Q_{max}, then the dimensionless heat loss per unit length is given by

$$\boxed{\frac{Q(t)}{Q_{\max}} = \frac{Q_{\text{loss}}}{\rho C_p(\pi r_o^2)(T_i - T_\infty)}} \tag{5.62}$$

In Fig. 5.15, Q/Q_{max} is plotted as a function of ($Bi^2\ Fo$) with Bi as the parameter. The procedure remains the same whether the solid is *heated* or *cooled*.

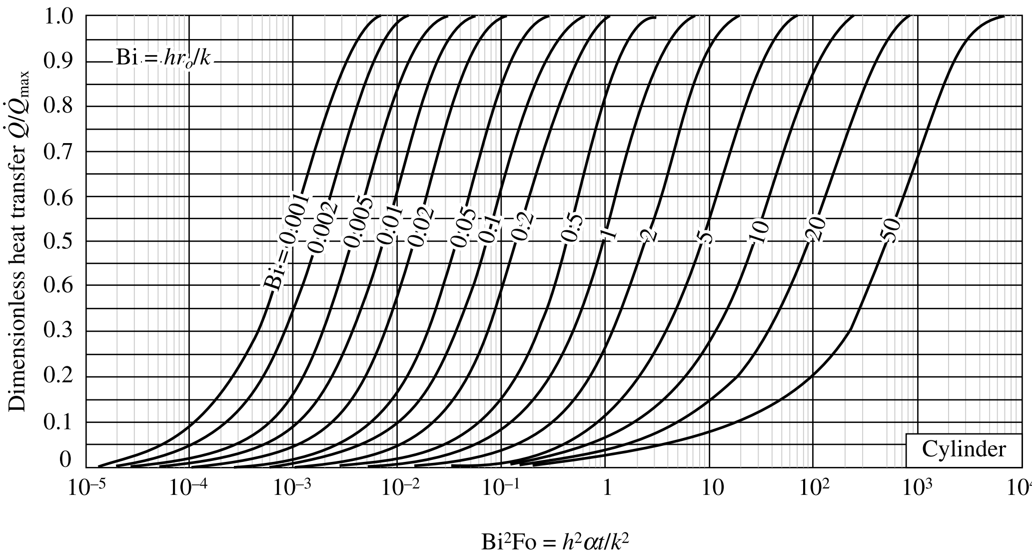


Fig. 5.15 *Dimensionless heat transfer for a long cylinder*

5.11.4 • Charts for a Solid Sphere

The dimensionless temperature at the centre (r = 0) is defined as

$$\boxed{\theta(0, t) = \frac{T(0, t) - T_\infty}{T_i - T_\infty}} \tag{5.63}$$

Figure 5.16 presents the dimensionless centre temperature for the sphere with uniform initial temperature T_i at time t = 0 which is exposed to the constant ambient temperature T_∞ with constant convection coefficient h at the surface. The centre temperature is a function of the Biot number hr_o/k and the Fourier number $\alpha t/r_o^2$.

The temperature of the surface of the sphere off the centre ($r \neq 0$) is obtained in exactly the same manner outlined earlier for an infinite cylinder. The temperature at any radial position r/r_o for the sphere can be obtained from Fig. 5.17 as a product of the position correction ratio and the dimensionless centre temperature.

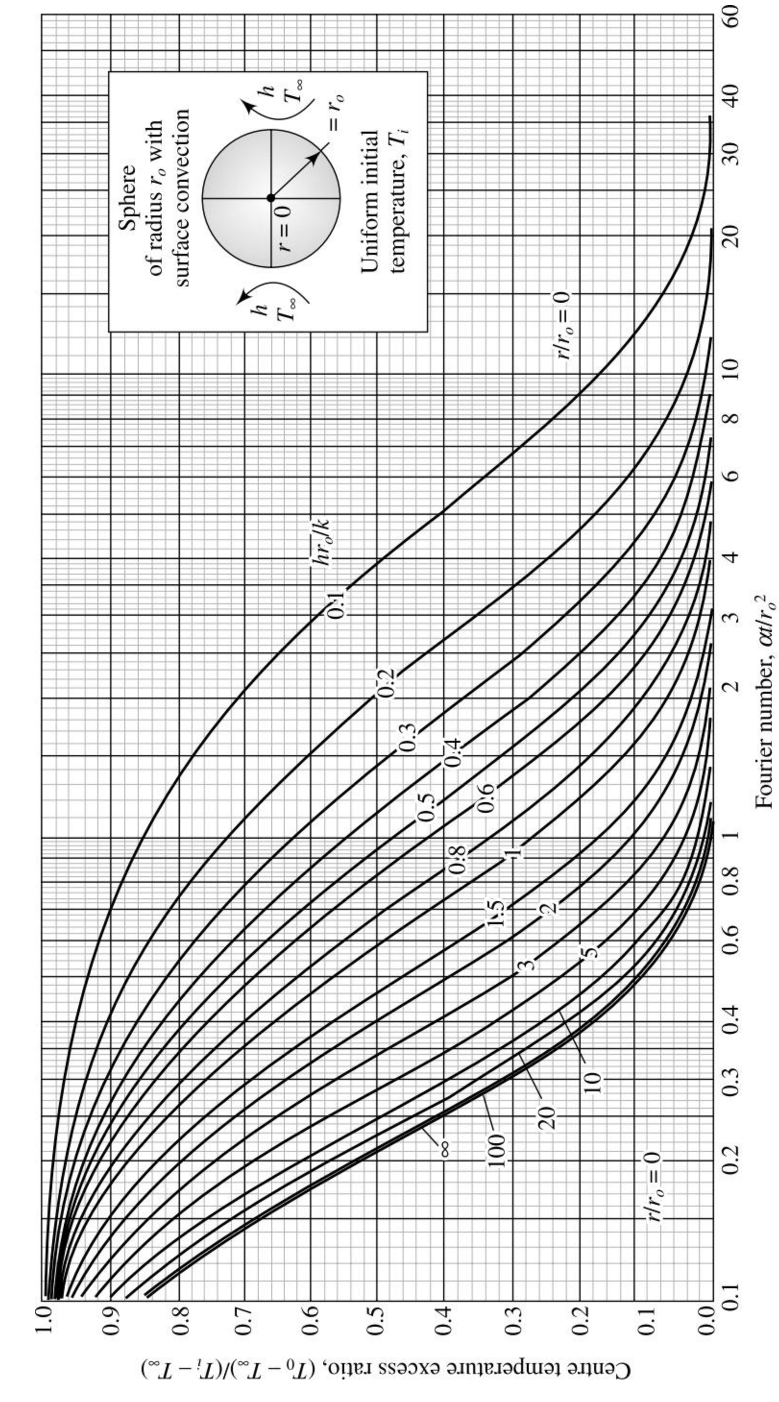


Fig. 5.16 *Dimensionless centre temperature for a sphere*

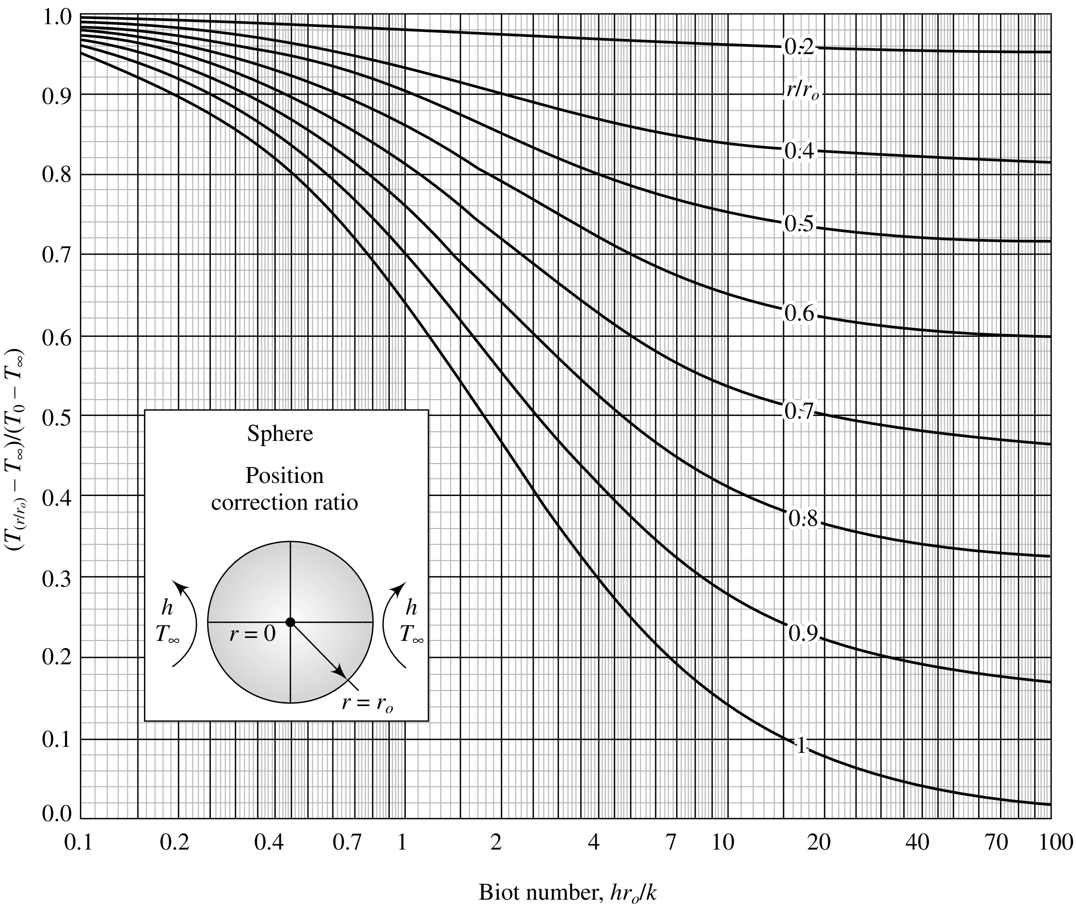


Fig. 5.17 *Position correction ratio for a sphere*

Thus, we get

$$\boxed{\left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\text{sphere}} = \underbrace{\left(\frac{T(0,t)-T_\infty}{T_i-T_\infty}\right)}_{\text{Fig. 6.16}}\underbrace{\left(\frac{T(r,t)-T_\infty}{T(0,t)-T_\infty}\right)}_{\text{Fig. 6.17}}} \tag{5.64}$$

Figure 5.18 presents the heat-loss ratio ($Q/Q_{\max}$), i.e., dimensionless heat transfer for various values of Biot numbers.

In order to decide the most satisfactory method of solution in such cases, the bottom line is

- If Bi < 0.1, the *lumped capacity analysis* is most appropriate.
- If Bi > 0.1, Fo < 0.05, *semi-infinite solid solution* is most appropriate. (explained in the next section).
- If Bi > 0.1 and 0.05 < Fo < 0.2, *complete series solution* is recommended.
- If Bi > 0.1 and Fo > 0.2, *the one-term approximate solution* or *chart solution* is preferable.

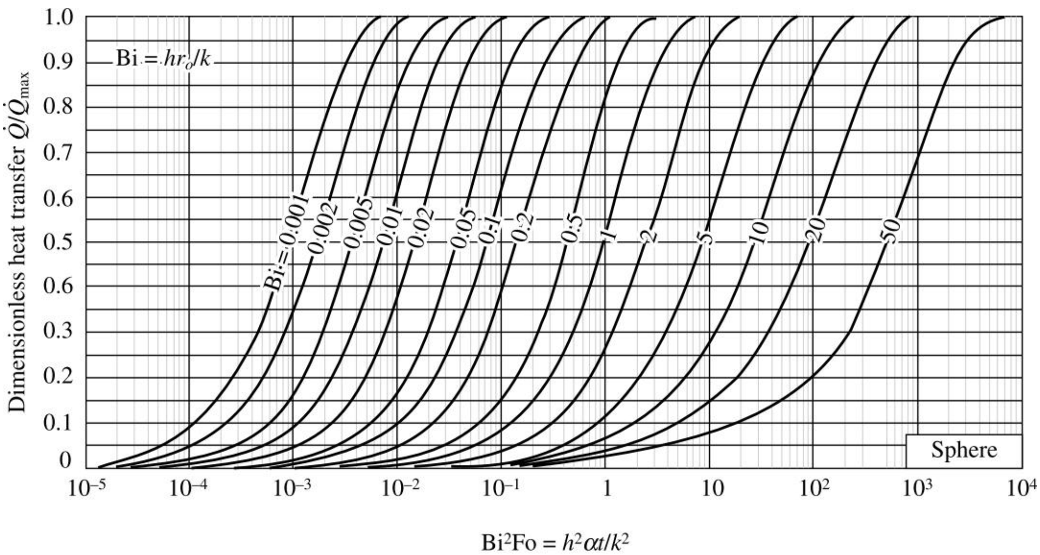


Fig. 5.18 *Dimensionless heat transfer for a sphere*

5.12 ❑ ONE-DIMENSIONAL TRANSIENT CONDUCTION IN A SEMI-INFINITE SOLID

A semi-infinite body is one in which at a given time there is always a portion of the body where the effect of heating (or cooling) is not felt at all and the temperature remains constant when a temperature change occurs on one of its boundaries. One example is the earth's crust. If the temperature on the earth's surface is changed, there is always some point below the surface that does not experience the effect of the change. Even at a distance of several metres below the surface, the surface temperature fluctuation may not be felt for a long time. Even the transient temperature distribution in a thick plane wall behaves like that of a semi-infinite solid until enough time has passed to allow any surface temperature changes to penetrate throughout the wall. Figure 5.19 represents such a semi-infinite solid.

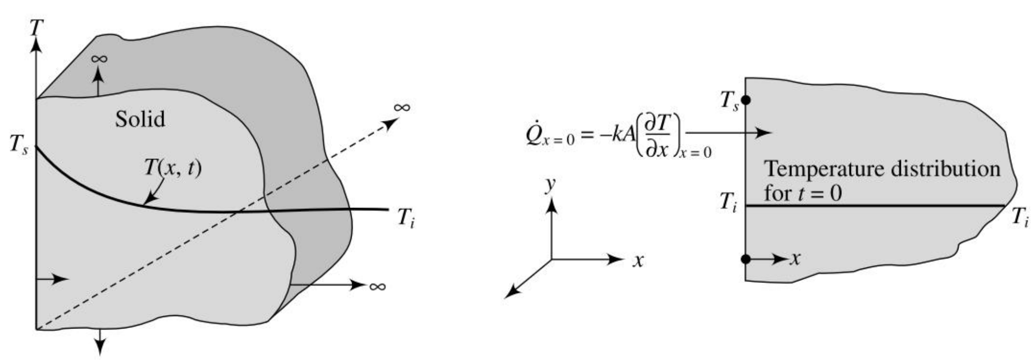


Fig. 5.19 *Schematic diagram and nomenclature for transient conduction in a semi-infinite solid.*

Many situations encountered in practice are such that the temperature changes do not penetrate far enough into the medium to have any effect on the conduction process. For example, case-hardening of tool steel involves fast quenching from a high temperature for a short duration. This results in rapid cooling and hardening of the metal close to the surface. However, the interior cools slowly after quenching and remains ductile. The process of conduction is, thus, limited to a region near the surface into which changes in temperature have been able to penetrate.

For different kinds of boundary conditions in such a semi-infinite medium, it is useful to know about the expressions giving solutions for the temperature distribution and heat-flow rate.

Closed-form solutions have been obtained for *three* types of changes in surface conditions, instantaneously applied at $t = 0$. These *three* cases are

- A sudden change in the surface temperature, $T_s \neq T_i$.
- A sudden application of a specified heat flux, q_s.
- A sudden exposure of the surface to a fluid at a different temperature through a uniform and constant heat transfer coefficient, h.

Temperature histories for the above *three* cases are illustrated qualitatively in Fig. 5.20. The solutions for all the three cases use the Gaussian error function erf (z) and complementary error function erfc(z) where z is a positive number. This standard mathematical function is tabulated in Table 5.4 for various values of z and is shown graphically in Fig. 5.21. The transient temperature solutions for a step change in (a) T = constant, (b) q_s = constant, and (c) h = constant are given below.

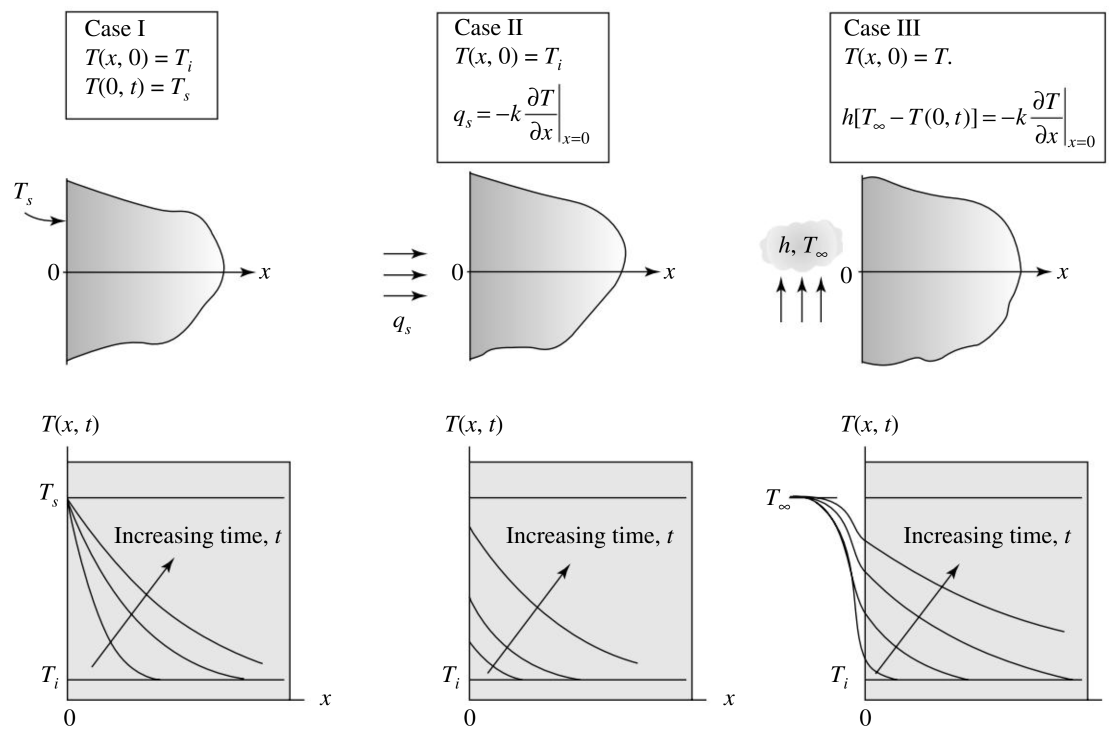


Fig. 5.20 *Three different boundary conditions for transient temperature profile of a semi-infinite solid: (a) Specified surface temperature (b) Specified surface heat flux (c) Specified convective heat-transfer coefficient*

Table 5.4 *The Gaussian complementary error functions*

z	erfc(z)	z	Erfc(z)	z	erfc(z)	z	erfc(z)	z	erfc(z)
0.00	1.0000	0.52	0.4621	1.04	0.1414	1.56	0.0274	2.08	0.00322
0.02	0.9774	0.54	0.4451	1.06	0.1339	1.58	0.0255	2.10	0.00298
0.04	0.9549	0.56	0.4284	1.08	0.1267	1.60	0.0237	2.12	0.00272
0.06	0.9324	0.58	0.4121	1.10	0.1198	1.62	0.0220	2.14	0.00247
0.08	0.9099	0.60	0.3961	1.12	0.1132	1.64	0.0204	2.16	0.00225
0.10	0.8875	0.62	0.3806	1.14	0.1069	1.66	0.0189	2.18 2.20	0.00205 0.00186
0.12	0.8652	0.64	0.3654	1.16	0.1009	1.68	0.0175	2.22	0.00169
0.14	0.8431	0.66	0.3506	1.18	0.0952	1.70	0.0162	2.26	0.00139
0.16	0.8210	0.68	0.3362	1.20	0.0897	1.72	0.0150	2.30	0.00114
0.18	0.7991	0.70	0.3222	1.22	0.0845	1.74	0.0139	2.34	0.00094
0.20	0.7773	0.72	0.3086	1.24	0.0795	1.76	0.0128	2.38 2.42	0.00076 0.00069
0.22	0.7557	0.74	0.2953	1.26	0.0748	1.78	0.0118	2.42	0.00060
0.24	0.7343	0.76	0.2825	1.28	0.0703	1.80	0.0109	2.46	0.00050
0.26	0.7131	0.78	0.2700	1.30	0.0660	1.82	0.0101	2.50	0.00041
0.28	0.6921	0.80	0.2579	1.32	0.0619	1.84	0.0093	2.55	0.00035
0.30	0.6714	0.82	0.2462	1.34	0.0581	1.86	0.0085	2.60	0.00024
0.32	0.6509	0.84	0.2349	1.36	0.0544	1.88	0.0078	2.65	0.00018
0.34	0.6306	0.86	0.2239	1.38	0.0510	1.90	0.0072	2.70	0.00013
0.36	0.6107	0.88	0.2133	1.40	0.0477	1.92	0.0066	2.75	0.00010
0.38	0.5910	0.90	0.2031	1.42	0.0446	1.94	0.0061	2.80	0.00008
0.40	0.5716	0.92	0.1932	1.44	0.0417	1.96	0.0056	2.85	0.00006
0.42	0.5525	0.94	0.1837	1.46	0.0389	1.98	0.0051	2.90	0.00004
0.44	0.5338	0.96	0.1746	1.48	0.0363	2.00	0.0047	2.95	0.00003
0.46	0.5153	0.98	0.1658	1.50	0.0339	2.02	0.0043	3.00	0.00002
0.48	0.4973	1.00	0.1573	1.52	0.0316	2.04	0.0039	3.20	0.00001
0.50	0.4795	1.02	0.1492	1.54	0.0294	2.06	0.0036	3.40	0.00000

Case 1: Constant Surface Temperature At time $t = 0$, the surface is at a uniform temperature T_i and the left face of the solid is suddenly raised to temperature T_s and held at that value.

See Fig. 5.21. The solid is initially at a uniform temperature T_i and for times $t > 0$, the boundary surface at $x = 0$ is maintained at temperature T_s. (At $x = 0$, $T = T_s$ for $t > 0$). For these boundary conditions, the non-dimensional temperature distribution in the solid is obtained as

$$\boxed{\frac{T(x,t) - T_s}{T_i - T_s} = erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-u^2)\,du} \tag{5.65}$$

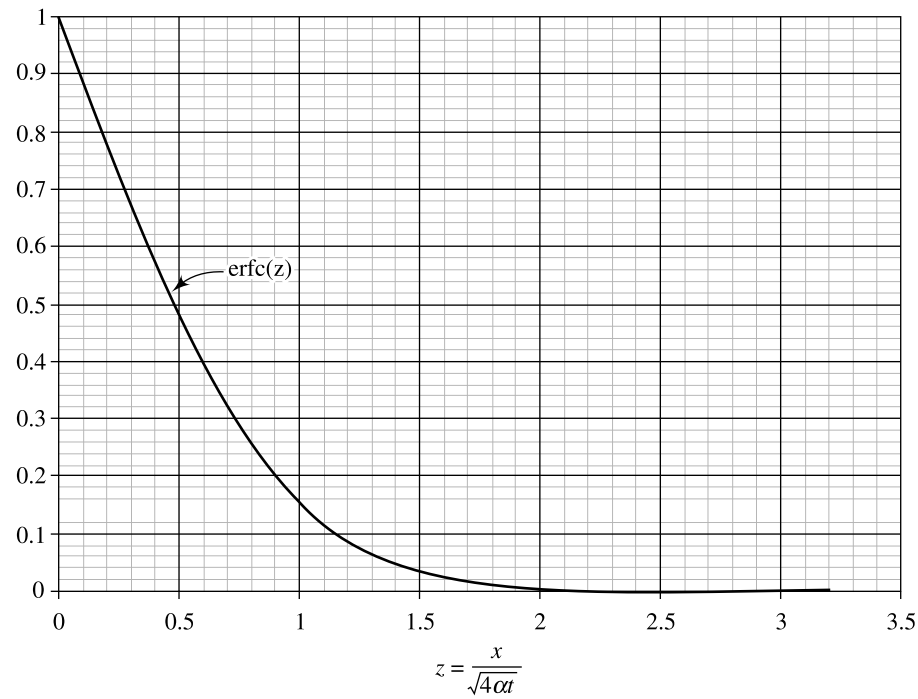


Fig 5.21 *Dimensionless temperature distribution for transient conduction in a semi-infinite solid whose surface is maintained at a constant temperature T_s.*

where the *similarity variable* $z = \dfrac{x}{2\sqrt{\alpha t}}$ and u is a *dummy variable*.

or
$$\boxed{\frac{T(x,t)-T_i}{T_s-T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \text{erfc}\left(\frac{1}{2\sqrt{Fo}}\right)} \text{ or erfc } (z) \tag{5.66}$$

where erfc is the *complementary error function* defined as erfc (z) = 1 – erf (z) and erf (z) is the *Gaussian error function.*

From Eq. (5.65),

$$\frac{d}{dx}[\text{erf}(z)] = \frac{d}{dx}\left[\frac{T(x,t)-T_s}{T_i-T_s}\right] = \frac{1}{(T_i-T_s)}\frac{dT}{dx} \tag{5.67}$$

and
$$\frac{dT}{dx} = -(T_s-T_i)\frac{d}{dx}[\text{erf}(z)]$$

The surface heat flux at any instant of time can be calculated from Fourier's law.

$$q_s = q_0 = -k\left(\frac{dT}{dx}\right)_{x=0} = k(T_s-T_i)\frac{d}{dx}[\text{erf}(z)]\bigg|_{x=0} \tag{5.68}$$

According to Taylor's theorem,

$$d[\text{erf}(z)] = d\left[\frac{2}{\sqrt{\pi}}\int_3^z e^{-u^2}du\right] = \frac{2}{\sqrt{\pi}}e^{-z^2}dz \Rightarrow \boxed{\frac{d}{dz}[\text{erf}(z)] = \frac{2}{\sqrt{\pi}}e^{-z^2}}$$

But $$z = \frac{x}{2\sqrt{\alpha t}} \quad \Rightarrow \quad \boxed{\frac{dz}{dx} = \frac{1}{2\sqrt{\alpha t}}}$$

We note that T is really a function of x and t so that what we want here is really the partial derivative of T with respect to x at some time t that we consider constant.

$$\frac{d}{dx}[\text{erf}(z)] = \frac{d}{dz}[\text{erf}(z)] \times \frac{dz}{dx} = \frac{2}{\sqrt{\pi}}e^{-z^2} \times \frac{1}{2\sqrt{\alpha t}} = \frac{e^{-z^2}}{\sqrt{\pi\alpha t}}$$

Substituting for $\frac{d}{dx}$ [erf (z)] in Eq. (5.67) and noting that $z^2 = \frac{x^2}{4\alpha t}$, we have

$$q_s = q_0 = k(T_s - T_i)\left.\frac{e^{-x^2/4\alpha t}}{\sqrt{\pi\alpha t}}\right|_{x=0}$$

∴ surface heat flux,

$$\boxed{q_s(t) = \frac{k(T_s - T_i)}{(\pi\alpha t)^{1/2}}} \qquad \text{(W/m}^2\text{)} \tag{5.69}$$

Heat-transfer rate at time t at the surface (x = 0) is given by

$$\dot{Q}_s = q_s \times A = kA\frac{T_s - T_i}{(\pi\alpha)^{1/2}} \times t^{-1/2} \qquad \text{(W)}$$

The total amount of energy Q which has entered the surface of the semi-infinite solid in the time interval 0 to t can be obtained as follows:

$$Q = \int_0^t \dot{Q}_s\,dt = kA\frac{T_s - T_i}{(\pi\alpha)^{1/2}}\int_0^t t^{-1/2}dt = kA\frac{T_s - T_i}{(\pi\alpha)^{1/2}}\left(\frac{t^{1/2}}{1/2}\right)_0^t$$

or $$\boxed{Q = 2kA\frac{T_s - T_i}{\sqrt{\pi\alpha/t}}} \qquad \text{(J)} \tag{5.70}$$

Case 2. Constant Surface Heat Flux At time t = 0, the surface is suddenly exposed to a constant heat flux q_s,

At $$x = 0, \quad q_s = q_0 = -k\frac{\partial T}{\partial x} \quad \text{for } t > 0$$

The temperature distribution is

$$\boxed{T(x,t) - T_i = \frac{2q_s\sqrt{(\theta t/\pi)}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q_s\,x}{k}\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)} \tag{5.71}$$

or $$\boxed{T - T_i = \frac{q_s}{k}\left[\left(\frac{4\alpha t}{\pi}\right)^{1/2} \exp(-x^2/4\alpha t) - x \operatorname{erfc}\frac{x}{(4\alpha t)^{1/2}}\right]}$$

Case 3: Constant Convective Heat-Transfer Coefficent At time $t = 0$, the surface is suddenly exposed to a fluid at temperature T_∞ with a convective heat-transfer coefficient, h.

At $$x = 0, \quad -k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0, t)]$$

The temperature distribution is

$$\boxed{\frac{T(x, t) - T_i}{T_\infty - T_i} = \left\{\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right\} - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]} \tag{5.72}$$

Note that the quantity $h^2\alpha t/k^2$ equals the product of the Biot number squared, $\left(Bi^2 = \frac{h^2x^2}{k^2}\right)$ and Fourier number ($Fo = \alpha t/x^2$).

In actual practice, Case 1 is approximated when there is a condensing or boiling fluid (with very high heat-transfer coefficient) brought in contact with the solid surface. Note that this case is equivalent to Case 3 (convective boundary condition) with $h \to \infty$ which results in the surface temperature, T_s equalling the fluid temperature T_∞.

Because for $h = \infty$, the second term on the right-hand side of Eq. (5.71) is zero, and the result is equivalent to Eq. (5.66) for Case 1.

Case 2 occurs when, for instance, on electric resistance heater is pressed against a surface so that a known and constant wall heat flux is imposed. This case is also an approximation of the situation when a high temperature radiation source is directed towards a surface with a much lower temperature.

Figure 5.22 shows the values for the temperature excess ratio calculated from Eq. (5.71) as a function of the dimensionless distance $x/2\sqrt{\alpha t}$ with the local Biot number hx/k as the parameter. When location and temperature ratio are specified, time can be obtained directly from the chart. Figure 5.23 presents the same temperature excess ratio plotted against $x/2\sqrt{\alpha t}$ with $h^2\alpha t/k^2$ as a parameter. If the excess temperature ratio and time are known, this chart should be used to obtain the location directly.

Penetration Depth *Penetration depth,* δ at any time t is the distance x from the surface where the temperature change is within 1% of the change in the surface temperature.

i.e., $$\frac{T(x, t) - T_i}{T_s - T_i} = \operatorname{erfc}(z) = 0.01 \text{ for which } z = \frac{x}{2\sqrt{\alpha t}} = 1.8 \qquad \text{(from Table 5.4)}$$

so that the *penetration depth*, δ is given by $\delta = 3.6\sqrt{\alpha t}$ (5.73)

We note that the penetration depth increases as the *square root of time*.

Penetration time, t_p is the time taken for the surface perturbations to be felt at that depth, δ.

Hence, $$\frac{\delta}{\sqrt{\alpha t_p}} = 3.6 \quad \Rightarrow \quad t_p = \frac{\delta^2}{13\alpha} \quad \text{or} \quad 0.077\,\delta^2/\alpha \tag{5.74}$$

If the penetration depth, δ, is small compared to the body dimensions, the asumption of a *semi-infinite solid* is valid.

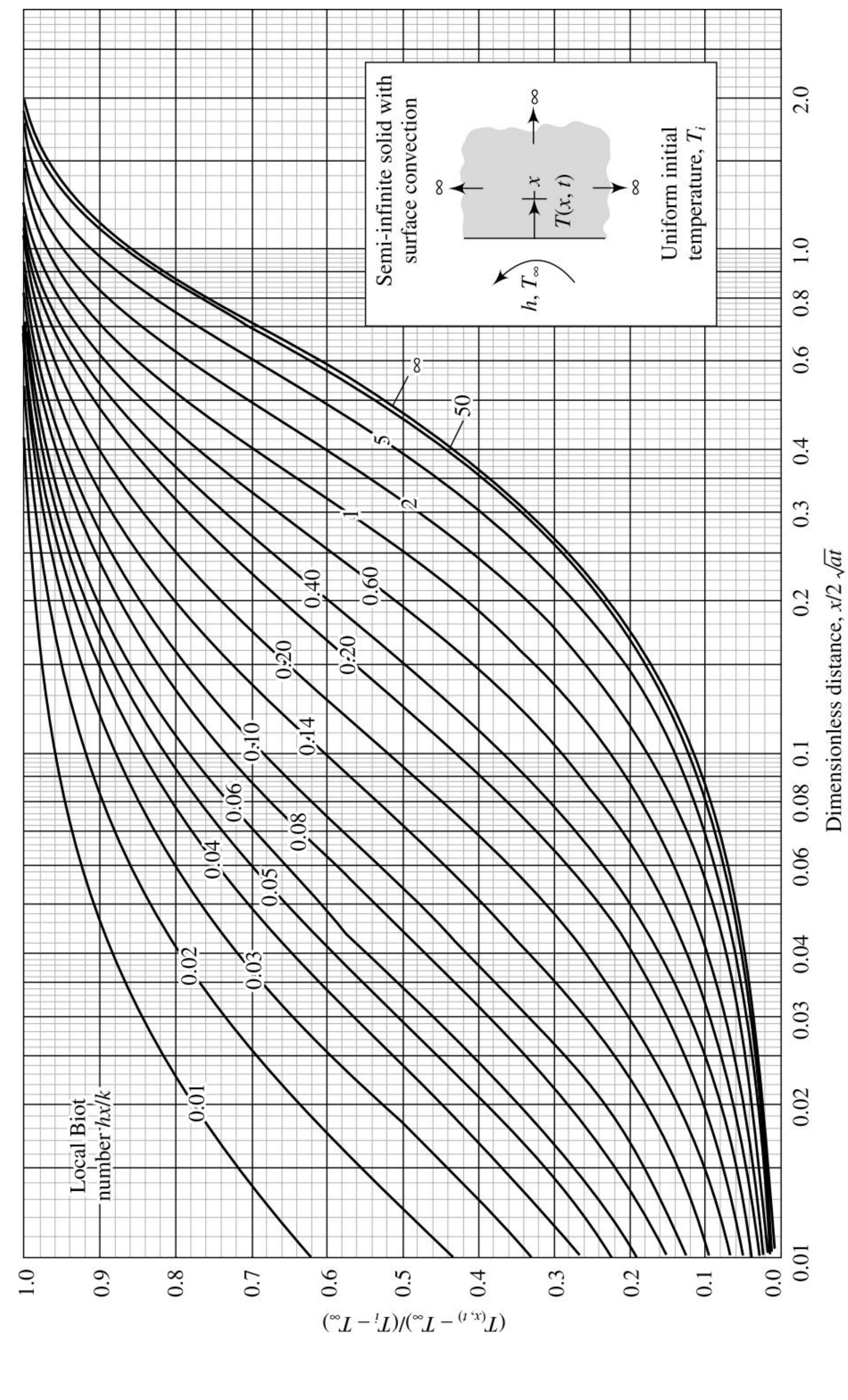


Fig. 5.22 *Transient temperature T(x, t) in a semi-infinite solid subjected to convection with hαt/k² as parameter*

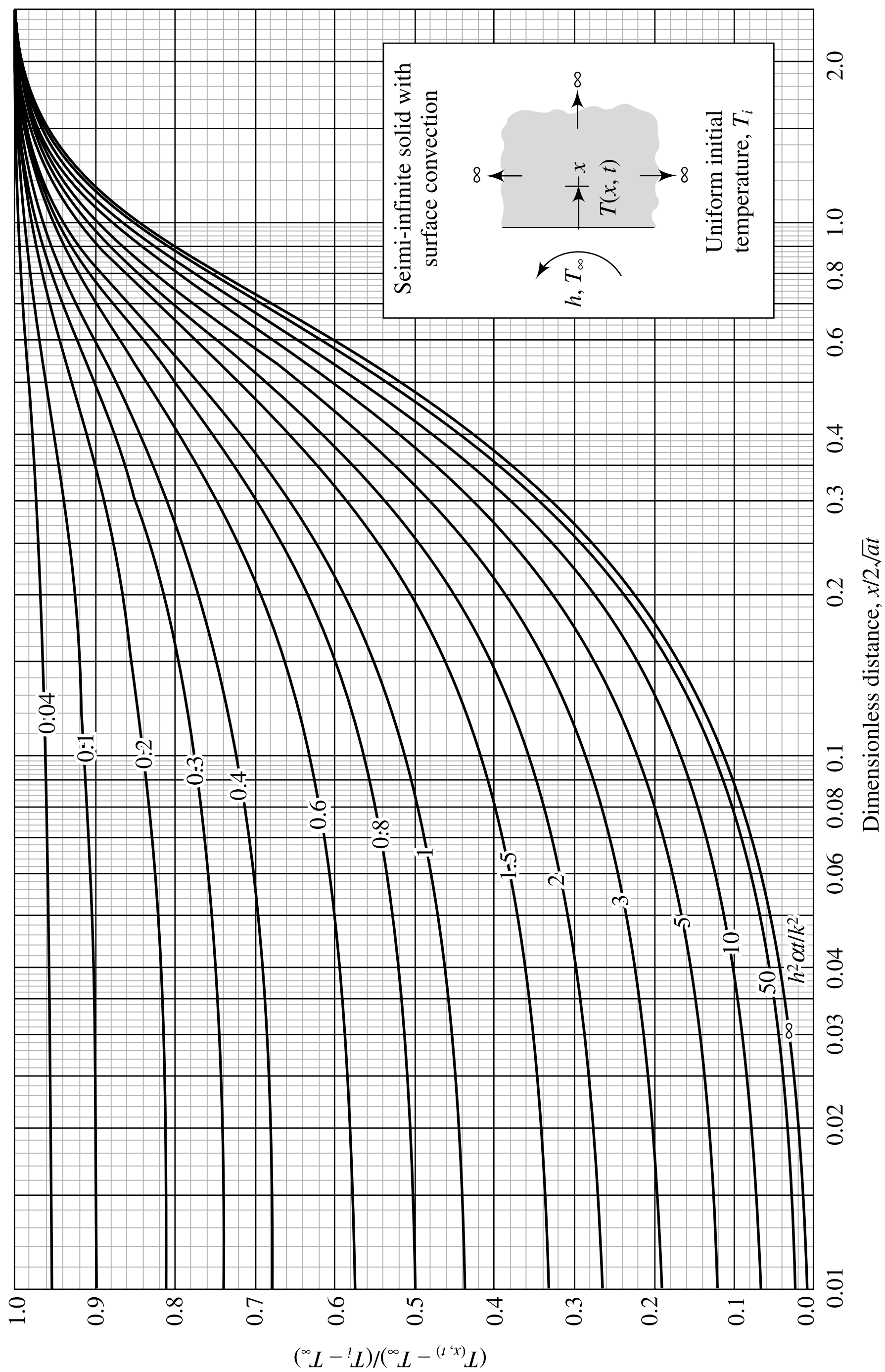


Fig. 5.23 *Transient temperature T(x, t) in a semi-infinite solid subjected to convection with hαt/k² as parameter*

5.13 ❑ MULTIDIMENSIONAL TRANSIENT HEAT CONDUCTION: PRODUCT SOLUTION

So far we have discussed only *one-dimensional* heat conduction problems related to a *large plane wall, long cylinder, a sphere and a semi-infinite medium*. However, in many practical situations, the assumption of one-dimensional conduction may not be valid. For instance, in a *short cylinder* whose length is comparable to diameter, clearly the temperture gradients will be significant in both the longitudinal and radial directions. The heat transfer will, therefore, be two-dimensional. Similarly, for a *long rectangular bar*, one can recognize that heat transfer will be significant in both *x*- and *y*-diretions, and in a *rectangular solid block* (*parallelepiped*), the heat transfer will be *three-dimensional*.

In such cases, for a *two-dimensional* system, with no internal heat generation, one can construct the solutions for dimensionless temperature distribution in transient heat conduction, by combining the solutions of dimensionless temperature distributions obtained for one-dimensional solutions of the individual systems which form the two-dimensional body by their intersection.

Hence, in general, one can write $\boxed{\theta_{\text{solid}} = \theta_{\text{system 1}} \times \theta_{\text{system 2}} \times \theta_{\text{system 3}}}$ for a *three-dimensional* body with System 1, System 2, and System 3 representing the *one-dimensional* systems which by their intersection make up the body. θ is the dimensionless temperature distribution of the one-dimensional system, which is available from the *charts* or *one-term approximation solutions*.

5.13.1 • Transient Heat Conduction in a Long Rectangular Bar

Consider the case of a long solid bar of rectangular cross section of width $2L_1$ and height $2L_2$ as shown in Fig. 5.24.

The bar is surrounded by a fluid at T_∞ which removes heat by convection with an associated convective heat-transfer coefficient *h*. Initially, the temperature distribution in the bar is specified as a given function of x_1 and x_2. The problem involves a temperature field which is a function of *three* variables—*two* space variables x_1 and x_2 and *one* time variable *t*.

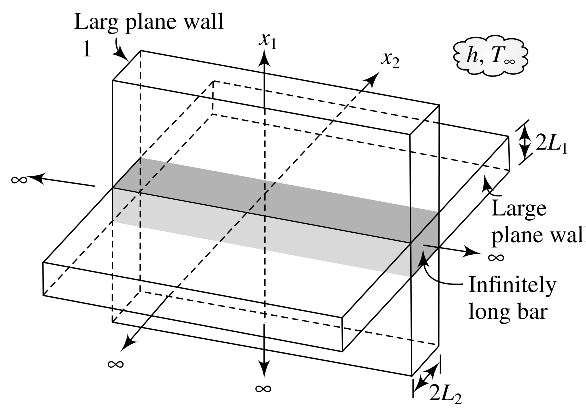


Fig. 5.24 *A long rectangular solid bar formed by the intersection of two plane walls*

$$\boxed{\begin{aligned}\frac{T(x_1, x_2, t) - T_\infty}{T_i - T_\infty} &= \theta_{\text{rect bar}}(x_1, x_2, t) \\ &= \theta_{\text{wall}}(x_1, t) \times \theta_{\text{wall}}(x_2, t)\end{aligned}} \tag{5.75}$$

5.13.2 • Transient Heat Conduction in a Short (*Finite*) Cylinder

Consider a short cylinder of height $2L$ as shown in Fig. 5.25.

We can imagine this body formed by the *intersection* of a large plane wall of thickness $2L$ and a long cylinder of radius r_o. We can have the convective heat-transfer coefficient, *h*, on the cylindrical face of the body which may be different from the convective heat transfer, h_p, at the top and bottom of the cylinder. However, for this case, T_∞ must be the same over all surfaces of the body. Placing the origin

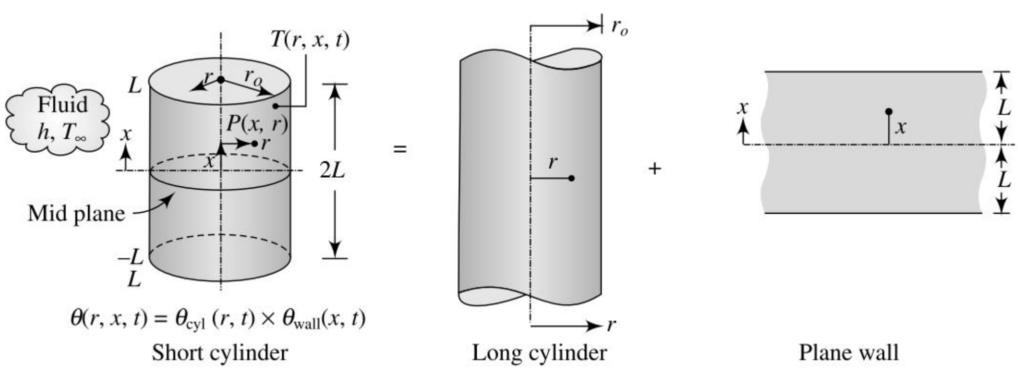


Fig. 5.25 *Geometric arrangement for product solution of a short cylinder*

at the midplane of the plane wall and at the axis of the cylinder. The temperature distribution in this body is given by

$$\boxed{\frac{T(r, x, t) - T_\infty}{T_i - T_\infty} = \theta_{\text{short cyl}}(r, x, t) = \theta_{\text{wall}}(x, t) \times \theta_{\text{cyl}}(r, t)} \tag{5.76}$$

where θ_{wall} (x, t) represents a transient solution for a *large plane wall* of thickness $2L$ and θ_{cyl} (r, t) represents a transient solution for a *long cylinder* of radius r_o.

5.13.3 • Transient Heat Conduction in a Rectangular Parallelepiped (Block)

Consider a rectangular block of a material as shown in Fig. 5.26. The block may be looked upon as being obtained by the intersection at right angles to one another of *three* plane walls of thicknesses $2L_1$, $2L_2$, and $2L_3$ parallel to the three coordinate directions. With proper boundary conditions, the product solution can be obtained in terms of three one-dimensional plane wall solutions. The one-term approximation or chart solution may be used. Three Biot numbers and *three* Fourier numbers are involved in this case, because *three* characteristic length dimensions are involved in this geometry. The solution may be represented by the product solution given by

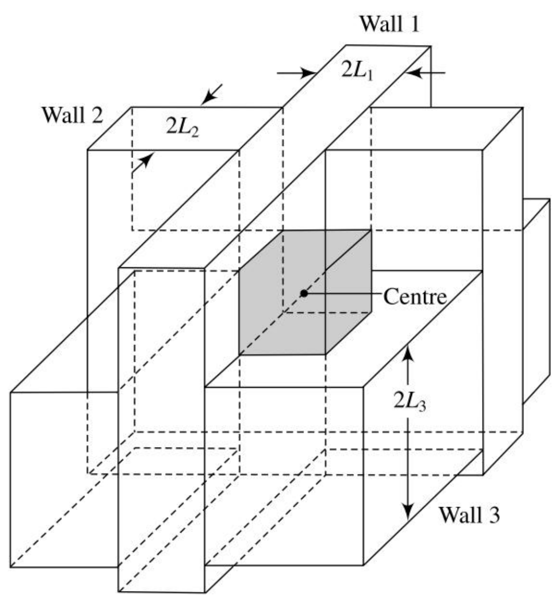
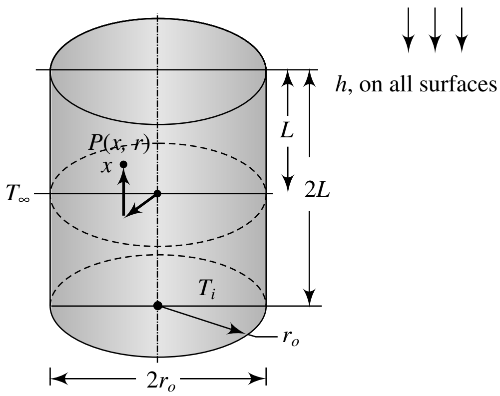


Fig. 5.26 *Three-dimensional transient heat conduction in a rectangular block*

$$\boxed{\begin{aligned}\frac{T(x_1, x_2, x_3, t) - T_\infty}{T_i - T_\infty} &= \theta(x_1, x_2, x_3, t) \\ &= \theta_{\text{wall}}(x_1, t) \times \theta_{\text{wall}}(x_2, t) \times \theta_{\text{wall}}(x_3, t)\end{aligned}} \tag{5.77}$$

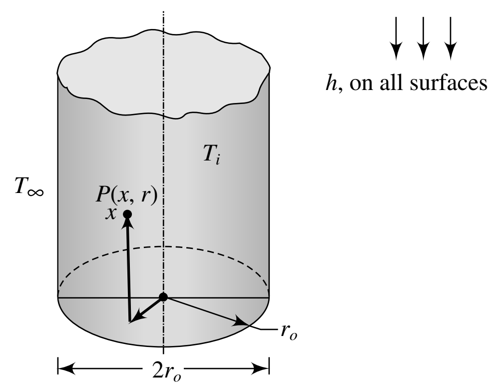
The geometries which can be analyzed by the product solution method are illustrated in Fig. 5.27 along with the corresponding product solutions for the cases shown. In performing the analysis, the following abbreviated nomenclature has been used.

$$\boxed{\begin{aligned}\theta_{\text{wall}(x,t)} &= \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\text{wall}}\\ \theta_{\text{cyl}(r,t)} &= \left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\text{long cyl}}\\ \theta_{\text{semi inf}(x,t)} &= \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\text{semi inf}}\end{aligned}} \tag{5.78}$$



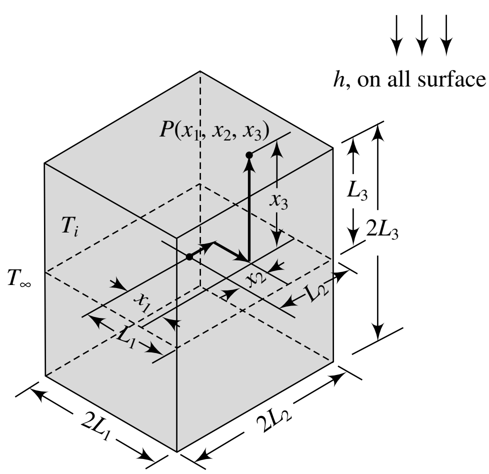
$\theta(x, r, t) = \theta_{\text{wall}}(x, t) \times \theta_{\text{cyl}}(r, t)$

(a) Short cylinder



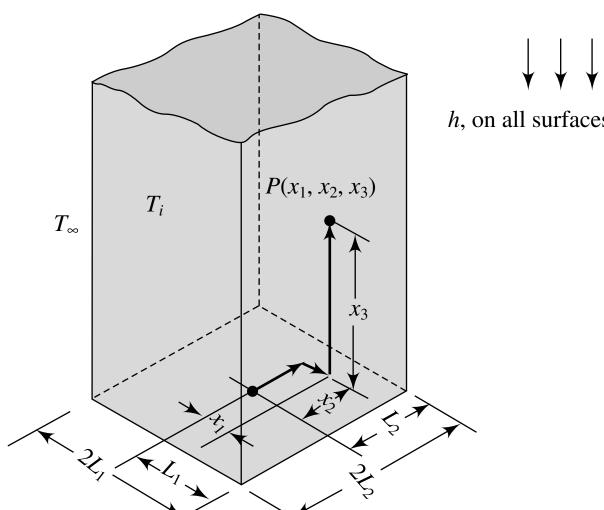
$\theta(x, r, t) = \theta_{\text{cyl}}(x, t) \times \theta_{\text{semi-inf}}(x, t)$

(b) Semi-infinite cylinder



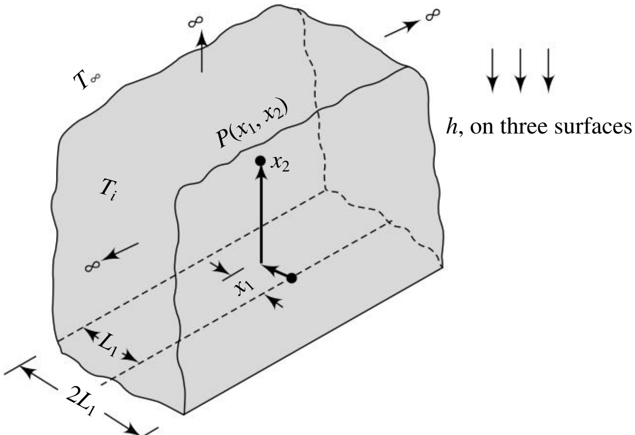
$\theta(x_1, x_2, x_3) = \theta_{\text{wall}}(x_1, t) \times \theta_{\text{wall}}(x_2, t) \times \theta_{\text{wall}}(x_3, t)$

(c) Rectangular parallelopiped



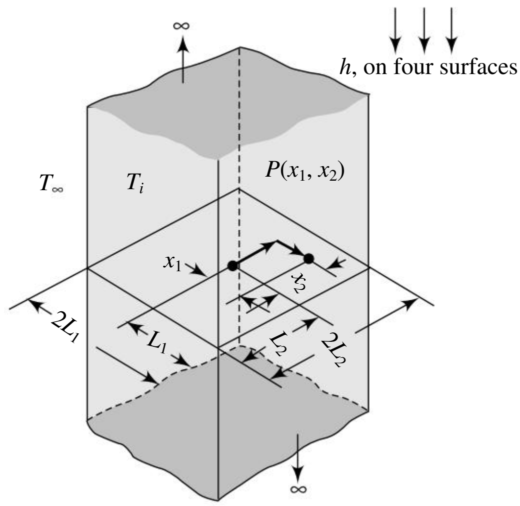
$\theta(x_1, x_2, x_3, t) = \theta_{\text{wall}}(x_1, t) \times \theta_{\text{wall}}(x_2, t) \times \theta_{\text{semi-inf}}(x_3, t)$

(d) Semi-infinite rectangular bar



$\theta(x_1, x_2, t) = \theta_{wall}(x_1, t) \times \theta_{semi\text{-}inf}(x_2, t)$

(e) Semi-infinite plate



$\theta(x_1, x_2, t) = \theta_{wall}(x_1, t) \times \theta_{wall}(x_2, t)$

(f) Infinite rectangular bar

Fig. 5.27 *Product solutions for temperatures in multi-dimensional systems: (a) Short cylinder (b) Semi-infinite cylinder (c) Rectangular parallelepiped (d) Semi-infinite rectangular bar (e) Semi-infinite plate (f) Infinite rectangular bar.*

5.14 ❑ HEAT TRANSFER IN TRANSIENT CONDUCTION IN MULTI-DIMENSIONAL SYSTEMS

The total heat transfer to or from a multidimensional solid after a certain period of time can be found using the one-dimensional solutions.

For a two-dimensional body formed by the intersection of *two* one-dimensional systems 1 and 2:

$$\boxed{\left(\frac{Q}{Q_{max}}\right)_{total} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right]} \tag{5.79}$$

For a three-dimensional body formed by the intersection of *three* one-dimensional systems 1, 2, and 3, one has:

$$\boxed{\begin{aligned}\left(\frac{Q}{Q_{max}}\right)_{total} &= \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right] \\ &\quad + \left(\frac{\dot{Q}}{Q_{max}}\right)_3 \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right]\left[1 - \left(\frac{Q}{Q_{max}}\right)_2\right]\end{aligned}} \tag{5.80}$$

The *product solutions* are ***not*** applicable when

- The initial temperature of the body is *not* uniform
- The fluid temperature T_∞ is *not* the same on all sides of the body
- The body involves heat generation
- The surface boundary conditions are of the second kind (*specified heat flux at the surface*)

It is important to recognize the following:

- Appropriate convective heat-transfer coefficient, h, associated with the surface to be analyzed must be used. It should be noted that the value of h may be different for the different geometries that make up the overall solution and that this approach is justified only for bodies whose initial temperature is uniform throughout.
- Dimensionless temperatures for the one-dimensional systems used to form the product solution for the two/three-dimensional body, must be chosen at the correct locations. In doing so, we must remember that for a *semi-infinite plate*, x is measured from the *surface* while for a plane wall, x is measured from the *mid-plane*.
- If the temperature is to be calculated after a given time for the *multi-dimensional* body, the solution is straightforward. However, if the time is to be calculated to attain a given temperature, then, a *trial and error* solution will be necessary.

5.15 ❑ PERIODIC VARIATION OF SURFACE TEMPERATURE

There are many systems in which the surface temperature varies periodically. Periodic heat flow occurs in *reciprocating internal combustion engines, cyclic regenerators* and *in the earth* due to diurnal (*daily*) cycle of the sun. These periodic changes are sinusoidal and complex. The surface temperature T_o of an a semi-infinite solid (*likely the earth's crust*) oscillates periodically about a mean temperature T_m in a sinusoidal fashion along the distance or depth from the surface at any instant as shown in Fig. 5.28.

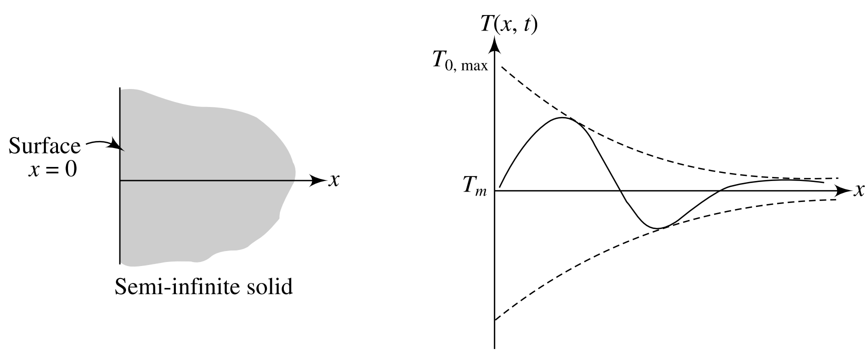


Fig. 5.28 *Periodic surface temperature variation in a semi-infinite solid.*

The nomenclature commonly used in such cases is indicated in Fig. 5.29 in which the temperature fluctuation is shown against time at the surface ($x = 0$) and at the depth x. It may be noticed that the amplitude of the temperature variation decays exponentially while a phase lag or time lag develops.

Let the variation of surface temperature T_0 of the solid in excess of the mean surface temperature be expressed as a cosine function of time as follows.

$$\theta_0 = (T_0 - T_m) = (T_0 - T_m)_{\max} (\cos \omega t)$$

$$= (T_0 - T_m)_{\max} (\cos 2 \pi f t) \qquad (\text{at } x = 0,\ t > 0)$$

or
$$\theta_0 = \theta_{0,\min} (\cos \omega t) \tag{5.81}$$

where T_0 = surface temperature of the solid at $x = 0$ at any time t.

T_m = surface mean temperature of the surface at $x = 0$.

$\theta_{0,\max} = T_{0,\max} - T_m$ = maximum excess of surface temperature over the mean value, i.e., amplitude of surface temperature variation at $x = 0$.

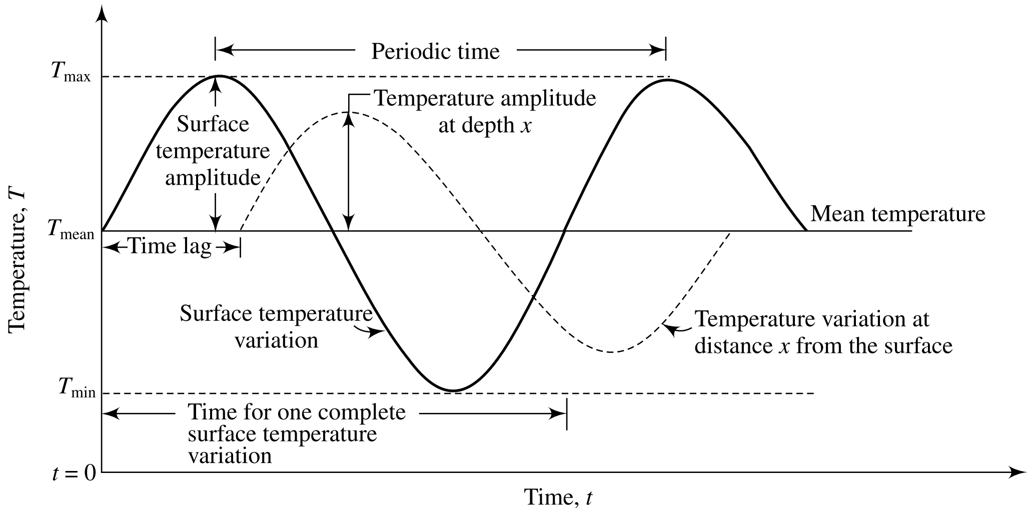


Fig. 5.29 *The nomenclature of periodic (long time) temperature response of a semi-infinite solid.*

$\theta_0 = (T_0 - T_m)$ = excess temperature at the surface at any time t.

$\omega = 2\pi f = 2\pi / P$

where $f \equiv$ frequency of temperature wave, i.e., number of complete changes per unit time P = period of oscillation = $1/f$.

Our objective is to determine the temperature at any depth x at any time t, that is, $T(x, t)$. Since T_m is a constant (*the value of which is known*) this problem can be equally well solved by finding $\theta\,(x, t)$.

The governing differential equation is $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial \mathrm{T}}{\partial \mathrm{t}}$.

or $$\frac{\partial^2 (T - T_m)}{\partial x^2} = \frac{1}{a}\frac{\partial^2 (T - T_m)}{\partial t} \qquad (\textit{since } T_m \textit{ is constant})$$

or $$\boxed{\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}} \tag{5.82}$$

The boundary conditions are:

$$\left.\begin{aligned} At\ t &= 0 \\ x &= 0 \end{aligned}\right\} \quad \theta_0 = \theta_{0,\max}$$

$$\left.\begin{aligned} At\ t &= 0 \\ x &= 0 \end{aligned}\right\} \quad \theta_0 = \theta_{0,\max} \cos \omega t$$

(a) Let us assume that the solution of Eq. (5.82) is of the form

$$\boxed{\theta_x = (\theta_{0,\max})\, e^{-mx} \cos(\omega t - mx)} \tag{5.83}$$

It can be easily seen that Eq. (5.83) satisfies the above boundary conditions. However, to verify whether Eq. (5.83) satisfies Eq. (5.82), let us differentiate Eq. (5.85) partially with respect to t and x. From Eq. (5.83),

$$\frac{\partial \theta}{\partial t} = (\theta_{0,\max}) e^{-mx} (-\omega) \sin(\omega t - mx) = -\omega(\theta_{0,\max}) e^{-mx} \sin(\omega t - mx) \tag{5.84}$$

$$\frac{\partial \theta}{\partial x} = -m(\theta_{0,\max}) e^{-mx} \cos(\omega t - mx) + m(\theta_{0,\max}) e^{-mx} \sin(\omega t - mx)$$

and
$$\frac{\partial^2 \theta}{\partial x^2} = m^2(\theta_{0,\max}) e^{-mx} \cos(\omega t - mx) - m^2(\theta_{0,\max}) e^{-mx}$$

$$\sin(\omega t - mx) - m^2(\theta_{0,\max}) e^{-mx} \sin(\omega t - mx) - m^2(\theta_{0,\max}) e^{-mx} \cos(\omega t - mx)$$

$$= -2m^2(\theta_{0,\max}) e^{-mx} \sin(\omega t - mx) \tag{5.85}$$

Substituting Eqs. (5.84) and (5.85) in Eq. (5.82),

$$-\omega(\theta_{0,\max}) e^{-mx} \sin(\omega t - mx) = -2\alpha m^2(\theta_{0,\max}) e^{-mx} \sin(\omega t - mx)$$

or
$$\boxed{\omega = 2\alpha m^2} \tag{5.86}$$

Thus, Eq. (5.83) satisfies Eq. (5.82) provided that

$$m^2 = \frac{\omega}{2\alpha} \quad \text{or} \quad \boxed{m = \pm\sqrt{\frac{\omega}{2\alpha}}} \tag{5.87}$$

Note that $m = -\sqrt{(\omega/2\alpha)}$ is rejected because this will make the amplitude $\theta_{x,\max}$ at depth x, infinite at large values of x. This is clearly not possible. Hence, the complete solution of Eq. (5.82) is

$$\boxed{\theta = (\theta_{0,\max}) e^{-\sqrt{(\omega/2\alpha)x}} \cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}\, x\right)} \tag{5.88}$$

The temperature variation with time at any depth x, given by Eq. (5.88) is shown in Fig. 5.30*,* while Fig. 5.31 shows the temperature distribution as a function of depth at $\omega t = 0$ and at $\omega t = \pi$.

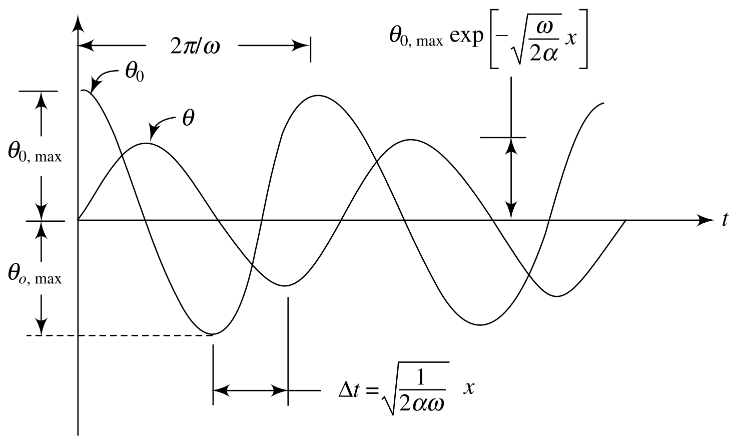


Fig. 5.30 *Periodic temperature variation with time.*

Several important conclusions can be drawn from Eq. (5.88).

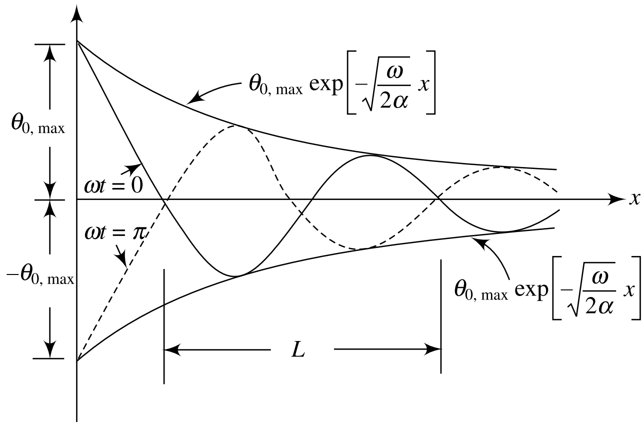


Fig. 5.31 *Periodic temperature distribution as a function of depth.*

Amplitude at any Depth x The maximum excess temperature or amplitude at any depth x, is obtained by putting:

$$\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x\right) = 1$$

Thus, the amplitude at a depth x is:

$$\boxed{\theta_{\max} = (\theta_{0,\max})\, e^{-\sqrt{(\omega/2\alpha)}\,x}} \tag{5.89}$$

Equation (5.89) shows that the amplitude of temperature variation decreases exponentially with increasing depth as illustrated by Fig. 5.31. The amplitude may become negligible at a certain depth.

The lower the thermal diffusivity, the smaller will be the depth or thickness of the wall at which the amplitude would be negligible. In the case of earth, at a depth of nearly 8 m, this happens and the temperature remains almost constant throughout the year. It is also noteworthy that the amplitude decreases if ω increases or α decreases.

Time Lag or Phase Difference at Depth x According to Eq. (5.81), the maximum fluctuation in surface temperature occurs when cos (ωt) = 1 or when t = 0, $2\pi/\omega$, $4\pi/\omega$,..., $2n\pi/\omega$ where n = 0, 1, 2, 3, … etc. Under these conditions $\theta_{0,t} = \theta_{0,\max}$. However, at any depth x, from Eq. (5.88) the maximum fluctuation in temperature will occur when

$$\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x\right) = 1 \text{ or } \omega t - \sqrt{\frac{\omega}{2\alpha}}x \ = 0,\ 2\pi,\ 4\pi,\ \ldots,\ \text{etc.} = 2n\pi$$

or

$$\boxed{t = \sqrt{\frac{1}{2\alpha\omega}}x + \frac{2n\pi}{\omega}} \tag{5.90}$$

With $\quad n = 0, \boxed{t = \sqrt{\frac{1}{2\alpha\omega}}\,x}$

Thus, while the excess temperature at the surface is maximum when $t = 2n\pi/\omega$ it is maximum at the depth x later when $t = (2n\pi/\omega + \sqrt{(1/2\,\alpha\omega)}\,x)$. This delay or phase difference is called *time lag*, Δt and is given by:

$$\boxed{\Delta t = \left\{\frac{2n\pi}{\omega} + \sqrt{\frac{1}{2\alpha\omega}}x\right\} - \frac{2n\pi}{\omega}} \quad \text{or} \quad \boxed{\Delta t = \sqrt{\frac{1}{2\alpha\omega}}x} \tag{5.91}$$

Significantly, the time lag increases with increasing x, and decreases with increasing α and ω. It is because of this time lag that the inside surface of the walls of a building attain the maximum temperature at about 6 p.m. whereas the outside surface exposed to the sun is hottest at about 3 p.m.

Form and Frequency of Wave at any Depth x Equation (5.89) shows that the form of the temperature wave does not change with depth. It remains a sinusoidal cosine wave. Also, from Eq. (5.90), it is seen that both the time interval between two successive maxima ($2\pi/\omega$), and the frequency of the wave ($\omega/2\pi$) do not change with depth.

Wavelength and Wave Velocity It is easily seen that the wavelength of a wave is the distance between two adjacent crests. Now, at $x = 0$, the maximum occurs at $t = 0$. The next maximum at $x = 0$ occurs when $t = t_c = 2\pi/\omega$ (Fig. 5.30). But at time equal to t_c, a maximum also occurs at depth x given by:

$$\cos\left(\omega t_c - \sqrt{\frac{\omega}{2\alpha}}x\right) = 1 \text{ or } \omega t_c - \sqrt{\frac{\omega}{2\alpha}}x = 0$$

or

$$x = \omega t_c \bigg/ \sqrt{\frac{\omega}{2\alpha}} \sqrt{2\omega\alpha t_c}$$

The distance x given by the last equation is, therefore, the wavelength (L) of the temperature.

$$\boxed{L = 2\pi\sqrt{\frac{2\alpha}{\omega}}} \tag{5.92}$$

Since, the velocity of a wave is the product of frequency and wavelength, the velocity V of the temperature wave is given by:

$$V = fL = \frac{L}{t_c} = \frac{\omega L}{2\pi} = 2\pi\sqrt{\frac{2\alpha}{\omega}}\left(\frac{\omega}{2\pi}\right)$$

or

$$\boxed{V = \sqrt{2\alpha\omega}}$$

Hence, the velocity with which the temperature wave penetrates the surface depends both on ω and α. The distance traversed by the wave during the time interval t_1 to t_2 is given by $V(t_2 - t_1) = (t_2 - t_1)\sqrt{2\alpha\omega}$.

Figure 5.32 shows the temperature distribution in the wall at two instants t_1 and t_2 where $t_2 > t_1$. The displacement of the wave at time ($t_2 > t_1$) is also shown. The upper and lower boundary curves represent $(\theta_{max}) = (\theta_{0,max})e^{-mx}$ and $-(\theta_{max}) = -(\theta_{0,max})e^{-mx}$ respectively.

Heat Flow Rate and Energy Storage Besides the temperature distribution, one is also interested in the rate of heat flow *into* or *out* of the plate at any instant and the energy stored in the wall every half cycle.

Since the rate at which heat enters the wall is

$$\dot{Q} = -kA\left(\frac{\partial T}{\partial x}\right)_{x=0} = -kA\left(\frac{\partial \theta}{\partial x}\right)_{x=0}$$

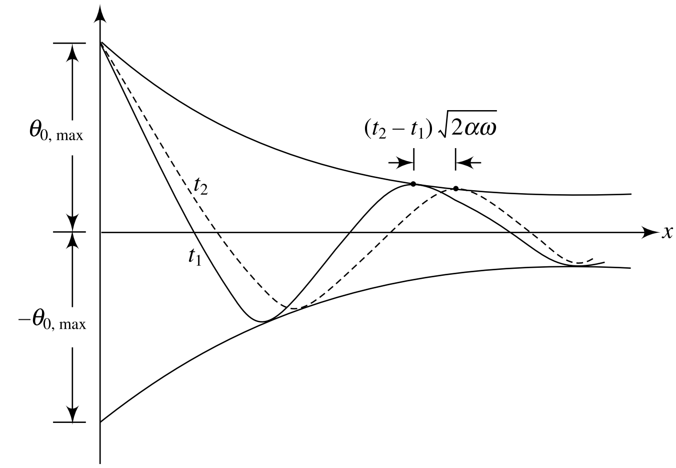


Fig. 5.32 *Displacement curve of a temperature wave.*

From Eq. (5.83), one gets

$$\dot{Q} = -kAm(\theta_{0,\max})e^{-mx}[\sin(\omega t - mx) - \cos(\omega t - mx)]_{x=0}$$

Heat flow at the surface is obtained by putting $x = 0$,

$$\dot{Q}_o = -kAm(\theta_{0,\max})(\cos\omega t - \sin\omega t)$$

$$= kA\sqrt{\frac{\omega}{2\alpha}}(\theta_{0,\max})(\cos\omega t - \sin\omega t)$$

or $$\dot{Q}_o = kA\sqrt{\frac{\omega}{\alpha}}(\theta_{0,\max})\left(\frac{\cos\omega t}{\sqrt{2}} - \frac{\sin\omega t}{\sqrt{2}}\right)$$

Hence, $$\boxed{\dot{Q}_o = kA\sqrt{\frac{\omega}{\alpha}}(\theta_{0,\max})\cos\left(\omega t + \frac{\pi}{4}\right)} \quad \text{(W)} \qquad (5.93)$$

Note that $\dot{Q}_0$ is *positive* in the limits $(\omega t + \pi/4) = -\pi/2$ to $+\pi/2$ and *negative* in the limits $(\omega t + \pi/4) = +\pi/2$ to $-\pi/2$. In other words, $\dot{Q}$ is *positive* between the limits:

$$t = \frac{1}{\omega}\left[-\frac{\pi}{2} - \frac{\pi}{4}\right] = -\frac{3\pi}{4\omega} \text{ and } t = \frac{1}{\omega}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = +\frac{\pi}{4\omega}$$

Integrating Eq. (5.94) between the two limits, we get the energy stored, Q, in a half-cycle.

$$Q = kA\sqrt{\frac{\omega}{\alpha}}(\theta_{0,\max})\int_{-3\pi/4\omega}^{\pi/4\omega}\cos(\omega t + \pi/4)\,dt$$

$$= kA\sqrt{\frac{\omega}{\alpha}}(\theta_{0,\max})\left[\frac{1}{\omega}\sin\left(\omega t + \frac{\pi}{4}\right)\right]_{3\pi/4\omega}^{\pi/4\omega} = kA\sqrt{\frac{\omega}{\alpha}}(\theta_{0,\max})\left(\frac{2}{\omega}\right)$$

or $$\boxed{Q = kA\frac{2}{\sqrt{\omega\alpha}}(\theta_{0,\max})} \quad \text{(J)} \qquad (5.94)$$

(b) Note that we had considered in Eq. (5.83) a cosine variation of temperature with time and obtained result as Eq. (5.88). If the surface temperature varies as a sine function, i.e.,

$$\boxed{\theta_0 = (\theta_{\theta,\max}) \sin \omega t}$$

It can be shown, by following an exactly identical procedure that

$$\boxed{\theta = (\theta_{0,\max}) e^{-\sqrt{(\omega/2\alpha)}x} \sin\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x\right)} \tag{5.95}$$

Furthermore, the time lag Δt, the amplitude at depth, x, $\theta_{\max}$ the wavelength L, the velocity V, and the energy stored in half-cycle Q, are all given by the same equations which were derived above for the cosine variation. Only in Eq. (5.95) for $\dot{Q}$ one has to replace cos ($\omega t - \pi/4$) by sin ($\omega t + \pi/4$).

Illustrative Examples

(A) Transient Temperature Variation

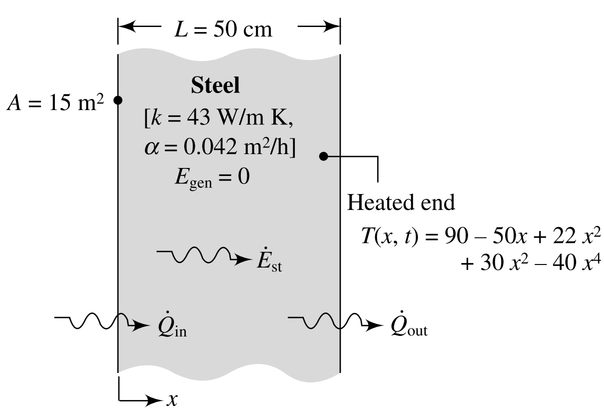
EXAMPLE 5.1 *At a certain instant the temperature distribution through a large plane steel wall [k = 43 W/m K, α = 0.042 m²/h], 50 cm thick and 15 m² area is expressed by the following equation:* $T = 90 - 50x + 22x^2 + 30x^3 - 40x^4$ *where* T *and* x *are measured in °C and m, respectively. Determine (a) the rate of heat transfer at* x = 0 *and* x = 0.5 m, *(b) the rate of thermal energy storage, (c) the rate of change of temperature at a distance of 30 cm from the heated end, and (d) the location where the rate of heating (or cooling) is maximum.*

Solution

Known Unsteady-state heat conduction through a large steel slab with prescribed temperature distribution.

Find (a) Rate of heat transfer at inlet and exit, $\dot{Q}_{in}$ and $\dot{Q}_{out}$ (kW), (b) Rate of energy stored, $\dot{E}_{st}$ (kW), (c) Rate of change of temperature, $\left.\frac{\partial T}{\partial t}\right|_{x=0.3m}$, (d) Location, $x(m)$ where $\dot{Q}$ is maximum.

Schematic



Assumptions (1) Constant thermal properties of the wall material. (2) One-dimensional transient heat conduction. (3) Convection and radiation effects are negligible. (4) Constant area of cross section. (5) There is no heat generation.

Analysis The temperature distribution across a large slab (plane wall) of steel at any instant t is given by:

$$T\,(x,\,t) = 90 - 50x + 22x^2 + 30x^2 - 40x^4$$

Differentiating the above expression with respect to x, we obtain

$$\frac{\partial T}{\partial x} = -\,50 + 44x + 90x^2 - 160x^3$$

Further differentiation yields, $\dfrac{\partial^2 T}{\partial x^2} = 44 + 180x - 480\,x^2$

Differentiating yet again, one has

$$\frac{\partial^3 T}{\partial x^3} = 180 - 960x$$

Heat-transfer rate in the x-direction is, $\dot{Q} = -kA\left(\dfrac{\partial T}{\partial \mathrm{x}}\right)$

(a) At $x = 0$,

$$\dot{Q}_{\text{in}} = -kA\left(\frac{\partial T}{\partial x}\right)_{x=0} = (-43\text{ W/m K})\,(15\text{ m}^2)\,(-50\text{ K/m})\left|\frac{1\text{ kW}}{10^3\text{ W}}\right|$$

$$= \mathbf{32.25\ kW} \qquad \textbf{(Ans.) (a)}$$

At $x = 0.5$ m,

$$\dot{Q}_{\text{out}} = -kA\left(\frac{\partial T}{\partial x}\right)_{x=L} = (-43\text{ W/m K})\,(15\text{ m}^2)$$

$$\times\{(-50) + 44(0.5) + 90(0.5)^2 - 160(0.5)^3\}(\text{K/m})\left|\frac{1\text{ kW}}{10^3\text{ W}}\right|$$

$$= \mathbf{16.45\ kW} \qquad \textbf{(Ans.) (a)}$$

(b) ***Energy balance: Control volume: Plane wall***

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_{\text{gen}} = \dot{E}_{\text{st}}$$

or Rate of thermal energy storage,

$$\dot{E}_{\text{st}} = \dot{E}_{\text{in}} - \dot{E}_{\text{out}} = \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}} = 32.25 - 16.45 = \mathbf{15.8\ kW} \qquad \textbf{(Ans.) (b)}$$

(c) For one-dimensional, unsteady-state heat conduction without internal heat generation,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$\therefore$ Rate of temperature change, $\dfrac{\partial T}{\partial t} = \alpha\dfrac{\partial^2 T}{\partial x^2} = \alpha\ \{44 + 180x - 480\,x^2\}$

At $x = 0.3$ m, $\dfrac{\partial T}{\partial t} = 0.042\text{ m}^2\text{/h}\ \{44 + (180)(0.3) - (470)(0.3)^2\}\ \text{K/m}^2$

$$= \mathbf{2.30\ K/h\ or\ {}^\circ C\ per\ hour} \qquad \textbf{(Ans.) (c)}$$

(d) To find the distance from the heated end for maximum rate of heating (or cooling), we have,

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right) = 0 \quad \text{or} \quad \frac{\partial}{\partial x}\left\{\alpha \frac{\partial^2 T}{\partial x^2}\right\} = 0$$

or $$\alpha \frac{\partial^3 T}{\partial x^3} = 0 \quad \text{i.e.,} \quad \frac{\partial^3 T}{\partial x^3} = 0$$

Hence, $180 - 960x = 0$

$\therefore$ $x = 0.1875$ m or **18.75 cm** **(Ans.) (d)**

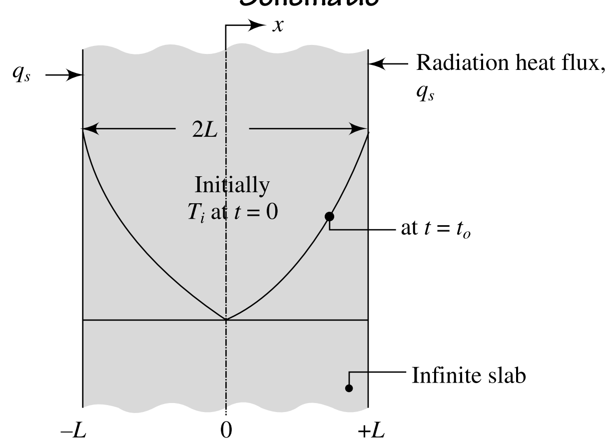
EXAMPLE 5.2 *Radiation heat flux of magnitude* q_s *falls perpendicularly on the two faces of an infinite slab of width* $2L$*, which is initially at a uniform temperature* T_i*. After a certain time has elapsed, the temperature distribution in the slab becomes parabolic and subsequently the temperature at any point in the slab increases linearly with time. Show that the temperature distribution at any time after the parabolic shape has been attained is* $T = T_i + \frac{q_s L}{2k}\left\{\left(\frac{x}{L}\right)^2 + \frac{2\alpha t}{L^2} - \frac{1}{3}\right\}$ [***NU: S 1992***]

Solution

Known A plane wall is subjected to radiant heat flux on both faces under specified conditions.

Find Temperature distribution, $T(x, t)$.

Schematic



Assumptions (1) Transient one-dimensional conduction. (2) Constant properties. (3) After $t = t_0$, temperature variation with time is linear.

Analysis The governing differential equation is

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{A}$$

with the boundary conditions:

$$\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = 0 \tag{B}$$

and $$k\frac{\partial T}{\partial x}\bigg|_{x=L} = q_s \quad \text{at} \quad x = L \tag{C}$$

and, the initial condition, $t = 0$, $T = T_i$. (D)

It is specified that after $t = t_0$, the temperature varies *linearly* with time.

$$\frac{\partial T}{\partial t} = \text{constant} \tag{E}$$

For $t > t_0$, by the first law of thermodynamics,

$$\int_0^L \rho C_p \frac{\partial T}{\partial t} dx = q_s \tag{F}$$

or $$\frac{\partial T}{\partial t} = \frac{q_s}{\rho C_p L} = \frac{\alpha}{kL} q_s \tag{G}$$

From Eq. (A), $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{q_s}{kL}$

Integrating with respect to x, we have $\dfrac{\partial T}{\partial x} = \dfrac{q_s}{kL} x + f_1(t)$

From Eq. (B), $f_1(t) = 0$

Then, $$\frac{\partial T}{\partial x} = \frac{q_s}{kL} x$$

At $x = L$, $k\left(\dfrac{\partial T}{\partial x}\right) = q_s$. This satisfies Eq. (C).

Integrating $\dfrac{\partial T}{\partial x}$ with respect to x, we get,

$$T = \frac{q_s}{kL}\frac{x^2}{2} + f_2(t) \tag{H}$$

Now, differentiating T with respect to t and substituting in Eq. (G), we have

$$\frac{\partial T}{\partial t} = 0 + \frac{d f_2(t)}{dt} = \frac{\alpha q_s}{kL}$$

Integrating with respect to the time t, we get

$$f_2(t) = \frac{\alpha t}{kL}(q_s) + C$$ where C is a constant of integration.

Substituting in Eq, (H),

$$T = \frac{q_s}{kL}\frac{x^2}{2} + \frac{\alpha t}{kL}(q_s) + C \tag{I}$$

At $t = t_0$, $T(t = t_0) = \dfrac{q_s}{kL}\dfrac{x^2}{2} + \dfrac{\alpha t_0}{kL}(q_s) + C$ and $\displaystyle\int_0^L \rho C_p (T_{t=t_0} - T_i)\, dx = (q_s) t_0$

or $$\rho C_p \int_0^L \left\{\frac{q_s}{kL}\frac{x^2}{2} + \frac{\alpha t_0 q_s}{kL} + C - T_i\right\} dx = q_s t_0$$

$$\rho C_p \left[\frac{q_s}{kL} \frac{x^3}{6} + \frac{\alpha t_0 q_s x}{kL} + Cx - T_i x \right]_0^L = q_s t_0$$

or

$$\rho C_p \left[\frac{q_s}{kL} \frac{L^3}{6} + \frac{\alpha t_0}{kL} L(q_s) + CL - T_i L \right] = q_s t_0$$

∴

$$\rho C_p \frac{q_s}{k} \frac{L^2}{6} + q_s t_0 + \rho C_p L(C - T_i) = q_s t_0 \quad \Rightarrow \quad C = T_i - \frac{q_s}{k} \frac{L}{6}$$

or

Substitution in Eq. (I) yields

$$T = \frac{q_s}{kL} \frac{x^2}{2} + \frac{\alpha t}{kL} q_s + T_i - \frac{q_s L}{6k} = T_i + \frac{q_s}{L} \left[\frac{x^2}{2k} + \frac{\alpha t}{k} - \frac{L^2}{6} \right]$$

$$= T_i + \frac{q_s L}{2k} \left[\left(\frac{x}{L} \right)^2 + \frac{2\alpha t}{L^2} - \frac{1}{3} \right]$$

or

$$\boxed{\boldsymbol{T = T_i + \frac{q_s L}{2k} \left[\left(\frac{x}{L} \right)^2 + \frac{2\alpha t}{L^2} - \frac{1}{3} \right]}}$$ **QED**

(B) Lumped Parameter Model

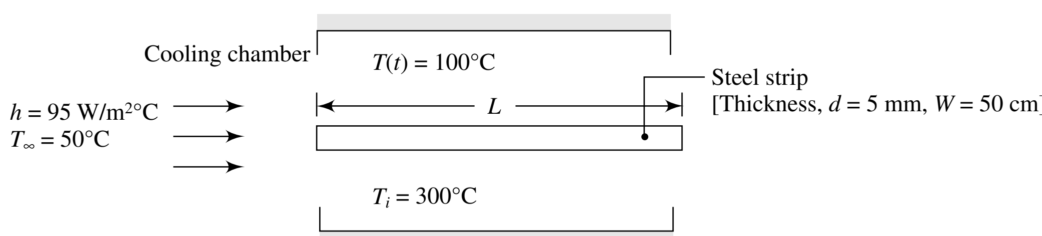
EXAMPLE 5.3 *A steel strip [ρ = 7900 kg/m³, C_p = 0.64 kJ/kg °C, k = 30 W/m °C], 5 mm thick, 50 cm wide, coming out of a rolling mill is passed through a cooling chamber maintained at 50°C. How long should the strip stay in the chamber if the temperature at no plane in the strip is to fall below 100°C, while the strip enters the chamber at 300°C. The surface heat-transfer coefficient is 95 W/m² °C.*

Solution

Known A steel strip from a rolling mill is allowed to cool in a cooling chamber.

Find Time required, t (s).

Schematic



Assumptions (1) Lumped-heat-capacity model is valid. (2) Constant properties and convection coefficient.

Analysis The Biot number for this problem is, $Bi = \dfrac{hL_c}{k}$.
where the characteristic length,

$$L_c = \frac{V}{A_s} = \frac{W\delta L}{2(WL + \delta L)} = \frac{W\delta}{2(W + \delta)} = \frac{50 \times 0.5 \text{ cm}^2}{2(50 + 0.5) \text{ cm}} \left(\frac{1 \text{ m}}{100 \text{ cm}} \right) = 2.475 \times 10^{-3} \text{m}$$

$$\therefore \qquad Bi = \frac{(95\ \text{W/m}^2\ ^\circ\text{C})\,(2.475 \times 10^{-3}\,\text{m})}{30\ \text{W/m}^\circ\text{C}} = 0.007\ 84 \qquad (<< 0.1)$$

∴ Internal temperature gradients can be neglected and lumped-parameter analysis is valid.

$$\text{Thermal diffusivity, } \alpha = \frac{k}{\rho C_p} = \frac{30\ \text{W/m}\ ^\circ\text{C}}{(7900\ \text{kg/m}^3)(640\ \text{J/kg}\ ^\circ\text{C})} = 5.934 \times 10^{-6}\ \text{m}^2/\text{s}$$

The temperature distribution is given by

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\,(-BiFo) \quad \text{or} \quad \ln \frac{T_i - T_\infty}{T(t) - T_\infty} = (Bi)\ (Fo)$$

$$\text{or} \qquad Fo = \frac{1}{Bi} \ln \frac{T_i - T_\infty}{T(t) - T_\infty} = \frac{1}{0.00784} \ln \frac{300 - 50}{100 - 50} = 205.3$$

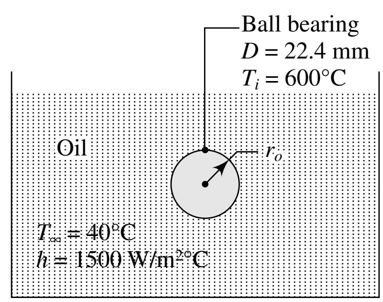
$$\text{But} \qquad Fo = \alpha t / L_c^2$$

$$\therefore \text{ Time required, } t = \frac{Fo \cdot L_c^2}{\alpha} = \frac{(205.3)(2.475 \times 10^{-3}\,\text{m})^2}{5.934 \times 10^{-6}\ \text{m}^2/\text{s}} = \mathbf{212\ s\ or\ 3.53\ min} \qquad \textbf{(Ans.)}$$

EXAMPLE 5.4 *Stainless-steel ball bearings (*k *= 22.2 W/m K, α = 4.85 × 10⁻⁶ m²/s) which have uniformly been heated to 850°C are hardened by quenching them in an oil bath that is maintained at 40°C. The ball diameter is 20 mm, and the convection coefficient associated with the oil bath is 600 W/m² K. (a) If quenching is to occur until the surface temperature of the balls reaches 100°C, how long must the balls be kept in the Oil? (b) If 10 000 balls are to be quenched per hour, what is the rate at which energy must be removed by the oil-bath cooling system in order to maintain its temperature at 40°C?*

Solution

Schematic



Known Diameter and initial temperature of ball bearings to be quenched in an oil bath.

Find (a) Time required for the balls to cool to 100°C, and (b) oil-bath cooling requirement.

Assumptions (1) Constant properties. (2) Internal temperature gradients can be neglected.

Analysis: (a) To determine whether the use of the lumped capacitance model is suitable, let us first compute the Biot number.

$$\text{With } L_c = \frac{V}{A} = \frac{r_o}{3} \text{ for a sphere, } Bi = \frac{h(r_o/3)}{k} = \frac{600\ \text{W/m}^2\text{K}\,(0.01\,m/3)}{22.2\ \text{W/mK}} = 0.09\ (< 0.1)$$

⇒ Lumped capacitance model is applicable.

$$\text{We note that } e^{-BiFo} = \frac{T(t) - T_\infty}{T_i - T_\infty} = \frac{100 - 40}{850 - 40} = 0.0741$$

Taking log on both sides $-BiFo$ = ln 0.0741

$$\therefore \text{ Fourier number, } Fo = \frac{-2.602}{-0.09} = 28.9$$

With $Fo = \dfrac{\alpha t}{L_c^2}$, the time required is

$$t = \frac{Fo(r_o/3)^2}{\alpha} = \frac{28.9 \times (0.01\text{ m}/3)^2}{4.85 \times 10^{-6}\text{ m}^2/\text{s}} = \mathbf{66.2\ s} \qquad \textbf{(Ans.) (a)}$$

The amount of energy transferred from a single ball during the cooling process is

$$Q = \rho C_p ¥ (T_i - T_\infty)[1 - e^{-BiFo}] = \frac{k}{\alpha}\left(\frac{4}{3}\pi r_o^3\right)(T_i - T_\infty)[1 - e^{-BiFo}]$$

$$= \left(\frac{22.2\text{ W/mK}}{4.85 \times 10^{-6}\text{ m}^2/\text{s}}\right) \times \frac{4}{3}\pi \times (0.01)^3\text{ m}^3 \times (850 - 40)\,K \times [1 - 0.0741]$$

$$= 14.38 \times 10^3\text{ J} \quad \text{or} \quad 14.38\text{ kJ}$$

Since 10 000 balls are to be quenched per hour, the oil bath cooling requirement is

$$Q = \frac{10\,000}{3600\text{ s}} \times 14.38\text{ kJ} = \mathbf{39.94\ kW} \qquad \textbf{(Ans.) (b)}$$

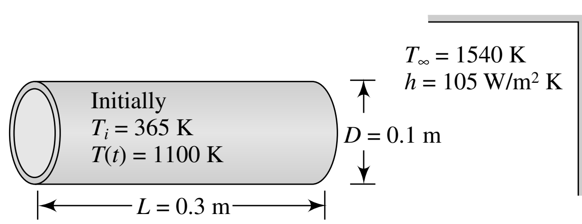
EXAMPLE 5.5 *A cylindrical stainless steel ingot (*k *= 23 W/m K, α = 5.0 × 10⁻⁶ m²/s) 0.1 m in diameter and 0.3 m long, passes through a heat treating furnace which is 6 m in length. The initial ingot temperature is 365 K, and it must reach 1100 K in preparation for working. The furnace gas is at 1540 K, and the combined radiant and convective surface coefficient is 105 W/m² K. In order that the required conditions be satisfied, what must be the maximum speed with which the ingot moves through the furnace?*

Solution

Known A cylindrical ingot getting heated in a furnace.

Find Maximum ingot speed.

Schematic



Assumptions (1) Lumped capocity analysis is valid. (2) Constant properties.

Analysis Initially, the Biot number is calculated to be

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-(Bi)(Fo)} \quad \text{or} \quad \ln\left[\frac{T_i - T_\infty}{T - T_\infty}\right] = (Bi)(Fo)$$

Characteristic length,

$$L_c = \frac{¥}{A} = \frac{\pi R^2 L}{2\pi R^2 + 2\pi RL} = \frac{RL}{2(R+L)} = \frac{0.05 \times 0.3}{2(0.05 + 0.3)} = 0.02143\text{ m}$$

$$\therefore \qquad Bi = \frac{hL_c}{k} = \frac{105\ \text{W/m}^2\ K \times 0.02143\ \text{m}}{23\ \text{W/m K}} = 0.098 \qquad (< 0.1)$$

Fourier number, $Fo = \dfrac{\alpha t}{L_c^2} = \dfrac{1}{Bi}\ln\left(\dfrac{T_i - T_\infty}{T - T_\infty}\right) = \dfrac{1}{0.098} \times \ln 2.67 = 0.9822/0.098 = 10.04$

Time required, $t = FoL_c^2/\alpha = \dfrac{10.04 \times 0.02143^2\, m^2}{5.0 \times 10^{-6}\ \text{m}^2/\text{s}} = 922\ \text{s}$

Maximum speed of ingot,

$$V = \frac{\text{Distance (m)}}{\text{Time, } t\text{(s)}} = \frac{6000\ \text{mm}}{922\ \text{s}} = \mathbf{6.5\ mm/s} \qquad \textbf{(Ans.)}$$

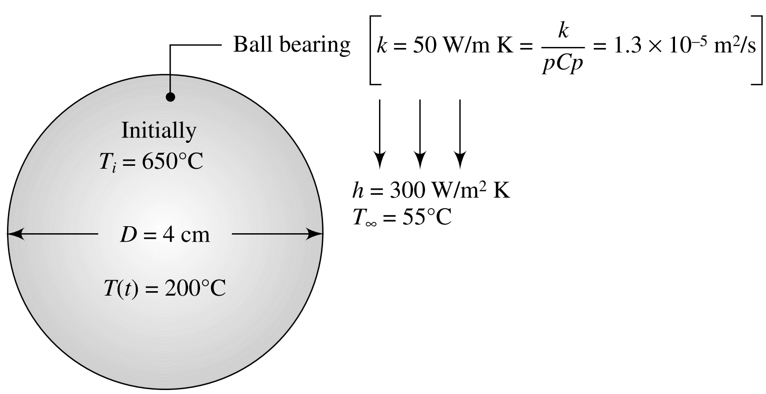
EXAMPLE 5.6 *Chromium-steel ball bearings (*k *= 50 W/m K,* α *= 1.3 × 10*$^{-5}$ *m*^{2}*/s) are to be heat treated. They are heated to a temperature of 650°C and then quenched in a vat of oil that has a temperature of 55°C. The ball bearings have a diameter of 4 cm. The heat-transfer coefficient between ball bearings and oil is 300 W/m*^{2} *K. Determine (a) The length of time the bearings must remain in oil before their temperature drops to 200°C, (b) The total amount of heat removed from each bearing during this time interval, and (c) The instantaneous heat-transfer rate from the bearings when they are first placed in the oil and when they reach 200°C.*

Solution

Known Steel balls are quenched in oil to cool to a specified temperature.

Find (a) Time to reach 200°C. (b) Amount of heat removed during this period. (c) Instantaneous heat flow rate (*initially and finally*).

Schematic



Assumptions (1) Lumped-capacity model is justified. (2) Constant properties. (3) Uniform heat-transfer coefficient.

Analysis Biot number,

$$Bi = \frac{hL_c}{k} \quad \text{where} \quad L_c = \frac{V}{A} = \frac{\pi D^3/6}{\pi D^2} = \frac{D}{6} \qquad (\textit{for spherical geometry})$$

$$\therefore \qquad Bi = \frac{(300 \text{ W/m}^2 \text{ K})(0.04/6 \text{ m})}{50 \text{ W/mK}} = 0.04$$

As the Biot number is less than 0.1, lumped-capacitance method can be used,

$$\text{(a)} \quad \frac{T(t) - T_\infty}{T_i - T_\infty} = \exp[-(Bi)(Fo)] = \exp\left(-\frac{hAt}{\rho C_p \text{V}}\right) = \exp\left(-\frac{ht}{\rho C_p L}\right) = \exp\left(-\frac{ht}{k/\alpha}\frac{6}{D}\right)$$

$$\therefore \qquad \frac{(200 - 55)°\text{C}}{(650 - 55)°\text{C}} = \exp\left(-\frac{300 \text{ W/m}^2 \text{ K}}{(50 \text{ W/m K}/1.3 \times 10^{-5} \text{ m}^2/\text{s})}\frac{6}{0.04 \text{ m}}\right) = \exp\,(-\,0.0117\,t)$$

$$\text{or} \qquad \frac{145}{595} = e^{-0.0117t} \quad \text{or} \qquad -\,0.0117\,t = \ln\,(145/595)$$

$\therefore$ Time required, $t = \dfrac{-1.4118}{-0.0117}$ = **120.67 s** **(Ans.) (a)**

(b) From energy balance: Total amount of heat removed from each bearing,

$$\text{Q}_{\text{conv,out}} = -\dot{E}_{\text{st}} = hA[T(t) - T_\infty]$$

$$\dot{Q}_{\text{out}} = \int_0^t \dot{Q}dt = \int_0^t hA[T(t) - T_\infty]\int_0^t -hA(T_i - T_\infty)\exp(-BiFo)\,dt$$

$$= \int_0^t hA(T_i - T_\infty)\exp\left[-\frac{hA}{\rho C_p \text{V}}t\right]dt = hA(T_i - T_\infty)\left[\exp\left(-\frac{h\,At}{\rho C_p \text{V}}\right) \times \left(-\frac{\rho C_p \text{V}}{hA}\right)\right]_0^t$$

$$= -\rho C_p \text{V}(T_i - T_\infty)\left[\exp\left(-\frac{hAt}{\rho C_p \text{V}}\right) - 1\right] = \int_0^t -hA(T_i - T_\infty)\,e^{-Bi\left(\frac{\alpha}{L_c^2}\right)t}\,dt$$

$$= -hA(T_i - T_\infty)\left[\frac{e^{Bi\left(\frac{dt}{L_c^2}\right)}}{-1}\right]_0^t = -\rho \text{C}_\text{p} \text{V}(T_i - T_\infty)[e^{-BiFo} - 1]$$

$$= -\frac{\text{k}}{\alpha}\frac{\pi}{6}D^3\,(T_i - T_\infty)[e^{-BiFo} - 1]$$

$$= -\frac{50 \text{ W/m K}}{1.3 \times 10^{-5} \text{ m}^2/\text{s}} \cdot \frac{\pi}{6}(0.04 \text{ m})^3\left(650 - 55\right)°\text{C or K}\left[e^{\left(\frac{-(0.04)(1.3\times 10^{-5} \text{ m}^2/\text{s})(120.67\text{s})}{(0.04/6)^2 \text{ m}^2}\right)} - 1\right]$$

= –76687 Ws ($e^{-1.4118}$ – 1) = 76687 J (1 – 0.2437) = 5.8 × 10^4 J = **58 kJ (Ans.)(b)**

(c) The instantaneous heat flow rate is given by the expression.

$$\dot{Q}_{\text{out}} = hA(T_i - T_\infty)\exp(-BiFo)$$

When the bearings are first placed in the oil, $t = 0$

Hence, $Fo = \dfrac{\alpha t}{L_c^2} = 0$

$\therefore \qquad \dot{Q}_{out} = hA(T_i - T_\infty)e^{-0} = -hA(T_i - T_\infty)$

$= (300 \text{ W/m}^2 \text{ K}) (\pi) (0.04 \text{ m})^2 (650 - 55)°\text{C or K} = \mathbf{897.24 \text{ W}}$ **(Ans.) (c)**

When the bearings reach the final temperature of 200°C,

$t = 120.67$ s

$$Fo = \frac{\alpha t}{L_c^2} = \frac{(1.3 \times 10^{-5} \text{ m}^2/\text{s})(120.67 \text{ s})}{(0.04/6 \text{ m}^2)^2} = 35.3$$

$e^{-BiFo} = \exp(-0.04 \times 35.3) = 0.2437$

$\therefore \qquad \dot{Q}_{out} = (300 \text{ W/m}^2 \text{ K})(\pi)(0.04 \text{ m})2 \ (650 - 55)°\text{C} \quad \text{or} \quad \text{K } (0.2437)$

$= (897.24)\ (0.2437) = \mathbf{218.65 \text{ W}}$ **(Ans.) (c)**

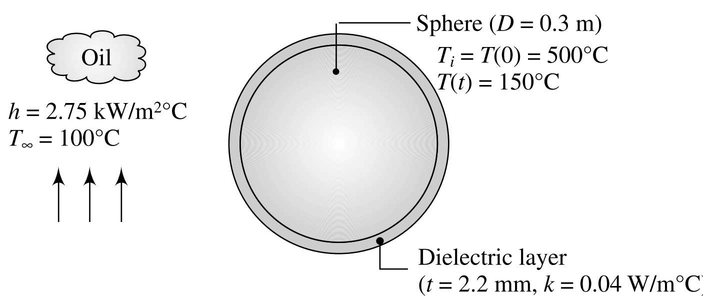
EXAMPLE 5.7 *A solid steel sphere [*k *= 48.8 W/m °C, ρ= 7832 kg/m*^{3}*,* C_p *= 0.559 kJ/kg °C] of 0.3 m diameter is cooled with a 2.2 mm thick layer of a dielectric material (k = 0.04 W/m °C). Calculate the time required for the coated sphere, initially at 500°C, to attain a temperature of 150°C when it is suddenly quenched in an oil bath maintained at 100°C with a convection coefficient of 2.75 kW/m*^{2}*°C.*

Solution

Known A sphere cooled with a dielectric material layer is quenched in an oil bath.

Find Time required for sphere to attain 150°C.

Schematic



Assumptions (1) Lumped-capacity model is valid. (2) Compared to steel sphere, the thermal capacitance of dielectric material is negligible. (3) Constant properties.

Analysis Thermal resistance to heat transfer from the sphere is due to conduction resistance of dielectric layer and the convective resistance.

$$\therefore \qquad R_{th} = \frac{t}{k_A} + \frac{1}{hA} = \frac{1}{UA}$$

$\therefore$ Overall heat-transfer coefficient,

$$U = \left[\frac{t}{k} + \frac{1}{h}\right]^{-1} = \left[\frac{2.2 \times 10^{-3} \text{m}}{0.04 \text{ W/m °C}} + \frac{1}{2.75 \times 10^{-3} \text{ W/m}^2\text{°C}}\right]^{-1} = 18.06 \text{ W/m}^2\text{°C}$$

The effective Biot number is, $Bi_e = \dfrac{UL_c}{k}$

where the characteristic length, $L_c = \dfrac{V}{A_s} = \dfrac{\pi D^3/6}{\pi D^2} = \dfrac{D}{6}$

$$\therefore \qquad Bi_e = \frac{UD}{6k} = \frac{18.06 \text{ W/m}^2 \text{ °C} \times 0.3 \text{ m}}{6 \times 48.8 \text{ W/m °C}} = 0.0185 \qquad (<< 0.1)$$

The lumped-capacity model is, therefore, applicable.

Time required for the sphere to reach 150°C is determined from

$$t = \frac{\rho C_p D}{6U} \ln\left[\frac{T_i - T_\infty}{T(t) - T_\infty}\right] = \frac{(7832 \text{ kg/m}^3)(559 \text{ J/kg °C})(0.3 \text{ m})}{6 \times 18.06 \text{ W/m}^2 \text{ °C}} \ln \frac{(500 - 100)\text{°C}}{(150 - 100)\text{°C}}$$

$= \mathbf{25021\ s\ or\ 7.0\ h}$ **(Ans.)**

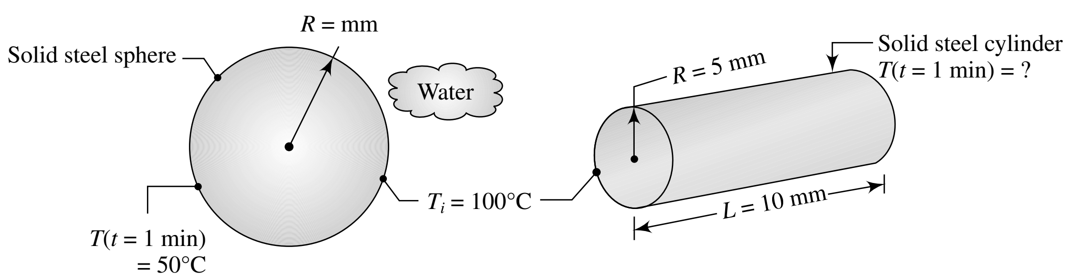
EXAMPLE 5.8 *A solid steel sphere of 10 mm radius and a solid steel cylinder of 5 mm radius and 10 mm length, both initially at a temperature of 100°C, are immersed in a large reservoir of cold water at 20°C. After 1 minute, the sphere is at a temperature of 50°C. Estimate the temperature of the cylinder after 1 minute. The thermal conductivity, specific heat and density of the steel are 13.4 W/m K are 0.468 kJ/kg K and 8238 kg/m³.*

Solution

Known A solid sphere and a solid cylinder of the same material and at the same initial temperature are exposed to the same ambient conditions.

Find Temperature of cylinder after 1 minute, given the sphere temperature after 1 minute.

Schematic



Assumptions (1) Constant material properties. (2) Lumped-capacity formulation is valid.

Analysis For a sphere, the characteristic length,

$$L_c = \frac{V}{A} = \frac{4/3\pi R^3}{4\pi R^2} = \frac{R}{3} = \frac{10}{3} \text{ mm}$$

For a cylinder, $L_c = \dfrac{V}{A} = \dfrac{\pi R^2 L}{2\pi RL + 2\pi R^2} = \dfrac{RL}{2(R+L)}$

Biot number, $Bi = \ Bi \equiv \dfrac{hL_c}{k} = \dfrac{5 \times 10}{2(5+10)} = \dfrac{50}{30} = \dfrac{5}{3}$ mm should be less than 0.1 for lumped-capacity model to be valid.

For the solid steel sphere, using the lumped-capacitance method,

$$\frac{T(t)-T_\infty}{T_i-T_\infty}=\exp\left(-\frac{ht}{\rho C_p L_c}\right) \quad \text{or} \quad \frac{50-20}{100-20}=\exp\left[-\frac{h(60)(3)}{(8238)(468)(10)(10^{-3})}\right]$$

or $$\ln\left(\frac{30}{80}\right)=-\frac{h}{214.19} \quad \text{or} \quad -0.98083 = -h/214.19$$

h = 210.08 W/m^2 K

For the solid-steel cylinder,

$$\frac{T(t)-20}{100-20}=\exp\left[-\frac{(210.08)(60)(3)}{(8238)(468)(5)(10^{-3})}\right]=0.1406 \quad = 0.1406$$

$\therefore$ $T(t)$ = 20 + 80 (0.1406) = 20 + 11.25 = **31.25°C** **(Ans.)**

Comment To verify the validity of lumped-capacitance model, *Bi* should be calculated and shown to be less than 0.1.

For sphere, $$Bi=\frac{hL}{k}=\frac{(210.08)(10/3)(10^{-3})}{13.4}=0.0523$$

For cylinder, $$Bi=\frac{(210.08)(5/3)(10^{-3})}{13.4}=0.026$$

In both the cases, *Bi* is less than 0.1

EXAMPLE 5.9 *A steel bar, 2 cm × 2 cm × 8 cm, is quenched from 400°C in a bath of oil at 50°C. Determine the immersion time so that the centre temperature of the bar reaches 100°C. Would the surface temperature be significantly different? Why?*

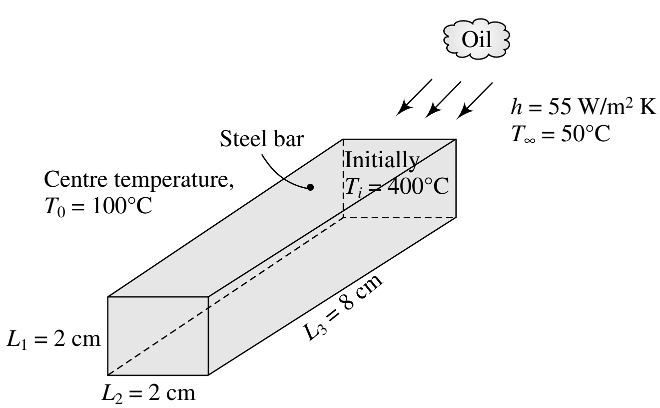
Given: h = *55 W/m^2 K For steel:* k = *50 W/m K,* C_p = *400 J/kg K,* ρ = *8100 kg/m^3*

Solution

Known A steel bar (*rectangular parallelopiped*) is quenched in an oil and exposed to convective cooling process.

Find Time required for the centre of the bar to reach the specified temperature.

Schematic



Assumptions (1) Constant properties and uniform heat-transfer coefficient. (2) Lumped capacity model is valid.

Analysis Characteristic dimension,

$$L_c = \frac{\forall}{A_s} = \frac{L_1 L_2 L_3}{2(L_1 L_2 + L_1 L_3 + L_2 L_3)} = \frac{(2 \times 2 \times 8)\ \text{cm}^3}{2[(2 \times 2) + (2 \times 8) + (2 \times 8)]\ \text{cm}^2} = 0.4444\ \text{cm}$$

Biot's number, $Bi = \dfrac{hL_c}{k} = \dfrac{(55\ \text{W/m}^2\ \text{K})(0.4444 \times 10^{-2}\ \text{m})}{50\ \text{W/mK}} = 0.0049 \quad (<< 0.1)$

Hence, the lumped-capacity model is appropriate. Internal temperature gradients can be neglected. *Therefore, the centre temperature will not be significantly different from the surface temperature.* **(Ans.)**

The transient temperature distribution is given by

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \frac{T_0 - T_\infty}{T_i - T_\infty} = \exp(-t/\tau) \qquad \text{(A)}$$

where τ is the thermal time constant defined as

$$\frac{\rho \forall C_p}{hA_s} = \frac{\rho C_p L_c}{h} = \frac{(8100\ \text{kg/m}^3)(400\ \text{J/kg K})(0.4444 \times 10^{-2}\ \text{m})}{55\ \text{W/m}^2\text{K}} = 261.8\ \text{s}$$

Substituting numerical values in Eq. (A), one gets

$$\frac{100 - 50}{400 - 50} = \exp\ (-t/261.8) \quad \text{or} \quad \ln 0.14286 = -\ t/261.8$$

Immersion time, t = (261.8 s) (1.95) = 510 s ≈ **8.5 min** **(Ans.)**

EXAMPLE 5.10 *A hot cylindrical ingot (*k *= 60 W/m °C,* ρ *= 7850 kg/m³ and* C$_p$ *= 0.430 kJ/kg °C) of 5 cm diameter and 25 cm length is removed from a furnace at 850°C and suddenly quenched in water at 20°C until its temperature drops to 550°C. Subsequently, the ingot is exposed to ambient air at 20°C and allowed to cool slowly to 100°C. The convection heat-transfer coefficient is 250 W/m² °C when the cooling medium is water and 25 W/m² °C with air is the cooling fluid. Estimate the total time required for cooling. State and estimate the total time required for cooling. State and justify any assumption made.*

Solution

Known Cooling of a hot cylindrical ingot in two stages: first in *water* and then in *air*.

Find Total time required for cooling.

Assumptions (1) Lumped-capacity formulation is valid. (2) Constant properties. (3) Uniform heat-transfer coefficient.

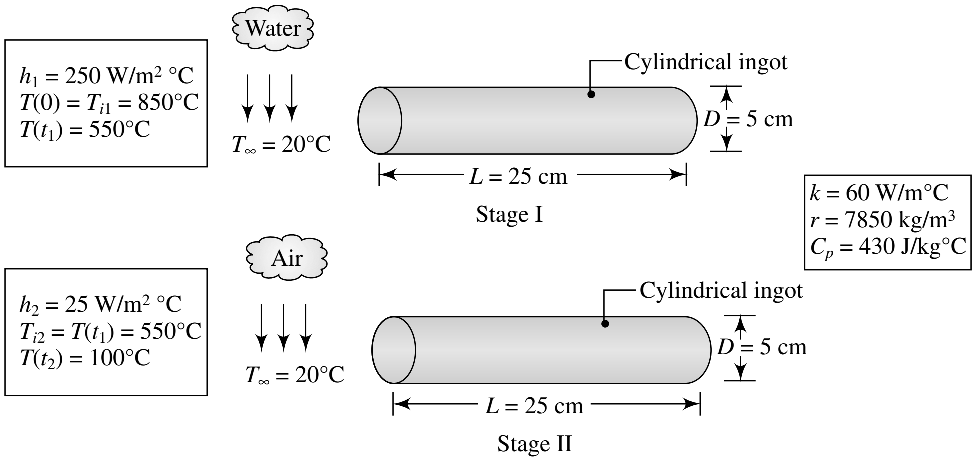
Analysis ***Stage I*: *Cooling in water at 20°C:***

The assumption of lumped capacity model is justified if the Biot number, Bi < 0.1,

$$Bi_1 = \frac{h_1 L_c}{k}$$

Characteristic length, $L_c = \dfrac{\forall}{A_s} = \dfrac{(\pi/4)D^2 L}{\underbrace{2 \times (\pi/4)D^2}_{\text{end surfaces}} + \underbrace{\pi DL}_{\text{curved surface}}} = \dfrac{DL}{2D + 4L} = \dfrac{DL}{2(D + 2L)}$

Schematic



$$= \frac{5 \times 25}{2(5 + 2 \times 25)} = 1.136 \text{ cm} \quad \text{or} \quad 1.136 \times 10^{-2} \text{ m}$$

Hence, $$Bi_1 = \frac{(250 \text{ W/m}^{2\,\text{o}}\text{C})(1.136 \times 10^{-2} \text{ m})}{60 \text{ W/m}\,^\text{o}\text{C}} = 0.04735$$

As $Bi_1 < 0.1$, the assumption is valid.

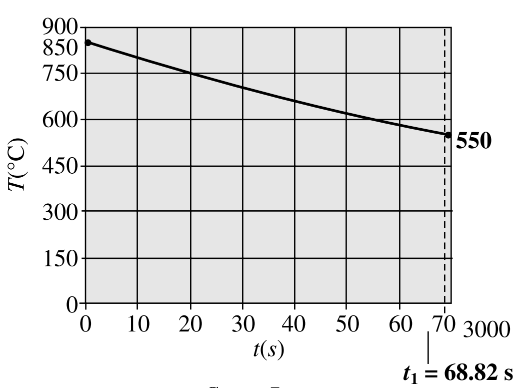
The temperature distribution is then given by

$$\frac{T(t_1) - T_\infty}{T_{i_1} - T_\infty} = e^{-h_1 A_s t_1/\rho C_p \forall} = \exp\left[-\frac{h_1 t_1}{\rho C_p L_c}\right] \text{or } \ln\left(\frac{T_{i_1} - T_\infty}{T(t_1) - T_\infty}\right) = \frac{h_1 t_1}{\rho C_p L_c}$$

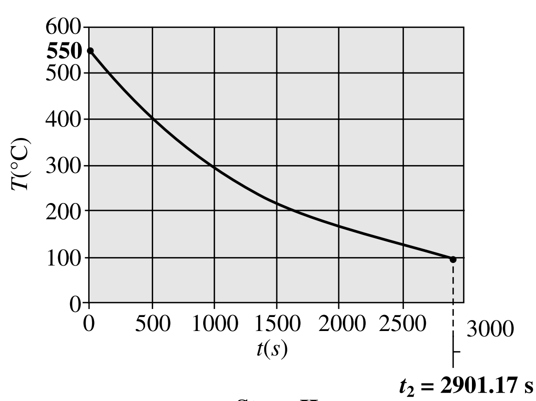
Now, $\rho C_p L_c = 7850 \text{ kg/m}^3 \times 430 \text{ J/kg °C} \times 0.01136 \text{ m} = 38.358 \times 10^3 \text{ J/m}^2 \text{ °C}$

Time required for cooling from 850°C to 550°C in water is

$$t_1 = \frac{\rho C_p L_c}{h_1} \ln\left\{\frac{T_{i1} - T_\infty}{T(t_1) - T_\infty}\right\} = \frac{38.358 \times 10^3 \text{ J/m}^{2\,\text{o}}\text{C}}{250 \text{ W/m}^{2\,\text{o}}\text{C}} \ln\left[\frac{850 - 20}{550 - 20}\right] = \mathbf{68.82\ s}$$



Stage-I



Stage-II

Stage II: Cooling in air at 20°C:

$$Bi_2 = \frac{h_2 L_c}{k} = \frac{25\ \text{W/m}^{2\,\text{o}}\text{C} \times 0.01136\ \text{m}}{60\ \text{W/m}\ ^\text{o}\text{C}} = 4.735 \times 10^{-3} \quad (<< 0.1)$$

The assumption of lumped-capacity model is valid in this case too.

Therefore,

$$t_2 = \frac{\rho C_p L_c}{h_2} \ln\left[\frac{T_{i_2} - T_\infty}{T(t_2) - T_\infty}\right] = \frac{38.358 \times 10^3}{25} \ln\left[\frac{550 - 20}{100 - 20}\right] = \mathbf{2901.17\ s}$$

Total time required, $t = t_1 + t_2 =$ **68.82 + 2901.17 = 2970 s or 49.5 min** **(Ans)**

EXAMPLE 5.11 *A thermocouple junction, which may be approximated as a sphere, is to be used to measure the temperature of a gas stream. The heat-transfer coefficient between the junction surface and the gas is 400 W/m² K, the thermal conductivity of the thermocouple is 20 W/m K, the density and specific heat of the couple are 8500 kg/m³ and 400 J/kg K respectively.*

Determine the junction diameter needed for the thermocouple to have a time constant of 1 second. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C.

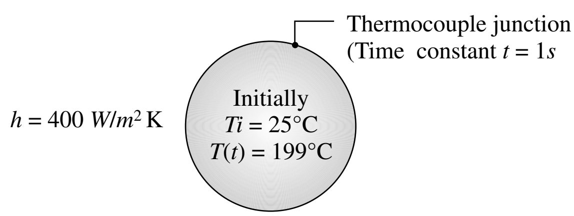
Solution

Known A thermocouple junction (*sphere*) is placed in a hot-gas stream.

Find Junction diameter for time constant of 1 s. Time to attain 199°C temperature.

Assumptions (1) Lumped-capacitance analysis is appropriate. (2) Constant thermophysical properties. (3) Uniform heat-transfer coefficient.

Schematic



Analysis The criterion for using the lumped capacitance method is that the Biot number, *Bi*, should be less than 0.1. For want of the junction diameter, *Bi* cannot be calculated. Hence, it is reasonable to take it for granted that internal temperature gradients can be neglected and lumped-capacitance model employed to find the diameter of the thermocouple junction. Subsequently, the criterion's validity can be verified.

Time constant,

$$\tau = \frac{\rho C_p \forall}{hA} = \frac{\rho C_p}{h}\left(\frac{\pi D^3/6}{\pi D^2}\right) = \frac{\rho C_p D}{6h}$$

$$\therefore \quad D = \frac{6h\tau}{\rho C_p} = \frac{(6)(400)\ \text{W/m}^2\,\text{K}\,(1\text{s})}{(8500\ \text{kg/m}^3)(400\ \text{J/kg K})} = 7.06 \times 10^{-4}\ \text{m} = 0.706\ \text{mm}\text{(Ans.)}$$

The characteristic length, $L_c = \frac{\forall}{A} = \frac{D}{6}$

$\therefore$ Biot number, $Bi = \dfrac{hL_c}{k} = \dfrac{hD}{6k} = \dfrac{(400\ \text{W/m}^2\text{K})(7.06 \times 10^{-4}\ \text{m})}{(6)(20\ \text{W/mK})} = 2.353 \times 10^{-3}$

Thus, $Bi < 0.1$, thereby proving the validity of the assumption.

To determine the time required for the junction to attain the temperature of 199°C, we have

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{hA}{\rho C_p V} t\right) \quad \text{or} \quad \frac{199 - 200}{25 - 200} = \exp\left(-\frac{ht}{\rho C_p L}\right)$$

or
$$\frac{1}{175} = \exp\left[-\frac{6ht}{\rho C_p D}\right] \quad \text{or} \quad -\frac{6ht}{\rho C_p D} = \ln\left(\frac{1}{175}\right)$$

$\therefore$ Time required, $t = \dfrac{\rho C_p D}{6h}(\ln 175)$

$\therefore$ $t = \dfrac{(8500\ \text{kg/m}^3)(400\ \text{J/kg K})(7.06 \times 10^{-4}\ \text{m})(5.1648)}{(6)(400\ \text{W/m}^2\ \text{K})} = \mathbf{5.17\ s}$ **(Ans.)**

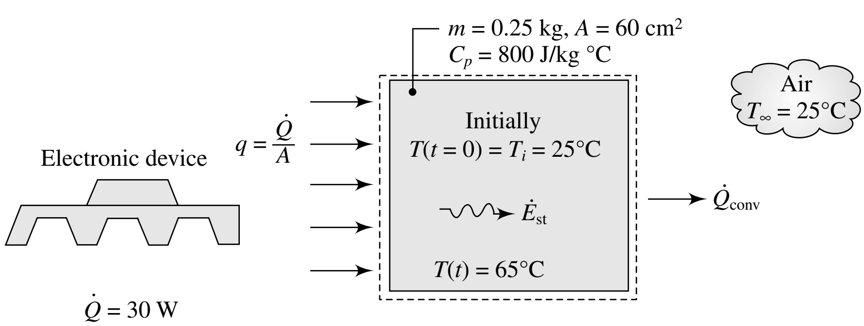
EXAMPLE 5.12 *An electronic device that dissipates 30 W and an attached heat sink have a combined mass of 0.25 kg, a surface area of 60 cm², and an effective specific heat of 0.8 kJ/kg °C. The device is initially at a uniform temperature of 25°C in air at 25°C with convection heat-transfer coefficient of 10 W/m² °C. The maximum permissible operating temperature is 65°C and at this temperature the device must be shut off. (a) Determine the steady (equilibrium) operating temperature. (b) Calculate the time required to reach the maximum operating temperature. (c) If a heat sink is to be added to the device so that the operating time is doubled, find the additional mass and area required assuming that the mass to area ratio of the added material is the same as that of the original device.*

Solution

Known Unsteady-state conduction in an electronic device dissipating heat with one face subjected to constant heat flux with the other exposed to convection.

Find (a) $T(\infty)$ °C), (b) t (s), (c) $(m^* - m)$(kg), $(A^* - A)$(cm²).

Schematic



Assumptions (1) Lumped heat-capacity model is valid. (2) Initial and ambient temperatures are same.

Analysis (a) For mixed boundary conditions, neglecting internal temperature gradients, the temperature variation is given by

$$\theta(t) = \theta_i e^{-at} + (b/a)(1 - e^{-at})$$

where $\theta(t) \equiv T(t) - T_\infty = 65 - 25 = 40\ ^\circ\text{C}$

$$\theta_i \equiv T_i - T_\infty = 25 - 25 = 0\ ^\circ\text{C}$$

$$a \equiv \frac{hA}{\rho\forall c} = \frac{hA}{mc} = \frac{(10\ \text{W/m}^2{}^\circ\text{C})(60 \times 10^{-4}\ \text{m}^2)}{(0.25\,kg)(800\ \text{J/kg}^\circ\text{C})} = 3 \times 10^{-4}\ \text{s}^{-1}$$

and $$b \equiv \frac{qA}{\rho\forall c} = \frac{\dot{Q}}{mc} = \frac{30\ \text{W}}{(0.25\ \text{kg})(800\ \text{J/kg}^o\text{C})} = 0.15\ \text{s}^{-1}\ ^\circ\text{C}$$

Equilibrium (steady-state) temperature (at $t = \infty$) is determined from

$$\theta(\infty) = 0 + \frac{b}{a}(1 - 0) = \frac{b}{a} = \frac{0.15}{3 \times 10^{-4}}\ ^\circ\text{C} = 500^\circ\text{C}$$

Hence, $T(\infty) = 25^\circ\text{C} + 500^\circ\text{C} =$ **525°C** **(Ans.) (a)**

(b) To find the time needed to attain the maximum allowable temperature of 65°C;

$$\theta(t) = T(t) - T_\infty = 65 - 25 = 40^\circ\text{C}$$

$$\theta_i(t) = T_i - T_\infty = 25 - 25 = 0^\circ\text{C}$$

$$b/a = 500^\circ\text{C}$$

It follows that, $40 = 0 + 500[1 - e^{-3\times10^{-4}}t]$

$$e^{-3\times10^{-4}t} = 1 - 0.08 \quad \text{or} \quad -3 \times 10^{-4}\ t = \ln 0.92$$

Time required, $t =$ **278 s** **(Ans.) (b)**

(c) Operating time, $t^* = 2t = 556$ s

In the changed situation, $\theta(t) = \theta_i(t) + \dfrac{b^*}{a^*}[1 - e^{-a^*t^*}]$

$$\frac{b^*}{a^*} = \frac{q^* A^*/m^* c}{hA^*/m^* c} = \frac{q^*}{h}$$

and $$a^* = \frac{hA^*}{m^* c} = \frac{hA}{mc} = 3 \times 10^{-4} \text{s}^{-1} \quad \left(\text{since } \frac{m}{A} = \frac{m^*}{A^*}, \text{ and } h \text{ and } c \text{ remain unchanged.}\right)$$

$$a^*\ t^* = 3 \times 10^{-4} \times 556 = 0.1668$$

Substituting values, $65 = 0 + \dfrac{b^*}{a^*}(1 - e^{-0.1668}) = 0.1536\, b^*/a^*$

or $b^* = a^* \times 423.18 = 3 \times 10^{-4} \times 423.18 = 0.127$

As $b^* = \dot{Q}/m^* c,\ m^* = \dfrac{\dot{Q}}{b^* c} = \dfrac{30}{0.127 \times 800} = 0.295\ \text{kg}$

Additional mass = (0.295 – 0.250) kg = **0.045 kg** **(Ans.) (c)**

$$A^* = A \times \frac{m^*}{m} = 60\ \text{cm}^2 \times \frac{0.295\ \text{kg}}{0.25\ \text{kg}} = 7 - 0.8\ \text{cm}^2$$

Additional area = (70.8 – 60)cm² = **10.8 cm²** **(Ans.) (c)**

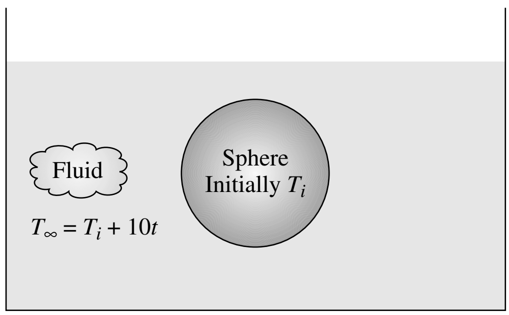
EXAMPLE 5.13 *A metallic sphere initially at a uniform temperature* T_i, *is immersed in a fluid which is heated by an electric heater such that* $T_\infty = T_i = 10t$. *Neglecting internal temperature gradients, derive an expression for the temperature of the sphere as a function of time and convective heat transfer coefficient.*

Solution

Known A metallic sphere with uniform temperature (location wise) is immersed in a fluid whose temperature *increases with time.*

Find Expression for temperature variation in the sphere with time.

Schematic



Assumptions (1) Internal temperature gradients are negligible (sphere is spatially isothermal). (2) Thermal properties of sphere and heat transfer coefficient are constant.

Analysis From energy balance: Rate of heat loss by convection = Rate of decrease of internal energy:

$$hA(T - T_\infty) = -\rho C_p \forall \frac{dT}{dt} \quad \text{or} \quad \frac{dT}{dt} + \frac{hA}{\rho C_p \forall} T = \frac{hA}{\rho C_p \forall}(T_i + 10t) \tag{A}$$

At $\quad t = 0,\ T = T_i,\ \displaystyle\int_{T_i}^{T} \frac{dT}{T - T_\infty} = -\frac{hA}{\rho C_p \forall}\int_0^t dt = -\frac{hAt}{\rho C_p \forall}$

Characteristic equation: $P + \dfrac{hA}{\rho C_p \forall} = 0$

Characteristic root: $P = -\dfrac{hA}{\rho C_p \forall}$

Complimentary solution: $T_c = C \exp\left(-\dfrac{hAt}{\rho C_p \forall}\right)$

The solution is of the form: $T_p = D + Bt$ where D and B are constants. Substituting T_p into Eq. (A), we have

$$\frac{d}{dt}(D+Bt)+\frac{hA}{\rho C_p \cancel{V}}(D+Bt)=\frac{hA}{\rho C_p \cancel{V}}(T_i+10t)$$

or $$\left(B+\frac{hA}{\rho C_p V}D\right)+\frac{hA}{\rho C_p V}Bt=\left(\frac{hA}{\rho C_p \cancel{V}}T_i\right)+\frac{hA}{\rho C_p V}10t \quad \text{(B)}$$

If the above equality is to hold, coefficients of like power on each side of the above equation must be equal.

$$B+\frac{hA}{\rho C_p V}D=\frac{hA}{\rho C_p \cancel{V}}T_i \quad \text{(C)}$$

and $$\frac{hA}{\rho C_p V}B=\frac{hA}{\rho C_p V}10 \quad \text{(D)}$$

Hence, $D = T_i-\left(\dfrac{\rho C_p V}{hA}\right)B$ from (C)

and $B = 10$ from (D)

$$T_p = D+Bt = T_i-\left(\frac{\rho C_p \cancel{V}}{hA}\right)10+10t$$

$\therefore$ $$T = T_c + T_p = C\exp\left(-\frac{hAt}{\rho C_p V}\right)+T_i-10\frac{\rho C_p V}{hA}+10t \quad \text{(E)}$$

When $t = 0$, $T = T_i$

$$T_i = C\exp(0)+T_i-10\frac{\rho C_p V}{hA} \quad \Rightarrow \quad C=\frac{10\rho C_p V}{hA}$$

Substituting this value of C in Eq. (E), we get

$$T = 10\frac{\rho C_p V}{hA}\left(e^{-\frac{hAt}{\rho C_p \cancel{V}}}-1\right)+T_i+10t$$

or $$\boxed{\boldsymbol{T-T_i=10\frac{\rho C_p V}{hA}\left(e^{-\frac{hAt}{\rho C_p V}}-1\right)+10t}} \quad \textbf{(Ans.)}$$

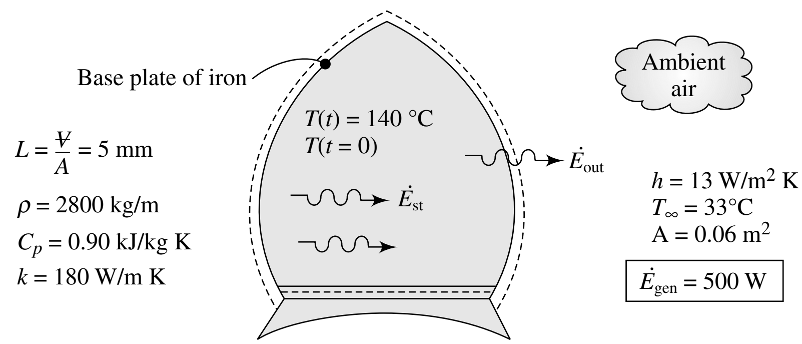
EXAMPLE 5.14 *The base plate of a 500 W household electric iron has a thickness of 5 mm and an ironing surface area of 0.06 m². Initially the iron is at a uniform temperature of 33°C, equal to the ambient air temperature. Suddenly the heating starts and the iron dissipates heat by convection with a surface heat-transfer coefficient of 13 W/m² K. Calculate how long it will take for the plate temperature to reach 140°C after the start of heating. The plate is made of an aluminium alloy with* ρ *= 2800 kg/m³,* C_p *= 0.90 kJ/kg K and* k *= 180 W/m K. (b) What would be the equilibrium temperature of the iron if the control did not switch off the current?*

Solution

Known An iron with a heating element operates under prescribed conditions.

Find (a) Time required by the iron to reach a temperature of 140°C since it was plugged in, (b) $T(t \to \infty)$ (°C).

Schematic



Assumptions (1) Constant plate properties. (2) Uniform heat-generation and heat-transfer coefficient. (3) Internal temperature gradients are neglected.

Analysis Thickness of the base plate, $L = \dfrac{\forall}{A} = 5 \times 10^{-3}$ m

$\therefore$ Volume, $\forall = \rho A$ = (5 × 10^{-3} m) (0.06 m^2) = 3 × 10^{-4} m^3

Mass of the plate, $m = \rho\forall$ = (2800 kg/m^3) (3 × 10^{-4} m^3) = 0.84 kg

Biot number, $Bi = \dfrac{hL}{k} = \dfrac{(13 \text{ W/m}^2 \text{ K})(5 \times 10^{-3} \text{ m})}{180 \text{ W/m K}}$ = 3.67 × 10^{-4} (<< 0.1)

Hence, lumped parameter analysis is justified.
Applying energy balance.

$$\dot{E}_{in} - \cancelto{0}{\dot{E}_{out}} + \dot{E}_{gen} = \dot{E}_{st} \quad \text{or} \quad -hA[T(t) - T_\infty] + \dot{E}_{gen} = mC_p \frac{dT}{dt}$$

With $\quad T(t) - T_\infty = \theta, d\theta = dT \quad \Rightarrow \quad \dfrac{d\theta}{dt} + \dfrac{hA}{mC_p}\theta = \dfrac{\dot{E}_{gen}}{mC_p}$

Let $\quad a \equiv \dfrac{hA}{mC_p} \quad$ and $\quad b \equiv \dfrac{\dot{E}_{gen}}{mC_p}$

Then $\quad \dfrac{d\theta}{dt} + a\theta = b$

The solution of this equation can be expressed as:

$$\boxed{\theta = \theta_i \exp(-at) + (b/a)[1 - \exp(-at)]}$$

where $\quad \theta_i = T(t = 0) - T_\infty = 0$

$$b/a = \frac{\dot{E}_{gen}}{hA} = \frac{500\ \text{W}}{(13\ \text{W/m}^2\ \text{K})(0.06\ \text{m}^2)}$$

$$a = \frac{hA}{mC_p} = \frac{(13\ \text{W/m}^2\ \text{K})(0.06\ \text{m}^2)}{(0.84\ \text{kg})(900\ \text{J/kg K})} = 1.032 \times 10^{-3}\ \text{s}^{-1}$$

$$\theta = T(t) - T_\infty = 140 - 33 = 107°\text{C}$$

Substituting the appropriate values in Eq. (1), we have

$$107 = 0 + 641\ [1 - \exp\ (-1.032 \times 10^{-3}\ t)]$$

or $$\exp[-\ 1.032 \times 10^{-3}\ t] = 1 - \left(\frac{107}{641}\right) = 0.833$$

or $$-1.032t \times 10^{-3} = \ln 0.833 = -\ 0.18263$$

∴ time required to reach 140°C,

$$t = \frac{0.18263 \times 10^3}{1.032} = \mathbf{177\ s} \text{ or } \mathbf{2.95\ min} \qquad \textbf{(Ans.) (a)}$$

Steady-state (*equilibrium*) temperature of the iron,

$$T(\infty) = \left(\frac{b}{a}\right) = 641°\text{C} + 33°\text{C} = \mathbf{674°C} \qquad \textbf{(Ans.) (b)}$$

EXAMPLE 5.15 *A batch reactor provided with submerged steam coil, contains 1000 kg mass of reactants having a specific heat of 3.8 kJ/kg K. The coil area is 1 m² and the steam is fed at 120°C. Assuming no heat loss to the surroundings, calculate the time taken to heat the material from 20°C to 90°C. The overall heat transfer coefficient is 600 W/m² K. If the external area of the vessel is 10 m² and the outside heat-transfer coefficient is 9 W/m² °C, what would be the time taken to heat the reactants over the same temperature range? What would be the maximum temperature the reactants could reach?*

Solution

Known Reactants are heated in a reactor equipped with a submerged steam coil.

Find Time required (a) Neglecting heat loss to surroundings and (b) Without neglecting it. (c) Maximum reactants' temperature

Assumptions (1) Reactants are well stirred. (2) Constant properties. (3) Constant heat-transfer coefficient.

Analysis Neglecting heat loss to be surroundings, the energy balance gives, $\dot{E}_{in} = \dot{E}_{st}$

i.e., heat transferred to reactants = rate of increase of internal energy of the reactants

or $$UA_{coil}(T_{steam} - T) = mC_p \frac{dT}{dt}$$

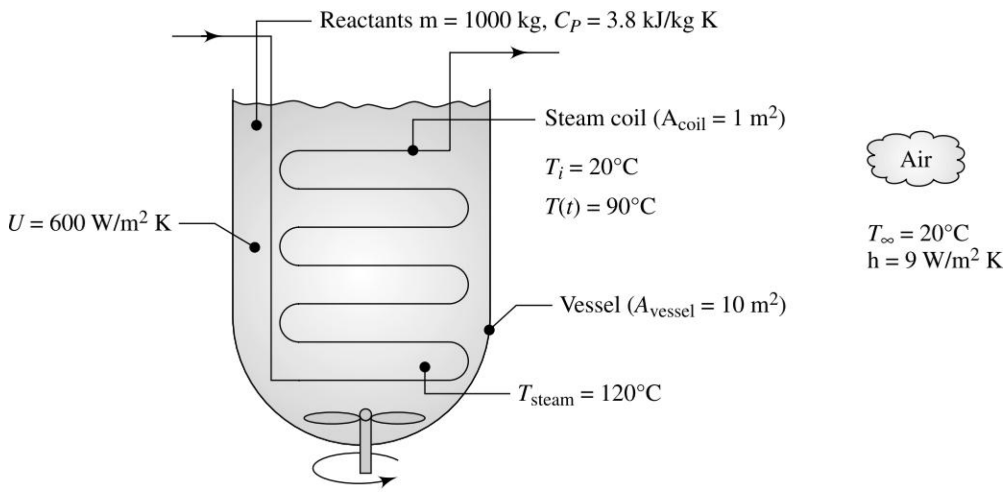
where T is the temperature of reactants which are heated from initial atmospheric temperature, $T_i = T_\infty = 20°\text{C}$ to the process temperature $T(t) = 90°\text{C}$.

or $$dt = \frac{mC_p dT}{UA_{coil}(T_{steam} - T)}$$

Integrating between the limits, are have

$$\int_0^t dt = \frac{mC_p}{UA_{coil}} \int_{T=T_i}^{T=T(t)} \frac{dt}{-(T - T_{steam})} \quad \text{or} \quad \boxed{t = \frac{mC_p}{UA_{coil}} \ln \frac{T_i - T_{steam}}{T(t) - T_{steam}}}$$

Schematic



Time taken is

$$t = \frac{(1000\text{ kg})(3.8\times 10^3\text{ J/kg K})}{(600\text{ W/m}^2\text{K})(1\text{ m}^2)}\ln\left(\frac{20-120}{90-120}\right) = \mathbf{7625\ s\quad or\quad 2.12\ h}\qquad \textbf{(Ans.)}$$

If we take into account the convective heat loss from the vessel to the surrounding air, the energy balance is

$$\dot{E}_{in} - \dot{E}_{out} = \dot{E}_{st} \quad \text{or} \quad UA_{coil}(T_{steam} - T) - hA_{vessel}(T - T_\infty) = mC_p\frac{dT}{dt}$$

Rearranging, $$dt = \frac{mC_p dT}{(UA_{coil}T_{steam} + hA_{vessel}T_\infty) - (UA_{coil} + hA_{vessel})T} = mC_p\frac{dT}{a - bT}$$

where $a \equiv UA_{coil}\,T_{steam} + hA_{vessel}\,T_\infty$ and $b \equiv (UA_{coil} + hA_{vessel})$

Integrating between proper limits, we get

$$\int_0^t dt = mC_p\int_{T_i}^{T(t)}\frac{dT}{a-bT} \quad \text{or} \quad t = -\frac{1}{b}mC_p\ln\left(\frac{a-bT}{a-bT_i}\right)$$

$$\boxed{t = \frac{mC_p}{b}\ln\left(\frac{a-bT_i}{a-bT}\right)}$$

With a = (600 W/m² K × 1 m² × 120°C) + (9 W/m² K × 10 m² × 20°C) = 73800 W

and b = (600 W/m² K × 1 m²) + (9 W/m² K × 10 m²) = 690 W/K

It follows that:

$$t = \frac{(1000\text{ kg})\,(3800\text{ J/kg K})}{(690\text{ W/K})}\left|\frac{1\text{ W}}{1\text{ J/s}}\right|\ln\left[\frac{73800\text{ W} - (690\text{ W/K}\times 20°\text{C})}{73800\text{ W} - (690\text{ W/K}\times 90°\text{C})}\right]$$

= 9003 s or **2.5 h** **(Ans.)**

Maximum temperature of the reactants will be achieved under steady operating conditions, i.e., when $\dot{E}_{in} = \dot{E}_{out}$

$$UA_{coil}\,(T_{steam} - T_{max}) = hA_{vessel}\,(T_{max} - T_\infty)$$

or $$120 - T_{max} = (T_{max} - 20)\left[\frac{9\times 10}{600\times 1}\right] = 0.15\,T_{max} - 3 \quad \Rightarrow \quad 1.15\,T_{max} = 123$$

$\therefore$ T_{max} = **107°C** **(Ans.)**

(C) Plane Wall

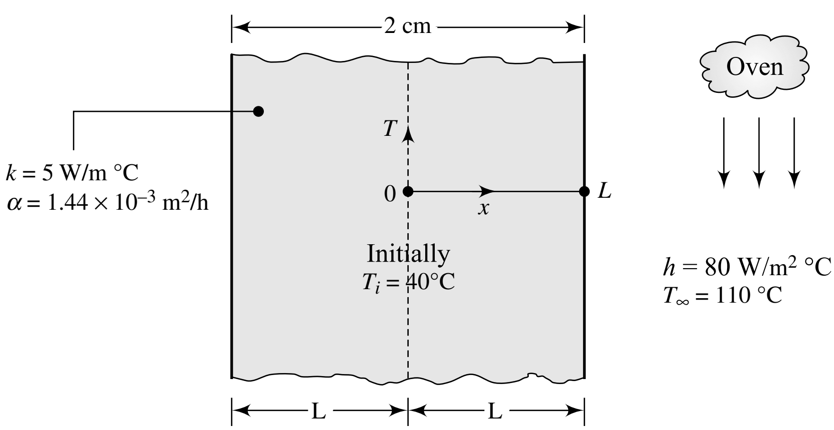
EXAMPLE 5.16 *A thermoplastic material [k = 5 W/m °C and α = 1.44 × 10⁻³ m²/ h] ought to be brought to at least 90°C for ease in moulding but its maximum allowable temperature at any point is 110°C. As a safeguard against possible damage, a sheet of this material of 2 cm thickness is placed in an oven which is maintained at 110°C. The sheet is initially at a uniform temperature of 40°C. The convection heat transfer coefficient is 80 W/m²°C. Determine the minimum time required for the sheet to reach 90°C everywhere.*

Solution

Known A thermoplastic sheet is heated in an oven in a convective environment.

Find Minimum time of immersion of the sheet.

Schematic



Assumptions (1) One-dimensional transient heat conduction. (2) Constant properties and uniform heat-transfer coefficient. (3) $Fo > 0.2$ so that one term analytical solution is justified.

Analysis For a plane wall, the characteristic length, L is half-thickness, i.e., 1 cm or 0.01 m.

$$\text{Biot number, } Bi = \frac{hL}{k} = \frac{(80\text{ W/m}^2\,^\circ\text{C})(0.01\text{ m})}{5\text{ W/m }^\circ\text{C}} = 0.16$$

From Table 5.2, with $Bi = 0.16$: $\lambda_1 = 0.3841$ rad and $A_1 = 1.0251$

The dimensionless centre temperature is

$$\frac{T(0,t) - T_\infty}{T_i - T_\infty} = \theta_{o,\text{wall}} = A_1 \exp(-\lambda_1^2 Fo) \quad \text{or} \quad \frac{90-110}{40-110} = 1.0251\exp(-0.3841^2 \times Fo)$$

or $\quad \ln\left[\dfrac{2/7}{1.0251}\right] = -(0.3841)^2 Fo \quad \Rightarrow \quad Fo = 8.66 = \alpha t/L^2$

Therefore, time required, $t = \dfrac{(8.66)(0.01\text{ m})^2}{1.44\times 10^{-3} m^2/h} = $ **0.60 h or 36 min** (Ans.)

Comment The minimum temperature will occur at the centre. Hence, once the centre attains 90°C, the entire sheet is at least at that temperature. The surface temperature will be

$$T(x/L = 1) = T_\infty + (T_i - T_\infty)\, Q_{o,\text{wall}} \cos \lambda_1$$

$$= 1104°\text{C} + (40 - 110)°\text{C} \times (2/7) \cos\left(0.3841 \times \frac{180°}{\pi}\right)$$

$= \mathbf{91.5°C}$, i.e., just 1.5°C above the midplane temperature.

(D) Long Cylinder

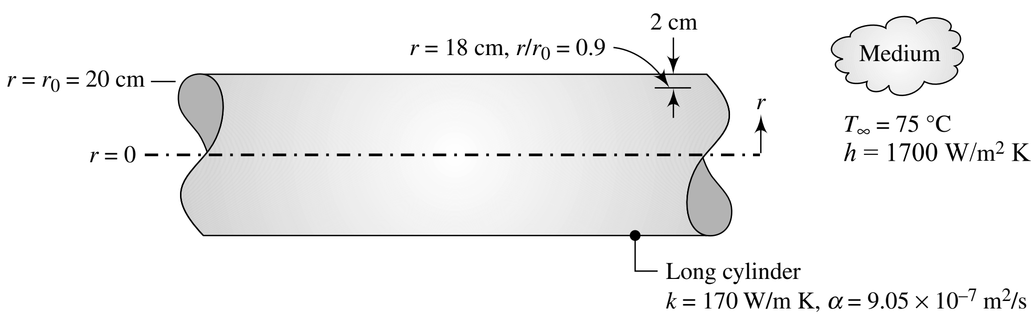
EXAMPLE 5.17 *A long 35 cm diameter cylindrical shaft made of stainless steel (k = 14.9 W/m K, ρ = 7900 kg/m³ and C_p = 0.477 kJ/kg K) comes out of an oven at a uniform temperature of 400°C. The shaft is then allowed to cool slowly in a chamber at 150°C with an average convection heat-transfer coefficient of 85 W/m² K. Determine (a) the temperature at the centre of the shaft 45 minutes after the start of the cooling process, (b) the temperature at the surface of the shaft after 45 minutes, and (c) the heat transfer per metre length of the shaft during this period.*

Solution

Known A long cylindrical shaft is cooled slowly under the given conditions.

Find (a) Centre temperature, T_0(°C), (b) Surface temperature, $T(r = r_o)$. (c) Heat transfer from the shaft per unit length, Q (kJ).

Schematic



Assumptions (1) The shaft is long and has a thermal symmetry about its centreline. Hence, heat conduction is one-dimensional. (2) Thermophysical properties of the shaft material and the heat transfer coefficient are constant. (3) The Fourier number, $Fo > 0.2$ so that the transient temperature charts and one term approximate solutions are applicable.

Analysis (a) The temperature within the shaft is a function of both radial distance r (from the axis) and time t. The Biot number for this problem is

$$Bi = \frac{hr_o}{k} = \frac{(85\text{ W/m}^2\text{ K})(0.175\text{ m})}{14.9\text{ W/m K}} = 0.9983$$

(> 0.1) ⇒ *Lumped-capacity model is not valid.*

Fourier number, $Fo = \dfrac{\alpha t}{r_o^2}$,

where $\alpha = \dfrac{k}{\rho C_p} = \dfrac{14.9 \text{ W/m K}\left|\dfrac{1 \text{ J/s}}{1 \text{ W}}\right|}{(7900 \text{ kg/m}^3)(0.477 \times 10^3 \text{ J/kg K})} = 3.954 \times 10^{-6} \text{ m}^2/\text{s}$

$\therefore \quad Fo = \dfrac{(3.954 \times 10^{-6} \text{ m}^2/\text{s})(45 \times 60 \text{ s})}{(0.175 \text{ m})^2} = 0.3486$

With $Bi = 0.9983$ and $Fo = 0.3486$, we obtain from the chart.

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = 0.7$$

and, the temperature at the centre of the shaft is

$T_0 = T_\infty + 0.7(T_i - T_\infty) = 150 + 0.7\ (400 - 150) = \mathbf{325°C}$ **(Ans.) (a)**

(b) The position correction factor for

$Bi = 0.9983$ and $r/r_o = 1$, $\dfrac{T(r/r_0 = 1) - T_\infty}{T_0 - T_\infty} = 0.64$ (*from the chart*)

$\therefore$ Surface temperature,

$T(r/r_o = 1) = T_\infty + (0.64)\ (0.7)\ (T_i - T_\infty) = 150 + 0.448 \times 250$

$= \mathbf{262°C}$ **(Ans.) (b)**

(c) The dimensionless heat-transfer ratio for a long cylinder is determined from the chart to be

$$\frac{Q}{Q_{max}} = \frac{Q}{Q_i} = 0.44$$

Hence, $Q = 0.44\ Q_{max}$

The maximum heat transfer from the cylinder per unit length is

$Q_{max} = mC_p(T_i - T_\infty) = \rho ⩝ C_p(T_i - T_\infty) = \rho(\pi r_o^2 L) C_p(T_i - T_\infty)$

$= (7900 \text{ kg/m}^3)\ (\pi \times 0.175^2 \text{ m}^2 \times 1 \text{ m})(0.477 \text{ kJ/kg K}) \times (400 - 150)°\text{C or K}$

$= \mathbf{90\ 638\ kJ}$

Heat transferred from the cylinder during 45 minutes,

$Q = (0.44)\ (90\ 638 \text{ kJ}) = \mathbf{39\ 880\ kJ}$ **(Ans.) (c)**

(E) Sphere

EXAMPLE 5.18 *Stainless-steel ball bearings [k = 21.9 W/m °C, ρ = 7900 kg/m³, C_p = 571 J/kg °C, $\alpha = 4.85 \times 10^{-6}$ m²/s] of 20 mm in diameter have an initial uniform temperature of 850°C and are suddenly quenched in an oil bath maintained at 40°C. The associated convective heat transfer coefficient is 1000 W/m² °C. (a) If the surface temperature of the balls after quenching is 100°C, how long must the balls be kept in oil? (b) What is the centre temperature of the balls? (c) If 20 000 balls are to be quenched per hour, find the oil bath cooling rate in order to maintain its temperature at 40°C.*

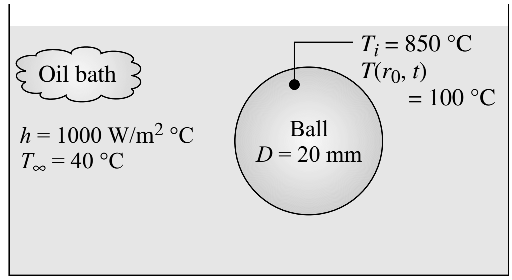
Solution

Known Ball bearings are hardened by quenching them in the oil bath.

Find (a) Time required, t (s) to attain $T(r_o, t) = 100°C$. (b) Centre temperature of balls, $T(0, t)$ (°C). (c) Oil-bath cooling rate, $\dot{Q}$ (W).

Assumptions (1) One-dimensional radial transient heat conduction. (2) Properties and convection coefficient are constant. (3) Fourier number, $Fo > 0.2$.

Schematic



Analysis (a) *Biot* number for a spherical ball,

$$Bi = \frac{hL_c}{k} = \frac{h(\cancel{V}/A)}{k} = \frac{h(\pi D^3/6/\pi D^2)}{k}$$

$$= \frac{hD}{6k} = \frac{(1000\ \text{W/m}^2\ ^\circ\text{C})(20\times 10^{-3}\ \text{m})}{6(21.9\ \text{W/m}\ ^\circ\text{C})} = 0.1522$$

As $Bi > 0.1$, the lumped-heat-capacity model is not quite appropriate. Transient temperature charts can be used for improved accuracy.

$$Bi = \frac{hr_o}{k} = \frac{(1000\ \text{W/m}^2\ ^\circ\text{C})(0.01\ \text{m})}{21.9\ \text{W/m}\ ^\circ\text{C}} = 0.457$$

Also $r/r_o = 1$ at the surface of the ball.

From the chart, with $Bi = 0.457$ and $r/r_o = 1$, $\dfrac{T(r_o, t) - T_\infty}{T_0 - T_\infty} = 0.80$

Now, $\theta_{\text{sph}}(r_o, t) = \dfrac{T_0 - T_\infty}{T_i - T_\infty} = \dfrac{100 - 40}{850 - 40} = 0.0741$

It follows that $\theta_{o,\text{sph}} = \dfrac{T(r_o, t) - T_\infty}{T_i - T_\infty} \Bigg/ \dfrac{T(r_o, t) - T_\infty}{T_0 - T_\infty} = \dfrac{0.0741}{0.8} = 0.0926$

With $Bi = 0.457$ and $\theta_0(t) = 0.0926$, from the chart. $Fo = \dfrac{\alpha t}{r_o^2} \cong 2.0$

Hence, the time required, $t = \dfrac{(2.0)(0.01\ \text{m})^2}{4.85\times 10^{-6}\ \text{m}^2/\text{s}}$ = **41 s** **(Ans.) (a)**

(b) $\theta_{o,\text{sph}}(t) = \dfrac{T_0 - T_\infty}{T_i - T_\infty} = 0.0926$

Centre temperature is

$T_0 = T_\infty + 0.0926\ (T_i - T_\infty) = 40°C + 0.0926\ (850 - 40) =$ **115°C** **(Ans.) (b)**

(c) $Bi^2 Fo = (0.457)^2\ (0.0741) = 0.418$

From the chart for a single ball, $\dfrac{Q}{Q_i} \cong 0.94$

Hence, $Q = 0.94 \text{ m } C_p(T_i - T_\infty) = 0.94 \; \rho \dfrac{\pi D^3}{6} C_p(T_i - T_\infty)$

$= (0.94)\,(7900 \text{ kg/m}^3) \left(\pi \times \dfrac{0.02^3}{6} m^3\right) (571 \text{ J/kg °C})\,(850 - 40)\text{°C}$

$= 14.4 \times 10^3$ J or 14.4 kJ

The oil-bath cooling rate, for 20 000 balls per hour

$$\dot{Q} = \frac{Q}{\Delta t} = \frac{14.4 \text{ kJ}}{3600 \text{ s}} \times 20000 = \mathbf{80\ kW} \qquad \textbf{(Ans.) (c)}$$

EXAMPLE 5.19 *An egg with a mean diameter of 40 mm and initially at a temperature of 20°C is placed in a saucepan of boiling water for 4 minutes and found to be boiled to the consumer's taste. For how long should a similar egg for the same consumer be boiled when taken from a refrigerator at a temperature of 5°C?*

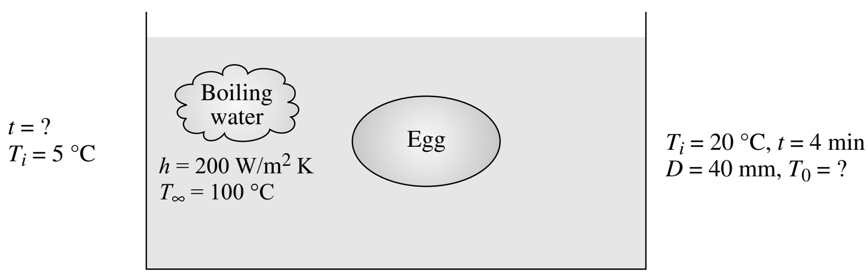
The egg properties are k = *2 W/m K,* ρ = *1200 kg/m³,* C_p = *2 kJ/kg K and the heat-transfer coefficient for the shell and shell water interface may be taken as 200 W/m² K. Compare the centre temperature attained with that computed by treating the egg as a lumped-heat-capacity system.*

Solution

Known An egg is cooked in boiling water under prescribed conditions.

Find Centre temperature of egg, $T(0, t)$ for t = 4 min and T_i = 20°C; Time required, t (s) for T_i = 5°C.

Schematic



Assumptions (1) The egg is spherical in shape with a radius of r_o = 20 mm. (2) Heat conduction in the egg is one-dimensional due to thermal symmetry about the centre point. (3) Constant egg properties and convection coefficient. (4) The Fourier number, Fo > 0.2 so that one term analytical approximate solution holds good.

Analysis The Biot number for this problem is

$$Bi = \frac{hr_o}{k} = \frac{(200 \text{ W/m}^2 \text{ K})(0.02 \text{ m})}{2 \text{ W/mK}} = 2$$

The coefficients λ_1 and A_1 for a sphere corresponding to Bi = 2 are from Table 5.2:

λ_1 = 2.0288 rad, A_1 = 1.4793.

The Fourier number, $Fo = \dfrac{\alpha t}{r_o^2} = \dfrac{(8.33 \times 10^{-7} \text{ m}^2/\text{s})(4 \times 60 \text{ s})}{0.02^2} = 0.5$

Substituting these and other values,

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo} = 1.4793 \exp[-(2.0288)^2 (0.5)] = 0.1889$$

Hence, the centre temperature of the egg after 4 min to the consumer taste is

$$T_0 = T_\infty + 0.1889(T_i - T_\infty) = 100°\text{C} + 0.1889\,(20 - 100)°\text{C} = \mathbf{84.9°C} \quad \textbf{(Ans.)}$$

A similar egg is cooked now with an initial uniform temperature of 5°C to the consumer's taste, i.e., T_0 = 84.9°C

Then, $$\frac{T_0 - T_\infty}{T_i - T_\infty} = \frac{84.9 - 100}{5 - 100} = 0.159$$

The value of $Bi = \dfrac{hr_o}{k}$ remains unchanged at Bi = 2 and so are the coefficients λ_1 and A_1.

Therefore, using the one-term analytical method,

$$0.159 = 1.4793 \exp[-(2.0288)^2 Fo]$$

where from $0.10745 = e^{-(2.0288)^2 Fo}$ or $-(2.0288)^2 Fo = \ln 0.10745 = -2.23075$

$\therefore$ $$Fo = 0.542 = \alpha t / r_o^2$$

And, the time required to reach the same temperature of 84.9°C now is

$$t = \frac{0.542(0.02\text{ m})^2}{8.33 \times 10^{-7}\text{ m}^2/\text{s}} = \mathbf{260\ s} \quad \text{or} \quad \mathbf{4\ min\ 20\ s} \qquad \textbf{(Ans.)}$$

The egg should, therefore, be boiled 20 seconds longer.

Alternatively: Using the *chart* (since Fo > 0.2), we get for Bi = 2 and Fo = 0.5:

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = 0.195 \quad \text{and} \quad T_0 = 100 + 0.195(20 - 100) = \mathbf{84.4°C}$$

From the *chart*, with

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = \frac{84.4 - 100}{5 - 100} = 0.164 \text{ and } Bi = 2, \text{ we read } Fo = 0.54$$

$\therefore$ $$t = 0.54\,(0.02\text{ m})^2/(8.33 \times 10^{-7}\text{ m}^2/\text{s}) \approx \mathbf{260\ s}$$

Lumped-Capacity Model:

Using the lumped-mass approximation method,

$$\frac{T_0 - T_\infty}{T_i - T_\infty} = \exp\left[-\frac{hAt}{\rho C_p V\!\!\!-}\right] = \exp\left[-\frac{ht}{\rho C_p} \times \frac{4\pi r_o^2}{4/3\pi r_o^3}\right] = \exp\left[-\frac{ht}{\rho C_p} \times \frac{3}{r_o}\right]$$

$$= \exp\left[-\frac{(200\text{ W/m}^2\text{K})(4 \times 60\text{ s})(3)}{(1200\text{ kg/m}^3)(2000\text{ J/kg K})(0.02\text{ m})}\right] = 0.05$$

Temperature of egg after 4 minutes is

$$T_0 = 100°\text{C} + (20 - 100)°\text{C}\,(0.05) = \mathbf{96°C}$$

The discrepancy is due mainly to the assumption of uniform temperature throughout the egg—an assumption in which internal temperature gradients are neglected and the system is treated as spacewise isothermal.

Comment The use of charts is certainly simpler but less accurate. For better accuracy, analytical solution is preferable, though a little more time consuming, especially when the reading error from the charts is likely to be considerable.

(F) Semi-Infinite Medium

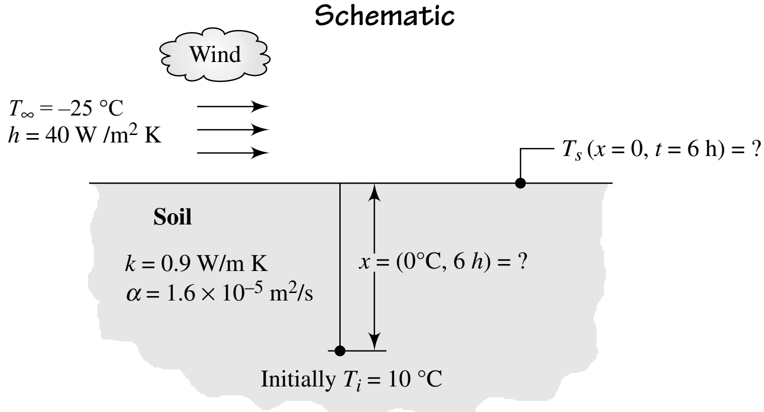
EXAMPLE 5.20 *During the harsh winter conditions in Kashmir, a sudden cold wave reduces the ambient air temperature to –25°C. Before the cold wave moved in, the earth at a location was initially at a uniform temperature of 10°C. High winds result in a convection heat-transfer coefficient of 40 W/m² K on the earth's surface for a period of 6 h. Taking the wet-soil properties at that location to be* k = *2.0 W/m K and* $\alpha = 1.72 \times 10^{-3}$ *m²/h, determine (a) the surface temperature of the earth at the end of 6 h, and (b) the distance from the earth's surface up to which the freezing effect will penetrate in this period.*

Solution

Known The earth's surface is subjected to very cold and windy conditions.

Find (a) Surface temperature of earth after 6 h, $T_s(x = 0, t = 6 \text{ h})$. (b) Depth of penetration, x $(T = 0°\text{C}, t = 6 \text{ h})$.

Schematic



Assumptions (1) The earth is idealized as a semi-infinite solid. (2) Latent-heat effects of freezing of moisture in soil are neglected. (3) Constant soil properties and heat-transfer coefficient.

Analysis: (a) At the surface of the earth, $x = 0$.

Hence, the Biot number, $Bi = \dfrac{hx}{k} = 0$

Also, $$\frac{h^2\alpha t}{k^2} = \frac{(40 \text{ W/m}^2\text{ K})^2(1.72 \times 10^{-3}\text{ m}^2/\text{h})(6h)}{(2.0 \text{ W/m K})^2} = 4.128 \quad \text{and} \quad \frac{h\sqrt{\alpha t}}{k} = 2.0317$$

The exact solution to this type of problem can be expressed as:

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right) \times \left[\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right] \quad \text{(A)}$$

Noting that $\dfrac{x}{2\sqrt{\alpha t}}$ = 0 at x = 0 and erfc $\left(\dfrac{x}{2\sqrt{\alpha t}}\right)$ = erfc (0) = 1

$$\frac{T(0,t)-T_i}{T_\infty - T_i} = 1 - e^{h^2\alpha t/k^2}\left[\operatorname{erfc}\left(\frac{h\sqrt{\alpha t}}{k}\right)\right]$$

$\therefore$ $\quad T\,(0,\,6\,h) = T_i + (T_\infty - T_i)\,[1 - e^{4.128}\{\operatorname{erfc}(2.0317)\}]$

$= 10°C + \{-25 - (10)\}\ [1 - (62.05 \times 0.00414)]$

$= 10 - 35(1 - 0.257) = 10 - (0.743)\,(35) =$ **–16°C** **(Ans.) (a)**

(b) $T\,(x,\,t = 6\text{ h})$ = Freezing temperature = 0°C

With $\dfrac{h\sqrt{\alpha t}}{k}$ = 2.0317 and $\dfrac{T(x,6h)-T_i}{T_\infty - T_i} = \dfrac{0-10}{-25-10}$ = 0.2857

Equation (A) can be solved by following a *trial-and-error* procedure. To satisfy this equation, we find:

The dimensionless distance, $\dfrac{x}{2\sqrt{\alpha t}}$ = **0.538**

Hence, the distance (depth) from the earth's surface is,

$x = 0.538 \times 2\sqrt{0.00172 \times 6}$ **0.11 m or 11 cm** **(Ans.) (b)**

EXAMPLE 5.21 *A passenger car travelling at a speed of 72 km/h is suddenly brought to rest within 5 seconds by applying the brakes. The braking system comprises four brakes (with each brake band of 350 cm² area) which press against the steel drums of equivalent area. The brake lining and the drum surfaces are at the same temperature and the heat generated during braking is dissipated across the drum surfaces. Determine the maximum rise in temperature assuming the mass of the car to be 1.5 t and the drums to be semi-infinite medium. Take for steel:* k *= 55 W/m K and* $\alpha = 15.2 \times 10^{-6}$ *m²/s*

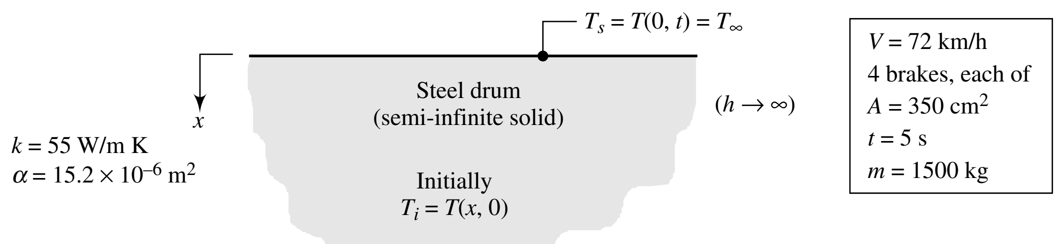
Solution

Known A car is brought to halt by suddenly applying the brakes.

Find Maximum temperature rise, $(T_s - T_i)$ (°C).

Assumptions (1) The drums are approximated as semi infinite slab with constant surface temperature. (2) Constant properties.

Schematic



Analysis Rate of decrease of kinetic energy when the brakes are applied,

$$\frac{1}{2}\dot{m}[V^2 - 0] = \frac{1}{2}\dot{m}V^2 \; \frac{1}{2}\frac{mV^2}{t} = \frac{1}{2}\frac{(1500\text{ kg})}{5\text{s}}\left(\frac{72 \times 1000\text{ m}}{3600\text{ s}}\right)^2$$

$$60 \times 10^3 \frac{\text{kg m}^2}{\text{s}^3}\left[\frac{1\text{ kJ/kg}}{10^3\text{ m}^2/\text{s}^2}\right] = 60\text{ kJ/s}$$

Energy balance: $\begin{pmatrix}\text{Kinetic energy} \\ \text{decrease}\end{pmatrix} = \begin{pmatrix}\text{Heat generated and dissipated} \\ \text{through drum surface}\end{pmatrix}$

Instantaneous heat-flow rate, at the surface ($x = 0$),

$$\dot{Q}_0 = \frac{kA(T_s - T_i)}{\sqrt{\pi\alpha t}} = \frac{1}{2}\dot{m}V^2 = 60\text{ kJ/s} = 60\ 000\text{ W}$$

Area for heat transfer

$$A = (350 \times 10^{-4}\text{ m}^2/\text{brake band})\ (4\text{ brakes}) = 0.14\text{ m}^2$$

Maximum temperature rise,

$$T_s - T_i = \frac{\dot{Q}_0\sqrt{\pi\alpha t}}{kA} = \frac{60000\text{ W} \times \sqrt{\pi \times 15.2 \times 10^{-6}\text{ m}^2/\text{s} \times 5\text{ s}}}{(55\text{ W/mK})(0.14\text{ m}^2)}$$

$$= \mathbf{120.4\ K\quad or\quad °C} \qquad \textbf{(Ans.)}$$

EXAMPLE 5.22 *A manufacturing process requires a thin veneer, 1 mm thick, to be glued onto a thick piece of particle board. The adhesive is thermal setting and requires a temperature of 120°C to fuse and adhere to the base. Consider the veneer and base to have essentially the same thermal conductivity* k *= 0.14 W/m °C and thermal diffusivity* $\alpha = 1.26 \times 10^{-7}$ *m²/s. (a) If the particle board and veneer are initially at 25°C, and the surface temperature is suddenly changed to 200°C by a heated copper pressure plate, how long would it take for the adhesive layer to fuse and set?*

(b) If the hot plate is assumed to act like an ambient environment of 200°C with a surface heat-transfer coefficient of 100 W/m² K, determine the time to bring the adhesive to the required temperature of 120°C. Also find the surface temperature and surface heat flux at this time.

Solution

Known A veneer and particle board have their surface temperature suddenly raised.

Find Time required for the adhesive layer to fuse and set, t (s) with (a) Sudden change in surface temperature, and (b) Convection boundary conditions.

Assumptions (1) The heated pressure plate makes intimate contact with the veneer layer and instantaneously raises the surface temperature. (2) The veneer and particle board are a semi-infinite solid. (3) Constant properties.

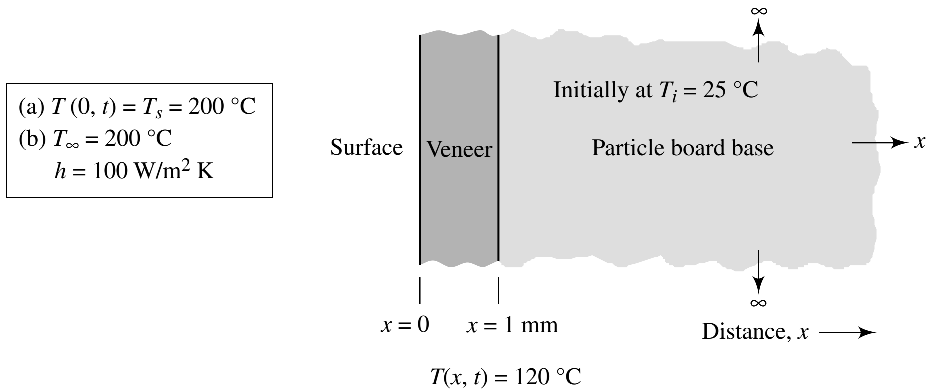
Analysis ***Case (a): Constant Surface Temperature***

For the sudden change in surface temperature, (due to extremely high heat-transfer coefficient, $h \to \infty$), the boundary and initial conditions are the following:

At $x = 0$ (the surface), $T = T_\infty$ for $t > 0$

At $t \le 0$, the temperature is T_i at all locations x into the solid.

Schematic



Local Biot number, $Bi = \dfrac{hx}{k} \to \infty \quad (\text{as } h \to \infty)$

The solution to the conduction equation under these conditions is,

$$\frac{T(x,t)-T_i}{T_s-T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$T(1 \text{ mm}, t) = 120°\text{C},\ T_s = 200°\text{C},\ T_i = 25°\text{C}$

Therefore, $\text{erfc}\left(\dfrac{x}{2\sqrt{\alpha t}}\right) = \dfrac{120-25}{200-25} = 0.543$

Table 5.3 gives a value of about 0.43 for the argument $x/2\sqrt{\alpha t}$.

Accordingly, $\dfrac{x}{2\sqrt{\alpha t}} = 0.43$ and solving for the time t yields

$$t = (x/0.86)^2/\alpha\{(1\times10^{-3}\text{m})/0.86\}^2/(1.26\times10^{-7}\text{m}^2/\text{s}) = \mathbf{10.7\ s} \qquad \textbf{(Ans.) (a)}$$

Case (b): Convection on the Surface

In this case, there is a finite surface heat-transfer coefficient, h = 100 W/m^2 K and an ambient temperature, T_∞ = 200°C.

The local Biot number, $Bi = \dfrac{hx}{k} = \dfrac{(100\ \text{W/m}^2\text{K})(0.001\ \text{m})}{0.14\ \text{W/mK}} = 0.714$

Also, $\dfrac{T_x - T_\infty}{T_i - T_\infty} = \dfrac{(120-200)°\text{C}}{(25-200)°\text{C}} = 0.457$

From the chart, the dimensionless distance $x/2\sqrt{\alpha t}$ is estimated to be 0.19 under the above conditions. The time then can be calculated as follows:

$$(0.19)^2 = \frac{x^2}{4\alpha t}$$

$$\therefore \qquad t = \frac{(x/0.19)^2}{4\alpha} = \frac{(0.001\ \text{m}/0.19)^2}{4\times1.26\times10^{-7}\ \text{m}^2/\text{s}} = \mathbf{55\ s} \qquad \textbf{(Ans.) (b)}$$

It is noteworthy that 55 s is a more realistic and exact estimate of time than 10.7 s obtained earlier with the assumption that h is almost infinite.

The surface temperature, $T(0, t)$ is given by

$$T_s = T(0,t) = T_\infty + (T_i - T_\infty)\left[e^{h^2\alpha t/k^2}\operatorname{erfc}\left(\frac{h\sqrt{\alpha t}}{k}\right)\right]$$

With $$\frac{h\sqrt{\alpha t}}{k} = \frac{100\ \text{W/m}^2\,\text{K}\sqrt{1.26\times10^{-7}\,\text{m}^2/\text{s}\times 55\ \text{s}}}{0.14\ \text{W/m K}} = 1.88\ ,$$

$$\operatorname{erfc}(1.88) = 0.00784, \text{ and } \exp(h^2 at/k^2) = e^{(1.88)^2} = 34.32$$

Surface temperature,

T_s = 200 + (25 – 200)[(34.32)(0.00784)] = **152.9°C** **(Ans.) (b)**

Surface heat flux,

$q_s(t) = h[T_\infty - T(0, t)]$ = 100 W/m²K[200 – 152.9]°C

or K = **4710 W/m²** **(Ans.) (b)**

(G) Short Cylinder/Semi-Infinite Cylinder

EXAMPLE 5.23 *A steel disc 30 cm in diameter and 10 cm thick, initially at a uniform temperature of 265°C is immersed in a liquid maintained at –15°C. The convective heat-transfer coefficients on the ends and on the cylindrical side are 340 and 1420 W/m² K, respectively. What is the temperature (a) at the centre of the disc, and (b) on the surface at the centre of one end after 5 minutes have elapsed? Rework the problem with the values of the convection coefficient interchanged.*

The relevant properties of steel are the following:

ρ = 7854 kg/m³, C_p = 487 J/kg K, k = 56.7 W/m K, $\alpha = 14.82 \times 10^{-6}$ m²/s

Solution

Known A steel disc (*a short cylinder*) with two convection coefficients for the ends and the lateral side is cooled in a liquid medium.

Find (a) Centre temperature, $T(0, 0, t)$ and (b) Temperature at the centre of end surface $T(0, L, t)$.

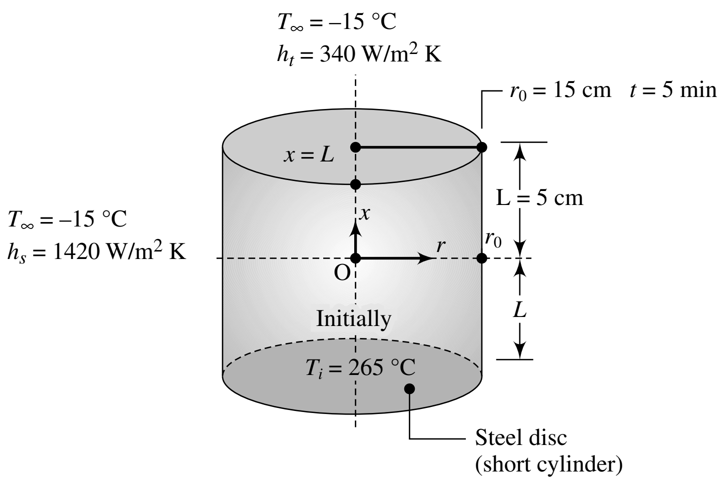
Assumptions (1) Two-dimensional transient heat conduction in r and x directions. (2) Constant properties. (3) Different heat-transfer coefficients for heat flow in radial and axial directions. (4) $Fo > 0.2$ so that Heisler charts can be used.

Analysis The temperature distribution in the steel disc is given by

$$\frac{T(r,x,t) - T_\infty}{T_i - T_\infty} = \theta_{\text{wall}}(x,t)\times\theta_{\text{cyl}}(r,t)$$

(a) To determine the centre temperature of the disc, we need to know $\theta_{\text{wall}}(0, t)$ and $\theta_{\text{cyl}}(0, t)$.

Schematic



Plane Wall:

$$L = 0.05 \text{ m}, h = h_t = 340 \text{ W/m}^2\text{K}$$

$$Bi = \frac{hL}{k} = \frac{(340W/m^2K)\,(0.05m)}{56.7W/mK} = 0.30$$

$$Fo = \frac{\alpha t}{L^2} = \frac{(14.82\times10^{-6}\,m^2/s)\,(5\times60\,s)}{(0.05m)^2} = 1.78$$

From the chart, with Bi = 0.30 and Fo = 1.78,

$$\theta_{\text{wall}}(0, t) = \left.\frac{T_o - T_\infty}{T_i - T_\infty}\right|_{\text{wall}} = 0.65$$

Long Cylinder:

$$r_o = 0.15 \text{ m}, h_s = 1420 \text{ W/m}^2 \text{ K}$$

$$r_o = 0.15 \text{ m}, h_s = 1420 \text{ W/m}^2\text{K}$$

$$Bi = \frac{hr_o}{K} = \frac{(1420 \text{ W/m}^2\text{K})(0.15 \text{ m})}{56.7 \text{ W/mK}} = 3.76$$

$$Fo = \frac{\alpha t}{r_o^2} = \frac{(14.82\times10^{-6} \text{ m}^2/\text{s})\,(5\times60 \text{ s})}{(0.15)^2} = 0.198$$

With Bi = 3.76 and Fo = 0.198. From the chart,

$$\theta_{\text{cyl}}(0, t) = \left.\frac{T_o - T_\infty}{T_i - T_\infty}\right|_{\text{cyl}} = 0.72$$

Therefore, $\theta_o\big|_{\text{short cylinder}} = \dfrac{T_o - T_\infty}{T_i - T_\infty} = \theta_{\text{wall}}(0, t) \times \theta_{\text{cyl}}(0, t) = (0.65)\,(0.72) = 0.468$

$\therefore$ the centre temperature of the disc is

$T(0,\ 0,\ t = 5\ \text{min}) = T_\infty + (0.468)(T_i - T_\infty)$

$= -15°\text{C} + \{265 - (-15\}°\text{C}\ (0.468) = \mathbf{116°C}$ **(Ans.) (a)**

(b) The centre of the one end, say, top surface of the disc is still at the centre of the long cylinder ($r = 0$), but at the outer surface of the plane wall ($x = L$). Hence, one must first find the surface temperature of the wall. We note that at the end,

$$x = L = 0.05\ \text{m, and}\ \frac{x}{L} = \frac{0.05\ \text{m}}{0.05\ \text{m}} = 1$$

With $Bi = 0.30$ and $x/L = 1.0$, from the chart, we get

$$\frac{T_{(x/L=1)} - T_\infty}{T_0 - T_\infty} = 0.86$$

$$\therefore \qquad \theta_{\text{wall}}(L, t) = \frac{T(L,t) - T_\infty}{T_i - T_\infty} = \frac{T(L,t) - T_\infty}{T_0 - T_\infty} \times \frac{T_o - T_\infty}{T_i - T_\infty} = (0.86)\times(0.65) = 0.56$$

Hence, $\left\{\dfrac{T(0, L, t) - T_\infty}{T_i - T_\infty}\right\}_{\text{short cylinder}} = \theta_{\text{wall}}(L,\ t) \times \theta_{\text{cyl}}(0,\ t) = (0.56)\ (0.72) = 0.403$

Temperature on the surface of the disc at the centre of one end after 5 min is

$$T(0,\ 5\ \text{cm},\ 5\ \text{min}) = -15°\text{C} + 0.403\ \{265 - (-15)\}°\text{C} = 98°\text{C}$$

Now, if the the values of h are interchanged then $h_t = 1420\ \text{W/m}^2\ \text{K}$ and $h_s = 340\ \text{W/m}^2\ \text{K}$.

Plane Wall: $Bi = \dfrac{hL}{k} = \dfrac{(1420\ \text{W/m}^2\ \text{K})(0.05\ \text{m})}{56.7\ \text{W/mK}} = 1.252$

$Fo = \dfrac{\alpha t}{L^2} = 1.78$ (same as before)

From the chart, $\theta_{\text{wall}}(0, t) = \left.\dfrac{T(0,t) - T_\infty}{T_i - T_\infty}\right|_{\text{wall}} = 0.255$

Long Cylinder: $Bi = \dfrac{hr_o}{k} = \dfrac{(340\ \text{W/m}^2\ \text{K})(0.15\ \text{m})}{56.7\ \text{W/mK}} = 0.90$

$Fo = \dfrac{\alpha t}{r_o} = 0.198$ (same as before)

From the chart, $\theta_{\text{cyl}}(0, t) = \left.\dfrac{T(0,t) - T_\infty}{T_i - T_\infty}\right|_{\text{cyl}} = 0.89$

$$\therefore \qquad T(0,\ 0,\ t) = T_\infty + T_i - T_\infty)(\theta_{\text{wall}}(0,\ t) \times \theta_{\text{cyl}}(0,\ t)\}$$

$$= -\ 15°\text{C} + \{265 - (-15)\}°\text{C} \times [0.255 \times 0.89]$$

$$= -\ 15°\text{C} + (280°\text{C})\ (0.227) = 48.5°\text{C}\theta_{\text{wall}}(L,\ t)$$

$$= \frac{T(L,t) - T_\infty}{T_i - T_\infty} = \frac{T(L,t) - T_\infty}{T_0 - T_\infty} \times \frac{T_0 - T_\infty}{T_i - T_\infty}$$

With Bi = 1.252 and x/L = 1, from the chart,

$$\frac{T(L,t)-T_\infty}{T_0-T_\infty}=0.59$$

∴ $\theta_{\text{wall}}(L,\ t) = (0.59)\ (0.255) = 0.151$

And $$\frac{T(0,L,t)-T_\infty}{T_i-T_\infty} = \theta_{\text{wall}}(L,\ t)\times\theta_{\text{cyl}}(0,\ t) = (0.151)(0.89) = 0.134$$

∴ $T(0,\ 5\ \text{cm},\ 5\ \text{min}) = T_\infty + 0.0134\ (T_i - T_\infty)$

$= -15°\text{C} + 0.134\ \{265 - (-15)\}°\text{C} =$ **22.5°C** **(Ans.) (b)**

Comment Notice that by interchanging the h-values, the answer changes significantly.

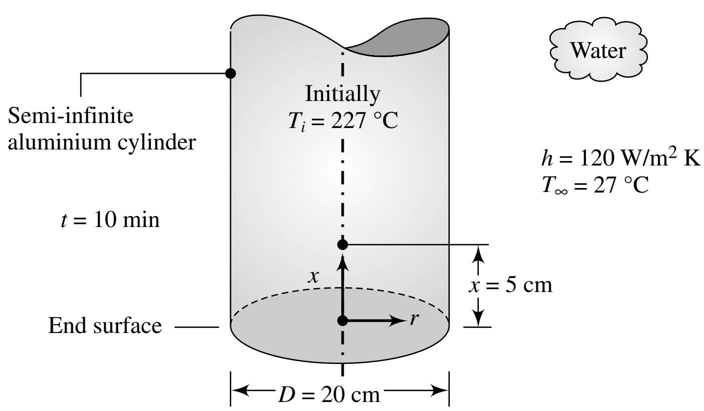
EXAMPLE 5.24 *A semi-infinite aluminium cylinder [k = 240 W/m K, $\alpha = 93.6 \times 10^{-6}$ m²/s] of 20 cm diameter is initially at a uniform temperature of 227°C. The cylinder is now placed in water at 27°C where the average convection heat transfer coefficient is 120 W/m² K. Determine the temperature at the centre of the cylinder 5 cm from the end surface 10 min after the start of cooling.*

Solution

Known A semi-infinite aluminium cylinder is cooled in water under the given conditions.

Find Temperature at the centre of the cylinder at x = 20 cm, $T(x, 0, t)$ (°C).

Schematic



Assumptions (1) Heat conduction is two-dimensional with $T = f(r, x, t)$. (2) Properties and convection coefficient are constant. (3) $Fo > 0.2$ so that approximate analytical solutions are applicable.

Analysis The semi-infinite cylinder can be considered as a two-dimensional body obtained by intersection of an infinite cylinder and a semi-infinite solid.

Infinite Cylinder: r_o = 0.1 m

$$Bi=\frac{hr_o}{k}=\frac{(120\ \text{W/m}^2\ \text{K})\ (0.1\ \text{m})}{240\ \text{W/mK}}=0.05$$

From Table 5.2, for $Bi = 0.05$

$$\lambda_1 = 0.3142 \text{ rad}, \; A_1 = 1.0124$$

$$Fo = \frac{\alpha t}{r_o^2} = \frac{(93.6 \times 10^{-6}\,\text{m}^2/\text{s})(10 \times 60\,\text{s})}{(0.1\,\text{m})^2} = 5.616$$

Hence, $\theta_0 = \theta_{cyl}\,(r = 0,\ t = 10 \text{ min}) = A_1 \exp(-\lambda_1^2 Fo)$

$$= 1.0124 \exp\,[-(0.3142 \text{ rad})^2\,(5.616)] = 0.5815$$

Semi-infinite Solid:

$$1 - \theta_{\text{semi-inf}}\,(x, t)\,\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right) \times \left\{\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right\}$$

Now,

$$\frac{x}{2\sqrt{\alpha t}} = \frac{0.05\,\text{m}}{2\sqrt{(93.6 \times 10^{-6}\,\text{m}^2/\text{s})(600\,\text{s})}} = 0.1055$$

$$\frac{h\sqrt{\alpha t}}{k} = \frac{(120\,\text{W/m}^2\text{K})\sqrt{(93.6 \times 10^{-6}\,\text{m}^2/\text{s})(600\,\text{s})}}{240\,\text{W/m K}} = 0.1185$$

$$\frac{hx}{k} = \frac{(120\,\text{W/m}^2\text{K})(0.05\,\text{m})}{240\,\text{W/m K}} = 0.025$$

$$\frac{h^2 \alpha t}{k^2} = (0.1185)^2 = 0.01404$$

Substituting these values, we have

$$\theta_{\text{semi-inf}}(x,\ t) = 1 - \text{erfc}(0.1055) + \exp(0.025 + 0.01404) \times \text{erfc}(0.1055 + 0.1185)$$

$$= 1 - 0.8865 + \exp(0.03904) \times \text{erfc}(0.224)$$

$$= 0.1135 + (1.0398)(0.7514) = 0.8948$$

Applying the product solution, we get

$$\frac{T(0, x, t) - T_\infty}{T_i - T_\infty} \theta_{\text{semi-inf}}\,(x, t) \times \theta_{cyl}(0, t) = (0.8948)(0.5815) = 0.52$$

Therefore, the temperature at the centre of the cylinder ($r = 0$) at a distance of 5 cm from the exposed bottom surface ($x = 5$ cm) is

$T(0,\ x,\ t) = 27°\text{C} + (0.52)(227 - 27)°\text{C} =$ **131°C** **(Ans.)**

(H) Multi-dimensional Systems

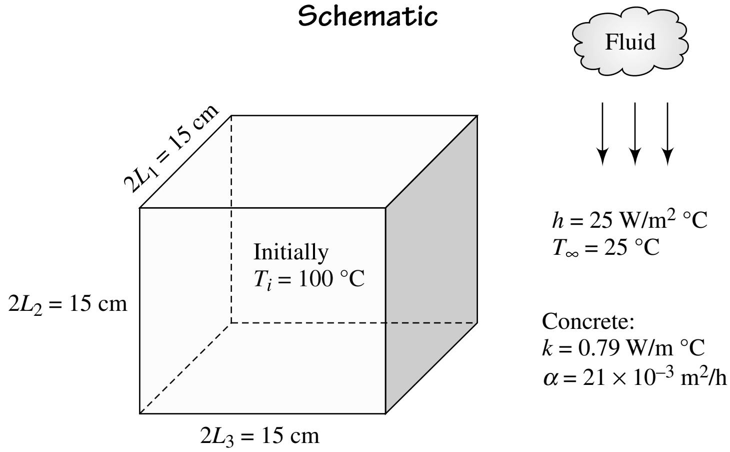
EXAMPLE 5.25 *A concrete cubical block (*k *= 0.79 W/m °C and* $\alpha = 2.1 \times 10^{-3}$ *m²/h) 15 cm on a side originally at a temperature of 100°C is suddenly immersed in a fluid at 25°C for which the convective heat transfer coefficient is 25 W/m² °C. Calculate (a) the temperature at the centre of the cube, and (b) at the midpoint of one face after one hour has passed.*

Solution

Known A concrete cube is allowed to cool in a convective environment.

Find (a) Centre temperature, $T(0, 0, 0, t)$. (b) Temperature at the midpoint of one face, $T(L_1, 0, 0, t)$.

Schematic



Assumptions (1) Three-dimensional conduction. (2) Constant properties. (3) Uniform heat-transfer coefficient on all exposed surfaces. (4) $Fo > 0.2$ to ensure the validity of one term approximate analytical solutions.

Analysis (a) A cube has all equal sides so that $2L_1 = 2L_2 = 2L_3 = 15$ cm. For this three-dimensional geometry, the product solution can be expressed as

$$\theta_{0,\text{cube}} = \frac{T(0,0,0,t) - T_\infty}{T_i - T_\infty} = [\theta_{\text{wall}}(0,t)]^3$$

where $$\theta_{\text{wall}}(0,t) = \frac{T(0,t) - T_\infty}{T_i - T_\infty}$$

To begin with, we must evaluate Bi of the number, Bi, and the corresponding values of the coefficients λ_1 and A_1 for one-term series solution.

$$Bi = \frac{hL}{k}$$ where L is the half-thickness of a side.

$\therefore$ $$Bi = \frac{(25\ \text{W/m}^{2\text{o}}\text{C})(7.5 \times 10^{-2}\,\text{m})}{0.79\ \text{W/m}\ ^\text{o}\text{C}} = 2.373$$

With $Bi = 2.373$, we have from Table 5.2 after linear interpolation :

$\lambda_1 = 1.120$ rad and $A_1 = 1.1903$

With $$Fo = \frac{\alpha t}{L^2} = \frac{(2.1 \times 10^{-3}\,\text{m}^2/\text{h})(1\,\text{h})}{(0.075\ \text{m})^2} = 0.373,$$

the dimensionless centre temperature is,

$$\theta_{0,\text{cube}} = [\theta_{0,\text{wall}}]^3 = [A_1\, e^{-\lambda_1^2 Fo}]^3 = [1.1903\exp(-1.12^2 \times 0.373)]^3$$
$$= (0.7455)^3 = 0.414$$

$\therefore$ Temperature at the centre of the cube after 1 h is

$T(0, 0, 0, t) = T_\infty + 0.414(T_i - T_\infty) = 25°\text{C} + 0.414\ (100 - 25)°\text{C}$
$= \mathbf{56°C}$ **(Ans.) (a)**

(b) Temperature at the midpoint of one face after 1 h has elapsed is,

$$\theta(L_1, 0, 0, t) = \theta(0, 0, 0, t) \times \theta(L_1, t)$$

where $\theta(L_1, t) = \theta(0, t) \cdot \cos \lambda_1 = 0.7455 \times \cos(1.12 \times 57.3°) = 0.325$

Hence, $T(L_1, 0, 0, t) = T_\infty + (0.414)(0.325)(T_i - T_\infty)$

$= 25°C + (0.134)(100 - 25)°C = \mathbf{35°C}$ **(Ans.) (b)**

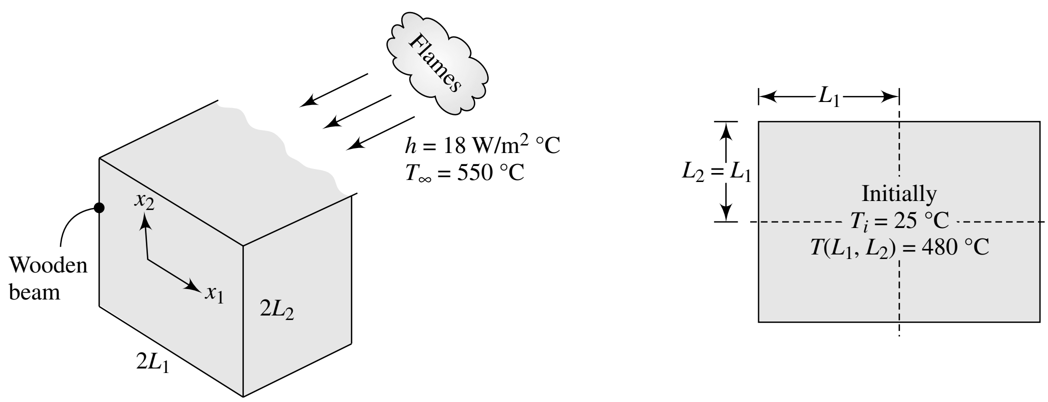
EXAMPLE 5.26 *A 10 cm × 10 cm long wooden beam [ρ = 800 kg/m³, C_p = 2.512 kJ/kg °C, k = 0.346 W/m °C) is initially at 25°C. It is suddenly exposed to flames at 550°C through a heat-transfer coefficient of 18 W/m² °C. (a) If the ignition temperature of the wood is 480°C, how much time will elapse before any portion of the wood starts burning? (b) Calculate the amount of energy transferred per unit length to the beam.*

Solution

Known A wooden beam of square cross section is subjected to flames in a convective environment.

Find (a) Time t(s) before any part of the beam starts burning. (b) Total energy transfer, $Q_{2D,\ \text{beam}}$ (J) per unit length.

Schematic



Assumptions (1) Two-dimensional conduction in x_1 and x_2 directions. (2) Constant properties and uniform heat-transfer coefficient.

Analysis (a) The beam is of square cross section with $2L_1 = 2L_2 = 10$ cm and $2L_3 \to \infty$, the transient temperature distribution is

$$\left.\frac{T(x_1, x_2, t) - T_\infty}{T_i - T_\infty}\right|_{2D} = \left.\frac{T(x_1, t) - T_\infty}{T_i - T_\infty}\right|_{\text{wall1}} \times \left|\frac{T(x_2, t) - T_\infty}{T_i - T_\infty}\right|_{\text{wall2}}$$

When the beam is exposed to sudden fire, its most vulnerable part will be its surface which will start burning first. The centre temperature $T(0, 0, t)$ will be *less* than the surface temperature $T(L_1, L_2, t)$. Time will have to be determined such that,

$$T_{\text{surface}} = T_{\text{ignition}} = 480°C = T(L_1, L_2, t)$$

Plane Wall 1: L_1 = 5 cm

$$(Bi)_1 = \frac{hL_1}{k} = \frac{(18\ \text{W/m}^2\ {}^\circ\text{C})(0.05\ \text{m})}{0.346\ \text{W/m}\ {}^\circ\text{C}} = 2.6$$

$$(Fo)_1 = \frac{\alpha t}{L_1^2} = \frac{k}{\rho C_p L_1^2} = \frac{0.346\ \text{W/m}\ {}^\circ\text{C}(t, s)\left[\dfrac{1\,\text{J/s}}{1\,\text{W}}\right]}{(800\ \text{kg/m}^3)(2512\ \text{J/kg}\ {}^\circ\text{C})\ (0.05\ \text{m})^2}\left(\frac{3600\ \text{s}}{1\,\text{h}}\right) = 0.258t\ (\text{h})$$

Plane Wall 2: $2L_2$ = $2L_1$ = 10 cm

$$(Bi)_2 = \frac{hL_2}{k} = 2.6 \quad \text{and} \quad (Fo)_2 = \frac{\alpha t}{L_2^2} = 0.248t(h)$$

At the surface of the beam, $\frac{x}{L} = 1$ for both plane walls 1 and 2.

To use the transient temperature charts or one-term approximation solution, *Fo* should be greater than 0.2. As time is unknown, let us consider this condition to be valid until confirmed later.

One-term approximate analytical series solution is more convenient to use here.

$$\theta(L_1, t)\big|_{\text{wall1}} [A_1 \exp(-\lambda_1^2 Fo) \cos \lambda_1]_{\text{wall1}} \tag{A}$$

$$\theta(L_2, t)\big|_{\text{wall2}} = [A_2 \exp(-\lambda_1^2 Fo) \cos \lambda_1]_{\text{wall2}}$$

As L_1 and L_2 are same, $\theta(L_1, t) = \theta(L_2, t)$

$\therefore$ $\theta(L_1, L_2, t) = [\theta(L_1, t)]^2$

Dimensionless temperature difference,

$$\theta(L_1, L_2, t) = \frac{T(L_1, L_2, t) - T_\infty}{T_i - T_\infty} = \frac{480 - 550}{25 - 550} = 0.133$$

$$\theta(L_1, t) = \sqrt{\theta(L_1, L_2, t)} = \sqrt{0.1333} = 0.365$$

At Bi = 2.6, from Table 5.2:

λ_1 = 1.1463 rad and A_1 = 1.1979

Hence, substituting numerical values in (A), we have,

$0.365 = 1.1979 \exp[-(1.1463)^2(0.248\ t(h))]\cos 1.1463$ rad

or $0.74 = \exp[-0.3259\ t(h)]$ or $-0.3259(h) = -0.3014$

Time required, t = **0.925 h or 55.5 min** **(Ans.) (a)**

(b) Total transient heat transfer for a two-dimensional geometry formed by the intersection of two one-dimensional geometries 1 and 2 is

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total},2D} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]$$

As $(Bi)_1 = (Bi)_2$, $(Fo)_1 = (Fo)_2$,

$(Q/Q_{\max})_1 = (Q/Q_{\max})_2$

$$\therefore \qquad \left(\frac{Q}{Q_{max}}\right)_{2D} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_1 \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right]$$

where $$\left(\frac{Q}{Q_{max}}\right)_1 = 1 - \theta_{o,\text{wall}_1} \frac{\sin \lambda_1}{\lambda_1}$$

$$\theta_{0,\text{wall}_1} = \frac{T(0, 0, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo_1}$$

where $Fo_1 = (0.248)(0.925) = 0.23 \qquad (> 0.2)$

$$\therefore \qquad \theta_{0,\text{wall}_1} = 1.1979 e^{-(1.1463)^2 (0.23)} = 0.8863$$

$$\left(\frac{Q}{Q_{max}}\right)_1 = 1 - (0.8863) \frac{\sin(1.1463 \text{ rad})}{1.1463} = 0.2954$$

Hence, $$\left(\frac{Q}{Q_{max}}\right)_{2D} = 0.2954 + 0.2954(1 - 0.2954) = 0.5035$$

The maximum amount of heat a body can gain

$$Q_{max} = Q_i = \rho \forall \; C_p(T_\infty - T_i) = \rho(2L_1)(2L_2)(\text{length, } L)(C_p)(T_\infty - T_i)$$

Per unit length,

$$Q_{max} = (800 \text{ kg/m}^3)\,(0.1 \text{ m} \times 0.1 \text{ m})\,(2.512 \text{ kJ/kg °C})\,(550 - 25)\text{°C} = 10\,550.4 \text{ kJ}$$

Heat transferred per unit length to the beam is

$$Q = (0.5035)Q_{max} = (0.5035)(10550.4) \text{ kJ} = \mathbf{5313 \text{ kJ}} \qquad \textbf{(Ans.) (b)}$$

Comment As $Fo > 0.2$, one-term approximation solutions obtained are valid. As $\theta_{0,\text{wall},2D} = (0.8863)^2$, the centre temperature, $T(0, 0, t) = 550 + (0.8863)^2 (25 - 550) = 137.6$°C while $T(L_1, L_2, t) = 480$°C. Hence, 55.5 min after the wooden beam, initially, at 25°C is exposed to the fire at 550°C, the outer surface of the beam will be at the ignition temperature of 480°C and the wood will start burning.

(P) Periodic Temperature Variation

EXAMPLE 5.27 *Experiments were performed in order to determine the thermal diffusivity of the soil by studying the heat flow during the usual diurnal (day–night) temperature changes. Assuming sinusoidal variation in surface temperature, calculate the thermal diffusivity of the soil if the thermocouples embedded at depths of 100 mm and 150 mm were found to record temperature of 9°C and 4°C, respectively.*

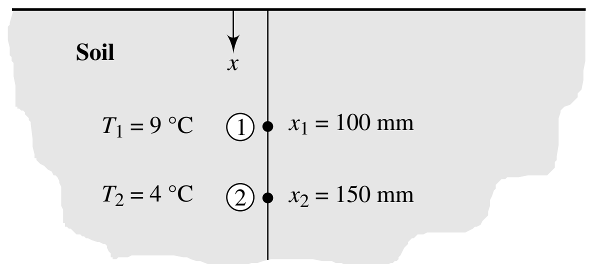
Solution

Known Measurements of temperatures are made at two identified depths in the soil during a day–night cycle.

Find Thermal diffusivity, α(m²/h).

Assumptions (1) Sinusoidal temperature variations. (2) The earth is considered a semi-infinite medium. (3) Constant thermophysical properties of soil.

Schematic



Analysis The amplitude of excess temperature at any depth is expressed as

$$\theta_{x,\text{amp}} = \theta_{0,\text{amp}} \exp\left\{-x\sqrt{\frac{\omega}{2\alpha}}\right\}$$

where α is the thermal diffusivity of the soil

$$\omega = 2\pi f = 2\pi \times \frac{1}{24} = \frac{\pi}{12}\text{rad/h}$$

At $x_1 = 0.1$ m, $\theta_{x_1,\text{amp}} = 9$°C

$x_2 = 0.15$ m, $\theta_{x_2,\text{amp}} = 4$°C

Substituting the values at the two locations in the above expression, we get

$$\theta_1 = \theta_0\, e^{-x_1\sqrt{\omega/2\alpha}} \tag{A}$$

and

$$\theta_2 = \theta_o e^{-x_2\sqrt{\omega/2\alpha}} \tag{B}$$

Dividing (B) by (A), we have

$$\frac{\theta_1}{\theta_2} = \exp\left[-x_1\sqrt{\frac{\omega}{2\alpha}} + x_2\sqrt{\frac{\omega}{2\alpha}}\right] \quad \text{or} \quad \frac{\theta_1}{\theta_2} = \exp\left[(x_2 - x_1)\sqrt{\frac{\omega}{2\alpha}}\right]$$

or

$$\ln\frac{\theta_1}{\theta_2} = \left[(x_2 - x_1)\sqrt{\frac{\omega}{2\alpha}}\right] \quad \text{or} \quad \frac{9}{4} = (0.15 - 0.1)m \times \sqrt{\frac{\pi(rad/h)}{12 \times 2\alpha(m^2/h)}}$$

or

$$\alpha = \frac{\pi}{\left[\ln\dfrac{2.25}{0.05}\right]^2 \times 24} = \frac{\pi}{263 \times 24}\text{m}^2/h\left|\frac{1h}{3600s}\right| = 0.138 \times 10^{-6}\,\text{m}^2/\text{s}$$

∴ Thermal diffusivity of the soil is,

α = **0.138 × 10^{-6} m^2/s** **(Ans.)**

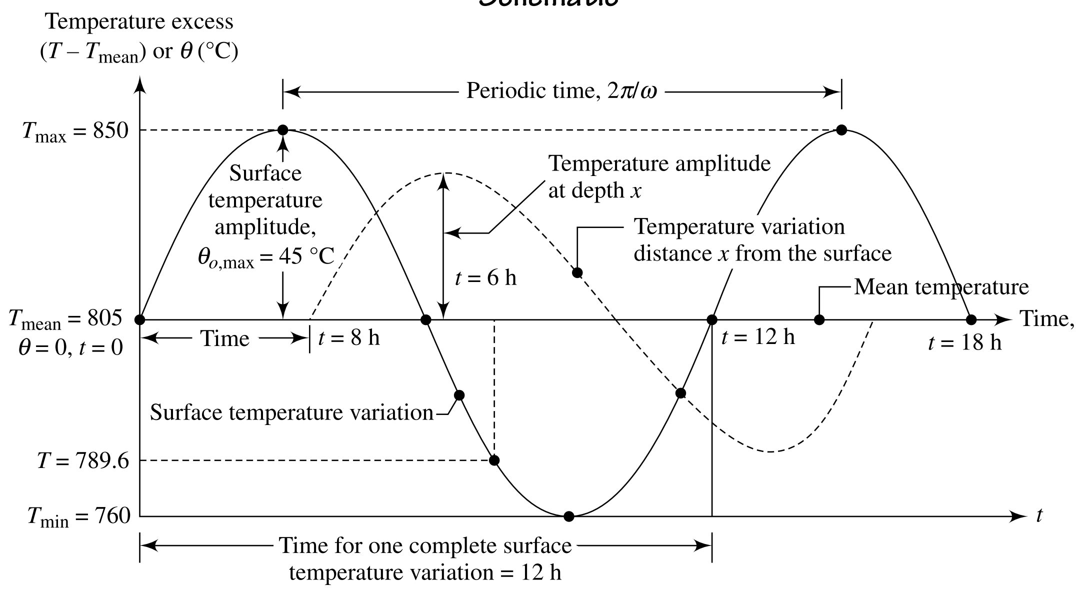
EXAMPLE 5.28 *The surface temperature of an annealing furnace varies periodically as a sine function from 760°C to 850°C. The period of oscillation is 12 hours. Determine for* k *= 1.75 W/m K and α = 8.5 × 10^{-7} m^2/s: (a) the amplitude of temperature variation at a depth of 10 cm, (b) the time lag of wave at a depth of 10 cm, (c) the surface temperature 18 hours after the surface temperature begins to exceed the mean value, and (d) the temperature at a depth of 10 cm, 8 hours after the surface temperature starts exceeding the mean value.*

Solution

Known An annealing furnace experiences periodic surface temperature variation over a given time interval.

Find (a) $\theta_{0,amp}$, (b) Δt for $x = 0.1$ m, (c) $\theta_{0,amp}$ at $t = 18$ h, (d) $\theta_{x,amp}$ at $x = 0.1$ m and $t = 8$ h.

Schematic



Assumptions (1) Constant properties. (2) Sinusoidal temperature oscillations. (3) One-dimensional heat conduction in x.

Analysis Period of oscillation, $t_p = 12$ h

Frequency of temperature wave, $f = \dfrac{1}{12} h^{-1}$

$$\therefore \qquad \omega = 2\pi f = \frac{2\pi}{12} \quad \text{or} \quad \frac{\pi}{6}\text{rad/h}$$

Thermal diffusivity,

$$\alpha = 8.5 \times 10^{-7} \times 3600 = 0.00306 \text{ m}^2/\text{h} = 3.06 \times 10^{-3} \text{ m}^2/\text{h}$$

$$\therefore \qquad \omega\alpha = \left(\frac{\pi}{6}\text{rad/h}\right)(0.003 \text{ m}^2/\text{h}) = 0.0016 \text{ m}^2/\text{h}^2$$

Depth, $x = 0.1$ m

(a) Amplitude of temperature variation at $x = 0.1$ m is

$$\theta_{x,\text{amp}} = \theta_{0,\text{amp}} \exp\{-x\sqrt{\omega/2a}\}$$

Mean temperature, $T_{\text{mean}} = \dfrac{850 + 760}{2} = 805°\text{C}$

Amplitude of surface temperature variation is

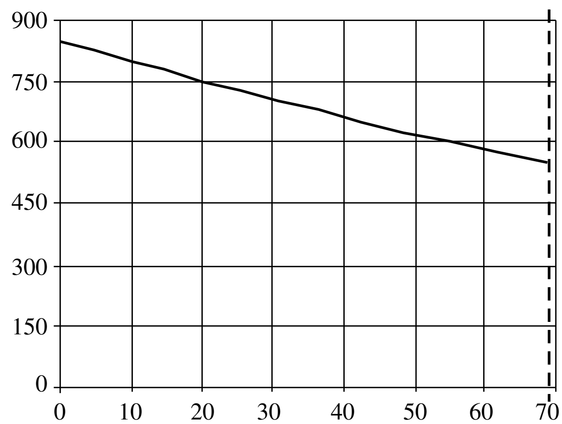
$$\theta_{0,\max} = T_{\max} - T_{\text{mean}} \quad \text{or} \quad T_{\text{mean}} - T_{\min}$$

$$= 850 - 805 = 45°\text{C} \quad \text{or} \quad 805 - 760 = 45°\text{C}$$

$$\therefore \qquad \theta_{x,\text{amp}} = 45°\text{C} \exp\left[-0.1\,m\sqrt{\frac{\pi \text{ rad/h}}{6 \times 2 \times 0.00306 \text{ m}^2/\text{h}}}\right] = \mathbf{17.84°C} \qquad \textbf{(Ans.) (a)}$$

(b) Time lag, $\Delta t = \frac{x}{\sqrt{2\omega\alpha}} = \frac{0.1m}{\sqrt{0.0016 \times 2m^2/h^2}}$ = **1.77 h** **(Ans.) (b)**

(c) Surface temperature after 18 h from the point when it begins to increase from its mean value, at $x = 0$ and $t = 18$ h is



$$\theta(0, t) = \theta_{0,\max} \exp\left[-x\sqrt{\frac{\omega}{2\alpha}}\right] \sin\left[\omega t - \sqrt{\frac{\omega\alpha}{2}}x\right]$$

$$= 45°\text{C} \exp\left[-0 \times \sqrt{\frac{\pi/6 \text{ rad/h}}{2 \times 0.00306 \text{ m}^2\text{/h}}}\right]$$

$$\times \sin\left[\left(\frac{\pi}{6} \times 18\right) - \sqrt{\frac{0.0016}{2}} \times 0\right]$$

$$= 45°\text{C} \exp(0) \times \sin 3\pi = 0°$$

∴ $T(x = 0, t = 18 \text{ h}) = T_{\text{mean}}$ = **805°C** **(Ans.) (c)**

(d) At $x = 0.1$ m and $t = 8$ h

$$\theta(0, t) = 45°\text{C} \exp[-0.1 \times 9.25) \sin\left[\left(\frac{8\pi}{6} - 0.1\sqrt{0.0008}\right)\right] = \mathbf{-15.4°C}$$

∴ $T(x = 0.10 \text{ m}, t = 8 \text{ h}) = T_{\text{mean}} + \theta(x, t) = 805 + (-15.4)°\text{C}$ = **789.6°C** **(Ans.) (d)**

Points to Ponder

- *Biot* number, $Bi = hL_c/k$ where k is the thermal conductivity of the solid, and L_c is the characteristic length $\forall/A_s$.
- If $Bi < 0.1$, the body can be regarded *spatially isothermal*.
- The temperature distribution when $T = f(t)$ is given by $\frac{T(t) - T_\infty}{T(0) - T_\infty} = \overline{e}^{(Bi)(Fo)}$.
- For lumped capacitance formulation, the characteristic length L_c of a cylinder of radius R and L is $RL/2\ (R + L)$.

- Neglecting internal thermal resistance, the convection coefficient, $h = \frac{kR}{3\alpha t}\ln\left(\frac{T_i - T_\infty}{T - T_\infty}\right)$ for a sphere of radius R.
- Thermal time constant $\tau = R_t C_t$ where R_t is the *convective thermal resistance*, and C_t is the *thermal capacitance*.
- The time required by a temperature measuring instrument after which the temperature difference is reduced to 63.2% of the initial temperature difference is called its *sensitivity*.
- Compared to water, air is the medium where the lumped system analysis is likely to be more applicable.
- In lumped-heat-capacity model, the temperature-time history of the body is regulated by the surface resistance with $Bi \to 0$.
- For all practical purposes, temperature readings should be taken after *four* time constants.
- Two significant dimensionless numbers in transient heat conduction are *Biot number* and *Fourier number*.
- In the lumped-capacity model of transient heat conduction, the product of *Bi*ot and Fourier numbers is the ratio of time required to thermal time constant.
- Fourier number, *Fo,* is the *dimensionless time*.
- Surface temperature of a solid changes more rapidly than temperatures in the solid if *Biot* number, $Bi \geq 10$.
- For high values of h and low values of k, large temperature differences occur between the inner and outer regions of a large solid.
- One term approximation for one-dimensional unsteady-state heat conduction is valid for *Fo greater* than 0.2.
- *Bi*ot number, *Bi,* for a sphere with convection coefficient h, thermal conductivity k, and diameter D equals $\frac{hD}{2k}$ for use in transient-temperature charts and is equal to $\frac{hD}{6k}$ for lumped capacity formulation.
- Heisler charts are applicable for plane wall ($2L$ thick), long cylinder of radius r_o and sphere of radius r_o, for $Bi\left(\equiv \frac{hL}{k}\Big/\frac{hr_o}{k}\right)$ greater than 0.1, and $Fo\left(\equiv \frac{\alpha t}{L^2}\Big/\frac{\alpha t}{r_o^2}\right)$ greater than 0.2.
- Consider a large plate of thickness L and a sphere of diameter $D = L$ made of the same material with the same convection coefficient. The sphere will reach the desired temperature level about *three* times as fast as the plate does.
- Heating or cooling of a road surface can be analyzed using the *semi-infinite slab* model.
- A semi-infinite solid is characterized by a *thick* slab with *low* thermal diffusivity and *short* exposure time.
- The case of specified surface temperature for analyzing transient conduction in a semi-infinite material can be handled by setting the convection coefficient to infinity.
- The solution for a multi-dimensional geometry is the *product* of the solutions of the one dimensional geometries whose intersection is the multi-dimensional body.
- Transient heat conduction in the *semi-infinite cylinder* is two-dimensional.
- In *a short cylinder*, the transient conduction is two-dimensional and the temperature varies in both x- and r-directions.
- The fraction of total heat transfer $\dot{Q}/\dot{Q}_{\max}$ up to a prescribed time t is determined using the *Gröber charts*.

- Heat flow in an internal combustion engine is an example of *periodic transient conduction*.
- In periodic unsteady-state conduction, the time lag is defined as $\Delta t = \sqrt{\dfrac{x}{2\alpha\omega}}$ where $\omega = 2\pi f$, x is depth, f is the frequency, and α is the thermal diffusivity.

GLOSSARY of Key Terms

• **Transient heat conduction**	Heat conduction in a solid in which the temperature variation depends not only on space coordinates but also time.
• **Lumped-heat-capacity analysis**	An idealized method in which the temperature distribution in a body is only a function of time with negligible internal temperature gradients and a uniform temperature throughout at a given time.
• **Biot number**	Ratio of internal thermal resistance to conduction in a solid to external thermal resistance to convection at the solid surface.
• **Fourier number**	A dimensionless measure of the time spent in a transient-state process.
• **Thermal time constant**	The product of thermal resistance (convective) and the thermal capacitance in a system which has the dimensions of time.
• **One-term approximation**	A simplified approach for simple geometries like plane wall, long cylinder, and sphere in the solution of one dimensional unsteady state problems for Fourier number greater than 0.2 with reasonable accuracy.
• **Transient-temperature charts**	The graphical solution for temperature distribution in geometrical configurations like an infinite slab, an infinite cylinder and a sphere involving one dimensional transient conduction for Fourier number greater than 0.2.
• **Semi-infinite solid (medium)**	A body in which at a given time, there is always a portion where the temperature remains unchanged when a temperature change occurs on one of its boundaries.
• **Penetration depth**	The limit of penetration of the temperature change at the interface in a semi-infinite medium beyond which the body is effectively at the initial temperature.
• **Penetration time**	The time taken for the surface perturbation of temperature to be felt in a semi-infinite solid at a given depth (point).
• **Short cylinder**	A finite cylinder of short length in which the unsteady-state temperature distribution becomes two-dimensional requiring product solution involving one-dimensional transients in a long cylinder and a plane wall.
• **Semi-infinite cylinder**	A cylinder which is a cross between a semi-infinite medium and a long cylinder with two-dimensional temperature distribution. The product solution involves one-dimensional transient in a semi-infinite body and a long cylinder.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

5.1 In transient heat conduction, the two relevant dimensionless numbers are those attributed to:
(a) Fourier and Biot (b) Reynolds and Prandtl
(c) Grashof and Prandtl (d) Biot and Nusselt

5.2 Consider the statements given below:
1. If the Biot number exceeds 0.1, the spatial effects in transient conduction analysis can be neglected.
2. Transient-temperature charts are reasonably adequate if the dimensionless time is greater than 0.2.
3. A short cylinder can be viewed as an intersection of infinite slab and infinite cylinder.
4. While using the Heisler–Grober charts for a sphere, the Biot number is $hr_o/3k$

(a) All the four statements are correct. (b) Only 1 is correct.
(c) 2 and 3 are correct. (d) 1, 2 and 3 are correct.

5.3 Identify the inappropriate expression in the context of transient heat conduction.
(a) $Bi^2 Fo = h^2 \alpha t/k^2$
(b) $(Q/Q_{\max})2D = (Q/Q_{\max})_1 + (Q/Q_{\max})_2[1 + (Q/Q_{\max})_1]$
(c) $\theta_{\text{semi-inf cyl}}\,(x, r, t) = \theta_{\text{semi-inf}}\,(x, t) \times \theta_{\text{cyl}}(r, t)$
(d) $\text{erfc}(x) = 1 - \dfrac{2}{\sqrt{\pi}} \displaystyle\int_o^x e^{-u^2}\, du$

5.4 Indicate the wrong statement:
(a) The characteristic length of a cube of each side of 12 cm for the lumped system analysis is 2 cm.
(b) The characteristic length of a spatially isothermal sphere of 4.8 kg mass with a material density of 2700 kg/m^3 is 2.5 cm.
(c) The characteristic length of a cylindrical steel ingot of length equal to diameter weighing 5.5 kg with a density of 7900 kg/m^3, neglecting internal temperature gradients, is 2.4 cm.
(d) The characteristic length of a long cylinder of 15 cm radius, ignoring internal conduction resistance, is 7.5 cm.

5.5 Fourier number may be expressed as:
(a) The ratio of buoyancy force to viscous force.
(b) The ratio of internal thermal resistance of a solid to the boundary layer thermal resistance.
(c) The ratio of gravitational and surface tension forces.
(d) The ratio of heat conduction rate to the rate of thermal energy storage in the solid.

5.6 Which one of the following dimensionless numbers is an indication of the ratio of internal (conduction) resistance to the surface (convection) resistance?
(a) Fourier number (b) Biot number (c) Nusselt number (d) Stanton number

5.7 Lumped parameter analysis for transient heat conduction is essentially valid for
(a) $Bi < 0.1$ (b) $0.1 < Bi < 0.5$ (c) $1 < Bi < 10$ (d) $Bi \to \infty$

5.8 Heisler charts are used to determine transient heat-flow rate and temperature distribution when:
(a) Solids possess infinitely large thermal conductivity.
(b) Internal conduction resistance is small and convective resistance is large.
(c) Internal conduction resistance is large and the convective resistance is small.
(d) Both conduction and convection resistances are almost of equal significance.

5.9 Heating or cooling of a road surface can be analyzed using:
(a) The semi-infinite medium formulation. (b) The lumped capacitance formulation.
(c) The infinite slab formulation. (d) None of these.

5.10 The time constant of a thermocouple is the time taken to attain:
(a) 50% of the value of the initial temperature difference.
(b) 63.3% of the value of the initial temperature difference.
(c) 63.2% of the value of the initial temperature difference.
(d) 98.8% of the value of the initial temperature difference.

5.11 Mostly, the unsteady heat flow occurs:
(a) Through the walls of a refrigerator. (b) During annealing of castings.
(c) Through the walls of a furnace. (d) Through the insulated pipe carrying steam.

5.12 Heisler and Gröber charts are used to find the transient temperature distribution in systems when:
(a) $Bi \to \infty$ (b) $Fo > 0.2$ (c) $Bi \to 0$ (d) $Fo \to \infty$

5.13 The temperature distribution, at a certain instant of time in a concrete slab during curing is given by $T = 3x^2 + 3x + 16$ where x is in cm and T is in kelvin. The rate of change of temperature with time would then be
(a) 0.009 K/s (b) 0.0045 K/s (c) –0.0012 K/s (d) –0.0018 K/s.
Assume the thermal diffusivity to be 0.003 cm^2/s

5.14 Match ***List I*** (*Material*) with ***List II*** (*Time lag*) according to the codes given below the lists in the context of a semi infinite medium with periodic variation in surface temperature:

List I (*Material*)	**List II** (*Time lag in h for depth 1 m and wave period 24 h*)
A. Steel	**1.** 23
B. Aluminium	**2.** 6.7
C. Wood	**3.** 2.5
D. Clay	**4.** 66

Codes:	**A**	**B**	**C**	**D**
(a)	2	3	4	1
(b)	1	2	5	3
(c)	3	1	2	4
(d)	4	2	1	3

5.15 A spherical thermocouple junction of diameter 0.706 mm is to be used for the measurement of temperature of a gas stream. The convective heat transfer coefficient on the bead surface is 400 W/m^2 K. Thermophysical properties of thermocouple material are: k = 20 W /m K, c = 400 J/kg K, and ρ = 8500 kg/m^3. If the thermocouple initially at 30°C is placed in a hot stream of 300°C, the time taken by the bead to reach 298°C, is
(a) 2.35 s (b) 4.9 s (c) 14.7 s (d) 29.4 s

5.16 In multi dimensional systems, if C denotes the solution for a long cylinder, and S indicates the solution for a semi-infinite medium, the surface temperature of a semi-infinite cylinder with an initial temperature of 200°C and the environment temperature of 50°C for C = 0.40 and S = 0.95 is:
(a) 252.5°C (b) 132.5°C (c) 107°C (d) 71.7°C

5.17 **Assertion (A):** In lumped-heat-capacity systems, the temperature gradient within the system is negligible.
Reason (R): In the analysis of lumped-capacity systems, the thermal conductivity of the system material is considered very high irrespective of the size of the system.
Codes:
(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not the correct explanation of A.
(c) A is true but R is false
(d) A is false but R is true. **[IES 2004]**

Answers

Multiple-Choice Questions

5.1 (a)	**5.2** (c)	**5.3** (b)	**5.4** (c)	**5.5** (d)	**5.6** (b)
5.7 (a)	**5.8** (d)	**5.9** (a)	**5.10** (c)	**5.11** (b)	**5.12** (b)
5.13 (d)	**5.14** (a)	**5.15** (b)	**5.16** (c)	**5.17** (a)	

REVIEW QUESTIONS

5.1 What do you understand by transient heat conduction? Briefly classify transient heat conduction.
5.2 Describe the method to calculate heat-flow rate in heating or cooling of bodies with known temperature distribution.
5.3 Define and explain the physical significance of *Bi*ot and Fourier numbers.
5.4 Explain the significance of time constant of a thermocouple.
5.5 Define thermal diffusivity and discuss its physical interpretation.
5.6 What is meant by *lumped-capacity analysis*? What is the criterion of its validity?
5.7 Neglecting internal temperature gradients, show that the ratio of excess temperature at any specific time ($T - T_\infty$) to the initial excess temperature ($T_i - T_\infty$) equals exp ($-t/R_tC_t$) where R_t and C_t are thermal resistance and thermal capacitance respectively.
5.8 What is the importance of *Heisler* and *Gröber* charts in solving one-dimensional transient conduction problems? What are their limitations?
5.9 What is *one-term approximation solution* in dealing with one-dimensional unsteady-state conduction with convective heating or cooling? What are the main assumptions and constraints?
5.10 Define a *semi-infinite solid*. In which situations is the assumption of a semi-infinite body applicable?
5.11 What is the *product solution method* in analyzing the multi-dimentional systems? What are the restrictions and approximations involved?
5.12 Explain the periodic temperature variation of a semi inifinite solid and mention its practical applications.

PRACTICE PROBLEMS

(A) Transient Temperature Variation

5.1 At a certain time, the temperature distribution in a long cylindrical fire tube, inner radius 30 cm and outer radius 50 cm, is given by $T = 800 + 1000\,r - 5000\,r^2$ where T is in °C and r in m. The thermal conductivity and thermal diffusivity of the tube material are 58 W/m K and 0.004 m^2 / h, respectively.

Find: (a) the rate of heat flow at inside and outside surfaces per unit length, (b) the rate of heat storage per unit length, (c) the rate of change of temperature at inner and outer surfaces.

[(a) 728.85 kW (b) 510.195 (c) –72 °C/h]

(B) Lumped Parameter Model

5.2 A large aluminium plate [ρ = 2707 kg/m^3, C_p = 0.896 kJ / kg °C, and k = 204 W/m °C] of thickness L = 0.10 m, initially at a uniform temperature T_i = 200°C, is cooled by exposing it to an air stream at T_∞ = 40°C. Determine the time required to cool the aluminium plate from 250 to 75°C if the heat transfer coefficient between the air stream and the surface is h = 80 W/m^2 °C. **[2716 s or 45 min 16 s]**

5.3 A copper block 1 cm × 2 cm × 3 cm is at 240°C when immersed in a fluid with h = 80 W/m^2 K and fluid temperature 20°C. Estimate the time required for the block to cool down to 40°C. Density, specific heat and thermal conductivity of copper are 8933 kg/m^3, 0.385 kJ / kg K and 389 W / m K, respectively. **[281.1 or 4 min 41 s]**

5.4 During a manufacturing process, two brass plates are to be joined face to face at a bonding temperature of 500 K. The dimensions of the plates are 0.6-cm by 5-cm by 10-cm and their initial temperature is 300 K. For bonding the plates have to be kept in an oven and the bonding time should not exceed 5 min. The mean convection heat transfer coefficient is 80 W/m^2 °C. Determine the minimum temperature of the oven if the bonded faces are in complete thermal contact. Properties of brass to be used are: k = 137 W/m°C, ρ = 8530 kg/m^3, C_p = 395 J/kg °C. **[550 K]**

5.5 A thin shell made of aluminium (k = 237 W/m °C, α = 97.1 × 10^{-6} m^2/s) of 8-mm diameter and 0.4-mm thickness falls off a conveyor vertically down to the ground. The shell temperature is initially at 95°C throughout. The uniform convection coefficient is 80 W/m^2 °C and the ambient air is 25°C as the shell gets cooled during the 15 m distance covered by it. Estimate the temperature finally reached when the shell eventually hits the ground. **[77.5°C]**

5.6 A plane wall, 10-mm thick, is fabricated from plain carbon steel [ρ = 7850 kg/m^3, C_p = 0.43 kJ/kg °C, k = 60 W/m °C] with an initial uniform temperature of 15°C. One side of the wall is exposed to furnace gases at 1000°C with a convection coefficient of 100 W/m^2 °C, while the other surface is in contact with water at 30°C with a heat transfer coefficient of 25 W/m^2 °C. How long will it take for the wall temperature to reach 720°C? **[599.2 s ≈ 10 min]**

5.7 A copper-constantan thermocouple junction which may be approximated as a sphere of 3-mm-diameter is used to measure the temperature of an air stream flowing with a velocity of 3 m/s. The initial temperature of the junction and air are at a temperature of 25°C. The air temperature suddenly changes to and is maintained at 200°C. (a) Calculate the time required for the thermocouple to indicate a temperature of 150°C. Also find the thermal time constant and the temperature indicated by the thermocouple at that instant. (b) Discuss the suitability of this thermocouple to measure unsteady state temperature of a fluid when the temperature variation in the fluid requires a time period of 3 s. Assume: h = 150 W/m^2 °C, ρ = 8685 kg/m^3, C_p = 377 J/kg °C, k = 29 W/m °C **[(a) 135.6]**

5.8 A potato ($V\!\!\!\!-$ = 3.1 × 10^{-4} m^3, A_s = 0.025 m^2), initially at a uniform temperature of 20°C, is placed in a microwave oven. The oven supplies 300 W of heat to the potato for 7.5 minutes. The temperature of the air in the oven is 200°C, and the convection heat transfer coefficient is 4 W/m^2 K. Determine: (a) the temperature of the potato after 7.5 minutes of heating, and (b) the rate of heat transfer from the air to the potato in the oven at that time. The thermophysical properties of the potato are:
ρ = 1055 kg/m^3, k = 0.498 W/m K, C_p = 3.64 kJ/kg K **[(a) 138°C (b) 6.2 W]**

5.9 A steel tube of length 20 cm with internal and external diameters of 10 and 12 cm is quenched from 500°C to 30°C in a large reservoir of water at 10°C. Below 100°C, the heat transfer coefficient is 1.5 kW/m^2 K. Above 100°C, the average value of heat transfer coefficient may be taken as 0.5 kW/m^2 K.

The density of steel is 7800 kg/m^3 and the specific heat is 0.47 kJ/kg K. Neglecting internal thermal resistance of the steel tube, determine the quenching time. **[17.50 s]**

5.10 An electronic device, which generates 60 W of heat, is mounted on an aluminium finned heat sink having a mass of 0.32 kg and specific heat of 935 J/kg °C. Under steady state conditions, it attains a temperature of 110°C in ambient air at 30°C. If the device is initially in equilibrium with the environment, calculate its temperature 7 minutes after the power is switched on. Derive the expression you use. The device and the heat sink may be assumed to be nearly isothermal. **[82.1°C]**

5.11 Pulverised coal pellets, approximated as spheres of 1-mm-diameter, are preheated by passing them through a cylindrical tube maintained at 1200°C before being injected into the furnace. The pellets suspended in air flow move with a speed of 3.5 m/s. The initial and final temperatures of the coal pellet are 30°C and 700°C. The dominant mode of heat transfer is radiation and the pellet is very small compared to the surface area of the tube. Determine the length of the tube required for this preheating. State clearly the assumptions made. The following thermophysical properties of coal can be used: ρ = 1350 kg/m^3, C_p = 1.26 kJ/kg °C, k = 0.26 W/ m °C. **[2.64 m]**

(C) Plane Wall

5.12 A steel pipeline of 1-m-diameter and 40-mm wall thickness is effectively insulated on its exterior surface. Hot oil at 60°C is pumped through the pipe with a convective heat transfer coefficient at its inner surface of 500 W/m^2 °C. Before the flow commences, the pipe walls are at a uniform initial temperature of –20°C. Determine, after a lapse of 8 min: (a) the temperature of the outer surface of the pipe wrapped by insulation, (b) the heat flux from the oil to the pipe, and (c) the energy transferred from the oil to the pipe per metre pipe length. The properties of the pipe material are:
ρ =7832 kg/m^3, C_p = 434 J/kg °C, k = 63.9 W/m °C.

[(a) 299.2°C (b) 7355 W/m^2 (c) 27.22 MJ]

5.13 Consider the nose of a missile re-entering the earth's relatively dense atmosphere with a very high velocity and creating huge heating effect. The nose section can be idealized as a 6-mm thick stainless steel plate with one surface adiabatic and the other exposed to convective environment. The uniform surface heat transfer coefficient is estimated to be 3390 W/m^2 K and the initial uniform temperature is 50°C. (a) Neglecting radiation effects, determine the maximum allowable time for heat dissipation if the effective air temperature in the vicinity of the nose section is estimated to be 2100°C. Metallurgical considerations limit the metal surface temperature to 1100°C. (b) Also compute the inside surface temperature under the specified conditions. The following thermophysical properties of steel may be used: ρ = 7900 kg/m^3, C_p = 611 J/kg K, k = 25.4 W/m K **[(a) 5.06 s (b) 678°C]**

5.14 A large steel plate, 75-cm-thick [k = 15 W/m °C, $\alpha = 3.96 \times 10^{-6}$ m^2/s] is initially at a uniform temperature of 50°C. Suddenly both of its surfaces are raised to and maintained at 450°C. Determine (a) the temperature at a plane 25-cm from the surfaces one hour after the sudden change in surface temperature, (b) the instantaneous heat transfer rate across the above plane per m^2 at 1 h, (c) the total heat flow across the above plane in 1 h, and (d) the temperature at the midplane of the plate after one hour. **[(a) 106.7 (b) 9.12 kW (c) 14.35 MJ (d) 71°C]**

(D) Long Cylinder

5.15 A long cylinder of radius 20 cm of a material with k = 170 W/m K and $\alpha = 9.05 \times 10^{-7}$ m^2/s is initially at a uniform temperature of 650°C. The cylinder is quenched in a medium at 75°C for heat treatment with h = 1700 W/m^2 K. Find the time at which the process should terminate to ensure that the temperature attained at a depth of 20 mm from its surface is 250°C. What will be the temperature on the axis of the cylinder at this time? Calculate the amount of energy transferred from the cylinder per metre length during this period. **[8.15×10^9 J]**

(E) Sphere

5.16 The glass beads ($k = 0.7$ W/m K, $C_p = 0.75$ kJ/kg K and $\rho = 2800$ kg/m^3) of 0.6-mm-diameter initially at 520°C throughout are sprayed into 20°C air and allowed them to harden as they fall to the ground with a constant downward velocity of 4.7 m/s as a part of the manufacturing process. The centre temperature of the beads should not exceed 40°C. The average convection heat transfer coefficient is 234 W/m^2 K. Calculate the surface temperature of the beads when the centre reaches 40°C, and (b) the distance through which the glass beads should fall from rest. Is the lumped capacity model valid?
[(a) 39°C (b) 14.0 m]

5.17 An 8-mm-diameter sphere [$k = 20$ W/m °C and $\alpha = 0.024$ m^2/h] is initially at a uniform temperature of 400°C. It is subjected to two-step cooling process. The sphere is first cooled in air at 25°C with a convection coefficient of 15 W/m^2 °C until a temperature of 335°C is reached. Subsequently, the sphere is cooled in a well-stirred water bath held at 25°C with a heat transfer coefficient of 6 kW/m^2 °C. Calculate: (a) the time required to accomplish the cooling process in both steps if the centre temperature of the sphere finally reaches 50°C, and (b) the surface temperature of the sphere at the end of the cooling process. **[(a) 2.4 s (b) 40°C]**

(F) Short Cylinder/Semi-Infinite Cylinder

5.18 A man is found dead at 7 o'clock in the morning in a room whose temperature is 17°C. The surface temperature on his waist is measured to be 24°C and the heat transfer coefficient is estimated to be 9 W/m^2 K. Considering the body as a 28-cm diameter and 1.8–m long cylinder, estimate the time of death of that man. Assume that the man was healthy when he died with an initial uniform temperature of 37°C. Take the properties of the body to be $k = 0.62$ W/m K and $\alpha = 0.00054$ m^2/h.
[7 h 24 min]

5.19 A short brass cylinder of 8-cm-diameter and 15-cm-height is initially at a uniform temperature of 400 K. The cylinder is now placed in atmospheric air at 300 K, where heat transfer takes place by convection with a heat transfer coefficient of 65 W/m^2 K. Determine (a) the centre temperature of the cylinder, (b) the centre temperature of the top surface of the cylinder, (c) the temperature at the mid-height of the side, and (d) the total heat transfer from the cylinder, 15 min after the start of the cooling.
[(a) 332.6 K (b) 331.94 K (c) 332.25 K (d) 164.63 kJ]

5.20 Short plastic cylinders, 6-cm long and 3-cm diameter, are uniformly heated in an oven as preparation for a pressing operation. Initially, the cylinders are at 25°C. The manufacturing requires that no portion of the plastic be below 200°C. The oven temperature is 250°C, and the heat transfer coefficient on the cylinders is 8 W/m^2 K. The plastic properties are: $k = 0.30$ W/m K, and $\rho C_p = 1040$ kJ/m^3 K. The supplier of the oven states that the plastic cylinders should be in the oven for 15 minutes. Determine: (a) the minimum temperature in the cylinder (in °C) after 15, min (b) the maximum temperature in the cylinder (in °C) after 15 min and (c) the rate at which heat must be added to the oven if 100 cylinders per minute are heated (in W). **[(a) 152°C, 193°C (c) 10.64 kW]**

5.21 A long Invar (36% nickel steel) rod [$k = 10.7$ W/m K and $\alpha = 0.0103$ m^2/h] 10-cm-diameter has one end machined flat perpendicular to the axis. The rod is initially at 320°C and is set to cool in 20°C air such that the heat transfer coefficient on the surface is 43 W/m^2 K. What is the temperature at a point 2 cm below the cylindrical surface and 4 cm from the flat end after 20 min? **[178.4°C]**

5.22 Two large bodies 1 and 2 with thermal conductivities and thermal diffusivities k_1 and k_2, and α_1 and α_2 are initially at temperatures T_1 and T_2 at their plane surfaces. If the plane surfaces are placed in contact with each other, show that under equilibrium conditions the contact surface temperature, T_s, is given by, $T_s = \dfrac{(k_1T_1/\sqrt{\alpha_1}) + (k_2T_2/\sqrt{\alpha_2})}{(k_1/\sqrt{\alpha_1}) + (k_2/\sqrt{\alpha_2})}$. A large mass of steel (k = 43 W/m °C and $\alpha = 1.17 \times 10^{-5}$ m²/s) at 100°C with one plane face, is dropped into water (k = 0.589 W/m K and $\alpha = 1.41 \times 10^{-7}$ m²/s) at 15°C. Determine (a) the temperature at the surface of contact and (b) How much time will elapse before the temperature at a location 2 m inside the surface will reach 95°C.

[(a) 90.57°C (b) 5 days]

5.23 (a) A very strong blizzard suddenly reduces the surface temperature of the earth in an open area to – 25°C and stays that way for a 36 h period of time. If the ground was initially at 15°C, estimate how far the freezing temperature would penetrate, neglecting any latent heat effects due to the soil moisture, using a value of thermal diffusivity of 0.83 mm²/s. (b) If the weather front moves in, drops the temperature quickly to – 25°C, then becomes calm instead of blizzard so that a heat transfer coefficient of 28 W/m² K exists at the surface. The thermal conductivity of the soil is 0.43 W/m K. Estimate the depth of penetration of freezing temperature under these conditions in 6 h.

[(a) 0.41 m (b) 0.15 m]

5.24 A thick aluminium block at 20°C is subjected to constant surface heat flux of 1000 W/m². Determine the temperature of the block at the surface and at a depth of 0.5 m from the surface after 20 minutes. If the block were made of concrete instead of aluminium, what would be the corresponding temperatures under identical conditions. Comment on the results obtained.

Material	*Thermal conductivity* (W/m² °C)	*Thermal diffusivity*(m²/s)
Aluminium	237	9.71×10^{-5}
Concrete	1.06	5.19×10^{-7}

[20°C]

(H) Multi-Dimensional Conduction

5.25 A cylindrical granite block (k = 2.5 W/m K and α = 1.15 mm²/s) of 50-mm-diameter as well as height is to be compared for transient thermal response with a cubical granite block with each side 50-mm long. Both blocks are initially at 25°C throughout and both are exposed to hot gases at 525°C in a furnace on all of their surfaces with a heat transfer coefficient of 50 W/m²K. Estimate the centre temperature of each geometry after 10, 20 and 60 min.

	***T* = 10 min**	***T* = 20 min**	***T* = 60 min**
$\theta_{o,\text{ short cyl}}$ Eq(a)	0.2803	0.0659	0.0002
$\theta_{o,\text{ cube}}$ Eq (c)	0.298	0.0725	0.000254
$T_{o,\text{ short cyl}}$ Eq. (b)	384.87	492.05	524.9
$T_{o,\text{ cube}}$ Eq (d)	376.0	488.75	524.87

5.26 On cold days, at a certain location, the temperature of the earth's surface varies sinusoidally between –20°C and 20°C within a period of 24 hours. Determine the amplitude of the temperature variation at a depth of 40 cm from the earth's surface. Also find the time lag of the temperature wave at a depth of 40 cm. What is the temperature at that depth 5 hours after the surface temperature reaches a minimum value? Determine the heat flowing into the surface per unit area every half cycle. For earth, take k = 0.63 W/m °C, C_p = 1.88 kJ/kg °C and ρ = 1600 kg/m³.

[(a) 0.103°C (b) 20.13 h (c) 0.07°C, 6457 kJ per m²]

5.27 A cold front moves in at a certain location for three weeks and causes a temperature drop of 20°C at the surface. The average thermal diffusivity of the soil is 0.15×10^{-6} m²/s. Estimate the temperature change at a distance of 0.80 m below the ground surface and the time lag for this case. Also find the burial depth you would recommend to avoid freezing while laying water mains if the specified base temperature is 5°C. **[0.240 m]**

6 Fundamentals of Convection

6.1 ❑ INTRODUCTION

In the preceding chapters, we had discussed the conduction mode of heat-transfer that involved a stationary object, be it solid or fluid. Convection heat-transfer, on the other hand, involves the process of carrying the thermal energy away from a solid surface to an adjacent *moving* fluid in the presence of temperature differences, or vice versa. The convection process has two contributing mechanisms: (1) the conduction of heat from a solid surface to a thin layer of adjacent fluid, and (2) the movement of hot fluid particles away from the solid surface, their place in turn being taken by relatively cold fluid particles. The movement of the fluid particles can be attributed to pressure changes, to buoyancy, or to a combination of both. Thus, the study of convective heat-transfer is intimately related to the study of fluid flow.

Fluid movement occurs due to two fundamentally different mechanisms: *natural* and *forced* convection. The resulting heat-transfer characteristics are significantly different. Consider convective heat-transfer from a hot surface to a cooler fluid. In *natural convection* (also called *free convection*), heat is conducted into the fluid near the hot surface, thereby raising the temperature of the fluid and decreasing its density. The surrounding colder fluid at a higher density then flows under the action of gravity to displace the hot fluid. Whenever the hot fluid rises, the colder fluid falls to fill the void. *An example of a circuit board cooled by natural convection is shown in* Fig. 6.1. Natural convection occurs with either a *hot surface/cold fluid or a cold surface/hot fluid* arrangement.

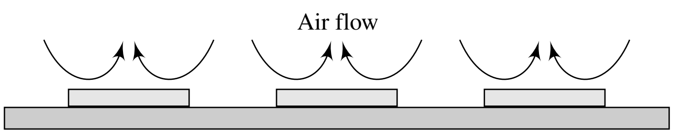


Fig. 6.1 *Cooling of electronic components by natural convection*

In *forced convection,* the fluid movement is caused by mechanical means, such as a fan or a blower, or a pump. However, it can also occur without external means being involved. For example, an ice skater racing across a lake experiences considerable forced convection as the air rushes past. As with natural convection, forced convection may involve either heating or cooling of a solid surface. Figure 6.2 shows the schematic of a computer chip on a circuit board that is cooled when the air flows over it. Heat from the chip is first conducted to the flowing air, and the movement of air then *convects* this heat away from the surface.

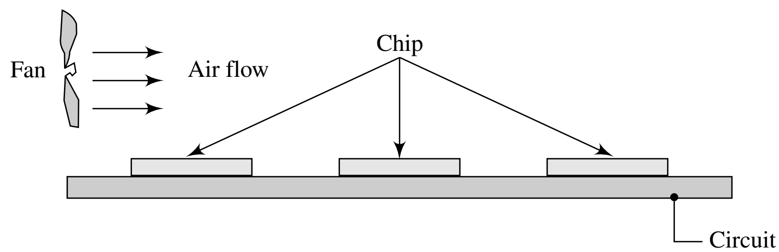


Fig. 6.2 *Cooling of electronic components by forced convection*

6.1.1 • Laminar and Turbulent Flows

We know that there are *two* types of flow: *laminar* and *turbulent*. The magnitude of the Reynolds number determines if the flow will be laminar or turbulent. The heat-transfer analysis for convection is, in general, more complicated than that for heat conduction, because the conservation of mass and momentum have to be satisfied in addition to the principle of energy conservation.

For turbulent flow, the analysis becomes extremely complex, owing to the transport of momentum and energy resulting from eddy motion. Most of the convective heat-transfer problems encountered in practice involve turbulent flow. Therefore, it is customary to rely heavily on experimental data and empirical correlations. On the other hand, laminar heat-transfer analyses are useful in presenting the basic principles of convection and provide results of some practical importance.

6.2 ❑ THE CONVECTIVE HEAT-TRANSFER COEFFICIENT

The rate of convective heat-transfer, $\dot{Q}$ from a solid surface to a surrounding fluid is given by

$$\boxed{\dot{Q} = hA(T_w - T_\infty)} \quad \text{(W)} \qquad \text{(Newton's law of cooling)} \tag{6.1}$$

where h = average convective heat-transfer coefficient, W/m^2 K
A = surface area for convective heat-transfer, m^2
T_w = temperature of the solid surface, °C
and T_∞ = temperature of the fluid sufficiently far from the solid surface so that it is not affected by the surface temperature, °C.

The heat-transfer coefficient, h, depends on several factors such as

- Fluid properties (*like density, viscosity, thermal conductivity, specific heat*)
- Type of flow (*laminar* or *turbulent*)
- Geometry (shape) of fluid passage (*circular, spherical,* or a *flat surface*)
- Nature of the surface (*rough/smooth*)
- Orientation of the heat-transfer surface

In fact, in determining the heat-transfer rate in convection, our main aim is to estimate the reliable and accurate value of the heat-transfer coefficient.

6.3 ❑ NUSSELT NUMBER

Consider a fluid flowing over a body. If the surface temperature is T_w and if the free-stream temperature is $T_{¥}$, the temperature of the fluid near the solid boundary will vary in some fashion as shown in Fig. 6.3

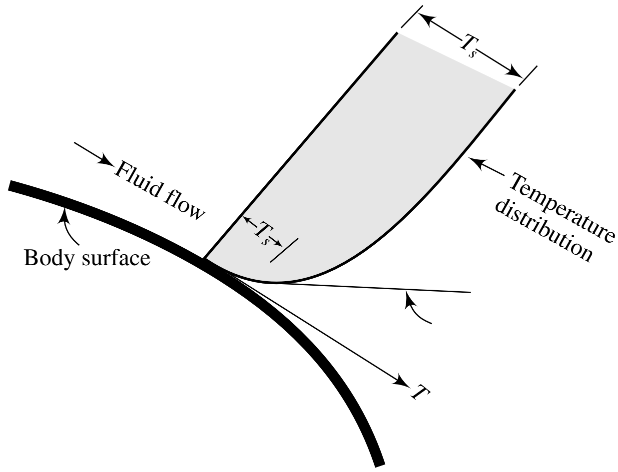


Fig. 6.3 *Fluid temperature gradient at the surface*

$$\boxed{\dot{Q} = -kA \left.\frac{\partial T}{\partial y}\right|_{y=0} = hA(T_w - T_\infty)} \tag{6.2}$$

where k = thermal conductivity of the fluid, W/m K, evaluated at $y = 0$, that is, the solid boundary-fluid interface, and $\left(\frac{\partial T}{\partial y}\right)_{y=0}$ = temperature gradient in the fluid at $y = 0$. The coordinate y is measured along the normal to the surface.

It follows that

$$\frac{h}{k} = -\frac{1}{(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

If a dimensionless distance η is defined as
$\eta = (y/L_c)$ where L_c is the characteristic length, we have

$$\frac{h}{k} = -\frac{1}{(T_w - T_\infty)L_c}\left(\frac{\partial T}{\partial \eta}\right)_{\eta=0}$$

or
$$Nu = \frac{hL_c}{k} = -\frac{1}{(T_w - T_\infty)}\left(\frac{\partial T}{\partial \eta}\right)_{\eta=0}$$

If a dimensionless temperature, θ is defined as $\theta = (T - T_\infty)/(T_w - T_\infty)$, the above equation can be written as

$$\boxed{Nu = \frac{hL_c}{k} = -\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0}} \tag{6.3}$$

The quantity (hL_c/k) in the above equation is a *dimensionless* parameter called the *Nusselt number*. *The Nusselt number is the dimensionless temperature gradient for the fluid evaluated at the wall-fluid interface*. The *Nusselt number* is similar to the *Biot number* with an important difference that k in the Nusselt number is the thermal conductivity of the fluid, while in the Biot number it represents that of the solid.

One can also interpret Nusselt number as the ratio of convection to conduction heat-transfer or as the ratio of conduction to convective thermal resistance.

Thus, $$Nu = \frac{hA\Delta T}{kA\Delta T/L} = \frac{\dot{Q}_{\text{conv}}}{\dot{Q}_{\text{cond}}} = hL/k$$

or $$Nu = \frac{R_{\text{cond}}}{R_{\text{cond}}} = \frac{L/kA}{1/hA} = \frac{hL}{k}$$

The convection coefficient, in general, varies along the flow direction. The average convective heat-transfer coefficient $\bar{h}$ for a surface is calculated by integrating the local convection coefficients h_x over the entire surface length L, as follows.

$$\boxed{\bar{h} = \frac{1}{L}\int_0^L h_x dx} \tag{6.4}$$

6.4 ❑ PRANDTL NUMBER

It is noteworthy that the relative sizes of the velocity and thermal boundary layers depend on *three* important physical properties: the *thermal conductivity*, *k; specific heat*, C_p; and *viscosity*, m. *Thermal conductivity* controls how easily heat is conducted in a fluid. *Specific heat* determines the temperature rise in the fluid due to conduction. And, *viscosity* influences the velocity field and thus the rate at which heat is convected. These significant physical parameters appear in a single dimensionless parameter, called Prandtl *number*, which is a *fluid property* itself.

$$\boxed{Pr = \frac{C_p\mu}{k}}$$

Figure 6.4 shows the typical ranges of Prandtl number for different fluids. *Liquid metals* have very *low* Prandtl numbers. For liquid metals, thermal transport is much more effective than momentum transport. *Gases* have Prandtl numbers ranging between 0.5 and 1, while *liquids* typically have Prandtl numbers greater than 1. It may be noted that there are no common fluids with Prandtl numbers of the order of 0.1.

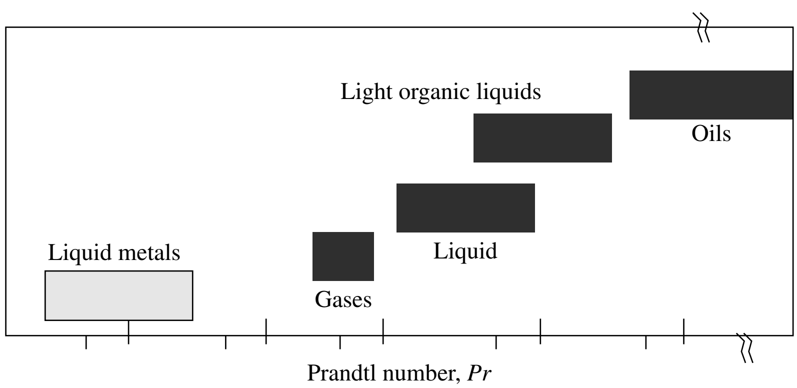


Fig. 6.4 *Typical Prandtl numbers for common fluids*

6.5 ❑ METHODS TO DETERMINE CONVECTIVE HEAT-TRANSFER COEFFICIENT

In convection heat-transfer analysis, the primary purpose is to determine the heat-transfer coefficient. Needless to say, the heat-transfer rate from the surface can easily then be evaluated.

There are generally *five* methods available to determine the convective heat-transfer coefficient:

- Dimensional analysis in conjunction with experimental data
- Exact mathematical solutions of boundary layer equations
- Integral-momentum approximate method
- Analogy between heat and momentum transfer
- Numerical methods

Of course, none of them can, by itself, solve all the problems we come across in practice, since each method has its own limitations.

- ***Dimensional analysis*** is mathematically simple, but it does not give any physical insight into the phenomenon. The method helps in the interpretation of the experimental data and expresses the data in terms of dimensionless groups.
- ***Exact solution of boundary-layer equations*** involve simultaneous solutions of differential equations derived for the boundary layer. The method is quite complicated and solutions are available for a few simple flow situations, such as flow over a flat plate, or a circular cylinder, in laminar flow. Describing the turbulent flow mathematically is very complex.
- ***Approximate integral-momentum method*** is relatively simple, and it is possible to get solutions to problems that cannot be treated by exact method of analysis. This method can also be applied to turbulent flow.
- ***Analogy between heat and momentum transfer*** is a useful method to deduce the convective heat-transfer coefficient from the knowledge of flow friction data, particularly for turbulent flows, without actually performing heat-transfer experiments. It is simpler to conduct flow (friction) experiments. The fact that the momentum and energy equations have the same form, under certain conditions, and therefore, the solutions also must have the same form is utilized.
- ***Numerical methods*** involves discretizing the differential equations and are therefore approximate. Solutions are obtained at discrete points in time and space rather than continuously; however, accuracy can be improved to acceptable levels by taking sufficiently close grids. The main advantage of the numerical methods is that the variation in the fluid properties and boundary conditions can be easily taken care of.

6.6 ❑ VELOCITY BOUNDARY LAYER

The concept of *boundary layer* was introduced by Ludwig Prandtl in the year 1904. Consider a thin, flat plate. Let a fluid approach the flat plate at a free stream velocity of u_∞. The fluid immediately in contact with the plate surface adheres to the surface and remains stationary (no slip condition). The region of flow which develops from the leading edge of the flat plate in which the viscous effects are observed is known as the *velocity boundary layer*, as shown in Fig. 6.5.

- The boundary layer divides the flow field into two regions: one, *the boundary-layer region*, where the viscosity effects are predominant and the velocity gradients are very steep, and, second, *the inviscid region* where the frictional effects are negligible and the velocity remains essentially constant at the free stream value.

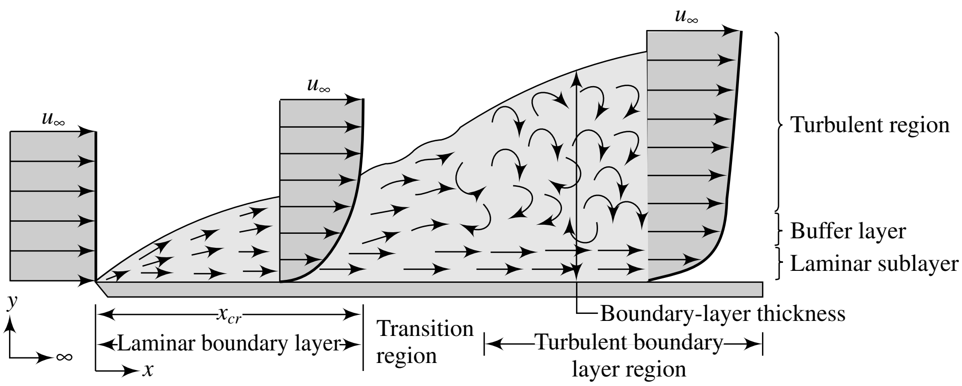


Fig. 6.5 *Transition from laminar to turbulent flow over a flat plate*

- The boundary-layer thickness (δ) is arbitrarily defined as that distance from the surface in the y-direction at which the velocity reaches 99% of the free stream velocity, u_∞. Since the flow over turbine blades and aerofoil sections of airplane wings can be approximated as flow over a flat plate.
- Since the fluid layers in the boundary layer travel at different velocities, the faster layer exerts a drag force (*or frictional force*) on the slower layer below it; the drag force per unit area is known as *shear stress* (τ). Shear stress is proportional to the velocity gradient at the surface. This is the reason why, the velocity profile has to be known to determine the frictional force exerted by a fluid on the surface. Shear stress is given by

$$\boxed{\tau_w = \mu\left(\frac{du}{dy}\right)_{y=0}} \quad (\text{N/m}^2) \tag{6.5}$$

Wall shear stress is determined in terms of the free stream velocity from the following relation:

$$\boxed{\tau_w = C_f \frac{\rho u_\infty^2}{2} \quad (\text{N/m}^2)} \tag{6.6}$$

where C_f is a *skin friction coefficient* or *drag coefficient*. ρ is the density of the fluid. C_f is determined experimentally in most cases. It varies along the length of the flat plate. Average skin friction coefficient $(\overline{C}_f)$ is calculated by suitably integrating the local value over the whole length of the plate and then the drag force over the entire plate surface is determined from

$$\boxed{F_D = \overline{C_f}\, A_s \frac{\rho u_\infty^2}{2}} \quad (N) \tag{6.7}$$

where A_s is the surface area, (m^2)

- Starting from the leading edge of the plate, for some distance along the length of the plate, the flow in the boundary layer is *laminar*, i.e., the layers of fluid are parallel to each other and the flow proceeds in a systematic, orderly manner. However, after some distance, disturbances appear in the flow and the flow becomes *turbulent*

There is intense mixing of fluid particles in the turbulent region. Heat-transfer is, therefore, more in turbulent flow compared to laminar flow. However, one has to pay increased power to pump the fluid.

Velocity profile in the laminar flow is approximately *parabolic*.

Turbulent region of boundary layer is preceded by transition region.

Turbulent boundary layer itself is made up of three layers: a very thin layer called *laminar sublayer*, then, a *buffer layer* and, finally, the *turbulent layer*.

Velocity profile in the laminar sublayer is approximately linear, whereas in the turbulent layer the velocity profile is somewhat flat.

Thickness of the boundary layer, δ is related to the Reynolds number as follows:

$$\boxed{\delta_{\text{lam}} = \frac{5.0x}{(Re_x)^{0.5}}} \qquad (\textit{laminar flow}) \qquad (6.8)$$

and, for *turbulent* flow region:

$$\boxed{\delta_{\text{turb}} = \frac{0.381x}{(Re_x)^{0.2}}} \qquad (\textit{turbulent flow}) \qquad (6.9)$$

where Re_x is the Reynolds number at any location x from the leading edge.

• Reynolds Number

Transition from *laminar* to *turbulent* flow depends primarily on the *free stream velocity*, *fluid properties, surface temperature*, and *surface roughness*, and is characterized by *Reynolds number*. Reynolds number is a dimensionless number, defined as

$$Re = (\text{forces/viscous forces})$$

or

$$\boxed{Re = \frac{u_\infty L_c}{\nu}} \qquad (6.10)$$

where u_∞ = free stream velocity, m/s

L_c = characteristic length, i.e., for a flat plate, it is the length along the plate in the flow direction, from the leading edge

ν = kinematic viscosity of fluid = μ/ρ, (m^2/s), where ρ is the density of fluid

When the Reynolds number is low, i.e., when the flow is laminar, inertia forces are small compared to viscous forces and the velocity fluctuations are *damped out* by the viscosity effects and the layers of fluid flow systematically, parallel to each other. When the Reynolds number is large, i.e. when the flow is turbulent, inertia forces are large compared to the viscous forces and the flow becomes chaotic. For a flat plate, in general, for practical purposes, the *critical Reynolds number*, Re_{cr} at which the flow changes from laminar to turbulent is taken as 5×10^5. It should be understood clearly that this is not a fixed value but depends on many parameters including the surface roughness.

6.7 ❑ THERMAL BOUNDARY LAYER

When the temperature of a fluid flowing on a surface is different from that of the surface, a *thermal boundary layer* develops on the surface, in a manner similar to the development of the velocity boundary layer.

Consider a fluid at a uniform velocity u_∞ and a uniform temperature of, approach the leading edge of a thin, flat plate. Let the plate be at a uniform temperature of T_w. The first layers that come in contact with the surface will adhere to the surface (*no-slip condition*) and reach thermal equilibrium with the surface

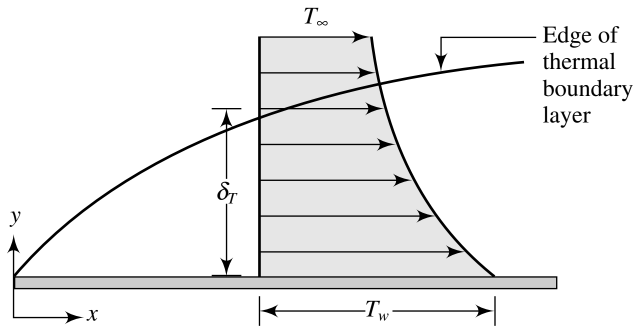


Fig. 6. 6 *Thermal boundary layer*

and attain a temperature of T_w. A temperature profile will develop in the flow field and the temperature will vary from T_w at the surface to T_∞ in the free stream. As shown in Fig. 6.6, at the surface, $(T_w - T) = 0$ and at the free stream condition $(T_w - T) = (T_w - T_\infty)$. The region in which the temperature variation in the y-direction is significant is known as *thermal boundary layer*. Thickness of the thermal boundary layer (δ_T) at any location is defined as that distance from the plate surface in the y-direction where the temperature difference between the fluid and the surface has reached 99% of the maximum possible temperature difference.

The thickness of the thermal boundary layer increases with increasing distance along the plate.

Regarding the relative growth of velocity and thermal boundary layers in a fluid, we may note the following:

- For *gases*, i.e., thicknesses of the hydrodynamic and thermal boundary layers are almost of the same order.
- For *liquid metals* ($Pr << 1$), i.e., the thermal boundary layer is *much thicker* than the hydrodynamic boundary layer.
- For *heavy oils* ($Pr >> 1$), i.e., the thermal boundary layer is *much thinner* than the hydrodynamic boundary layer.

For laminar flow, the thickness of the thermal boundary layer is related to that of the hydrodynamic boundary layer, as follows:

$$\boxed{\frac{\delta_T}{\delta} \approx \frac{1}{Pr^{1/3}}}$$ where Pr is the Prandtl number.

6.8 ❑ DIFFERENTIAL EQUATIONS FOR THE BOUNDARY LAYER

In convection studies, since there is a fluid flow, we are interested in the shear stress and the friction coefficient. To determine these, we need the velocity gradient at the surface. To determine the velocity gradient at the surface, we apply the equation of conservation of momentum (*in conjunction with the equation of conservation of mass*) to a differential volume element in the boundary layer. And, to determine the temperature gradient at the surface, we apply the equation of conservation of energy to a differential volume element in the boundary layer. We start with the application of equation for conservation of mass:

6.9 ❑ CONSERVATION OF MASS—THE CONTINUITY EQUATION FOR THE BOUNDARY LAYER

Consider a differential control volume, of section ($dx\ dy$) and unit depth, within the boundary layer, as shown in Fig. 6.7.

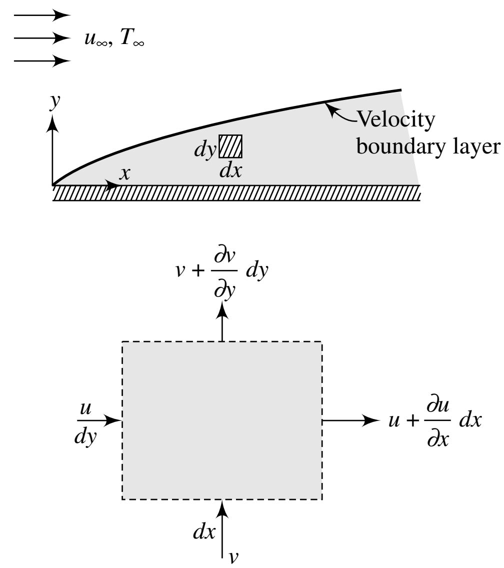


Fig. 6.7 *Differential control volume used in the derivation of the continuity equation for the velocity boundary layer on a flat surface.*

Assumptions

- Steady incompressible, two-dimensional fluid flow
- Constant fluid properties
- Shear force in the y-direction is negligible

Let u and v be the velocity components in the x and y-directions. Then, noting that the mass flow rate is given by (density × mean velocity × area of cross section) and that the depth is unity, we can write

Mass flow *entering* the control volume in the x-direction $= \rho u dy$

Mass flow *leaving* the control volume in the x-direction

$$= \rho\left[u + \left(\frac{\partial u}{\partial x}\right)dx\right]dy$$

The, *net* mass flow *into* the element in the x-direction $= -\rho\left(\frac{\partial u}{\partial x}\right)dx\,dy$

Also, the *net* mass flow *into* the control volume in the y-direction $= -\rho\left(\frac{\partial v}{\partial y}\right)dy\,dx$

Since the net mass flow into the control volume, in steady state, must be equal to zero, we can write:

$$-\rho\left\{\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right)\right\}dx\,dy = 0$$

For an incompressible fluid, ρ is constant. Then, it follows that

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \tag{6.11}$$

This is the mass *continuity equation* for the boundary layer.

6.10 ❑ CONSERVATION OF MOMENTUM EQUATION FOR THE BOUNDARY LAYER

Consider again the differential control volume of section $dx \times dy$ and unit depth within the boundary layer as depicted in Fig. 6.8.

With no pressure variation in the y-direction and with the assumption that viscous shear forces, in the y-direction are negligible.

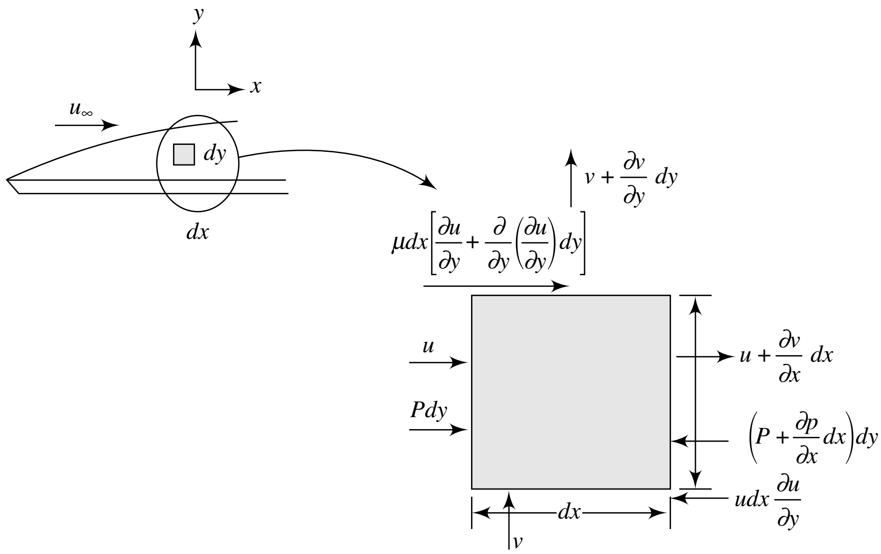


Fig. 6.8 *Momentum equation*

Assumptions

No pressure variations in the y-direction. Viscous shear forces in the y-direction are negligible.
The momentum flux in the x-direction *entering* the left face = $(\rho u dy)u = \rho u^2 dy$

The momentum flux in the x-direction *leaving* the right face = $\rho\left[u + \left(\frac{\partial u}{\partial x}\right)dx\right]^2 dy$

$$= \rho u^2 dy + 2\rho\left(\frac{\partial u}{\partial x}\right)dx\,dy + \left[\left(\frac{\partial u}{\partial x}\right)dx\right]^2$$

The momentum flux in the x-direction *entering* the bottom face = $(\rho v dx)u = \rho uvdx$

The momentum flux in the x-direction *leaving* the top face $= \rho\left(v + \frac{\partial v}{\partial y}dy\right)\left(u + \frac{\partial u}{\partial y}dy\right)dx$

$$= \rho dx\left[uv + u\frac{\partial v}{\partial y}dy + v\frac{\partial u}{\partial y}dy + \cancelto{0}{\frac{\partial u \partial v}{\partial y^2}dy^2}\right]$$

$$= \rho uvdx + \rho u\left(\frac{\partial v}{\partial y}\right)dx\,dy + \rho v\left(\frac{\partial v}{\partial y}\right)dx\,dy$$

Hence,

$$\begin{pmatrix}\text{Net momentum flux}\\ \textit{entering}\text{ the }x\text{-direction}\end{pmatrix} = \begin{pmatrix}\text{Momentum flux }\textit{leaving}\\ \text{the right and top faces}\end{pmatrix} - \begin{pmatrix}\text{Momentum flux }\textit{entering}\text{ the}\\ \text{left and bottom faces}\end{pmatrix}$$

$$= \left[\rho u^2 dy + 2\rho u\left(\frac{\partial u}{\partial x}\right)dx\,dy\right] + \left[\rho u\,v\,dx + \rho u\left(\frac{\partial v}{\partial y}\right)dx\,dy + \rho v\left(\frac{\partial u}{\partial y}\right)dx\,dy\right] - \rho u^2 dy - \rho uvdx$$

$$= 2\rho u\left(\frac{\partial v}{\partial x}\right)dx\,dy + \rho u\left(\frac{\partial v}{\partial y}\right)dx\,dy + \rho v\left(\frac{\partial u}{\partial y}\right)dx\,dy$$

$$= \rho\left[u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right)\right]dx\,dy + \rho u\left\{\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right)\right\}dx\,dy$$

Now, from the continuity equation, we have $\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0$

The net momentum transfer in the x-direction $= \rho\left\{u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right)\right\}dx\,dy$ (6.12)

The forces considered in this analysis are those due to viscous shear and the pressure forces on the control volume.

Pressure force on the left face is $P\ dy$ and over the right face is $-\left[P + \left(\frac{\partial P}{\partial x}\right)dx\,dy\right]$

The net pressure force in the direction of motion is $-\left(\frac{\partial P}{\partial x}\right)dx\,dy$ (6.13)

Viscous shear force on the bottom face is $-\mu\left(\frac{\partial u}{\partial y}\right)dx$

and the shear force on the top face is $\mu\left[\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)dy\right]dx$

The net viscous shear force in the direction of motion is

$$\left[\mu\left(\frac{\partial u}{\partial y}\right) + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)dy\right]dx - \mu\left(\frac{\partial u}{\partial y}\right)dx = \mu\left(\frac{\partial^2 u}{\partial y^2}\right)dx\,dy \tag{6.14}$$

Equating the sum of the viscous-shear and pressure forces to the net momentum transfer in the x-direction, we get

$$-\left(\frac{\partial P}{\partial x}\right)dx\,dy + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)dx\,dy$$

$$\rho\left\{u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right)\right\}dx\,dy$$

Simplifying,

$$\boxed{\rho\left\{u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right)\right\} = \mu\left(\frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial p}{\partial x}\right)} \tag{6.15}$$

This is the *momentum equation* of the laminar boundary layer for two-dimensional, steady flow of an incompressible fluid.

If the pressure gradients in the x-direction is negligible, which is relevant for flow over a flat plate since $\left(\frac{\partial u_\infty}{\partial x}\right) = 0$, the above equation is reduced to

$$\boxed{u\left[\left(\frac{\partial u}{\partial x}\right)+v\left(\frac{\partial u}{\partial y}\right)=\nu\left(\frac{\partial^2 u}{\partial y^2}\right)\right]} \tag{6.16}$$

where $\nu = \mu/\rho$ = kinematic viscosity

6.11 ❑ CONSERVATION OF ENERGY EQUATION FOR THE BOUNDARY LAYER

We shall now apply the energy conservation requirement for the laminar thermal boundary layer to develop the energy equation.

Consider the elemental control volume shown in Fig. 6.9 which shows the rate at which energy is conducted and convected *into* and *out*.

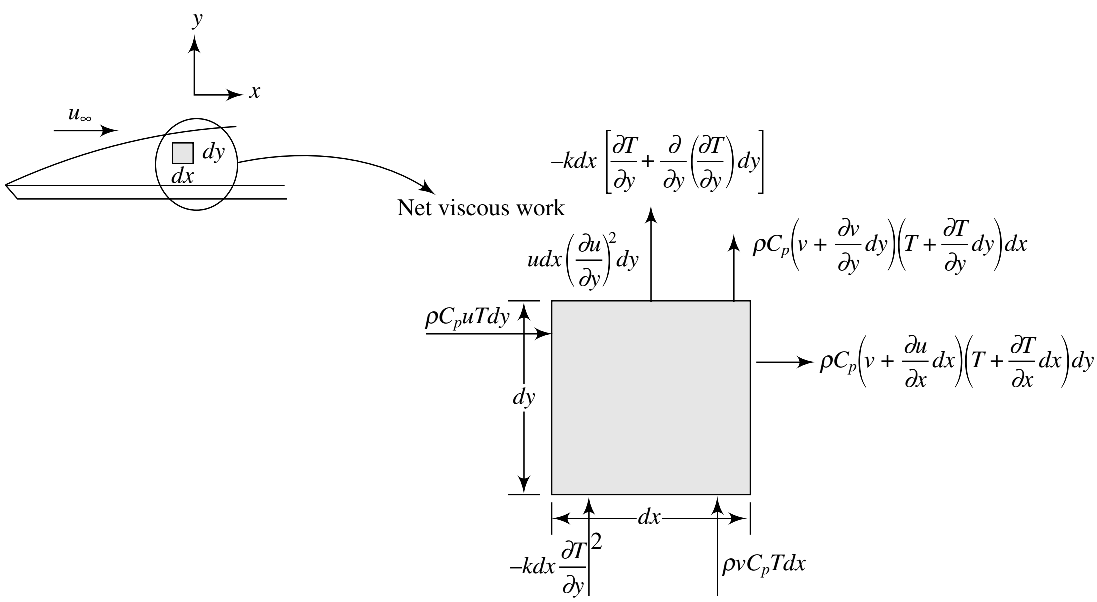


Fig. 6.9 *Energy equation*

Assumptions

- Steady, incompressible two-dimensional flow.
- Conduction is only in the y-direction, and negligible conduction in the direction of flow (x-direction).
- Constant fluid properties: viscosity, thermal conductivity, and specific heat.
- Negligible body forces: viscous energy dissipation is negligible.
- Note that in addition to the *conductive* terms, there are *four convective* terms.

Convection in the x-direction Energy *into* the control volume = $\rho\ C_p\ u\ T\ dy$

Energy *out* of the control volume $=\rho C_p\left\{u+\left(\frac{\partial u}{\partial x}\right)dx\right\}\times\left\{T+\left(\frac{\partial T}{\partial x}\right)dx\right\}dy$

Therefore, neglecting second-order differentials, the net energy convected *out* in the *x*-direction is

$$\rho C_p \left\{ v\left(\frac{\partial T}{\partial y}\right) + T\left(\frac{\partial u}{\partial x}\right) \right\} dx\, dy \tag{6.17}$$

In a similar fashion, the net energy convected *out* in the *y-direction* is

$$\rho C_p \left\{ v\left(\frac{\partial T}{\partial y}\right) + T\left(\frac{\partial v}{\partial y}\right) \right\} dx\, dy \tag{6.18}$$

Conduction in the y-direction Net conduction *into* the control volume in the *y*-direction is

$$-k\, dx\left(\frac{\partial T}{\partial y}\right) - \left[-k\, dx\left\{ \left(\frac{\partial T}{\partial y}\right) + \left(\frac{\partial^2 T}{\partial y^2}\right) dy \right\} \right] = k\left(\frac{\partial^2 T}{\partial y^2}\right) dx\, dy$$

Likewise, the net conduction *into* the control volume in the *x*-direction is

$$k\left(\frac{\partial^2 T}{\partial y^2}\right) dx\, dy$$

When the viscous effects are neglected, the energy balance can be expressed as

(*Net energy in*) = (*Net energy out*)

$$\rho C_p \left\{ \left[u\left(\frac{\partial T}{\partial x}\right) + T\left(\frac{\partial u}{\partial x}\right) \right] dx\, dy + \rho C_p \left[v\left(\frac{\partial T}{\partial y}\right) + T\left(\frac{\partial v}{\partial y}\right) \right] \right\} dx\, dy = k\left(\frac{\partial^2 T}{\partial x^2}\right) dx\, dy + k\left(\frac{\partial^2 T}{\partial y^2}\right) dx\, dy$$

or

$$\rho C_p \left\{ u\left(\frac{\partial T}{\partial x}\right) + T\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial T}{\partial y}\right) + T\left(\frac{\partial v}{\partial y}\right) \right\} dx\, dy = k\left[\left(\frac{\partial^2 T}{\partial x^2}\right) + \left(\frac{\partial^2 T}{\partial y^2}\right) \right] dx\, dy$$

or

$$\rho C_p \left\{ u\left(\frac{\partial T}{\partial x}\right) + v\left(\frac{\partial T}{\partial y}\right) \right\} + T\left\{ \left[\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial v}\right) \right] \left[k\left(\frac{\partial^2 T}{\partial x^2}\right) + \left(\frac{\partial^2 T}{\partial y^2}\right) \right] \right\}$$

Now, from continuity equation, for incompressible fluid flow: $\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0$

Since the boundary layer is very thin, $\left(\frac{\partial T}{\partial y}\right) >> \left(\frac{\partial T}{\partial x}\right)$ (i.e., conduction in the *x*-direction is negligible) and $\frac{\partial^2 T}{\partial x^2} = 0$.

Dividing by ρC_p, the energy equation of the laminar boundary layer becomes

$$u\left(\frac{\partial T}{\partial x}\right) + v\left(\frac{\partial T}{\partial y}\right) = \left(\frac{k}{\rho C_p}\right)\left(\frac{\partial^2 T}{\partial y^2}\right) \quad \text{or} \quad \boxed{u\left(\frac{\partial T}{\partial x}\right) + v\left(\frac{\partial T}{\partial y}\right) = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right)} \tag{6.19}$$

where $\alpha = \frac{k}{\rho C_p} =$ thermal diffusivity

This is the energy equation of the laminar boundary layer.

You might have noticed the striking similarity between the momentum balance equation and the energy balance equation.

If $\nu = \alpha$, i.e., $Pr = 1$, the solution to the momentum and energy equations will have exactly the some form.

When the viscous dissipation cannot be neglected, as for high velocity flow or in the case of very viscous fluids (e.g., in journal bearings), or when the fluid shear rate is extremely high, an additional term for *viscous dissipation*, ϕ appears on the LHS of the energy balance. ϕ is defined as

$$\boxed{\phi = \mu\left[\left(\frac{\partial u}{\partial y}\right)+\left(\frac{\partial v}{\partial x}\right)\right]^2 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]-\left(\frac{2}{3}\right)\left[\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial v}{\partial y}\right)\right]^2} \tag{6.20}$$

6.12 ❑ THE LAMINAR FLOW IN PIPE

Flow in circular tubes is of great practical importance. As shown in Fig. 6.10, fluid enters with a uniform velocity across the cross-sectional area of the tube, as indicated by arrows of equal length. Once the fluid comes in contact with the stationary wall of the tube, it is reduced to zero velocity and a layer of slow-moving layer of the fluid, called boundary layer, forms near the tube wall. The boundary-layer thickness increases as the fluid moves downstream. In the entrance region, the velocity changes in both *radial* and *axial* directions. Eventually, the boundary layer extends to the centre of the tube and the velocity profile assumes a rounded shape. The profile no longer changes further downstream beyond this entrance length. All flow is in *axial* direction and there is no component of velocity in the radial direction. This is called the *fully developed* flow region, where velocity profile is *independent* of the distance from the tube entrance.

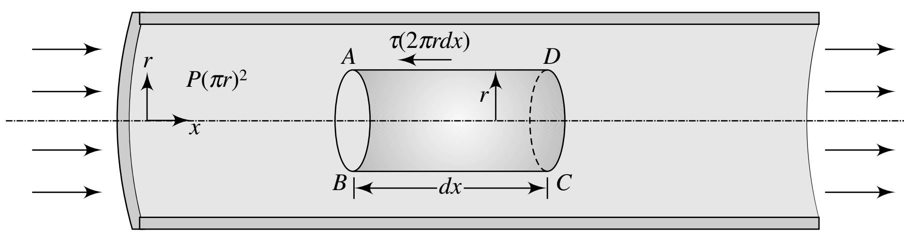


Fig. 6.10 *Differential control volume force balance for laminar flow through a circular tube*

Consider fully developed, steady, incompressible, laminar flow in a circular tube (the *Hagen–Poiseuille* flow).

Let us analyze the equilibrium of forces on a concentric cylindrical element of length dx and radius r, at a distance y from the tube surface.

Force balance for fully developed flow is given by

$$P(\pi r^2) = (P + dP)(\pi r^2) + (2\pi r dx)\tau$$

or
$$-\frac{dP}{dx} = \frac{2\tau}{r}$$

or
$$\text{Shear stress, } \tau(r) = \frac{r}{2}\left(-\frac{dP}{dx}\right)$$

At the tube wall ($r = R$), the shear stress,

$$\boxed{\tau_{(r=R)} = \tau_w = \frac{R}{2}\left(-\frac{dP}{dx}\right)} \tag{6.21}$$

(**Note:** $\left(-\dfrac{dP}{dx}\right)$ is the *negative* pressure gradient for a horizontal pipe or tube. But for a tube inclined at an angle θ with the horizontal, the pressure gradient would be $\dfrac{dP^*}{dx}$ where $P^* = P + \rho gz$, known as *piezometric* pressure.)

From Newton's law of viscosity,

$$\tau(r) = \mu\frac{du}{dy} = -\mu\frac{du}{dr} \quad [y = R - r,\, dy = -dr]$$

Therefore,

$$-\mu\frac{du}{dr} = \frac{r}{2}\left(-\frac{dP}{dx}\right)$$

or
$$du = \frac{r}{2\mu}\left(\frac{dP}{dx}\right)dr$$

Integrating with respect to r, we have

$$u(r) = \frac{1}{2\mu}\left(\frac{dP}{dx}\right)\cdot\frac{r^2}{2} + A$$

The constant of integration A can be evaluated from the boundary condition:

At $\quad r = R,\ u = 0$

It follows that

$$0 = \frac{R^2}{4\mu}\left(\frac{dP}{dx}\right) + A \quad \Rightarrow \quad A = \frac{-R^2}{4\mu}\left(\frac{dP}{dx}\right)$$

Thus, the velocity at any point is given by

$$\therefore \quad \boxed{u(r) = \frac{1}{4\mu}[R^2 - r^2]\left(-\frac{dP}{dx}\right)} \tag{6.22}$$

Negative sign is because of the pressure drop in the direction of flow.

Maximum velocity occurs at the centreline.

Hence,
$$\boxed{u_{\max} = u(r = 0) = \frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)} \tag{6.23}$$

The velocity distribution is obtained by dividing $u(r)$ by $u_{\max}$.

$$\frac{u(r)}{u_{\max}} = \frac{R^2 - r^2}{R^2}$$

or
$$\boxed{\frac{u(r)}{u_{\max}} = 1 - \left(\frac{r}{R}\right)^2} \tag{6.24}$$

Average velocity,

$$V = \frac{2}{R^2}\int_0^R u(r)rdr$$

or $$V = \frac{2}{R^2}\int_0^R u_{\max}\cdot r\left(\frac{R^2 - r^2}{R^2}\right)dr = \frac{2u_{\max}}{R^4}\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^R$$

$\therefore$ $$V = \frac{2u_{\max}}{R^4}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = 2u_{\max}\times\frac{1}{4} = \frac{1}{2}u_{\max}$$

This shows that the average velocity is half the maximum velocity, i.e.,

$$\boxed{V = \frac{u_{\max}}{2}} \tag{6.25}$$

Substituting for $u_{\max}$ equal to $2V$, the *parabolic velocity profile* is

$$\boxed{u(r) = 2V\left[1 - \left(\frac{r}{R}\right)^2\right]} \tag{6.26}$$

Volumetric flow rate or discharge through the tube is

$$\dot{\forall} = \frac{\dot{m}}{\rho} = A_c V = A_c\frac{u_{\max}}{2} = \frac{R^2}{8\mu}\left(-\frac{dP}{dx}\right)\pi R^2$$

or $$\boxed{\dot{\forall} = \frac{\pi R^4}{8\mu}\left(-\frac{dP}{dx}\right)} \tag{6.27}$$

For a tube of length L with pressure dropping from P_1 to P_2, the pressure gradient is

$$-\frac{dP}{dx} = -\left(\frac{P_2 - P_1}{L - 0}\right) = \frac{P_1 - P_2}{L}$$

The pressure drop is then expressed as

$$\frac{\Delta P}{L} = \frac{P_1 - P_2}{L} = \frac{8\mu\dot{\forall}}{\pi R^4}$$

In terms of tube diameter $D = 2R$, we get

$$\boxed{\Delta P = \frac{128\mu\dot{\forall}L}{\pi D^4}} \tag{6.28}$$

Note: For an inclined pipe, δP will be replaced with

$$\begin{aligned}\Delta P^* &\equiv (p_1 - p_2) + \rho g(z_1 - z_2)\\ &= (P_1 - P_2) - \rho g L\sin\theta = \Delta P - \rho g L\sin\theta\end{aligned}$$

This is known as *Hagen–Poiseuille* equation.
Head lost due to friction,

$$h_f = \frac{\Delta P}{\rho g} = \frac{128\mu\dot{\forall}L}{\rho g\pi D^4}$$

Since $$\dot{\forall} = (\pi R^2)V = \frac{\pi D^2}{4}\times V$$

$$\boxed{h_f = \frac{32\mu VL}{\rho g D^2}} \quad \text{(m)} \tag{6.29}$$

We note that

$$\tau_w = \frac{R}{2}\left(-\frac{dP}{dx}\right)$$

For the pipe of length L,

$$\tau_w = \frac{P_1 - P_2}{L}\frac{D}{4}$$

Head loss due to friction,

$$h_f = h_L = \frac{\Delta P}{\rho g} = \frac{4\tau_w L}{\rho g d}$$

We define Fanning friction factor, f_F,

$$f_F = \frac{\tau_w}{\rho V^2/2}$$

$$\tau_w = \frac{\Delta P D}{4L} = \frac{\rho g h_f D}{4L}$$

$$= \frac{32\mu VL}{D^2} \times \frac{D}{4L} = 8\frac{\mu V}{D}$$

$$\therefore \qquad f_F = \frac{2\tau_w}{\rho V^2} = \frac{2 \times 8\mu V}{D\rho V^2} = \frac{16\nu}{VD} = \frac{16}{Re}$$

Use or this friction factor in the UK is more prevalent. In the USA, however, the Darcy–Weisbach friction factor, f is commonly used which is 4 times f_F. Hence, $\boxed{f = \frac{64}{Re}}$ (6.30)

Pumping power requirement,

$$\boxed{\wp = \Delta P \cdot \dot{\forall} = \rho g \dot{\forall} h_f = \dot{m} g h_f} \quad \text{(W)} \tag{6.31}$$

6.13 ❑ FRICTION FACTOR FOR FULLY DEVELOPED TURBULENT FLOW IN PIPES

In the case of *smooth* pipes, where relative roughness is zero, the friction factor is given by *Petukhov* as

$$\boxed{\begin{aligned} &f = (0.79 \ln Re - 1.64)^{-2} && \textit{turbulent flow, smooth wall} \\ \text{or} \quad &f = (1.82 \log Re - 1.64)^{-2} && 3000 < Re_D < 5 \times 10^6 \end{aligned}}$$

Two other simple, but less accurate, correlations for smooth tubes applicable over a more limited range of Reynolds number are

$$\boxed{\begin{array}{ll} & \textit{turbulent flow, smooth wall} \\ f = 0.316/Re^{0.25} & 3000 < Re_D \le 2 \times 10^4 \\ f = 0.184/Re^{0.2} & Re_D \ge 2 \times 10^4 \end{array}}$$

When the tube surfaces are not smooth, the friction factor is a function of the Reynolds number, Re_D as well as relative roughness (ε/D).

One accurate way of determining the friction factor for a tube with a rough surface is to use the correlation given below:

$$\boxed{\frac{1}{\sqrt{f}} = -2.0 \log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right) \quad \textit{turbulent flow}} \tag{6.32}$$

The above *implicit* correlation for f is known as the *Colebrook equation*. The parameter, ε, in this relation is the roughness of the pipe wall. It is the mean height (*typically small*) of naturally occurring protrusions on the wall. Table 6.1 gives the values of relative roughness, ε/D for some common pipe surfaces. Note that ε/D is an additional non-dimensional parameter.

The *Colebrook* equation is rather complex as it involves iterative calculation. To simplify calculations, an approximate *explicit* relation for f was given by *Haaland* in 1983:

Table 6.1 *Equivalent roughness for clean pipes*

Equivalent roughness, ε	
Pipe	**Millimetres (mm)**
Riveted steel	0.9–9.0
Concrete	0.3–3.0
Cast iron	0.26
Galvanized iron	0.15
Commercial steel	0.045
Wrought iron	0.046
Stainless steel	0.002
Copper or brass tubing	0.0015
Plastic, glass	0.0 (smooth)

$$\boxed{\frac{1}{\sqrt{f}} \approx -1.8 \log\left[\left(\frac{\varepsilon/D}{3.7}\right)^{1.11} + \frac{6.9}{Re}\right] \quad \textit{turbulent flow}} \tag{6.33}$$

This relation gives values within 2% of those obtained from the *Colebrook* equation.

Given the uncertainties in the pipe-wall condition, the *Haaland relation* is often of sufficient accuracy and may be used instead of the *Colebrook equation*.

6.14 ❑ LAMINAR AND TURBULENT FLOWS

Reynolds number determines the character of the flow. Experiments show that the flow regime in a typical pipe is a function of Reynolds number according to

$Re_D < 2300$	*Laminar*
$2300 < Re_D < 4000$	*Transitional*
$Re_D > 4000$	*Turbulent*

These ranges are approximate and depend on various factors such as the roughness of the pipe wall and the nature of the inlet flow. Transitional flow is characterized by bursts of turbulence, which are eventually damped out. The disturbance typically starts near the wall and is carried into the interior of the flow, where it is smoothed out into laminar flow. If the Reynolds number is high enough, however, the disturbances are not damped out. *The critical Reynolds number at which transitional flow occurs depends on the geometry and the flow situation*.

For a given fluid (ρ and μ) and characteristic length, increasing velocity tends to destabilize the flow. Higher velocity implies a higher Reynolds number, since velocity appears in the numerator of the Reynolds number. A high Reynolds number is characteristic of turbulent flow.

The characteristic length, L_c, appears in the numerator of the Reynolds number. For a given flow rate and fluid, small diameters tend to be stabilizing and lead to lower Reynolds numbers. Disturbances that start at the wall can be damped by the presence of the other wall. If there is enough separation between the walls, a packet of perturbed flow has time to develop into full-blown turbulence before getting near the other wall, where it can be stabilized by viscous forces.

The final parameters in the Reynolds number are the *fluid properties*, *density*, and *viscosity*. These are not independent but are a function of the type of fluid. With a given velocity and characteristic length, high viscosity is a stabilizing influence in the flow. It is very difficult to perturb a viscous fluid out of its flow pattern. Conversely, higher densities tend to destabilize the flow. Once a high-density packet of fluid is perturbed, it is difficult to force it back into a smooth, laminar flow pattern.

The Reynolds number can be viewed as the ratio of *inertial* forces to viscous *forces*. To understand this concept, imagine a small cube of fluid where each side has a length L. To accelerate this cube against inertia, a force equal to the mass times the acceleration is applied, that is, Inertia force, $F_i = ma = m\dfrac{dV}{dt}$

For *non-circular* pipes, the hydraulic diameter given by $\boxed{D_h = 4A_c/P}$ should be used in place of the actual diameter in the Reynolds number and head loss relations. A_c is the cross-sectional area and P, the wetted perimeter.

6.15 ❑ DIMENSIONAL ANALYSIS

Dimensional analysis is an important mathematical tool in the study of fluid mechanics and heat-transfer. It is a mathematical technique which makes use of the study of dimensions as an aid to the solution of many engineering problems. The main advantage of dimensional analysis of a problem is that it reduces the number of variables in the problem by combining dimensional variables to form nondimensional parameters.

It is based on the principle of dimensional homogeneity, and uses the dimensions of relevant parameters affecting the phenomenon.

The Buckingham Π-theorem states that if, in a dimensionally homogeneous equation, there are n dimensional variables which are completely described by m fundamental dimensions (such as M, L, t, etc.) they may be grouped into $(n–m)$ Π-terms. Each Π–term is a dimensionless parameter.

The required number of Π-terms is fewer than the number of original variables by m, where m is determined by the minimum number of reference dimensions required to describe the original list of variables.

The functional relation among the independent, dimensionless π-parameters must be determined experimentally.

Once the dimensionless groups are formed, these dimensionless groups are arranged in the form of a suitable equation. This equation would contain some constants. The values of these constants are obtained by performing experiments on models. A *model* is a representation of a physical system that may be used to predict the behaviour of the system in some desired respect. The physical system for which the predictions are to be made is called the *prototype*.

The process of identifying the non-dimensional or Π-parameters controlling the physical phenomenon is carried out in *two* steps:

First, the number of variables, n, influencing the physical process are listed. The variables listed must be such that none is derivable by an algebraic combination of the others. The variables for example can include μ and ρ or ν and ρ; but not all the three.

Secondly, the number of fundamental dimensions, m, required to specify the units of the variables involved are identified. The main dimensions are mass (M), length (L), time (t), and temperature (T). Then by Buckingham Pi theorem, the number of non-dimensional parameters will be $p = (n - m)$. There are *two* methods available for dimensional analysis:

- Rayleigh method, and
- Buckingham-Π theorem.

The expressions of some of the common non-dimensional groups are listed in Table 6.2.

Table 6.2 *Physical interpretation of some dimensionless groups*

Dimensionless Number	Expression	Physical Significance
Biot (*Bi*)	$\frac{hL_c}{k}$	$\frac{\text{Internal resistance to heat conduction}}{\text{External resistance to heat convection}}$
Eckert (*E*)	$\frac{V^2}{C_p(T_w - T_\infty)}$	$\frac{\text{Kinetic energy}}{\text{Thermal energy}}$
Euler (*Eu*)	$\frac{\Delta P}{\frac{1}{2}\rho V^2}$	$\frac{\text{Pressure forces}}{\text{Inertia forces}}$
Fourier (*Fo*)	$\frac{\alpha t}{L_c^2} = \frac{kt}{\rho C_p L_c^2}$	Dimensionless time for transient condition
Froude (*Fr*)	$\frac{V^2}{gL_c}$	$\frac{\text{Inertia forces}}{\text{Gravity forces}}$
Grashof (Gr)	$\frac{g\beta L_c^3(T_s - T_\infty)}{\nu^2}$	$\frac{\text{(Buoyancy forces)(Inertia forces)}}{\text{(Viscous forces)}^2}$
Lewis (*Le*)	$\frac{\alpha}{D}$	$\frac{\text{Thermal diffusivity}}{\text{Mass diffusivity}}$
Mach (*M*)	$\frac{V}{a}$	$\frac{\text{Flow velocity}}{\text{Sonic velocity}}$
Nusselt (*Nu*)	$\frac{hL_c}{k}$	$\frac{\text{Convection heat transfer}}{\text{Conduction heat transfer}}$
Peclet (*Pe*)	$\frac{C_p \rho V L_c}{k} = Re\ Pr$	$\frac{\text{Bulk heat transfer}}{\text{Conduction heat transfer}}$
Prandtl (*Pr*)	$\frac{C_p \mu}{k} = \frac{\nu}{\alpha}$	$\frac{\text{Momentum diffusivity}}{\text{Thermal diffusivity}}$
Rayleigh (*Ra*)	$\frac{g\beta L_c^3(T_s - T_\infty)}{\nu\alpha} = Gr\ Pr$	$\frac{\text{Forces due to buoyancy and inertia}}{\text{Forces due to viscosity and thermal diffusion}}$

contd.

contd.

Reynolds (*Re*)	$\dfrac{\rho V L_c}{\mu} = \dfrac{V L_c}{\nu}$	$\dfrac{\text{Inertia forces}}{\text{Viscous forces}}$
Schmidt (*Sc*)	$\dfrac{\mu}{\rho D} = \dfrac{\nu}{D}$	$\dfrac{\text{Momentum diffusivity}}{\text{Mass diffusivity}}$
Sherwood (*Sh*)	$\dfrac{h_D L_c}{D}$	Ratio of concentration gradients
Friction factor (*f*)	$\dfrac{h_f}{(L/D)(V^2/2g)}$	$\dfrac{\text{Frictional head loss}}{\text{Velocity head}}$
Skin friction coefficient (C_f)	$\dfrac{\tau_w}{SV^2}$	$\dfrac{\text{Wall shear stress}}{\text{Dynamic pressure}}$
Stanton (*St*)	$\dfrac{\text{h}}{\rho C_p V} = \dfrac{Nu}{Re\, Pr} = \dfrac{Nu}{Pe}$	$\dfrac{\text{Convective heat transfer}}{\text{Thermal capacity (storage)}}$

Typically, *eight* steps listed below are involved for determining the Π-parameters using the Buckingham Π theorem.

Step 1 List all the variables that are involved in the problem. (Let n be the number of parameters.)

Step 2 Express each of the variables in terms of basic dimensions. The basic dimensions will be M, L, and t. Add temperature T in heat-transfer problems (Let m be the number of basic dimensions.)

Step 3 Determine the required number of Π-terms. [$p = n - m$]

Step 4 Select a number of repeating variables, where the number required is equal to the number of reference dimensions.
The following are the *three* general guidelines for selecting repeating variables.

1. In a dimensional analysis of any physical system, the repeating variables must include among them all of the m main dimensions.
2. Preferably, a wise mass density choice, a characteristic velocity would be to elect, and characteristic length need as repeating variables. We can add thermal conductivity as the fourth repeating variable if temperature is the fourth fundamental dimension.
3. Do not choose the *dependent variable* as one of the repeating variables.

Step 5 Set up dimensional equation combining the repeated parameters, selected in Step 4 with each of the other parameters in turn to form dimensionless groups.

Essentially, each Π-term will be of the form $A_1^{a_i} A_2^{b_i} A_3^{c_i} A_4^{d_i} \; A_i$ where A_i is one of the non- repeating variables. A_1, A_2, and A_3 are the repeating variables and the exponents a_i, b_i, c_i and d_i are determined so that the combination is dimensionless.

Step 6 Repeat Step 5 for each of the remaining non-repeating variables.

Step 7 Check all the resulting Π-terms to make sure they are dimensionless.

Step 8 Express the final form as a relationship among the Π-terms, and think about what it means.

Typically, the final form can be written as

$$\boxed{\Pi_1 = \phi(\Pi_2, \Pi_3, \Pi_4 \ldots \Pi_{n-m})}$$

The model may be larger, smaller, or even of the same size as the prototype. For complete similarity between the prototype and the model, every dimensionless parameter referring to the conditions in the model must have the same numerical value as the corresponding parameter referring to the prototype. Thus, the model and the prototype must be completely similar:

- geometrically,
- kinematically, and
- dynamically.

Geometric similarity is the similarity of *shape*, kinematic similarity is the similarity of *motion*, and dynamic similarity is the similarity of *forces* acting on the body.

Illustrative Examples

A: Boundary-Layer Concept: Laminar and Turbulent Flow

EXAMPLE 6.1 *The heat-transfer coefficient for a gas flowing over a thin flat plate 3 m long and 0.3 m wide, varies with distance from the leading edge according to,* h(x) = $10x^{-1/4}$ *W/m² K*

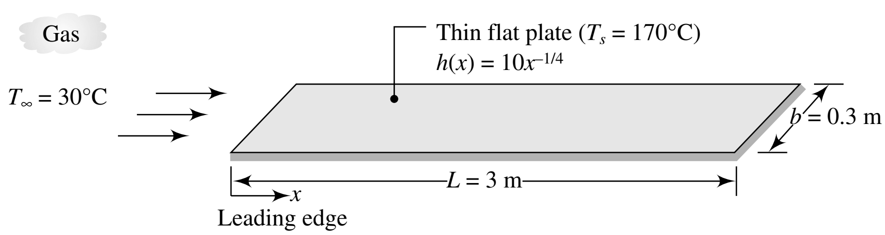
Calculate (a) the average heat-transfer coefficient, (b) the rate of heat-transfer between the plate and the gas if the plate is at 170°C and the gas is at 30°C, and (c) the local heat flux 2 m from the leading edge. **[IES 1991]**

Solution

Known Gas flows over a flat plate. Variation of h(x) with x is given.

Find (a) $\bar{h}_L(\text{W/m}^2\,\text{K})$, (b) $\dot{Q}(\text{W})$, (c) $q_{(x=2\text{m})}(\text{W/m}^2)$.

Schematic



Assumptions (1) Steady-state conditions. (2) Isothermal surface.

Analysis Average heat-transfer coefficient for the entire plate of length L is

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x\,dx = \frac{1}{L}\int_0^L 10x^{-1/4}\,dx$$

$$= \frac{10}{L}\left[\frac{x^{-1/4+1}}{1-1/4}\right]_0^L = \frac{40}{3L}[L^{3/4}]$$

$$= \frac{40/3}{L^{1/4}} = \frac{40}{3}L^{-1/4} = \frac{40}{3}\times 3^{-0.25} = \mathbf{10.13\ W/m^2K} \qquad \textbf{(Ans) (a)}$$

Heat-transfer rate,

$$\dot{Q} = \bar{h}_L A(T_s - T_\infty) = (10.13 \text{ W/m}^2 \text{ K}) (3 \text{ m} \times 0.3 \text{ m}) (170 - 30) \text{ K}$$

$$= \mathbf{1276.5 \text{ W}} \quad \textbf{(Ans.)}$$

Local heat-transfer coefficient,

$$h_{(x\,=\,2\text{m})} = 10(2)^{-1/4} = 8.41 \text{ W/m}^2 \text{ K}$$

Average heat-transfer coefficient at 2 m from the leading edge is

$$\bar{h}_x = \frac{4}{3} \times 8.41 = 11.21 \text{ W/m}^2 \text{ K}$$

Local heat flux at 2 m from the leading edge is

$$q(x) = \bar{h}_x(T_s - T_\infty) = (11.21 \text{ W/m}^2 \text{ K}) (170 - 30) \text{ K}$$

$$= \mathbf{1570 \text{ W/m}^2} \quad \textbf{(Ans.) (c)}$$

Comment We note that as the distance from the leading edge, x increases, the value of $h(x)$ and $\bar{h}$ decrease.

EXAMPLE 6.2 *Experimental results for the local heat-transfer coefficient* h_x *for flow over a 2.5 m long and 0.7 m wide thin flat plate were found to fit the relation,* $h_x(x) = 10x^{-1/4}$ *W/m² °C where* x *is the distance from the leading edge of the plate.*

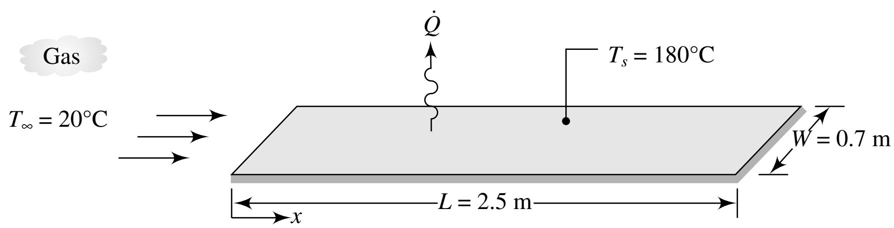
(a) Develop an expression for the ratio of the average heat-transfer coefficient $\bar{h}_x$ *to the local heat-transfer coefficient* h_x. *(b) Sketch the variation of* h_x *and* $\bar{h}_x$ *with* x. *(c) Determine the rate of heat-transfer between the plate and the gas if the plate and gas temperatures are 180°C and 20°C respectively. (d) What will be the local heat flux at a distance of 1.25 m from the leading edge?*

Solution

Known Variation of local heat-transfer coefficient for flow over a flat plate.

Find (a) Derivation to find $\bar{h}(x)/h(x)$. (b) Plot of $\bar{h}(x)$ and $h(x)$ with x. (c) Heat rate, $\dot{Q}$. (d) Local heat flux, $q(x)$.

Schematic



Assumption Steady operating conditions prevail.

Analysis (a) The average value of the convection heat-transfer coefficient over the plate from x = 0 (leading edge) to $x = x$ (any specified distance from the leading edge) is given by

$$\bar{h}_x = \bar{h}_x(x) = \frac{1}{x}\int_0^x h_x(x)dx$$

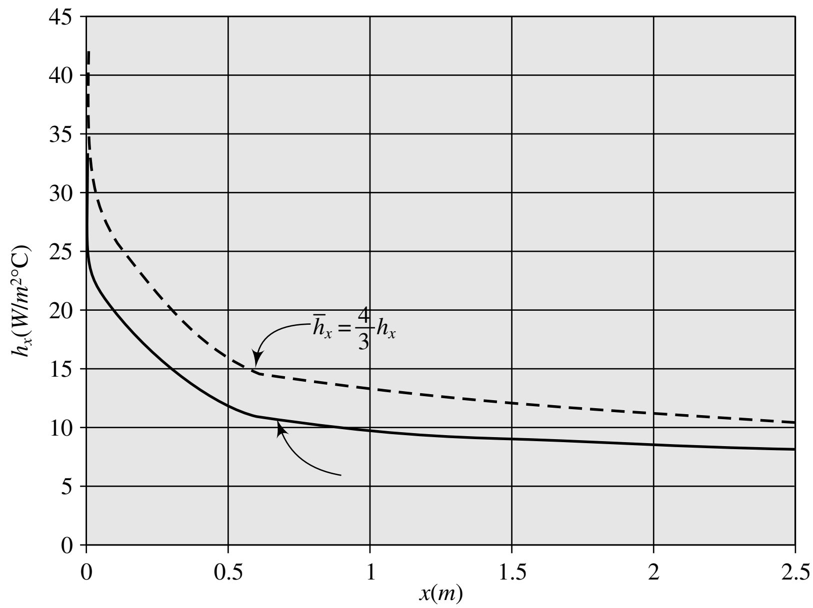
Substituting the expression for the local heat-transfer coefficient, $h_x(x) = 10x^{-1/4}$ and integrating with respect to x between the limits 0 and x.

$$\bar{h}_x = \frac{1}{x}\int_0^x 10x^{-1/4}\,dx = \frac{10}{x}\int_0^x x^{-1/4}\,dx = \frac{10}{x}\left[\frac{x^{\left(-\frac{1}{4}+1\right)}}{-\frac{1}{4}+1}\right]$$

$$= \frac{4}{3}\times\frac{10}{x}x^{3/4} = \frac{4}{3}\times(10x^{-1/4}) = \frac{4}{3}h_x(x)$$

∴ The ratio $\frac{\bar{h}_x}{h_x(x)} = \mathbf{\frac{4}{3}}$ **(Ans.) (a)**

(b) The variation of h_x and $\bar{h}_x$ with x is sketched below:



(c) The rate of heat-transfer from the hotter plate to the colder gas is determined from the Newton's law of cooling given by $\dot{Q} = \bar{h}A_s(T_s - T_\infty)$
where $\bar{h}$ is the *average* heat-transfer coefficient for the entire length of the plate (in the direction of fluid flow). As is the total surface area of the plate, and T_s and $-T_\infty$ are the uniform surface temperature, and the ambient fluid temperature respectively. $(T_s - T_\infty)$ can be expressed either in °C or K. Average heat-transfer coefficient is

$$\bar{h} = \bar{h}_L = \frac{4}{3}h_{x=L} = \frac{4}{3}\times 10L^{-1/4} = \frac{40}{3}(2.5\text{ m})^{-0.25} = 10.6\text{ W/m}^2\text{ °C}$$

Surface area, A_s = LW = (2.5 m)(0.7 m) = 1.75 m^2 (*for one surface only*)
Temperature difference, $(T_s - T_\infty)$ = 180 – 20 = 160°C
Therefore, the heat-transfer rate is

$$\dot{Q} = \left(10.6\frac{\text{W}}{\text{m}^2\text{°C}}\right)(1.75\text{ m}^2)\ (160\text{ °C}) = \mathbf{2968\ W}$$ **(Ans.) (c)**

(d) Local heat flux at x = 1.25 m is

$$q(x) = \frac{\dot{Q}(x)}{A} = h_x(T_s - T_\infty) = 10(1.25)^{-0.25} \frac{\text{W}}{\text{m}^2\,°\text{C}}(180 - 20)°\text{C}$$

$= \mathbf{1513\ W/m^2}$ **(Ans.) (d)**

(B) Internal Flow: Flow Through Closed Conduits

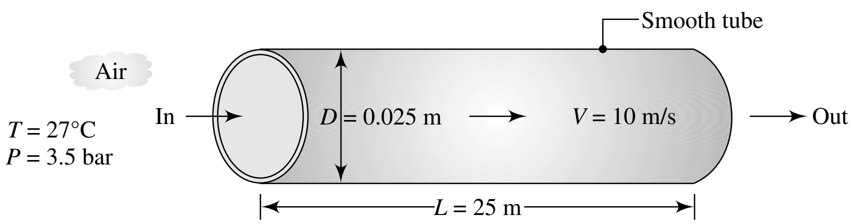
EXAMPLE 6.3 *Air at 3.5 bar and 27°C flows in a smooth 2.5 cm ID tube with a bulk velocity of 10 m/s; the tube is 25 m long. What is the pressure drop and power required to move the air through the tube?*

Solution

Known Air flows through a tube under specified conditions.

Find Pressure drop, (Pa) and Power required (W).

Schematic



Assumptions (1) Steady-state conditions. (2) Air is an ideal gas.

Analysis From property tables for air at 27°C: μ = 18.46 × 10^{-6} kg/m s

Density, $$\rho = \frac{P}{RT} = \frac{350\ \text{kPa}}{(0.287\ \text{kJ/kg K})(300.15\ \text{K})}\left|\frac{1\ \text{kJ}}{1\ \text{kPa m}^3}\right| = 4.063\ \text{kg/m}^3$$

Reynolds number, $$Re_D = \frac{\rho VD}{\mu} = \frac{(4.063\ \text{kg/m}^3)(10\ \text{m/s})(0.025\ \text{m})}{18.46 \times 10^{-6}\ \text{kg/m s}} = 5.5 \times 10^4 \quad (> 2300)$$

∴ Flow is *turbulent.*

Friction factor, $f = 0.184\,(Re_D)^{-0.2} = 0.184\,(5.5 \times 10^4)^{-0.2} = 0.0207$

Else: $f = [0.79 \ln(5.5 \times 10^4) - 1.64]^{-2} = 0.0205$

Pressure drop, $$\Delta P = f\frac{\rho V^2}{2}\frac{L}{D} = 0.0205 \times \frac{(4.063\ \text{kg/m}^3)(10\ \text{m/s})^2}{2} \times \frac{25\ \text{m}}{0.025\ \text{m}}$$

= 4166 Pa or **4.17 kPa** **(Ans.)**

Power required, $$\wp = \Delta P\left(\frac{\dot{m}}{\rho}\right) = \Delta P\dot{\forall} = \Delta P\left(\frac{N}{m^2}\right)A_c(m^2)V\left(\frac{m}{s}\right)$$

$$= (4166\ \text{Pa})\left(\frac{\pi}{4} \times 0.025^2\ \text{m}^2\right)(10\ \text{m/s})\left|\frac{1\ \text{W}}{1\ \text{Pa m}^3/\text{s}}\right| = \mathbf{20.45\ W}$$ **(Ans.)**

EXAMPLE 6.4 *Calculate the Darcy friction factor and the pressure drop per unit length when air at atmospheric pressure at 30°C flows through a 2 cm square tube with a mass flow rate of 0.7 kg/min. Assume that the flow is fully developed.*

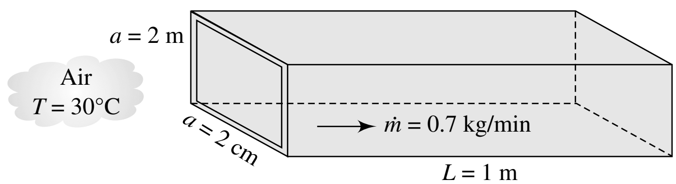
At 30°C, for air at 1 atm: $\rho = 1.164$ *kg/m³,* $\nu = 16.08 \times 10^{-6}$ *m²/s*

Solution

Known Air flows through a tube of square cross-section.

Find Friction factor, f and Pressure drop, δP (Pa).

Schematic



Assumptions (1) Fully developed flow. (2) Air is an ideal gas.

Analysis Equivalent diameter,

$$D_e = 4\frac{A_c}{P} = \frac{4 \times a \times a}{4a} = a = 0.02 \text{ m}$$

$$Re_D = \frac{\rho V D_e}{\mu} = \frac{V D_e}{\nu}$$

$$V = \frac{\dot{m}}{\rho A_c} = \frac{(0.7/60)\text{kg/s}}{1.164 \text{ kg/m}^3 \times 2 \times 2 \times 10^{-4} \text{ m}^2} = 25.0 \text{ m/s}$$

$$Re_D = \frac{25 \text{ m/s} \times 0.02 \text{ m}}{16.08 \times 10^{-6} \text{ m}^2/\text{s}} = 31165$$

With $f = (0.79 \ln Re_D - 1.64)^{-2} = (0.79 \ln 31165 - 1.64)^{-2} = 0.0234$

Pressure drop per unit length,

$$\Delta P = \left(\frac{fL}{D_e}\right)\left(\frac{\rho V^2}{2}\right) = \frac{0.0234 \times 1 \text{ m} \times 1.164 \text{ kg/m}^3 \times 25^2 \text{ m}^2/\text{s}^2}{0.02 \text{ m} \times 2} = 426 \frac{\text{kg}}{\text{m s}^2}\left|\frac{1 \text{ N s}}{1 \text{ kg m}}\right|^2$$

$$= \mathbf{426 \text{ N/m}^2 \text{ or } 426 \text{ Pa}}$$ **(Ans.)**

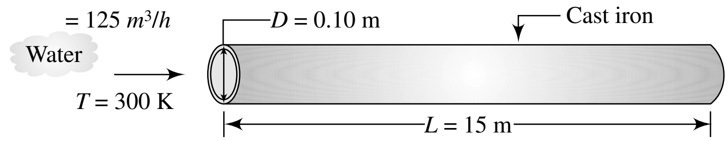
EXAMPLE 6.5 *A horizontal cast-iron pipe (*$\varepsilon = 0.26$ *mm) of 10 cm diameter carries 125 m³/h of water. The length of the pipe is 15 m. Calculate the pressure drop and pumping power requirement. The water is at a temperature of 300 K (*$\rho = 997$ *kg/m³),* $\mu = 855 \times 10^{-6}$ *kg/m s.*

Solution

Known Water flows through a CI pipe with a specified flow rate.

Find Pressure drop and pumping power.

Schematic



Assumptions (1) The flow is incompressible. (2) The flow is fully developed.

Analysis Let us first determine whether the flow is *laminar* or *turbulent*. For this, we calculate the Reynolds number. Once the flow regime is known, the appropriate friction factor can be evaluated and the pressure drop calculated with $\delta P = \rho\, g\, h_L$, where

$$h_L = f\left(\frac{L}{D}\right)\frac{V^2}{2g}$$

The Reynolds number is defined as, $Re = \dfrac{\rho V D}{\mu}$

The velocity can be found from

$$V = \frac{\dot{V}}{A_c} = \frac{(125/3600)\text{m}^3/\text{s}}{(\pi/4)(0.1\,\text{m})^2} = 4.42\text{ m/s}$$

The Reynolds number is

$$Re = \frac{(997\text{ kg/m}^3)(4.42\text{ m/s})(0.1\text{ m})}{855\times10^{-6}\text{ kg/m s}} = 5.155\times10^5$$

The Reynolds number is greater than 4000; therefore, the flow is *turbulent*. Now we check the relative roughness. The surface roughness of a cast-iron pipe is given as

$$\varepsilon = 0.26\text{ mm}$$

The relative roughness is then

$$\frac{\varepsilon}{D} = \frac{0.26\text{ mm}}{100\text{ mm}} = 0.0026$$

With $Re = 5.155 \times 10^5$ and $\varepsilon/D = 0.0026$,

$$\frac{1}{\sqrt{f}} \approx -1.8\log\left[\left(\frac{\varepsilon/D}{3.7}\right)^{1.11} + \frac{6.9}{Re}\right]$$

$$\approx -1.8\log\left[\left(\frac{0.0026}{3.7}\right)^{1.11} + \frac{6.9}{5.155\times10^5}\right] = 6.267 \quad\Rightarrow\quad f \approx 0.0255$$

The head loss may now be calculated as

$$h_L = f\left(\frac{L}{D}\right)\frac{V^2}{2g} = (0.0255)\left(\frac{15\text{ m}}{0.1\text{ m}}\right)\left(\frac{4.42^2\text{ m}^2/\text{s}^2}{2\times9.81\text{ m/s}^2}\right)$$

$$= 3.80\text{ m}$$

Pressure drop in a horizontal pipe is

$$\Delta P = \rho g h_L = (997\ \text{kg/m}^3)\left(9.81\frac{\text{N}}{\text{kg}}\right)(3.80\ \text{m})$$

$$= 37.2 \times 10^3\ \text{N/m}^2 \quad \text{or} \quad \mathbf{37.2\ kPa} \qquad \textbf{(Ans.)}$$

Pumping power,

$$\wp = \Delta P\,\dot{V} = (37.2\ \text{kPa})(125/3600\ \text{m}^3/\text{s})$$

$$= \mathbf{1.29\ kW} \qquad \textbf{(Ans.)}$$

(C) Dimensional Analysis

EXAMPLE 6.6 *Experimental investigations are conducted on a long cylinder of 100 mm diameter, 0.25 W/m K thermal conductivity with uniform volumetric internal heat generation. Using dimensional analysis, develop the relation between the steady-state temperature at the centre of the cylinder, T_o, the diameter D, the thermal conductivity k, and the rate of heat generation, $\overline{q}$ taking the wall temperature as reference. Determine the equation for the centre temperature if the difference between centre and wall temperature is 40°C when the rate of uniform heat generation is 4 kW/m³.*

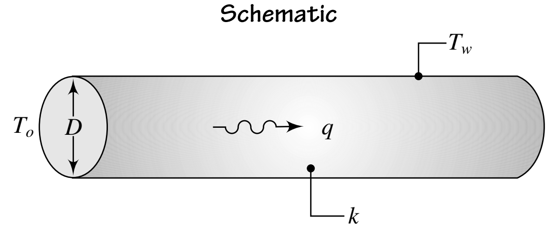
Solution

Known A long solid cylinder of diameter D and thermal conductivity k experiences heat generation at the rate of $\overline{q}$.

Find Non-dimensional relationship between $(T_o - T_w)$, D, k and $\overline{q}$.

Assumptions (1) Steady operating conditions. (2) One-dimensional (radial) heat conduction

Schematic



Analysis *Rayleigh Method*:

- Total number of parameters, $n = 4$, viz.,

$(T_o - T_w)(°\text{C})$ [T]

$k\left(\text{W/m K or } \dfrac{J}{\text{s m K}} \text{ or } \dfrac{\text{Nm}}{\text{s m K}} \text{ or } \dfrac{\text{kg m}}{\text{s}^2}\dfrac{\text{m}}{\text{s m K}} \text{ or kg m/s}^3\ \text{K}\right)$ $[\text{MLt}^{-3}\text{T}^{-1}]$

$\overline{q}\left(\text{W/m}^3 \text{ or } \dfrac{\text{Nm}}{\text{s m}^3} \text{ or } \dfrac{\text{kg m}}{\text{s}^2}\dfrac{\text{m}}{\text{s m}^3} \text{ or kg/m s}^3\right)$ $[\text{ML}^{-1}\text{t}^{-3}]$

$D(\text{m})$ [L]

- $(T_o - T_w) = A(D)^a (k)^b (\overline{q})^c$
- $[\text{T}] = \text{A}[\text{L}]^a [\text{MLt}^{-3}\text{T}^{-1}]^b [\text{ML}^{-1}\text{t}^{-3}]^c$
- Equating powers on both sides for

$$\begin{aligned} \text{M} &: \quad 0 = b + c \\ \text{L} &: \quad 0 = a + b - c \quad \Rightarrow \quad \boxed{a = 2} \end{aligned}$$

$$\text{T} \quad : \quad 0 \quad = \quad -3b - 3c \quad \Rightarrow \quad \boxed{c = 1}$$

$$\text{T} \quad : \quad 1 \quad = \quad -b \quad \Rightarrow \quad \boxed{b = -1}$$

- The non-dimensional relationship is

$$\boxed{T_o - T_w = A(D)^2 (k)^{-1} (\overline{q})}$$

- Constant, $A = (T_o - T_w)(D)^{-2}(k)^{+1}(\overline{q})^{-1}$

Substituting values, $A = \dfrac{(T_o - T_w)k}{D^2 \overline{q}} = \dfrac{(40\text{ K})(0.2\text{ W/m K})}{(0.1\text{ m})^2 (4000\text{ W/m}^3)} = 0.2$

Therefore, $T_o - T_w = 0.2\dfrac{D^2 \overline{q}}{k}$

or $\boxed{\boldsymbol{T_o = T_w + (0.2\, D^2 \overline{q} / k)}}$ **(Ans.)**

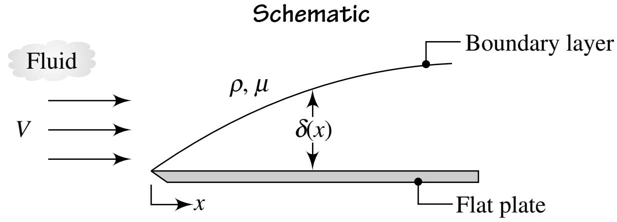
EXAMPLE 6.7 *The boundary-layer thickness, δ, on a smooth flat plate in an incompressible flow without pressure gradients depends on the free-stream velocity, V, the fluid density, ρ, the fluid viscosity, μ, and the distance from the leading edge of the plate, x. Use x, V, and ρ as repeating variables to develop a set of pi terms that could be used to describe this problem.*

Solution

Known Velocity boundary layer thickness, $\delta = fcn\ [V, \rho, \mu, x]$

Find π-groups. Non-dimensional relationship.

Schematic



Analysis The variables involved are: δ, V, ρ, μ and x $\Rightarrow$ $n = 5$
Primary (reference) dimensions are: M, L and t $\Rightarrow$ $m = 3$
Applying Buckingham–π theorem:
Dimensionless numbers,

$$p = n - m = 5 - 3 = 2 \quad \Rightarrow \quad \pi_1 = f(\pi_2)$$

The dimensions in the M Lt system of all parameters are:

S.No.	Variable	Units	Dimensions
1	δ	m	[L]
2	V	m/s	[Lt^{-1}]
3	ρ	kg/m^3	[ML^{-3}]
4	μ	kg/m s	[$ML^{-1}t^{-1}$]
5	x	m	[L]

The three repeating variables are: x, V, and ρ

Setting up dimensional equations: $\pi_1 = \delta(x)^a (V)^b (\rho)^c$

$$[\mathrm{M}^0\,\mathrm{L}^0\,\mathrm{t}^0] = [\mathrm{L}][\mathrm{L}]^a[\mathrm{Lt}^{-1}]^b[\mathrm{ML}^{-3}]^c$$

Equating indices:

For M : $0 = c \quad \Rightarrow \quad c = 0$
For L : $0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -1$
For t : $0 = -b \quad \Rightarrow \quad b = 0$

$\therefore \quad \delta(x)^{-1}\,(V)^{\circ}(\rho)^{\circ} \quad \Rightarrow \quad \boxed{\pi_1 = \dfrac{\delta}{x}}$

$$\pi_2 = \mu(x)^a (V)^b (\rho)^c$$

$$[\mathrm{M}^{\circ}\,\mathrm{L}^{\circ}\,\mathrm{t}^{\circ}] = [\mathrm{ML}^{-1}\,\mathrm{t}^{-1}][\mathrm{L}]^a[\mathrm{Lt}^{-1}]^b[\mathrm{ML}^{-3}]^c$$

Equating indices:

For M : $0 = 1 + c \quad \Rightarrow \quad c = -1$
For L : $0 = -1 + a + b - 3c \quad \Rightarrow \quad a = -1$
For t : $0 = -1 - b \quad \Rightarrow \quad b = -1$

$\therefore \quad \pi_2 = \mu(x)^{-1}(V)^{-1}(\rho)^{-1} = \dfrac{\mu}{\rho V x}$

Rearranging, the reciprocal of π is Reynolds number,

$\therefore \quad \boxed{\pi_2^* = \dfrac{\rho V x}{\mu}\ Re}$

Hence, $\boxed{\dfrac{\delta}{x} = f(\rho V x/\mu)}$ or $\boxed{\boldsymbol{\delta = x\phi(Re)}}$ **(Ans.)**

Points to Ponder

- Laminar sublayer is a region near the flat surface in a turbulent zone where velocity variation is nearly linear.
- For $Pr < 1$, the thermal boundary layer is thicker than the velocity boundary layer.
- Liquids exhibit the largest spread in Prandtl number (Pr) values with lower Pr at higher temperatures.
- The power dissipated in moving water through a pipe at a flow rate of $\dot{V}$ with a pressure drop of δP is $\delta P\,\dot{V}$.
- The friction factor for flow through a pipe of constant diameter is inversely proportional to Reynolds number.
- The friction factor for fully turbulent flow through a circular tube in the rough regime depends only on relative roughness.
- The dimensional analysis will yield correct result even if an extra variable is included by mistake.
- The boundary-layer velocity profile for turbulent flow parallel to a flat plate can be represented by $\dfrac{u}{u_\infty} = \left(\dfrac{y}{\delta}\right)^{1/7}$.

- The wall shear stress is much smaller for laminar flow than that for turbulent flow.
- The critical Reynolds number for flow in smooth pipes is 2300.
- The value of Prandtl number for gases is of the order of 0.7.
- The maximum velocity in a fully developed turbulent flow through a circular tube is 1.25 times the average velocity.
- In fully developed laminar flow in a circular pipe, the head loss due to friction is directly proportional to square of the mean velocity.

GLOSSARY of Key Terms

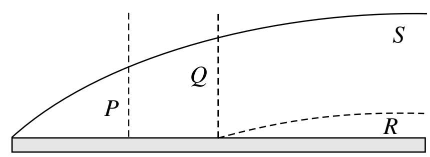
• **Boundary condition**	The condition that the flow velocity must equal the surface velocity at the surface.
• **Boundary layer**	At high Reynolds numbers, relatively thin 'boundary layers' exist in the flow adjacent to surfaces where the flow is brought to rest. Boundary layers are characterized by high shear with the highest velocities away from the surface. Frictional force, viscous stress, and vorticity are significant in boundary layers. The associated approximation based on the existence of thin boundary layers surrounded by irrotational or inviscid flow is called the boundary-layer approximation.
• **Boundary-layer thickness**	The full thickness of the viscous layer that defines the boundary layer, from the surface to the edge. The 'edge' of the boundary layer is often defined as the point where the boundary layer velocity is a large fraction of the free-stream velocity is 99 percent of the free-stream velocity.
• **Displacement thickness**	Displacement thickness (δ^*) is a measure of the thickness of the mass flow rate deficit layer. In all boundary layers, $\delta^* < \delta$.
• **Momentum thickness**	A measure of the layer of highest deficit in momentum flow rate adjacent to the surface as a result of frictional resisting force (shear stress). Momentum thickness θ is proportional to will shear stress. In all boundary layers, $\theta < \delta^*$.
• **Buckingham–Pi theorem**	A mathematical theorem used in dimensional analysis which predicts the number of dimensionless groups that must be functionally related from a set of dimensional parameters which are believed to be functionally related.
• **Buffer layer**	That part of a turbulent boundary layer, close to the wall, which lies between the viscous and inertial sublayers.
• **Friction drag**	The part of the drag on an object resulting from integrated surface shear stress in the direction of flow relative to the object.
• **Flow separation**	A phenomenon where a boundary layer adjacent to a surface is forced to leave, or 'separate' from, the surface due to 'adverse' pressure forces (i.e., increasing pressure) in the flow direction. Flow separation occurs in regions of high surface curvature, for example, circular cylinders and spheres.

- **Similarity:** The principle that allows one to quantitatively relate one flow to another when certain conditions are met. Geometric similarity, for example, must be true before one can hope for kinematic or dynamic similarity.
- **Dynamic similarity:** If two objects are geometrically and kinematically similar, then if the ratios of all forces (pressure, viscous stress, gravity force, etc.) between a point in the flow surrounding one object, and the same point scaled appropriately in the flow surrounding the other object, are all the same at all corresponding pairs of points, the flow is dynamically similar.
- **Geometric similarity:** Two objects of different size are geometrically similar if they have the same geometrical shape [i.e., if all dimensions of one are a constant multiple of the corresponding dimensions of the other).
- **Kinematic similarity:** If two objects are geometrically similar then if the ratios of all velocity components between a point in the flow surrounding one object, and the same point scaled appropriately in the flow surrounding the other object, are all the same at all corresponding pairs of points, the flow is kinematically similar.

OBJECTIVE-TYPE QUESTIONS

Multiple-Choice Questions

6.1 The development of boundary layer zones labelled P, Q, R and S over a flat plate is shown in the given figure Based on this figure, match ***List I*** (boundary-layer zones) with ***List II*** (type of boundary layer) and select the correct answer using the codes given below the lists:



List I	List II
A. P	1. Transitional
B. Q	2. Laminar viscous sublayer
C. R	3. Laminar
D. S	4. Turbulent

Codes:

	A	B	C	D
(a)	3	1	2	4
(b)	3	2	1	4
(c)	4	2	1	3
(d)	4	1	2	3

6.2 The boundary-layer thickness at a distance x from the leading edge in a laminar boundary layer is given by

(a) $5.0\ Re^{-1/2}$ (b) $4.64\ Re^{-1/2}$ (c) $5.0\,(x^{1/2}\nu^{1/2})u_\infty^{-1/2}$ (d) none of the above

6.3 Stanton number is given by

(a) $\dfrac{\text{Nusselt number}}{\text{Reynolds number} \times \text{Prandtl number}}$ (b) $\dfrac{\text{Reynolds number}}{\text{Nusselt number} \times \text{Prandtl number}}$

(c) $\dfrac{\text{Prandtl number} \times \text{Reynolds number}}{\text{Nusselt number}}$ (d) $\dfrac{\text{Prandtl number} \times \text{Nusselt number}}{\text{Reynolds number}}$

6.4 Prandtl number will be least for
(a) water (b) liquid metal (c) aqueous solution (d) engine oil

6.5 The ratio of the energy transferred by convection to that by conduction is called
(a) Stanton number (b) Nusselt number (c) Biot number (d) Peclet number

6.6 The advantages of dimensional analysis are:
(a) lays the foundation of an efficient experimental programme
(b) indicates a possible form of semi-empirical correlation
(c) suggests logical grouping of quantities for presenting the results
(d) all of the above

6.7 Choose the correct (*or most appropriate*) answer:
Friction factor for flow through a smooth pipe depends on
(a) Reynolds number only
(b) both Reynolds number and relative roughness
(c) Relative roughness only
(d) Neither on Reynolds number nor on relative roughness

6.8 For laminar flow through a long pipe, the pressure drop per unit length increases
(a) in linear proportion to the cross-sectional area
(b) in direct proportion to the diameter of the pipe
(c) in inverse proportion to the cross-sectional area
(d) in inverse proportion to the square of the cross-sectional area

6.9 A flow field which has only convective acceleration is
(a) a steady uniform flow (b) an unsteady uniform flow
(c) a steady non-uniform flow (d) an unsteady non-uniform flow

6.10 Atmospheric air at 40°C (ν = 17.2 mm^2/s) flows over a flat plate with a velocity of 20 m/s. The distance from the leading edge at which transition from laminar to turbulent flow will occur ($Re_{\text{trans}} = 5 \times 10^5$) is
(a) 0.43 m (b) 0.258 m (c) 0.05 m (d) 1.27 m

6.11 Consider the turbulent flow of a fluid through a circular pipe of diameter, D. Identify the correct pair of statements.
I. The fluid is well-mixed II. The fluid is unmixed
III. $Re_D < 2300$ IV. $Re_D > 2300$
(a) I, III (b) II, IV (c) I, II (d) I, IV

6.12 Match Group A with Group B:

Group A	Group B
P. Biot number	1. Ratio of buoyancy to viscous force
Q. Grashof number	2. Ratio of inertia force to viscous force
R. Prandtl number	3. Ratio of momentum to thermal diffusivities
S. Reynolds number	4. Ratio of internal thermal resistance to boundary-layer thermal resistance

A. P – 4, Q - 1, R – 3, S – 2 B. P – 4, Q - 3, R – 1, S – 2
C. P – 3, Q - 2, R – 1, S – 4 D. P – 2, Q - 1, R – 3, S – 4

Answers

Multiple-Choice Questions

6.1 (a)	**6.2** (c)	**6.3** (a)	**6.4** (b)	**6.5** (b)	**6.6** (d)
6.7 (a)	**6.8** (c)	**6.9** (c)	**6.10** (a)	**6.11** (d)	**6.12** (a)

REVIEW QUESTIONS

6.1 What is the difference between local and average convection heat-transfer coefficients?

6.2 What is the generally accepted value of the Reynolds number above which the flow is turbulent for (a) flow (*external*) over a flat plate, and (b) for flow (*internal*) through a smooth tube?

6.3 What is hydraulic diameter? How is it defined? What is it equal to for a circular tube of diameter D?

6.4 With the help of a diagram, explain the difference between (a) a laminar boundary layer, (b) turbulent boundary layer, and (c) Laminar sub-layer.

6.5 What do you understand by (a) critical length, (b) hydrodynamic boundary-layer thickness, and (c) thermal boundary-layer thickness for flow of a fluid along a heated flat surface?

6.6 Derive the momentum and energy equations for laminar boundary layer on a flat plate in two dimensions.

6.7 What is the physical significance of Reynolds number and Mach number? Give examples of their area of applications.

6.8 What is the primary purpose of dimensional analysis in problems involving heat-transfer by convection? What is the most serious limitation of such an analysis?

6.9 Discuss the importance of dimensional analysis and explain clearly Buckinghan's π theorem method of dimensional analysis.

6.10 Give the expressions for the following dimensionless numbers and highlight their physical significance.

(a) Prandtl number (b) Nusselt number (c) Peclet number
(d) Stanton number (e) Rayleigh number (f) Mach number

PRACTICE PROBLEMS

(A) Boundary Layer Concept. Laminar and Turbulent Flow

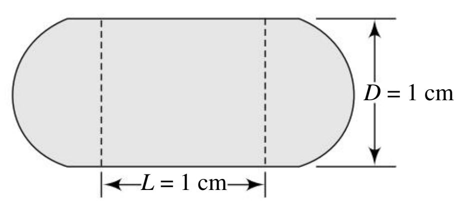
6.1 A small thermocouple probe is placed in a laminar boundary layer near a flat plate at 55°C over which the water at 35°C and 0.20 m/s flows. The probe is used to establish the temperature profile which is found to be well represented by $\theta = \dfrac{T - T_w}{T_\infty - T_w} = \dfrac{3}{2}\left(\dfrac{y}{\delta_t}\right) - \dfrac{1}{2}\left(\dfrac{y}{\delta_t}\right)^3$

At a location where the thermal boundary layer thickness δ_t is 21 mm, determine (a) the heat flux from plate to water, and (b) the heat transfer coefficient.

Properties of water at 45°C : k = 0.637 W/m °C **[108.47 W]**

(B) Internal Flow: Flow Through Closed Conduits

6.2 Water at 30°C (ρ = 995.6 kg/m^3, ν = 0.782 × 10^{-6} m^2/s) flows with a velocity of 1.5 m/s through a tube. The cross-section of the tube is shown in the figure. Determine the pressure drop per unit length. **[1933 Pa/m]**



6.3 It was found during a test in which water flowed with a velocity of 2.44 m/s through a tube (2.54-cm-ID and 6.08-m-long) that the head lost due to friction was 1.22 m of water. Estimate the surface heat transfer coefficient based on Reynolds analogy.

Take ρ = 998 kg/m^3 and C_p = 4.187 kJ/kg K **[21.4 kW/ m^2 K]**

6.4 A fluid is flowing through a 10-cm-inner diameter pipe in which velocity is uniform over the cross section of the pipe but the temperature varies linearly from 90°C at the pipewall to 0°C at the centreline. Determine the heat transfer coefficient based on the mean fluid temperature if the heat flow rate from the wall is 9280 W/m^2. **[309.33 W/m^2 °C]**

(C) Dimensional Analysis

6.5 Show by dimensional analysis that for forced convection heat-transfer, $Nu = \phi\left(Re, Pr, \frac{U^2}{C_p T}\right)$ when frictional heating in the fluid cannot be neglected. Discuss physical significance of each term.

6.6 Atmospheric air at 25°C flows through a 25-mm-ID, 2-m-long smooth circular tube. The Reynolds number is 16 000. Determine the average air velocity and the mass flow rate. The pressure drop is measured to be 130 Pa. If the fluid flowing through the tube is now changed to water at 25°C without changing the Reynolds number, estimate the pressure drop. The heat transfer coefficient with air is estimated to be 47 W/m^2°C. The Prandtl number of the fluid is believed to influence the Nusselt number by a factor $Pr^{0.36}$. Estimate the heat-transfer coefficient with water in place of air.
The following properties can be used:

Fluid	ρ(kg/m^3)	ν(m^2/s)	k(W/m °C)	Pr
Air (1 atm, 25°C)	1.184	15.62×10^{-6}	0.02551	0.7296
Saturated water (25 °C)	997.0	8.94×10^{-7}	0.607	6.14

[2405 W/m^2 °C]

6.7 The average Reynolds number of air passing in turbulent flow over a 1.5-m-long flat plate is 2.4×10^6. Under these conditions the average Nusselt number was found to be equal to 4150. Determine the average heat transfer coefficient for an oil having a thermal conductivity of 0.1316 W/m °C, a specific heat of 2.194 kJ/kg °C, and a viscosity of 13.2 kg/m h at the same Reynolds number in flow over the same plate. **[1625 W/m^2 °C]**

6.8 It is required to estimate the heat transfer from a cylinder of 50-mm-diameter and of surface temperature 140°C when kept in a cross flow of air at a velocity of 4 m/s and temperature 20°C. For this purpose, scale-model experiments are performed using a 1/5 scale-model with the same surface and air temperature but different velocities. The following results are obtained from the experiments on the model:

Velocity of air (m/s)	2.0	5.0	10.0	20.0
Heat transfer coefficient (W/m^2 K)	39.5	71.2	106.5	165.3

Calculate the heat transfer coefficient and the rate of heat transfer per metre length of the actual cylinder. **[165.3 W/m^2 K, 311.5 83 W]**

6.9 Consider a tube of 25-mm-diameter and surface temperature of 50°C losing heat by natural convection to still air at 15°C. In order to estimate the heat loss, model tests with a wire heated electrically to 270°C are to be carried out in compressed air at 15°C. The wire is to be 2.5 mm in diameter. Determine the pressure that would be required, assuming that Nu = ϕ (Ra) with properties evaluated at the film temperature. **[24.0 bar]**

7

Forced Convection: External Flow

7.1 ❑ INTRODUCTION

Heat-transfer to or from a surface in *external flow* has numerous engineering applications. In such a flow, there will always be a boundary-layer region in which velocity and temperature gradients exist and another region outside the boundary layer in which both the fluid velocity and temperature will remain constant. The growth of the boundary layer will be without restrictions. In this chapter, our main aim will be to determine the drag force and the heat-transfer rate for different flow geometries like parallel flow on a flat plate, and the flow across curved surfaces such as cylinders and spheres under forced convection conditions without phase change. Convection correlations for a wide range of flow conditions will be presented and methodology for solving practical problems will be outlined.

7.2 ❑ FLAT PLATE IN PARALLEL FLOW

The simplest case of forced convection in external flow is that of a flat plate in parallel flow. When any fluid flows near a stationary surface, a velocity boundary layer is formed. The edge of the velocity boundary layer is arbitrarily defined as the point where the local velocity, u is 99% of the free-stream velocity, u_∞. If the fluid is at a different temperature than the plate temperature, a thermal boundary layer is also formed. The edge of the thermal boundary layer is arbitrarily defined as the point where the difference between the fluid temperature and the surface temperature $(T - T_w)$ is 99% of the difference between the free-stream temperature and the surface temperature $(T_\infty - T_w)$. Heat is conducted from the surface into the fluid and then is swept downstream with the flow (*bulk transport or advection*). Heat is added to the fluid all along the plate, and the thermal boundary layer also grows in thickness with the distance along the plate. As discussed in the previous chapter, the laminar boundary layer begins at $x = 0$ (the leading edge) as shown in Fig. 7.1. The transition to turbulence occurs at a downstream location x_c at which the critical or transition Reynolds number Re_c (usually 5×10^5) is attained.

7.3 ❑ EXACT SOLUTION OF LAMINAR BOUNDARY LAYER OVER A FLAT PLATE

The analytical solution involves the solution of the *continuity, momentum and energy* equations for the boundary layer on a flat plate given below:

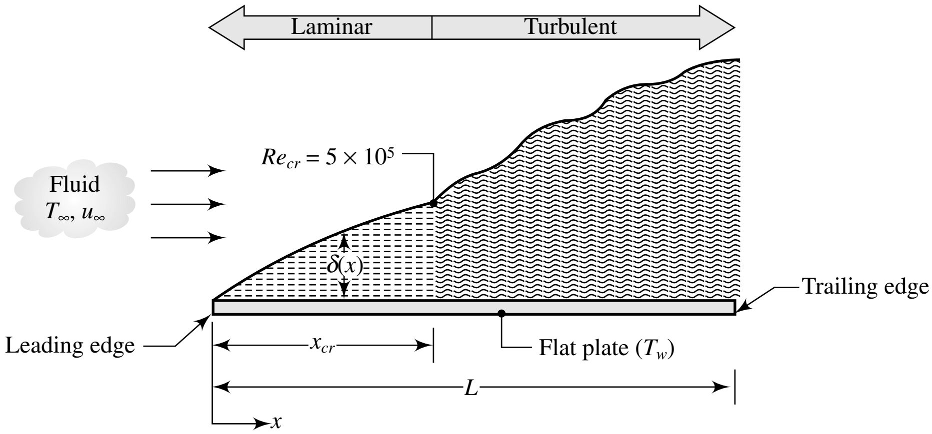


Fig. 7.1 *Laminar and turbulent regions of the boundary layer during flow over a flat plate*

$$\boxed{\begin{aligned} &\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0 && \text{(Continuity)} \\ &u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right) = \nu\left(\frac{\partial^2 u}{\partial y^2}\right) && \text{(Momentum)} \\ &u\left(\frac{\partial T}{\partial x}\right) + v\left(\frac{\partial T}{\partial y}\right) = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right) && \text{(Energy)} \end{aligned}} \tag{7.1}$$

The assumptions involved are

1. Steady, incompressible, laminar, two-dimensional flow.
2. Constant property, incompressible fluid.
3. Zero pressure gradient.
4. Negligible viscous heat dissipation.

The mathematical solution is quite complex and only the major significant convection parameters obtained. By Blasius are presented below:

Boundary-layer thickness $$\boxed{\delta \approx \frac{5x}{\sqrt{Re_x}}} \tag{7.2}$$

Thermal boundary-layer thickness, $$\boxed{\delta_T = \frac{5x}{\sqrt{Re_x}\, Pr^{1/3}}} \tag{7.3}$$

Hence, $$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}} \tag{7.4}$$

For three cases of fluids, the velocity and temperature profiles are shown in Figure 7.2. Note that for liquid metals $Pr << 1$ and $\delta << \delta_T$, for gases $Pr \approx 1$ ($\delta \approx \delta_T$) and for oils, $Pr >> 1$, i.e., $\delta >> \delta_T$.

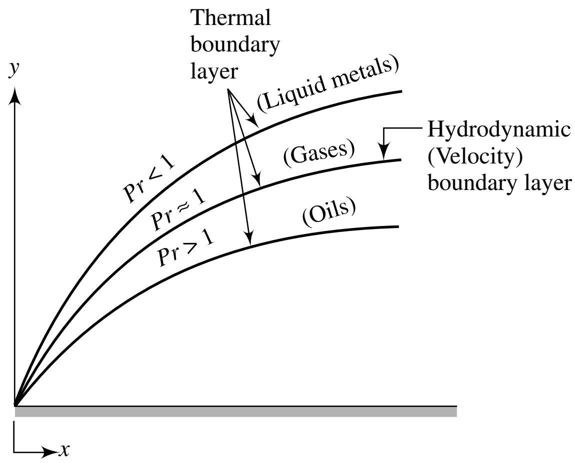


Fig. 7.2 *Hydrodynamic and thermal boundary layers for different Prandtl numbers*

Wall shear stress,

$$\boxed{\tau_w = \mu\left(\frac{du}{dy}\right)_{y=0} = 0.332\,\mu\frac{u_\infty}{x}\sqrt{Re_x}} \tag{7.5}$$

For laminar flow, the local skin friction coefficient at a distance x from the leading edge is

$$\boxed{C_{fx} = \frac{\tau_w}{(1/2)\,\rho u_\infty^2} = 0.664\,Re_x^{-1/2} \qquad Re_x \geq 10^3} \tag{7.6}$$

where the subscript x on the Reynolds number again indicates that Reynolds number is based on the length x from the leading edge. The average shear stress for a plate of length L can be obtained from the average skin-friction coefficient,

$$\boxed{\bar{C}_f = \frac{1}{L}\int_0^L C_{fx}\,dx = 1.328\,Re_L^{-1/2} \qquad 10^3 < Re_L \leq 5\times10^5} \tag{7.7}$$

where the Reynolds number is now based on the plate length, L.
Thus, for laminar flow over a flat plate, the average skin-friction coefficient is *twice* the value of the local skin friction coefficient at $x = L$.
The equation for the local Nusselt number is

$$\boxed{Nu_x = \frac{h_x x}{k} = 0.332\,Re_x^{1/2}\,Pr^{1/3} \qquad \begin{array}{l} Re_x < 5\times10^5 \\ Pr > 0.6 \\ \text{isothermal plate} \end{array}} \tag{7.8}$$

where h_x is the local heat-transfer coefficient, that is, the heat-transfer coefficient at a distance x from the leading edge of the plate. The local heat-transfer coefficient, depends on the local Reynolds number, Re_x, defined as

$$\boxed{Re_x = \frac{\rho u_\infty x}{\mu}} \tag{7.9}$$

To obtain a correlation for the average heat-transfer coefficient, we cannot simply average the Nusselt number, since it contains x. Instead, we must evaluate

$$\bar{h} = \frac{1}{L}\int_0^L h_x dx = \frac{1}{L}\int_0^L \left(\frac{k}{x}\right)(0.332)\left(\frac{u_\infty x}{\nu}\right)^{1/2} Pr^{1/3} dx \tag{7.10}$$

or

$$\bar{h} = \frac{0.332\,k(u_\infty)^{1/2}\,Pr^{1/3}}{L\nu^{1/2}}\int_0^L \frac{dx}{x^{1/2}} = \frac{0.332\,k\,Pr^{1/3}}{L}\left(\frac{u_\infty}{\nu}\right)^{1/2} \times 2[x^{1/2}]_0^L$$

or

$$= \frac{0.664\,k\,Pr^{1/3}}{L}\left(\frac{u_\infty L}{\nu}\right)^{1/2} = \frac{k}{L}0.664\,Re_L^{1/2}Pr^{1/3}$$

Average Nusselt number is

$$\boxed{\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.664\,Re_L^{1/2}Pr^{1/3} \qquad \begin{array}{l} Re_L < 5\times 10^5 \\ Pr > 0.6 \\ \text{laminar-flow} \\ \text{isothermal plate} \end{array}} \tag{7.11}$$

In the above equations, the fluid properties are evaluated at the mean temperature between the free-steam temperature and the plate-surface temperature, i.e., at the *film temperature* given by

$$\boxed{T_f = \frac{T_w + T_\infty}{2}} \tag{7.12}$$

If $x > x_c$, the boundary layer becomes *turbulent*. The *local* Nusselt number in a turbulent boundary layer is given by the following empirical correlation:

$$\boxed{Nu_x = \frac{h_x x}{k} = 0.0296\,Re_x^{4/5}Pr^{1/3} \qquad \begin{array}{l} Re_x > 5\times 10^5 \\ 0.6 < Pr < 60 \\ \text{isothermal plate} \end{array}} \tag{7.13}$$

$$\boxed{Nu_x = 0.185\,Re_x(\log_{10} Re_x)^{-2.584}\,Pr^{1/3} \qquad \begin{array}{l} Re_x > 10^7 \\ 0.6 < Pr < 60 \\ \text{isothermal plate} \end{array}} \tag{7.14}$$

Fluid properties in this equation are also evaluated at the *film temperature*.
Heat flux is related to the heat-transfer coefficient by

$$\boxed{q(x) = h_x(T_w - T_\infty)} \quad (\text{W/m}^2) \tag{7.15}$$

where $q(x)$ is the local heat flux at a location x. For the isothermal plate, heat flux is large near the leading edge, where the heat-transfer coefficient is high, and smaller downstream, where the heat-transfer coefficient is relatively lower. In most practical applications, the average heat flux over the entire plate is the quantity of interest. To determine the average heat flux, we need an average heat-transfer coefficient which can be determined from the local heat-transfer coefficient using

$$\boxed{\bar{h} = \frac{1}{L}\int_0^L h_x dx} \quad (\text{W/m}^2\text{K}) \tag{7.16}$$

where L is the total length of the plate.

and $$\overline{Nu_L} = \frac{\bar{h}L}{k} = \int_0^L \frac{1}{x} Nu_x\, dx$$

[Note that $Nu_L \neq (1/L)\int_0^L Nu_x dx$, as you might like to believe]

Three cases of practical interest are shown in Fig. 7.3. In Fig. 7.3(a), the boundary layer is laminar over the entire length of the plate. In Fig. 7.3(b), the boundary layer is laminar on the first part of the plate and turbulent on the rest of the plate. In Fig. 7.3(c), the boundary layer is essentially turbulent over the entire plate.

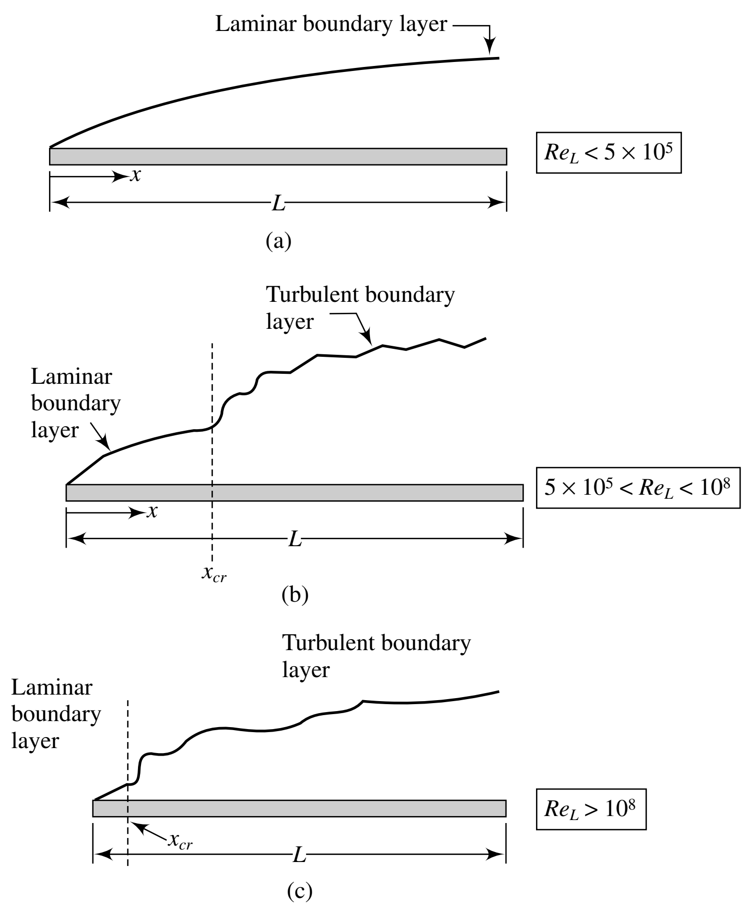


Fig. 7.3 *Three cases of boundary-layer development on an isothermal flat plate. (a) The boundary layer is laminar over the entire plate. (b) The laminar and turbulent boundary layers are of comparable size. (c) The turbulent boundary layer extends over almost the entire plate.*

In this case, one can neglect the small part of the plate covered by a laminar boundary layer and assume that the boundary layer is turbulent over the entire plate.

In many practical cases, the flow is actually turbulent starting from the leading edge. This can occur if the boundary layer is disturbed at the leading edge. For example, if the plate has a finite thickness, the corner of the leading edge can trip the boundary layer into turbulence.

- **Constant Heat Flux**

All the foregoing correlations for external flows are valid for an *isothermal surface*. Correlations are also available for a *uniform wall heat flux* (e.g., electrically heated surface) for which the local Nusselt numbers tend to be *higher* than those for an isothermal surface. For *laminar* flow along a uniformly heated flat plate, the local Nusselt number is given by

$$\boxed{Nu_x = 0.453\, Re_x^{1/2} Pr^{1/3} \quad Pr > 0.5} \tag{7.17}$$

In terms of surface heat flux and temperature difference, we can write

$$Nu_x = \frac{q_w x}{k(T_w - T_\infty)} \tag{7.18}$$

Average temperature difference along the plate for this case is obtained by performing the integration:

$$\overline{(T_w - T_\infty)} = \frac{1}{L}\int_0^L (T_w - T_\infty)\, dx$$

Substituting for $(T_w - T_\infty)$ from Eq. (7.1) and performing the integration, we get

$$\overline{(T_w - T_\infty)} = \frac{q_w \dfrac{L}{k}}{0.6795\, Re_L^{1/2} Pr^{1/3}} \quad \text{and} \quad q_w = \frac{3}{2} h_L \overline{(T_s - T_\infty)} \tag{7.19}$$

Again, for the constant heat flux case, for very wide range of Prandtl numbers, the modified correlation is

$$\boxed{Nu_x = \frac{0.4637\, Re_x^{1/2}\, Pr^{1/3}}{\left[1 + \left(\dfrac{0.02052}{Pr}\right)^{2/3}\right]^{1/4}} \qquad Re_x Pr > 100 \qquad \text{Constant heat flux}} \tag{7.20}$$

Fluid properties are still evaluated at the film temperature.

In all cases, average Nusselt number is $\boxed{\overline{Nu} = 2Nu_L}$

For *turbulent* gas flow along a uniformly heated flat plate, *Kays and Crawford* recommend for the local Nusselt number

$$\boxed{Nu_x = 0.0308\, Re_x^{4/5} Pr^{1/3} \qquad 0.5 < Pr < 400, \quad 5\times10^5 < Re_x < 5\times10^6} \tag{7.21}$$

For the *laminar* case, the local heat-transfer coefficient is *36% higher* $\left\{\left(\dfrac{0.453 - 0.332}{0.332}\right)\times 100\right\}$ on a *constant heat-flux* plate than that on an *isothermal* plate. The difference is much smaller *in turbulent* flow, amounting to just about 4%.

Equation (7.41) is *not* valid for *liquid metals* ($Pr << 1$)
The following correlation is suggested by *Kays* for *liquid metals*.

$$\boxed{Nu_x = 0.565\, Pe_x^{0.5}} \qquad ...(Pr < 0.005) \qquad (7.22)$$

where $Pe_x = Re_x\, Pr$ = Peclet number

7.4 ❑ THE INTEGRAL METHOD: LAMINAR FORCED CONVECTION ON A FLAT PLATE

Assumptions (1) Steady, laminar, two-dimensional, and incompressible flow. (2) Viscosity is uniform with temperature.

Consider a control volume *ABCD* of length dx and thickness δ at a distance x from the leading edge.

Rate at which mass flows through the face $AB = \rho \int_0^\delta u dy$

Rate at which mass flows out through the face $DC = \rho \int_o^\delta u dy + \frac{d}{dx}\left[\rho \int_o^\delta u dy\right] dx$

Hence, to satisfy the equation of continuity, the rate at which mass flows in through the face *AD*

$$= \left[\rho \frac{d}{dx} \int_0^\delta u dy\right] dx$$

Rate at which the x-direction momentum enters through the face $AB = \rho \int_0^\delta u^2 dy$.

Rate at which the x-direction momentum enters through the face *AD*

$$= u_\infty \left[\rho \frac{d}{dx} \int_o^\delta u dy\, dx\right]$$

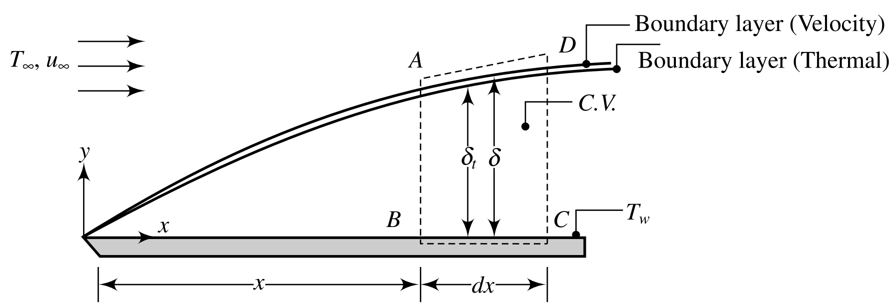


Fig. 7.4 *Local heat-transfer coefficient and total rate of heat-transfer along a flat plate*

Rate at which the x-direction momentum leaves through the face *DC*

$$= \rho \left[\int_0^\delta u^2 dy + \frac{d}{dx}\left\{\int_0^\delta u^2 dy\right\} dx\right]$$

Consider the x-direction forces exerted by the surroundings on the fluid inside the control volume (CV). Of the surface forces, we have to consider the x-direction component of the pressures on the faces AB, AD, and DC and the viscous forces on the faces BC and AD. The x-direction pressure forces on AB and AD are equal and opposite to those on DC $\left(\text{since pressure gradient } \frac{dP}{dx} = 0\right)$ and cancel each other out. The viscous force on the face AD is zero since it is at the edge of the boundary layer.

Hence, the only surface force in the x-direction exerted on the control volume by the surroundings is the viscous force at the surface of the flat plate and is given by

$$\tau_w \, dx = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} dx$$

Applying Newton's second law of motion for steady flow through the CV, we get

$$\begin{pmatrix}\text{Rate at which } x\text{-direction} \\ \text{momentum leaves the CV}\end{pmatrix} - \begin{pmatrix}\text{Rate at which } x\text{-direction} \\ \text{momentum enters the CV}\end{pmatrix} = \begin{pmatrix}\text{Net } x\text{-direction forces exerted} \\ \text{on the CV by the surroundings}\end{pmatrix}$$

$$\tau_w dx = \rho\left\{\frac{d}{dx}\int_0^\delta u^2 dy\right\}dx + \rho\int_0^\delta u^2 dy\Bigg\} - u_\infty \rho\left\{\frac{d}{dx}\int_0^\delta u dy\right\}dx - \rho\int_0^\delta u^2 dy$$

$$= \rho\left\{\frac{d}{dx}\int_0^\delta (u^2 - uu_\infty) dy\right\}dx$$

Therefore,

$$\boxed{\tau_w = \rho u_\infty^2 \frac{d}{dx}\left[\int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy\right]} \tag{7.23}$$

Equation (7.23) is known as *vön Karman integral-momentum* equation for the boundary layer.
Let the velocity distribution be a polynomial of the form:

$$u = a + by + cy^2 + dy^3$$

where a, b, c, and d are constants. The constants may be found by applying the known boundary conditions.

- At $y = 0$, $u = 0$ (*no-slip condition at the plate surface*)
- At $y = \delta$, $u = u_\infty$ (*free-stream velocity at the edge of the boundary layer*)
- At $y = 0$, both u and v are zero. Hence, $\frac{\partial^2 u}{\partial y^2} = 0$ (*from momentum equation*)
- At $y = \delta$, $\frac{\partial u}{\partial y} = 0$ (no shear stress a the edge of the boundary layer)

Substituting these conditions,

$$a = 0,\ b = \frac{3}{2}\frac{u_\infty}{\delta},\ c = 0,\ d = -\frac{u_\infty}{2\delta^3}$$

The velocity profile can then be expressed as

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Inserting this expression in the integral momentum equation derived above, we have

$$\frac{d}{dx}\left\{\rho u_\infty^2 \int_o^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy\right\} = \mu \frac{du}{dy}\bigg]_{y=0} = \frac{3}{2}\frac{u_\infty}{\delta}$$

Carrying out the integration leads to

$$\frac{d}{dx}\left(\frac{39}{280}\rho u_\infty^2 \delta\right) = \frac{3}{2}\frac{u_\infty}{\delta}$$

Since ρ and u_∞ are constants, the variables may be separated to give

$$\delta d\delta = \frac{140}{13}\frac{\nu}{u_\infty}dx \quad \text{and} \quad \delta^2 = \frac{280}{13}\frac{\nu x}{u_\infty} + C$$

$$C = 0 \quad \text{as} \quad \delta = 0 \quad \text{at} \quad x = 0 \qquad \Rightarrow \quad \delta^2 = \frac{280}{13}\frac{\nu x}{u_\infty}$$

Hence, $$\delta = 4.64\sqrt{\frac{\nu x}{u_\infty}} \quad \text{or} \quad \boxed{\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}} \tag{7.24}$$

where $Re_x = \dfrac{u_\infty x}{\nu}$ is called the *Reynolds number*.

Clearly, the boundary-layer thickness δ for parallel flow over a flat plate will further *decrease* with an increase in the free-stream velocity u_∞ (and, hence, the Reynolds number) and *increase* with an increase in the distance from the leading edge of the plate, and the kinematic viscosity of the fluid.

Let the temperature distribution be a polynomial of the form

$$T = a + by + cy^2 + dy^3$$

The constants can be evaluated by applying the following boundary conditions:

- At $y = 0$, $T = T_w$
- At $y = \delta_t$, $T = T_\infty$
- At $y = 0$, $\partial^2 T/\partial y^2 = 0$
- At $y = \delta_t$, $\left(\dfrac{\partial T}{\partial y}\right) = 0$

Substituting these conditions,

$$a = T_w,\ b = 3(T_\infty - T_w)/2\delta_t$$

$$c = 0,\ d = -(T_\infty - T_w)/2\delta_t^3$$

The temperature profits can then be expressed as

$$T = T_w + \frac{3}{2\delta_t}(T_\infty - T_w)y$$

$$-\frac{1}{2\delta_t^3}(T_\infty - T_w)y^3$$

or $$\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3$$

where δ_t is the thermal boundary-layer thickness. The integral energy equation of the boundary layer for constant properties and constant free-stream temperature T_∞ is

$$\frac{d}{dx}\left[\int_o^{\delta_t} u(T_\infty - T)dy = \alpha\frac{\partial T}{\partial y}\right]_{y=0}$$

Now, $T_\infty - T = (T_\infty - T_w) - (T - T_w)$

$$\therefore \quad \frac{T_\infty - T}{T_\infty - T_w} = 1 - \left(\frac{T - T_w}{T_\infty - T_w}\right) = 1 - \frac{3}{2}\left(\frac{y}{\delta_t}\right) + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3$$

Substituting the temperature and velocity distribution equations, we have

$$\frac{d}{dx}\left\{\int_o^{\delta_t} u_\infty\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right](T_\infty - T_w)\left[1 - \frac{3}{2}\frac{y}{\delta_t} + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]dy\right\} = \frac{3\alpha(T_\infty - T_w)}{2\delta_t}$$

Let $\eta \equiv \dfrac{\delta_t}{\delta}$

Carrying out the integration leads to

$$u_\infty\frac{d}{dx}\left[\delta\left(\frac{3}{20}\eta^2 - \frac{3}{280}\eta^4\right)\right] = \frac{3}{2}\frac{\alpha}{\delta\eta}$$

Assuming $\delta_t < \delta$ and $\eta < 1$, η^4 is too small to be neglected. Then

$$\frac{3}{20}u_\infty\frac{d}{dx}(\delta\eta^2) = \frac{3\alpha}{2\eta\delta}$$

Performing the differentiation yields

$$\frac{1}{10}u_\infty\left[2\delta\eta\frac{d\eta}{dx} + \eta^2\frac{d\delta}{dx}\right] = \frac{\alpha}{\delta\eta}$$

or $$\frac{u_\infty}{10}\left[2\delta^2\eta^2\frac{d\eta}{dx} + \eta^3\delta\frac{d\delta}{dx}\right] = \alpha$$

But $$\delta d\delta = \frac{140}{13}\frac{\nu}{u_\infty}dx \quad \text{and} \quad \delta^2 = \frac{280}{13}\frac{\nu x}{u_\infty}$$

so that, we have

$$\eta^3 + 4x\eta^2\frac{d\eta}{dx} = \frac{13}{14}\frac{\alpha}{\nu}$$

or $$\eta^3 + \frac{4}{3}x3\eta^2\frac{d\eta}{dx} = \frac{13}{14Pr} \quad \text{as} \quad \frac{\nu}{\alpha} = Pr$$ (*Prandtl number*)

or $$\eta^3 + \frac{4}{3}x\frac{d}{dx}(\eta^3) = \frac{13}{14Pr}$$

This is a linear differential equation of the first order in η^3, and the solution is

$$\eta^3 = \frac{13}{14Pr} + Cx^{-3/4}$$

With the thermal boundary layer commencing at a distance x_o from the leading edge, Fig. 7.5 boundary condition $\eta = 0$ at $x = x_0$ may be applied to yield

$$\eta^3 = \frac{13}{14Pr}\left[1 - \left(\frac{x}{x_0}\right)^{-3/4}\right] \tag{7.25}$$

For the case when it commences at the leading edge,

$$\eta = 0 \text{ (or } \delta_T = 0) \text{ at } x = 0,\ C = 0$$

$$\eta = \frac{\delta_T}{\delta} = \left(\frac{13}{14Pr}\right)^{1/3} \quad \text{or} \quad \boxed{\frac{\delta_T}{\delta} = \frac{1}{1.026} Pr^{-1/3} = \frac{0.976}{Pr^{1/3}}} \tag{7.26}$$

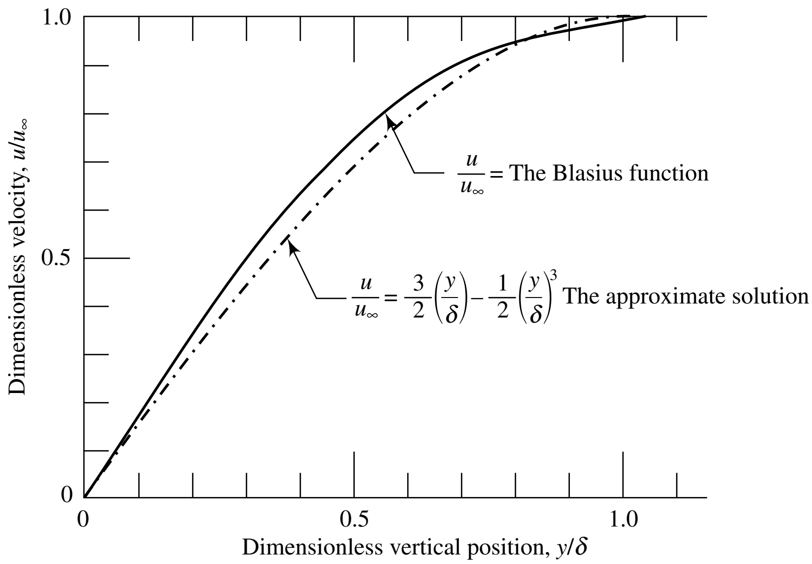


Fig. 7.5 *Comparison of third-degree polynomial fit with the exact boundary-layer velocity profile*

Note: These results are valid for laminar boundary-layer conditions only.
The approximation velocity profile is compared with the exact *Blasius* profile in Fig. 7.6 and the two prove to be equal within a maximum error of 8%.

7.4.1 • Mass Flow through the Boundary

If we consider a section at any distance x from the leading edge, the mass flow rate through that section is given by:

$$\dot{m}_x = \int [\text{Density} \times \text{Velocity} \times \text{Cross-sectional area}]$$

Integration is performed within the limits 0 to δ.

or $$\dot{m}_x = \int_0^\delta \rho u dy$$

Assuming the *cubic velocity profile* as done earlier, substituting for u, we get

$$\dot{m}_x = \int_0^\delta \rho \left[u_\infty \left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\right] dy$$

$$\dot{m}_x = \rho u_\infty \left[\frac{3}{4} \times \frac{y^2}{\delta} - \frac{1}{8} \times \frac{y^4}{\delta^3}\right]_0^\delta = \rho u_\infty \delta \left[\frac{3}{4} - \frac{1}{8}\right]$$

or $$\dot{m}_x = \frac{5}{8} \rho u_\infty \delta$$

Mass entrained between two sections at x_1 and x_2 can be calculated as:

$$\boxed{\Delta \dot{m} = \frac{5}{8}\rho u_\infty (\delta_2 - \delta_1)} \tag{7.27}$$

where δ_1 and δ_2 are the thicknesses of boundary layer at sections x_1 and x_2 respectively.

7.4.2 • Local Heat Transfer Coefficient (h_x)

We obtain h_x from the relation: $h_x = \dfrac{-k\left(\dfrac{dT}{dy}\right)_{y=0}}{T_w - T_\infty}$

7.4.3 • Unheated Starting Length

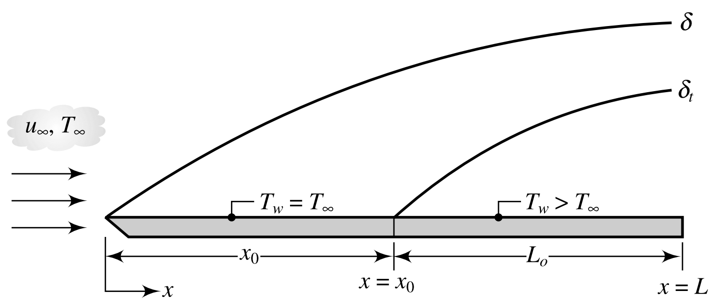


Fig. 7.6 *Boundary-layer development on a flat plate with unheated starting length*

Getting dT/dy from Eq. (7.26), and taking the values of δ and (δ_T/δ), we get for laminar flow over a flat plate

$$h_x = 0.332\frac{k}{x}Re_x^{1/2}Pr^{1/3}\frac{1}{\left[1-\left(\dfrac{x_0}{x}\right)^{3/4}\right]^{1/3}}$$

and, in terms of non-dimensional Nusselt number, we write

$$\boxed{Nu_x = \frac{h_x x}{k} = \frac{0.332\,Re_x^{1/2}Pr^{1/3}}{\left[1-\left(\dfrac{x_0}{x}\right)^{3/4}\right]^{1/3}}} \qquad (\textit{unheated starting length}) \tag{7.28}$$

If the plate is heated over the entire length, $x_0 = 0$, and we get

$$h_x = 0.332\frac{k}{x}Re_x^{1/2}Pr^{1/3}$$

and

$$\boxed{Nu_x = \frac{h_x x}{k} = 0.332\,Re_x^{1/2}Pr^{1/3}} \tag{7.29}$$

Note that Eq. (7.22) is in excellent agreement with the value obtained with *exact analysis*.

For unheated starting length (x_o), for turbulent flow over a flat plate

$$\boxed{Nu_x = \frac{Nu_x|_{x=0}}{\left[1-\left(\frac{x_o}{x}\right)^{9/10}\right]^{1/9}}} \tag{7.30}$$

7.4.4 • Shear Stress and Drag Coefficient

Wall shear stress, $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[u_\infty\left\{\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}\right] = u_\infty\left[\frac{3}{2\delta} - \frac{3}{2}\frac{y^2}{\delta^3}\right]$$

$\therefore$
$$\tau_w = \mu u_\infty\left[\frac{3}{2\delta} - \frac{3y^2}{2\delta^3}\right]_{y=0} = \frac{3}{2}\frac{\mu u_\infty}{\delta}$$

or
$$\tau_w = \frac{3\mu u_\infty}{2} \times \frac{\sqrt{\rho u_\infty}}{4.64 \times \sqrt{\mu x}}$$

$$= \frac{3\mu u_\infty}{2\times 4.64} \times \frac{1}{\sqrt{x}}\frac{\sqrt{\rho u_\infty}}{\sqrt{\mu}} \times \frac{\sqrt{\rho u_\infty}}{\sqrt{\rho u_\infty}} = 0.323\frac{\rho u_\infty^2}{\sqrt{\rho u_\infty x/\mu}}$$

Shear stress,
$$\boxed{\tau_w = \frac{0.323\rho u_\infty^2}{\sqrt{Re_x}}} \tag{7.31}$$

Thus, the shear stress is proportional to $x^{-1/2}$ in the laminar boundary layer.
Local skin-friction coefficient,

$$C_{f,x} = \frac{\tau_w}{(1/2)\rho u_\infty^2} = \frac{2\times 0.323}{\sqrt{Re_x}} = \frac{0.646}{\sqrt{Re_x}}$$

Average skin-friction coefficient,

$$\overline{C_{f,L}} = \frac{1}{L}\int_0^L C_{f,x}dx = \frac{1}{L}\times 0.646\int_0^L \sqrt{\frac{\nu}{u_\infty}}x^{-1/2}dx$$

$$= \frac{0.646}{L}\sqrt{\frac{\nu}{u_\infty}}\left[\frac{x^{1/2}}{1/2}\right]_0^L = 1.292\sqrt{\frac{\nu}{u_\infty L}}$$

Drag coefficient,

$$\boxed{C_D = \overline{C_{fL}} = \frac{1.292}{\sqrt{Re_L}}} \tag{7.32}$$

Drag force, $F_D = \tau_w(bL)$
where b is the width of the plate.
Equation (7.31) is valid for fluids with Prandtl numbers varying from 0.6 to 50, i.e., it is not applicable to *liquid metals* for which $Pr << 0.6$.

For a wide range of Prandtl numbers, *Churchill and Ozoe* have given the following correlation, for *laminar* flow on an *isothermal* flat plate:

$$\boxed{Nu_x = \frac{0.3387\,Re_x^{1/2}\,Pr^{1/3}}{\left[1+\left(\dfrac{0.0468}{Pr}\right)^{2/3}\right]^{1/4}} \qquad Re\,Pr > 100} \tag{7.33}$$

Based on the exact analysis and confirmed by experiment, the skin friction coefficient is given by

$$\boxed{C_{fx} = 0.0592\,Re_x^{-1/5}} \qquad 10^5 < Re_x < 10^7 \tag{7.34a}$$

$$\boxed{C_{fx} = 0.0296\,Re_x^{-1/7}} \qquad 10^5 < Re_x < 10^7 \tag{7.34b}$$

In these expressions, x is the distance measured from the leading edge.
For heat-transfer across the turbulent boundary layer, the local Nusselt number is given by

$$\boxed{Nu_x = \frac{(C_{fx}/2)Re_x Pr}{1+12.7(C_{fx}/2)^{1/2}(Pr^{2/3}-1)}} \tag{7.35}$$

With C_{fx} given by Eq. (7.33), this form is valid for $0.5 < Pr < 2000$, $5 \times 10^5 < Re_x < 10^7$.
Alternatively, there is a simpler power law expression recommended by *Whitaker*,

$$\boxed{Nu_x = 0.0296\,Re_x^{0.8}\,Pr^{1/3}} \tag{7.36}$$

which is valid for $0.7 < Pr < 400$, $5 \times 10^5 < Re_x < 3 \times 10^7$.

7.5 ❑ TURBULENT BOUNDARY LAYER CONDITION FOR FLOW OVER AN ISOTHERMAL FLAT PLATE

The velocity and temperature profiles for steady, turbulent boundary layer on an isothermal plate of temperature T_w are of the form $\dfrac{u}{u_\infty} = \left(\dfrac{y}{\delta}\right)^{1/7}$ and $\dfrac{T_w - T}{T_w - T_\infty} = \left(\dfrac{y}{\delta_t}\right)^{1/7}$. It is known experimentally that the wall (surface) shear stress is related to the boundary-layer thickness by an expression of the form $\tau_w = 0.0228\rho u_\infty^2 \left(\dfrac{u_\infty \delta}{\nu}\right)^{-1/4}$.

Assumptions (1) Steady, incompressible two-dimensional flow. (2) Constant properties. (3) Negligible viscous dissipation. (4) Isothermal flat plate. (5) Fully turbulent boundary layer. The momentum integral equation is

$$\rho u_\infty^2 \frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy = \tau_w$$

Substituting the expression for the shear stress, we get

$$\rho u_\infty^2 \frac{d}{dx}\int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left(1 - \left(\frac{y}{\delta}\right)^{1/7}\right) dy = 0.0228\rho u_\infty^2 \left(\frac{u_\infty \delta}{\nu}\right)^{-1/4}$$

But

$$\frac{d}{dx}\int_0^\delta \left\{\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right\} dy = \frac{d}{dx}\left[\frac{7}{8}\frac{y^{8/7}}{\delta^{1/7}} - \frac{7}{9}\frac{y^{9/7}}{\delta^{2/7}}\right]_0^\delta$$

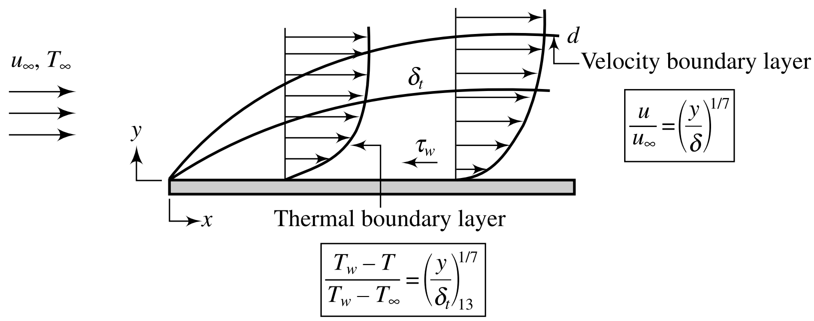


Fig. 7.7 *Isothermal flat plate*

Therefore,

$$\frac{d}{dx}\left[\frac{7}{8}\delta - \frac{7}{9}\delta\right] = 0.0228\left(\frac{u_\infty \delta}{\nu}\right)^{-1/4} \quad \text{or} \quad \frac{7}{72}\frac{d\delta}{dx} = 0.0228\left(\frac{\nu}{u_\infty}\right)^{1/4}\delta^{-1/4}$$

Separating the variables and integrating, we have

$$\frac{7}{72}\int_0^\delta \delta^{1/4}\, d\delta = 0.0228\left(\frac{\nu}{u_\infty}\right)^{1/4}\int_0^x dx \quad \text{or} \quad \frac{7}{72}\times\frac{4}{5}\delta^{5/4} = 0.0228\left(\frac{\nu}{u_\infty}\right)^{1/4} x$$

or $$\delta = \left(\frac{72\times 5}{7\times 4}\times 0.0228\right)^{4/5}\left(\frac{\nu}{u_\infty}\right)^{\frac{4}{5}\times\frac{1}{4}}\times x^{4/5} \quad \text{or} \quad \delta = 0.376\left(\frac{u_\infty}{\nu}\right)^{-1/5}\times x^{-1/5}\times x$$

or $$\frac{\delta}{x} = 0.376\left(\frac{u_\infty x}{\nu}\right)^{-1/5} = 0.376/Re_x^{1/5}$$

∴ $$\boxed{\boldsymbol{\frac{\delta}{x} = \frac{0.376}{Re_x^{1/5}}}} \tag{7.37}$$

Substituting for δ in the expression for τ_s, we obtain

$$\tau_w = 0.0228\rho u_\infty^2\left(\frac{u_\infty}{\nu}\right)^{-1/4}[0.376\, x\, Re_x^{-1/5}]^{-1/4}$$

Friction coefficient at a distance x from the leading edge ($x = 0$) is

$$C_{f,x} = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2} = 2\times 0.0228\left[0.376x\frac{u_\infty}{\nu}\frac{u_\infty^{-1/5}x^{-1/5}}{\nu^{-1/5}}\right]^{-1/4}$$

or $$C_{f,x} = 0.0456\times 0.376^{-0.25}\times\left[\frac{u_\infty^{4/5}x^{4/5}}{\nu^{4/5}}\right]^{-1/4}$$

$$= 0.0592\left(\frac{u_\infty x}{\nu}\right)^{4/5\times\frac{-1}{4}} = 0.0592\left(\frac{u_\infty x}{\nu}\right)^{-1/5}$$

Local friction coefficient,

$$\therefore \qquad \boxed{\overline{C_{f,x}} = \frac{\mathbf{0.0592}}{\boldsymbol{Re_x^{1/5}}}} \tag{7.38}$$

Average skin-friction coefficient is

$$\overline{C_{f,x}} = \frac{1}{x}\int_0^x C_{f,x}\,dx = \frac{1}{x}0.0592\left(\frac{u_\infty}{\nu}\right)^{-1/5}\int_0^x x^{-1/5}dx$$

$$= \frac{1}{x}0.0592\left(\frac{u_\infty}{\nu}\right)^{-1/5} x^{4/5}\left(\frac{5}{4}\right) = 0.074\left(\frac{u_\infty x}{\nu}\right)^{-1/5}$$

$$\therefore \qquad \boxed{\overline{C_{f,x}} = \mathbf{0.074}(\boldsymbol{Re_x})^{-1/5}} \tag{7.39}$$

From Colburn analogy,

$$\overline{St}\,Pr^{2/3} = \overline{C_{f,x}}/2 = \frac{1}{2}(0.074\,Re_x^{-1/5})$$

where $\overline{St}$ is the average Stanton number $\equiv \dfrac{\overline{Nu_x}}{Re_x Pr}$

or

$$\overline{Nu_x} = Pr^{-2/3} \times 0.037\,Re_x^{-1/5}\,Re_x Pr$$

$$= 0.037\,Re_x^{4/5}Pr^{1/3}$$

$$\boxed{\overline{Nu_x} = \mathbf{0.037}\,\boldsymbol{Re_x^{4/5}Pr^{1/3}}} \tag{7.40}$$

For gases with $\delta_1 \approx \delta$, the expressions for Nu_x and $\overline{Nu_L}$ can be developed using the integral method. The energy integral equation for turbulent flow is

$$\frac{d}{dx}\int_0^{\delta_t} u(T_\infty - T)dy = -\frac{q_w}{\rho C_p} = -\frac{h(T_w - T_\infty)}{\rho C_p}$$

Therefore,

$$u_\infty \frac{d}{dx}\int_0^{\delta_t}\frac{u}{u_\infty}\left(\frac{T - T_\infty}{T_w - T_\infty}\right)dy = \frac{h}{\rho C_p}$$

or

$$u_\infty \frac{d}{dx}\int_0^{\delta_t}\frac{u}{u_\infty}\left[1 - \left(\frac{T_w - T}{T_w - T_\infty}\right)\right]dy = \frac{h}{\rho C_p}$$

or

$$u_\infty \frac{d}{dx}\int_0^{\delta_t}\left(\frac{y}{\delta}\right)^{1/7}\left[1 - \left(\frac{y}{\delta_t}\right)^{1/7}\right]dy = \frac{h}{\rho C_p}$$

or

$$u_\infty \frac{d}{dx}\int_0^{\delta_t}\left\{\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y^2}{\delta\delta_t}\right)^{1/7}\right\}dy = \frac{h}{\rho C_p}$$

or

$$u_\infty \frac{d}{dx}\left[\frac{7}{8}\frac{y^{8/7}}{\delta^{1/7}} - \frac{7}{9}\frac{y^{9/7}}{\delta^{1/7}\delta_t^{1/7}}\right]_0^{\delta_t} = \frac{h}{\rho C_p}$$

or $$u_\infty \frac{d}{dx}\left[\frac{7}{8}\frac{\delta_t^{8/7}}{\delta^{1/7}} - \frac{7}{9}\frac{\delta_t^{9/7}}{\delta^{1/7}\,\delta_t^{1/7}}\right] = \frac{h}{\rho C_p}$$

with $$\frac{\delta_t}{\delta} \equiv \eta,$$

$$u_\infty \frac{d}{dx}\left[\frac{7}{8}\delta(\eta)^{8/7} - \frac{7}{9}\delta(\eta)^{8/7}\right] = \frac{h}{\rho C_p}$$

or $$u_\infty \frac{d}{dx}\left[\frac{7}{8} - \frac{7}{9}\right]\delta\eta^{8/7} = \frac{h}{\rho C_p}$$

or $$u_\infty \frac{d}{dx}\left(\frac{7}{72}\delta\eta^{8/7}\right) = \frac{h}{\rho C_p}$$

with $\delta_t \approx \delta$, i.e. $\eta \approx 1$, and $\delta = x(0.376)(u_\infty x/\nu)^{-1/5}$, we have

$$\frac{7}{72}u_\infty \frac{d\delta}{dx} = \frac{h}{k/\alpha} \quad \text{where } \alpha \equiv \frac{k}{\rho C_p}$$

or $$\frac{7}{72}u_\infty (0.376)\left(\frac{u_\infty}{\nu}\right)^{-1/5} \frac{d}{dx}(x^{4/5}) = \frac{h\alpha}{k}$$

or $$0.037\frac{u_\infty^{4/5}}{\nu^{-1/5}} \cdot \frac{\nu}{\nu} \cdot \frac{4}{5}x^{-1/5} \cdot \frac{x}{x} = \frac{h\alpha}{k}$$

or $$\left(0.037 \times \frac{4}{5}\right)\left(\frac{u_\infty x}{\nu}\right)^{4/5}\left(\frac{\nu}{\alpha}\right) = \frac{hx}{k}$$

Local Nusselt number,

$$\boxed{\boldsymbol{Nu_x = \frac{hx}{k} = 0.0296\, Re_x^{4/5}\, Pr}} \tag{7.41}$$

where h is the local convection heat-transfer coefficient.
Hence, the average heat-transfer coefficient is

$$\bar{h}_x = \frac{1}{x}\int_0^x h\,dx = \frac{1}{x}\int_0^x 0.0296\frac{k}{x}Pr\left(\frac{u_\infty}{\nu}\right)^{4/5} x^{4/5}\,dx$$

$$= \frac{1}{x} \times 0.0296\,k\,Pr\left(\frac{u_\infty}{\nu}\right)^{4/5}\int_0^x x^{-1/5}\,dx$$

or $$\frac{\bar{h}_x x}{k} = 0.0296\left(\frac{u_\infty}{\nu}\right)^{4/5}\frac{5}{4}x^{4/5}\,Pr$$

$\therefore$ $$\overline{Nu}_x = 0.037\left(\frac{u_\infty x}{\nu}\right)^{4/5} Pr$$

$\therefore$ $$\boxed{\boldsymbol{\overline{Nu}_x = 0.037\,(Re_x)^{4/5}\, Pr}} \tag{7.42}$$

7.6 ❑ MIXED-BOUNDARY LAYER CONDITION

• Flow Along a Flat Plate

Figure 7.7 shows the schematic of flow along a flat plate. A laminar boundary layer forms from the leading edge, and transition to turbulent flow usually occurs at a value of $Re_x = \frac{u_\infty x}{\nu}$ where u_∞ is the free-stream velocity, in the range 50 000–500 000, for x measured from the leading edge. Higher values are associated with careful wind tunnel tests, and lower values are more characteristic of practical situations where such factors as surface roughness and vibration are present.

In many cases, we are equally interested in the average values for a flat plate of finite length, L. Such average values may be found by appropriately integrating over the total plate length. However, in such instances, one must account for the laminar boundary layer which occurs on the surface between the leading edge and the point at which the laminar-turbulent transition is assumed to take place. If the presence of a transition region is ignored, laminar flow is up to the distance, x_c (*at which transition takes place*) and between x_c and the plate length L, the flow is turbulent. The distance x_c is calculated from

$$Re_c = \frac{u_\infty x_c}{\nu} \quad \Rightarrow \quad x_c = \frac{Re_c \nu}{u_\infty}$$

where Re_c is the *critical Reynolds number*.

Average Heat-Transfer Coefficient with Partly Laminar and Partly Turbulent Flow We look at the calculation of the average heat-transfer coefficient for flow past a flat plate wherein the flow is *partly laminar* and *partly turbulent*. Assuming that the transition takes place abruptly at $x = x_c$, the *average* heat-transfer coefficient may be defined as

$$\bar{h} = \frac{1}{L}\left[\int_0^{x_c} h_{\text{lam}}\, dx + \int_{x_c}^{L} h_{\text{turb}}\, dx\right]$$

The heat-transfer coefficients appearing on the right-hand side of the above equation are given by

$$h_{\text{lam}} = \frac{Nu_x k}{x} = \frac{u_\infty k}{\nu}\frac{Nu_x}{Re_x} = \frac{u_\infty k}{\nu}\frac{0.332\ Re_x^{1/2} Pr^{1/3}}{Re_x} \qquad (\textit{Blasius solution})$$

$$h_{\text{turb}} = \frac{Nu_x k}{x} = \frac{u_\infty k}{\nu}\frac{Nu_x}{Re_x} = \frac{u_\infty k 0.0296\ Re_x^{0.8} Pr^{1/3}}{\nu\, Re_x} \qquad (\textit{Colburn analogy})$$

We substitute these two expressions in and simplify to get

$$\frac{\bar{h}L}{k} = \overline{Nu}_L = 0.332\, Pr^{1/3} \int_0^{Re_{x_c}} \frac{d\,Re_x}{Re_x^{1/2}} + 0.0296\, Pr^{1/3} \int_{Re_{x_c}}^{Re_L} \frac{d\,Re_x}{Re_x^{0.2}}$$

$$\overline{Nu}_L = 0.332\, Pr^{1/3} \frac{Re_c^{1/2}}{1/2} + 0.0296\, Pr^{1/3} \frac{(Re_L^{0.8} - Re_c^{0.8})}{0.8}$$

After integration, we get the following expression for the average Nusselt number

$$\overline{Nu}_L = [0.664\, Re_c^{1/2} + 0.037\, Re_L^{0.8} - 0.037\, Re_c^{0.8}]Pr^{1/3}$$

or $$\overline{Nu}_L = Pr^{1/3}[0.037\,Re_L^{4/5} - (0.037\,Re_c^{4/5} - 0.664\,Re_c^{1/2})]$$

$$\overline{Nu}_L = [(0.037\,Re^{4/5} - B)]Pr^{1/3}$$

where $$\boxed{B \equiv 0.037\,Re_c^{4/5} - 0.664\,Re_c^{1/2}} \tag{7.43}$$

With $Re_c = 5 \times 10^5$, we get

$$B = 0.037(5 \times 10^5)^{0.8} - 0.664(5 \times 10^5)^{0.5} = 871.3$$

$$\boxed{\overline{Nu}_L = (0.037\,Re_L^{0.8} - 871.3)\,Pr^{1/3}} \tag{7.44}$$

An alternative equation recommended by *Whitaker* may give better results with some liquids due to the viscosity-ratio term:

$$\boxed{\overline{Nu}_L = 0.036\,Pr^{0.43}(Re_1^{0.8} - 9200)(\mu_\infty / \mu_w)^{0.25}} \tag{7.45}$$

$$[0.7 < Pr < 380;\, 2 \times 10^5 < Re_L < 5.5 \times 10^6;\, 0.26 < \mu_\infty / \mu_w < 3.5]$$

All fluid properties except μ_w should be at the free-stream temperature and μ_w at the wall surface temperature. For the gases, the viscosity ratio term is omitted and the properties are evaluated at the film temperature.

Average Skin-Friction Coefficient with Partly Laminar and Partly Turbulent Flow To determine the total drag, an average skin-friction coefficient is required. If transition is assumed to occur abruptly at x_c, the average shear stress on a plate of length L is

$$\bar{\tau}_w = \frac{1}{L}\left[\int_0^{x_c} \tau_w(\text{laminar})\,dx + \int_{x_c}^{L} \tau_w\,(turbulent)\,dx\right]$$

Dividing by (1/2) ρu_∞^2,

$$\overline{C}_{f,L} = \frac{1}{L}\left[\int_0^{x_c} C_{fx}(\text{laminar})\,dx + \int_{x_c}^{L} C_{fx}(turbulent)dx\right]$$

And substituting for local skin friction coefficient,

$$\overline{C}_{f,L} = \frac{1}{L}\left[\int_0^{x_c} 0.664\,Re_x^{-1/2}dx + \int_{x_c}^{L} 0.0592\,Re_x^{-1/5}\,dx\right]$$

It is convenient to integrate with respect to Re_x rather than x:

$$Re_x = \frac{u_\infty x}{\nu}; \quad d\,Re_x = \frac{u_\infty}{\nu}dx \quad \text{or} \quad dx = \frac{\nu}{u_\infty}d\,Re_x$$

$$\overline{C}_{f,L} = \frac{\nu}{u_\infty L}\left[\int_0^{Re_c} 0.664\,Re_x^{-1/2}d\,Re_x + \int_{Re_c}^{Re_L} 0.0592\,Re_x^{-1/5}\,d\,Re_x\right]$$

$$= \frac{1}{Re_L}[(2)(0.664)\,Re_c^{1/2} + (5/4)(0.0592)(Re_L^{4/5} - Re_c^{4/5})]$$

$$= \frac{1}{Re_L}\left[1.328\,Re_c^{-1/2}Re_c + 0.074\,Re_L^{4/5}\left[1 - \left(\frac{Re_c}{Re_L}\right)^{4/5}\right]\right]$$

$$\boxed{\overline{C}_{f,L} = 0.074\,Re_L^{-1/5} + Re_L^{-1}[1.328\,Re_c^{1/2} - 0.074\,Re_c^{4/5}]} \tag{7.46}$$

or
$$\boxed{\overline{C}_{f,L} = \frac{0.074}{Re_L^{1/2}} - \frac{A}{Re_L}} \tag{7.47}$$

where
$$\boxed{A = 0.074\,Re_c^{4/5} - 1.328\,Re_c^{1/2}} \tag{7.48}$$

The above equation is accurate for $Re_L < 10^7$. The average skin friction coefficient $\overline{C}_{f,L}$ is also the drag coefficient, C_D. The total viscous drag force on a plate of width b and length L is $F = C_D(1/2)\rho u_\infty^2 bL$.

The quantity A depends on the transition Reynolds number, Re_C, and is given in Table 7.1.

Table 7.1 *Values of A for different critical Reynolds number, Re_c*

Re_c	3×10^5	5×10^5	1×10^6	3×10^6
A	1055	1743	3341	8944

For higher Reynolds numbers, *Schlichting* approximates the drag coefficient with the following empirical formula

$$\boxed{C_D = 0.455(\log_{10} Re_L)^{-2.584} - A\,Re_L^{-1}} \quad (Re_C < Re_L < 10^9) \tag{7.49}$$

7.7 ❑ LIQUID-METAL HEAT-TRANSFER ACROSS A FLAT PLATE

Consider a *liquid metal* flowing across a flat plate. The Prandtl number for liquid metals is very low, of the order of 0.01. The thermal boundary-layer thickness, therefore, is very large compared to hydrodynamic boundary-layer thickness. This is due to high values of thermal conductivity of liquid metals. Since $\delta/\delta \approx P_r^{1/3}$, $\delta << \delta_T$, the velocity profile has a very blunt shape over most of the length of the plate as shown in Fig. 7.8. As a first approximation, then, we can assume a *slug flow* model and a cubic temperature profile to calculate the heat-transfer rate. It follows that

$$\boxed{u = u_\infty} \tag{7.50}$$

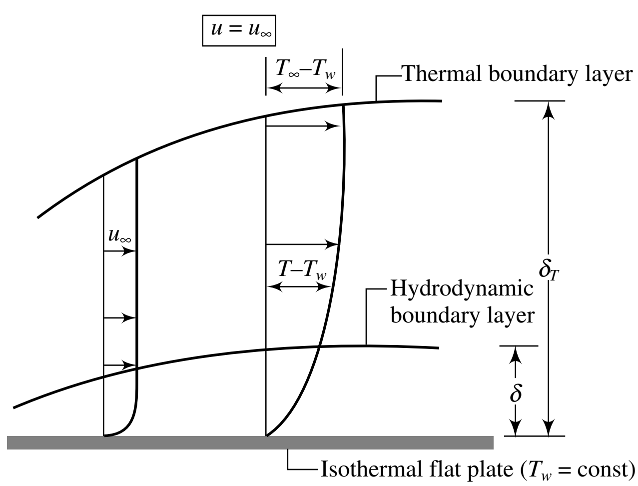


Fig. 7.8 *Boundary-layer regimes for analysis of liquid metal heat-transfer*

$$\boxed{\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\frac{y}{\delta_T} - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3} \tag{7.51}$$

Applying the integral energy equation,

$$\frac{d}{dx}\left[\int_0^{\delta_T} (T_\infty - T)u\,dy\right] = \alpha \frac{\partial T}{\partial y}\bigg]_{y=0}$$

Substituting, $u = u_\infty$

and $$T_\infty - T = (T_\infty - T_w) - (T - T_w)$$

$$= (T_\infty - T_w)\left[1 - \frac{T - T_w}{T_\infty - T_w}\right]$$

In the above equation, we have

$$(T_\infty - T_w)u_\infty \frac{d}{dx}\left\{\int_0^{\delta_T}\left[1 - \frac{3}{2}\frac{y}{\delta_T} + \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3\right]dy\right\}$$

$$= (T_\infty - T_w)\alpha \frac{\partial}{\partial y}\left[\frac{3}{2}\frac{y}{\delta_T} - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3\right]_{y=0}$$

or $$u_\infty \frac{d}{dx}\left[y - \frac{3}{2}\frac{y^2}{2\delta_T} + \frac{1}{2}\frac{y^4}{4\delta_T^3}\right]_0^{\delta_T} = \alpha\left[\frac{3}{2\delta_T} - \frac{3y^2}{2\delta_T^3}\right]_{y=0}$$

or $$\frac{d}{dx}\left[\delta_T - \frac{3}{4}\delta_T + \frac{1}{8}\delta_T\right] = \frac{\alpha}{u_\infty} \times \frac{3}{2\delta_T}$$

or $$\frac{3}{8}\frac{d\delta_T}{dx} = \frac{3}{2}\frac{\alpha}{u_\infty \delta_T}$$

or $$\delta_T d\delta_T = \frac{4\alpha}{u_\infty}dx$$

Integrating, $$\frac{\delta_T^2}{2} = \frac{4\alpha}{u_\infty}x \quad \Rightarrow \quad \delta_T = \sqrt{\frac{8\alpha x}{u_\infty}}$$

or $$\frac{\delta_T}{x} = \sqrt{\frac{8\alpha}{xu_\infty}} = 0.53\frac{k}{x}\sqrt{\frac{u_\infty x}{\nu}} \times \sqrt{\frac{\nu}{\alpha}}$$

$$h_x = \frac{-k\,(\partial T/\partial y)_{y=0}}{T_w - T_\infty} = \frac{3k}{2\delta_T} = \frac{3k}{2}\frac{\sqrt{u_\infty}}{\sqrt{8\alpha x}}$$

$$= \frac{1.5\sqrt{x}\sqrt{x}\sqrt{u_\infty}\sqrt{u}}{\sqrt{8}x\sqrt{\alpha x}} \times \frac{\sqrt{\nu}}{\sqrt{\nu}}$$

Hence, the local Nusselt number is,

$$Nu_x = \frac{h_x x}{k} = 0.53 \frac{k}{x} \sqrt{Re_x} \sqrt{Pr} \; 0.530(Re_x Pr)^{1/2}$$

or $$\boxed{Nu_x = 0.530\, Pe^{1/2}} \qquad (7.52)$$

where Pe is the Peclet number

This is in close correspondence with the exact solution

$$\boxed{Nu_x = 0.564\, Pe^{1/2}} \qquad (7.53)$$

The hydrodynamic boundary-layer thickness, with cubic velocity profile,

$$\frac{\delta}{x} = \frac{4.64}{Re_x^{1/2}} = 4.64 \sqrt{\frac{\nu}{u_\infty x}}$$

Then $$\frac{\delta}{\delta_T} = 4.64 \sqrt{\frac{\nu}{u_\infty x}} \times \sqrt{\frac{x u_\infty}{8\alpha}} = \frac{4.64}{\sqrt{8}} \sqrt{\frac{\nu}{\alpha}}$$

or $$\boxed{\frac{\delta}{\delta_T} = 1.64 \sqrt{Pr}} \qquad (7.54)$$

Using $Pr \sim 0.01$, we get

$$\frac{\delta}{\delta_T} \sim 0.16, \text{ i.e., } \delta << \delta_T$$

This is in reasonable agreement with our *slug-flow* model.

7.8 ❑ METHODOLOGY FOR CALCULATION OF CONVECTION COEFFICIENT

1. Identify the flow situation (parallel flow/cross flow) and geometry (flat plate/ cylinder/sphere/ tube bank).
2. Evaluate fluid properties at the reference temperature which is usually the film temperature (mean of surface and fluid temperatures) but in certain cases, the free-stream temperature T_∞.
3. Calculate the Reynolds number to find if the flow is laminar or turbulent. Be careful about the appropriate characteristic length.
4. Ascertain the boundary condition (constant surface temperature/constant surface heat flux).
5. Select the proper latest empirical correlation, noting the conditions of its applicability and determine the heat-transfer coefficient as desired (local or average). For heat-transfer rate, always use the average convection coefficient.

7.9 ❑ CROSSFLOW OVER A CYLINDER

A velocity boundary layer forms on the *upstream* side of the cylinder. This boundary layer separates from the cylinder at some location, and a wake forms on the *downstream* side of the cylinder. Due to velocity variations around the circumference, the convective heat-transfer coefficient also varies, as shown in Fig. 7.9. However, in most cases, only the *circumferentially averaged* heat-transfer coefficient is required.

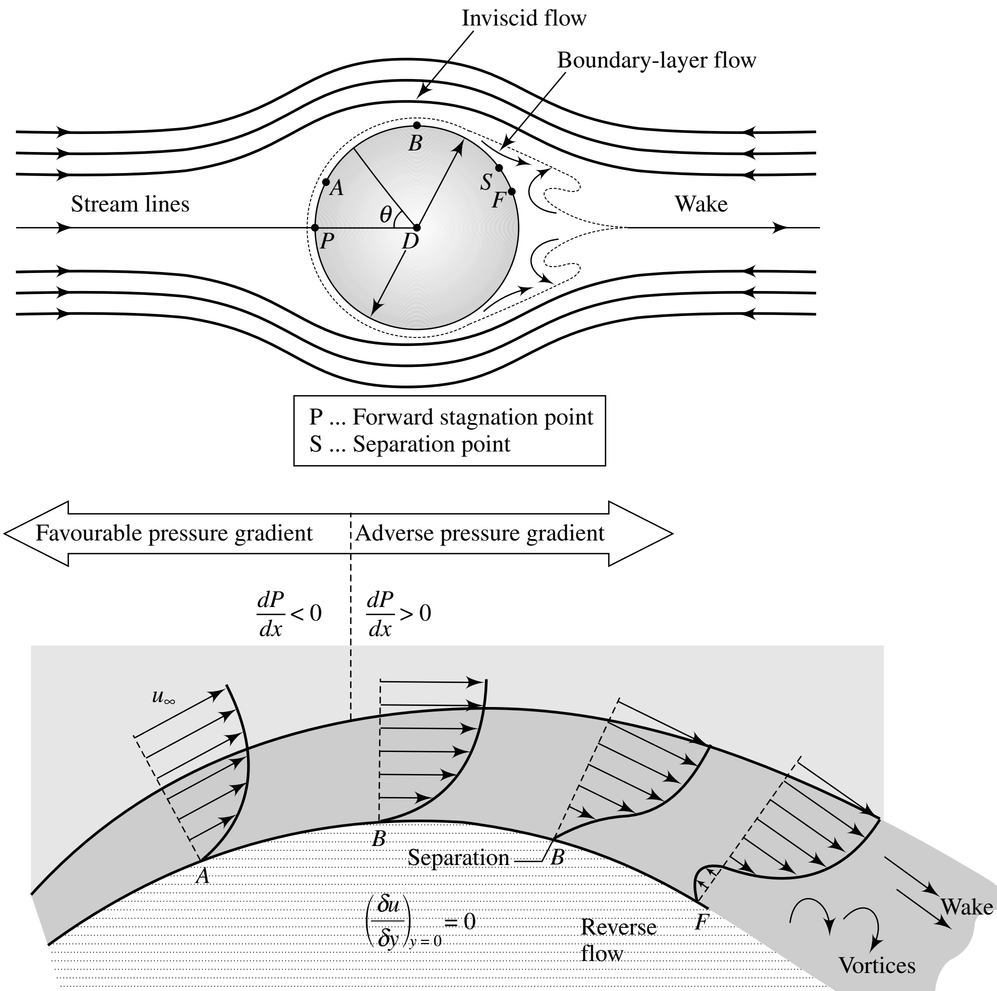


Fig. 7.9 *Flow past a circular cylinder and velocity profiles at various locations on the cylinder*

Flow over a *flat plate* with zero pressure gradient was an example of external-flow forced convection in which the growing boundary layer remained attached to the surface all along. However, in the case of curved surfaces like *cylinders* and *spheres*, the pressure gradient will result in flow separation affecting the drag force. The fluid flow across a circular cylinder is shown in Fig. 7.9.

The flow over a cylinder strongly depends on the Reynolds number, VD/ν where V is the velocity of the undisturbed flow. Different flow patterns occur at different Reynolds numbers. As a result, it is difficult to find a simple equation for the convective heat-transfer coefficient that applies to all ranges of Reynolds number.

The thickness of the boundary increases as the flow proceeds along the body and velocity gradient near the wake keeps decreasing. At a certain location, the velocity gradient actually becomes zero and flow separation takes place. Beyond the separation point, the fluid separates with a reversed flow region. The pressure variation departs from the ideal-flow pressure variation as shown by the dashed line in Fig. 7.9. Pressure variation is very little beyond the separation point. Turbulent flow normal to a cylinder also shows flow separation as indicated in the figure.

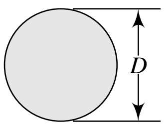
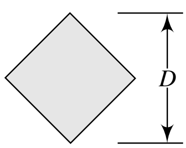
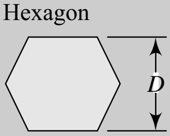
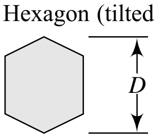
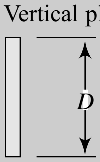
Determination of the drag coefficient and the heat-transfer coefficient is quite complicated due to complexity of flow patterns around the cylinder.

For cross flow over a cylinder, the average Nusselt number is correlated by an equation of the form

$$\boxed{Nu_D = \frac{hD}{k} = C\, Re_D^m\, Pr^{1/3}} \tag{7.55}$$

Table 7.2 gives correlations for *circular* and *non-circular* cylinders in cross flow.

Table 7.2 *Empirical correlations for the average Nusselt number for forced convection over circular and non-circular cylinders in cross flow*

Geometry	Fluid	Range of Reynolds Number	Nusselt number
Circle 	Gas or liquid	0.4–4 4–40 40–4000 4000–40 000 40 000–400 000	$Nu = 0.989\, Re^{0.330}\, Pr^{1/3}$ $Nu = 0.911\, Re^{0.385}\, Pr^{1/3}$ $Nu = 0.683\, Re^{0.466}\, Pr^{1/3}$ $Nu = 0.193\, Re^{0.618}\, Pr^{1/3}$ $Nu = 0.027\, Re^{0.805}\, Pr^{1/3}$
Square 	Gas	2500–8000 5000–100 000	$Nu = 0.177\, Re^{0.699}\, Pr^{1/3}$ $Nu = 0.102\, Re^{0.675}\, Pr^{1/3}$
Square (tilted 45°) 	Gas	2500–7500 5000–100 000	$Nu = 0.289\, Re^{0.624}\, Pr^{1/3}$ $Nu = 0.246\, Re^{0.588}\, Pr^{1/3}$
Hexagon 	Gas	5000–100 000	$Nu = 0.153\, Re^{0.638}\, Pr^{1/3}$
Hexagon (tilted 45°) 	Gas	5000–19 500 19 500–100 000	$Nu = 0.160\, Re^{0.638}\, Pr^{1/3}$ $Nu = 0.0385\, Re^{0.782}\, Pr^{1/3}$
Vertical plate 	Gas	4000–15 000	$Nu = 0.228\, Re^{0.731}\, Pr^{1/3}$

The Reynolds number and the Nusselt number are usually based on the cylinder diameter. The average Nusselt number is correlated by an equation of the form

$$\boxed{Nu_D = C\,Re_D^m\,Pr^n \left(\frac{Pr_\infty}{Pr_w}\right)^{1/4}} \tag{7.56}$$

Table 7.3 *Constants C and n*

Re_D **range**	C	m
1–40	0.75	0.4
40–1000	0.51	0.5
10^3–2×10^5	0.26	0.6
$2 \times 10^5 - 10^6$	0.076	0.7

Recently, *Zhukauskas* has given the set of C, m, and n that are to be used in and the range of validity of these. Table 7.3 gives the values of the constants along with the ranges of applicability.

The above correlation is valid for $0.7 < Pr < 500$, $1 < Re_D < 10^6$. n is specified as 0.37 for $Pr \le 10$ and as 0.36 for $Pr \ge 10$. All properties are calculated at the mean temperature $\frac{(T_w + T_\infty)}{2}$. Pr_∞ and Pr_w are evaluated at free-stream temperature T_∞ and wall temperature T_w respectively.

The characteristic length, for a circular cylinder to calculate the Reynolds number is the *external* diameter D. And the Reynolds number is defined as

$$\boxed{Re_D = \frac{VD}{\nu}} \tag{7.57}$$

where ν is the uniform velocity of flow as it approaches the cylinder.
The critical Reynolds number for flow across the cylinder is

$$\boxed{Re_{cr} = 2 \times 10^5} \tag{7.58}$$

Up to $Re_{cr} = 2 \times 10^5$, the boundary layer remains laminar and beyond this value, the boundary layer becomes turbulent.

Flow patterns for a flow across a cylinder are shown in Fig. 7.10. Fluid particles at the mid-plane of a stream approaching the cylinder strike the cylinder at the *stagnation point* and come to a halt, thus increasing the pressure. The rest of the fluid branches around the cylinder forming a boundary layer that embraces the cylinder walls. Pressure decreases in the flow direction and the velocity increases. At very low free stream velocities ($Re < 4$), the fluid completely wraps around the cylinder. As the velocity increases, the boundary layer detaches from the surface at the rear, forming a wake behind the cylinder. This point is called *separation point*. Flow separation occurs at about $\theta = 80°$ measured from the stagnation point when the boundary layer is *laminar* and at about $\theta = 140°$ when the boundary layer is *turbulent*.

Drag force for a cylinder in cross flow is primarily due to two effects: one, *friction drag* due to the shear stress at the surface, and the other, *pressure drag* due to the pressure difference between the stagnation point and the wake. At low Reynolds number ($Re_D < 4$), friction drag is predominant, and at high Reynolds numbers ($Re_D > 5000$), pressure drag is significant. At the intermediate values of Re_D, both effects contribute to the drag.

Average drag coefficient C_D for cross flow over a *cylinder* is shown in Fig. 7.12. Then, the drag force acting on the body in cross flow is obtained from

$$\boxed{F_D = \frac{C_D \rho A u_\infty^2}{2}} \quad \text{(N)} \tag{7.59}$$

where A is the *frontal area*, i.e., area normal to the direction of flow.

[A = LD for a *cylinder* and $A = \frac{\pi D^2}{4}$ for a *sphere*]

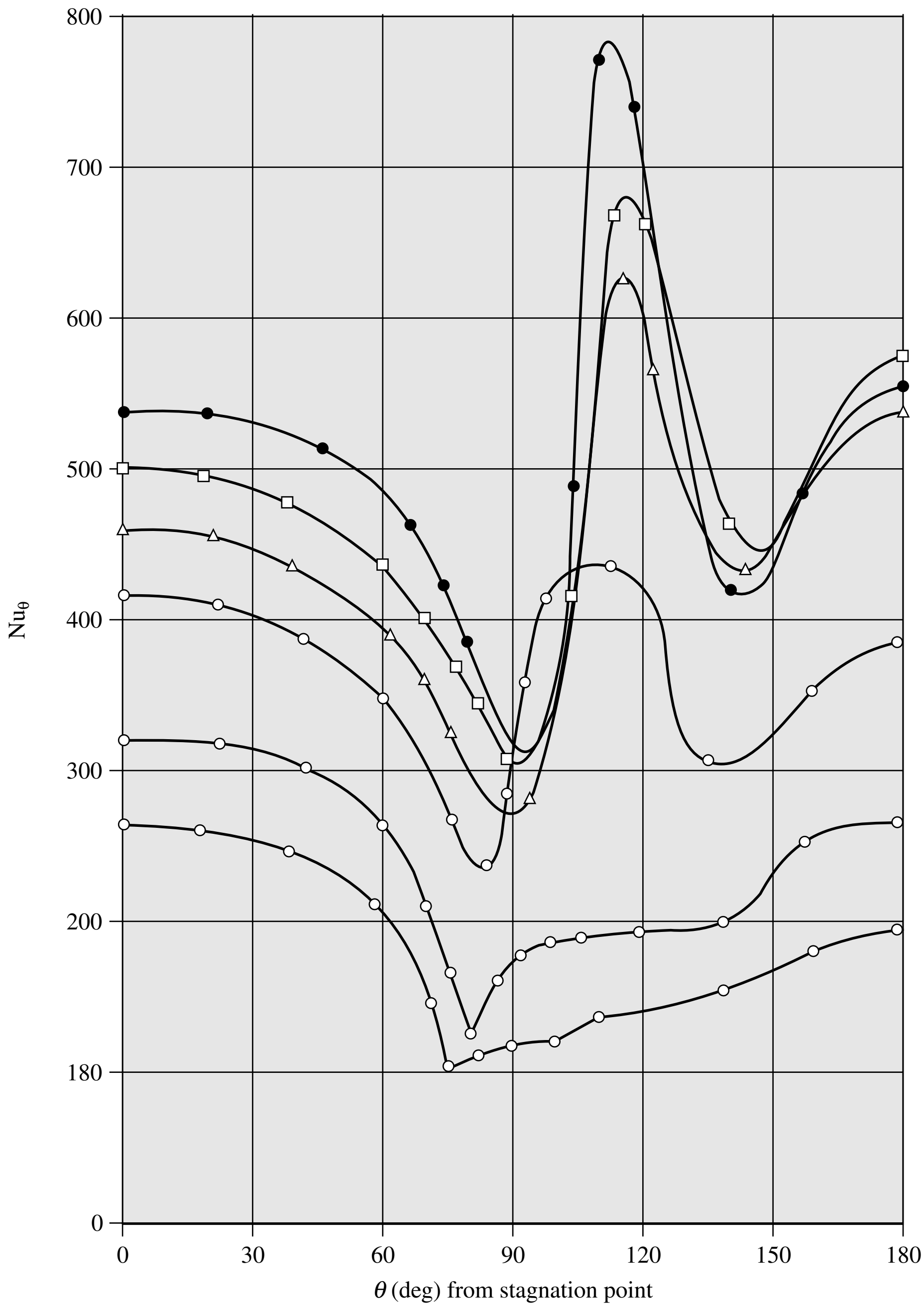


Fig. 7.10 *Local Nusselt number for a circular cylinder in cross flow of air*

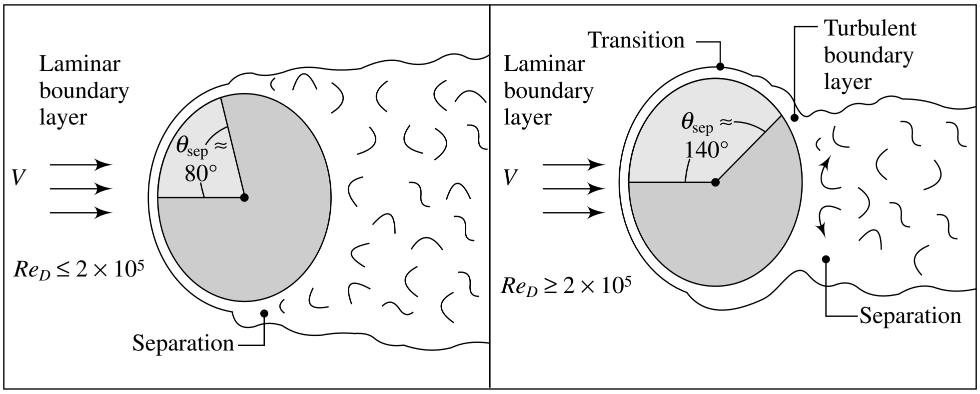


Fig. 7.11 *The effect of turbulence on separation in cross flow over a circular cylinder*

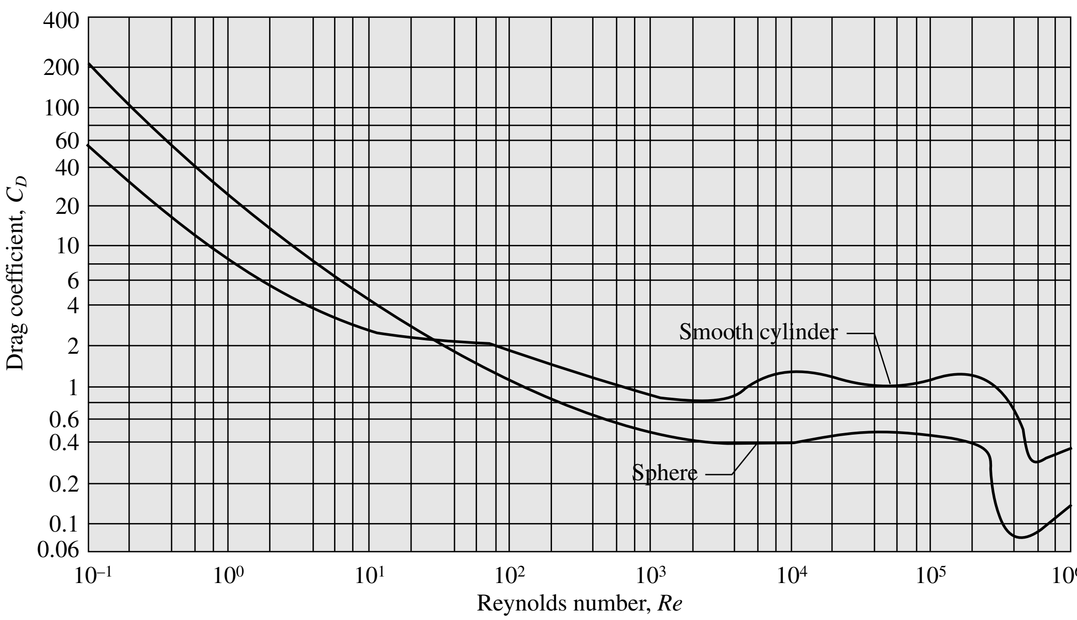


Fig. 7.12 *Drage coefficient for a smooth circular cylinder and for a sphere in cross flow as a function of the Reynolds number*

Another important correlation due to *Whitaker* applicable for *gases* and *liquids* is

$$\boxed{\overline{Nu}_D = (0.4\, Re_D^{0.5} + 0.06\, Re_D^{0.66})\, Pr^{0.4} (\mu_\infty/\mu_w)^{0.25}} \tag{7.60}$$

$$\boxed{\begin{array}{l} 10 < Re_D < 100\,000 \\ 0.67 < Pr < 300 \\ 0.25 < (\mu_\infty/\mu_w) < 5.2 \end{array}} \tag{7.61}$$

All the properties are evaluated at the *free stream temperature* T_∞ except μ_w which is at the surface temperature T_w.

Fand has suggested the following correlation for calculating the heat-transfer coefficient from a liquid to the cylinder in cross flow

$$\boxed{Nu_D = (0.35 + 0.56\,Re_D^{0.52})Pr^{0.3}} \quad (\text{valid for } 10^{-1} < Re_D < 10^5) \tag{7.62}$$

For the *average* Nusselt number for flow across a circular cylinder, the following comprehensive correlations are suggested by *Churchill and Bernstein*.

$$\boxed{\overline{Nu}_D = 0.3 + \frac{0.62\,Re_D^{1/2}\,Pr^{1/3}}{[1+(0.4/Pr)^{2/3}]^{1/4}}\left[1+\left(\frac{Re_D}{282\,000}\right)^{1/2}\right] \quad Re_D Pr > 0.2, \quad 2\times10^4 < Re_D < 4\times10^5} \tag{7.63}$$

$$\boxed{\overline{Nu}_D = 0.3 + \frac{0.62\,Re_D^{1/2}\,Pr^{1/3}}{[1+(0.4/Pr)^{2/3}]^{1/4}}\left[1+\left(\frac{Re_D}{282\,000}\right)^{5/8}\right]^{4/5} \quad Re_D Pr > 0.2, \quad 4\times10^5 < Re_D < 5\times10^6} \tag{7.64}$$

These correlations are valid for the *constant wall temperature case*. Fluid properties are to be evaluated at the mean film temperature $\frac{1}{2}(T_w + T_\infty)$.

7.10 ❑ FLOW ACROSS TUBE BANKS

Cross flow over tube banks and the heat-transfer associated with it occur in numerous heat-transfer equipment such as the *condensers* and *evaporators* of power plants, *refrigerators*, and *air conditioners*. In such applications, one fluid moves through the tubes while the other moves over the tubes in a perpendicular direction.

The two frequently used geometrical arrangements *in-line* (*aligned*) or *staggered* tube in the direction of flow, velocity V are shown in Fig. 7.13. The outer tube diameter D is taken as the characteristic length. The configuration in the tube bank is characterized by the *transverse pitch* S_T, between tubes in a row (the rows are perpendicular to the flow direction) and *longitudinal pitch* S_L (*in the flow direction between adjacent rows of tubes*). The diagonal pitch S_D between tube centres in the diagonal row is used in the staggered arrangement.

As the fluid enters the tube bank, the flow area decreases from $A_1 = S_T L$ to $A_T = (S_T - D)L$ between the tubes, and thus flow velocity increases. In staggered arrangement, the velocity may increase further in the diagonal region if the tube rows are very close to each other. In tube banks, the flow characteristics are dominated by the maximum average in the tube bank velocity V_{max} in the tube bank rather than the approach velocity V. Therefore, the Reynolds number for heat-transfer and pressure drop calculations is then given by

$$\boxed{Re_D = \frac{\rho V_{max} D}{\mu} = \frac{V_{max} D}{\nu}} \tag{7.65}$$

The maximum velocity is determined from the conservation of mass requirement for steady *incompressible* flow.

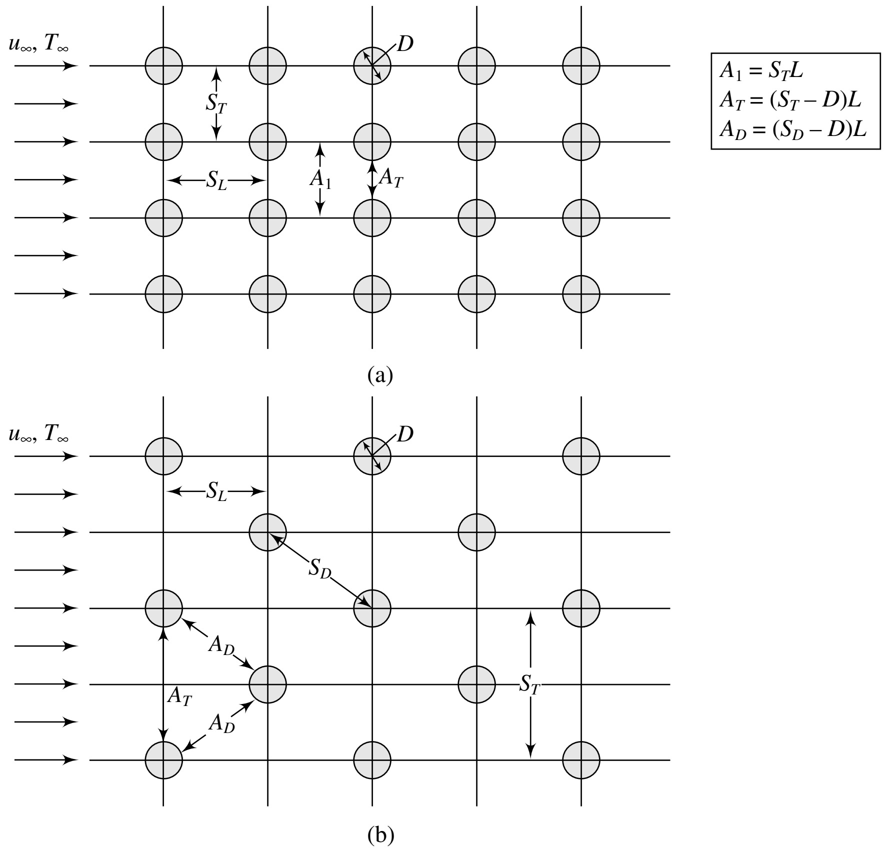


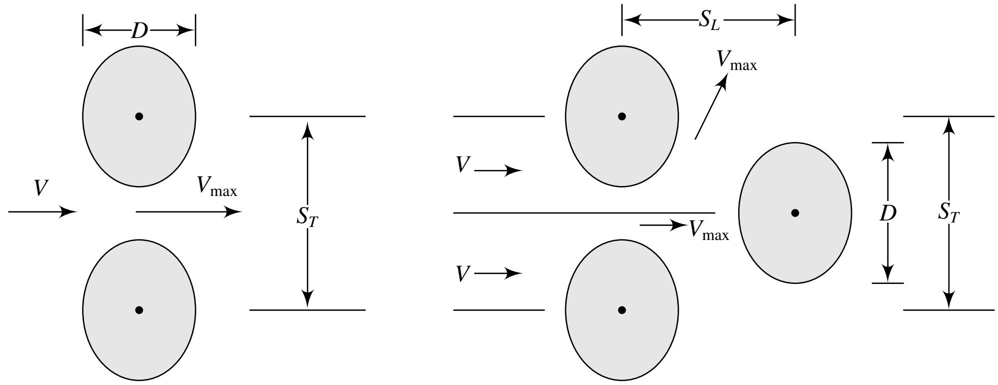
Fig. 7.13 *Tube banks: (a) aligned (b) staggered*

For the aligned (*in-line*) *arrangement*, the *maximum velocity* occurs at the minimum flow area between the tubes, and from the conservation of mass $\boxed{\rho V A_1 = \rho V_{\max} A_T}$ or $VS_T = V_{\max}(S_T - D)$. Then the maximum velocity $V_{\max}$ occurs at the transverse plane.

$$\boxed{V_{\max} = \frac{S_T}{S_T - D} V} \tag{7.66}$$

For the *staggered* configuration, the fluid passes through the area A_T and then through the area $2A_D$. The maximum velocity can occur either at the transverse plane A_T or the diagonal plane A_D of Fig. 7.14. If $2A_D > A_T$, the maximum velocity will occur at A_T between the tubes and thus the $V_{\max}$ relation for in-line comfiguration can also be used for staggered tube banks. But if $2A_D < A_T$ i.e. if $2(S_D - D) < (S_T - D)]$ the maximum velocity will occur at the diagonal plane, and the maximum velocity in this case becomes Staggered and $S_D < (S_T + D)/2$:

$$\boxed{V_{\max} = \frac{S_T}{2(S_D - D)} V} \tag{7.67}$$



$$V_{max} = \left(\frac{ST}{S_T - D}\right)V$$

(a)

For $S_T - D < 2\sqrt{(S_T/2)^2 + S_L^2} - 2D$

$$V_{max} = [S_T/(S_T - D)]V$$

For $S_T - D > 2\sqrt{(S_T)^2 + S_L^2} - 2D$

$$V_{max} = \left(\frac{\sqrt{(1 + 4(S_L/S_T)^2}}{2D}\right)S_T$$

(b)

Fig. 7.14 *Visualization of maximum velocity location in (a) in-line, and (b) staggered tube bank arrangements*

where $$S_D = \left[S_L^2 + \left(\frac{S_T}{2}\right)^2\right]^{1/2} < \frac{S_T + D}{2} \tag{7.68}$$

since $\rho V A_1 = \rho V_{max}(2A_D)$ or $VS_T = 2V_{max}(S_D - D)$

Note that the factor 2 is because the fluid is divided into two streams as it moves from A_T to A_D planes. *Zhukauskas* has recently proposed a correlation of the general form

$$\boxed{Nu_D = \frac{hD}{k} = C\, Re_D^m\, Pr^n\, (Pr/Pr_w)^{0.25}} \tag{7.69}$$

where the values of the constants C, m, and n depend on the Reynolds number. Such correlations are presented given in Table 7.3 for $0.7 < Pr < 500$ and $0 < Re_D < 2 \times 10^6$. All properties except Pr_w are to be evaluated at the arithmetic mean of the fluid inlet temperature and exit temperature.

The average Nusselt number relations in Table 7.4 are for tube banks with 16 or more rows. For $N_L < 16$, a correction factor F is applied such that

$$\boxed{Nu_{D,N_L} = FNu_D} \tag{7.70}$$

Table 7.4 *Correcton factor F to be used in $Nu_{D,N_L} = FNu_D$ for $N_L < 16$ and $Re_D > 1000$*

N_L	1	2	3	4	5	7	10	13
In-line	0.70	0.80	0.86	0.90	0.93	0.96	0.98	0.99
Staggered	0.64	0.76	0.84	0.89	0.93	0.96	0.98	0.99

The values of F are given in Table 7.5. The correction factor itself varies with the tube arrangement and the Reynolds number. If $Re_D > 1000$, the correction factor is independent of Reynolds number.

Table 7.5 *Nusselt number correlations for cross flow over tube banks for N > 16 and 0.7 < Pr < 500*

Configuration	Range of Re_D	Correlation
In-line	0–100	$Nu_D = 0.9\,Re_D^{0.4}Pr^{0.36}(Pr/Pr_w)^{0.25}$
	100–1000	$Nu_D = 0.52\,Re_D^{0.5}Pr^{0.36}(Pr/Pr_w)^{0.25}$
	1000–2 × 10⁵	$Nu_D = 0.27\,Re_D^{0.63}Pr^{0.36}(Pr/Pr_w)^{0.25}$
	2 × 10⁵ – 2 × 10⁶	$Nu_D = 0.033\,Re_D^{0.8}Pr^{0.4}(Pr/Pr_w)^{0.25}$
Staggered	0–500	$Nu_D = 1.04\,Re_D^{0.4}Pr^{0.36}(Pr/Pr_w)^{0.25}$
	500–1000	$Nu_D = 0.71\,Re_D^{0.5}Pr^{0.36}(Pr/Pr_w)^{0.25}$
	1000–2 × 10⁵	$Nu_D = 0.35(S_T/S_L)^{0.2}Re_D^{0.6}Pr^{0.36}(Pr/Pr_w)^{0.25}$
	2 × 10⁵ – 2 × 10⁶	$Nu_D = 0.031(S_T/S_L)^{0.2}Re_D^{0.8}Pr^{0.36}(Pr/Pr_w)^{0.25}$

Once the Nusselt number and thus the average heat-transfer coefficient for the entire tube bank is known, the heat-transfer rate can be determined from Newton's law of cooling using the logarithmic mean temperature difference ΔT_{lm} defined as

$$\boxed{\Delta T_{lm} = \frac{(T_w - T_i) - (T_w - T_e)}{\ln[(T_w - T_i)/(T_w - T_e)]} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}} \tag{7.71}$$

We also show that the exit temperature of the flow fluid T_e can be determined from

$$\boxed{T_e = T_w - (T_w - T_i)\exp\left(-\frac{hA_s}{\dot{m}C_p}\right)} \tag{7.72}$$

where $\boxed{A_s = N\pi DL}$ is the heat-transfer surface area and $\boxed{\dot{m} = \rho V(N_T S_T L)}$ is the mass flow rate of the fluid. Here, N is the total number of tubes in the bank, N_T is the number of tubes in a transverse plane, L is the length of the tubes, and V is the velocity of the fluid just before entering the tube bank. Then the heat-transfer rate can be determined from

$$\boxed{\dot{Q} = hA_s\Delta T_{lm} = \dot{m}C_p(T_e - T_i)} \tag{7.73}$$

All properties except Pr_w are evaluated at the arithmetic mean of the inlet and exit temperatures of the fluid. Pr_w is evaluated at T_w.

7.10.1 • Pressure Drop

The pressure drop ΔP for flow across a banks of N_L tubes can be calculated

$$\boxed{\Delta P = N_L f X \frac{\rho V_{\max}^2}{2}} \quad \text{(Pa)} \tag{7.74}$$

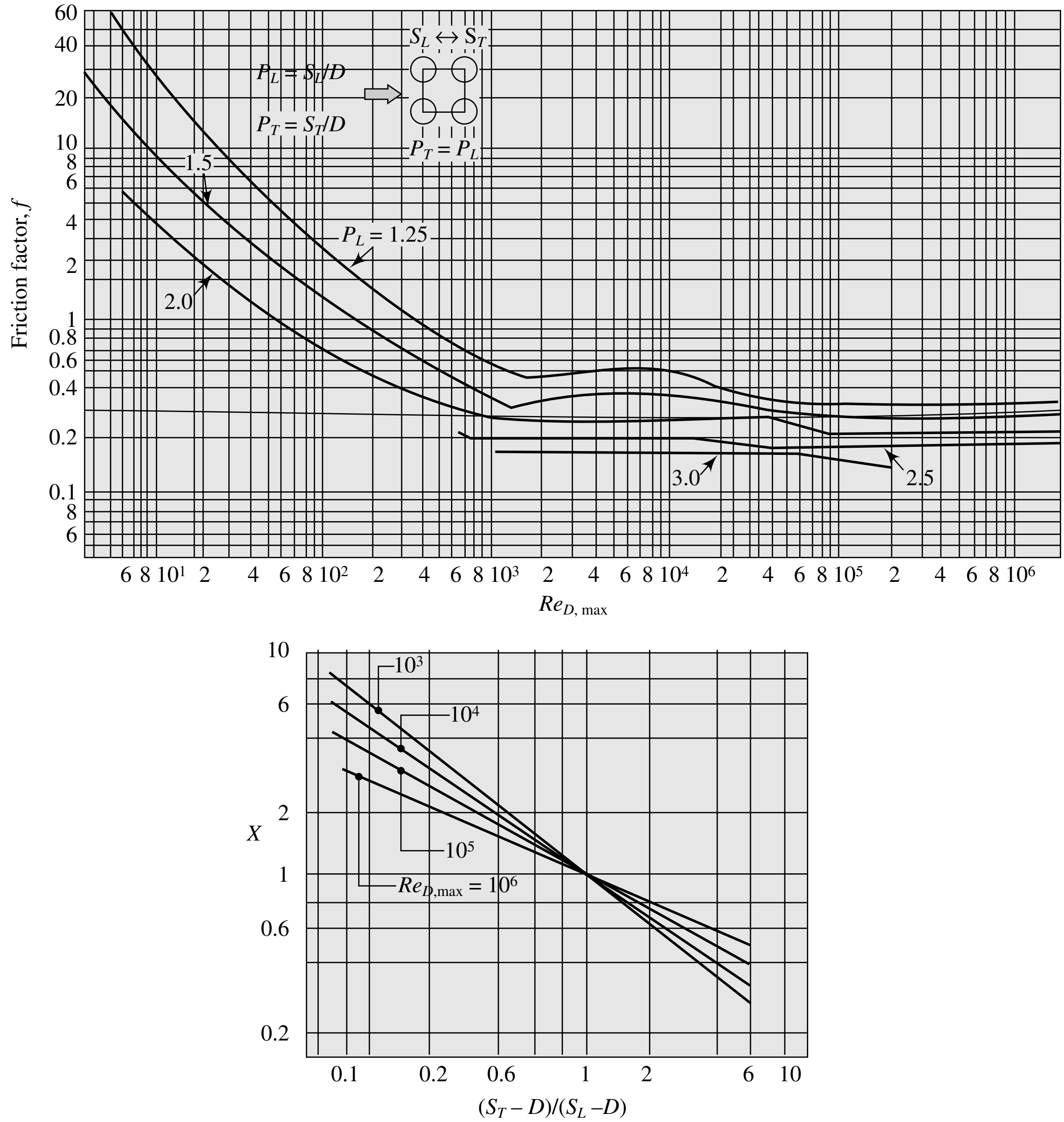


Fig. 7.15 *(a) Friction factor, f and (b) Correction factor X for calculating pressure drop in an in-line tube-ank configuration*

where f is the friction factor and X is the correction factor, both plotted in Fig. 7.15(a) and 7.15(b) against the Reynolds number based on the maximum fluid velocity $V_{\max}$. The friction factor in Fig. 7.16(a) is for a *square*, *in-line tube bank* ($S_T = S_L$), and the correction factor is used to account for the effects of deviation of rectangular in-line arrangements from the square arrangement. Similarly, the friction factor in Fig. 7.16(a) is applicable for an *equilateral staggered tube bank* ($S_T = S_D$), and the correction factor Fig. 7.16(b) is to account for the effects of departure from equilateral arrangement. Note that $X = 1$ for both *square* and *equilateral* triangle arrangements. Also, pressure drop occurs in the flow direction, and thus we used N_L (*the number of rows*) in the ΔP relation.

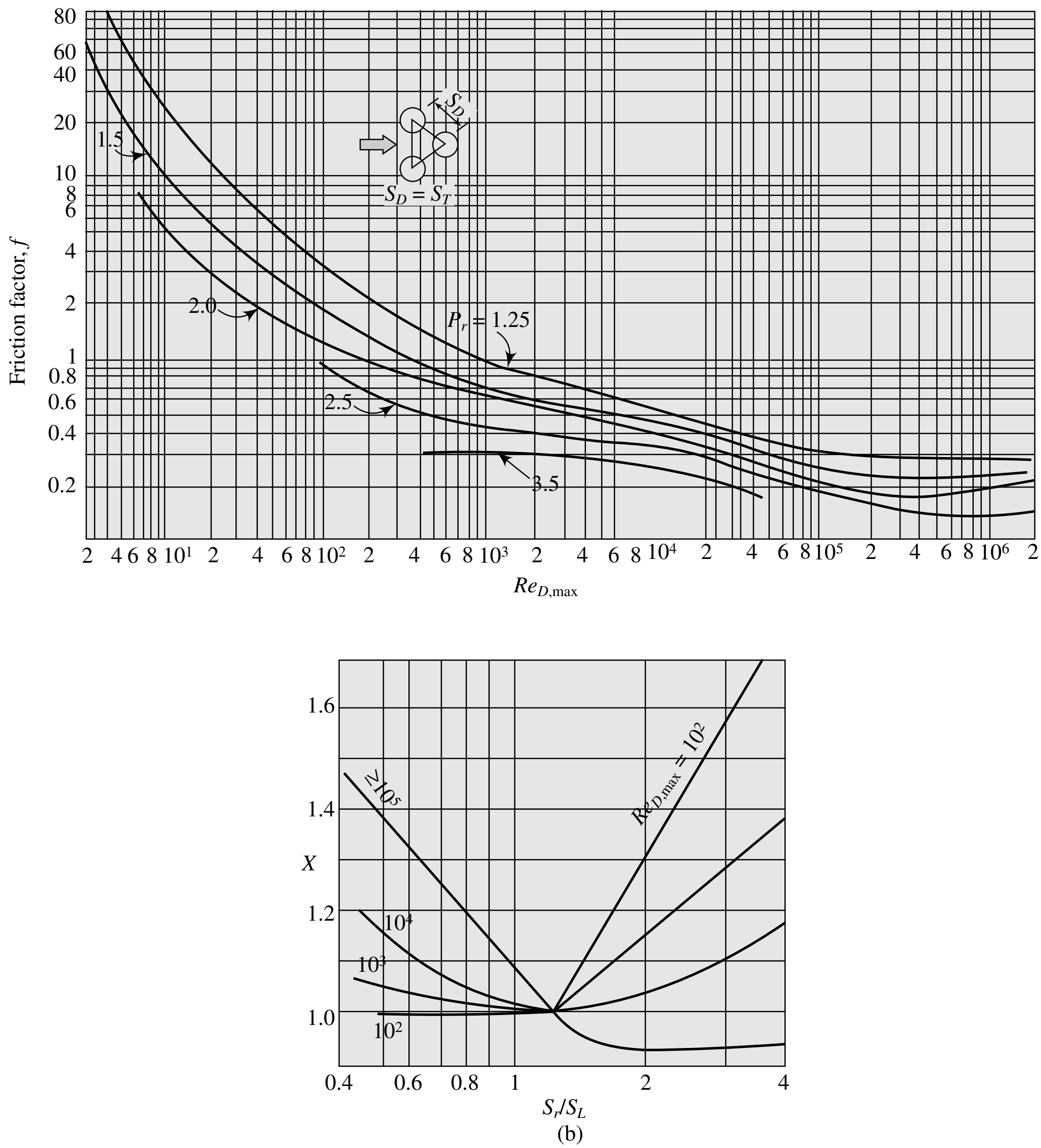


Fig. 7.16 *(a) Friction factor, f and (b) Correction factor X for calculating pressure drop in a staggered tube-bank configuration*

The pumping power requirement can be obtained from

$$\boxed{\wp_{\text{pump}} = (\Delta P)\dot{\forall} = (\dot{m}/\rho)\Delta P} \quad \text{(W)} \tag{7.75}$$

where $\dot{m} = \rho\dot{\forall} = \rho\forall(N_T S_T L)$ is the *mass flow rate* of the fluid through the bank of tubes. The *power required* is proportional to the pressure drop. Hence, the benefits of increasing heat-transfer in a tube bank should be compared against the cost of additional power requirement.

7.11 ❑ FLOW OVER A SPHERE

The flow pattern around a sphere is somewhat similar to that for flow normal to a circular cylinder except that it does not exhibit the same regular eddy shedding phenomena. Fig. 7.17 shows the variation of drag coefficient C_D as a function of Reynolds number Re_D. For very low Reynolds numbers (*creeping flow*), the Stokes law is valid and C_D is inversely proportional to Re.
It is expressed as

$$\boxed{C_D = \frac{24}{Re_D} \quad Re_D < 0.5} \tag{7.76}$$

For higher Reynolds numbers, the appropriate correlation is

$$\boxed{C_D \approx \frac{24}{Re_D}\left(1 + \frac{Re_D^{2/3}}{6}\right) \quad 2 < Re_D < 500} \tag{7.77}$$

At still higher Reynolds numbers, C_D is constant, equal to approximately 0.44 in the range $500 < Re_D < 2 \times 10^5$.
McAdams recommends the following correlation for heat-transfer from spheres to a flowing gas

$$Nu_D = 0.37\,Re_D^{0.6} \qquad (\textit{for } 25 < Re_D < 100\,000) \tag{7.78}$$

For forced convection over a *sphere*, the following correlation for average Nusselt number is recommended by *Whitaker* which is applicable for gases and liquids.

$$\boxed{\overline{Nu}_D = \frac{\bar{h}D}{k} = 2 + [0.4\,Re_D^{1/2} + 0.06\,Re_D^{2/3}]Pr^{0.4}\left(\frac{\mu_\infty}{\mu_w}\right)^{1/4}} \tag{7.79}$$

$$\begin{bmatrix} 3.5 < Re_D < 8 \times 10^4 \\ 0.7 < Pr < 380 \\ 1.0 < \dfrac{\mu_\infty}{\mu_w} < 3.2 \end{bmatrix}$$

All properties are evaluated at the *fluid temperature* T_∞ except μ_w which is evaluated at the *surface temperature*, T_w.

As the Reynolds number (Re_D) approaches zero, the Nusselt number approaches 2 when the surface temperature simultaneously approaches fluid temperature and the sphere is in an infinite fluid with conduction as the only mechanism of heat-transfer.

Note that the lower limit of $\overline{Nu_D} = 2$ in the above expression corresponds to conduction heat-transfer from a sphere into stationary, infinite surroundings. Figure 7.17 illustrates this important observation.

We know that heat-flow rate across a spherical shell is given by

$$\dot{Q} = \frac{4\pi k(T_1 - T_2)}{(1/r_1) - (1/r_2)}$$

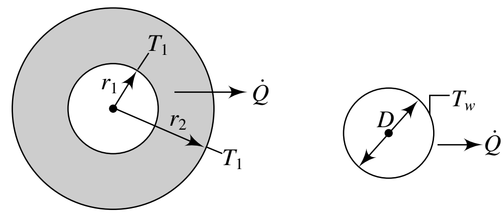


Fig. 7.17 *Conduction across a spherical shell, and from a sphere into stationary infinite surroundings*

Area of inner surface, $A_i = 4\pi r_1^2$

For $D = 2r_1$, and very large r_2 ($r_2 \to \infty$),

$$\dot{Q} = \frac{2k}{D}\pi D^2(T_1 - T_2)$$

$$= h(\pi D^2)(T_w - T_\infty)$$

Hence, the heat-transfer coefficient is

$$h = \frac{2k}{D} \text{ (since } A = \pi D^2\text{)}$$

and $$\boxed{Nu_D = \frac{hD}{k} = 2}$$

7.12 ❑ ANALOGY BETWEEN HEAT AND MOMENTUM TRANSFER

7.12.1 • Reynolds Analogy

Reynolds has developed a relation between the heat-transfer and skin-friction coefficient. Near the surface the heat-transfer is always due to conduction because the flow here is stationary.

Wall shear stress in laminar flow in the normal direction to the plate

$$\tau_w = \mu \frac{du}{dy} \qquad \text{(\textit{Newton's law of viscosity})} \tag{7.80}$$

Heat-transfer rate along the y-direction (*Fourier's rate equation*)

$$\dot{Q} = -kA\frac{dT}{dy} \tag{7.81}$$

Temperature and velocity profiles are identical when the dimensionless Prandtl number is unity which is approximately the case for most gases ($0.6 < Pr < 1.0$).

$$Pr = \frac{C_p \mu}{k} = 1 \quad \text{or} \quad \frac{k}{\mu} = C_p \tag{7.82}$$

Combination of expressions (7.80), (7.81) and (7.82) yields

$$\dot{Q} = -C_p A \tau_w \frac{dT}{du}$$

Separating the variables and integrating within the limits,

- $T = T_w$ when $u = 0$ at the plate surface
- $T = T_\infty$ when $u = u_\infty$ at the outer edge of the boundary layer

$$\frac{\dot{Q}}{C_p A \tau_w} \int_0^{u_\infty} du = -\int_{T_w}^{T_\infty} dT$$

or $$\frac{\dot{Q}}{C_p A \tau_w} u_\infty = (T_w - T_\infty) \quad \text{or} \quad \frac{\dot{Q}}{A(T_w - T_\infty)} = \tau_w \frac{C_p}{u_\infty}$$

The left-hand side represents the heat-transfer coefficient h_x. Also, from the definition of skin friction coefficient, we have $C_{fx} = \tau_w \Big/ \frac{1}{2}\rho u_\infty^2$. Making these substitutions, we obtain

$$h_x = C_{fx} \times \frac{1}{2}\rho u_\infty^2 \times \frac{C_p}{u_\infty} = \frac{C_{fx}}{2}(\rho C_p u_\infty)$$

In the dimensionless form,

$$\boxed{\frac{h_x}{\rho C_p u_\infty} = \frac{C_{fx}}{2}} \tag{7.83}$$

The dimensionless group of terms $h_x/(\rho C_p u_\infty)$ is called the *Stanton number* St_x and it represents the *Nusselt number* divided by the product of the *Reynolds* and *Prandtl numbers*, i.e.,

$$\boxed{\frac{Nu_x}{Re_x Pr} = St_x = \frac{C_{fx}}{2}} \tag{7.84}$$

The physical significance of Stanton number is

$$St = \frac{h_x \Delta t}{\rho C_p u_\infty \Delta t} = \frac{\text{Actual heat flux to the fluid}}{\text{Heat flux capacity of the fluid flow}}$$

Equation (7.84) is called the *Reynolds analogy* and is an excellent example of the similar nature of heat and momentum transfer. This inter-relationship can be used directly to infer heat-transfer data from the measurement of shear stress.

7.12.2 • Colburn Analogy

We will now establish a relationship between the local heat-transfer coefficient and the local drag coefficient in laminar boundary layer flow over a flat plate

$$Nu_x = \frac{h_x x}{k} = 0.332(Re_x)^{1/2}(Pr)^{1/3}$$

$$C_{fx} = \frac{0.664}{(Re_x)^{1/2}}$$

Dividing both sides of expression for the by product $Re_x Pr^{1/3}$, we get

$$\frac{Nu_x}{Re_x Pr^{1/3}} = \frac{0.332}{(Re_x)^{1/2}} = \frac{C_{fx}}{2}$$

The left-hand side of this equality can be rearranged as

$$\frac{Nu_x}{Re_x Pr^{1/3}} = \frac{Nu_x}{Re_x Pr} Pr^{2/3} = St_x Pr^{2/3}$$

The inter-relationship between heat and momentum transfer then becomes

$$\boxed{St_x Pr^{2/3} = \frac{C_{fx}}{2}} \tag{7.85}$$

Similarly, on an average basis,

$$\overline{St}\, Pr^{2/3} = C_f/2$$

The above correlation for laminar boundary layer on a flat plate was applied by *Colburn* to a wide range of data for flow configurations of all types and found to be quite accurate provided that (a) there is no *form or pressure drag*, and (b) $0.6 < Pr < 60$.

Equation (7.85) has been designated as *Colburn analogy*. It is noteworthy that for $Pr = 1$, the *Colburn* and *Reynolds* analogies are the same.

The Colburn analogy is valid for the case of laminar boundary-layer flow over a flat plate because the pressure gradient term $(dP/dx) = 0$. It is not valid for other situations of laminar flow over curved surfaces or laminar flows inside tubes where the pressure gradient in the flow direction is non-zero. However, in the case of turbulent flow, conditions are less sensitive to pressure gradient. Thus, the Colburn analogy equation can be shown to be approximately valid for local coefficients in turbulent flow over a flat plate. The corresponding equation on an average basis is also valid if the boundary layer is turbulent from the leading edge.

7.12.3 • The Reynolds Analogy for Turbulent Heat Transfer

In the case of turbulent flow past a flat surface, the total apparent shear stress and heat flux are given by

$$\tau = \rho(\nu + \varepsilon)\frac{\partial u}{\partial y},$$

$$q = -\rho C_p(\alpha + \varepsilon_H)\frac{\partial T}{\partial y}$$

Reynolds assumed, as a simple model, that the turbulent boundary layer consisted of only the fully turbulent zone that is; he presumed that the laminar sublayer and the buffer zone are negligible. Thus, in the boundary layer $\varepsilon >> \nu$ and $\varepsilon_H >> \alpha$, so that ν and α may be taken as negligible and the ratio of the two expressions given above yields

$$\frac{q}{\tau} = -C_p \frac{\varepsilon_H}{\varepsilon}\frac{\partial T}{\partial u} \equiv \frac{C_p}{Pr_t}\frac{\partial T}{\partial u},$$

in which $Pr_t = \varepsilon/\varepsilon_H$ is *the turbulent Prandtl number*. Reynolds further assumed that since the eddy viscosity and eddy diffusivity arise from the same mechanism of transverse fluctuation, the $Pr_t \approx = 1$, i.e., $\varepsilon \approx \varepsilon_H$, so that

$$\frac{q}{\tau} = -C_p \frac{\partial T}{\partial u}$$

The similarity between heat and momentum transfer discussed for laminar flow in the preceding section noted that the ratio (q/τ) was constant. Reynolds assumed that this same similarity exists for the fully turbulent boundary layer. Intagrating from $T = T_w$, $u = 0$ to $T = T_\infty$, $u = u_\infty$ while taking q/τ as constant and equal to the wall values, we obtain

$$\frac{q_0}{\tau_0} = -C_p \frac{T_\infty - T_w}{u_\infty}$$

or

$$\frac{h(T_w - T_\infty)}{\tau_o} = C_p \frac{(T_w - T_\infty)}{u_\infty}$$

$$h = \frac{\tau_0 C_p}{u_\infty}$$

While this latter expression appears to be the same as that for laminar flow, one should note that although it has been presumed that the turbulent Prandtl number is unity, $Pr_t \approx 1$, it has not been assumed that the molecular Prandtl number $Pr = \nu/\alpha$ is 1. Thus, the above statement may also be written

$$h = \frac{\tau_0 k Pr}{\mu u_\infty} = C_{f,x} \frac{1}{2} \rho u_\infty^2 k\, Pr / \mu u_\infty = \frac{C_{fx}}{2x} \frac{\rho u_\infty k Pr_x}{\mu}$$

or
$$\boxed{Nu_x = \frac{h_x}{k} = \frac{C_{fx}}{2} Re_x Pr} \tag{7.86}$$

or Stanton number,
$$\boxed{St = \frac{Nu_x}{Re_x Pr} = \frac{C_{fx}}{2}} \tag{7.87}$$

This is known as the *Reynolds analogy* for turbulent heat-transfer on a flat surface.

Illustrative Examples

(A) Flat Plate in Parallel Flow: Laminar and Turbulent

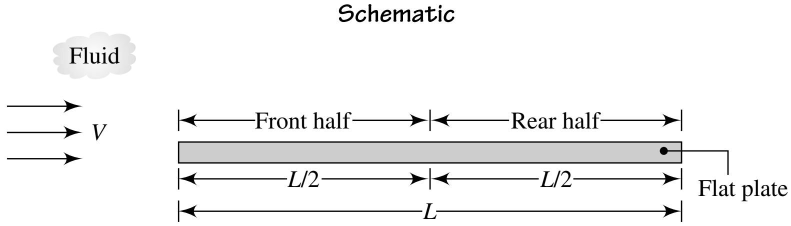
EXAMPLE 7.1 *Assuming the boundary layer to be turbulent over the entire plate, determine the ratio of the drag force acting on the front half of a flat plate to that on the rear half if the plate is oriented at zero incidence angle in a free stream of uniform velocity of fluid flowing past it.*

Solution

Known Turbulent flow of a fluid over the entire flat plate.

Find $\dfrac{F_{D,\text{front half}}}{F_{D,\text{rear half}}}$

Schematic



Assumptions (1) Turbulent flow over the plate. (2) Zero incidence angle.

Analysis Drag force, $F_D = \bar{C}_f = \bar{C}_{f,L} \dfrac{1}{2} \rho A V^2$ where V is the free stream fluid velocity

$$\bar{C}_{f,L} = \frac{0.074}{Re_L^{1/5}} = \frac{0.074}{(VL/\nu)^{1/5}}$$

Then, per unit width, for the entire plate,

$$F_D = 0.074 \left[\frac{\nu}{VL}\right]^{0.2} \times 0.5 \rho \times L \times 1 \times V^2$$

For the front half of the plate,

$$F_{D,I} = 0.074\left[\frac{\nu}{V(L/2)}\right]^{0.2} \times 0.5\rho \times \frac{L}{2} \times 1 \times V^2 = \frac{2^{0.2}}{2} \times F_D = 2^{-0.8}\,F_D$$

For the rear half, $F_{D,II} = F_D - F_{D,I} = F_D[1 - 2^{-0.8}]$

$$\therefore \qquad \frac{F_{D,\text{front half}}}{F_{D,\text{rear half}}} = \frac{2^{-0.8}F_D}{(1-2^{-0.8})F_D} = \frac{2^{-0.8}}{1-2^{-0.8}} = \mathbf{1.349} \qquad \textbf{(Ans.)}$$

EXAMPLE 7.2 *Air at a free stream temperature of 5°C and velocity of 10 m/s is in parallel flow over a flat plate of 3 m length and 1 m width which is maintained at a temperature of 85°C. Assuming the transition Reynolds number of 5 × 10⁵ and neglecting the first 5 cm for heat-transfer calculations, calculate the rate of heat loss from both sides of the plate and identify the location at which maximum heat loss occurs along the length of the plate.*

Properties of air at the film temperature of 45°C are

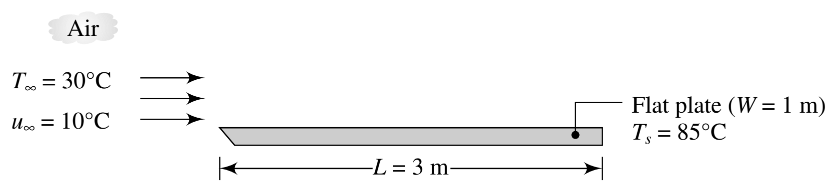
k = *0.02763 W/m °C, ν = 1.77 × 10⁻⁵ m²/s, Pr = 0.7045*

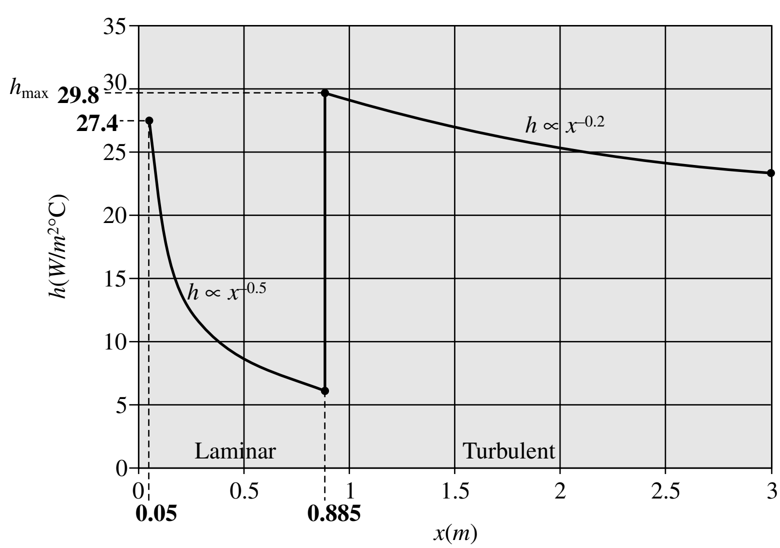
Solution

Known Plate dimensions and temperature. Air velocity and temperature. Flow over a flat plate.

Find Heat loss from both surfaces of the plate. Location of maximum heat loss on the plate.

Schematic





Assumptions (1) Steady operating conditions. (2) Critical Reynolds number is 5×10^5. (3) Constant properties. (4) First 5 cm of plate is ignored from the heat-transfer standpoint.

Analysis • For *laminar flow* over a flat plate $(Re < Re_{cr})$

$$Nu_x = \frac{hx}{k} = 0.332(Re_x)^{1/2} Pr^{1/3} \propto (Re_x)^{0.5}$$

Since $$Re_x = \frac{u_\infty x}{\nu},\ h = \frac{k}{x} \times 0.332\left(\frac{u_\infty}{\nu}\right)^{0.5} \times x^{0.5} \times Pr^{1/3} \propto x^{-0.5}$$

• For *turbulent flow* over a flat plate $(Re > Re_{cr})$

$$Nu_x = \frac{hx}{k} = 0.0296(Re)^{4/5}(Pr)^{1/3} \propto (Re_x)^{0.8}$$

$\therefore$ $$h = \frac{k}{x} \times 0.0296\left(\frac{u_\infty}{\nu}\right)^{0.8} \times x^{0.8} \times Pr^{1/3} \propto x^{-0.2}$$

Thus, we note that the local Nusselt number varies as $Re_x^{1/2}$ for the *laminar* case and $Re_x^{4/5}$ for the *turbulent* case. The local convection heat-transfer coefficient varies as $x^{-1/2}$ for laminar flows and $x^{-1/5}$ for turbulent flows. In both cases, as x increases, the value of h and, hence, the heat transfer rate $(\dot{Q} = hA_s\Delta T)$ decreases along the plate.

The maximum value of h in the laminar part of the boundary layer will occur at x = 0.05 m (since for the first 5 cm from the leading edge of the plate, heat transfer is not considered). Properties to be used should be at the film temperature.

$$T_f = \frac{1}{2}(T_s - T_\infty) = (85 + 5)/2 \text{ i.e. } 45°\text{C}.$$

Local heat-transfer coefficient at x = 0.05 m is

$$h = Nu_x \frac{k}{x} = \frac{k}{x} \times 0.332(Re_x)^{1/2}(Pr)^{1/3}$$

$$Re_x = \frac{u_\infty x}{\nu} = \frac{10 \text{ m/s} \times 0.05 \text{ m}}{17.7 \times 10^{-5} \text{ m}^2/\text{s}} = 2.825 \times 10^4$$

$\therefore$ $$h = \frac{0.02763 \text{ W/m °C}}{0.05 \text{ m}} \times 0.332 \times (2.825 \times 10^4)^{1/2} \times (0.7045)^{1/3} = 27.4 \text{ W/m}^2 \text{ °C}$$

The value of h *decreases* with distance from the leading edge x because the boundary layer thickness, δ *increases*.

$$\delta(x) \approx \frac{5.0x}{(Re_x)^{1/2}} = 5x\sqrt{\frac{\nu}{u_\infty x}} \propto x^{0.5}$$

The temperature gradient at the wall $\left[\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{h(T_s - T_\infty)}{-k} \propto h\right]$ *decreases* with *increasing* x and *decreasing* h.

At the transition point where the boundary layer becomes turbulent, the temperature gradient near the wall is *steeper* even though the boundary layer is *thicker*.

Hence, the local heat-transfer coefficient suddenly *increases* at the transition location. To identify the location of the transition point, we note that

$$Re_{cr} = 5 \times 10^5 = \frac{u_\infty x_{cr}}{\nu}$$

Hence, $$x_{cr} = \frac{(5 \times 10^5)(1.77 \times 10^{-5}\ \text{m}^2/\text{s})}{10\ \text{m/s}} = 0.885\ \text{m}$$

At $x = x_{cr}$, the boundary layer is *turbulent*. Hence, the local convection coefficient is calculated from

$$h = Nu_x \frac{k}{x_{cr}} = \frac{k}{x_{cr}} \times 0.0296 \times (Re_{cr})^{0.8} (Pr)^{1/3}$$

$$= \frac{0.02763\ \text{W/m}\,°\text{C}}{0.885\ \text{m}} \times 0.0296(5 \times 10^5)^{0.8} \times (0.7045)^{1/3} = 29.8\ \text{W/m}^2\,°\text{C}$$

This is greater than 27.4 W/m^2 °C for *laminar* flow at x = 5 cm. As x *increases* beyond x_{cr}, h will *decrease* as $x^{-0.2}$. Clearly, the maximum local heat-transfer coefficient (and, hence, the maximum heat loss) will occur at

x = 0.885 m **(Ans.)**

Heat-transfer rate for the entire plate from both *top* and *bottom* surfaces is determined from

$$\dot{Q} = \bar{h}(2LW)(T_s - T_\infty)$$

where $$\bar{h} = Nu_L \frac{k}{L} = \frac{k}{L} \times [0.037\, Re_L^{4/5} - 871] Pr^{1/3}$$

$$= \frac{0.02763\ \text{W/m}\,°\text{C}}{3\ \text{m}} \times \left[0.037 \times \left(\frac{10\ \text{m/s} \times 3\ \text{m}}{1.77 \times 10^{-5}\ \text{m}^2/\text{s}}\right)^{0.8} - 871\right] (0.7045)^{1/3}$$

$$= 22\ \text{W/m}^2\,°\text{C}$$

Total heat loss is

$\therefore$ $\dot{Q} = (22\ \text{W/m}^2\,°\text{C})(2 \times 3\ \text{m} \times 1\ \text{m})(85 - 5)°\text{C}$ = **10.58 × 10³ W or 10.58 kW (Ans.)**

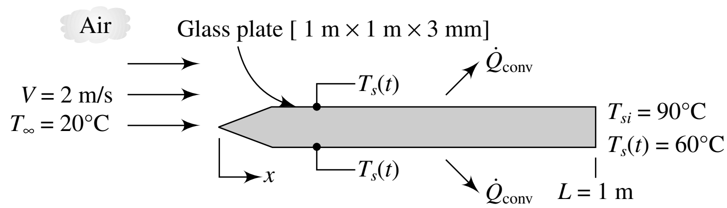
EXAMPLE 7.3 *In a certain glass-making process, a plate of glass (1 m square, 3 mm thick) heated uniformly to 90°C is cooled by air at 20°C flowing over both sides parallel to the plate at 2 m/s. Calculate the initial rate of cooling of the plate and the time required for the plate to cool to 60°C. Neglect temperature gradients in the glass plate and consider only forced convection effects. Properties of glass:* ρ *= 2500 kg/m³,* C_p *= 0.67 kJ/kg K. The following properties of air may be used:* k *= 27.6 ×* 10^{-3} *W/m K,* Pr *= 0.709,* ν *= 17.7 ×* 10^{-6} *m²/s.*

Solution

Known Glass plate of prescribed dimensions and temperature in air stream at given velocity and temperature.

Find Initial heat-transfer (cooling) rate and time required to attain new plate temperature.

Schematic



Assumptions (1) Steady operating conditions. (2) Uniform plate surface temperature. (3) Negligible radiation effects.

Analysis Film temperature, $T_f = \dfrac{T_{s,av} + T_\infty}{2} = \dfrac{1}{2}\left[\left(\dfrac{90+60}{2}\right) + 20\right] = 47.5°\text{C}$

Initial rate of cooling, $\dot{Q} = \bar{h} A_s (T_{si} - T_\infty)$

where $A_s = 2 \times 1 \text{ m} \times 1 \text{ m} = 2 \text{ m}^2$ [*considering both surfaces*]

and $T_{si} - T_\infty = 90 - 20 = 70°\text{C}$

$$Re_D = \frac{VL}{\nu} = \frac{2 \text{ m/s} \times 1 \text{ m}}{17.7 \times 10^{-6} \text{ m}^2/\text{s}} = 1.13 \times 10^5 \quad (< 5 \times 10^5)$$

Hence, the air flow is *laminar*.

The appropriate correlation is

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.664 (Re_L)^{1/2} (Pr)^{1/3}$$

Hence, $$\bar{h} = \frac{0.664 \times 27.6 \times 10^{-3} \text{ W/m K}}{1 \text{ m}} \times (1.13 \times 10^5)^{1/2} (0.709)^{1/3} = 5.5 \text{ W/m}^2 \text{ K}$$

Initial cooling rate,

$$\dot{Q} = \bar{h} A_s (T_{si} - T_\infty) = (5.5 \text{ W/m}^2 \text{ K})(2 \text{ m}^2)(70°\text{C}) = \mathbf{770 \text{ W}} \qquad \textbf{(Ans.)}$$

To find the time required for cooling of the plate to 60°C, one must ascertain if the lumped capacity model is valid.

$$Bi = \frac{\bar{h} L_c}{k} \quad \text{where} \quad L_c = \frac{\forall}{A} = \frac{(1 \times 1 \times 3 \times 10^{-3}) \text{m}^3}{2 \text{ m}^2} = 1.5 \times 10^{-3} \text{ m}$$

$\therefore$ $$Bi = \frac{5.5 \times 1.5 \times 10^{-3}}{1.4} = 5.893 \times 10^{-3} \; (<< 0.1) \; (\text{Assuming } k_{\text{glass}} = 1.4 \text{ W/m K})$$

Thus, the validity of the assumption that internal temperature gradients can be neglected is established.

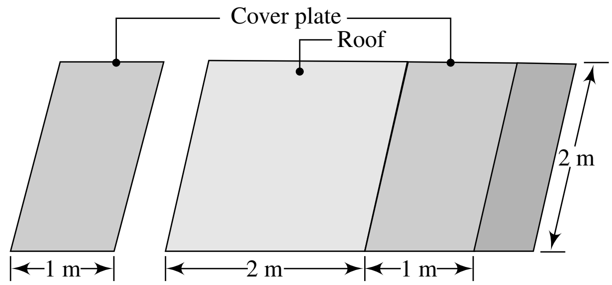
Time required, $t = \dfrac{\rho C_p L_c}{\bar{h}} \ln \dfrac{T_{si} - T_\infty}{(T_s(t) - T_\infty)}$

$$= \frac{2500 \text{ kg/m}^3 \times 670 \text{ J/kg K} \times 1.5 \times 10^{-3} \text{ m}}{5.5 \text{ W/m}^2\text{K}} \ln\left[\frac{(90-20)°\text{C}}{(60-20)°\text{C}}\right]$$

= 256 s or 4 min 16 s **(Ans.)**

EXAMPLE 7.4 *A flat-plate solar collector has a cover plate at 45°C exposed to ambient air at 25°C in parallel flow over the plate with free stream velocity of 8 km/h. (a) Compute the heat-loss rate from the plate. (b) If the plate is installed 2 m from the leading edge of a roof and flush with the roof surface, calculate the rate of heat loss.*
Properties of air at 35°C are
k = *0.02625 W/m°C,* ν = *16.55 × 10^{-6} m^2/s,* Pr = *0.7268*

Schematic

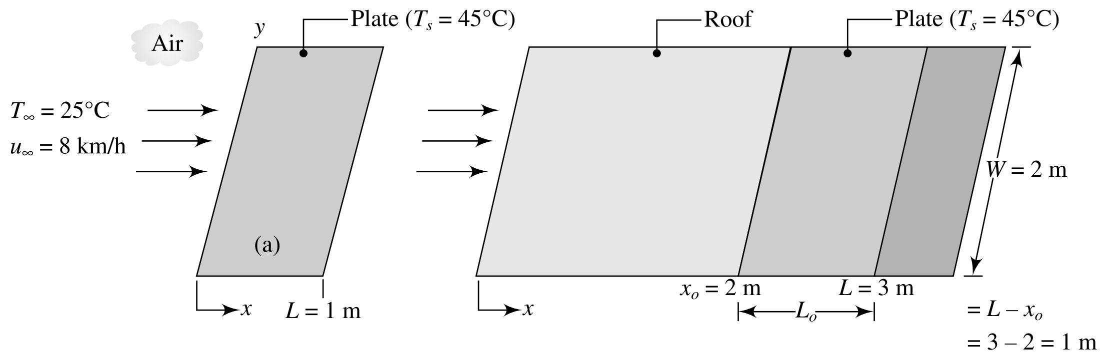


Solution

Known Air flow conditions. Dimensions and temperature of cover plate of a flat plate solar collector.

Find (a) Heat loss *without* unheated starting length. (b) Heat loss *with* unheated starting length.

Schematic



Assumptions (1) Steady-state conditions. (2) $Re_{cr} = 5 \times 10^5$. (3) Radiation effect is neglected. (4) Constant properties. (5) Boundary layer unruffled by roof plate interface.

Analysis (a) Reynolds number,

$$Re_L = \frac{u_\infty L}{\nu} = \frac{(8/3.6)\,\text{m/s} \times 1\,\text{m}}{16.55 \times 10^{-6}\,\text{m}^2/\text{s}} = 1.343 \times 10^5 < Re_{cr}$$

For laminar flow,

$$\overline{Nu}_L = 0.664\, Re_L^{1/2}\, Pr^{1/3} = 0.664(1.343 \times 10^5)^{1/2}(0.7268)^{1/3} = 218.8$$

Average heat-transfer coefficient,

$$\bar{h} = \overline{Nu}_L \frac{k}{L} = 218.8 \times \frac{0.02625\ \text{W/m°C}}{1\,\text{m}} = \mathbf{5.74\ W/m^2\,°C}$$

Heat-loss rate,

$$\dot{Q} = \bar{h}A_s(T_s - T_\infty) = \bar{h}(WL)(T_s - T_\infty)$$
$$= (5.74\ \text{W/m}^2\,°\text{C})(2\,\text{m} \times 1\,\text{m})(45 - 25)°\text{C} = \mathbf{229.7\ W}$$ **(Ans.) (a)**

(b) In this case, L = 3 m and x_o = 2 m.

$$Re_L = \frac{(8/3.6)\,\text{m/s} \times 3\,\text{m}}{16.55 \times 10^{-6}\,\text{m}^2/\text{s}} = 4.028 \times 10^5 < Re_{cr}$$

Therefore, laminar boundary-layer conditions exist over the entire roof plate surface (idealized as a flat plate).

For obtaining average convection coefficient of an isothermal plate with an *unheated starting length,* we have

$$\bar{h} = \frac{k}{L} \times \frac{L}{L_o} \times \left[1 - \left(\frac{x_o}{L}\right)^{3/4}\right]^{2/3} \times 0.664 (Re_L)^{1/2}\, Pr^{1/3}$$

$$= \frac{0.02625\,\text{W/m}\,°\text{C}}{1\,\text{m}} \times \left[1 - \left(\frac{2}{3}\right)^{3/4}\right]^{2/3} \times 0.664 (4.028 \times 10^5)^{1/2} (0.7268)^{1/3}$$

$$= 4.08\,\text{W/m}^2\,°\text{C}$$

Heat loss, $\dot{Q} = \bar{h}(L_o W)(T_s - T_\infty)$

$$= (4.08\,\text{W/m}^2\,°\text{C})(1\,\text{m} \times 2\,\text{m})(45 - 25)°\text{C} = \mathbf{163\,W}$$ **(Ans.) (b)**

Comment It is worth noting that with *unheated starting length*, the prior development of velocity boundary layer decreases $\bar{h}$ resulting in reduced heat rate.

EXAMPLE 7.5 *A Maruti Swift is travelling at a steady speed of 108 km/h on a highway where the ambient air temperature is 18°C. The hood of the vehicle is at 42°C. The hood may be approximated as a 1.2 m square flat plate. Determine the rate of heat loss per kilometre. The following properties of air at the film temperature of 30°C are*

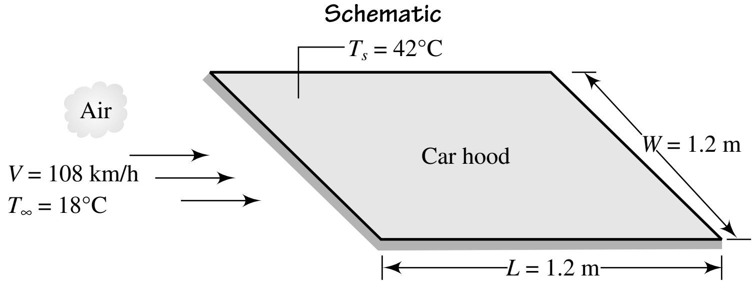
$$C_p = 1.007\ \text{kJ/kg K},\ \mu = 18.72 \times 10^{-6}\ \text{kg/m s},\ k = 0.025\ 88\ \text{W/m K}$$

Solution

Known Car hood as a square flat plate. Surface and ambient temperature. Car speed and hood dimensions.

Find Heat dissipation rate per km.

Schematic



Assumptions (1) Steady operating conditions. (2) Air is an ideal gas. (3) Constant properties.

Analysis Film temperature, $T_f = \frac{1}{2}(T_s + T_\infty) = \frac{1}{2}(42 + 18)$ = 30°C or 303.15 K

$$Re_L = \frac{\rho VL}{\mu}$$

where $$\rho = \frac{P}{RT} = \frac{101.325 \text{ kPa}}{0.287 \text{ kJ/kg K} \times 303.15 \text{ K}}\left|\frac{1 \text{ kJ}}{1 \text{ kPa m}^3}\right| = 1.1646 \text{ kg/m}^3$$

and $$V = 108\frac{\text{km}}{\text{h}}\left|\frac{10^3 \text{ m}}{1 \text{ km}}\right|\frac{1 \text{ h}}{3600 \text{ s}}\right| = 30 \text{ m/s}$$

$$Re_L = \frac{(1.1646 \text{ kg/m}^3)(30 \text{ m/s})(1.2 \text{ m})}{18.72 \times 10^{-6} \text{ kg/m s}} = 2.24 \times 10^6$$

Prandtl number, $$Pr = \frac{C_p \mu}{k} = \frac{(1.007 \times 10^3 \text{ J/kg K})(18.72 \times 10^{-6} \text{ kg/m s})}{0.02588 \text{ W/m K}} = 0.7284$$

With $5 \times 10^5 \le Re_L \le 10^8$ and $0.6 < Pr < 60$,

$$Nu = [0,037\,Re^{0.8} - 871]Pr^{1/3} = \frac{hL}{k}$$

Average convection heat-transfer coefficient is

$$\bar{h} = \frac{k}{L}Nu = \frac{k}{L}(0.037\,Re^{0.8} - 871)Pr^{1/3}$$

$$= \frac{0.02588 \text{ W/m K}}{1.2 \text{ m}} \times [0.037(2.24 \times 10^6)^{0.8} - 871](0.7284)^{1/3} = 69.45 \text{ W/m}^2 \text{ K}$$

Rate of heat loss per km,

$$\frac{\dot{Q}}{V} = \frac{\bar{h}(WL)(T_s - T_\infty)}{V}$$

$$= \frac{(69.45 \text{ W/m}^2 \text{ K})(1.2 \text{ m} \times 1.2 \text{ m})(42 - 18)°\text{C}}{(108 \text{ km/h})(1 \text{ h}/3600 \text{ s})}\left|\frac{1 \text{ J/s}}{1 \text{ W}}\right|\left|\frac{1 \text{ kJ}}{10^3 \text{ J}}\right|$$

$$= \mathbf{80.0 \text{ kJ/km}}$$ **(Ans.)**

EXAMPLE 7.6 *Air at 1 atm and 20°C is flowing over both sides of a flat plate at a free stream velocity of 3 m/s. The plate is 0.3 m square and is maintained at a uniform surface temperature of 60°C. Assuming cubic velocity and temperature profiles and the critical Reynolds number = 5 × 10⁵, determine the following quantities at the trailing edge of the plate: (a) Hydrodynamic boundary-layer thickness (b) Local skin-friction coefficient (c) Average skin-friction coefficient (d) Local shear stress (e) Thermal boundary-layer thickness (f) Local heat-transfer coefficient (g) Average heat-transfer coefficient (h) Rate of heat-transfer from the entire plate (i) Total drag force on the plate (j) Total mass flow rate entering the velocity boundary layer (k) Distance from the leading edge of the plate where transition from laminar to turbulent flow occurs.*

Thermophysical properties of air at 1 atm and 40°C are

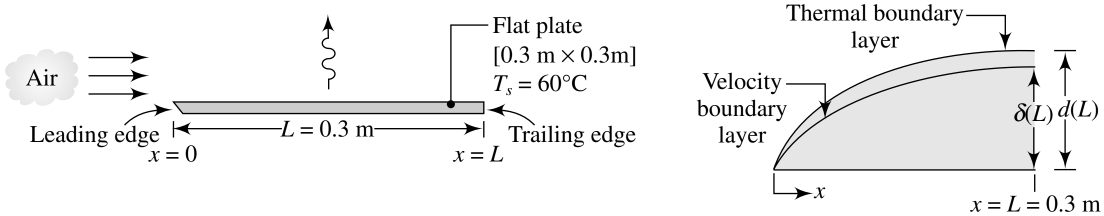
k = *0.02662 W/m K,* C_p = *1.007 kJ/kg K,* μ = *0.06905 kg/m h*

Solution

Known Air-flow conditions on an isothermal flat plate.

Find (a) δ (mm); (b) $C_{f,L}$; (c) $\overline{C_{f,L}}$; (d) $\tau(x = L)$ (N/m²); (e) δ_1 (mm); (f) $h_{(x = L)}$ (W/m² K); (g) $\bar{h}$ (W/m² K); (h) $\dot{Q}$(W); (i) F_D(N); (j) $\dot{m}$ (kg/s); (k) x_{cr} (m).

Schematic



Assumptions (1) Steady operating conditions exist. (2) Air is an ideal gas. (3) $Re_{cr} = 5 \times 10^5$. (4) Constant properties.

Analysis (a) Reynolds number, $Re = \dfrac{\rho u_\infty L}{\mu}$

Film temperature, $T_f = (T_s + T_\infty)/2 = (60 + 20)/2 = 40°C$ or 313.15 K

Density of air, $\rho = \dfrac{P}{RT} = \dfrac{101.325 \text{ kPa}}{0.287 \text{ kJ/kg K} \times 313.15 \text{ K}}\left|\dfrac{1 \text{ kJ}}{1 \text{ kPa m}^3}\right| = 1.1274 \text{ kg/m}^3$

$$\therefore \qquad Re = \frac{(1.1274 \text{ kg/m}^3)(3 \text{ m/s})(0.3 \text{ m})}{0.069\,05 \text{ kg/m h}}\left|\frac{3600 \text{ s}}{1 \text{ h}}\right| = 52\,900 \quad (< Re_{cr} = 5 \times 10^5)$$

Hence, the velocity boundary layer is *laminar*.
Boundary layer thickness at $x = 0.3$ m is given by

$$\delta(x = L) = \frac{4.64L}{\sqrt{Re}} = \frac{(4.64)(0.3 \text{ m})}{\sqrt{52\,900}}\left|\frac{10^3 \text{ mm}}{1 \text{ m}}\right| = \mathbf{6.052 \text{ mm}} \qquad \textbf{(Ans.) (a)}$$

(b) Local skin-friction coefficient,

$$C_{f(x = L)} = 0.664(Re_{(x = L)})^{-1/2} = 0.664\,[52\,900]^{-0.5} = 0.00289 \qquad \textbf{(Ans.) (b)}$$

(c) Average skin-friction coefficient,

$$\overline{C_{f,L}} = 2C_{f,L} = 2 \times 0.00289 = \mathbf{0.00578} \qquad \textbf{(Ans.) (c)}$$

(d) Local shear stress,

$$\tau_{(x=L)} = C_{f(x=L)}\frac{1}{2}\rho u_\infty^2 = (0.00289)\left(\frac{1}{2} \times 1.1274 \text{ kg/m}^3 \times 3^2 \text{ m}^2/\text{s}^2\right)\left|\frac{1 \text{ N}}{1 \text{ kg m/s}^2}\right|$$

$$= \mathbf{0.01465 \text{ N/m}^2} \qquad \textbf{(Ans.) (d)}$$

(e) Thermal boundary layer thickness, $\delta_t = \dfrac{\delta}{1.026\,Pr^{1/3}}$

Prandtl number,

$$\therefore \qquad Pr = \frac{C_p\mu}{k} = \frac{(1007\ \text{J/kg K})(0.06905/3600)\text{kg/m s}}{0.02662\ \text{W/m K}}\left|\frac{1\ \text{W}}{1\ \text{J/s}}\right|\left|\frac{1\ \text{N}}{1\ \text{kg m/s}^2}\right|\left|\frac{1\ \text{J}}{1\ \text{Nm}}\right| = 0.7256$$

$$\delta_t = \frac{6.052\ \text{mm}}{1.026(0.7256)^{1/3}} = \mathbf{6.564\ mm} \qquad \textbf{(Ans.) (e)}$$

(f) Local Nusselt number for *laminar* flow over a flat plate is

$$Nu_x = 0.332\, Re_x^{1/2} Pr^{1/3} = h_x x / K$$

At $x = L$,

$$h_{(x=L)} = \frac{k}{L} \times 0.332\, Re_L^{1/2} Pr^{1/3} = \frac{0.02662\ \text{W/m K}}{0.3\ \text{m}} \times 0.332 \times \sqrt{52900} \times 0.7256^{1/3}$$

$$= \mathbf{6.09\ W/m^2} \qquad \textbf{(Ans.) (f)}$$

(g) Average heat transfer coefficient,

$$\bar{h} = 2h_{(x=L)} = 2 \times 6.09 = \mathbf{12.18\ W/m^2\ K} \qquad \textbf{(Ans.) (g)}$$

(h) Rate of heat transfer from *both* sides of the whole plate,

$$\dot{Q} = \bar{h}(2WL)(T_s - T_\infty) = (12.18\ \text{W/m}^2\text{K})\ (2 \times 0.3\ \text{m} \times 0.3\ \text{m})\ (60 - 20)°\text{C or K}$$

$$= \mathbf{87.68\ W} \qquad \textbf{(Ans.) (h)}$$

(i) Total drag force on the plate,

$$F_D = \overline{C_{f,L}} \times 2 \times \frac{1}{2}\rho A u_\infty^2 = (0.00578)\ (1.1274\ \text{kg/m}^3)\ (0.3 \times 0.3\ \text{m}^3)\ (3\ \text{m/s})^2$$

$$= \mathbf{0.0053\ N} \qquad \textbf{(Ans.) (i)}$$

(j) Mass flow that enters the boundary layer from the free stream from $x = 0$ to $x = 0.3$ m is

$$\dot{m} = \int_0^\delta \rho u\, dy = \rho \int_0^\delta u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy$$

$$= \rho u_\infty \left[\frac{3}{2}\frac{y^2}{2\delta} - \frac{1}{2}\frac{y^4}{4\delta^3}\right]_0^\delta = \rho u_\infty \delta\left[\frac{3}{4} - \frac{1}{8}\right] = \frac{5}{8}\rho u_\infty \delta$$

$\therefore$ Mass flow of air through the boundary layer,

$$\dot{m} = \frac{5}{8} \times 1.1274\ \text{kg/m}^3 \times 3\ \text{m/s} \times 6.052 \times 10^{-3}\ \text{m} = \mathbf{0.0128\ kg/s} \qquad \textbf{(Ans.) (j)}$$

(k) Critical or transition Reynolds number,

$$Re_{cr} = \frac{\rho u_\infty x_{cr}}{\mu} = 5 \times 10^5$$

$$\therefore \qquad x_{cr} = \frac{(5 \times 10^5)(0.06905/3600)\ \text{kg/m s}}{(1.1274\ \text{kg/m}^3)(3\ \text{m/s})} = \mathbf{2.835\ m} \qquad \textbf{(Ans.) (k)}$$

EXAMPLE 7.7 *Hydrogen at 20°C and at a pressure of 1 atm is flowing along a flat plate at a velocity of 3 m/s. If the plate is 0.3 m wide and at 70°C, calculate the following quantities at* x *= 0.3 m and at the distance corresponding to the transition point, i.e.,* Re_x *= 5 × 10*^{5}*.*

(a) Hydrodynamic boundary-layer thickness, in cm (b) Local friction coefficient (c) Average friction coefficient (d) Drag force in N (e) Thickness of thermal boundary layer in cm (f) Local convective heat-transfer coefficient, in W/m² °C (g) Average convective heat-transfer coefficient, in W/m² °C, (h) Rate of heat-transfer, in W

Properties of hydrogen at 1 atm and mean film temperature of 45 °C are

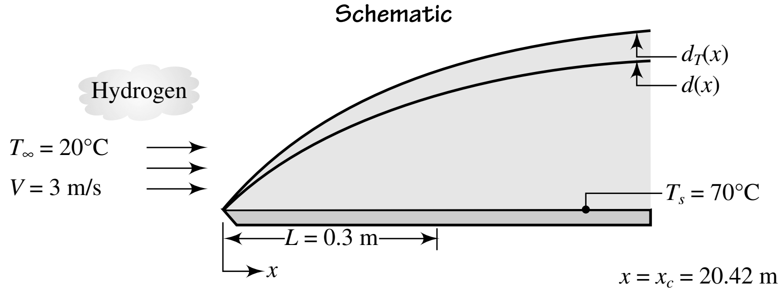
$$\rho = 0.0772 \text{ kg/m}^3,\ k = 0.191 \text{ W/m °C},\ \nu = 122.5 \times 10^{-6} \text{ m}^2\text{/s},\ Pr = 0.701$$

Solution

Known Hydrogen at atmospheric pressure flows along an isothermal flat plate under specified conditions.

Find (a) δx and $\delta_{x,c}$ (cm), (b) $C_{f,x}$ and $C_{f_{x,c}}$, (c) $\overline{C_{f,x}}$ and $\overline{C_{f,x,c}}$, (d) $F_D(N)$, (e) $\delta_{T,x}$ and $\delta_{T_{x,c}}$ (cm), (f) h_x and $h_{x,c}$, (g) $\bar{h}_x$ and $\bar{h}_{x,c}$ (W/m² °C), (h) $\dot{Q}$ and $\dot{Q}_{x,c}$ (W).

Schematic



Assumptions (1) Steady operating conditions. (2) The edge effects are negligible. (3) Critical Reynolds number $Re_{x,c} = 5 \times 10^5$. (4) Hydrogen is an ideal gas. (5) The local atmospheric pressure is 1 atm.

Analysis Critical (*transitional*) distance from the leading edge

$$x_c = \frac{Re_{x,c}\nu}{v} = \frac{(5 \times 10^5)(122.5 \times 10^{-6} \text{ m}^2\text{/s})}{3 \text{ m/s}} = 20.42 \text{ m}$$

(a) Hydrodynamic boundary layer thickness:

$$\delta_x = \frac{5.0x}{\sqrt{Re_x}} = \frac{5.0 \times 0.3 \text{ m}}{\sqrt{7346}} = \mathbf{0.0175 \text{ m or } 1.75 \text{ cm}} \qquad \textbf{(Ans.) (a)}$$

$$\delta_{x,c} = \frac{5.0x_c}{\sqrt{Re_{x,c}}} = \frac{5 \times 20.42 \text{ m}}{\sqrt{5 \times 10^5}} = \mathbf{0.144 \text{ m}} \text{ or } \mathbf{14.4 \text{ cm}} \qquad \textbf{(Ans.) (a)}$$

(b) Local friction coefficient:

$$C_{f,x} = \frac{0.664x}{\sqrt{Re_x}} = \frac{0.664}{\sqrt{7346}} = \mathbf{0.00775} \qquad \textbf{(Ans.) (b)}$$

$$C_{f,x,c} = \frac{0.664}{\sqrt{5 \times 10^5}} = \mathbf{9.39 \times 10^{-4}} \qquad \textbf{(Ans.) (b)}$$

(c) Average friction coefficient:

$$\overline{C_{f,x}} = \frac{1.328}{\sqrt{Re_x}} = 2C_{f,x} = 2 \times 0.00775 = \mathbf{0.0155} \qquad \textbf{(Ans.) (c)}$$

$$\overline{C_{f,x_c}} = \frac{1.328}{\sqrt{Re_{x,c}}} = 2C_{f,x_c} = 2 \times 9.39 \times 10^{-4} = \mathbf{1.878 \times 10^{-3}} \qquad \textbf{(Ans.) (c)}$$

(d) Drag force:

$$F_D = \overline{C_{f,x}}\ A_x \frac{1}{2}\rho V^2 = \overline{C_{f,x}}\,(xW)\frac{1}{2}\rho V^2$$

$$= (0.0155)(0.3 \text{ m} \times 0.3 \text{ m})\frac{1}{2}(0.0772 \text{ kg/m}^3)(3 \text{ m/s})^2$$

$$= \mathbf{4.846 \times 10^{-4}\ N} \qquad \textbf{(Ans.) (d)}$$

$$F_D(x_c = 20.42 \text{ m}) = (1.878 \times 10^{-3})(20.42 \text{ m} \times 0.3 \text{ m})$$

$$\times \frac{1}{2}(0.0772 \text{ kg/m}^3)(3 \text{ m/s})^2 \qquad \textbf{(Ans.) (d)}$$

$$= \mathbf{0.004\ N}$$

(e) Thermal boundary layer thickness:

$$\delta_T = \frac{\delta}{Pr^{1/3}} = 1.75 \text{ cm}(0.701)^{-1/3} = \mathbf{1.97\ cm} \qquad \textbf{(Ans.) (e)}$$

$$\delta_{T(x_c)} = 14.4 \text{ cm}(0.701)^{-1/3} = \mathbf{16.21\ cm} \qquad \textbf{(Ans.) (e)}$$

(f) Local heat-transfer coefficient:

$$h_x = Nu_x \frac{k}{x} = \frac{k}{x} \times 0.332(Re_x)^{1/2}(Pr)^{1/3}$$

$$= \frac{0.191 \text{ W/m°C}}{0.3 \text{ m}} \times 0.332(7346)^{1/2}(0.701)^{1/3} = \mathbf{16.1\ W/m^2\ °C} \qquad \textbf{(Ans.) (f)}$$

$$h_{x,c} = \frac{0.191 \text{ W/m°C}}{20.42 \text{ m}} \times 0.332(5 \times 10^5)^{1/2}(0.701)^{1/3} = \mathbf{1.95\ W/m^2\ °C} \qquad \textbf{(Ans.) (f)}$$

(g) Average heat transfer coefficient:

$$\bar{h}_x = 2h_x = 2 \times 16.1 = \mathbf{32.2\ W/m^2\ °C} \qquad \textbf{(Ans.) (g)}$$

$$\bar{h}_{x,c} = 2h_{x,c} = 2 \times 1.95 = \mathbf{3.90\ W/m^2\ °C} \qquad \textbf{(Ans.) (g)}$$

(h) Heat transfer rate:

$$\dot{Q} = \bar{h}_x(xW)(T_s - T_\infty) = (32.2 \text{ W/m}^2\text{ °C})(0.3 \text{ m} \times 0.3 \text{ m})(70 - 20)\text{°C}$$

$$= \mathbf{145\ W} \qquad \textbf{(Ans.) (h)}$$

$$\dot{Q}_{(x_c)} = \bar{h}_{x,c}(x_cW)(T_s - T_\infty) = (3.90 \text{ W/m}^2\text{ °C})(20.42 \text{ m} \times 0.3 \text{ m})(70 - 20)\text{°C}$$

$$= \mathbf{1195\ W} \qquad \textbf{(Ans.) (h)}$$

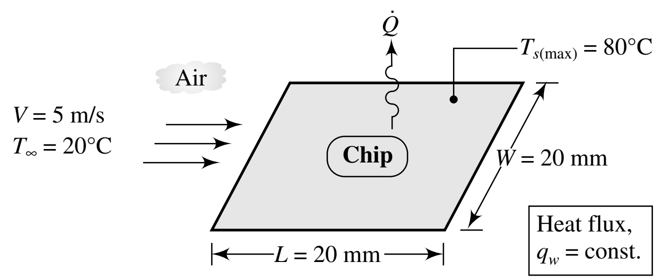
EXAMPLE 7.8 *Computer chips generate heat during their operation, their failure rate increasing with increasing operating temperature. A square computer chip (20 mm × 20 mm) is cooled by blowing air at local atmospheric pressure (the location is at an elevation of 1600 m above the mean sea level) and 20°C over the surface with a fan. The air approaches the chip with a velocity of 5 m/s. The chip construction is such that the electrical power dissipated in the chip results in a uniform heat flux over the chip surface. Any heat transfer from the chip's bottom surface to the circuit board may be neglected. The maximum allowable chip surface temperature is 80°C. Determine the rate of heat dissipation permissible from the computer chip.*

Solution

Known Dimensions and maximum possible chip temperature. Air-flow conditions.

Find Maximum heat dissipation rate, $\dot{Q}$[W].

Schematic



Assumptions (1) Steady-state conditions. (2) Critical Reynolds number, $Re_{x,cr} = 5 \times 10^5$. (3) Radiation effects are negligible. (4) Heat transfer from the bottom surface is ignored. (5) Uniform heat flux.

Analysis Heat-transfer rate, $\dot{Q} = hA_s(T_s - T_\infty) = q_w A_s$

Uniform heat flux, $q_w = h(x)[T_s(x) - T_\infty]$

Maximum surface temperature = $T^s(x = L)$ and the corresponding heat-transfer coefficient, $h(x = L)$ is *minimum.*

Mean film temperature, $T_f = (T_{s(\max)} + T_\infty)/2 = (80 + 20)/2 = 50°C$

The properties of air at 1 atm and 50°C are:

$$k = 0.028 \text{ W/m K}, \quad Pr = 0.704, \quad \nu = 18.2 \times 10^{-6} \text{ m}^2\text{/s}$$

Except for ν and α, all other properties of air like C_p, μ, k, and Pr are *independent* of pressure.

The altitude (*elevation*) of the location, $Z = 1600$ m

Temperature of air at altitude Z is, $T = T_0 - LZ$

where L is the uniform temperature lapse rate,

And, local atmospheric pressure at that height will be, $P = P_0(T/T_0)^{g/RL}$

where $P_0 = 101.325$ kPa $T_0 = 288.15$ K (*Mean sea level conditions*)

With $L = 6.5$ K/km, $R = 287$ J/kg K and $g = 9.81$ m/s²

$$g/RL = \frac{9.81 \text{ N/kg} \times 1000 \text{ m}}{287 \text{ J/kg K} \times 6.5 \text{ K}}\left|\frac{1 \text{ J}}{1 \text{ Nm}}\right| = 5.26$$

Temperature of air at Z = 1.6 km,

$$T = 288.15 - (6.5)(1.6) = 277.75 \text{ K}$$

Also, $$P = 101.325 \text{ kPa}\left(\frac{277.75 \text{ K}}{288.15 \text{ K}}\right)^{5.26} = 83.5 \text{ kPa}$$

$$= \left(\frac{83.5 \text{ kPa} \times 1 \text{ atm}}{101.325 \text{ kPa}}\right) = 0.824 \text{ atm}$$

Note that $\rho = \dfrac{P}{RT}$ and $\rho = f(P)$. Hence, $\nu \propto \dfrac{1}{P}$

At 1 atm and 50°C, $\nu = \dfrac{\mu}{\rho}$ = 18.2 × 10^{-6} m²/s

∴ At 0.824 atm and 50°C: $\nu = \dfrac{18.2}{0.824} \times 10^{-6}$ = 22.08 × 10^{-6} m²/s

Reynolds number, $Re_L = \dfrac{VL}{\nu} = \dfrac{(5 \text{ m/s})(0.02 \text{ m})}{22.08 \times 10^{-6} \text{ m}^2/\text{s}}$

= 4528.5 (< 5 × 10^5) ⇒ The flow is ***laminar***.

As Pr > 0.6 and the flow is *laminar*, for *uniform heat flux*:

$$Nu_x = 0.453\ (Re_x)^{1/2}\ (Pr)^{1/3}$$

∴ $$Nu_L = \frac{h_L L}{k} = 0.453\ (4.528.5)^{1/2}\ (0.704)^{1/3} = 27.12$$

Hence, the heat-transfer coefficient at the tip of the chip,

$$h_L = Nu_L \frac{k}{L} = 27.12 \times \frac{0.028 \text{ W/m K}}{0.02 \text{ m}} = 37.968 \text{ W/m}^2 \text{ K}$$

Maximum permissible rate of heat dissipation is

$$\dot{Q} = h_L A_s (T_{s(L)} - T_\infty) = (37.968 \text{ W/m}^2 \text{ K})(0.02 \times 0.02 \text{ m}^2)(80 - 20)\text{°C or K}$$
$$= \mathbf{0.911 \text{ W}}$$ **(Ans.)**

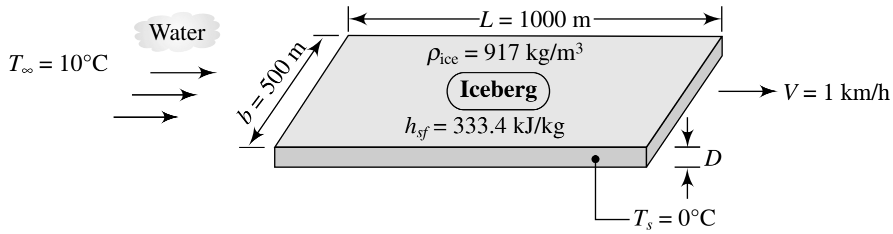
EXAMPLE 7.9 *Several schemes have been proposed to supply arid (desert) regions with fresh water. One plan involves towing icebergs from the polar regions to dry regions that require fresh water. Consider an icebergs 1000 m long and 500 m wide, which is towed through 10°C water at a velocity of 1 km/h. The density of ice is 333.4 kJ/kg. Determine (a) the average rate at which the flat bottom of the iceberg will melt (in mm/h), (b) the amount of ice in million tonnes that will melt if the voyage is 5000 km long.*
Properties of water at 5°C are
ρ = *999.9 kg/m³*, μ = *1.519 × 10^{-3} kg/m s*, k = *0.571 W/m °C*, Pr = *11.2*
Ice: ρ = *917 kg/m³*, h_{sf} = *333.4 kJ/kg*

Solution

Known An iceberg of given dimensions is being towed through water.

Find (a) Rate of melting of ice (mm/h), (b) Mass of ice melted over a distance of 5000 km (*million tones*).

Schematic



Assumptions Heat transfer is essentially between the bottom surface of the iceberg and the surrounding water.

Analysis ***Energy balance***:

Heat transferred by forced convection from water to ice = Heat received by ice for its melting

i.e., $$\bar{h}_L A(T_\infty - T_s) = \dot{m}_{ice} h_{sf} = \frac{d}{dt}(\rho_{ice} AD) h_{sf} = (\rho_{ice} A h_{sf})\frac{dD}{dt}$$

where D is the depth (thickness) of the iceberg and $\frac{dD}{dt}$ is the rate of melting or recession rate

Hence, $$\frac{dD}{dt} = \frac{\bar{h}_L (T_\infty - T_s)}{\rho_{\text{ice}} h_{sf}} \tag{A}$$

Reynolds number, $$Re_L = \frac{\rho V L}{\mu} = \frac{(999.9 \text{ kg/m}^3)(1 \times 10^3/3600 \text{ m/s})(1000 \text{ m})}{1.519 \times 10^{-3} \text{ kg/m s}}$$

$$= 182.85 \times 10^6 \ (> 10^8) \quad \Rightarrow \quad \textit{Turbulent flow}$$

The appropriate correlation is

$$\overline{Nu}_L = 0.037(Re_L)^{0.8} (Pr)^{1/3} = 0.037 \ (182.85 \times 10^6)^{0.8} \ (11.2)^{1/3}$$

$$= 337 \times 10^3 = \frac{\bar{h}_L L}{k}$$

Average convective heat-transfer coefficient is

$$\bar{h}_L = \overline{Nu}_L \frac{k}{L} = \frac{337 \times 10^3 \times 0.571}{1000} = 192.42 \text{ W/m}^2 \text{ K}$$

Substituting this value in Eq. (A), one gets

$$\frac{dD}{dt} = \frac{(192.42)(10-0)}{917 \times 333.4 \times 10^3}\left[\frac{\text{W}}{\text{m}^2\text{K}} \times \frac{\text{K}}{\text{kg}} \times \frac{\text{m}^3}{\text{J}} \times \text{kg}\left(\frac{1 \text{ J}}{1 \text{ Ws}}\right) \times \frac{3600 \text{ s}}{1 \text{ h}} \times \frac{10^3 \text{ mm}}{1 \text{ m}}\right]$$

$= \mathbf{22.66\ mm/h}$ **(Ans.) (a)**

Amount of ice melted,

$$m_{\text{ice}} = \dot{m}_{\text{ice}} \Delta t = \dot{m}_{\text{ice}} \times \frac{\text{Distance traversed}}{\text{Velocity}} = (\rho_{\text{ice}} A)\frac{dD}{dt} \times \frac{s}{V}$$

$$= 917 \text{ kg/m}^3 \times 1000 \text{ m} \times 500 \text{ m} \times 22.66 \times 10^{-3} \text{ m/h} \times \frac{5000 \text{ km}}{1 \text{ km/h}}$$

$$= 5.2 \times 10^{10} \text{ kg}\left(\frac{1 \text{ tonne}}{10^3 \text{ kg}}\right)\left(\frac{1 \text{ million tonne}}{10^6 \text{ tonne}}\right) = \textbf{52 million tonnes}$$ **(Ans.) (b)**

EXAMPLE 7.10 *A refrigeration truck is travelling at 90 km/h on a national highway in Rajasthan where the ambient temperature is 40°C. The refrigerated compartment of the truck can be considered to be a 3 m wide, 2 m high, and 6 m long rectangular box. The refrigeration system of the truck can provide 33.5 tons of refrigeration (1 ton of refrigeration = 211 kJ/min). Assume the air flow over the entire outer surface of the compartment to be turbulent and the heat-transfer coefficient to be same for all sides, determine the average outer surface temperature of the truck's refrigerated compartment, neglecting radiation effects. The following property table may be used:*

Temperature (°C)	20 °C	25 °C	30 °C	35 °C	40 °C
k(W/m °C)	0.025 14	0.025 51	0.025 88	0.026 25	0.026 62
$\nu \times 10^6$ (m²/s)	15.16	15.62	16.08	16.55	17.02
Pr	0.7309	0.7296	0.7282	0.7268	0.7255

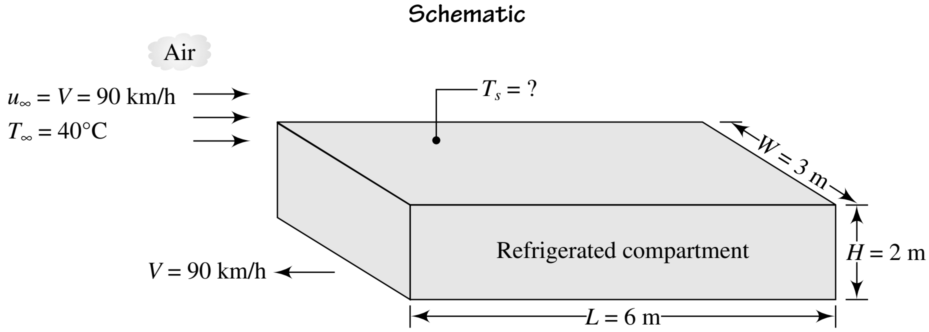
If the truck has to pass through a desert region where the ambient air temperature is 50°C, by how much will the tonnage of the chiller increase for the same surface temperature?

Solution

Known Dimensions of the refrigerated rectangular compartment of a travelling truck. Ambient temperature and cooling capacity.

Find Average surface temperature of the compartment, T_s(°C).

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Entire outer surface is turbulent. (4) Radiation effects are negligible.

Analysis Cooling load of the refrigerated compartment,

$$\dot{Q} = 33.5 \text{ TR } (211/60) \text{ kW/1 TR} = 117.8 \text{ kW}$$

Total surface area, A_s = 2[LH + LW +WH = 2[(6 × 2) + (6 × 3) + (3 +2)]m² = 72 m²

Neglecting radiation heat transfer, from energy balance:

$$\begin{pmatrix}\text{Rate of heat extraction from} \\ \text{the refrigerated compartment}\end{pmatrix} = \begin{pmatrix}\text{Heat transferred from} \\ \text{the air to the surface}\end{pmatrix}$$

That is, $\dot{Q} = \bar{h}A_s(T_\infty - T_s)$

where h is the average heat transfer coefficient.

Hence, the heat flux is

$$q_s = \bar{h}(T_\infty - T_s) = \frac{\dot{Q}}{A_s} = \frac{117.8 \times 10^3 \text{ W}}{72 \text{ m}^2} = 1636 \text{ W/m}^2$$

Average surface temperature,

$$T_s = T_\infty - \frac{q_s}{\bar{h}} = 40 - \frac{1636}{\bar{h}}$$

For forced convection heat transfer over a flat plate, if the boundary layer is *turbulent,* the appropriate correlation is

$$\overline{Nu}_L = 0.037(Re_L)^{4/5}(Pr)^{1/3} = \frac{\bar{h}L}{k}$$

$$\therefore \qquad \bar{h} = 0.037\frac{k}{L}(Re_L)^{0.8}(Pr)^{1/3}$$

Since $\bar{h}$ is dependent on the properties of air to be evaluated at the film temperature, $T_f = \frac{1}{2}(T_s + T_\infty)$ but T_s is unknown, we have to go in for *trial-and-error* approach. Clearly, T_s has to be less than T_∞ and looking at the cooling capacity of 33.5 TR, let

T_s = 10°C as a first guess.

Then $$T_f = \frac{1}{2}(T_s + T_\infty) = \frac{10+40}{2} = 25°\text{C}$$

From the property table provided, at 25°C:

k = 0.025 51 W/m °C, Pr = 0.7296 and $\nu = 15.62 \times 10^{-6}$ m²/s

Hence, $$Re_L = \frac{VL}{\nu} = \frac{u_\infty L}{\nu} = \frac{(90/3.6)\text{m/s} \times 6 \text{ m}}{15.62 \times 10^{-6} \text{ m}^2/\text{s}} = 9.6 \times 10^6$$

$$\therefore \qquad \bar{h} = \frac{0.037 \times 0.02551 \text{ W/m °C}}{6 \text{ m}}(9.6 \times 10^6)^{0.8}(0.7296)^{1/3} = 54.6 \text{ W/m}^2 \text{ °C}$$

Surface temperature,

T_s = 40 – (1636/54.6) = **10°C** **(Ans.)**

As the assumed and calculated values of T_s are same, no more trial is warranted.

If T_∞ and $T_s = 10°\text{C}, T_f = \frac{1}{2}(T_s + T_\infty) = 30°\text{C}$. Using the properties given in the table at 30°C,

$$\bar{h} = \frac{0.02588 \text{ W/m °C}}{6 \text{ m}} \times 0.037\left(\frac{(90/3.6)\text{m} \times 6 \text{ m}}{16.08 \times 10^{-6} \text{ m}^2/\text{s}}\right)^{0.8} (0.7282)^{1/3} = 54 \text{ W/m}^2 \text{ °C}$$

Cooling load, $\dot{Q} = 54 \text{ W/m}^2 \text{ °C} \times 72 \text{ m}^2 \times (50 - 10)°\text{C} = 155.5$ kW = 44.22 TR

Percent increase in the tonnage of the chiller

$$= \frac{44.22 - 33.5}{33.5} \times 100 = \mathbf{32\%}$$ **(Ans.)**

EXAMPLE 7.11 *Consider two air streams in parallel flow over opposite surfaces of a thin, 2 m long flat plate. One air stream has a temperature of 180°C and a free stream velocity of 50 m/s while the other air stream has a temperature of 30°C and a free stream velocity of 5 m/s. Determine (a) the heat flux between the two air streams at the midpoint of the plate, and (b) the plate temperature.*
Properties of atmospheric air:

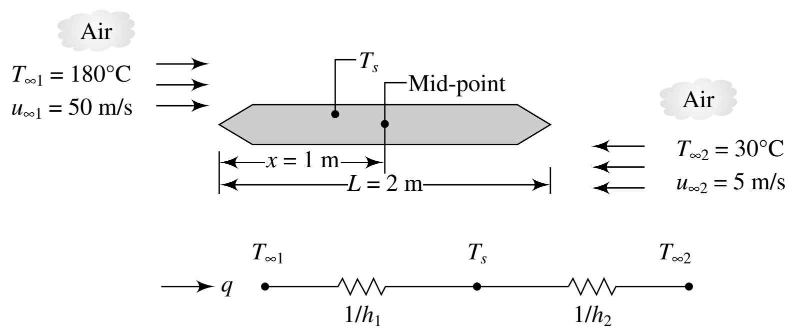
T (°C)	k (W/m °C)	ν (m²/s)	Pr
180	0.036 46	32.12×10^{-6}	0.6992
30	0.025 88	16.08×10^{-6}	0.7282

Solution

Known Prescribed velocity and temperature on the opposite sides of a flat plate.

Find (a) Heat flux between the streams at midpoint of the plate. (b) Plate temperature.

Schematic



Assumptions (1) Air streams are at 1 atm. (2) Critical Reynolds number, $Re_{cr} = 5 \times 10^5$. (3) Axial conduction along the plate is negligible.

Analysis Let us first evaluate Reynolds number at midpoint ($x = L/2$).

Stream 1: $$Re_1 = \frac{u_{\infty_1} x}{\nu_1} = \frac{50 \text{ m/s} \times 1.0 \text{ m}}{32.12 \times 10^{-6} \text{ m}^2/\text{s}}$$

$$= 1.557 \times 10^6 \qquad (> 5 \times 10^6) \qquad \Rightarrow \qquad \textit{Turbulent}$$

Stream 2: $$Re_2 = \frac{u_{\infty_2} x}{\nu_2} = \frac{5 \text{ m/s} \times 1.0 \text{ m}}{16.08 \times 10^{-6} \text{ m}^2/\text{s}}$$

$$= 3.11 \times 10^5 \qquad (< 5 \times 10^5) \qquad \Rightarrow \qquad \textit{Laminar}$$

Hence, at the midpoint stream 1 is *turbulent* and stream 2 is *laminar*. Using the appropriate empirical correlations, the local heat-transfer coefficients at the midpoint for the two cases are

$$h_1 = \frac{k_1}{x} Nu_{x,1} = \frac{k_1}{x}(0.0296) Re_{x,1}^{4/5}\, Pr_1^{1/3}$$

$$= \frac{0.03646 \text{ W/m °C}}{1 \text{ m}}(0.0296)(1.55 \times 10^6)^{0.8}(0.6992)^{1/3} = 86.13 \text{ W/m}^2 \text{ °C}$$

$$h_2 = \frac{k_2}{x} Nu_{x,2} = \frac{k_2}{x}(0.332) Re_{x,2}^{1/2} \, Pr_2^{1/3}$$

$$= \frac{0.02588 \text{ W/m °C}}{1 \text{ m}} (0.332)(3.11 \times 10^5)^{1/2} (0.7282)^{1/3} = 4.31 \text{ W/m}^2 \text{ °C}$$

Referring to the thermal circuit, the heat flux at the midpoint of the plate is

$$q = \frac{\Delta T_{\text{overall}}}{R_{\text{total}}} = \frac{T_{\infty 1} - T_{\infty 2}}{\left(\frac{1}{h_1}\right) + \left(\frac{1}{h_2}\right)} = \frac{(180 - 30)\text{°C}}{\left\{\left(\frac{1}{86.13}\right) + \left(\frac{1}{4.31}\right)\right\} \frac{\text{m}^2 \text{°C}}{\text{W}}} = \mathbf{616 \ W/m^2} \quad \textbf{(Ans.) (a)}$$

We note that

$$q = h_1 (T_{\infty 1} - T_s) = h_2 (T_s - T_{\infty 2}) \quad \Rightarrow \quad \left(1 + \frac{h_2}{h_1}\right) T_s = T_{\infty 1} + \frac{h_2}{h_1} T_{\infty 2}$$

With $\quad \dfrac{h_2}{h_1} = \dfrac{4.31}{86.13} = 0.05,$

Plate temperature,

$$T_s = \frac{T_{\infty_1} + \frac{h_2}{h_1} \cdot T_{\infty_2}}{1 + \frac{h_2}{h_1}} = \frac{180 + (0.05) \times (30)}{1 + 0.05} = \mathbf{173°C} \qquad \textbf{(Ans.) (b)}$$

EXAMPLE 7.12 *A flat plate, 1.5 m wide and 2.5 m long, is to be maintained at 100°C in air with a free stream temperature of 20°C. Determine the velocity at which the air should flow over the flat plate so that the energy dissipation rate from the plate is 4.5 kW.*

Properties of air at 60°C:

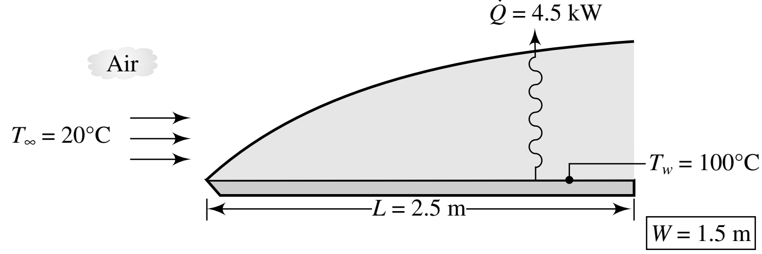
k = *28.08 × 10⁻³ W/m K,* Pr = *0.7202,* ν = *18.96 × 10⁻³ m²/s*

Solution

Known The top surface of a flat plate is to be cooled by forced air.

Find Air velocity, u_∞ (m/s).

Schematic



Assumptions (1) Steady-state conditions. (2) Constant air properties at the film temperature. (3) The critical Reynolds number is 5×10^5.

Analysis The heat-dissipation rate is

$$\dot{Q} = \bar{h} A_s (T_s - T_\infty)$$

Hence, the average heat-transfer coefficient is

$$\bar{h} = \frac{\dot{Q}}{A_s(T_s - T_\infty)} = \frac{4.5 \times 10^3 \text{ W}}{[(1.5 \times 2.5)\text{m}^2][(100-20)\text{K}]} = 15 \text{ W/m}^2\text{ K}$$

Assuming the flow to be *turbulent,* $(5 \times 10^5 \le Re_L \le 10^7)$, the average Nusselt number over the entire plate is determined to be

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = (0.037\, Re_L^{4/5} - 871) Pr^{1/3}$$

or
$$\left\{ \left(\frac{\bar{h}L}{k} \right) Pr^{-1/3} + 871 \right\} \frac{1}{0.037} = Re_L^{4/5}$$

or
$$Re_L^{4/5} = \left[\frac{(15 \text{ W/m}^2\text{ K})(2.5 \text{ m})}{0.02808 \text{ W/m K}} (0.7202)^{-1/3} + 871 \right] \frac{1}{0.037} = 63\,807$$

or
$$Re_L = 1.014 \times 10^6 > Re_{L,cr}$$

The assumption of *turbulent* flow is thus justified. Therefore, the free stream velocity is

$$u_\infty = \frac{Re_L \nu}{L} = (1.014 \times 10^6)(18.96 \times 10^{-6} \text{ m}^2/\text{s})/(2.5 \text{ m}) = \mathbf{7.69 \text{ m/s}} \qquad \textbf{(Ans.)}$$

EXAMPLE 7.13 *The crankcase of an automobile is approximately 0.6 m long, 0.2 m wide and 0.1 m deep. Assuming that the surface temperature of the crankcase is 350 K, estimate the rate of heat flow from the crankcase to atmospheric air at 276 K at a road speed of 30 m/s. Assume that the vibration of the engine and the chassis induce the transition from laminar to turbulent flow so near to the leading edge that, for practical purposes, the boundary layer is turbulent over the entire surface. Neglect radiation and use for the front and rear surfaces the same average convective heat-transfer coefficient as for the bottom and sides. Properties of air at the mean film temperature of 313 K:*

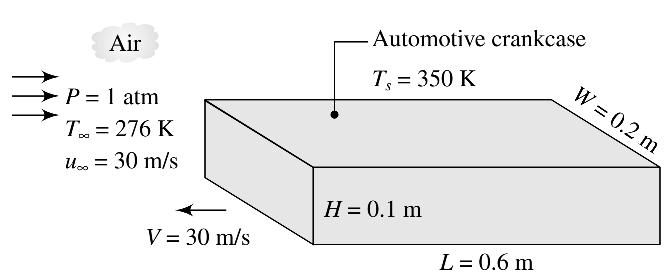
ρ = 1.127 kg/m^3, μ = 19.18 × 10^{-6} kg/ms, Pr = 0.7255, k = 0.02662 W/m K

Solution

Known Crankcase of an automobile loses heat by forced convection to atmospheric air.

Find Heat-loss rate, $\dot{Q}$(W).

Schematic



Assumptions (1) Steady operating conditions. (2) Constant air properties. (3) Radiation is neglected. (4) Flow is fully turbulent from the leading edge.

Analysis Reynolds number,

$$Re_L = \frac{\rho u_\infty L}{\mu} = \frac{1.127\ \text{kg/m}^3 \times 30\ \text{m/s} \times 0.6\ \text{m}}{19.18 \times 10^{-6}\ \text{kg/m s}} = 1.058 \times 10^6$$

The average Nusselt number is

$$\overline{Nu}_L = 0.037\, Re_L^{0.8}\, Pr^{1/3} = 0.037(1.058 \times 10^6)^{0.8}(0.7255)^{1/3} = 2194$$

and, the average convective heat transfer coefficient is

$$\bar{h} = \overline{Nu}_L \frac{k}{L} = \frac{2194 \times 0.02662\ \text{W/m K}}{0.6\ \text{m}} = 97.3\ \text{W/m}^2\ \text{K}$$

The total surface area that dissipates heat is

$$A_s = 2[(LW) + (LH) + (WH)]$$
$$= 2[(0.6\ \text{m} \times 0.2\ \text{m}) + (0.6\ \text{m} \times 0.1\ \text{m}) + (0.2\ \text{m} \times 0.1\ \text{m})] = 0.40\ \text{m}^2$$

The rate of heat loss from the crankcase is

$$\dot{Q} = \bar{h}A_s(T_s - T_\infty) = (97.3\ \text{W/m}^2\ \text{K})(0.40\ \text{m}^2)(350 - 276)\text{K} = \mathbf{2880\ W} \qquad \textbf{(Ans.)}$$

EXAMPLE 7.14 *The surface temperature of a thin flat plate located parallel to an air stream is 90°C. The free stream velocity is 60 m/s and its temperature is 0°C. The plate is 60 cm wide and 45 cm long in the direction of the air stream. Assume that the flow in the boundary layer changes abruptly from laminar to turbulent at a transition Reynolds number of 4 × 10^5. Neglecting the end effect of the plate, determine*

(a) the average heat-transfer coefficient in the laminar and turbulent regions, (b) the rate of heat transfer for the entire plate, considering both sides, (c) the average friction coefficient in the laminar and turbulent regions, (d) the total drag force.

Properties of air at 1 atm and 45 °C are

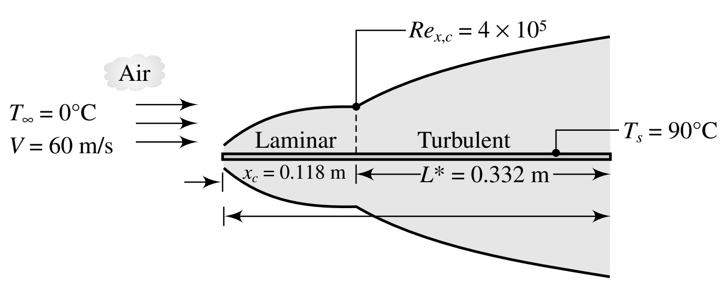
$$k = 27.63 \times 10^{-3}\ \text{W/m°C},\quad \nu = 17.7 \times 10^{-6}\ \text{m}^2/\text{s},\quad Pr = 0.704$$

Solution

Known Air flow over an isothermal flat plate under specified conditions.

Find (a) $\bar{h}_{\text{lam}}, \bar{h}_{\text{turb}}$; (b) $\dot{Q}$; (c) $\bar{C}_{f,\text{lam}}, \bar{C}_{f,\text{turb}}$; (d) F_D.

Schematic



Assumptions (1) Steady operating conditions. (2) Air is an ideal gas. (3) Constant properties. (4) End effects of the plate are negligible. (5) Transition Reynolds number, $Re_c = 4 \times 10^5$.

Analysis (a) Laminar boundary layer exists upto critical $Re = 4 \times 10^5$

$$Re_{xc} = \frac{Vx_c}{\nu} = 4 \times 10^5$$

Distance from the leading edge up to which the flow will be *laminar* is determined to be

$$x_c = \frac{(4 \times 10^5)(17.7 \times 10^{-6}\ \text{m}^2/\text{s})}{60\ \text{m/s}} = 0.118\ \text{m} \quad \text{or} \quad 11.8\ \text{cm}$$

Average heat-transfer coefficient in the *laminar region* is

$$\bar{h}_{\text{lam}} = \overline{Nu}\frac{k}{x_c} = 0.664\frac{k}{x_c}(Re_{x,c})^{1/2}\, Pr^{1/3}$$

$$= 0.664 \times \frac{0.02763\ \text{W/m °C}}{0.118\ \text{m}} \times (4 \times 10^5)^{1/2}(0.704)^{1/3}$$

$$= \mathbf{87.48\ W/m^2\ °C} \qquad \textbf{(Ans.) (a)}$$

Turbulent flow ranges between $x = x_c = 0.118$ m and $x = L = 0.45$ m. Average heat-transfer coefficient is

$$\bar{h}_{\text{turb}} = \frac{1}{L^*}\int_{x_c}^{L} h_{\text{turb}}\, dx \quad \text{where} \quad L^* = L - x_c$$

where $$h_{\text{turb}} = Nu_x\frac{k}{x} = 0.0296\, Re_x^{0.8} Pr^{1/3}\frac{k}{x}$$

$\therefore$ $$\bar{h}_{\text{turb}} = \frac{1}{L^*} \times 0.0296\, Pr^{1/3}\left(\frac{V}{\nu}\right)^{0.8} k\int_{x_c}^{L} x^{-0.2}dx$$

$$= \frac{1}{L^*} \times 0.0296\, Pr^{1/3} \times k\left(\frac{V}{\nu}\right)^{0.8}\left[\frac{x^{0.8}}{0.8}\right]_{x_c}^{L}$$

$$= 0.037\,\text{Pr}^{1/3}\,\frac{k}{L^*}\left(\frac{V}{\nu}\right)^{0.8}[L^{0.8} - x_c^{0.8}]$$

$$= 0.037 \times 0.704^{1/3} \times \frac{0.02763}{0.332} \times \left(\frac{60}{17.7 \times 10^{-6}}\right)^{0.8}[0.45^{0.8} - 0.118^{0.8}]$$

$$= \mathbf{159.26\ W/m^2\ °C} \qquad \textbf{(Ans.) (a)}$$

(b) Rate of heat transfer for the entire plate considering *both* sides is

$$\dot{Q} = \dot{Q}_{\text{lam}} + \dot{Q}_{\text{turb}} = (\bar{h}_{\text{lam}}A_{\text{lam}} + \bar{h}_{\text{turb}}A_{\text{turb}})(T_s - T_\infty)$$

$$= [\{(87.48)(2 \times 0.6 \times 0.118)\} + \{(159.26)(2 \times 0.6 \times 0.332)\}](90 - 0)$$

$$= \mathbf{6825\ W} \qquad \textbf{(Ans.) (b)}$$

(c) Average friction coefficient, $\bar{C}_f$ in the laminar region (up to $x_c = 0.118$ m) is

$$\bar{C}_{f,\text{lam}} = \frac{1.328}{\sqrt{Re_{\text{lam}}}} = \frac{1.328}{(4\times10^5)^{1/2}} = \mathbf{2.1\times10^{-3}} \qquad \textbf{(Ans.) (c)}$$

$$\bar{C}_{f,\text{turb}} = \frac{1}{L^*}\int_{x_c}^{L} C_{f,x}\,dx = \frac{1}{L^*}\times 0.0592\int_{x_c}^{L}\left(\frac{V}{\nu}\right)^{-0.2} x^{-0.2}dx$$

$$= \frac{0.0592}{0.332}\times\left(\frac{60}{17.7\times10^{-6}}\right)^{-0.2}\times\frac{1}{0.8}[L^{0.8}-x_c^{0.8}]$$

$$= 8.814\times10^{-3}(0.45^{0.8}-0.118^{0.8}) = \mathbf{3.823\times10^{-3}} \qquad \textbf{(Ans.) (c)}$$

(d) The force experienced by the plate is

$$F_D = \bar{C}_{f,L}\frac{1}{2}\rho A_s V^2$$

where $\rho = \dfrac{P}{RT} = \dfrac{101.325\text{ kPa}}{(0.287\text{ kJ/kg K})(45+273.15)\text{K}} = 1.11\text{ kg/m}^3$

For both sides of the plate,

$$F_D = \bar{C}_{f,\text{lam}}\frac{1}{2}\rho A_{\text{lam}}V^2 + \bar{C}_{f,\text{turb}}\frac{1}{2}\rho A_{\text{turb}}V^2$$

$$= \frac{1}{2}\rho V^2[\bar{C}_{f,\text{lam}}A_{\text{lam}} + \bar{C}_{f,\text{turb}}A_{\text{turb}}]$$

$$= \frac{1}{2}\times1.11\times60^2[\{(2.1\times10^{-3})(2\times0.6\times0.118)+(3.823\times10^{-3})(2\times0.6\times0.332)\}]$$

$$= \mathbf{3.637\text{ N}} \qquad \textbf{(Ans.) (d)}$$

EXAMPLE 7.15 *Consider the flow of air at* T_∞ *= 24°C and* P *= 1 bar with a free stream velocity of* V *= 3 m/s along a flat plate of length* L *(*equal to its width, W*) with its surface held at a uniform temperature of 130°C. The drag force on the plate is measured experimentally to be 15.7 mN. Assuming laminar flow and using the Chilton Colburn analogy, calculate the rate of heat transfer from the plate to the air.*

Properties of air at 1 bar and 77°C are:

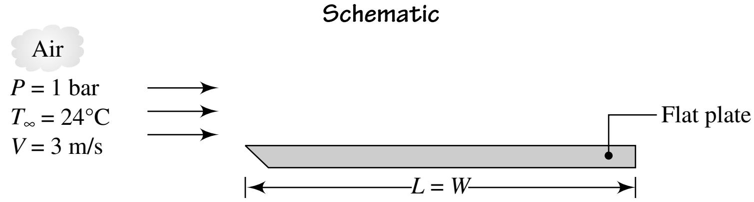
ρ = 0.995 kg/m^3, ν = 20.92 × 10^{-6} m^2/s, Pr = 0.700, C_p = 1.009 kJ/kg°C

Solution

Known Temperature and velocity of free stream air flowing over a square flat plate. Plate temperature and drag force.

Find Heat-transfer rate.

Schematic



Assumptions (1) Steady, incompressible flow. (2) Uniform plate surface temperature, (3) Pressure drag is negligible.

Analysis Drag force acting on the entire plate is, $F_D = \overline{C}_{f,L} A_s \dfrac{\rho V^2}{2}$

For laminar flow over the entire plate, the average friction coefficient or drag coefficient is

$$C_D = \overline{C_{f,L}} = 1.328\, Re_L^{-1/2} = 1.328\left(\frac{VL}{\nu}\right)^{-1/2}$$

Also $A_s = LW = L^2$ (*since* $L = W$)

Therefore, $F_D = 1.328\left(\dfrac{\nu}{VL}\right)^{1/2} \times L^2 \times \dfrac{1}{2}\rho V^2$

or $$L^{3/2} \times 1.328\left(\frac{20.92\times10^{-6}}{3}\right)^{0.5} \times 0.5 \times 0.995 \times 3^2 = 0.0157$$

Plate length is then, $L = \left(\dfrac{0.0157}{0.0157}\right)^{2/3} = 1.0 \text{ m}$

$$\overline{C_{f,L}} = 1.328\left(\frac{3 \text{ m/s}\times 1 \text{ m}}{20.92\times10^{-6} \text{ m}^2/\text{s}}\right)^{-1/2} = 3.507\times10^{-3}$$

Using the *Chilton–Colburn* analogy, $\overline{St}\, Pr^{2/3} = \dfrac{\overline{C_{f,L}}}{2}$

where $$\overline{St} = \frac{\overline{Nu}}{Re\, Pr} = \frac{\overline{h}L}{k}\frac{\mu}{\rho VL}\frac{k}{C_p\mu} = \frac{\overline{h}}{\rho V C_p}$$

It follows that

$$\overline{h} = \frac{\rho V C_p \overline{C_{f,L}}}{2\, Pr^{2/3}} = \frac{0.995\times3\times1.009\times10^3\times3.507\times10^{-3}}{2\times(0.7)^{2/3}} = 6.7 \text{ W/m}^2\,°\text{C}$$

Heat-transfer rate from the plate is

$$\dot{Q} = \overline{h}(LW)(T_s - T_\infty) = (6.7 \text{ W/m}^2\,°\text{C})(1 \text{ m}\times1 \text{ m})(130-24)°\text{C} = \mathbf{710\ W} \quad \text{(Ans.)}$$

EXAMPLE 7.16 *Electronic components mounted on a circuit board dissipating 30 mW are cooled by atmospheric air at 27°C and 12 m/s. A chip of 4 mm length and 4 mm width is located 12.5 cm from the leading edge. Estimate the chip surface temperature. The appropriate correlation for this situation is,* $\text{Nu}_x = 0.04\, \text{Re}_x^{0.85}\, \text{Pr}^{0.333}$

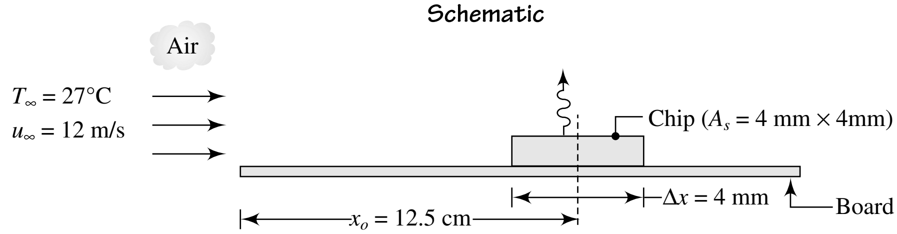
Properties of air at 35°C: $k = 0.0269$ W/m °C, $\nu = 16.7\times10^{-6}$, Pr = 0.706

Solution

Known A chip dissipating heat is cooled by forced air flowing over it.

Find Surface temperature, T_s.

Schematic



Assumptions (1) Heat is transferred from the upper surface of the chip. (2) Constant air properties. (3) Average convection coefficient over the chip length equals the local value at the centre position.

Analysis Heat-transfer rate by forced convection from the chip's top surface to the ambient air is

$$\dot{Q}_{\text{chip}} = \bar{h}_{\text{chip}} A_s (T_s - T_\infty) \quad \Rightarrow \quad T_s = T_\infty + \frac{\dot{Q}_{\text{chip}}}{\bar{h}_{\text{chip}} A_s}$$

Using the given correlation,

$$Nu_x = 0.04\, Re_x^{0.85}\, Pr^{0.333} = 0.04 \left[\frac{u_\infty x}{\nu}\right]^{0.85} (Pr)^{0.333}$$

Average convection coefficient

$$\therefore \qquad \bar{h} = h_{x_o} = Nu_x \frac{k}{x_o} = 0.04 \left[\frac{12 \text{ m/s } 0.125 \text{ m}}{16.7 \times 10^{-6} \text{ m}^2/\text{s}}\right]^{0.85} (0.706)^{0.333} \times \frac{0.0269 \text{ W/m °C}}{0.125 \text{ m}}$$

$$= 124.4 \text{ W/m}^2 \text{ °C}$$

$\therefore$ Surface temperature,

$$T_s = 27°\text{C} + \frac{30 \times 10^{-3} \text{ W}}{(0.004 \text{ m})^2 (124.4 \text{ W/m}^2 \text{ °C})} = \mathbf{42°C} \qquad \textbf{(Ans.)}$$

Comment T_f = (42 + 27)/2 = 34.5 °C. Properties given were at 35°C. **Hence OK.**

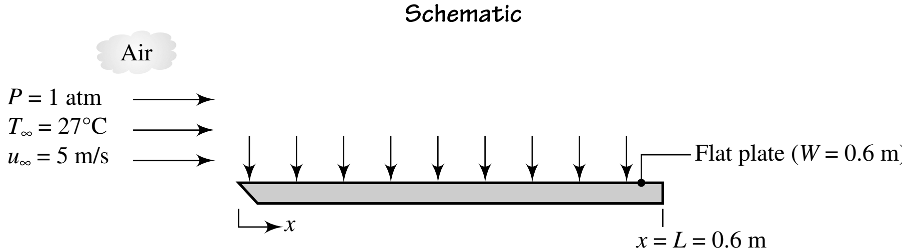
EXAMPLE 7.17 *A 0.6 m × 0.6 m electrically heated plate with the heater rating of 1 kW is placed in an air stream at 1 atm 27°C with a free stream velocity of 5 m/s. Calculate (a) the average temperature difference along the plate, (b) the average convection heat-transfer coefficient, and (c) the temperature of the plate at the trailing edge.*

Solution

Known Flat plate with constant heat flux exposed to flowing air stream.

Find (a) $\overline{T_s - T_\infty}$; (b) $\bar{h}$; (c) $T_{s,L}$

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Air is an ideal gas.

Analysis In this case, the plate is subjected to constant heat flux,

$$q_s = \dot{Q}/A_s = \frac{1000 \text{ W}}{0.6 \text{ m} \times 0.6 \text{ m}} = 2.78 \times 10^3 \text{ W/m}^2$$

Since the plate temperature is not known, we cannot evaluate the air properties at the film temperature $\left[T_f = \frac{1}{2}(T_s + T_\infty)\right]$.

Hence, initial calculations will be based on the free stream air temperature, of 27°C or 300 K.
At 300 K, k = 0.0263 W/m °C, ν = 15.89 × 10^{-6} m²/s, Pr = 0.707
Reynolds number,

$$Re_L = \frac{u_\infty L}{\nu} = \frac{(5 \text{ m/s})(0.6 \text{ m})}{15.89 \times 10^{-6} \text{ m}^2/\text{s}} = 1.888 \times 10^5 \quad (< 5 \times 10^5) \quad \Rightarrow \quad \text{Laminar flow}$$

The average temperature difference is

$$\overline{T_s - T_\infty} = \frac{q_s L/k}{0.6795\, Re_L^{1/2} Pr^{1/3}}$$

$$= \frac{[(2.78 \times 10^3 \text{ W/m}^2)(0.6 \text{ m})/0.0263 \text{ W/m °C}]}{0.6795(1.888 \times 10^5)^{1/2}(0.707)^{1/3}} = 240.9\text{°C}$$

or T_s = 240.9 + 27 = 267.9°C

Film temperature, $T_f = \frac{T_s + T_\infty}{2} = \frac{267.9 + 27}{2} = 147.45\text{°C} \approx 420 \text{ K}$

Properties at 420 K:

k = 0.0352 W/m °C, ν = 28.85 × 10^{-6} m²/s, Pr = 0.688

$$Re_L = \frac{(0.6)(5)}{28.85 \times 10^{-6}} = 1.04 \times 10^5$$

$$\overline{T_s - T_\infty} = \frac{(2.78 \times 10^3)(0.6)/0.0352}{0.6795(1.04 \times 10^5)^{1/2}(0.688)^{1/3}} = 244.5\text{°C}$$

For laminar flow over a flat plate subjected to constant heat flux,

$$T_f = \frac{244.5 + 27 + 27}{2} = 149\text{°C} \approx 422 \text{ K}$$

Properties of air at 422 K are

k = 0.0354 W/m °C, ν = 29 × 10^{-6} m²/s, Pr = 0.688

$$Re_L = \frac{0.6 \times 5}{29 \times 10^{-6}} = 1.035 \times 10^6$$

$$\overline{T_s - T_\infty} = \frac{(2.78 \times 10^3 \times 0.6/0.0354)}{0.6795(1.035 \times 10^5)^{1/2}(0.688)^{1/3}} = 244\text{°C}$$

Since this value is not much different from 244.5°C, we have
Average temperature difference along the plate

$$\overline{T_s - T_\infty} = \mathbf{244°C} \qquad \textbf{(Ans.) (a)}$$

Local heat-transfer coefficient at $x = L$ is

$$h_{x=L} = Nu_L \frac{k}{L} = 0.453(Re_L)^{1/2}(Pr^{1/3})\frac{k}{L}$$
$$= 0.453(1.035 \times 10^6)^{1/2}(0.688)^{1/3}(0.0354 \text{ W/m °C})/0.6 \text{ m} = 7.59 \text{ W/m}^2 \text{ °C}$$

Average convection heat-transfer coefficient is

$$\bar{h} = 2h_{x=L} = 2 \times 7.59 \text{ W/m}^2 \text{ °C} = \mathbf{15.18\ W/m^2\ °C} \qquad \textbf{(Ans.) (b)}$$

Temperature of the plate at the trailing edge is

$$T_{s,L} = T_\infty + \frac{q_s L}{Nu_L k} = T_\infty + \frac{q_s}{h_{x=L}} = 27\text{°C} + \left(\frac{2.78 \times 10^3 \text{ W/m}^2}{7.59 \text{ W/m}^2 \text{ °C}}\right) = \mathbf{393°C} \quad \textbf{(Ans.) (c)}$$

EXAMPLE 7.18 *Air at atmospheric pressure and a temperature of 225°C flows over a flat plate with a velocity of 6 m/s. The plate is 15 cm wide and is maintained at a temperature of 75°C. Calculate (a) the thickness of the velocity, (b) thermal boundary layers, and (c) the local heat-transfer coefficient at a distance of 0.5 m from the leading edge. (d) Also calculate the drag force exerted on the plate, and (e) the rate of heat transfer to the plate over a length of 0.5 m. Assume that the flow is over both sides of the plate. Assume cubic velocity and temperature profile and the integral method of analysis.*
Properties of air at 1 atm and 150°C:

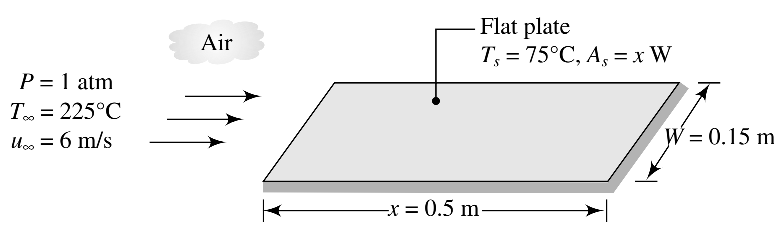
k = *0.03541 W/m K,* ν = *29.16 × 10⁻⁶ m²/s,* Pr = *0.688,* ρ = *0.8346 kg/m³*

Solution

Known Plate dimensions and temperature. Velocity and temperature of air flowing over the plate.

Find (a) δ ; (b) δ_T ; (c) $h_{(x\,=\,0.5\text{ m})}$; (d) F_D ; (e) $\dot{Q}$;

Schematic



Assumptions (1) Air is an ideal gas. (2) Uniform plate-surface temperature. (3) Constant air properties. (4) Radiation heat transfer is not considered. (5) Critical Reynolds number is, $Re_{cr} = 5 \times 10^5$. (6) Cubic velocity profile, $\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$, and integral approximation method.

Analysis Reynolds number at x = 0.0.5 m is

$$Re_x = \frac{u_\infty x}{\nu} = \frac{6 \text{ m/s} \times 0.5 \text{ m}}{29.16 \times 10^{-6} \text{ m}^2/\text{s}} = 1.029 \times 10^5 \qquad (< 5 \times 10^5)$$

∴ *Laminar* boundary layer exists.

(a) Velocity boundary-layer thickness,

$$\delta = \frac{4.64x}{(Re_x)^{1/2}} = \frac{4.64 \times 0.5 \text{ m}}{(1.029 \times 10^5)^{1/2}} = 7.23 \times 10^{-3} \text{ m} = \mathbf{7.23 \text{ mm}} \qquad \textbf{(Ans.) (a)}$$

(b) Thermal boundary-layer thickness,

$$\delta_T = \frac{\delta}{1.026\, Pr^{1/3}} = \frac{7.23 \text{ mm}}{1.026 \times 0.688^{1/3}} = 7.98 \times 10^{-3} \text{ m} = \mathbf{7.98 \text{ mm}} \qquad \textbf{(Ans.) (b)}$$

(c) Local heat-transfer coefficient,

$$h_x = \frac{3k}{2\delta_T} = \frac{3 \times 0.03541 \text{ W/m K}}{2 \times 7.98 \times 10^{-3} \text{ m}} = \mathbf{6.656 \text{ W/m}^2 \text{ K}} \qquad \textbf{(Ans.) (c)}$$

Average heat-transfer coefficient,

$$\bar{h} = 2h_x = 2 \times 6.656 = 13.312 \text{ W/m}^2 \text{ K}$$

(d) Average skin friction coefficient,

$$\bar{C}_{f,L} = 1.292\, Re_L^{-1/2} = (1.292)\,(1.029 \times 10^5)^{-1/2} = 0.004$$

Drag force exerted on both top and bottom surfaces of the plate,

$$\therefore \qquad F_D = 2\left(\frac{1}{2}\rho A_s u_\infty^2\right)\bar{C}_{f,L}$$

$$= 2 \times \frac{1}{2} \times 0.8346 \text{ kg/m}^3 \times 0.15 \text{ m} \times 0.5 \text{ m} \times 6^2 \text{ m}^2/\text{s}^2 \times 0.004 \left|\frac{1 \text{ N}}{1 \text{ kg m/s}^2}\right|$$

$$= \mathbf{0.009 \text{ N}} \qquad \textbf{(Ans.) (d)}$$

(e) Rate of heat transfer to the plate

$$\dot{Q} = \bar{h}(2A_s)(T_\infty - T_s) = (13.312 \text{ W/m}^2 \text{ K})\,(2 \times 0.5 \text{ m} \times 0.15 \text{ m})\,(225 - 75)\ °\text{C or K}$$

$$= \mathbf{299.5 \text{ W}} \qquad \textbf{(Ans.) (e)}$$

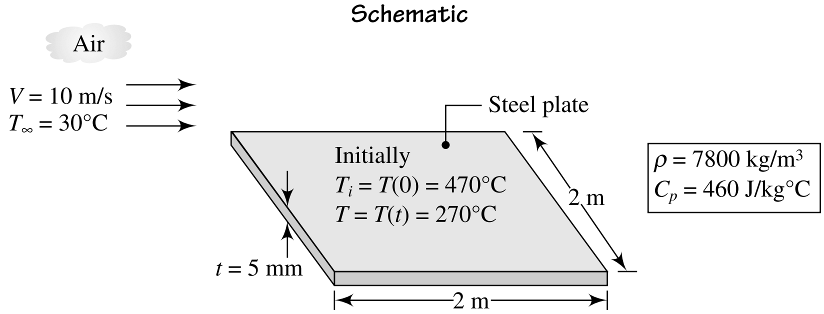
EXAMPLE 7.19 *A 2 m × 2 m steel plate of 5 mm thickness at 470°C is being cooled by blowing air at 30°C and a velocity of 10 m/s over both its surfaces. Find the time required to cool the plate to 270°C. Neglect temperature gradients in the plate and heat transfer by radiation. Assume average properties of steel as: ρ = 7800 kg/m³, C_p = 460 J/kg °C.*
Properties of air at 200°C are: k = 38.87 × 10^{-3} W/m K, ν = 35.35 × 10^{-6} m²/s, Pr = 0.685

Solution

Known Cooling of plate under forced convection conditions.

Find Time required to cool the plate.

Schematic



Assumptions (1) Lumped capacitance method is valid and internal temperature gradients are neglected. (2) Negligible radiation effects. (3) Constant properties.

Analysis Mean temperature at which properties of air are to be evaluated is, $T_f = \dfrac{\overline{T}_s + T_\infty}{2}$
where T_∞ is the ambient air temperature = 30°C
$\overline{T}_s$ is the average surface temperature of plate

$$= \frac{1}{2}[T(0) + T(t)] = \frac{1}{2}(470 + 270)°\text{C} = 370°\text{C}$$

$\therefore$ Film temperature, $T_f = \dfrac{1}{2}(370 + 30)°\text{C} = 200°\text{C}$

The properties of air at this temperature are given in the problem statement.
To determine the convection coefficient, h, let us first determine the Reynolds number to find whether the air flow is *laminar* or *turbulent*.
Reynolds number,

$$Re_L = \frac{VL}{\nu} = \frac{(10\text{ m/s})(2\text{ m})}{35.35 \times 10^{-6}\text{ m}^2/\text{s}} = 5.658 \times 10^5 \qquad (> 5 \times 10^5) \Rightarrow \textit{Turbulent flow}$$

The appropriate correlation for forced convection (*external turbulent flow*) is

$$\overline{Nu}_L = \frac{\overline{h}L}{k} = (0.037\,Re_L^{4/5} - 871)Pr^{1/3}$$

for the range $5 \times 10^5 < Re_L < 10^8$
Substituting the proper numerical values, we have

$$\overline{h} = \frac{k}{L}\overline{Nu} = \frac{k}{L}(0.037\,Re_L^{4/5} - 871)\,Pr^{1/3}$$

$$= \frac{38.87 \times 10^{-3}\text{ W/m K}}{2\text{ m}}\{0.037(5.658 \times 10^5)^{0.8} - 871\}(0.685)^{1/3} = 10.44\text{ W/m}^2\text{ K}$$

Neglecting temperature gradients in the plate, we have
Heat loss rate from *both* sides of the plate = Rate of decrease of stored (internal) energy

$$\therefore \qquad \dot{Q}_{\text{out}} = -\dot{E}_{st} = -mC_p\frac{dT}{dt} \quad \text{or} \quad \overline{h}A_s(T - T_\infty) = -\rho\forall C_p\frac{dT}{dt}$$

As $\qquad d(T - T_\infty) = dT$, one can write

$$\therefore \qquad \int_{T=T_i}^{T=T} \frac{d(T - T_\infty)}{T - T_\infty} = \frac{-\bar{h}A_s}{\rho ¥ C_p} \int_{t=0}^{t=t} dt$$

$$\ln \frac{T - T_\infty}{T_i - T_\infty} = -\frac{hA_s}{\rho ¥ C_p} t$$

Time required is

$$t = \frac{\rho ¥ C_p}{hA_s} \ln \frac{T_i - T_\infty}{T - T_\infty}$$

$$= \frac{(7800 \text{ kg/m}^3)(460 \text{ J/kg °C})(2 \text{ m} \times 2 \text{ m} \times 0.005 \text{ m})}{(10.44 \text{ W/m}^2 \text{ K})(2 \times 2 \text{ m} \times 2 \text{ m})} \times \ln \frac{(470 - 30)\text{°C}}{(270 - 30)\text{°C}}$$

$$= \mathbf{521 \text{ s}} \qquad \textbf{(Ans.)}$$

Comment Many a time, one is inclined to use the correlation $Nu = 0.037\ Re^{4/5}\ Pr^{1/3}$ if Re exceeds $Re_{cr} = 5 \times 10^5$. But there are constraints we need to pay heed to. If $Re > 10^8$, then only it is appropriate to use the above correlation meant for fully turbulent conditions. If we had used this correlation,

$$\bar{h} = \frac{0.03887}{2} \times 0.037(5.658 \times 10^5)(0.686)^{1/3}$$

$$= 25.36 \text{ W/m}^2 \text{ K as against } 10.44 \text{ W/m}^2 \text{ K}$$

Also, the time required,

$$t = \frac{\rho C_p ¥}{hA_s} \ln \frac{T_i - T_\infty}{T - T_\infty} = \left(859.4 \times \frac{10.44}{25.36}\right) \ln \frac{440}{240} = \mathbf{214 \text{ s}} \quad \text{as against } \mathbf{521 \text{ s}}$$

EXAMPLE 7.20 *Air at 27°C is blown over a 8 m long, 1.25 m wide flat plate with a fan at a velocity of 12 m/s. Air flows parallel to the 8 m long side. The average temperature of the plate is not to exceed 227°C. The location is at an elevation of 2400 m where the atmospheric pressure is 75.6 kPa. Determine (a) the rate of heat transfer between the plate and the air considering both laminar and turbulent boundary layers, (b) the percentage error involved if the boundary layer is assumed to be turbulent right from the leading edge of the plate. (c) If the air flows parallel to the 1.25 m side, find the rate of heat loss. Assume the heat transfer from the back side of the plate to be negligible and disregard radiation.*

Thermophysical properties of air at 1 atm, and 127°C:

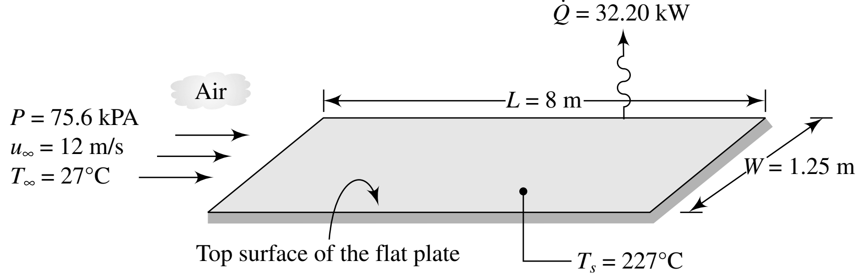
$$k = 33.8 \times 10^{-3} \text{ W/m K}, \ \nu = 26.41 \times 10^{-6} \text{ m}^2\text{/s}, \ Pr = 0.690$$

Solution

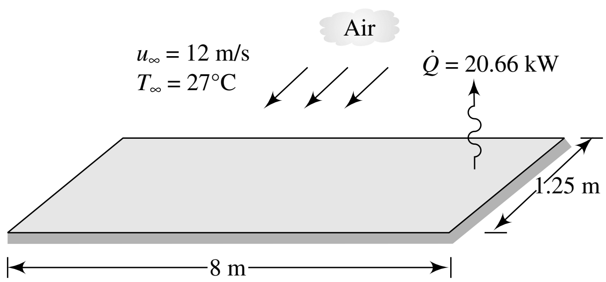
Known The top surface of a flat plate is cooled by forced air at a specified altitude.

Find Heat-transfer rate, $\dot{Q}$: (a) Combining *laminar* and *turbulent* flow, (b) Considering only *turbulent* flow, (c) Flow parallel to 1.25 m side.

Schematic



(a) Flow along the longer side



(b) Flow along the shorter side

Assumptions (1) Steady operating conditions prevail. (2) The critical Reynolds number is 5×10^5. (3) Radiation effects are disregarded. (4) Air is an ideal gas.

Analysis (a) Note that ν is inversely proportional to density and thus to pressure for an ideal gas.

$$\nu = \frac{\mu}{\rho} \text{ and } \rho = \frac{P}{RT}$$

$$\nu_{@1\text{atm}} = 26.41 \times 10^{-6} \text{ m}^2/\text{s}$$

$$P = 75.6 \text{ kPa} \times \frac{1 \text{ atm}}{101.325 \text{ kPa}} = 0.746 \text{ atm}$$

$$\therefore \qquad \upsilon_{@0.746\text{atm}} = \frac{\nu_{@1\text{atm}} \times 1 \text{ atm}}{0.746 \text{ atm}} = \frac{26.41 \times 10^{-6} \text{ m}^2/\text{s}}{0.746} = 35.4 \times 10^{-6} \text{ m}^2/\text{s}$$

Reynolds number,

$$Re_L = \frac{u_\infty L}{\nu} = \frac{(12 \text{ m/s})(8 \text{ m})}{35.4 \times 10^{-6} \text{ m}^2/\text{s}} = 2.712 \times 10^6 \qquad (> 5 \times 10^5 \text{ and } < 10^8)$$

Average Nusselt number for the *entire* plate, considering both *laminar* and *turbulent* flows, is determined to be

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = (0.037\, Re_L^{0.8} - 871) Pr^{1/3} = [0.037(2.712 \times 10^6)^{0.8} - 871](0.690)^{1/3}$$

$$= 3812.8$$

Then $$\bar{h} = \frac{k}{L}\overline{Nu_L} = \frac{33.8 \times 10^{-3}\ \text{W/m K}}{8\ \text{m}}(3812.8) = 16.1\ \text{W/m}^2\ \text{K}$$

Area of top surface of plate, $A_s = WL = 1.25\ \text{m} \times 8\ \text{m} = 10\ \text{m}^2$

Rate of heat transfer,

$$\dot{Q} = \bar{h}A_s(T_s - T_\infty) = (16.1\ \text{W/m}^2\ \text{K})(10\ \text{m}^2)(227 - 27)°\text{C or K}$$

$$= \mathbf{32.2 \times 10^3\ W \quad or \quad 32.2\ kW}$$ **(Ans.) (a)**

(b) Disregarding the *laminar* region and assuming *turbulent* flow over the entire plate,

$$\overline{Nu} = 0.037\,Re^{0.8}\,Pr^{1/3} = \frac{\bar{h}L}{k}$$

∴ Average heat-transfer coefficient is

$$\bar{h} = \frac{k}{L}0.037\,Re^{4/5}\,Pr^{1/3} = \frac{0.0338\ \text{W/m K}}{8\ \text{m}}(0.037)(2.712 \times 10^6)^{0.8}(0.69)^{1/3}$$

$$= 19.36\ \text{W/m}^2\ \text{K}$$

Heat-transfer rate is

$$\dot{Q} = \bar{h}A_s(T_s - T_\infty) = (19.36\ \text{W/m}^2\ \text{K})\ (10\ \text{m}^2)\ (227 - 27)°\text{C or K}$$

$$= 38.72 \times 10^3\ \text{W or } 38.72\ \text{kW}$$

Percentage error involved

$$= \frac{38.72 - 32.2}{32.2} \times 100 = \mathbf{20.2\%}$$ **(Ans.) (b)**

(c) When air flow takes place along the 1.25 m side, L = 1.25 m, the Reynolds number at the end of the plate is

$$Re_L = \frac{u_\infty L}{\nu} = \frac{(12\ \text{m/s})(1.25\ \text{m})}{35.4 \times 10^{-6}\ \text{m}^2/\text{s}} = 4.237 \times 10^5 \qquad (< 5 \times 10^5)$$

Hence, the flow is *laminar* over the entire length of the plate. The *average* Nusselt number is given by the following correlation.

$$\overline{Nu} = 0.664\,Re^{1/2}\,Pr^{1/3} = \frac{\bar{h}L}{k}$$

∴ Average heat-transfer coefficient is

$$\bar{h} = \frac{0.0338\ \text{W/m K}}{1.25\ \text{m}} \times 0.664 \times (4.237 \times 10^5)^{0.5}(0.69)^{1/3} = 10.33\ \text{W/m}^2\ \text{K}$$

Heat-loss rate is

$$\dot{Q} = \bar{h}A_s(T_s - T_\infty) = (10.33\ \text{W/m}^2\ \text{K})(10\ \text{m}^2)(227 - 27)°\text{C or K}\left(\frac{1\ \text{kW}}{10^3\ \text{W}}\right)$$

$$= \mathbf{20.66\ kW}$$

This is considerably *less* than that determined in Case (a). **(Ans.) (c)**

EXAMPLE 7.21 *Engine oil at 20°C is forced over a 10 cm square plate at a velocity of 10 m/s. The plate is heated to a uniform temperature of 100°C. Calculate (a) the drag force, and (b) the heat lost by the plate.*

Properties of engine oil at the film temperature of 60°C are

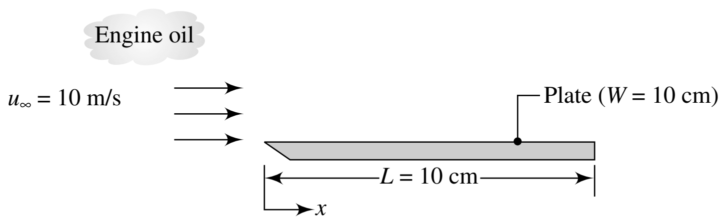
$$k = 0.1404 \text{ W/m°C}, \text{ Pr} = 1080, \nu = 8.5656 \times 10^{-5} \text{ m}^2\text{/s}, \rho = 863.9 \text{ kg/m}^3$$

Solution

Known Engine oil flows across a square plate with a specified velocity.

Find (a) Drag force, F_D (N), (b) Heat lost, $\dot{Q}$(W).

Schematic



Assumptions (1) Steady operating conditions exist. (2) The critical Reynolds number, $Re_{cr} = 5 \times 10^5$.

Analysis Reynolds number at the end of the plate at $x = L = 0.10$ m is

$$Re_x = \frac{u_\infty x}{\nu} = \frac{(10 \text{ m/s})(0.10 \text{ m})}{8.565 \times 10^5 \text{ m}^2/\text{s}} = 11\ 675\ (<5 \times 10^5) \quad \Rightarrow \quad \textit{Laminar flow}$$

(a) Drag coefficient,

$$C_D = 1.328\, Re_L^{-1/2} = 1.328(11\,675)^{-1/2} = 0.0123$$

Surface area, $A_s = WL = 0.1 \text{ m} \times 0.1 \text{ m} = 0.01 \text{ m}^2$

Drag force,

$$F_D = C_D \frac{1}{2} \rho A_s u_\infty^2 = 0.0123 \times \frac{1}{2} \times 863.9 \text{ kg/m}^3 \times 0.01 \text{ m}^2 \times (10 \text{ m/s})^2$$

$= \mathbf{5.31\ N}$ **(Ans.) (a)**

(b) Using Churchill–Ozoe correlation for laminar flow with $Pe_x \geq 100$ and applicable for all Prandtl numbers, the local Nusselt number is

$$Nu_x = \frac{0.3387\, Re_x^{1/2}\, Pr^{1/3}}{[1 + (0.0468/Pr)^{2/3}]^{1/4}} = \frac{0.3387(11675)^{1/2}(1080)^{1/3}}{[1 + (0.0468/1080)^{2/3}]^{1/4}} = 375.37$$

Local heat-transfer coefficient at $x = L = 0.10$ m is

$$h_x = Nu_x \frac{k}{x} = 375.37 \times \frac{0.1404 \text{ W/m°C}}{0.10 \text{ m}} = 527 \text{ W/m}^2\,\text{°C}$$

Average heat-transfer coefficient,

$$\overline{h} = 2h_x = 1054 \text{ W/m}^2\,\text{°C}$$

Rate of heat loss from one side of the plate is

$$\dot{Q} = \bar{h} A_s (T_s - T_\infty) = (1054 \text{ W/m}^2\,°\text{C})(0.01 \text{ m}^2)(100 - 20)°\text{C}$$
$$= \mathbf{843.3\ W} \qquad \textbf{(Ans.) (b)}$$

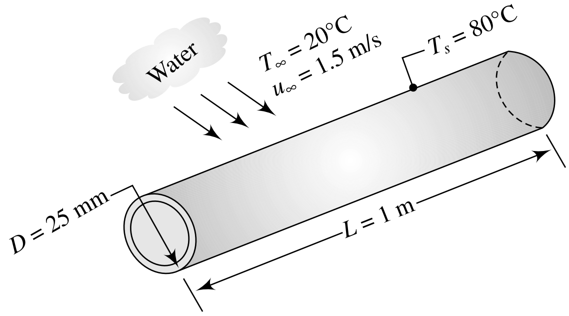
(B) Cylinder In Cross Flow

EXAMPLE 7.22 *A copper tube of 20 mm outside diameter is losing heat at a rate of 90 W/m of its length due to convection alone to a stream of air flowing across it. If the surface temperature is 90°C and the air temperature is 30°C, determine the velocity of air. Properties of air at 60°C:*

$k = 28.08 \times 10^{-3}$ W/m K, $\nu = 18.96 \times 10^{-6}$ m²/s, Pr = 0.7202

Solution

Schematic



Known Air flows across a copper tube under specified conditions.

Find Velocity of air, u_∞ (m/s).

Assumptions (1) Steady-state conditions. (2) Air is an ideal gas with constant properties. (3) Constant temperature around the boundary.

Analysis With a heat loss rate of 90 W/m and tube diameter, D = 20 mm = 0.02 m, it follows that

$$\text{Heat flux, } q = \frac{\dot{Q}}{\pi D L} = \frac{90 \text{ W}}{\pi \times 0.02 \text{ m} \times 1 \text{ m}} = 1432.4 \text{ W/m}^2$$

Convective heat-transfer coefficient,

$$h = \frac{q}{T_s - T_\infty} = \frac{1432.4 \text{ W/m}^2}{(90 - 30)°\text{C or K}} = 23.87 \text{ W/m}^2 \text{ K}$$

Film temperature

$$T_f = \frac{T_s + T_\infty}{2} = \frac{90 + 30}{2} = 60°\text{C}$$

Using the specified properties of air at this temperature, we have Nusselt number,

$$Nu = \frac{hD}{k} = \frac{23.87 \text{ W/m}^2 \text{ K} \times 0.02 \text{ m}}{0.02808 \text{ W/m K}} = 17.0$$

Let us assume that Reynolds number lies between 40 and 4000
The appropriate correlation for this range is

$$Nu = C\, Re^m\, Pr^{1/3} \text{ where } C = 0.683,\ m = 0.466 \qquad [\text{for } 40 < Re < 4000]$$

Then $\quad 17.0 = 0.683\,(Re)^{0.466}\,(0.7202)^{1/3}$

∴ Reynolds number,

$$Re = (27.768)^{1/0.466} = 1252$$

As the calculated value of *Re* falls in the range 40 to 4000, our initial guess was OK.

Hence, $Re = 1252 = \dfrac{u_\infty D}{\nu}$

Velocity of air is then determined to be,

$$u_\infty = \frac{Re\,\nu}{D} = \frac{1252}{0.02\ \text{m}} \times 18.96 \times 10^{-6}\ \text{m}^2/\text{s} = \mathbf{1.19\ m/s} \qquad \text{(Ans.)}$$

EXAMPLE 7.23 *A steel wire, 2 mm in diameter, is being cooled from 200°C to 100°C by passing it through a 2 m wide air stream at 1 bar and 30°C. The velocity of air is 50 m/s in a direction perpendicular to the wire. Determine the velocity of wire for this purpose. For steel, use: Density (ρ) = 8000 kg/m³; Specific heat* (C_p) *= 0.5 kJ/kg K; Thermal conductivity* (k) *= 50 W/m K.*

Properties of steel:

$\rho_s = 8000\ \text{kg/m}^3$, $C_{ps} = 500\ \text{J/kg K}$, $k_s = 50\ \text{W/m K}$

Properties of air at 90°C:

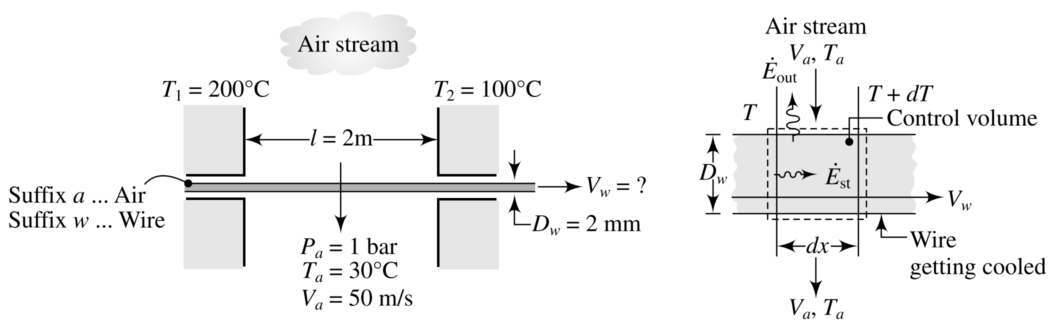
$\nu_a = 22.35 \times 10^{-6}\ \text{m}^2/\text{s}$, $k_a = 0.031\ \text{W/m K}$, $Pr_a = 0.697$

Solution

Known A steel wire is cooled by passing through an air stream perpendicular to it.

Find Velocity of wire, V_w (m/s).

Schematic



Assumptions (1) The situation is crossflow over a cylinder. (2) Internal temperature gradients across the cross section of the wire can be neglected. Hence there is uniform temperature at any cross section. (3) Negligible axial conduction.

Analysis Mean wire temperature, $\overline{T}_s = \dfrac{200 + 100}{2} = 150°\text{C}$

∴ Mean film temperature, $(\overline{T}_s + T_a)/2 = \dfrac{1}{2}(150 + 30) = 90°\text{C}$

Reynolds number,

$$Re = \frac{V_a D_w}{\nu_a} = \frac{50\ \text{m/s} \times 0.002\ \text{m}}{22.35 \times 10^{-6}\ \text{m}^2/\text{s}} = 4474.3$$

The appropriate correlation is

$$Nu = 0.3 + \frac{0.62\,Re^{1/2}\,Pr^{1/3}}{\left\{1+\left(\frac{0.4}{Pr}\right)^{2/3}\right\}^{1/4}}\left[1+\left(\frac{Re}{282000}\right)^{5/8}\right]^{4/5}$$

$$= 0.3 + \frac{0.62(4474.3)^{1/2}(0.697)^{1/3}}{\left\{1+\left(\frac{0.4}{0.697}\right)^{2/3}\right\}^{1/4}}\left[1+\left(\frac{4474.3}{282000}\right)^{5/8}\right]^{4/5} = 34.47$$

Convection heat-transfer coefficient is

$$h = \frac{Nu \cdot k_a}{D_w} = 34.47 \times \frac{0.031\ \text{W/m K}}{0.002\ \text{m}} = 534.3\ \text{W/m}^2\,\text{K}$$

Now applying energy balance to an elemental control volume in the moving wire, one gets

$$\cancelto{0}{\dot{E}_{\text{in}}} - \dot{E}_{\text{out}} + \cancelto{0}{\dot{E}_{\text{gen}}} = \dot{E}_{st} \quad \Rightarrow \quad \dot{E}_{\text{out}} = -\dot{E}_{st}$$

As the wire is getting cooled when exposed to the cold air stream, heat is transferred out, resulting in a decrease in the rate of thermal energy storage. Therefore,

$$\dot{Q}_{\text{out,conv}} = -\dot{m}_w C_{p_w} dT$$

where $\quad \dot{m}_w = \rho_w A_c V_w = \rho_w \frac{\pi}{4} D_w^2 V_w$

and $\quad \dot{Q}_{\text{conv,out}} = hA_s(T - T_a) = h(\pi D_w dx)(T - T_a)$

or $\quad -\rho_w V_w \cdot \frac{\pi D_w^2}{4} C_{p_w} dT = h(\pi \cdot D_w \cdot dx)(T - T_a)$

or $\quad -V_w \int_{T_1}^{T_2} \frac{dT}{T - T_a} = \frac{4h}{\rho_w C_{p_w} D_w} \int_o^l dx \quad$ or $\quad V_w \cdot \ln \frac{T_1 - T_a}{T_2 - T_a} = \frac{4hl}{\rho_w C_{p_w} D_w}$

Velocity of wire, $\quad V_w = \dfrac{4hl}{\rho_w C_{p_w} D_w \ln \dfrac{(T_1 - T_a)}{(T_2 - T_a)}}$

$$= \frac{4 \times 534.3\ \text{W/m}^2\,\text{K} \times 2.0\ \text{m}}{8000\ \text{kg/m}^3 \times 500\ \text{J/kg K} \times 0.002\ \text{m} \times \ln\{(200 - 30°\text{C}/100 - 30)\}°\text{C}}$$

= **0.60 m/s** **(Ans.)**

Check: For wire, the Biot number is

$$Bi = \frac{hD_w}{k_w} = \frac{534\ \text{W/m}^2\,\text{K} \times 0.002\ \text{m}}{50\ \text{W/m K}} = 0.021 \approx 0$$

∴ Assumption (2) is justified.

Also, $\quad V_w << V_a$

∴ Assumption (1) is justified.

EXAMPLE 7.24 *A copper busbar of 20 mm diameter used for electrical transmission is cooled by air at 1 bar and 27°C flowing past it in crossflow with a free stream velocity of 2 m/s. The maximum allowable busbar surface temperature is 77°C. Determine the heat-transfer coefficient and the permissible current the busbar can carry if the electrical resistivity of copper is 0.0175 micro ohm per metre length. Use the following correlation for flow across a single cylinder:*

$$\mathrm{Nu} = \frac{\mathrm{hD}}{\mathrm{k}} = [0.4\,\mathrm{Re}^{1/2} + 0.06\,\mathrm{Re}^{2/3}]\mathrm{Pr}^{0.4}\left[\frac{\mu_\infty}{\mu_s}\right]^{1/4}$$

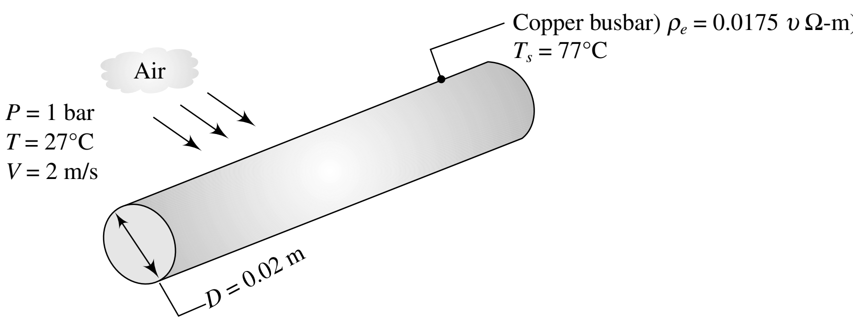
Thermophysical properties of air at 1 atm:
At T_∞ *= 27°C:* k *= 0.0263 W/m K,* ν *= 15.89 × 10⁻⁷ kg/m s,* μ *= 184.6 × 10⁻⁷ kg/m s,* Pr *= 0.707*
At T_s *= 77°C:* μ_s *= 208.2 × 10⁻⁷ kg/m s*

Solution

Known Circular busbar in cross flow of air at specified velocity.

Find Convection coefficient, h (W/m^2 K);

Schematic



Assumptions (1) Steady-state conditions. (2) Constant properties. (3) Air is an ideal gas. (4) Radiation is not considered.

Analysis Reynolds number,

$$Re = \frac{VD}{\nu} = \frac{(2\text{ m/s})(0.02\text{ m})}{15.89\times10^{-6}\text{ m}^2/\text{s}} = 2517.3$$

$$(\mu_\infty/\mu_s) = \frac{184\times10^{-7}}{208.2\times10^{-7}} = 0.8866$$

Nusselt number,

$$Nu = [0.4(2517.3)^{0.5} + 0.06(2517.3)^{2/3}](0.707)^{0.4}(0.8866)^{0.25} = 26.33$$

Therefore, the heat transfer coefficient is

$$h = Nu\frac{k}{D} = 26.33\times\frac{0.0263\text{ W/m K}}{0.02\text{ m}} = \mathbf{34.6\ W/m^2\ K} \qquad \text{(Ans.)}$$

Rate of heat transfer to air is

$$\dot{Q} = hA_s(T_s - T_\infty) = h(\pi DL)(T_s - T_\infty)$$
$$= (34.6\text{ W/m}^2\text{ K})(\pi\times0.02\times1\text{ m})(77-27)\text{K} = 108.7\text{ W}$$

The heat dissipation equals the rate of heat generation, i.e.,

$$\dot{Q} = \dot{E}_{\text{gen}} = I^2 R_e = I^2 \rho \frac{L}{A_c} = I^2 \rho \frac{L \times 4}{\pi D^2}$$

Maximum permissible current carried by the busbar is

$$I = \sqrt{\frac{\dot{Q} \cdot \pi D^2}{4 \rho L}} = \left[\frac{108.7 \times \pi \times 0.02^2}{4 \times 0.0175 \times 10^{-6} \times 1} \right]^{1/2} = \mathbf{1397\ A} \qquad \textbf{(Ans.)}$$

C: Spheres

EXAMPLE 7.25 *Air at 1 atm and 30°C blows across a 15 mm diameter sphere at a free stream velocity of 5 m/s. A small heater inside the sphere maintains the surface temperature at 80°C. Compute the rate of heat lost by the sphere.*

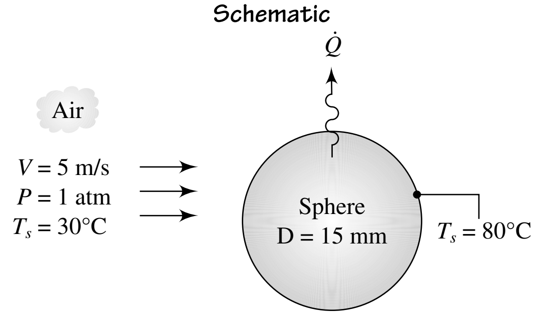
Solution

Known A sphere exposed to forced air loses heat.

Find Heat loss rate, $\dot{Q}$(W).

Assumptions (1) Steady-state conditions. (2) Radiation effects are negligible. (3) Air is an ideal gas.

Schematic



Properties The relevant properties of air are

$\mu_s = \mu_{@80°C} = 2.096 \times 10^{-5}$ kg / m s

At T_∞ = 30°C and 1 atm:

$\mu_\infty = 1.872 \times 10^{-5}$ kg/ms $\qquad Pr = 0.7282$

$k = 0.02588$ W/m °C $\qquad \rho = 1.164$ kg/m^3

Analysis The appropriate empirical correlation is

$$Nu_{\text{sph}} = \frac{hD}{k} = 2 + \{0.4\, Re^{1/2} + 0.06\, Re^{2/3}\} Pr^{0.4} \left(\frac{\mu_\infty}{\mu_s} \right)^{1/4}$$

Note that in this case, the fluid properties are evaluated at the *free stream temperature T∞*, except for μ_s which is evaluated at the *surface temperature*.

The Reynolds number is determined from

$$Re_D = \frac{\rho V D}{\mu} = \frac{(1.164\ \text{kg/m}^3)(5\ \text{m/s})(0.015\ \text{m})}{1.872 \times 10^{-5}\ \text{kg/ms}} = 4663.46$$

The Nusselt number is

$$Nu = 2 + \{0.4 \times (4663.46)^{1/2} + 0.06 \times (4663.46)^{2/3}\}(0.7282)^{0.4} \times \left(\frac{1.872 \times 10^{-5}}{2.096 \times 10^{-5}} \right)^{0.25}$$

$$= 37.73$$

Then the average convection heat-transfer coefficient becomes,

$$h = \frac{k}{D} Nu = \left(\frac{0.02588 \text{ W/m °C}}{0.015 \text{ m}} \right) (37.73) = 65.1 \text{ W/m}^2 \text{ °C}$$

The rate of heat loss is then

$$\dot{Q} = hA_s (T_s - T_\infty)$$

where A^s is the heat-transfer surface area equal to $\pi D^2 = \pi (0.015)^2 \text{ m}^2$

Hence, $$\dot{Q} = \left(65.1 \frac{\text{W}}{\text{m}^2 \text{ °C}} \right) (\pi \times 0.015^2 \text{ m}^2) (80 - 30) \text{°C} = \mathbf{2.3 \text{ W}}$$ **(Ans.)**

Comment Note that except for μ_s which is determined at surface temperature, all other thermophysical properties of fluid (air) are to be evaluated for a *sphere* at the *free stream temperature* T_∞ and *not* the film temperature, T_f.

EXAMPLE 7.26 *In a powder-based surface-coating process, aluminium particles are heated in a very hot stream of argon gas at 1400 K before deposition on the substrate. The spherical powder particles of average diameter of 100 microns are injected into the stream at a temperature of 20°C. The velocity of the gas is 5 m/s. The particles will attain a temperature of 30°C below the melting point of aluminium. Determine (a) the Nusselt number, (b) the time constant, (c) the time required to reach the final temperature, (d) the Biot number.*

The following correlation and the properties may be used:

$$\text{Nu}_\text{D} = 2 + (0.4 \, \text{Re}_\text{D}^{1/2} + 0.06 \, \text{Re}_\text{D}^{2/3}) \text{Pr}^{0.4} \left(\frac{\mu}{\mu_\text{w}} \right)^{1/4}$$

Properties: Aluminium: Melting point = 660°C; $\text{T}_\text{w} = \frac{1}{2}(630 + 20)$ *°C = 325°C:* ρ *= 2702 kg/m³,* k *= 218 W/m K,* C_p *= 0.903 kJ/kg K*

Argon $[\text{T}_\infty$ *= 1400 K]:* k *= 0.0535 W/m K,* Pr *= 0.638,* $\mu = 6.56 \times 10^{-5}$ *kg/ m s,* ρ *= 0.348 kg/m³*

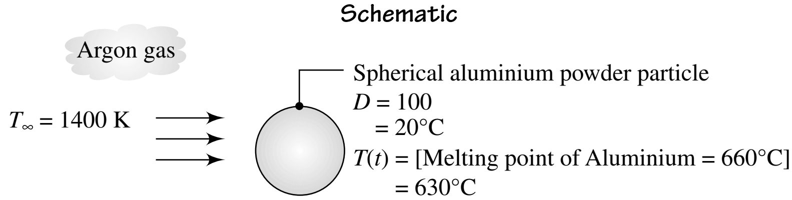
$$\mu_w [T_w = 325°C] = 3.83 \times 10^{-5} \text{ kg/m s} \approx 600 \text{ K}$$

Solution

Known Aluminium particles of prescribed diameter and temperature injected in an argon gas stream at the given temperature and velocity to attain the final temperature.

Find (a) Nusselt number, *Nu*, (b) Time constant, τ, (c) Time required, *t*, and (d) Biot number, *Bi*.

Schematic



Assumptions (1) Lumped capacity model is valid and internal temperature gradients are negligible. (2) Negligible radiation effects.

Analysis (a) Reynolds number,

$$Re_D = \frac{\rho VD}{\mu} = \frac{0.348\ \text{kg/m}^3 \times 5\ \text{m/s} \times 100 \times 10^{-6}\ \text{m}}{6.56 \times 10^{-5}\ \text{kg/m s}} = 2.6524$$

∴ Nusselt number,

$$Nu_D = 2 + [0.4 \times (2.6524)^{1/2} + 0.06 \times (2.6524)^{2/3}] \times (0.638)^{0.4} \times \left(\frac{6.56 \times 10^{-5}}{3.83 \times 10^{-5}}\right)^{1/4}$$

$$= \mathbf{2.7325} \qquad \textbf{(Ans.) (a)}$$

(b) Convective heat-transfer coefficient,

$$h = Nu_D \frac{k}{D} = (2.7325)\left(\frac{0.0535\ \text{W/m K}}{100 \times 10^{-6}\ \text{m}}\right) = 1462\ \text{W/m}^2\ \text{K}$$

Time constant, $\tau = \dfrac{\rho ∀ C_p}{hA} = \dfrac{\rho C_p L_c}{h}$

where, for a sphere, $L_c = \dfrac{∀}{A} = \dfrac{\pi D^3/6}{\pi D^2} = \dfrac{D}{6}$

$$\therefore \qquad \tau = \frac{\rho C_p D}{6h} = \frac{2702\ \text{kg/m}^3 \times 903\ \text{J/kg K} \times 100 \times 10^{-6}\ \text{m}}{6 \times 1462\ \text{W/m}^2\ \text{K}}\left[\frac{1\ \text{W}}{1\ \text{J/s}}\right] = \mathbf{0.0278\ s} \qquad \textbf{(Ans.) (b)}$$

(c) Time required,

$$t = \frac{\rho C_p D}{6h}\ln\frac{T_i - T_\infty}{T - T_\infty} = \tau \ln\frac{T_i - T_\infty}{T - T_\infty} = (0.0278\ \text{s})\ln\frac{20 - 1400}{630 - 1400}$$

$$= \mathbf{0.0162\ s} \quad \textbf{or} \quad \mathbf{16.2\ ms} \qquad \textbf{(Ans.) (c)}$$

(d) $Bi = \dfrac{hL_c}{k} = \dfrac{hD}{6k} = \dfrac{1462\ \text{W/m}^2\ \text{K} \times 100 \times 10^{-6}\ \text{m}}{6 \times 218\ \text{W/m K}}$

$$= \mathbf{1.118 \times 10^{-4}} \quad \textbf{(< 0.1)} \qquad \textbf{(Ans.) (d)}$$

Hence, solid temperature variation is negligible and the lumped-capacity model is valid.

EXAMPLE 7.27 *A cross-flow heat exchanger for heating air consists of a 15 × 15 square aligned array of tubes for which* $S_L = S_T$ *= 24 mm,* D *= 12 mm, and the length of each tube = 1.5 m. The air is at atmospheric pressure and 30°C and flows with a velocity of 3.5 m/s. It is heated by means of hot water flowing inside the tubes. Calculate the total rate of heat transfer to the air if the average surface temperature of the tubes is 74°C.*

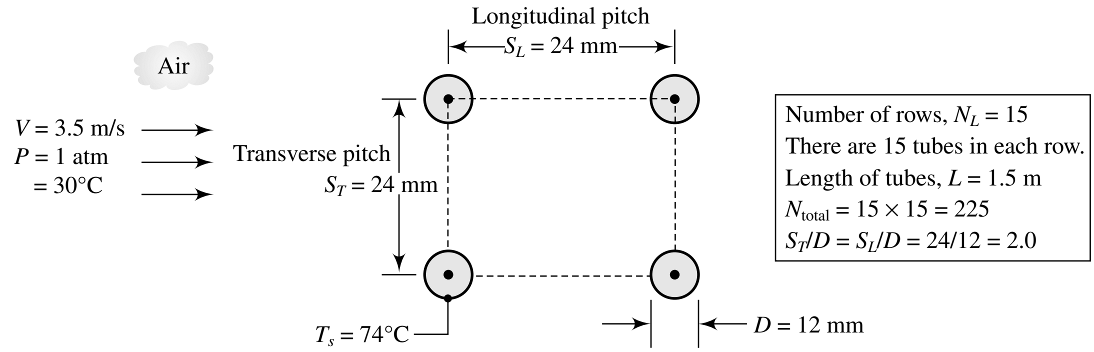
Where is the controlling thermal resistance when heat flows from the water to the air? What modifications would you suggest in the design of the heat exchanger if the total rate of heat transfer is to be increased? The total number of tubes, their diameter and length, and the given temperatures and velocity are fixed. What penalty would one pay for making these modifications?

Solution

Known Geometry and operating conditions of a tube bank.

Find Heat-transfer rate to air, $\dot{Q}$[W].

Schematic



Assumptions (1) Steady operating conditions exist. (2) Negligible effect of change in air temperature across tube bank on air properties. (3) Negligible radiation effects.

Properties Initial estimate of T_e = 50°C. At $T_m = \frac{1}{2}(T_i + T_e) = \frac{30+50}{2}$ = 40°C:

ρ = 1.164 kg/m^3, ν = 17.02 × 10^{-6} m^2/s, Pr = 0.7255, C_p = 1.007 kJ/kg °C

$Pr_{74°C}$ = 0.7168

Analysis ***Square aligned array of tubes***

Maximum air velocity within the tube bank is,

$$V_{\max} = \frac{S_T}{S_L - D}V = \left(\frac{24 \text{ mm}}{24 \text{ mm} - 12 \text{ mm}}\right)(3.5 \text{ m/s}) = 7 \text{ m/s}$$

Reynolds number,

$$Re_{D,\max} = \frac{\rho V_{\max} D}{\mu} = \frac{V_{\max} D}{\nu} = \frac{7 \text{ m/s} \times 0.012 \text{ m}}{17.02 \times 10^{-6} \text{ m}^2/\text{s}} = 4935$$

For N_L = 15, correction factor, F = 0.997
and for $Re_{D,\max}$ = 4935, the average Nusselt number is

$$\overline{Nu}_D = F \times 0.27(Re_{D,\max})^{0.63} Pr^{0.36} (Pr/Pr_s)^{0.25}$$

$$= 0.997 \times 0.27 \times (4935)^{0.63} (0.7255)^{0.36} \left(\frac{0.7255}{0.7168}\right)^{0.25} = 51$$

$$\therefore \quad \bar{h} = \overline{Nu}_D \frac{k}{D} = 51 \times \frac{0.02662 \text{ W/m°C}}{0.012 \text{ m}} = 113.2 \text{ W/m}^2 \text{ °C}$$

Exit air temperature,

$$T_e = T_s - (T_s - T_i)\exp\left[-\frac{\bar{h}\pi DLN_L N_T}{\rho_i V(N_T S_T L)C_p}\right]$$

$$= 74°\text{C} - (74 - 30)°\text{C}\exp\left[-\frac{(113.2 \text{ W/m °C})(\pi \times 0.012 \text{ m} \times 15 \times 15)}{(1.164 \text{ kg/m}^3)(3.5 \text{ m/s})(15 \times 0.024 \text{ m})(1007 \text{ J/kg °C})}\right]$$

$$= 51°\text{C}$$

$$T_m = \frac{1}{2}(T_i + T_e) = \frac{30 + 51}{2} = 40.5°\text{C}$$

$\therefore$ Our initial estimate was correct.

Log mean temperature difference,

$$\Delta T_{lm} = \frac{(T_s - T_i) - (T_s - T_e)}{\ln[(T_s - T_i)/(T_s - T_e)]} = \frac{(74 - 30) - (74 - 51)}{\ln(44/23)} = 32.37°\text{C}$$

$\therefore$ $$\dot{Q} = \bar{h}(\pi DL)\Delta T_{lm} = (113.2 \text{ W/m}^2\text{ °C})(15 \times 15 \times \pi \times 0.012 \text{ m} \times 1.5 \text{ m})\ (32.37°\text{C})$$

$= 46.6 \times 10^3$ W**= 46.6 kW**

Else, $$\dot{m} = 1.164 \text{ kg/m}^3 \times 3.5 \text{ m/s} \times 15 \times 0.024 \text{ m} \times 1.5 \text{ m} = 2.2 \text{ kg/s}$$

$$\dot{Q} = \dot{m}C_p(T_e - T_i)$$

$= 2.2$ kg/s $\times$ 1.007 kJ/kg °C $\times$ (51 – 30)°C **= 46.5 kW** **(Ans.)**

Points to Ponder

- The essential boundary conditions for any velocity distribution in a laminar boundary layer are:
- At $y = 0$, $u = v = 0$, $\dfrac{\partial^2 u}{\partial y^2} = 0$
- At $y = \delta$ $u = u_\infty$.
- Convection heat transfer due to pressure drop induced velocities is called forced convection.
- For forced flow external to surfaces, the laminar boundary layer persists to a local Reynolds number of approximately 5×10^5.
- The local heat-transfer coefficient for turbulent flows past a flat plate varies according to the relation $h = Ax^{-0.2}$ where A is a constant of appropriate dimensions and x is the distance measured from the leading edge. The average Nusselt number is 1.25 times the local Nusselt number.
- In the case of flow normal to tube banks, there are two general types of orientation—in-line and staggered geometry. For minimum pressure drop, one would use in line arrangement, and for maximum heat transfer staggered arrangement.
- The film temperature is usually defined as $\frac{1}{2}(T_s + T_\infty)$.
- In respect of cross flow over tube banks:

  S_L = Longitudinal pitch S_T = Transverse pitch

  S_D = Diagonal pitch.

  The diagonal pitch between tube centres is determined from $S_D = \sqrt{S_L^2 + (S_T/2)^2}$.

- The separation point, given in terms of the angle θ from the front stagnation point for a cylinder in cross flow in about 80° for laminar flow and about 140° for turbulent flow.
- For gases with $Pr \approx 1$, the recovery factor for high-speed flow over a flat plate is $Pr^{1/2}$ for laminar and $Pr^{1/3}$ for turbulent flow.
- The factor of two is included to account for both sides of the plate in problems pertaining to flow over a flat plate for calculating heat transfer or drag force.
- The expressions for velocity boundary layer thickness $\delta(x)$ and friction coefficient and C_{fx}, for three typical cases are velocity profile:

	Linear	Cubic	Exact solution
$C_{f,x}$	$0.578 / \sqrt{Re_x}$	$0.646\, Re_x^{-1/2}$	$0.664\, Re_x^{-1/2}$
$\delta(x)$	$3.46x / \sqrt{Re_x}$	$4.64x\, Re_x^{-1/2}$	$4.96x\, Re_x^{-1/2}$

Linear profile results underpredict those associated with the exact solution.

- The average heat-transfer coefficient $\overline{h}_L$ decreases with increasing $Re_{x,crit}$ as more of the surface becomes covered with a laminar boundary layer, over a flat plate. For mixed boundary layer condition, h_x varies as $x^{-1/2}$ and $x^{-1/5}$ in laminar and turbulent flow, respectively.

  Properties are generally evaluated at the mean film temperature $\frac{1}{2}(T_s + T_\infty)$.
- The unheated starting length acts to decrease the value of h_x at any location on the heated plate. However, its effect decreases with decreasing x_o and increasing x.
- Properties of the fluid at the free stream temperature T_∞ must be evaluated for the Zhukauskas correlation for flow over a cylinder. In Whitaker correlation for cylinder and sphere too, note carefully that fluid properties are evaluated at the free-stream fluid temperature T_∞.
- If the tube bank is made more compact, it has the desired effect of increasing the convection coefficient and therefore the heat transfer rate. However, it has the adverse effect of increasing the pressure drop and, hence, the fan power requirement.

GLOSSARY of Key Terms

- **Stagnation point** — The point where the fluid stream approaching the cylinder strikes the cylinder and comes to a halt, thus increasing the pressure.
- **Separation point** — The point on the cylinder where the boundary layer detaches from the surface. Flow separation occurs at $\theta = 140°$ when the boundary layer is turbulent, θ is the angle measured clock wise from the stagnation point.
- **Drag coefficient** — Non-dimensional drag given by the drag force on an object nondimensionalized by dynamic pressure of the free stream flow times frontal area of the object:

$$\boxed{C_D \equiv \frac{F_D}{\frac{1}{2}\rho A V^2}}$$

- **Drag force** — The force on an object opposing the motion of the object.
- **Laminar flow** — A stable well-ordered state of fluid flow in which all pairs of adjacent fluid particles move alongside one another forming laminates. A flow that is not laminar is either turbulent or transitional to turbulence, which occurs above a critical Reynolds number.

- **Skin Friction** Coefficient of surface shear stress τ_w non-dimensionalized by an appropriate dynamic pressure $\frac{1}{2}\rho V^2$. $\boxed{C_f = \tau_w \Big/ \frac{1}{2}\rho V^2}$.

OBJECTIVE-TYPE QUESTIONS

- **Multiple-Choice Questions**

7.1 When a hot fluid is flowing over a cold flat plate, the temperature gradient
(a) is zero at the surface
(b) is negative at the surface
(c) is zero at the edge of the thermal boundary layer
(d) none of the above

7.2 For turbulent flow over a flat plate, the average Nusselt number is given by $\overline{Nu}_L = 0.037(Re_L)^{0.8}(Pr)^{1/3}$
Which of the following is then a wrong statement?
The average heat-transfer coefficient increases as the
(a) 4/5 power of free stream velocity
(b) 2/3 power of thermal conductivity
(c) 1/5 power of length of flat plate
(d) 1/3 power of specific heat capacity

7.3 For laminar flow over a flat plate, the average Nusselt number is given by $\overline{Nu}_L = 0.664\sqrt{Re_L}\ Pr^{1/3}$
Which of the following is a wrong statement? To double the heat transfer coefficient,
(a) viscosity is to be decreased sixty four times
(b) density is to be increased four times
(c) thermal conductivity is to be increased eight times
(d) specific heat is to increased eight times

7.4 The temperature gradient in the fluid flowing over a heated plate is
(a) zero at the plate surface
(b) positive at the plate surface
(c) very steep at the plate surface
(d) zero at the edge of the thermal boundary layer

7.5 For laminar flow over a flat plate, the local skin friction coefficient decreases with the distance from the leading edge x in the flow direction by a factor of
(a) $x^{1/2}$ (b) $x^{-1/2}$ (c) $x^{1/3}$ (d) $x^{-1/3}$

7.6 The local heat-transfer coefficient in laminar flow over a flat plate
(a) increases with the distance from the leading edge
(b) increases with the viscosity of the fluid
(c) increases with the velocity of the free stream
(d) decreases with the density of the fluid

7.7 Choose the correct statement:
For a given value of Nusselt number, the convective heat-transfer coefficient, h
(a) decreases with increasing thermal conductivity of fluid
(b) increases with distance x form the leading edge
(c) increases with the thermal conductivity of fluid
(d) is independent of the distance x

7.8 The expression for average Nusselt number in the case of partly laminar and partly turbulent flow past a plate with transition Reynolds number = 3×10^5 is:

(a) $Nu_L = (0.037\,Re_L - 871.3)Pr^{1/3}$ (b) $Nu_L = 0.664\,Re_L^{1/2}\,Pr^{1/3}$

(c) $Nu_L = (0.037\,Re_L^{0.8} - 572.4)Pr^{1/3}$ (d) $Nu_L = 0.0296\,Re_L^{0.8}\,Pr^{1/3}$

7.9 When there is a flow of fluid over a flat plate of length 'L', the average heat-transfer coefficient, $\bar{h}$ (other symbols have the usual meaning):

(a) $\int_0^L h_x dx$ (b) $\frac{d}{dx}(h_x x)$ (c) $\frac{1}{L}\int_0^L h_x dx$ (d) $\frac{k}{L}\int_0^L Nu_x dx$

7.10 The laminar boundary layer thickness δ at any point x for flow over a flat plate is given by

(a) $\frac{\delta}{x} = \frac{0.664}{\sqrt{Re_x}}$ (b) $\frac{\delta}{x} = \frac{1.328}{\sqrt{Re_x}}$ (c) $\frac{\delta}{x} = \frac{1.75}{\sqrt{Re_x}}$ (d) $\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}$

7.11 Water (Prandtl number ≅ 6) flows over a flat plate which is heated over the entire length. Which one of the following relationships between the hydrodynamic boundary layer thickness (δ) and the thermal boundary layer thickness (δ_t) is true?

(a) $\delta_t > \delta$ (b) $\delta t < \delta$ (c) $\delta t = \delta$ (d) cannot be predicted.

7.12 Based on the heat transfer tests on PCBs (printed circuit boards), the local Nusselt number is correlated by

$$Nu_x = 0.02\,Re_x^{0.9} \quad \text{for} \quad Re_x > 2 \times 10^4$$

The average Nusselt number is expressed by

(a) $\overline{Nu}_x = 2Nu_x$ (b) $\overline{Nu}_x = Nu_x/0.9$ (c) $\overline{Nu}_x = \frac{4}{3}Nu_x$ (d) $\overline{Nu}_x = Nu_x$

7.13 The accepted criterion for transition from laminar to turbulent flow across a circular cylinder in terms of the Reynolds number ($Re_D \equiv u_\infty D/\nu$) is

(a) 2300 (b) 3×10^4 (c) 2×10^5 (d) 5×10^5

7.14 The heat transfer by conduction from a spherical surface to a stationary infinite medium around the surface can be obtained by putting Nu equal to

(a) 2.0 (b) 1.8 (c) 1.6 (d) 1.0

7.15 The critical value of Reynolds number for transition from laminar to turbulent boundary layer in external flow over a flat surface is usually taken as

(a) 2300 (b) 4000 (c) 5×10^5 (d) 5×10^6

7.16 A fluid flowing over a flat plate has the following properties:

Dynamic viscosity: 25×10^{-6} kg/ms

Specific heat: 2.0 kJ/kg K

Thermal conductivity: 0.05 W/m K

The hydrodynamic boundary-layer thickness is measured to be 0.5 mm. The thickness of thermal boundary layer would be

(a) 0.1 mm (b) 0.5 mm (c) 1.0 mm (d) None of the above

7.17 The transition Reynolds number for flow over a flat plate is 5×10^5. What is the distance from the leading edge at which transition will occur for flow of water with a uniform velocity of 1 m/s? [For water, the kinematic viscosity, $\nu = 0.858 \times 10^{-6}$ m²/s].

(a) 1 m (b) 0.43 m (c) 43 m (d) 103 m

7.18 Experimental results indicate that the local heat-transfer coefficient h_x for flow over a plate with an extremely rough surface is approximated by the relation $h_x = a\,x^{-0.14}$ where a is a constant coefficient and x is the distance from the leading edge of the plate. The relation between this local heat-transfer coefficient and the average heat-transfer coefficient $\bar{h}$ for a plate of length x is

(a) $\bar{h} = 0.568\,h_x$ (b) $\bar{h} = 0.852h_x$ (c) $\bar{h} = 1.163h_x$ (d) $\bar{h} = 1.988h_x$

7.19 The non-dimensional fluid temperature profile near the surface of a convectively cooled flat plate is given by, $\dfrac{T_w - T}{T_w - T_\infty} = a + b\dfrac{y}{L} + c\left(\dfrac{y}{L}\right)^2$, where y is measured perpendicular to the plate, L is the plate length, and a, b, and c are arbitrary constants. T_w and T_∞ are wall and ambient temperatures, respectively. If the thermal conductivity of the fluid is k and the wall heat flux is q^n, the Nusselt number, $Nu = \dfrac{q''}{T_w - T_\infty}\dfrac{L}{k}$ is equal to

(a) a (b) b (c) $2c$ (d) $(b + 2c)$

7.20 For laminar forced convection over a flat plate, if the free stream velocity increases by a factor of 2, the average heat transfer coefficient.

(a) remains same (b) decreases by factor of $\sqrt{2}$
(c) rises by a factor of $\sqrt{2}$ (d) rises by a factor of 4

Answers

Multiple-Choice Questions

7.1 (c)	**7.2** (c)	**7.3** (c)	**7.4** (d)	**7.5** (b)	**7.6** (c)
7.7 (c)	**7.8** (c)	**7.9** (c)	**7.10** (d)	**7.11** (b)	**7.12** (b)
7.13 (c)	**7.14** (a)	**7.15** (c)	**7.16** (b)	**7.17** (b)	**7.18** (c)
7.19 (c)	**7.20** (c)				

PRACTICE PROBLEMS

(A) Flat Pate In Parallel Flow: Laminar and Turbulent

7.1 Atmospheric air at 30°C is flowing over a flat plate with an approach velocity of 4 m/s. Determine (a) the distance from the leading edge of the plate where the air velocity is 3.96 m/s at a vertical distance of 1 cm from the plate surface. (b) the vertical distance from the plate surface at the same distance from the leading edge where the temperature will be 30.3°C. The uniform plate surface temperature is 60°C. Use the following properties of air at 30°C:
$\rho = 1.1646$ kg/m^3, $\mu = 1.86 \times 10^{-5}$ Ns/m^2, $C_p = 1.007$ kJ/kg °C, $k = 0.0265$ W/m °C

[(a) 1.0 m (b) 1.12 cm]

7.2 Air at velocity of 3 m/s and at 20°C flows over a flat plate along its length. The length, width and thickness of the plate are 100 cm, 50 cm and 2 cm, respectively. The top surface of the plate is maintained at 100°C. Calculate the heat lost by the plate and the temperature of the bottom surface of the plate under steady-state conditions. The thermal conductivity of the plate may be taken as 23 W/m K. The properties of air as:
$\rho = 1.06$ kg/m^3, $\nu = 18.97 \times 10^{-6}$ m^2/s, $k = 0.028\,94$ W/m K, $Pr = 0.696$

[100.4°C]

7.3 Air at atmospheric pressure and temperature of 20°C flows over a plate (80 cm × 40 cm) at a velocity of 2 m/s. The plate is maintained at 100°C. If the flow is along the 80 cm side, calculate the heat transfer rate from (a) the first half of the plate (b) the full plate, and (c) the next half of the plate.
The properties of air at the mean film temperature of 60°C are
$k = 28.74 \times 10^{-3}$ W/m°C, $\nu = 19.21 \times 10^{-6}$ W/m², $Pr = 0.7024$

[(a) 110.8 W (b) 156.7 W (c) 45.9 W]

7.4 An air-cooled motorcycle engine has fins which may be approximated as individual flat plates of length $L = 0.2$ m. Disturbances in the free stream cause the transition to occur at $Re_{x,\text{trans}} = 2 \times 10^5$. Determine, for a speed of 140 km/h, the average heat transfer coefficient from the fin surface permitting separate laminar and turbulent regions. Compare this with the result obtained by considering turbulent flow right from the leading edge. Use the following properties of air:
$k = 0.0263$ W/m°C, $\nu = 15.89$ mm²/s, $Pr = 0.707$ **[114 W/m²°C, 155 W/m°C]**

7.5 A refrigerated truck is travelling on the highway at the speed of 80 km/h in a desert area where the temperature is 70°C. The body of the truck is considered to be a rectangular box 3 m wide, 2 m high and 4.5 m long. The heat transfer from the front and back end of the box is neglected. Assume no separation of the air stream from the surface and consider the boundary layer as turbulent over the entire surface. The heat transfer coefficient for the sides may be assumed to be same as top and bottom. The temperature at the surface is uniformly at 10°C. For every heat loss, one ton capacity of the refrigerating unit is needed. Calculate: (a) the heat loss from the four surfaces, (b) the tonnage of the refrigerating unit and (c) the power required to overcome resistance acting on the four sides.
Properties of air at 40°C:
$k = 0.0276$ W/m °C, $\nu = 16.96 \times 10^{-6}$ m²/s, $Pr = 0.699$

[(a) 142 kW (b) 40.4 TR (c) 14.44 kW]

7.6 Air at 30°C and 1 atm pressure flows over a flat plate at a velocity of 2 m/s. Calculate the boundary layer thickness at distances of 20 cm and 40 cm from the leading edge of the plate. (a) Compute also the mass flow which enters the boundary layer between x = 20 cm and x = 40 cm per metre depth of the plate. The plate is heated and maintained at a temperature of 90°C over its entire length. (b) Compute the heat transferred in the first 40 cm of the plate per metre depth. (c) Determine the drag force exerted on the first 40 cm of the plate using the analogy between fluid friction and heat transfer.
Properties of air at 60°C are:
$C_p = 1.007$ kJ/kg K, $\rho = 1.059$ kg/m³, $k = 0.02808$ W/m K, $\nu = 18.96 \times 10^{-6}$ m²/s,
$Pr = 0.7202$
Properties of air at 30°C are: $\rho = 1.164$ kg/m^3, $\nu = 16.08 \times 10^{-6}$ m^2/s

[(a) 3.55 × 10⁻³ kg/s (b) 206 W (c) 5.48 mN]

7.7 Atmospheric air at a free-stream velocity of 12 m/s and 30°C flows parallel to a 3 m long flat plate maintained at a temperature of 150°C. Calculate the heat transfer coefficient at the trailing edge of the plate based on the following analogies: (a) Colburn analogy, (b) Prandtl analogy and (c) Von Kármán analogy. The properties of air at the film temperature of 90°C are:
$k = 30.24 \times 10^{-3}$ W/m°C, $Pr = 0.7132$, $\nu = 22.01 \times 10^{-6}$ m²/s

[(a) 24.93 W/m² °C (b) 21.15 W/m² °C (c) 22.49 W/m² °C]

(B) Cylinder In Cross Flow

7.8 Air at 2 atm and 300 K flows across a circular cylinder of 50 mm OD with a velocity of 16 m/s. The cylinder is maintained at a temperature of 350 K. Determine (a) the drag force and (b) the rate of heat transfer per metre length of the cylinder.
Properties of atmospheric air at the film temperature of 325 K are:

$k = 0.0282$ W/m °C, $Pr = 0.703$, $\nu = 1.841 \times 10^{-5}$ m²/s **[(a) 15.3 N (b) 983.7 W]**

7.9 Water at 20°C flows normal to the axis of a circular tube with a velocity of 1.5 m/s. The diameter of the tube is 25 mm. Calculate the average heat-transfer coefficient if the tube surface is maintained at a uniform temperature of 80°C. Also estimate the heat-transfer rate per unit length of the tube.

Properties of water at $T_f = \dfrac{20+80}{2} = 50°\text{C}$ are:

Specific heat, $C_p = 4.1813$ kJ/kg K Kinematic viscosity, $\nu = 0.568 \times 10^{-6}$ m²/s
Thermal conductivity, $k = 0.6395$ W/m K Prandtl number, $Pr = 3.68$
Density, $\rho = 990$ kg/m³
Dynamic viscosity of water:

μ_w at $80°\text{C} = 3.5456 \times 10^{-4}$ kg/m s, μ at $20°\text{C} = 1.006 \times 10^{-3}$ kg/m s

Use the correlation: $\overline{Nu} = (0.4\,Re^{0.5} + 0.06\,Re^{2/3})\,Pr^{0.4}\left(\dfrac{\mu}{\mu_w}\right)^{1/4}$ **[52.9 kW]**

(C) Spheres

7.10 The decorative plastic film on a copper sphere of 10 mm diameter is cured in an oven at 75°C. After removal from the oven, the sphere is exposed to an air stream at 10 m/s and 23°C. Estimate the time taken to cool the sphere to 35°C. Use the following correlation:

$$Nu = 2 + [0.4\,Re^{1/2} + 0.06\,Re^{2/3}]Pr^{0.4}\left(\frac{\mu_\infty}{\mu_s}\right)^{1/4}$$

Properties:
Copper: $\rho = 8933$ kg/m³, $k = 400$ W/m K, $C_p = 380$ J/kg K
Air (at $T_\infty = 23°\text{C}$): $\mu = 18.16 \times 10^{-6}$ N s/m², $\nu = 15.36 \times 10^{-6}$ m²/s, $Pr = 0.709$, $k = 0.0258$ W/m K
$\mu_{s@35°\text{C}} = 19.78 \times 10^{-6}$ N s/m² **[67.9 s]**

(D) Tube Bank

7.11 Air at atmospheric pressure and 20°C flows across a bank of tubes 12 rows high and 6 rows deep at an undisturbed velocity of 10 m/s measured upstream of the tubes. The surfaces of the tubes are maintained at 100°C. The diameter of the tubes is 26 mm, and the tubes are arranged in a staggered fashion. The centres of the tubes form corners of an equilateral triangle of 45 mm sides.(a) Determine the total heat transfer rate if the tubes are 4 m long. (b) Calculate the heat transfer rate if the tubes are arranged in an in-line fashion with the same row spacing. Compare it to the staggered arrangement. (c) Compare the pressure drops across the tube bundle for the two orientations.

[(a) 299 kW (b) 307.6 kW (c) 558 Pa, 478 Pa]

8 Forced Convection—Internal Flow

8.1 ❑ INTRODUCTION

The fluid flow through circular pipes or tubes and even of non-circular geometries for heating or cooling applications in the industry under forced convection when a pump or fan is used to force the fluid flow is of great practical importance. Besides heat-transfer considerations, one is often interested in the evaluation of pressure drop along the tube length since it has a direct bearing on the pumping power requirements. Determination of heat-transfer coefficient is also important to calculate the required pipe length for the desired heat-transfer rate in both laminar and turbulent flows. In the following sections, we will analyze these issues and present some important empirical correlations.

8.2 ❑ HYDRODYNAMIC CONSIDERATIONS

In external-flow forced convection, such as for flow over a flat plate, the boundary-layer development was allowed to continue without any restriction since the fluid has a free surface. However, in internal flow, the flowing fluid is completely confined by the inside surfaces of the pipes or tubes. This imposes a limit on the development of boundary layer.

When a fluid, initially of uniform velocity, enters a closed conduit (*a pipe or tube*), a boundary layer builds up along the surface of the pipe. But the flow cannot be the same as in the case of a flat plate due to the presence of the opposite wall—where a boundary layer is also developing. As the flow proceeds down the pipe, the boundary layer growing along the pipe wall gets thicker, eventually growing together from opposite sides and filling the pipe with *boundary-layer flow*.

Figure 8.1 shows the development of the hydrodynamic (*velocity*) boundary layer in a pipe. At the pipe entrance, the fluid at the centre is at the undisturbed uniform free-stream velocity, but the fluid at the wall has zero velocity (*no-slip condition*). A velocity boundary layer starts building up on each wall, just as it does for external flow over a flat plate. With lower velocities near the pipe wall, the centreline velocity must increase to satisfy the principle of conservation of mass. At some distance from the entrance, the boundary layers become so thick that they extend to the centre of the pipe and meet at a point. At that point, the velocity profile no longer changes shape further downstream, and the flow is then considered hydrodynamically fully developed.

The pipe length required to establish this fully developed flow is called the *starting* or *entrance* length. In the starting length, the velocity distribution across a diameter consists of a potential core region near the centre of the pipe (*the velocity being uniform*) which joins the boundary-layer region at each surface where the velocity varies from the potential core value to zero at the wall. As one moves along the pipe in the starting region, the viscous or boundary layer portion of the velocity distribution curve

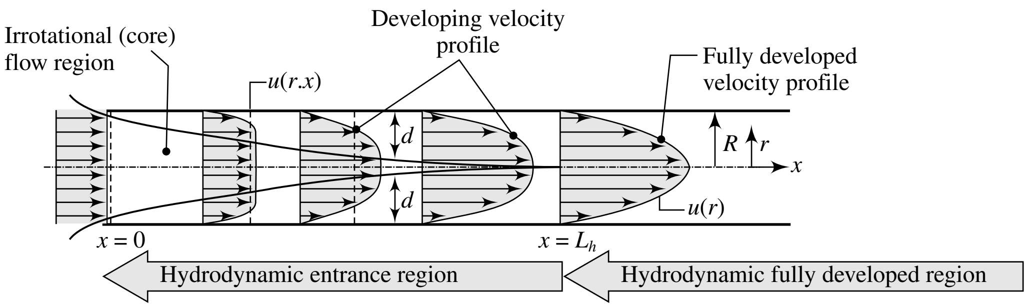


Fig. 8.1 *Development of the laminar, velocity boundary layer in a circular pipe*

increases while the potential core portion decreases. The law of conservation of mass requires that the mean velocity across the cross section (*mass-flow rate divided by density and cross-sectional area*) must remain constant for incompressible flow. Hence, the velocity of the potential core region must *increase* as the flow proceeds down the *starting length*.

For fully developed pipe flow, the Reynolds number based on length (as in the external flow along a flat plate) loses its significance, and it is customary to employ the *Reynolds number,* based on the pipe diameter Re_D, defined as

$$\boxed{Re_D = \frac{\rho V D}{\mu} = \frac{VD}{\nu}} \tag{8.1}$$

Here, D denotes the pipe diameter and V is the mean velocity of the flow in the pipe. Experiments indicate that a critical, or transition, Reynolds number of 2300 may be used. For values of $Re_D < 2300$, laminar flow may be expected, whereas greater values indicate the presence of turbulent flow. This critical value of $Re_D = 2300$ is not to be treated as a precise value because certain extraneous conditions may result in the transition to take place at other values too.

The *starting* length for laminar pipe flow is

$$\boxed{\frac{L_h}{D} = 0.0575\, Re_D \approx 0.05\, Re_D} \tag{8.2}$$

This formula must be treated as approximate since the fully developed condition is reached asymptotically, and hence the starting length, L_h, is difficult to define. No adequate theory exists which will predict the starting length in *turbulent flow* since the nature of the pipe entrance, the pipe roughness, etc., will have a serious effect on the growth and transition of the boundary layer in the pipe. Starting lengths of **25 to 40 pipe diameters** are typical for turbulent flow.

8.3 ❑ THERMAL CONSIDERATIONS

If the pipe wall is heated or cooled, a thermal boundary layer is also formed as shown in Fig. 8.2. It may be thicker, thinner, or of the same size as the velocity boundary layer, depending on the Prandtl number. Like the velocity boundary layer, the thermal boundary layer too grows from the wall until it reaches the centre of the pipe. At that point, the temperature profile becomes fully developed and its shape no longer changes further downstream *although its temperature level does change*.

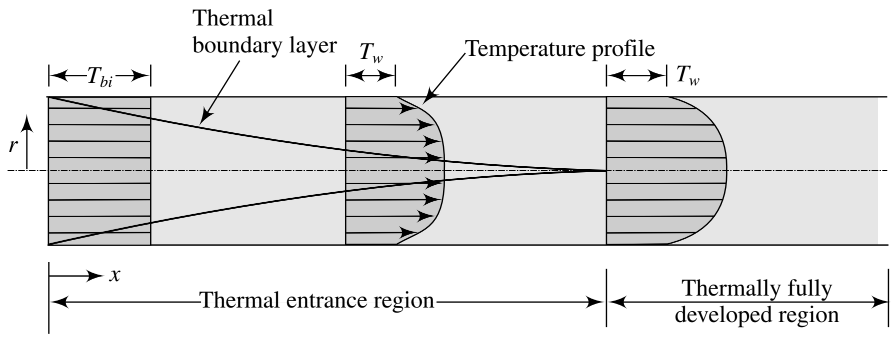


Fig. 8.2 *Development of the thermal boundary layer in a circular tube*

If a fluid enters the tube at a uniform temperature, which is less than the tube-wall temperature, convection heat-transfer occurs and a thermal boundary layer begins to develop. A thermally fully developed condition is finally reached. The shape of the fully developed temperature profile will depend on whether a uniform wall temperature or uniform heat flux is maintained. In both cases, the fluid temperature constantly increases with increasing distance from the entrance.

For *laminar flow,* the *thermal entry length* may be expressed as

$$\boxed{\frac{L_t}{D} \approx 0.05\, Re_D Pr} \tag{8.3}$$

If $Pr > 1$, the hydrodynamic boundary layer develops much faster than the thermal boundary layer. For $Pr < 1$, the opposite is true. For very large Prandtl number fluids, such as lubricating oils ($Pr \geq 100$), we can assume a fully developed velocity profile throughout the thermal entrance region. In the case of turbulent flow, conditions are more or less independent of Prandtl number, and to a first approximation, we can assume $L/D \approx 10$.

8.3.1 • Bulk Temperature

In a confined (*internal*) flow through a tube, there is obviously no free stream velocity and we use a mean velocity instead. Similarly, unlike external flow, there is no fixed free stream temperature in internal flow. Hence, we must define a mean bulk temperature. The bulk temperature T_b is a convenient reference temperature playing much the same role as the free stream temperature T_∞ for external flows. The mean (or bulk) temperature of the fluid at a given cross section is defined in terms of the thermal energy transported by the fluid as it moves along the tube. The rate of thermal energy transport, may be found by integrating the product of the mass flux (ρu) and the internal energy per unit mass (C_pT) over a given cross section.

$$\dot{E} = \int_{A_c} \rho u C_p T\, dA_c$$

We define the bulk temperature such that

$$\dot{E} \equiv \dot{m} C_p T_b$$

Then $$T_b = \frac{\int_{A_c} \rho u C_p T\, dA_c}{\dot{m} C_p} = \frac{\rho C_p \int uT\, dA_c}{\rho A_c V C_p} = \frac{\int_0^R uT(2\pi r dr)}{\pi R^2 V}$$

For *incompressible* flow in a *circular* tube with constant C_p, it follows that

$$\boxed{T_b = \frac{2}{VR^2}\int_0^R uTr\,dr} \tag{8.4}$$

8.3.2 • Fully Developed Conditions

If the flow is hydrodynamically developing, the velocity does change with both r and x but $(\partial u/\partial x) = 0$ in the *fully developed region*. In contrast, if there is heat-transfer, (dT_b/dx), as well as $(\partial T/\partial x)$ at any radius r, is not zero. Accordingly, the temperature profile $T(r)$ is continuously changing with x, and it would seem that the fully developed condition could never be reached. We define the dimensionless temperature difference of the form $(T_w - T)/(T_w - T_b)$ to simplify the analysis. Although the temperature profile $T(r)$ continues to change with x, the relative shape of the profile does not change in a *thermally fully developed flow*. The requirement for such a condition is expressed as

$$\boxed{\frac{\partial}{\partial x}\left[\frac{T_w(x) - T(r,x)}{T_w(x) - T_b(x)}\right] = 0} \tag{8.5}$$

where, T_w is the tube-wall temperature, T is the local fluid temperature, and T_b is the bulk temperature of the fluid over the cross section of the tube.

This condition is finally reached in a tube for which there is either a *uniform wall heat flux* (q_w = constant) or a *uniform wall temperature* (T_w = constant). These conditions occur in many engineering applications. For example, a *constant wall heat flux* would exist if the tube wall were *electrically heated* or if the outer surface were *uniformly irradiated*. *Constant wall temperature* condition would be present when *phase change* (due to boiling or condensation) were taking place at the outer surface. We must note that it is impossible to simultaneously impose both conditions of constant wall heat flux and constant wall temperature. If q_w is constant, T_w must vary with x, and if T_w is constant, q_w must change with x.

In the thermally fully developed flow of a fluid with constant properties, the local convection coefficient is a constant, independent of x.

8.4 ❑ LAMINAR-FLOW HEAT-TRANSFER IN A CIRCULAR PIPE: FULLY DEVELOPED FLOW

Laminar convective heat-transfer is analyzed with two different boundary conditions. One condition is a *Constant and Uniform Heat Flux (UHF)* on the wall and the other condition is a *Constant and Uniform Wall Temperature (UWT)*. The fluid temperature develops quite differently in *these* two cases.

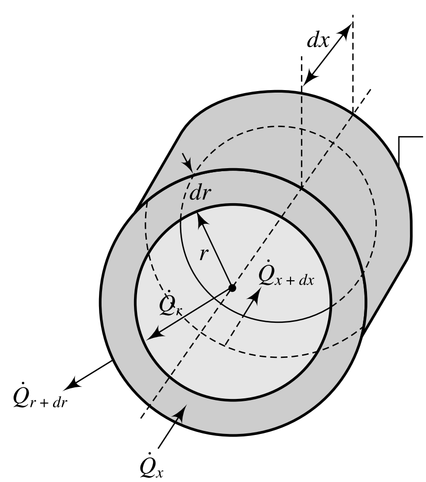


Fig. 8.3 *Differential control volume energy balance for fluid flow through a circular tube*

8.4.1 • Constant Wall Heat Flux

Let us first consider heating of a constant property fluid in a pipe subjected a constant heat flux at the wall. In this case, the temperature field varies throughout the cross section of the pipe, as shown in Fig. 8.3. The flow field and the temperature field are both *fully developed*; that is, the profile shape does not vary with downstream location. There is one important difference between the fully developed velocity and fully developed temperature fields. The average velocity does not change with downstream location. However, due to heat addition, the average temperature *does* increase. Although the average temperature changes, the *shape* of the temperature profile does not.

8.4.2 • Energy Equation for Laminar Flow in Tubes

Heat is conducted *radially* and convected *axially* in an annular control volume within the flow, as shown in Fig. 8.3. The radial heat flow *into* the control volume by conduction is given by

$$\dot{Q}_r = -k(2\pi r dx)\frac{\partial T}{\partial r}$$

and *out of* the control volume is, $\dot{Q}_{r+dr} = \dot{Q}_r + \frac{\partial}{\partial r}(\dot{Q}_r)dr$

There is no heat conduction in the axial direction.
The net rate at which heat is conducted into the control volume is given by

$$\begin{aligned}\dot{Q}_{\text{cond, net, in}} &= \dot{Q}_r - \dot{Q}_{r+dr} = -\frac{\partial}{\partial r}(\dot{Q}_r)dr \\ &= -\frac{\partial}{\partial r}\left\{-k(2\pi r dx)\frac{\partial T}{\partial r}\right\}dr \\ &= 2\pi k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)dr\,dx \end{aligned} \tag{8.6}$$

The rate at which energy is convected or carried into the control volume can be expressed in terms of the mass-flow rate of the fluid entering the control volume and the energy associated with it.

Heat convected into the control volume in the axial direction (*the radial component of velocity is zero for fully developed flow*) is given by

$$\dot{Q}_{\text{conv},x} = d\dot{m}C_pT = \rho u(2\pi r dr)C_pT$$

The heat convected out of the control volume is

$$\dot{Q}_{\text{conv},x+dx} = \dot{Q}_{\text{conv},x} + \frac{\partial}{\partial x}(\dot{Q}_{\text{conv},x})dx$$

The net heat flow out of the control volume by axial convection in the x-direction is

$$\begin{aligned}\dot{Q}_{\text{conv, net out}} &= \dot{Q}_{\text{conv},x+dx} - \dot{Q}_{\text{conv},x} \\ &= \rho u(2\pi r dr)C_p\left(\frac{\partial T}{\partial x}\right)dx\end{aligned} \tag{8.7}$$

The energy balance then yields $\dot{Q}_{\text{cond, net in}} = \dot{Q}_{\text{conv, net out}}$

Substitution from Eq. (8.6) and (8.7) gives

$$2\pi k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)dr\,dx = \rho u(2\pi r dr)C_p\left(\frac{\partial T}{\partial x}\right)dx$$

or

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \left(\frac{\rho C_p}{k}\right)u\frac{\partial T}{\partial x} = \frac{u}{\alpha}\frac{\partial T}{\partial x}} \tag{8.8}$$

For fully developed laminar flow, the velocity distribution across a pipe cross section is given by the well-known parabolic form,

$$\boxed{\frac{u}{V} = 2\left[1 - \left(\frac{r}{R}\right)^2\right]} \tag{8.9}$$

where u is the local velocity at the radial location r, and R is the pipe radius.
Substituting for u in Eq. (8.7), we have

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{2V}{\alpha}\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial x}} \tag{8.10}$$

where V is the mean velocity in the pipe and R is the pipe radius. $\partial T/\partial x$ is the rate of change of temperature with axial position. Figure 8.4 shows the temperature profile at two axial locations. The bulk fluid temperature, $T_b(x)$, increases with downstream location, since heat is added at the wall. The wall temperature, $T_w(x)$, also increases, but the *difference* between the bulk temperature and wall temperature, $(T_w - T_b)$ remains the same. This is because the shape of the temperature profile *does not* change in the fully developed region.

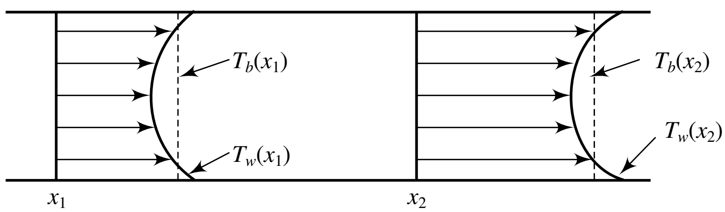


Fig. 8.4 *Temperature profiles in fully developed laminar convection*

For a circular tube, referring to Fig. 8.5, one can also write

$$q_w(\pi D x) = \dot{m}C_p[T_b(x) - T_{bi}]$$

Hence,
$$T_b(x) = T_{bi} + \frac{q_w(\pi D x)}{\dot{m}C_p} \tag{8.11}$$

Differentiating with respect to x,

$$\frac{dT_b(x)}{dx} = \frac{q_w(\pi D)}{\dot{m}C_p}$$

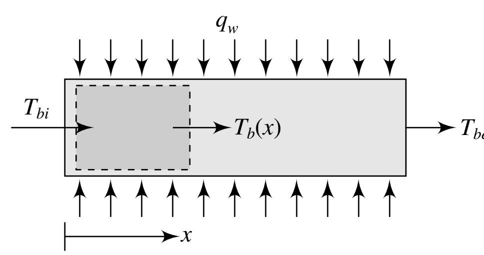


Fig. 8.5 *A control volume that extends a distance x along the inside of a pipe with constant heat flux on the walls.*

Noting that $\dot{m} = \rho V A_c$, where A_c is the cross-sectional area of the pipe,

$$\frac{dT_b(x)}{dx} = \frac{q_w(\pi D)}{\rho V A_c C_p} = \frac{4q_w(\pi D)}{\rho V \pi D^2 C_p} = \frac{4q_w}{\rho V D C_p} = \frac{2q_w}{\rho V R C_p} \tag{8.12}$$

where R is the radius of the pipe. Combining Eq. (8.12), Eq. (8.8), and Eq. (8.7) gives

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{4q_w}{\alpha\rho R C_p}\left[1 - \left(\frac{r}{R}\right)^2\right]} \tag{8.13}$$

We note that $\partial T/\partial r$ is only a function of r and $\alpha = k/\rho C_p$. One may then rewrite this as the following ordinary differential equation:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{4q_w}{kR}\left[1-\left(\frac{r}{R}\right)^2\right]} \tag{8.14}$$

or
$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{4q_w}{kR}\left[r-\frac{r^3}{R^2}\right]$$

Separating the variables and integrating, we have

$$\left[r\frac{\partial T}{\partial r} = \frac{4q_w}{kR}\left(\frac{r^2}{2}-\frac{r^4}{4R^2}\right)\right]+C_1 \quad \text{or} \quad \frac{\partial T}{\partial r} = \frac{4q_w}{kR}\left[\frac{r}{2}-\frac{r^3}{4R^2}\right]+\frac{C_1}{r}$$

Integrating again, we get

$$\boxed{T(r) = \frac{4q_w}{kR}\left[\frac{r^2}{4}-\frac{r}{16R^2}\right]+C_1\ln r + C_2} \tag{8.15}$$

To evaluate the two constants of integration C_1 and C_2, we require two boundary conditions. The *first* boundary condition is that the temperature must be finite at the pipe centre where $r = 0$; therefore, $C_1 = 0$. The *second* boundary condition is that the temperature at the wall is T_w at $r = R$. Applying the second boundary condition gives.

$$T_w = \frac{4q_w}{kR}\left[\frac{R^2}{4}-\frac{R^4}{16R^2}\right]+C_2$$

$$\boxed{C_2 = T_w - \frac{4q_w}{kR}\left(\frac{3R^2}{16}\right)} \tag{8.16}$$

Thus, the temperature profile in a fully developed laminar flow with constant wall heat flux and constant fluid properties is

$$T(r) = T_w - \frac{4q_w}{kR}\left[\frac{3}{16}R^2 - \frac{1}{4}\left(\frac{r}{R}\right)^2 \times R^2 + \frac{1}{16}\left(\frac{r}{R}\right)^4 \times R^2\right]$$

or
$$\boxed{T(r) = T_w - \frac{4q_w R}{k}\left[\frac{3}{16}-\frac{1}{4}\left(\frac{r}{R}\right)^2+\frac{1}{16}\left(\frac{r}{R}\right)^4\right]} \tag{8.17}$$

Convection heat flux is given by

$$q_w = \frac{\dot{Q}_w}{A} = h(T_w - T_b)$$

where T_b is the bulk temperature.

$$T_b = \frac{\int_A uT(r)dA}{VA}, V = \frac{\int_A u dA}{A}$$

The differential area $dA = 2\pi r\ dr$. Then the bulk (mean) temperature may be written as

$$\boxed{T_b = \frac{\int_0^R u(r)T(r)2\pi r dr}{V\pi R^2} = \frac{2\int_0^R u(r)T(r)rdr}{VR^2}} \tag{8.18}$$

At this point, the velocity field for a fully developed flow, $u(r)$ as given by Eq. (8.16), and the temperature field for a fully developed flow, $T(r)$ as given by Eq. (8.17), are substituted into Eq. (8.18) to get

$$T_b = \frac{\int_0^R 4V\left(1-\frac{r^2}{R^2}\right)\left\{T_w - \frac{4q_w R}{k}\left[\frac{3}{16}-\frac{1}{4}\left(\frac{r}{R}\right)^2+\frac{1}{16}\left(\frac{r}{R}\right)^4\right]\right\}rdr}{VR^2}$$

Carrying out the integration yields

$$T_b = \frac{4}{R^2}\int_0^R\left[\left(r-\frac{r^3}{R^2}\right)T_w - \frac{4q_w R}{k}\left(r-\frac{r^3}{R^2}\right)\times\left\{\frac{3}{16}-\frac{r^2}{4R^2}+\frac{1}{16}\frac{r^4}{R^4}\right\}\right]dr$$

or $$T_b = \frac{4}{R^2}\left[T_w\left(\frac{R^2}{2}-\frac{R^4}{4R^2}\right) - B\int_0^R\left\{\frac{3}{16}\left(r-\frac{r^3}{R^2}\right)-\frac{1}{4R^2}\left(r^3-\frac{r^5}{R^2}\right)+\frac{1}{16R^4}\left(r^5-\frac{r^7}{R^2}\right)\right\}\right]$$

where $$B = \frac{4q_w R}{k}$$

or $$T_b = \frac{4}{R^2}$$

$$\left[R^2T_w\left(\frac{1}{2}-\frac{1}{4}\right) - B\left[\frac{3}{16}\left(\frac{R^2}{2}-\frac{R^4}{4R^2}\right)-\frac{1}{4R^2}\left(\frac{R^4}{4}-\frac{R^6}{6R^2}\right)+\frac{1}{16R^4}\left(\frac{R^6}{6}-\frac{R^8}{8R^2}\right)\right]\right]$$

or $$T_b = T_w - \frac{4}{R^2}B\left[\frac{3}{16}\frac{R^2}{4}-\frac{1}{4\times 12}\frac{R^4}{R^2}+\frac{1}{16}\frac{R^6}{24R^4}\right] = T_w - B\left[\frac{3}{16}-\frac{1}{12}+\frac{1}{96}\right]$$

$$= T_w - B\left[\frac{18-8+1}{96}\right] = T_w - \frac{11}{96}B$$

or $$T_b = T_w - \frac{11}{96}\times\frac{4q_w R}{k} \quad\Rightarrow\quad T_w - T_b = \frac{11}{96}\times\frac{2q_w(2R)}{k}$$

or $$T_w - T_b = \frac{11}{48}\frac{q_w D}{k}$$

But $$q_w = h(T_w - T_b)$$

$\therefore$ $$h(T_w - T_b) = \frac{48}{11}(T_w - T_b)k/D \quad \text{or} \quad \frac{hD}{k} = \frac{48}{11} = 4.364$$

$$\boxed{Nu_D = \frac{hD}{k} = 4.364 \quad Re_D < 2300\ \textit{Laminar, constant heat flux}} \tag{8.19}$$

In fully developed laminar flow, the Nusselt number is a constant and the heat-transfer coefficient, h, is not a function of x. However, both the wall temperature and the bulk mean temperature vary with x.

The bulk (*mean*) fluid temperature increases *linearly* along the pipe starting at the initial temperature as illustrated in Fig. 8.6. The heat flux at the surface is related to the difference between the wall temperature and the bulk (*mean*) temperature through

$$q_w = \frac{\dot{Q}}{A} = h(T_w - T_b)$$

and, the wall temperature is

$$\boxed{T_w = \frac{q_w}{h} + T_b} \quad (8.20)$$

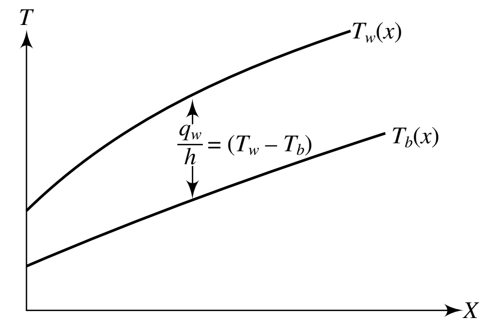


Fig. 8.6 *Fluid and wall (surface) temperature in a pipe with constant heat flux*

The heat-transfer $\dot{Q}$ is related to the difference in temperatures at the tube inlet and outlet. Fluid moves at a constant mass flow rate $\dot{m}$, and convection heat-transfer occurs at the inner surface (tube wall). The fluid kinetic and potential energy changes, as well as energy transfer by conduction in the axial direction can be ignored.

The rate of convection heat-transfer to the fluid must, therefore, equal the rate at which the fluid enthalpy increases.

$$d\dot{Q} = \dot{m}C_p dT_b$$

or

$$\boxed{\dot{Q} = \dot{m}C_p(T_{be} - T_{bi})} \quad (8.21)$$

8.4.3 • Constant Wall Temperature

Another common boundary condition we come across in practice is *constant wall temperature*. It is quite possible analytically to find the temperature profile in fully developed laminar flow with a constant wall temperature. Similar procedure is followed as in the case of constant wall heat flux, but the analysis becomes more complicated. To obtain the heat-transfer coefficient in a tube with constant wall temperature, fully developed laminar flow, and constant fluid properties, we use

$$\boxed{Nu_D = \frac{hD}{k} = 3.66 \quad Re_D < 2300 \text{ Laminar, constant wall temperature}} \quad (8.22)$$

The properties of the fluid in these equations are based on the bulk mean temperature, T_{bm}, which is the mean of the *inlet* and *exit* bulk fluid temperatures, i.e. $T_{bm} = \frac{1}{2}(T_{bi} + T_{be})$.

8.5 ❑ TURBULENT FLOW CONVECTION IN PIPES: EMPIRICAL CORRELATIONS

Completely analytical solutions for turbulent flow in pipes are simply not available. We have to depend on experimental and numerical investigations. Several empirical correlations are available, and some of the more common ones for pipe flows are presented here. Unlike laminar flow, in which the boundary condition (*constant wall temperature* or *constant wall heat flux*) changes the heat-transfer coefficient, turbulent correlations are generally valid for both situations except in the case of *liquid-metal heat-transfer*.

By far the most widely used correlation for turbulent flow in pipes is the *Dittus and Boelter equation:*

$$\boxed{\begin{array}{l} Nu_D = 0.023\, Re_D^{0.8}\, Pr^n \quad \left.\begin{array}{l} Re_D > 10\,000 \\ 0.7 < Pr \le 160 \\ \dfrac{L}{D} \ge 10 \end{array}\right\} \text{turbulent} \\ n = 0.3 \quad \text{cooling} \quad T_w < T_b \\ n = 0.4 \quad \text{heating} \quad T_w > T_b \end{array}} \tag{8.23}$$

All fluid properties are evaluated at the bulk mean temperature, which is the average of the inlet and outlet fluid temperatures. The exponent on the Prandtl number depends on whether the fluid is getting heated or cooled.

The effect of property variations in the radial direction for laminar flow of liquids is accounted for by correcting the mean Nusselt number using the following expression:

$$\boxed{\overline{Nu}_{\text{corr}} = \overline{Nu}\left(\frac{\mu}{\mu_w}\right)^{0.14}} \tag{8.24}$$

where the subscript w indicates the wall temperature.
When there are large temperature differences present in the fluid flow, wide variations in the fluid properties between the tube wall and the centreline are possible. We note that the viscosity of a liquid decreases with an increase in temperature while the viscosity of a gas increases with an increase in temperature. Such appreciable changes cause deviation in the velocity profile for isothermal flow as indicated in Fig. 8.7.

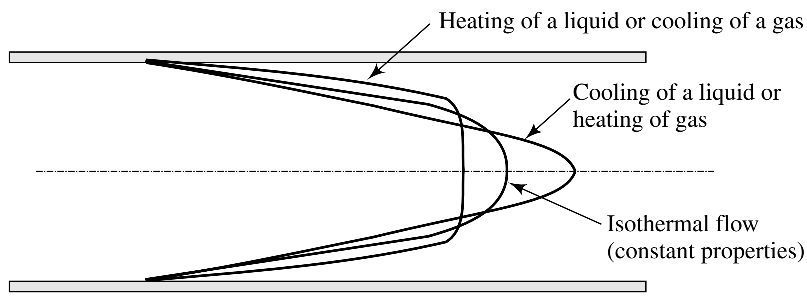


Fig. 8.7 *Influence of heating on velocity profile for laminar flow in a circular tube*

When temperature differences become large, property variations may be considerable, as mentioned above. In that case, the following correlation by *Seider and Tate* should be used:

$$\boxed{\begin{array}{l} Nu_D = 0.027\, Re_D^{0.8} Pr^{1/3}\left(\dfrac{\mu}{\mu_w}\right)^{0.14} \quad \left.\begin{array}{l} Re_D > 10\,000 \\ 0.7 < Pr \le 16\,700 \\ \dfrac{L}{D} \ge 10 \end{array}\right\} \text{turbulent} \\ \qquad n = 0.3 \quad \text{cooling} \quad T_w < T_b \\ \qquad n = 0.4 \quad \text{heating} \quad T_w > T_b \end{array}} \tag{8.25}$$

All fluid properties are evaluated at the bulk mean temperature except μ_w, which is the dynamic viscosity evaluated at the wall temperature T_w. This correlation may be used for both constant wall temperature and constant wall heat flux conditions.

Petukhov's correlation for forced convection heat-transfer in smooth or rough pipes gives excellent estimates of the heat-transfer coefficient for smooth pipes and reasonable estimates for rough pipes.

$$\boxed{\begin{array}{ll} Nu_D = \dfrac{(f/8)Re_D Pr}{1.07 + 12.7\sqrt{(f/8)}\,(Pr^{2/3} - 1)}\left(\dfrac{\mu}{\mu_w}\right)^n & \left.\begin{array}{l} 10^4 < Re_D < 5\times10^6 \\ 0.5 < Pr < 2000 \\ 0.08 < \mu/\mu_w < 40 \end{array}\right\} \\ & n = 0 \text{ constant heat flux} \\ & n = 0.11 \quad T_w > T_b \\ & n = 0.25 \quad T_b > T_w \end{array}} \tag{8.26}$$

For the friction factor, use *Petukhov's* correlation for smooth pipes, which is given as

$$\boxed{\begin{array}{ll} f = (0.79 \ln Re - 1.64)^{-2} & \text{turbulent flow, smooth wall} \\ & 3000 < Re_D < 5\times10^6 \end{array}} \tag{8.27}$$

The above correlations apply in the fully turbulent regime, where Re > 10 000. When 2300 < Re_D < 10 000, the flow may be in transition between laminar and turbulent. A useful correlation by *Gnielinski* for low Reynolds number is

$$\boxed{\begin{array}{ll} Nu_D = \dfrac{(f/8)(Re_D - 1000)Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)} & 3000 < Re_D < 5\times10^6 \\ & 0.5 < Pr < 2000 \end{array}} \tag{8.28}$$

This correlation is also equally applicable for constant wall temperature or constant heat flux situations. Properties of the fluid are evaluated at the average temperature of the fluid at inlet and exit. It has the added advantage of being valid for both smooth and rough tubes.

8.6 ❑ HEAT-TRANSFER CONSIDERATIONS: CONSTANT WALL TEMPERATURE

Consider a tube-wall surface held at an almost constant temperature. For example, if a fluid is boiling or condensing at constant pressure on the outside of a tube or pipe, the fluid is at the saturation temperature all along the pipe. Since high heat-transfer coefficients are associated with boiling and condensation, the wall temperature is approximately equal to the fluid temperature due to negligible thermal resistance ($1/hA_s \approx 0$) and is also uniform along the pipe.

To calculate the fluid temperature as a function of x, the distance along the pipe, let us begin with a differential control volume in Fig. 8.8. The fluid enters the left face of the control volume, exchanges heat with the wall, and leaves at the right face.

Assuming fully developed conditions, the only quantity that is a function of x is bulk fluid temperature, T_b. The differential area may be written in terms of the perimeter, P, and the length of the control volume, dx.

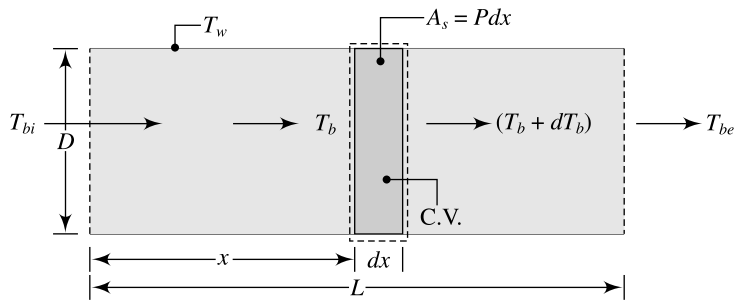


Fig. 8.8 *Differential control volume in a pipe with a constant wall temperature*

$$\text{Energy Balance:}\begin{pmatrix}\text{Rate of heat transfer to the}\\ \text{fluid in the control volume}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of enthalpy of the}\\ \text{fluid flowing through the tube}\end{pmatrix}$$

$$hP(T_w - T_b)dx = \dot{m}C_p dT_b$$

Separating the variables, we have

$$dx = \frac{\dot{m}C_p}{hP}\left(\frac{dT_b}{T_w - T_b(x)}\right)$$

Integrating this equation for a pipe of length L with inlet temperature T_{bi} and exit temperature T_{be} gives

$$\int_0^L dx = \frac{-\dot{m}C_p}{hP}\int_{T_{bi}}^{T_{be}} \frac{dT_b}{T_b(x) - T_w}$$

Performing the integration,

$$L = \frac{-\dot{m}C_p}{hP}[\ln(T_{be} - T_w) - \ln(T_{bi} - T_w)] = \frac{-\dot{m}C_p}{hP}\ln\left(\frac{T_{be} - T_w}{T_{bi} - T_w}\right)$$

To determine the length of the pipe for the specified exit temperature of the fluid, this expression is very useful.

Length of the pipe required,

$$\boxed{L = \frac{\dot{m}C_p}{hP}\ln\left(\frac{T_w - T_{bi}}{T_w - T_{be}}\right)} \tag{8.29}$$

where $P = \pi D$ for a circular pipe with internal diameter D.

Since the surface area of the pipe is length times perimeter, i.e., $A_s = PL$, we can also write

$$\ln\left(\frac{T_{be} - T_w}{T_{bi} - T_w}\right) = -\frac{hA_s}{\dot{m}C_p} \tag{8.30}$$

or

$$\boxed{\frac{T_{be} - T_w}{T_{bi} - T_w} = \exp\left(-\frac{hA_S}{\dot{m}C_p}\right)}$$

Solving for the exit fluid temperature, we get

$$\boxed{T_{be} = (T_{bi} - T_w)\exp\left(-\frac{hA_s}{\dot{m}C_p}\right) + T_w} \tag{8.31}$$

This result is shown in Fig. 8.9. At the inlet, the fluid temperature rises sharply towards the wall temperature, then approaches the wall temperature asymptotically. Near the outlet of the pipe, the difference between the wall temperature and bulk fluid temperature is smaller and less heat is transferred. From the first law, the total heat-transferred between the wall and fluid is

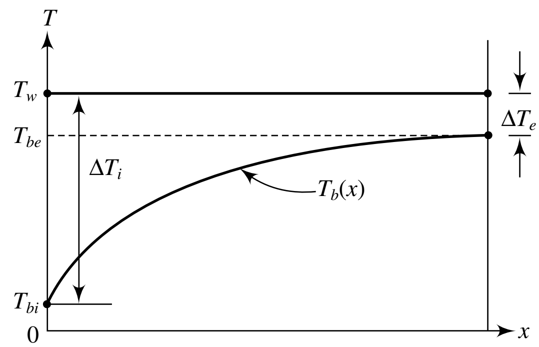


Fig. 8.9 *Fluid and surface temperatures in a pipe with constant wall temperature.*

$$\boxed{\dot{Q} = \dot{m}C_p(T_{be} - T_{bi})} \tag{8.32}$$

Rearranging, $\dot{m}C_p = \dfrac{\dot{Q}}{T_{be} - T_{bi}}$

Let us define:

$$\Delta T_i \equiv T_w - T_{bi}$$
$$\Delta T_e \equiv T_w - T_{be}$$

It follows that

$$\dot{m}C_p = \frac{\dot{Q}}{T_{be} - T_{bi}} = \frac{\dot{Q}}{(T_w - T_{bi}) - (T_w - T_{be})} = \frac{\dot{Q}}{\Delta T_i - \Delta T_e}$$

Substituting this into Eq. (8.29) and expressing the left-hand side in terms of ΔT_i and ΔT_e, gives

$$\ln\left(\frac{\Delta T_e}{\Delta T_i}\right) = \frac{-hA_s(\Delta T_i - \Delta T_e)}{\dot{Q}}$$

or

$$\boxed{\dot{Q} = \frac{hA_s(\Delta T_i - \Delta T_e)}{\ln(\Delta T_i/\Delta T_e)} = hA_s\Delta T_{lm}} \tag{8.32a}$$

where $\Delta T_{\ln}$ is called LMTD (*log mean temperature difference*), and can be expressed as

$$\boxed{\Delta T_{\text{lm}} = \frac{(\Delta T_i - \Delta T_e)}{\ln\left(\dfrac{\Delta T_i}{\Delta T_e}\right)}} \tag{8.33}$$

In many engineering applications, the temperature of an external fluid, rather than the tube wall temperature, is fixed. In such a case, we can still use the above result if T_w is replaced by T_∞ (the free stream temperature of the external fluid) and h is replaced by U (the average overall heat-transfer coefficient).

Let us consider an insulated pipe of length L with convection on both inside and outside, as shown in Fig. 8.10. The fluid on the outside of the pipe is at a constant temperature, T_∞, along the pipe length. The thermal circuit for this geometry consists of *four* resistances in series:

1. the convection resistance on the inside,
2. the conduction resistance through the wall,
3. the conduction resistance through the insulation, and
4. the convection resistance on the outside.

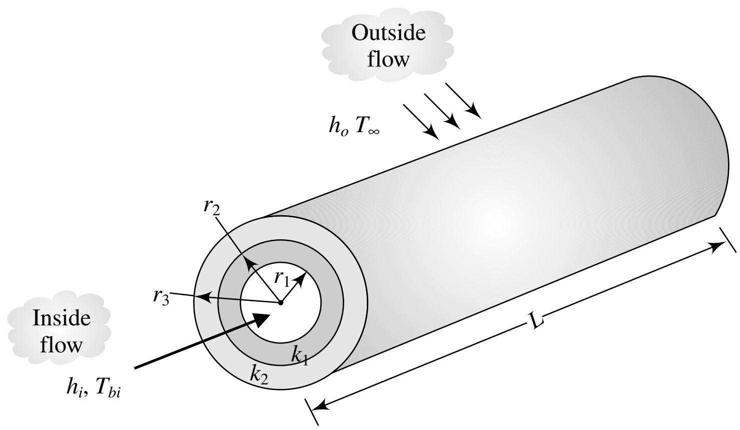


Fig. 8.10 *An insulated pipe with convection on both inside and outside*

The total thermal resistance is

$$\boxed{R_{\text{total}} = \frac{1}{h_i A_i} + \frac{\ln(r_2/r_1)}{2\pi k_1 L} + \frac{\ln(r_3/r_2)}{2\pi k_2 L} + \frac{1}{h_o A_o}} \tag{8.34}$$

The areas for convection on the inside and outside are $A_i = 2\pi r_1 L$ and $A_o = 2\pi r_3 L$, respectively. The total thermal resistance is then given by

$$\boxed{R_{\text{total}} = \frac{1}{2\pi r_1 L h_i} + \frac{\ln(r_2/r_1)}{2\pi k_1 L} + \frac{\ln(r_3/r_2)}{2\pi k_2 L} + \frac{1}{2\pi r_3 L h_o}} \tag{8.35}$$

For this length of pipe, L, the fluid temperature is T_b and the heat-transfer rate is

$$\boxed{\dot{Q} = \frac{T_b - T_\infty}{R_{\text{total}}}} \tag{8.36}$$

In terms of *overall heat-transfer coefficient, U*, one can write

$$\boxed{\dot{Q} = UA(T_b - T_\infty)} \quad \text{(W)}$$

The overall heat-transfer coefficient may be based on either the *inside area* or the *outside* area, that is,

$$\boxed{\dot{Q} = U_i A_i (T_b - T_\infty) = U_o A_o (T_b - T_\infty)} \tag{8.37}$$

Arbitrarily using the inside area as an example, we get

$$\boxed{U_i A_i = \frac{1}{R_{\text{total}}}} \tag{8.38}$$

Note: The advantage of defining an overall heat-transfer coefficient is that the preceding equations for flow in a pipe with an *isothermal* wall can be applied. All that we have to do is to simply replace T_w by T_∞ and hA by UA. The total heat-transferred is, $\boxed{\dot{Q} = UA_s \Delta T_{lm}}$. This is applicable only if T_∞ does not vary along the pipe.

8.7 ❑ NON-CIRCULAR CONDUITS: FULLY DEVELOPED TURBULENT FLOW

So far we have discussed internal flows in circular tubes but many engineering applications involve non-circular cross section. In fully developed turbulent flow in tubes, (Re_D > 2300), the heat-transfer and pressure-drop characteristics are governed largely by the extremely thin viscous sublayer next to the tube wall. The velocity and temperature distribution over most of the cross-sectional flow area is relatively flat. Under these circumstances, the heat-transfer rate from the tube wall would be almost independent of the cross-sectional shape of the tube in turbulent flow. Circular tubes can, therefore, be used even for non-circular ducts with reasonable accuracy provided $Pr > 0.7$. The only difference is that the diameter of the circular tube will be replaced by hydraulic diameter (D_h), also known as *equivalent diameter* (D_e), defined as follows:

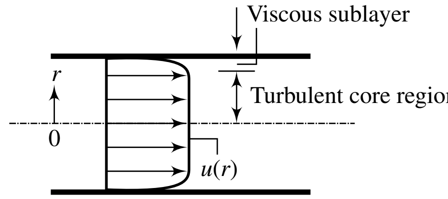


Fig. 8.11 *Velocity profile for turbulent flow in a tube*

$$\boxed{D_h(\text{or } D_e) = \frac{4A_c}{P}} \tag{8.39}$$

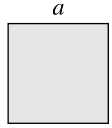
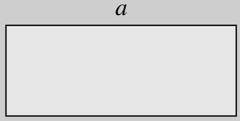
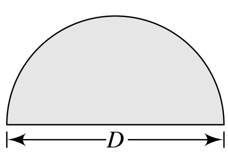
where A_c = area of cross section

P = wetted perimeter

Incidentally, for a circular tube $A_c = \dfrac{\pi}{4}D^2$ and $P = \pi D$ so that $D_h = 4 \times \dfrac{\pi}{4} D^2/\pi D = D$.

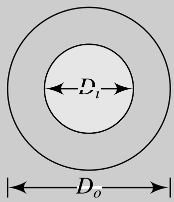
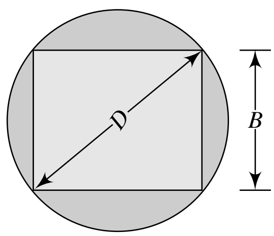
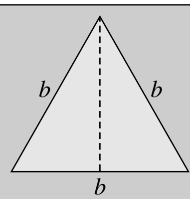
The values of D_h for some non-circular ducts are presented in the following table.

Table 8.1 *Hydraulic diameter for some non-circular conduits*

S.No.	Cross Section of Geometry	Hydraulic Diameter (D_h)
1.	a  a Square	A
2.	a  b Rectangular	$\dfrac{2ab}{a+b}$
3.	 D Semicircular	$\dfrac{\pi D}{\pi + 2}$

contd.

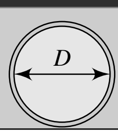
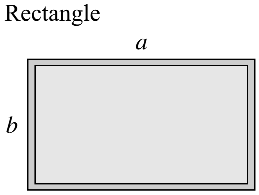
contd.

4.	 Concentric annlus	$D_o - D_i$
5.	 Circular tube with square inserted inside	$\dfrac{D(\pi-2)}{\pi+\sqrt{8}}$
6.	 Triangular	$\dfrac{b}{\sqrt{3}}$

8.7.1 • Fully Developed Laminar Flow

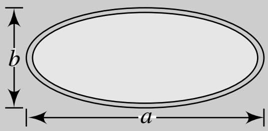
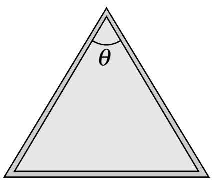
For laminar flow through various *non-circular* tubes, the Nusselt number relations are given in Table 8.2 for both *constant heat flux* and *constant wall temperature* boundary conditions.

Table 8.2 *Nusselt number and friction factor for fully developed laminar flow in tubes of various cross sections ($D_h = 4A_c/P$, $Re = VD_h/v$, and $Nu = hD_h/k$).*

Tube Geometry (Cross Section of Tube)	a/b or θ^o	Nusselt Number		Friction Factor, f
		T_w = Const.	q_w = Const	
Circle 	—	3.66	4.36	64.00/Re
Rectangle 	a/b			
	1	2.98	3.61	56.92/Re
	2	3.39	4.12	62.20/Re
	3	3.96	4.79	68.36/Re
	4	4.44	5.33	72.92/Re
	6	5.14	6.05	78.80/Re
	8	5.60	6.49	82.32/Re
	∞	7.54	8.24	96.00/Re

contd.

contd.

Ellipse 	a/b			
	1	3.66	4.36	64.00/*Re*
	2	3.74	4.56	67.28/*Re*
	4	3.79	4.88	72.96/*Re*
	8	3.72	5.09	76.60/*Re*
	16	3.65	5.18	78.16/*Re*
Triangle Isoceles 	$\underline{\theta}$			
	10^0	1.61	2.45	50.80/*Re*
	30^0	2.26	2.91	52.28/*Re*
	60^0	2.47	3.11	53.32/*Re*
	90^0	2.34	2.98	52.60/*Re*
	120^0	2.00	2.68	50.96/*Re*

8.7.2 • Flow Through the Concentric Tube Annulus: Laminar flow

In many heat-transfer applications, one fluid passes through an annular space formed by the inner and outer surfaces of two concentric tubes while the other fluid passes through the inner tube.

Consider a concentric annulus of inner diameter D_i and outer diameter D_o. The hydraulic diameter of the annulus is

$$D_h = \frac{4A_c}{P} = \frac{4\pi(D_o^2 - D_1^2)/4}{\pi(D_o + D_i)} = D_o - D_i \tag{8.40}$$

Annular flow is associated with two Nusselt numbers—Nu_i, on the *inner tube* surface and Nu_o on the *outer tube* surface – since it may involve heat-transfer on both surfaces. The Nusselt numbers for fully developed laminar flow are with one surface isothermal (*constant temperature*) and the other surface adiabatic (*insulated*). The convection coefficients for the inner and the outer surface can be obtained from the Nusselt numbers given in Table 8.3.

Table 8.3 *Nusselt number for fully developed laminar flow in an annulus with one surface insulated and the other at constant temperature.*

D_i / D_o	Nu_i	Nu_o
0	—	3.66
0.05	17.46	4.06
0.10	11.56	4.11
0.25	7.37	4.23
0.50	5.74	4.43
1.00	4.86	4.86

$$\boxed{h_i = \frac{Nu_i k}{D_h} \text{ and } h_o = \frac{Nu_o k}{D_h}} \tag{8.41}$$

We must note that for *fully developed turbulent flow*, the inner and outer convection coefficients may be assumed *equal* to each other, and the tube annulus can be considered a *non-circular duct* with a hydraulic diameter of $D_h = D_o - D_i$. The Nusselt number in this case can be determined from an appropriate turbulent flow relation like the Dittus–Boelter or Gnielinski correlation.

Thus, for a circular pipe, the hydraulic diameter reduces to the ordinary diameter. For turbulent flows, any of the empirical correlations for circular tube give reasonably accurate results provided the diameter in both the Nusselt and Reynolds numbers is replaced by the hydraulic diameter.

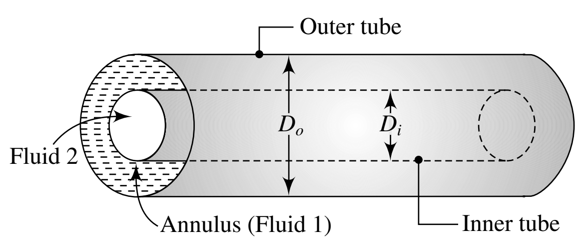


Fig. 8.12 *A double-pipe heat exchanger comprising two concentric pipes*

8.8 ❑ ENTRANCE EFFECTS IN FORCED CONVECTION: LAMINAR FLOW

The empirical correlations for developing laminar flow in the entrance region for the average Nusselt number under constant wall temperature through a circular tube are now presented below:

8.8.1 • Hydrodynamically Developed, Thermally Developing Flow

When the velocity profile is fully developed in an unheated starting length, a correlation due to Hausen, the average heat-transfer coefficient is a circular pipe with constant wall temperature is often used:

$$\boxed{Nu = 3.66 + \frac{0.0668\,Gz}{1 + 0.04\,Gz^{2/3}} \quad \text{Entrance region, constant wall temperature} \quad \text{Unheated starting length } Re < 2300} \tag{8.42}$$

where Graetz number $Gz = Re\,Pr = \dfrac{D}{L}$

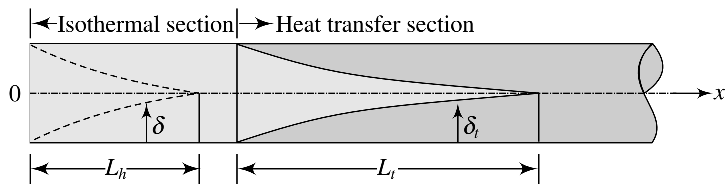


Fig. 8.13 *Development of boundary layers with an unheated starting length*

Properties are evaluated at the *bulk mean fluid temperature*. If the pipe is long, the Graetz number becomes very small and the correlation approaches Nu = 3.66, which is the result for fully developed flow.

If the hydrodynamic development length, $L_h \approx 0.05\ Re_D D$ is much smaller than the length of the tube, the flow is assumed to be fully developed and the velocity profile is already developed. Then the thermal development length, $L_t \approx 0.05\ Re_D\ Pr_D$ should be calculated. If it is comparable to or much greater than the tube length, then the temperature profile is still developing.

8.8.2 • Hydrodynamically and Thermally Developing Flow

$$\boxed{\mathrm{Nu} = 1.86\,Gz^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14} \quad \left.\begin{array}{l} Re < 2300 \\ 0.48 < Pr < 16700 \\ 0.0044 < \left(\dfrac{\mu}{\mu_w}\right) < 9.75 \end{array}\right\} \quad \left(\frac{Re_D Pr}{L/D}\right)^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14} > 2} \tag{8.43}$$

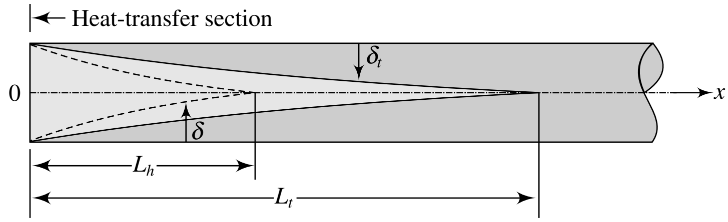


Fig. 8.14 *Simultaneously developing velocity and thermal profiles.*

All properties in this equation are evaluated at the *bulk mean fluid temperature* except μ_w, which is the viscosity evaluated at the *pipe-wall temperature*. This equation does not reduce to the correct limit for long pipes. It should be used only when it gives Nusselt numbers larger than 3.66. If the Nusselt number predicted by the equation falls below 3.66 then the flow may be presumed to be fully developed and a constant value of 3.66 should be used.

If the hydrodynamic development length, $L_h \approx 0.05\ Re_D D$ is much smaller than the length of the tube, the flow is assumed to be fully developed and the velocity profile is already developed. Then the thermal development length, $L_t \approx 0.05\ Re_D\ Pr\ D$ should be calculated. If it is comparable to much the greater temperature profile is still developing.

The correlation should, therefore, be used only when it gives Nusselt numbers larger than 3.66. If the Nusselt number predicted by the equation falls below 3.66, then the flow may be presumed to be fully developed and a constant value of 3.66 should be used. Figure 8.15 shows the variation of local Nusselt number with the reciprocal of Graetz number for constant heat flux and constant wall temperature conditions.

We can see that the local Nusselt number decreases with increasing values of x and that in both situations, it attains the asymptotic value of 4.364 for constant heat flux and 3.657 for constant wall temperature corresponding to fully developed temperature profile.

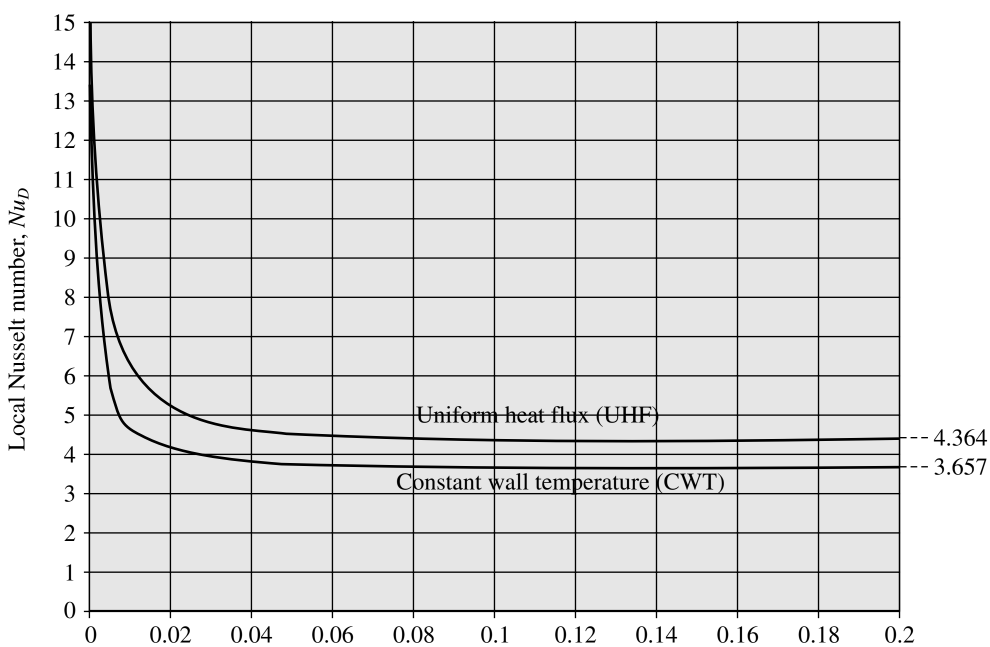


Fig. 8.15 *Local Nusselt number variation in the thermal entrance region of a circular tube for laminar flow. The velocity profile is fully developed.*

Figures 8.16 and 8.17 depict the developing and fully developed temperature profiles in pipe flow for both cases. For constant property fluids, the Nusselt number and heat-transfer coefficient are conetant in fully developed flow.

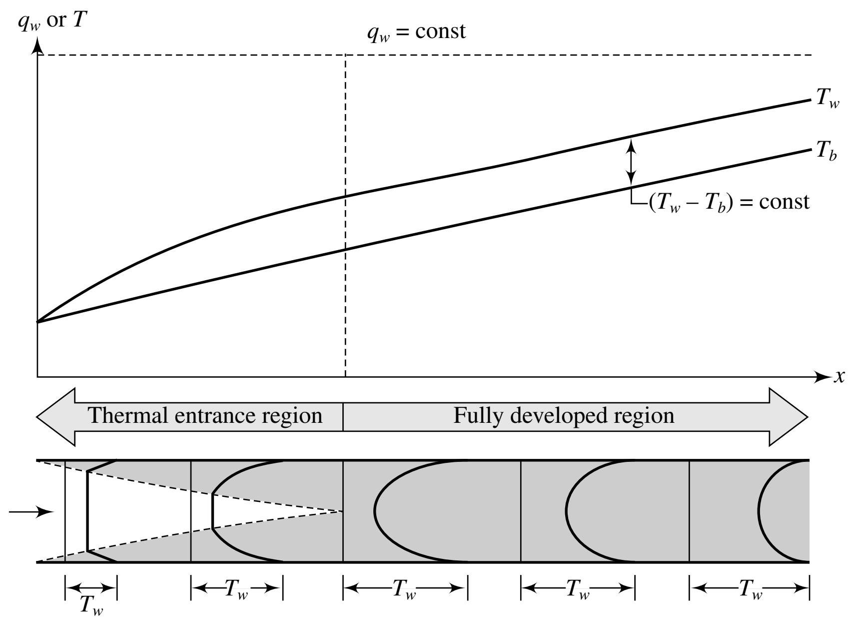


Fig. 8.16 *Developing and fully developed temperature profiles in pipe flow: Constant wall heat flux*

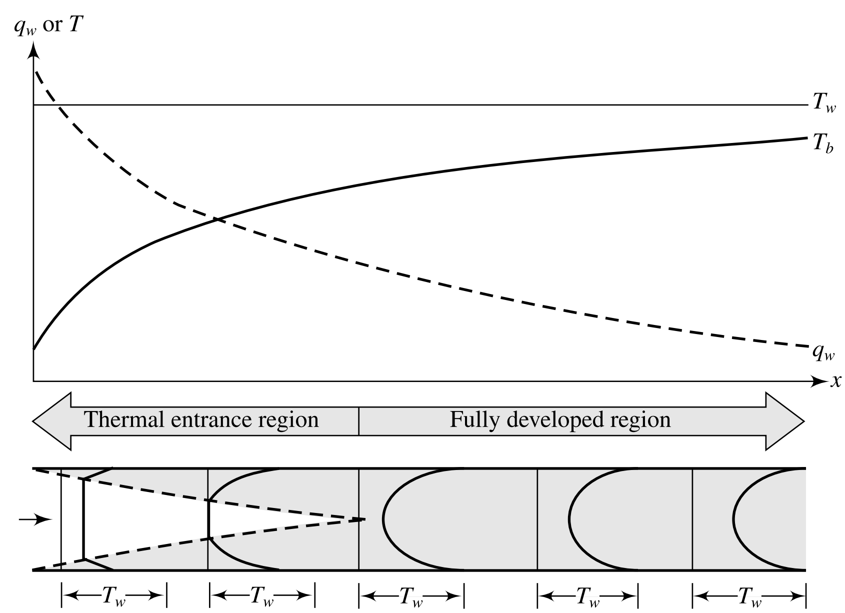


Fig. 8.17 *Developing and fully developed temperature profiles in pipe flow : Constant wall temperature*

We present here some equations to calculate the local and average Nusselt numbers for both cases of laminar tube flow.

Constant wall heat flux:

Local Nusselt Number Nu_D = $1.302(x^*)^{-1/3} - 0.5$ for $0.00005 < x^* < 0.0015$

$$Nu_D = 1.302\,(x^*)^{-1/3} - 0.5 \qquad \text{for } 0.00005 < x^* < 0.0015$$

$$= 4.364 + 8.68(1000x^*)^{-0.506}\, e^{-41x^*} \qquad \text{for } x^* > 0.0015$$

Average Nusselt Number $\overline{Nu}_D = 1.953(x^*)^{-1/3}$ for $x^* < 0.03$

$= 4.364 + 0.0722/x^*$ for $x^* > 0.03$

x^* = Dimensionless axial distance = $(x/D)/Re_D\, Pr$

Thermal entrance length $(L_t/D) = 0.04305\, Re_D\, Pr$

Constant wall temperature:

Local Nusselt Number Nu_D = $1.077(x^*)^{-1/3} - 0.7$ for $x^* < 0.01$

$$= 3.657 + 6.874(1000x^*)^{-0.488}\, e^{-57.2}x^* \qquad \text{for } x^* > 0.01$$

Average Nusselt Number $\overline{Nu}_D = 1.615(x^*)^{-1/3} - 0.2$ for $0.005 < x^* < 0.03$

$= 3.657 + 0.0499/x^*$ for $x^* > 0.03$

Thermal entrance length $(L_t/D) = 0.03347\, Re_D\, Pr$

8.9 ❑ LIQUID-METAL HEAT TRANSFER

Liquid metals with very low Prandtl numbers (of the order of 0.01) have high thermal conductivities and high heat-transfer rates can be obtained with these fluids. High rates of heat removal from a limited space make them suitable for applications like nuclear reactor. Compared to water and other organic coolants, liquid metals continue to remain in the liquid state even at high temperatures. This can be helpful in compact heat exchanger design. The only drawback is that these are difficult to handle due to their corrosive nature and violent action if they come in contact with water or air.

It is important to note that due to very small ratio of δ/δ_t (thermal boundary layer thickness being much larger than velocity boundary layer thickness), and $Pr < 0.5$, the usual correlations for turbulent flow through tubes are not applicable for liquid metals.

For fully developed turbulent flow of liquid metals in smooth tubes, the following correlations are recommended.

- $\boxed{Nu_D = 0.625(Re_D Pr)^{0.4}}$

[constant wall heat flux, properties at bulk mean temperature, $L/D > 60$ and $10^2 < Re_D\, Pr < 10^4$]

- $\boxed{Nu_D = 5.0 + 0.025(Re_D Pr)^{0.8}}$

[constant wall temperature, properties at bulk mean temperature, $L/D > 60$ and $Re_D\, Pr > 10^2$]

- $\boxed{Nu_D = 4.82 + 0.0185(Re_D Pr)^{0.827}}$

[constant heat flux, properties at bulk mean temperature, $3.6 \times 10^3 < Re_D < 9.05 \times 10^5$ and $10^2 < Re_D\, Pr < 10^4$]

The more recent correlations recommended for calculating the local Nusselt number for fully developed turbulent flows for fluids with $Pr < 0.1$ are due to *Notter and Sleicher*. They are

- $\boxed{Nu_D = 4.8 + 0.0156\,Re_D^{0.85}Pr^{0.93}}$ (constant wall temperature boundary condition)
- $\boxed{Nu_D = 6.3 + 0.067\,Re_D^{0.85}Pr^{0.93}}$ (constant heat flux boundary condition)

These correlations are valid for $10^4 < Re_D < 10^6$ and for $0.004 < Pr < 0.1$. Properties are evaluated at the local bulk mean fluid temperature.

8.10 ❑ ANALOGY BETWEEN MOMENTUM AND HEAT-TRANSFER

In a turbulent flow, the transport of heat is similar to the transfer of momentum. Eddies transport both momentum and heat.

8.10.1 • Heat-transfer for turbulent Flow in Circular Tubes

The equation for shear stress in the fluid was then written as

$$\tau = \rho(\varepsilon_M + \nu)\frac{du}{dy} \tag{8.44}$$

where the subscript M denotes momentum transfer. When convective heat-transfer takes place in a turbulent flow, a significant contribution to the heat diffusion in the fluid is made by the macroscopic transport owing to the eddy motion. This contribution is characterized by the *eddy diffusivity for heat*, ε_H, which is defined by

$$q = \rho C_p(\varepsilon_H + \alpha)\frac{dT}{dy} \tag{8.45}$$

The quantity $(\rho C_p \varepsilon_H)$ may be considered as an increase in the conductivity of the fluid due to turbulence. The ratio, $(\varepsilon_M/\varepsilon_H)$, is referred to as the turbulent Prandtl number, Pr_t.

If the Prandtl number equals unity ($\alpha = \nu$) and if the eddy diffusivities for heat and momentum are assumed equal ($Pr_t = 1$) then the velocity profile and the temperature profile will be similar. The similarity of the velocity profile and the temperature profile results in a heat flux distribution, $q(y)$, which is identical to the shear stress distribution, $\tau(y)$. This is known as the *Reynolds analogy*.

Then dividing Eq. (8.45) by Eq. (8.44) gives

$$\frac{q}{\tau} = \frac{C_p(\varepsilon_H + \alpha)dT}{(\varepsilon_M + \nu)}\frac{dT}{du}$$

with $\varepsilon_H = \varepsilon_M$ and $\alpha = \nu$, we have

$$\frac{q}{C_p \tau}du = -dT \tag{8.46}$$

Integrating from the tube surface to the mean bulk condition, and assuming q/τ is constant, equal to q_w/τ_w at the surface, we have

$$\frac{q_w}{C_p \tau_w}\int_0^V du = \int_{T_w}^{T_b} -dT \quad \text{or} \quad \frac{q_w V}{C_p \tau_w} = T_w - T_b \tag{8.47}$$

The heat flux at the wall can be written as

$$q_w = h(T_w - T_b) \quad \Rightarrow \quad T_w - T_b = q_w/h \tag{8.48}$$

and the shear stress at the wall can be expressed as

$$\tau_w = \frac{\Delta P}{L}\frac{A_c}{P} = \frac{\Delta P(\pi D^2)}{4\pi DL} = \frac{\Delta P}{4}\cdot\frac{D}{L}$$

where ΔP = the pressure drop $= f\dfrac{L}{D}\cdot\rho\dfrac{V^2}{2}$

where V is the mean flow velocity

so that $\tau_w = \dfrac{f}{8}\rho V^2$

From equations (8.47) and (8.48),

$$\frac{q_w}{h} = \frac{q_w V}{C_p \tau_w} \quad \Rightarrow \quad h = C_p \frac{f}{8}\rho\frac{V^2}{V}$$

or
$$\frac{h}{\rho C_p V} = \frac{f}{8}$$

Stanton number is defined as

$$St = \frac{Nu_D}{Re_D Pr} = \frac{hD}{k}\frac{\mu}{\rho VD}\frac{\rho}{\mu}\frac{k}{\rho C_p} = \frac{h}{\rho C_p V}$$

$\therefore$
$$\boxed{St = \frac{f}{8} = \frac{C_f}{2}} \tag{8.49}$$

This expression as called the *Reynolds analogy* for tube flow and is valid for $Pr \approx 1$. It gives reasonably good results for fluids with Pr close to unity, such as gases. It is valid for both laminar and turbulent flows.

For fluids with Pr different from unity, the dependence on Prandtl number is of the order $Pr^{2/3}$. The Reynolds analogy then may be modified by this factor to yield in terms of the Stanton number.

$$\boxed{St\, Pr^{2/3} = \frac{f}{8} = \frac{Nu_D}{Re_D Pr}\cdot Pr^{2/3} = \frac{C_f}{2}} \tag{8.50}$$

All the fluid properties in this equation are evaluated at $(T_b + T_w)/2$ except C_p in the Stanton number which is evaluated at the bulk mean temperature of the fluid. The above expression is called the *Colburn analogy*.

The analogy between momentum and heat-transfer in tube flow has enable to obtain a relation between the heat-transfer coefficient, h, and the fluid friction factor, f. Thus, h can be evaluated from the knowledge of f and vice versa.

Illustrative Examples

(A) Circular and Non-circular Tubes: Laminar Flow

EXAMPLE 8.1 *A fully developed laminar velocity profile exists in a circular pipe. A constant heat flux* q_w *is added to the fluid, which is initially at a uniform temperature* T_o. *Show that the length of pipe required to achieve a fully developed temperature profile is given by*

$$L = \frac{7}{384}\frac{u_{max} D^2}{\alpha}$$

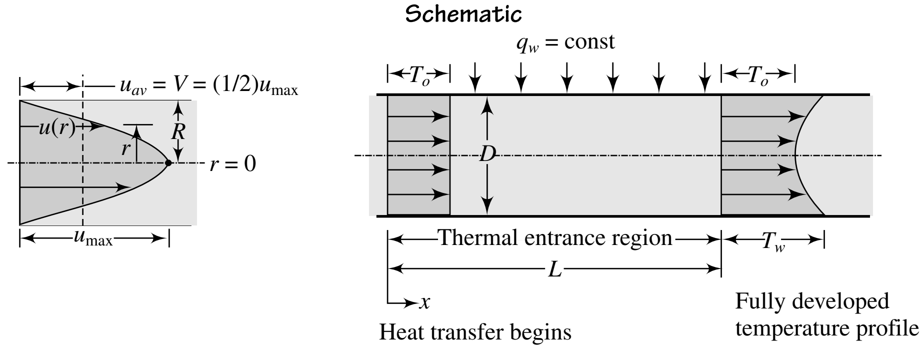
[IIT, Roorkee]

Solution

Known Fully developed laminar velocity profile in a circular pipe. Constant wall heat flux is added to the fluid.

Find Length of pipe required to achieve fully developed temperature profile. (*Thermal entrance length*).

Schematic



Assumptions (1) Steady, incompressible, laminar flow. (2) Fully developed velocity profile. (3) Uniform surface heat flux.

Analysis For constant wall heat flux boundary condition, $\dfrac{\partial T}{\partial x} = \text{const} = \dfrac{4\alpha q_w}{u_{max} kR}$

The temperature distribution is given by

$$T - T_o = \frac{1}{\alpha}\left(\frac{\partial T}{\partial x}\right)u_{max}\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] \text{ or } T = T_o + \frac{4\alpha q_w}{u_{max} kR}\cdot\frac{u_{max}}{2\alpha}\left(\frac{r^2}{2} - \frac{r^4}{8R^2}\right)$$

$$\therefore \qquad T = T_o + \frac{2q_w}{kR}\left(\frac{r^2}{2} - \frac{r^4}{8R^2}\right)$$

Enthalpy flowing out

$$= \int_o^R \rho u_{max}\left(1 - \frac{r^2}{R^2}\right) C_p \left\{T_o + \frac{2q_w}{kR}\left(\frac{r^2}{2} - \frac{r^4}{8R^2}\right)\right\} 2\pi r dr$$

$$= \rho u_{max} C_p 2\pi \int_o^R \left[T_o r + \frac{2q_w}{kR}\left(\frac{r^3}{2} - \frac{r^5}{8R^2}\right) - \frac{T_o r^3}{R^2} - \frac{2q_w}{kR}\left(\frac{r^5}{2R^2} - \frac{r^7}{8R^4}\right)\right] dr$$

$$= \rho u_{max} C_p \cdot 2\pi \left[\frac{T_o R^2}{2} + \frac{2q_w}{kR}\left(\frac{R^4}{8} - \frac{R^6}{48R^2}\right) - \frac{T_o}{R^2}\frac{R^4}{4} - \frac{2q_w}{kR}\left(\frac{R^6}{12R^2} - \frac{R^8}{64R^4}\right)\right]$$

$$= \rho u_{\max} C_p 2\pi \left[T_o \frac{R^2}{4} + \frac{2q_w}{kR} \cdot R^4 \left(\frac{1}{8} - \frac{1}{48} - \frac{1}{12} + \frac{1}{64} \right) \right]$$

$$= \rho u_{\max} C_p 2\pi \left[T_o \frac{R^2}{4} + \frac{2q_w R^3}{k} \cdot \frac{14}{384} \right]$$

Enthalpy flowing in

$$= \int_o^R \rho u_{\max} \left(1 - \frac{r^2}{R^2} \right) C_p T_o 2\pi r dr = \rho u_{\max} C_p 2\pi T_o \frac{R^2}{4}$$

Enthalpy flowing out; Enthalpy flowing in

$$= \rho u_{\max} C_p 2\pi \left[\frac{2q_w R^3}{k} \cdot \frac{14}{384} \right] = \text{Heat transferred into the fluid,}$$

$$\dot{Q} = 2\pi RL \cdot q_w$$

Thermal entrance length is then determined from

$$L = \frac{\rho u_{\max} C_p}{k} 2R^2 \cdot \frac{14}{384} = \frac{u_{\max} R^2}{\alpha} \cdot \frac{28}{384} \qquad [\text{since } \alpha \equiv k/\rho C_p]$$

$$= \frac{u_{\max} D^2}{4\alpha} \cdot \frac{28}{384}$$

Hence, $$\boxed{L = \frac{7}{384} \frac{u_{\max} D^2}{\alpha}}$$ **QED**

EXAMPLE 8.2 *One concept used for solar-energy collection involves placing a tube at the focal point of a parabolic reflector and passing a fluid through the tube. This arrangement can be approximated as a tube of 60 mm diameter on a sunny day with constant surface heat flux of 1900 W/m². Pressurized water enters the tube at 20°C with a flow rate of 0.01 kg/s. (a) Calculate the tube length required to obtain an outlet temperature of 60°C. (b) Assuming fully developed conditions, determine the surface temperature at the tube exit. Properties of water are:*

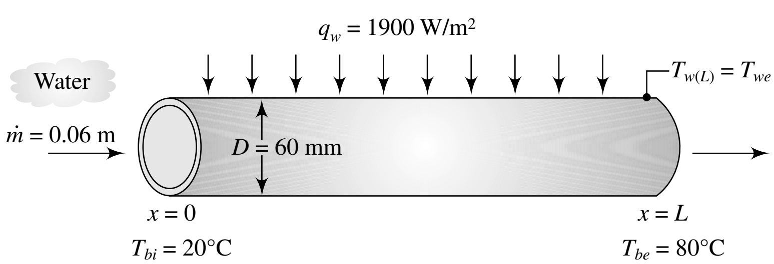
C_p = *4181 kJ/kg°C,* k = *0.670 W/m°C ,* μ = *0.355 × 10⁻³,* Pr = *2.22*

Solution

Known Flow of water through a tube with uniform wall heat flux.

Find (a) Tube length, L (m); (b) Tube surface temperature at exit, T_w (L) (°C).

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Fully developed conditions at tube exit. (4) Constant surface heat flux.

Analysis (a) For constant wall heat flux condition:

$$\dot{Q} = q_w(\pi DL) = \dot{m}C_p(T_{be} - T_{bi})$$

Length of the tube required is

$$L = \frac{\dot{m}C_p}{q_w(\pi D)}(T_{be} - T_{bi}) = \frac{(0.01\ \text{kg/s})(4181\ \text{J/kg °C})}{(1900\ \text{W/m}^2)(\pi \times 0.06\ \text{m})}(80 - 20)\text{°C}$$

$$= \mathbf{7.0\ m} \qquad \textbf{(Ans.) (a)}$$

(b) Reynolds number,

$$Re = \frac{4\dot{m}}{\pi D\mu} = \frac{4 \times 0.01\ \text{kg/s}}{\pi \times 0.06\ \text{m} \times 0.355 \times 10^{-3}\ \text{kg/m s}} = 598 \qquad (< 2300)$$

The flow is *laminar*.

For fully developed laminar flow through a tube with constant wall heat flux condition, we have

$$Nu = \frac{48}{11} = \frac{hD}{k}$$

Hence, the convection heat-transfer coefficient is

$$h = \frac{48}{11}\frac{k}{D} = \frac{48}{11} \times \frac{0.670\ \text{W/m °C}}{0.06\ \text{m}} = 48.73\ \text{W/m}^2\ \text{°C}$$

The tube surface temperature at the exit is determined from

$$q_w = h(T_{we} - T_{be})$$

Therefore,

$$T_{we} = T_w(L) = T_{be} + \frac{q_w}{h} = 80\text{°C} + \frac{1900\ \text{W/m}^2}{48.73\ \text{W/m}^2} = \mathbf{119.0°C} \qquad \textbf{(Ans.) (b)}$$

Comment Thermal entry length, $L_t \approx 0.05\ Re_D\ PrD = 0.05 \times 598 \times 2.22 \times 0.06 = 3.98$ m. Since the tube length $L = 7.0$ m is greater than $L_t = 3.98$ m, the assumption of fully developed conditions is appropriate.

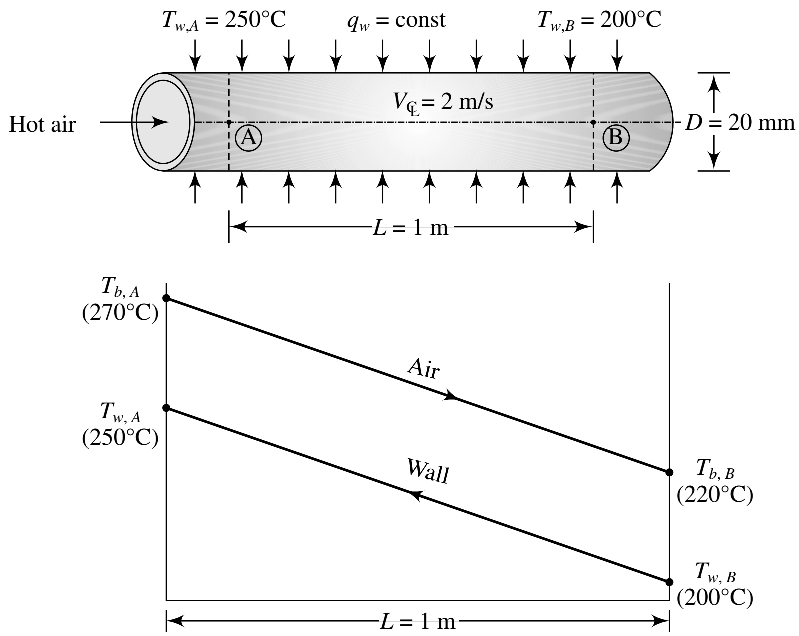
EXAMPLE 8.3 *Hot air flowing through a metal pipe of 20 mm diameter is cooled at a constant rate per unit length of pipe. At a particular section, denoted (A), the air velocity in the centre of the pipe is found to be 2 m/s and the wall temperature, as measured by a thermocouple on the inner surface of the pipe, is 250°C. At a section, denoted (B), situated 1 m downstream from (A), the wall temperature is found to be 200°C. Estimate the mean air temperature at section (B), using the following properties of air at 245°C:* $C_p = 1.032$ *kJ/kg °C,* $k = 0.0407$ *W/m °C,* $\rho = 0.6817$ *kg/m*^{3}*,* $\mu = 2.7417 \times 10^{-5}$ *kg/m s*

Solution

Known Hot air is being cooled as it flows through a metal pipe under constant heat flux conditions.

Find Mean air temperature at section (B), $T_{b,B}$.

Schematic



Assumptions (1) Steady flow conditions exist. (2) The wall heat flux is uniform. (3) Air is an ideal gas. (4) The inner surfaces of the pipe are smooth. (5) Flow is fully developed.

Analysis An indication of whether the flow is laminar or turbulent is given by the Reynolds number $Re = \dfrac{\rho VD}{\mu}$ where V is the mean fluid velocity. The centreline velocity of air, i.e., maximum flow velocity is 2 m/s. The mean flow velocity will depend on the velocity profile. For *laminar* flow:

$$u = V_{\max}\left(1 - \frac{r^2}{R^2}\right) \text{ and } u = V_{\max} \text{ at } r = 0 \qquad \text{(centreline velocity, } V_℄ \text{)}$$

The average velocity, $V = \dfrac{1}{2}V_{\max} = \dfrac{1}{2}V_℄ = 1\text{ m/s}$

Assuming *laminar* flow, $Re_D = \rho V\, D/\mu$

$$\therefore \qquad Re_D = \frac{(0.6817\text{ kg/m}^3)(1\text{ m/s})(20 \times 10^{-3}\text{ m})}{2.7417 \times 10^{-5}\text{ kg m/s}} = 497.28$$

This confirms (*as Re_D is less than Re_{crit} of about 2300*) that the flow is *laminar.*

As q_w is given as constant along the length of the pipe,

$$q_w = q_A = q_a = h(T_b - T_w) = h(T_b - T_w)_B$$

or $$T_{b,A} - T_{b,B} = T_{w,A} - T_{w,B} = (250 - 200)°\text{C} = 50°\text{C}$$

The mean heat flux through the pipe wall between sections (A) and (B) equals the change in the enthalpy of air.

$$q = q_W = \frac{\dot{Q}}{A_s} = \frac{\dot{m}C_p[T_{b,A} - T_{b,B}]}{A_s}$$

But, the mass-flow rate of air,

$$\dot{m} = \rho A_c V = \rho \frac{\pi}{4} D^2 V \text{ and } A_s = \pi DL$$

Hence,

$$q = \frac{\rho \times \frac{\pi}{4} D^2 V \times C_p[T_{b,A} - T_{b,B}]}{\pi DL} = \frac{\rho DVC_p(T_{w,A} - T_{w,B})}{4L}$$

$$= \frac{(0.6817 \text{ kg/m}^3)(0.02 \text{ m})(1 \text{ m/s})(1032 \text{ J/kg °C})(250 - 200)\text{°C}}{4 \times 1 \text{ m}} = \mathbf{176 \text{ W/m}^2}$$

The heat-transfer coefficient may be found from the equation $Nu_D = \frac{hD}{k} = \frac{48}{11}$ for laminar flow through a pipe under uniform surface heat flux conditions.

At section (B), $T_{b,B} = \frac{q}{h} + T_{w,B}$

where $h = Nu_D \frac{k}{D} = \frac{48}{11} \times 0.0407 \frac{\text{W}}{\text{m°C}} \times \frac{1}{0.02 \text{ m}} = 8.88 \text{ W/m}^2 \text{ °C}$

The mean air temperature at section (B) is then

$$T_{b,B} = \frac{176 \text{ W/m}^2}{8.88 \text{ W/m}^2 \text{ °C}} + 200\text{°C} = \mathbf{220°C} \qquad \textbf{(Ans.)}$$

Comment We note that

$$T_{b,A} = 50 + 220 = 270\text{°C and } T_{bm} = \frac{1}{2}(270 + 220) = 245\text{°C}$$

The properties were given at 245°C indeed.

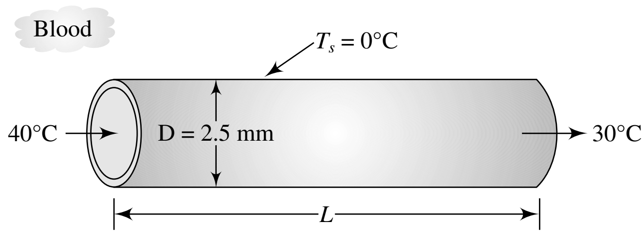
EXAMPLE 8.4 *A single-tube heat exchange device is to be designed to cool blood bypassed from a patient from 40 to 30°C by passing the fluid through a coiled tube placed in a water ice mixture. The tube diameter is 2.5 mm and the volume flow rate is 0.1 litre per minute. Neglecting entrance effects and assuming fully developed conditions, calculate (a) the Prandtl number, and (b) the total heat transfer rate lost from the blood while passing through the tube. Use the following properties of blood at the bulk mean temperature of 35°C: ρ = 1000 kg/m^3, $\nu = 7 \times 10^{-7}$ m^2/s, k = 0.5 W/m K, C_p = 4.0 kJ/kg K If the free convection effects on the outside of the tube are included, the mean overall heat transfer coefficient between the blood and the water ice mixture is estimated to be 300 W/m^{2}K. (c) Determine the tube length required to obtain the exit blood temperature.*

Solution

Known Blood is cooled while passing through a tube placed in water ice mixture (constant tube surface temperature).

Find (a) Prandtl number. (b) Heat-transfer rate. (c) Tube length if U = 300 W/m^2 K.

Schematic



Assumptions (1) Steady operating conditions. (2) Entrance effects are negligible and fully developed flow prevails. (3) Constant properties and heat transfer coefficient.

Analysis Reynolds number,

$$Re_D = \frac{4\dot{m}}{\pi D\mu} = \frac{4\dot{m}}{\pi D\nu\rho} = \frac{4\dot{\forall}}{\pi D\nu} = \frac{4(0.1\times 10^{-3}\ \text{m}^3/\text{min})(1\ \text{min}/60\ \text{s})}{\pi(2.5\times 10^{-3}\ \text{m})(7\times 10^{-7}\ \text{m}^2/\text{s})}$$

$$= 1212.6 \quad (< 2300)$$

Hence, the flow is *laminar*.

For laminar and fully developed flow conditions for constant tube wall temperature:

Average heat-transfer coefficient,

$$\bar{h} = \frac{3.66(0.5\ \text{W/m K})}{2.5\times 10^{-3}\ \text{m}} = 732\ \text{W/m}^2\ \text{K}$$

Prandtl number,

$$Pr = \frac{C_p\mu}{k} = \frac{C_p\nu\rho}{k} = \frac{(400\ \text{J/kg K})(7\times 10^{-7}\ \text{m}^2\text{s})(1000\ \text{kg/m}^3)}{0.5\ \text{W/m K}} = \mathbf{5.60} \qquad \textbf{(Ans.)}$$

The total heat-transfer rate can be obtained from an overall energy balance.

Knowing the inlet and exit temperatures of blood, the rate of heat transfer *from* the blood is determined to be

$$\dot{Q} = -\dot{m}C_p(T_{be} - T_{bi}) = \rho\dot{\forall}C_p(T_{bi} - T_{be})$$

$$= (1000\ \text{kg/m}^3)\ [(0.1\times 10^{-3}/60)\text{m}^3/\text{s}]\ (4000\ \text{J/kg K})\ [(40 - 30)\ \text{K}]$$

$$= \mathbf{66.67\ W} \qquad \textbf{(Ans.)}$$

Also, $\dot{Q} = U\,A\,(LMTD)$

$$LMTD = \frac{\Delta T_i - \Delta T_e}{\ln\left(\dfrac{\Delta T_i}{\Delta T_e}\right)} = \frac{(40 - 0) - (30 - 0)}{\ln(40/30)} = 34.76°\text{C}$$

and, surface area, $A = \pi DL$

Therefore, the tube length required is, $L = \dfrac{\dot{Q}}{L\pi D(LMTD)}$

Incorporating free convection effects on the tube surface, the mean overall heat transfer coefficient is specified as 300 W/m² K.

Hence, $$L = \frac{66.67 \text{ W}}{(300 \text{ W/m}^2\text{K})(\pi \times 2.5 \times 10^{-3} \text{ m})(34.76°\text{C})} = \mathbf{0.814 \text{ m}}$$ **(Ans.)**

B: Circular and Non-circular Tubes: Turbulent Flow

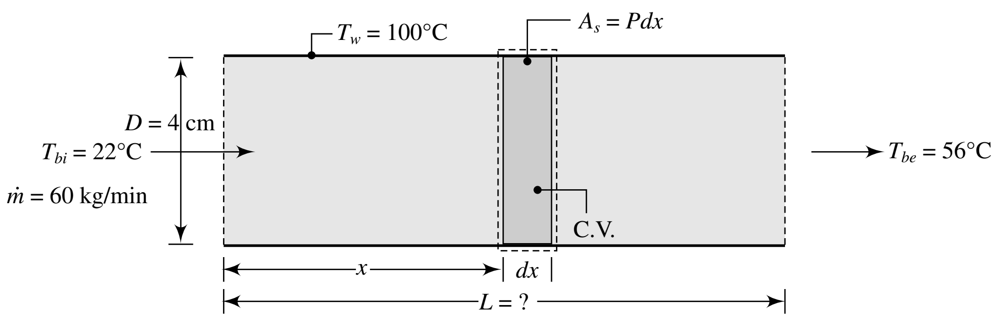
EXAMPLE 8.5 *Oil is heated from 22°C to 56°C by passing through a tube of 4 cm diameter. Find out the length of the tube required, for an oil flow rate of 60 kg/min, if the surface temperature of the tube wall is maintained at 100°C. Assume the following properties of oil at the mean bulk temperature:* $\rho = 895$ *kg/m³,* k = 0.151 *W/m K,* $\nu = 0.40 \times 10^{-6}$ *m²/s,* $C_p = 2.177$ *kJ/kg K [IES 1997]*

Solution

Known Oil is heated in a tube under constant wall temperature condition.

Find Tube length required, L (m).

Schematic



Assumptions (1) Steady operating conditions. (2) Fully developed turbulent flow. (3) Constant oil properties.

Analysis Reynolds number, $Re_D = \dfrac{VD}{\nu}$, Mass flow rate, $\dot{m} = \dfrac{\pi}{4}D^2\rho V$

$\therefore$ Mean velocity, $V = \dfrac{4\dot{m}}{\rho\pi D^2}$ and $Re_D = \dfrac{4\dot{m}D}{\rho\pi D^2\nu} = \dfrac{4\dot{m}}{\rho\pi D\nu}$

$$\therefore \quad Re_D = \frac{4(60/60 \text{ kg/s})}{(895 \text{ kg/m}^3)(\pi \times 0.04 \text{ m})(0.4 \times 10^{-6} \text{ m}^2/\text{s})}$$

$$= 88\,913.3 \quad (> 2300) \quad \Rightarrow \quad \text{The flow is } \textit{turbulent}.$$

$$Pr = \frac{C_p\mu}{k} = \frac{C_p\nu\rho}{k} = \frac{(2177 \text{ J/kg K})(0.4 \times 10^{-6} \text{ m}^2/\text{s})(895 \text{ kg/m}^3)}{0.151 \text{ W/m K}}$$

$$= 5.16 \quad (0.7 < Pr < 100)$$

One can use the *Dittus–Boelter* equation to determine h.

$$Nu_D = \frac{hD}{k} = 0.023(Re_D)^{0.8}(Pr)^{0.4}$$ (*n = 0.4, as the oil is being heated*)

$$\therefore \qquad h = \left(\frac{0.151 \text{ W/m K}}{0.04 \text{ m}}\right)(0.023)(88913.3)^{0.8}(5.16)^{0.4} = 1523.6 \text{ W/m}^2 \text{ K}$$

To find out the tube length, we write the energy balance on the elemental control volume (CV) at a distance x from one end as shown in the schematic.

$$\dot{m}C_p dT_b = h(Pdx)(T_w - T_b)$$

or
$$\frac{hP}{\dot{m}C_p}\int_0^L dx = \int_{T_{bi}}^{T_{be}} \frac{dT_b}{T_w - T_b} \quad \text{or} \quad \frac{hPL}{\dot{m}C_p} = \ln\left(\frac{T_w - T_{bi}}{T_w - T_{be}}\right)$$

Tube length required,

$$L = \frac{\dot{m}C_p}{h\pi D}\ln\left(\frac{T_w - T_{bi}}{T_w - T_{be}}\right) \qquad (P = \pi D \textit{ for a circular tube})$$

$$= \frac{(60/60)\text{kg/s}(2177 \text{ J/kg K})}{(1523.6 \text{ W/m K})(\pi \times 0.04 \text{ m})}\ln\left(\frac{100-22}{100-56}\right) = \mathbf{6.5\ m} \qquad \textbf{(Ans.)}$$

Check*:** $\dfrac{L}{D} = \dfrac{6.5}{0.04} = 162.5 > 50$ ***Hence, OK.

EXAMPLE 8.6 *(a) Consider the flow in a circular tube of diameter* D *and length* L *whose wall is maintained at a constant temperature* T_w *with the fluid inlet and exit temperatures of* T_{bi} *and* T_{be} *respectively. Show that,* $T_{be} = T_w - (T_w - T_{bi})e^{-4\text{St } L/D}$ *where* St *is the Stanton number.*

(b) Ethylene glycol is cooled from 65 to 40°C while flowing at 10 m/s through a 30 mm ID pipe with a constant wall temperature of 20°C. Determine the length of the pipe required and the resulting pressure drop.

Properties of ethylene glycol at 52.5°C are

ρ *= 1092.5 kg/m*^{3}*,* μ *= 0.649* $\times$ *10*$^{-2}$*,* Pr *= 63.3*

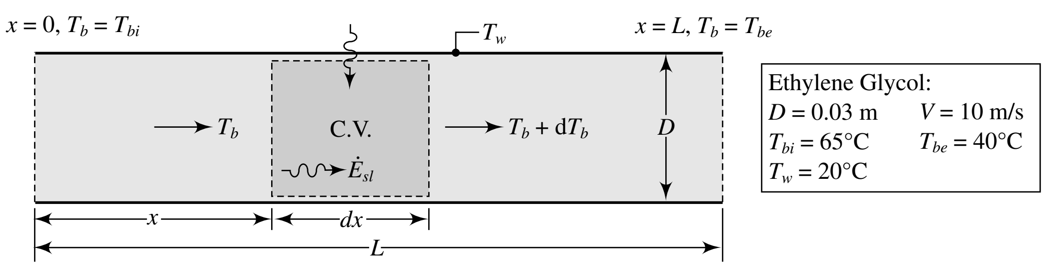
At 20°C, μ_s *= 2.2* $\times$ *10*$^{-2}$ *kg/m s*

Solution

Known Fluid flow through a tube with constant wall temperature.

Find (a) To prove that $T_{be} = T_w - (T_w - T_{bi})\exp[-4\, St\, L/D]$, (b) Length of pipe and pressure drop.

Schematic



Assumption (1) Steady operating conditions. (2) Constant properties. (3) Uniform and constant tube surface temperature. (4) Fully developed turbulent flow.

Analysis (a) Energy balance for an elemental control volume shown in the schematic gives

$$d\dot{Q} = \dot{m}C_p dT_b = h(x)(\pi D dx)(T_w - T_b)$$

Rearranging,

$$\frac{dT_b}{(T_w - T_b)} = \frac{\pi D}{\dot{m}C_p} h(x)dx$$

Integrating this between, x = 0 and $x = L$, one gets

$$\int_{T_{bi}}^{T_{be}} \frac{dT_b}{T_w - T_b} = \frac{\pi D}{\dot{m}C_p} \int_0^L h(x)dx$$

$$\ln\left[\frac{T_w - T_{bi}}{T_w - T_{be}}\right] = \frac{\pi DhL}{\dot{m}C_p} = \frac{hA_s}{\dot{m}C_p} \qquad (\because \pi DL = A_s)$$

where the average heat-transfer coefficient has been defined as $h = \frac{1}{L}\int_0^L h(x)dx$.
It follows that

$$\frac{T_w - T_{be}}{T_w - T_{bi}} = \exp\left[\frac{-hA_s}{\dot{m}C_p}\right] \tag{A}$$

where A_s is the total heat transfer surface area.

For the tube, $\dot{m} = \rho\left(\frac{\pi D^2}{4}\right)V$ and, hence,

$$\frac{hA_s}{\dot{m}C_p} = \frac{h(\pi DL)}{\rho(\pi D^2/4)VC_p} = \frac{4h}{\rho VC_p}\frac{L}{D} = 4St\frac{L}{D}$$

where St stands for Stanton number, a *non-dimensional parameter*. The Stanton number is related to three dimensionless parameters as shown below:

$$St = \frac{Nu}{Re\,Pr} = \frac{hD}{k}\frac{\mu}{\rho VD}\frac{k}{\mu C_p} = \frac{h}{\rho VC_p}$$

Equation (A) can then be expressed as

$$\boxed{\boldsymbol{T_{be} = T_w - (T_w - T_{bi})e^{-4\,St\,L/D}}}$$ ***Hence proved.***

(b) Using the *Sider–Tate* correlation:

$$Nu = 0.027\,\mathrm{Re}^{0.8}\,Pr^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14}$$

where $$Re = \frac{\rho VD}{\mu} = \frac{(1092.5\text{ kg/m}^3)(10\text{ m/s})(0.03\text{ m})}{0.649\times10^{-2}\text{ kg/m s}}$$

$= 50\ 500 \qquad (< 2300) \qquad \Rightarrow \qquad$ *Turbulent flow*

$\therefore$ $$Nu = 0.027(50\,500)^{0.8}(63.3)^{1/3}\left(\frac{0.649\times10^2}{2.2\times10^{-2}}\right)^{0.14} = 525$$

$$St = \frac{Nu}{Re\,Pr} = \frac{525}{50\,500\times63.3} = 0.000164$$

We note that

$$\frac{T_{be} - T_w}{T_{bi} - T_w} = \exp\left[\frac{-4\,St\,L}{D}\right] \text{ or } \ln\left(\frac{40-20}{65-20}\right) = -\frac{4 \times 0.000164 \times L}{0.03}$$

The pipe length required,

$$L = \frac{-0.03 \times \ln(20/45)}{4 \times 0.000164} = \mathbf{37.0\ m}$$

(Ans.) (b)

Pressure drop, $\Delta P = f\left(\frac{L}{D}\right)\frac{1}{2}\rho V^2$

where $f = (0.79 \ln Re - 1.64)^2 = [0.79 \ln 50\,500 - 1.64]^2 = 0.0209$

$\therefore$ $$\Delta P = 0.0209\left(\frac{37}{0.03}\right)\frac{1092.5 \times 10^2}{2}(Pa) = \mathbf{1409\ kPa}$$ **(Ans.) (b)**

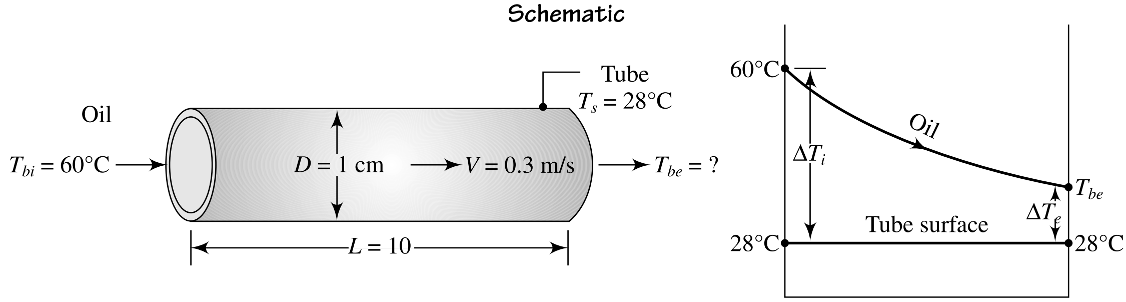
EXAMPLE 8.7 *Lubricating oil at a temperature of 60°C enters a 1 cm diameter tube with a velocity of 0.3 m/s. The tube surface is maintained at 28°C. Assuming that the oil has the following average properties, calculate for the tube length of 10 m, (a) the outlet temperature of the oil, (b) the heat-transfer rate, (c) the log mean temperature difference, and (d) the arithmetic mean temperature difference.*
$\rho = 865\ kg/m^3$ $C_p = 1.78\ kJ/kg°C$, $k = 0.14\ W/m\ K$, $\nu = 9 \times 10^{-6}\ m^2/s$

Solution

Known Oil flows through a tube under specified conditions.

Find (a) T_{be} (°C), (b) $\dot{Q}$(W), (c) ΔT_{lm} and (d) ΔT_{am}.

Schematic



Assumptions (1) Steady-state conditions prevail. (2) Constant tube wall temperature. (3) Constant properties. (4) Thermally developing, laminar flow.

Analysis Reynolds number

$$Re_D = \frac{VD}{\nu} = \frac{0.3\ \text{m/s} \times 0.01\ \text{m}}{9 \times 10^{-6}\ \text{m}^2/\text{s}} = 333.3$$

Prandtl number,

$$Pr = \frac{C_p \nu \rho}{k} = \frac{1780\ \text{J/kg K} \times 9 \times 10^{-6}\ \text{m}^2/\text{s} \times 865\ \text{kg/m}^3}{0.14\ \text{W/m K}} = 98.98$$

Hydrodynamic entry length,

$$L_h = 0.0575\, Re_D D = 0.0575 \times 333.3 \times 0.01 \text{ m} = \mathbf{0.192\ m}$$

Thermal entry length,

$$L_t = 0.03347\ Re_D Pr\, D = 0.03347 \times 333.3 \times 98.98 \times 0.01 \text{ m} = \mathbf{11\ m}$$

For $L_h << L_t$, the velocity profile may be assumed to be already developed and only the temperature profile is developing.
Accordingly to evaluate the convection coefficient, we use the Hausen's correlation,

$$\overline{Nu_D} = f(Re_D, Pr, D/L) = 3.66 + \frac{0.0668(D/L)\, Re_D Pr}{1 + (0.04)[(D/L) Re_D Pr]^{2/3}} = \frac{\bar{h} D}{k}$$

Substituting proper values,

$$\overline{Nu_D} = 3.66 + \frac{0.0668\,(0.01/10)(333.3)(98.98)}{1 + (0.04)[(0.01/10)(333.3)(98.98)]^{2/3}}$$

$$= 3.66 + 1.56 = 5.22$$

Average heat transfer coefficient,

$$\bar{h} = \frac{k}{D}\overline{Nu_D} = \frac{0.14 \text{ W/m K} \times 5.22}{0.01 \text{ m}} = \mathbf{73.1\ W/m^2\ K}$$

Mass-flow rate of oil flowing through the tube,

$$\dot{m} = \rho \frac{\pi}{4} D^2 V = (865 \text{ kg/m}^3)\left(\frac{\pi}{4} \times 0.01^2\right)\text{m}^2 (0.3 \text{ m/s}) = 0.02038 \text{ kg/s}$$

From energy balance:

$$\dot{Q} = \dot{m} C_p (T_{bi} - T_{be}) = \bar{h}(A_s)\Delta T_{lm} = \bar{h}(\pi D L)\frac{[(T_{bi} - T_w) - (T_{be} - T_w)]}{\ln\left(\dfrac{T_{bi} - T_w}{T_{be} - T_w}\right)}$$

or

$$\dot{m} C_p \ln\left(\frac{T_{bi} - T_w}{T_{be} - T_w}\right) = \bar{h}(\pi D L)$$

Exit oil temperature,

$$T_{be} = T_w + (T_{bi} - T_w)\exp\left(-\frac{\bar{h}\pi D L}{\dot{m} C_p}\right)$$

$$= 28°\text{C} + (60 - 28)°\text{C} \exp\left(\frac{-73.1 \text{ W/m}^2 \text{ K} \times 0.01 \text{ m} \times 10 \text{ m}}{0.02038 \text{ kg/s} \times 1780 \text{ J/kg K}}\right)$$

$$= \mathbf{45°C}$$ **(Ans.) (a)**

Heat-transfer rate,

$\dot{Q}$ = (0.02038 kg/s) (1.78 × 10^3 J/kg °C) (60 – 45)°C = **544 W** **(Ans.) (b)**

Log mean temperature difference,

$$\Delta T_{lm} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i / \Delta T_e)}$$

where $\Delta T_i \equiv T_{bi} - T_w = (60 - 28)°C = 32°C$ and $\Delta T_e \equiv T_{be} - T_w = (45 - 28)°C = 17°C$

$$\therefore \quad \Delta T_{lm} = \frac{32 - 17}{\ln(32/17)} = \mathbf{23.7°C} \qquad \textbf{(Ans.) (c)}$$

Arithmetic mean temperature difference,

$$\Delta T_{am} = \frac{1}{2}(\Delta T_i + \Delta T_e) = \frac{32 + 17}{2} = \mathbf{24.5°C} \qquad \textbf{(Ans.) (d)}$$

Comment ***Check:*** $Q = h(\pi DL)\Delta T_{lm}$ = (73.1 W/m² K) (π × 0.01 m × 10 m) (237°C) = 544 W

EXAMPLE 8.8 *Air at 2 bar and 200°C is heated as it flows through a 30 mm diameter tube at a velocity of 15 m/s. (a) Compute the heat-transfer rate per metre length of the tube if a constant wall heat flux condition is maintained and the wall temperature is 20°C above the air temperature throughout the length of the tube. (b) Determine the increase in the bulk temperature over a 3 m length of the tube Thermophysical properties of air to be used are:*

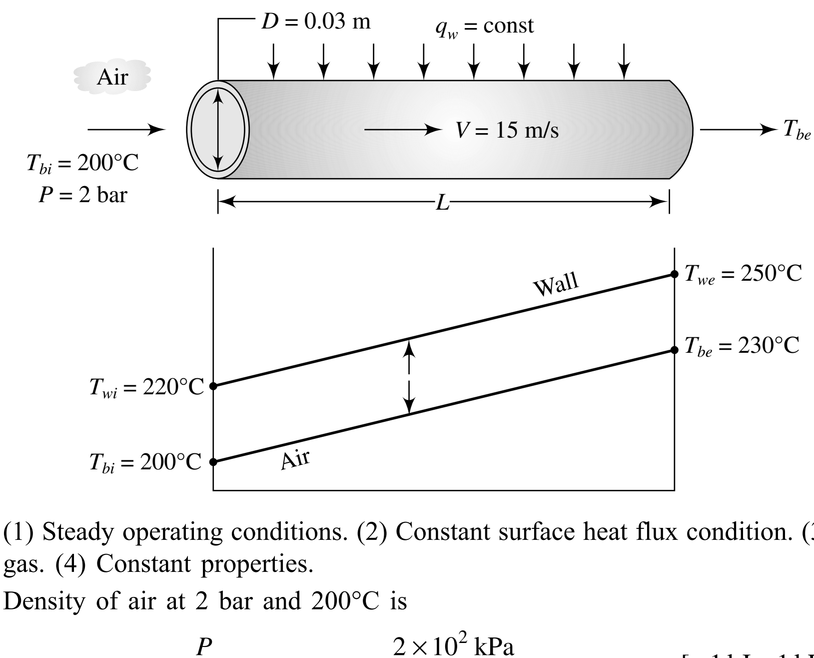
k = *0.03779 W/m K,* μ = *25.77 ×* 10^{-6} kg/ms, C_p = *1.023 kJ/kg K,* Pr = *0.6974*

Solution

Known Heating of air flowing through a tube with prescribed velocity under constant wall heat flux condition.

Find (a) Heat rate per *m* tube length, $\frac{\dot{Q}}{L}$; (W/m) (b) Increase in bulk temperature over 3 m length tube, ΔT_b (°C).

Schematic



Assumptions (1) Steady operating conditions. (2) Constant surface heat flux condition. (3) Air is an ideal gas. (4) Constant properties.

Analysis Density of air at 2 bar and 200°C is

$$\rho = \frac{P}{RT} = \frac{2 \times 10^2 \text{ kPa}}{(0.287 \text{ kPa m}^3/\text{kg K})(200 + 273.15)\text{K}} \quad [\because 1 \text{ kJ} = 1 \text{ kPa m}^3]$$

$$= 1.4728 \text{ kg/m}^3$$

Reynolds number,

$$Re_D = \frac{\rho VD}{\mu} = \frac{(1.4728 \text{ kg/m}^3)(15 \text{ m/s})(0.03 \text{ m})}{25.77 \times 10^{-6} \text{ kg/ms}} = \mathbf{25\,718}$$

Thus the flow is *turbulent.*
Using the Dittus–Boelter correlation,

$$Nu_D = \frac{hD}{k} = 0.023(Re_D)^{0.8}(Pr)^{0.4} \qquad (n = 0.4 \text{ as air is being heated.})$$

or $$h = \frac{(0.023)(0.03779 \text{ W/m K})}{0.03 \text{ m}}(25718)^{0.8}(0.6974)^{0.4} = 84.65 \text{ W/m}^2 \text{ K}$$

Constant wall heat flux implies that

$$q_w = \frac{\dot{Q}}{A} = h(T_w - T_b) = \text{const} \quad \Rightarrow \quad (T_w - T_b) = \text{const}$$

In this case, $(T_w - T_b)$ = 20°C and is constant all along the length of the tube.
Hence, the heat transfer rate per metre length of the tube is

$$\frac{\dot{Q}}{L} = h(\pi D)(T_w - T_b) = (84.65 \text{ W/m}^2 \text{ K})(\pi \times 0.03 \text{ m})(20 \text{ K})$$

$$= \mathbf{159.6 \text{ W/m}}$$ **(Ans.) (a)**

The heat-transfer rate is also given by

$$\dot{Q} = \dot{m}C_p(T_{be} - T_{bi})$$

or $$\left(\frac{\dot{Q}}{L}\right)L = \dot{m}C_p \Delta T_b \quad \Rightarrow \quad \Delta T_b = \frac{(\dot{Q}/L)L}{\dot{m}C_p}$$

Mass-flow rate of air through the tube is

$$\dot{m} = \rho A_c V = (1.4728 \text{ kg/m}^3)\left[\left(\frac{\pi}{4} \times 0.03^2\right)\text{m}^2\right](15 \text{ m/s}) = 0.0156 \text{ kg/s}$$

Bulk temperature increase over 3 m length of the tube is

$$\Delta T_b = \frac{(159.6 \text{ W/m})(3 \text{ m})}{(0.0156 \text{ kg/s})(1.023 \times 10^3 \text{ J/kg K})} = \mathbf{30°C}$$ **(Ans.) (b)**

Comment

- Note that the Dittus–Boelter correlation holds good for both *constant wall temperature* and *constant wall heat flux* conditions.
- It is significant that for *constant wall temperature* condition:

$$\frac{h(\pi DL)}{\dot{m}C_p} = \ln\left(\frac{T_w - T_{bi}}{T_w - T_{be}}\right)$$

or $$T_{be} = T_w - (T_w - T_{bi})\exp[-h(\pi DL)/\dot{m}C_p]$$

[$(T_w - T_b)$ is ***not*** constant throughout the tube length.]

- Air properties are evaluated at the bulk mean temperature, $T_{bm}\left(\equiv\frac{1}{2}(T_{bi}+T_{be})\right)$. As T_{be} was not known initially, the properties of air at 200°C were used. Since ΔT_b comes out to be 30°C, $T_{bm} = \frac{1}{2}(200 + 230) = 215°C$. With properties of air at 215°C, calculations can be repeated but without any significant change in the results.

EXAMPLE 8.9 *Water is heated while flowing through a 1.5 cm × 3.5 cm rectangular cross section duct at a velocity of 1.25 m/s. The water enters the duct at 40°C. The duct wall surface temperature is maintained at 85°C. Determine the length of the duct required to raise the temperature of water by 30°C. Use the following correlation:*

$$\text{Nu} = \frac{(f/8)(\text{Re}-1000)\text{Pr}}{1+12.7\sqrt{f/8}(\text{Pr}^{2/3}-1)} \quad \textit{where } f = (0.79 \ln \text{Re} - 1.64)^{-2}$$

The properties of water at T_{bm} = 55°C are

$\rho = 985.2$ kg/m³ $\qquad C_p = 4.183$ kJ/kg K

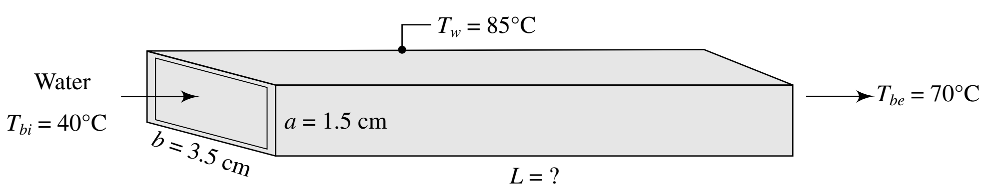
$k = 0.649$ W/m K $\qquad \mu = 0.504 \times 10^{-3}$ kg/ms

Solution

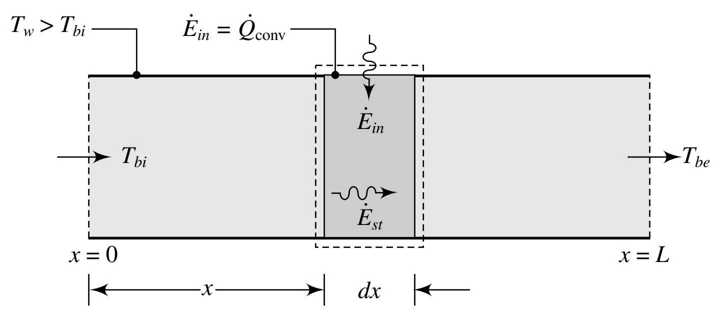
Known Water flowing through a rectangular cross-section duct is heated under constant wall temperature condition.

Find Length of the duct (m).

Schematic



Rectangular cross section



Control volume energy balance

Assumptions (1) Constant wall temperature. (2) Fully developed flow. (3) Constant properties. (4) Steady state conditions.

Analysis Reynolds number, $Re = \dfrac{\rho V D_e}{\mu}$

where D_e (*equivalent diameter*) $= \dfrac{2ab}{a+b} = \dfrac{3(3.5\text{ cm})(1.5\text{ cm})}{(3.5+1.5)\text{cm}}$ = 2.1 cm

$$Re_D = \frac{(985.2\text{ kg/m}^3)(1.25\text{ m/s})(2.1\times10^{-2}\text{ m})}{0.504\times10^{-3}\text{ kg/m s}} = 51312.5 \qquad (>3000)$$

The flow is therefore *turbulent*.
Prandtl number,

$$Pr = \frac{C_p\mu}{k} = \frac{4.183\times10^3\text{ J/kg K}\times0.504\times10^{-3}\text{ kg/m}^3}{0.649\text{ W/m K}} = 3.25$$

Darcy friction factor,

$$f = (0.79 \text{ le } Re - 1.64)^{-2} = (0.79 \ln 51\,312.5 - 1.64)^{-2} = 0.02083$$

Nusselt number,

$$Nu = \frac{f/8(Re-1000)Pr}{1+12.7\sqrt{f/8}(Pr^{2/3}-1)} = \frac{(0.02083/8)(51312.5-1000)(3.25)}{1+12.7\sqrt{\dfrac{0.02083}{8}}(3.25^{2/3}-1)} = 240$$

Convection coefficient,

$$h = Nu\frac{k}{D_e} = 240\times\frac{0.649\text{ W/m K}}{0.021\text{ m}} = 7420.8\text{ W/m}^2\text{ K}$$

∴

Control volume energy balance:

$$\dot{E}_{\text{in}} = \dot{E}_{st}$$

$$h(P\,dx)(T_w - T_b) = \dot{m}C_p dT_b \quad \text{or} \quad dx = \frac{\dot{m}C_p}{hP}\frac{dT_b}{(T_w - T_b)}$$

Integrating,

$$\int_{x=0}^{x=L} dx = \frac{\dot{m}C_p}{hP}\int_{T_b=T_{bi}}^{T_b=T_{be}} \frac{dT_b}{(T_w-T_b)}$$

$$L = \frac{\dot{m}C_p}{hP}\ln\left(\frac{T_w - T_{bi}}{T_w - T_{be}}\right)$$

where $P = 2(a + b) = 2(3.5 + 1.5)$ cm = 10 cm or 0.1 m
and $\dot{m} = \rho A_c V = \rho(ab)V = 985.2\text{ kg/m}^3 \times 1.5\times3.5\times10^{-4}\text{ m}^2\times1.25\text{ m/s} = 0.6465\text{ kg/s}$

Length of the duct required,

$$L = \frac{(0.6465\text{ kg/s})(4183\text{ J/kg K})}{(7420.8\text{ W/m}^2\text{ K})(0.1\text{ m})}\ln\left(\frac{85-40}{85-70}\right) = \mathbf{4.0\ m} \qquad \textbf{(Ans.)}$$

Comment Since $L/D_e = 4.0/0.021 = 190.5$ which is much greater than 10, the assumption of fully developed flow is valid.

EXAMPLE 8.10 *Cold atmospheric air at 10°C enters a semi circular cross sectioned channel having a diameter of 15 cm. The channel is 6 m long. Determine the uniform temperature of the inside surface of the channel if the mean velocity of the air passing through the channel is 9 m/s. The temperature of the air exiting the channel is 30°C.*

Properties of air at 1 atm and 20°C:

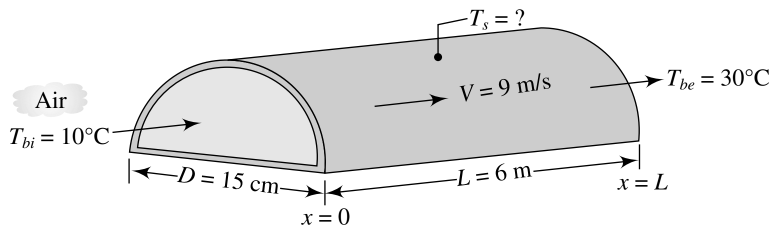
C_p = *1.007 kJ/kg°C,* k = *0.02514 W/m K,* ν = *15.16 × 10^{-6} kg/m s,* Pr = *0.7309*

Solution

Known Air is heated while flowing through a passage of semi circular cross section under constant wall temperature conditions.

Find Surface temperature of channel, T_s (°C).

Schematic



Assumptions (1) Steady operating conditions. (2) Fully developed velocity and temperature profiles. (3) Constant thermophysical properties of air.

Analysis For a *non-circular* cross section, equivalent or hydraulic diameter, $D_h = 4\dfrac{A_c}{P}$ should be used as a *characteristic dimension* for internal flow, forced convection conditions.

Cross-sectional area,

$$A_c = \frac{1}{2}\frac{\pi D^2}{4} = \frac{\pi}{8} \times 0.15^2 \text{ m}^2 = 8.836 \times 10^{-3} \text{ m}^2$$

(Wetted perimeter, P) = (Diameter, D) + (Half the circumference, π D/2)

$$= D\left(1 + \frac{\pi}{2}\right) = 0.15\left(1 + \frac{\pi}{2}\right)(\text{m}) = 0.3856 \text{ m}$$

$$\therefore \qquad D_h = \frac{4A_c}{P} = \frac{4 \times 8.836 \times 10^{-3} \text{ m}^2}{0.3856 \text{ m}} = 0.0917 \text{ m}$$

Reynolds number,

$$Re = \frac{VD_h}{\nu} = \frac{(9 \text{ m/s})(0.0917 \text{ m})}{15.16 \times 10^{-6} \text{ m}^2/\text{s}} = 54\,439 \quad \Rightarrow \quad \text{Turbulent flow}$$

Using the *Dittus–Böelter* correlation,

$$Nu = 0.023(Re)^{0.8}(Pr)^{0.4} = 0.023(54439)^{0.8}(0.7309)^{0.4} = 124.74$$

$$\therefore \qquad h = \frac{k}{D_h}Nu = \frac{0.02514 \text{ W/m K}}{0.0917 \text{ m}}(124.74) = 34.2 \text{ W/m}^2 \text{ K}$$

Mass-flow rate of air,

$$\dot{m} = \rho_i A_c V = \frac{P}{RT} A_c V = \frac{101.325 \text{ kPa}}{0.287 \text{ kJ/kg K} \times 283.15 \text{ K}} \times 8.836 \times 10^{-3} \text{ m}^2 \times 9 \text{ m/s}$$

$$= 0.0992 \text{ kg/s}$$

Energy balance for constant channel surface temperature:

$$h(PL)(\Delta T_{lm}) = \dot{m} C_p (T_{be} - T_{bi})$$

where $$\Delta T_{lm} = \frac{(T_s - T_{bi}) - (T_s - T_{be})}{\ln\left(\dfrac{T_s - T_{bi}}{T_s - T_{be}}\right)} = \frac{T_{be} - T_{bi}}{\ln\left(\dfrac{T_s - T_{bi}}{T_s - T_{be}}\right)}$$

Hence, $$\frac{hPL}{\dot{m} C_p} = \ln \frac{T_s - T_{bi}}{T_s - T_{be}}$$

or $$\exp\left(-\frac{hPL}{\dot{m} C_p}\right) = \frac{T_s - T_{be}}{T_s - T_{bi}} = M \quad \Rightarrow \quad T_s - T_{be} = MT_s - MT_{bi}$$

Inner surface temperature of the channel is then given by

$$T_s = \left(\frac{T_{be} - M\,T_{bi}}{1 - M}\right)$$

where $$M = \exp\left[-\frac{34.2 \text{ W/m}^2 \text{ K} \times 0.3856 \text{ m} \times 6 \text{ m}}{0.0992 \text{ kg/s} \times 1007 \text{ J/kg °C}}\right] = 0.453$$

$\therefore$ $$T_s = \frac{30 - (0.453)(10)}{1 - 0.453} = \mathbf{46.6°C}$$ **(Ans.)**

EXAMPLE 8.11 *Blood at 32°C enters a 2.5 mm inside diameter steel tube with a volumetric flow rate of 15 mL/s. The tube surface is electrically heated to impart a uniform heat flux. The tube wall temperature must not exceed 44°C to avoid damage to the blood. Calculate the minimum length of the tube required to warm the blood to 37°C. Blood properties may be approximated as those of water. Properties of water at the bulk mean temperature of 34.5°C are*

$\rho = 994.0 \text{ kg/m}^3 \quad \mu = 7.32 \times 10^{-4} \text{ Ns/m}^2, \quad k = 0.624 \text{ W/m K},$

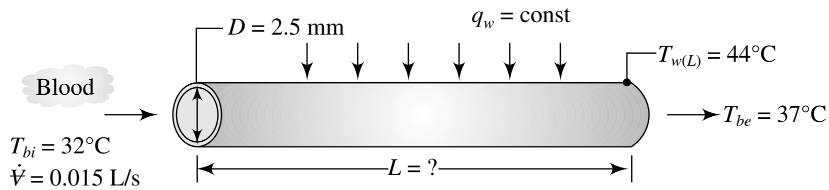
$\text{Pr} = 4.91, \; C_p = 4.178 \text{ kJ/kg K}$

Solution

Known Blood is warmed in a tube under constant heat flux conditions.

Find Tube length required.

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Blood properties are same as water properties. (4) Smooth tube. (5) Fully developed flow. (6) Uniform heat flux.

Analysis Reynolds number for a circular tube is, $Re = \dfrac{\rho VD}{\mu} = \dfrac{4\dot{m}}{\pi D\mu}$

where $\dot{m} = \rho\dot{V} = (994 \text{ kg/m}^3)(0.015 \times 10^{-3} \text{ m}^3\text{/s}) = 0.01491 \text{ kg/s}$

$$\therefore \quad Re = \frac{4 \times 0.0149 \text{ kg/s}}{\pi \times 0.0025 \text{ m} \times 7.32 \times 10^{-4} \text{ Ns/m}^2} = 10374 \quad \Rightarrow \quad \textit{Turbulent flow}$$

Using the Dittus–Bolter equation with n = 0.4, since the fluid (blood) is being heated,

$$Nu = \frac{hD}{k} = 0.023(Re)^{0.8}(Pr)^{0.4}$$

$$\therefore \quad h = \frac{0.023 \times 0.624 \text{ W/m °C}}{0.0025 \text{ m}}(10374)^{0.8}(4.91)^{0.4} = 17.7 \times 10^3 \text{ W/m}^2 \text{ K}$$

Since q_w = const, $(T_w - T_h)$ is same throughout,

$$(T_w - T_b)_{x=L} = (T_w - T_h)_{\max} = (44 - 37)°\text{C} = 7°\text{C or K}$$

$$\begin{pmatrix}\text{Heat transferred from the}\\ \text{tube surface to the blood}\end{pmatrix} = \begin{pmatrix}\text{Rate of increase of stored}\\ \text{energy of the blood}\end{pmatrix}$$

$$\therefore \quad \dot{Q} = \dot{m}C_p(T_{be} - T_{bi}) = (0.01491 \text{ kg/s})(4178 \text{ J/kg K})(37 - 32)°\text{C} \quad \text{or} \quad \text{K} = 311.5 \text{ W}$$

Furthermore, $\dot{Q} = h(\pi DL)(T_w - T_b)$

Therefore, the tube length required,

$$L = \frac{\dot{Q}}{h(\pi D)(T_w - T_b)} = \frac{311.5 \text{ W}}{17.7 \times 10^3 \text{ W/m}^2 \text{ K}(\pi \times 2.5 \times 10^{-3} \text{ m})(7 \text{ K})}$$

$= \mathbf{0.32 \text{ m}}$ **(Ans.)**

Comment Since L/D = 0.32 m/0.0025 m = 128 (> 60) the assumption of fully developed flow is justified.

EXAMPLE 8.12 *A thick walled tube (20 mm ID and 40 mm OD) with its outer surface insulated is heated electrically to provide a uniform heat generation rate of 10^6 W/m^3. Water enters the tube at a temperature of 25°C with a mass flow rate of 325 kg/h.*

(a) How long must the tube be to achieve the desired exit temperature of 40°C?

(b) Determine the local convection heat transfer coefficient at the tube outlet if the inner wall temperature of the tube at the outlet is 55°C.

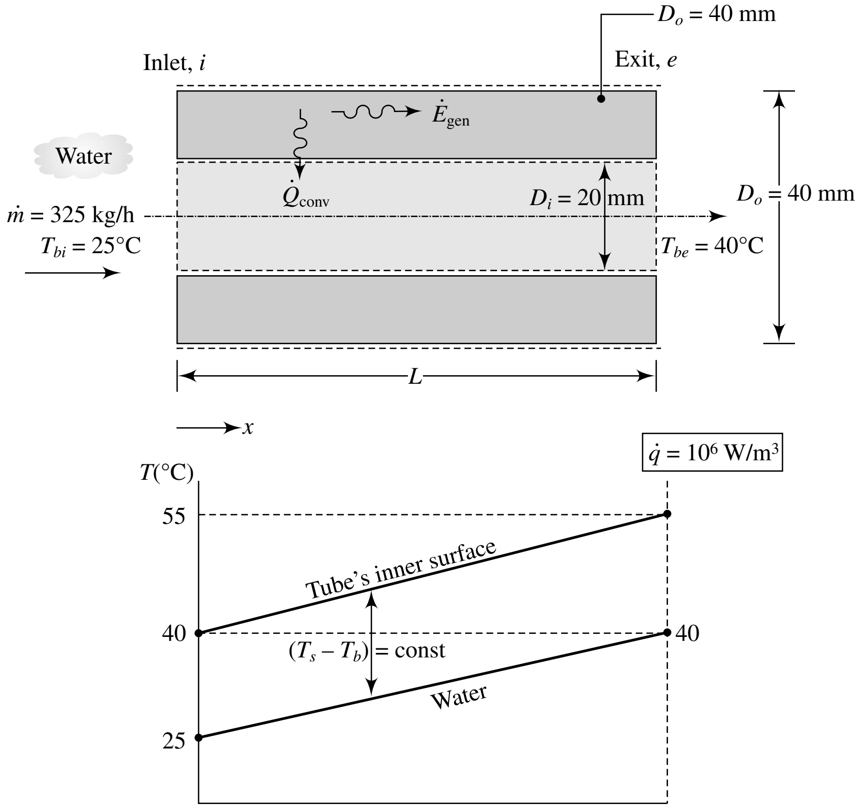
(c) If the flow conditions are fully developed over the entire tube, calculate the tube's inner surface temperature at the inlet.

Solution

Known Water heating system comprising thick walled tube with uniform heat generation.

Find (a) Length, L; (b) Local convection coefficient; $h_{(x = L)}$; (c) Tube inner surface temperature, $T_{s,i}$.

Schematic



Fully developed flow

Assumptions (1) Steady operating conditions. (2) Uniform wall heat flux. (3) Insulated outer tube surface. (4) Constant fluid properties.

Properties $C_{p\text{water}}$ at T_{bm} = (25 + 40)/2 = 32.5°C = 4.178 kJ/kg °C.

Analysis From energy balance:

$$\begin{pmatrix}\text{Heat generated}\\ \text{within the tube wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of heat transfer to water}\\ \text{flowing through the tube}\end{pmatrix}$$

(*since the outer surface of the tube is adiabatic*).

$$\overline{q}\,\forall = q_s A_s = \dot{m} C_p (T_{be} - T_{bi})$$

where A_s is the inner surface area of the tube ($\equiv \pi\, D_i\, L$)

and $\forall$ is the volume in which electrical heating takes place $\left(\equiv \dfrac{\pi}{4}(D_o^2 - D_i^2)L\right)$.

It follows that, $\overline{q}\dfrac{\pi}{4}(D_o^2 - D_i^2)L = \dot{m} C_p (T_{be} - T_{bi})$

Hence, the tube length required is

$$\boxed{L = \frac{4\dot{m}C_p(T_{be} - T_{bi})}{\bar{q}\pi(D_o^2 - D_i^2)}}$$

or $$L = \frac{4[(325/3600)\text{kg/s}](4178\text{ J/kg°C})(40-25)\text{°C}}{(10^6\text{ W/m}^3)(\pi)(0.04^2 - 0.02^2)\text{m}^2} = \mathbf{6.0\ m}$$ **(Ans.) (a)**

Constant surface heat flux,

$$q_s = \frac{\bar{q}\forall}{A_s} = \frac{\bar{q}\frac{\pi}{4}(D_o^2 - D_i^2)L}{\pi D_i L} = \frac{\bar{q}(D_o^2 - D_i^2)}{4D_i} = \frac{10^6\text{ W/m}^3}{4}\frac{(0.04^2 - 0.02^2)\text{m}^2}{0.02\text{ m}}$$

$$= 15000\text{ W/m}^2$$

Since $\dot{Q}_{\text{conv}} = q_s A_s = hA_s(T_s - T_b)$,

$$\boxed{q_s = \text{const} = h_{(x=L)}[T_{s,i} - T_{b_{(x=L)}}]}$$

Local convection heat-transfer coefficient at the outlet,

$$h_{(x=L)} = \frac{q_s}{(T_{s,i})_{\text{exit}} - T_{be}} = \frac{15000\text{ W/m}^2}{(55-40)\text{°C}} = \mathbf{1000\ W/m^2\,°C}$$ **(Ans.) (b)**

In fully developed conditions for the whole tube h as well as $(T_{si} - T_b)$ are constant throughout.

Hence, $T_{s,i(\text{inlet})} - T_{bi} = T_{s,i(\text{exit})} - T_{be}$

Inner tube surface temperature at inlet is

$$T_{s,i(\text{exit})} = 25\text{°C} + (55 - 40)\text{°C} = \mathbf{40°C}$$ **(Ans.) (c)**

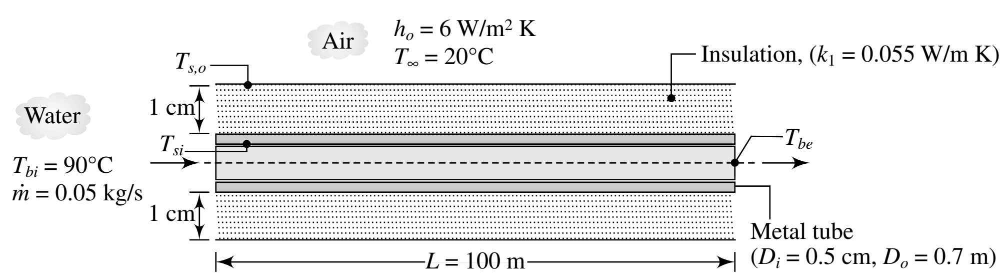
EXAMPLE 8.13 *Water at 90°C enters a metal tube of 100 m length, inside diameter 0.5 cm and outside diameter 0.7 cm. The mass flow rate of the water is 0.05 kg/s. A 1 cm thick layer of insulation (*k *= 0.055 W/m K) is fixed on the outside of the tube. Assuming that heat is lost only by free convection from the outer surface of the insulation, that the free convection heat transfer coefficient is 6 W/m² K, and that the ambient air is at a temperature of 20 °C, calculate the exit bulk mean temperature. Calculate also the pressure drop.*

Solution

Known Water-flow rate and inlet temperature for an insulated tube of specified dimensions.

Find Exit bulk mean temperature, T_{be}; Pressure drop, ΔP.

Schematic



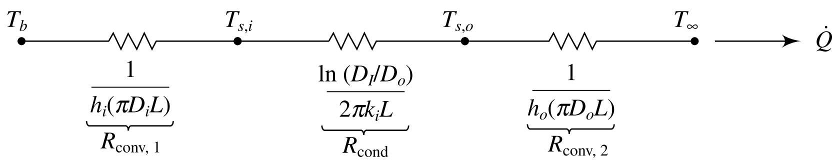
Assumptions (1) Steady operating conditions. (2) Fully developed flow throughout the tube. (3) Negligible tube wall conduction resistance. (4) Uniform free convection heat transfer coefficient.

Properties ***Water (T_{bm} = 87°C):***

C_p = 4.203 kJ/kg K, k = 0.674 W/m K, $\mu = 0.324 \times 10^{-3}$, Pr = 2.02

$\rho_{@90°C}$ = 965.3 kg/m³

Analysis Neglecting temperature drop across the metal tube wall, the thermal circuit associated with heat transfer from the water is given below:



where D_i = 0.005 m, D_o = 0.007 m and D_I = diameter of insulated tube = $D_o + 2t$ = (0.007 + 2 × 0.01) m = 0.027 m, L = 100 m, k_I = 0.055 W/m K, h_o = 6 W/m² K.

Calculation of thermal resistances:

- Conduction resistance due to insulation is

$$R_{\text{cond}} = \frac{\ln(0.027/0.007)}{2\pi(0.055\text{ W/m K})(100\text{ m})} = \mathbf{0.039\,063\text{ K/W}}$$

- Resistance due to free convection at the outer surface of the insulated tube is

$$R_{\text{conv,2}} = \frac{1}{6\text{ W/m}^2\text{ K}(\pi \times 0.027\text{ m} \times 100\text{ m})} = \mathbf{0.019\,65\text{ K/W}}$$

- To calculate $R_{conv,1}$, we need to determine inside (water side) convection heat transfer coefficient. For internal flow through a circular tube of inner diameter D_i, we have

$$Re_D = \frac{4\dot{m}}{\pi D_i \mu} = \frac{4 \times 0.05\text{ kg/s}}{\pi(0.005\text{ m})(0.324 \times 10^{-3}\text{ kg/m s})} = 39\,297.5 \quad (> 2300)$$

The flow is *turbulent.*

Friction factor, $f = (0.79 \ln Re_D - 1.64)^{-2}$ = {(0.79 ln (39297.5) – 1.64}$^{-2}$ = 0.02216

Using the following correlation,

$$Nu_D = \frac{(f/8)(Re_D - 1000)Pr}{1 + 12.7\sqrt{f/8}(Pr^{2/3} - 1)} = 153.1 = \frac{h_i D_i}{k}$$

Hence, $$h_i = \frac{k}{D_i} Nu_D = \frac{0.674\text{ W/m K}}{0.005\text{ m}}(153.1) = 20639\text{ W/m}^2\text{ K}$$

Convective resistance (*water side*) is

$$R_{\text{conv,1}} = \frac{1}{(20639\text{ W/m}^2\text{ K})(\pi \times 0.005\text{ m} \times 100\text{ m})} = \mathbf{3.0845 \times 10^{-5}\text{ K/W}}$$

Total thermal resistance,

$R_{\text{total}} = (UA_s)^{-1}$ = (0.039063 + 0.01965 + 3.0845 × 10^{-5}) = 0.05874 K/W

And, UA_s = 17.023 W / K

We note that, $\dot{Q} = UA_s \Delta T_{lm} = \dot{m} C_p (T_{bi} - T_{be})$

where $\Delta T_{lm} = \dfrac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}$

Since $\Delta T_i - \Delta T_e = (T_{bi} - T_\infty) - (T_{be} - T_\infty) = T_{bi} - T_{be}$,

$$\frac{UA_s}{\dot{m}C_p} = \ln\frac{\Delta T_i}{\Delta T_e} \quad \text{or} \quad \frac{\Delta T_e}{\Delta T_i} = \exp\left(-\frac{UA_s}{\dot{m}C_p}\right)$$

or
$$\frac{T_{be} - T_\infty}{T_{bi} - T_\infty} = \exp\left[\frac{-17.023 \text{ W/K}}{0.05 \text{ kg/s} \times 4203 \text{ J/kg K}}\right]$$

or $T_{be} = 20°\text{C} + (90 - 20)°\text{C} \times \exp(-0.081) = \mathbf{84.55°C}$ **(Ans.)**

Average bulk mean temperature,

$$T_{bm} = \frac{1}{2}(T_{bi} - T_{be}) = \frac{1}{2}(90 + 84.55) = 87.27°\text{C}$$

Since the properties of water were at 87°C, no iteration is necessary.

Mean flow velocity,

$$V = \frac{\dot{m}}{\rho_i\left(\frac{\pi}{4}D_i^2\right)} = \frac{0.05 \text{ kg/s}}{965.3 \text{ kg/m}^3\left\{\frac{\pi}{4} \times 0.005^2 \text{ m}^2\right\}} = 2.638 \text{ m/s}$$

Pressure drop,

$$\Delta P = f\left(\frac{L}{D_i}\right)\left(\frac{1}{2}\rho V^2\right) = 0.02216\left(\frac{100 \text{ m}}{0.005 \text{ m}}\right)\left(\frac{1}{2} \times 966.7\frac{\text{kg}}{\text{m}^3} \times 2.638^2 \text{ m}^2/\text{s}^2\right)$$

$$= 1.49 \times 10^6 \text{ Pa} = \mathbf{14.9 \text{ bar}}$$ **(Ans.)**

EXAMPLE 8.14 *Water at 50°C flows through a circular pipe of length 2.5 m and diameter 0.02 m with a mass flow rate of 0.06 kg/s. Heat is supplied to the water and the wall heat flux varies sinusoidally with axial distance x as follows:* $q_w = 50\,000 \sin\left(\frac{\pi x}{2.5}\right)$

where q_w *is in W/m², and x is measured in metres from the inlet.*

Calculate the variation of the bulk mean temperature of the water and the wall temperature with axial distance. Assume that the Dittus–Boelter equation holds for this situation.

Solution

Known Water flows through a circular tube and is heated by wall heat flux varying sinusoidally with axial distance.

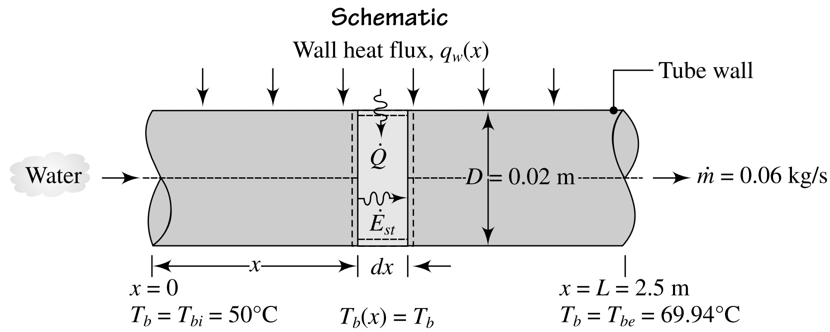
Find Variation of water (*bulk*) temperature, $T_b(x)$ and of wall temperature, $T_w(x)$.

Assumptions (1) Steady operating conditions exist. (2) Constant properties. (3) Heat flux varies sinusoidally with axial distance x.

Analysis Consider a control volume of length dx.

$$\begin{pmatrix}\text{Rate of heat transfer by convection}\\ \text{from tube wall to flowing water}\end{pmatrix} = \begin{pmatrix}\text{Rate of increase of thermal energy}\\ \text{storage of water}\end{pmatrix}$$

$$\dot{Q}_{\text{in,conv}} = \dot{E}_{st} \quad \text{or} \quad \dot{m}C_p dT_b = q_w \pi D\, dx = \pi D\left(50\,000 \sin\frac{\pi x}{2.5} dx\right)$$



Integrating from 0 to x and T_{bi} to T_b, we have

$$\int_{T_{bi}}^{T_b(x)} dT_b = \frac{50\,000\pi D}{\dot{m}C_p}\int_o^x \sin\frac{\pi x}{2.5}dx$$

or $$T_{b(x)} - T_{bi} = \frac{50\,000\pi D}{\dot{m}C_p} \times \frac{2.5}{\pi}\left[-\cos\frac{\pi x}{2.5}\right]_o^x$$

or $$T_{b(x)} - T_{bi} = \frac{50\,000 \times 2.5 \times D}{\dot{m}C_p}\left[1-\cos\frac{\pi x}{2.5}\right]$$

or $$T_{b(x)} = 50°\text{C} + \frac{50\,000\text{ W/m}^2 \times 2.5\text{ m} \times 0.02\text{ m}}{0.06\text{ kg/s} \times 4.179 \times 10^3\text{ J/kg K}}\left[1-\cos\frac{\pi x}{2.5}\right]$$

$$\boxed{\boldsymbol{T_b(x) = 50 + 9.97\left[1-\cos\left(\frac{\pi x}{2.5}\right)\right]}}$$ **(Ans.)**

At the tube exit, the water temperature is

$$x = L = 2.5\text{ m}$$

$$T_{be} = 50 + 9.970 \times 2 = 69.94°\text{C}$$

Properties will be evaluated at $\dfrac{50 + 69.94}{2} \cong 60°\text{C}$

$$\mu = 466.5 \times 10^{-6}\text{ Ns/m}^2,\ Pr = 2.98,\ k = 0.654\text{ W/m K}$$

$\therefore$ $$Re_D = \frac{4\dot{m}}{\pi\mu D} = \frac{4 \times 0.06\text{ kg/s}}{\pi \times 466.5 \times 10^{-6}\text{ kg/m s} \times 0.02\text{ m}} = 8188\,8$$

Using the *Dittus–Boelter* correlation:

$$Nu_D = 0.023 \times 8188^{0.8} \times 2.98^{0.4} = 48.08 = \frac{hD}{k}$$

$\therefore$ Convection coefficient,

$$h = 48.08 \times \frac{0.654\text{ W/m K}}{0.02\text{ m}} = 1572.2\text{ W/m}^2\text{ K}$$

$$(T_w - T_b)\text{K} \times 1572.2\text{ W/m}^2\text{ K} = q_w = 50\,000 \sin\frac{\pi x}{2.5}(\text{W/m}^2)$$

$\therefore$ $$\boxed{T_w = 31.80 \sin\frac{\pi x}{2.5} + T_b}$$

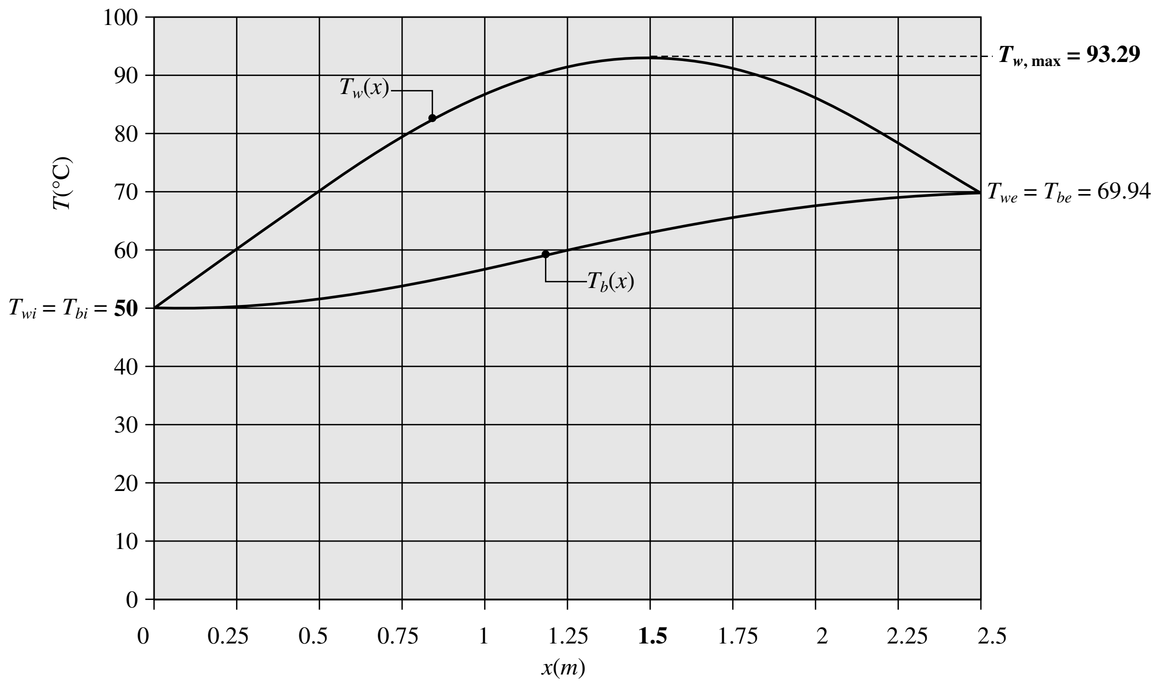
or $$\boxed{\boldsymbol{T_w(x) = 31.80 \sin\frac{\pi x}{2.5} + 50 + 9.97\left[1 - \cos\frac{\pi x}{2.5}\right]}}$$ **(Ans.)**

Wall temperature at tube exit ($x = L = 2.5$ m) is

$$T_{we} = 31.80 \sin\frac{\pi(2.5)}{2.5} + T_{be} = T_{be} = 69.94°C$$

$$T_{wi} = T_{bi} = 50°C$$

The variation of $T_b(x)$ and $T_w(x)$ is graphically represented below:



EXAMPLE 8.15 *Hot air at 1 atm and 90°C enters a 10 m long uninsulated square duct of cross section 0.25 m × 0.25 m which passes through the attic of a house at a volumetric flow rate of 0.20 m³/s and leaves at 80°C. Estimate (a) the duct surface temperature, and (b) the rate of heat loss to the air space in the attic. Properties of air at* $T_{bm} = \frac{1}{2}(90 + 80) = 85°C$: $\rho = 0.9862$ kg/m³, $v = 21.46 \times 10^{-6}$ m²/s, $C_p = 1008.8$ J/kg K, $k = 30.24 \times 10^{-3}$ W/m K

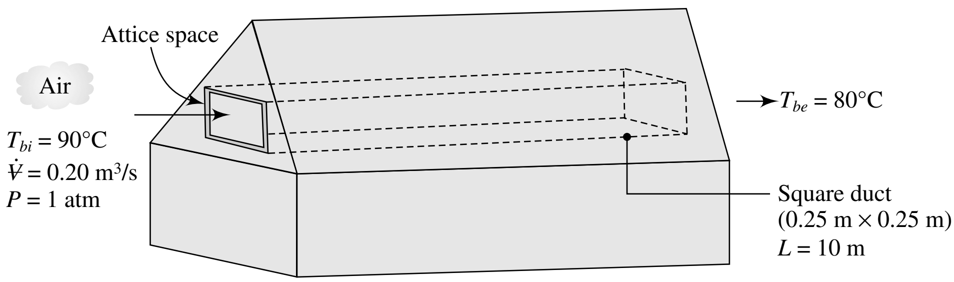
Solution

Known Hot air passes thorough a square duct of specified length with constant wall temperature.

Find (a) Duct surface temperature, T_w (°C) and (b) Heat loss rate, $\dot{Q}$ (W).

Assumptions (1) Steady operating conditions. (2) Air is an ideal gas. (3) Smooth inner surface of the duct.

Schematic



Analysis Mass-flow rate of air,

$$\dot{m} = \rho\dot{V} = (0.9862)(0.2) = 0.1972 \text{ kg/s}$$

Reynolds number, $Re = \dfrac{\rho V D_e}{\mu} = \dfrac{\rho D_e}{\mu}\left(\dfrac{\dot{m}}{\rho A_c}\right) = \dfrac{D_e}{\nu}\dfrac{\dot{V}}{A_c}$

Equivalent diameter, $D_e = \dfrac{4A_c}{P} \quad \Rightarrow \quad \dfrac{D_e}{A_c} = \dfrac{4}{P} = \dfrac{4}{4a} = \dfrac{1}{a} = \dfrac{1}{0.25} = 4$

Hence, $Re = \dfrac{4\dot{V}}{\nu} = \dfrac{4 \times 0.2}{21.46 \times 10^{-6}} = 37279 \quad (> 2300)$

Prandtl number, $Pr = \dfrac{C_p \nu \rho}{k} = \dfrac{1008.8 \times 21.46 \times 10^{-6} \times 0.9862}{30.24 \times 10^{-3}} = 0.706$

The Dittus–Boelter equation with n = 0.3 [as $T_w < T_{bi}$]:

$$Nu = 0.023(\text{Re})^{0.8}\,(\text{Pr})^{0.3} = 0.023(37279)^{0.8}\,(0.706)^{0.3} = 94.09$$

Average convection heat-transfer coefficient,

$$h = Nu\frac{k}{D_e} = \frac{(94.09)(30.24 \times 10^{-3})}{0.25} = 11.38 \text{ W/m}^2\text{ K}$$

Energy balance for a differential control volume of length dx with the bulk temperature decrease of $-dT_b$ is: $-\dot{E}_{\text{out}} = \dot{E}_{st}$ or $\dot{Q}_{\text{out,conv}} = -\dot{E}_{st}$

or $\qquad h(P\,dx)(T_b - T_w) = -\dot{m}C_p dT_b$

Separating the variables and integrating, we have

$$-\frac{hP}{\dot{m}C_p}\int_0^L dx = \int_{T_{bi}}^{T_{be}} \frac{dT_b}{T_b - T_w} \quad \text{or} \quad -\frac{hPL}{\dot{m}C_p} = \ln\left(\frac{T_{be} - T_w}{T_{bi} - T_w}\right)$$

or $\qquad \exp\left[-\dfrac{hPL}{\dot{m}C_p}\right] = \dfrac{T_{be} - T_w}{T_{bi} - T_w}$

With $P = 4a = 4 \times 0.25 = 1$ m

$$\frac{hPL}{\dot{m}C_p} = \frac{(11.38 \text{ W/m}^2\text{ K})(1 \text{ m})(10 \text{ m})}{(0.1972 \text{ kg/s})(1008.8 \text{ J/kg K})} = 0.572,$$

$$\frac{80 - T_w}{90 - T_w} = e^{-0.572} = 0.5644 \quad \text{or} \quad 80 - T_w = 50.8 - 0.5644T_w$$

or $$T_w = \frac{29.2}{0.4356} = \mathbf{67°C} \qquad \textbf{(Ans.) (a)}$$

Heat-loss rate,

$$\dot{Q} = \dot{m}C_p(T_{bi} - T_{be}) = (0.1972 \text{ kg/s})\ (1008.8 \text{ J/kg K})\ (90 - 80)°\text{C}$$

$$= \mathbf{1990\ W} \qquad \textbf{(Ans.) (b)}$$

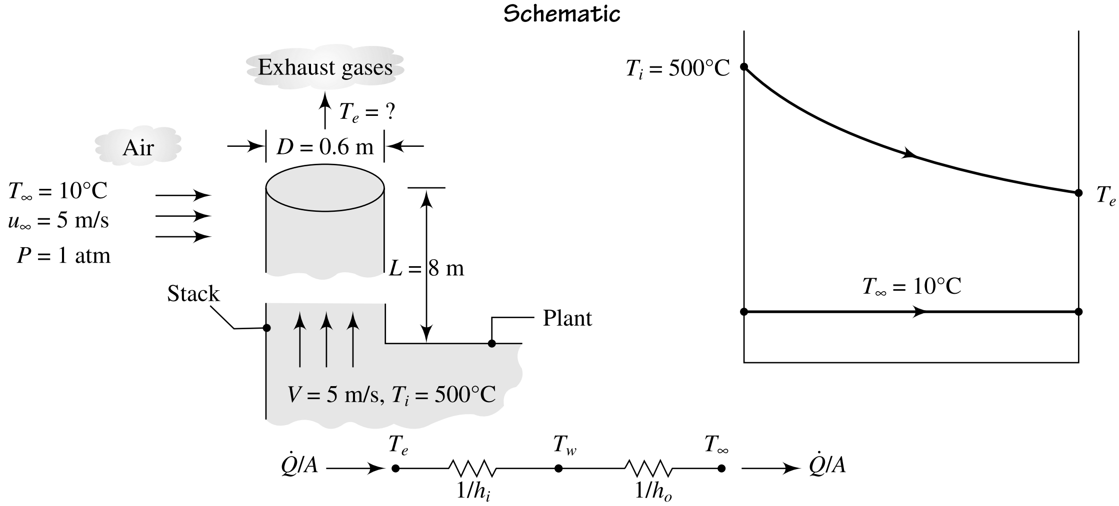
EXAMPLE 8.16 *Hot exhaust gases are discharged by a plant to the atmosphere through a vertical, cylindrical, thin-walled chimney (stack) 0.6 m diameter and 8 m high. The gases at 500°C and 5 m/s enter the stack at about atmospheric pressure and may be assumed to have the properties of air. Wind at 10°C blows over the exterior surface of the stack at a free stream velocity of 5 m/s. Determine the outlet gas and stack surface temperature.*

Solution

Known Exhaust gases are discharged through a stack to the environment under specified conditions.

Find Exit temperature of gas, and outer surface temperature of stack.

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Negligible radiation effects. (4) Fully developed flow. (5) The stack gases have properties of air.

Properties
Air (1 atm, 450°C): k = 0.05298 W/m °C, $\upsilon = 34.15 \times 10^{-6}$ kg/m s, C_p = 1.081 kJ/kg °C, ρ = 0.488 kg/m³, Pr = 0.6965

Air (1 atm, 10°C): k = 0.02439 W/m °C, $\mu = 17.78 \times 10^{-6}$ kg/m s, Pr = 0.7336, ρ = 1.246 kg/m³

Air (1 atm, 200°C): Pr = 0.6974

Analysis The stack surface temperature depends upon the exhaust gas outlet temperature. To determine exit gas temperature, we use a constant temperature condition. Since the surface temperature along the height of the stack is not uniform, the stack wall temperature can be replaced by external ambient air temperature of 10°C which is constant.

The thermal circuit below can help in understanding this analysis better:

Overall heat-transfer coefficient, $U = \left[\dfrac{1}{h_i} + \dfrac{1}{h_o}\right]^{-1}$

Internal Flow:

Properties are evaluated at the mean gas temperature of 450°C (assumed) with gas properties as those of air.

Roynolds number, $Re_D = \dfrac{\rho VD}{\mu} = \dfrac{(0.488 \text{ kg/m}^3)(5 \text{ m/s})(0.6)}{34.15 \times 10^{-6} \text{ kg/m s}} = 42\,870 \qquad (> 2300)$

The flow is *turbulent* and $L/D = \dfrac{8 \text{ m}}{0.6 \text{ m}} = 13.3$ (> 10). Hence, the Dittus–Böelter correlation can be used.

$$Nu_D = 0.023\ (Re_D)^{0.8}\ (Pr)^{0.3}$$

We note that $n = 0.3$ because the exhaust gases are being cooled in the stack.
Inside heat-transfer coefficient,

$$h_i = \frac{k}{D} Nu_D = \frac{(0.05298 \text{ W/m °C})}{0.6 \text{ m}} \times 0.023(42\,870)^{0.8}(0.6965)^{0.3} = 9.25 \text{ W/m}^2 \text{ °C}$$

External Flow:

For cross flow over a cylinder, the *Zhukauskas* relation is appropriate. Properties are evaluated at T_∞ = 10°C except μ_w which is at T_w = 200°C (*assumed*). The stack is thin-walled. Hence, $D_o = D_i = D$.

$$Re_D = \frac{\rho VD}{\mu} = \frac{1.245 \text{ kg/m}^3 \times 5 \text{ m/s} \times 0.6 \text{ m}}{17.78 \times 10^{-6} \text{ kg/m s}} = 210\,236$$

$$Nu_D = C\,Re_D^m\,Pr^n \left(\frac{Pr}{Pr_s}\right)^{1/4}$$

For the range $2 \times 10^5 < Re_D < 10^6$, $C = 0.076$, $m = 0.7$ and $n = 0.37$ since $Pr \le 10$

$$Nu_D = 0.076\ (210\ 236)^{0.7}\ (0.7336)^{0.37} \left[\frac{0.7336}{0.6974}\right]^{0.25} = 365.1$$

Therefore,

$$h_o = \frac{k}{D} Nu_D = \frac{0.02439 \text{ W/m °C}}{0.6 \text{ m}} \times 365.1 = 14.84 \text{ W/m}^2 \text{ °C}$$

$$\therefore \qquad U = \left[\frac{1}{9.25} + \frac{1}{14.84}\right]^{-1} = 5.7 \text{ W/m}^2 \text{ °C}$$

Mass-flow rate of exhaust gases,

$$\dot{m} = \rho A_c V = (0.488 \text{ kg/m}^3)\left(\frac{\pi}{4} \times 0.6^2 \text{ m}^2\right)(5 \text{ m/s}) = 0.69 \text{ kg/s}$$

For constant temperature case: $\dot{Q} = UA(\text{LMTD}) = \dot{m}C_p(T_e - T_i)$

where $\quad \text{LMTD} = \dfrac{\Delta T_i - \Delta T_e}{\ln \dfrac{\Delta T_i}{\Delta T_e}} = \dfrac{(T_i - T_\infty) - (T_e - T_\infty)}{\ln(T_i - T_\infty)/(T_e - T_\infty)} = \dfrac{T_i - T_e}{\ln\left[\dfrac{T_i - T_\infty}{T_e - T_\infty}\right]}$

It follows that

or $$UA = \dot{m}C_p \ln\left[\frac{T_i - T_\infty}{T_e - T_\infty}\right]$$

$$-\frac{UA}{\dot{m}C_p} = \ln\left(\frac{T_e - T_\infty}{T_i - T_\infty}\right) \Rightarrow \left[\frac{T_e - T_\infty}{T_i - T_\infty}\right] = \exp\left[-\frac{UA}{\dot{m}C_p}\right]$$

Exit gas temperature,

$$T_e = T_\infty + (T_i - T_\infty)\exp(-U\pi DL/\dot{m}C_p)$$

$$= 10°\text{C} + (500 - 10)°\text{C} \exp\left[-\frac{5.7 \text{ W/m}^2\,°\text{C} \times \pi \times 0.6 \text{ m} \times 8 \text{ m}}{0.69 \text{ kg/s} \times 1081 \text{ J/kg}\,°\text{C}}\right]$$

$$= \mathbf{446.7°C}$$ **(Ans.)**

Bulk mean temperature, $T_m = \frac{1}{2}(T_i + T_e) = \frac{1}{2}(500 + 446.7) = 473.3°\text{C}$

Assumed value = 450°C. The comparison is thus reasonable.
From the thermal circuit, we can write

$$\frac{T_e - T_w}{1/h_i} = \frac{T_w - T_\infty}{1/h_o} \text{ or } h_i T_e - h_i T_w = h_o T_w - h_o T_\infty$$

or $$T_w(h_o + h_i) = h_i T_e + h_o T_\infty$$

Surface temperature of stack at exit,

$$T_w = \frac{(9.25 \times 446.7) + (14.84 \times 10)}{14.84 + 9.25} = \mathbf{177.7°C}$$ **(Ans.)**

(C) Liquid Metal Heat Transfer

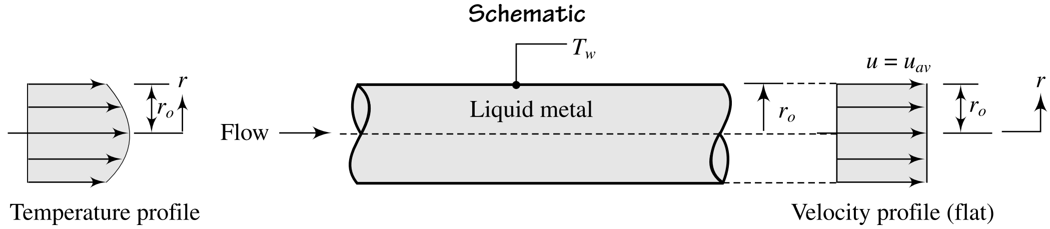
EXAMPLE 8.17 *Consider the constant property laminar forced flow of a liquid metal in a circular pipe. At the entrance, both the velocity and temperature profiles are flat and the boundary condition of a constant heat flux is applied. Since the Prandtl number of the liquid metal is low and of the order of 0.01, the velocity profile develops very slowly as compared to the temperature profile. As a first approximation, one may therefore assume that the velocity profile stays flat. Make this approximation and show that the Nusselt number after the temperature profile is fully developed is constant and equal to 8.*

Solution

Known Form of velocity and temperature profiles for liquid metal flow in a circular pipe.

Find Nusselt number, *Nu*.

Schematic



Assumptions (1) Steady, incompressible flow. (2) Slug flow (flat velocity profile). (3) Constant wall heat flux. (4) Fully developed, laminar flow. (5) Uniform properties.

Analysis For fully developed laminar flow in a circular pipe, the energy equation is

$$\boxed{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)} \qquad \text{(For fully developed flow, } v = 0\text{)} \qquad \text{(A)}$$

In the case of constant wall heat flux (q_w= *const*),

$$\frac{\partial T}{\partial x} = \frac{dT_m}{dx}$$

where T_m is the mean temperature of liquid metal over the cross section of the tube.

For very low Prandtl number fluid like a liquid metal $\delta << \delta_t$, the velocity profile develops very slowly. One, therefore, has $u(t) = u_0$ (*slug flow*)

Equation (A) becomes

$$u_0\frac{dT_m}{dx} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \quad \Rightarrow \quad \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{u_0}{\alpha}\frac{dT_m}{dx}r$$

Integrating with respect to r,

$$r\frac{\partial T}{\partial r} = \left(\frac{u_0}{\alpha}\frac{dT_m}{dx}\right)\frac{r^2}{2} + C_1$$

Boundary condition # 1:

$$\frac{\partial T}{\partial r} = 0 \text{ at } r = 0$$

$\therefore$ $C_1 = 0$

Then

$$\boxed{\frac{\partial T}{\partial r} = \left(\frac{u_0}{\alpha}\frac{dT_m}{dx}\right)\frac{r}{2}} \qquad \text{(B)}$$

Integrating again,

$$T(r) = \frac{u_0}{\alpha}\frac{dT_m}{dx}\frac{r^2}{4} + C_2$$

Boundary condition # 2:

$$T(r = r_o) = T_w$$

$$\therefore \qquad C_2 = T_w - \frac{u_0}{\alpha}\frac{dT_m}{dx}\frac{r_o^2}{4}$$

Temperature distribution is then determined to be

$$\boxed{\boldsymbol{T(r) = T_w - \frac{u_0}{4\alpha}\frac{dT_m}{dx}(r_o^2 - r^2)}} \qquad \textbf{(Ans.)}\ \text{(C)}$$

Bulk or mean temperature is defined as

$$T_b \text{ or } T_m = \frac{\int \rho u dA_c C_v T}{\dot{m}C_v} = \frac{\rho u_m C_v \int 2\pi r T\, dr}{\rho u_m \pi r_o^2 C_v} = \frac{\int_0^{r_o} 2rT\, dr}{r_o^2}$$

or $$\boxed{T_m = \frac{2}{r_o^2}\int_0^{r_o} rT\,dr} \qquad \text{(D)}$$

Substituting Eq. (C) into Eq. (D), we get

$$T_m = \frac{2}{r_o^2}\int_0^{r_o}\left\{rT_w - \frac{u_o}{4\alpha}\frac{dT_m}{dx}(rr_o^2 - r^3)\right\}dr$$

$$= \frac{2}{r_o^2}\left[T_w\frac{r_o^2}{2} - \frac{u_0}{4\alpha}\frac{dT_m}{dx}\left(\frac{r_o^4}{2} - \frac{r_o^4}{4}\right)\right] = T_w - \frac{u_0 r_o^2}{8\alpha}\frac{dT_m}{dx}$$

or $$\boxed{T_w - T_m = \frac{u_0 r_o^2}{8\alpha}\frac{dT_m}{dx}} \qquad \text{(E)}$$

We note that at $r = r_o$, $q_w = h(T_w - T_m)$ = const

Also, from Fourier's law, $\dfrac{\dot{Q}}{A} = q_w = k\left.\dfrac{\partial T}{\partial r}\right|_{r=r_o}$

∴ Convection coefficient is

$$h = \frac{k\left.\dfrac{\partial T}{\partial r}\right|_{r_o}}{T_w - T_m}$$

Therefore, substituting for $\left.\dfrac{\partial T}{\partial r}\right|_{r=r_o}$ and $(T_w - T_m)$ from equations (B) and (E),

$$h = \frac{k\left(\dfrac{u_0 r_o}{2\alpha}\right)\dfrac{dT_m}{dx}}{\dfrac{u_0 r_o^2}{8\alpha}\dfrac{dT_m}{dx}} = \frac{4k}{r_o} = \frac{8k}{D}$$

∴ Nusselt number, $$\boxed{\boldsymbol{Nu_D = \frac{hD}{k} = 8}} \qquad \textbf{(Ans.)}$$

Comment Note that in this case, $\dfrac{\partial T}{\partial r}$ was *positive*. Hence, $q_w = k\left.\dfrac{\partial T}{\partial r}\right|_{r=r_o}$

EXAMPLE 8.18 *Liquid sodium is to be heated from 120 to 149°C at a rate of 2.3 kg/s. A 2.5 cm diameter electrically heated tube is available. If the tube wall temperature is not to exceed 200°C, calculate the minimum tube length required. Use the correlation:*

$$\text{Nu} = 6.3 + 0.0169\,\text{Re}^{0.85}\,\text{Pr}^{0.93}$$

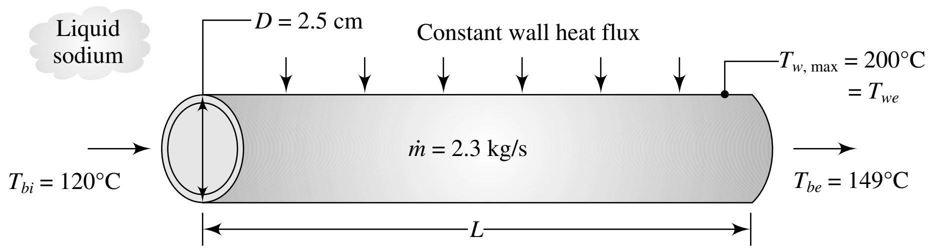
Properties of liquid sodium at the bulk mean temperature of 134.5°C are
Density = 918.7 kg/m³ Specific heat = 1368 J/kg K
Viscosity = 6.372 × 10⁻⁴ kg/m s Thermal conductivity = 84.12 W/m K

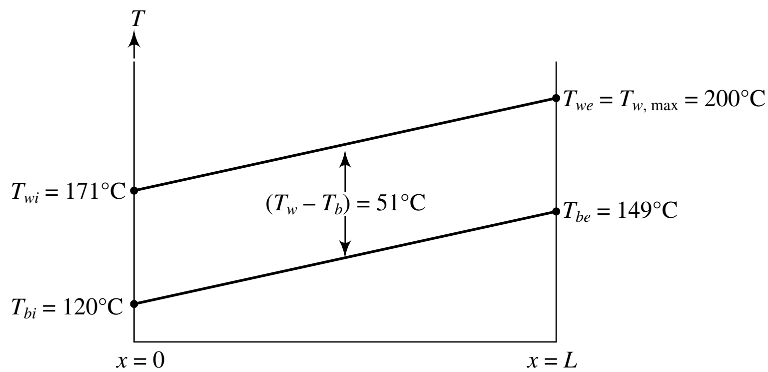
Solution

Known Liquid sodium is heated in an electrically heated tube under constant heat flux conditions.

Find Minimum tube length.

Schematic





Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Constant heat flux. (4) Fully developed turbulent flow.

Analysis Reynolds number,

$$Re = \frac{\rho VD}{\mu} = \frac{4\dot{m}}{\pi D\mu} = \frac{4 \times 2.3 \text{ kg/s}}{\pi(0.025 \text{ m})(6.372 \times 10^{-4} \text{ kg/m s})} = 183\,832$$

Prandtl number, $Pr = \dfrac{C_p\mu}{k} = \dfrac{(1368 \text{ J/kg K})(6.372 \times 10^{-4} \text{ kg/m s})}{84.12 \text{ W/m K}} = 0.0103$

Nusselt number, $Nu = 6.3 + 0.0167(183.832)^{0.85}(0.0103)^{0.93} = 13.4$

Convection coefficient, $h = Nu\dfrac{k}{D} = 13.4 \times \dfrac{84.12 \text{ W/m K}}{0.025 \text{ m}} = 45120 \text{ W/m}^2 \text{ K}$

Heat transfer rate, $\dot{Q} = h(\pi DL)(T_{we} - T_{be}) = \dot{m}C_p(T_{be} - T_{bi})$

$$= (2.3 \text{ kg/s})(1368 \text{ J/kg K})(149 - 120)\text{K} = 91246 \text{ W}$$

Maximum tube wall temperature will be at the tube exit. For constant heat flux case, $(T_w - T_b)$ is constant throughout the tube. Hence, the minimum tube length required is

$$L = \frac{\dot{Q}}{h(\pi D)(T_{we} - T_{be})} = \frac{91246 \text{ W}}{(45120 \text{ W/m}^2 \text{ K})(\pi \times 0.025 \text{ m})(200 - 149)\text{K}}$$

$= \mathbf{0.505\ m}$ or **50.5 cm** **(Ans.)**

EXAMPLE 8.19 *Sodium–potassium (22% Na + 78% K) liquid metal mixture with a flow rate of 1 kg/s is heated in a stainless steel tube of 2.5 cm ID. The tube wall temperature is maintained at 450°C. The tube length is 60 cm and the fluid inlet temperature is 300°C. Calculate the exit temperature of the medium. Use the following correlation:*

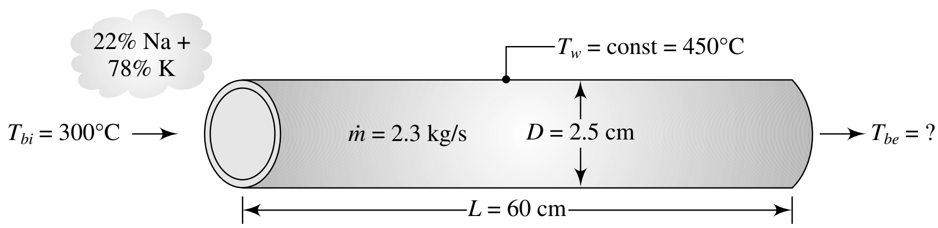
$$\mathrm{Nu} = 4.8 + 0.0156\ \mathrm{Re}^{0.85}\mathrm{Pr}^{0.93} \qquad (T_w = const)$$

Solution

Known Sodium–potassium (22/78) liquid metal mixture flows through a tube and is heated under constant wall temperature.

Find Exit temperature of the medium, T_{be}.

Schematic



Assumptions (1) Steady operating conditions. (2) Constant tube surface temperature. (3) Fully developed, turbulent flow.

Analysis Properties are to be evaluated at the bulk mean temperature, $T_{bm} = \dfrac{1}{2}(T_{bi} + T_{be})$

However, since T_{be} is to be determined, let T_{be} = 400°C with T_w = 450°C. Then the properties of (22% Na + 78% K) at (300 + 400)/2 = 350°C are:

k = 27.2 W/m°C $\qquad\mu = 2.912 \times 10^{-4}$ kg/m s

C_p = 889.8 J/kg°C $\qquad Pr = 9.57 \times 10^{-3}$

Reynolds number for a circular tube is

$$Re = \frac{4\dot{m}}{\pi D \mu} = \frac{4(1\text{ kg/s})}{\pi(0.025\text{ m})(2.912 \times 10^{-4}\text{ kg/m s})} = 1.75 \times 10^5$$

Nusselt number, $Nu = 4.8 = 0.0156\ (1.75 \times 105)^{0.85}(9.57 \times 10^{-3})^{0.93} = 10.71$

Heat-transfer coefficient, $h = Na\dfrac{k}{D} = 10.71 \times \dfrac{27.2\text{ W/m°C}}{0.025\text{ m}} = 11655\text{ W/m}^2\text{ °C}$

From energy balance: $h(\pi DL)\dfrac{(T_w - T_{bi}) - (T_w - T_{be})}{\ln\left[\dfrac{T_w - T_{bi}}{T_w - T_{be}}\right]} = \dot{m}C_p(T_{be} - T_{bi})$

or $\qquad \boxed{\ln\left[\dfrac{T_w - T_{bi}}{T_w - T_{be}}\right] = \dfrac{h\pi DL}{\dot{m}C_p}}$ $\quad$ or $\quad T_{be} = T_w - (T_w - T_{bi})\left[-\dfrac{h\pi DL}{\dot{m}C_p}\right]$

$$= 450\text{°C} - (450 - 300)\text{°C} \exp\left[-\frac{11655\text{ W/m}^2\text{ °C}(\pi \times 0.025\text{ m})(0.6\text{ m})}{(1\text{ kg/s})(889.8\text{ J/kg°C})}\right]$$

= 369°C **(Ans.)**

Comment With the calculated value of T_{be} = 369°C, $T_{bm} = \frac{1}{2}(300 + 369) = 334.5°\text{C}$. This is not too far from 350°C that we had assumed. Since the properties do not significantly change over such a small temperature difference, no more trial is necessary.

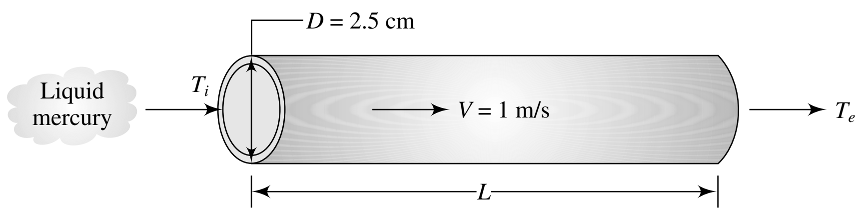
EXAMPLE 8.20 *Liquid mercury flows through a long tube (2.5 cm ID) with a velocity of 1 m/s. Calculate the local heat-transfer coefficient for (a) the constant wall temperature boundary condition, and (b) the constant heat flux boundary condition. Assume the following properties for mercury:*
Density = 12 870 kg/m³
Viscosity = 0.863 × 10^{3} kg/m s
Specific heat = 134 J/kg K
Thermal conductivity = 14.0 W/m K

Solution

Known Liquid mercury flows with a prescribed velocity through a circular tube.

Find $\bar{h}(\text{W/m}^2\,\text{K})$ for (a) CWT, and (b) CHF conditions.

Schematic



Assumptions (1) Steady-state conditions. (2) Tube's inner surface is smooth.

Analysis For a circular cross section,
Reynolds number,

$$Re_D = \frac{\rho V D}{\mu} = \frac{(12870\ \text{kg/m}^3)(1\ \text{m/s})(0.025\ \text{m})}{0.863 \times 10^{-3}\ \text{kg/m s}} = 372\,827$$

$$Pr = \frac{C_p \mu}{k} = \frac{(134\ \text{J/kg K})(0.863 \times 10^{-3}\ \text{kg/m s})}{14.0\ \text{W/m K}} = 0.00826$$

(a) ***Constant wall temperature***: For this boundary condition, the relevant correlation is

$$Nu_D = 4.8 + 0.0156\ Re_D^{0.85}\,Pr^{0.93} = 4.8 + 0.0156\ (372827)^{0.85}\ (0.00826)^{0.93}$$
$$= 4.8 + (0.0156)\ (628.9) = 14.611$$

Local heat-transfer coefficient,

$$h = 14.611 \times \frac{14.0\ \text{W/m K}}{0.025\ \text{m}} = \mathbf{8182\ W/m^2\ K}$$ **(Ans.) (a)**

(b) ***Constant heat flux***: Since Re_D is between 10^4 and 10^6 Pr between 0.004 and 0.1, the appropriate correlation for this boundary condition is

$$Nu_D = 6.3 + 0.0167\ Re_D^{0.85}\,Pr^{0.93} = 6.3 + 0.0167\ (372827)^{0.85}\ (0.00826)^{0.93}$$
$$= 16.804 = \frac{hD}{k}$$

Local heat-transfer coefficient,

$$h = Nu_D \frac{k}{D} = 16.804 \times \frac{14.0 \text{ W/m K}}{0.025 \text{ m}} = \mathbf{9410\ W/m^2\ K} \qquad \textbf{(Ans.) (b)}$$

Points to Ponder

- For a developing velocity profile in the case of steady laminar fluid flow in a straight circular tube, $u = u(x, r)$ and for developed velocity profile, $u = u(r)$.
- The heat flux and pressure drop are higher in the entrance regions of a pipe.
- Unlike laminar flow, the heat-transfer coefficient and the friction factor in turbulent flow depend strongly on the surface roughness.
- The local Nusselt number in the thermal entrance region for laminar flow in a circular pipe with fully developed velocity profile is always *less* for constant wall temperature case than for constant wall heat flux condition.
- For a fully developed constant property laminar flow of a fluid in a circular pipe, the local and average heat-transfer coefficients are same.
- In non-circular tubes, the characteristic length is a hydraulic diameter which is defined as four times the area of cross section by the wetted perimeter.
- The hydraulic diameter for flow through tube annulus with D_i and D_o as inner and outer diameters respectively is $(D_o - D_i)$.
- In a fluid flow through a non-circular duct the Reynolds number based on the hydraulic diameter, D_h cannot be expressed as $Re = \dfrac{4\dot{m}}{\pi D_h \mu}$.
- The Dittus–Boelter equation for fully developed turbulent pipe flow is applicable for both constant wall temperature and constant wall heat flux conditions.
- The Dittus–Boelter and Sieder–Tate equations for turbulent flow through a pipe cannot be applied for liquid metals.
- The Graetz number is defined as $Gz = (L/D)/RePr$.
- Liquid metals have Prandtl numbers in the range $0.1 \geq Pr \geq 0.001$.

GLOSSARY of Key Terms

- **Hydraulic (equivalent) diameter** The characteristic length used in calculating parameters such as Reynolds and Nusselt numbers in forced convection heat transfer through non-circular tubes in place of the internal diameter of a circular tube. This equivalent or effective diameter is 4 times the flow area of cross section over the wetted perimeter.
- **Uniform heat flux condition** Constant heat-transfer rate per unit surface area over the entire length of a tube during forced convection internal flow, e.g., in an electrically heated outer surface of a tube/pipe. The difference between the wall temperature and the bulk fluid temperature in this case remains constant along the tube axis in the fully developed region.
- **Constant wall (surface) temperature condition** The tube wall temperature is held constant throughout the tube length, e.g., in a steam heated tube surface. The fluid temperatuer increases or decreases (depending on the wall temperature) exponentially and the effective temperature difference for computing heat-transfer rate is the logarithmic mean temperature difference (LMTD).

● **Hydrodynamic entrance length**	The distance required for the friction factor to decrease to within 5% of its fully deeloped value along the tube length.
● **Thermal entrance length**	The distance required for the Nusselt number to decrease to within 5% of its fully developed value along the tube length.
● **Fully developed flow (hydrodynamically)**	The shape of the velocity profile remains unchanged and the friction factor has a constant value along the tube length.
● **Fully developed flow (thermally)**	The shape of the temperature profile is unchanging, and the Nusselt number has a constant value along the tube length.
● **Turbulent flow**	The flow is turbulent when the Reynolds number Re_D exceeds 2300, although turbulence becomes fully established for $Re_D > 10\,000$.
● **Laminar flow**	The flow is laminar when the Reynolds number Re_D is less than 2300 with a characteristic parabolic velocity profile.

OBJECTIVE-TYPE QUESTIONS

● Multiple-Choice Questions

8.1 For fully developed laminar flow and heat transfer in a uniformly heated long tube, if the flow velocity is doubled and the tube diameter is halved, the heat-transfer coefficient will be

(a) double the original value (b) half of the original value
(c) same as before (d) four times the original value

8.2 The bulk temperature of a fluid flowing through a tube or duct is defined as

(a) $T_b = \dfrac{1}{\dot{m}} \int \rho u T \, dA$ (b) $T_b = \dfrac{1}{A} \int h \, dA$

(c) $T_b = \dfrac{1}{A} \int T \, dA$ (d) $T_b = \dfrac{1}{\dot{m}} \int \rho u T \, dA$

8.3 Graetz number is defined as**:**

(a) $Re\ Pr$ (b) $Re\ Pr$ (c) $Re\ Pr/(x/D)$ (d) $Gr\ Pr$

8.4 In laminar flow through a tube, under fully developed conditions:

(1) $(\partial u/\partial x) = 0$
(2) $(\partial T/\partial x)$ at any radius r is not zero
(3) the temperature profile $T(r)$ continuously changes with x
(4) for constant tube wall temperature, surface heat flux is constant

Of the above statements:

(a) only 1 and 2 are correct (b) 1, 2 and 3 are correct
(c) 1, 2, 3 and 4 are correct (d) 1 and 4 are correct

8.5 Nusselt number for fully developed turbulent flow in a pipe is given by $Nu = C\,(Re)^a\,(Pr)^b$. The values of a and b are:

(a) $a = 0.5$ and $b = 0.33$ for both heating and cooling.
(b) $a = 0.5$ and $b = 0.4$ for heating and $b = 0.3$ for cooling.
(c) $a = 0.8$ and $b = 0.4$ for heating and $b = 0.3$ for cooling.
(d) $a = 0.8$ and $b = 0.3$ for heating and $b = 0.4$ for cooling.

8.6 For steady, uniform flow through pipes with constant heat flux supplied to the wall, what is the value of Nusselt number?

(a) 48/11 (b) 11/48 (c) 24/11 (d) 11/24

8.7 An uninsulated air conditioning duct of 1m × 0.5 m rectangular cross section, carrying air at 20°C with a velocity of 10 m/s, is exposed to an ambient at 30°C. Neglect the effect of the duct construction material. For air in the range of 20 30°C, the data are as follows:

Thermal conductivity = 0.025 W/m K, Viscosity = 18 μPa s, Prandtl number = 0.73, Density = 1.2 kg/m^3.

The laminar-flow Nusselt number (Nu) is 3.4 for constant wall temperature conditions and for turbulent flow: $Nu = 0.023\ Re^{0.8}\ Pr^{1/3}$.

(A) The Reynolds number for the flow is

(a) 444 (b) 890 (c) 4.44×10^5 (d) 5.33×10^5

(B) The heat transfer per metre length of the duct, in watts, is

(a) 3.8 (b) 5.3 (c) 89 (d) 769

8.8 The average heat-transfer coefficient for turbulent flow through a tube is prescribed by the correlation,

$$\overline{Nu} = 0.023(Re)^{0.8}(Pr)^{1/3}$$

If the fluid properties and the flow velocity remain unchanged, a 2.5 times increase in the tube diameter will decrease the heat-transfer coefficient by a factor of

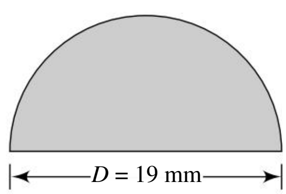
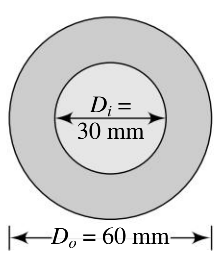
(a) 1.20 (b) 1.32 (c) 1.86 (d) 2.1

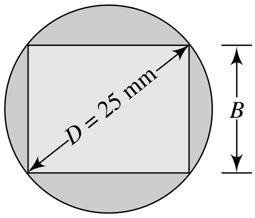
8.9 A fluid of 1.0 W/m K thermal conductivity flows in fully developed flow with Reynolds number of 1500 through a pipe of diameter 10 cm. The heat transfer coefficients for uniform heat flux and uniform wall temperature boundary conditions are, respectively

(a) 36.57 and 43.64 W/m^2 K (b) 43.64 and 36.57 W/m^2 K

(c) 43.64 W/m^2 K for both the cases (d) 36.57 W/m^2 K for both the cases

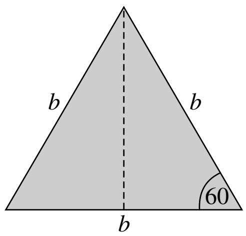
8.10 Match ***List I*** with ***List II*** according to the codes given below:

List I (***Cross section of the geometry***)	**List II** (***Hydraulic diameter***)
A.  Semicircular	**1.** 30 mm
B.  Concentric annulus	**2.** 4.78 mm

C. 

Circular tube with
square insert inside

3. 8.66 m

D. 

Triangular

4. 11.61 mm

Codes:	**A**	**B**	**C**	**D**
(a)	1	2	3	4
(b)	2	3	4	1
(c)	4	1	2	3
(d)	4	2	1	3

8.11 Water flowing at 2 kg/s through a 40 mm diameter tube is being heated from 30°C to 65°C by maintaining the tube wall temperature at 90°C.
Physical properties of water at its mean bulk temperature are
$\mu = 577 \times 10^{-6}$ kg/m s, $k = 0.640$ W/m K, $C_p = 4180$ J/kg K, $Pr = 3.77$
$Nu = 0.023\ (Re)^{0.8}\ (Pr)^{0.4}$

(A) The heat-transfer coefficient (kW/m^2 K)
(a) 6.77 (b) 67.7 (c) 8.95 (d) 5.24

(B) The required tube length for these conditions is
(a) 4.3 m (b) 8.6 m (c) 0.86 m (d) 86 m

Answers

Multiple-Choice Questions

8.1 (a) **8.2** (d) **8.3** (c) **8.4** (b) **8.5** (c) **8.6** (a)
8.7(A) (c) **(B)** (d) **8.8** (a) **8.9** (b) **8.10** (c)
8.11(A) (a) **(B)** (b)

REVIEW QUESTIONS

8.1 Explain the importance of reference temperature in convection heat transfer, and define clearly the term *mean bulk temperature*. Show that for flow through a circular tube of constant radius, R and average fluid velocity, V, $T_b = \dfrac{2}{VR^2}\int_0^R u(r)T(r)r\,dr$

8.2 Define the hydrodynamic entrance length and thermal entrance length for fluid flow in a tube? Draw the changing temperature profiles with constant wall temperature and constant heat flux conditions.

8.3 Consider the flow of (a) liquid metal, (b) a gas, and (c) an oil in a tube. How will the hydrodynamic and thermal entry lengths compare if the flow is laminar and if the flow is turbulent?

8.4 How is the friction factor for fluid flow through a tube related to the pressure drop? How is the pressure drop related to the pumping power requirement for a given mass flow rate?

8.5 How is characteristic length for a tube of non-circular cross section defined?

8.6 What are the key features of fully developed flow inside closed conduits?

8.7 How does the bulk fluid temperature and the wall heat flux vary with distance from the tube entrance for heating or cooling of a fluid under constant wall temperature condition?

8.8 How does surface roughness affect the heat transfer in a tube if the fluid flow is turbulent? What would be the effect if the flow in the tube were laminar?

8.9 What does the *logarithmic mean temperature difference* represent for flow in a tube for constant wall temperature?

8.10 What is the *Dittus–Boelter* equation? What are the restrictive conditions? How is *Sieder–Tate* correlation different from the *Dittus–Boelter* equation?

8.11 What is the physical significance of *Graetz* number?

8.12 Show that $St = \frac{f}{8}$ for turbulent flow through a tube by Reynolds analogy where f is the Darcy's friction factor.

PRACTICE PROBLEMS

(A) Circular Tubes: Laminar Flow

8.1 A small air-cooled condenser is to be designed. The air passes through a number of small circular ducts which have essentially a uniform surface temperature. The ducts are 5 mm in diameter and 40-mm long. The air enters with a mean flow velocity of 4.75 m/s at 24°C and the wall temperature is 52°C. Calculate the average heat transfer coefficient and estimate the exit air temperature.

The following properties of air at 27°C may be used:

$\nu = 15.89 \times 10^{-6}$ m²/s, $k = 0.0263$ W/m K, $Pr = 0.707$, $C_p = 1.007$ kJ/kg K, $\mu_{w@52°C} = 19.64 \times 10^{-6}$ kg/m s

[49.4 W/m² K]

8.2 In a solar water heating system, water at a temperature of 27°C enters a 20-mm internal diameter tube at a Reynolds number of 1200. The tube is heated by concentrated sun rays such that constant tube surface heat flux condition exists. If the bulk temperature and the centreline temperature at the tube exit are 67°C and 46°C, respectively, determine (a) the constant heat flux supplied to the tube, (b) the length of the tube, and (c) whether boiling occurs in any part of the tube.

Properties: Saturated water $[1\,\text{atm}, T_{bm} = \frac{1}{2}(T_{bi} + T_{be}) = (27 + 67)/2 = 47°\text{C}]$:

$\rho = 989$ kg/m³, $C_p = 4.18$ kJ/kg K, $k = 0.640$ W/m K, $\mu = 0.577 \times 10^{-3}$ kg/m s, $Pr = 3.77$

[(a) 4608 W/m² (b) 6.28 m (c) 100°C]

8.3 Water at 10°C is to be heated to 40°C in a thin-walled tube of 2-cm-ID at a mass flow rate of 0.01 kg/s. The outer surface of the tube is subjected to a uniform and constant wall heat flux of 15 kW/m². Neglecting any entrance effects, determine the (a) Reynolds number, (b) heat-transfer coefficient, (c) length of the pipe needed, (d) inner tube surface temperature at the exit, (e) friction factor, (f) pressure drop in the pipe, and (g) pumping power required if the pump efficiency is 50%.

[(a) 714.5 (b) 132.4 W/m²°C (c) 1.33 m (d) 153.3°C (e) 0.0896 (f) 0.0 Pa (g) 6.1 × 10⁻⁵ W]

(B) Turbulent Flow: Circular and Non-circular Tubes

8.4 Hot exhaust gases leaving a stationary diesel engine at 450°C enter a 0.15-m-diameter pipe with a mean flow velocity of 5 m/s. The pipe surface temperature is 175°C. Using the properties of air for exhaust gases and the Dittus–Boelter correlation, determine the pipe length if the exhaust gases leave the pipe at 250°C. Properties: At the bulk mean air temperature of $\frac{1}{2}(450 + 250)$°C = 350°C:
$k = 0.0491$ W/m°C, $\nu = 55.46 \times 10^{-6}$ m²/s, $Pr = 0.676$, $C_p = 1059$ J/kg°C, $\rho = 0.566$ kg/m³ **[10.81 m]**

8.5 A solar concentrator focuses sunlight on a bank of molybdenum alloy tubes which are 16-mm-ID, and 360 mm long. A pyrometer scan indicates an average surface temperature of 1650°C for a tube and the air enters the tube at 60°C with a mass flow rate of 0.035 kg/s. Estimate the exit air temperature. **[338.8°C]**

8.6 Water at 20°C is to be heated by passing it through the tube. The surface of the tube is maintained at 90°C. The diameter of the tube is 1.3 cm while its length is 9 m. Find the mass flow rate so that the exit temperature of water will be 60°C. The properties of water are:
$\rho = 895$ kg/m³, $C_p = 4.174$ kJ/kg K, $k = 0.64$ W/m K, $\nu = 0.62 \times 10^{-6}$ m²/s, $\beta = 4.25 \times 10^{-3}$ K^{-1}
Use the correlation: $Nu = 0.023\, Re^{0.8} Pr^{0.3}$ **[2.59 kg/s]**

8.7 Hot air at atmospheric pressure and 80°C enters an 8-m-long uninsulated square duct of cross section 0.2 × 0.2 m that passes through the attic of a house at a rate of 0.15 m³/s. The duct is observed to be nearly isothermal at 60°C. Determine the exit temperature of the air and the rate of heat loss from the duct to the attic space. **[71.2°C, 1325 W]**

8.8 Water at a temperature of 50°C enters a 1-m-long square tube with 15-mm by 15-mm cross section. The water mass flow rate is 13.5 kg/min, and the tube is maintained at a constant surface temperature of 90°C. Determine (a) the water outlet temperature, (b) the heat transfer rate from the tube to the water, and (c) the pressure drop.

T (°C)	ρ(kg/m³)	C_p(kJ/kg°C)	k(W/m°C)	μ(kg/ms)	Pr
325	987.2	4.182	0.645	528×10^{-6}	3.42
330	984.3	4.184	0.650	489×10^{-6}	3.15
335	982.3	4.186	0.656	453×10^{-6}	2.88

Use the Gnielinski correlation: $Nu = \dfrac{(f/8)(Re - 1000)Pr}{1 + 12.7\sqrt{f/8}\,(Pr^{2/3} - 1)}$ where $f = (0.79 \ln Re - 1.64)^{-2}$

[(a) 63.8°C (b) 13.8 kW (c) 796 N/m²]

(C) Liquid-Metal Heat Transfer

8.9 Liquid bismuth flows at a rate of 4.5 kg/s through a 5-cm-diameter stainless steel tube. The bismuth enters at 415°C and is heated to 440°C as it passes through the tube. If the tube wall is at a temperature 20°C higher than the bismuth bulk temperature by maintaining constant heat flux along the tube, calculate the length of the tube required to effect the heat transfer.
Use the following properties at 427.5°C:
$\mu = 1.34 \times 10^{-3}$ kg/m s, $k = 15.6$ W/m K, $C_p = 0.149$ kJ/kg K, $Pr = 0.013$
Use the following correlation (for $10^4 < Re < 10^6$) for q_w = const: $Nu = 6.3 + 0.0167\, Re^{0.85}\, Pr^{0.93}$
[1.57 m]

9 Natural (or Free) Convection Heat Transfer

9.1 ❑ INTRODUCTION

In the previous two chapters, we focused on the *forced convection* heat transfer involving the mechanical device or external mechanism to create motion. Free convection heat transfer, on the other hand, is caused by the influence of a body force (*gravitational, centrifugal, electrical*, *or magnetic*) on a fluid. *One example of free convection is in the interior cooling of gas-turbine blades that spin at high rotational speed.* But by far the most common form of natural convection encountered in engineering practice, is caused by the density variations in a fluid due to heating or cooling by a surface. Even in the case of incompressible fluid, density differences in the presence of temperature differences between the surface and the fluid create a flow due to buoyancy effects which is normally negligible in the case of forced convection. In this chapter, we concentrate primarily on temperature-induced buoyancy from a single surface in a quiescent (*still*) fluid.

In free convection, fluid movement is caused because of density differences in the fluid due to temperature differences under the influence of gravity. Density differences cause a *buoyancy force* which, in turn, causes the fluid circulation by *convection currents*. Buoyancy force is the upward force exerted by a fluid on a completely or partially immersed body and is equal to the *weight of the fluid* displaced by the body.

Obviously, fluid velocity in natural convection is *low* as compared to that in forced convection. Velocities associated with natural convection are relatively small, not much more than 2 m/s. As a result, the natural convection heat-transfer coefficients tend to be much smaller than those for forced convection. For gases, these coefficients are of the order of only 5 W/m^2 K. And yet, natural convection is one of the important modes of heat transfer used in practice since there are no moving parts. One must of course be careful to always check if simultaneous radiation heat transfer is significant to the thermal design.

All real fluids are viscous. Fluids past a solid surface result in the formation of a boundary layer. The thickness of the boundary layer is very small compared to the characteristic dimension of the solid surface. The velocity of fluid particles is *zero* at the solid surface and is *equal to the free stream* velocity at the edge of the boundary layer.

The boundary layer may be *laminar or turbulent*. In *laminar flow*, the fluid particles follow a smooth and continuous path and do not have a macroscopic mixing between successive layers. In a *turbulent flow*, there is a random macroscopic mixing of fluid particles across successive layers of fluid flow.

Natural convection requires (a) a solid–fluid interface, (b) a temperature difference between the temperature of the solid and the surrounding fluid, and (c) mixing motion of fluid particles due to the density difference created by the temperature gradient.

Natural convection flows can be either *external* or *internal*. *External flows* include flow up a heated wall and the plume rising above a power-plant stack. *Internal flows* are found between the cover plate and absorbing surface of a solar collector and inside hollow insulating walls.

Since there is no obvious characteristic velocity of a natural convection flow, the *Reynolds* number of forced convection does not play any role. It is replaced by the *Grashof* number or *Rayleigh* number.

Application Areas Natural convection heat transfer finds extensive applications in the following areas:

- Cooling of transformers, transmission lines, and rectifiers
- Heating of houses by steam or electrical radiators
- Heat loss from the steam pipelines in power plants and heat gain in the refrigerant pipelines in air-conditioning applications
- Cooling of electronic devices (*chips, transistors, etc.*) by finned heat sinks, etc.
- Buoyant plume rising from a smokestack

9.2 ❑ PHYSICAL MECHANISM OF NATURAL CONVECTION

Consider the familiar example of a heated, vertical plate kept hanging in quiescent (*stagnant*) air. Let the temperature of the heated surface be T_w and that of the surrounding air, T_∞ ($T_w > T_\infty$). A layer of air in the immediate vicinity of the plate will get heated up by conduction and the density of this heated air layer decreases. As a result, the heated air rises and the cold air from the surroundings moves in to take its place. This layer, in turn, gets heated up, moves up and is again replaced by cooler denser air. Thus, *convection currents* are set up causing the heat to be carried away from the hot surface. The fluid near the plate experiences an upward force due to the effect of buoyancy.

On the other hand, if the temperature of the solid surface is lower than that of the surrounding fluid ($T_w < T_\infty$), the fluid near the surface gets cooled and its density increases. As a result, the fluid starts moving downwards. Both situations are illustrated in Fig. 9.1.

During the temperature-induced flow, a boundary layer is set up along the length of the plate as shown. With the x-axis taken along the vertical length of the plate, and the y-axis perpendicular to it, the velocity and temperature profiles are shown in Fig. 9.2. As far as the velocity profile is concerned, at the plate surface, the fluid velocity is zero due to *no-slip* condition; then, the velocity increases to a maximum value and then, drops to zero at the outer edge of the boundary layer since the surrounding air is assumed to be quiescent. Note the difference in this velocity profile as compared to that in the case of forced convection. The boundary layer is *laminar* for some distance along the length, and then depending on the fluid properties and the temperature difference between the wall and the fluid, the boundary layer becomes *turbulent*.

The forces that exist near the surface due to density differences caused by heat transfer make the fluid move along the surface. The resulting movement *convects* energy to or from the surface. One may infer that the buoyancy-induced velocities extend into the fluid as far as the temperature changes do. Such a conclusion is valid for fluids whose Prandtl number is less than or equal to about 1.

One is basically interested in the events near the surface of a vertical plate in a gravitational field because the events are easier to visualize; the general nature of the phenomenon also applies to other geometries. The nature of the flow in the boundary layer will be either laminar (*layer-like*) or slightly turbulent. The nature or character of the free convection turbulent boundary layer is nowhere near as

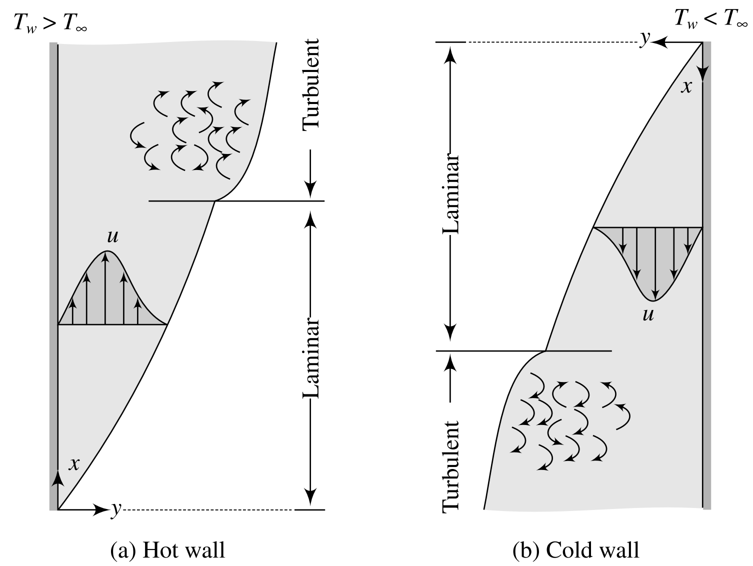


Fig. 9.1 *Flow patterns during heating and cooling of a vertical flat plate by free convection*

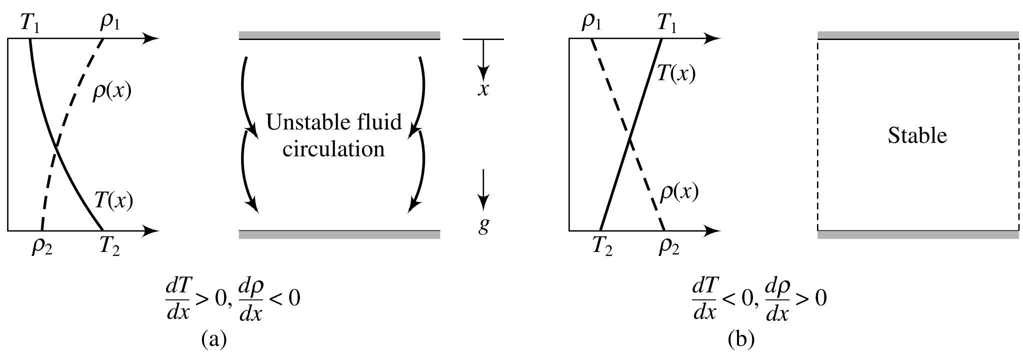


Fig. 9.2 *Conditions in a fluid between large horizontal plates at different temperatures. (a) Unstable temperature gradient. (b) Stable temperature gradient.*

strong compared to what is observed in forced convection. In fact, it would be more appropriate to call the flow *wavy* rather than *turbulent*.

The flow in the boundary layer at the beginning, i.e., $x = 0$ (*bottom* for $T_w > T_\infty$ and *top for* $T_w < T_\infty$), is always *laminar* and as velocities along the plate increase, transition will occur when the flow becomes unstable and starts wavering. The heat-transfer characteristics will be more pronounced in the turbulent region of the boundary layer than in the laminar region.

The physical properties in the dimensionless parameters that comprise the empirical correlations are usually evaluated at the average film temperature $T_f = (T_w + T_\infty)/2$ unless otherwise specifically specified to use another temperature.

In *pure* free convection, the fluid outside the boundary layer must be at rest (*quiescent*), not in motion ($u_\infty = 0$). When a small finite u_∞ exists, the effects of forced flow becomes important when the square of the Reynolds number is about the same magnitude as the Grashof number (i.e., $Re^2 \approx Gr$). Mixed free and forced convection will be discussed later.

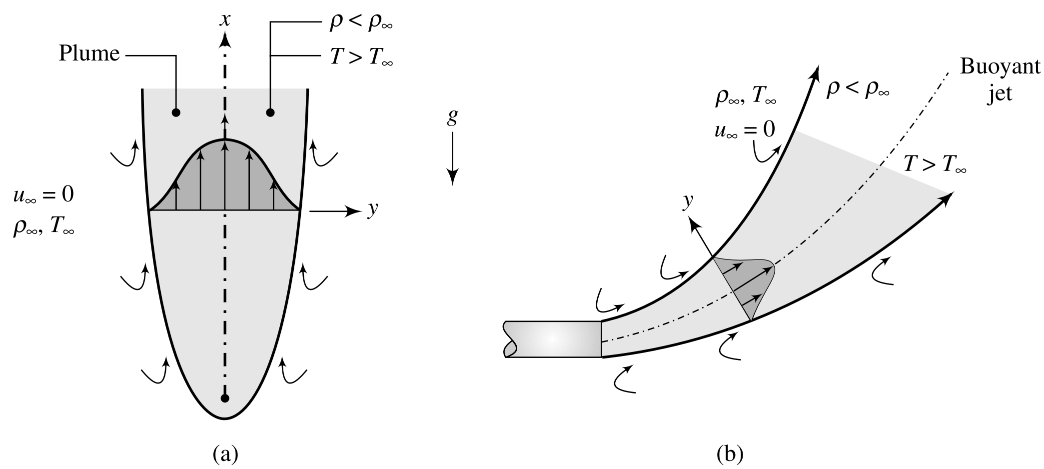


Fig. 9.3 *Buoyancy-driven free boundary layer flows in an extensive, quiescent medium. (a) Plume formation above a heated wire. (b) Buoyant jet associated with a heated discharge.*

It needs to be recognized that the presence of density gradient does not always ensure free convection. Referring to Fig. 9.2, consider two parallel horizontal plates at different temperatures and the fluid (say air) between them. If the upper plate is hotter, the fluid at the top becomes lighter and being bounded by the upper wall, it can't move and the heat transfer is only by conduction through the fluid. However, if the top surface is colder, the fluid will be heavier and convection currents will set up if the buoyancy forces overcome the viscous forces.

Free convection can also occur without the entire fluid being bounded by a wall.

In Fig. 9.3(a), the heating of a fluid is shown by a heater wire of small cross section submerged in it. The fluid will start rising entraining the quiescent fluid in the vicinity along. This will create a plume as shown. The plume will spread and get dispersed. The other example of a submerged buoyant jet is illustrated in Fig. 9.3(b).

9.3 ❑ ANALYTICAL SOLUTION OF LAMINAR FREE CONVECTION OVER A VERTICAL FLAT PLATE

Analytical solution of natural convection heat transfer is a little more complicated than forced convection since velocity field is coupled to the temperature field because the flow is induced by temperature differences. This coupling of velocity and temperature fields is peculiar to natural convection flows. In forced convection, the momentum equation can be solved in isolation from the energy equation. In free convection, however, the buoyancy force which is temperature dependent appears in the momentum equation. Hence, it is not possible to solve the momentum and energy equations separately, but together and simultaneously.

Schmidt and Beckmann analyzed the problem of laminar free convection on an isothermal vertical wall as shown in Fig. 9.4.

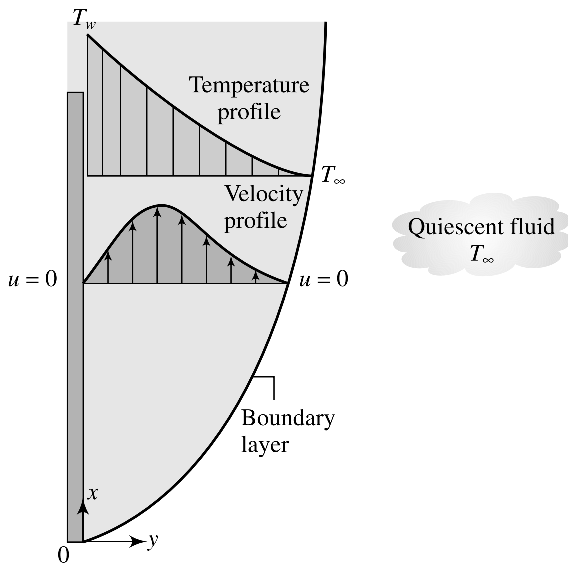


Fig. 9.4 *Typical velocity and temperature profiles for natural convection flow over a hot vertical plate at temperature T_s inserted in at temperature T_∞.*

The momentum and energy equations, viz.,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

are partial differential equations.

These are made ordinary differential equations by introducing the following parameters:

$$\eta = C\frac{y}{x^{1/4}} \quad \text{(dimensionless similarity variable)} \tag{9.1}$$

and $$\phi = \frac{T - T_\infty}{T_w - T_\infty} \quad \text{(dimensionless temperature)} \tag{9.2}$$

where $$C = \left[\frac{g\beta(T_w - T_\infty)}{4\nu^2}\right]^{1/4}$$

and ϕ is a function of η only.

Thus, $$\eta = \left[\frac{g\beta(T_w - T_\infty)}{4\nu^2}\right]^{1/4}\frac{y}{x^{1/4}} = \left(\frac{1}{4}\right)^{1/4}\left[\frac{g\beta(T_w - T_\infty)x^3}{\nu^2}\right]^{1/4}\frac{y}{x}$$

or $\boxed{\eta = \left(\frac{Gr_x}{4}\right)^{1/4}\frac{y}{x}}$ where Gr_x is the local Grashofnumber which represents the ratio

$$\frac{\text{(Buoyancy forces)(Inertia forces)}}{\text{(Viscous forces)}^2} \tag{9.3}$$

Let us introduce another term, $f(\eta) = \dfrac{u}{4\nu^2 C\sqrt{2x}}$ (9.4)

where $f(\eta)$ is a function of only η.

The momentum and energy equations now take the following forms:

$$\boxed{\frac{d^3 f}{d\eta^3} + 3f\frac{d^2 f}{d\eta^2} - 2\left(\frac{df}{d\eta}\right)^2 + \phi = 0} \tag{9.5a}$$

and
$$\boxed{\frac{d^2\phi}{d\eta^2} + 3Pr\, f\frac{d\phi}{d\eta} = 0} \tag{9.5b}$$

The boundary conditions are as follows:

At $y = 0$, $T = T_w$, $v = u = 0$ or $\eta = 0$; $\dfrac{df}{d\eta} = 0, \phi = 1$,

At $y = \infty$, $T = T_\infty$, $v = u = 0$ or $\eta = \infty$; $\dfrac{df}{d\eta} = 0, \phi = 0$

The solutions to the preceding equations were given by *Ostrach*, and the resulting temperature and velocity distributions are given in Figures 9.5 and 9.6.

The *local* heat-transfer coefficient may be evaluated from

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = h_x(T_w - T_\infty)$$

$$\therefore \qquad h_x = -\frac{k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}}{T_w - T_\infty} = \frac{-k\left(\dfrac{\partial\phi}{\partial\eta}\times\dfrac{d\eta}{dy}\right)_{y=0}(T_w - T_\infty)}{T_w - T_\infty}$$

or
$$h_x = -k\frac{C}{x^{1/4}}\left(\frac{d\phi}{d\eta}\right)_{y=0} = -k\left[\frac{g\beta(T_w - T_\infty)}{4\nu^2}\right]^{1/4}\frac{1}{x^{1/4}}\left(\frac{d\phi}{d\eta}\right)_{y=0}$$

$$= -\frac{k}{x}\left[\frac{g\beta(T_w - T_\infty)x^3}{4\nu^2}\right]^{1/4}\left(\frac{d\phi}{d\eta}\right)_{y=0}$$

or
$$\bar{h}_x = \frac{1}{-\sqrt{2}}\frac{k}{x}(Gr_x)^{1/4}\left(\frac{d\phi}{d\eta}\right)_{y=0} \tag{9.6}$$

The average value of the heat-transfer coefficient is given as

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x\,dx = \frac{4}{3}(h_x)_{x=L}$$

or
$$\bar{h}_L = -\frac{4}{3}\frac{1}{\sqrt{2}}\frac{k}{L}(Gr_L)^{1/4}\left(\frac{d\phi}{d\eta}\right)_{y=0} \tag{9.7}$$

The value of $(d\theta/d\eta)_{y=0}$ is still unknown. Its empirical value is given below:

$$\boxed{-\left(\frac{d\phi}{d\eta}\right)_{y=0} = \frac{0.676Pr^{1/2}}{(0.861+Pr)^{1/4}}} \tag{9.8}$$

Therefore, $$h_x = \frac{1}{\sqrt{2}}\frac{k}{x}(Gr_x)^{1/4}\frac{0.676Pr^{1/2}}{(0.861+Pr)^{1/4}}$$

Local Nusselt number,

$$Nu_x = \frac{h_x x}{k} = \frac{1}{\sqrt{2}}\frac{0.676Pr^{1/2}}{(0.861+Pr)^{1/4}}(Gr_x)^{1/4}$$

$$\Rightarrow \quad \boxed{Nu_x = 0.478(Gr_x)^{1/4}Pr^{1/2}(0.861+Pr)^{-1/4}} \tag{9.9}$$

or $$\frac{h_x x}{k} = (\text{const.})(x^3)^{1/4} \qquad \left[\text{since} \quad Gr_x = \frac{g\beta\Delta T x^3}{\nu^2}\right]$$

or $$h_x = (\text{const.})\frac{x^{3/4}}{x} = (\text{const.})x^{-1/4}$$

Average heat-transfer coefficient,

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x dx = \frac{(\text{const.})}{L}\int_0^L x^{-1/4}\,dx$$

$$= \frac{(\text{const.})}{L}x^{3/4}\Bigg|_{x=L} \times \frac{4}{3} = \frac{4}{3}h_{x=L}$$

$$\therefore \quad \overline{Nu}_L = \frac{\bar{h}_L L}{k} = \frac{4}{3}\frac{1}{\sqrt{2}}\frac{0.676Pr^{1/2}}{(0.861+Pr)^{1/4}}(Gr_x)^{1/4}$$

$$\Rightarrow \quad \boxed{\overline{Nu}_L = 0.637(Gr_x)^{1/4}Pr^{1/2}(0.861+Pr)^{-1/4}} \tag{9.10}$$

For air, with $Pr = 0.714$, we have

$$\boxed{Nu_x = 0.360Gr_x^{1/4}} \quad \text{and} \quad \boxed{Nu_L = 0.480Gr_L^{1/4}} \tag{9.11}$$

The foregoing results apply irrespective of whether $T_w > T_\infty$ or $T_w < T_\infty$. If $T_w < T_\infty$, conditions are inverted from those of Fig. 9.4, the leading edge is at the top of the plate, and positive x is defined in the direction of the gravity force convection.

> **Note:** The coefficient $\frac{4}{3}$ in Eq. (9.10) for free convection on a vertical surface compares with a coefficient of 2 for laminar forced convection over a flat plate. Also in the laminar flow, the convection coefficient h varies as $x^{-1/4}$ *for free convection*, compared with $x^{-1/2}$ *for forced convection*.

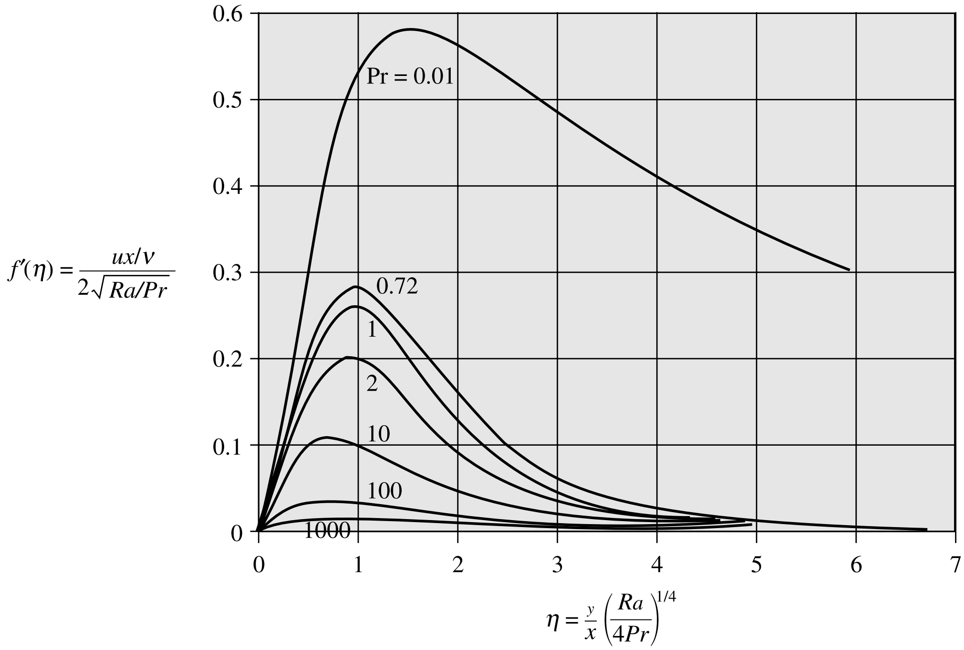


Fig. 9.5 *Analytical results for velocity profile for laminar free convection on a flat plate*

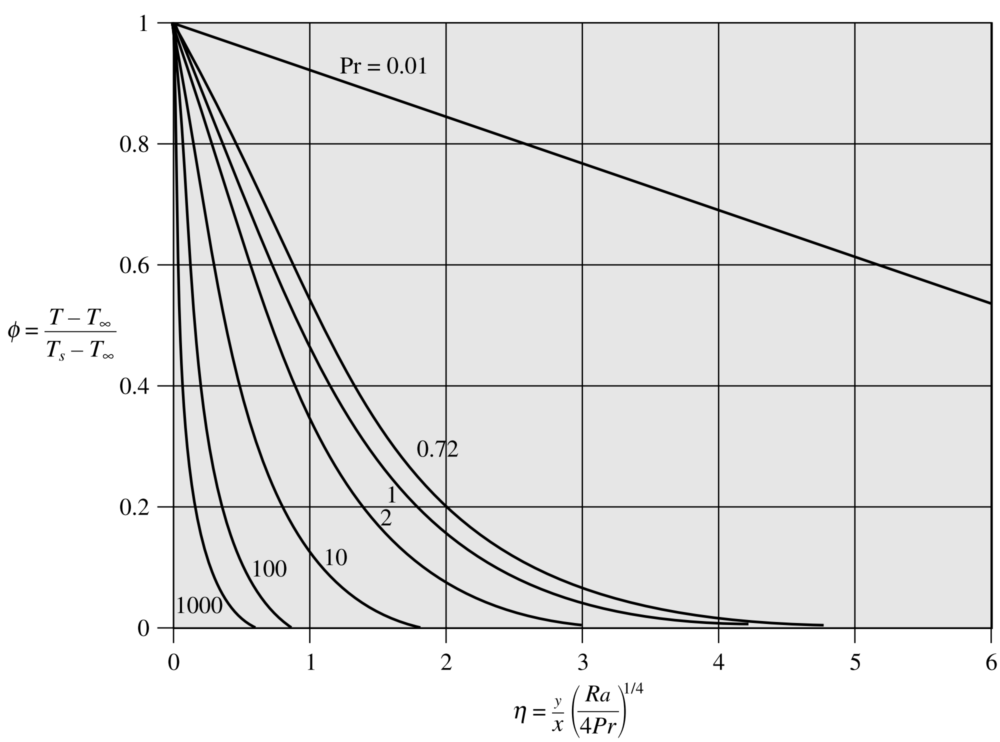


Fig. 9.6 *Analytical results for temperature profile in laminar free convection on a vertical flat plate*

9.4 ❑ INTEGRAL METHOD FOR NATURAL CONVECTION HEAT TRANSFER ON A VERTICAL FLAT PLATE

The most frequently studied case of natural convection is that of a fluid adjacent to a plane vertical wall. Consider a heated vertical plate of length L from which heat is lost by natural convection.

The plate is maintained at a temperature T_w and the temperature of the surrounding stagnant fluid is T_∞. The motion of the fluid is confined to a region close to the surface.

Here we calculate the heat-transfer coefficient if the boundary layer is *laminar*. Let the thickness of the velocity boundary layer be δ and that of the thermal boundary layer δ_T.

The velocity and temperature profiles are shown in Fig. 9.7. The velocity in the x-direction, both at the wall and at the edge of the plate, has to be zero. At the wall the velocity is zero because of *no-slip condition*, it increases to some maximum value and again decreases to zero at the edge of the boundary layer since the *free stream* conditions are quiescent (*still*) in a free convection system. Both boundary layers remain laminar until some distance from the bottom edge and then turn turbulent. For simplicity, let us assume $\delta_T = \delta$, since in free convection the flow pattern exists because of the temperature difference and the two boundary layers may be expected to be of the same thickness. For $Pr > 1$, $\delta > \delta_T$ and the typical velocity and temperature profiles are illustrated in Fig. 9.8.

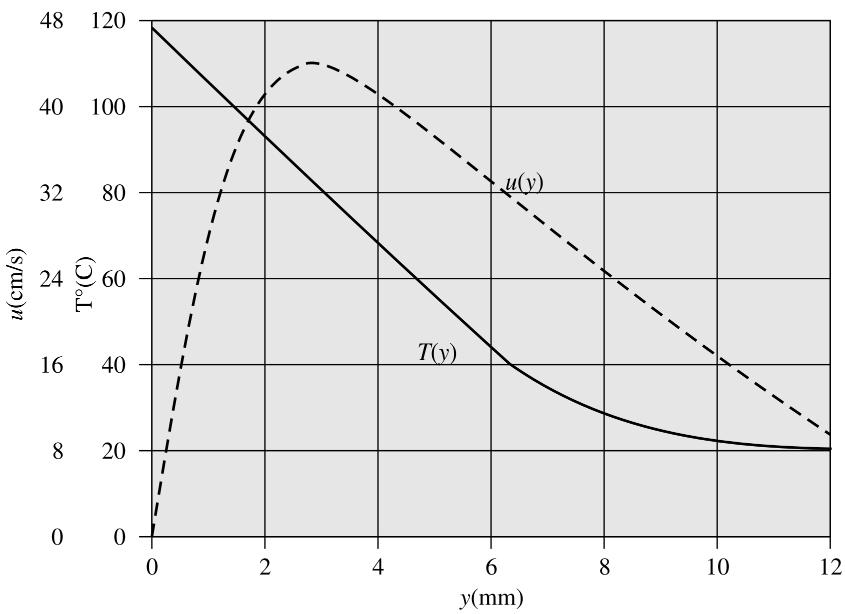


Fig. 9.7 *Temperature and velocity distributions in the vicinity of a heated flat plate placed vertically in still air*

Assumptions (1) Steady, incompressible, laminar fluid flow. It may be emphasized here that the term *incompressible* implies constant density but it is through variations in fluid density that the fluid is driven in free convection. In the analysis, only in the buoyancy-force term the density changes are taken into account but elsewhere the density is considered constant.

(2) Prandtl number is near unity which implies that $\delta \approx \delta_T$.

(3) Buoyancy effects are limited to the boundary-layer region and $u_\infty = 0$.

(4) Viscous heat dissipation is neglected since the magnitude of velocity is small.

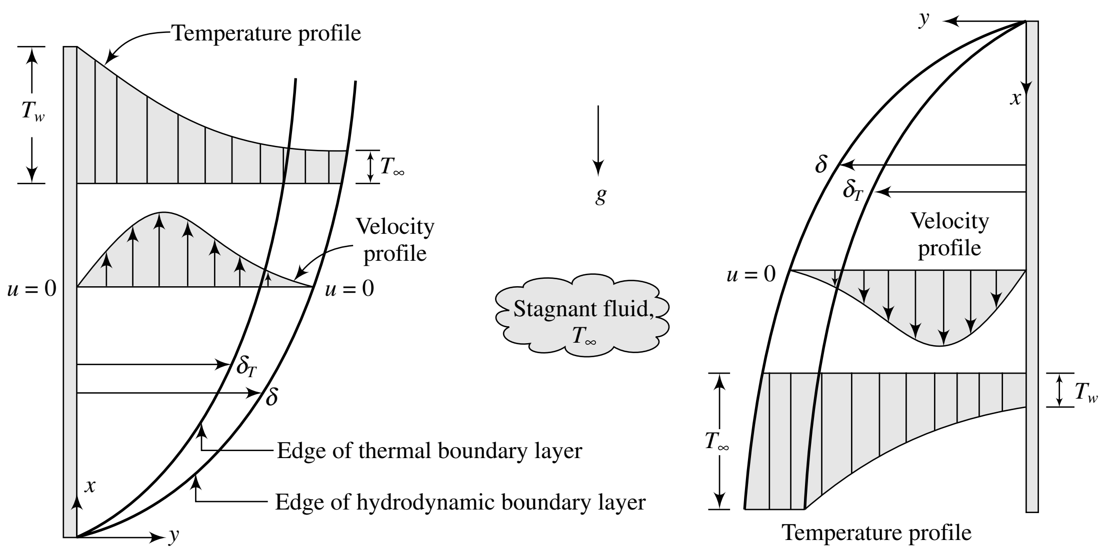


Fig. 9.8 *Typical velocity and temperature profiles in the boundary layer for natural convection heat transfer from (a) heated, and (b) cooled vertical plate in a stationary fluid (Pr > 1).*

Consider the control volume *ABCD* shown in Fig. 9.9. Here, we are assuming $\delta_T = \delta$. Applying Newton's second law of motion to flow through control volume (C. V). Length of CV is dx and height is δ, width is unity.

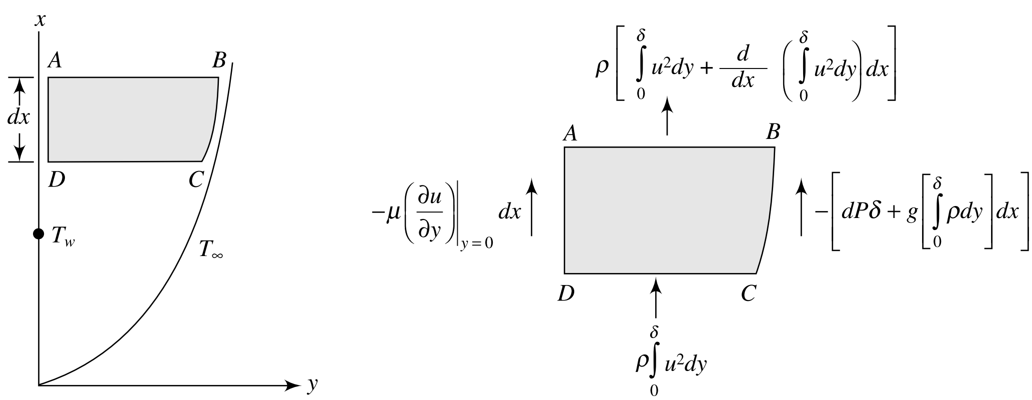


Fig. 9.9 *Elemental control volume for the integral method application of Newton's second law to a free convection boundary layer*

$$\underbrace{\begin{pmatrix}\text{Rate at which the } x\text{-direction}\\ \text{momentum per unit width}\\ \text{leaves the CV}\end{pmatrix}}_{\text{(i)}} - \underbrace{\begin{pmatrix}\text{Rate at which the } x\text{-direction}\\ \text{momentum per unit width}\\ \text{enters the CV}\end{pmatrix}}_{\text{(ii)}} = \underbrace{\begin{pmatrix}\text{Pressure forces exerted}\\ \text{in the } x\text{-direction by the}\\ \text{surroundings on CV}\end{pmatrix}}_{\text{(iii)}}$$

$$+ \underbrace{\begin{pmatrix}\text{Viscous forces exerted in the } x\text{-direction}\\ \text{by the surroundings on CV}\end{pmatrix}}_{\text{(iv)}} + \underbrace{\begin{pmatrix}\text{Body forces in the } x\text{-direction}\\ \text{acting on the fluid inside the CV}\end{pmatrix}}_{\text{(v)}} \tag{9.12}$$

Each term in the above equation will have units of rate of change of momentum per unit width $= \frac{\text{kg m}}{\text{s. s m}} = \frac{\text{kg}}{\text{s}^2}$

Let us write the terms one by one,

(i) $\rho\left[\int_0^\delta u^2\,dy + \frac{d}{dx}\left(\int_0^\delta u^2 dy\right)dx\right]$ (ii) $\rho\int_0^\delta u^2 dy$

(iii) $-dP\ \delta$ (iv) $-\mu\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0} dx$

(v) $-\left(g\int_0^\delta \rho\,dy\right)dx$

Note that (iii), (iv), and (v) have *negative* signs because these forces are acting on the control volume and the sum of the terms of RHS of the equation (9.12) is equal and opposite to that of LHS, so that the summation of all forces is zero. Substituting in the equation (9.12) gives

$$\rho\frac{d}{dx}\left[\int_0^\delta [u^2 dy]\right]dx = -dP\,\delta - \mu\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0} dx - \left[g\int_0^\delta \rho\ dy]\right]dx$$

Dividing by dx,

$$\rho\frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = -\frac{dP}{dx}\delta - \mu\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0} - g\int_0^\delta \rho\ dy \tag{9.13}$$

Coefficient of volumetric thermal expansion β is defined as

$$\beta = -\frac{(\rho-\rho_\infty)}{\rho(T-T_\infty)} \tag{9.14a}$$

$$\frac{dP}{dx} = -\rho_\infty g \tag{9.14b}$$

where ρ_∞ is the density of the surrounding fluid at the temperature T_∞.

Substituting the values from equations (9.14a) and (9.14b) in the equation (9.13) and dividing by ρ, we get

$$\frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = \frac{\rho_\infty g}{\rho}\delta - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0} - g\frac{\int_0^\delta \rho\,dy}{\rho}$$

Since, $\delta = \int_0^\delta dy$ and ρ is a function of y it must be written under the integral sign, we can write

$$\frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = g\int_0^\delta \frac{(\rho_\infty-\rho)}{\rho} - dy - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0} \tag{9.15}$$

Using Eq. (9.14a) we get, $\dfrac{\rho_\infty - \rho}{\rho} = -\beta(T_\infty - T)$

Hence, replacing ρ by temperature by introducing β, the integral momentum equation can be expressed as

$$\boxed{\frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = -g\beta\int_0^\delta (T_\infty - T)\,dy - \nu\left(\frac{\partial u}{\partial y}\right)\Bigg|_{y=0}} \tag{9.16}$$

The integral energy equation can be obtained by integrating the energy equation over the boundary layer

$$\boxed{\frac{d}{dx}\int_0^\delta (T_\infty - T)u\,dy = \alpha\left(\frac{\partial T}{\partial y}\right)\Bigg|_{y=0}} \tag{9.17}$$

Choice of Temperature and Velocity Profiles Equations (9.16) and (9.17) require velocity and temperature profiles.

For temperature distribution, we have the following boundary conditions.

At $y = 0$, $T = T_w$ (a)

At $y = \delta, T = T_\infty, \dfrac{\partial T}{\partial y} = 0$ (b) and (c)

With these *three* boundary conditions, the temperature distribution can be expressed with *three* arbitrary constants.

Let $T = C_1 + C_2 y + C_3 y^2$

Boundary condition (a) gives, $C_1 = T_w$

Boundary condition (c) gives, $\dfrac{\partial T}{\partial y} = 0 = C_2 + 2C_3\delta$ (9.18)

Boundary condition (b) gives, $T_\infty = T_w + C_2\delta + C_3\delta^2$ (9.19a)

Multiplying Eq. (9.18) by δ, $0 = C_2\delta + 2C_3\delta^2$ (9.19b)

Solving Eqs. (9.19a) and (9.19b),

$$\boxed{C_3 = \frac{T_w - T_\infty}{\delta^2}} \quad \text{and} \quad \boxed{C_2 = \frac{-2(T_w - T_\infty)}{\delta}} \tag{9.20}$$

Substituting the three constants, the temperature distribution is given by

$$T = T_w - 2\frac{(T_w - T_\infty)}{\delta}y + \frac{(T_w - T_\infty)}{\delta^2}y^2$$

or
$$\frac{T - T_w}{T_w - T_\infty} = -2\frac{y}{\delta} + \frac{y^2}{\delta^2}$$

Adding 1 on both sides,

$$\frac{T - T_w}{T_w - T_\infty} + 1 = 1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}$$

The resulting temperature profile becomes,

$$\boxed{\frac{T - T_\infty}{T_w - T_\infty} = \left(1 - \left(\frac{y}{\delta}\right)\right)^2} \tag{9.21}$$

For the velocity distribution, we have the following boundary conditions:

At $y = 0$, $u = 0$ (A)

At $y = \delta$, $u = 0$ (B)

At $y = \delta$, $\frac{\partial u}{\partial y} = 0$ (C)

At $y = 0$, $\frac{\partial^2 u}{\partial y^2} = \text{constant, say} = C_0$ (D)

With these *four* boundary conditions, the velocity distribution can be assumed to be a cubic polymanial with *four* arbitrary constants.

Let $$\frac{u}{u_x} = C_1 + C_2 y + C_3 y^2 + C_4 y^3$$

where, u_x is some arbitrary function of x and has dimensions of u.

Boundary condition (A) gives, $C_1 = 0$

Boundary condition (B) gives,

$$C_2\delta + C_3\delta^2 + C_4\delta^3 = 0 \tag{9.22}$$

Boundary condition (C) gives,

$$\frac{\partial u}{\partial y} = 0 = C_2 + 2C_3\delta + 3C_4\delta^2 \tag{9.23}$$

Boundary condition (D) gives,

$$\frac{\partial^2 u}{\partial y^2} = C_0 = 2C_3 \quad \Rightarrow \quad \boxed{C_3 = C_0/2}$$

Solving Eqs. (9.22) with (9.23) multiplied by δ, we have

$$-0 = C_2\delta + C_3\delta^2 + C_4\delta^3 + \frac{0 = C_2\delta + 2C_3\delta^2 + 3C_4\delta^2}{0 = C_3\delta^2 + 2C_4\delta^3}$$

$\therefore$ $$C_4 = -\frac{-C_3\delta^2}{2\delta^3} = -\frac{C_0}{2\times 2\delta} \quad \Rightarrow \quad C_4 = \frac{C_0}{4\delta}$$

From Eq. (9.22),

$$C_2 = -\frac{-C_3\delta^2 - C_4\delta^3}{\delta} = \frac{-\frac{C_0}{2}\delta^2 + \frac{C_0\delta^2}{4\delta}\delta^3}{\delta} = \frac{-C_0\delta^2}{4\delta}$$

$$\Rightarrow \qquad \boxed{C_2 = -\frac{C_0\delta}{4}}$$

Solution of Equations Substituting the constants, the velocity distribution is given by

$$\frac{u}{u_x} = \frac{-C_0\delta}{4}y + \frac{C_0}{2}y^2 - \frac{C_0}{4\delta}y^3$$

or $$\frac{u}{u_x} = \frac{y}{\delta}\left[\frac{\delta}{y}\left[\frac{-C_0}{4}\delta y + \frac{C_0}{2}y^2 - \frac{C_0}{4\delta}y^3\right]\right]$$

or $$\frac{u}{u_x} = \frac{y}{\delta}\left[\frac{-C_0}{4}\delta^2 + \frac{C_0}{2}\delta y - \frac{C_0}{4}y^2\right]$$

or $$\frac{u}{u_x} = -C_0\frac{y}{\delta}\left[\frac{\delta^2}{4} - \frac{1}{2}\delta y + \frac{1}{4}y^2\right] \tag{9.24}$$

or $$\frac{u}{u_x} = \left[\frac{-C_0\delta^2}{4}\right]\frac{y}{\delta}\left[1 - 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]$$

or $$u = \left[\frac{-C_0\delta^2 u_x}{4}\right]\frac{y}{\delta}\left[1 - \left(\frac{y}{\delta}\right)\right]^2$$

Let $$\frac{-C_0\delta^2 u_x}{4} = u'_x \qquad (\textit{another constant})$$

The velocity profile can now be written in a more compact form as

$$\boxed{\frac{u}{u'_x} = \frac{y}{\delta}\left[1 - \left(\frac{y}{\delta}\right)\right]^2} \tag{9.25}$$

or $$u = u'_x\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2$$

where u'_x is an unknown quantity with the dimensions of velocity.

or $$u = u'_x\left[\frac{y}{\delta}\left(1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right)\right] \Rightarrow u = u'_x\left[\frac{y}{\delta} - 2\frac{y^2}{\delta^2} + \frac{y^3}{\delta^3}\right]$$

Differentiating with respect to y and equating it to zero, we have

$$\frac{du}{dy} = 0 = u'_x\left[\frac{1}{\delta} - \frac{4y}{\delta^2} + \frac{3y^2}{\delta^3}\right] \qquad \text{or} \quad 3y^2 - 4y\delta + \delta^2 = 0$$

Solving this quadratic equation,

$$y = \frac{-(-4\delta) \pm \sqrt{(-4\delta)^2 - 4(3)(\delta)^2}}{2 \times 3}$$

$$= +\frac{4}{6}\delta \pm \sqrt{\frac{16\delta^2 - 12\delta^2}{36}} = +\frac{2}{3}\delta \pm \frac{\delta}{3} = \delta \quad \text{or} \quad \frac{1}{3}\delta$$

The velocity will be maximum at $y = \dfrac{\delta}{3}$ since at $y = \delta$, $u = 0$

because $\quad u_{\max} = u_x' \times \dfrac{1}{3}\left(1 - \dfrac{1}{3}\right)^2 = \dfrac{4}{27}u_x'$

and this maximum is

$$u_{\max} = \frac{4}{27}u_x' \quad \text{at} \quad y/\delta = \frac{1}{3}$$

The velocity distribution can thus be expressed as

$$\boxed{u = \frac{27}{4}u_{\max}\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2} \tag{9.26}$$

The mean or average velocity is given by

$$u_{av} = \frac{1}{\delta}\int_0^{\delta} u\,dy = \frac{u_x'}{\delta}\int_0^{\delta}\left(\frac{y}{\delta} - \frac{2y^2}{\delta^2} + \frac{y^3}{\delta^3}\right)dy$$

$$= \frac{u_x'}{\delta}\left[\frac{y^2}{2\delta} - \frac{2y^3}{3\delta^2} + \frac{y^4}{4\delta^3}\right]_0^{\delta} = \frac{u_x' \cdot \delta}{\delta}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= u_x'\frac{1}{12} = \frac{27}{4}u_{\max}\frac{1}{12}$$

$$\therefore \qquad \boxed{u_{av} = \frac{27}{48}u_{\max}} \tag{9.27}$$

Substituting the velocity and temperature distributions from equations (9.23) and (9.24) in equations (9.9) and (9.10), we have

$$\underbrace{\frac{d}{dx}\left[\int_0^{\delta} u_x'^2 \frac{y^2}{\delta^2}\left(1 - \frac{y}{\delta}\right)^4 dy\right]}_{\text{I}} = \underbrace{g\beta\int_0^{\delta}(T_w - T_\infty)\left(1 - \frac{y}{\delta}\right)^2 dy}_{\text{II}} \underbrace{-\nu\left[\frac{\partial}{\partial y}\left(u_x'\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)\right)^2\right]_{y=0}}_{\text{III}}$$

Solving terms in the equation above one by one separately,

First Term

$$\frac{d}{dx}\left[\int_0^{\delta} u_x'^2 \frac{y^2}{\delta^2}\left(1 - \frac{y}{\delta}\right)^4 dy\right]$$

We note that

$$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

$$\left(1-\frac{y}{\delta}\right)^4 = 1 - 4\frac{y}{\delta} + \frac{12}{2}\frac{y^2}{\delta^2} - \frac{24}{6}\frac{y^3}{\delta^3} + \frac{24}{24}\frac{y^4}{\delta^4} = 1 - 4\frac{y}{\delta} + 6\frac{y^2}{\delta^2} - 4\frac{y^3}{\delta^5} + \frac{y^4}{\delta^4}$$

Hence,
$$\frac{y^2}{\delta^2}\left[1-\frac{y}{\delta}\right]^4 = \frac{y^2}{\delta^2} - 4\frac{y^3}{\delta^3} + 6\frac{y^4}{\delta^4} - 4\frac{y^5}{\delta^5} + \frac{y^6}{\delta^6}$$

$\therefore$
$$\frac{d}{dx}\left[\int_0^\delta u_x'^2\left(\frac{y^2}{\delta^2} - 4\frac{y^3}{\delta^3} + 6\frac{y^4}{\delta^4} - 4\frac{y^5}{\delta^5} + \frac{y^6}{\delta^6}\right)dy\right] = \frac{d}{dx}\left[u_x'^2\left[\frac{\delta}{3} - \delta + \frac{6}{5}\delta - \frac{4}{6}\delta + \frac{1}{7}\delta\right]\right]$$

$$\frac{d}{dx}\left[u_x'^2\delta\left[\frac{35-105+126-70+15}{105}\right]\right] = \frac{1}{105}\frac{d}{dx}u_x'^2\delta \tag{9.28}$$

Second Term

$$g\beta\int_0^\delta (T_w - T_\infty)\left(1-\frac{y}{\delta}\right)^2 dy = g\beta(T_w - T_\infty)\int_0^\delta\left(1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right)dy = g\beta(T_w - T_\infty)\frac{\delta}{3} \tag{9.29}$$

Third Term

$$\nu\frac{\partial}{\partial y}\left[u_x'\frac{y}{\delta}\left[1-\left(\frac{y}{\delta}\right)\right]^2\right]\Bigg|_{y=0} = \nu\frac{\partial}{\partial y}u_x'\left[\frac{y}{\delta} - 2\frac{y^2}{\delta^2} + \frac{y^3}{\delta^3}\right]_{y=0} = \frac{\nu u_x'}{\delta} \tag{9.30}$$

Substituting terms from equations (9.25), (9.26), and (9.27),

$$\boxed{\frac{1}{105}\frac{d}{dx}u_x'^2\delta = \frac{1}{3}g\beta(T_w - T_\infty)\delta - \frac{\nu u_x'}{\delta}} \tag{9.31}$$

For Eq. (9.27), substituting the relevant terms,

$$\frac{d}{dx}\int_0^\delta (T_\infty - T)u\,dy = \alpha\frac{\partial T}{\partial y}\bigg|_{y=0}$$

Solving LHS first,

$$\text{LHS} = \frac{d}{dx}\int_0^\delta -(T_w - T_\infty)\left(1-\frac{y}{\delta}\right)^2 u_x'\frac{y}{\delta}\left[1-\left(\frac{y}{\delta}\right)\right]^2 dy$$

$$= \frac{d}{dx}\left[-(T_w - T_\infty)u_x'\int_0^\delta \frac{y}{\delta}\left(1-\left(\frac{y}{\delta}\right)\right)^4 dy\right]$$

$$= -(T_w - T_\infty)\frac{d}{dx}\left[u_x'\int_0^\delta\left[\frac{y}{\delta} - 4\frac{y^2}{\delta^2} + 6\frac{y^3}{\delta^3} - 4\frac{y^4}{\delta^4} + \frac{y^5}{\delta^5}\right]dy\right]$$

$$= -(T_w - T_\infty)\frac{d}{dx}u_x'\left[\frac{\delta}{2} - \frac{4}{3}\delta + \frac{6}{4}\delta - \frac{4}{5}\delta + \frac{1}{6}\delta\right]$$

$$= -(T_w - T_\infty)\frac{d}{dx}u_x'\left[\frac{15 - 40 + 45 - 24 + 5}{30}\right]\delta$$

$$= -(T_w - T_\infty)\frac{d}{dx}\frac{\delta}{30}u_x' = -\frac{1}{30}(T_w - T_\infty)\frac{d}{dx}u_x'\,\delta \tag{9.32}$$

$$\text{RHS} = \alpha\left(\frac{\partial T}{\partial y}\right)\bigg|_{y=0} = \alpha\frac{\partial}{\partial y}\left[(T_w - T_\infty)\left(1 - \left(\frac{y}{\delta}\right)\right)^2 + T_\infty\right]\bigg|_{y=0}$$

$$= \alpha(T_w - T_\infty)\frac{\partial}{\partial y}\left[1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right]\bigg|_{y=0} = \alpha(T_w - T_\infty)\left(-\frac{2}{\delta}\right) \tag{9.33}$$

Substituting in Eq. (9.27),

$$\boxed{\frac{1}{30}\frac{d}{dx}u_x'\delta = \frac{2\alpha}{\delta}} \tag{9.34}$$

Equations (9.28) and (9.31) need to be solved now. For this, the variation in u_x' and δ with respect to x should be known. The condition is: Both u_x' and δ are zero, at $x = 0$.
Assume the relationships in the form,

$$\boxed{u_x' = C_1 x^m} \tag{9.35a}$$

$$\boxed{\delta = C_2 x^n} \tag{9.35b}$$

Substituting in Eqs. (9.28) and (9.31),

$$\frac{1}{105}\frac{d}{dx}C_1^2 x^{2m} C_2 x^n = \frac{1}{3}g\beta(T_w - T_\infty)C_2 x^n - \frac{\nu C_1 x^m}{C_2 x^n}$$

or
$$\boxed{\frac{(2m+n)}{105}C_1^2 C_2 x^{(2m+n-1)} = \frac{1}{3}g\beta(T_w - T_\infty)C_2 x^n - \nu\frac{C_1}{C_2}x^{(m-n)}} \tag{9.36}$$

For this equation to be dimensionally balanced, the indices of x on the RHS must equal the indices of x on the LHS.

$\therefore$ $\quad 2m + n - 1 = n \;\Rightarrow\; m = \frac{1}{2}$

and $\quad 2m + n - 1 = m - n \;\Rightarrow\; n = \frac{1}{4}$ $\quad$ (also $n = m - n$)

Substituting in Eq. (9.36),

$$\frac{5}{4 \times 105}C_1^2 C_2 x^{1/4} = \frac{1}{3}g\beta(T_w - T_\infty)C_2 x^{1/4} - \nu\frac{C_1}{C_2}x^{1/4}$$

or
$$\frac{1}{84}C_1^2 C_2 = \frac{1}{3}g\beta(T_w - T_\infty)C_2 - \frac{C_1\nu}{C_2}$$

or $$\frac{1}{30}\frac{d}{dx}C_1x^mC_2x^n = \frac{2\alpha}{C_2x^n}$$

or $$\boxed{\frac{(m+n)}{30}C_1C_2x^{(m+n-1)} = \frac{2\alpha}{C_2}x^{-n}} \tag{9.37}$$

Equating the indices of x, $m + n - 1 = -n$

$$\boxed{m = \frac{1}{2} \quad \text{and} \quad n = \frac{1}{4}}$$

These values also satisfy ($m + n - 1 = -n$).
Substituting for m and n in Eq. (9.37),

$$\frac{3}{4\times 30}C_1C_2x^{-1/4} = \frac{2\alpha}{C_2}x^{-1/4}$$

or $$\frac{1}{40}C_1C_2 = \frac{2\alpha}{C_2} \quad \text{or} \quad C_1C_2^2 = 80\alpha$$

$\therefore$ $$C_1 = \frac{80\alpha}{C_2^2} \tag{9.38}$$

Substituting in Eq. (9.37) to obtain the values of C_1 and C_2.

$$\frac{1}{84}\left(\frac{80\alpha}{C_2^2}\right)^2 C_2 = \frac{1}{3}g\beta(T_w - T_\infty)C_2 - \frac{80\alpha}{C_2^2C_2}\nu$$

Dividing by C_2, we get

$$\frac{1}{84}\frac{80^2\alpha^2}{C_2^4} = \frac{1}{3}g\beta(T_w - T_\infty) - \frac{80\alpha\nu}{C_2^4}$$

or $$80\left[\frac{80\alpha^2}{84C_2^4}\right] + 80\frac{\alpha\nu}{C_2^4} = \frac{1}{3}g\beta(T_w - T_\infty)$$

or $$3\times 80\left[\frac{20}{21}\alpha^2 + \alpha\nu\right] = g\beta(T_w - T_\infty)C_2^4$$

Dividing by α^2 throughout, we get

$$240\left[\frac{20}{21} + \frac{\nu}{\alpha}\right] = \frac{g\beta}{\alpha^2}(T_w - T_\infty)C_2^4$$

or $$C_2 = (240)^{1/4}\left(\frac{20}{21} + Pr\right)^{1/4}\left(\frac{g\beta(T_w - T_\infty)}{\alpha^2}\right)^{-1/4} \qquad \left(\text{since } \frac{\nu}{\alpha} = Pr\right)$$

or $$C_2 = 3.93(0.952 + Pr)^{1/4}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{-1/4}\left(\frac{\nu^2}{\alpha^2}\right)^{-1/4}$$

or $$\boxed{C_2 = 3.93(0.952 + Pr)^{1/4}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{-1/4}(Pr)^{-1/2}} \tag{9.39}$$

and $$C_1 = \frac{80\alpha}{C_2^2} = \frac{80\alpha}{\left[3.93(0.952 + Pr)^{1/4}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{-1/4}(Pr)^{-1/2}\right]^2}$$

or $$C_1 = \frac{80\alpha}{15.44(0.952 + Pr)^{1/2}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{-1/2}\left[\left(\frac{\nu}{\alpha}\right)^{-1/2}\right]^2}$$

or $$\boxed{C_1 = 5.18(0.952 + Pr)^{-1/2}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{1/2}\nu} \tag{9.40}$$

Hence, $$\boxed{u_x' = C_1 x^m = 5.18(0.952 + Pr)^{-1/2}\nu\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{1/2}x^{1/2}} \tag{9.41a}$$

and $$\boxed{\delta = C_2 x^n = 3.93(0.952 + Pr)^{1/4}(Pr)^{-1/2}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{-1/4}x^{1/4}} \tag{9.41b}$$

or $$\frac{\delta(x)}{x} = 3.93(0.952 + Pr)^{1/4}(Pr)^{-1/2}\left(\frac{g\beta(T_w - T_\infty}{\nu^2}\right)^{-1/4}x^{-3/4}$$

or $$\boxed{\frac{\delta}{x} = 3.93\left(\frac{0.952 + Pr}{Pr^2}\right)^{1/4}Gr_x^{-1/4}} \tag{9.42}$$

where $Gr_x = \dfrac{g\beta(T_w - T_\infty)x^3}{\nu^2}$ is the *local* Grashof number.

9.4.1 • Mass-flow Rate Through the Boundary

Mass-flow rate through the boundary layer per unit width at any location x is

$$\dot{m} = \rho u_{av}(\delta \times 1) = \rho u_x' \frac{\delta}{12} = \rho\frac{C_1}{12}x^{1/2} \times C_2 x^{1/4} = \frac{\rho C_1 C_2}{12}x^{3/4}$$

Total mass-flow rate from $x = 0$ to $x = L$ per metre width through the boundary will be

$$\dot{m}_{\text{total}} = \frac{1}{L}\int_0^L \frac{\rho C_1 C_2}{12}x^{3/4}dx = \frac{\rho C_1 C_2}{12L}\cdot\left[\frac{x^{7/4}}{7/4}\right]_0^L = \frac{\rho C_1 C_2}{12L} \times \frac{4}{7}L^{7/4}$$

$$= \frac{\rho L^{7/4}}{21L} \times 5.18 \times 3.93(0.952 + Pr)^{-1/4}\nu\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{1/4}(Pr)^{-1/2}$$

$$= \left(\frac{5.18 \times 3.93}{21}\right)\rho\nu\frac{(g\beta(T_w - T_\infty)L^3/\nu^2)}{Pr^{1/2}(0.952 + Pr)^{1/4}}$$

$$\therefore \qquad \boxed{\dot{m}_{\text{total}} = 0.97\rho\nu\left[\frac{Gr_L}{Pr^2(0.952+Pr)}\right]^{1/4}} \tag{9.43}$$

9.4.2 • Local Nusselt Number

The local heat-transfer coefficient is given by

$$h_x = \frac{-k\left(\dfrac{\partial T}{\partial y}\right)\bigg|_{y=0}}{(T_w - T_\infty)}$$

Since
$$T = T_\infty + (T_w - T_\infty)\left[1 - \left(\frac{y}{\delta}\right)\right]^2$$

$$\left(\frac{\partial T}{\partial y}\right)\bigg|_{y=0} = \left[0 + (T_w - T_\infty)2\left[1 - \frac{y}{\delta}\right] \times \left\{0 - \frac{1}{\delta}\right\}\right]_{y=0} = \frac{-2(T_w - T_\infty)}{\delta}$$

$$h_x = \frac{2k}{\delta} = \frac{2k}{3.93(0.952+Pr)^{1/4}(Pr)^{-1/2}\left(\dfrac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{-1/4} x^{1/4}} = Cx^{-1/4}$$

Local Nusselt number,

$$Nu_x = \frac{h_x x}{k} = \frac{2}{3.93}(0.952+Pr)^{-1/4}(Pr)^{1/2}\left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{1/4} x^{-1/4}\, x$$

$$Nu_x = 0.509(0.952+Pr)^{-1/4}(Pr)^{1/2}\left(\frac{g\beta(T_w - T_\infty)x^3}{\nu^2}\right)^{1/4} \tag{9.44}$$

$$\boxed{Nu_x = 0.509(0.952+Pr)^{-1/4}(Pr)^{1/2}(Gr_x)^{1/4}} \qquad [Gr_x = \textit{local Grashof number}] \tag{9.45}$$

9.4.3 • Average Nusselt Number

Average heat-transfer coefficient is given by

$$\overline{h_L} = \frac{\int_0^L h_x dx}{\int_0^L dx} = \frac{1}{L}\int_0^L h_x dx = \frac{C}{L}\int_0^L x^{-1/4} dx \quad \text{or} \quad \bar{h}_L = \frac{C}{L}\left(\frac{4}{3}L^{3/4}\right) = \frac{4}{3}CL^{-1/4}$$

or
$$\boxed{\bar{h}_L = \frac{4}{3}h_{(x=L)}}$$

The *average* heat-transfer coefficient is thus 4/3 times the *local* heat-transfer coefficient.

It follows that

$$\bar{h}_L = \frac{4}{3} \times 0.509\, k(0.952 + Pr)^{-1/4}\, (Pr)^{1/2} \left(\frac{g\beta(T_w - T_\infty)}{\nu^2} \right)^{1/4} L^{-1/4}$$

$$\boxed{\bar{N}u_L = \frac{\bar{h}_L L}{k} = 0.6786(0.952 + Pr)^{-1/4}\, (Pr)^{1/2}\, (Gr_L)^{1/4}} \tag{9.46a}$$

or

$$\boxed{\bar{N}u_L = 0.6786 \left(\frac{Pr}{0.952 + Pr} \right)^{1/4} Ra_L^{1/4}} \tag{9.46b}$$

where Ra_L is the Rayleigh number defined as

$$Ra_L = \frac{g\beta(T_w - T_\infty)L^3}{\nu\alpha} \quad \text{and} \quad Gr_L = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2} = \text{Grashof number}$$

In the above discussion, the plate was hotter than the surroundings, (i.e., $T_w > T_\infty$). If $T_w < T_\infty$, the same results will be valid but the flow near the plate will be in the downward direction.

9.5 ❑ INTEGRAL METHOD FOR TURBULENT FREE CONVECTION PAST A VERTICAL SURFACE

In a long vertical wall, the *laminar* boundary layer goes through a transition to turbulence in free convection. It is found that for $GrPr > 10^9$, heat transfer from a vertical isothermal plate is controlled by *turbulent* flow. Obviously, the results of laminar-flow free convection on a vertical plate are no longer valid in this case.

Turbulent free convection can be analyzed by an integral method as suggested by *Eckert and Jackson.* Let us rewrite the momentum and energy equations as follows:

$$\frac{d}{dx}\int_0^\delta u^2 dy = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} + g\beta\int_0^\delta (T - T_\infty)dy$$

$$\frac{d}{dx}\int_0^\delta u(T - T_\infty)dy = -\alpha\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

The thicknesses of the velocity and temperature boundary layers are assumed constant. To solve the above two equations, the following expressions for velocity and temperature profiles are used:

$$u = u_x \eta^{1/7}(1 - \eta)^4 \quad \text{and} \quad \frac{T - T_\infty}{T_w - T_\infty} = 1 - \eta^{1/7}$$

where $\quad \eta = y/\delta.$

Replacing the term $-\nu[\{\partial u/\partial y\}_{y\,=\,0}]$ by $-\ \tau_w/\rho$ in the momentum equation and the term $-\alpha[\{\partial T/(\partial y)\}_{y\,=\,0}]$ by $(q_w)/(\rho C_p)$, we have

$$\frac{d}{dx}\int_0^\delta u^2 dy = -\frac{\tau_w}{\rho} + g\beta\int_0^\delta (T - T_\infty)\,dy$$

and

$$\frac{d}{dx}\int_0^\delta (T - T_\infty)u\,dy = \frac{q_s}{\rho C_p}$$

We use the Blasius expression for the shear stress

$$\tau_w = 0.0225\,\rho u_x^2 \left(\frac{\nu}{u_x \delta}\right)^{1/4} \qquad \text{where } u_x \text{ is the characteristic velocity.}$$

Also, using the modified Reynolds' analogy, we get

$$Nu_x = \frac{1}{2} Pr^{1/3}\, Re_x C_{f,x} \tag{9.47}$$

or
$$\frac{q_w x}{k(T_w - T_\infty)} = \frac{1}{2} Pr^{1/3} \left(\frac{u_x x}{\nu}\right)\left(\frac{\tau_w}{\frac{1}{2}\rho u_x^2}\right)$$

or
$$\frac{q_w}{\rho C_p} = Pr^{-2/3} \frac{T_w - T_\infty}{u_x}\left(\frac{\tau_w}{\rho}\right) \tag{9.48}$$

Substituting Eqs. (9.44) and (9.45) into Eqs. (9.46a) and (9.46b), we get

$$0.0523 \frac{d}{dx}(u_x^2 \delta) = 0.125\ g\beta(T_w - T_\infty)\delta - 0.0225 u_x^2 \left(\frac{\nu}{u_x \delta}\right)^{1/4}$$

and
$$0.0336 \frac{d}{dx}(u_x \delta) = 0.0225\ Pr^{-2/3}\, u_x \left(\frac{\nu}{u_x \delta}\right)^{1/4} \tag{9.49}$$

The solutions to the preceding equations may be assumed as

$$u_x = C_1 x^{m_1} \text{ and } \delta = C_2 x^{m_2}$$

Substituting these values in Eqs. (9.48) and (9.49), we get

$$0.0523 \frac{d}{dx}(C_1^2 x^{2m_1} C_2 x^{m_2}) = 0.125\, g\beta(T_s - T_\infty) C_2 x^{m_2} - 0.0225 C_1^2 x^{2m_1} \times \left(\frac{\nu}{C_1 x^{m_1} C_2 x^{m_2}}\right)^{1/4}$$

and
$$0.0366 \frac{d}{dx}(C_1 x^{m_1} C_2 x^{m_2}) = 0.0225\ Pr^{-2/3} (C_1 x^{m_1}) \left(\frac{\nu}{C_1 x^{m_1} C_2 x^{m_2}}\right)^{1/4} \tag{9.50}$$

or
$$0.0523\ C_1^2 C_2 (2m_1 + m_2) x^{2m_1 + m_2 - 1}$$
$$= 0.125\ g(T_w - T_\infty) C_2 x^{m_2} - 0.0225 (C_1 x^{m_1})^{-1/4} (C_2 x^{m_2})^{-1/4} \nu^{1/4} \tag{9.51a}$$

and
$$0.0366\, C_1 C_2 (m_1 + m_2) x^{m_1 + m_2 - 1} = 0.0225\, Pr^{-2/3} (C_1 x^{m_1})^{3/4} (C_2 x^{m_2})^{-1/4} \nu^{1/4} \tag{9.51b}$$

Equating the exponents of x in Eqs. (9.50) and (9.51a), we obtain

$$2m_1 + m_2 - 1 = \frac{7}{4} m_1 - \frac{1}{4} m_2$$

$$m_1 + m_2 - 1 = \frac{3}{4} m_1 - \frac{1}{4} m_2 \quad \text{or} \quad m_1 = \frac{1}{2} \quad \text{and} \quad m_2 = 0.7$$

Substituting these values in Eqs. (9.50) and (9.51a) and solving for C_1 and C_2, we have

$$C_1 = 0.0689\, C_2^{-5}\, \nu Pr^{-8/3}$$

and
$$C_2 = \left[0.00338 \frac{\nu^2}{g\beta(T_w - T_\infty)}(1 + 0.494\, Pr^{2/3} Pr^{-16/3})\right]^{1/10}$$

Using
$$u_x = C_1 x^{1/2} \quad \text{and} \quad Gr_x = \frac{g\beta(T_w - T_\infty)x^3}{\nu^2} \tag{9.52}$$

we finally get $u_x = 1.185 \dfrac{\nu}{x} Gr_x^{1/2} (1 + 0.494\, Pr^{2/3})^{-1/2}$

Similarly,

$$\boxed{\frac{\delta}{x} = 0.565 Gr_x^{-0.1}\, Pr^{-8/15} (1 + 0.494\, Pr^{2/3})^{1/10}} \tag{9.53}$$

$$Re_x = \frac{u_x x}{\nu} = 1.185 Gr_x^{1/2} (1 + 0.494\, Pr^{2/3})^{-1/2}$$

and
$$\frac{1}{2} C_{f,x} = \frac{\tau_w}{\rho u_x^2} = 0.0225 \left(\frac{\nu}{u_x x}\right)^{1/4} \left(\frac{x}{\delta}\right)^{1/4}$$

Substituting the values of Re_x and (1/2) $C_{f,x}$ in Eq. (9.47), the resulting *Nusselt number* is found to be

$$\boxed{Nu_x = 0.0295 Gr_x^{2/5}\, Pr^{7/15} (1 + 0.494\, Pr^{2/3})^{-2/5}} \tag{9.54a}$$

or
$$\boxed{Nu_x = 0.0295 \left[\frac{Pr^7}{(1 + 0.494 Pr^{2/3})^6}\right]^{1/15} Gr_x^{2/5}} \tag{9.54b}$$

This analysis suggests that h varies as $x^{0.2}$.

To obtain an average heat-transfer coefficient, $\bar{h}_L = \dfrac{1}{L}\displaystyle\int_0^L h_x dx$

The average value of Nusselt's number can be obtained as

or
$$\boxed{\begin{aligned} \overline{Nu}_L &= \frac{Nu_{x=L}}{1.2} = 0.0246 Gr_L^{2/5}\, Pr^{7/15} (1 + 0.494 Pr^{2/3})^{-2/5} \\ &= 0.0246 \left[\frac{Ra_L Pr^{1/6}}{1 + 0.494 Pr^{2/3}}\right]^{2/5} \end{aligned}} \tag{9.55}$$

For air, taking Pr = 0.71, we get

$$\boxed{\overline{Nu}_L = 0.0183 Gr_L^{2/5}} \tag{9.56}$$

This equation agrees well with the experimental results.

9.6 ❑ TRANSITION AND TURBULENCE IN NATURAL CONVECTION

We must recognize that the transition form laminar to turbulent condition in both free and forced convection are caused by hydrodynamic and thermal instabilities. In free convection on a vertical plate, transition occurs when the critical Rayleigh number,

$$Ra_{cr} = Gr\,Pr = \frac{g\beta(T_w - T_\infty)x_{cr}^3}{\nu\alpha} \approx 10^9$$

Thus, the distance from the leading edge at which transition occurs is given by

$$\boxed{x_{cr} = 10^9 \times \left[\frac{\nu\alpha}{g\,\beta(T_w - T_\infty)}\right]^{1/3}} \tag{9.57}$$

This transition is illustrated in Fig. 9.10.

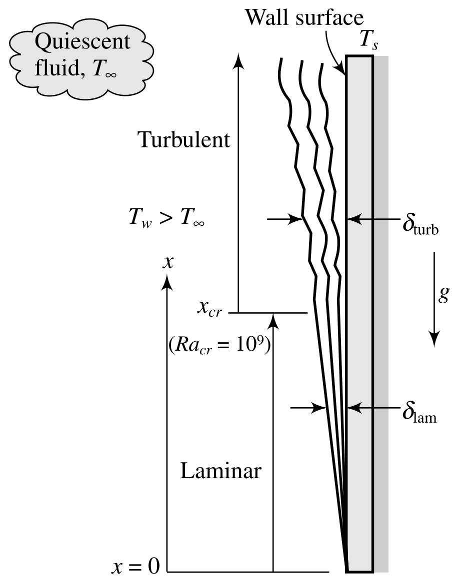


Fig. 9.10 *Transition from laminar flow to turbulent flow in the boundary layer on a vertical flat plate with natural convection on the surface.*

9.7 ❑ VERTICAL PLATE AT CONSTANT TEMPERATURE

Vertical plate is an important geometry since heat transfer from the walls of a furnace can be calculated by the relations applicable to a vertical plate.

McAdams suggested the following relations for fluids whose Prandtl number is close to unity, i.e., for air and other gases, generally:

$$\boxed{\overline{Nu}_L = 0.59\,Ra_L^{1/4}} \quad (10^4 < Ra_L < 10^9) \qquad (\textit{laminar flow}) \tag{9.58}$$

and

$$\boxed{\overline{Nu}_L = 0.10\,Ra_L^{1/3}} \quad (10^9 < Ra_L < 10^{12}) \qquad (\textit{turbulent flow}) \tag{9.59}$$

Properties of fluid at $T_f = \dfrac{1}{2}(T_w + T_\infty)$.

In a plot of *Nu* against *Ra*, the slope is 1/4 in laminar flow and is 1/3 in the turbulent flow as shown in Fig. 9.11.

Note that the average heat-transfer coefficient in the *turbulent* flow regime is independent of the length *L*. *Churchill and Chu* recommend the following more accurate correlations with the length (or height) *L* as the characteristic length and valid for all values of the Prandtl numbers:

$$\boxed{Nu_L = 0.68 + \frac{0.670\,Ra_L^{1/4}}{\left[1 + \left(\dfrac{0.492}{Pr}\right)^{9/16}\right]^{4/9}} \quad (10^{-1} < Ra_L < 10^9)} \qquad (\textit{laminar flow}) \tag{9.60}$$

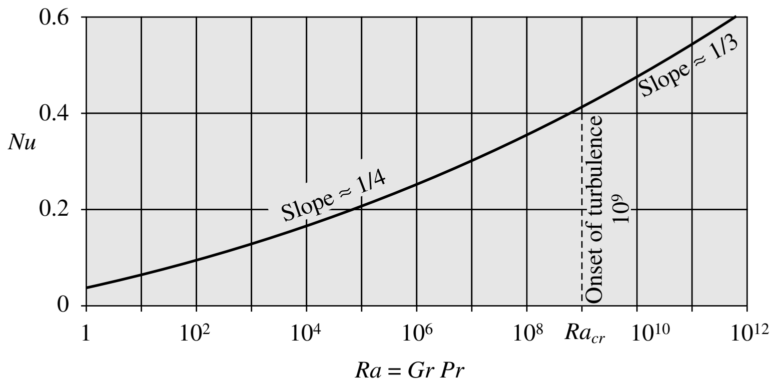


Fig. 9.11 *Free convection from a vertical wall*

$$\boxed{\overline{Nu}_L = \left\{0.825 + \frac{0.387\, Ra_L^{1/6}}{\left[1+\left(\dfrac{0.492}{Pr}\right)^{9/16}\right]^{8/27}}\right\}^2} \quad (\textit{laminar and turbulent flow}) \tag{9.61}$$

(for the entire range of Ra_L up to 10^{12})

For the laminar flow range ($Ra_L < 10^9$), however, slightly more accurate results are obtained if Eq. (9.60) is used instead of Eq. (9.61).

Fluid properties are evaluated at the film temperature $T_f = (T_w + T_\infty)/2$.

- **Vertical Cylinder at Constant Temperature**

A vertical cylinder or tube can be treated as a *vertical plate* (with length or height, L of cylinder as the characteristic dimension), and the correlations for the vertical plate can be applied if the cylinder diameter is large compared to the boundary-layer thickness. To satisfy this condition, the following criterion is satisfied:

$$\boxed{\frac{D}{L} \ge \frac{35}{Ra^{1/4}}} \tag{9.62}$$

9.8 ❑ VERTICAL PLATE WITH CONSTANT HEAT FLUX

The boundary layer is laminar for $Ra < 10^9$. Modified Grashof number, $\boxed{Gr^* = Gr\, Nu}$ is significant in evaluating h under constant heat flux condition.

$$Gr_x^* = Gr_x Nu_x = \frac{g\beta(T_w - T_\infty)x^3}{\nu^2}\,\frac{q_w x}{T_w - T_\infty} = \frac{g\beta q_x x^4}{k\nu^2}$$

and the local Nusselt number, $Nu_x = \dfrac{h_x x}{k}$ where q_w is the constant wall heat flux

The following two relations are recommended for **local** heat-transfer coefficients in *laminar* and *turbulent* ranges respectively:

$$\boxed{Nu_x = 0.60(Gr_x^* Pr)^{0.2}} \quad \text{for} \quad 10^5 < Gr_x^* Pr < 10^{11} \quad (laminar) \tag{9.63}$$

and

$$\boxed{Nu_x = 0.17(Gr_x^* Pr)^{0.25}} \quad \text{for} \quad 2 \times 10^{13} < Gr_x^* Pr < 10^6 \quad (turbulent) \tag{9.64}$$

The average heat-transfer coefficient in the *laminar* region is obtained by integration over the entire height L of the plate as $\boxed{\bar{h} = \frac{5}{4} h_{x=L}}$ and, for the *turbulent* region, h_x is independent of x. $\bar{h} = h_{x=L}$

9.9 ❑ HORIZONTAL PLATE: UNIFORM WALL TEMPERATURE

Figure 9.12 shows the flow pattern from the surface of a horizontal plate under different conditions in free (natural) convection

Upper Surface of Heated Plate/Lower Surface of Cooled Plate

The characteristic length in the Nusselt and Rayleigh numbers for horizontal plate is evaluated as

$$\boxed{L_c = \frac{\text{Plate surface area}}{\text{Perimeter}} = \frac{A_s}{P}}$$

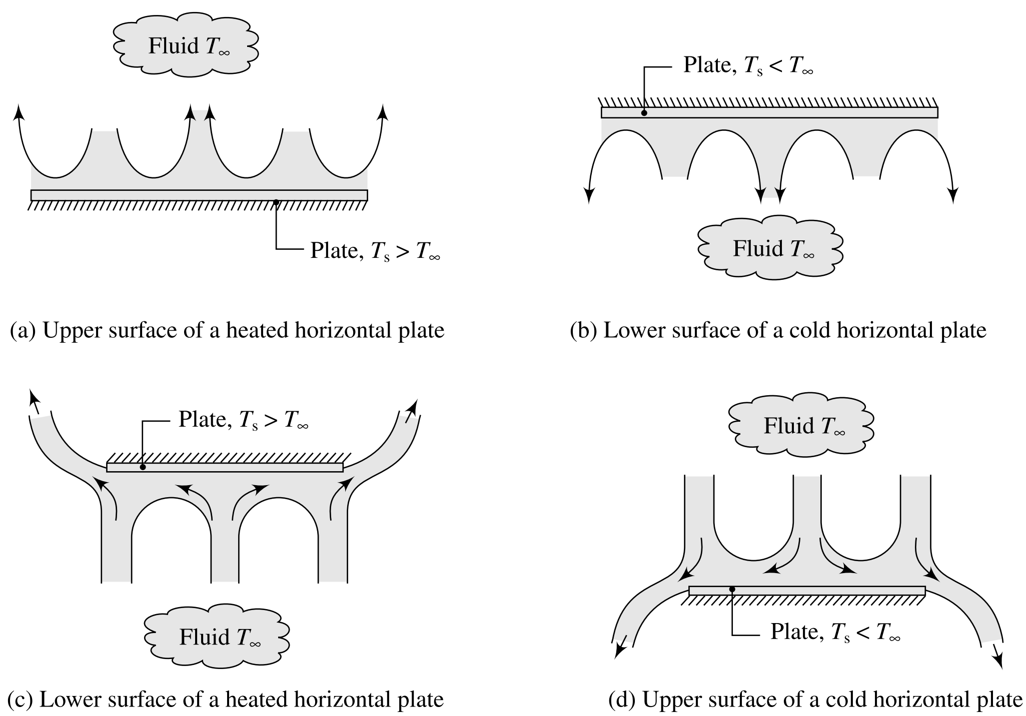


Fig. 9.12 *Natural convection flow patterns for a horizontal plate under different conditions.*

A flat circular disk, for example, would have L_c = D/4 where D is the disk diameter. The correlations recommended by *McAdams* are

$$\boxed{\begin{aligned}\overline{Nu}_L &= 0.54\,Ra_L^{1/4} \quad 2.6 < 10^4 < Ra_L < 10^7 \\ \overline{Nu}_L &= 0.15\,Ra_L^{1/3} \quad 10^7 < Ra_L < 3\times10^{10}\end{aligned}}$$

(*laminar flow*) (9.65)

(*turbulent flow*) (9.66)

Lower Surface of a Heated Horizontal Plate/Upper Surface of a Cooled Plate

$$\boxed{Nu_L = 0.27\,Ra_L^{1/4} \quad 3\times10^5 < Ra_L < 3\times10^{10}} \quad (\textit{laminar flow}) \tag{9.67}$$

$$\left(\text{Properties evaluated at } T_f = \frac{T_w + T_\infty}{2}\right)$$

9.10 ❑ HORIZONTAL PLATE: UNIFORM WALL HEAT FLUX

Heated Surface Facing Upwards

$$\boxed{\begin{aligned}\overline{Nu}_L &= 0.13(Ra_L)^{1/3} \quad Ra_L < 2\times10^8 \\ \overline{Nu}_L &= 0.16(Ra_L)^{1/3} \quad 5\times10^8 < Ra_L < 10^{11}\end{aligned}} \tag{9.68}$$

Heated Surface Facing Downwards

$$\boxed{\overline{Nu}_L = 0.58(Ra_L)^{1/5} \quad 10^6 < Ra_L < 10^{11}} \tag{9.69}$$

Properties to be evaluated at

$$T_f = T_w - 0.25(T_w - T_\infty)$$

except β which should be found at $\frac{1}{2}(T_w + T_\infty)$

9.11 ❑ FREE CONVECTION ON AN INCLINED PLATE

The heat-transfer coefficient for free convection on an inclined plate can be predicted by the vertical plate correlations if the gravitational term in the Grashof number is adjusted to accommodate the effect of the inclination. Thus, for inclined plates (inclined at an angle θ to the vertical), vertical plate relations can be used by replacing g by (g cos θ) for $Ra < 10^9$. *Inclined length L* is the *characteristic dimension*. The orientation of the inclined surface, whether the surface is facing *upwards* or *downwards*, is also a factor that affects the Nusselt number. To make a distinction in the orientation of the surface, we designate the sign of the angle θ that the surface makes with the vertical as follows:

1. The angle θ is considered *negative* if the hot surface is *facing up*, as illustrated in Fig. 9.13(a).
2. The angle θ is considered *positive* if the hot surface is *facing down*, as illustrated in Fig. 9.13(c).

Figure 9.13(b) illustrates the limiting cases of $\theta \to -90°$, the horizontal plate with *hot surface facing upwards*, and $\theta \to +90°$, the horizontal plate with *hot surface facing downwards*.

- **Uniform Wall Heat Flux**

Here, we present the heat-transfer correlations based on the extensive experimental investigations for free convection from an inclined plate subjected to approximately uniform wall flux to water.

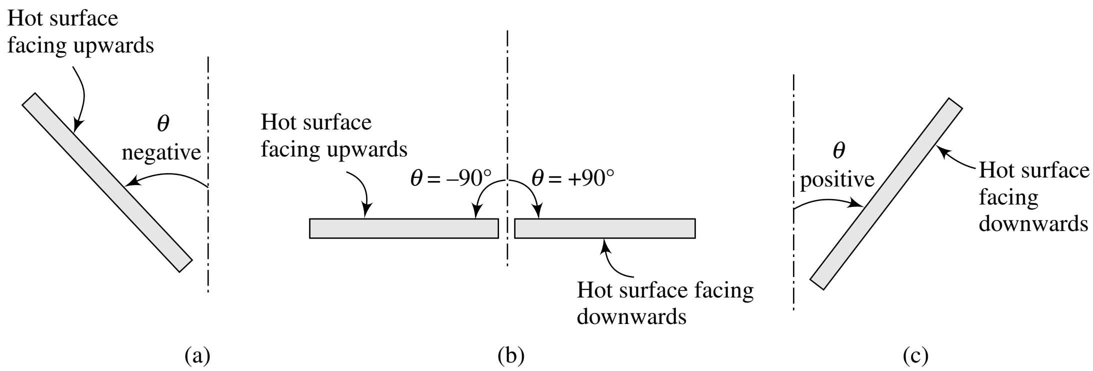


Fig. 9.13 *The concept of positive and negative inclination angles from the vertical to define the orientation of the hot surface.*

For an inclined plate with the *heated surface facting downwards*:

$$\boxed{\overline{Nu} = 0.56(Gr_L\,Pr\cos\theta)^{1/4} \quad \text{for} \quad +\theta < 88°, 10^5 < Gr_L Pr < 10^{11}} \tag{9.70}$$

For the plate slightly inclined with the horizontal (that is, 88° < θ < 90°) and the **heated surface facing downwards**, Eq. (9.70) is applicable. The $\frac{1}{4}$ power in Eq. (9.65) implies that the flow is always in the laminar regime.

For the inclined plate with the *heated surface facing upwards*, the heat-transfer correlation has been developed with the following considerations. It is assumed that Eq. (9.65) is applicable in the laminar flow regime of $Gr_L\,Pr < Gr_c\,Pr$, where Gr_c is is the critical Grashof number at which the transition from laminar to turbulent flow takes place. In the turbulent regime, it is assumed that Eq. (10-68) is applicable if $Gr_L\,Pr$ is replaced by $Gr_L\,Pr\cos\theta$. With this consideration (*from the experimental data of Fujii and Imura*), two expressions can be developed, one involving the coefficient 0.13 based on the results of experiments with the 30 cm test plate and the other involving the coefficient 0.16 based on a 5 cm test plate. Here, we present the average of these two results and give the correlation of free convection on an inclined plate with the *heated surface facing upwards as*

$$\boxed{\overline{Nu} = 0.145[(Gr_L\,Pr)^{1/3} - (Gr_c\,Pr)^{1/3}] + 0.56(Gr_c\,Pr\cos\theta)^{1/4}} \tag{9.71}$$

for $Gr_L\,Pr < 10^{11}$, $Gr_L > Gr_c$, and –15° < θ < –75°. Here, the value of the transition Grashof number Gr_c depends on the angle of inclination θ, as listed in Table 9.1.

In Eqs. (9.65) and (9.66), all physical properties are evaluated at the mean temperature

$$\boxed{T_m = T_w - 0.25(T_w - T_\infty)}$$

and β is evaluated at $\boxed{T_\infty + 0.25(T_w - T_\infty)}$

Table 9.1 *Transition Grashof number (see Fig. 9.12 for the definition of θ)*

θ, degrees	Gr_c
–15	5 × 109
–30	10^9
–60	10^8
–75	10^6

9.12 ❑ HORIZONTAL CYLINDERS

Correlations proposed by *McAdams* with the diameter *D* as the characteristic length are

$$\boxed{\begin{aligned} \overline{Nu}_D &= 0.53(Ra_D)^{1/4} \quad 10^4 < Ra_D < 10^9 \\ \overline{Nu}_D &= 0.13(Ra_D)^{1/3} \quad 10^9 < Ra_D < 10^{12} \end{aligned}} \tag{9.72}$$

The flow pattern on the outside of a long horizontal heated cylinder at the surface temperature of T_s (or T_w) is shown in Fig. 9.14.

The following correlations suggested by *Morgan* for a horizontal cylinder with constant surface temperature are suitable for fluids with Prandtl number, Pr varying between about 0.7 and 3:

$$\boxed{\begin{array}{ll} \overline{Nu}_D = 0.675 Ra_D^{0.058} & 10^{-10} < Ra_D < 10^{-2} \\ \overline{Nu}_D = 1.02 Ra_D^{0.148} & 10^{-2} < Ra_D < 10^{2} \\ \overline{Nu}_D = 0.85 Ra_D^{0.188} & 10^{2} < Ra_D < 10^{4} \\ \overline{Nu}_D = 0.48 Ra_D^{0.25} & 10^{4} < Ra_D < 10^{7} \\ \overline{Nu}_D = 0.125 Ra_D^{1/3} & 10^{7} < Ra_D < 10^{12} \end{array}} \quad (9.73)$$

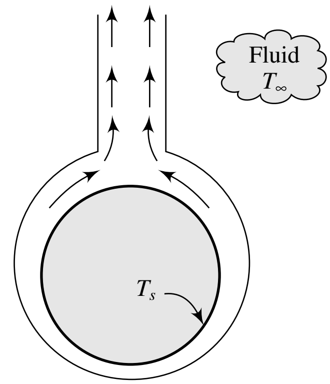


Fig. 9.14 *Natural convection flow over a heated horizontal cylinder*

The following *Churchill–Chu* correlations for the *constant wall temperature* case in the case of a horizontal cylinder with the outer diameter as the characteristic dimension, are the most appropriate and valid for all values of the Prandtl number.

$$\boxed{\overline{Nu}_D = 0.36 + \frac{0.518 Re_D^{1/4}}{[1 + (0.559/Pr)^{9/16}]^{4/9}} \quad (\text{for } 10^{-6} < Ra_D < 10^{9})} \quad (9.74)$$

$$\boxed{\overline{Nu}_D = \left\{ 0.60 + \frac{0.387 Ra_D^{1/6}}{\left\{ 1 + \left(\frac{0.559}{Pr} \right)^{9/16} \right\}^{8/27}} \right\}^2 \quad (\text{for } 10^{9} < Ra_D < 10^{12})} \quad (9.75)$$

Properties in the above equations are evaluated at the film temperature, $T_f \equiv \frac{1}{2}(T_w + T_\infty)$.

For thin wires (D = 0.2 mm to 1 mm), the Rayleigh number is usually very small and a film type of flow pattern is observed. The following correlation is usually used:

$$\boxed{\overline{Nu}_D = 1.18 (Ra_D)^{1/8}} \quad (9.76)$$

Heat transfer form the horizontal cylinders to *liquid metals* may be calculated from

$$\boxed{\overline{Nu}_D = 0.53 (Gr_D Pr^2)^{1/4}} \quad (9.77)$$

9.13 ❑ SPHERE

In the case of a sphere, the diameter D is the characteristic dimension in all correlations. An empirical correlation reported by *Yuge* is

$$\boxed{\overline{Nu}_D = 2 + 0.43 Ra_D^{1/4} \quad 1 < Ra_D < 10^5} \quad \left[\text{Properties at } T_f = \frac{1}{2}(T_w + T_\infty) \right] \quad (9.78)$$

For free convection on a single isothermal sphere in water, the following correlation is proposed:

$$\boxed{\overline{Nu}_D = 2 + 0.50 Ra_D^{1/4} \quad \text{for} \quad 3 \times 10^5 < Ra_D < 3 \times 10^8} \tag{9.79}$$

The average Nusselt number over the entire surface of an isothermal sphere can be determined more accurately from the following *Churchill–Chu* correlation:

$$\boxed{\overline{Nu}_D = 2 + \frac{0.589 Ra_D^{1/4}}{[1 + (0.469/Pr)^{9/16}]^{4/9}}} \qquad \text{provided } Ra_D < 10^{11} \text{ and } Pr > 0.7 \tag{9.80}$$

$$\left[\text{Properties evaluated at } T_f = \frac{1}{2}(T_w + T_\infty)\right]$$

It is worth noting that as ΔT approaches zero, the Nusselt number approaches the value 2 which is the case for pure conduction from the spherical surface in a very large extent of fluid.

9.14 ❑ OBJECTS OF ARBITRARY SHAPE

For a laminar free convection boundary layer on an object of any arbitrary shape, in fluids other than those for which $Pr \ll 1$, *Lienhard* recommends that the average Nusselt number is approximately given by

$$\boxed{\overline{Nu}_L = 0.52 Ra_L^{1/4}} \tag{9.81}$$

where the characteristic length L is the length of the boundary layer, for example, $L = \pi D/2$ for a *cylinder* or *sphere*.

9.15 ❑ FREE CONVECTION FROM RECTANGULAR BLOCKS AND SHORT CYLINDERS

Here, L_H is the horizontal dimension and L_V is the vertical dimension. The appropriate heat-transfer correlation for rectangular blocks is

$$\boxed{Nu_L = 0.55 (Ra_L)^{1/4}} \quad (10^4 < Ra_L < 10^9) \tag{9.82}$$

where the characteristic length L is defined as

$$\boxed{\frac{1}{L} = \frac{1}{L_V} + \frac{1}{L_H} \quad \text{or} \quad L = \frac{L_H L_V}{L_H + L_V}}$$

For *short cylinders* ($D = L$), $\boxed{Nu = 0.775 (Ra)^{0.208}}$ (9.83)

9.16 ❑ SIMPLIFIED EQUATIONS FOR AIR

Since air is the common fluid in most of the free convection problems encountered in practice, it is useful to have simplified relations for those situations.

The simplified air correlations are tabulated in Table 9.2. These correlations can be extended for pressures other than atmospheric (*higher or lower*) by multiplying expressions for h by the following factors where P is in bar:

Laminar: $\left(\dfrac{P}{1.01325}\right)^{1/2}$

Turbulent: $\left(\dfrac{P}{1.01325}\right)^{2/3}$

Table 9.2 *Simplified equations for free convection to air at atmospheric pressure (constant wall temperature)*

Geometry	Characteristic dimension, L_c	Heat-transfer coefficient	Laminar (Range of *Gr Pr*)	Heat-transfer coefficient	Turbulent (Range of *Gr Pr*)
Vertical plate or cylinder	Height L	$h = 1.42\left(\dfrac{\Delta T}{L}\right)^{1/4}$	10^4 to 10^9	$h = 1.31(\Delta T)^{1/3}$	10^9 to 10^{12}
Horizontal cylinder	Outer diameter *D*	$h = 1.32\left(\dfrac{\Delta T}{D}\right)^{1/4}$	10^4 to 10^9	$h = 1.24(\Delta T)^{1/3}$	10^9 to 10^{12}
Horizontal plate: (*Heated surface facing upwards or cooled surface facing downwards*)	Area/ Parameter *A*/*P*	$h = 1.32\left(\dfrac{\Delta T}{L}\right)^{1/4}$	10^5 to 2×10^7	$h = 1.52(\Delta T)^{1/3}$	2×10^7 to 3×10^{10}
Harizontal plate: (*Heated surface facing downwards or cooled surface facing upwards*)	Area/ Parameter (*A*/*P*)	$h = 0.59\left(\dfrac{\Delta T}{L}\right)^{1/4}$	3×10^5 to 3×10^{10}	—	—
Spheres		$h = [2 + 0.392 Gr_D^{1/4}]\dfrac{k}{D}$ for $1 < Gr_D < 10^5$			

9.17 ❑ CORRELATIONS FOR FREE CONVECTION IN ENCLOSED SPACES

So far, our discussion was confined to the fluid that extended indefinitely from the surface—far enough so that the confining surfaces exerted no influence on the heat-transfer situations. In this section, we will consider some correlations for heat exchange between surfaces that are close together so that the boundary layers interact, enclosures in which there are internal natural flows of considerable engineering importance.

The situation is complicated because the fluid in the enclosure usually does not remain stationary. In a vertical enclosure, for example, the fluid adjacent to the hotter surface rises and the fluid adjacent to the cooler one falls, setting off a *rotationary* motion within the enclosure that increases heat transfer through the enclosure.

The characteristics of heat transfer through a horizontal enclosure depend on whether the hotter plate is at the top or at the bottom, as shown in Fig. 9.15. When the hotter plate is at the top, no convection currents develop in the enclosure, since the lighter fluid is always on top of the heavier fluid. Heat transfer in this case is by pure conduction, and we have Nu = 1. When the hotter plate is at the bottom, the heavier fluid will be on top of the lighter fluid, and there will be a tendency for the lighter fluid to

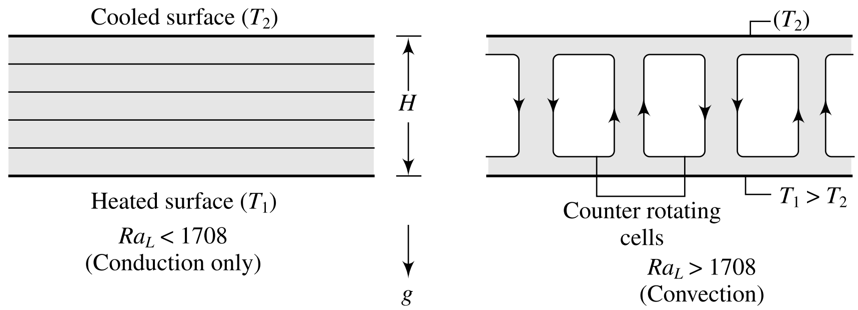


Fig. 9.15 *The flow structure in a horizontal enclosure heated from below*

topple the heavier fluid and rise to the top, where it comes in contact with the cooler plate and cools down. Until that happens, however, heat transfer is still by pure conduction and $Nu = 1$. When $Ra > 1708$, the buoyancy force overcomes the fluid viscous resistance and initiates natural convection currents, which are observed to be in the form of hexagonal cells called **Benard cells**. For $Ra > 3 \times 10^5$, the cells break down and the fluid motion becomes turbulent.

The Rayleigh number for an enclosure is determined from

$$\boxed{Ra_L = \frac{g\beta(T_1 - T_2)L^3}{\nu^2} Pr}$$

where the characteristic length L is the distance between the hot and cold surfaces, and T_1 and T_2 are the temperatures of the hot and cold surfaces, respectively. All fluid properties are evaluated at the average fluid temperature, $\bar{T} = (T_1 + T_2)/2$.

• Effective Thermal Conductivity

When the Nusselt number is known, the rate of heat transfer through the enclosure is

$$\dot{Q} = hA(T_1 - T_2) = kNuA\frac{T_1 - T_2}{L}$$

since $h = k\ Nu\ /\ L$. The rate of steady-state heat conduction across a layer of thickness L, area A, and thermal conductivity k is expressed as

$$\boxed{\dot{Q}_{\text{cond}} = kA\frac{T_1 - T_2}{L}} \tag{9.84}$$

where T_1 and T_2 are the temperatures on the two sides of the layer. A comparison of this relation with Eq. (9.77) reveals that the convection heat transfer in an enclosure is analogous to heat conduction across the fluid layer in the enclosure provided that the thermal conductivity k is replaced by kNu. That is, the fluid in an enclosure behaves like a fluid whose thermal conductivity is kNu as a result of convection currents. Therefore, the quantity kNu is called the *effective thermal conductivity* of the enclosure.

That is,

$$\boxed{k_{\text{eff}} = kNu} \tag{9.85}$$

Note that for the special case of $Nu = 1$, the effective thermal conductivity of the enclosure becomes equal to the thermal conductivity of the fluid. This is expected since this case corresponds to pure conduction.

9.18 ❑ HORIZONTAL RECTANGULAR ENCLOSURES

For horizontal enclosures containing air, based on the plate spacing L, *Jakob* recommends the following correlations:

$$\boxed{\begin{array}{ll} \overline{Nu}_L = 0.195\, Ra_L^{1/4} & 10^4 < Ra_L < 4 \times 10^5 \\ \overline{Nu}_L = 0.068\, Ra_L^{1/3} & 4 \times 10^5 < Ra_L < 10^7 \end{array}} \tag{9.86}$$

These relations can also be used for other gases with $0.5 < Pr < 2$. Using *water*, *silicone oil*, and *mercury* in their experiments, *Globe and Dropkin* suggested the following empirical correlation for horizontal enclosures heated from below.

$$\boxed{\overline{Nu}_L = 0.069\, Ra_L^{1/3}\, Pr^{0.074}} \tag{9.87}$$

(valid for $3 \times 10^5 < Ra_L < 7 \times 10^9$)

The space between the plates, L is the characteristic dimension. All thermophysical properties are evaluated at the average of the two plate temperatures, i.e., $\frac{1}{2}(T_1 + T_2)$. The ratio (L/H) must be sufficiently small so that the effects of the side walls is negligible.

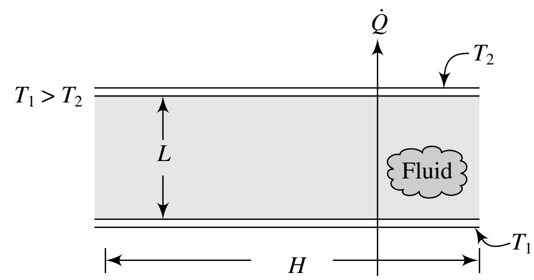


Fig. 9.16 *A horizontal rectangular enclosure with isothermal surfaces*

Based on experiments with air, *Hollands et al*. recommend this correlation for horizontal enclosures,

$$\boxed{\overline{Nu}_L = 1 + 1.44\left[1 - \frac{1708}{Ra_L}\right]^{+} + \left[\frac{Ra_L^{1/3}}{18} - 1\right]^{+} \quad Ra_L < 10^8} \tag{9.88}$$

The notation [ ]+ indicates that if the quantity in the bracket is *negative*, it should be set equal to zero.

9.18.1 • Vertical Rectangular Enclosures

Consider a vertical enclosure in which a fluid contained between two parallel plates of height H and separated by a distance L, as illustrated in Fig. 9.16. The plates are maintained at uniform temperatures T_1 and T_2. H/L is the aspect ratio.

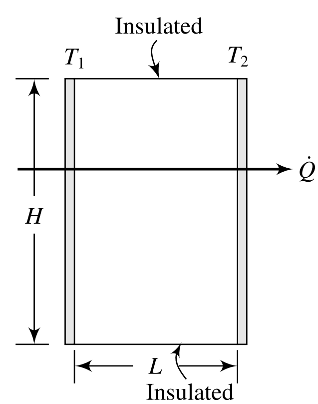


Fig. 9.17 *Vertical enclosure with isothermal surfaces*

- For *small aspect ratios*, with the horizontal surfaces insulated, the following correlations due to *Berkovsky* and *Polevikov* may be used *for fluids of any Prandtl number*.

$$\boxed{\overline{Nu}_L = 0.22\left(\frac{Pr}{0.2 + Pr} Ra_L\right)^{0.28} \left(\frac{H}{L}\right)^{-1/4} ; Ra_L < 10^{10}, 2 < H/L < 10, Pr < 10^5} \tag{9.89}$$

$$\boxed{\overline{Nu}_L = 0.18\left(\frac{Pr}{0.2+Pr}Ra_L\right)^{0.29};\left(\frac{Pr}{0.2+Pr}Ra_L\right) > 10^3, 1 < H/L < 2, 10^{-3} < Pr < 10^5} \tag{9.90}$$

For *larger aspect ratios*, the following correlations due to can be used [*McGregor and Emery*]

$$\boxed{\begin{array}{ll}\overline{Nu}_L = 0.42\,Ra_L^{1/4}\,Pr^{0.012}\left(\dfrac{H}{L}\right)^{-0.3} & 10 < H/L < 40 \\ & 1 < Pr < 2\times 10^4 \\ & 10^4 < Ra_L < 10^7\end{array}} \tag{9.91}$$

$$\boxed{\begin{array}{ll}\overline{Nu}_L = 0.046 Ra_L^{1/3} & 1 < H/L < 40 \\ & 1 < Pr < 20 \\ & 10^6 < Ra_L < 10^9\end{array}} \tag{9.92}$$

- For very large aspect ratios ($5 < H/L < 110$), the correlation due to El Shirbiny et al. given below is recommended.
  Choose the largest of the following, i.e.,

$$\left.\begin{array}{ll} & \overline{Nu}_D = \max\{Nu_1, Nu_2, Nu_3\} \\ \text{where} & \boxed{Nu_1 = 0.0605\,Ra_L^{1/3}} \\ & \boxed{Nu_2 = \left\{1+\left[\dfrac{0.104\,Ra_L^{0.293}}{1+(6310\,Ra_L)^{1.36}}\right]^3\right\}^{1/3}} \\ & \boxed{Nu_3 = 0.242\left(\dfrac{Ra_L}{H/L}\right)^{0.272}}\end{array}\right\} \tag{9.93}$$

These are valid for $10^2 < Ra_L < 2 \times 10^7$

9.18.2 • Inclined Rectangular Enclosures

Air spaces between two inclined parallel plates are typically encountered in flat-plate solar collectors (between the glass cover and the absorber plate). Heat transfer through an inclined enclosure depends on the aspect ratio H / L as well as the tilt angles θ from the horizontal as shown in Fig. 9.17.

- For large aspect ratios, the following correlation due to *Hollands et al.* correlates experimental data extremely well.

$$\boxed{\begin{array}{l}\overline{Nu}_L = 1 + 1.44\left[1-\dfrac{1708}{Ra_L\cos\theta}\right]^+ \\ \left\{1-\dfrac{1708(\sin 1.8\theta)^{1.6}}{Ra_L\cos\theta}\right\}+\left[\left(\dfrac{Ra_L\cos\theta}{5830}\right)^{1/3}-1\right]^+\end{array}} \tag{9.94}$$

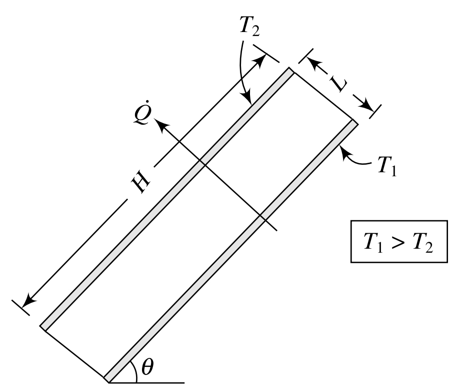


Fig. 9.18 *An inclined rectangular enclosure with large aspect ratio. The angle θ is measured from the horizontal*

For $Ra_L < 10^5$, $0° < \theta < 70°$, and $H / L \geq 12$. Any quantity in $[\ ]^+$ should be set equal to zero if it is *negative*. This is to ensure that $Nu = 1$ for $Ra_L \cos\theta < 1708$. Note that this relation reduces to Eq. (9.88) for *horizontal enclosures* for $\theta = 0°$, as expected.

- For enclosures with smaller aspect ratios ($H / L < 12$), the next correlation can be used provided that the tilt angle is *less* than the critical value θ^* listed in Table 9.3.

$$\boxed{\overline{Nu}_L = \overline{Nu}_{L\,\theta=0°}\left(\frac{\overline{Nu}_{L\,\theta=90°}}{\overline{Nu}_{L\,\theta=0°}}\right)^{\theta/\theta^*}(\sin\theta^*)^{\theta/4\theta^*}} \qquad 0° < \theta < \theta^* \tag{9.95}$$

Table 9.3 *Critical inclination angles for different aspect ratios in inclined rectangular enclosures*

Aspect ratio, (H/L)	1	3	6	12	> 12
Critical tilt angle (θ^*)	25°	53°	60°	67°	70°

For tilt angles *greater* than the critical value ($\theta^* < \theta < 90°$), the Nusselt number can be obtained by multiplying the Nusselt number for a vertical enclosure by $(\sin\theta)^{1/4}$.

$$\boxed{\overline{Nu}_L = \overline{Nu}_{L\,\theta=90°}(\sin\theta)^{1/4} \qquad \theta^* < \theta < 90°, \text{any } H/L} \tag{9.96}$$

- For enclosures tilted more than 90°, the recommended relation is

$$\boxed{\overline{Nu}_L = 1 + (\overline{Nu}_{L\,\theta=90°} - 1)\sin\theta \qquad 90° < \theta < 180°, \text{any } H/L} \tag{9.97}$$

The variation of Nusselt number as a function of tilt angle is qualitatively shown in Fig. 9.18.

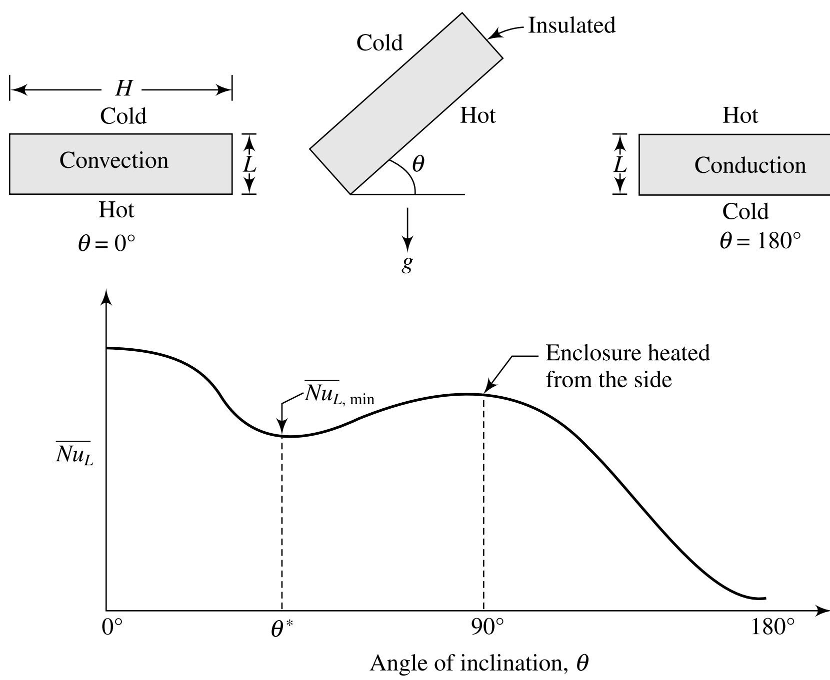


Fig. 9.19 *Effect of inclination angle on natural convection in an inclined enclosure*

9.19 ❑ CONCENTRIC CYLINDERS AND SPHERES

Natural convection in enclosures formed between concentric cylinders and concentric spheres when the gap is filled with various fluids such as air, water, and oils have been correlated by *Railthby and Hollands*.

9.19.1 • Concentric Cylinders

Consider a fluid contained in the annular space between two long concentric horizontal cylinders maintained at uniform but different temperatures of T_i and T_o, as shown in Fig. 9.19. The diameters of the inside and outside cylinders are D_i and D_o, respectively, and the characteristic length is the gap width between the cylinders, $L = \frac{1}{2}(D_o - D_i)$.

$$\boxed{\begin{aligned} \frac{k_{\text{eff}}}{k} &= 0.386\left(\frac{Pr}{0.861+Pr}\right)^{1/4} Ra_{\text{cyl}}^{1/4} \quad 10^2 < Ra_{\text{cyl}} < 10^7 \\ Ra_{\text{cyl}} &= \frac{[\ln(D_o/D_i)]^4}{L^3(D_i^{-3/5}+D_o^{-3/5})^5} Ra_L \end{aligned}} \tag{9.98}$$

where

9.19.2 • Concentric Spheres

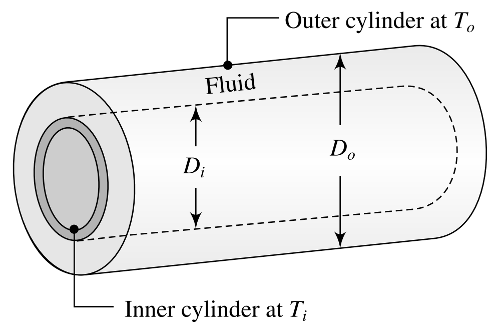


Fig. 9.20 *Natural convection in a concentric cylinder*

For concentric spheres of outside and inside diameters of D_o and D_i and held at uniform temperatures of T_o and T_i, respectively as shown in Fig. 9.20, the total heat-transfer rate through the gap or spacing between the spheres by free convection is given by

$$\boxed{\dot{Q} = k_{\text{eff}} \frac{\pi D_i D_o}{L}(T_i - T_o) \quad \text{(W)}} \tag{9.99}$$

where $L = (D_o - D_i)/2$ is the characteristic length, and the effective thermal conductivity is given by

$$\boxed{\begin{aligned} \frac{k_{\text{eff}}}{k} &= 0.74\left(\frac{Pr}{0.861+Pr}\right)^{1/4} Ra_{\text{sph}}^{1/4} \\ Ra_{\text{sph}} &= \frac{L}{(D_o D_i)^4}\frac{Ra_L}{(D_i^{-7/5}+D_o^{-7/5})^5} \quad 10^2 < Ra_{\text{sph}} < 10^4 \end{aligned}} \tag{9.100}$$

Again, if $k_{\text{eff}} / k < 1$, we should use $k_{\text{eff}} = k$

The heat-transfer rate through the annulus between the cylinders by free convection per unit length is expressed as

$$\boxed{\dot{Q} = \frac{2\pi k_{\text{eff}}}{\ln(D_o/D_i)}(T_i - T_o) \quad \text{(W/m)}} \tag{9.101}$$

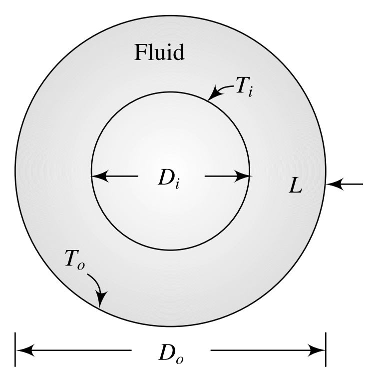


Fig. 9.21 *Two concentric isothermal spheres*

where the effective thermal conductivity k_{eff} is given by the following correlation.

For $Ra_{L,\text{ cyl}} < 100$, natural convection currents are negligible and if k_{eff}/k is less than one then the process is one of pure conduction in the fluid and $k_{\text{eff}} = k$ should be used. The fluid properties are to be evaluated at the average temperature $(T_i + T_o)/2$.

9.20 ❑ NATURAL CONVECTION IN TURBINE ROTORS, ROTATING CYLINDERS, DISKS AND SPHERES

Significance in the thermal *shafting, flywheels, turbine rotors*, and machinery. In high-speed rotating components of compressors and turbines, the centrifugal force field is much larger compared to gravitational force components of rotating blades.

Cooling of Turbine Blades Free convection can occur even under the influence of centrifugal force. Note that the acceleration due to gravity g is replaced by the centrifugal acceleration $(\omega^2 r_m)$.

Grashof number is

$$\boxed{Gr_L = \frac{(\omega^2 r_m)\beta \Delta T L^3}{\nu^2}} \tag{9.102}$$

where L is the length of the cooling passage.

The following correlation is used to estimate the heat-transfer coefficient

$$\boxed{\overline{Nu} = \frac{\bar{h}L}{k} = 0.0246\left[\frac{Pr^{1.17}\,Gr_L}{1+0.495\,Pr^{2/3}}\right]^{0.4}} \quad \text{(turbulent flow) } Gr_L > 10^{12} \tag{9.103}$$

The heat-transfer rate is given by

$$\boxed{\dot{Q} = \bar{h}(\pi D L)(T_w - T_\infty)} \tag{9.104}$$

where D and L are the diameter and length of the hole drilled from the root to the tip of the blade for cooling, T_w is the surface temperature of the hole, and T_∞ the coolant temperature.

Rotating Cylinders Natural convection from a heated horizontal rotating cylinder is characterized by the Grashof number,

$$\boxed{Gr_D = g\beta(T_w - T_\infty)D^3/\nu^2}, \quad \text{which becomes the controlling parameter.}$$

At speeds greater than critical ($Re_\omega > 8000$ in air), where the peripheral-speed Reynolds number: $Re_\omega = \dfrac{\omega \pi D^2}{\nu}$

The following empirical correlation is used for the average Nusselt number in natural convection above critical velocity:

$$\boxed{\overline{Nu}_D = \frac{\bar{h}D}{k} = 0.11(0.5\,Re_\omega^2 + Gr_D\,Pr)^{0.35}} \tag{9.105}$$

Rotating Disk At rotational speeds, Reynolds number $\omega\, D^2/\nu$ is below 10^6. The boundary layer on the disk is laminar and is of uniform thickness. D is the disk diameter.

The average *Nusselt number* for a disk rotating in air in laminar flow regime is

$$\boxed{Nu_D = \frac{hD}{k} = 0.36\left(\frac{\omega D^2}{\nu}\right)^{1/2} \qquad (\text{for } \omega D^2/\nu < 10^6)} \tag{9.106}$$

For *turbulent* flow regime, the local *Nusselt number* at a radius r is given approximately by

$$\boxed{Nu_r = \frac{hr}{k} = 0.0195\left(\frac{\omega r^2}{\nu}\right)^{0.8}} \tag{9.107}$$

Laminar flow is between $r = 0$ and $r = r_c$, and turbulent flow between $r = r_c$ and $r = r_o$, in the outer ring. The average Nusselt number can be estimated from

$$\boxed{\overline{Nu}_r = \frac{\bar{h}\, r_o}{k} = 0.015\left(\frac{\omega r_o^2}{\nu}\right)^{0.8} - 100\left(\frac{r_c}{r_o}\right)^2} \tag{9.108}$$

Rotating Sphere For $Pr > 0.7$, in laminar flow regime ($Re_\omega = \omega D^2/\nu < 5 \times 10^4$), the average Nusselt number can be evaluated from

$$\left.\begin{array}{ll} & \boxed{\overline{Nu}_D = 0.43\, Re_\omega^{0.5}\, Pr^{0.4} \qquad (Re_\omega < 5 \times 10^4)} \\ \text{and} & \boxed{\overline{Nu}_D = 0.066\, Re_\omega^{0.67}\, Pr^{0.4} \quad (5 \times 10^4 < Re_\omega < 7 \times 10^5)} \end{array}\right\} \tag{9.109}$$

9.21 ❑ COMBINED NATURAL AND FORCED CONVECTIONS

In many practical situations, natural and forced convection may occur together. At high velocities, forced convection may be predominant, but at low velocities, the effect of natural convection also must be included. Furthermore, natural and forced convections may occur in the same direction or they may act in the opposite directions. The following criteria are used to determine if the combined free and forced convection is to be considered.

$$Gr_L/(Re_L^2) << 0.1 \qquad (\textit{forced convection regime})$$

$$Gr_L/(Re_L^2) >> 10 \qquad (\textit{free convection regime})$$

$$0.1 < Gr_L/(Re_L^2) \approx 10 \qquad (\textit{mixed (free and forced) convection regime})$$

The effect of buoyancy in the forced convection heat transfer is profoundly influenced by the direction of the buoyancy force relative to that of the forced flow. *Three* special cases are presented here as shown in Fig. 9.22.

Assisting Flow (e.g., *Upward Forced Flow over a Hot Surface*)
In this case, the buoyancy-induced motion is in the *same* direction as the forced motion. Thus, both free and forced convection help each other to increase heat transfer.

Opposing Flow (e.g., *Upward Forced Flow over a Cold Surface*). In this case, the buoyancy-induced motion is in the direction *opposite* to that of forced motion. Thus, free convection opposes forced convection resulting in reduced heat transfer.

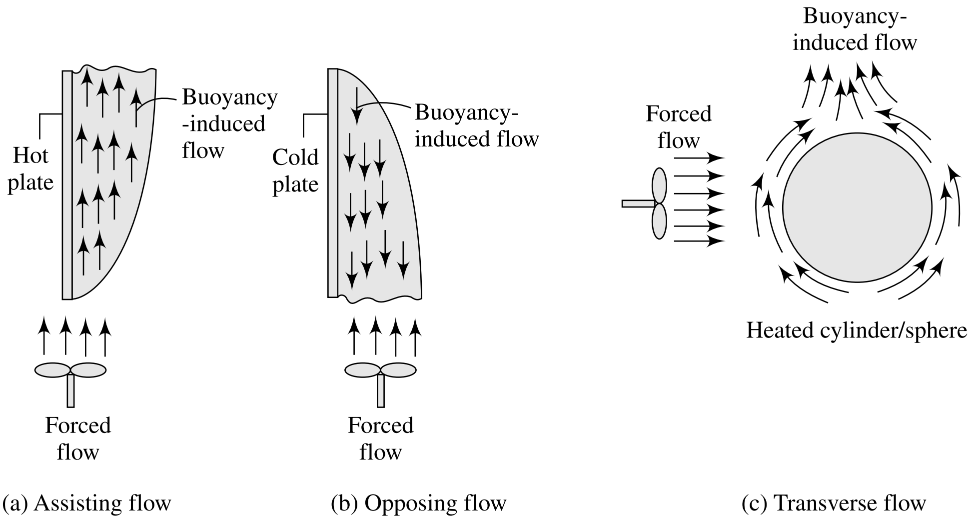


Fig. 9.22 *Natural convection can increase or prevent heat transfer, depending on the relative directions of buoyancy-induced motion and the forced convection motion.*

Transverse Flow (e.g., *Horizontal Forced Flow over a Hot or Cold Cylinder/Sphere*). In this case, the buoyancy-induced motion is perpendicular to the forced motion. This type of flow facilitates fluid mixing, thereby improving heat transfer.

In the mixed convection regime, the following expression is used to calculate the Nusselt number:

$$\boxed{Nu^m = Nu^m_{\text{forced}} \pm Nu^m_{\text{free}}} \tag{9.110}$$

where the *first* and *second* terms on the right-hand side are the Nusselt numbers for the *forced* and *free* convection respectively. A value of $m = 3$ is generally recommended although a value of $m = 3.5$ or 4 is preferred for transverse flows. *Positive* sign is used for assisting and transverse flow while the negative sign is used for opposing flow.

Internal Flow The following correlations are used for the average Nusselt number in the case of internal flow.

For mixed convection through a horizontal tube with *laminar flow* ($Re_D \le 2000$): (correlation due to *Brown and Gauvin*):

$$\boxed{Nu_D = 1.75[Gz + 0.012(Gz\,Gr_D^{1/3})^{4/3}]^{1/3}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}} \tag{9.111}$$

where μ_b and μ_w are the viscosities of the fluid at the bulk mean temperature and the wall (*surface*) temperature respectively, and Gz is the Graetz number, given by $Gz = Re_D Pr\left(\frac{D}{L}\right)$

For mixed convection *turbulent flow* (correlation due to *Metais and Eckert*):

$$\boxed{Nu_D = 4.69\,Re_D^{0.27}\,Pr^{0.21}\,Gr_D^{0.07}\left(\frac{D}{L}\right)^{0.36}} \tag{9.112}$$

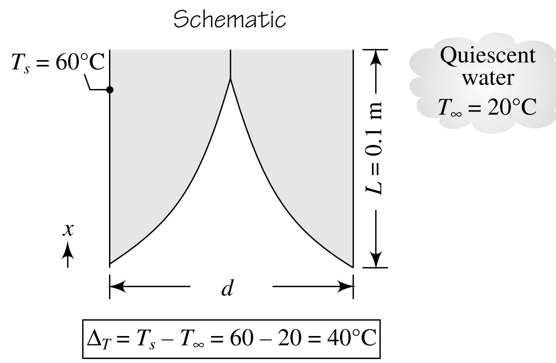
Illustrative Examples

(A) Exact Analysis, Integral Solution

EXAMPLE 9.1 *Thin vertical plates, 10 cm long, initially at 60°C are suspended in a water bath maintained at 20°C. What minimum spacing would prevent interference between their free convection boundary layers?*

Properties of water at 40°C are
$\nu = 0.659 \times 10^{-6}$ m²/s, $\beta = 3.87 \times 10^{-4}$ K⁻¹, Pr = 4.31

Solution



Known Thin vertical plates are cooled in a water bath.

Find Minimum spacing, d, to prevent interference between boundary layers.

Assumptions (1) Water bath is still (*quiescent*). (2) Isothermal plates.

Analysis The minimum separation distance will be *twice* the thickness of the boundary layer at $x = L = 0.10$ m.

Grashof number,

$$Gr_x = \frac{g\beta\Delta T x^3}{\nu^2} = \frac{9.81\,\text{m/s}^2 \times 3.87 \times 10^{-4}\,\text{K}^{-1} \times 40\,\text{K} \times 0.1^3\,\text{m}^3}{(0.659 \times 10^{-6}\,\text{m}^2/\text{s})^2} = 3.497 \times 10^8$$

Transition occurs at $Gr\,Pr \cong 10^9$

$$Gr_{x=L}\,Pr = 3.497 \times 10^8 \times 4.31 = 1.507 \times 10^9$$

This value is greater than 10^9, indicating turbulent boundary layer.

For *turbulent* boundary layer (*by integral method*),

$$\delta_{\text{turb}} = \frac{0.565\,x[1 + 0.494\,Pr^{2/3}]^{1/10}}{Gr^{0.1}\,Pr^{8/15}} = \frac{0.565 \times 0.1[1 + 0.494(4.31)^{2/3}]^{0.1}}{(3.497 \times 10^8)^{0.1} \times (4.31)^{8/15}}$$

$$= 3.94 \times 10^{-3}\,\text{m} \quad \text{or} \quad 3.94\,\text{mm}$$

∴ Minimum spacing, $d = 2\delta$ = **7.88 mm** **(Ans.)**

EXAMPLE 9.2 *A 25 cm long glass plate maintained at 350 K is suspended vertically in the still atmospheric air at 300 K. Estimate (a) the boundary layer thickness at the trailing edge of the plate, and (b) the average heat-transfer coefficient.*

If a similar plate is placed in a wind tunnel and air at 1 atm is blown over it at a free steam velocity of 5 m/s, under identical steady operating conditions, estimate (c) the boundary layer thickness at the trailing edge of the plate, and (d) the average heat-transfer coefficient.

The properties of air at 1 atm and 325 K are
k = 0.0282 W/m K, $\nu = 18.41 \times 10^{-6}$ m²/s, Pr = 0.703

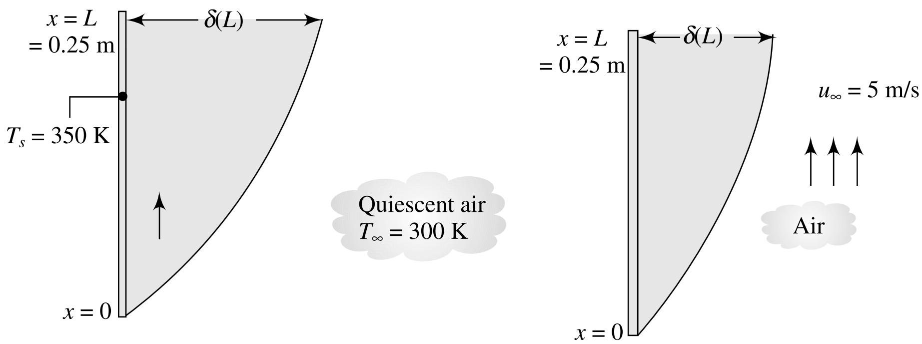
Solution

Known A vertical heated plate losing heat under natural convection and forced convection conditions.

Find ***Free convection*:** (a) $\delta(L)$, mm (b) $\overline{h}_L$, W/m^2 K

***Forced convection*:** (c) $\delta(L)$, mm (d) $\overline{h}_L$, W/m^2 K

Schematic



(a) Free convection (b) Forced convection

Assumptions (1) Steady-state conditions. (2) Air is an ideal gas. (3) Negligible radiation. (4) Buoyancy effects are negligible when u_∞ = 5 m/s.

Analysis ***Free convection*: (*Still air*)**

Film temperature, $T_f = \frac{1}{2}(T_s + T_\infty) = \frac{1}{2}(350 + 300) = 325$ K

With $\beta_{\text{ideal gas}} = \frac{1}{T_f(\text{K})} = \frac{1}{325}(\text{K}^{-1})$, for the quiescent air,

the Grashof number is

$$Gr_x = \frac{g\beta x^3(T_s - T_\infty)}{\nu^2} = \frac{(9.81\,\text{m/s}^2)[1/(325\,\text{K})](x, m)^3(350 - 300)\text{K}}{(18.41\times10^{-6}\,\text{m}^2/\text{s})^2} = 4.453\times10^9\times x^3$$

At the trailing edge of the plate, $x = L = 0.25$ m.

$$\therefore \quad Gr_{x=L} = (4.453\times10^9)(0.25)^3 = 6.958\times10^7$$

Rayleigh number, $Ra_L = Gr_L Pr = (6.958\times10^7)(0.703) = 4.891\times10^{7}$

Since $Ra_L < 10^9$, the free convection boundary layer is *laminar*.

The thickness of boundary layer is given by, $\frac{\delta}{x} = \frac{3.93(0.952 + Pr)^{1/4}}{Gr_x^{1/4}\,Pr^{1/2}}$

$\therefore$ At the trailing edge of the plate,

$$\delta(L) = \frac{3.93L(0.952 + Pr)^{1/4}}{Gr_L^{1/4}\quad Pr^{1/2}} = \frac{3.93\times0.25(0.952 + 0.703)^{1/4}}{(6.958\times10^7)^{1/4}(0.703)^{1/2}} = 0.0145\text{ m}$$

$= \mathbf{14.5\ mm}$ **(Ans.) (a)**

Local heat-transfer coefficient at $x = L$ (*At the trailing edge of the plate*) is

$$h_{x=L} = \frac{k}{L} Nu_{x=L} = \frac{k}{L} \times \frac{0.508\, Pr^{1/2} Gr_L^{\ 1/4}}{(0.952 + Pr)^{1/4}}$$

$$= \frac{0.0282 \text{ W/m°C}}{0.25 \text{ m}} \times \frac{0.508(0.703)^{1/2}(6.958 \times 10^7)^{1/4}}{(0.952 + 0.703)^{1/4}} = 3.87 \text{ W/m}^2\text{°C}$$

Average heat-transfer coefficient,

$$\bar{h}_L = \frac{4}{3} h_{x=L} = \frac{4}{3} \times 3.87 = \mathbf{5.16 \ W/m^2\ K} \qquad \textbf{(Ans.) (b)}$$

***Forced Convection*: ** Reynolds number, for air flow at u_∞ = 5 m/s is

$$Re_L = \frac{u_\infty L}{\nu} = \frac{(5 \text{ m/s})(0.25 \text{ m})}{18.41 \times 10^{-6} \text{ m}^2\text{/s}} = 6.79 \times 10^4 \qquad (< 5 \times 10^5)$$

and the boundary layer is *laminar*.

Boundary-layer thickness at the trailing edge is

$$\delta(L) \approx \frac{5L}{Re_L^{1/2}} = \frac{(5.0)(0.25 \text{ m})}{\sqrt{6.79 \times 10^4}} = \mathbf{4.8 \times 10^{-3} \ m \quad or \quad 4.8\ mm} \qquad \textbf{(Ans.) (c)}$$

Average Nusselt number, $\bar{N}u_L = 0.664\, Re_L^{1/2} Pr^{1/3} = \dfrac{\bar{h}L}{k}$

$\therefore$ Average heat-transfer coefficient,

$$\bar{h}_L = \bar{N}u_L \frac{k}{L} = \frac{0.0282 \text{ W/m K}}{0.25 \text{ m}} \times 0.664(6.79 \times 10^4)^{1/2}(0.703)^{1/3}$$

$$= \mathbf{17.35 \ W/m^2\ K} \qquad \textbf{(Ans.) (d)}$$

It is obvious that the boundary-layer thickness in forced convection is much *smaller* than that in free convection while the average heat-transfer coefficient in forced convection is *much larger* than that in free convection.

Comment The assumption of negligible buoyancy effects for u_∞ = 5 m/s is justified because

$$(Gr_L / Re_L^2) = \frac{6.958 \times 10^7}{(6.79 \times 10^4)^2} = 0.015 << 1.$$

EXAMPLE 9.3 *Consider a hot vertical plate, 30 cm high and 1.0 m wide, with a uniform surface temperature of 127°C suspended in quiescent air at 27°C and atmospheric pressure. Using the integral (approximate) solution, estimate the following:*

(a) Maximum velocity at 15 cm from the lower edge of the plate (b) Mean velocity at 15 cm from the lower edge of the plate (c) Boundary-layer thickness at 15 cm from the lower edge of the plate (d) Local heat-transfer coefficient at 15 cm from the lower edge of the plate (e) Average heat-transfer coefficient over the entire surface of the plate (f) Total mass flow through the boundary (g) Total heat loss from both sides of the plate (h) Rise in temperature of the air passing through the boundary.

Properties of dry air at atmospheric pressure and film temperature of 77°C are

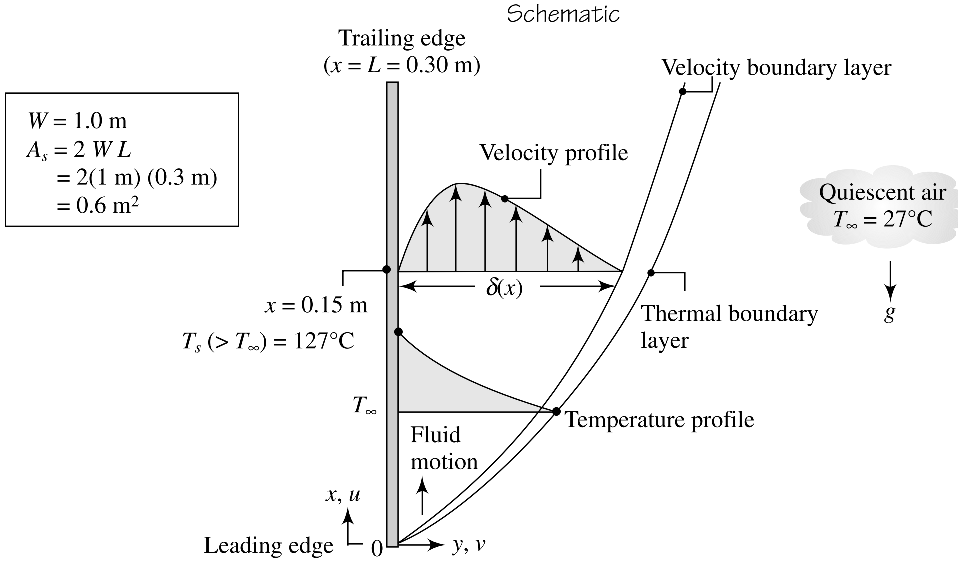
k = *0.03 W/m K,* Pr = *0.700,* ρ = *0.955 kg/m³,* ν = *20.92 × 10⁻⁶ m²/s,* C_p = *1.005 kJ/kg K*

Solution

Known A hot vertical plate is exposed to quiescent air.

Find (a) u_{max}, (b) u_{av}, (c) δ (x), (d) h (x) (at x = 0.15 m). (e) h_L, (f) $\dot{m}_{tot}$, (g) $\dot{Q}$, (h) ΔT_{air}

Schematic



Assumptions (1) Steady operating conditions. (2) Air is an ideal gas. (3) Local atmospheric pressure is 1 atm.

Analysis (a) With $\beta_{\text{ideal gas}} = \dfrac{1}{T_f(K)}$, Grashof number (at x = 0.15 m),

$$Gr_x = \frac{g\beta x^3(T_s - T_\infty)}{\nu^2} = \frac{(9.81\text{ m/s}^2)[1/(77+273.15)\text{K}](0.15\text{ m})^3(127-27)°\text{C} \quad \text{or} \quad \text{K}}{(20.92\times 10^{-6}\text{ m}^2/\text{s})^2}$$

$$= 2.16 \times 10^7$$

and $$Gr_L(\text{at } x = L = 0.30\text{ m}) = 2.16\times 10^7 \times \left(\frac{0.30}{0.15}\right)^3 = 1.728\times 10^8$$

Maximum velocity based on integral solution is, $u_{max} = \dfrac{4}{27}u_x$

where $$u_x = 5.17\left[\frac{g\beta(T_s - T_\infty)}{Pr + 0.952}\right]^{1/2} x^{1/2} = 5.17\sqrt{Gr_x}\times\frac{\nu}{x}\times\frac{1}{(Pr+0.952)^{1/2}}$$

$$= 5.17\sqrt{2.16\times 10^7}\times\frac{20.92\times 10^{-6}}{0.15}\times\frac{1}{(0.700+0.952)} = 2.03\text{ m/s}$$

Hence, $$u_{max} = \frac{4}{27}\times 2.03 = \mathbf{0.30\ m/s}$$ **(Ans.) (a)**

(b) Mean velocity,

$$u_{av}(\text{at } x = 0.15 \text{ m}) = \frac{27}{48}u_{\max} = \left(\frac{27}{48}\right)(0.30) = \mathbf{0.169 \ m/s} \qquad \textbf{(Ans.) (b)}$$

(c) Boundary layer thickness, $\delta(x)$ (at x = 0.15 m) is

$$\delta(x) = 3.93x\left[\frac{0.952 + Pr}{Gr_x \ Pr^2}\right]^{1/4} = 3.93 \times 0.15 \text{ m} \times \left[\frac{0.952 + 0.700}{2.16 \times 10^7 \times 0.7^2}\right]^{1/4}$$

$$= \mathbf{0.0117 \ m \quad or \quad 11.7 \ mm} \qquad \textbf{(Ans.) (c)}$$

(d) Local Nusselt number,

$$Nu_x = 0.508\,Pr^{1/2}(0.952 + Pr)^{-1/4}\,Gr_x^{1/4} = \frac{2x}{\delta} = \frac{h_x x}{k}$$

Hence, local heat-transfer coefficient at x = 0.15 m,

$$h_x = \frac{2k}{\delta} = \frac{2 \times 0.03 \text{ W/m K}}{0.0117 \text{ m}} = \mathbf{5.12 \ W/m^2 \ K} \qquad \textbf{(Ans.) (d)}$$

(e) Average heat-transfer coefficient over the entire plate,

$$\bar{h}_L = \overline{Nu}_L \frac{k}{L}$$

$$\overline{Nu}_L = 0.677\,Gr_L^{1/4} Pr^{1/2}(0.952 + Pr)^{-1/4}$$

$$= 0.677(1.728 \times 10^8)^{1/4}(0.7)^{1/2}(0.952 + 0.7)^{-1/4} = 57.3$$

$$\therefore \qquad \bar{h}_L = 57.3 \times \frac{0.03 \text{ W/m K}}{0.30 \text{ m}} = \mathbf{5.73 \ W/m^2 \ K} \qquad \textbf{(Ans.) (e)}$$

(f) Total mass flow through the boundary is

$$\dot{m}_{\text{total}} = 1.7\rho\nu W\left[\frac{Gr_L}{Pr^2(Pr + 0.952)}\right]^{1/4}$$

$$= 1.7 \times 0.955 \text{ kg/m}^3 \times 20.92 \times 10^{-6} \text{ m}^2\text{/s} \times 1.0 \text{ m}\left[\frac{1.728 \times 10^8}{0.7^2(0.7 + 0.952)}\right]^{1/4}$$

$$= \mathbf{0.0041 \ kg/s} \qquad \textbf{Ans (f)}$$

(g) Total heat loss from *both* sides of the plate is

$$\dot{Q} = \bar{h}_L(2WL)(T_s - T_\infty) = (5.73 \text{ W/m}^2 \text{ K})(2 \times 0.3 \text{ m} \times 1.0 \text{ m})(127 - 27)°\text{C} \quad \text{or} \quad \text{K}$$

$$= \mathbf{343.8 \ W} \qquad \textbf{(Ans.) (g)}$$

(h) Rise in temperature of air passing through the boundary is calculated from

$$\dot{Q} = \dot{m}_{\text{total}} C_p \Delta T_{\text{air}}$$

$$\therefore \qquad \Delta T_{\text{air}} = \frac{343.8 \text{ W or J/s}}{0.0041 \text{ kg/s} \times 1009 \text{ J/kg K}} = \mathbf{83.1°C} \qquad \textbf{(Ans.) (h)}$$

(B) Vertical Plate/Tube

EXAMPLE 9.4 *A 50 cm high vertical plate is at a constant surface temperature of 120°C and exposed to quiescent air at 30°C. Determine the rate of heat-transfer per unit width by (a) the exact solution, (b) the integral solution, (c) the McAdam's correlation, and (d) the Churchill Chu correlation.*

Properties of air at the mean film temperature of 75°C are

k = *0.029 17 W/m °C*, ν = *20.46 × 10⁻⁶ m²/s*, Pr = *0.7166*

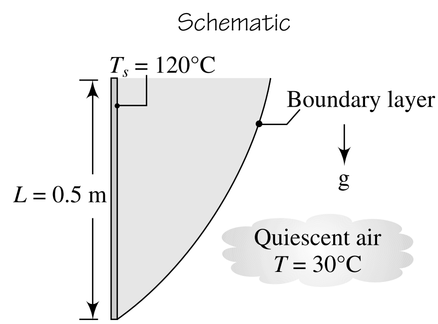
Solution

Known An isothermal vertical plate exposed to still cold air.

Find Heat transfer per metre width by four different methods.

Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Air is an ideal gas at 1 atm.

Analysis Isobaric coefficient of volumetric thermal expansion, for an ideal gas,



$$\beta = \frac{1}{T_f} = \frac{1}{(75+273.15)\text{K}} = \frac{1}{348.15}\text{K}^{-1}$$

Grashof number, with plate height as the characteristic length, L is

$$Gr_L = \frac{g\beta L^3(T_s - T_\infty)}{\nu^2} = \frac{(9.81\,\text{m/s}^2)[1/(348.15\,\text{K})](0.5\,\text{m})^3(120-30)°\text{C or K}}{(20.46\times10^{-6}\,\text{m}^2/\text{s})^2}$$

$$= 7.5726\times10^8$$

and $$Ra_L = Gr_L Pr = (7.5726\times10^8)(0.7166) = 5.4265\times10^8$$

The value of Ra_L is *less* than 10^9. Hence the flow is *laminar*.

(a) ***Exact solution*:** The average Nusselt number for a vertical plate of length L is given by

$$\overline{Nu}_L = 0.64\,Gr_L^{1/4}Pr^{1/2}(0.861+Pr)^{-1/4}$$

$$= 0.64(7.5726\times10^8)^{0.25}(0.7166)^{0.5}(0.861+0.7166)^{-0.25} = 80.2$$

Average heat-transfer coefficient,

$$\bar{h} = \overline{Nu}_L\frac{k}{L} = 80.2\times\frac{0.02917\,\text{W/m°C}}{0.5\,\text{m}} = 4.68\,\text{W/m}^2\,°\text{C}$$

Heat-transfer rate per m width is

$$\dot{Q} = \bar{h}(L\times1)(T_s - T_\infty) = (4.68\,\text{W/m}^2\,°\text{C})(0.5\,\text{m}\times1\,\text{m})(120-30)°\text{C}$$

$$= \mathbf{210.6\,W}$$ **(Ans.) (a)**

(b) ***Integral solution*:**

$$\overline{Nu}_L = 0.68\,Gr_L^{1/4}Pr^{1/2}(0.952+Pr)^{-1/4}$$

$$= 0.68(7.5726\times10^8)^{1/4}(0.7166)^{1/2}(0.952+0.7166)^{-1/4} = 83.9$$

Heat rate per unit width is

$$\dot{Q} = \bar{h} L (T_s - T_\infty) = \overline{Nu_L} \frac{k}{L} L (T_s - T_\infty) = \overline{Nu}\, k (T_s - T_\infty)$$

$$= (83.9)(0.02917)(120 - 30) = \mathbf{220.3\ W} \qquad \textbf{(Ans.) (b)}$$

(c) ***McAdam's correlation***:

$$\overline{Nu_L} = 0.59 (Ra_L)^{1/4} = 0.59 (5.4265 \times 10^8)^{0.25} = 90.0$$

∴ Heat transfer per metre width is

$$\dot{Q} = \overline{Nu_L}\, k (T_s - T_\infty) = (90.0)(0.02917)(90) = \mathbf{236.3\ W} \qquad \textbf{(Ans.) (c)}$$

(d) ***Churchill–Chu correlation***:

$$\overline{Nu_L} = 0.68 + \frac{0.67 Ra_L^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{4/9}}$$

$$\overline{Nu_L} = 0.68 + \frac{0.67 (5.4265 \times 10^8)^{1/4}}{[1 + (0.492/0.7166)^{9/16}]^{4/9}} = 79.25$$

∴ Heat-loss rate per *m* width is

∴ $$\dot{Q} = \overline{Nu_L}\, k (T_s - T_\infty) = (79.25)(0.02917)(120 - 30) = \mathbf{208.0\ W} \qquad \textbf{(Ans.) (d)}$$

Comment Values of heat-transfer rate predicted by the *exact analysis* and the *Churchill–Chu correlation* are quite close to each other, and are more accurate and appropriate. The approximate *integral method* and the *McAdam's correlation* predict higher values, and their use should not be preferred.

EXAMPLE 9.5 *A vertical plate, 10 cm in height and 30 cm long, is used to cool some electronic components mounted on it. The energy to be dissipated is 10 W and the plate temperature should not exceed 50°C. Assuming that the plate loses heat from only one side to ambient air at 30°C, determine whether a blower is necessary or not.*

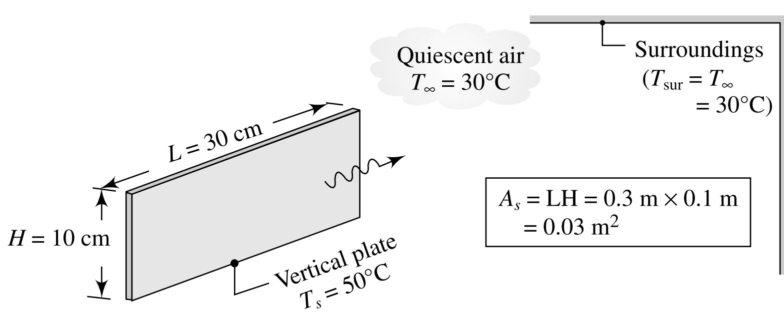
Properties of air at 40°C: k *= 0.02662 W/m °C,* ν *= 17.02 × 10*$^{-6}$ *m*^{2}*/s,* Pr *= 0.7255*

Solution

Known A vertical plate is to dissipate 10 W heat from its one side to ambient air to ensure maximum surface temperature of 50°C.

Find If blower is required.

Schematic



Assumptions (1) Steady-state conditions. (2) Isothermal plate with one side insulated. (3) Air is an ideal gas.

Analysis Blower (*forced convection*) will be necessary if the free convection and radiation heat-transfer from the plate is *less* than the stipulated dissipation rate of energy.

Let us first check if *free convection alone* is sufficient to carry away heat at the rate of 10 W or more.

Rayleigh number, $Ra_L = \dfrac{g\beta L_c^3 (T_s - T_\infty) Pr}{\nu^2}$

With $\beta_{\text{ideal gas}} = \dfrac{1}{(40 + 273.15)\text{K}} = 1/(313.15\text{ K})$

and the characteristic length, L_c for vertical plate = Height, H = 0.1 m, we have

$$Ra_L = \frac{(9.81\text{ m/s}^2)[1/(313.15\text{ K})](0.1\text{ m})^3[(50 - 30)°\text{C}](0.7255)]}{(17.02 \times 10^{-6}\text{ m}^2/\text{s})^2} = 1.569 \times 10^6 (< 10^9)$$

Therefore, $Nu = 0.68 + \dfrac{0.67\, Ra^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{4/9}} = 18.925$

∴ Convection coefficient,

$$\bar{h} = Nu\frac{k}{L_c} = 18.925 \times \frac{0.02662\text{ W/m °C}}{0.1\text{ m}} = 5.04\text{ W/m}^2\text{ °C}$$

Rate of heat loss, $\dot{Q}_{\text{conv}} = \bar{h}A_s(T_s - T_\infty)$ = (5.04 W/m² °C) (0.03 m²) (50 – 30)°C

= **3.023 W < 10 W**

Hence, free convection alone is *not* sufficient.

Let us now estimate the contribution made by radiant heat exchange too.

Radiation heat-transfer, $\dot{Q}_{\text{rad}} = \sigma A_s \varepsilon_s (T_s^4 - T_{\text{sur}}^4)$

Assuming ε_s = 1 and T_{sur} = T_∞ = 30°C or 303.15 K,

$$\dot{Q}_{\text{rad,max}} = (5.67 \times 10^{-8}\text{ W/m}^2\text{ K}^4)(0.03\text{ m}^2)(1)[(323.15^4 - 303.15^4)\text{K}^4] = 4.183\text{ W}$$

Total heat-dissipation rate,

$\dot{Q}_{\text{max (no blower)}} = \dot{Q}_{\text{conv}} + \dot{Q}_{\text{rad,max}}$ = (3.023 + 4.183) W = **7.206 W** (< 10 W) **(Ans.)**

A blower is therefore necessary.

EXAMPLE 9.6 *The heat-transfer rate due to natural convection from a rectangular plate maintained at a uniform temperature of 400 K to quiescent air at 300 K with its shorter side held vertically is 27 per cent higher than when the larger side is vertical. Neglecting heat-transfer by radiation, estimate (a) the dimensions of the plate of 25 cm² surface area, and (b) the rate of heat loss from the plate in the two cases. The relevant empirical correlation and the thermophysical properties of at the film temperature of 350 K are*

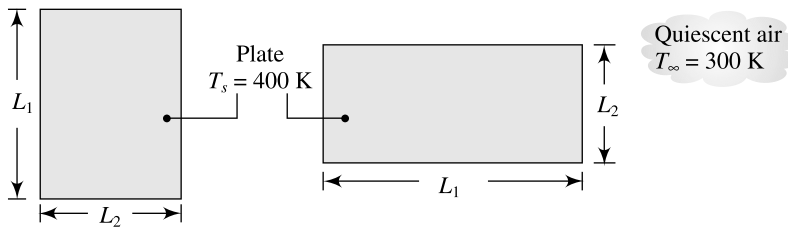
T (*K*)	*k* (W/m K)	ν (m²/s)	Pr
350	0.030	20.92×10^{-6}	0.700

Solution

Known Surface area and temperature of a rectangular plate exposed to still atmospheric air. Heat transfer with smaller side vertical 27% greater than with longer side vertical.

Find (a) Dimensions of plate, $L_1 \times L_2$ (cm); (b) Heat-transfer rate in the two cases, $\dot{Q}_1$ and $\dot{Q}_2$ (W).

Schematic



Assumptions (1) Steady operating conditions. (2) The plate is isothermal. (3) Air is an ideal gas. (4) Radiation effects are not considered.

Analysis The empirical correlation is, $Nu = 0.59(Ra)^{1/4}$

where $\quad Nu = \dfrac{hL_c}{k} \quad$ and $\quad Ra = \dfrac{g\beta L_c^3 (T_s - T_\infty)Pr}{\nu^2}$

The characteristic length L_c for an isothermal vertical plate transferring heat by free convection is the height of the plate.

The dimensions of the rectangular plate are $L_1 \times L_2$ where L_1 is the length (*longer side*) and L_2 is the width (*smaller side*).

The rate of free convection heat loss from the hot plate.

$$\dot{Q} = hA_s(T_s - T_\infty) = Nu\frac{k}{L_c}A_s(T_s - T_\infty)$$

$$= 0.59\left[\frac{g\beta L_c^3(T_s - T_\infty)Pr}{\nu^2}\right]^{1/4} \times \frac{k}{L_c}A_s(T_s - T_\infty)$$

$$= (\text{constant})[L_c^3]^{1/4} \times \frac{1}{L_c} = (\text{constant})[L_c]^{-0.25}$$

Ratio of heat-transfers,

$$\frac{\dot{Q}_2}{\dot{Q}_1} = \frac{\text{Heat loss with shorter side vertical}}{\text{Heat loss with larger side vertical}} = \left(\frac{L_2}{L_1}\right)^{-0.25}$$

$$= (L_1/L_2)^{1/4} = 1.27 \qquad [\text{Since } \dot{Q}_2 \text{ is 27\% more than } \dot{Q}_1]$$

It follows that, $\dfrac{L_1}{L_2} = (1.27)^4 = 2.60$

Since the surface area, $A_s = L_1L_2 = 25\ \text{cm}^2$,

$$(2.6L_2)(L_2) = 25\ \text{cm}^2$$

or $\quad L_2 = \sqrt{25/2.6}\ \text{cm} = \mathbf{3.1\ cm}$ and $L_1 = 2.6 \times 3.1 = \mathbf{8.06\ cm}$ **(Ans.) (a)**

With film temperature, $T_f = \frac{1}{2}(T_s + T_\infty) = (400 + 300)/2 = 350 \text{ K}, \beta_{\text{ideal gas}} = \frac{1}{T_f} = \frac{1}{350 \text{ K}}$

Free convection heat-transfer rate with shorter side vertical ($L_c = L_2$) is determined to be

$$\dot{Q}_2 = 0.59\left[\frac{(9.81 \text{ m/s}^2)[1/(350 \text{ K})](0.031 \text{ m})^3[(400-300)\text{K}](0.7)}{(20.92\times 10^{-6} \text{ m}^2/\text{s})^2}\right]^{1/4}$$

$$\times \frac{0.03 \text{ W/m K}}{0.031 \text{ m}} \times (25\times 10^{-4} \text{ m}^2)(400-300)\text{K} = \mathbf{2.73\ W} \quad \textbf{(Ans.) (b)}$$

and $$\dot{Q}_1 = \frac{\dot{Q}_2}{1.27} = \mathbf{2.15\ W} \quad \textbf{(Ans.) (b)}$$

EXAMPLE 9.7 *A nuclear reactor with its core constructed of parallel vertical plates 2.5 m high and 1.5 m wide has been designed on free convection heating of liquid bismuth. The maximum temperature of the plate surfaces is limited to 1200 K while the lowest permissible temperature of bismuth is 600 K. Determine the maximum possible heat dissipation from both sides of each plate.*

Properties of liquid bismuth at 900 K:

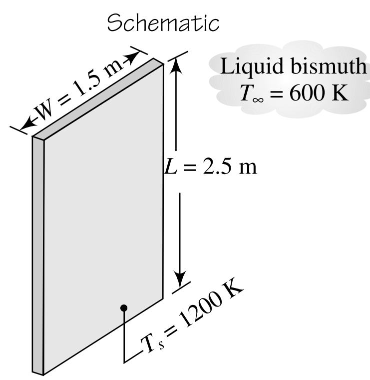
k = *15.6 W/m K,* Pr = *0.0099,* ν = *9.96 × 10⁻⁸ m²/s,* β = *0.00012 K⁻¹,* α = *1.02 × 10⁻⁵ m²/s*

Solution

Known Nuclear reactor core idealized as vertical flat plate of prescribed dimensions. Temperatures of surface (*plate*) and surrounding fluid (*liquid bismuth*).

Find Heat dissipation from both sides of the plate.

Assumptions (1) Liquid bismuth is quiescent. (2) Plate is isothermal. (3) Constant thermophysical properties.



Analysis For vertical flat plate, the characteristic length is the height of the plate.

∴ $L_c = L = 2.5$ m

Rayleigh number, $Ra_L = Gr_L Pr = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2}\cdot\frac{\nu}{\alpha}$

$$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha}$$

$$= \frac{9.81 \text{ m/s}^2 \times 0.00012 \text{ K}^{-1} \times (1200-600)\text{K} \times 2.5^3 \text{ m}^3}{(9.96\times 10^{-8} \text{ m}^2/\text{s})(1.02\times 10^{-5} \text{ m}^2/\text{s})}$$

$$= 1.1\times 10^{13} \qquad (>10^9)$$

Using the Churchill–Chu correlation:

Nusselt number,

$$\overline{Nu}_L = \left\{0.825 + \frac{0.387\, Ra_D^{1/6}}{[1+(0.492/Pr)^{9/16}]^{8/27}}\right\}^2 = \left\{0.825 + \frac{0.387\times(1.1\times 10^{13})^{1/6}}{[1+(0.492/0.0099)^{9/16}]^{8/27}}\right\}^2$$

$$= 897.3$$

Average heat-transfer coefficient,

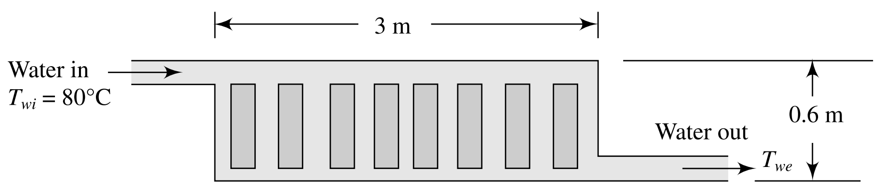
$$\bar{h} = 897.3 \times \frac{15.6 \text{ W/m K}}{2.5 \text{ m}} = 5599 \text{ W/m}^2 \text{ K}$$

Heat-transfer rate,

$$\dot{Q} = \bar{h} A_s (T_s - T_\infty) = \left(5599 \frac{\text{W}}{\text{m}^2 \text{ K}}\right)(2 \times 2.5 \times 1.5)\text{m}^2 \times (1200 - 600)\text{K} \left(\frac{1 \text{ MW}}{10^6 \text{ W}}\right)$$

$= \mathbf{25.2\ MW}$ **(Ans.)**

EXAMPLE 9.8 *A domestic hot water radiator is situated in a room at a temperature of 20°C. The radiator is basically constructed of two pressed steel plates fitted together to form a number of water channels between them and has overall dimensions and heat-transfer areas as shown in the figure. The air-side heat-transfer coefficient h_r due to radiation is 5 W/ m² K, the water-side coefficient is 1000 W/ m² K and the water mass-flow rate and inlet temperature are 0.05 kg/s and 80°C respectively. Estimate the total heat-transfer rate to the room.*



Hot water radiator

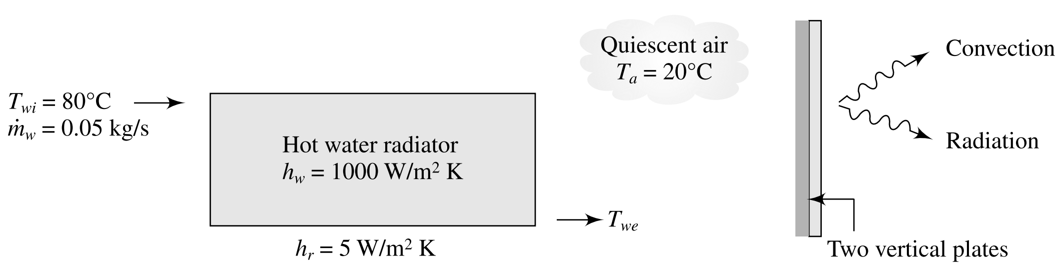
Air temperature, T_a *= 20°C*
Air-side area ≅ 2 × 3 × 0.6 = 3.6 m²
Water side area, A_w *= 2 m²*

Solution

Known The hot-water radiator is transferring heat to the surrounding air by both natural convection and radiation.

Find Rate of heat loss, $\dot{Q}$(W).

Schematic



Assumptions (1) Steady-state conditions. (2) Constant fluid properties. (3) Radiation effects are taken care of by specifying the radiation heat-transfer coefficient. (4) Air is an ideal gas.

Analysis The two sides of the radiator may be treated as two vertical plates for free convection purposes. Free convection air side heat-transfer coefficient can be found by determining the Rayleigh number which in turn involves a knowledge of the temperature difference between the radiator and the surrounding air. The radiator metal temperature and the water temperature are almost the same owing to the large heat-transfer coefficient on the water side compared to the air side. The radiator temperature varies over its length between the inlet and exit water temperatures.

The average radiator surface temperature is then equal to mean water temperature, $\frac{1}{2}(T_{wi}+T_{we})$. But exit water temperature is not known. Hence, let us assume that T_{we} is less than T_{wi} by 8°C. The mean radiator surface temperature, $\overline{T}_s \cong \frac{1}{2}(T_{wi}+T_{we}) = \frac{1}{2}$ [80 + (80 – 8)] = 76°C. The required surface to air temperature difference $\overline{T}_s - T_\infty$ = 76 – 20 = 56°C. It may be added that the estimated temperature difference (T_{wi} – T_{we}) is not going to materially affect ΔT, i.e., $\overline{T}_s - T_\infty$.

Film temperature, $T_f = \frac{1}{2}(\overline{T}_s + T_\infty)$

Properties of atmospheric air at 48°C are

$$k = 0.02721 \text{ W/m °C}, \quad Pr = 0.7233$$

$$\nu = 1.779 \times 10^{-6} \text{ m}^2\text{/s}, \ \beta = 1/T_f(\text{K}) = \frac{1}{(48+273.15)\text{K}}$$

Rayleigh number,

$$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} Pr = \frac{(9.81 \text{ m/s}^2)\times 1 \times (76-20)°\text{C} \times (0.6 \text{ m})^3 \times 0.7233}{321.15 \text{ K} \times (1.779 \times 10^{-5} \text{ m}^2\text{/s})^2}$$

$$= 0.845 \times 10^9 \qquad (\leq 10^9)$$

Using the Churchill–Chu correlation, the average Nusselt number is

$$\overline{Nu} = 0.68 + 0.67 Ra^{1/4}\left[1+\left(\frac{0.492}{Pr}\right)^{\frac{9}{16}}\right]^{\frac{-4}{9}}$$

$$= 0.68 + 0.67\ (0.845 \times 10^9)^{1/4} \times [1 + (0.492/0.7233)^{9/16}]^{-4/9} = 88.53$$

Air-side heat-transfer coefficient

$$h_a = \frac{k}{L}\overline{Nu} = \frac{0.02721 \text{ W/m °C}}{0.6 \text{ m}} \times 88.53 = 4.0 \text{ W/m}^2 \text{ °C}$$

Total heat-transfer rate may be determined from

$$\dot{Q} = UA\Delta T$$

where $\quad \Delta T = \overline{T}_s - T_\infty = 55°\text{C}$ and $\dfrac{1}{UA} = \dfrac{1}{h_a^* A_a} + \dfrac{1}{h_w A_w}$

The term h_a^* is the total air-side heat-transfer coefficient comprising convection and radiation components.

$$h_a^* = h_a + h_r = 4 + 5 = 9.0 \text{ W/m}^2 \text{ K}$$

Thus, $\quad \dfrac{1}{UA} = \dfrac{1}{9.0 \times 3.6} + \dfrac{1}{1000 \times 2}\left(\dfrac{\text{m}^2\text{ K}}{\text{W} \times \text{m}^2}\right)$

$\therefore \quad UA = 32$ W/K

Therefore, $\dot{Q}$ = (32 W/K) (55°C or K) = **1760 W or 1.76 kW** **(Ans.)**

This heat-transfer rate yields the water temperature difference ($T_{wi} - T_{we}$) given by

$$\dot{Q} = \dot{m}_w C_{p_w}(T_{wi} - T_{we})$$

$$\therefore \quad T_{wi} - T_{we} = \frac{\dot{Q}}{\dot{m}_w C_{p_w}} = \frac{1.76 \text{ kW}}{0.05 \text{ kg/s} \times 4.19 \text{ kJ/kg K}} = 8.4°\text{C}$$

Initially, we had assumed ($T_{wi} - T_{we}$) = 8°C which is now justified

Comment It is noteworthy that the heat-exchange surface area is not the same on *air* and *water sides*. Hence, area is included with the convection coefficient while computing *UA*.

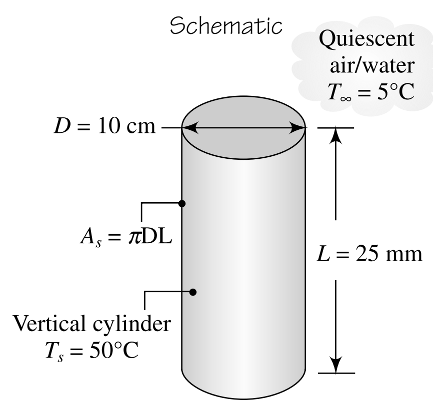
EXAMPLE 9.9 *A vertical cylinder is 10 cm in diameter and 25 cm in height. The surface temperature is maintained at 50°C and the ambient fluid temperature is 5°C at a pressure of 1 atm. Determine the heat lost by the cylindrical surface due to free convection effects. Verify the validity of the usual assumption that the outer surface of a vertical cylinder can be treated as a vertical plate.*

At 27.5°C and 1 atm, the properties are the following:

Air: k = 0.0257 W/m °C, ν = 1.585 × 10^{-5} m²/s, Pr = 0.7289

Water: k = 0.611 W/m °C, ν = 8.474 × 10^{-7} m²/s, β = 0.271 × 10^{-3} K^{-1}, Pr = 5.78

Solution



Known A vertical cylinder loses heat by natural convection to the surrounding atmospheric air/ water.

Find Heat-transfer rate, $\dot{Q}$(W) to air/water.

Assumptions (1) Steady state conditions. (2) Air is an ideal gas. (3) Correlation for vertical plate is applicable. (4) Radiation effects are neglected.

Analysis ***Air :*** Let us first determine the Rayleigh number to enable us to select the appropriate empirical correlation. The characteristic length for a vertical surface (plate or cylinder) is the height, i.e., L = 0.25 m

Grashof number

$$Gr_L = g\beta L^3(T_s - T_\infty)/\nu^2$$

where $\beta = \frac{1}{T_f}(K^{-1})$ for an ideal gas.

Film temperature, $T_f = \frac{T_s + T_\infty}{2} = \frac{50 + 5}{2}$ = 27.5°C or 300.65 K

$$Gr_L = \frac{(9.81 \text{ m/s}^2)[1/(300.65 \text{ K})](0.25 \text{ m})^3(50 - 5)°\text{C}}{(1.585 \times 10^{-5} \text{ m}^2/\text{s})^2} = 91.323 \times 10^6$$

The correlation for vertical plate is applicable for a vertical cylinder if

$$D \geq 35\, L / Gr_L^{1/4}$$

$$\frac{35L}{Gr_L^{1/4}} = \frac{35 \times 0.25}{(91.323 \times 10^6)^{0.25}} = 0.0895 \text{ m} \quad \text{or} \quad 8.95 \text{ cm}$$

D = 10 cm which is *greater* than 8.95 cm. Hence, the vertical plate correlations are valid.

$$Ra_L = Gr_L\, Pr = (91.323 \times 10^6)\,(0.7289) = 66.565 \times 10^6 \qquad (< 10^9)$$

Therefore the boundary layer is *laminar*. Using the Churchill–Chu correlation for this range,

$$Nu_L = 0.68 + 0.67\, Ra_L^{1/4} \left\{ 1 + \left(\frac{0.492}{Pr} \right)^{9/16} \right\}^{-4/9}$$

$$= 0.68 + 0.67(66.565 \times 10^6)^{1/4} \left\{ 1 + \left(\frac{0.492}{0.7289} \right)^{9/16} \right\}^{-4/9} = 47.266$$

Heat-transfer coefficient,

$$h = \frac{k}{L} Nu = \frac{0.0257 \text{ W/m °C}}{0.25 \text{ m}} (47.266) = 4.86 \text{ W/m}^2 \text{ °C}$$

Rate of heat-transfer by free convection from the vertical cylinder to the surrounding air is

$$\dot{Q}_{\text{air}} = hA_s(T_s - T_\infty) = h(\pi DL)(T_s - T_\infty)$$

$$= (4.86 \text{ W/m}^2 \text{ °C})(\pi \times 0.1 \text{ m} \times 0.25 \text{ m})(50 - 5)\text{°C} = \mathbf{17.2\ W} \qquad \textbf{(Ans.)}$$

Water: $Gr_L = \dfrac{g\beta L^3 (T_s - T_\infty)}{\nu^2} = \dfrac{9.81(0.271 \times 10^{-3})(0.25)^3(50 - 5)}{(8.474 \times 10^{-7})^2} = 2.6 \times 10^9$

Vertical plate approximation is valid if, $D \geq 35\, L\, Gr_L^{-1/4}$

We note that D = 10 cm and $35 \times 25 \text{ cm} \times (2.6 \times 10^9)^{-1/4} = 3.87 \text{ cm}$

Since D is greater than 3.87 cm, the assumption is justified.

$$Ra_L = Gr_L Pr = (2.6 \times 10^9)(5.78) = 1.505 \times 10^{10} \qquad (> 10^9)$$

The boundary layer is, therefore, *turbulent*.

$$Nu_L = \left[0.825 + \frac{0.387\, Ra_L^{1/6}}{[1 + (0.492/Pr)^{9/16}]^{8/27}} \right]^2$$

$$= \left[0.825 + \frac{0.387(1.505 \times 10^{10})^{1/6}}{[1 + (0.492/5.78)^{9/16}]^{8/27}} \right]^2 = 354.3$$

$$\therefore \qquad h = Nu_L \frac{k}{L} = 354.3 \left(\frac{0.611 \text{ W/m°C}}{0.25 \text{ m}} \right) = 865.9 \text{ W/m}^2 \text{ °C}$$

Heat-transfer rate to water,

$$\dot{Q}_{\text{water}} = (865.9)(\pi \times 0.10 \times 0.25)(50 - 5) = \mathbf{3060\ W} \qquad \textbf{(Ans.)}$$

Comment We must recognize the fact that besides this heat transfer, the surface will also radiate to the surroundings an amount depending upon the nature of the surface, the geometrical orientation with respect to other surfaces nearby, and the temperature level of those other surfaces.

EXAMPLE 9.10 *A hot plate 1 m × 0.5 m at 180°C is kept vertically in still air at 20°C. Find (a) the heat-transfer coefficient, (b) the initial rate of cooling of the plate in °C / min, and (c) the time required for cooling the plate from 180°C to 80°C if the heat-transfer is due to convection only.*

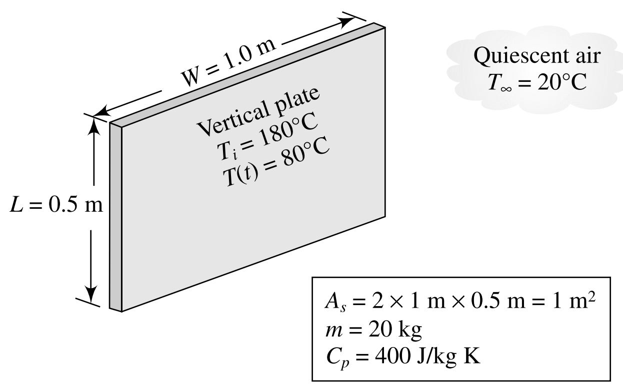
Mass of the plate is 20 kg and specific heat is 400 J/kg K. Assume that the 0.5 m side is vertical, that the heat-transfer coefficient calculated in (a) remains constant and that the heat transfer takes place from both sides of the plate.

Solution

Known A hot vertical plate dissipates heat to still air from both sides.

Find (a) Convection coefficient, h (W/m^2 K); (b) Initial cooling rate, $\dfrac{dT}{dt}\left(\dfrac{°\text{C}}{\text{min}}\right)$; (c) Time required for cooling, $t(s)$.

Schematic



Assumptions (1) Air is an ideal gas. (2) Constant properties. (3) Radiation effects are negligible. (4) Lumped-parameter analysis is valid for transient conduction.

Properties Film temperature, $T_f = \dfrac{180 + 20}{2} = 100°\text{C}$

At 100°C and 1 atm, for air:

$$k = 0.03095 \text{ W/m K},\ \nu = 2.306 \times 10^{-5} \text{ m}^2/\text{s},\ Pr = 0.7111,\ \beta = \frac{1}{(373.15\text{ K})}$$

Analysis Characteristic dimension is the height of the plate, $L = 0.5$ m

Rayleigh number,

$$Ra_L = \frac{g\beta L^3 (T_s - T_\infty) Pr}{\nu^2}$$

$$Ra_L = \frac{(9.81\ \text{m/s}^2)(1/373.15)\text{K}^{-1}(0.5\ \text{m})^3(180-20)°\text{C}(0.7111)}{(2.306\times 10^{-5}\ \text{m}^2/\text{s})^2}$$

$= 703.12 \times 10^6$ $(< 10^9)$ $\Rightarrow$ Laminar boundary layer.

The appropriate correlation is

$$Nu = 0.68 + 0.67\, Ra_L^{1/4}\left[1+\left(\frac{0.492}{Pr}\right)^{9/16}\right]^{-4/9}$$

$$= 0.68 + (0.67)(703.12\times 10^6)^{1/4}\left[1+\left(\frac{0.492}{0.7111}\right)^{9/16}\right]^{-4/9} = 84.43 = \frac{hL}{k}$$

Natural convection heat-transfer coefficient,

$$h = \frac{k}{L}Nu = \frac{0.03095\ \text{W/m K}}{0.5\ \text{m}}(84.43) = \mathbf{5.23\ W/m^2\ K} \qquad \textbf{(Ans.) (a)}$$

Applying control volume energy balance:

$$\dot{E}^0_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}^0_{\text{gen}} = \dot{E}_{st} \quad \Rightarrow \quad \dot{E}_{\text{out}} = -\dot{E}_{st}$$

i.e., Heat dissipated by the plate by convection from *both* sides = Rate of *decrease* of internal energy

$$\dot{Q}_{\text{out}} = -mC_p\frac{dT}{dt}$$

$\therefore$ Cooling rate initially in °C/min,

$$-\frac{dT}{dt} = \frac{hA_s(T_s - T_\infty)}{mC_p} = \frac{(5.23\ \text{W/m}^2\ \text{K})(1\ \text{m}^2)(180-20)°\text{C or K}}{(20\ \text{kg})(400\ \text{J/kg K})}\left|\frac{1\ \text{J/s}}{1\ \text{W}}\right|\left|\frac{60\ \text{s}}{1\ \text{min}}\right|$$

$$= \mathbf{6.27°C \quad or \quad K/min} \qquad \textbf{(Ans.) (b)}$$

Lumped heat capacity model for *transient (unsteady state)* conduction [$T = f(t)$] is assumed to be valid. Then, the time required for cooling the plate is

$$t = \frac{\rho C_p L_c}{h}\ln\frac{\theta_i}{\theta}$$

where $L_c = \dfrac{V}{A_s}$, $\theta_i = T_i - T_\infty$, $\theta = T(t) - T_\infty$

Time required for cooling is

$$t = \frac{\rho C_p V}{A_s h}\ln\frac{\theta_i}{\theta} = \frac{mC_p}{hA_s}\ln\frac{T_i - T_\infty}{T - T_\infty} = \frac{(20\ \text{kg})(400\ \text{J/kg K})}{(5.23\ \text{W/m}^2\ \text{K})(1\ \text{m}^2)}\ln\frac{(180-20)°\text{C}}{(80-20)°\text{C}}$$

$$= \mathbf{1500\ s \quad or \quad 25\ min} \qquad \textbf{(Ans.) (c)}$$

(C) Horizontal Plate

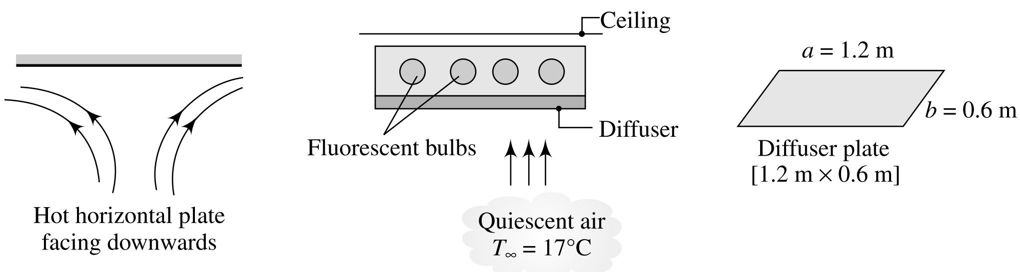
EXAMPLE 9.11 *A fluorescent light is covered with a diffuser, which is a sheet of translucent plastic of 1.2 m by 0.6 m size. The electronics controlling the light are temperature sensitive and must be kept cool. If 65 W of heat is dissipated by the light and removed by free convection from the bottom surface of the diffuser to room air at 17°C, estimate the surface temperature of the diffuser.*

Solution

Known A horizontal rectangular plastic plate (*diffuser*) with heated bottom downwards loses heat by natural convection.

Find Surface temperature of diffuser, T_s(°C).

Schematic



Assumptions (1) Steady operating conditions exist. (2) Air is an ideal gas. (3) Radiation effects are neglected. (4) All the heat generated flows through the diffuser.

Analysis Heat is dissipated from the fluorescent bulbs and transferred to the diffuser. The heated diffuser dissipates this heat (65 W) to the surrounding still air from its bottom surface. The appropriate correlation for this case is

Hot horizontal plate facing downwards:

$$Nu = hL/k = 0.27\ Ra^{1/4} \quad \text{for} \quad 3 \times 10^5 < Ra < 3 \times 10^{10}$$

where $$Ra = Gr\ Pr = \frac{g\beta L^3 (T_s - T_\infty) Pr}{\nu^2}$$

All thermophysical properties of air in the above correlation are to be evaluated at the film temperature, $T_f = \frac{1}{2}(T_s + T_\infty)$. The characteristic length, L = Plate area, A_s/Plate perimeter, P.

However, the plate surface temperature is unknown. Hence, a *trial-and-error* procedure is necessary.

The rate of heat transfer by convection is $\dot{Q} = hA_s(T_s - T_\infty)$.

where $h = \frac{k}{L}\ Nu = \frac{k}{L} 0.27\ (Ra)^{1/4}$. The calculated value of $\dot{Q}$ based on the assumed value of T_s must be 65 W.

Now, $$h = \frac{\dot{Q}}{A_s(T_s - T_\infty)} = 0.27\frac{k}{L}\left[\frac{g\beta L^3 (T_s - T_\infty) Pr}{\nu^2}\right]^{1/4}$$

or $$\left[\frac{\dot{Q}L}{kA_s(T_s - T_\infty)}\right]^4 = (0.27)^4\left[\frac{g\beta L^3(T_s - T_\infty)Pr}{\nu^2}\right]$$

or $$\boxed{(T_s - T_\infty)^5 = \left[\frac{\dot{Q}}{kA_s(0.27)}\right]^4 \frac{L\nu^2}{g\beta Pr}}$$

or $$\boldsymbol{T_s - T_\infty = \left[\frac{\dot{Q}}{kA_s(0.27)}\right]^{4/5}\left(\frac{L\nu^2}{g\beta Pr}\right)^{1/5}} \qquad \text{(A)}$$

For the diffuser, $L = \dfrac{A_s}{P} = \dfrac{ab}{2(a+b)} = \dfrac{1.2 \times 0.6\ \text{m}^2}{2(1.2 + 0.6)\text{m}} = 0.2$ m

Let us assume the film temperature to be 35°C (*to be modified later if necessary*).
At 35°C and 1 atm, the properties of air are

$$\beta = \frac{1}{(35 + 273.15)\text{K}}, \qquad k = 0.02625\ \text{W/m K}$$

$$\nu = 1.655 \times 10^{-5}\,\frac{\text{m}^2}{\text{s}}, \qquad Pr = 0.7268$$

Substituting these values into Eq. (A), one has

$$T_s - T_\infty = \left[\frac{65}{(0.02625)(1.2 \times 0.6)(0.27)}\right]^{0.8} \times \left[\frac{0.2 \times (1.655 \times 10^{-5})^2}{9.81 \times (1/308.15)(0.7268)}\right]^{0.2}$$
$$= 1923.4 \times 0.01883 = 36.2°\text{C}$$

∴ Surface temperature, T_s = 17 + 36.2 = **53.2°C** **(Ans.)**

And, the film temperature predicted by the above calculation is

$$T_f = \frac{1}{2}(53.2 + 17) = 35.1°\text{C}$$

The initial guess was T_f = 35°C. No more trial is therefore required. Hence, the diffuser surface temperature is T_s = 53.2°C **(Ans.)**

Comment To justify the validity of the correlation, one must find the Rayleigh number, *Ra* with the computed value of T_s.

$$Ra = \frac{(9.81)(1/308.15)(0.2)^3(53.2 - 17)(0.7268)}{(1.655 \times 10^{-5})^2} = 2.445 \times 10^7$$

As this value is well within the recommended range [3×10^5 to 3×10^{10}], the estimated surface temperature is correct.

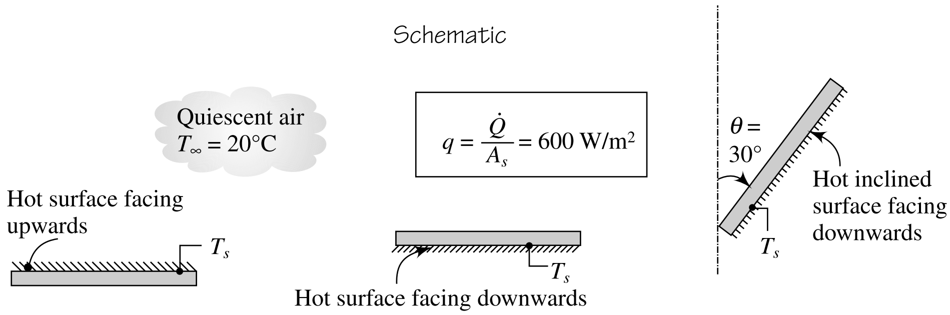
EXAMPLE 9.12 *A 0.4 × 0.4 m square plate receives a constant wall heat flux at a rate of 600 W/m² and is placed in the quiescent room air at 20°C. Determine the average natural convection heat-transfer coefficient, and the mean surface temperature of the plate for the following three orientations of the hot surface: (a) Hot surface, horizontal, facing upwards (b) Hot surface, horizontal, facing downwards (c) Heated surface, inclined at an angle of 30°C with the vertical, facing downwards*

Solution

Known A square plate subjected to constant heat flux is exposed in still air.

Find Convection coefficient, h(W/m^2 °C) and surface temperature, T_s(°C) for three different geometries.

Schematic



(a) (b) (c)

Assumptions (1) Constant-wall heat flux conditions. (2) Constant properties. (3) Radiation effects are neglected.

Analysis Characteristic length for a square plate is

$$L_c = \frac{A_s}{P} = \frac{a \times a}{4a} = \frac{a}{4} = \frac{0.4\text{ m}}{4} = 0.1\text{ m}$$

Reference temperature, $T_e = T_s - 0.25(T_s - T_\infty)$

Since T_s is not known, let us guess h = 6 W/m^2 °C

so that $q = h(T_s - T_\infty) \quad \Rightarrow \quad T_s = T_\infty + \frac{q}{h} = 20°\text{C} + \frac{600\text{ W/m}^2}{6\text{ W/m}^2\text{ °C}} = 120°\text{C}$

Then T_e = 120 – 0.25(120 – 20) = 95°C

Properties of air at 95°C: k = 0.0306 W/m °C, $\nu = 2.254 \times 10^{-5}$ m^2/s, Pr = 0.7122

Rayleigh number, $Ra_L = \dfrac{g\beta L^3(T_s - T_\infty)Pr}{\nu^2}$

where $\beta = [T_\infty + 0.50(T_s - T_\infty)]^{-1}(\text{K})^{-1} = \{[20 + 0.5(120 - 20)] + 273.15\}^{-1} = 0.002914\text{ K}^{-1}$

(a) ***Hot horizontal surface facing upwards***:

$$Ra_L = \frac{(9.81\text{ m/s}^2)(0.002914\text{ K}^{-1})(0.1\text{ m})^3(120 - 20)\text{K}(0.7122)}{(2.254 \times 10^{-5}\text{ m}^2/\text{s})^2}$$

$$= 4.0 \times 10^6 \qquad (< 2 \times 10^8)$$

$$\therefore \qquad \overline{Nu}_L = 0.13(Ra_L)^{1/3} = 0.13(4.0 \times 10^6)^{1/3} = 20.65 = \frac{\bar{h}L}{k}$$

The average convection heat-transfer coefficient,

$$\bar{h} = \overline{Nu}_L \frac{k}{L} = 20.65 \times \frac{0.0306\text{ W/m °C}}{0.1\text{ m}} = 6.32\text{ W/m}^2\text{ °C}$$

This is slightly *more* than the assumed value of h = 6 W/m^2 °C.

Surface temperature, $T_s = T_\infty + \dfrac{q}{h} = 20 + \dfrac{600}{6.32} = 115°\text{C}$

Then $T_e = T_s - 0.25(T_s - T_\infty) = 15 - 0.25(115 - 20) = 91.3°\text{C}$

At 91.3°C: k = 0.0303 W/m °C, ν = 2.214 × 10^{-5} m^2/s, Pr = 0.7129

$$\beta = [273.15 + \{20 + 0.5(115 - 20)\}]^{-1} = 0.00294\ \text{K}^{-1}$$

$$\therefore \quad Ra_L = \frac{9.81 \times 0.00294 \times 0.1^3 (115 - 20)(0.7129)}{(2.214 \times 10^{-5})^2} = 3.98 \times 10^6$$

$$\therefore \quad \overline{Nu}_L = 0.13(3.98 \times 10^6)^{1/3} = 20.6$$

Hence, $\bar{h} = 20.6 \times \dfrac{0.0303}{0.1}$ = 6.24 W/m^2 °C

This value of h is quite close to the one obtained earlier.

Hence, $\boldsymbol{\bar{h} = 6.24\ \text{W/m°C}}$ **and** $\boldsymbol{T_s = 115°\text{C}}$ **(Ans.) (a)**

(b) ***Hot horizontal surface facing downwards*:**

In this configuration, the value of $\bar{h}$ is likely to be less.

Let $\bar{h} = 4\ \text{W/m}^2\ °\text{C}$.

$$\therefore \quad T_s = T_\infty + \frac{q}{h} = 20 + \frac{600}{4} = 170°\text{C}$$

$$T_e = T_s - 0.25(T_s - T_\infty) = 170 - 0.25(170 - 20) = 132.5°\text{C}$$

At 132.5°C, k = 0.0324 W/m °C, ν = 2.529 × 10^{-5} m^2/s Pr = 0.7072

$$\beta = [273.15 + \{20 + 0.5(170 - 20)\}]^{-1} = 0.0027\ \text{K}^{-1}$$

$$Ra_L = \frac{9.81 \times 0.0027 \times 0.1^3 \times (170 - 20) \times 0.7072}{(2.529 \times 10^{-5})^2} = 4.42 \times 10^6$$

$$\overline{Nu}_L = 0.58(Ra_L)^{1/5} \quad \text{for} \quad 10^6 < Ra_L < 10^{11} = \frac{\bar{h} L}{k}$$

$$\therefore \quad \bar{h} = 0.58 \frac{k}{L}(Ra_L)^{0.2} = 0.58 \times \frac{0.0324}{0.1} \times (4.42 \times 10^6)^{0.2} = 4.0\ \text{W/m}^2\ °\text{C}$$

This value is same as assumed. Hence, no more trial is necessary.

$\boldsymbol{\bar{h} = 4\ \text{W/m}^2\ °\text{C}}$ **and** $\boldsymbol{T_s = 170°\text{C}}$ **(Ans.) (b)**

(c) ***Heated surface, inclined at 30°C with the vertical, facing downwards*:**

Assuming $\bar{h} = 5\ \text{W/m}^2\ °\text{C}$

$$T_s = T_\infty + \frac{q}{h} = 20 + \frac{600}{5} = 140°\text{C}$$

$$T_s = T_\infty - 0.25(T_s - T_\infty) = 140 - 0.25(140 - 20) = 110°\text{C}$$

At 110°C, $\beta = [273.15 + \{20 + 0.5(140 - 20)\}]^{-1}$ = 0.00283 K^{-1}

$$k = 0.03165 \text{ W/m°C}, \quad \nu = 2.414 \times 10^{-6} \text{ m}^2/\text{s}, \quad Pr = 0.7092$$

$$Ra_L = \frac{9.81 \times 0.00283 \times 0.1^3 \times (140 - 20) \times 0.7092}{(2.414 \times 10^{-5})^2} = 4.057 \times 10^6$$

$$\overline{Nu}_L = 0.56(Ra_L \cos\theta)^{1/4} \quad \text{for} \quad \theta < 88°, \quad 10^5 < Ra_L < 10^{11}$$

$$\therefore \qquad \bar{h} = 0.56\frac{k}{L}(Ra_L \cos\theta)^{0.25} = 0.56\frac{0.03165}{0.1} \times (4.057 \times 10^6 \cos 30°)^{0.25} = 7.67 \text{ W/m}^2 \text{ °C}$$

Then $\quad T_s = 20 + \dfrac{600}{7.67} \approx 98°\text{C}$

At 78.5°C, $\quad T_e = 98 - 0.25(98 - 20) = 78.5°\text{C}$

$$\beta = [273.15 + \{20 + 0.5(98 - 20)\}]^{-1} = 0.003 \text{ K}^{-1}$$

$$k = 0.0294 \text{ W/m°C}, \nu = 2.082 \times 10^{-5} \text{ m}^2/\text{s}, Pr = 0.7157$$

$$Ra_L = \frac{9.81 \times 0.003 \times 0.1^3 \times (98 - 20) \times 0.7157}{(2.082 \times 10^{-5})^2} = 3.79 \times 10^6$$

$$\therefore \qquad \bar{h} = 0.56 \times \frac{0.0294}{0.1} \times [3.79 \times 10^6 \times \cos 30°]^{1/4} = 7.0 \text{ W/m}^2 \text{ °C}$$

Now, $\quad T_s = 20 + \dfrac{600}{7} \approx 105°\text{C}$

$$T_e = 105 - 0.25(105 - 20) = 84°\text{C}$$

At 84°C, $\quad \beta = [273.15 + \{20 + 0.5(105 - 20)\}]^{-1} = 0.00298 \text{ K}^{-1}$

$$k = 0.0298 \text{ W/m°C}, \nu = 2.139 \times 10^{-5} \text{ m}^2/\text{s}, Pr = 0.7145$$

$$Ra_L = \frac{9.81 \times 0.00298 \times 0.1^3 \times (105 - 20) \times 0.7145}{(2.139 \times 10^{-5})^2} = 3.88 \times 10^6$$

$$\therefore \qquad \bar{h} = 0.56 \times \frac{0.0298}{0.1} \times [3.88 \times 10^6 \times \cos 30°]^{1/4} = 7.15 \text{ W/m}^2 \text{ °C}$$

This value is close enough to the one previously obtained.

Hence, $\quad \boldsymbol{\bar{h} = 7.15 \text{ W/m}^2 \text{ °C}}$ **and** $\boldsymbol{T_s = 105°\text{C}}$ **(Ans.) (c)**

EXAMPLE 9.13 *In a wind tunnel, 15°C air at 5 m/s flows over a flat plate (1 m × 0.8 m) that is aligned parallel to the flow direction. The plate temperature is 35°C. One of the sides of the plate is arranged to be parallel to the flow direction, such that the heat transfer is less. Estimate (a) The rate of heat transfer from the plate from one side (top surface), and (b) The initial rate of cooling (dT/dt) (°C/h) if the mass of the plate is 5 kg and the specific heat of the plate is 875 J/kg K. (c) If the air flow is turned off, compute the heat flow rate from the upper surface of the plate in still air at 15°C. (d) What is the percentage change in the heat-flow rate?*

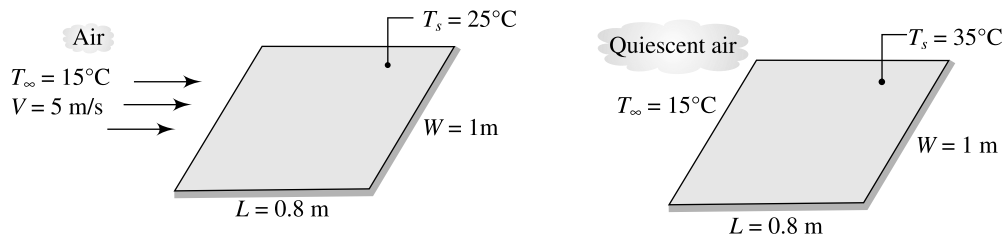
Properties of air at 25°C: k = *0.0261 W/m °C,* ν = *15.71 × 10*$^{-6}$ *m*^{2}*/s,* Pr = *0.71*

Solution

Known Cooling of the top surface of a flat plate by forced air.

Find (a) $\dot{Q}_{\text{forced}}(\text{W})$; (b) $\dfrac{-dT}{dt}(°\text{C/h})$; (c) $\dot{Q}_{\text{free}}(\text{W})$; (d) % change in heat transfer.

Schematic



Forced convection ***Free convection***

Assumptions (1) Steady operating conditions exist. (2) The critical Reynolds number is 5×10^5. (3) Radiation effects are negligible. (4) Air is an ideal gas.

Analysis (a) ***Forced convection:***

Heat transfer from the plate will be less if the air flows parallel to the 0.8 m side. Hence, the characteristic length, $L_c = L = 0.8$ m.

Film temperature, $T_f = \dfrac{1}{2}(T_s + T_\infty) = (35 + 15)°\text{C}/2 = 25°\text{C}$

The Reynolds number at the end of the plate is

$$Re_L = \frac{VL}{\nu} = \frac{(5\text{ m/s})(0.8\text{ m})}{15.71 \times 10^{-6}\text{ m}^2/\text{s}} = 2.546 \times 10^5$$

which is less than the critical Reynolds number of 5×10^5. The average Nusselt number for the entire top surface of the plate is then determined to be

$$Nu = \frac{hL}{k} = 0.664\, Re_L^{1/2}\, Pr^{1/3} = 0.664(2.546 \times 10^5)^{1/2}(0.71)^{1/3} = 298.9$$

Then,
$$h = Nu\frac{k}{L} = (298.9)\frac{0.0261\text{ W/m °C}}{0.8\text{ m}} = 9.75\text{ W/m}^2\text{ °C}$$

$$A_s = WL = 1\text{ m} \times 0.8\text{ m} = 0.8\text{ m}^2$$

Heat-transfer rate,

$$\dot{Q}_{\text{forced}} = hA_s(T_s - T_\infty) = (9.75\text{ W/m}^2\text{ °C})(0.8\text{ m}^2)(35 - 15)°\text{C} = \mathbf{156.0\ W}$$ **(Ans.) (a)**

(b) ***Initial rate of cooling,***

$$\frac{-dT}{dt} = \frac{\dot{Q}}{mC_p} = \frac{156.0\text{ W}}{(5\text{ kg})(875\text{ J/kg K})}\left|\frac{1\text{ J/s}}{1\text{ W}}\right|$$

= 0.03566 K/s or 128.4 K (or °C) per hour **(Ans.) (b)**

(c) ***Natural convection:*** When the air flow is turned off, the upper surface of the plate will be cooled in still (*quiescent*) air under natural (*free*) convection conditions.

For free convection heat transfer from the top surface of the heated horizontal plate, the characteristic length,

$$L_c = \frac{A_s}{P} = \frac{WL}{2(W+L)} = \frac{1\text{ m}\times 0.8}{2(1+0.8)\text{m}} = 0.222\text{ m}$$

With $$\beta_{\text{ideal gas}} = \frac{1}{T_f(\text{K})} = \frac{1}{(25+273.15)\text{K}} = \frac{1}{298.15\text{ K}}$$

$$Ra = \frac{g\beta L_c^3(T_s - T_\infty)Pr}{\nu^2} = \frac{(9.81\text{ m/s}^2)[1/(298.15\text{ K})](0.222\text{ m})^3(35-15)°\text{C}(0.71)}{(15.71\times 10^{-6}\text{ m}^2/\text{s})^2}$$

$$= 2.077\times 10^7$$

which is greater than 10^7.

The appropriate correlation is, $Nu = 0.15(Ra)^{1/3} = \dfrac{hL_c}{k}$

Then $h = \dfrac{k}{L_c}\times 0.15(Ra)^{1/3} = \dfrac{0.0261\text{ W/m °C}}{0.222\text{ m}}\times 0.15(2.077\times 10^7)^{1/3} = 4.843\text{ W/m}^2\text{ °C}$

and, the heat loss rate is

$$\dot{Q}_{\text{free}} = hA_s(T_s - T_\infty) = (4.843\text{ W/m}^2\text{ °C})(0.8\text{ m}^2)(35-15)°\text{C} = \mathbf{77.5\ W} \qquad \textbf{(Ans.) (c)}$$

Percentage change (*decrease*) in heat-transfer after the flow is turned off is

$$\frac{(156.0-77.5)\text{W}}{156.0\text{ W}}\times 100 = \mathbf{50.3\%} \qquad \textbf{(Ans.) (d)}$$

EXAMPLE 9.14 *A flat-plate solar collector, 1.5 m wide and 6 m long, is placed horizontally on the flat roof of a house. The average temperature of the exposed surface of the collector is 50°C. Determine the rate of heat loss from the collector by natural convection on a calm day when the ambient air temperature is 20°C. Also calculate the heat loss by radiation, assuming the collector surface emissivity to be 0.9 with an effective sky temperature of – 30°C.*

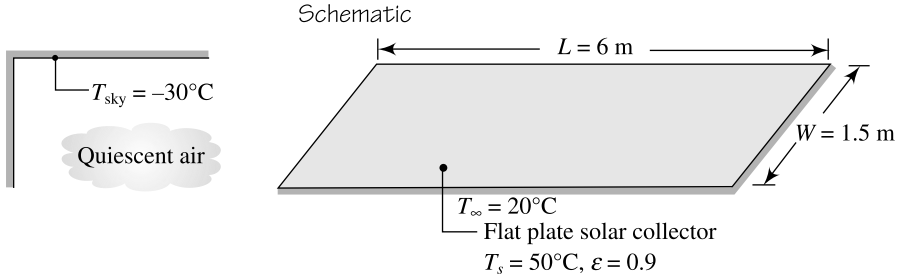
Properties of air at 35°C: k = *0.02625 W/m K,* ν = *1.655 × 10^{–5} m²/s, Pr = 0.7268*

Solution

Known A flat-plate solar collector loses heat by convection and radiation under specified conditions.

Find Rate of heat-transfer, $\dot{Q}_{\text{conv}}$ (W) and $\dot{Q}_{\text{rad}}$ (W).

Schematic



Assumptions (1) Steady operating conditions exist. (2) Quiescent air.

Analysis ***Heat loss by convection:***

For a horizontal plate with a hot surface facing up, the characteristic length,

$$L = \frac{A_s}{P} = \frac{LW}{2(L+W)} = \frac{6\text{ m} \times 1.5\text{ m}}{2(6+1.5)\text{m}} = 0.6\text{ m}$$

The Rayleigh number is, $Ra_L = \dfrac{g\beta(T_s - T_\infty)L^3}{\nu^2}Pr$ where $\beta = \dfrac{1}{T_f(\text{K})}$

Film temperature, $T_f = \dfrac{T_s + T_\infty}{2} = \dfrac{50+20}{2}$ = 35°C or 308.15 K $\Rightarrow$ $\beta = \dfrac{1}{308.15}\text{K}^{-1}$

$$\therefore \qquad Ra_L = \frac{(9.81\text{ m/s}^2)[1/(308.15\text{ K})](50-20)\text{K}(0.6\text{ m})^3}{(1.655 \times 10^{-5}\text{ m}^2/\text{s})^2}(0.7268)$$

$$= 5.474 \times 10^8 \qquad [10^7 < Ra_L < 10^{11}]$$

The natural convection Nusselt number can be found from

$$Nu_L = 0.15\,(Ra_L)^{1/3} = 0.15\,(5.474 \times 10^8)^{1/3} = 122.7$$

Then, $h = \dfrac{k}{L}Nu_L = \dfrac{0.02625\text{ W/m K}}{0.6\text{ m}}$ (122.7) = 5.368 W/m² K

Surface area of the solar collector, $A_s = LW$ = 6 m × 1.5 m = 9.0 m²

and, the convection heat transfer is

$\dot{Q} = hA_s(T_s - T_\infty)$ = (5.368 W/m² K) (9 m²)(50 – 20)°C or **K = 1449 W** **(Ans.)**

Heat loss by radiation: The radiation heat-transfer in this case is determined to be

$$\dot{Q}_{\text{rad}} = \sigma A_s \varepsilon (T_s^4 - T_{\text{sur}}^4)$$

$$= (5.67 \times 10^{-8}\text{ W/m}^2\text{ K}^4)(9\text{ m}^2)(0.9)[(50 + 273.15\text{ K})^4 - (20 + 273.15\text{ K})^4]$$

$$= \mathbf{1616\ W}$$ **(Ans.)**

The total heat loss is estimated to be

(1449 + 1616) W = 3065 W.

Comment Heat loss by radiation is *greater* than that by convection. It is therefore significant that radiation heat-transfer cannot be neglected in natural convection problems.

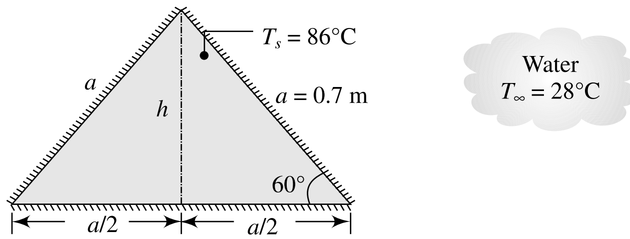
EXAMPLE 9.15 *An equilateral triangular plate of 0.7 m side is placed horizontally in a water tank and held at a temperature of 86°C. The water temperature is 28°C. The hotter side of the plate faces downwards. Calculate the rate of heat transfer. Properties of water at* T_f *= 57°C are* k *= 0.650 W/m K,* ν *= 0.497 × 10⁻⁶ m²/s,* β *= 504 × 10⁻⁶ K⁻¹,* Pr *= 3.15*

Solution

Known Triangular plate with hot surface facing down loses heat to water by free convection.

Find Rate of convective heat loss, $\dot{Q}$[W].

Schematic



Assumptions (1) Constant properties. (2) The plate is isothermal.

Analysis Characteristic length, $L_c = \frac{A_s}{P}$

where $A_s = \frac{1}{2}ah = \frac{1}{2}a \times a \sin 60° = \frac{1}{2} \times 0.7^2 \times 0.866 = 0.2122 \text{ m}^2$

and $P = 3a = 3 \times 0.7 = 2.1 \text{ m} \quad \Rightarrow \quad L_c = 0.2122 \text{ m}^2/2.1 \text{ m} = 0.101 \text{ m}$

Rayleigh number,

$$Ra_L = \frac{g\beta L_c^3 (T_s - T_\infty)Pr}{\nu^2}$$

$$= \frac{(9.81 \text{ m/s}^2)(504 \times 10^{-6} \text{ K}^{-1})(0.101 \text{ m})^3(86-28)\text{K}(3.15)}{(0.497 \times 10^{-6} \text{ m}^2/\text{s})^2}$$

$$= 3.77 \times 10^9 \qquad (10^5 < Ra_L < 10^{10})$$

The appropriate correlation is

$$\overline{Nu}_L = 0.27(Ra_L)^{1/4} = 0.27(3.77 \times 10^9)^{1/4} = 66.9 = \frac{\bar{h}L}{k}$$

Average convection heat-transfer coefficient,

$$\bar{h} = \overline{Nu}_L \frac{k}{L} = 66.9 \times \frac{0.650 \text{ W/m K}}{0.101 \text{ m}} = 430.6 \text{ W/m}^2 \text{ K}$$

Rate of heat transfer,

$$\dot{Q} = \bar{h}A_s(T_s - T_\infty) = (430.6 \text{ W/m}^2 \text{ K})(0.2122 \text{ m}^2)(86-28)\text{K}$$

$= \mathbf{5300 \text{ W}}$ **or** $\mathbf{5.3 \text{ kW}}$ **(Ans.)**

EXAMPLE 9.16 *Water at 40°C flows across a horizontal cylinder (50 mm diameter) maintained at a uniform surface temperature of 20°C with a velocity of 0.05 m/s.*
(a) Calculate the value of the heat-transfer coefficient. (b) If the flow is stopped, to what value would the heat-transfer coefficient reduce? (c) Calculate the Rayleigh number for case (b). (d) Have you any comments to offer after obtaining the heat-transfer coefficients in (a) and (b)?
Properties of water at 30°C.

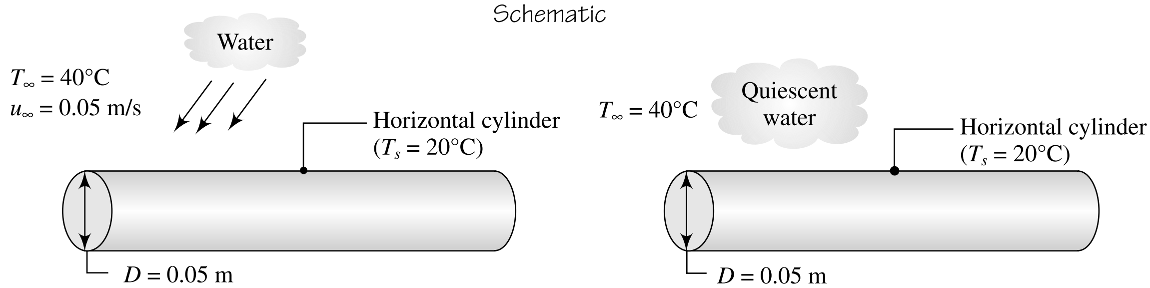
ρ = *996 kg/m³*, μ = *798 × 10⁻⁶ kg/m s*, k = *0.615 W/m K*,
Pr = *5.42*, β = *2.94 × 10⁻⁴ K⁻¹*, ν = *0.8012 × 10⁻⁶ m²/s*

Solution

Known Flow of water across a horizontal isothermal cylinder.

Find (a) h_{forced}(W/m² K); (b) h_{free}(W/m² K); (c) Ra_D; (d) Comment.

Schematic



(a) (b)

Assumptions (1) Steady-state conditions. (2) Constant properties.

Analysis (a) The characteristic length in the case of horizontal cylinder is its diameter.

Reynolds number, $$Re_D = \frac{\rho u_\infty D}{\mu} = \frac{(996\ \text{kg/m}^3)(0.05\ \text{m/s})(0.05\ \text{m})}{798 \times 10^{-6}\ \text{kg/m s}} = 3120$$

Using Fand's correlation: $$\overline{Nu}_D = (0.35 + 0.56\, Re_D^{0.52})Pr^{0.3} \qquad [\text{for } 10^{-1} < Re_D < 10^5]$$
$$= (0.35 + 0.56(3120)^{0.52})(5.42)^{0.3} = 61.6$$

Average heat-transfer coefficient,

$$\therefore \qquad \bar{h} = \overline{Nu}_D \frac{k}{D} = \frac{(61.6)(0.615\ \text{W/m K})}{0.05\ \text{m}} = \mathbf{757.5\ W/m^2\ K} \qquad \textbf{(Ans.) (a)}$$

(b) If there is *no flow*, the heat transfer will be by *free* convection.

Rayleigh number, $$Ra_D = \frac{g\beta(T_\infty - T_s)D^3\, Pr}{\nu^2}$$
$$= \frac{(9.81\ \text{m/s}^2)(2.94 \times 10^{-4}\ \text{K}^{-1})(40 - 20)\text{K}(0.05\ \text{m})^3(5.42)}{(0.8012 \times 10^{-6}\ \text{m}^2/\text{s})^2}$$
$$= \mathbf{6.09 \times 10^7} \qquad \textbf{(Ans.) (c)}$$

Using Churchill–Chu correlation: (for $Ra_D < 10^9$)

$$\overline{Nu}_D = 0.36 + \frac{0.518\, Ra_D^{1/4}}{[1 + (0.559/Pr)^{9/16}]^{4/9}} = 0.36 + \frac{0.518(6.09 \times 10^7)^{1/4}}{[1 + (0.559/5.42)^{9/16}]^{4/9}} = 41.38$$

Average heat-transfer coefficient,

$$\bar{h} = \overline{Nu}_D \frac{k}{D} = 41.38 \times \frac{0.615\ \text{W/m K}}{0.05\ \text{m}} = \mathbf{509\ W/m^2\ K} \qquad \textbf{(Ans.) (b)}$$

Thus,

$$\boxed{\begin{aligned} \bar{h}_{\text{forced}} &= \mathbf{757.5\ W/m^2\ K} \\ \bar{h}_{\text{free}} &= \mathbf{509.0\ W/m^2\ K} \end{aligned}}$$

Comment In case (a), there will be some effect of *free convection*, leading to a heat-transfer coefficient higher than 757.5 m^2 K. **(Ans.) (d)**

(D) Horizontal Cylinder

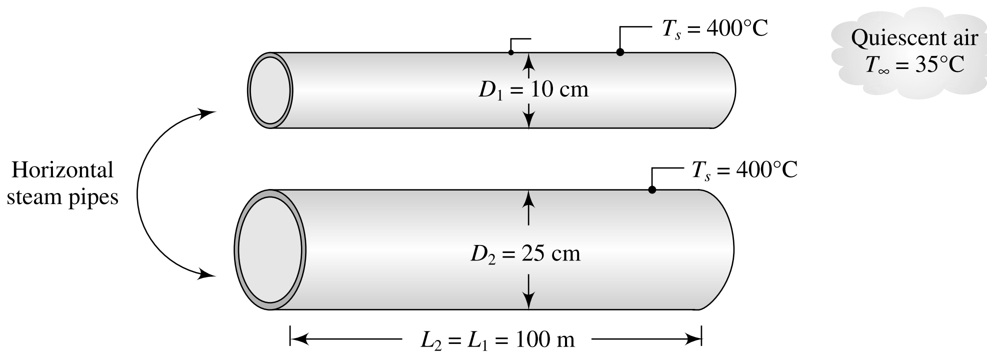
EXAMPLE 9.17 *In a boiler house, there are two horizontal steam pipes of diameters 10 cm and 25 cm. The two have the same surface temperature of 400°C. The temperature of the surrounding air is 35°C. The pipes are sufficiently far from each other so that the thermal boundary layers do not interfere. Determine (a) the heat loss from each pipe per 100 m length, (b) the ratio of heat-transfer coefficients, and (c) the ratio of heat-transfer rates.*

Solution

Known Two horizontal pipes of different diameters in a boiler house.

Find Heat loss from each pipe per 100 m length.

Schematic



Assumptions (1) Steady operating conditions. (2) Air is an ideal gas. (3) Constant properties.

Properties At the film temperature, $T_f = \frac{1}{2}(T_s + T_\infty) = \left(\frac{400+35}{2}\right)$°C = 217.5°C or 490.65 K, the atmospheric air properties are:

$$k = 40.06 \times 10^{-3}\ \text{W/m°C},\ \nu = 37.59 \times 10^{-6}\ \text{m}^2/\text{s},\ Pr = 0.6844,\ \beta = \frac{1}{T_f(\text{K})}$$

Analysis **Case I: D_1 = 10 cm**: Rayleigh number with the pipe diameter as the characteristic length is, $Ra_1 = \dfrac{g\beta D_1^3 (T_s - T_\infty)Pr}{\nu^2}$

$$= \frac{(9.81\ \text{m/s}^2)[1/(490.65\ \text{K})](0.10\ \text{m})^3(400-35)\text{°C or K}(0.6844)}{(37.59 \times 10^{-6}\ \text{m}^2/\text{s})^2}$$

$$= 3.535 \times 10^6 \qquad (< 10^9)$$

Using the *Churchill–Chu* correlation:

$$Nu_1 = \frac{h_1 D}{k} = 0.36 + \frac{0.518\, Ra^{1/4}}{[1 + (0.559/Pr)^{9/16}]^{4/9}} = 17.28$$

$$\therefore \qquad h_1 = Nu_1 \frac{k}{D_1} = 17.28 \times \frac{40.06 \times 10^{-3}\ \text{W/m°C}}{0.10\ \text{m}} = 6.92\ \text{W/m}^2\ \text{°C}$$

$$\therefore \quad \dot{Q} = h_1(\pi D_1 L)(T_s - T_\infty) = (6.92 \text{ W/m}^2\,°\text{C})(\pi \times 0.10 \text{ m} \times 100 \text{ m})(400 - 35)°\text{C}$$

$= \mathbf{79.35 \times 10^3}$ **W or 79.35 kW** **(Ans.) (a)**

Case II: D_2 = 25 cm:

$$Ra_2 = Ra_1\left(\frac{D_2}{D_1}\right)^3 = 3.535 \times 10^6 \times \left(\frac{25 \text{ cm}}{10 \text{ cm}}\right)^3 = 5.523 \times 10^7$$

$$Nu_2 = 0.36 + \frac{0.518(5.523 \times 10^7)^{1/4}}{[1 + (0.559/0.6844)^{9/16}]^{4/9}} = 33.99$$

$$\therefore \quad h_2 = Nu_2 \frac{k}{D_2} = 33.99 \times \frac{40.06 \times 10^{-3}}{0.25} = 5.45 \text{ W/m}^2\,°\text{C}$$

$$\therefore \quad \dot{Q}_2 = h_2(\pi D_2 L)(T_2 - T_\infty) = (5.45)(\pi \times 0.25 \times 100)(400 - 35)$$

$= \mathbf{156.15 \times 10^3}$ **W or 156.15 kW** **(Ans.) (a)**

Ratio of heat-transfer coefficients, $\dfrac{h_1}{h_2} = \dfrac{6.92}{5.45} = 1.27$ **(Ans.) (b)**

Ratio of heat-transfer rates, $\dfrac{\dot{Q}_1}{\dot{Q}_2} = \dfrac{79.35}{156.15} = \mathbf{0.508}$ **(Ans.) (c)**

EXAMPLE 9.18 *Estimate the free convection heat-transfer coefficient for a horizontal fine wire of 2 mm diameter exposed to atmospheric air at 20°C if the wire surface is maintained at 300°C. Also calculate the maximum allowable current intensity if the wire resistance is 8 ohms per metre. Use the correlation: $Nu_D = 1.18(Ra_D)^{1/8}$.*

Properties of air at 160°C: k = 35.25×10^{-3} W/m K, ν = 29.75×10^{-6} m²/s, Pr = 0.701

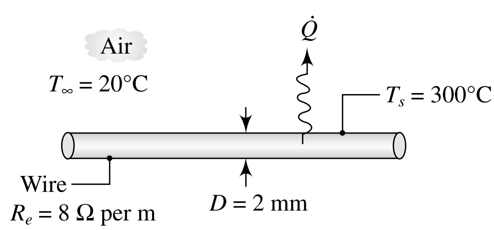
Solution

Known A horizontal fine wire is exposed to air.

Find Free convection heat-transfer coefficient, *h*. Maximum current, *I*.

Assumptions (1) Steady-state conditions exist. (2) Air is quiescent fluid and behaves as an ideal gas.

Schematic



Analysis Film temperature,

$$T_f = \frac{T_s + T_\infty}{2} = \frac{300 + 20}{2} = 160°\text{C}$$

$\beta_{\text{ideal gas}} = 1/(160 + 273.15) = 1/433.15$ K

Rayleigh number,

$$Ra_D = \frac{g\beta(T_s - T_\infty)D^3\,Pr}{\nu^2} = \frac{(9.81 \text{ m/s}^2)(1)(300 - 20)°\text{C}(0.002)^3 \text{ m}^3(0.701)}{(433.15)K(29.75 \times 10^{-6})^2 \text{ m}^2/\text{s}^2} = 40.18$$

For a fine wire: Nusselt number,

$$\overline{Nu}_D = 1.18(Ra_D)^{1/18} = 1.18\,(40.18)^{1/8} = 1.872$$

Average convection coefficient,

$$\bar{h} = \overline{Nu}_D\, k/D = (1.872)\ (35.25 \times 10^{-3}\ \text{W/m K})/0.002\ \text{m} = \mathbf{33.0\ W/m^2\ K} \qquad \textbf{(Ans.)}$$

Heat-transfer rate per metre length of wire,

$$\dot{Q} = \bar{h}\ (\pi D)\ (T_s - T_\infty)$$

$$= (33\ \text{W/m}^2\text{K})\ (\pi \times 0.002\ \text{m})\ (300 - 20)°\text{C} = 58.06\ \text{W} \qquad \textbf{(Ans.)}$$

But $\dot{Q} = I^2 R_c$ where I is the maximum current intensity.

$$I = \sqrt{\frac{\dot{Q}}{R_e}} = \left(\frac{58.06}{8}\right)^{1/2} = \mathbf{2.69\ A} \qquad \textbf{(Ans.)}$$

EXAMPLE 9.19 *An uninsulated horizontal pipe of 20 cm diameter carrying high pressure steam is exposed to quiescent atmospheric air at 20°C. The surroundings are also at 20°C. The outer-surface temperature of the pipe is 180°C. The total heat loss per metre length of the pipe by both convection and radiation is 1700 W. Determine (a) the radiation heat-transfer coefficient, and (b) the pipe surface emissivity.*

The properties of air at 100°C are the following:

Thermal conductivity: 0.0321 W/m K, Kinematic viscosity: 23.13 × 10⁶ m²/s, Prandtl number: 0.688

The following relation for free convection around a long horizontal cylinder may be used:

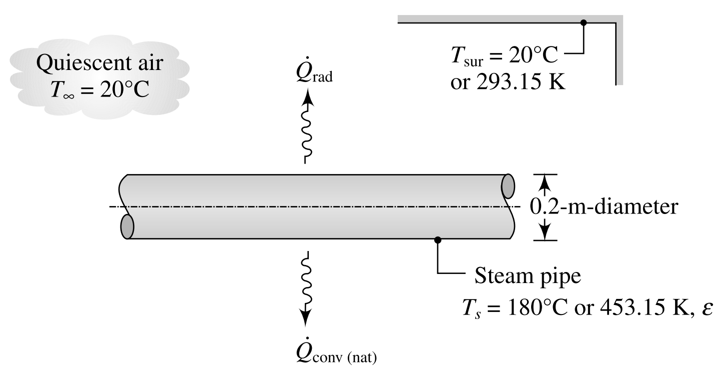
$$\text{Nu} = \left\{ 0.60 + 0.387\, \text{Ra}_\text{D}^{1/6} \left[1 + \left(\frac{0.559}{\text{Pr}}\right)^{9/16} \right]^{-8/27} \right\}^2$$

Solution

Known A horizontal steam pipe loses heat by convection and radiation.

Find h_{rad} and ε.

Schematic



Assumptions (1) Steady operating conditions. (2) Diffuse-gray isothermal pipe surface. (3) Air is idealized to behave as a prefect gas.

Analysis Total heat loss *per metre length* of the pipe is

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{nat conv}} + \dot{Q}_{\text{rad}} = h_{\text{total}}\ (\pi D)\ (T_s - T_\infty) = (h_{\text{conv}} + h_{\text{rad}})\ (\pi D)\ (T_s - T_\infty)$$

The natural convection heat-transfer coefficient is obtained from the given correlation.

$$Nu = \frac{hD}{k} = \{0.60 + 0.387\, Ra_D^{1/6}\, [1 + (0.559/\mathrm{Pr})^{9/16}]^{-8/27}\}^2$$

where $$Ra_D = \frac{g\beta D^3 (T_s - T_\infty)\mathrm{Pr}}{\nu^2}$$

With $$\beta_{\text{ideal gas}} = \frac{1}{T_f\,(K)} = \left[\frac{453.15 + 293.15}{2}\right]^{-1} = 0.00268\ \mathrm{K}^{-1}$$

$$Ra_D = \frac{(9.81)(0.00268)(0.2)^3(180-20)(0.688)}{(23.13\times 10^{-6})^2} = 4.327\times 10^7$$

$$h_{\text{conv}} = Nu\frac{k}{D} = \frac{0.0321}{0.2}\times\left\{0.60 + 0.387(4.327\times 10^7)^{1/6}\left[1+\left(\frac{0.559}{0.688}\right)^{9/16}\right]^{-8/27}\right\}^2$$

$= 70\ \mathrm{W/m^2K}$

Per unit length: $\dot{Q}_{\text{total}} = 1700\ \mathrm{W} = h_{\text{total}}\ (\pi \times 0.2)\ (180 - 20)$

$$\therefore \qquad h_{\text{total}} = \frac{1700}{\pi \times 0.2 \times 160} = 16.91\ \mathrm{W/m^2\,K} = h_{\text{conv}} + h_{\text{rad}}$$

Radiation heat-transfer coefficient,

$h_{\text{rad}} = 16.91 - 7.00 = \mathbf{9.91\ W/m^2\ K}$ **(Ans.) (a)**

We note that $h_{\text{rad}} = \varepsilon\sigma(T_s + T_{\text{sur}})(T_s^2 + T_{\text{sur}}^2)$

Pipe surface emissivity is

$$\varepsilon = \frac{9.91}{(5.67\times 10^{-8})(453.15+293.15)(453.15^2+293.15^2)} = \mathbf{0.804}$$ **(Ans.) (b)**

Comment The contribution of radiation heat loss is $(h_{\text{rad}}/h_{\text{total}})(100) = (9.61/16.91)\ (100) = 58.6\%$. The radiation effects should, therefore, be considered in free convection heat-transfer situations.

(E) Sphere

EXAMPLE 9.20 *A 60 W incandescent light bulb of 80 mm diameter placed in a room at 25°C is able to convert only a fraction of the electrical power it consumes into light. The remaining power is dissipated through convective and radiative heat transfer to the surroundings. The emissivity of the glass of the lightbulb is 0.90 and the equilibrium temperature of the glass bulb is 169°C. The interior surfaces of the room may be assumed to be at the room temperature.*

Determine (a) the percentage of power consumed by the bulb converted into light, and (b) the radiation heat-transfer coefficient. Use the following correlation

$$\overline{\mathrm{Nu}}_{\mathrm{D}} = 2 + 0.589\,\mathrm{Ra}_{\mathrm{D}}^{1/4}[1 + (0.469/\mathrm{Pr})^{9/16}]^{-4/9}$$

The following properties of air at 370 K may be used:

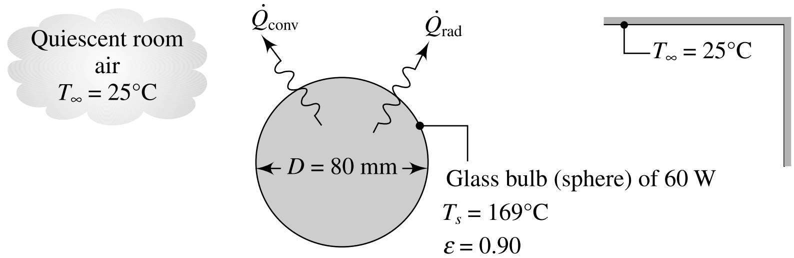
$k = 31.5 \times 10^{-3}$ W / m K, $\nu = 23.12 \times 10^{-6}\ \mathrm{m^2/s}$, $\mathrm{Pr} = 0.696$

Solution

Known A glass bulb dissipates heat by convection and radiation to the room air.

Find (a) Percentage of lightbulb power converted into light. (b) Radiation heat-transfer coefficient, h_r(W/m^2 K).

Schematic



Assumptions (1) Steady operating conditions exist. (2) Air is an ideal gas. (3) Constant properties.

Analysis The glass bulb can be idealized as a sphere of diameter $D = 0.08$ m

Total heat transfer,

$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{rad} = \bar{h} A_s (T_s - T_\infty) + \sigma A_s \varepsilon [T_s^4 - T_{sur}^4]$$

To find the convective heat-transfer coefficient, $\bar{h}$, we need to determine the Rayleigh number. For a *sphere*, the characteristic dimension is the diameter D.

$$Ra_D = \frac{g\beta D^3 (T_s - T_\infty) Pr}{\nu^2}$$

Film temperature, $T_f = \frac{1}{2}(T_s + T_\infty) = \frac{1}{2}(169 + 25) = 97°C$

$$\therefore \qquad \beta_{\text{ideal gas}} = \frac{1}{T_f(K)} = \frac{1}{(97 + 273.15\text{ K})} = 2.70 \times 10^{-3}\text{ K}^{-1}$$

$$Ra_D = \frac{(9.81\text{ m/s}^2)(2.70 \times 10^{-3}\text{K}^{-1})(0.08\text{ m})^3 (169 - 25)°C(0.696)}{(23.12 \times 10^{-6}\text{ m}^2/\text{s})^2}$$

Nusselt number, $\overline{Nu}_D = 2 + 0.589(2.544 \times 10^6)^{1/4}\left[1 + \left(\frac{0.469}{0.696}\right)^{9/16}\right]^{-4/9} = 20.11$

The convection coefficient, $h = \frac{k}{D} Nu_D = \left(\frac{31.5 \times 10^{-3}\text{ W/mK}}{0.08\text{ m}}\right)(20.11) = 7.92\text{ W/m}^2\text{ K}$

Heat dissipated by convection,

$\dot{Q}_{conv} = \bar{h}\,(\pi D^2)\,(T_s - T_\infty) = (7.92\text{ W/m}^2\text{ K})(\pi \times 0.08)^2\text{ m}^2(169 - 25)°C$ or K $= 22.9$ W

Heat dissipated by radiation, $\dot{Q}_{rad} = \sigma A_s \varepsilon (T_s^4 - T_{sur}^4)$

$= (5.67 \times 10^{-8}\text{ W/m}^2\text{ K}^4)\,(\pi \times 0.08^2)\text{ m}^2\,(0.9)\,[(169 + 273.15)^4 - (25 + 273.15)^4]\text{K}^4$

$= 31.1$ W

Total heat-dissipation rate, $\dot{Q}$ = (22.9 + 31.1) W = 54 W

The glass bulb heats up quickly as a result of absorbing electrical energy converted into heat. This heat is then dissipated to the surroundings by both *convection* and *radiation*.

$\therefore$ Percentage of power converted into heat = $\dfrac{54\text{ W}}{60\text{ W}} \times 100 = \mathbf{90\%}$

Hence, the incandescent lightbulb is able to convert only **10%** of the bulb power it consumes into light. **(Ans.) (a)**

Total (convection + radiation) heat-transfer coefficient,

$$\bar{h} = \bar{h}_c + \bar{h}_r = \frac{\dot{Q}}{A_s(T_s - T_\infty)} = \frac{54.0\text{ W}}{\pi(0.08\text{ m})^2(169-25)°\text{C or K}} = 18.65\text{ W/m}^2\text{ K}$$

The convective heat-transfer coefficient, $\bar{h}_c$ = 7.92 W/m² K

Radiation heat-transfer coefficient,

$$\bar{h}_r = \bar{h} - \bar{h}_c = 18.65 - 7.92 = \mathbf{10.73\ W/m^2\ K} \qquad \textbf{(Ans.) (b)}$$

Comment The incandescent lightbulb is inexpensive but very inefficient means of converting electrical energy into light. The fluorescent tube is much more efficient.

(F) Enclosures (Channels/Cavities/Concentric Cylinders/Concentric Spheres)

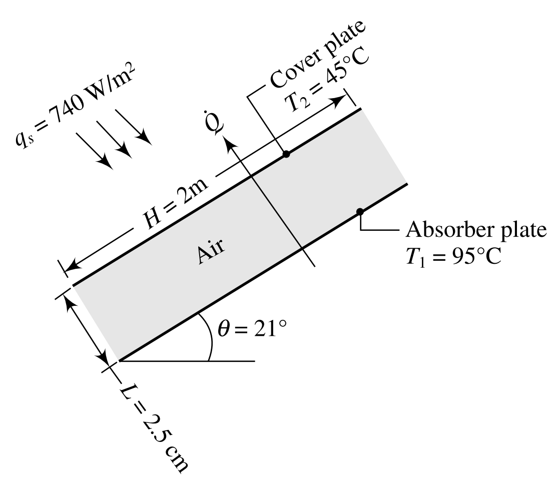
EXAMPLE 9.21 *A flat-plate solar collector 2 m long and 1 m wide has its glass cover plate separated from the absorber plate by an air gap of 2.5 cm. The average temperature of the absorber plate is 95°C, while that of the glass cover is 45°C. The 2 m side of the collector is inclined at an angle of 21° to the horizontal. (a) Determine the rate at which heat flows by free convection from the absorber plate to the glass cover. (b) If the incident solar heat flux to the system is 740 W/m², what is the percentage heat loss by convection?*

Solution

Known Absorber plate and glass-cover temperatures and geometry for a flat-plate solar collector.

Find Heat-transfer rate by free convection, $\dot{Q}[W]$.

Assumptions (1) Air is an ideal gas at 1 atm in the spacing. (2) Radiation effects are not considered.



Properties *Air:* $\left(1\text{ atm}, T_m = \frac{1}{2}(T_1 + T_2) = 70°\text{C}\right)$

k = 0.02881 W/m °C, ν = 19.95 × 10^{-6} m²/s, $Pr = 0.7177$, $\beta = \dfrac{1}{343.15}\text{K}^{-1}$

Analysis For the inclined enclosure, $Ra_L = \dfrac{g\beta(T_1 - T_2)L^3}{\nu^2} Pr$

$$Ra_L = \frac{(9.81\ \text{m/s}^2)[1/(343.15\ \text{K})](95-45)°\text{C}\,(0.025\ \text{m})^3(0.7177)}{(19.95\times10^{-6}\ \text{m}^2/\text{s})^2} = 40275$$

Aspect ratio, $\dfrac{H}{L} = \dfrac{2\ \text{m}}{0.025\ \text{m}} = 80$

With $\dfrac{H}{L} > 12$ and tilt angle θ < 70°, the appropriate correlation is

$$\overline{Nu}_L = 1 + 1.44\left[1 - \frac{1708}{Ra_L\cos\theta}\right]^*\left\{1 - \frac{1708(\sin 1.8\,\theta)^{1.6}}{Ra_L\cos\theta}\right\} + \left[\frac{(Ra_L\cos\theta)^{1/3}}{18} - 1\right]^*$$

$$= 1 + 1.44\left[1 - \frac{1708}{(40275\cos 21°)}\right]^*\left\{1 - \frac{1708\left(\sin 1.8\times 21\right)^{1.6}}{40275\cos 21°}\right\} + \left[\frac{(40275\cos 21°)^{1/3}}{18} - 1\right]^*$$

$$= 1 + 1.44(0.9546)(0.9792) + 0.86 = 3.21 = \frac{\bar{h}L}{k}$$

Average convection heat-transfer coefficient,

$$\bar{h} = \overline{Nu}_L\frac{k}{L} = 3.21\times\frac{0.02881\ \text{W/m°C}}{0.025\ \text{m}} = 3.7\ \text{W/m}^2\,°\text{C}$$

Hence, the rate of heat transfer is

$$\dot{Q} = \bar{h}A_s(T_1 - T_2) = \bar{h}(HW)(T_1 - T_2)$$

$$= (3.7\ \text{W/m}^2°\text{C})(2\ \text{m}\times 1\ \text{m})(95-45)°\text{C} = \mathbf{370\ W}$$ **(Ans.) (a)**

Percentage heat loss by convection is calculated as

$$\frac{\dot{Q}}{\dot{Q}_{\text{in}}} = \frac{\dot{Q}}{q_sA_s} = \frac{370\ \text{W}}{740\ \text{W/m}^2\times 2\ \text{m}^2} = \mathbf{0.25}\quad\textbf{or}\quad\mathbf{25\%}$$ **(Ans.) (b)**

EXAMPLE 9.22 *Liquid nitrogen is stored in a thin-walled spherical vessel of 1 m diameter. The vessel is placed concentrically within a larger, thin-walled spherical container of 1.1 m diameter, and the annular space is filled with atmospheric helium. Under normal operating conditions, the inner and outer surface temperatures are –196°C and 22°C, respectively. If the latent heat of evaporation of nitrogen is 201 kJ/kg, determine the mass-flow rate of gaseous nitrogen vented from the system.*

Properties of helium (P = 1 atm, T = – 87°C) are

ν = 57.56 × 10^{-6} m²/s, Pr = 0.6955, k = 0.1115 W/m K, α = 0.8285 × 10^{-4} m²/s

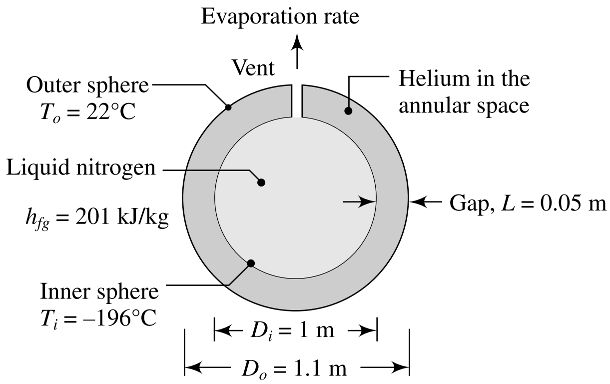
Solution

Known In a concentric spherical container, liquid nitrogen is contained in the inner sphere while atmospheric helium fills up the annular space.

Find Rate of evaporated liquid nitrogen vented off, $\dot{m}$

Assumptions (1) Helium is a quiescent ideal gas. (2) Constant properties. (3) Radiation effects are not considered. (4) Concentric spheres are isothermal.

Schematic



Analysis The average temperature is

$$T_{\text{mean}} = \frac{T_i + T_o}{2} = \frac{-196 + 22}{2} = -87°\text{C} = 186.15 \text{ K}$$

$$\beta_{\text{ideal gas}} = \frac{1}{186.15}\text{K}^{-1} = 5.372 \times 10^{-3} \text{ K}^{-1}$$

Energy balance for a control surface about the liquid nitrogen gives: $\dot{Q} = \dot{m}h_{fg} = \dot{Q}_{\text{conv}}$
From the *Raithby and Hollands* correlations for free convection between concentric spheres, we have

$$\dot{Q}_{\text{conv}} = k_{\text{eff}}\pi\left(\frac{D_i D_o}{L}\right)(T_o - T_i) \tag{1}$$

where the characteristic length, $L = \frac{1}{2}(D_o - D_i)$

The *Raithby and Hollands* relation is

$$\frac{k_{\text{eff}}}{k} = 0.74\left[\frac{Pr}{(0.861 + Pr)}\right]^{1/4} (Ra_s^*)^{1/4} \tag{2a}$$

where $$Ra_{sph}^* = \left[\frac{L}{(D_o D_i)^4}\frac{Ra_L}{(D_i^{-7/5} + D_o^{-7/5})^5}\right] \tag{2b}$$

Rayleigh number is

$$Ra_L = \frac{g\beta(T_o - T_i)L^3}{\nu\alpha}$$

$$= \frac{9.81 \text{ m/s}^2 \times 0.005372 \text{ K}^{-1} \times [22 - (-196)]°\text{C}(0.05 \text{ m})^3}{57.56 \times 10^{-6} \text{m}^2/\text{s} \times 0.8285 \times 10^{-4} \text{m}^2/\text{s}} = 3.01 \times 10^5$$

Using Eq. (2b),

$$Ra_{sph}^{*} = \left\{\frac{0.05\text{ m}}{(1\text{ m}\times 1.1\text{ m})^4}\right\}\left\{\frac{3.01\times 10^5}{[(1)^{-7/5}+(1.1)^{(-7/5)}]^5\text{m}^{-7}}\right\} = \frac{0.05\times 3.01\times 10^5}{(1.1)^4(1+1.1^{-1.4})^5} = 433.66$$

Effective thermal conductivity is then obtained using Eq. (2a):

$$k_{\text{eff}} = 0.74(0.1115\text{ W/mK})\left[\frac{0.6955}{0.861+0.6955}\right]^{1/4}(443.66)^{1/4} = 0.3096\text{ W/mK}$$

And, the heat-transfer rate is found from Eq. (1).

$$\dot{Q}_{\text{conv}} = (0.3096\text{ W/mK})(\pi)\left[1\text{ m}\times\frac{1.1\text{ m}}{0.05\text{ m}}\right]\ [22-(-196)]°\text{C} = 4665\text{ W}$$

The rate at which the stored liquid nitrogen is vented after evaporation is

$$\dot{m} = \frac{\dot{Q}_{\text{conv}}}{h_{fg}} = \frac{4665\text{ W or J/s}}{201\times 10^3\text{ J/kg}} = \mathbf{0.0232\ kg/s\quad or\quad 83.55\ kg/h} \qquad \textbf{(Ans.)}$$

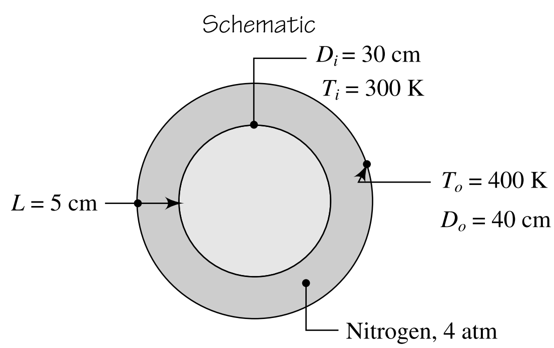
EXAMPLE 9.23 *An annulus formed by two concentric, horizontal tubes is filled with nitrogen at 4 atm. The inner and outer surface temperatures and diameters are 300 K, 30 cm and 400 K, 40 cm respectively. Calculate the heat-transfer rate by convection per unit length of the tubes.*

Properties of nitrogen at 1 atm, 350 K are

$\rho = 0.9625\text{ kg/m}^3$, $C_p = 1.042\text{ kJ/kg K}$, $k = 0.0293\text{ W/m K}$, $\mu = 200\times 10^{-7}\text{ N s/m}^2$

Solution

Known The space between concentric cylinders with specified diameters and temperatures is filled with nitrogen at 4 atm.



Find $\dot{Q}/L$ (W/m).

Assumptions (1) Ideal-gas behaviour of nitrogen. (2) Density is proportional to pressure. (3) Properties like k, μ, and Pr are independent of pressure.

Properties

$Pr = C_p\,\mu/k = (1.042\times 10^3)\times(200\times 10^{-7})/0.0293 = 0.711$

With $\rho \propto P$, $\rho_{(4\text{ atm})} = 4\times 0.9625\text{ kg/m}^3$,

$$\nu = \mu/\rho = \frac{200\times 10^{-7}}{4\times 0.9625} = 5.195\times 10^{-6}\text{ m}^2/\text{s}$$

$$\alpha = k/\rho C_p = 0.0293/(4\times 0.9625\times 1042) = 7.304\times 10^{-6}\text{ m}^2/\text{s}$$

Analysis For the annular region, $Ra_L = \dfrac{g\beta(T_o - T_i)L_c^3}{\nu\alpha}$

where $L = (D_o - D_i)/2 = \frac{1}{2}(0.4 - 0.3)\text{ m} = 0.05\text{ m}$

and $\beta = \frac{1}{T} = \frac{1}{(400+300)/2} = \frac{1}{350}\text{K}^{-1}$

Then $Ra_L = \frac{(9.81\text{ m/s}^2)[1/(350\text{ K})](400-300)\,K(0.05\text{ m})^3}{(5.195\times10^{-6}\text{m}^2/\text{s})(7.304\times10^{-6}\text{ m}^2/\text{s})} = 9.233\times10^6$

and $Ra^*_{\text{cyl}} = [\ln(D_o/D_i)]^4\, Ra_L/L_c^3\,(D_i^{-3/5} + D_o^{-3/5})^5$

$$= [\ln(40/30)]^4 \times (9.233\times10^6)/(0.05\text{ m})^3\,(0.3^{-0.6} + 0.4^{-0.6}) = 645\,135$$

Furthermore,

$$\frac{k_{\text{eff}}}{k} = 0.386\left[\frac{Pr}{0.861+Pr}\right]^{1/4}(Ra^*_{\text{cyl}})^{1/4} = 0.386\left[\frac{0.711}{0.861+0.711}\right]^{1/4}(645\,135)^{1/4} = 8.97$$

Therefore, the convective heat-transfer rate is

$$\frac{\dot{Q}}{L} = \frac{2\pi k_{\text{eff}}}{\ln(D_o/D_i)}(T_o - T_i) = \frac{2\pi\times8.97\times0.0293\text{ W/m K}\times(400-300)\text{ K}}{\ln(40/30)}$$

$= \mathbf{574\ W/m}$ **(Ans.)**

Comment Heat rate by convection is *about nine* times that for conduction. Radiation heat exchange will also be significant in this case. An increase in nitrogen pressure reduces ν which in turn increases Ra_L, thereby increasing the free convection heat-transfer.

(G) Rotating Cylinders/Discs/Spheres

EXAMPLE 9.24 *A vertical hot oven door, 0.6 m high, is at 200°C and is exposed to atmospheric pressure air at 40°C. If the door is subjected to an upward forced flow of air, find the minimum free stream air velocity for which free convection effects may be neglected. Use the following properties of air at 120°C and 1 atm.*

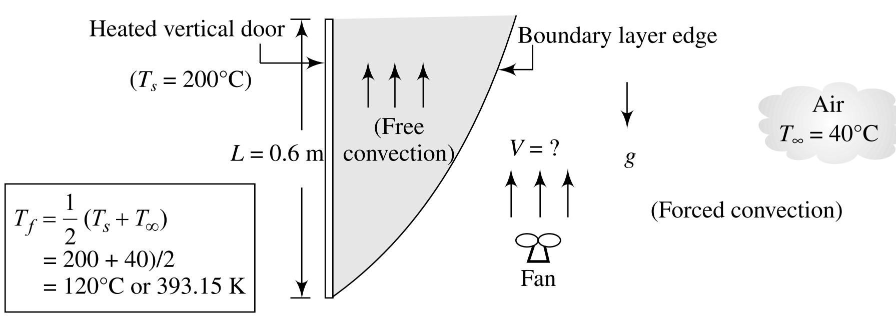
$k = 0.03235$ W/m K, $\nu = 25.22\times10^{-6}$ m²/s, $Pr = 0.7073$

Solution

Known A vertical door exposed to atmospheric air is subjected to mixed free and forced convection.

Find Minimum air velocity when natural convection is neglected, V(m/s).

Schematic



Assumptions (1) Air is an ideal gas at 1 atm. (2) Radiation heat transfer is not considered. (3) Steady operating conditions.

Analysis We note that the *Richardson* number, *Ri* should be less than 0.1 if natural convection effect is to be neglected.

To find minimum free stream air velocity to ensure pure forced convection, we equate

$$Ri = \frac{Gr_L}{Re_L^2} = 0.1$$

Reynolds number, $Re_L = \left[\frac{Gr_L}{0.1}\right]^{1/2} = \sqrt{\frac{10\,g\beta L^3 (T_s - T_\infty)}{\nu^2}}$

where $\beta_{\text{ideal gas}} = \frac{1}{T_f\,(\text{K})} = \frac{1}{(393.15\ \text{K})} = 0.00254\ \text{K}^{-1}$

$$Re_L = \frac{VL}{\nu} = \sqrt{\frac{10 \times 9.81\ \text{m/s}^2 \times 0.00254\ \text{K}^{-1} \times (0.6\ \text{m})^3 \times (200 - 40)°\text{C}}{(25.22 \times 10^{-6}\ \text{m}^2/\text{s})^2}} = 116.44 \times 10^3$$

Velocity of air is

$$V = \frac{116.44 \times 10^3 \times 25.22 \times 10^{-6}\ \text{m}^2/\text{s}}{0.6\ \text{m}} = \mathbf{4.90\ m/s} \qquad \textbf{(Ans.)}$$

EXAMPLE 9.25 *Air at 1 atm and 300 K is forced through a horizontal 30 mm diameter and 0.5 m long tube at a mean velocity of 0.3 m/s. The tube wall is maintained at a constant temperature of 400 K. Determine the average heat-transfer coefficient for this situation and compare it with that obtained for strictly laminar forced convection using the Sieder–Tate correlation.*

Properties of atmospheric air at T_f = 350 K:

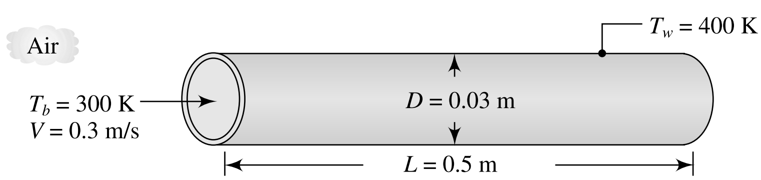
$k = 0.03$ W/m K, $\nu = 20.92 \times 10^{-6}$ m²/s, Pr = 0.7
$\mu_{b(300\ K)} = 18.46 \times 10^{-6}$ kg/ms, $\mu_{f(350\ K)} = 20.82 \times 10^{-6}$ kg/ms
$\mu_{w(400\ K)} = 23.01 \times 10^{-6}$ kg/m s

Solution

Known Air flows through a horizontal tube with a prescribed velocity.

Find Average heat-transfer coefficient, $\bar{h}$, for mixed convection and for forced convection.

Schematic



Assumptions (1) Steady state conditions. (2) Constant properties. (3) Air is an ideal gas.

Analysis Grashof number, $Gr_D = \frac{g\,\beta\,D^3 (T_w - T_\infty)}{\nu^2}$

where $$\beta_{\text{ideal gas}} = \frac{1}{T_f} = \frac{1}{\frac{1}{2}(300+400)\,\text{K}} = \frac{1}{350\,\text{K}}$$

$$\therefore \qquad Gr_D = \frac{(9.81\,\text{m/s}^2)\,[1/(350\,\text{K})](0.03\,\text{m})^3\,(400-300)\,\text{K}}{(20.92\times10^{-6}\,\text{m}^2/\text{s})^2} = 1.729\times10^5$$

Reynolds number,

$$Re_D = \frac{VD}{\nu} = \frac{(0.3\,\text{m/s})\,(0.03\,\text{m})}{20.92\times10^{-6}\,\text{m}^2/\text{s}} = 430.2 \;\; (< 2300) \qquad \Rightarrow \textit{Laminar flow}$$

$$\frac{Gr_D}{Re_D^2} = \frac{1.729\times10^5}{(430.2)^2} = 0.934$$

Since $\dfrac{Gr_D}{Re_D^2}$ is between 0.1 and 10, the situation is one *of mixed convection flow*.

■ ***Combined Free and Forced Convection:***

Graetz number, $Gz = Re_D\, Pr\dfrac{D}{L}$ = (430.2) (0.7) (0.03 m/0.5 m) = 18.07

The appropriate correlation is

$$\overline{Nu}_D = 1.75[Gz + 0.012(Gz\,Gr_D^{1/3})^{4/3}]^{1/3}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

$$= 1.75[18.07 + 0.012\{18.07\times(1.729\times10^5)^{1/3}\}^{4/3}]^{1/3}\left[\frac{18.46\times10^{-6}}{23.01\times10^{-6}}\right]^{0.14} = 8.79$$

The average heat-transfer coefficient is

$$\bar{h} = \overline{Nu}_D\,\frac{k}{D} = 8.79\times\frac{0.03\,\text{W/m°C}}{0.03\,\text{m}} = \mathbf{8.79\,W/m^2\,°C} \qquad \textbf{(Ans.)}$$

■ ***Only Forced Convection:***

Using the Sieder–Tate correlation:

$$\overline{Nu}_D = 1.86(Re_D Pr)^{1/3}\left(\frac{\mu_f}{\mu_w}\right)^{0.14}\left(\frac{D}{L}\right)^{1/3} = 1.86\,Gz^{1/3}(\mu_f/\mu_w)^{0.14}$$

$$= 1.86(18.07)^{1/3}\left(\frac{20.82\times10^{-6}}{23.01\times10^{-6}}\right)^{0.14} = \mathbf{4.81\,W/m^2\,°C} \qquad \textbf{(Ans.)}$$

Comment If the calculations are made on the basis of only laminar forced convection, the percentage error involved would be $\left(\dfrac{8.79-4.81}{8.79}\right)(100)$ = **45.3%.**

Points to Ponder

- Convection heat transfer from buoyancy caused velocities is called *free convection*.
- Grashof number represents the ratio of [(buoyancy forces) × (inertia forces)] to (viscous forces).
- The coefficient of volumetric thermal expansion β is the reciprocal of the absolute film temperature *only for an ideal gas*.
- The volume expansivity, β of liquid metals is *not* $\dfrac{1}{T_f(K)}$.
- In the case of a vertical isothermal plate in natural convection, the fluid velocity at the surface and at the edge of the boundary layer is zero.
- Rayleigh number is a product of Grashof number and Prandtl number.
- The larger the value of the (*GrPr*) product, the more one would expect natural convection effects to prevail.
- In the case of free convection over vertical surfaces, where the linear dimension is the height, the flow in the boundary layer becomes turbulent for a Rayleigh number of about 10^9.
- The location of transition point from the leading edge of a flat plate for flow of a fluid of kinematic viscosity ν over it at a free stream velocity of u_∞ is $Re_{cr}\ \nu/u_\infty$.
- The maximum velocity within the natural convection boundary layer with cubic velocity profile based on the integral solution for a vertical plate is given by $u_{\max}$ = 4/27 U.
- The natural convection correlations for vertical plate (both laminar and turbulent flows) may be applied to a vertical cylinder (diameter *D* and height *L*) if $\dfrac{D}{L} >> Ra_L^{-1/4}$.
- Free convection is of primary importance if the Richardson number defined as the ratio of *Gr* and Re^2 is greater than 10.
- For a horizontal plate, the characteristic length *L* used in the correlations for free convection heat transfer is [Area, *A*/Perimeter, *P*].
- For $Gr_x/\text{Re}_x^2 >> 1$, the flow should be considered as entirely natural convection.
- The effective thermal conductivity of an enclosure equals the product of thermal conductivity of the fluid and the Nusselt number.

GLOSSARY of Key Terms

Natural convection	Heat transfer in which the flow is a buoyancy-induced motion resulting from body forces acting on density gradients which, in turn, arise from temperature gradients in the quiescent fluid.
Grashof number	Dimensionless number interpreted as the ratio of busoancy forces to the viscous forces.
Mixed convection	Combined forced and free convection characerized by the ratio Gr/Re^2 in the range of 0.3 to 16.
Volumetric coefficient of thermal expansion	Ratio of density difference to temperature variation times the original density at constant pressure for an ideal gas, this parameters is reciprocal of absolute film temperature.

Critical Rayleigh number	A value of the product numbers which indicates the transition from laminar to turbulent flow in natural convection heat transfer.
Boussinesq approximation	For free convection flows, it neglects all variable property effects except for density in the momentum equation and approximates the density difference terms with a simplifed equation of state.
Benard cells	A pattern of hexagonal cells created when the enclosed fluid layer when two horizontal plates are spaced δ two apart and the lower surface is hotter than the upper surface with $Ra_\delta > 1708$.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

9.1 In the laminar region, the rate of heat transfer by free convection over a vertical plate is proportional to
(a) $(T_s \sim T_\infty)^{1/4}$ (b) $(T_s \sim T_\infty)^{5/4}$ (c) $(T_s \sim T_\infty)^{3/4}$ (d) $(T_s \sim T_\infty)$

9.2 In the turbulent region, the heat-transfer rate of a vertical surface is proportional to
(a) ΔT (b) $\Delta T^{1/3}$ (c) $\Delta T^{4/3}$ (d) $\Delta T^{3/5}$
where $\Delta T \equiv (T_{\text{surface}} \sim T_{\text{fluid}})$

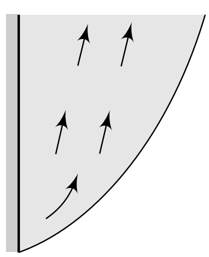
9.3 Free convection flow depends on all of the following except
(a) density (b) coefficient of viscosity
(c) gravitational force (d) velocity

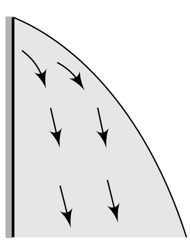
9.4 In natural convection heat transfer, the maximum velocity in the laminar boundary layer occurs when y is approximately equal to
(a) 0 (b) $\delta/3$ (c) $\delta/2$ (d) δ
where δ is the boundary-layer thickness.

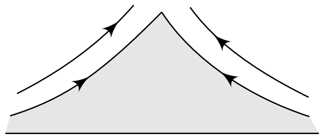
9.5 In natural convection heat transfer, the boundary layer becomes turbulent when (*Gr Pr*) is
(a) equal to 10^9 (b) equal to 10^{10} (c) greater than 10^9 (d) greater than 6×10^{10}

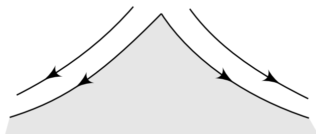
9.6 Match ***List I*** with ***List II*** and select the correct answer using the codes given below the lists:

List I (***Flow pattern***) List II (***Situation***)

A.  **1.** Heated horizontal plate

B.  **2.** Cooled horizontal plate

C.  **3.** Heated vertical plate

D.  **4.** Cooled vertical plate

Codes:	**A**	**B**	**C**	**D**
(a)	4	3	2	1
(b)	3	4	1	2
(c)	3	4	2	1
(d)	4	3	1	2

9.7 When we have simultaneous free and forced convection, Gr/Re^2 is
(a) greater than 1 (b) almost equal to 1
(c) less than 1 (d) equal to Peclet number

9.8 For calculation of heat transfer by natural convection from a horizontal cylinder, what is the characteristic length in Grashof number?
(a) Diameter of the cylinder
(b) Length of the cylinder
(c) Circumference of the base of the cylinder
(d) Half the circumference of the base of the cylinder

9.9 The average Nusselt number in laminar natural convection from a vertical wall at 180°C with still air at 20°C is found to be 48. If the wall temperature becomes 30°C, all other parameters remaining the same, the average Nusselt number will be
(a) 8 (b) 16 (c) 24 (d) 32

Answers

Multiple-Choice Questions

9.1 (b) **9.2** (c) **9.3** (d) **9.4** (b) **9.5** (c) **9.6** (b) **9.7** (b) **9.8** (a)
9.9 (c)

REVIEW QUESTIONS

9.1 What is natural convection? How is it different from forced convection? In which mode of heat transfer will the convection heat-transfer coefficient usually be higher, and why?
9.2 Enumerate some examples of free convection flow.
9.3 What does the Grashof number represent? How is it different from the Reynolds number?
9.4 Show that the coefficient of volumetric thermal expansion of an ideal gas is $\beta = 1/T$, where T is the absolute temperature.
9.5 How does the Rayleigh number differ from the Grashof number?
9.6 What is the criterion for transition from laminar to turbulent boundary layer in free convection on a vertical flat plate?

9.7 Why is an analytical solution of a free convection problem more involved than that of the forced convection problem? What is meant by Boussinesq approximation?

9.8 How is a modified Grashof number defined for a constant heat-flux condition on a vertical flat plate?

9.9 What is the relation between local heat-transfer coefficient and average heat-transfer coefficient for free convection from a vertical flat plate under laminar and turbulent flow conditions?

9.10 Sketch the velocity distribution and temperature distribution for free convection heat transfer on a vertical flat plate when the plate is heated and when it is cooled.

9.11 Specify the boundary conditions which must be satisfied by the velocity and the temperature distribution for a free convection boundary layer on a vertical flat plate.

9.12 Which condition should be satisfied so that the outer surface of a vertical cylinder can be treated as a vertical plate in natural convection calculations?

9.13 A hot horizontal plate whose back side is insulated loses heat to the surroundings. Of the two options, viz., hot surface facing up and hot surface facing down, in which case the plate will cool faster and why?

9.14 Derive an expression for the maximum velocity in the natural convection boundary layer on a vertical isothermal flat plate. At what location in the boundary layer does this maximum velocity occur?

9.15 Mention a few applications of natural convection heat transfer in enclosures. How is *aspect ratio* defined for an enclosure?

9.16 What is meant by the effective thermal conductivity of an enclosure? How are the effective conductivity, fluid thermal conductivity, and the Nusselt number related to each other?

9.17 Under what conditions does natural convection assist forced convection, and under what conditions does it oppose forced convection?

9.18 When neither free nor forced convection is negligible, is it appropriate to calculate Nusselt number for each independently and add the two to determine the total convection heat transfer? If not, how does one evaluate heat-transfer in a mixed convection regime?

9.19 Discuss the criterion for ascertaining the type of convection regime (*free, mixed or forced*) in a specified situation.

PRACTICE PROBLEMS

(A) Exact Analysis, Integral Solution

9.1 (a) A 0.03-m-long glass plate is hung vertically in the air at 27°C while its temperature is maintained at 77°C. Calculate the boundary layer thickness at the trailing edge of the plate and the average heat transfer coefficient. (b) If the same plate is placed in a wind tunnel and the air is blown over it at a velocity of 4 m/s, estimate the boundary layer thickness at its trailing edge and the average heat transfer coefficient.

Properties of air at 52°C:

$\beta = 3.07 \times 10^{-3}$ K^{-1}, $k = 28.15 \times 10^{-3}$ W/m K, $\nu = 18.41 \times 10^{-6}$ m^2/s, $Pr = 0.700$

Use for free convection: $\delta = 3.93 \times (0.952 + \mathrm{Pr})^{1/4} \times \dfrac{1}{Pr^{1/2}\, Gr_x^{1/4}}$

$$Nu_x = 0.508\, Pr^{1/2}\,(0.952 + Pr)^{-1/4} \times Gr_x^{1/4}$$

Use for forced convection: $Nu = 0.664\, Re_L^{1/2}\, Pr^{1/3}$

[(a) 8.58 mm, 8.73 W/m² °C (b) 1.86 mm, 44.66 W/m² °C]

(B) Vertical Plate/Tube

9.2 Calculate the rate of heat input into a vertical steel plate that loses heat by free convection to the ambient air. The plate is 30 cm by 50 cm and is maintained at 150°C. The ambient air temperature is 18°C. Rework the problem if the surrounding fluid is engine oil. Properties of air and engine oil at the film temperature of 84°C are:

Fluid	k [W/m K]	ν [m²/s]	Pr	β[K⁻¹]
Air	**30.5×10^{-3}**	**21.69×10^{-6}**	0.699	0.0028
Engine oil	**138×10^{-3}**	**33.3×10^{-6}**	440.3	**0.70×10^{-3}**

Use the empirical correlation: $\overline{Nu_L} = [0.68 + A/B]$

where $A = 0.67\, Re_L^{1/4}$ $\quad B = [1 + (0.492/Pr)^{9/16}]^{4/9}$

[Air: 220 W, Engine oil: 3348 W]

9.3 A thin electrically heated vertical plate, 25×25 cm, is immersed in a large tank of water. The electrical energy supplied to the plate was measured and found to be 2.5 kW. The average water temperature is 25°C. Assuming constant surface heat flux condition for both vertical surfaces, estimate the maximum surface temperature of the plate.

The following correlations may be used:

$$Nu_x = 0.60(Ra_x^*)^{0.25} \qquad 10^5 < Ra_x^* < 10^{11}$$

$$Nu_x = 0.17(Ra_x^*)^{0.25} \qquad 2 \times 10^{13} < Ra_x^* < 10^{16}$$

where Ra_x^* is the modified Rayleigh number.

The following properties of water (at 40°C) may be used in the calculations:

k(W/m K)	$\nu \times 10^6$(m²/s)	$\alpha \times 10^7$(m²/s)	$\beta \times 10^4$(K⁻¹)
0.631	0.658	1.524	3.77

[56°C]

9.4 A 25 W plate heater has a 20-cm-square area and is 3-mm-thick. It is held vertically and loses heat by free convection from both faces to the surrounding air at 25°C. Neglecting internal temperature gradients, determine the steady-state temperature attained by the heater. Properties of air at atmospheric pressure:

T(K)	$k \times 10^3$ (W/m °C)	$\nu \times 10^6$ (m²/s)	$\alpha \times 10^6$ (m²/s)	Pr
300	26.3	15.89	22.5	0.707
350	30.0	20.92	29.9	0.700
400	33.8	26.41	38.3	0.690

[82.9°C]

9.5 A thin, square vertical steel plate with sides 0.4 m by 0.4-m-long, 4 mm thickness, density 7900 kg/m³, and specific heat 477 J/kg °C is initially at a uniform temperature of 225°C and is exposed to quiescent atmospheric air at 25°C. Heat transfer from the plate to the air takes place by laminar free convection. Using the simplified correlation for free convection in air and assuming that the lumped-system analysis is applicable, develop an expression for the temperature of the plate as a function of time. Estimate the time required for the plate to cool to 115°C. **[992 s]**

9.6 A window, 0.30-m-high and 0.45-m-wide, is centred in an oven door, 0.50-m-high and 0.75-m-wide. During steady operating conditions, when the room and surroundings temperature is 25°C, the window reaches a temperature of 45°C and the door surface attains a temperature of 35°C. Assuming the surface emissivity of unity for both the door and the window, estimate the heat transfer rate (a) from the door and window, and (b) from the door if the door did not have a window.
Properties of dry air at 1 atm:

Temperature (°C)	$k \times 10^2$ (W/m K)	$\nu \times 10^6$ (m²/s)	Pr
30	2.67	16.0	0.701
40	2.76	16.96	0.699

Use the following correlation: $Nu_L = 0.68 + \dfrac{0.670\, Ra_L^{1/4}}{[1+(0.492/\text{Pr})^{9/16}]^{4/9}}$

[(a) 50.34 W (b) 34.5 W]

(C) Horizontal Plate

9.7 Air flows through a long rectangular (30-cm height × 60-cm width) air conditioning duct maintains the duct's outer surface temperature at 15°C. If the duct is uninsulated and exposed to air at 25°C, calculate the heat gained by the duct per metre length, assuming it to be horizontal. Use the following correlations:
(a) Upper surface heated or lower surface cooled: $NU_L = 0.15Ra_L^{1/3}$
(b) Lower surface heated or upper surface cooled: $\overline{Nu}_L = 0.27RA_L^{1/4}$
(c) Vertical surfaces: $\overline{Nu}_L = 0.59RA_L^{1/4}$
Take the properties of air at 20 °C as ρ = 1.205 kg/m³, k = 25.93 × 10⁻³ W/m K, ν = 15.06 × 10⁻⁶ m²/s

[(a) 3.71 W/m²K (b) 3.94 W/m²K (c) 1.70 W/m²K, 56.1 W/m]

(D) Horizontal Cylinder

9.8 A 15-m-long, horizontal copper pipe of 2.5-cm-OD conveys saturated steam at 1.43 bar (T_{sat} = 110°C, h_{fg} = 2230 kJ/kg). The pipe is contained within an environmental testing chamber in which the ambient temperature is maintained at 30°C, and the ambient air pressure can be changed from 0.5 atm to 2 atm. Investigate the effect of this pressure change on the rate of condensation.
Properties of air at 1 atm and 70°C: k = 0.0297 W/m K, $\nu = 20.02 \times 10^{-6}$ m²/s, Pr = 0.694
Use the correlations: $Nu = 0.53(Ra_D)^{1/4}$ $\quad 10^4 < Ra_D < 10^9$
$Nu = 0.13(Ra_D)^{1/3}$ $\quad 10^9 < Ra_D < 10^{12}$

[0.5 atm: 1.067 kg/h, 2 atm: 3.02 kg/h]

9.9 (a) The suction line of a refrigeration system has the refrigerant R-12 (boiling point: – 30°C) inside a copper tubing of 12.5-mm-outside diameter, 2 m of which is exposed to dry stagnant air at 44°C. Calculate the heat gain by free convection. (b) If the suction line is subjected to a cross draft of 5 m/s, what will be the heat gain?
Use the following correlations and properties of dry air at atmospheric air at 7°C:

k = 0.0247 W/m °C, $\nu = 14.11 \times 10^{-6}$ m²/s, Pr = 0.712

Free convection: $Nu = [0.60 + 0.387\, Ra_D^{1/4}\{1+(0.559/\text{Pr})^{9/16}\}^{-8/27}]^2$

Forced convection: $Nu = 0.3 + \dfrac{0.62\, Re_D^{1/2} Pr^{1/3}}{[1+(0.4/Pr)^{2/3}]^{1/4}}\left[1+\left(\dfrac{\mathrm{Re}_D}{282000}\right)^{5/8}\right]^{4/5}$

[(a) 58.0 W (b) 397.0 W]

(E) Sphere

9.10 A 15-cm-diameter polished copper sphere ($\varepsilon = 0.045$) with a surface temperature of 200°C is kept in a room where the air and surrounding objects including the room structure are at 36°C. Determine the heat loss from the sphere, and also the percentage of this loss due to radiation. Use the correlation: $\overline{Nu}_D = 2 + A/B$

where $A = 0.589\, Ra_D^{0.25}$

$B = [1 + (0.469/Pr)^{0.5625}]^{0.4444}$

Properties of air at the film temperature of 118°C:

$k = 0.03312$ W/m °C, $\nu = 25.42 \times 10^{-6}$ m²/s, $Pr = 0.692$ **[84.54 W]**

(F) Enclosures

9.11 A flat-plate solar collector inclined at an angle of 19° to the horizontal has the absorber plate and the glass cover plate maintained at the average temperatures of 70°C and 24°C, respectively. Estimate the convection heat loss from the absorber plate to the glass if the spacing between the two plates is (a) 10 mm and (b) 30 mm. Properties of air: $k = 0.02778$ W/m°C, $\nu = 17.9 \times 10^{-6}$ m²/s, $Pr = 0.7042$

[(a) 420 W (b) 362.5 W]

(H) Combined Free and Forced Convection

9.12 Consider a 0.5-m-wide and 0.4-m-high window with an interior surface temperature of 10°C. A small fan creates a slight upward movement of air over the window with its velocity being of the order of 1 m/s. Determine the rate of heat transfer through the window if the room air temperature is 20°C. The following are the thermophysical properties of air at 15°C:

$k = 0.02476$ W/m °C, $\nu = 1.47 \times 10^{-5}$ m²/s, $Pr = 0.7323$ **[11.74 W]**

10 Heat Transfer with Change in Phase

10.1 ❑ INTRODUCTION

Our foregoing discussions on convection heat transfer were focussed on *homogeneous, single-phase* systems. Convection processes associated with the *change in phase* of a fluid are also of great importance. In this chapter, we will consider those processes which can take place at a *solid-liquid* interface, viz., *boiling* and *condensation*. When a liquid is in contact with a solid surface which is maintained at a temperature greater than the saturation temperature of the liquid, the change from the liquid to the vapour state due to *boiling* can occur. Conversely, *condensation* from a vapour to the liquid phase takes place when a vapour strikes a solid surface that is at a temperature below the corresponding saturation temperature. Very high heat-transfer rates accompanied by quite *small temperature differences* make both these heat-transfer mechanisms of profound practical significance. Clearly, phase-change heat transfer is always associated with very high heat-transfer coefficients. In these cases, latent heat effects associated with the phase change are important. The change from the liquid to the vapour state due to boiling is sustained by heat transfer from the solid surface. Condensation of a vapour to the liquid phase results in heat transfer to the solid surface.

The energy required for phase change for some liquids like water is very high. At a saturation pressure of 1 atm and the corresponding saturation temperature of 100°C, the latent heat of vaporization is 2257 kJ/kg. As the temperature increases, the energy needed to vaporize reduces and the saturation pressure increases. At the peak of the saturation dome, when the specific volumes of liquid and vapour are same. The latent heat is zero and there is no longer any phase change. The phase-change processes for some liquids (like boiling and condensation) occur under the saturation dome at a specific pressure shown in Fig. 10.1. During change of phase, the temperature remains constant at equilibrium. During a dynamic process, however, small temperature changes do exist which cause huge quantity of energy transfer to or from the surface. Naturally, very high heat-transfer coefficients are achieved and the process is an effective means of heat transfer.

Boiling and *condensation* are usually classified as *forms of convective heat transfer mechanism* and the heat transfer coefficient h is used to describe the heat flux q as a function of the temperature difference between the solid surface and the surrounding fluid ΔT_e. This is due to the fact that both involve *fluid motion*.

There are added considerations too that are unique to these phase change processes. It is important to note that the heat transfer to or from a fluid during a phase change can occur almost without affecting the fluid temperature.

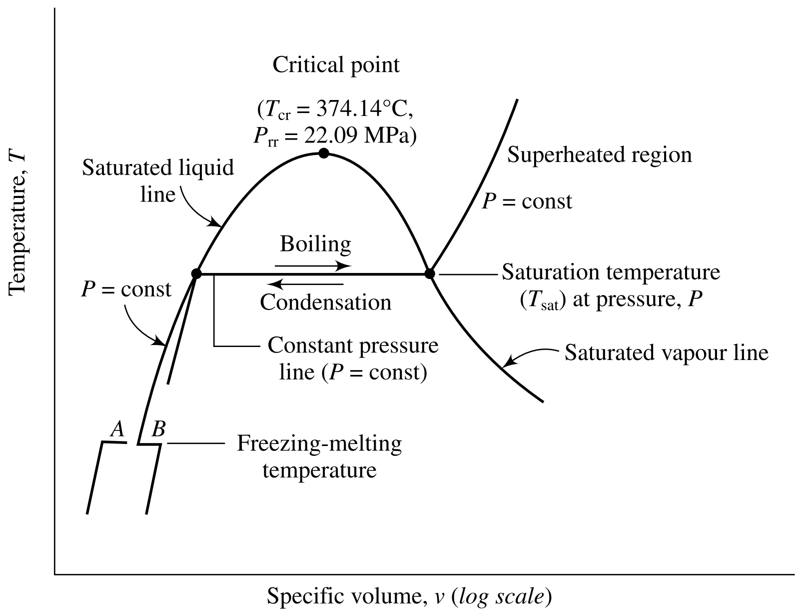


Fig. 10.1 *Saturation dome (temperature-specific volume diagram) indicating phase-change processes for a pure substance which (A) contracts on freezing, and (B) expands on freezing (e.g., water)*

The important parameters which characterize boiling and condensation are

- *Latent heat* (of evaporation or condensation, as the case may be)
- *Surface characteristics*
- *Surface tension* between the liquid-vapour interface, σ, and
- *Density difference* between the two phases (*liquid and vapour*).

Buoyancy force owing to the density difference and proportional to $g(\rho_l - \rho_v)$ combined with *latent heat* h_{fg} contribute to much higher heat-transfer coefficients than in single phase convection systems. The above-mentioned additional parameters make these phase-change processes extremely complex and difficult to analyze.

10.2 ❑ APPLICATIONS

There are several engineering applications in which heat transfer occurs with a phase change. For instance, in a modern steam power plant, the pressurized liquid is converted into vapour (*steam*) in the *boiler* or *steam generator*, and the exhaust steam after expansion in the turbine is restored to the liquid state in the *condenser* for reuse so that the condensed liquid can be pumped back to the boiler to complete the vapour power cycle. In vapour-compression and vapour-absorption refrigeration systems, the *evaporator* where boiling of the liquid refrigerant occurs, and the *condenser,* are both crucial components. Besides, the *petrochemical*, *chemical,* and *metallurgical plants* also employ such heat-exchange equipment, the sound design of which calls for a thorough understanding of the processes of boiling and condensation. The potential of high heat-flux transport with modest temperature differential has also found a recent application in *heat pipes.*

Nuclear reactors, rocket nozzles, spacecraft, etc., are all high-performance machines and release a very large quantity of heat (of the order of 3×10^6 to 3×10^7 W/m^2). Boiling heat transfer is used to cool the structural components in these machines. Hence, understanding its accurate mechanism and limitations is very essential.

10.3 ❑ NON-DIMENSIONAL PARAMETERS IN BOILING AND CONDENSATION

We have already learnt how the appropriate non-dimensional parameters can be obtained by using the Buckingham-pi theorem. For heat transfer with phase change, the heat-transfer coefficient, h depends on the following:

- Difference between the surface and saturation temperatures, $\Delta T \equiv (T_w \sim T_{sat})$.
- Body force arising from the liquid-vapour density difference, $g\,(\rho_l - \rho_v)$.
- Latent heat of vaporization, h_{fg}.
- Surface tension, σ.
- Characteristic length, L.
- Thermo-physical properties of the liquid or vapour: Density ρ, specific heat C_p, thermal conductivity k and dynamic viscosity μ.

It follows that:

$$\boxed{h = f\{\Delta T, g(\rho_l - \rho_v), h_{fg}, \sigma, L, \rho, C_p, k, \mu\}}$$

We note that there are 10 variables in 5 dimensions (namely, mass, length, time, temperature, and energy). Clearly, there will be (10 – 5) = 5 dimensionless or pi-groups, which can be identified as follows:

$$\underbrace{\frac{hL}{k}}_{(1)} = f\left[\underbrace{\frac{\rho g(\rho_l - \rho_v)L^3}{\mu^2}}_{(2)}, \underbrace{\frac{C_p\,\Delta T}{h_{fg}}}_{(3)}, \underbrace{\frac{C_p\,\mu}{k}}_{(4)}, \underbrace{\frac{g(\rho_l - \rho_v)L^2}{\sigma}}_{(5)}\right]$$

The functional relationship involving dimensionless parameters is given by:

$$\boxed{Nu_L = f\left[\frac{\rho g(\rho_l - \rho_v)L^3}{\mu^2}, Ja, Pr, Bo\right]} \tag{10.1}$$

The *Nusselt* and *Prandtl* numbers are fairly familiar by now. The new dimensionless parameters are the *Jacob number* Ja, the *Bond* number, *Bo*, and an unnamed parameter which is similar to the *Grashof number*, and represents the effect of buoyancy-induced fluid motion on heat transfer.

The *Jacob number* is the ratio of the maximum sensible energy absorbed by the liquid (vapour), i.e., $C_p\Delta T$ to the latent energy absorbed by the liquid during condensation (boiling), i.e., h_{fg}. In many applications, the sensible energy is much less than the latent energy and Ja has a small numerical value.

The *Bond number* is the ratio of the buoyancy force, i.e., $(\rho_l - \rho_v)L^3\,g$ to the surface tension force, i.e., σL.

10.4 ❑ TYPES OF BOILING

Boiling heat transfer is defined as a mode of heat transfer with change in phase from liquid to vapour at the solid-liquid interface. Boiling is the process of intensive vaporization in the whole volume of a liquid that may be at the saturation temperature or slightly superheated, accompanied by the vapour bubble formation. Boiling is possible in the entire temperature range right from triple point to the critical point (0.01°C to 374.14°C for water). There is a popular misconception that evaporation and boiling are the same phenomena. The fact of the matter is that while boiling occurs at the *solid-liquid interface* characterised by vapour bubble formation and growth at the heating surface the evaporation process takes place at a *liquid vapour boundary*.

The principal parameters of interest are:

- The temperature of the metal surface T_w.
- The saturation temperature T_{sat} corresponding to the liquid pressure.
- The heat flux q at the surface.

Heat is transferred from the solid surface to the liquid according to the following equation:

$$\boxed{q = h(T_w - T_{\text{sat}}) = h\Delta T_e} \tag{10.2}$$

where h is the boiling heat-transfer coefficient, and $\Delta T_e \equiv (T_w - T_{sat})$ is the *driving potential* called the *excess temperature* of the surface above the saturation temperature of the liquid. Both T_w and q can be determined experimentally.

10.4.1 • Pool Boiling and Flow Boiling

Boiling may be classified as either *pool boiling* or a forced convection flow known as *flow boiling*. *Pool boiling* is boiling on a heating surface submerged in a *pool* of initially quiescent, i.e., still liquid. Heat is supplied to a stationary quantity of fluid from an immersed heating surface. The fluid motion near the surface is due to natural convection and to mixing caused by bubble growth and detachment. Although there is a sharp decline in the liquid temperature close to the solid surface, the temperature through most of the liquid remains slightly above saturation. Bubbles generated at the *liquid-solid* interface, therefore, rise to and are transported across the liquid-vapour interface. A familiar example is the boiling of water placed in a kettle on the top of a stove or heater. The heat can also be transferred through a heating coil.

In forced convection boiling or flow boiling, the fluid motion is induced by external agencies, (*like a pump*) as well as by natural convection and bubble-induced mixing. Flow boiling occurs when there is a forced flow of a stream of fluid over the surface where the heating surface may be the tube or channel wall confining the flow as the boiler tubes. A boiling flow is composed of a mixture of liquid and vapour and is a type of *two-phase* flow. Figure 10.2 shows pool boiling as distinguished by flow boiling.

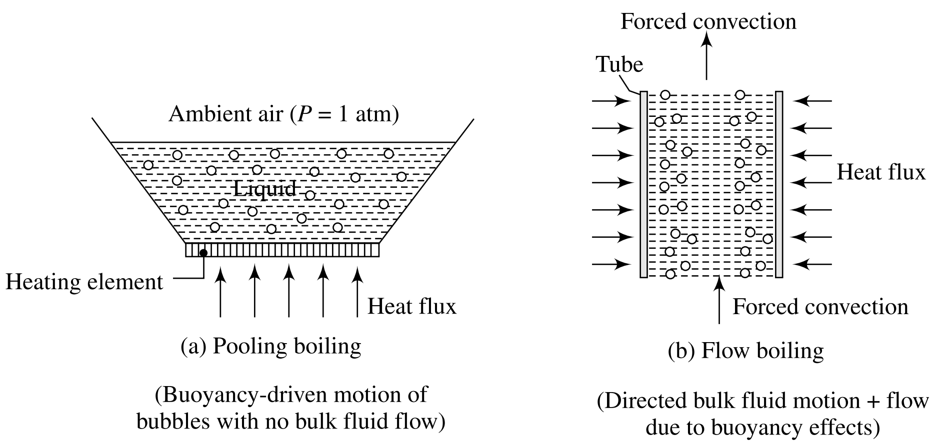


Fig. 10.2 *Types of boiling based on the bulk fluid motion*

10.4.2 • Subcooled and Saturated Boiling

Boiling is also categorized depending on whether it is *subcooled* or *saturated* (Fig. 10.3). In *subcooled or local boiling*, the temperature of the pool of the liquid is less than the saturation temperature corresponding to the pressure of the liquid. The bubbles formed at the surface eventually condense in the liquid. The temperature of the liquid is slightly more than the saturation temperature in *saturated or bulk boiling*. Bubbles formed at the surface rise through the liquid by buoyancy forces, finally escaping from the free surface. Figure 10.4 depicts the temperature distribution in saturated (*bulk*) pool boiling. Temperature profile in subcooled (*local*) pool boiling with a liquid-vapour interface is shown in Fig. 10.5.

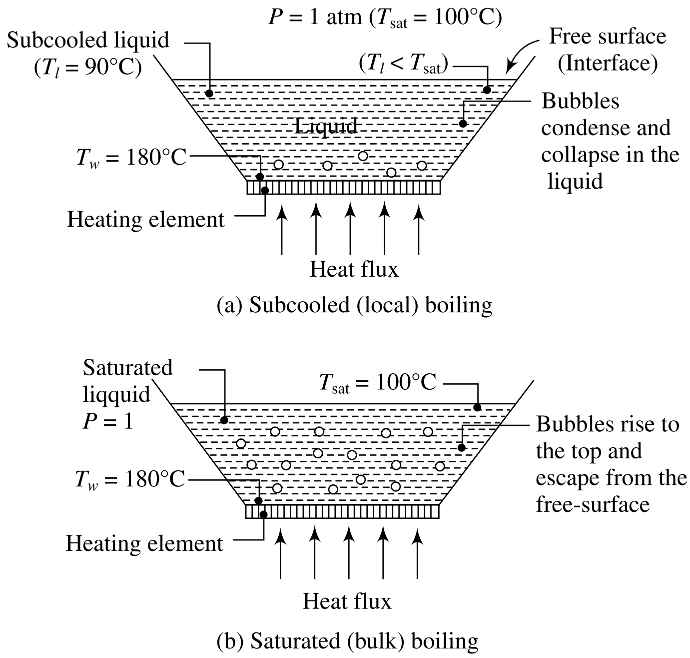


Fig. 10.3 *Types of boiling based on bulk fluid temperature*

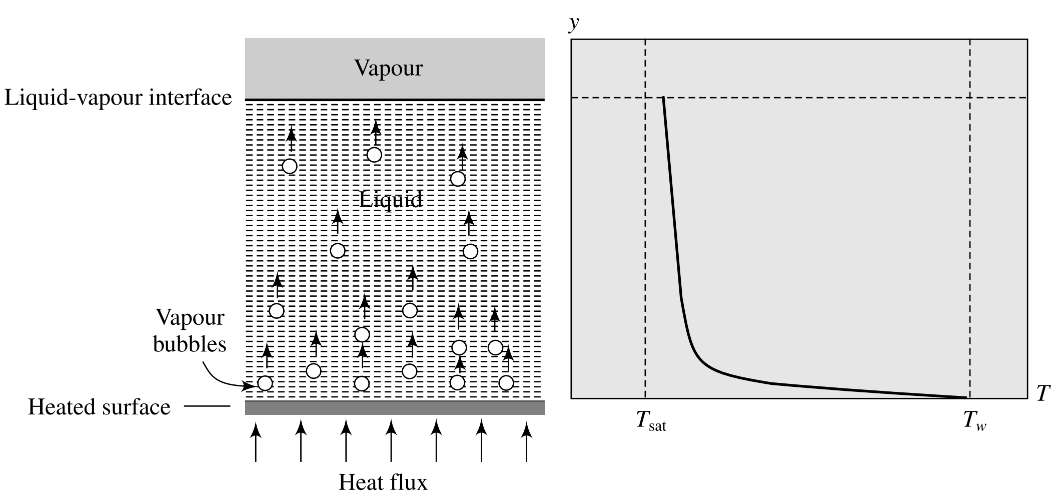


Fig. 10.4 *Temperature profile in saturated pool boiling [$T_l \geq T_{sat}$]*

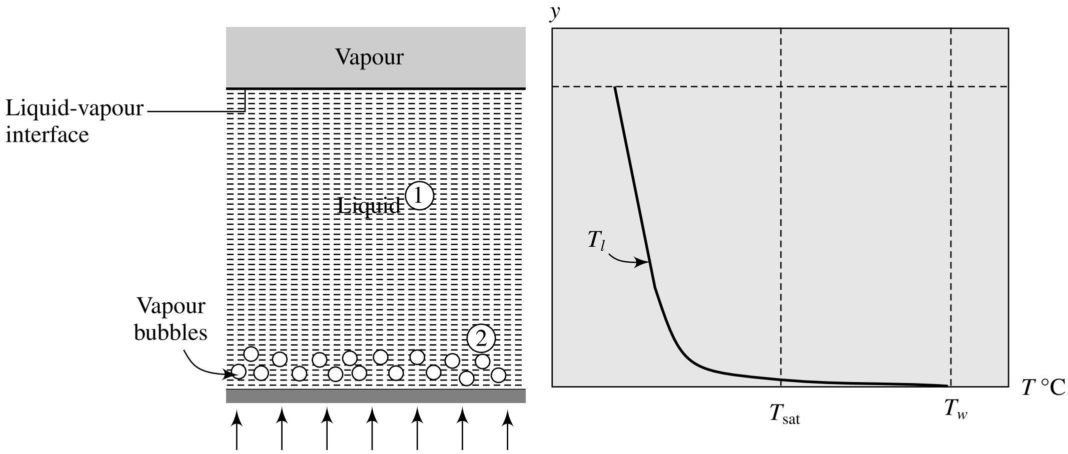


Fig. 10.5 *Temperature profile in subcooled liquid: (1) Single-phase liquid; (2) Boiling boundary layer*

10.5 ❑ THE BOILING CURVE

Nukiyama (1934) of Japan was the first to properly identify different regimes of pool boiling. He calculated both the heat flux and the temperature from a horizontal *nichrome* wire (melting point 1500 K) immersed in saturated water by measuring the current flow and voltage drop. In *power-controlled* heating used by him, the wire temperature T_w (*hence, the excess temperature* ΔT_e) is the dependent variable and the power setting (hence, the heat flux) is the *independent* variable. Figure 10.6 shows a typical boiling curve in which the heat flux is plotted against the excess temperature ($\Delta T_e \equiv T_w - T_{sat}$) when the controlling parameter is the power input to the heater. The effect of increasing pressure is also shown in the figure.

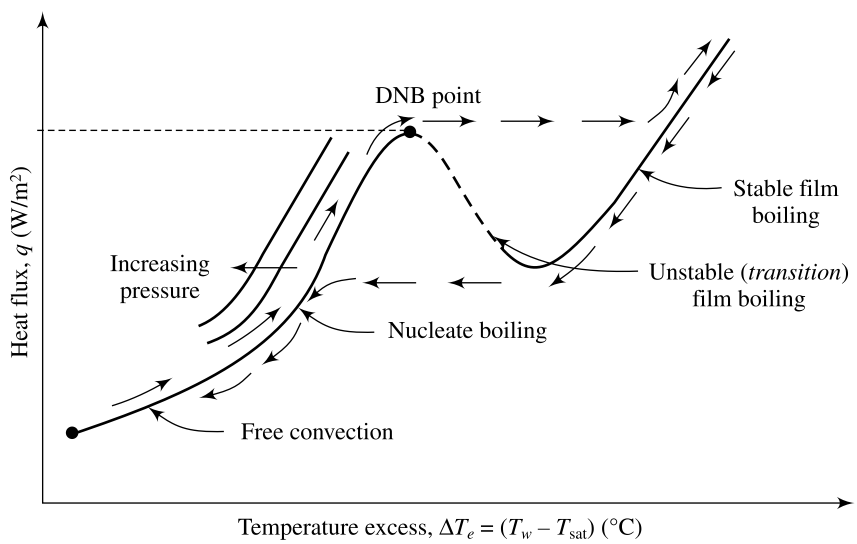


Fig. 10.6 *Nukiyama boiling curve with power input as the controlling parameter, also indicating the effect of increasing pressure*

It is noteworthy that the heat transfer from a heated surface (say at 110°C) to a pool of water at 1 atm is essentially the same no matter what the bulk temperature of water is (it could be 85°C, 90°C or 95°C). It is for this reason that the driving potential is (T_w – T_{sat}) or ΔT_e rather than the difference between the wall surface temperature and the fluid bulk temperature. One can see that as power is applied, the heat flux increases, initially slowly and then very rapidly, with excess temperature.

Boiling starts and bubbles appear when $\Delta T_e \approx 5$°C. With increase in power input the heat flux increases to a very high level, q_{max}. The situation becomes unstable *suddenly*, for a value slightly larger than q_{max}. The wire (*wall*) temperature jumps to the melting point resulting in *burn-out*.

Nukiyama repeated the experiment with a *platinum* wire with higher melting point (2045 K) and could maintain heat fluxes above q_{max} without burn-out. After reducing the power (and the heat flux), the cooling curve followed. When the heat flux reached the minimum point q_{min}, the situation became unstable again and a further decrease in q caused the excess temperature to drop suddenly, and the process followed the original heating curve back to the saturation point.

Nukiyama believed that this *hysteresis effect* was the result of the power-controlled mode of heating, where ΔT_e is a *dependent* variable.

However, if the heater surface temperature T_w (or excess temperature ΔT_e) is the controlling parameter then five distinct regimes of boiling can be identified by examining the pool-boiling curve. Figure 10.7 shows different regimes of the boiling curve for saturated pool boiling of water at the atmospheric pressure. This curve indicates the relation between heat flux, q, and excess temperature ΔT_e. The slope of the curve represents the heat-transfer coefficient, h.

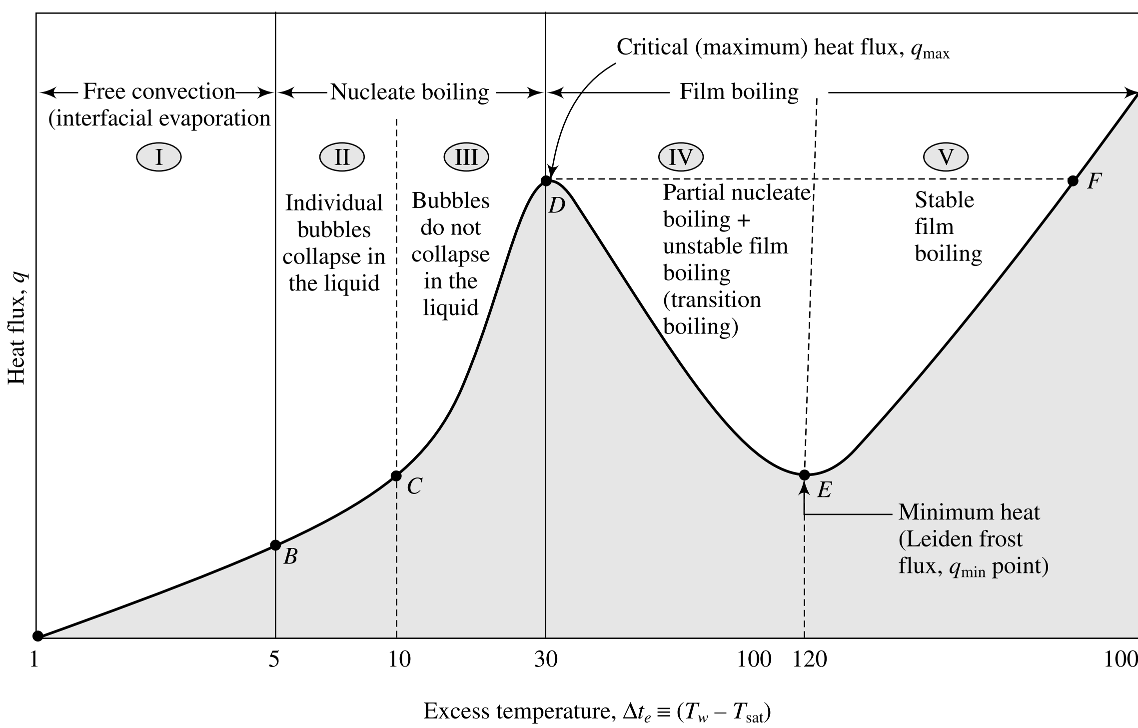


Fig. 10.7 *Nukiyama's boiling curve at atmospheric pressure and saturation temperature with heater surface temperature as the controlling parameter*

Free Convection Boiling [ΔT_e < 5°C] (Figure 10.8) In the range *A-B*, as the surface (wall) temperature rises above the liquid saturation temperature, natural or free convection currents are induced causing fluid motion. In this regime, the liquid is slightly superheated (liquid temperature above the saturation temperature) near the heated surface. There is no phase change at the heater surface and the superheated liquid near the heater rises to the free surface where evaporation occurs.

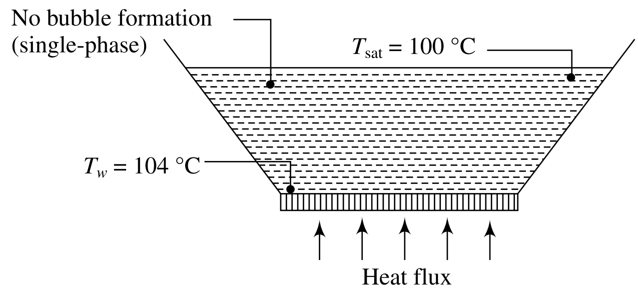


Fig. 10.8 *Free convection boiling (A-B)*

In *single-phase* free convection, the heat flux q is proportional to $\Delta T_e^{5/4}$ (*for laminar flow*) and the $\Delta T^{4/3}$ (*for turbulent flow*). And, $h \propto \Delta T_e^{1/4}$ (*laminar*) and $h \propto \Delta T_e^{1/3}$ (*turbulent*) because $q = h\ \Delta T_e$. Figure 10.9 displays the free convection (non-boiling) curve *B* for subcooled pool boiling which is slightly higher than the curve for saturated pool boiling *A*. The curve is further raised as indicated by the curve *C* in case of forced convection.

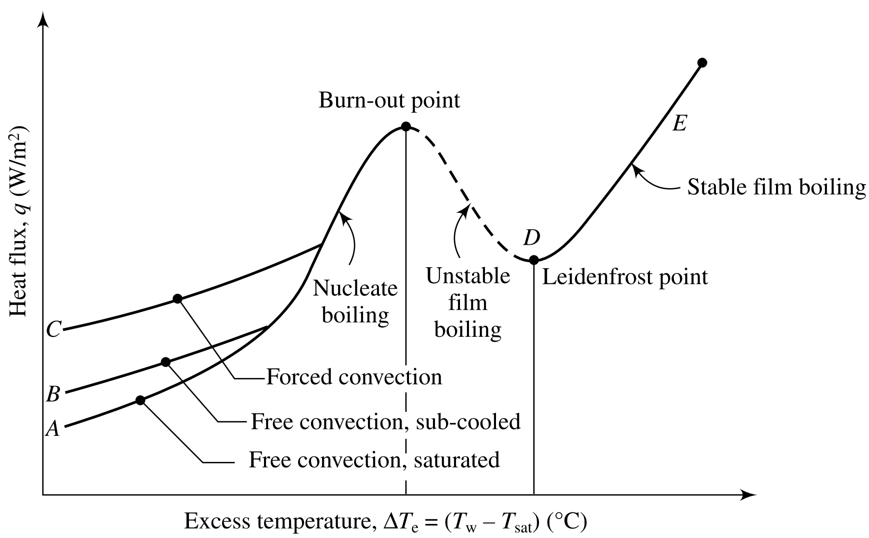


Fig. 10.9 *Boiling curve showing non-boiling curves for subcooled and saturated pool boiling, and the forced convection curve*

Nucleate Boiling [5°C < ΔT_e < 30°C] (Figure 10.10) Vapour bubbles begin to form at the nucleation sites such as the tiny cavities, pits or scratches on the submerged surface. The bubbles transport the latent heat of vaporization and cause a rapid increase in heat transfer by agitating the liquid near the heating surface. The mechanism in this range is called *nucleate boiling.* There are *two subregions* in Nucleate boiling:

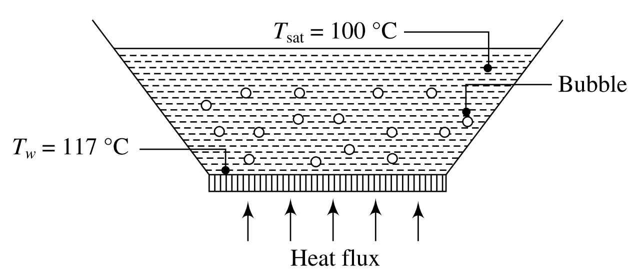


Fig. 10.10 *Nucleate boiling (B-C and C-D)*

- *Local boiling or subcooled boiling* which is a nucleate boiling where the bubbles formed at the heating surface tend to condense in the cooler liquid as they rise and collapse before they reach the free surface of the liquid.
- *Bulk boiling* which is a nucleate boiling in a saturated liquid in which the columns of bubbles do not collapse and rise to the interface.

In the nucleate boiling range, q varies as ΔT_e^n, where n generally ranges from 2 to 5.

In the region *B-C*, *isolated bubbles* originate at nucleation sites and then detach from the surface. Both the heat flux and the convection coefficient increase as a consequence. As a result of detachment there is agitation and stirring of the fluid near the surface. In the region *C-D*, the vapour bubbles leave the surface as continuous stream of *jets* or *columns*. When the bubble population becomes too high at some high heat flux point *D*, the outgoing bubbles may pose hurdles in the path of incoming liquid. The vapour thus forms an insulating blanket covering the heating surface and thereby raises the surface temperature. This is referred to as the *boiling crisis*. It is also called *burn-out, DNB* (*Departure from Nucleate Boiling*) or *CHF* (*Critical Heat Flux*). In water at atmospheric pressure, the value of the peak heat flux is more than 1 MW/m^2.

It is noteworthy that nucleate boiling is characterized by very high heat-transfer rates with relatively small temperature differences. It is, therefore, preferable to operate many engineering devices in this range. Heat-transfer coefficients in excess of 10 kW/m^2 K can be obtained in this region, which are significantly larger than those with convection without phase change.

Transition Boiling [30°C < ΔT_e < 120°C] (Figure 10.11) If the surface temperature is increased beyond that corresponding to the *critical heat flux*, boiling becomes unstable and the process enters the region of *partial film boiling or transition boiling* (range *D-E*). It becomes increasingly difficult for the liquid to reach the heated surface because the bubble formation is very vigorous and a vapour film or a bubble blanket begins to form on the surface. At any point on the surface, conditions oscillate between *film* and *nucleate boiling*, but the fraction of the total surface covered by the film increases with an increase in ΔT_e. The thermal conductivity of the vapour being much less than that of the liquid, the surface temperature increases rapidly to reach the point *E*, called the *Leidenfrost point*, where the heat flux is a minimum, $q_{\min}$, and the surface is completely covered by a *vapour blanket*.

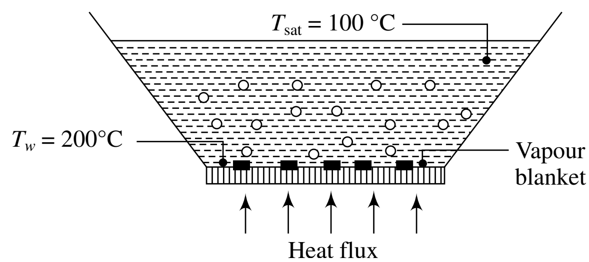


Fig. 10.11 *Transition boiling (D-E)*

Stable Film Boiling [ΔT_e > 120°C] (Figure 12.12) At the point *E*, heat transfer from the surface to the liquid takes place by conduction through the poorly conducting vapour film. As the surface temperature is further increased, radiation through the vapour film becomes increasingly significant and the heat flux increases with increasing ΔT_e. Eventually, a point *F* is reached at which the surface temperature may reach or exceed the melting point of the heater material.

So far, ΔT_e was the *independent* variable. However, in many applications that involve controlling heat flux (e.g., in a *nuclear reactor* or in an *electric resistance heating device*) q is the *independent* variable. The value of ΔT_e and, hence, the value of T_w will also increase, following the boiling curve to the point *D*. However, any increase in q beyond this point will induce a radical departure from the boiling curve in which the surface temperature will increase dramatically. Since, T_w may even exceed the melting point of the solid, destruction or failure of the system may occur. The point *D* is, therefore, often called the

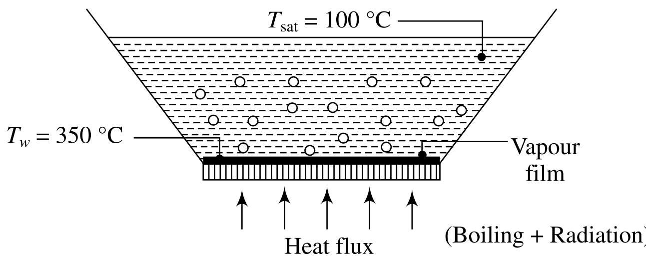


Fig. 10.12 *Stable film boiling (E-F)*

burn-out point or the *boiling crisis*, and accurate knowledge of the *Critical Heat Flux* (*CHF*) is important. It is, therefore, desirable to operate a heat transfer surface close to this value, but one must not exceed this value.

10.6 ❑ HEAT-TRANSFER CORRELATIONS

10.6.1 • Pool Boiling Correlations

In this section, the correlations for the boiling regions and the maximum and minimum heat fluxes as indicated in Fig. 10.13 are presented.

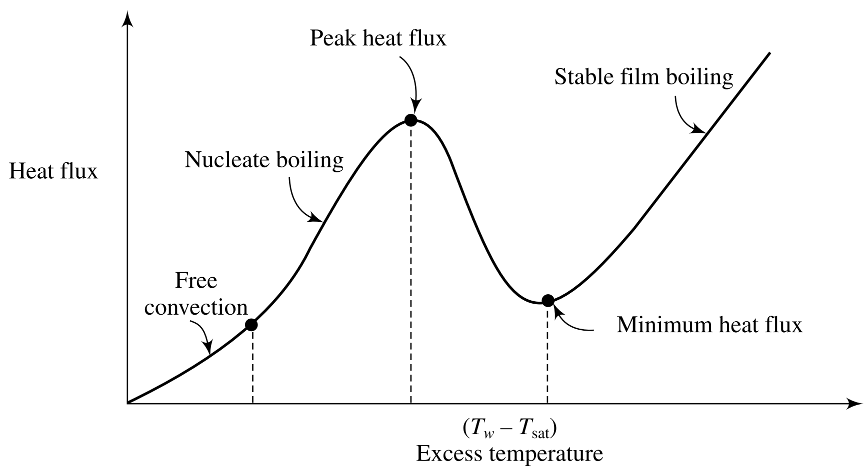


Fig. 10.13 *Pool-boding correlations are used to determine the heat flux in different boiling regimes*

Natural Convection As stated earlier, only superheated liquid near the surface is formed and evaporation occurs at the surface of the pool in the natural convection region *A-B* of the boiling curve. Appropriate free convection correlations can be used to estimate heat-transfer coefficients and heat-transfer rates. Some of the more widely used correlations are given below:

$$q = \frac{k}{D}(T_w - T_{sat})\left\{0.36 + \frac{0.518\,Ra_D^{1/4}}{[1+(0.559/Pr)^{9/16}]^{4/9}}\right\} \tag{10.3}$$

$$\text{for } 10^{-6} < Ra_D < 10^9\ (laminar)$$

$$q = \frac{k}{D}(T_w - T_{\text{sat}})\left\{0.60 + \frac{0.387\,Ra_D^{1/6}}{[1+(0.559/Pr)^{9/16}]^{8/27}}\right\} \tag{10.4}$$

$$\text{for } 10^9 < Ra_D < 10^{12}\ (turbulent)$$

The properties are evaluated at the mean film temperature $(T_w + T_{\text{sat}})/2$.

10.6.2 • Nucleate Pool Boiling

Nucleate boiling is probably the most important regime of boiling. Similar to the other types of convection heat transfer, Nusselt number in this regime is also a function of appropriate Reynolds and Prandtl numbers. The characteristic length and velocity in this case are related to the *bubble diameter*, *bubble velocity* and *the number of bubbles*. The bubble diameter is controlled by the surface tension, and the bubble velocity, by the buoyancy force and the viscosity of the liquid.

The dependence of q on ΔT_e characterizes the most popular correlation for nucleate pool boiling, which was developed by *Rohsenow*.

$$\boxed{q = \mu_l h_{fg}\left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2}\left(\frac{C_{pl}\Delta T_e}{C_{sf}h_{fg}Pr_l^n}\right)^3} \tag{10.5}$$

Alternatively,

$$\boxed{\frac{C_{pl}\Delta T_e}{h_{fg}P_{rl}^n} = C_{sf}\left[\frac{q}{\mu_l h_{fg}}\left\{\frac{\sigma}{g(\rho_l - \rho_v)}\right\}^{1/2}\right]^{1/3}} \tag{10.6}$$

where q = Surface heat flux (W/m^2)

μ_l = Liquid dynamic viscosity (kg/m s)

h_{fg} = Latent heat of vaporization (J/kg)

ρ_l = Density of saturated liquid (kg/m^3)

ρ_v = Density of saturated vapour (kg/m^3)

g = Gravitational acceleration (m/s^2)

σ = Surface tension at the liquid vapour interface (N / m).

Pr_l = Prandtl number of saturated liquid

C_{pl} = Specific heat of saturated liquid

C_{sf} = An empirical constant depending on surface-fluid combination.

n = 1 for water, n = 1.7 for other liquids

The subscript l refers to saturated liquid and subscript v refers to saturated vapour.

The representative values of the coefficient C_{sf} and the exponent n are listed in Table 10.1. Values of the surface tension and the latent heat of vaporization for water are given in Table 10.2 and the saturation pressure of water in Table 10.3. Note that the empirical correlation contains the dimensionless parameters, viz., Jacob number $Ja = C_p\Delta T_e/h_{fg}$ and the Prandtl number. Thus, $q \propto Ja^3\,Pr_l^{-3n}$.

Table 10.1 *Values of correlation constant C_{sf} for various fluid-surface combinations*

Fluid-Surface Combination	C_{sf}
Water–Nickel	0.0060
Water–Platinum	0.0130
Water–Copper (*scored surface*)	0.0068
Water–Copper (*polished surface*)	0.0130
Water–Brass	0.0060
Water–Stainless steel (*polished mechanically*)	0.0132
Carbon tetrachloride–Copper	0.013
Benzene–Chromium	0.0100
n-Pentane–Copper (*lapped surface*)	0.0049
n-Pentane–Copper	0.0154
n-Pentane–Chromium	0.015
Ethyl alcohol–Chromium	0.0027
Iso-Propyl Alcohol–Copper	0.0025
35% Potassium Carbonate–Copper	0.0054
50% Potassium Carbonate–Copper	0.0027
n-Butyl Alcohol–Copper	0.0030
Water–stainless steel (*Teflon-coated*)	0.0058
Water–stainless steel (*ground and polished*)	0.0080
Water–stainless steel (*chemically etched*)	0.0133

Table 10.2 *Surface tension and latent heat of vaporization values for water*

Saturation temperature T_{sat} (°C)	Surface tension σ ($\times 10^3$ N/m)	Latent heat of vaporization h_{fg} (kJ/kg)
0	75.5	2501.3
20	72.9	2454.1
40	69.5	2406.7
60	66.1	2358.5
80	62.7	2308.8
100	58.9	2257.0
150	48.7	2114.3
200	37.8	1940.7
250	26.1	1716.2
300	14.3	1404.9
350	3.6	893.4

Table 10.3 *Saturation pressure of water at various temperatures*

Temperature, °C	Saturation Pressure, Pa	Temperature, °C	Saturation Pressure, Pa
–40	13	10	1228
–36	20	15	1 705
–32	31	20	2 339
–28	47	25	3 169
–24	70	30	4 246
–20	104	35	5 628
–16	151	40	7 384
–12	218	50	12 349
– 8	310	100	101 325
– 4	438	200	1.55×10^6
0	611	300	8.58×10^6
5	872	374.14*	$22.09 \times 10^{6*}$

*Critical Point

The Rohsenow correlation is applicable only to clean surfaces. When it is used to estimate the heat flux, errors as large as ± 100% can occur. However, since $\Delta T_e\, \alpha (q)^{1/3}$, this error is reduced by a factor of 3 when the expression is used to find ΔT_e from the knowledge of q. Furthermore, since $q \propto h_{fg}^{-2}$ and h_{fg} decreases with increasing saturation pressure (and *temperature*), the nucleate boiling heat flux will increase as the liquid is pressurized.

10.6.3 • Peak Heat Flux

In several engineering applications, heat fluxes rather than wall temperatures are specified during heat transfer processes. In nuclear power plants, for instance, the heat flux from the radioactive fuel is generally prescribed in the boiler, and one must be careful to prevent the heat flux from reaching the burn-out level to avoid a severe accident.

The critical heat flux $q_{cr} = q_{\max}$ is an important quantity because one wants to operate a boiling process close to this point, but the possibility of dissipating heat exceeding it is dangerous. *Kutateladze*, through dimensional analysis, and *Zuber*, through a hydrodynamic stability analysis, obtained the following correlation:

$$\boxed{q_{\max} = \frac{\pi}{24} \rho_v h_{fg} \left[\frac{\sigma g(\rho_l - \rho_v)}{\rho_v^2} \right]^{1/4} \left(\frac{\rho_l + \rho_v}{\rho_l} \right)^{1/2}} \quad (\text{W/m}^2) \tag{10.7}$$

which, as a first approximation, is independent of the surface material and depends only weakly on the geometry. If we replace the Zuber constant $(\pi/24) = 0.131$ by an experimental value of 0.149 and approximate the last term in parentheses by unity since $\rho_v << \rho_l$, then

$$\boxed{q_{\max} = 0.149 \rho_v h_{fg} \left[\frac{\sigma (\rho_l - \rho_v) g}{\rho_v^2} \right]^{1/4}} \quad (\text{W/m}^2) \tag{10.8}$$

In principle, this expression is applicable to a *horizontal heater surface of infinite extent*, and there is *no characteristic length* but in practice, the expression can be used if the characteristic length is large compared to the mean bubble diameter parameter.

The peak heat flux for other heater geometries can be evaluated by the following expression:

$$q_{\max} = C_{\max}\, h_{fg}[\sigma\rho_v^2(\rho_l - \rho_v)\,g]^{1/4}$$

The values of C_{max} and the characteristic length are given in Table 10.4.

Table 10.4 *Values of the correction factor C_{cr} or C_{max} giving the critical (maximum) heat flux. Dimensionless characteristic length parameter, $L^* = L\sqrt{g(\rho_l - \rho_v)/\sigma}$.*

No.	Heater geometry	C_{cr} or C_{max}	Characteristic heater dimension	Range of L^*
1.	Large horizontal flat plate	0.149	Width or diameter	$L^* \geq 27$
2.	Small horizontal flat plate	18.9 K^+	Width or diameter	$9 \leq L^* \leq 20$
3.	Large horizontal cylinder	0.12	Radius	$L^* > 1.2$
4.	Small horizontal cylinder (*Lienhard*)	$0.12\, L^{*-0.25}$	Radius	$0.15 \leq L^* \leq 1.2$
5.	Horizontal cylinder	$\exp\,[-3.44\sqrt{L^*}]$	Radius	$L^* \geq 0.15$
6.	Large sphere	0.11	Radius	$L^* \geq 4.26$
7.	Small sphere	$0.227\, L^{*-0.5}$	Radius	$0.15 \leq L^* \leq 4.26$
8.	Any large finite body	0.118	$L = \dfrac{\text{Volume (∀)}}{\text{Surface area (A)}}$	$L^* \geq 4$

$^+K = \sigma/[g(\rho_l - \rho_v)A]$

10.6.4 • Transition Boiling and Minimum Heat Flux

Usually, in practice, transition boiling is not of much interest since the heat flux is the controlling parameter and for a prescribed value, either nucleate or stable film boiling is obtained. Hence, no satisfactory correlation is known for this regime.

The minimum heat flux, however, is of importance because it signifies the beginning of the stable film-boiling regime. If the heat flux drops below this minimum, the film will collapse, causing the surface to cool and nucleate boiling to be re-established.

It is interesting to mention that in both peak heating flux and minimum heat flux (i.e., q_{max} and q_{min}), the gravitational acceleration g is an important factor. Both heat fluxes are proportional to $g^{1/4}$, all other factors remaining constant.

The expression for the *minimum heat flux* q_{min}, for *large horizontal plates* is

$$\boxed{q_{\min} = 0.09\rho_v h_{fg}\left[\frac{\sigma g(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2}\right]^{1/4}} \quad (\text{W/m}^2) \qquad (10.9)$$

All properties are evaluated at the liquid saturation temperature.

10.7 ❑ FILM BOILING

Film boiling is characterized by a vapour film covering the heating surface. At excess temperatures beyond the *Leidenfrost point*, a continuous vapour film blankets the surface and there is no contact between the liquid phase and the surface. Since the vapour has a lower thermal conductivity relative to the liquid, very large temperature differences are necessary to transfer heat at a rate approaching the nucleate boiling region. Film boiling is, therefore, used only if circumstances make it unavoidable. Examples of such situations are when liquid gases such as oxygen or hydrogen are boiling at ordinary temperatures. It is, however, often encountered in chemical process equipment and in cryogenic systems. It occurs in cooling systems of chemical fuel and nuclear rocket engines.

The absolute values of heat-transfer coefficients become considerable at high pressures, which is why boiler tubes do not burn out, even though the temperature difference between wall and liquid increases significantly. This enables us to use film boiling process in many steam-generating installations.

The stable film boiling region was studied experimentally and analytically by *Bromley* for *horizontal tubes* and *vertical plates* and suggested the following equation:

$$\boxed{h_c = C\left[\frac{k_v^3\rho_v(\rho_l-\rho_v)g(h_{fg}+0.4C_{pv}\Delta T)}{D\mu_v(T_w-T_{\text{sat}})}\right]^{1/4}} \tag{10.10}$$

or

$$\boxed{Nu_D = \frac{h_c D}{k_v} = C\left[\frac{g(\rho_l-\rho_v)h'_{fg}D^3}{\nu_v k_v(T_w-T_{\text{sat}})}\right]^{1/4}} \quad \text{where } \nu_v = \frac{\mu_v}{\rho_v} \tag{10.11}$$

where C is the correlation constant. For a very large diameter tube of diameter D or a horizontal surface, $C = \left(0.59 + \dfrac{0.69\lambda}{D}\right)$ and $L = \lambda$,

and $$\lambda = 2\pi\left[\frac{\sigma}{g(\rho_l-\rho_v)}\right]^{1/2}$$

- For a horizontal surface, $C = 0.59$, $D \to \infty$.
- For a horizontal cylinder, $C = 0.62$ and $L = D$.
- For a sphere, $C = 0.67$ and $L = D$.

All vapour properties are evaluated at the mean film temperature, $T_f = (T_w + T_{\text{sat}})/2$. The latent heat h_{fg}, and the liquid density ρ_l are taken at the saturation temperature at the given pressure.

Thus, Eq. (10.13) is true only if the heat transfer is taking place by conduction through the film and does not include the effects of radiation. At higher surface temperatures ($T_w \geq 300°\text{C}$), radiation heat transfer across the vapour film becomes significant. Bromley suggested the following relation for total heat-transfer coefficient.

$$h = h_c\left(\frac{h_c}{h}\right)^{1/3} + h_r \tag{10.12}$$

Here, h_c is the film boiling heat-transfer coefficient *without* the radiation effects and h_r is the radiation heat-transfer coefficient that can be calculated from the radiation heat exchange between two parallel planes.

$$\boxed{h_r = \frac{1}{(1/\varepsilon) + (1/\alpha) - 1} \cdot \frac{\sigma(T_w^4 - T_{\text{sat}}^4)}{T_w - T_{\text{sat}}}} \tag{10.13}$$

where α = Absorptivity of liquid ($\approx$ 1)

ε = Emissivity of heater surface

σ = Stefan–Boltzmann constant (not surface tension) = 5.67×10^{-8}(W/m^2 K^4)

T_w = Wall temperature (K)

T_{sat} = Saturation temperature of liquid (K)

The effective radiation heat-transfer coefficient h_r is then given by:

$$\boxed{h_r = \frac{\sigma\varepsilon(T_w^4 - T_{\text{sat}}^4)}{(T_w - T_{\text{sat}})}} \tag{10.14}$$

If $h_r < h_c$, the following simpler form can be used:

$$\boxed{h = h_c + (3/4)h_r} \tag{10.15}$$

10.8 ❑ FACTORS AFFECTING NUCLEATE BOILING

The following factors influence nucleate boiling:

1. Material, Shape, and Condition of the Heating Surface The boiling heat-transfer coefficient depends greatly on the material of the heating surface. Under identical conditions of pressure and temperature difference, it is different for different metals (viz., copper has a high value than steel, zinc, and chromium).

The heat-transfer rates are also influenced by the conditions of heating surface. A rough surface gives a better heat transmission than when the surface is either smooth or has been coated (smoothness weakens the metal tendency to get wetted).

The shape of the heating surface also affects the transmission of heat.

2. Liquid Properties Through experiments, it has been observed that the size of the bubble increases with the dynamic viscosity of the liquid. With an increase in the bubble size, the frequency of bubble formation decreases which results in reduced heat transfer.

Moreover, high thermal conductivity of the liquid improves the rate of heat transfer.

3. Pressure The pressure influences the rate of bubble growth and in turn also affects the temperature difference ($T_{\text{sat}} - T_\infty$) causing heat flow. For a boiling liquid, the maximum allowable heat flux first increases with pressure until critical pressure is reached and thereafter it declines.

4. Mechanical Agitation Experiments have shown that the heat-transfer rate increases with the increase in the degree of agitation.

10.9 ❑ INTERNAL FLOW BOILING

Internal *flow boiling* or *two-phase flow with heat transfer* occurs in many practical situations. For example, in steam generators where the fluid enters a bundle of tubes as water, heat is transferred through the tube walls and the fluid leaves as steam. Flow boiling is considerably more complex than pool boiling because there is no free surface for the vapour to escape and the liquid and vapour have to flow together inside the tubes.

Consider a vertical tube heated uniformly over its length with constant heat flux and fed with subcooled liquid at its base at such a rate that the liquid is totally evaporated over the length of the tube. Figure 10.14 shows the different flow patterns encountered over the length of the tube depending on the dryness fraction, the fluid properties and the flow rate. The variation of wall and fluid temperatures are also shown.

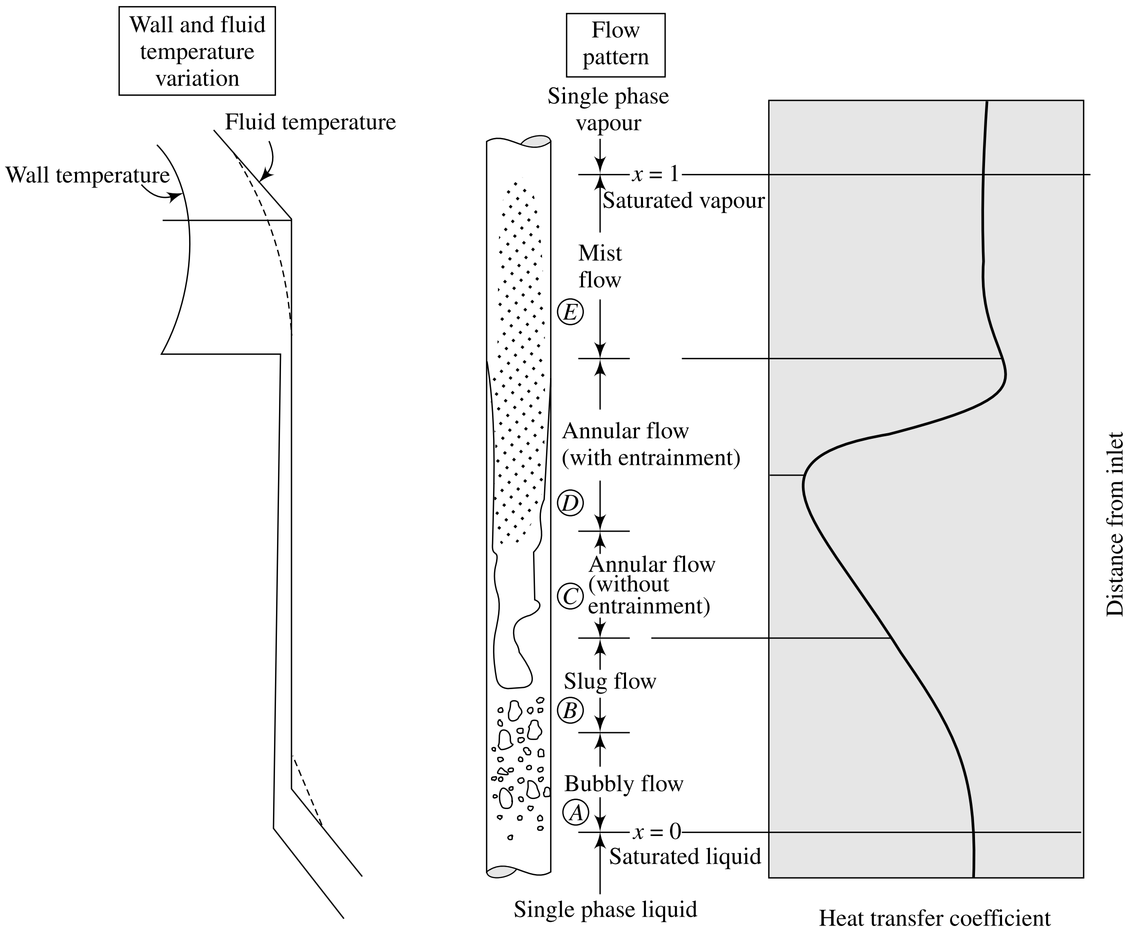


Fig. 10.14 *Development of flow boiling (two-phase flow) in a vertical tube with a uniform wall heat flux (not to scale)*

Pure saturated liquid ($x = 0$) flows upwards through the tube. In this case, single-phase pipe-flow correlations can be used to estimate the heat transfer rate.

While the liquid is being heated up to the saturation temperature and the wall temperature remains below that necessary for nucleation, the process of heat transfer in single phase convective heat transfer to the liquid phase (*Region A*). At some point along the tube, the conditions adjacent to the wall are such that the formation of vapour from nucleation sites can occur. Initially, vapour formation takes place in the presence of subcooled liquid (*Region B*) and this heat transfer mechanism is known as *subcooled nucleate boiling*.

In the subcooled boiling region *B*, the wall temperature remains essentially constant a few degrees above the saturation temperature, the mean bulk fluid temperature is increasing to the saturation temperature. The amount by which the wall temperature exceeds the saturation temperature is known as *degree of superheat* and the difference between the saturation and local bulk fluid temperature is known as *degree of subcooling*.

The transition between regions *B* and *C*, the subcooled nucleate boiling region and the saturated nucleate boiling regions is clearly defined from the thermodynamic point of view. It is the point at which the liquid reaches the saturation temperature ($x = 0$). In the region *C* to *D*, the variable characterising the heat transfer, (*the mechanism*) is the thermodynamic mass quality, x, of the fluid.

Bubble-flow Regime Nucleate boiling begins from the section *A* onwards. Vapour bubbles start to form at the tube wall and are carried along with the liquid. The flow pattern obtained is called *bubbly flow* (*Region I*). The dryness fraction of the fluid in this region is very low, *less than 0.05*.

Slug-flow Regime Further along the tube, there is an increase in the production rate of vapour bubbles which coalesce to form larger bubbles called *slugs*. The flow pattern is now called *slug flow (Region II)*. The dryness fraction in this region is still low, usually *less than 0.1*.

Annular-flow Regime Increased heating leads to further production of vapour. The volume of vapour relative to the liquid becomes large and the vapour begins to flow in a central core with the liquid flowing in an annulus around it. This flow is called *annular flow* (*Region III*). In this region, the liquid annulus gradually reduces in thickness along the length, becomes a film and eventually disappears. Vapour bubbles continue to be nucleated in the liquid at the tube wall over much of the annular flow region and join the vapour core at the liquid-vapour interface. The annular flow region exists over a large length during which the dryness fraction *increases from approximately 0.1 to about 0.8 or 0.9.*

In the later part of the annular flow region, the vapour core exerts a large shear force on the liquid film and causes liquid droplets to be entrained in the vapour. This is referred to as *entrainment*.

Mist-flow Regime At vapour qualities (*dryness fractions*) of 0.25 or more, the annular liquid film disappears resulting in a vapour flow with fine entrained liquid drops. This is characterized as *mist flow* (*Region IV*). The liquid drops gradually evaporate along the length and finally one obtains a *single-phase vapour flow* at the section *B* with vapour quality of 100%. From this section onwards, the vapour is superheated.

It is of interest to study the variation of the heat transfer coefficient and the wall temperature along the length of the tube. Before the heat transfer begins, the temperature of the liquid in the single-phase region is the saturation temperature corresponding to the pressure of the liquid. The wall is also at the same temperature. In ***regions I, II, and III***, the value of the heat-transfer coefficient increases along the length and is relatively high. On the other hand, in the ***region IV***, the value is low. Consequently, because of the constant heat flux boundary condition, the temperature difference between the wall and the fluid is low up to the end of the annular flow region and increases suddenly when the transition occurs from annular flow to mist flow. Since the fluid temperature is constant at the saturation value, the wall temperature also suddenly increases as seen in Fig. 10.14. The transition point is also referred to as *dry-out* point and is of significance because of the jump in wall temperature. This condition of *dry-out* often puts an effective limit on the amount of evaporation that can be allowed to take place in a tube at a particular value of heat flux. It is extremely important in the design of evaporators, steam boilers, nuclear reactors and other devices cooled by forced convection boiling.

At the end of the mist-flow region, in the single-phase vapour flow region, both the fluid and the wall temperature increase linearly with the length.

10.10 ❑ FORCED CONVECTION-BOILING CORRELATIONS

For forced convection boiling in smooth tubes, *Rosenhow and Griffith* (1955) recommended that total heat flux should be computed by adding the nucleate pool-boiling heat flux to the forced convection effect calculated from the Dittus–Boelter equation *but* with the coefficient *0.023 replaced by 0.019*. Thus,

$$\boxed{q_{\text{total}} = q_{\text{nucleate boiling}} + q_{\text{forced convection}}}$$

or

$$\boxed{h_{\text{two-phase}} = h_{\text{nucleate pool boiling}} + h_{\text{forced convection}}} \tag{10.16}$$

When one finds the heat-transfer coefficient in forced convection, only the mass-flow rate of the liquid at the cross section should be considered.

Chen also assumed that the mechanisms of nucleate boiling and flow boiling were additive, but suggested that the heat transfer coefficients should be multiplied by parameters S and F before these are added. Then

$$\boxed{h_{\text{two-phase}} = S\,h_{\text{nucleate pool boiling}} + F\,h_{\text{forced convection}}} \tag{10.17}$$

where S is called the *suppression* factor (*between 0 and 1*) to account for the presence of the *forced flow*, and F is *correction* factor (*greater than 1*) to take care of the presence of *nucleation* and *a film in the annular region*.

Chen presented correlations for calculating the values of S and F, and suggested that the *Forster–Zuber* correlation be used for calculating the heat transfer coefficient in nucleate pool boiling. Chen's correlation is most commonly used.

For horizontal tubes, *McAdams et al.* suggest the following relation for *low pressure* boiling water:

$$\boxed{q = 2.253(\Delta T_e)^{3.96}\ (\text{W/m}^2) \quad \text{for } 0.2 < P < 0.7 \text{ MPa}} \tag{10.18}$$

For higher pressures, *Levy* recommends the relation

$$\boxed{q = 283.2P^{4/3}(\Delta T_e)^3\ (\text{W/m}^2) \quad \text{for } 0.7 < P < 14 \text{ MPa}} \tag{10.19}$$

In these equations, the pressure P is in MPa and ΔT_e is in °C.

• Simplified Relations for Boiling-Heat-Transfer Coefficients with Water at Atmospheric Pressure $\Delta T_e = (T_w - T_{\text{sat}})$(°C).

Some of the simplest empirical correlations for water boiling on the submerged surfaces are presented in Table 10.5 for a quick estimate of the boiling-heat-transfer coefficient.

Table 10.5 *Boiling-heat-transfer coefficient*

Surface	Heat flux q (kW/m²)	Heat-transfer coefficient h (W/m² K)	Approximate range of ΔT_e (K)	Approximate range of h (W/m² K)
Horizontal	$q < 16$	$1042(\Delta T_e)^{1/3}$	0–7.76	0–2060
	$16 < q < 240$	$5.56(\Delta T_e)^3$	7.32–14.4	2180–16 600
Vertical	$q < 3$	$537(\Delta T_e)^{1/7}$	0–4.51	0–670
	$3 < q < 63$	$7.96(\Delta T_e)^3$	4.41–9.43	680–6680

To account for the effect of pressure, the heat-transfer coefficients estimated by using the above simplified relations may be modified by using the following empirical relation:

$$\boxed{h_p = h_1 (P/P_1)^{0.4}} \tag{10.20}$$

where h_p = heat-transfer coefficient at some pressure P

h_1 = heat-transfer coefficient at atmospheric pressure as obtained from Table 10.6

P = system pressure

P_1 = standard atmospheric pressure

For forced-convection local boiling inside vertical tubes, the following relation may be used:

$$\boxed{h = 2.54(\Delta T_e)^3 \times \exp(P/1.551)} \quad (\text{W/m}^2\,\text{K}) \tag{10.21}$$

where ΔT_e = temperature difference between the surface and saturated liquid in K (or °C) and P is the pressure in MPa. This equation can be used over a pressure range of 5 to 170 atm.

10.11 ❑ CONDENSATION

Condensation is defined as the removal of heat from a system in such a manner that vapour is converted into liquid. When the temperature of a vapour is decreased below its saturation temperature, the vapour condenses. After coming into contact with a cool surface, the vapour's latent heat is released, and heat is transferred to the surface, resulting in the formation of condensate. The condensate will accumulate on the *horizontal* surface, till the whole surface is covered by the liquid. On a *vertical* or *inclined* surface, however, the condensate will flow downwards along the surface under the influence of gravity.

10.12 ❑ TYPES OF CONDENSATION

Whenever a saturated vapour comes in contact with a surface at a lower temperature, condensation occurs. Irrespective of the orientation of the surface (*horizontal, vertical,* or *inclined*), the vapour can condense in either of the two ways: *filmwise* or *dropwise.* In filmwise condensation [Fig. 10.15(a)], the dominant mode, a thin continuous liquid film covers the entire condensing surface and the condensate flows off the surface under the action of gravity. This occurs when the surface is clean and uncontaminated, and the vapour is relatively free of impurities.

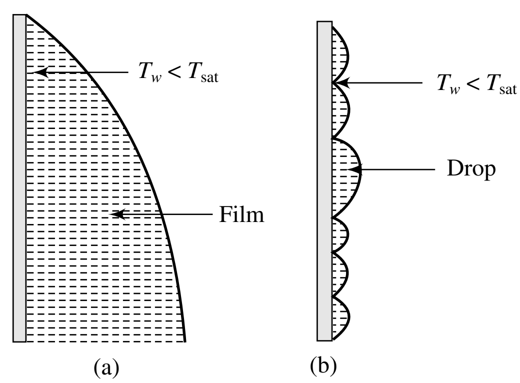


Fig. 10.15 *Modes of condensation: (a) Filmwise, and (b) Dropwise on a vertical surface*

The second type of condensation, known as *dropwise condensation* [Fig. 10.15(b)] is observed to occur when the surface is coated with a substance that inhibits wetting or is contaminated with oil or other fatty acids or is highly polished. In this case, individual drops are formed on the condensing surface. These drops grow in size and coalesce (*combine with one another*), as they roll down in some random fashion, to form larger ones, thus leaving the surface exposed for the formation of a new drop.

In the *film-condensation* process, the surface is covered by the film, which grows in thickness as it moves down the surface. The presence of a liquid film over the surface constitutes thermal resistance to heat transfer. In *dropwise condensation,* a significantly large part of the surface is directly exposed to the vapour. Of the two, higher condensation and heat-transfer rates are experienced in dropwise condensation. For this reason, many surface coatings and vapour additives like oleic acid have been used to promote and maintain dropwise condensation. There is no film barrier to heat transfer in dropwise condensation, and a portion of the cool surface is always in contact with the vapour without the insulating influence of the liquid layer. This accounts for higher heat-transfer coefficients (up to 290 kW/m^2 K) associated with dropwise condensation which is certainly preferable for industrial applications. However, commercially, dropwise condensation is difficult to attain and maintain for long periods of time (even the promoters employed to prolong this mode 1ose their effectiveness in course of time). Hence, all condensing equipments are usually designed by assuming that film condensation will exist.

No matter which mode of condensation occurs (*filmwise* or *dropwise*), the resistance to heat flow between the vapour and the cool surface increases with flow direction. It is, therefore, advisable to use either short vertical surfaces or horizontal cylinders in situations involving film condensation.

Film condensation being more common and amenable to analysis than dropwise condensation on which very little literature is available, more attention will be focused on film-condensation. Table 10.6 brings out briefly the salient features of film and dropwise condensation.

Table 10.6 *Comparison between film condensation and dropwise condensation*

Film Condensation	Dropwise Condensation
1. In *film condensation*, the condensate wets the surface and forms a liquid film on the surface that slides down under the influence of gravity.	**1.** In *dropwise condensation,* the condensed vapour forms countless droplets of varying diameters on the surface instead of a continuous film.
2. Relatively less heat-transfer coefficients are associated with film condensation.	**2.** Higher heat-transfer coefficients (about 5.10 times greater than those in film condensation) can be achieved.
3. On a rusty or etched plate, the vapour is condensed in a continuous film over the entire wall	**3.** With a polished surface, the condensate is formed in drops which rapidly grow in size (up to 3 mm in diameter) and roll down the surface.
4. The condensate itself forms a film (layer) on the surface which imposes some extra thermal resistance.	**4.** Droplets provide very little thermal resistance.
5. Practical condenser design assumes film condensation since the cost of non-wetting agents can outweigh the benefits from the increase in heat-transfer coefficient.	**5.** Suitable coatings and non-wetting agents like oleic acid are introduced into the vapour.

10.13 ❑ LAMINAR-FILM CONDENSATION ON A VERTICAL PLATE

Nusselt had first proposed the fundamental analysis of condensation in 1916 applicable for vertical plates as well as vertical tubes. To illustrate this approach, let us consider Fig. 10.16. The film originates at the top of the vertical plate and flows down under the influence of gravity. The plate temperature is maintained at T_w while the vapour at the edge of the film (*the liquid-vapour interface*) is at the saturation temperature T_{sat}(T_w < T_{sat}). The film thickness and the condensate mass flow rate is increased with increasing x (*positive direction of x measured downwards*) because of continuous condensation. The flow is assumed *laminar* everywhere. The thickening is due almost entirely to the addition of condensing vapour and not to retardation of the fluid in the film. The streamlines in the film can thus be assumed vertical.

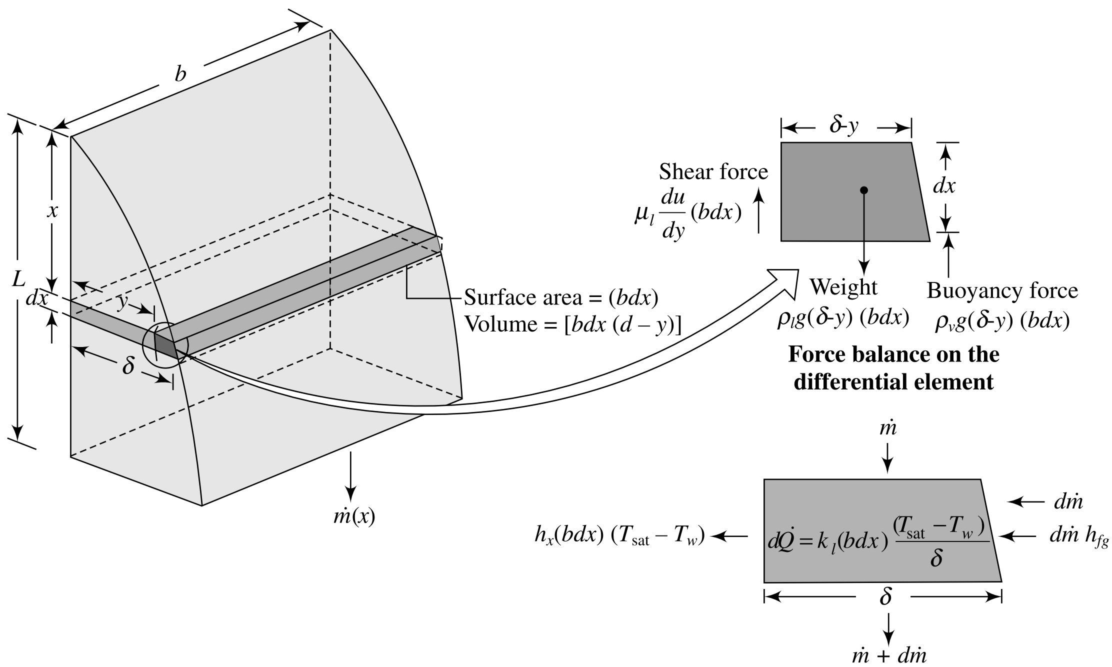


Fig. 10.16 *Film condensation on an isothermal vertical plate*

No heat can flow from the film towards the vapour as the surface of the film is essentially at the same temperature as the vapour. The entire heat transfer resulting from liberation of energy from the vapour condensing on the liquid film, is towards the wall. And this heat transfer is entirely by conduction because the stream lines are vertical. The temperature gradient in the film is thus constant assuming constant liquid thermal conductivity within the film.

10.13.1 • Nusselt's Analysis

Consider a vertical flat plate of length (*height*) L with width b maintained at a constant temperature of T_w. Vapour at temperature, T_{sat} condenses over the plate surface. To develop the expression for the average Nusselt number during the laminar-film condensation on the vertical surfaces, the following *assumptions* are made:

- The condensate flow is laminar and the fluid's thermophysical properties are constant.
- The vapour is pure, stationary and saturated, and the plate is at a constant temperature, T_w.
- There is no thermal resistance at the liquid-vapour interface which is at the saturation temperature, T_{sat}.
- The shear stress at the liquid-vapour interface is negligible.
- The heat transfer across the film occurs only by conduction and thus the liquid temperature distribution is linear.
- Momentum transfer in the condensate film is negligible as the flow velocity associated with the liquid film is low. There is thus no acceleration of the fluid in the condensate layer. It follows that there is only a static balance of forces.
- Only viscous shear, buoyancy and gravitational forces are assumed to act on the fluid.
- Enthalpy changes associated with subcooling are negligible.

Figure 10.3 depicts the temperature profile and velocity distribution at a position x below the top of the plate after invoking the above assumptions.

Velocity Distribution Consider the differential control volume of the fluid ($dxdy$) with width at a distance x from the top of the vertical plate. Neglecting pressure gradient in the vertical direction, the force balance on the control volume gives

(Weight, or gravitational force) = (Viscous (*shear*) force) + (Buoyancy force)

$$\rho_l\, g(\delta - y) b\, dx = \mu_l \frac{du}{dy}(b\, dx) + \rho_v\, g(\delta - y) b\, dx$$

or
$$\mu_l \frac{du}{dy} = g(\rho_l - \rho_v)(\delta - y)$$

where u is the liquid velocity in the x-direction.
It follows that

$$du = \frac{(\rho_l - \rho_v) g}{\mu_l}(\delta - y) dy$$

Integrating,
$$u(y) = \frac{(\rho_l - \rho_v) g}{\mu_l}\left[\delta y - \frac{y^2}{2}\right] + C$$

At the wall, because of no-slip condition for a real fluid, the boundary condition is as follows:

At $\quad y = 0,\ u = 0$

Hence, $\quad C = 0$

After rearranging, the velocity distribution is given by

$$\boxed{u(y) = \frac{(\rho_l - \rho_v) g \delta^2}{\mu_l}\left[\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]} \tag{10.22}$$

Equation (10.22) gives the velocity profile, u (y), which is seen to be *parabolic*.
Clearly, the maximum flow velocity will be at the film surface and is obtained by setting $y = \delta$.

Then
$$u_{\max} = \frac{(\rho_l - \rho_v) g}{\mu_l}[\delta^2 - (\delta^2/2)] = \frac{(\rho_l - \rho_v) g \delta^2}{2\mu_l}$$

Average Film Velocity

$$u_{av} = \frac{1}{\delta}\int_0^{\delta} u(y)\, dy = \frac{1}{\delta}\int_0^{\delta} \frac{(\rho_l - \rho_v) g}{\mu_l}\int_0^{\delta}\left(\delta y - \frac{y^2}{2}\right) dy$$

$$= \frac{(\rho_l - \rho_v) g}{\mu_l \delta}\left[\delta \cdot \frac{y^2}{2} - \frac{y^3}{6}\right]_0^{\delta} = \frac{(\rho_l - \rho_v) g \delta^2}{\mu_l}\left[\frac{1}{2} - \frac{1}{6}\right]$$

$$\boxed{u_{av} = \frac{(\rho_l - \rho_v) g \delta^2}{3\mu_l}} \tag{10.23}$$

Mass-flow Rate The mass-flow rate of the condensate then, at any distance x from the top of the plate, is given by

$$\dot{m}(x) = \rho_l A_c u_{av} = \rho_l (b\delta)\frac{(\rho_l - \rho_v)g\delta^2}{3\mu_l}$$

or $$\boxed{\dot{m} = \frac{\rho_l(\rho_l - \rho_v)gb\delta^3}{3\mu_l}} \tag{10.24}$$

Film Thickness Differentiating the above expression,

$$d\dot{m} = \frac{\rho_l(\rho_l - \rho_v)gb(3\delta^2 d\delta)}{3\mu_l} = \frac{\rho_l(\rho_l - \rho_v)gb\delta^2 d\delta}{\mu_l}$$

where $d\dot{m}$ is the incremental mass-flow rate of condensate added as the flow proceeds from x to $(x + dx)$, and the film thickens from δ to $(\delta + d\delta)$.

The heat-transfer rate, $d\dot{Q}$, resulting from condensation of vapour must equal the incremental mass-flow rate, $d\dot{m}$ times the latent heat (*enthalpy*) of vaporization (or *condensation*) h_{fg}, because the vapour is saturated and there is *no subcooling*. Thus,

$$\boxed{d\dot{Q} = d\dot{m}\, h_{fg} = \frac{\rho_l(\rho_l - \rho_v)gb\delta^2 d\delta}{\mu_l}\cdot h_{fg}} \tag{10.25}$$

This amount of heat which is transferred from the vapour during condensation should be equal to the heat transfer by conduction across the film to the wall which is given by the Fourier law:

$$\boxed{d\dot{Q} = k_l(bdx)\frac{(T_{\text{sat}} - T_w)}{\delta}} \tag{10.26}$$

where (bdx) is the area of cross section of the control volume
Equating the two expressions for $d\dot{Q}$, we have

$$\frac{k_l(bdx)(T_{\text{sat}} - T_w)}{\delta} = \frac{\rho_l(\rho_l - \rho_v)bg\delta^2 d\delta}{\mu_l}\cdot h_{fg}$$

or $$\delta^3 d\delta = \frac{\mu_l k_l(T_{\text{sat}} - T_w)dx}{\rho_l(\rho_l - \rho_v)gh_{fg}}$$

Integrating between the top surface ($x = 0$) down to x, we obtain

$$\frac{\delta^4}{4} = \frac{\mu_l k_l(T_{\text{sat}} - T_w)x}{\rho_l(\rho_l - \rho_v)g\,h_{fg}} + C$$

where C is a constant of integration and is equal to zero because of the boundary condition:

$\delta = 0$ at $x = 0$,

so that $$\boxed{\delta = \left[\frac{4\mu k(T_{\text{sat}} - T_w)x}{\rho_l(\rho_l - \rho_v)gh_{fg}}\right]^{1/4}} \tag{10.27}$$

Local Heat-transfer Coefficient We also note that

$$d\dot{Q} = h_x(bdx)(T_{sat} - T_w) = k_l(bdx)\frac{(T_{sat} - T_w)}{\delta} \Rightarrow \boxed{h_x = \frac{k_l}{\delta}}$$

where h_x is the local heat-transfer coefficient.
Substituting the value of δ, we get

$$\boxed{h_x = \left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3 h_{fg}}{4\mu_l x(T_{sat} - T_w)}\right]^{1/4}} \quad (10.28)$$

Both $\delta(x)$ and h_x vary down the vertical wall. We recognize that the film thickness $\delta(x)$ varies as $x^{1/4}$ and h_x is proportional to $x^{-1/4}$. Figure 10.17 illustrates the variation of $\delta(x)$ and h_x with x.

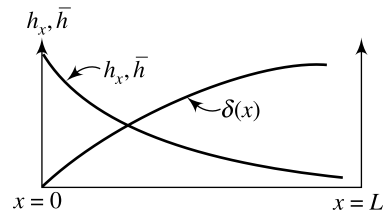


Fig. 10.17 *Variation of heat-transfer coefficient and of film thickness for condensate in laminar film condensation along a vertical surface*

Average Heat-transfer Coefficient The average condensation heat-transfer coefficient $\bar{h}_L$ can be obtained by integrating over the entire length of the plate, L.

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x dx = \frac{1}{L}C\int_0^L x^{1/4} dx$$

$$= \frac{4}{3}\frac{C}{L}[x^{3/4}]_0^L = \frac{4}{3}\frac{C}{L}L^{3/4} = \frac{4}{3}CL^{-1/4}$$

where $$C = \left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3 h_{fg}}{4\mu_l (T_{sat} - T_w)}\right]^{1/4}$$

or $$\bar{h}_L = \frac{4}{3(4)^{0.25}}\left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3 h_{fg}}{\mu_l L(T_{sat} - T_w)}\right]^{1/4}$$

or $$\boxed{\bar{h}_L = 0.943\left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3 h_{fg}}{\mu_l L(T_{sat} - T_w)}\right]^{1/4}} \quad (10.29)$$

Thus, the average heat-transfer coefficient over the entire length of the vertical surface (*tube or plate*) L is 4/3 times the local heat-transfer coefficient at $x = L$.

If the buoyancy force due to the displaced vapour were neglected, since $\rho_v << \rho_l$ except near the critical point, then

$$\boxed{\bar{h}_L = 0.943\left[\frac{\rho_l^2 gk_l^3 h_{fg}}{\mu_l L(T_{sat} - T_w)}\right]^{1/4}} \quad (10.30)$$

A comparison of this theoretical result with the results of experiments has indicated that the measured heat-transfer coefficient is about 20% higher than that predicted by the theoretical analysis. McAdams recommends that the constant 0.943 in the above expression be multiplied by 1.2 and replaced by 1.13.

In that case,

$$\boxed{\bar{h}_L = 1.13\left[\frac{\rho_l(\rho_l-\rho_v)gk_l^3h_{fg}}{\mu_l L(T_{\text{sat}}-T_w)}\right]^{1/4}} \tag{10.31}$$

where the suffix l in the properties ρ, k, and μ unambiguously shows that these values are for the liquid. This is more realistic in the *undulating* or *wavy-laminar* flow, i.e., $30 < Re_f < 1800$.

All the thermophysical properties in the equation for $\bar{h}_L$ are to be evaluated at the mean film temperature, i.e., $\frac{1}{2}(T_{\text{sat}}+T_w)$ while h_{fg} and ρ_v should be taken at the saturation temperature, T_{sat}.

It must be remembered that the heat-transfer coefficient derived above for a *vertical plate* is also applicable for laminar film condensation on the *outside* or *inside* surface of a *vertical tube*, provided the condensate film thickness δ is much smaller than the tube diameter, D (i.e., $\delta << D$). A vertical tube can be looked upon as a vertical plate folded about a vertical axis. However, this relation may not be used for *inclined tubes*.

Some recent refined analyses take into account a *non-linear* temperature profile in the film and an additional energy to cool the film below the saturation temperature (i.e., *the effect of subcooling*). Both the effects are taken care of if we replace h_{fg} with h^*_{fg} where

$$\boxed{h^*_{fg} = h_{fg} + 0.68C_{pl}(T_{\text{sat}}-T_w)} \tag{10.32}$$

where C_{pl} is the specific heat of the liquid, and $(T_{\text{sat}} - T_w) = (T_{\text{saturated vapour}} - T_{\text{surface}})$

Since *Jacob number*, $Ja = C_{pl}\dfrac{(T_{\text{sat}}-T_w)}{h_{fg}}$, one can also write

$$\boxed{h^*_{fg} = h_{fg}(1+0.68\,Ja)}$$

The *Jacob number* is a measure of the relative magnitude of the film subcooling.

Hence, the refined version of the correlation for the average heat-transfer coefficient can be expressed as

$$\boxed{\bar{h}_L = 0.943\left[\frac{\rho_l(\rho_l-\rho_v)gh^*_{fg}}{\mu_l L(T_{\text{sat}}-T_w)}\right]^{1/4}} \tag{10.33}$$

where all properties are to be evaluated at the mean film temperature except h_{fg} and ρ_v which are at T_{sat}. While there have been many refinements and extensions of Nusselt's analysis, the above expression is sufficiently accurate in most of the cases involving laminar film condensation.

The heat-transfer coefficient can also be expressed in terms of a dimensionless Nusselt number.

$$\boxed{Nu_L = \frac{\bar{h}_L L}{k} = 0.943\left[\frac{\rho_l(\rho_l-\rho_v)gL^3h_{fg}}{\mu_l k_l(T_{\text{sat}}-T_w)}\right]^{1/4}} \tag{10.34}$$

This can be rearranged as

$$\overline{Nu}_L = 0.943\left[\frac{(\rho_l-\rho_v)gL^3}{\rho_l\nu_l^2}\times\frac{C_p\nu_l\rho_l}{k_l}\times\frac{h_{fg}}{C_{pl}(T_{\text{sat}}-T_w)}\right]^{1/4}$$

where $\dfrac{g(\rho_l-\rho_v)}{\rho_l}\dfrac{L^3}{\nu_l^2}$ is *Grashof number*, Gr_L

$$\frac{C_{pl} \nu_l \rho_l}{k_l} \qquad \text{is } \textit{Prandtl number, Pr}$$

and
$$\frac{C_{pl}(T_{\text{sat}} - T_w)}{h_{fg}} \qquad \text{is } \textit{Jacob number, Ja}$$

Hence, one can write

$$\boxed{\boldsymbol{Nu_L = 0.943\,[Gr_L\,Pr/Ja]^{1/4}}} \tag{10.35}$$

The relationship is valid as long as the condensate flow is *laminar*. This holds good if the value of the film Reynolds number ($4\dot{m}/\mu_l$), is less than 1800.

Heat-transfer Rate The average heat-transfer rate and the heat flux can now be calculated from

$$\boxed{\dot{Q} = \bar{h}_L A (T_{\text{sat}} - T_w)} \text{ and } \boxed{q = \bar{h}_L (T_{\text{sat}} - T_w)} \tag{10.36a}$$

The area, A is equal to (bL) for a plate of width b while that for a vertical cylinder of diameter D will be (πDL).

Also, the condensate mass-flow rate is

$$\boxed{\dot{m} = \frac{\dot{Q}}{h_{fg}} = \frac{\bar{h}_L (T_{\text{sat}} - T_w)}{h_{fg}}} \tag{10.36b}$$

10.14 ❑ LAMINAR-FILM CONDENSATION ON AN INCLINED SURFACE

If a plate or cylindrical tube is inclined at an angle θ with the *vertical*, the net effect of the above analysis is to replace the body (*gravitational*) force with its component parallel to the heat-transfer surface, as shown in Fig. (10.18).

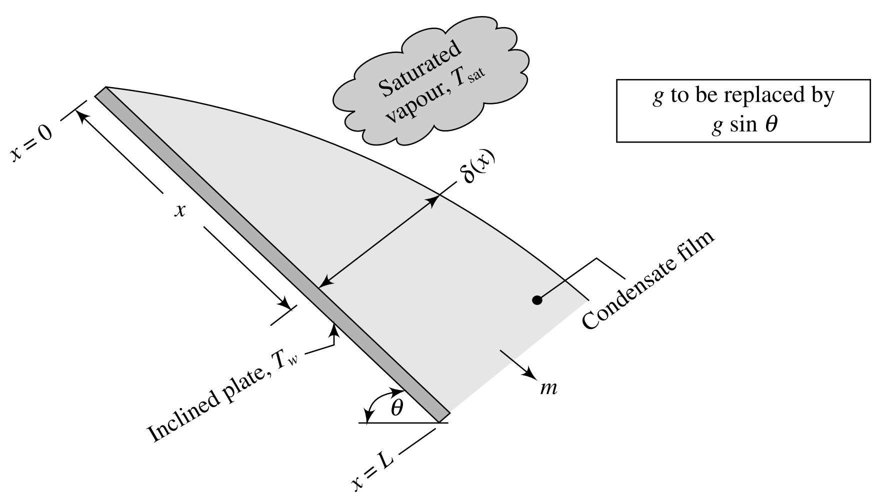


Fig. 10.18 *Film condensation on an inclined plate*

Thus, $\boxed{g' = g\cos\theta}$

And if this inclination is $\theta°$ with the *horizontal*,

Then $\boxed{g' = g\sin\theta}$

The average condensation heat-transfer coefficient, $\bar{h}$ can be is now rewritten by simply substituting g' in place of g.

It follows that

$$\boxed{\bar{h}_L = 0.943\left[\frac{\rho_l^2 (g\cos\theta) k_l^3 h_{fg}}{\mu_l L (T_{\text{sat}} - T_w)}\right]^{1/4}} \quad (\rho_l << \rho_l) \tag{10.37}$$

where L is the length or height measured along the plate surface and θ is the angle of tilt with the vertical.

10.15 ❑ FILM REYNOLDS NUMBER

The ratio of the momentum (*inertia*) forces to the viscous forces is determined by the Reynolds number of the condensate film. In film condensation on a vertical surface, the momentum forces increase as the falling film thickness and velocity increase. When these forces are much greater than the viscous (*shear*) forces, ripples or waves appear on the cascading condensate film which becomes turbulent. This profoundly affects the heat-transfer characteristics through the condensate layer.

Since the film Reynolds number, Re_f is an indicator of the condensate flow conditions, it is often convenient to express the average convection coefficient $\bar{h}_L$ directly in terms of the Reynolds number. The Reynolds number of a falling film is defined as

$$\boxed{Re_f = \frac{\rho_l D_h V}{\mu_l} = \frac{\rho_l}{\mu_l}\left(\frac{4A_c}{P}\right)\left(\frac{\dot{m}}{\rho_l A_c}\right) = \frac{4\dot{m}}{P\mu_l}} \tag{10.38}$$

where D_h = hydraulic diameter (m)

A_c = cross-sectional area of the condensate flow film (m^2)

P = wetted perimeter (m)

V = average flow velocity (m/s)

μ = dynamic viscosity (kg/m s)

For a vertical plate of width b, $\boxed{P = b}$

and, for a vertical tube or cylinder of diameter D, $\boxed{P = \pi D}$

For a horizontal cylinder of diameter D and length L, $\boxed{P = 2L}$

Hydraulic Diameter (D_h) For a vertical plate, $D_h = \dfrac{4A_c}{P} = \dfrac{4(b\delta)}{b} = 4\delta$

For a vertical tube, $D_h = \dfrac{4A_c}{P} = \dfrac{4(\pi D\delta)}{\pi D} = 4\delta$

For a horizontal tube, $D_h = \dfrac{4A_c}{P} = \dfrac{4(2L\delta)}{2L} = 4\delta$

The transition from *laminar* to *turbulent* condensation takes place at the critical Reynolds number of approximately 1800. Thus, the criterion for *laminar flow* is

$$\boxed{Re_f < 1800}$$

Since the film Reynolds number is generally used to characterize the film flow, we can identify three regions of film condensation on a vertical surface based on its value. These regions (shown in Fig. 10.19) are the following:

Wave-free laminar ($Re_f \le 30$), wavy-laminar ($30 \le Re_f \le 1800$) and Turbulent ($Re_f > 1800$).

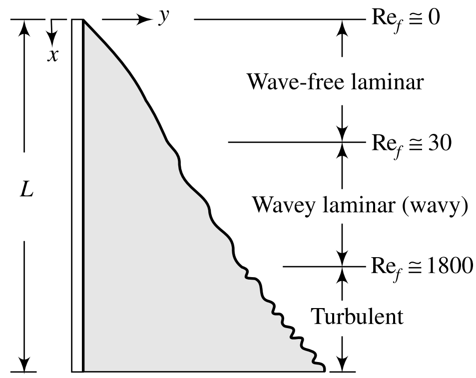


Fig. 10.19 *Three regions of film condensation on a vertical surface.*

We note that

$$\dot{Q} = \dot{m} h_{fg} = \bar{h} A (T_{sat} - T_w),$$

Hence,
$$\dot{m} = \frac{\bar{h} A (T_{sat} - T_w)}{h_{fg}}$$

Then

$$\boxed{Re_f = \frac{4\dot{m}}{P\mu_l} = \frac{4\bar{h}_L A (T_{sat} - T_w)}{P h_{fg} \mu_l} = \frac{4\bar{h}\pi D L (T_{sat} - T_w)}{(\pi D) h_{fg} \mu_l} = \frac{4hL(T_{sat} - T_w)}{h_{fg}\mu_l}}$$

(*for a vertical tube*) (10.39)

$$\boxed{Re_f = \frac{4\bar{h}(bL)(T_{sat} - T_w)}{b h_{fg} \mu_l} \text{ or } \frac{4\bar{h} L (T_{sat} - T_w)}{h_{fg}\mu_l}}$$

(*for a vertical or inclined plate*) (10.40)

10.16 ❑ CONDENSATION NUMBER OR MODIFIED NUSSELT NUMBER

Local mass-flow rate, $\dot{m} = \dfrac{\rho_l(\rho_l - \rho_v) g b \delta^3}{3\mu_l}$

Local film Reynolds number,

$$Re_f = \frac{4\dot{m}}{P\mu_l} = \frac{4\dot{m}}{b\mu_l}$$

$$\therefore \quad \frac{Re_f b \mu_l}{4} = \frac{\rho_l(\rho_l - \rho_v) g b \delta^3}{3\mu_l}$$

or
$$\delta = \left[\frac{3 Re_f \mu_l^2}{4\rho_l(\rho_l - \rho_v) g}\right]^{1/3}$$

Local heat-transfer coefficient

$$h = \frac{k_l}{\delta} = \left[\frac{4\rho_l(\rho_l - \rho_v) g k_l^3}{3 Re_f \mu_l^2}\right]^{1/3}$$

Average heat-transfer coefficient,

$$\bar{h} = \frac{4}{3}h = \frac{4}{3}\left[\frac{4}{3}\frac{\rho_l(\rho_l - \rho_v)gk_l^3}{Re_f \mu_l^2}\right]^{1/3} = \left(\frac{4}{3}\right)^{4/3}\left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3}{Re_f \mu_l^2}\right]^{1/3}$$

Usually, $\rho_v << \rho_l$ and $(\rho_l - \rho_v) \approx \rho_l$.

Then $$\bar{h} = 1.47\, Re_f^{-1/3}\left[\frac{\rho_l^2 k_l^3 g}{\mu_l^2}\right]^{1/3}$$

or $$\boxed{\bar{h} = 1.47\, k_l \left[\frac{g}{\nu_l^2}\right]^{1/3} Re_f^{-1/3}} \Rightarrow \boxed{\frac{\bar{h}}{k_l}\left(\frac{\nu_l^2}{g}\right)^{1/3} = 1.47\, Re_f^{-1/3}}$$

The quantity on the left-hand side is dimensionless $\left[\frac{\text{W}}{\text{m}^2\text{K}}\frac{\text{m K}}{\text{W}}\left\{\frac{m^4}{s^2}\frac{s^2}{m}\right\}^{1/3}\right]$ and is known as *condensation number*, *Co*.

$$\therefore \qquad \boxed{Co = 1.47\, Re_f^{-1/3}} \tag{10.41}$$

Since $(\nu_l^2/g)^{1/3}$ has the unit of length, $Co = \frac{\bar{h}L_c}{k_l}$ where characteristic length, $L_c = \left(\frac{\nu_l^2}{\text{g}}\right)^{1/3}$

This can be called *modified Nusselt number*

Now, let us define a dimensionless number called *Condensation number* (Co), or *modified Nusselt number* as follows:

$$Co = \left[\frac{\mu_l^2}{\rho_l(\rho_l - \rho_v)g}\right]^{1/3} \cdot \frac{\bar{h}}{k_l}$$

For simplicity, we assume: $\rho_v << \rho_l$. *Condensation number* can then be simplified as

$$\boxed{Co = \frac{\bar{h}}{k_l}\left(\frac{\mu_l^2}{\rho_l^2 g}\right)^{1/3} = \frac{\bar{h}}{k_l}\left(\frac{\nu_l^2}{g}\right)^{1/3}} \tag{10.42}$$

Laminar Flow (Wave-Free) ($Re_f \le 30$)

$$\boxed{\bar{h} = 1.47 k_l \left(\frac{g}{\nu_l^2}\right)^{1/3} Re_f^{-1/3}}$$

$$\boxed{Co = 1.47\, Re_f^{-1/3}}$$

where Re_f is the film Reynolds number at the bottom of the vertical surface ($x = L$).

Laminar-wavy Region ($30 < Re_f < 1800$) Kutateladz proposed a correction factor of $0.8\,(Re_f/\,4)^{0.11}$ to be multiplied to local heat-transfer coefficient to account for the rippling effect, and recommended the following correlation:

$$\boxed{Re_f = \left[4.81 + \frac{3.7Lk_l\,(T_{\text{sat}} - T_w)}{\mu_l h_{fg}}\left(\frac{g}{\nu_l^2}\right)^{1/3}\right]^{0.82}} \tag{10.43}$$

and
$$\boxed{\bar{h} = \frac{Re_f k_l}{1.08\, Re_f^{1.22} - 5.2}\left(\frac{g}{\nu_l^2}\right)^{1/3}} \tag{10.44}$$

$$\boxed{Co = \frac{Re_f}{1.08\, Re_f^{1.22} - 5.2}} \tag{10.45}$$

Turbulent Region ($Re_f > 1800$) In many practical applications, the vertical surface (*plate or tube*) may be long enough so that the condensate film becomes sufficiently thick to cause transition to turbulence which is characterised by $Re_f = 1800$. For *turbulent* flow (i.e., $Re_f > 1800$), the average convective heat-transfer coefficient is given by the following empirical correlation for a vertical plate after the onset of turbulence, as suggested by McAdams.

$$\boxed{\bar{h}_L = 0.0077\, Re_f^{0.4}\left[\frac{\rho_l^2 g k^3}{\mu_l^2}\right]^{1/3}} \tag{10.46}$$

Incorporating the effect of buoyancy force of the displaced vapour, the same expression becomes

$$\boxed{\bar{h}_L = 0.0077\, Re_f^{0.4}\left[\frac{\rho_l(\rho_l - \rho_v) g k_l^3}{\mu_l^2}\right]^{1/3}} \tag{10.47}$$

All physical properties are to be evaluated at the mean film temperature, i.e., $(T_{\text{sat}} + T_w)/2$.
Labuntsov recommends the following correlation for turbulent film condensation valid for Prandtl numbers around unity:

$$Re_f = \left[\frac{0.069 L k_l Pr_l^{0.5}(T_{\text{sat}} - T_w)}{\mu_l h_{fg}^*}\left(\frac{g}{\nu_l^2}\right)^{1/3} - 151\, Pr_l^{0.5} + 253\right]^{4/3} \tag{10.48}$$

and
$$\boxed{\bar{h} = \frac{Re_f k_l}{8750 + 58\, Pr_l^{-0.5}(Re_f^{0.75} - 253)}\left(\frac{g}{\nu_l^2}\right)^{1/3}} \tag{10.49}$$

$$\boxed{Co = \frac{Re_f}{8750 + 58(Re_f^{0.75} - 253)Pr_l^{-0.5}}} \tag{10.50}$$

A universal correlation for both *wavy* and *turbulent* regimes is proposed by Chen *et al.* (1987) as follows:

$$\boxed{Co = \frac{\bar{h}}{k_l}\left(\frac{\nu_l^2}{\text{g}}\right)^{1/3} = [Re_f^{-0.44} + 5.82 \times 10^{-6}\, Re_f^{0.8}\, Pr_l^{1.3}]^{1/2}} \quad (Re_f \geq 30) \tag{10.51}$$

It may be observed that in the *turbulent* film condensation region, the *condensation number* depends also on the liquid Prandtl number, Pr_l apart from the film Reynolds number, Re_f.

A vertical tube can be looked upon as a vertical plate folded about a vertical axis.

However, this relation *may not be* used for *inclined* tubes.

10.17 ❑ FILM CONDENSATION ON A HORIZONTAL TUBE

In the case of *laminar*-film condensation of *pure*, *saturated*, and *stationary* vapour on the *outside* of a *horizontal tube* or cylinder as illustrated in Fig. 10.20, the condensate film forms at the top of the cylinder and flows around it, thickening as it grows symmetrically and the angle θ *measured from the vertical*), increases. At any polar location θ, where $\theta = x/(D/2)$, the problem can be analysed by representing the condensate film as that along a vertical surface inclined at an angle $\phi\left(\phi = \dfrac{\pi}{2} - \theta\right)$ with the horizontal. Condensation takes place on a bank of horizontal tubes, in shell-and-tube heat exchangers commonly used in power plants and the process industries. Figure 10.20 depicts laminar-film condensation on a single horizontal tube.

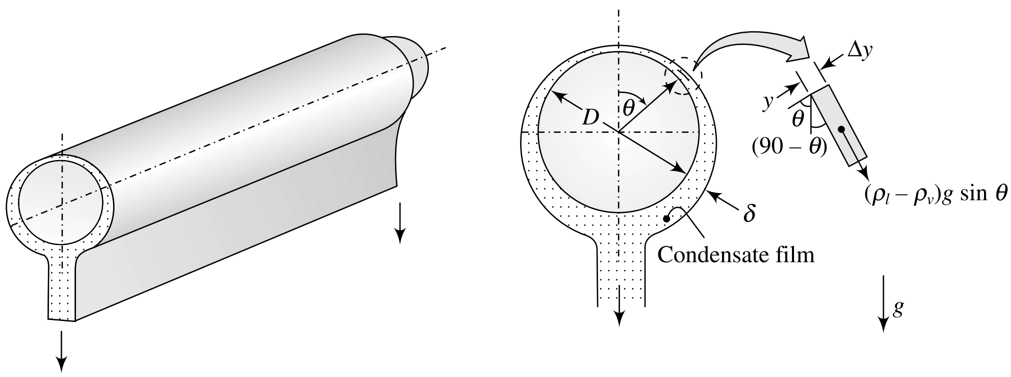


Fig. 10.20 *Film condensation on a single horizontal tube with a continuous sheet of condensate*

For the average heat-transfer coefficient for laminar-film condensation on the outside of a single horizontal tube, Nusselt obtained the following relation by proceeding in a manner similar to that followed for the vertical plate. The average heat-transfer coefficient is given by

$$\boxed{\bar{h}_D = 0.729\left[\frac{\rho_l^2 g k_l^3 h_{fg}}{\mu_l D(T_{\text{sat}} - T_w)}\right]^{1/4}} \qquad (\textit{laminar flow}) \qquad (10.52)$$

where D is the outside diameter of the tube.

This result can also be expressed in terms of the film Reynolds number and condensation number as follows:

$$\boxed{Co = \frac{\bar{h}}{k_l}\left(\frac{\nu_l^2}{g}\right)^{1/3} = 1.51(Re_f)^{-1/3}} \qquad (10.53)$$

where Re_f is the film Reynolds number at the bottom of the tube (θ = 180°).

This equation is also valid for filmwise condensation on the *inside* of a horizontal tube provided the film does not separate and the tube is not very long.

If the vapour density is non-negligible, ρ_l^2 can be replaced by $\rho_l(\rho_l - \rho_v)$ and h_{fg} can be replaced by, h_{fg}^* for better accuracy as in the case of vertical surfaces.

Thus, we have

$$\boxed{\bar{h}_D = 0.729\left[\frac{\rho_l(\rho_l-\rho_v)gk^3h_{fg}}{\mu_l D(T_{\text{sat}}-T_w)}\right]^{1/4}} \qquad (\textit{laminar flow}) \tag{10.54}$$

All physical properties should be evaluated at the mean film temperature, i.e., $(T_{\text{sat}} + T_w)/2$ except h_{fg} and C_{pl} which should be taken at the saturation temperature (*or saturation pressure*).

Comparing the relative merits of horizontal and vertical tubes, we have, as the ratio of the respective Nusselt's numbers:

$$\boxed{\frac{\overline{Nu}_H}{\overline{Nu}_V}=\frac{\overline{Nu}_D}{\overline{Nu}_L}=\frac{0.729}{0.943}\left(\frac{D^3}{L^3}\right)^{1/4}=\frac{\bar{h}_D}{\bar{h}_L}\left(\frac{D}{L}\right)} \quad \text{or} \quad \boxed{\frac{\bar{h}_D}{\bar{h}_L}=0.773\left(\frac{L}{D}\right)^{1/4}} \tag{10.55}$$

If $\bar{h}_D = \bar{h}_L$ then

$$\boxed{\frac{L}{D}=\left(\frac{1}{0.773}\right)^4 = 2.8} \tag{10.56}$$

Thus, when $(L/D) = 2.8$, the values of $\bar{h}_D$ and $\bar{h}_L$ for the *horizontal* and *vertical* surfaces are the same.

Interestingly, if for example, L = 1 m and D = 20 mm, i.e., (L/D) = 50, it follows that $\bar{h}_D$ = 2.05 $\bar{h}_L$. Thus, almost twice as much steam will be condensed if a vertical tube is arranged in a horizontal position. In commercial condensers, the ratio (L/D) is usually in the range of 50 to 100 or even more. This justifies the preference for a horizontal surface as against a vertical surface. The higher condensing capacity of a horizontal tube is due to the fact that much thinner film forms in the horizontal position. Note that the condensate film on a single horizontal tube travels only a distance of $(\pi D/2)$ before it falls off the surface at the bottom of the tube.

Condensation on a single horizontal tube seldom changes into turbulent flow. Hence, **Eq. (10.54)** holds good for all practical purposes.

Laminar-Film Condensation on a Sphere The following correlation is recommended for laminar-film condensation on the outer surface of a sphere (Fig. 10.21):

$$\boxed{\bar{h}_D = 0.826\left[\frac{\rho_l(\rho_l-\rho_v)gh_{fg}k_l^3}{\mu_l D(T_{\text{sat}}-T_w)}\right]^{1/4}} \tag{10.57}$$

Note that the constant 0.729 for a horizontal tube is replaced with the constant 0.826 for a sphere.

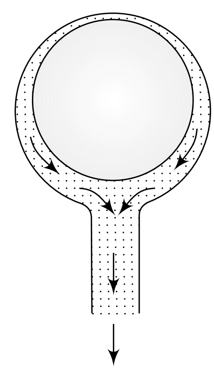


Fig. 10.21 *Film condensation on a sphere*

10.18 ❑ LAMINAR FILM CONDENSATION ON A VERTICAL BANK OF HORIZONTAL TUBES

Most commercial condensers condense steam on a huge bundle of horizontal tubes, with condensate dripping from one tube to the next. Condenser design usually involves horizontal tubes arranged in a vertical tier or bank as illustrated in Fig. 10.22. In such a case, gravity controls the condensate run-off from the bottom of one tube onto the top of the tube below. If it is assumed that the drainage from one horizontal tube flows smoothly onto the next tube below, then for a vertical tier of N tubes, each of diameter D, the average condensation film coefficient, $\bar{h}_D$ is obtained by replacing D by (ND) in Eq. (10.54). Thus, for a vertical tier of N horizontal tubes, the average heat-transfer coefficient is given by

$$\boxed{\bar{h}_{D,N} = 0.729\left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3 h_{fg}^*}{\mu_l(ND)(T_{\text{sat}} - T_w)}\right]^{1/4}} \tag{10.58}$$

or

$$\boxed{\bar{h}_{D,N} = \frac{1}{N^{1/4}}[\bar{h}_D]}$$

Here again, the Reynolds number must be checked for laminar flow, i.e., $Re_f < 1800$.
The heat-transfer rate is calculated from

$$\boxed{\dot{Q} = \bar{h}_{D,N}\,(N_{\text{total}}\,\pi DL)(T_{\text{sat}} - T_w)} \tag{10.59}$$

where N_{total} is the total number of tubes, and (N_{total} πDL) is the total heat-transfer surface area.

If the tubes are arranged in a rectangular array with N rows and M columns then $N_{\text{total}} = M \times N$. In the case of square matrix, $N_{\text{total}} = N^2$.

The expression for $\bar{h}_{D,N}$ in a vertical bank of horizontal tubes given above presumes that the condensate drips from the top tube to the next tube below smoothly and completely. Experiments however reveal that the value of $\bar{h}_{D,N}$ based on the above analysis yields conservative results. Actual heat-transfer coefficients (*and, hence, the condensate flow rates*) are greater because of the fact that when condensate drips from one tube to the other some splashing does occur between the tubes resulting in smaller film thicknesses, less thermal resistance and large quantity of condensate. Besides, some condensation occurs on the sheets or condensate streams between tubes.

According to *Chen*, for N horizontal tubes stacked vertically, the expression is

$$\bar{h}_{D,N} = 0.729(1 + 0.2\,\overline{CT})\left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3 h_{fg}^*}{\mu_l(ND)\left(T_{\text{sat}} - T_w\right)}\right] \tag{10.60}$$

where

$$\overline{CT} = (N-1)C_{pl}(T_{\text{sat}} - T_w)/h_{fg}$$

If the parameter $\overline{CT}$ is less than 2, and $Pr_l > 1$, the agreement with experimental results is found to be satisfactory.

For the case of a vertical bank of N horizontal tubes, the ratio of the average heat-transfer coefficient for N tubes ($\bar{h}_{D,N}$) to that for the top tube ($\bar{h}_1$) is given by k_{ern} as

$$\boxed{\frac{\bar{h}_N}{\bar{h}_1} = N^{-1/6}} \tag{10.61}$$

The ratio of the average heat-transfer coefficient for a tube in the n^{th} *row* to that for a tube in the *first row* is expressed as

$$\boxed{\frac{\bar{h}_n}{\bar{h}_1} = n^{5/6} - (n-1)^{5/6}} \tag{10.62}$$

where $\bar{h}_n$ is the average heat-transfer coefficient in the n^{th} row and $\bar{h}_1$ that for the topmost row in the bank.

While Eq. (10.27) is valid for laminar-film condensation ($Re_f < 1800$), transition to turbulent mode may be possible at the bottom tubes of a vertical tier of horizontal tubes. The film Reynolds number, $Re_f = \dfrac{4\dot{m}}{\mu_l P}$, where $\dot{m}$ is the condensate mass-flow rate at the lowest part of the tube bank and P, the wetted perimeter, for horizontal tubes, each of length L arranged in a vertical bank, is given by $P = 2L$.

Most condensers use horizontal tube bundles through which cold water flows while the vapour condenses outside. Generally the tubes are staggered vertically to impede too great a build-up of film on the lower tubes as the liquid drips off the upper tubes.

Staggering the condenser tube columns illustrated in Fig. 10.22 has a distinct advantage because it decreases the proportion of the tube covered with condensate from the tube above.

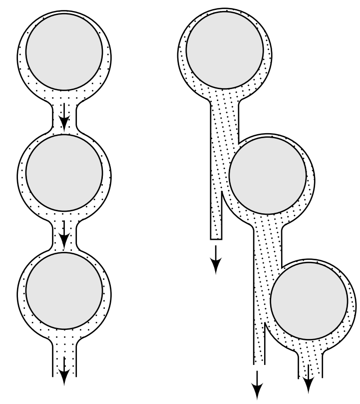


Fig. 10.22 *Film condensation on horizontal tubes (cylinders) with dripping condensate: (a) a vertical bank of aligned (in-line) tubes, and (b) a bank of tubes in staggered arrangement*

The heat-transfer coefficient depends on the number of rows starting from the top. The ratio $(\bar{h}_n/\bar{h}_1)$ is plotted against the number of rows in Fig. 10.23 for both in-line and staggered banks where h_n is the heat-transfer coefficient for the n^{th} row and h_1, for the top (first) row.

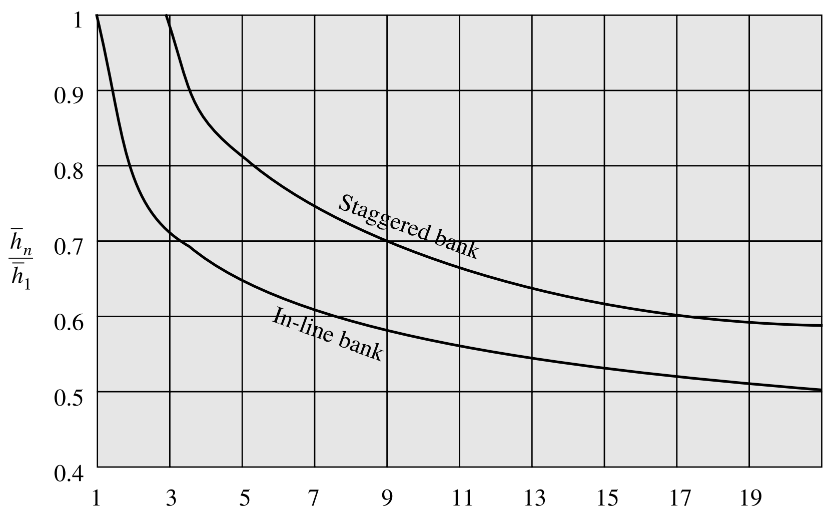


Fig. 10.23 *Variation of heat-transfer coefficient with the number of rows for an in-line bank and a staggered bank of tubes*

10.19 ❑ FORCED FLOW CONDENSATION: HORIZONTAL CYLINDER

The vapour outside the condensate film was so far assumed to be stationary, i.e., the condensation process was characterized by natural convection. When condensing vapour is forcibly pumped over a horizontal tube (*cylinder*) of diameter D with a free-stream velocity u_∞, the following correlation (*Shekriladze and Gomelauri, 1996*) is recommended:

$$\boxed{\frac{\bar{h}_D D}{k_l} = 0.64\, Re_D^{1/2}\left[1+\left\{1+\frac{1.69\, g h_{fg}^* \mu_l D}{u_\infty^2 k_l (T_{\text{sat}}-T_w)}\right\}^{1/2}\right]^{1/2}} \quad (\textit{valid} \text{ for } Re_D < 10^6) \tag{10.63}$$

where $\qquad Re_D = \rho_l u_\infty D/\mu_l$

10.20 ❑ EFFECT OF SUPERHEATED VAPOUR ON FILM CONDENSATION

When saturated vapour condenses on a cool solid surface, the heat-transfer process is essentially one of mass transfer from vapour to liquid and the simultaneous release of latent heat of condensation. The mechanism of heat transfer in condensation is however qualitatively different when the condensing vapour is in a superheated state. Unlike the saturated vapour, the superheated vapour is not in direct contact with the condensing film. First, it has to be cooled to be condensed. The temperature falls gradually in the intermediate layer between the superheated vapour and the liquid film.

This intermediate layer acts as a boundary layer. Condensation takes place only at the interface between the intermediate layer and the liquid film. For film condensation of superheated vapour depends on the degree of superheat, i.e., $(T_{\text{sup}} - T_{\text{sat}})$ seldom exceeds that for the saturated vapour at the same pressure. The temperature profile from the superheated vapour to the solid surface is depicted in Fig. 10.24.

In this case, the amount of heat liberated per unit mass of vapour is

$$\boxed{h'_{fg} = \underbrace{h_{fg}}_{\text{latent heat}} + \underbrace{C_{p_{\text{sup}}}(T_{\text{sup}} - T_{\text{sat}})}_{\text{enthalpy of superheat}}} \tag{10.64}$$

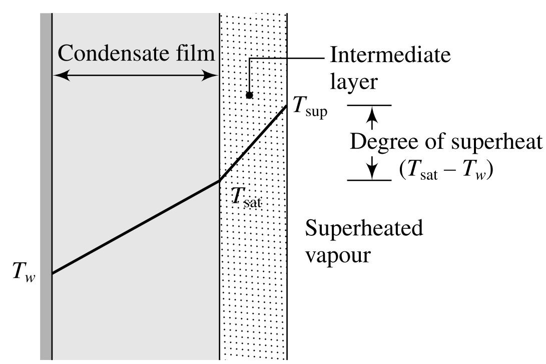


Fig. 10.24 *Temperature profile in condensation of superheated vapour on a vertical surface*

where $C_{p_{\text{sup}}}$ and T_{sup} are the specific heat and temperature of the superheated vapour, respectively. If we assume that the liquid-vapour interface is at the saturation temperature then Eq. (10.10) holds good in this case too except that h_{fg} needs to be replaced by $[h_{fg} + C_{p_{\text{sup}}}(T_{\text{sup}} - T_{\text{sat}})]$. Thus for stationary superheated vapour condensing on an isothermal vertical plate or tube,

$$\boxed{\bar{h}_L = 0.943\left[\frac{\rho_l^2 g k_l^3 \{h_{fg} + C_{p_{\text{sup}}}(T_{\text{sup}} - T_{\text{sat}})\}}{\mu_l L (T_{\text{sat}} - T_w)}\right]^{1/4}} \quad (\rho_v << \rho_l) \tag{10.65}$$

Similarly, for a superheated vapour condensing on a horizontal tube,

$$\boxed{\bar{h}_D = 0.729\left[\frac{\rho_l^2 g k_l^3 h'_{fg}}{\mu_l D (T_{\text{sat}} - T_w)}\right]^{1/4}} \quad (\rho_v << \rho_l)$$

The heat-transfer rate is

$$\boxed{\dot{Q} = \bar{h}_L A (T_{\text{sat}} - T_w)} \tag{10.66}$$

and the condensate mass-flow rate is

$$\boxed{\dot{m} = \frac{\dot{Q}}{[\bar{h}_{fg} + C_{p_{\text{sup}}}(T_{\text{sup}} - T_{\text{sat}})]}} \tag{10.67}$$

The ratio of heat-transfer coefficients for superheated vapour to that with saturated vapour is

$$\boxed{\frac{\bar{h}_{\text{sup}}}{\bar{h}_{\text{sat}}} = \left[\frac{h_{fg} + C_{p\,\text{sup}}(T_{\text{sup}} - T_{\text{sat}})}{h_{fg}}\right]^{1/4}} \tag{10.68}$$

Depending on the degree of superheat, there is usually a slight improvement in condensation heat-transfer coefficient for *liquid non-metals*. Of course, condensation is possible only when the temperature of the cold surface is below the saturation temperature.

In most of the situations in engineering practice, the contribution of enthalpy of superheat is usually very small as compared to the latent enthalpy of condensation.

For instance, let us consider superheated steam at the atmospheric pressure (T_{sat} = 100°C) and 110°C temperature. The enthalpy of superheat ≈ 2.1 (110 – 100) = 21 kJ/kg. This is far too small in comparison with h_{fg} which is equal to 2257 kJ/kg at 1 atm (*not even 1 percent of the latent heat*). In condensing steam with 100°C superheat, the rate of heat transfer was experimentally found to be only 3 percent higher than that for saturated steam at the same pressure and with the same wall temperature.

10.21 ❑ EFFECT OF NON-CONDENSABLE GASES ON FILM CONDENSATION

A significant decrease in the condensation heat transfer coefficient results from the presence of very small amounts of non-condensable gas. In most condensing equipment, such as power plant steam condensers, provision is, therefore, made to bleed off non-condensable gases that leak into the system. For instance, the average condensation-film coefficient for saturated steam condensing at 1 atm is reduced by as much as 50% in the presence of just 1% (by mass) of air. The non-condensable gas envelops the cooling surface and can only contribute to increased thermal resistance, thereby inhibiting or impeding the heat transfer. Furthermore, when a vapour containing non-condensable inert gas condenses, the non-condensable gas is left at the surface and the incoming condensable vapour has to diffuse through this body of vapour-gas mixture collected in the proximity of the condensate surface before it can reach the cool surface to condense. If high heat-transfer rates are aimed at, it is a sound suggestion to make provision at the design stage itself to vent the non-condensable gases, accumulating inside the condenser.

10.22 ❑ FORCED CONVECTION INSIDE HORIZONTAL TUBES

For *forced* condensation *inside horizontal tubes* (Fig. 10.25) at low vapour velocities valid for Reynolds number, $Re_v = (\rho_v u_v D/\mu_v)_{\text{inlet}} < 3.5 \times 10^4$, Chato (1962) recommends the following correlation:

$$\boxed{\bar{h}_{D(\text{int})} = 0.555\left[\frac{\rho_l^2 g k_l^3 h'_{fg}}{\mu_l D(T_{\text{sat}} - T_w)}\right]^{1/4}} \tag{10.69}$$

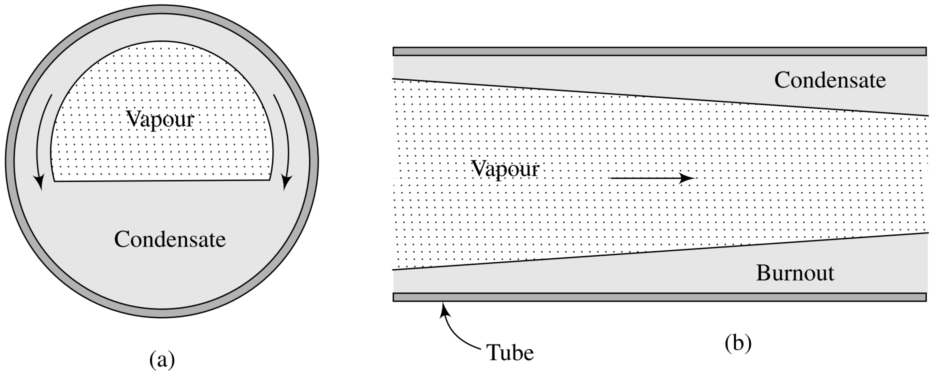


Fig. 10.25 *Condensate flow in a horizontal tube with (a) low vapour velocities, and (b) large vapour velocities*

where $h'_{fg} \equiv h_{fg} + \frac{3}{8} C_{pl} (T_{sat} - T_w)$, D is the inside diameter of the tube and Re_v the Reynolds number of the vapour is to be evaluated at the tube inlet conditions using the internal tube diameter as the characteristic length.

10.23 ❑ DROPWISE CONDENSATION

In dropwise condensation, as stated earlier, the vapour condenses into small liquid droplets of various sizes which fall down the surface in random fashion. The drops form in cracks and pits on the surface, grow in size, break away from the surface, knock off other droplets and eventually run off the surface, without forming a film, under the influence of gravity.

For dropwise condensation of steam on copper surfaces, the correlations to be used are given by

$$\boxed{h_{\text{dropwise}} = \begin{matrix} 511\,04 + 2044 T_{\text{sat}} \\ 255\,310 \end{matrix}\Bigg\}} \quad \begin{matrix} 22°\text{C} < T_{\text{sat}} < 100°\text{C} \\ T_{\text{sat}} > 100°\text{C} \end{matrix} \tag{10.70}$$

where T_{sat} is in °C and the heat-transfer coefficient h_{dropwise} is in W/m^2 °C or its equivalent w/m^2 K.

Illustrative Examples

(A) Nucleate Boiling and Peak Heat Flux

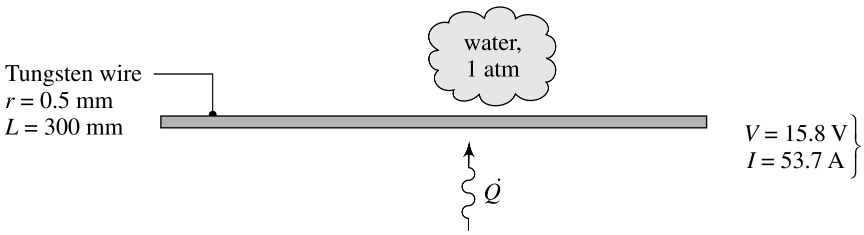
EXAMPLE 10.1 *A tungsten wire immersed horizontally in a water bath at atmospheric pressure is heated electrically with a steady-state applied voltage drop of 15.8 V and a current of 53.7 A. The wire has a radius of 0.5 mm and is 300 mm long. Determine (a) the heat flux, and (b) the wire surface temperature if the boiling heat-transfer coefficient is estimated to be 45 kW/m^2 K.*

Solution

Known Tungsten wire, electrically heated is submerged in water at 1 atm.

Find (a) Heat flux, q [W/m^2], (b) Wire-surface temperature, T_w [°C].

Schematic



Assumptions (1) Steady state prevails. (2) Negligible heat loss from the heater surface.

Analysis (a) Steady-state heat flux,

$$q = \frac{\dot{Q}}{A_s} = \frac{VI}{2\pi rL} = \frac{(15.8)(53.7)}{2\pi(0.5\times10^{-3}\text{ m})(300\times10^{-3}\text{ m})}\left(\frac{1\text{ kW}}{10^3\text{ W}}\right)$$

$$= \frac{848.46}{9.4248\times10^{-6}} = \mathbf{900.24\ kW/m^2} \qquad \textbf{(Ans.) (a)}$$

(b) Also, the heat-transfer rate is

$$\dot{Q} = hA_s\Delta T_e$$

The excess temperature,

$$\Delta T_e = T_w - T_{\text{sat}} = \frac{\dot{Q}}{hA_s} = \frac{q}{h} = \frac{900.24\text{ kW/m}^2}{45\text{ kW/m}^2\text{ K}}$$

Hence, the wire-surface temperature is

$$T_w = T_{\text{sat}} + \Delta T_e = 100 + 20 = \mathbf{120°C} \qquad \textbf{(Ans.) (b)}$$

EXAMPLE 10.2 *Estimate the critical heat flux for boiling water at 1 atm on the moon's surface, where the gravitational acceleration is one sixth that on the earth.*

Properties: Saturated water, 1 atm (T_{sat} = 100°C):

ρ_v = *0.5978 kg/m³* $\qquad$ ρ_l = *957.9 kg/m³*

h_{fg} = *2257 kJ/kg* $\qquad$ σ = *0.0589 N/m*

Solution

Known Boiling water at 1 atm on the moon where the gravitational field is 1/6 th that of the earth.

Find $q_{\max}$(W/m²).

Assumptions (1) Nucleate pool boiling.

Analysis Critical heat flux, on the moon at 1 atm,

$$q_{\max} = 0.149\,h_{fg}[g\sigma\rho_v^2(\rho_l - \rho_v)]^{1/4}$$

Here $\qquad g = \frac{1}{6}\times 9.81\text{ m/s}^2$

Hence, $\qquad q_{\max} = 0.149(2257\times10^3)\left[\frac{9.81}{6}\times 0.0589(0.5978)^2(957.9 - 0.5978)\right]^{1/4}$

$$= 805.7\times10^3\text{ W/m}^2 = \mathbf{805.7\ kW/m^2} \qquad \textbf{(Ans.)}$$

EXAMPLE 10.3 *n-butyl alcohol is to be boiled at atmospheric pressure on a 30 cm diameter horizontal copper heating plate. (a) Determine the critical heat flux that can be attained in the nucleate pool-boiling regime and the corresponding heater surface temperature. (b) Also calculate the minimum heat flux. (c) If the surface temperature is 130°C, find the rate of evaporation.*

Properties of n-butyl alcohol at $T_{sat\ @\ 1\ atm}$ *= 117.5°C:*

σ = 0.0183 N/m	h_{fg} = 591.5 kJ/kg
ρ_l = 737 kg/m^3	ρ_v = 2.3 kg/m^3
μ_l = 0.39 × 10^{-3} kg/ms	Pr = 6.9
n = 1.7	C_{sf} = 0.00305
C_{pl} = 2.876 kJ/kg K	

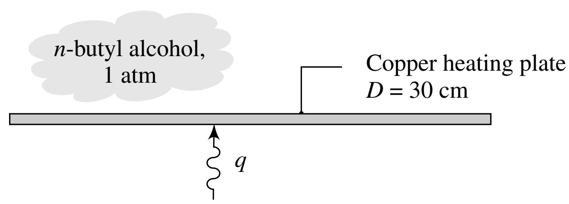
Solution

Known Nucleate pool boiling on a copper heater surface in *n*-butyl alcohol.

Find (a) Maximum heat flux and corresponding heater surface temperature. (b) Minimum heat flux.

Assumptions (1) Polished copper surface. (2) Nucleate boiling regime.

Schematic



Analysis The dimensionless parameter is

$$L^* = L\left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2} \quad \text{where} \quad L = D = 0.3\text{ m}$$

$$= 0.3\left[\frac{9.81(737 - 2.3)}{0.0183}\right]^{1/2} = 188.27$$

As $L^* > 27$, $C_{cr} = 0.149$

Hence, the critical heat flux is

$$q_{\text{nucleate,max}} = 0.149\, h_{fg}[\sigma g \rho_v^2(\rho_l - \rho_v)]^{1/4}$$

$$= 0.149(591.5 \times 10^3)[0.0183 \times 9.81 \times (2.3)^2(737 - 2.3)]^{1/4}$$

$$= 453 \times 10^3\ \text{W/m}^2 = \mathbf{453\ kW/m^2}$$ **(Ans.) (a)**

But

$$q_{\text{nucleate}} = \mu_l h_{fg}\left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2}\left[\frac{C_{pl}(T_w - T_{\text{sat}})}{C_{sf} h_{fg} Pr_l^n}\right]^3$$

$$= (0.39 \times 10^3)(591.5 \times 10^3) \times \left[\frac{9.81(737 - 2.3)}{0.0183}\right]^{1/2}$$

$$\times \left[\frac{2.876(T_w - T_{\text{sat}})}{0.00305 \times 591.5 \times (6.9)^{1.7}}\right]^3$$

$$= (T_w - T_{\text{sat}})^3 \times 30.914$$

∴ Excess temperature,

$$\Delta T_e = T_w - T_{sat} = \left[\frac{453000}{30.914}\right]^{1/3} = 24.47°C$$

Hence, the heater surface temperature is

$$T_w = 117.50 + 24.47 = \mathbf{141.97°C} \qquad \textbf{(Ans.) (a)}$$

Minimum heat flux is

$$q_{\min} = 0.09\,\rho_v\,h_{fg}\left[\frac{\sigma g(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2}\right]^{1/4}$$

$$= 0.09 \times 2.3 \times 591500 \times \left[\frac{0.0183 \times 9.81 \times (737 - 2.3)}{(737 + 2.3)^2}\right]^{1/4}$$

$$= 15.3 \times 10^3 \text{ W/m}^2 = \mathbf{15.3\ kW/m^2} \qquad \textbf{(Ans.) (b)}$$

This is **2.6%** of the peak heat flux.
For excess temperature,

$$\Delta T_e = T_w - T_{sat} = 130 - 117.5 = 12.5°C$$

The heat flux can be evaluated as follows

$$q = (\text{const})(\Delta T_e)^3 \text{ where constant} = 30.914$$

$$\therefore \qquad q = 30.914 \times (12.5)^3 = 60.4 \times 10^3 \text{ W/m}^2 = 60.4 \text{ kW/m}^2$$

Heat-transfer rate is

$$\dot{Q} = q\left(\frac{\pi D^2}{4}\right) = (60.4 \times 10^3)\left(\frac{\pi}{4} \times 0.3^2\right) = \mathbf{4268\ W}$$

And, the rate of evaporation is

$$\dot{m} = \frac{\dot{Q}}{h_{fg}} = \frac{4268}{591500} \times 3600 = \mathbf{26.0\ kg/h} \qquad \textbf{(Ans.) (c)}$$

EXAMPLE 10.4 *Copper tubes 25 mm in diameter and 75 cm long are to be employed for boiling saturated water at 1 atm. If the tubes are to be operated at three-fourth of the critical heat flux (CHF), find the number of tubes required to provide an evaporation rate of 850 kg/h. What is the surface temperature of the tubes under these conditions?*

Solution

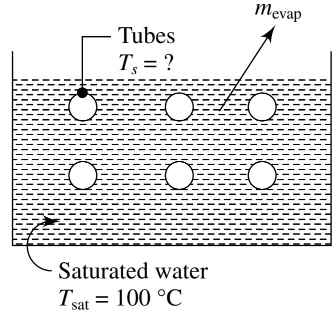
Known Copper tubes in boiling water at 1 atm operating at 3/4th of the CHF.

Find Number of tubes for $\dot{m}_{evap}$ = 850 kg/h. Tube surface temperature, T_w(°C).

Assumptions (1) Steady-state conditions. (2) Water exposed to 1 atm and uniform temperature at 100°C.

Properties Saturated water at 100°C (T_{sat} at 1 atm):
ρ_l = 957.9 kg/m³ h_{fg} = 2257 kJ/kg

Schematic



C_{pl} = 4.217 kJ/kg K μ_l = 0.282 × 10^{-3} kg/m s
Pr_l = 1.75 σ = 0.0589 N/m
ρ_v = 0.6 kg/m^3

Analysis Since $q < q_{max}$, i.e., CHF, nucleate boiling is indicated.

Heat flux, $q_{\text{nucleate}} = \frac{3}{4} \times$ Critical Heat Flux

$= (0.75)\,(1.26)$ MW/m^2 = **0.945 × 10^6 W/m^2**

If N is the number of tubes required,

$$(q_{\text{nucleate}})(\pi DL)(N) = (\dot{m})(h_{fg})$$

$$\therefore \quad N = \frac{(850\text{ kg/h})(1\text{ h}/3600\text{ s})(2257 \times 10^3\text{ J/kg})}{(0.945 \times 10^6\text{ W/m}^2)(\pi \times 0.025\text{ m} \times 0.75\text{ m})}$$

$= \mathbf{9.6 \cong 10}$ **(Ans.)**

The tube-surface temperature can be determined by using the Roshsenow correlation:

$$T_w - T_{\text{sat}} = \frac{C_{sf} h_{fg} Pr_l^n}{C_{pl}} \left[\frac{q_w}{\mu_l h_{fg}} \sqrt{\frac{\sigma}{g(\rho_l - \rho_v)}} \right]^{1/3}$$

For *polished copper-water* combination, C_{sf} = 0.013 and n = 1.0 for water.
Substitution of relevant numerical values, one has

$$T_w - T_{\text{sat}} = \frac{(0.013)(2257 \times 10^3\text{ J/kg})(1.75)}{(4217\text{ J/kg K})}$$

$$\times \left[\frac{(0.945 \times 10^6\text{ W/m}^2)}{(0.282 \times 10^{-3}\text{ kg/m s})(2257 \times 10^3\text{ J/kg})} \sqrt{\frac{0.0589\text{ N/m}}{(9.81\text{ m/s}^2)(957.9 - 0.6)\text{kg/m}^3}} \right]^{1/3}$$

$= \mathbf{18.9°C}$

Hence, the tube-surface temperature,

$T_w = T_{\text{sat}}$ **+ 18.9 = 100 + 18.9 = 118.9°C** **(Ans.)**

EXAMPLE 10.5 *Water at atmospheric pressure boils in a stainless steel kitchen pan with an excess temperature of 8°C. (a) Estimate the heat flux if water were to be boiled in a pressure cooker at a pressure of 2.0 bar. (b) If the excess temperature were to be increased to 14°C at 1 atm pressure, what would be the heat flux? Also find the percentage change in the boiling heat flux in both cases.*

Solution

Known Boiling of water in a kitchen pan at 1 atm and surface temperature 7°C above the saturation temperature.

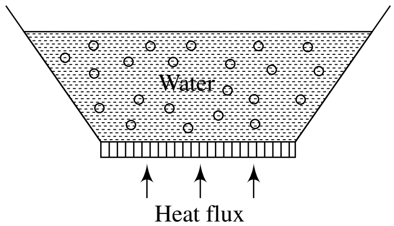
Find (a) q with P = 2 bar, ΔT_e = 8°C, (b) q with, P = 1 atm, ΔT_e = 14°C.

Assumptions (1) Steady operating conditions. (2) Constant properties.

Analysis (a) Anticipating nucleate boiling, for a horizontal surface we use the simplified relation: h = 5.56 $(\Delta T_e)^3$ = 5.56 $(8)^3$ = 2846.7 W/m^2 K

Heat flux = $h\Delta T_e = (2846.7\text{ W/m}^2\text{ K})(8°\text{C or K}) = 22.77\text{ kW/m}^2$

Schematic



(a) P_1 = 1 atm, P = 2 bar

$\Delta T_e = T_w - T_{sat}$ = 8 °C

(b) $P^* = P_1$ = 1 atm

Δt_e^* = 14 °C

Since the heat flux so calculated lies between the applicable range of $16 < q < 240$ W/m², the empirical relation used was correct.

When the pressure is increased to 2 bar,

$$h_p = h_1\left(\frac{P}{P_1}\right)^{0.4} = 2846.7 \text{ W/m}^2\text{ K}\left[\frac{2\text{ bar}}{1.01325\text{ bar}}\right]^{0.4}$$

$$= 3736.5 \text{ W/m}^2 \text{ K}$$

And the heat flux = (3736.5 W/m² K) (8 K) = **29.9 kW/m²**

Percentage increase in the heat flux = [(29.9 – 22.77) / 22.77] (100) = **27%** **(Ans.) (a)**

(b) If P^* = 1 atm and ΔT_e = 14°C then

$$h^* = 5.56\ (14)^3 = 15256.4 \text{ W/m}^2 \text{ K}$$

and, the heat flux = (15256.4) (14) (10^{-3}) = 213.6 kW/m²

Percentage increase = [(213.6 – 22.77) / 22.77)] (100) = **838 %** **(Ans.) (b)**

(B) Film Boiling

EXAMPLE 10.6 *A horizontal, heated aluminium cylinder (emissivity ε = 0.45) of 25 mm diameter and 25 cm long at 500°C is immersed in a liquid nitrogen bath at –196°C. Neglecting end effects, determine (a) the initial heat-transfer rate, and (b) the initial heat flux.*

Properties: Liquid nitrogen at 1 atm, –196°C:

h_{fg} = *201 kJ/kg* ρ_l = *800 kg/m³*

Nitrogen vapour at film temperature of 152°C and 1 atm:

ρ_v = *0.8034 kg/m³* k_v = *0.0343 W/m°C*

C_{pv} = *1.043 kJ/kg°C* μ_v = *2.308 × 10^{-5} kg/m s*

Solution

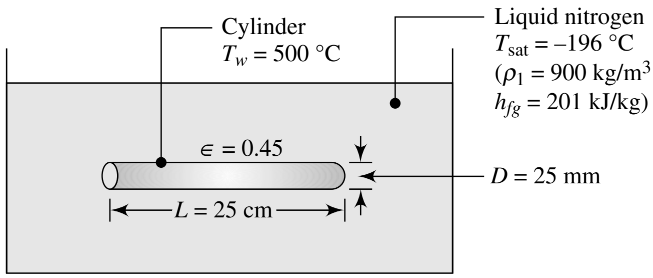
Known A horizontal aluminium cylinder heated to 500°C is immersed in a liquid nitrogen bath.

Find Initial rate of heat transfer.

Assumptions (1) Steady operating conditions prevail. (2) Constant properties. (3) Heat transfer from the end surfaces is neglected.

Analysis Excess temperature, $T_w - T_{sat}$ = 500 – (–196) = 696°C. Clearly, with such a large temperature difference, *stable film boiling* will occur.

Schematic



Film temperature,

$$T_f = \frac{1}{2}(T_w + T_{\text{sat}}) = \frac{1}{2}(500 + (-196)) = 152°\text{C}$$

$$h'_{fg} = h_{fg} + 0.4C_{p_v}(T_w - T_{\text{sat}}) = 201 + (0.4 \times 1.043 \times 696) = 491.37\ \text{kJ/kg}$$

For a horizontal cylinder, the film boiling heat-transfer coefficient

$$h_c = 0.62\left[\frac{\rho_v(\rho_l - \rho_v)gk_v^3 h'_{fg}}{\mu_v D(T_w - T_{\text{sat}})}\right]^{1/4}$$

$$= 0.62\frac{[0.8034\ \text{kg/m}^3(800 - 0.8034)\text{kg/m}^3(9.81\ \text{m/s}^2)(0.0343\ \text{W/m °C})(491.37 \times 10^3\ \text{J/kg})]}{[(2.308 \times 10^{-5}\ \text{kg/m s})(0.025\ \text{m})(696\ °\text{C})]}$$

$$= 82.33\ \text{W/m2 °C}$$

Radiation heat-transfer coefficient is

$$h_r = \sigma\varepsilon(T_w - T_{\text{sat}})(T_w^2 + T_{\text{sat}}^2)$$

$$= (5.67 \times 10^{-8}\ \text{W/m}^2\,\text{K}^4)(0.45)(273.15 + 500) + (273.15 - 196)\text{K}$$

$$[(273.15 + 500)^2 + (273.15 - 196)^2\ \text{K}] = 13.1\ \text{W/m}^2\ °\text{C}$$

Since $h_r < h_c$, the overall heat-transfer coefficient is

$$h = h_c + \frac{3}{4}h_r = 82.33 + \frac{3}{4} \times 13.1 = \mathbf{92.2\ W/m^2\ °C}$$

Initial heat-transfer rate is

$$\dot{Q} = hA_s(T_w - T_{\text{sat}}) = (92.2\ \text{W/m}^2\ °\text{C})(\pi \times 0.025\ \text{m} \times 0.25\ \text{m})(696°\text{C})$$

$$= \mathbf{1260\ W}$$ **(Ans.) (a)**

Initial heat flux is

$$q = \frac{\dot{Q}}{A_s} = (92.2\ \text{W/m}^2\ °\text{C})(696°\text{C}) = \mathbf{64170\ W/m^2}$$ **(Ans.) (b)**

(C) Forced Convection Boiling

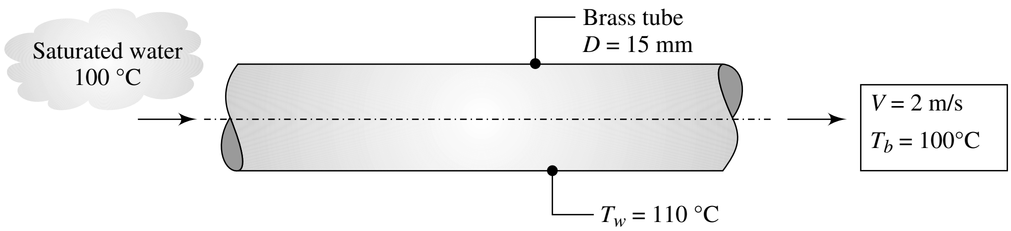
EXAMPLE 10.7 *Saturated water at 1 atm flows through a 15 mm diameter smooth brass tube where the wall is maintained at 110°C. The mean flow velocity of water is 2 m/s. Determine the heat-transfer rate per unit length of the tube.*

Solution

Known Forced convection and nucleate boiling processes occur in a smooth tube with specified water velocity and tube wall temperature.

Find $\dot{Q}/L$(W/m).

Schematic



Assumptions (1) Fully developed flow. (2) Nucleate boiling conditions prevail on the inner wall of the tube. (3) Forced convection and boiling effects can be separately determined.

Properties ***Saturated water (T_{sat} = 100°C):***

$k_l = 0.679$ W/m K	$\rho_v = 0.5978$ kg/m^3
$\rho_l = 957.9$ kg/m^3	$h_{fg} = 2257$ kJ/kg
$\mu_l = 0.282 \times 10^{-3}$ kg/m s	$Pr_l = 1.75$
$C_{pl} = 4.217$ kJ/kg K	$\sigma = 0.0589$ N/m
$C_{sf} = 0.006$	$n = 1.0$

Analysis The heat-transfer rate comprises two parts: forced convection and nucleate boiling. That is,

$$\dot{Q} = \dot{Q}_{\text{conv}} + \dot{Q}_{\text{boiling}} = \pi DL[q_{\text{conv}} + q_{\text{boiling}}]$$

Forced convection:

Reynolds number,

$$Re = \frac{\rho VD}{\mu} = \frac{957.9(2.0)(0.015)}{0.282 \times 10^{-3}} = 101904\ (> 2300)$$

Using Dittus–Boelter correlation:

$$Nu = 0.023(Re)^{0.8}(Pr)^{0.4} = 0.023(101904)^{0.8}(1.75)^{0.4} = 292.08$$

Hence, $$h_{\text{conv}} = Nu\frac{k_l}{D} = (292.08)\left(\frac{0.679}{0.015}\right) = 13\ 221\ \text{W/m}^2\ \text{K}$$

Heat flux,

$$q_{\text{conv}} = h_{\text{conv}}(T_w - T_b) = 13\ 221\ (110 - 100) = 132.21 \times 10^3\ \text{W/m}^2$$

Nucleate boiling:

$$q_{\text{nucleate}} = \mu_l h_{fg}\left[\sqrt{\frac{g(\rho_l-\rho_v)}{\sigma}}\right]\times\left[\frac{C_{pl}(T_w - T_{\text{sat}})}{C_{sf}h_{fg}Pr_l^n}\right]^3$$

$$= (0.282\times10^{-3})(2257\times10^3)\left[\sqrt{\frac{9.81(957.9-0.5978)}{0.0589}}\right]\times\left[\frac{4.217(110-100)}{(0.006)(2257)(1.75)}\right]^3$$

$$= 1.432\times10^6 \text{ W/m}^2$$

Therefore, heat rate per unit tube length is

$$\frac{\dot{Q}}{L} = \pi D[q_{\text{conv}} + q_{\text{boiling}}] = \pi(0.015)[(132.21\times10^3) + (1.432\times10^6)](\text{W/m})$$

$= \mathbf{73.7\ kW/m}$ **(Ans.)**

(P) Condensation

EXAMPLE 10.8 *Show that the condensation Reynolds number for laminar-film condensation on a vertical plate can be expressed as*

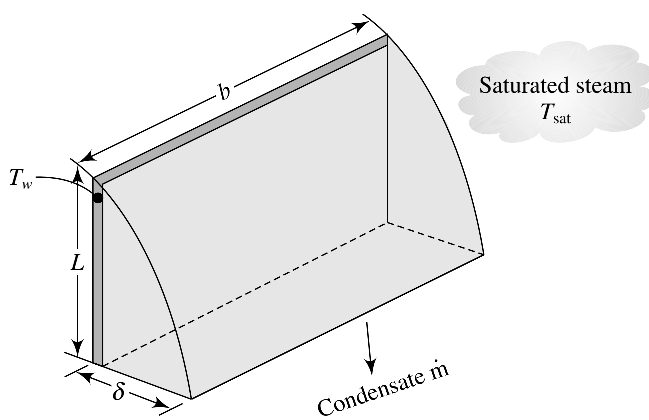
$$\text{Re}_\text{f} = 3.771\left[\frac{\rho_\text{l}(\rho_\text{l}-\rho_\text{v})\text{g}\,\text{k}_\text{l}^3\text{L}^3(\text{T}_{sat}-\text{T}_\text{w})^3}{\mu_\text{l}^5\text{h}_\text{fg}^3}\right]^{1/4}$$

Solution

Known Laminar-film condensation on a vertical plate.

Find Expression for film Reynolds number.

Schematic



Assumptions (1) Steady operating conditions are established. (2) The condensate flow is laminar. (3) The plate is isothermal.

Analysis Film Reynolds number,

$$Re_f = \frac{4\dot{m}}{P\mu_l} = \frac{4\dot{m}}{b\mu_l}$$

(*where perimeter P = width b for a vertical plate*)
Mass-flow rate of condensation,

$$\dot{m} = \frac{\rho_l(\rho_l - \rho_v)bg\delta^3}{3\mu_l}$$

where film thickness,

$$\delta = \left[\frac{4\mu_l k_l L(T_{\text{sat}} - T_w)}{\rho_l(\rho_l - \rho_v)gh_{fg}}\right]^{1/4}$$

Substituting for $\dot{m}$, one has

$$Re_f = \frac{4\rho_l(\rho_l - \rho_v)bg}{3\mu_l}\left[\frac{4\mu_l k_l L(T_{\text{sat}} - T_w)}{\rho_l(\rho_l - \rho_v)g\,h_{fg}}\right]^{3/4} \cdot \frac{1}{b\mu_l}$$

$$= \frac{4}{3}\left[\frac{\rho_l(\rho_l - \rho_v)g}{\mu_l}\right]^{1/4} \times (4)^{3/4} \times \left[\frac{k_l^3 L^3(T_{\text{sat}} - T_w)^3}{h_{fg}^3}\right]^{1/4} \times \left(\frac{1}{\mu_l^4}\right)^{1/4}$$

$$= \frac{(4)^{7/4}}{3}\left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3 L^3(T_{\text{sat}} - T_w)^3}{\mu_l^5 h_{fg}^3}\right]^{1/4}$$

or $$\boxed{\boldsymbol{Re_f = 3.771\left[\frac{\rho_l(\rho_l - \rho_v)g\,k_l^3 L^3(T_{\text{sat}} - T_w)^3}{\mu_l^5 h_{fg}^3}\right]^{1/4}}}$$ **(Ans.)**

Hence, proved.

EXAMPLE 10.9 *A square pan with its bottom surface maintained at 77°C is exposed to steam at 100°C and 1 atm. The pan has a lip all around so that the condensate formed cannot flow away. How deep will the condensate film be after 10 minutes have elapsed at this condition? Use the following properties of water:*

$$T_{\text{sat}} = 100°\text{C} : h_{fg} = 2257 \text{ kJ/kg}$$

$$T_f = \frac{1}{2}(100 + 77) = 88.5°\text{C} :$$

$$\rho = 966 \text{ kg/m}^3 \qquad k = 0.675 \text{ W/m K}$$

Solution

Known Condensation of atmospheric steam on a square pan surface at specified temperature.

Find Condensate film thickness, δ.

Assumptions (1) Laminar-film condensation. (2) Linear temperature profile with no subcooling of the condensate.

Analysis Energy balance at the liquid vapour interface is

$$\dot{Q} = \dot{m}h_{fg} = \left[\rho A\frac{d\delta}{dt}\right]h_{fg}$$

or Heat flux, $q = \frac{\dot{Q}}{A} = \rho h_{fg} \frac{d\delta}{dt}$

Also, $q = h_x (T_{\text{sat}} - T_w) = \frac{k}{\delta}(T_{\text{sat}} - T_w)$

$\therefore$ $\rho h_{fg} \frac{d\delta}{dt} = \frac{k}{\delta}(T_{\text{sat}} - T_w)$

Separating the variables and integrating,

$$\int_0^\delta \delta\, d\delta = \frac{k(T_{\text{sat}} - T_w)}{\rho h_{fg}} \int_0^t dt \quad \text{or} \quad \frac{\delta^2}{2} = \frac{kt(T_{\text{sat}} - T_w)}{\rho h_{fg}}$$

Film thickness,

$$\delta = \left[\frac{2kt(T_{\text{sat}} - T_w)}{\rho h_{fg}}\right]^{1/2} = \left[\frac{2 \times 0.675 \times (10 \times 60) \times (100 - 77)}{966 \times 2257 \times 10^3}\right]^{1/2}$$

$= 2.92 \times 10^{-3}$ m = **2.92 mm** **(Ans.)**

EXAMPLE 10.10 *Determine (a) the average heat-transfer coefficient, (b) the heat-transfer rate, and (c) the amount of condensate dripping off the bottom of a 0.25 m square vertical plate that is exposed to saturated steam on one side at 0.125 MPa pressure. The plate surface is maintained at a temperature of 105°C. (d) Plot the values of film thickness and local heat-transfer coefficient against different locations from the top of the plate. (e) Calculate the average velocity of the condensate film at the bottom of the plate, and (f) the average condensation heat-transfer coefficient over the entire plate if the plate is inclined at 30° with the vertical. (g) Show that the film condensation is wave free laminar.*

The following thermophysical properties may be used:

Saturated steam: ($T_{sat\ @\ 0.125\ MPa}$ = 106°C):

ρ_v = 0.7273 kg/m³ h_{fg} = 2241 kJ/kg

Saturated water: $\left\{T_f = \frac{1}{2}(106 + 105) = 105.5°C\right\}$:

ρ_l = 953.9 kg/m³ k_l = 0.88 W/m K

C_{pl} = 4.224 kJ/kg K $\mu_l = 0.267 \times 10^{-3}$ kg/m s

Solution

Known Condensation of saturated steam at 0.125 MPa on a vertical plate (0.25 × 0.25 m) at 105°C.

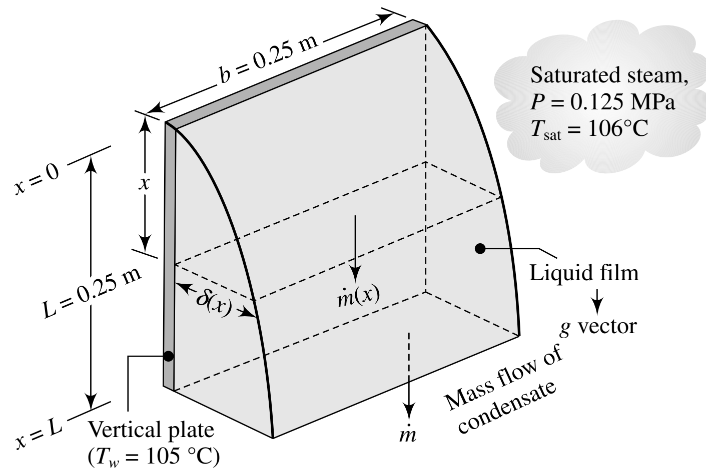
Find (a) $\bar{h}_L$ (b) $\dot{Q}$ (c) $\dot{m}(L)$ (d) $h(x)$ and $\delta(x)$ vs x (e) $u_{av(x\,=\,L)}$ (f) $\bar{h}_{L,\text{inclined}}$ (g) Re_f.

Assumptions (1) Steady operating conditions prevail. (2) Laminar-film condensation. (3) Isothermal plate. (4) Effect of non-condensable gases on steam is negligible.

Analysis The modified latent heat of condensation is

$$h_{fg}^* = h_{fg}(1 + 0.68\ \text{Ja})$$

Schematic



where

$$\text{Ja} = \frac{C_{pl}(T_{\text{sat}} - T_w)}{h_{fg}} = \frac{4.224(106 - 105)}{2241}$$

$$= 1.885 \times 10^{-3}$$

$$\therefore \qquad h_{fg}^* = 2241[1 + (0.68)(1.885 \times 10^{-3})] = 2243.87 \text{ kJ/kg}$$

Condensate film thickness,

$$\delta(x) = \left[\frac{4\mu_l k_l x(T_{\text{sat}} - T_w)}{\rho_l(\rho_l - \rho_v)g h_{fg}^*}\right]^{1/4} = \left[\frac{4 \times 0.267 \times 10^{-3} \times 0.681 \times x \times (106 - 105)}{953.9 \times (953.9 - 0.7273)(9.81)(2243.87 \times 10^3)}\right]^{1/4}$$

$$= \left[\frac{4 \times 0.267 \times 10^{-3} \times 0.681 \times x \times (106 - 105)}{953.9 \times (953.9 - 0.7273)(9.81)(2243.87 \times 10^3)}\right]^{1/4}$$

$$= 7.764 \times 10^{-5} \times x^{1/4} (\text{m}) = 0.7764\, x^{1/4} (\text{mm})$$

Local heat-transfer coefficient,

$$h_x = \frac{k_l}{\delta(x)} = \frac{0.681}{7.764 \times 10^{-5}\, x^{1/4}} = 8.771 \times x^{-1/4} \text{ W/m}^2 \text{ K}$$

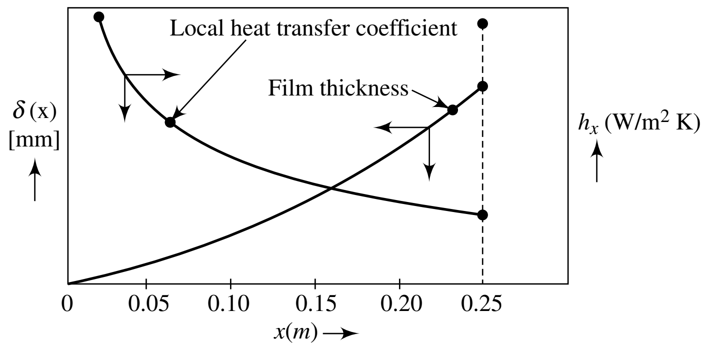
The values of $\delta(x)$ and h_x are tabulated below for different values of x:

Sr. No.	x(m)	$\delta(x)$ (mm)	h_x(W/m² K)
1	0	0	∞
2	$\frac{1}{5}L = 0.05$ m	0.036 71	18 550
3	$\frac{2}{5}L = 0.10$ m	0.043 66	15 598

contd.

contd.

4	$\frac{3}{5}L = 0.15\text{ m}$	0.048 32	14 094
5	$\frac{4}{5}L = 0.20\text{ m}$	0.051 92	13 116
6	$\frac{5}{6}L = 0.25\text{ m}$	0.054 90	12 404



(Ans.) (d)

Local condensation coefficient, at the bottom of the tube,

$$h_{x=L} = \frac{k_l}{\delta(L)} = \frac{0.681}{0.549 \times 10^{-3}} = 12404\text{ W/m}^2\text{ K}$$

Average heat-transfer coefficient,

$$\bar{h}_{L,\text{vertical}} = \frac{4}{3} h_{x=L} = \frac{4}{3} \times 12404 = \mathbf{16539\ W/m^2\ K} \quad \textbf{(Ans.) (a)}$$

Average heat-transfer coefficient at θ = 30° with the vertical,

$$\bar{h}_{L,\text{inclined}} = \bar{h}_{L,\text{vertical}} \times (\cos\theta)^{1/4} = 16539(\cos 30°)^{1/4}$$

$$= \mathbf{15955\ W/m^2\ K} \quad \textbf{(Ans.) (f)}$$

Average flow velocity of the condensate film at $x = L$ is

$$u_{av} = \frac{\dot{m}_{(x=L)}}{\rho_l (b\delta_L)} = \frac{\rho_l(\rho_l - \rho_v) g \delta_L^3 b}{(\rho_l b \delta_L) 3\mu_l} = \frac{(\rho_l - \rho_v) g \delta_L^2}{3\mu_l}$$

$$= \frac{(953.9 - 0.7273)(9.81)(0.0549 \times 10^{-3})^2}{3 \times 0.267 \times 10^{-3}}$$

$$= \mathbf{0.03518\ m/s \quad or \quad 35.18\ mm/s} \quad \textbf{(Ans.) (e)}$$

Rate of condensation over the entire plate,

$$\dot{m} = \rho_l (b\delta_L) u_{av(x=L)} = 953.9 \times 0.25 \times 0.0549 \times 10^{-3} \times 0.03518$$

$$= 4.606 \times 10^{-4}\text{ kg/s} = \mathbf{1.658\ kg/h} \quad \textbf{(Ans.) (c)}$$

Heat-transfer rate, $\dot{Q} = \dot{m} h_{fg}^* = (4.606 \times 10^{-4})(2243.87 \times 10^3)$

$$= \mathbf{1033.6\ W} \quad \textbf{(Ans.) (b)}$$

And, the rate of condensate production,

$$\dot{m} = \frac{\dot{Q}}{h_{fg}^{*}} = \frac{1033.6}{2243.87 \times 10^{3}} = 4.606 \times 10^{-4} \text{ kg/s} \quad \text{or} \quad 1.658 \text{ kg/h}$$

Film Reynolds number,

$$Re_f = \frac{4\dot{m}}{b\mu_l} = \frac{4 \times 4.606 \times 10^{-4}}{0.25 \times 0.267 \times 10^{-3}} = \mathbf{27.6} \qquad \textbf{(< 30)} \quad \textbf{(Ans.) (g)}$$

Hence, the laminar-film condensation is *smooth and wave free.*

EXAMPLE 10.11 *Saturated steam at 0.08208 bar (*T_{sat} *= 42°C) is exposed to a 60 cm square vertical plate having a uniform surface temperature of 32°C.*

Estimate

(a) the film thickness, the mean and maximum flow velocity of the condensate, and the local heat transfer coefficient at 20 cm from the bottom of the plate

(b) the average heat-transfer coefficient and the total heat-transfer rate to the plate surface after applying McAdam's correction.

(c) the total condensation rate and the film Reynolds number

(d) the average heat-transfer coefficient if the plate is inclined at 60°C with the vertical

Use the following properties of saturated water:

At 37°C:

$k = 0.628$ W/m °C

$\rho = 993$ kg/m³

$\mu = 0.695 \times 10^{-3}$ kg/m s

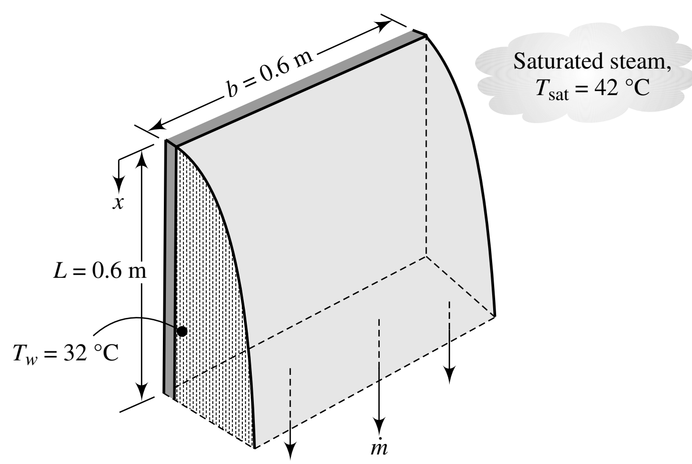
$h_{fg@}$ 42°C = 2401.9 kJ/kg

Solution

Known Vertical square plate exposed to condensing steam.

Find (a) δ, u_{mean}, u_{max} and h_x at $x = 0.4$ m, (b) $\overline{h}_L, \dot{Q}$, (c) $\dot{m}, Re_f$, (d) $\overline{h}_{inclined}$.

Schematic



Assumptions (1) Laminar-film condensation on a vertical surface. (2) Negligible concentration of non-condensables in steam. (3) $\rho >> \rho_v$.

Analysis (a) At 20 cm from the bottom of the plate, i.e., the length x measured from the top equal to (60 – 20) = 40 cm or 0.4 m, we have

Film thickness at x = 0.4 m,

$$\delta = \left[\frac{4\mu k(T_{\text{sat}} - T_w)x}{\rho^2 g h_{fg}}\right]^{1/4}$$

$$= \left[\frac{4(0.695 \times 10^{-3} \text{ kg/m s})(0.628 \text{ W/m °C})(42 - 32)\text{°C}(0.4 \text{ m})}{(993 \text{ kg/m}^3)^2(9.81 \text{ m/s}^2)(2401.9 \times 10^3 \text{ J/kg})}\right]^{1/4}$$

= **0.132 × 10^{-3} m** or **0.132 mm** **(Ans.) (a)**

Mean velocity,

$$u_{\text{mean}} = \frac{\dot{m}}{\rho A_c} = \frac{\dot{m}}{\rho b\delta} = \frac{\rho^2 g\delta^3 b}{3\mu\rho b\delta} = \frac{\rho g\delta^2}{3\mu} = \frac{(993 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(0.132 \times 10^{-3} \text{ m})^2}{3(0.695 \times 10^{-3} \text{ kg/m s})}$$

= **0.0814 m/s** or **8.14 cm/s** **(Ans.) (a)**

Maximum velocity,

$$u_{\max} = \frac{\rho g\delta^2}{2\mu} = \frac{3}{2}u_{\text{mean}} = 1.5 \times 8.14 \text{ cm/s} = \mathbf{12.21 \text{ cm/s}}$$ **(Ans.) (a)**

Local heat-transfer coefficient

$$h_x = \frac{k}{\delta} = \frac{0.628 \text{ W/m°C}}{0.132 \times 10^{-3} \text{ m}} = \mathbf{4757.6 \text{ W/m}^2 \text{ °C}}$$ **(Ans.) (a)**

(b) Average heat-transfer coefficient,

$$\bar{h}_L = \frac{4}{3}h_{(x=L)} = 0.943\left[\frac{\rho^2 g k^3 h_{fg}}{\mu L(T_{\text{sat}} - T_w)}\right]^{1/4}$$

Applying McAdam's correction,

$$\bar{h}_L = 1.2 \times 0.943\left[\frac{\rho^2 g k^3 h_{fg}}{\mu L(T_{\text{sat}} - T_w)}\right]^{1/4}$$

$$= 1.13\left[\frac{(993)^2(9.81)(0.628)^3(2401.9 \times 10^3)}{(0.695 \times 10^{-3})(0.6 \text{ m})(42 - 32)\text{°C}}\right]^{1/4}$$

= **6887 W/m² °C** **(Ans.) (b)**

Total heat-transfer rate,

$$\dot{Q} = \bar{h}_L(bL)(T_{\text{sat}} - T_w)$$

$$= (6887 \text{ W/m}^2 \text{ °C})(0.6 \text{ m} \times 0.6 \text{ m})(42 - 32)\text{°C} = 24.8 \times 10^3 \text{ W}$$

= **24.8 kW** **(Ans.) (b)**

(c) Condensation rate,

$$\dot{m} = \frac{\dot{Q}}{h_{fg}} = \frac{24.8 \times 10^3 \text{ W}}{2401.9 \times 10^3 \text{ J/kg}} \left| \frac{3600 \text{ s}}{1 \text{ h}} \right| \left| \frac{1 \text{ J/s}}{1 \text{ W}} \right| = \mathbf{37 \text{ kg/h}}$$

(Ans.) (c)

Film Reynolds number,

$$Re_f = \frac{4\dot{m}}{b\mu} = \frac{4 \times (37/3600)\text{kg/s}}{0.6 \text{ m} \times 0.695 \times 10^{-3} \text{ kg/m s}} = \mathbf{99}$$

(Ans.) (c)

(d) $\overline{h}_{\text{inclined}} = \overline{h}_{\text{vertical}} \times (\cos\theta)^{1/4} = 6887 \times (\cos 60°)^{1/4}$

$= \mathbf{5791 \text{ W/m}^2 \text{ °C}}$ **(Ans.) (d)**

EXAMPLE 10.12 *Saturated steam at atmospheric pressure condenses on a vertical square plate (width equal to height) that is maintained at 98°C. The rate at which the condensate drips off the plate at the bottom is 1.85 kg/h. Calculate the dimensions of the plate and the average heat transfer coefficient if Nusselt's solution is valid. Show that the film condensation is laminar. Assume linear temperature distribution and negligible enthalpy changes associated with liquid sub cooling.*

The following properties of saturated water may be used:

ρ = *958.6 kg/m³* k = *0.679 W/m °C*

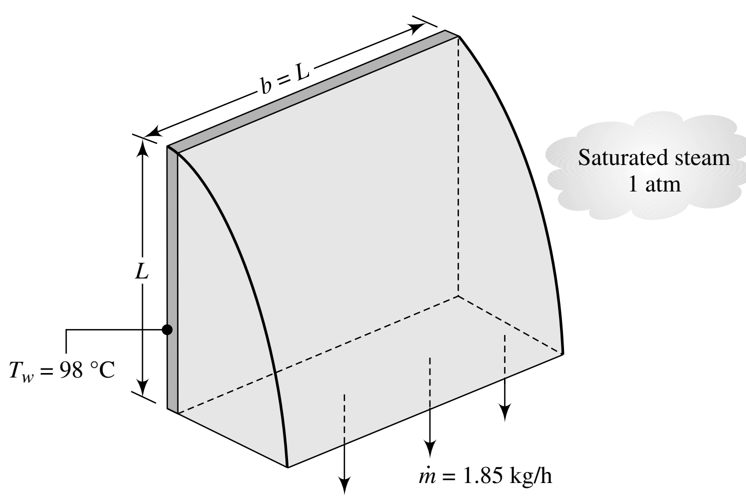
μ = *0.285 × 10⁻³ N s/m²* h_{fg} = *2257 kJ/kg*

Solution

Known Saturated steam at 1 atm condenses on a vertical square plate.

Find Dimensions of plate. Heat-transfer coefficient.

Schematic



Assumptions (1) Steady operating conditions. (2) Laminar-film condensation. (3) Negligible effect of sub cooling, non-condensable gases and non-linear temperature variation.

Analysis The average heat transfer coefficient for laminar film condensation of saturated vapour on a vertical isothermal plate (*Nusselt solution*) is determined from

$$\bar{h}_L = 0.943\left[\frac{\rho^2 g k^3 h_{fg}}{\mu L(T_{\text{sat}} - T_w)}\right]^{1/4} = BL^{-1/4}$$

where

$$B = 0.943\left[\frac{958.6^2 \times 9.81 \times 0.679^3 \times 2257 \times 10^3}{0.285 \times 10^{-3} \times (100 - 98)}\right]^{0.25} = 9695.4$$

Heat-transfer rate during the condensation process is

$$\dot{Q} = \bar{h}_L (bL)(T_{\text{sat}} - T_w) = h_L (L^2)(T_{\text{sat}} - T_w) \qquad (\because b = L)$$

$$= (9695.4)\ (L^{-1/4})(L^2)$$

$$(100 - 98) = 19390.8\ L^{7/4}\ \text{(W)} \qquad \text{(A)}$$

(a) The rate of condensation of steam is

$$\dot{m} = \frac{1.85}{3600}\ \text{kg/s}$$

Latent heat of condensation,

$$h_{fg} = 2257 \times 10^3\ \text{J/kg}$$

Therefore,

$$\dot{Q} = \dot{m} h_{fg} = (1.85/3600)\ (2257 \times 10^3) = 1160\ \text{(W)} \qquad \text{(B)}$$

Equating (A) and (B), one gets

$$19\,390.8\ L^{7/4} = 1160\ \text{(W)}$$

It follows that

$$L = b = \left[\frac{1160}{19390.8}\right]^{4/7} = 0.2\ \text{m or } 20\ \text{cm}$$

The plate dimensions are **20 cm × 20 cm** **(Ans.)**

And, $\bar{h}_L = BL^{-1/4}$ = (9695.4) $(0.2)^{-1/4}$

= **14500 W/m² °C** **(Ans.)**

Film Reynolds number,

$$Re_f = \frac{4\dot{m}}{P\mu} = \frac{4\dot{m}}{b\mu} = \frac{4(1.85/3600)}{0.2 \times 0.285 \times 10^{-3}} = 36 \quad \Rightarrow \quad \text{Laminar}$$

Comment Since Re_f is marginally higher than the wave free laminar limit of 30, the effect of ripples will be vanishingly small.

(R) Horizontal Cylinder

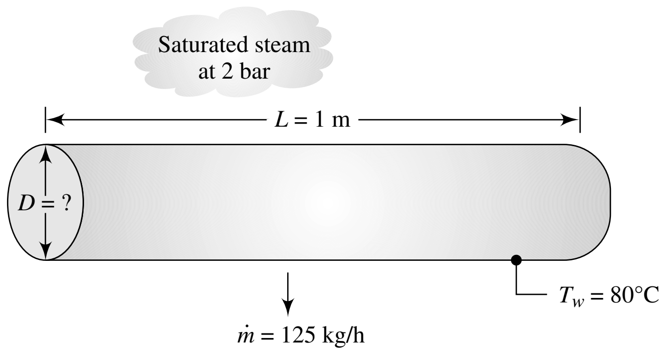
EXAMPLE 10.13 *Determine the diameter of a horizontal tube, 1 m long, used to condense saturated steam at a pressure of 2.0 bar. The tube surface is maintained at a constant temperature of 80°C and the rate of condensation required to be achieved is 125 kg/h.*

Solution

Known A horizontal tube of given length and surface temperature exposed to condensing steam at a prescribed pressure.

Find Tube diameter to achieve a specified condensation rate.

Schematic



Assumptions (1) Laminar film condensation. (2) Negligible concentration of non-condensable gases in steam.

Properties Saturated vapour ($T_{\text{sat @ 2 bar}}$ = 120.2°C):

$$h_{fg} = 2201.9 \text{ kJ/kg}$$

$$\rho_v = \frac{1}{0.8857} = 1.129 \text{ kg/m}^3$$

Saturated water $\left(T_f = \frac{T_{\text{sat}} + T_w}{2} = \frac{120.2 + 80}{2} \cong 100°\text{C}\right)$:

$$\rho_l = 958.1 \text{ kg/m}^3 \qquad C_{pl} = 4.215 \text{ kJ/kg K}$$

$$k_l = 0.6775 \text{ W/m K} \qquad \mu_l = 0.2822 \times 10^{-3} \text{ kg/m s}$$

Analysis Modified latent heat of condensation,

$$h^*_{fg} = h_{fg} + C_{pl}(T_{\text{sat}} - T_w)(0.68) = 2201.9 + 4.215\ (120.2 - 80)\ (0.68)$$

$$= 2317.12 \text{ kJ/kg}$$

Average heat-transfer coefficient for horizontal orientation is

$$\bar{h} = 0.729\left[\frac{(\rho_l - \rho_v)\rho_l\, g\, k_l^3\, h^*_{fg}}{\mu_l D(T_{\text{sat}} - T_w)}\right]^{1/4}$$

or $$\bar{h} = 0.729\left[\frac{(958.1 - 1.129)(958.1)(9.81)(0.6775)^3(2317.12 \times 10^3)}{0.2822 \times 10^{-3} \times D \times (120.2 - 80)}\right]^{1/4} = 3564\ D^{-1/4}$$

Also, heat-transfer rate,

$$\dot{Q} = \dot{m}\, h^*_{fg} = \bar{h}(\pi D L)(T_{\text{sat}} - T_w) \quad \text{or} \quad \bar{h}D = \frac{\dot{m}\, h^*_{fg}}{\pi L(T_{\text{sat}} - T_w)}$$

or $$3564\, D^{-1/4} D = \frac{125 \times 2317.12 \times 10^3}{3600\pi \times 1 \times (120.2 - 80)} \quad \text{or} \quad D^{3/4} = \frac{125 \times 2317.12}{3.6\pi \times 40.2 \times 3564}$$

$\therefore$ Tube diameter,

$D = (0.1786)^{4/3} = 0.10$ m = **100 mm or 10 cm** **(Ans.)**

EXAMPLE 10.14 *Saturated steam at 1 atm condenses on a 2.5 cm* OD *vertical tube at a rate of 12.5 kg/h. The tube surface is maintained at 90°C by circulating cooling water. (a) Determine the required tube length. (b) If the tube were oriented horizontally, what would be the length of the tube for the same rate of condensation?*

Use the following properties of saturated water at 95°C:

$\rho = 961.5$ kg/m³ $\quad \mu = 0.297 \times 10^{-3}$ kg/m s

k = 0.677 W/m °C $\quad C_p = 4.212$ kJ/kg °C

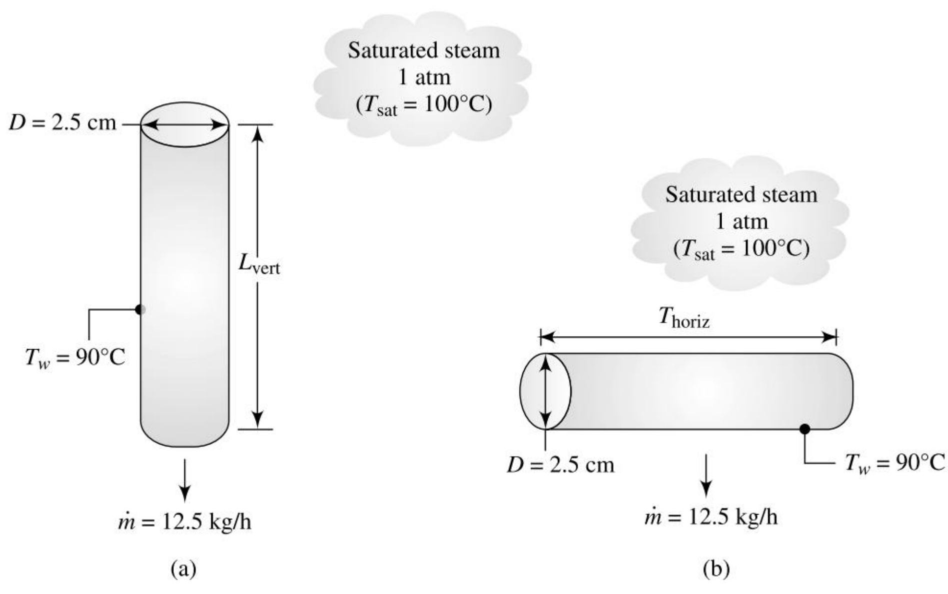
At 100°C (1 atm): $\quad h_{fg} = 2257$ kJ/kg

Solution

Known Vertical tube of 2.5 cm diameter with surface temperature of 90°C used to condense steam at 1 atm at a rate of 12.5 kg/h.

Find (a) Length of tube, L (m). (b) L_{horiz} (m) for same condensation rate.

Schematic



Assumptions (1) Steady operating conditions exist. (2) The tube is isothermal. (3) The flow of condensate is wavy laminar over the entire tube (to be verified later). (4) The density of vapour is much smaller than the density of liquid ($\rho_v << \rho_l$).

Analysis (a) ***Vertical tube:***

The modified latent heat of vaporization is

$$h_{fg}^* = h_{fg} + 0.68C_p(T_{\text{sat}} - T_w)$$
$$= 2257 + (0.68)\ (4.212)\ (100 - 90) = 2285.6 \text{ kJ/kg}$$

For wavy laminar flow, the Reynolds number and the condensation heat-transfer coefficient are determined to be

$$Re = \left[4.81 + \frac{3.70\,Lk(T_{\text{sat}} - T_w)}{\mu h_{fg}^*}\left(\frac{g}{\nu^2}\right)^{1/3}\right]^{0.82} \tag{A}$$

where $\left(\dfrac{g}{\nu^2}\right)^{1/3} = \left[g\left(\dfrac{\rho}{\mu}\right)^2\right]^{1/3} = \left[9.81\dfrac{m}{s^2}\left(\dfrac{961.5 \text{ kg/m}^3}{0.297\times10^{-3}\text{ kg/m s}}\right)^2\right]^{1/3} = 46847 \text{ m}^{-1}$

Film Reynolds number,

$$Re = \frac{4\dot{m}}{\pi D\mu} = \frac{4\times(12.5/3600)\text{kg/s}}{\pi(0.025\text{ m})(0.297\times10^{-3}\text{ kg/m s})} = 595.42$$

Substituting values in Eq. (A), and rearranging

$$(595.42)^{1.22} - 4.81 = \frac{3.7L\times0.677\text{ W/m°C}(100-90)\text{°C}}{0.297\times10^{-3}\text{ kg/m s}\times2285.6\times10^3\text{ J/kg}}\times46847(\text{m}^{-1})$$

or $2423.3 = 1728.7L$

∴ Tube length required,

$$L_{\text{vert}} = L = \frac{2423.3}{1728.7} = \mathbf{1.40\ m} \qquad \textbf{(Ans.) (a)}$$

(b) ***Horizontal tube:***

The condensation heat-transfer coefficient,

$$h = 0.729\left[\frac{\rho^2 g k^3 h_{fg}^*}{\mu D(T_{\text{sat}} - T_w)}\right]^{1/4} = 0.729\left[\frac{(961.5)^2(9.81)(0.677)^3(2285.6\times10^3)}{(0.297\times10^{-3})(0.025)(100-90)}\right]^{0.25}$$
$$= 12506.5 \text{ W/m}^2\text{ °C}$$

Heat-transfer rate,

$$\dot{Q} = h(\pi DL)(T_{\text{sat}} - T_w) = \dot{m}h_{fg}^*$$

Therefore, the tube length required,

$$L_{\text{horiz}} = L = \frac{\dot{m}h_{fg}^*}{h(\pi D)(T_{\text{sat}} - T_w)} = \frac{(12.5/3600)\text{kg/s}\times(2285.6\times10^3\text{ J/kg})}{(12506.5\text{ W/m}^2\text{ °C})(\pi\times0.025\text{ m})(100-90)\text{°C}}$$
$$= \mathbf{0.808\ m} \qquad \textbf{(Ans.) (b)}$$

Comment For vertical tube, Re = 595.4 ($30 \le Re \le 1800$). Hence, the assumption of *wavy laminar flow* was valid.

EXAMPLE 10.15 *Saturated steam at 1 atm condenses on the outer surface of a thin horizontal tube of 6 mm diameter and 1 m long. The tube is cooled by water at 17°C flowing through inside at a flow rate of 0.054 kg/s.*

Calculate:

(a) the inside heat-transfer coefficient, (b) the outside (condensation) heat-transfer coefficient, (c) the tube-wall temperature, (d) the total heat-transfer rate, (e) the condensation rate, and (f) the film Reynolds number

For water on the tube inside, take:

k = *0.598 W/m K,* μ = *1.08 ×10*$^{-3}$ *Pa s,* C_p = *4.184 kJ/kg K*

For steam at 1 atm, take:

h_{fg} = *2257 kJ/kg*

For the condensate, take:

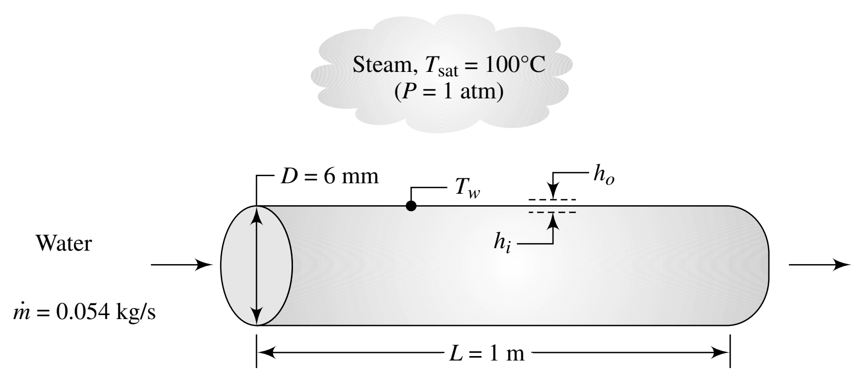
ρ = *957.85 kg/m*^{3}*,* C_p = *4.217 kJ/kg K,* k = *0.680 W/m K,* μ = *1.0044 kg/m h*

Solution

Known Condensation of steam on the outside of a thin tube cooled by internal water flow.

Find Inside and outside convection coefficients, tube surface temperature, rate of condensation, heat-transfer rate, and film Reynolds number.

Schematic



Assumptions (1) Laminar-film condensation. (2) Fully developed flow in the tube. (3) Thermal resistance of tube wall is negligible. (4) Negligible concentration of non-condensable gases in the steam.

Analysis From an energy balance on the inner tube with a constant wall temperature:

$$h_o(T_{sat} - T_w) = h_i(T_w - \overline{T}_b)$$

Internal flow in the tube:

Using the Properties of Water Given:

$$Re_D = \frac{4\dot{m}}{\pi D \mu} = \frac{4 \times 0.054}{\pi \times 0.006 \times 1.08 \times 10^{-3}}$$

$$= 10610 \qquad (> 2300) \qquad \Rightarrow \quad \textit{Turbulent flow}$$

$$Pr = \frac{C_p \mu}{k} = \frac{(4.184 \times 10^3)(1.08 \times 10^{-3})}{0.598} = 7.56$$

Dittus–Boelter correlation:

$$Nu_D = \frac{h_i D}{k} = 0.023(Re_D)^{0.8}(Pr)^{0.4}$$

Hence, $h_i = \dfrac{(0.023)(0.598)}{0.006} \times (10610)^{0.8}(7.56)^{0.4}$

= **8556 W/m² K** **(Ans.) (a)**

Condensation on the outer surface of the tube:

$$h_o = 0.729\left[\frac{\rho^2 g k^3 h_{fg}}{\mu D(T_{\text{sat}} - T_w)}\right]^{1/4} = 0.729\left[\frac{957.85^2 \times 9.81 \times 0.680^3 \times 2257 \times 10^3}{(1.0044/3600) \times 0.006 \times (100 - T_w)}\right]^{0.25}$$

$$= 32219.5(100 - T_w)^{-1/4}$$

From the energy balance:

$$\{32219.5(100 - T_w)^{-1/4}\}(100 - T_w) = 8556(T_w - 17)$$

or $$(100 - T_w)^{0.75} = (T_w - 17)(0.2656)$$

By *trial and error:*

T_w [°C]	LHS	RHS
70	12.82	14.08
68	13.45	13.55
67.8	**13.52**	**13.50**

Thus, T_w = **67.8°C** **(Ans.) (c)**

∴ $h_o = 32219.5\ (100 - 67.8)^{-0.25}$ = **13 525 W/m² K** **(Ans.) (b)**

Total heat-transfer rate,

$$\dot{Q} = h_o(\pi DL)(T_{\text{sat}} - T_w) = h_i(\pi DL)(T_w - \overline{T}_b)$$

$$= (13525)(\pi \times 0.006 \times 1)(100 - 67.8)$$

= **8209 W or 8.21 kW** **(Ans.) (d)**

Condensation rate,

$$\dot{m} = \frac{\dot{Q}}{h_{fg}} = \frac{8.21\ \text{kW}}{2257\ \text{kJ/kg}}\left(\frac{3600\ \text{s}}{1\ \text{h}}\right) = \mathbf{13.09\ kg/h}$$ **(Ans.) (e)**

Film Reynolds number,

$$Re_f = \frac{4\dot{m}}{P\mu} = \frac{4\dot{m}}{2L\mu} = \frac{4 \times (13.09/3600)}{2 \times (1.0044/3600)} = \mathbf{26}$$ **(< 30) (Ans.) (f)**

The flow of film is, therefore, *wave-free laminar.*

(S) Tube Array

EXAMPLE 10.16 *Determine the outer diameter of the 12 horizontal tubes arranged in a rectangular array, 3 tube high and 4 tube wide, in order to condense 300 kg/h of saturated Refrigerant –12 (CCl_2F_2) at 50°C on them per unit length of the tubes. The surface of the tubes is maintained at a uniform temperature of 40°C.*

R *12 has the following thermophysical properties:*

At 50°C: h_{fg} = *121.43 kJ/kg*

At 45°C: ρ_ℓ = *1236.55 kg/m³* C_ℓ = *1.01175 kJ/kg °C*

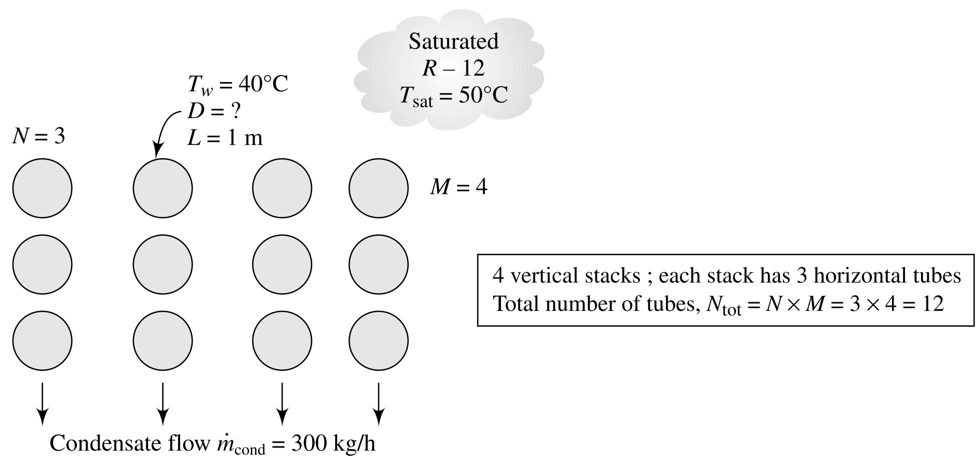
μ_ℓ = *2.362 × 10⁻⁴ kg/ms* k_ℓ = *0.068 W/m °C*

Solution

Known Saturated R-12 vapour condenses over 12 horizontal tubes arranged in a 3 × 4 rectangular array.

Find Tube outside diameter, D.

Schematic



Assumptions (1) Steady operating conditions. (2) Laminar-film condensation.

Analysis Heat-transfer rate, $\dot{Q} = \dot{m}_{cond}\, h_{fg}^*$

where $h_{fg}^* = h_{fg}\ (1 + 0.68\ Ja)$

Jacob number, $Ja = C_{pl}\ \Delta T / h_{fg}$

With $\Delta T \equiv T_{sat} - T_w = 50 - 40 = 10°C$, $C_{pl} = 1.01175$ kJ/kg °C and $h_{fg} = 121.43$ kJ/kg,

$$Ja = \frac{1.01175 \times 10}{121.43} = 0.08332$$

$$h_{fg}^* = 121.43\ (1 + 0.68 \times 0.08332) = 128.31 \text{ kJ/kg}$$

Hence, $$\dot{Q} = \left(\frac{300 \text{ kg}}{3600 \text{ s}}\right)(128.31 \text{ kJ/kg}) = 10.7 \text{ kW} \quad \text{(A)}$$

Average condensation heat-transfer coefficient for a vertical tier of $N = 3$ tubes (valid for all $M = 4$ tiers) is determined from

$$\bar{h}_{D,N} = 0.729 \left\{ \frac{\rho_l(\rho_l - \rho_v) g\, k_l^3\, h_{fg}^*}{\mu_l (ND)(T_{sat} - T_w)} \right\}^{1/4}$$

As ρ_v is not given, we assume $\rho_v << \rho_l$ and $\rho_l(\rho_l - \rho_v) \approx \rho_l^2$. Substituting the appropriate numerical values, except the diameter D, which is unknown, we have

$$\bar{h}_{D,N} = 0.729 \left\{ \frac{(1236.55)^2 (9.81)(0.068)^3 (128.31 \times 10^3)}{(2.362 \times 10^{-4})(3D)(50 - 40)} \right\}^{0.25}$$

$$= 349\ D^{-0.25}\ (\text{W/m}^2\ °\text{C})$$

Heat-transfer rate,

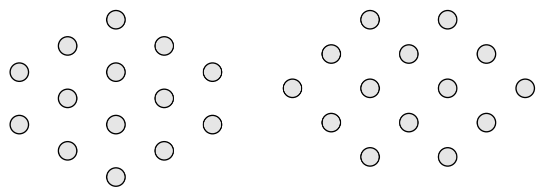
$$\dot{Q} = \bar{h}_{D,N}\, A(T_{sat} - T_w) = \bar{h}_{D,N}\,(N_{tot}\pi DL)(T_{sat} - T_w)$$

$$= (394\ D^{-0.25})(12 \times \pi \times D \times 1)(50 - 40)\ (\text{W}) = 148.5\ \text{D}^{0.75}\ \text{kW} \qquad \text{(B)}$$

Equating (A) and (B), we get

$$148.5\ D^{0.75} = 10.7 \text{ or } D = \left(\frac{10.7\ \text{kW}}{148.5\ \text{kW}}\right)^{4/3} = 0.03\ \text{m} = \mathbf{3.0\ cm} \qquad \textbf{(Ans.)}$$

EXAMPLE 10.17 *A cross section through a bundle of horizontal condenser tubes is shown in Fig. (a). The same bundle is rotated through 90° and is shown in the adjoining Fig. (b). Assuming laminar film condensation on the outside surface of the tubes, find the ratio of the condensate rates in the two tube configurations.*



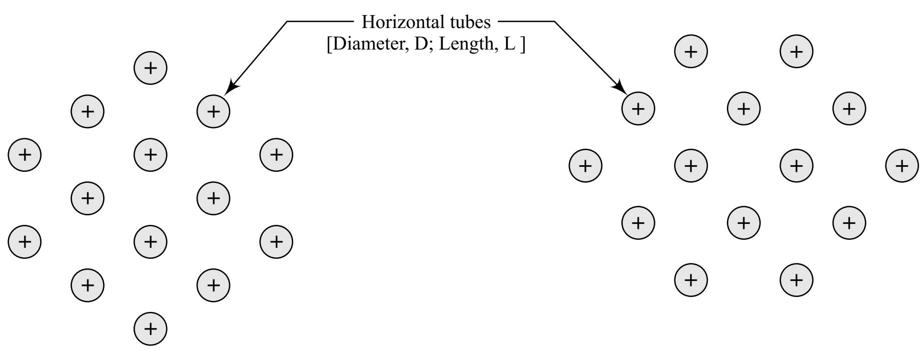
(a) (b)

Solution

Known A bundle of horizontal condenser tubes is rotated through 90°.

Find Ratio of the rates of condensate production in the two geometries.

Schematic



Assumptions (1) Laminar-film condensation. (2) Steady operating conditions.

Analysis The average heat-transfer coefficient for each column of N horizontal tubes is given by

$$\bar{h}_{D,N} = 0.729\left[\frac{\rho_l(\rho_l - \rho_v)\, g\, k_l^3\, h_{fg}^*}{\mu_l (ND)(T_{sat} - T_w)}\right]^{1/4}$$

and $\dot{Q} = \bar{h}_{D,N}(N_{tot}\,\pi DL)(T_{sat} - T_w) = CN^{3/4}$ (*where C is a constant*).

This is valid for the same dimensions and fluid properties.

For tube configuration (a):

Column #	N	$\dot{Q}$
1	2	$C \cdot (2^{3/4})$
2	3	$C \cdot (3^{3/4})$
3	4	$C \cdot (4^{3/4})$
4	3	$C \cdot (3^{3/4})$
5	2	$C(2^{3/4})$

Total rate of heat transfer = $C[2^{3/4} + 3^{3/4} + 4^{3/4} + 3^{3/4} + 2^{3/4}]]$ = 10.751 C

For tube configuration (b):

Column #	N	$\dot{Q}$
1	1	C
2	2	$C \cdot (2^{3/4})$
3	3	$C \cdot (3^{3/4})$
4	2	$C \cdot (2^{3/4})$
5	3	$C \cdot (3^{3/4})$
6	2	$C \cdot (2^{3/4})$
7	1	C

Total rate of heat transfer = $C[1 + 2^{3/4} + 3^{3/4} + 2^{3/4} + 3^{3/4} + 2^{3/4} + 2^{3/4} + 1] = 11.604\ C$

$$\text{Ratio of condensation rates} = \frac{\dot{m}(b)}{\dot{m}(a)} = \frac{\dfrac{\dot{Q}(b)}{h_{fg}^*}}{\dfrac{\dot{Q}(a)}{h_{fg}^*}} = \frac{\dot{Q}(b)}{\dot{Q}(a)} = \frac{11.604\,\text{C}}{10.751\,\text{C}} = \mathbf{1.079}$$

Comment Stacking of individual tubes should be confined to as small a number as possible for each column to obtain greater condensate production.

EXAMPLE 10.18 *Saturated steam at 0.05 bar [T_{sat} = 32.87°C, h_{fg} = 2423.0 kJ/kg, ρ_v = 0.0355 kg/m^3] condenses on a coil whose surface temperature is maintained at 20°C. The tube diameter is 15 mm, the coil diameter is 300 mm, and the pitch of the coil is 40 mm. There are 10 turns in the coil and its axis is vertical. Determine the rate of condensation.*

Properties of saturated vapour at $T_f \equiv \frac{1}{2}(T_{sat} + T_w)$ *are the following:*

ρ_l = 996.7 kg/m^3 $\qquad$ k_l = 0.609 W/m °C

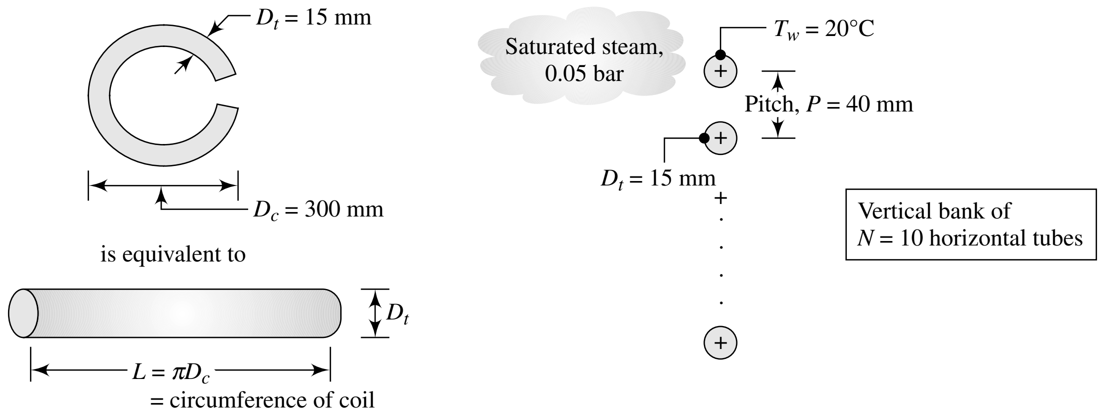
μ_l = 0.864 × 10^{-3} kg/m s $\qquad$ C_{pl} = 4.179 kJ/kg °C

Solution

Known Saturated steam condenses on a 10-turn coil with its axis vertical.

Find Condensation rate, $\dot{m}_{\text{cond}}$.

Schematic



Assumptions (1) Steady operating conditions exist. (2) Laminar film condensation. (3) Negligible effect of non-condensables on steam.

Analysis From the problem statement it is clear that the coil with 10 turns and vertical axis can be looked upon as a vertical tier of 10 horizontal tubes with each tube having a diameter D_t = 15 mm, length equal to circumference of one turn of coil, $L = (\pi D_c)$ where D_c is the coil diameter equal to 300 mm, and centre to centre distance between two consecutive tubes, i.e., pitch is p = 40 mm as shown in the schematic.

The average heat-transfer coefficient for laminar-film condensation of saturated vapour on a vertical stack of N horizontal tubes is determined from

$$\bar{h}_{D,N} = 0.729\left[\frac{\rho_l(\rho_l - \rho_v)gk_l^3 h_{fg}^*}{\mu_l(ND)(T_{\text{sat}} - T_w)}\right]^{1/4}$$

where $h_{fg}^* = h_{fg} + 0.68 C_{pl}(T_{\text{sat}} - T_w)$

$= 2423.0 + (0.68)\,(4.179)\,(32.87 - 20) = 2459.57$ kJ/kg

Substituting numerical values, one has

$$\bar{h}_{D,N} = 0.729\left[\frac{996.7(996.7 - 0.0355)(9.81)(0.609)^3(2459.57 \times 10^3)}{(0.864 \times 10^{-3})(10 \times 0.015)(32.87 - 20)}\right]^{1/4}$$

$= 5502.44$ W/m^2 °C

The surface area for all 10 tubes is

$$A_s = N\pi D_t(\pi D_c) = 10\pi^2 \times 0.015 \times 0.3 = 0.444 \text{ m}^2$$

Heat-transfer rate during this condensation process is

$$\dot{Q} = \bar{h}_{D,N} A_s (T_{\text{sat}} - T_w) = (5502.44)\,(0.444)\,(32.87 - 20) = 31442.48 \text{ W}$$

The rate of condensation is determined to be

$$\dot{m}_{\text{cond}} = \frac{\dot{Q}}{h_{fg}^*} = \frac{31442.48 \text{ J/s}}{2459.57 \times 10^3 \text{ J/kg}}\left|\frac{3600 \text{ s}}{1 \text{ h}}\right| = \mathbf{46.0\ kg/h} \qquad \textbf{(Ans.)}$$

EXAMPLE 10.19 *Horizontal tubes, 2.5 cm OD and 0.6 m long, are arranged in a vertical bank. Saturated steam at 105°C condenses on these tubes maintained at 95°C. The condensation rate is 26.7 kg/h. How many tubes are required?*

The following properties of saturated water may be used:

T_{sat} = *105°C*	h_{fg} = *2243.7 kJ/kg*	ρ_v = *0.7045 kg/m³*
T_f = *100°C*	ρ_l = *957.9 kg/m³*	k_l = *0.679 W/m°C*
μ_l = *0.282 × 10⁻³ kg/m s*	C_{pl} = *4.217 kJ/kg°C*	

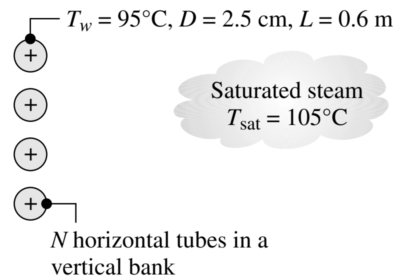
Solution

Known Steam condenses on a vertical tier of horizontal tubes.

Find Number of tubes, N, in the vertical bank.

Assumptions (1) Steady-state conditions, (2) Laminar-film condensation.

Schematic



Analysis Modified latent heat of vaporization (*to account for non-linear temperature distribution and enthalpy change due to subcooling*) is

$$h_{fg}^* = h_{fg} + 0.68C_{pl}(T_{sat} - T_w)$$
$$= 2243.7 + (0.68)(4.217)(105 - 95) = 2272.38 \text{ kJ/kg}$$

Average condensation heat-transfer coefficient for a *single* horizontal tube is

$$\bar{h}_{\text{horiz,1tube}} = 0.729\left[\frac{\rho_l(\rho_l - \rho_v)g\,k_l^3 h_{fg}^*}{\mu_l D(T_{sat} - T_w)}\right]^{1/4}$$
$$= 0.729\left[\frac{(957.9)(957.9 - 0.7045)(9.81)(0.679)^3(2272.38 \times 10^3)}{(0.282 \times 10^{-3})(0.025)(105 - 95)}\right]^{1/4}$$
$$= 12653.2 \text{ W/m}^2\,°\text{C}$$

For a *vertical* bank of N *horizontal* tubes, the average heat-transfer coefficient will be

$$\bar{h}_{\text{horiz},N\text{ tubes}} = \bar{h}_{\text{horiz,1tube}} \times N^{-1/4}$$

Heat-transfer rate during the condensation process,

$$\dot{Q} = \bar{h}_{\text{horiz},N\text{ tubes}}(N\pi DL)(T_{sat} - T_w)$$
$$= (12653.2)(N)^{-1/4}(N \times \pi \times 0.025 \times 0.6)(105 - 95) = 5962.68\text{ N}^{3/4}\text{ (W)}$$

Also, $$\dot{Q} = \dot{m}_{cond}\,h_{fg}^* = \left(\frac{26.7}{3600}\frac{\text{kg}}{\text{s}}\right)\left(2272.38 \times 10^3\,\frac{\text{J}}{\text{kg}}\right) = 16853.5\text{ (W)}$$

Equating the two expressions for $\dot{Q}$, we get

$$5962.68\,N^{3/4} = 16853.5$$

∴ Number of tubes required,

$$N = \left[\frac{16853.5}{5962.68}\right]^{4/3} = 3.996 \simeq \mathbf{4}$$ **(Ans.)**

Points to Ponder

- The pool boiling curve is a plot of wall heat flux, q against the excess temperature, $\Delta T_{e,}$ i.e., surface saturation temperature difference on log scale.
- There are four distinct regimes in a typical boiling curve: (1) Free convection boiling ($\Delta T_e < 10°C$), (2) Nucleate boiling ($10°C < \Delta T_e < 30°C$), (3) Transition boiling ($30°C < \Delta T_e < 120$ °C), and (4) Film boiling ($\Delta T_e > 120°C$).
- Typically, the peak heat flux for boiling of water on a large, flat heater at a saturation temperature of 100°C is 1.27 MW/m^2.
- In the heat-input-controlled pool boiling, the boiling curve indicates unstable heating paths.
- Nucleate boiling is most significant because we can obtain very high heat-transfer rates with modest temperature differences.
- In nucleate boiling, the surface heat flux is inversely proportional to the cube of constant C_{sf} which depends on the heating surface-fluid combination.
- Peak heat flux or critical heat flux is also known as DNB (departure from nucleate boiling) or burn-out point or boiling crisis.
- Nucleate boiling is strongly dependent on the surface condition but is not so sensitive to the surface geometry. However, peak heat flux and film boiling are influenced by surface geometry.
- Knowledge of peak heat flux is necessary to prevent the failure of the heating dements.
- Stable film boiling has many applications in the boiling of cryogenic fluid.
- In internal flow forced convection boiling, two modes of heat transfer dominate the overall process: nucleate boiling and forced convection.
- In film condensation, the vapour condenses into a continuous film covering the entire surface.
- In dropwise condensation, the vapour condenses into small liquid droplets of different sizes.
- For the same temperature difference, the heat transfer rate in dropwise condensation is much greater than that in film condensation. Hence, promoters like oleic acid or surface coatings with materials like 'Teflon' are used but droplet condensation is difficult to maintain.
- In laminar-film condensation on a vertical plate, the average velocity of falling film is 2/3rd the maximum velocity, and the average heat-transfer coefficient is 4/3 times the local heat-transfer coefficient at the specified distance from the top of the plate.
- Equations for condensation on a vertical plate are also applicable in the case of a vertical tube as long as the diameter of the tube is much larger than the film thickness.
- The presence of small concentrations of non-condensable gases like air in vapour severely reduces the rate of condensation.

GLOSSARY of Key Terms

• **Boiling**	A phase change process that occurs at the solid-liquid boundary when the submerged solid surface is at a temperature greater than the saturation temperature of the liquid.
• **Burn-out**	A damaging phenomenon when the peak heat flux is exceeded and the surface temperature shoots up to the melting point.
• **Condensation**	A phase-change process when a vapour is in contact with a surface which is slightly below the corresponding saturation temperature.

• **Critical heat flux**	At the end of the nucleate boiling regime, the heat flux reaches a maximum due to agitation of the liquid caused by vigorous production of vapour bubbles.
• **Dropwise condensation**	The condensate coalesces into droplets, which when forced roll off the condensing surface under the action of gravity. There is no film barrier to heat transfer.
• **Evaporation**	A phase-change process that occurs at the liquid-vapour boundary.
• **Excess temperature**	The difference between the wall temperature of the heated surface and the saturation temperature of the liquid.
• **Film boiling**	The state in which the heated surface is covered with a vapour film so that the liquid does not touch the heated surface.
• **Film condensation**	The condensate is formed as a layer of liquid film condensing on the surface. The film represents a thermal resistance to heat transfer.
• **Film Reynolds number**	A criterion which determines if the condensate flow is laminar or turbulent.
• **Flow boiling**	Boiling of a liquid forced over a heated surface as two-phase flow with heat transfer.
• **Free convection boiling**	No bubbles are formed even through the liquid may be slightly superheated. The vapour is formed by evaporation at the liquid surface.
• **Minimum heat flux**	It is marked by the end of transition boiling and the beginning of the film boiling regime. It occurs at the Leidenfrost temperature.
• **Nucleate boiling**	The process in which bubbles begin to form on the submerged heated surface, get detached from and are initially dissipated in the liquid but finally rise rapidly to break through the surface of the liquid.
• **Pool boiling**	When the heated surface is submerged below the free surface of a liquid which has no bulk motion.
• **Saturated boiling**	The process of pool boiling when the liquid is maintained at the saturation temperature.
• **Subcooled boiling**	The process of pool boiling when the liquid is below the saturation temperature.
• **Transition boiling**	Boiling regime in which parts of the heated surface are covered by the vapour film while nucleate boiling occurs over the remaining parts.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

10.1 Heat-transfer coefficients for free convection in gases, forced convection in gases and vapours and for boiling water lie, respectively, in the ranges of

(a) 5 – 15, 20 – 200, and 3000 – 50000 W/m^2 K
(b) 20 – 50, 200 – 500, and 50000 – 10^5 W/m^2 K
(c) 50 – 100, 500 – 1000, and 10^5 – 10^6 W/m^2 K
(d) 20 – 100, 200 – 1000, and a constant 10^6 W/m^2 K

10.2 In spite of large heat-transfer coefficients in boiling liquids, fins are used advantageously when the entire surface is exposed to

(a) nucleate boiling (b) film boiling (c) transition boiling (d) all modes of boiling

10.3 In spite of large heat-transfer coefficients in boiling liquids, cavities are used advantageously when the entire surface is exposed to

(a) nucleate boiling (b) film boiling (c) transition boiling (d) all regimes of boiling

10.4 In pool boiling the highest heat-transfer coefficient occurs in

(a) subcooled boiling zone (b) nucleate boiling zone
(c) partial film-boiling zone (d) film-boiling zone

10.5 The burn-out heat flux in the nucleate boiling regime is a function of which of the following properties?

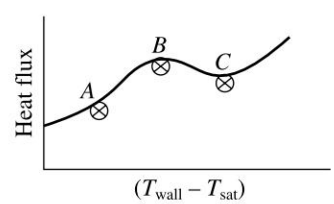
1. Heat of evaporation
2. Temperature difference
3. Density of vapour
4. Density of liquid
5. Vapour-liquid surface tension

Select the correct answer using the codes given below:

Codes:

(a) 1, 2, 4, and 5 (b) 1, 3, 4, and 5 (c) 1, 2, 3, and 5 (d) 2, 3, and 4

10.6 The figure given shows a pool-boiling curve. Consider the following statements in this regard:



1. Onset of nucleation causes a marked change in slope.
2. At the point *B*, the heat-transfer coefficient is maximum.
3. In an electrically heated wire submerged in the liquid, film heating is difficult to achieve.
4. Beyond the point *C*, radiation becomes significant.

Of these statements,

(a) 1, 2, and 4 are correct (b) 1, 3, and 4 are correct
(c) 2, 3, and 4 are correct (d) 1, 2, and 3 are correct

10.7 Consider the following statements regarding nucleate boiling:

1. The temperature of the surface is greater than the saturation temperature of the liquid.
2. Bubbles are created by the expansion of entrapped gases or vapour at small cavities in the surface.
3. The temperature is greater than that of film boiling.
4. The heat transfer from the surface of the liquid is greater than that in film boiling.

Of these statements:

(a) 1, 2, and 4 are correct (b) 1 and 3 are correct
(c) 1, 2, and 3 are correct (d) 2, 3, and 4 are correct

10.8 When water is boiling on the outside surface of a submerged body (ΔT = excess temperature):

1. The heat-transfer coefficient is proportional to $(\Delta T)^3$.
2. The heat flux is proportional to $(\Delta T)^4$.
3. The heat-transfer coefficient increases with increasing pressure.

Of these statements,

(a) 1, 2, and 3 are correct (b) only 1 and 2 are correct
(c) 1 and 3 are correct (d) only 3 is correct

10.9 Heat flux increases with excess temperature beyond the Leidenfrost point due to

(a) occurrence of subcooled boiling (b) promotion of nucleate boiling
(c) radiation effect becomes significant (d) vapour space becomes large

10.10 Consider the following statements:

If a surface is pockmarked with a number of cavities then as compared to a smooth surface,

1. radiation will increase
2. nucleate boiling will increase
3. condensation will increase
4. convection will increase.

Of these statements,

(a) 1, 2, and 3 are correct
(b) 1, 2, and 4 are correct
(c) 1, 3, and 4 are correct
(d) 2, 3, and 4 are correct

10.11 The surface temperature of a 0.5 m long and 2 mm diameter electric resistance wire submerged in water at 1 atm is experimentally measured to be 124°C. The wattmeter reading for power consumption is 3.5 kW. The boiling heat-transfer coefficient is then

(a) 68216 W/m^2 K (b) 46420 W/m^2 K (c) 255 W/m^2 K (d) 10120 W/m^2 K

10.12 The boiling heat-transfer coefficient for an excess temperature of 10°C associated with water at 1 atm and 100°C in a nicked container (C_{sf} = 0.006) is 1.5×10^5 W/m^2 °C. For a polished copper surface (C_{sf} = 0.013), the boiling heat flux would be

(a) 147.5 kW/m^2 (b) 1475 kW/m^2 (c) 47.5 kW/m^2 (d) 4.75 kW/m^2

10.13 With an increase in the excess temperature, the boiling heat flux

(a) increases continuously
(b) decreases and then increases
(c) decreases, then increases and again decreases
(d) increases, then decreases and again increases

10.14 Consider the following statements regarding condensation heat transfer:

1. For a single tube, the horizontal position is preferred over vertical position for better heat transfer.
2. Heat-transfer coefficient decreases if the vapour stream moves at high velocity.
3. Condensation of steam on an oily surface is dropwise.
4. Condensation of pure benzene vapour is always dropwise.

Of these statements,

(a) 1 and 2 are correct
(b) 2 and 4 are correct
(c) 1 and 3 are correct
(d) 3 and 4 are correct

10.15 During film condensation, the film thickness Δ and the heat-transfer coefficient h, vary with the distance x from the leading edge as

(a) Δ decreases, h increases
(b) both Δ and h increase
(c) Δ increases, h decreases
(d) both Δ and h decrease

10.16 Consider the following statements:

1. If a condensing liquid does not wet a surface, dropwise condensation will take place.
2. Dropwise condensation has a higher heat-transfer coefficient than film condensation.
3. Reynolds number of condensing liquid is calculated based on its mass-flow rate.
4. Suitable coating is used to promote film condensation.

Of these statements,

(a) 1, 2, 3, and 4 are correct
(b) 2, 3, and 4 are correct
(c) 1, 2, and 3 are correct
(d) 1 and 4 are correct

10.17 The average heat-transfer coefficient in case of laminar-film condensation over a vertical plate is equal to

(a) $\frac{1}{2}h_{x=L}$ (b) $0.4\ h_{x=L}$ (c) $\frac{4}{3}h_{x=L}$ (d) $2\ h_{x=L}$

10.18 For laminar-film condensation of a saturated vapour on a vertical, isothermal flat plate, the rate of heat transfer is proportional to

(a) $(T_{sat} - T_w)^{1/4}$ (b) $(T_{sat} - T_w)$ (c) $(T_{sat} - T_w)^{2/3}$ (d) $(T_{sat} - T_w)^{3/4}$

10.19 Mark the wrong statement regarding laminar-film condensation on a vertical plate:

(a) The rate of condensation heat transfer is maximum at the upper edge of the plate and progressively decreases as the lower edge is approached.

(b) At a definite point on the heat-transfer surface, the film-heat-transfer coefficient is directly proportional to the thermal conductivity and inversely proportional to the thickness of the film at the point.

(c) The average heat-transfer coefficient is two-third of the local heat-transfer coefficient at the lower edge of the plate.

(d) The film thickness increases as the fourth root of the distance from the top edge.

10.20 The rate of condensate production, $\dot{m}_{cond}$ in case of laminar-film condensation over a horizontal pipe

(a) increases with T_w

(b) decreases with T_w

(c) does not change with T_w

(d) increases, then decreases and finally increases again with T_w

10.21 For film condensation, the heat-transfer coefficients will be equal, whether the tube is horizontal or vertical, when the ratio of length to diameter is

(a) 1.3 (b) 2.8 (c) 0.77 (d) more than 10

10.22 Steam condenses at 50°C on a 2 m^2 vertical flat plate held at 40°C. The average condensation heat-transfer coefficient is 5250 W/m^2 K and the latent heat of condensation is 2383 kJ/kg. The rate at which condensate is being formed is

(a) 0.033 kg/s (b) 0.28 kg/s (c) 0.044 kg/s (d) 0.15 kg/s

10.23 Saturated water vapour at 95°C(h_{fg} = 2270 kJ/kg) condenses on the outer surface of a 2 m long horizontal tube with an outside diameter of 6 cm. The outer surface of the tube is maintained at 85°C. The rate of condensation is 50 kg/h. The condensation heat-transfer coefficient is

(a) 8363 W/m^2 K (b) 5144 W/m^2 K (c) 7258 W/m^2 K (d) 10252 W/m^2 K

10.24 In laminar-film condensation of saturated steam on a vertical tube, the thermal conductivity of water at the film temperature 0.661 W/m °C and the film thickness at the bottom of the tube is 0.108 mm. The average heat transfer coefficient in W/m^2 °C is

(a) 6880 (b) 8160 (c) 9500 (d) 10 100

Answers

Multiple-Choice Questions

10.1 (a)	**10.2** (b)	**10.3** (a)	**10.4** (b)	**10.5** (b)	**10.6** (a)
10.7 (a)	**10.8** (a)	**10.9** (c)	**10.10** (b)	**10.11** (b)	**10.12** (a)
10.13 (d)	**10.14** (c)	**10.15** (c)	**10.16** (c)	**10.17** (c)	**10.18** (d)
10.19 (c)	**10.20** (b)	**10.21** (b)	**10.22** (c)	**10.23** (a)	**10.24** (b)

REVIEW QUESTIONS

10.1 Discuss the physical mechanism of boiling.

10.2 Why do bubbles form on the heating surface?

10.3 Distinguish between *evaporation* and *boiling*.

10.4 What is excess temperature? What do you mean by ONB and DNB?

10.5 Why does a drop of oil put on a hot plate dance over the plate before finally evaporating?

10.6 Distinguish between *pool boiling* and *flow boiling*.

10.7 Distinguish between *subcooled* and *saturated boiling*.

10.8 Draw the Nukiyama boiling curve. Identify and indicate the different boiling regimes. Also discuss the characteristics of each regime.

10.9 How is *film boiling* different form *nucleate boiling*? Is the boiling heat flux invariably higher in the *stable film boiling* regime compared to the case of *nucleate boiling*?

10.10 Discuss various factors affecting *nucleate boiling*.

10.11 Explain how *burn-out* is caused. Indicate the *burn-out point* and the *burn-out flux* on a boiling curve.

10.12 What are the different boiling regimes in a vertical tube during *forced convection boiling*?

10.13 Distinguish between *film* and *dropwise* condensation. Which do you think is a more effective mechanism of heat transfer? In practice, how can the dropwise condensation be obtained?

10.14 What is the *modified latent heat of vaporization*? What is *Jacob number*?

10.15 Discuss the effect of the presence of a *non-condensable gas* in a vapour on the condensation heat transfer?

10.16 Define Reynolds number for film condensation.

10.17 In film condensation, how does the local heat-transfer coefficient vary with the distance from the top of the plate?

10.18 Does the local heat-transfer coefficient for film condensation increase with increasing temperature difference between the condensing vapour and the plate?

10.19 What is the critical Reynolds number for transition from laminar flow to turbulent flow in condensation of vapours on a vertical plate?

10.20 How is the mass-flow rate of the condensate related to the average heat-transfer coefficient?

10.21 What is the relation between Reynolds number and average heat-transfer coefficient for film condensation on a vertical plate?

10.22 Is it possible to use the relations which were developed for condensation of saturated vapours to predict the average heat-transfer coefficients for condensation of superheated vapours?

PRACTICE PROBLEMS

(A) Nucleate Boiling and Peak Heat Flux

10.1 A resistance heater made of 2-mm-diameter nickel wire is used to heat water at atmospheric pressure. Estimate the highest temperature at which this heater can operate safely without the risk of burn-out. **[109.6°C]**

10.2 A pool of saturated water at 130°C is boiling off a horizontal brass plate at a temperature of 140°C. Assuming that the boiling is of the nucleate type, calculate the heat flux. Take the following properties of water at 130°C:

C_{pl} = 4.263 kJ/kg K, h_{fg} = 2174 kJ/kg, $\mu_l = 0.213 \times 10^{-3}$ kg/m s, σ = 0.053 N/m,
ρ_l = 934.6 kg/m^3, ρ_v = 1.496 kg/m^3, Pr_l = 1.33, C_{sf} = 0.006 **[1.57×10^6 W/m^2]**

(B) Film Boiling

10.3 Estimate the heat transfer rate per unit length of a polished stainless steel bar horizontally immersed in water at 27°C at 1 atm. The bar is of 5-cm diameter with surface emissivity of 0.17. The surface temperature of the bar is maintained at 250°C. **[3.44 kW/m]**

(C) Forced Convection Boiling

10.4 Water at 4 bar flows through a vertical tube of 50-mm-inside diameter. Local boiling occurs when the tube wall is 15°C above the saturation temperature. Estimate the boiling heat transfer rate per metre length of the tube. The correlation to be used is, $h = 2.54(\Delta T_e)^3 \exp(P/15.3)$ where P is in bar.

[28.0 kW/m]

10.5 Saturated water at 120°C enters a 2.5-cm-diameter copper tube with a mass flow rate of 50 kg/min. The tube wall is subjected to a constant and uniform heat flux. Calculate the two-phase heat transfer coefficient and the heat flux at a location where the quality $x = 0.25$ and the wall temperature is 130°C. Use the Rohsenow's correlation for nucleate boiling of saturated liquid and the Dittus-Boelter equation for single-phase forced convection liquid flow. **[32.2 kW/m² °C, 322 kW/m²]**

(P) Condensation

10.6 Saturated steam at 120°C condenses on a 2-cm-OD vertical tube which is 20-cm- long. The tube wall is maintained at a temperature of 119°C. Calculate the average heat transfer coefficient and the thickness of the condensate film at the base of the tube. Assume that Nusselt's solution is valid.
Given: $P_{\text{sat}} = 1.985$ bar
$k_w = 0.686$ W/ m K, $\rho_w = 943$ kg / m³, $\mu_w = 237.3 \times 10^{-6}$ N s / m², $h_{fg} = 2202.2$ kJ/kg

[17 925 W/m² K]

10.7 A vertical cooling fin approximating a flat plate 40-cm in height is exposed to saturated steam at atmospheric pressure ($T_{\text{sat}} = 100$°C, $h_{fg} = 2257$ kJ/kg). The fin is maintained at a temperature of 90°C. Estimate the following: (a) the thickness of the film at the bottom edge of the fin, (b) the average heat transfer coefficient, and (c) the heat transfer rate after incorporating McAdam's correction. The relevant fluid properties are:
$\rho_l = 965.3$ kg/m³, $k_l = 0.680$ W/m K, $\mu_l = 3.153 \times 10^{-4}$ N s/m²

The following relations may be used: $\delta_L = \left[\dfrac{4 k_l \mu_l (T_{\text{sat}} - T_s)\, x}{g\, h_{fg}\, \rho_l (\rho_l - \rho_v)}\right]^{1/4}, \bar{h} = \dfrac{4}{3}\dfrac{k_l}{\delta_l}$

[(a) 0.1136 mm (b) 7984 W/m² K (c) 38.32 kW]

10.8 Saturated steam at 1.43 bar and 110°C condenses on a 25-mm OD vertical tube which is 500-mm long. The tube wall is maintained at 100°C. Calculate the average heat transfer coefficient and the rate of condensation. The properties of condensate at 105°C are:
$\rho = 954.7$ kg/m³, $\mu = 0.271 \times 10^{-3}$ kg/m s, $h_{fg@110°\text{C}} = 2230$ kJ/kg, $k = 0.68$ W/ m K

[7777.4 W/m² K, 4.93 kg/h]

10.9 The outer surface of a cylindrical drum of 25-mm-OD is exposed to saturated steam at 1.25 bar. The surface of the vertical drum is maintained at 94°C. Determine: (a) the height of the drum, and (b) the average heat transfer coefficient, and (c) the thickness of the condensate film at the base of the drum to condense 25 kg/h of steam.
Use the following properties of water:
Water vapour (1.25 bar, $T_{\text{sat}} = 106$°C): $\rho_v = 0.7273$ kg/m³, $h_{fg} = 2241.0$ kJ/kg
Water liquid $\left(T_f = \dfrac{1}{2}(106 + 94) = 100°\text{C}\right)$: $\rho_l = 957.9$ kg/m³, $k_l = 0.679$ W/m K
$C_{pl} = 4.217$ kJ/kg K, $\mu_l = 0.282 \times 10^{-3}$ N s/m²

[(a) 15.75 (b) 10644 W/m² K (c) 0.094 mm]

10.10 Saturated Freon-12 (R-12) vapour at a pressure of 2.61 bar condenses on a 12.5-mm-OD, 1.25-m long vertical tube which is maintained at a surface temperature of –15°C. Find: (a) the mean film heat transfer coefficient, (b) the total heat transfer rate, and (c) the rate of condensation at the tube surface. The following thermophysical properties of Freon – 12 may be used:

Saturation temperature at 2.61 bar, $T_{\text{sat}} = -5°\text{C}$

Latent heat of condensation at $-5°\text{C}$, $h_{fg} = 153.823$ kJ/kg

Vapour density at $-5°\text{C}$, $\rho_v = 15.393$ kg/m^3

Liquid thermal conductivity, $k_{l\,\text{at}\,T_f=-10°\text{C}} = 0.073$ W/m K

Liquid viscosity, $\mu_{l\,\text{at}\,-10°\text{C}} = 3.158 \times 10^{-4}$ kg/m s

Liquid specific heat, $C_{pl} = 920.3$ J/kg K

Liquid density, $\rho_{l\,\text{at}\,-10°\text{C}} = 1429$ kg/m^3

[(a) 834.4 W/m² K (b) 204.8 W (c) 4.7 kg/h]

(R) Horizontal Cylinder

10.11 Saturated ammonia vapour at 13.12 bar is exposed to a 13-mm-outside diameter horizontal tube, 1.2-m long. The tube wall is maintained at a uniform temperature of 26°C. Estimate the condensation rate in kg/h. Properties: Saturated vapour: [$T_{\text{sat @ 13.12 bar}} = 34°\text{C}$]

h_{fg} = 11.27.6 KJ/kg, ρ_v = 10.142 kg/m^3

Saturated liquid: $\left[T_f = \dfrac{T_{\text{sat}} + T_w}{2} = \dfrac{34 + 26}{2} = 30°\text{C}\right]$

ρ_l = 596.37 kg/m^3, C_{pl} = 4.89 kJ/kg K, $\mu_l = 208.13 \times 10^{-6}$ kg/m s, k_l = 0.507 W/m K

[11.08 kg/h]

10.12 Saturated steam at 101.325 kPa is condensed on a 19-mm-diameter tube whose surface is maintained at 347 K. Find the value of the heat transfer coefficient if the tube is 0.6- m-long, and oriented (a) vertically, and (b) horizontally. What will be the condensation rate in each of the two cases? Assume laminar film condensation. Properties: Condensate at 360 K:

Density (ρ_l)… 967.12 kg/m^3 Viscosity (μ_l)… 0.324×10^{-3} kg/m s

Specific heat (C_{pl})... 4.203 kJ/kg K Thermal conductivity (k_l)... 0.674 W/m K

Saturated steam at 373.15 K: Vapour density (ρ_v)... 0.596 kg/m^3

Latent heat of condensation (h_{fg}) ..... 2257 kJ/kg

[(a) 7.7 kg/h (b) 14.1 kg/h]

10.13 Water flows through a horizontal copper tube (3-cm-ID, 2-mm-thick) with a mass flow rate of 20 kg/min. Saturated steam at 2 bar condenses on the outside of the tube. The entering and leaving temperatures of the water are 35°C and 85°C, respectively. Determine the tube length required, making appropriate assumptions. **[6.36 m]**

(S) Vertical Array of Horizontal Tubes

10.14 100 tubes of 12-mm-diameter each are arranged in a square array and are exposed to steam at atmospheric pressure. Calculate the mass of steam condensed per unit length of tubes if the tube wall temperature is maintained at 98°C. For water film at mean temperature, take:

ρ = 960 kg/m^3, $\mu = 282 \times 10^{-6}$ kg/ m s, k = 0.68 W/m K, $h_{fg\,@100°\text{C}}$ = 2257 kJ/kg

[154 kg/h m]

10.15 (a) Saturated Freon – 12 (R-12) vapour at 60°C is condensed over a horizontal 1.27-cm-OD tube of 0.7 m length. The tube wall is maintained at 27°C by an internal flow of cool air. Determine the heat transfer coefficient. (b) If the saturated R-12 vapour at 60°C were to condense on a bank of 1.27-cm-OD tubes, 10 in each vertical column and 12 in each horizontal row, estimate the total rate of condensation. Properties of Refrigerant – 12:

Saturated vapour: (T_{sat} = 60°C): h_{fg} = 126.33 kJ/kg ρ_v = 59.84 kg/m^3

Saturated liquid: $\left(T_f = \frac{1}{2}(T_{sat} + T_w)\right)$: ρ_l = 1239.48 kg/m^3, C_{pl} = 1.0283 kJ/kg K

k_l = 0.06835 W/m K μ_l = 2.34.4 × 10^{-6} Pa s

[(a) 1184.3 W/m^{2}K (b) 1775 kg/h]

11 Heat Exchangers

11.1 ❑ INTRODUCTION

Heat exchanger is a thermal device which exchanges heat from one fluid stream to the other (one or more) fluid stream. It is a process equipment whose sole aim is to transfer the energy stored in one fluid to another fluid. Simply stated, the device used to transfer thermal energy from a hotter fluid to a colder fluid is called a *heat exchanger*. In other words, heat exchangers are facilitators in ensuring effective exchange of heat between two fluids at different temperatures and prevention of mixing of these fluids through a separating wall. Strictly speaking, the word *heat exchanger* is a misnomer. Perhaps the more correct word would be *heat transferer*.

A large variety of heat exchangers are used extensively in engineering practice. Each of the heat exchanger types has *advantages* and *disadvantages*. The choice of which configuration to use will depend on the following factors: (1) application, (2) types of fluids, (3) pressure and temperature levels, (4) modes of heat transfer, (5) pressure-drop restrictions, (6) maintenance and cleaning requirements, (7) cost, (8) size, (9) weight, (10) construction materials needed, and so on. However, at the end of the day, the basic consideration is whether or not the chosen heat exchanger will handle the heat-transfer rate required in the particular application.

Applications include not only *process industries, petrochemical industries*, *refrigeration and air-conditioning systems, waste heat recovery, chemical plants, space radiators,* and *thermal power plants* but also the *domestic sector*. *Room air conditioners* and *household refrigerators* belong to the latter category. Capacity varies from a few hundred watts (*as in the case of chillers in photographic processing units*) to 500 MW or more for a single unit (*as in the case of condensers of thermal power plants*). The wide spectrum of capacity and application has naturally resulted in the evolution of many designs.

11.2 ❑ CLASSIFICATION OF HEAT EXCHANGERS

The word *heat exchanger* a broad refers to spectrum of devices which can be classified in several different ways. Figure 11.1 illustrates the classification of heat exchangers at a glance.

Heat exchangers are manufactured in various *sizes*, *types*, and *configurations*. Broadly, one can classify heat exchangers based on

(a) heat-transfer process
(b) heat-transfer mechanism
(c) flow arrangement
(d) degree of compactness
(e) constructional features

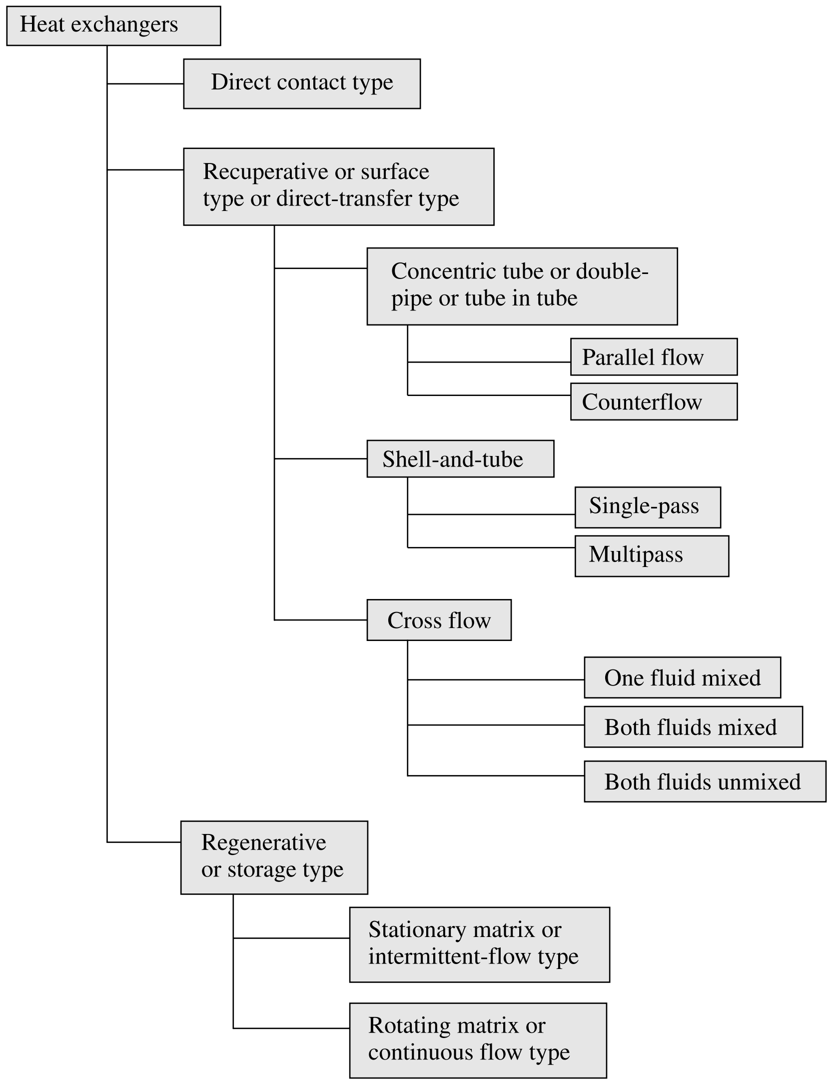


Fig. 11.1 *Types of heat exchanger*

11.2.1 • Classification Based on Heat-Transfer Process

Heat exchangers can be of *direct contact* or *indirect contact type.*

Direct-contact Heat Exchanger In direct-contact heat exchangers the fluids are allowed to come into contact directly. This is possible only when the fluids involved are immiscible like oil and water and can be easily separated from each other. *Open-feed water heaters, desuperheaters*, *cooling towers*, and *jet condensers* are some typical examples.

Direct-contact heat exchangers are more commonly used where simultaneous heat and mass transfer is involved. One such exchanger is a *water heater* (shown in Fig. 11.2) in which steam is bubbled through

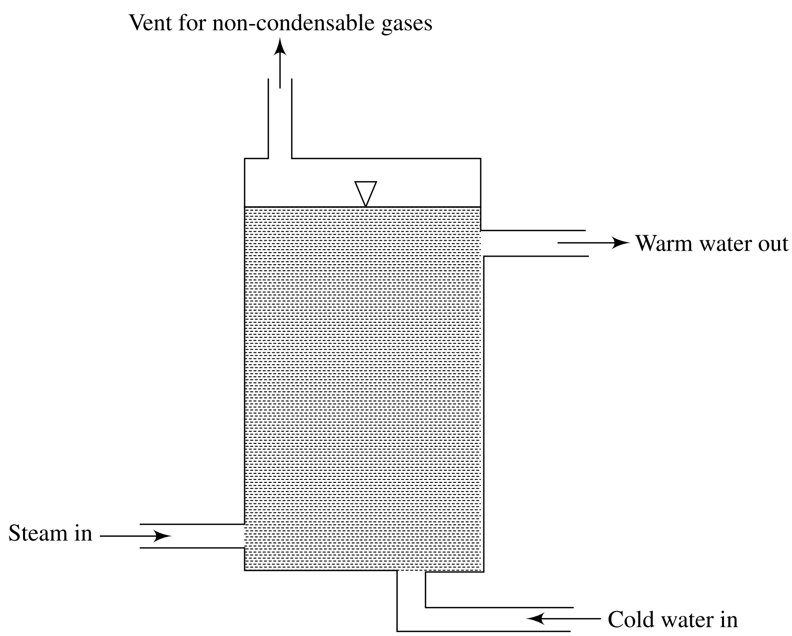


Fig. 11.2 *Water heater: a direct-contact heat exchanger*

the water. The steam is condensed and mixes with the water. The two fluid streams (steam and water) enter separately but leave as a single stream (warm water). Cooling towers are heat exchangers in which direct contact takes place between the hot fluid (water) and the cold fluid (air).

Advantage The cost of heat-transfer surface and problems of fouling are avoided.

Disadvantage The separation of the two liquids at each end of the exchanger is quite often difficult.

Indirect-contact Heat Exchanger If heat transfer is brought about through a wall that separates the two fluids, such devices are called *indirect-contact* type. There is no mixing of the two fluids. In such cases such as *automobile radiators*, the hot and cold fluids are separated by a solid partition. These are also called *surface heat exchangers*.

Indirect-contact heat exchangers are further subdivided into two categories, viz., *recuperators* (*heat exchangers without storage*) and *regenerators* (*heat exchangers with storage*).

Recuperative Heat Exchangers In *recuperators*, a solid surface separates the two fluids at different temperatures and the operation is in steady-state mode. Heat is transferred by both convection and conduction through the dividing wall.

Regenerative Heat Exchangers In *regenerators*, on the other hand, one heating surface is exposed at certain intervals of time, first to a hot fluid and then to a cold one. The regenerator surface first removes heat from the hot fluid and is itself heated in the process. Then the surface gives up this heat to the cold fluid. The heat-transfer process in the regenerator is always transient (*unsteady sate*) unlike recuperators which typically operate in steady-state conditions.

Unlike recuperators, the flow of heat in regenerative heat exchangers is not steady and continuous but intermittent. These can be either *static* (*fixed bed*) or *dynamic* (*rotary*). The *static* type has no moving parts and consists of a porous mass (*matrix*) of considerable heat-storage capacity through which the hot and cold fluid streams flow alternately (Fig. 11.3(a)). A flow-switching device regulates the periodic flowing of the two fluids. During the flow of the hot fluid, the heat is transferred from the hot fluid to the matrix where it is stored, making the matrix temperature rise. Later, this stored heat is given up to the cold fluid, when, in turn, it passes through the matrix, thus causing its temperature to fall. Then the hot fluid flow is switched off, and the cold fluid flow is switched on. Static-type (fixed bed) regenerators can be non-compact and are used extensively in high-temperature applications (900 to 1500°C), such as *air preheaters* for *coke manufacturing* and *glass melting tanks*.

Typically, in a *regenerative rotary* heat exchanger, when a portion of the heat exchanger is in the hot stream, the solid wall of the regenerator absorbs heat. This section then rotates into the cold stream, where the hot wall releases heat to the cold air. Thus, each section of the heat exchanger operates in a periodic (*transient*) mode as it rotates continuously [Fig. 11.3(b)].

The overall operating conditions of the heat exchanger can however be considered steady. Heat exchangers with a rotating matrix are also being used in gas liquefaction plants and in energy conservation systems.

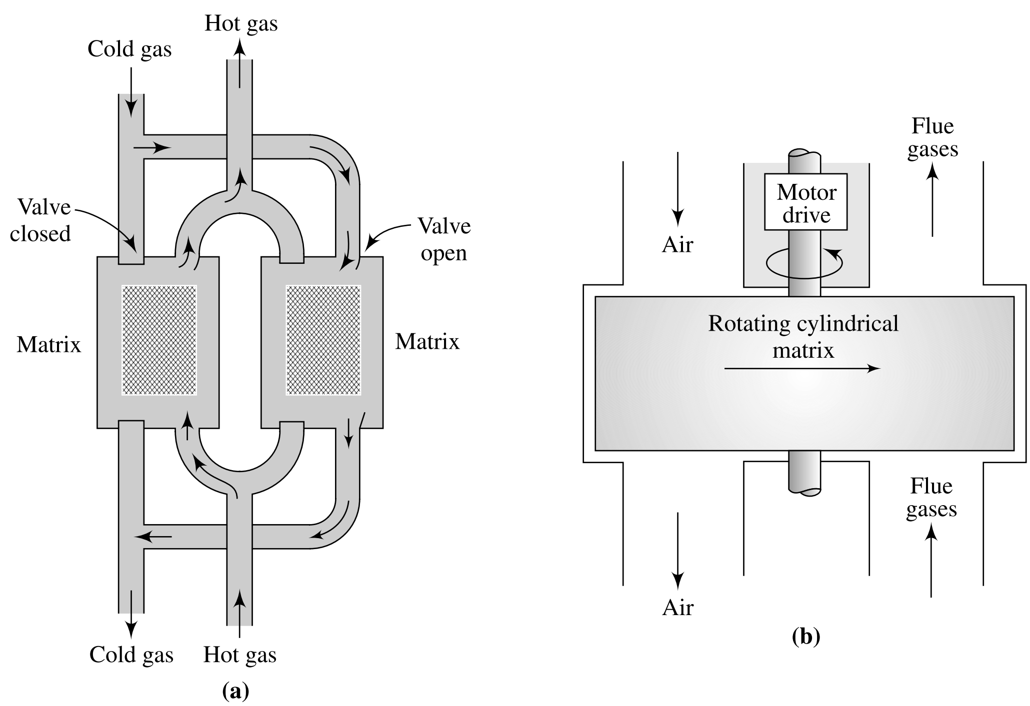


Fig. 11.3 *Regenerative-type heat exchangers: (a) Fixed bed (b) Rotary*

11.2.2 • Classification Based on Heat-Transfer Mechanism

These possibilities for the heat-transfer mechanism include a combination of any two of the following:

- Two-stream heat exchangers (single phase-forced or free convection)
- Single-steam heat exchangers (phase change-boiling or condensation)
- Radiation or combined convection and radiation

Two-stream Heat Exchangers Usually, the mode of heat transfer is single-phase forced convection on both sides of the two-stream heat exchanger. The temperatures of both fluid streams (*hot and cold*) change in the exchanger.

Examples Automobile radiators, aircraft oil coolers. Such exchangers can be further classified as follows:

Heat exchange can be *liquid-to-liquid, liquid-to-gas*, or *gas-to-gas*. *Liquid-to-liquid* heat exchangers are by far the most common. Both fluids are pumped through the exchanger. The heat transfer on both the tube-and-shell sides is by forced convection. As the heat-transfer coefficients are quite large in liquid flow, fins are rarely used.

In the *liquid-to-gas* exchanger, the fins are generally provided on the gas side, where the heat transfer coefficient is relatively low.

Gas-to-gas exchangers are used in the air preheaters, cryogenic gas-liquefaction systems, and steel furnaces. Internal and external fins generally are used in the tubes to improve heat transfer.

Single-stream Heat Exchangers Of the two fluids, one fluid undergoes *phase change*. The other fluid stream only experiences temperature variation in the exchanger. Condensers, boilers, and radiators for space power plants involve condensation, boiling, and radiation, respectively, on one of the surfaces of the heat exchanger. Condensers are used in *steam power plants, chemical processing plants, and nuclear power plants.*

Examples Evaporators and condensers in thermal power plants and refrigeration systems.

Figure 11.4 shows a few examples of the *heat-exchange* processes involved in recuperative heat exchangers.

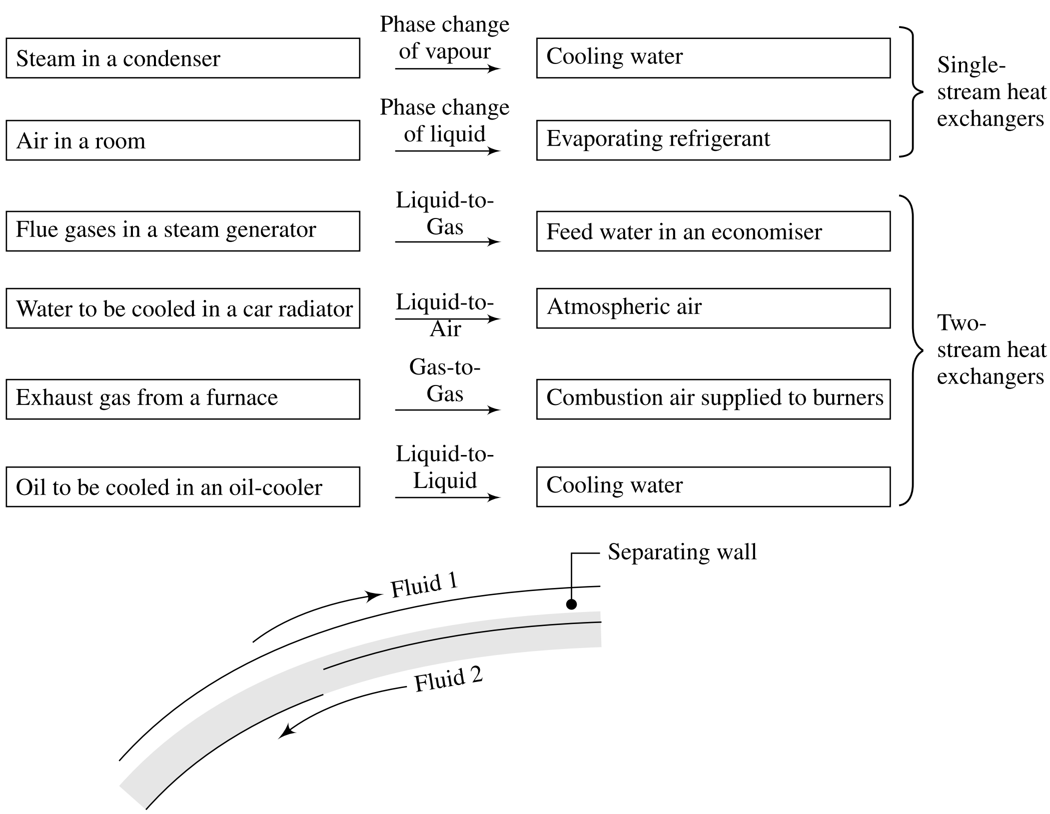


Fig. 11.4 *Some examples of single-stream and two-stream heat exchange processes*

It is good to remember that there are two fluid streams—hot and cold—even in condensers and evaporators, classified as single stream exchangers. The term *single-stream* simply means that the temperature of only one stream varies along the exchanger; the other one has essentially the same temperature throughout.

Space Radiators The only way the waste heat from a condenser can be dissipated in a space vehicle is by thermal radiation.

11.2.3 • Classification Based on Flow Arrangement

Depending on the relative orientation of the flow direction of the two fluids exchanging heat, many possibilities exist in heat exchangers.

Parallel-flow (Co-current) [Fig. 11.5(a)] Both the hot and cold fluids enter the heat exchanger at the same end, and flow through in the same direction, and leave together at the other end (the terms *hot* and *cold* are used in a relative sense).

Counterflow (Counter-current) [Fig. 11.5(b)] The hot and cold fluids enter the heat exchanger at the opposite ends and flow through in opposite directions. Heat is transferred continuously from the hotter to the colder fluid along the length of the exchanger.

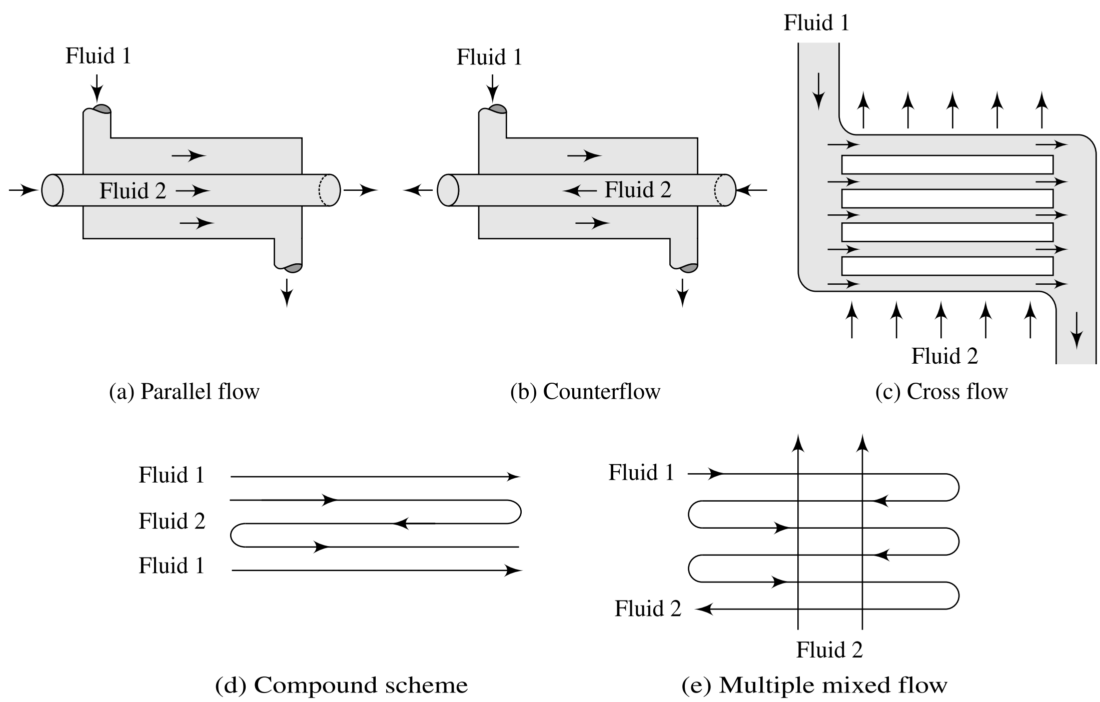


Fig. 11.5 *Typical arrangements for fluid flow in heat exchangers*

Cross Flow In the cross-flow exchanger, the two fluids usually flow at right angles to each other. There are a number of parallel paths for the fluid flowing in the tubes as shown in Fig. 11.5(c) and each path is separated physically from the neighbouring paths. The fluid in the tubes is called *unmixed*. For the fluid flowing in the upward direction there are no passages for the fluid flowing across the tubes. This fluid is said to be *mixed*.

Figure 11.5(d) shows simultaneous parallel and counterflow configuration, while
Figure 11.5(e) indicates a complex multiple mixed-flow arrangement.

Depending on the design, there are *three* flow patterns in any cross-flow heat exchanger as shown below (Fig. 11.6). One must clearly distinguish between *mixed flow* and *unmixed flow*. If the entire fluid stream is spread out across the heat exchanger, it is referred to as *mixed flow* while if the fluid stream is divided across the heat exchanger by making it pass through confined passages, it is known as *unmixed flow* (Fig. 11.7).

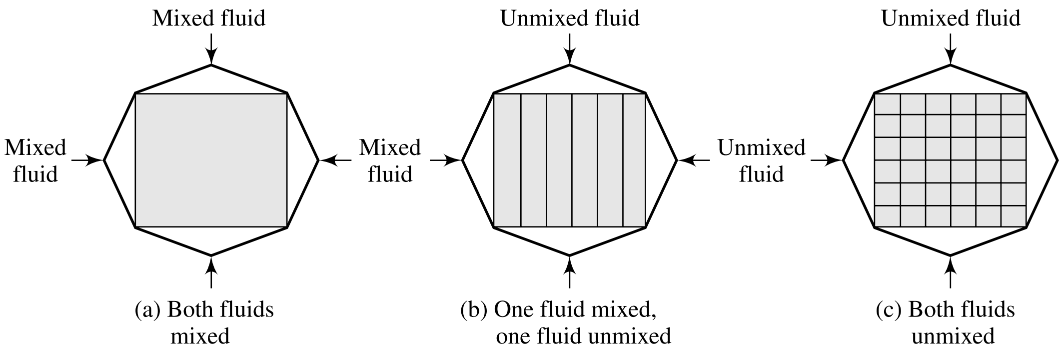


Fig. 11.6 *Flow variations for a cross-flow heat exchanger*

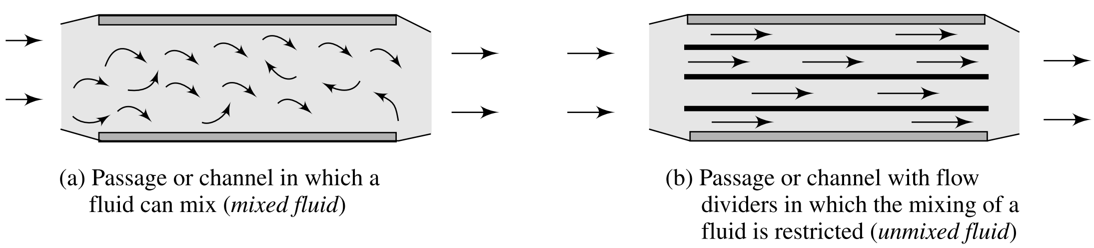


Fig. 11.7 *Mixed and unmixed fluid flow in cross-flow heat exchangers*

11.2.4 • Classification based on Degree of Compactness

Compactness is a favourable feature for any heat exchanger. In *gas-to-gas* or *gas-to-liquid* heat exchangers, where at least one fluid is a gas with its associated low heat transfer coefficient, it is necessary to pack more surface area to compensate for it. This gives large (hA) product on the gas side. A measure of compactness is the *area density*, B. It is defined as the ratio of *the heat-transfer surface area of the heat exchanger and its volume*. If B is greater than 700 m^2/m^3, the heat exchangers are classified as *compact* irrespective of their structural design. The compact heat exchanger itself may not be small. Table 11.1 and Fig. 11.8 illustrate the ranges of characteristic hydraulic diameter, D_h and the area density for a number of different types of heat exchangers.

A couple of examples of compact heat exchangers are *automobile radiators* with an area density of about 1100 m^2/m^3 and the *glass ceramic heat exchangers* for some automotive gas-turbine engines with an area density of the order of 6600 m^2/m^3. Incidentally, the human lungs, with an area density of nearly 20 000 m^2/m^3, are by far the most compact ones.

Double pipe and *shell-and-tube type* exchangers have an area density of the order of 70 to 500 m^2/m^3. Clearly, these are not considered compact.

Table 11.1 *Surface area-to-volume ratios of different heat exchangers*

Type of heat exchanger	Hydraulic diameter (mm)	Surface area/Volume (m^2/m^3)
Plain tube, shell-and-tube	40 to 6	60–600
Plate heat exchangers	20 to 10	180–350
Strip fin and louvred fin heat exchangers	10 to 0.5	350–7100
Automobile radiators	5 to 2.5	710–1500
Cryogenic heat exchangers	3.7 to 1.7	1000–2500
Gas turbine rotary regenerators	1.2 to 0.5	3000–7100
Matrix types, wire screen, sphere bed, corrugated sheets	2.5 to 0.2	1500–18000
Human lungs	0.2 to 0.15	18000–25000

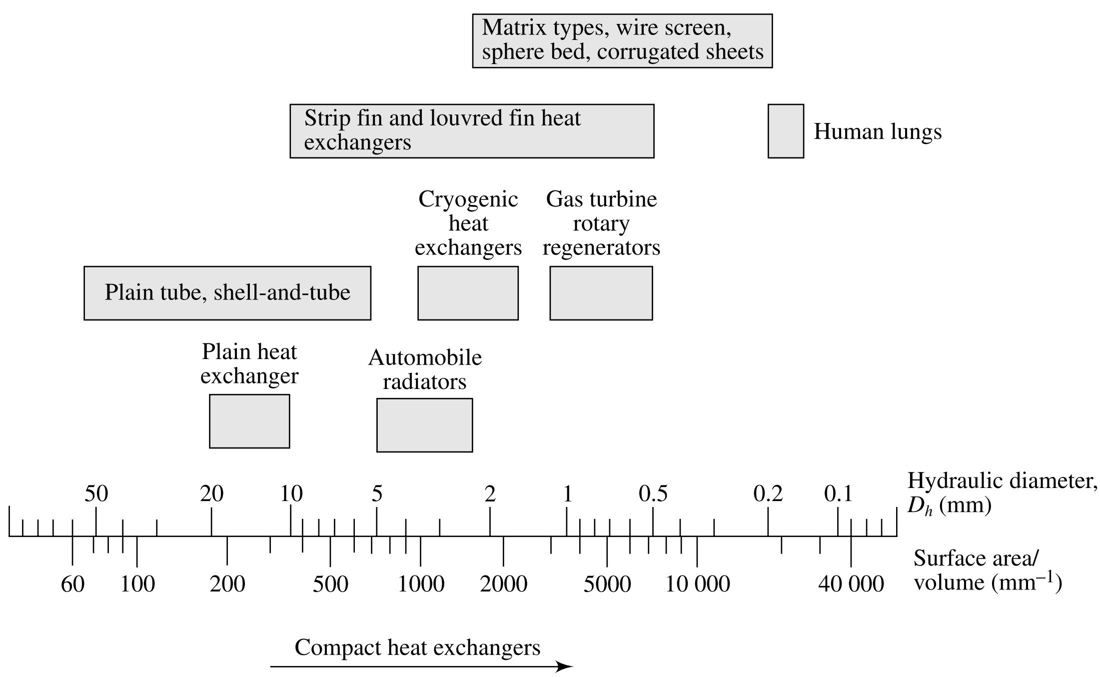


Fig. 11.8 *Classification of heat exchangers based on compactness (surface-area-to-volume ratio)*

Compact heat exchangers are generally of the cross-flow configuration (Fig. 11.9). These are used when the pressure drops available for pumping the fluids are limited and the heat exchanger load is moderate. Small flow passages and laminar flow usually characterize the compact heat exchangers.

A high degree of compactness is desirable when the weight and physical size become important. To improve the effectiveness or the compactness of heat exchangers, arrays of finned tubes or plates are usually employed. In a *gas-to-liquid* heat exchanger, for example, the heat-transfer coefficient on the *gas* side is much *lower* than that on the *liquid* side. Fins are, therefore, used on the *gas side*. The extended heat transfer surface on the gas side becomes much more compact. Some of the applications include *automobiles, marine, aircraft, or aerospace, cryogenic systems,* and *refrigeration and air conditioning*.

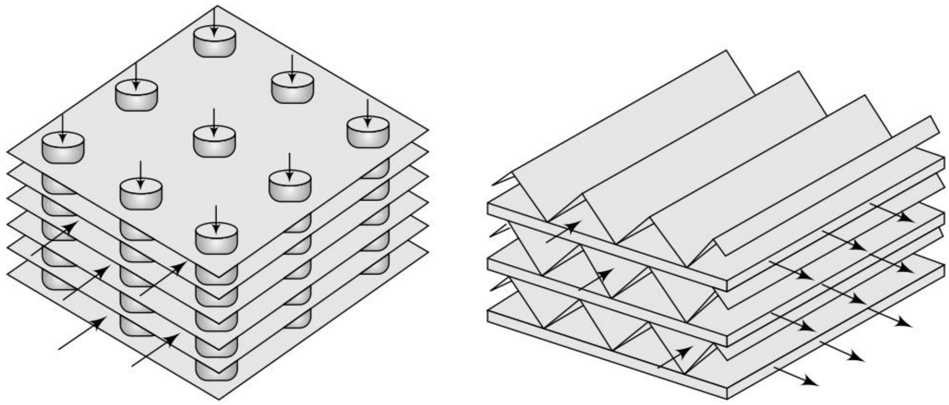


Fig. 11.9 *Cross-flow compact heat exchangers*

11.2.5 • Classification Based on Constructional Features

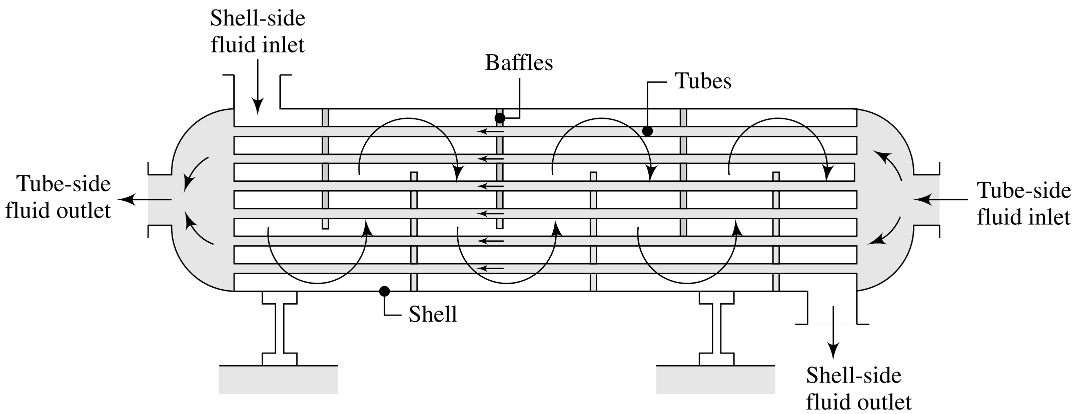
Heat exchangers can also be categorized according to their constructional features.

Tubular Heat Exchangers Tubular exchangers are manufactured in several sizes, flow arrangements, and types. They can operate over a wide range of operating pressures and temperatures. Due to ease of manufacturing and relatively low cost these exchangers find their widespread use in engineering applications.

Concentric-Tube Heat Exchangers The simplest heat-exchanger geometry is the *tube-in-tube* (also known as the co-axial, *concentric tube* or the *double-pipe*) heat exchanger. The parallel and counterflow patterns generally involve the *tube-in-tube* configuration. One fluid flows in the central (inner) tube and the other fluid flows through the annulus. A truly co-current or counter-current exchanger is rarely found in practical applications.

Shell-and-tube Heat Exchangers The commonly used *shell-and-tube* exchanger consists of round tubes mounted on a cylindrical shell with their axes parallel to that of the shell. Several types of shell-and-tube exchanger are available based on the flow configurations and constructional details. When large quantities of fluid are to be handled as in the *chemical process industry*, the *food industry*, or the *petroleum industry*, pumping power (*pressure drop*) consideration is more important than space limitation. It is advisable to keep the equipment size reasonable from the point of view of capital cost. The shell and tube exchangers can meet these requirements. The fluid flowing outside the tubes is mixed. The baffles are installed to support the tubes, direct the flow on the shell side and to promote turbulence (*mixing*) in addition to keeping the velocity across the tubes high and nearly at right angles. It is a *cross-counterflow* mode of operation. Figure 11.10(a) shows an arrangement with a single pass for the fluid inside the tubes through the exchanger and a *single pass* on the shell side as compared to Fig. 11.10(b) where there are *two* passes through the exchanger on the *tube side*. Many other configurations can be obtained by combining and modifying the types shown in Fig. 11.10 and using them in series- or parallel-flow circuits. As the number of passes increases, the effectiveness approaches that of an ideal counterflow heat exchanger.

If the tube-side fluid makes a single traverse through the exchanger, it is called *one-tube-pass*. The one-shell-pass, two-tube-pass shell-and-tube heat exchanger is more popular because only one end of the exchanger requires perforation to lead the tubes in and out. The first tube pass gives parallel flow, and



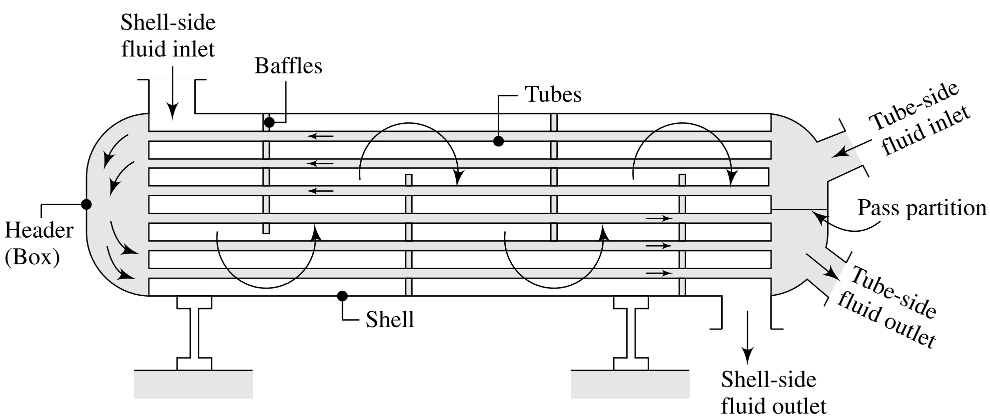


Fig. 11.10 *Shell-and-tube heat exchanger: (a) One-shell-pass and one-tube-pass (b) One-shell-pass and two-tube-pass*

the second gives counterflow. *Shell-and-tube* heat exchangers are commonly used in *power plants, oil refineries*, and *chemical processing plants*.

The major components are *tubes, tube sheets, baffles, front- end head, rear-end headers*, and the *shell*. Some of these are standardized (by the TEMA, i.e., *Tubular Exchanger Manufacturers Association*).

In Fig. 11.11, situations are depicted in which there is more than one pass for the two fluids. The fluid flowing through the tubes is called the *tube fluid* while the fluid flowing outside the tubes is known as the *shell fluid*. The baffles (Fig. 11.12) serve to create turbulence and thereby enhance the heat-transfer rate.

Cross-flow Heat Exchangers (Fig. 11.13) An increase in complexity occurs if *cross flow* is used. The two fluids follow flow paths that are perpendicular to each other. Each fluid flow must stay in a prescribed path and is not permitted to *mix* to the right or left. Usually, the *mixed* fluid case has the least pressure drop associated with the flow, because there is less surface for fluid shear to take place. But less surface is not so desirable from the point of view of heat transfer. Heat exchangers with both fluids *unmixed* are very efficient in transferring energy, because of the large amount of surface area in a compact volume.

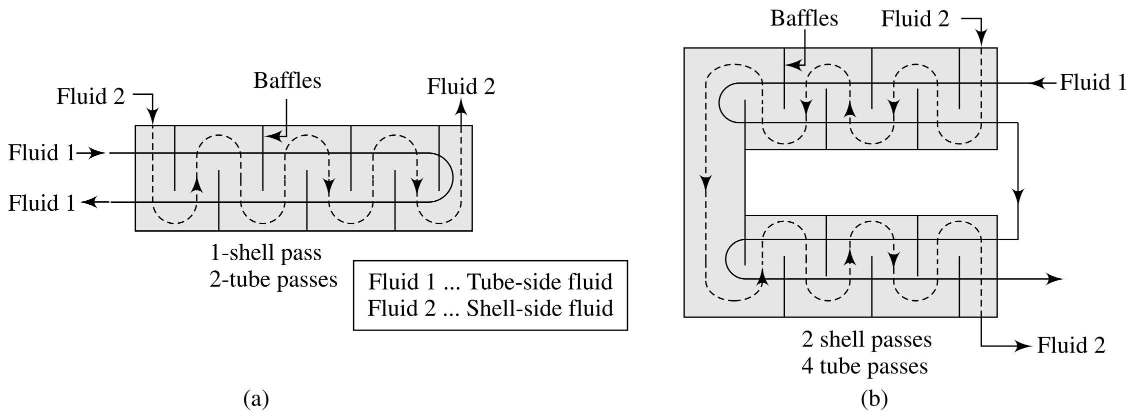


Fig. 11.11 *Two kinds of shell-and-tube heat exchangers*

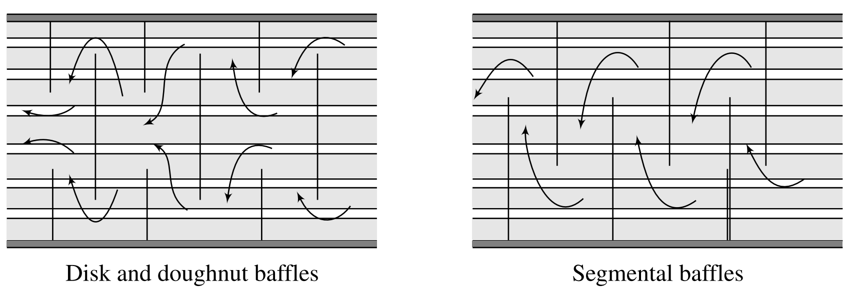


Fig. 11.12 *Types of shell baffles*

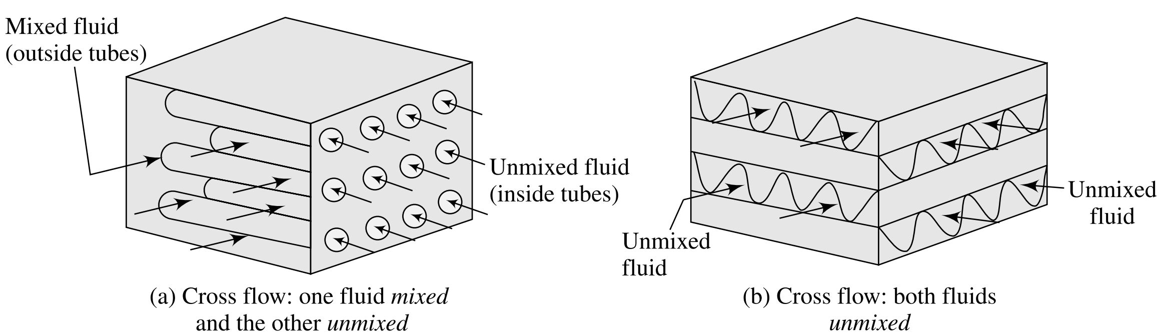


Fig. 11.13 *Mixed and unmixed fluid flow in cross-flow heat exchangers*

Cross-flow heat exchangers are more economical in space utilization compared to the concentric-tube type. Aircraft oil coolers are a typical example of the application of cross-flow exchangers which require minimum space. Sometimes these are preferable when it makes the layout of ducts and piping more convenient and accessible.

Plate-heat Exchangers The plate-heat exchangers (Fig. 11.14) usually are constructed of rectangular metal thin plates with about 5 mm spacing. The plates may be smooth or may have some form of corrugation. Corrugations on the plates promote turbulence and very high heat-transfer rates. Since the plates are generally arranged for general countercurrent flow, very close approach temperatures are obtained. Due to these advantages, the plate-heat exchanger is being used extensively in an increasing number of industrial applications. The area density of plate exchangers is in the range of 100 to 200 m^2 / m^3. The plates can be easily dismantled. These are normally used in the *food-processing* industry. The pressure and temperature are however limited to 20 bar and 150°C.

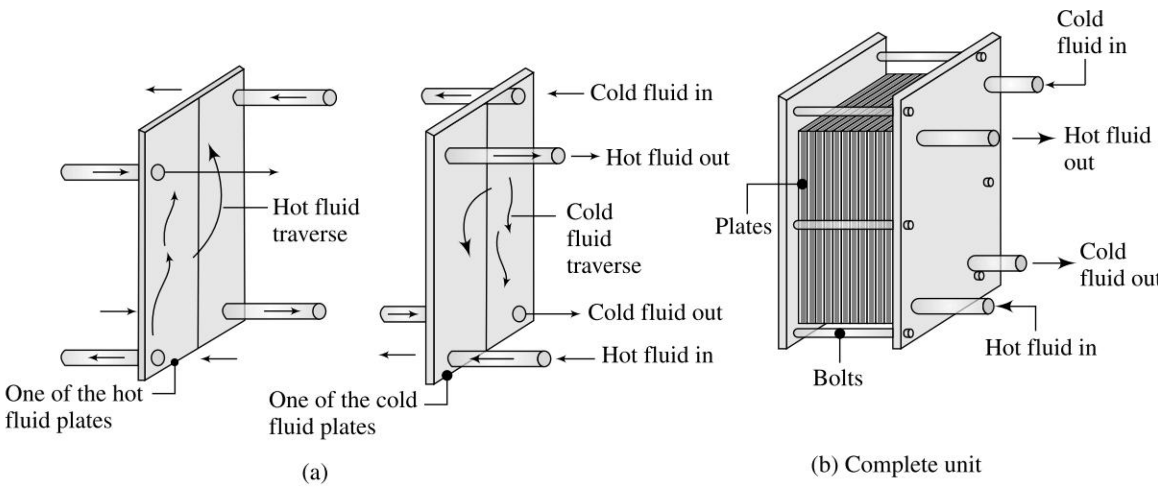


Fig. 11.14 *Plate-heat exchanger*

Extended-Surface Heat Exchangers

Plate-fin Heat Exchangers (Fig. 11.15) The degree of compactness can be significantly improved (up to about 6000 m^2/m^3) by using the plate-fin type of configuration. Louvred or corrugated fins are sandwiched between an array of parallel flat plates. Plate-fin exchangers are generally used for *gas-to gas* applications, but they are meant for *low-pressure* applications usually below 10 bar. The operating temperatures are limited to about 800°C. Plate-fin heat exchangers also find use in cryogenic applications.

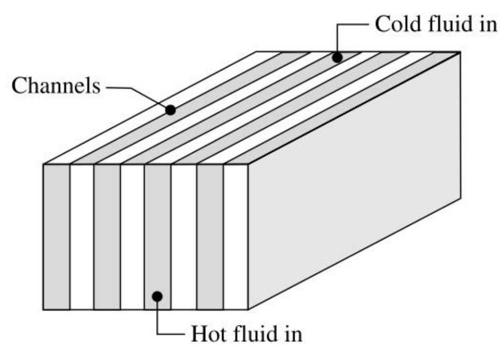


Fig. 11.15 *Plate-fin heat exchanger*

Tube-fin Heat Exchangers These are used usually in *gas-to-liquid* heat exchange with liquid being the tube-side fluid. Figure 11.16 illustrates a typical configuration. Tube-fin exchangers can withstand a wide range of tube fluid operating pressures usually not above 30 bar and operating temperatures from ultra low cryogenic applications to about 870°C. These exchangers have an area density of about 330 m^2 / m^3 which is much less than that of the plate-fin exchangers.

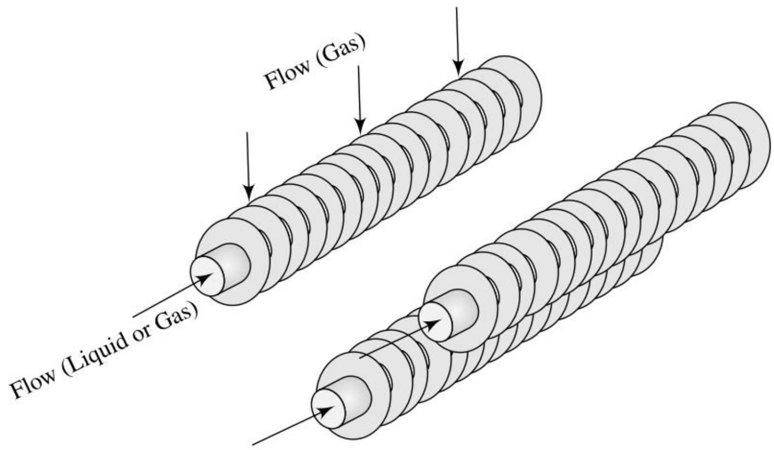


Fig. 11.16 *Cross-flow heat exchangers with extended surfaces (tube-fin)*

11.3 ❑ TEMPERATURE DISTRIBUTION IN HEAT EXCHANGERS

The heat-exchanger fluid temperatures vary with different geometric configurations and flow conditions. At different locations in the heat exchanger, the temperature differences between the two fluids are different. The geometry and operating conditions are taken into account when we consider a specific heat exchanger. The temperature of each fluid varies along the length of the heat exchanger depending on the heat capacity rate that is the mass flow rate-specific heat product $(\dot{m}C_p)$ or C. Usually, the temperature distribution is plotted as a function of the distance along the length of the heat exchanger.

Figure 11.17(a) shows a parallel-flow arrangement in which both fluids flow in the same direction, with the cold fluid experiencing a temperature rise and the hot fluid a temperature drop. The exit temperature of the cold fluid cannot exceed of the hot fluid. The driving temperature differential ΔT, i.e., $(T_{hot} - T_{cold})$ *decreases* continuously from end to end in the *parallel-flow* mode. Figure 11.17(b) illustrates a counterflow arrangement in which fluids flow in opposite directions. The exit temperature of the cold fluid can be higher than that of the hot fluid. Ideally, the exit temperature of one fluid may approach the inlet temperature of the other. Note that the temperature difference ΔT varies *much less* from end to end in the counterflow mode. This has significant influence on the exchanger performance. Note carefully how the relative values of the heat capacity rates of the two fluids affect the nature of temperature variation in both parallel and counterflow heat exchangers.

Figure 11.18(a) corresponds to a situation in which the hot fluid condenses and heat is transferred to the cold fluid, causing its temperature to rise along the path of flow. In Fig. 11.18(b) cold liquid is evaporating and the hot fluid is being cooled along its path of flow. Note that in the case of condensing vapour and evaporating liquid, the temperature remains essentially constant since during phase change, the heat-capacity rate becomes almost infinite. Figure 11.18(c) characterizes a counterflow heat exchanger in which the temperature rise in the cold fluid is equal to the temperature drop in the hot fluid; thus the temperature difference ΔT between the hot and cold fluids is constant throughout. However, in all other cases, the temperature difference ΔT between the hot and cold fluids varies with position along the direction of flow.

In multipass and cross-flow arrangements, the temperature distribution in the heat exchanger display a more complex pattern. The temperature profiles for a typical one-shell-pass and two-tube-pass heat

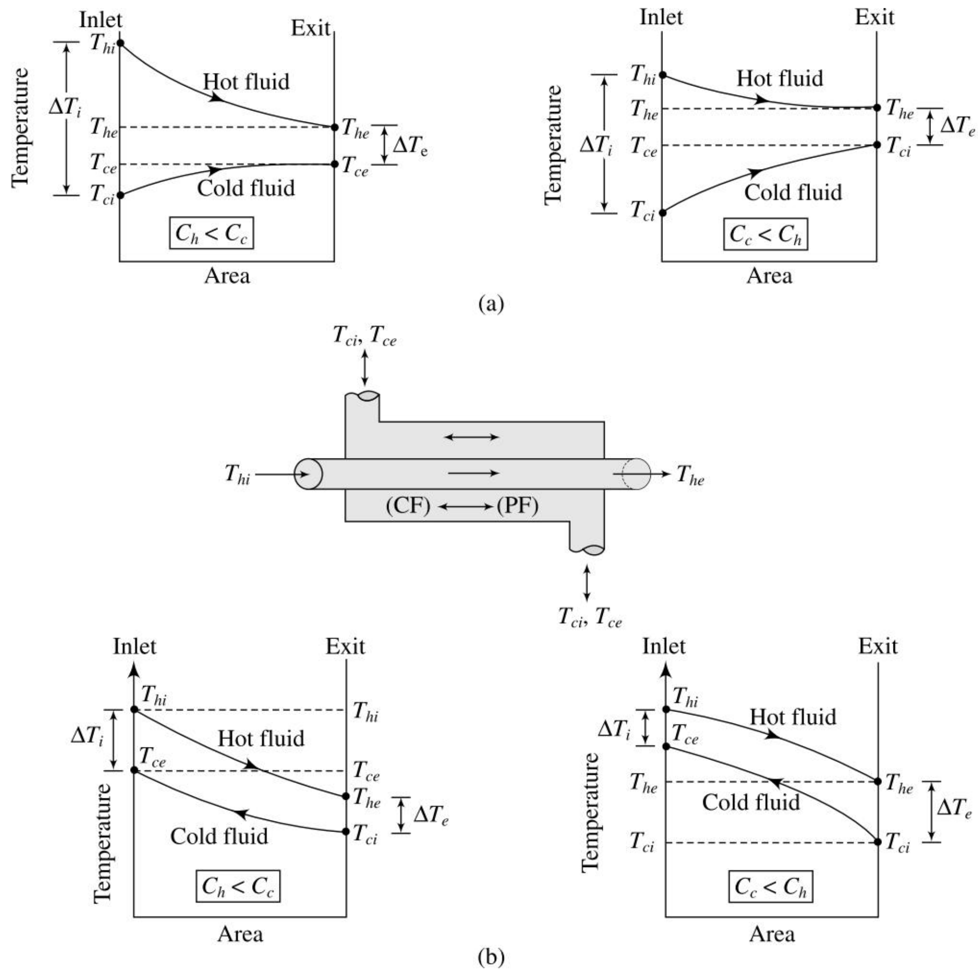


Fig. 11.17 *Temperature distributions in (a) parallel flow (PF), and (b) counterflow (CF), concentric tube (double pipe) heat exchangers*

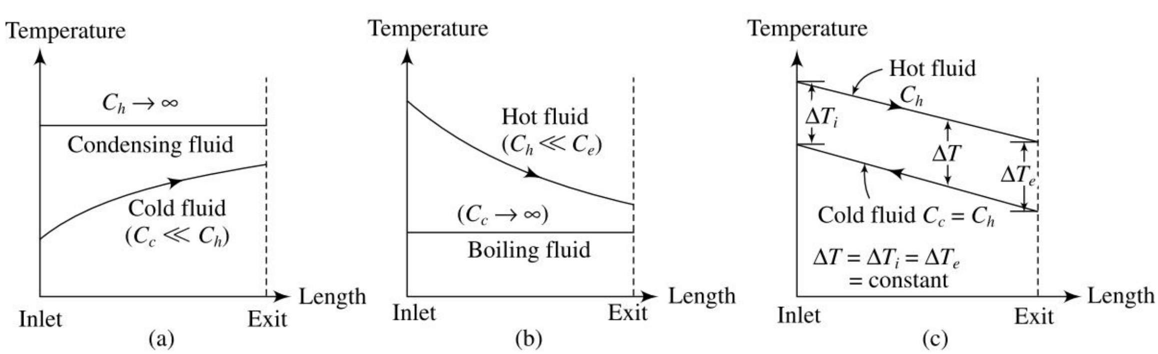


Fig. 11.18 *Temperature profiles for a heat exchanger with (a) $C_h \to \infty$ (condenser), (b) $C_c \to \infty$ (evaporator), and (c) $C_c = C_h$. (balanced heat exchanger)*

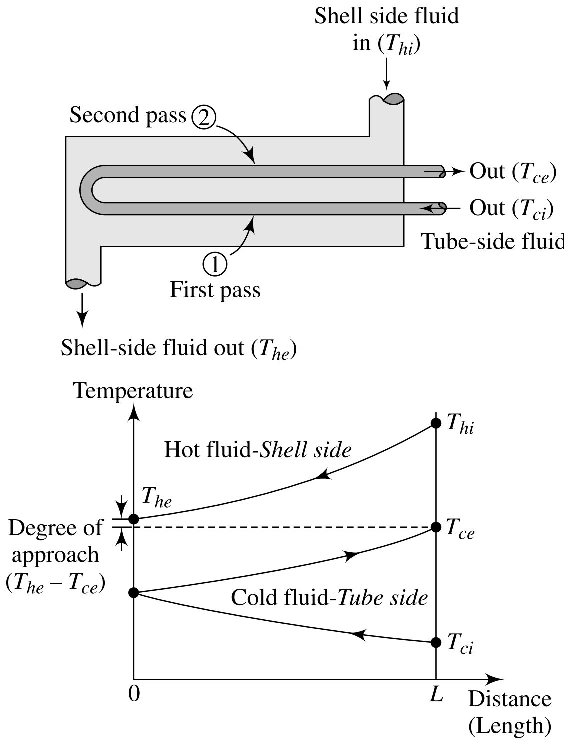


Fig. 11.19 *(a) Temperature variation in a one-shell-pass and two-tube-pass heat exchanger (the hot and cold fluid streams enter at the same end).*

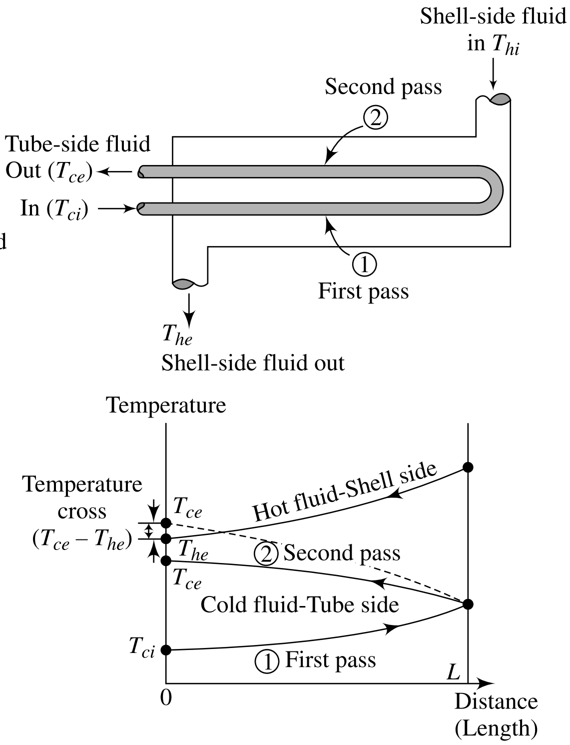


Fig. 11.19 *(b) Temperature variation in a one-shell-pass and two-tube-pass heat exchanger (the hot and cold fluid streams enter at the opposite ends).*

exchanger are illustrated in Fig. 11.19(a) and (b) depending on whether the two fluids enter at the same end or at the opposite ends. In this context, the two terms *approach* and *cross* are generally used which are explained below.

• Approach and Cross

The *degree of approach* of a heat exchanger is the *minimum local temperature difference* between hot and cold fluids along its length. Depending on the flow configuration, heat duty, single-stream or two-stream etc, the temperature approach could occur anywhere in the exchanger. For two-stream heat exchangers with constant specific heats the approach is *min* $(T_h - T_c)$. In the case of *parallel-flow* and *cross-flow* exchangers, the temperature approach is $(T_{he} - T_{ce})$. In the case of *counterflow*, if $C_c < C_h$, the approach is $(T_{hi} - T_{ce})$ while it is $(T_{he} - T_{ce})$ for $C_h < C_c$. In *one-shell* and *two-tube-pass* exchanger, $(T_{he} - T_{ce})$ is the approach (when both fluid streams enter at the same end). Sometimes T_{ce} may be greater than T_{he} as may happen when hot and cold fluids enter at different ends. In that case $(T_{ce} - T_{he})$ is referred to as *temperature cross*.

In the case of a cross-flow heat exchanger, the temperature varies in two directions as illustrated in Fig. 11.20 for the hot and cold fluids. The temperature variation in the case of a fixed-bed regenerator in which the temperature is plotted along time is indicated for storage and retrieval periods in Fig. 11.21. For a rotary regenerator, the temperature changes are indicated in Fig. 11.22.

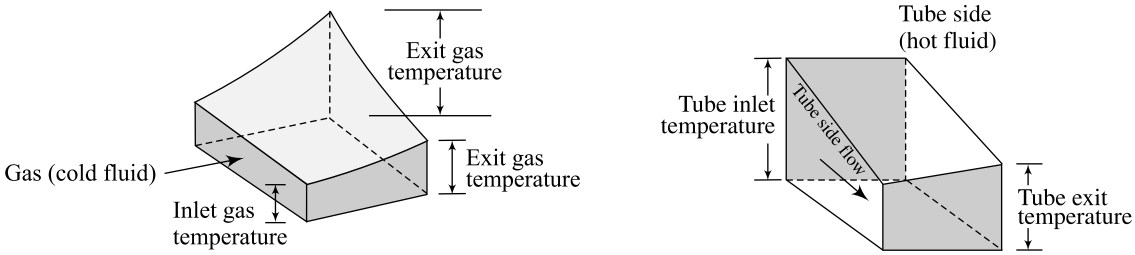


Fig. 11.20 *Typical temperature profiles for a cross-flow heat exchanger with both fluids unmixed*

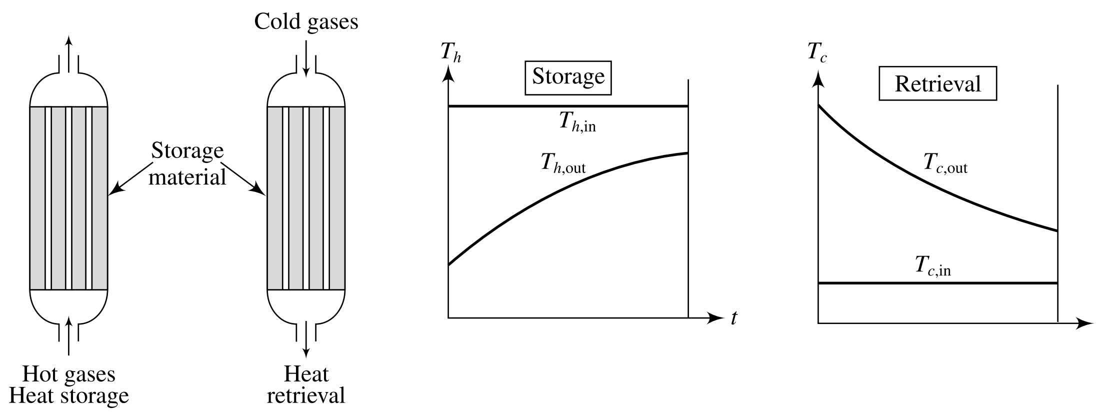


Fig. 11.21 *Fixed-bed regenerator*

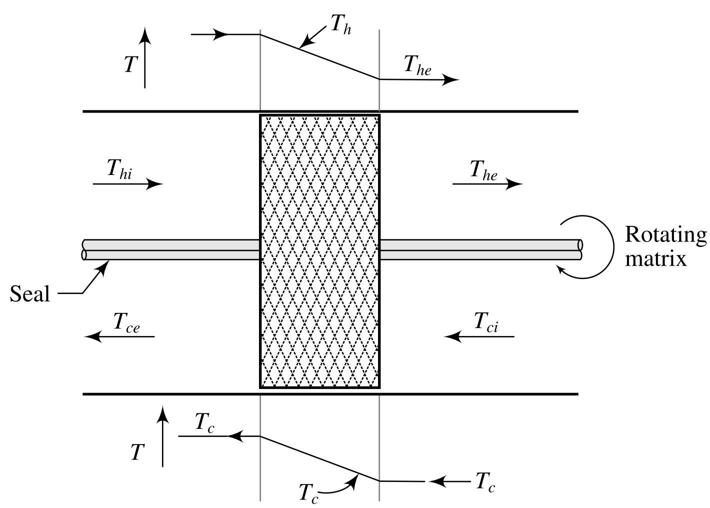


Fig. 11.22 *A rotary regenerative heat exchanger*

11.4 ❑ OVERALL HEAT-TRANSFER COEFFICIENT AND FOULING FACTORS

Most heat exchangers involve heat transfer from one fluid to another across a plane wall or a tube wall—more commonly a tube wall.

Consider the two fluids maintained at different constant temperatures $T_{\infty i}$ and $T_{\infty o}$ flowing on the inside and outside of a separating plane metal wall of thermal conductivity k, thickness L and the associated heat-transfer coefficients on the two sides are h_i and h_o. The quantity of heat to be transferred is $\dot{Q}$ and the surface area is A. Referring to the schematic and the thermal resistance network shown in Fig. 11.23, the heat-transfer rate can be written as

$$\dot{Q} = h_i A(T_{\infty i} - T_1) = \frac{kA(T_1 - T_2)}{L} = h_o A(T_2 - T_{\infty o})$$

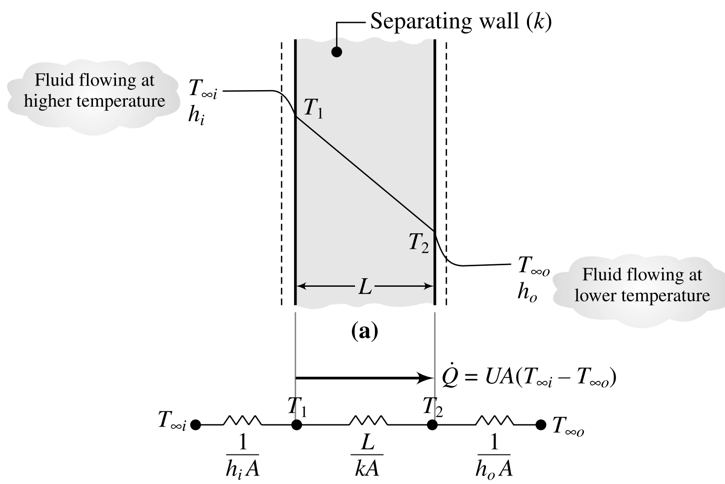


Fig. 11.23 *Overall heat transfer through a plane-wall heat exchanger*

In terms of the overall temperature difference, one can write

$$\dot{Q} = \frac{\Delta T_{\text{overall}}}{\Sigma R_{\text{th}}} = \frac{T_{\infty i} - T_{\infty o}}{R_{\text{conv},i} + R_{\text{cond,wall}} + R_{\text{conv},o}}$$

Also $\dot{Q} = UA\Delta T_{\text{overall}}$

where $$\boxed{\frac{1}{UA} = \sum R_{\text{th}} = \frac{1}{h_i A} + \frac{L}{kA} + \frac{1}{h_o A}}$$

Hence, for this *clean, unfinned* plane wall heat exchanger, the overall heat-transfer coefficient, U is given by

$$\boxed{\frac{1}{U} = \frac{1}{h_i} + \frac{L}{k} + \frac{1}{h_o}}$$

Note that in this case the area in contact with both hot and cold fluid is the same, but in other types of exchangers such as concentric tube or shell-and-tube types, the *inside* and *outside* areas are different.

It is noteworthy that the overall (*total*) thermal resistance to heat transfer between the two fluids in a heat exchanger depends not only on the thermal resistances due to convective heat transfer of both fluids, and conduction through the solid wall, but also due to additional scale deposits called *fouling*, that may have adhered to the wall surfaces. For a particular application, some or all of these resistances may be negligible. In general, there are *five* contributors to the overall thermal resistance:

- *two* convective resistances,
- *two* fouling resistances, and
- *a* wall resistance,

Over and above these resistances, the effect of finned surfaces (*on either or both sides*) in reducing the convective resistance should also be taken into account in evaluating the overall heat-transfer coefficient.

Figure 11.24 illustrates a typical double-pipe heat exchanger and the corresponding thermal circuit and temperature variation.

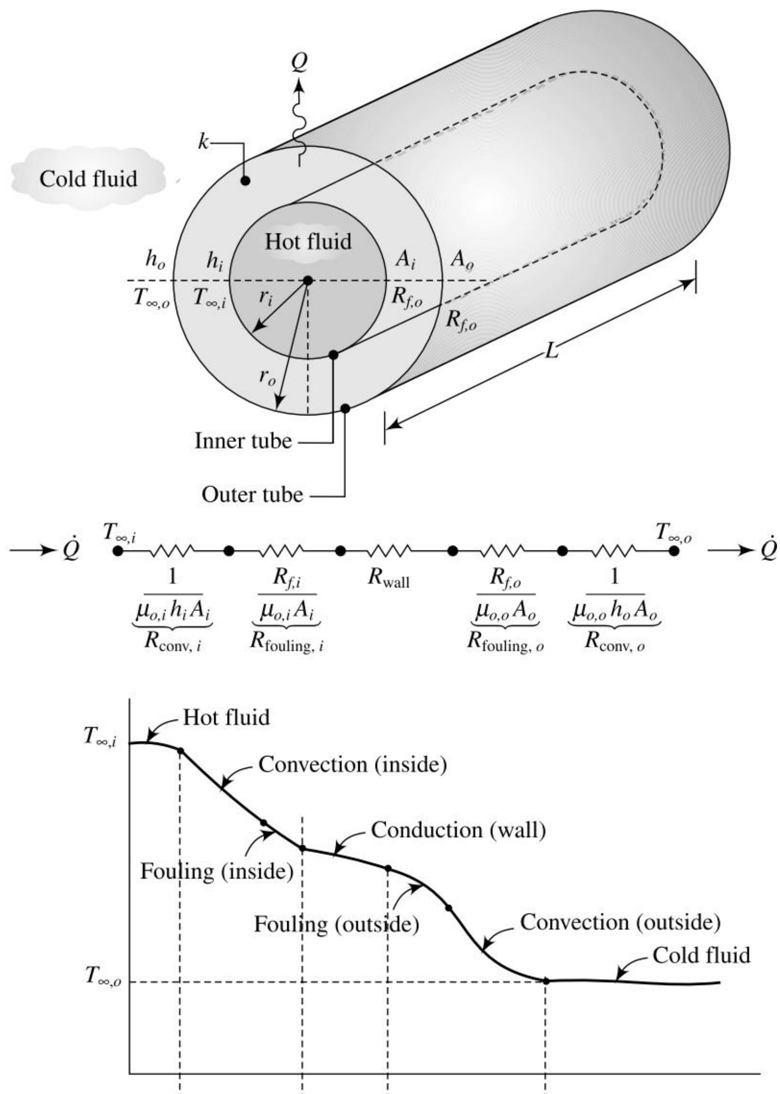


Fig. 11.24 *Thermal circuit and temperature profile associated with heat transfer in a concentric tube heat exchanger.*

The following parameters affect the performance of any heat exchanger:

$\dot{m}_h$ = Mass-flow rate of the hot fluid (kg/s)
C_{ph} = Specific heat capacity of the hot fluid (J/kg K)
$\dot{m}_c$ = Mass-flow rate of the cold fluid (kg/s)
C_{pc} = Specific heat capacity of the cold fluid (J/kg K)
T_{hi} = Hot-fluid inlet temperature (°C)
T_{he} = Hot-fluid exit (outlet) temperature (°C)
T_{ci} = Cold-fluid inlet temperature (°C)
T_{ce} = Cold-fluid exit temperature (°C)
k = Thermal conductivity of the separating wall material (W/m K)
R_{fi} = Fouling resistance (*factor*) due to scaling and deposits on the inside of the separating wall (m^2 K/W).
R_{fo} = Fouling resistance (*factor*) due to scaling and deposits on the outside of the separating wall (m^2 K/W).
h_i = Convective heat-transfer coefficient on the inner surface of the separating wall (m^2 K/W)
h_o = Convective heat-transfer coefficient on the outer surface of the separating wall (m^2 K/W)
$\eta_{o,i}$ = Overall fin efficiency on the inside surface
$\eta_{o,o}$ = Overall fin efficiency on the outside surface

Evaluation of the overall heat-transfer coefficient is central to thermal design. Wall *resistances* are known from the material's thermal conductivity and size. Fouling *resistances* are assumed based on experience. The inside and outside heat-transfer coefficients are obtained from the appropriate correlations after obtaining the relevant data.

Representing the heat-transfer rates in terms of thermal resistances, we have

$$\boxed{\dot{Q} = \frac{\Delta T_{\text{overall}}}{R_{\text{total}}}}$$

where $$R_{\text{total}} = R_{\text{conv},i} + R_{\text{fouling},i} + R_{\text{wall}} + R_{\text{fouling},o} + R_{\text{conv},o}$$

Heat-transfer rate is

$\dot{Q} = UA(\Delta T_m)$, where U is *the overall heat-transfer coefficient.*

and $R_{\text{total}} = \Sigma R = \dfrac{1}{UA}$, U being based on the average heat-transfer coefficient on either side and the mean temperature difference between the streams. A refers to the surface area of the side on which it is based.

Now, $$\boxed{\frac{1}{UA} = \frac{1}{(hA)_o} + \frac{R_{fo}}{A_o} + \frac{\ln(D_o/D_i)}{2\pi k_{\text{wall}} L} + \frac{R_{fi}}{A_i} + \frac{1}{(hA)_i}} \tag{11.1}$$

so that the overall heat-transfer coefficient based on the *hot side* (*say outside tube*) is

$$\boxed{\frac{1}{U_o} = \frac{1}{h_o} + R_{fo} + \frac{D_o \ln(D_o/D_i)}{2k_{\text{wall}}} + \left(R_{fi} + \frac{1}{h_i}\right)\frac{D_o}{D_i}} \tag{11.2}$$

If the fins are used then the resulting surface area has to be considered by including *area- weighted-fin* efficiency, or *overall fin efficiency*, η_o.

$$\boxed{R_{\text{total}} = \frac{1}{\eta_{o,i}h_i A_i} + \frac{R_{fi}}{\eta_{o,i}A_i} + R_{\text{wall}} + \frac{R_{fo}}{\eta_{o,o}A_o} + \frac{1}{\eta_{o,o}h_o A_o}} \tag{11.3}$$

For *finned* surfaces, the overall fin efficiencies $\eta_{o,o}$ and $\eta_{o,i}$ for fins on the outer and inner surfaces, respectively are to be used. When no fins are present, these efficiencies become unity.

An *overall heat-transfer coefficient*, *U*, is then defined through the relation

$$\boxed{\frac{1}{UA} = \left[\frac{1}{\eta_{o,i}h_i A_i} + \frac{R_{fi}}{\eta_{o,i}A_i} + R_{\text{wall}} + \frac{R_{fo}}{\eta_{o,o}A_o} + \frac{1}{\eta_{o,o}h_o A_o}\right]} \tag{11.4}$$

The area used in the *UA* product can be either the inside area, A_i, or the outside area, A_o, of the tubes or channels in a heat exchanger. However, the product $UA = U_i A_i = U_o A_o$ is a constant because a heat exchanger has only one total thermal resistance:

$$\boxed{R_{\text{total}} = \frac{1}{UA} = \frac{1}{U_i A_i} = \frac{1}{U_o A_o}} \tag{11.5}$$

If the wall resistance is negligible (*due to higher thermal conductivity of wall material or due to small wall thickness*) fouling in negligible and fins are not present on either side, then with $A_i \approx A_o$,

$$\boxed{U = \left[\frac{1}{h_i} + \frac{1}{h_o}\right]^{-1} = \frac{h_i h_o}{h_i + h_o}} \tag{11.6}$$

The value of *U* will change, depending on whether the inside area is used to define it (U_i) or the outside area is used (U_o). For example, consider a circular tube with fins on its outside, the outside heat transfer area is significantly different from that on the inside of the tube. Hence, U_i will be different from U_o. The magnitude of *U* can be determined by evaluating each term in **Eq. (11.3)** using information about the fluids, the flow rates, and the geometry of a specified heat exchanger. The convective resistances, $R_{\text{conv}} = 1/\eta_o hA$, are evaluated using appropriate correlations for the convective heat-transfer coefficients, *h*. The overall surface efficiency, η_o is given by

$$\boxed{\eta_o = 1 - \frac{A_{\text{fin}}}{A_{\text{tot}}}(1 - \eta_{\text{fin}})}$$

where A_{fin} is the surface area of the fins and η_{fin} is the fin efficiency. A_{tot} is the total heat-transfer surface area including the fins and bare (*exposed*) surface.

η_o, is unity if the surface has *no* fins or is evaluated with information about the efficiency of fins/extended surfaces. Alternatively, one can calculate, $R_{\text{conv}} = \dfrac{1}{[h\eta_{\text{fin}} A_{\text{fin}} + hA_{\text{unfin}}]}$.

The wall resistance, R_w, is due to steady one-dimensional *conduction* through the solid wall.

$$R_{\text{wall}} = \frac{\ln(r_o/r_i)}{2\pi kL} \quad \text{(circular tube)} \tag{11.7}$$

$$R_{\text{wall}} = \frac{L}{kA} \quad \text{(plane wall)} \tag{11.8}$$

One must note that in the *wall* thermal resistance, it is the *wall* thermal conductivity (not the *fluid* thermal conductivity) that is to be used.

11.4.1 • Fouling Factors

It is a well-known fact that the inner surfaces of the tubes of a heat exchanger cannot remain clean after many months of operation. When a heat exchanger has been in operation over a long period, deposits are likely to accumulate on the tube surfaces in contact with the hot and cold fluids or there could be deterioration of surfaces themselves due to corrosion. These effects, over a period of time, adversely affect the exchanger performance and the heat-transfer surface so affected is said to be *fouled*. In an extreme situation, with the passage of time, there may even be clogging of flow passages. Typical fouling materials are *mineral deposits, corrosion products, dirt, biological growth, deposits caused by chemical reactions in the fluid,* and *sedimentation*.

Usually, fouling is classified into *six* categories:

- *Precipitation fouling*, or scaling when the solution of dissolved materials is crystallized on the heat transfer surface.
- *Particulate fouling*, when the finely divided solids suspended in the fluid are collected on the heat transfer surface.
- *Chemical reaction fouling*, when the deposits due to chemical reaction are formed on the heat transfer surface.
- *Corrosion fouling*, when the corrosion products are deposited on the heat-transfer surface.
- *Biological fouling*, when the micro-organisms get attached to a heat-transfer surface.
- *Solidification fouling*, when pure liquid or a component from the liquid phase is crystallized on a sub-cooled heat transfer surface.

If the fouling thickness becomes too high, it can result in

- Increased fluid flow resistance (*pressure drop*)
- Oversized or redundant equipment to accommodate the increased thermal resistance resulting in higher capital cost
- Use of special materials of construction to minimize the effects of fouling
- Increased cleaning requirements resulting in increased downtime and loss of production
- Heat losses due to thermal inefficiencies
- Costs incurred due to periodic cleaning of heat exchangers

The effects of fouling are usually expressed in terms of a *fouling resistance* or *fouling factor*, R_f, which has units of m^2 K/W. Fouling has the effect of adding a resistance of the order of 10^{-4} m^2 K/W in series. The degree of fouling depends on the type of fluid—*specific type of fluid heat exchanger materials, operating surface temperatures, fluid flow velocities* and the length of service *age*.

It is noteworthy that if U is much less than 10 000 W/m^2 K fouling may be insignificant as it will introduce only small resistances in series. Clearly, in a *water-to-water* heat exchanger, for instance, with U of the order of 2000, fouling could be important; but in a *finned-tube gas-to-gas* heat exchanger in which U may be around 200, fouling will be rather irrelevant. Hence, fouling is crucial or otherwise depending on the order of magnitude of U.

11.4.2 • Comparison of Clean and Fouled Heat Exchangers

With a *new* or *clean* surface,

$$\dot{Q}_{\text{clean}} = U_{\text{clean}}\, A_o(\text{LMTD}) = \frac{\text{LMTD}}{R_{\text{total, new}}}$$

Similarly, for a *fouled dirty* surface,

$$\dot{Q}_{\text{dirty}} = U_{\text{dirty}} A_o (\text{LMTD}) = \frac{\text{LMTD}}{R_{\text{total, fouled}}}$$

Now, $$\boxed{R_{\text{total, new}} = \frac{1}{U_{\text{clean}} A} = \left[\frac{1}{h_i A_i} + \frac{1}{2\pi k L} \ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_o A_o} \right]} \tag{11.9}$$

The overall heat-transfer coefficient for a *clean* heat-transfer surface (U_{clean}) is

$$\boxed{U_{\text{clean}} = \left[\frac{r_o}{r_i} \frac{1}{h_i} + \frac{r_o}{k} \ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_o} \right]^{-1}} \tag{11.10}$$

The expression of U for a *fouled* heat-transfer surface that incorporates fouling factors (U_{dirty}) is

$$\boxed{U_{\text{dirty}} = \left[\frac{r_o}{r_i} \frac{1}{h_i} + \frac{r_o}{r_i} R_{fi} + \frac{r_o}{k} \ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_o} + R_{fo} \right]^{-1}} \tag{11.11}$$

with $$\boxed{R_{\text{total, fouled}} = \frac{1}{U_{\text{dirty}} A} = \left[\frac{1}{h_i A_i} + \frac{R_{fi}}{A_i} + \frac{1}{2\pi k L} \ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_o A_o} + \frac{R_{fo}}{A_o} \right]} \tag{11.12}$$

When the tubes are new, one can compute the value of U_{clean} from the convective heat-transfer coefficients on their *inside* and *outside* surfaces, and the thermal resistance due to conduction through the *tube wall*. The value of U_{clean} will be reduced to U_{dirty} after the surfaces of the tubes are covered with deposits of dirt, ash, soot, scale, or other substances over a period of operation. Then

$$\boxed{\frac{1}{U_{\text{dirty}}} = \frac{1}{U_{\text{clean}}} + R_{fi}\left(\frac{r_o}{r_i}\right) + R_{fo}} \quad \text{or} \quad \boxed{\frac{1}{U_{\text{dirty}}} = \frac{1}{U_{\text{clean}}} + R_{\text{fouling}}} \tag{11.13}$$

For a *finned* heat-transfer surface, fouling thermal resistance, $R_{\text{fouling}} = R_f/\eta_o A$.

11.4.3 • Dominant Resistance

It is possible that one of the resistances is significantly larger than the others. That resistance is called the *dominant* or *controlling* resistance. This might happen with a gas flow (*low heat transfer coefficient*) on one side of a *thin-walled* heat exchanger and boiling (*high heat-transfer coefficient*) on the other side. If the areas on the two sides are comparable with negligible wall conduction resistance, then the gas-side thermal resistance would dominate the total thermal resistance. In the heat exchanger design, a better estimate of the exchanger area can be obtained if more effort is directed to quantify the dominant resistance rather than concentrating on a more accurate estimate of a minor resistance.

Consider a horizontal steel pipe carrying hot water and exposed to atmospheric air. The overall heat-transfer coefficient U can be estimated from the expression $U = \left[\frac{1}{h_i} + \frac{1}{h_o}\right]^{-1}$, neglecting pipe-wall resistance and assuming thin-walled pipe. Typically, in such a case h_i (*water-side*) = 2000 W/m^2 K and h_o (*air-side*) = 10 W/m^2 K, with U = 9.95 W/m^2 K. Take another case of a steam condenser in which the water-side inside heat-transfer coefficient is 3000 W/m^2 K and the outside condensation heat-transfer coefficient is 15000 W/m^2 K. It follows that the value of U will be 2500 W/m^2 K which is less than the lesser of the two values of individual heat transfer coefficients. (Clearly, U is almost completely controlled by the lower value of the heat-transfer coefficient associated with higher thermal resistance)

11.5 ❑ ANALYSIS OF HEAT EXCHANGERS

In the analysis of a heat exchanger involving heat exchange between two fluids separated by a solid wall, when the temperatures of the hot and cold fluids are held constant, the rate of heat transfer is

$$\boxed{\dot{Q} = \frac{T_{\text{hot}} - T_{\text{cold}}}{R_{\text{total}}}} \tag{11.14}$$

Usually, however, the fluid temperatures are not constant along the separating wall. The analysis must therefore take into account the varying temperature difference between the two fluids along the heat exchanger.

Consider the heat transfer across a differential element of a heat exchanger of area dA with temperature drop dT_h and temperature rise dT_c for the hot and cold fluids respectively.
From energy balance, we have

$$d\dot{Q} = -(\dot{m}C_p)_h \, dT_h \tag{11.15}$$

$$= +(\dot{m}C_p)_c \, dT_c \tag{11.16}$$

$$= U dA \; (\Delta T) \tag{11.17}$$

where subscripts

h = hot, c = cold

$\dot{m}$ = mass flow rate, C_p ¼ specific heat capacity

U = local overall heat-transfer coefficient

Integrating between appropriate temperature limits, one has

$$\dot{Q} = (\dot{m}C_p)_h (T_{hi} - T_{he}) = C_h (T_{hi} - T_{he}) \tag{11.18}$$

$$\dot{Q} = (\dot{m}C_p)_c (T_{ce} - T_{ci}) = C_c (T_{ce} - T_{ci}) \tag{11.19}$$

$$\dot{Q} = UA\Delta T_m \tag{11.20}$$

where subscripts i and e denote inlet and exit respectively where C_h and C_c are the heat capacity rates of the hot and cold fluid, respectively, ΔT_m is the appropriate effective mean temperature difference
The above equations can now be represented in the functional form of dependent and independent variables as

$$\boxed{\dot{Q}, T_{he}, T_{ce} \; = \text{function } (C_h, C_c, T_{hi}, T_{ci}, U, A, \text{flow arrangement})} \tag{11.21}$$

Any two of the dependent variables will determine the remaining ones. The first *four* of the *independent* variables are the operating conditions and the remaining *three* parameters are the ones under designer's control.

In the case of condensers and evaporators, involving phase change the heat transfer rate will be given by $\dot{Q} = \dot{m} h_{fg}$.
Energy balance is crucial in the heat-exchanger analysis.

• Sizing and Rating Analyses

The analysis of heat exchangers generally involves the use of energy balance, the heat transfer rate equation, and the evaluation of the total thermal resistance between the two fluid streams.

The objective of the analysis is either (1) to develop the information needed to build a new heat exchanger that will transfer a given heat transfer rate (*a design or sizing problem*), or (2) to determine the heat transfer rate (or *heat duty*) possible with a given heat exchanger (a *rating* problem).

Sizing problem [Fig. 11.25(a)] deals with determining the surface area (*and geometric configuration*) for the specified service, i.e., heating/cooling load within the constraints of pressure loss, etc. In this case, the type of heat exchanger is known, along with the two fluids and their flow rates. In addition, the inlet temperatures of the two fluids and the required heat duty are known. (This is equivalent to specifying the exit temperatures of the two fluids.) The task is to determine the heat exchanger *size* or area needed to provide a specified heat-transfer rate.

Rating problem [Fig. 11.25(b)] deals with determining the performance (*exit temperatures, heat transfer rates, pressure loss*) for specified inlet conditions of a specific design. For a *rating problem*, geometry (*number, size, spacing,* and *layout of tubes, fin geometry, shell geometry, etc.*) and heat exchanger type (*shell*-and-*tube*, *plate-fin*, etc.) are known, and the two fluids, the flow rates, and the inlet temperatures are given. The task is to determine the overall heat-transfer rate. (This is equivalent to determining the exit temperatures of the two fluids.).

Two analyses of heat transfer in a heat exchanger are commonly used: the *Log Mean Temperature Difference* (LMTD) method and the *effectiveness-NTU* (ε-NTU) method. While either approach can be used for either a *sizing* or a *rating*-type problem, typically the LMTD method is used for *sizing* problems and the ε-NTU approach is used for *rating*. In both methods of analysis, *the overall heat transfer coefficient is required.*

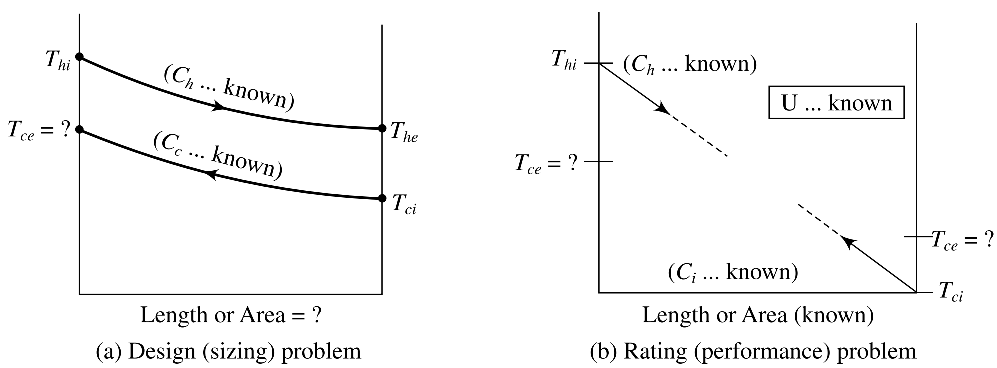


Fig. 11.25 *Sizing and Rating analyses*

11.6 ❑ THE LMTD METHOD

In this section, we develop a relationship between the effective mean temperature difference and the heat-exchanger configuration and operating conditions.

11.6.1 • The Parallel-Flow Heat Exchanger

The hot and cold fluid temperature profiles are shown in Fig. 11.26. The temperature difference $\Delta T \equiv (T_h - T_c)$ is initially large but decreases rapidly with increasing x, approaching zero asymptotically. For

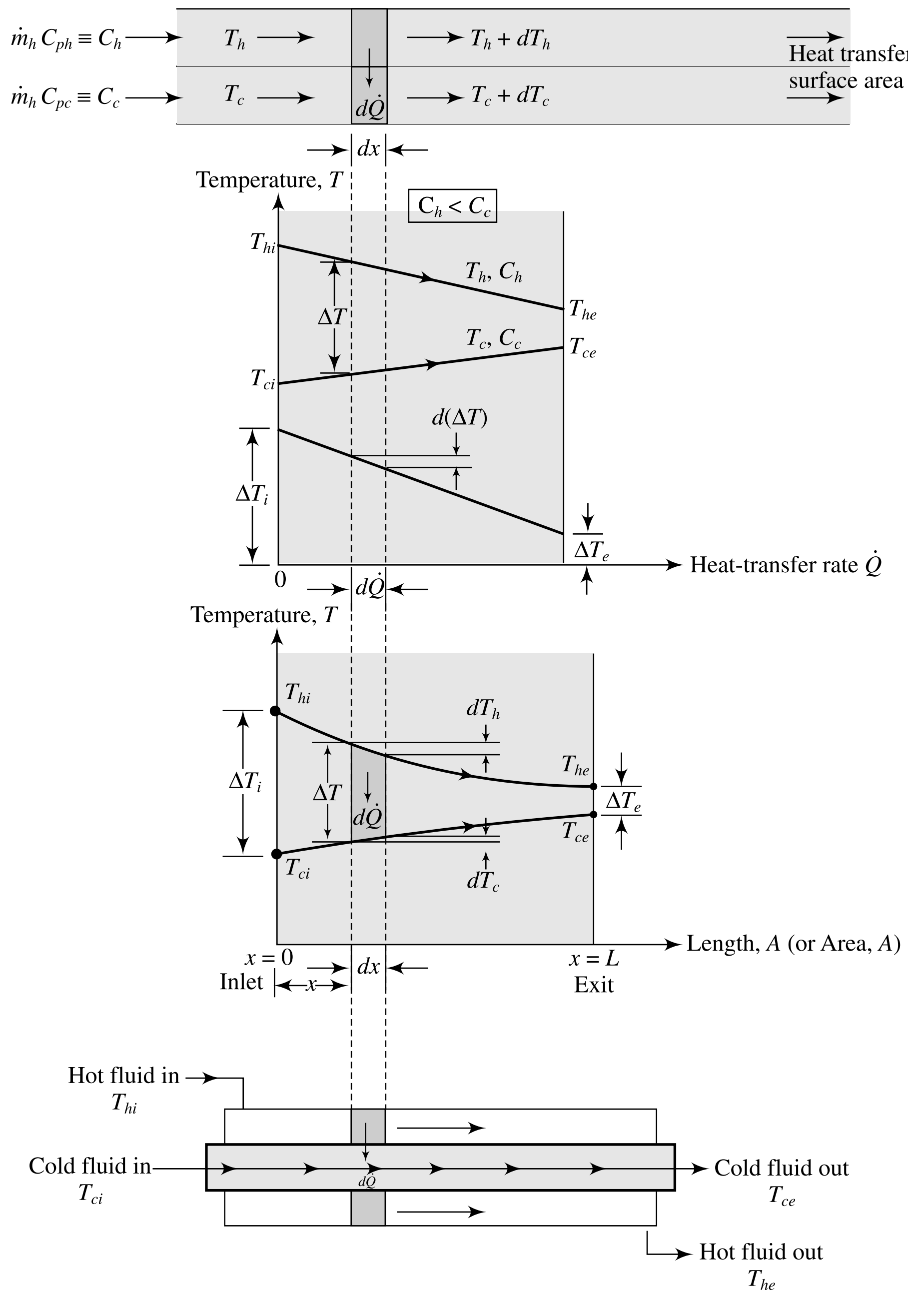


Fig. 11.26 *Temperature distribution in a counter-flow heat exchanger*

such an exchanger, the exit temperature of cold fluid can never exceed that of the hot fluid. The subscripts i and e designate inlet and exit ends of the heat exchanger.

We apply an energy balance to differential elements in the hot and cold fluids. Each element is of length dx and heat-transfer surface area dA. The main assumptions involved in the analysis are the following:

1. Steady operating conditions exist.
2. The heat exchanger is insulated from its surroundings. The only heat exchange is between the hot and cold fluids.
3. Axial conduction along the tubes can be ignored.
4. Changes in potential and kinetic energy are negligible.
5. The specific heats of the two fluids are constant.
6. The overall heat transfer coefficient is constant and uniform.

Let us remember that the specific heats do change with temperature variations, and the overall heat-transfer coefficient may also change because of variations in fluid properties and flow conditions. Fortunately, in most of the applications such variations are insignificant, and average values of C_{pc}, C_{ph} and U for the heat exchanger are mostly acceptable.

Applying an energy balance to each of the differential elements, one gets

$$d\dot{Q} = -\dot{m}_h C_{ph} dT_h \equiv -C_h dT_h \quad \text{and} \quad = C_h(T_{hi} - T_{he}) d\dot{Q} = \dot{m}_c C_{pc} dT_c \equiv -C_c dT_c$$

where C_h and C_c are the hot and cold fluid heat-capacity rates, respectively. One can integrate these expressions across the heat exchanger to get the overall energy balance.

$$\dot{Q} = \dot{m}_h C_{ph}(T_{hi} - T_{he}) = C_h(T_{hi} - T_{he}) \quad \text{and} \quad \dot{Q} = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) = C_c(T_{ce} - T_{ci})$$

The heat transfer across the surface area dA can also be expressed as

$$d\dot{Q} = U \Delta T \, dA$$

where $\Delta T = (T_h - T_c)$ is the *local* temperature difference between the hot and cold fluids.

But $$dT_h = \frac{-d\dot{Q}}{C_h} \text{ and } dT_c = \frac{-d\dot{Q}}{C_c}$$

Now $$d(\Delta T) = dT_h - dT_c$$

$\therefore$ $$d(\Delta T) = -d\dot{Q}\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

Substituting for $d\dot{Q} \equiv U dA \Delta T$ and integrating across the heat exchanger, one has

$$\int_i^e \frac{d(\Delta T)}{\Delta T} = U\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\int_o^A dA \quad \text{or} \quad \ln\left(\frac{\Delta T_e}{\Delta T_i}\right) = -UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

Substituting for C_h and C_c equal to, $\dot{Q}/(T_{hi} - T_{he})$ and $\dot{Q}/(T_{ce} - T_{ci})$, respectively it follows that

$$\ln\left(\frac{\Delta T_e}{\Delta T_i}\right) = -UA\left(\frac{T_{hi} - T_{he}}{\dot{Q}} + \frac{T_{ce} - T_{ci}}{\dot{Q}}\right) = -\frac{UA}{\dot{Q}}[(T_{hi} - T_{ci}) - (T_{he} - T_{ce})]$$

For the parallel-flow heat exchanger, we note that, $\Delta T_i = (T_{hi} - T_{ci})$ and $\Delta T_e = (T_{he} - T_{ce})$

Then

$$(T_{hi} - T_{ci}) - (T_{he} - T_{ce}) = \Delta T_i - \Delta T_e$$

and $$\dot{Q} = UA\frac{\Delta T_e - \Delta T_i}{\ln(\Delta T_e/\Delta T_i)} \quad \text{or} \quad \dot{Q} = UA\frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}$$

Thus, the appropriate mean temperature difference is a *log mean temperature difference*, ΔT_{lm} or LMTD. Accordingly,

$$\dot{Q} = UA\Delta T_{lm} \quad \text{or} \quad UA \text{ (LMTD)}$$

where
$$\boxed{\text{LMTD or } \Delta T_{lm} = \frac{\Delta T_e - \Delta T_i}{\ln(\Delta T_e/\Delta T_i)} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}} \tag{11.22}$$

11.6.2 • The Counterflow Heat Exchanger

Shown in Fig. 11.27 are a counterflow heat exchanger and the hot and cold fluid temperatures. The appropriate temperature difference to use is obtained by applying conservation of energy and the heat-transfer rate equation to the differential segment shown in the figure. The positive x-direction is from the left end to the right end of the heat exchanger. The main assumptions involved in the analysis are the following:

1. Steady-state conditions prevail.
2. Constant specific heats if the flow is single phase. If there is a phase change (*boiling or condensation*), it occurs at a constant temperature (*constant pressure*).
3. Constant overall heat-transfer coefficient over the complete heat exchanger.
4. If there are multiple tubes, each tube has the same mass-flow rate. Also, the flow outside the tubes is evenly distributed across the heat exchanger.
5. Temperatures and velocities are uniform over all cross-sectional flow areas.
6. The two fluids exchange heat only with each other, and there is no shaft work or heat generation.
7. Potential and kinetic energy effects are ignored.
8. Axial conduction along the solid surfaces is ignored.
9. Heat conduction in the longitudinal direction of the walls is negligible.
10. No heat loss from the heat exchanger.

At the differential element in the heat exchanger, the heat-transfer rate can be expressed in terms of the overall heat-transfer coefficient U as

$$d\dot{Q} = U(T_h - T_c)dA \tag{11.23}$$

where $d\dot{Q}$ is the differential heat-transfer rate in the differential area dA, which stretches from x to (x + dx). The temperature difference (T_h – T_c) is the *local* temperature difference, which varies all along the heat exchanger, and U is the overall heat-transfer coefficient.

Conservation of energy applied to the hot fluid gives

$$\dot{Q} = \dot{m}_h C_{ph}(T_{hi} - T_{he}) = C_h(T_{hi} - T_{he}) \tag{11.24}$$

where we define $C \equiv \dot{m}C_p$, the heat-capacity rate. Similarly, for the cold fluid,

$$\dot{Q} = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) = C_c(T_{ce} - T_{ci}) \tag{11.25}$$

Both the hot and cold fluid temperatures *decrease* in the *positive* x-direction. Thus, the differential heat transfer rates between x and (x + dx) are

$$d\dot{Q} = \dot{m}_h C_{ph}(-dT_h) = -C_h dT_h \tag{11.26}$$

$$d\dot{Q} = \dot{m}_c C_{pc}(-dT_c) = -C_c dT_c \tag{11.27}$$

$$\therefore \qquad d\dot{Q} = -C_h dT_h = -C_c dT_c \tag{11.28}$$

We note that
$$d(T_h - T_c) = dT_h - dT_c \tag{11.29}$$

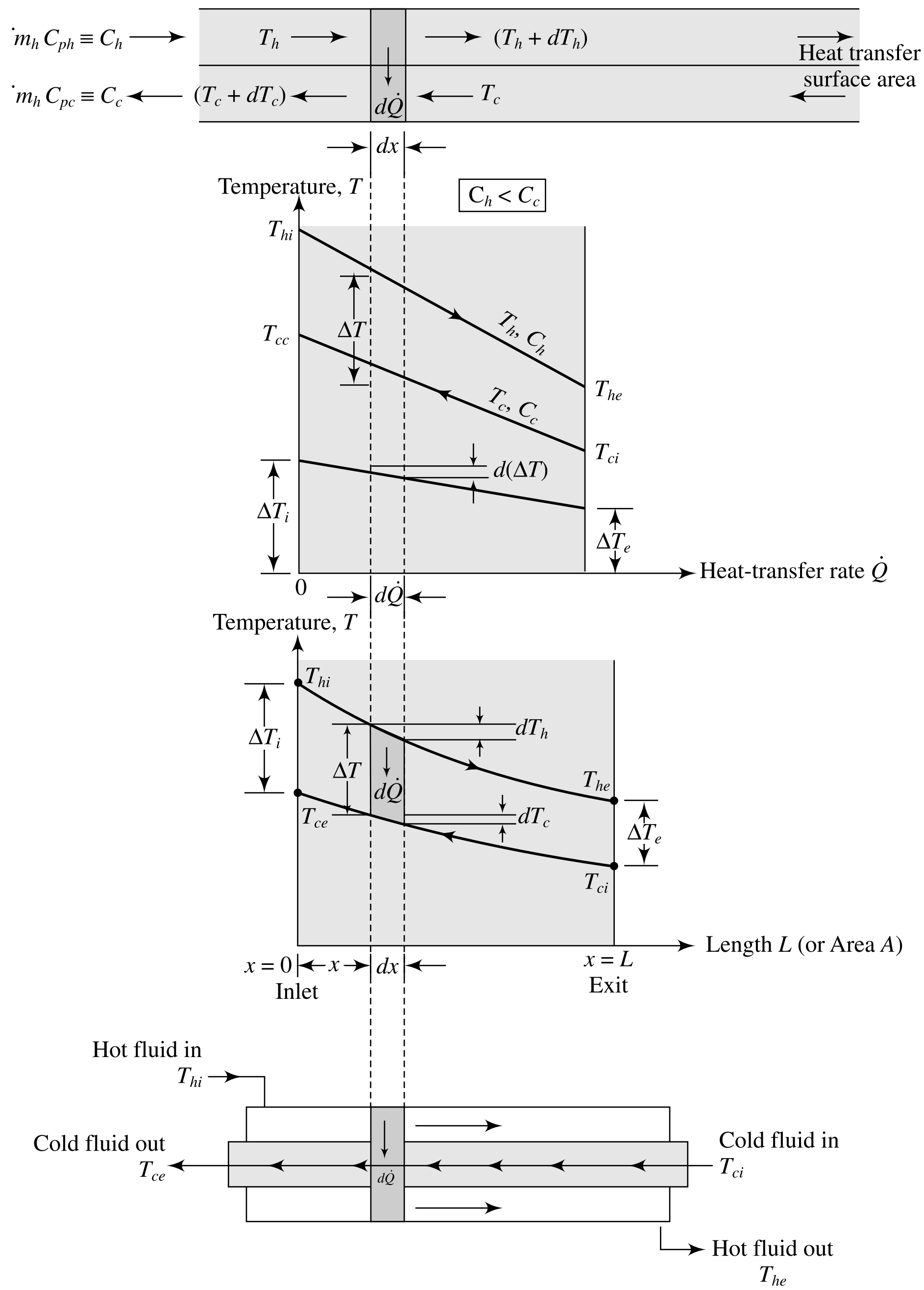


Fig. 11.27 *Temperature distribution in a counter-flow heat exchanger*

$$dT_h = \frac{-d\dot{Q}}{C_c} \quad \text{and} \quad dT_c = \frac{-d\dot{Q}}{C_c}$$

It follows that

$$d(T_h - T_c) = \frac{-d\dot{Q}}{C_h} + \frac{d\dot{Q}}{C_c} = -d\dot{Q}\left(\frac{1}{C_h} - \frac{1}{C_c}\right) \tag{11.30}$$

Substituting for $d\dot{Q}$ and rearranging, we get

$$\frac{d(T_h - T_c)}{(T_h - T_c)} = -U\left[\frac{1}{C_h} + \frac{1}{C_c}\right]dA$$

Integrating over the heat exchanger length from end i to end e gives

$$\int_i^e \frac{d(T_h - T_c)}{T_h - T_c} = -\left(\frac{1}{C_h} - \frac{1}{C_c}\right)U\int_0^A dA \tag{11.31}$$

or
$$\ln\left(\frac{\Delta T_e}{\Delta T_i}\right) = -\left(\frac{1}{C_h} - \frac{1}{C_c}\right)UA \tag{11.32}$$

where for this counterflow heat exchanger,

$$\boxed{\begin{array}{ll} \Delta T_i \equiv T_{hi} - T_{ce} & \\ & \text{counterflow} \\ \Delta T_e \equiv T_{he} - T_{ci} & \end{array}} \tag{11.33}$$

We note that

$$\frac{1}{C_h} = (T_{hi} - T_{he})/\dot{Q} \text{ and } \frac{1}{C_c} = (T_{ce} - T_{ci})/\dot{Q}$$

Substituting these expressions, we have

$$\ln\left(\frac{\Delta T_e}{\Delta T_i}\right) = \ln\left(\frac{T_{he} - T_{ci}}{T_{hi} - T_{ce}}\right) = -UA\left[\left(\frac{T_{hi} - T_{he}}{\dot{Q}}\right) - \left(\frac{T_{ce} - T_{ci}}{\dot{Q}}\right)\right] \tag{11.34}$$

or
$$\ln\left(\frac{\Delta T_i}{\Delta T_e}\right) = \frac{UA}{\dot{Q}}[(T_{hi} - T_{ce}) - (T_{he} - T_{ci})]$$

Rearranging, we obtain

$$\dot{Q} = UA\frac{\Delta T_i - \Delta T_e}{\ln\left(\dfrac{\Delta T_i}{\Delta T_e}\right)} = UA\Delta T_{lm} \tag{11.35}$$

where the quantity ΔT_{lm} is called the *log mean temperature difference,* abbreviated *or* LMTD and is defined as

$$\boxed{\Delta T_{lm} = \frac{\Delta T_i - \Delta T_e}{\ln\left(\dfrac{\Delta T_i}{\Delta T_e}\right)} = \frac{\Delta T_e - \Delta T_i}{\ln\left(\dfrac{\Delta T_e}{\Delta T_i}\right)}} \tag{11.36}$$

This expression for LMTD is identical to that obtained for *parallel-flow* arrangement. The only difference is in the evaluation of the temperature difference at the two ends of the heat exchanger. For the *parallel-flow* heat exchanger, the end temperature differences are:

$$\boxed{\begin{array}{ll}\Delta T_i \equiv T_{hi} - T_{ci} & \text{parallel flow} \\ \Delta T_e \equiv T_{he} - T_{ce} & \end{array}} \tag{11.37}$$

In parallel-flow arrangement, ΔT_i is always greater than ΔT_e. Hence, $\Delta T_i \equiv \Delta T_{\max}$ and $\Delta T_e \equiv \Delta T_{\min}$. However, it is possible that in a counterflow exchanger ΔT_i could be *less* than ΔT_e depending on the flow rates and heat capacities of the fluids. Equation (11.35) will still give a *positive* value for ΔT_{lm} since $(\Delta T_i - \Delta T_e)$ is *negative* and $\ln (\Delta T_i / \Delta T_e)$ will also be *negative*.

The average arithmetic temperature difference, ΔT_{am} is always larger than the LMTD or ΔT_{lm}. If the ratio $\Delta T_i/\Delta T_e \le 2$, ΔT_{am} approaches ΔT_{lm}. If there is slight variation in the temperatures of the two fluids along the length of the heat exchanger, the temperature difference can be calculated as the arithmetic mean of the inlet and exit differences:

$$\boxed{\Delta T_{am} = \frac{1}{2}(\Delta T_i + \Delta T_e) \quad \text{or} \quad = \frac{\Delta T_e}{2}\left(\frac{\Delta T_i}{\Delta T_e} + 1\right)} \tag{11.38}$$

In brief, the comparison of parallel-flow and counterflow heat exchangers is given below:

Parallel-flow	Counterflow
◆ For the given mass-flow rates and temperature changes, the counterflow heat exchanger requires less surface area than its parallel flow counter part. ◆ The temperature $T_{c,\text{ out}}$ can at best be equal to $T_{h,\text{ out}}$ when the exchanger has infinite area. $T_{c,\text{ out}}$ greater than $T_{h,\text{ out}}$ is impossible. ◆ Less quantity of heat is transferred.	◆ In this case with infinite area, one of the fluid streams would leave at the entering temperature of the other. ◆ The heat-transfer potential for a given pair of fluids in a given heat exchanger is greater for counterflow than for parallel flow. ◆ The temperature $T_{c,\text{ out}}$ can be greater than $T_{h,\text{ out}}$. ◆ Greater quantity of heat can be transferred operating under otherwise similar conditions.

11.6.3 • Special Case: Balanced Heat Exchanger

In a balanced counterflow heat exchanger where $\dot{m}_h C_{p_h} = \dot{m}_c C_{p_c}$ (Fig. 11.28), the value of ΔT is the same all along the exchanger length. It can be proved that both T_h and T_c vary linearly with the same slope and that the temperature profiles of the two fluids are *linear* and *parallel*. The necessary proof is as follows:

Heat-transfer rate.

$$\dot{Q} = \dot{m}_h C_{p_h}(T_{hi} - T_{he}) = \dot{m}_c C_{p_c}(T_{ce} - T_{ci})$$

Since $\quad \dot{m}_h C_{p_h} = \dot{m}_c C_{p_c}$,

$$T_{hi} - T_{he} = T_{ce} - T_{ci} \quad \text{or} \quad T_{hi} - T_{ce} = T_{he} - T_{ci}$$

or $\quad \boxed{\Delta T_i = \Delta T_e}$

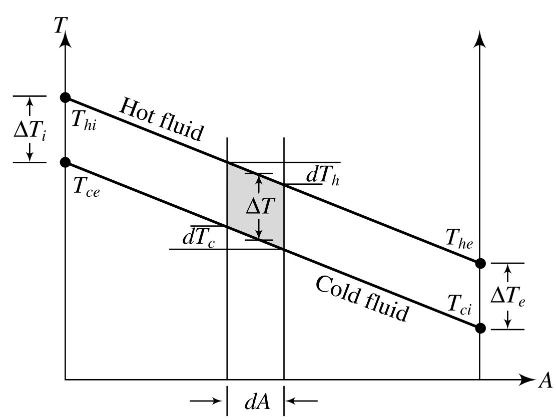


Fig. 11.28 *Temperature variation in a balanced counterflow heat exchanger*

Also, $d\dot{Q} = \dot{m}_h C_{p_h}(-dT_h) = \dot{m}_c C_{p_c}(-dT_c)$

$\therefore$ $dT_h = dT_c$

$$d(\Delta T) = d(T_h - T_c) = dT_h - dT_c = 0$$

$\therefore$ $\Delta T = \text{const}$

Thus, $$\boxed{\Delta T = \Delta T_i = \Delta T_e = \text{constant}} \tag{11.39}$$

The difference between temperatures of hot and cold fluids is constant throughout the length of the heat exchanger. Hence, the two temperature profiles are parallel to each other.

Furthermore, with dA as the differential area of the exchanger and U as the overall heat transfer coefficient.

$$\cancel{d}\dot{Q} = UdA\Delta T = \dot{m}_h C_{p_h}(-dT_h) = \dot{m}_c C_{p_c}(-dT_c)$$

The slopes of the temperature profiles of the two fluids are

$$\frac{d}{dA}(T_h) = \frac{dT_h}{dA} = -\frac{U\Delta T}{\dot{m}C_{p_h}}$$

$$\frac{d}{dA}(T_c) = \frac{dT_c}{dA} = -\frac{U\Delta T}{\dot{m}_c C_{p_c}}$$

Since $\dot{m}_h C_{p_h} = \dot{m}_c C_{p_c}$, $\dfrac{dT_h}{dA} = \dfrac{dT_c}{dA}$

It follows that the two temperature profiles are *linear* and *parallel* with *constant* and *equal* but *negative* slopes.

Since the temperature differences on either end of a counterflow heat exchanger, ΔT_i and ΔT_e, are equal

$\text{LMTD} = \dfrac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)} = \dfrac{0}{0}$, i.e., indeterminate or mathematically absurd

Hence, it is necessary to use L' Hospital's rule to find the effective mean temperature difference.

Let us define $\dfrac{\Delta T_i}{\Delta T_e}$ as x. Then, as $\Delta T_i \to \Delta T_e$, $x \to 1$.

It follows that

$$\text{LMTD} = \lim \Delta T_i = \Delta T_e \frac{\Delta T_e\left[\dfrac{\Delta T_i}{\Delta T_e} - 1\right]}{\ln\dfrac{\Delta T_i}{\Delta T_e}}$$

or $$\lim(x \to 1)\frac{\Delta T_e[x-1]}{\ln x}$$

Differentiating both the numerator and denominator with respect to x, one has

$$\text{LMTD} = \lim_{x\to 1}\frac{\dfrac{d}{dx}[\Delta T_e(x-1)]}{\dfrac{d}{dx}(\ln x)} = \frac{\Delta T_e(1-0)}{(1/x)|_{x=1}} = \Delta T_e$$

$\therefore$ $\boxed{\text{LMTD} = \Delta T_e = \Delta T_i}$

Clearly, the effective ΔT must equal ΔT_i or ΔT_e.

11.6.4 • Special Case: Variable Overall Heat-Transfer Coefficient

In case the overall heat-transfer coefficient, U varies appreciably from one end of the heat exchanger to the other, the usual assumption of representing U by an average, constant value is no longer valid.

In a double-pipe heat exchanger if the overall heat-transfer coefficient is a linear function of the temperature difference, the heat-transfer rate can be determined as follows.

The temperature differences at the two ends are

$$\Delta T_i \equiv T_{hi} - T_{ci} \quad \text{and} \quad \Delta T_e \equiv T_{he} - T_{ce}$$

The overall heat-transfer coefficient varies linearly with the temperature difference, $\Delta T (\equiv T_h - T_c)$.

Then $$U = a + b\Delta T$$

where symbols a and b denote constants.

The heat exchanged in an incremental length of the exchanger (of surface area dA_s) may be expressed in the following three ways:

$$d\dot{Q} = U dA_s \Delta T = (a + b\Delta T) dA_s \Delta T$$

$$d\dot{Q} = -\dot{m}_h C_{ph} dT_h \qquad \text{(negative slope)}$$

$$d\dot{Q} = \dot{m}_c C_{pc} dT_c$$

Solving for dT_h and dT_c yields

$$dT_h = -\frac{1}{\dot{m}_h C_{ph}} d\dot{Q}$$

$$dT_c = \frac{1}{\dot{m}_c C_{pc}} d\dot{Q}$$

Then $$dT_h - dT_c = d(T_h - T_c) = d(\Delta T) = -d\dot{Q}\left[\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_h C_{pc}}\right]$$

$$\frac{d(\Delta T)}{d\dot{Q}} = -\left[\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_h C_{pc}}\right]$$

We note that

$$\dot{Q} = \dot{m}_h C_{ph}(T_{hi} - T_{he}) \;\Rightarrow\; \frac{1}{\dot{m}_h C_{ph}} = \frac{T_{hi} - T_{he}}{\dot{Q}}$$

$$\dot{Q} = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) \;\Rightarrow\; \frac{1}{\dot{m}_c C_{pc}} = \frac{T_{ce} - T_{ci}}{\dot{Q}}$$

It follows that

$$\frac{d(\Delta T)}{(a + b\Delta T) dA_s \Delta T} = -\left[\frac{T_{hi} - T_{he} + T_{ce} - T_{ci}}{\dot{Q}}\right]$$

or $$\frac{d(\Delta T)}{\Delta T(a+b\Delta T)} = -\frac{dA_s}{\dot{Q}}[(T_{hi}-T_{ci})-(T_{he}-T_{ce})] = -\frac{dA_s}{\dot{Q}}(\Delta T_i - \Delta T_e)$$

Integrating between limits, we have

$$\int_{\Delta T_i}^{\Delta T_e} \frac{-d(\Delta T)}{\Delta T(a+b\Delta T)} = \frac{\Delta T_i - \Delta T_e}{\dot{Q}}\int_0^{A_s} dA_s$$

or $$\frac{(\Delta T_i - \Delta T_e)A_s}{\dot{Q}} = \frac{1}{a}\ln\left[\frac{\Delta T}{a+b\Delta T}\right]_{\Delta T_e}^{\Delta T_i}$$

$$= \frac{1}{a}\ln\left[\frac{\Delta T_i}{a+b\Delta T_i} - \frac{\Delta T_e}{a+b\Delta T_e}\right] = \frac{1}{a}\ln\left[\frac{\Delta T_i}{\Delta T_e}\left(\frac{a+b\Delta T_e}{a+b\Delta T_i}\right)\right]$$

or $$\frac{(\Delta T_i - \Delta T_e)A_s}{\dot{Q}} = \frac{1}{a}\ln\left[\frac{\Delta T_i}{\Delta T_e}\frac{U_e}{U_i}\right]$$

Since $U_i = a + b\Delta T_i$ and $U_e = a + b\Delta T_e$, the constant a can be found as follows.

$$U_i - U_e = b(\Delta T_i - \Delta T_e)$$

$\therefore$ $$b = (U_i - U_e)/(\Delta T_i - \Delta T_e)$$

Then $$a = U_i - b\Delta T_i = U_i - \frac{(U_i - U_e)}{(\Delta T_i - \Delta T_e)}\Delta T_i$$

or $$a = \frac{U_i\Delta T_i - U_i\Delta T_e - U_i\Delta T_i + U_e\Delta T_i}{\Delta T_i - \Delta T_e}$$

$\therefore$ $$\frac{1}{a} = \frac{\Delta T_i - \Delta T_e}{U_e\Delta T_i - U_i\Delta T_e}$$

Substituting for $1/a$, we get

$$A_s = \frac{(\Delta T_i - \Delta T_e)}{\dot{Q}} = \frac{(\Delta T_i - \Delta T_e)}{U_e\Delta T_i - U_i\Delta T_e} \times \ln\left[\frac{\Delta T_i U_e}{\Delta T_e U_i}\right]$$

The rate of heat transfer is

$$\boxed{\boldsymbol{\dot{Q} = A_s\left\{\frac{U_e\Delta T_i - U_i\Delta T_e}{\ln\left(\dfrac{U_e\Delta T_i}{U_i\Delta T_e}\right)}\right\}}}$$

It may be noted that this expression holds good even for the counterflow heat exchanger.

11.7 ❑ LMTD CORRECTION FACTOR

So far we have analyzed heat exchangers with simple geometry. For more complex configurations, like multipass and cross-flow heat exchangers the effective mean temperature difference ΔT_m is *not* equal to $(\text{LMTD})_{\text{CF}}$ and is to be modified by a correction factor F.

The counterflow is used as a reference as it provides the best condition for heat transfer. The physical limitations however do not always allow pure counterflow arrangement. The deviation from the counterflow is taken care of in terms of LMTD correction factor F defined as

$$F = \frac{\Delta T_m}{(\Delta T_{lm})_{\mathrm{CF}}} \tag{11.40}$$

so that $$\dot{Q} = \mathrm{UAF}(\Delta T_{lm})_{\mathrm{CF}} \tag{11.41}$$

The correction factor F depends on *three* pieces of information:

$$\boxed{F = \text{function } (P,\ R,\ \text{heat exchanger geometry and flow arrangement})}$$

where P, a relative measure of the tube-side temperature change compared to the inlet temperature difference, is defined by

$$\boxed{P = \frac{T_{\text{tube,exit}} - T_{\text{tube,inlet}}}{T_{\text{shell,inlet}} - T_{\text{tube,inlet}}} \quad \text{or} \quad \frac{T_{ce} - T_{ci}}{T_{hi} - T_{ci}} \quad \text{or} \quad \frac{\text{Temperature gain of the cold fluid}}{\text{Largest temperature difference}}} \tag{11.42}$$

The quantity R is a heat-capacity ratio, expressed as the tube-side fluid heat capacity rate divided by the shell-side heat capacity rate. It is related to the temperatures through the use of conservation of energy:

$$\boxed{R = \frac{C_{\text{tube}}}{C_{\text{shell}}} = \frac{T_{\text{shell,inlet}} - T_{\text{shell,exit}}}{T_{\text{tube,exit}} - T_{\text{tube,inlet}}} \quad \text{or} \quad \frac{T_{hi} - T_{he}}{T_{ce} - T_{ci}} \quad \text{or} \quad \frac{\text{Temperature drop of the hot fluid}}{\text{Temperature gain of the cold fluid}}} \tag{11.43}$$

Thus, for example, the LMTD correction factor for 1-2 (*one-shell pass* and *two-tube pass*) arrangement is given by.

$$\boxed{F = \frac{(R^2+1)^{1/2}\ln\left[\dfrac{1-P}{1-RP}\right]}{(R-1)\ln\left[\dfrac{2-P(R+1-(R^2+1)^{1/2}}{2-P(R+1+(R^2+1)^{1/2}}\right]}} \tag{11.44}$$

Because of the complex nature of the dependence of F on P and R, it is usually represented in the graphical form as a family of curves of F versus P with R as a parameter.
Figure 11.29 is an example of how the LMTD correction is applied in such cases.

Figure 11.30 shows curves of F for some common heat-exchanger configurations. It may be noted that it is irrelevant whether the hot or cold fluid flows through the shell or the tubes.

The log mean temperature difference for counterflow ($\Delta T_{lm,\ \mathrm{CF}}$) is used as the *reference* temperature difference, because a counterflow heat exchanger provides the *greatest* mean temperature difference between the two fluids with specified inlet and outlet temperatures. All other heat exchangers have a ΔT_{mean} smaller than that of a counterflow heat exchanger. Hence, for a given heat duty and overall heat-transfer coefficient, the *counterflow* heat exchanger will require the *smallest* surface area. Note that for a counterflow heat exchanger, F =1. For all other heat exchangers, $F < 1$.

Consider the shape of the curves in Fig. 11.30 when R becomes very small, that is, when $R \to 0$. In this situation for all heat exchanger types, flow arrangements, and values of P, the value of $F \to 1$. This can occur in two situations:

- When there is a phase change (*boiling or condensing*) on one side of a heat exchanger. The enthalpy changes, even though the temperature remains essentially constant. The definition of specific heat is $C_p = \partial h/\partial T]_p$, where h is the enthalpy. Hence, when $C_p \to \infty$ during a constant-pressure phase change, the heat-capacity rate $(\dot{m}C_p)$ tends towards infinity. This causes. $R \equiv C_{\text{tube}}/C_{\text{shell}}\ C_{\min}/C_{\max} \to 0$.

In complicated heat exchangers like the 2-shell-pass, 6-tube-pass exchanger shown:

$\Delta T_{mean} = F(\Delta T_{lm})_{\text{counter flow}}$

F = Correction factor

$$(\Delta T_{lm})_{\text{counter flow}} = \frac{(T_{hi} - T_{ce}) - (T_{he} - T_{ci})}{\ln\left(\dfrac{T_{hi} - T_{ce}}{T_{he} - T_{ci}}\right)}$$

$$\dot{Q} = UA\ (F.\Delta T_{lm,\ CF})$$

$$(F < 1)$$

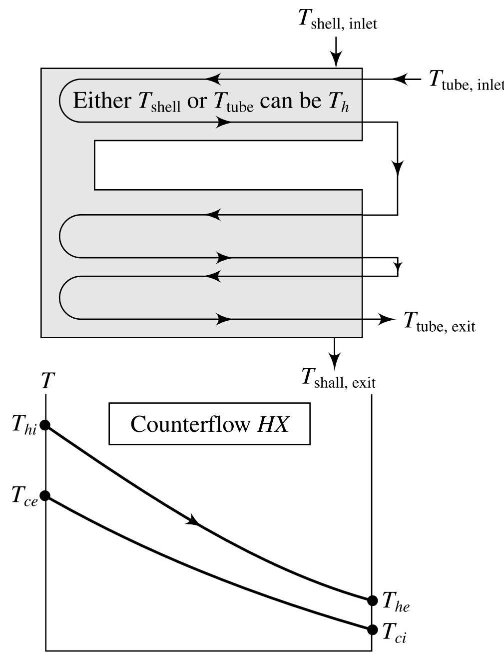
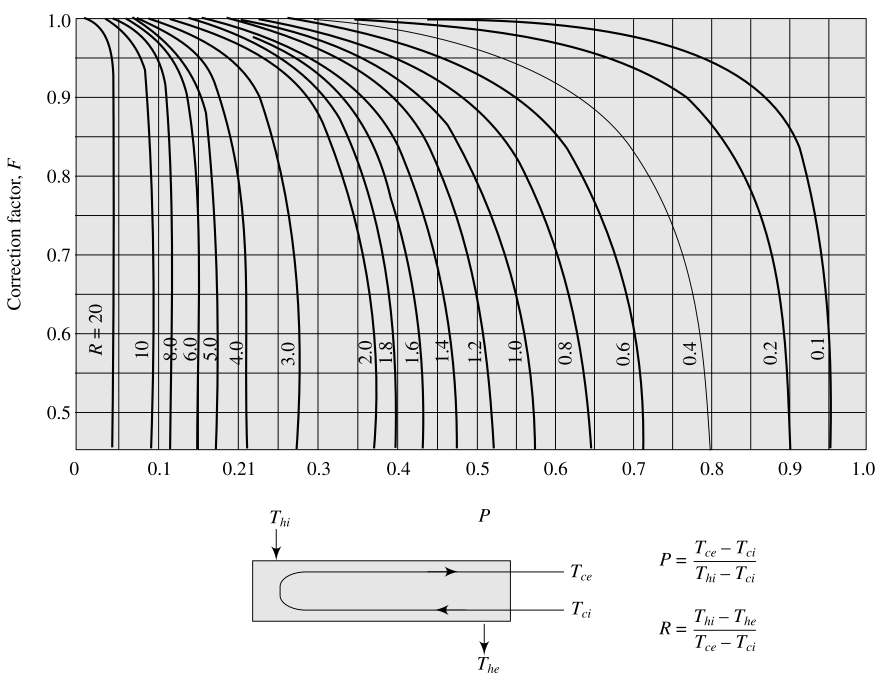
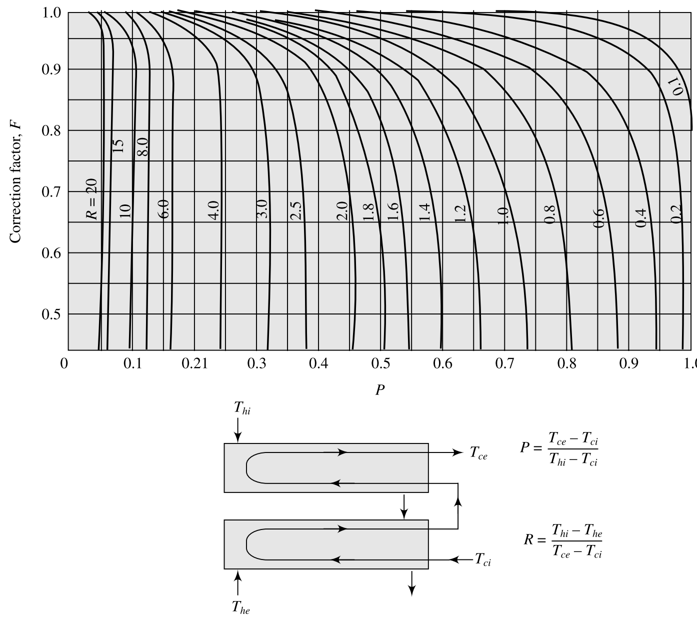


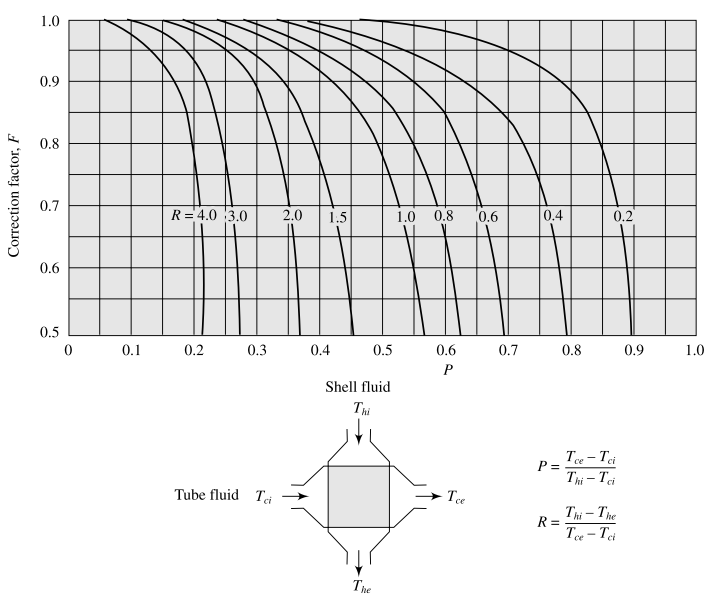
Fig. 11.29 *The basis of LMTD in a multipass exchanger, prior to LMTD correction*



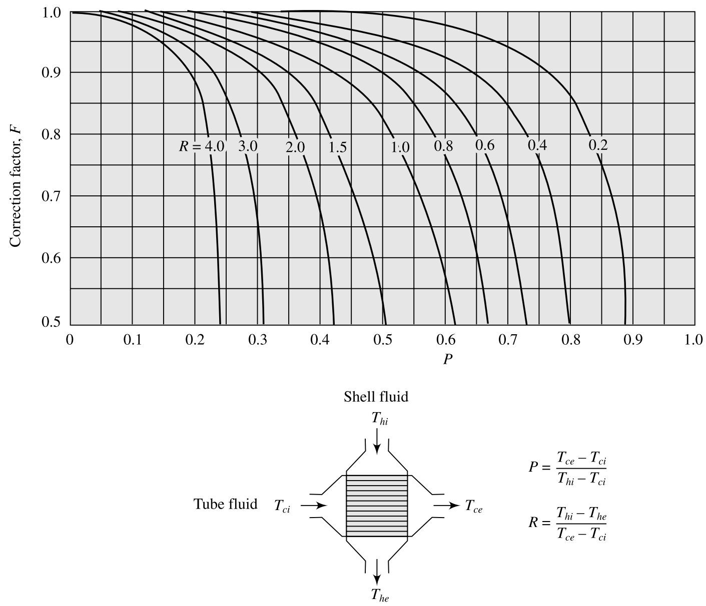
(a) Correction factor F for a one-shelf-pass, two-tube-pass heat exchanger



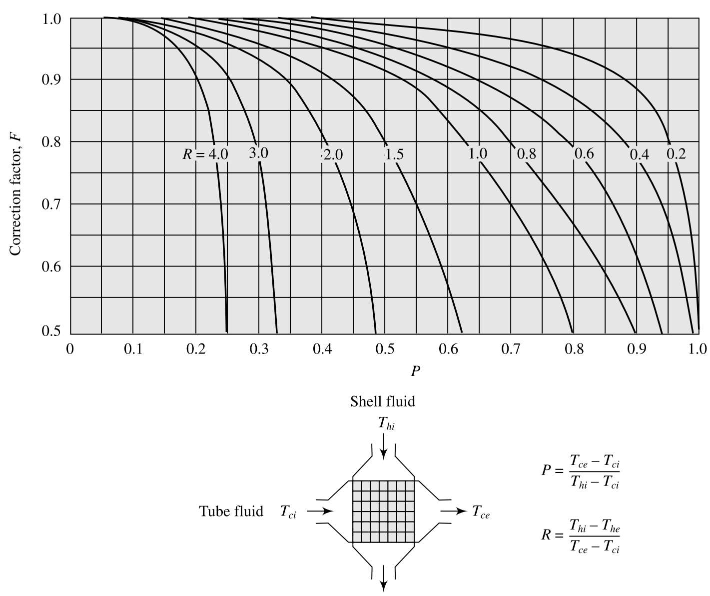
(b) Correction factor F for a two-shelf-pass, four-tube-pass heat exchanger



(c) Correction factor F for a cross-flow heat exchanger (single-pass) both fluids mixed



(d) Correction factor F for a cross-flow heat exchanger (single-pass): one fluid mixed and the other unmixed



(e) Correction factor F for a cross-flow heat exchanger (single-pass): both fluids unmixed

Fig. 11.30 *LMTD correction factors F for several heat-exchanger types. (a) Shell-and-tube with one shell pass and any multiple of two tube passes (two, four, etc., tube passes). (b) Shell-and-tube with two shell passes and any multiple of four tube passes (four, eight, etc., tube passes). (c) Single-pass, cross flow with both fluids unmixed. (d) Single-pass, cross flow with one fluid mixed and the other unmixed. (e) Single pass, cross flow with both fluids mixed.*

- When the heat-capacity rate on one side of a heat exchanger is *very large* relative to the other side, perhaps due to a large difference in mass-flow rates, the heat-capacity ratio again approaches *zero*, $R \to 0$.

In both these situations, the temperature of one of the fluids remains constant and the heat exchanger configuration becomes irrelevant in finding the value of F.

The curves drawn are only for $R \leq 1$. For $R > 1$, one may find F using a simple reciprocal rule. As long as a heat exchanger has a uniform heat-transfer coefficient and the fluid properties are constant.

$$\boxed{F(P, R) = F(PR, 1/R)} \tag{11.45}$$

Thus, if R is greater than unity, one can still evaluate F using PR instead of P and $1/R$ in place of R. Table 11.2 describes the steps needed to design (*size*) or rate a heat exchanger using the LMTD method. Design is a straightforward calculation using the LMTD method. Rating requires an *iterative* procedure.

Table 11.2 *Designing (sizing) and rating a heat exchanger using the LMTD method*

Design (*Sizing*) Problem	**Rating (*Performance Prediction*) Problem**
Known: The type of heat exchanger and basic configuration (*e.g., diameter and wall thickness of tubes*); the two fluids and their flow rates; the inlet temperatures of the two fluids; the required heat transfer rate (*or the two exit temperatures*).	**Known:** The geometry (*number, size, spacing, and layout of tubes, fin geometry, shell geometry, etc.*) and type of heat exchanger (shell-and-tube, plate-fin, fin-tube, etc.); the two fluids and their flow rates; the two inlet temperatures.
Find: The surface area required, A_s.	**Find:** The overall heat-transfer rate, $\dot{Q}$ or the two fluid exit temperatures, T_{ce} and T_{he}.
Procedure:	**Procedure:**
1. Evaluate the heat-transfer coefficients on each side of the heat exchanger using the specified geometry, fluid properties, and flow rates.	1. Evaluate the heat-transfer coefficients on each side of the heat exchanger using the specified geometry, fluid properties, and flow rates.
2. Calculate the wall resistance and estimate the fouling factors, if required.	2. Calculate the wall resistance and estimate the fouling factors, if required.
3. Determine the overall heat-transfer coefficient, U.	3. Determine the overall heat-transfer coefficient, U.
4. Calculate the dimensionless parameters P and R.	4. Calculate the capacity-rate ratio, R.
5. Evaluate the LMTD correction factor, F for the heat exchanger geometry using a relevant chart or equation.	5. Assume a value of one of the exit temperatures, calculate the other exit temperature and calculate P, or assume a value of P and calculate the fluid exit temperatures.
6. Use the given heat-transfer rate to find the exit temperatures, or calculate the heat duty $\dot{Q}$ from the given temperatures.	6. Evaluate the correction factor F for the heat exchanger geometry using a relevant plot or equation.
7. Calculate $\Delta T_{lm,\,\mathrm{CF}}$, i.e., LMTD for counterflow arrangement.	7. Calculate $\Delta T_{lm,\,\mathrm{CF}}$, i.e., LMTD for counterflow arrangement.
8. Calculate the surface area using Eq. (11.21). $A = \dfrac{\dot{Q}}{UF(\Delta T_{lm,\mathrm{CF}})}$	8. Calculate the heat-transfer rate, $\dot{Q}$.

contd.

contd.

	9. Calculate the fluid exit temperatures $T_{out} = T_{in} \pm (\dot{Q}/\dot{m}C_p)$. Compare them with those assumed in Step 5.
	10. Repeat steps 5 through 9 until the solution converges. (Use the temperatures calculated in Step 9 as the next assumed temperature.)

11.8 ❑ THE EFFECTIVENESS—NTU METHOD

Number of Transfer Units and Heat-Exchanger Effectiveness The concept of the Logarithmic Mean Temperature Difference (LMTD) can be applied to determine the heat-transfer rate or to determine the exchanger surface area required for a given heat transfer rate provided that the inlet and exit temperatures of the two fluids are known. If only the inlet temperatures of the two fluids are known, it becomes necessary to adopt a *trial-and-error* method avoided by using the *effectiveness-Number of Transfer Units* (ε-*NTU*) *method*. The ε-NTU method is developed with the same energy and rate equations as used in the LMTD method, but the equations are manipulated differently to obtain a different but analogous result.

In all heat-exchanger configurations and flow arrangements, the exchanger effectiveness depends on three pieces of information:

$$\boxed{\varepsilon = \text{function } (R, \text{ NTU, heat exchanger geometry and flow arrangement})}$$

The heat-exchanger geometry considerations include the type of construction (*counterflow*, *parallel flow, etc*.), number of fluid passes, and mixed or unmixed fluids. These considerations are identical to those taken into account when evaluating the correction factor F in the LMTD method.

The performance of heat exchangers can be evaluated by their effectiveness, ε, defined as

$$\boxed{\varepsilon = \frac{\text{Actual heat-transfer rate}}{\text{Maximum possible heat-transfer rate}} = \frac{\dot{Q}_{\text{actual}}}{\dot{Q}_{\max}}} \tag{11.46}$$

The magnitude of the effectiveness ranges between 0 (*no heat transfer at all*) and 1 (maximum possible heat transfer for the given fluid inlet temperatures, the mass flow rates, and the specific heats). The actual heat transfer rate can be calculated as follows:

$$\dot{Q}_{\text{actual}} = \dot{m}_h C_{ph}(T_{hi} - T_{he}) = C_h(T_{hi} - T_{he})$$

or

$$\dot{Q}_{\text{actual}} = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) = C_c(T_{ce} - T_{ci}) \tag{11.47}$$

$\dot{Q}_{\max}$ is obtained from an *idealized* heat exchanger of *infinite surface area* with *counterflow arrangement*

Thus

$$\boxed{\varepsilon = \frac{C_c(T_{ce} - T_{ci})}{C_{\min}(T_{hi} - T_{ci})} = \frac{C_h(T_{hi} - T_{he})}{C_{\min}(T_{hi} - T_{ci})}} \tag{11.48}$$

The main equation for the *effectiveness-NTU* (ε-NTU) method of analyzing heat exchangers is

$$\dot{Q}_{\text{actual}} = \varepsilon \dot{Q}_{\max} = \varepsilon C_{\min}(T_{hi} - T_{ci})$$

The parameter needed to characterize the performance of a heat exchanger is the effectiveness, ε. The effectiveness is dependent on the *heat exchanger geometry* (*configuration*), the *number of transfer units*, NTU, and the ratio of heat capacity rates.

The NTU has been defined as the ratio of the product UA to the minimum capacity rate $C_{\min}$ and is the dimensionless quantity. Thus,

$$\boxed{\text{NTU} = \frac{UA}{C_{\min}}} \tag{11.49}$$

This represents the non-dimensional thermal size of the heat exchanger (*but does not necessarily imply the physical size*). This dimensionless group can be looked upon as a comparison of the *heat capacity of the heat exchanger*, expressed in W/K or W/°C, with *the heat capacity of the fluid flow*. NTU can also be looked upon as the ratio of the larger of the two fluid temperature differences and the LMTD.

Since $\qquad \dot{Q} = UA(\text{LMTD}) = (\dot{m}C_p)_h(T_{hi} - T_{he}) = (\dot{m}C_p)_c(T_{ce} - T_{ci})$

or $\qquad \text{LMTD} = \dfrac{C_h(T_{hi} - T_{he})}{UA} = \dfrac{C_c(T_{ce} - T_{ci})}{UA}$

$$C_c = (\dot{m}C_p)_c \text{ and } C_h = (\dot{m}C_p)_h$$

where $\qquad \text{NTU}_h = \dfrac{T_{hi} - T_{he}}{\text{LMTD}} \text{ and } \text{NTU}_c = \dfrac{T_{ce} - T_{ci}}{\text{LMTD}}$

For a given LMTD, the driving force, a well-designed heat exchanger should give maximum possible change in the fluid temperature.

Hence, $\qquad \text{NTU}_h = \dfrac{UA}{C_h} \text{ and } \text{NTU}_h = \dfrac{UA}{C_c}$

A minimum-temperature fluid, i.e., the fluid with $C_{\min}$ will yield larger temperature difference.

Hence, $\qquad \text{NTU} = \dfrac{UA}{C_{\min}}$

The greater the number of transfer units, the more effective will be the heat exchanger.
A second non-dimensional parameter is the heat-capacity ratio, R.

$$\boxed{R = \frac{C_{\min}}{C_{\max}} = \frac{(\dot{m}C_p)_{\min}}{(\dot{m}C_p)_{\max}} = \frac{(\dot{m}C_p)_{\text{smaller}}}{(\dot{m}C_p)_{\text{bigger}}}} \tag{11.50}$$

where $C_{\min}$ is the heat-capacity rate of the fluid with the smaller value of the product of mass flow rate and specific heat, and $C_{\max}$ is that of the fluid with the larger value.

To calculate the maximum possible heat-transfer rate (*without violating the second law of thermodynamics*), we must examine how the temperatures of the two single-phase fluids behave. The maximum possible heat-transfer rate would occur when one of the fluids undergoes the maximum possible temperature rise (*or fall*). That is, the highest possible exit temperature for the cold fluid would equal T_{hi} and the lowest possible exit temperature for the hot fluid would equal T_{ci}. Hence, the maximum possible temperature change of either fluid would be $(T_{hi} - T_{ci})$, i.e., $(T_{h,\text{in}} - T_{c,\text{in}})$.

Which fluid could undergo this maximum temperature change? We have defined the heat capacity rate as $C = \dot{m}C_p$. In general, this product is different for the hot (C_h) and cold (C_c) fluids flowing through a heat exchanger; that is, C_h need not be equal to C_c. For example, if $C_h > C_c$, then we designate the larger heat-capacity rate $C_{\max} = C_h$ and the smaller heat-capacity rate $C_{\min} = C_c$. Since the energy balance must be satisfied, we have

$$\boxed{\dot{Q}_{\text{actual}} = C_h\Delta T_h = C_c\Delta T_c} \tag{11.51}$$

As a result, the cold fluid would undergo a larger actual temperature change (ΔT_c) than the hot fluid (ΔT_h); that is, $\Delta T_c > \Delta T_h$.

Clearly, the fluid with the smaller heat-capacity rate (C_{min}) will undergo a greater temperature change than the fluid with the larger heat capacity rate (C_{max}). Thus, only the fluid with the heat capacity rate, C_{min}, could experience the maximum possible temperature change ($T_{hi} - T_{ci}$) that would lead to the maximum possible heat-transfer rate:

$$\boxed{\dot{Q}_{\max} = C_{\min}(T_{hi} - T_{ci})} \tag{11.52}$$

The same result is obtained if we had assumed $C_c > C_h$.

11.8.1 • Effectiveness of a Parallel-flow Heat Exchanger

The fluid temperature variation along the length of a two-stream steady-flow coaxial parallel flow heat exchanger is shown in Fig. 11.31. An energy balance on an elemental area dA gives

$$d\dot{Q} = -C_h dT_h = +C_c dT_c = U\,dA(T_h - T_c) \quad \text{or} \quad UdA\Delta T$$

where $$C_c = \dot{m}_c C_{p_c}, C_h = \dot{m}_h C_{p_h}, \Delta T \equiv T_h - T_c \tag{11.53}$$

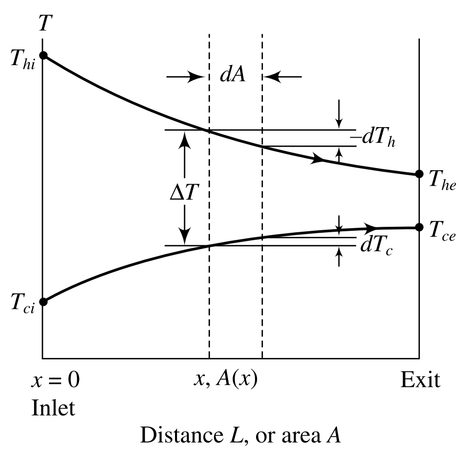


Fig. 11.31

Dividing through by $C_c\Delta T$, one finds

$$\frac{U\,dA}{C_c} = -\frac{C_h}{C_c}\frac{dT_h}{\Delta T} = \frac{dT_c}{\Delta T}$$

We note that if

$$\frac{C_1}{C_2} = \frac{A_1}{A_2} = \frac{B_1}{B_2} \quad \Rightarrow \quad \frac{C_1}{C_2} = \frac{A_1 + B_1}{A_2 + B_2}$$

It follows that

$$\frac{U\,dA}{C_c} = \frac{-(dT_h - dT_c)}{[(C_c/C_h + 1)](\Delta T)} = -\frac{d(T_h - T_c)}{(\Delta T) \times [1 + (C_c/C_h)]} \tag{11.54}$$

or $$\frac{U}{C_c}\left(1+\frac{C_c}{C_h}\right)dA = -\frac{d(\Delta T)}{\Delta T}$$

Integrating, we get

$$\frac{-U}{C_c}\left(1+\frac{C_c}{C_h}\right)\int_0^A dA = \int_{\Delta T_i}^{\Delta T_e} \frac{d(\Delta T)}{\Delta T} \tag{11.55}$$

or $$\frac{-UA}{C_c}\left(1+\frac{C_c}{C_h}\right) = \ln\frac{\Delta T_e}{\Delta T_i}$$

or $$\frac{\Delta T_e}{\Delta T_i} = \frac{T_{he}-T_{ce}}{T_{hi}-T_{ci}} = \exp\left[-\frac{UA}{C_c}\left(1+\frac{C_c}{C_h}\right)\right] \tag{11.56}$$

Effectiveness is defined as

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{C_c(T_{ce}-T_{ci})}{C_{\min}(T_{hi}-T_{ci})} = \frac{C_h(T_{hi}-T_{he})}{C_{\min}(T_{hi}-T_{ci})}$$

wherefrom

and $$\left.\begin{aligned} T_{ce} &= T_{ci} + \varepsilon\frac{C_{\min}}{C_c}(T_{hi}-T_{ci}) \\ T_{he} &= T_{hi} - \varepsilon\frac{C_{\min}}{C_h}(T_{hi}-T_{ci}) \end{aligned}\right\} \tag{11.57}$$

Substituting for T_{he} and T_{ce} in Eq. (11.56)., we have

$$\frac{(T_{hi}-T_{ci}) - \varepsilon(T_{hi}-T_{ci})\left[\frac{C_{\min}}{C_h}+\frac{C_{\min}}{C_c}\right]}{T_{hi}-T_{ci}} = \exp\left[-\frac{UA}{C_c}\left(1+\frac{C_c}{C_h}\right)\right]$$

or $$1-\varepsilon\left[\frac{C_{\min}}{C_h}+\frac{C_{\min}}{C_c}\right] = \exp\left[-\frac{UA}{C_c}\left(1+\frac{C_c}{C_h}\right)\right]$$

Rearranging,

$$\boxed{\varepsilon = \frac{1-\exp\left[\frac{-UA}{C_c}\left(1+\frac{C_c}{C_h}\right)\right]}{\left[\frac{C_{\min}}{C_h}+\frac{C_{\min}}{C_c}\right]}} \tag{11.58}$$

- If $\boldsymbol{C_c < C_h}$, then $C_c = C_{\min}$,

$$C_h = C_{\max}, \; R = \frac{C_{\min}}{C_{\max}} = \frac{C_c}{C_h} \quad \text{and} \quad \text{NTU} = \frac{UA}{C_{\min}} = \frac{UA}{C_c}$$

Substituting in Eq. (11.58), the expression for ε can be written more conveniently as Effectiveness (*cold fluid minimum*),

$$\varepsilon_{\text{CFM}} = \frac{1-\exp\left[-\dfrac{UA}{C_{\min}}\left(1+\dfrac{C_{\min}}{C_{\max}}\right)\right]}{1+\dfrac{C_{\min}}{C_{\max}}} = \frac{1-\exp[-\text{NTU}(1+R)]}{1+R}$$

- If $\boldsymbol{C_h < C_c}$, $C_h = C_{\min}, C_c = C_{\max},\ R = \dfrac{C_{\min}}{C_{\max}} = \dfrac{C_h}{C_c}$ and $\text{NTU} = \dfrac{UA}{C_{\min}} = \dfrac{UA}{C_h}$

Equation (11.58) can also be written as

$$\varepsilon = \frac{1-\exp\left[-\dfrac{UA}{C_c}\times\dfrac{C_h}{C_h}\left(1+\dfrac{C_c}{C_h}\right)\right]}{\dfrac{C_{\min}}{C_h}+\dfrac{C_{\min}}{C_c}} = \frac{1-\exp\left[-\dfrac{UA}{C_h}\left(\dfrac{C_h}{C_c}+1\right)\right]}{\dfrac{C_{\min}}{C_h}+\dfrac{C_{\min}}{C_c}}$$

Effectiveness (*hot fluid minimum*),

$$\varepsilon_{\text{HFM}} = \frac{1-\exp\left[-\dfrac{UA}{C_{\min}}\left(1+\dfrac{C_{\min}}{C_{\max}}\right)\right]}{1+\dfrac{C_{\min}}{C_{\max}}} = \frac{1-\exp[-\text{NTU}(1+R)]}{1+R}$$

Thus, we find that $\varepsilon_{\text{HFM}} = \varepsilon_{\text{CFM}} = \varepsilon$. Hence, no matter which fluid (*hot or cold*) is the *minimum* fluid, the effectiveness of a *parallel-flow heat exchanger* is given by

$$\boxed{\varepsilon_{\text{PF}} = \frac{1-\exp[-\text{NTU}(1+R)]}{1+R}} \tag{11.59}$$

The effectiveness lies between 0 and 1.
In other words,

$$\boxed{\varepsilon_{\text{PF}} = \text{function}[\text{NTU}, R]} \tag{11.60}$$

11.8.2 • Effectiveness of Counterflow Heat Exchanger

The fluid temperature variation along the length of a two-stream, steady-flow heat exchanger is shown in Fig. 11.32. The energy balance on an exchanger element of area dA can be expressed as

$$d\dot{Q} = U\Delta T\, dA = -C_h dT_h = -C_c dT_c \tag{11.61}$$

Dividing through by $C_c\ \Delta T$, one gets

$$\frac{UdA}{C_c} = -\frac{C_h}{C_c}\frac{dT_h}{\Delta T} = -\frac{dT_c}{\Delta T}$$

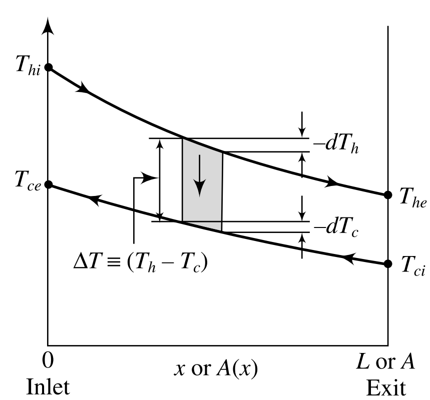


Fig. 11.32

We note that if

$$\frac{C_1}{C_2} = \frac{A_1}{A_2} = \frac{B_1}{B_2} \quad \Rightarrow \quad \frac{C_1}{C_2} = \frac{A_1 - B_1}{A_2 - B_2} \tag{11.62}$$

It follows that

$$\frac{UdA}{C_c} = -\frac{C_h dT_h}{C_c \Delta T} + \frac{dT_c}{\Delta T}$$

or
$$\frac{UdA}{C_c} = -\frac{1}{\Delta T}\left[\frac{dT_h - dT_c}{(C_c/C_h) - 1}\right] = \frac{d(T_h - T_c)}{\Delta T(1 - C_c/C_h)}$$

or
$$\frac{U}{C_c}\left[1 - \frac{C_c}{C_h}\right] dA = \frac{d(\Delta T)}{\Delta T}$$

Integrating, one gets

$$\frac{U}{C_c}\left[1 - \frac{C_c}{C_h}\right]\int_0^A dA = \int_{\Delta T_i}^{\Delta T_e} \frac{d(\Delta T)}{\Delta T}$$

or
$$\frac{UA}{C_c}\left[1 - \frac{C_c}{C_h}\right] = \ln\frac{\Delta T_e}{\Delta T_i} \tag{11.63}$$

or
$$\ln\frac{\Delta T_i}{\Delta T_e} = \frac{-UA}{C_c}\left[1 - \frac{C_c}{C_h}\right]$$

or
$$\frac{\Delta T_i}{\Delta T_e} = \frac{T_{hi} - T_{ce}}{T_{he} - T_{ci}} = \exp\left[-\frac{UA}{C_c}\left(1 - \frac{C_c}{C_h}\right)\right] \tag{11.64}$$

Effectiveness is defined as

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{C_c(T_{ce} - T_{ci})}{C_{\min}(T_{hi} - T_{ci})} = \frac{C_h(T_{hi} - T_{he})}{C_{\min}(T_{hi} - T_{ci})}$$

wherefrom

$$\left.\begin{aligned} T_{ce} &= T_{ci} + \varepsilon\frac{C_{\min}}{C_c}(T_{hi} - T_{ci}) \\ T_{he} &= T_{hi} - \varepsilon\frac{C_{\min}}{C_h}(T_{hi} - T_{ci}) \end{aligned}\right\} \tag{11.65}$$

Substituting for T_{ce} and T_{he} in Eq. (11.66), we get

$$\frac{(T_{hi} - T_{ci}) - \varepsilon\dfrac{C_{\min}}{C_c}(T_{hi} - T_{ci})}{(T_{hi} - T_{ci}) - \varepsilon\dfrac{C_{\min}}{C_h}(T_{hi} - T_{ci})} = \exp\left[-\frac{UA}{C_c}\left(1 - \frac{C_c}{C_h}\right)\right]$$

or
$$\frac{1 - \varepsilon\dfrac{C_{\min}}{C_c}}{1 - \varepsilon\dfrac{C_{\min}}{C_h}} = \exp\left[-\frac{UA}{C_c}\left(1 - \frac{C_c}{C_h}\right)\right] = K \text{ (say)}$$

or $$K - K\varepsilon\frac{C_{\min}}{C_h} = 1 - \varepsilon\frac{C_{\min}}{C_c} \quad \text{or} \quad \varepsilon\left[\frac{C_{\min}}{C_c} - K\frac{C_{\min}}{C_h}\right] = 1 - K$$

$\therefore$ $$\boxed{\varepsilon = \frac{1-K}{\dfrac{C_{\min}}{C_c} - K\dfrac{C_{\min}}{C_h}}} \quad \text{where} \quad K = \exp\left[-\frac{UA}{C_c}\left(1 - \frac{C_c}{C_h}\right)\right] \tag{11.66}$$

- If $C_c < C_h$, $C_c = C_{\min}$, $C_h = C_{\max}$

With $$R = \frac{C_{\min}}{C_{\max}} = \frac{C_c}{C_h} \quad \text{and} \quad \text{NTU} = \frac{UA}{C_{\min}} = \frac{UA}{C_c}$$

and substituting these dimensionless parameters in Eq. (11.68), one gets
Effectiveness (*cold fluid minimum*),

$$\varepsilon_{\text{CFM}} = \frac{1 - \exp\left[-\dfrac{UA}{C_{\min}}\left(1 - \dfrac{C_{\min}}{C_{\max}}\right)\right]}{1 - \dfrac{C_{\min}}{C_{\max}}\exp\left[-\dfrac{UA}{C_{\min}}\left(1 - \dfrac{C_{\min}}{C_{\max}}\right)\right]}$$

or $$\boxed{\varepsilon_{\text{CFM}} = \frac{1 - \exp[-\text{NTU}(1-R)]}{1 - R\exp[-\text{NTU}(1-R)]}}$$

- If $C_h < C_c$, $C_h = C_{\min}$, $C_c = C_{\max}$

With $R = \dfrac{C_{\min}}{C_{\max}} = \dfrac{C_h}{C_c}$ and $\text{NTU} = \dfrac{UA}{C_{\min}} = \dfrac{UA}{C_h}$, and substituting these dimensionless parameters in Eq. (11.66), one has
Effectiveness (*hot fluid minimum*),

$$\varepsilon_{\text{HFM}} = \frac{1 - \exp\left[-\dfrac{UA}{C_{\max}}\left(1 - \dfrac{C_{\max}}{C_{\min}}\right)\right]}{\dfrac{C_{\min}}{C_{\max}} - \exp\left[-\dfrac{UA}{C_{\max}}\left(1 - \dfrac{C_{\max}}{C_{\min}}\right)\right]}$$

Multiplying both numerator and denominator by $\exp\left[+\dfrac{UA}{C_{\max}}\left(1 - \dfrac{C_{\max}}{C_{\min}}\right)\right]$, we have

$$\varepsilon_{\text{HFM}} = \frac{\exp\left[\dfrac{UA}{C_{\max}}\dfrac{C_{\min}}{C_{\min}}\left(1 - \dfrac{C_{\max}}{C_{\min}}\right)\right] - 1}{\dfrac{C_{\min}}{C_{\max}}\exp\left[\dfrac{UA}{C_{\max}}\dfrac{C_{\min}}{C_{\min}}\left(1 - \dfrac{C_{\max}}{C_{\min}}\right)\right] - 1} = \frac{1 - \exp\left[+\dfrac{UA}{C_{\min}}\left(\dfrac{C_{\min}}{C_{\max}} - 1\right)\right]}{1 - \dfrac{C_{\min}}{C_{\max}}\exp\left[+\dfrac{UA}{C_{\min}}\left(\dfrac{C_{\min}}{C_{\max}} - 1\right)\right]}$$

or $$\boxed{\varepsilon_{\text{HFM}} = \frac{1 - \exp[-\text{NTU}(1-R)]}{1 - R\exp[-\text{NTU}(1-R)]}}$$

Thus, we find that $\boxed{\varepsilon_{\text{HFM}} = \varepsilon_{\text{CFM}} = \varepsilon}$

Hence, no matter which fluid (*hot or cold*) is the *minimum* fluid, the effectiveness of the *counterflow heat exchanger* is given by

$$\boxed{\varepsilon_{CF} = \frac{1 - \exp^{[-NTU(1-R)]}}{1 - R\exp^{[-NTU(1-R)]}}}$$

Note that NTU is indicative of the *physical size* (i.e., surface area) and the *quality* of heat-transfer surfaces (i.e., *U*) of the heat exchanger. The larger the value of NTU, the closer the heat exchanger reaches thermodynamic limit of operation. It may be noted that NTU ≥ 0 and $0 \leq R \leq 1$.

For example, the high-performance sodium heat exchangers used with fast breeder reactors have an NTU of about 6.

Special Cases:
For all heat exchangers:

With $R = 0$, $\varepsilon_{PF} = \dfrac{1 - \exp[-NTU(1+0)]}{1+0} = 1 - \exp(-NTU)$

With $R = 0$, $\varepsilon_{CF} = \dfrac{1 - \exp[-NTU(1-0)]}{1 - 0 \times \exp[-NTU(1-0)]} = 1 - \exp(-NTU)$

Thus, when $R \equiv \dfrac{C_{min}}{C_{max}} = 0$, involving *phase change* of *one* fluid:

$$\boxed{\boldsymbol{\varepsilon = 1 - \exp(-NTU)}} \qquad \text{for all heat exchangers.}$$

Parallel flow $C_{min} = C_{max}$ or $R = 1$ (*Balanced heat exchanger*) When the heat-capacity rates of hot and cold fluids are same, $\dfrac{C_{min}}{C_{max}} = R = 1$ and the expression for effectiveness becomes

$$\boxed{\varepsilon_{PF(R=1)} = \frac{1}{2}[1 - \exp(-2\,NTU)]}$$

The maximum effectiveness in parallel-flow arrangement will be for very large values of NTU, i.e., NTU $\to \infty$ is 50 %.

$$\varepsilon_{\max(PF)(R=1)} = \frac{1}{2}[1 - \exp(-\infty)] = 0.5$$

Counterflow $C_{min} = C_{max}$ or $R = 1$(Balanced Heat Exchanger) In a gas-to-gas heat exchanger with comparable flow rates on either side, the ratio R is nearly one, since all gases have heat capacities of the same order of magnitude. In this limit, the effectiveness of counterflow heat exchanger becomes.

$$\varepsilon_{CF} = \frac{1 - \exp[-NTU(1-1)]}{1 - R\exp[-NTU(1-1)]} = \frac{0}{0}, \text{ i.e., indeterminate}$$

Using L' Hospital's rule,

$$\varepsilon_{CF} = \lim_{R \to 1}\left\{\frac{1 - \exp[-NTU(1-R)]}{1 - R\exp[-NTU(1-R)]}\right\} = \lim_{R \to 1}\left\{\frac{f(R)}{g(R)}\right\} = \lim_{R \to 1}\left\{\frac{f'(R)}{g'(R)}\right\}$$

$$f'(R) = \frac{d}{dr}f(R) = 0 - \exp[-NTU(1-R)] \times (NTU)$$

and $$g'(R) = \frac{d}{dr} g(R) = 0 - (1)\exp[-\text{NTU}(1-R)] - R\exp[-\text{NTU}(1-R)] \times (\text{NTU})$$
$$= -\exp[-\text{NTU}(1-R)](1 + R\,\text{NTU})$$

Therefore,

$$\varepsilon_{\text{CF}} = \lim_{R\to 1}\left\{\frac{-\exp[-\text{NTU}(1-R)]\times \text{NTU}}{\exp[-\text{NTU}(1-R)][1+R\,\text{NTU}]}\right\}$$

$$\therefore \quad = \lim_{R\to 1}\left\{\frac{e^{-0}\times \text{NTU}}{e^{-0}[1+R\,\text{NTU}]}\right\} = \frac{\text{NTU}}{1+\text{NTU}}$$

$$\boxed{\varepsilon_{\text{CF}(R=1)} = \frac{\text{NTU}}{1+\text{NTU}}}$$

Maximum effectiveness in this case will be for very large NTU, i.e., max $\varepsilon_{\text{CF}(R=1)} = \frac{\text{NTU}_{(\to\infty)}}{1+\text{NTU}_{(\to\infty)}} = 1$

In a *counterflow* mode, the effectiveness of *gas-gas exchanger* can thus be 100% while for *parallel flow* it is only 50%. Hence, *gas-to-gas* heat exchangers are invariably of *counterflow* type.

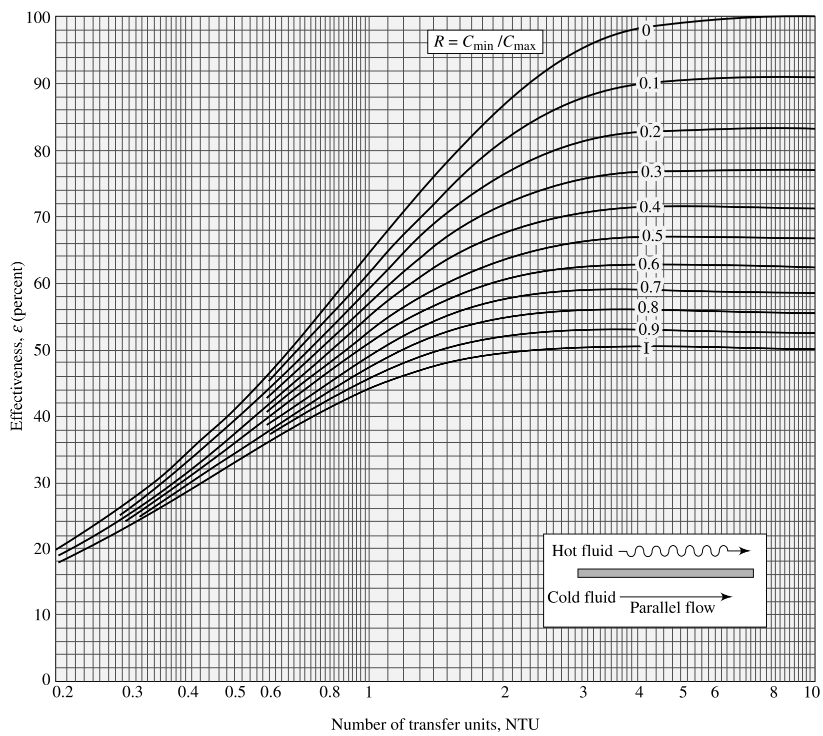
Table 11.3 *Common relations for effectiveness and NTU for heat exchangers:* *$[R = C_{min}/C_{max} = (\dot{m}C_p)_{smaller}/(\dot{m}C_p)_{bigger}$; $NTU = UA_s/C_{min}]$*

Type of Heat Exchanger	Effectiveness Relations	NTU Relations
■ All exchangers with R = 0 (*Evaporators* and *Condensers*)	$\varepsilon = 1 - \exp(-\text{NTU})$	$\text{NTU} = -\ln(1-\varepsilon)$
■ ***Double pipe*** (*Concentric tube*) Parallel flow	$\varepsilon = \frac{1-\exp[-\text{NTU}(1-R)]}{1-R\exp[-\text{NTU}(1-R)]}$	$\text{NTU} = \frac{1}{R-1}\ln\left(\frac{\varepsilon-1}{R\varepsilon-1}\right)$
Counterflow (R < 1)	$\varepsilon = \frac{1-\exp[-\text{NTU}(1+R)]}{1+R}$	$\text{NTU} = \frac{-\ln[1-\varepsilon(1+R)]}{1+R}$
Counterflow (R = 1)	$\varepsilon = \frac{\text{NTU}}{1+\text{NTU}}$	$\text{NTU} = \frac{\varepsilon}{1-\varepsilon}$
■ ***Shell-and-tube***: One-shell pass; 2, 4, 6, … tube passes	$\varepsilon_1 = 2\left\{(1+R) + \sqrt{(1+R^2)} \times \frac{(1+\exp[-\text{NTU}\sqrt{1+R^2}])}{(1-\exp[-\text{NTU}\sqrt{1+R^2}])}\right\}^{-1}$	$\text{NTU} = -\frac{1}{\sqrt{(1+R^2)}}\ln\left(\frac{E-1}{E+1}\right)$ $E = \frac{2/\varepsilon_2 - (1+R)}{\sqrt{(1+R^2)}}$
Multiple shell passes, 2n 4n, 6n, … tube passes (n = *number of shell passes*, ε_1 = *effectiveness of each shell-pass*)	$\varepsilon = \left[\left(\frac{1-\varepsilon_1 R}{1-\varepsilon_1}\right)^n - 1\right]\left[\left(\frac{1-\varepsilon_1 R}{1-\varepsilon_1}\right)^n - R\right]^{-1}$	Use the above two equations with $\varepsilon_2 = \frac{F-1}{F-R}$, $F = \left(\frac{\varepsilon R - 1}{\varepsilon - 1}\right)^{1/n}$

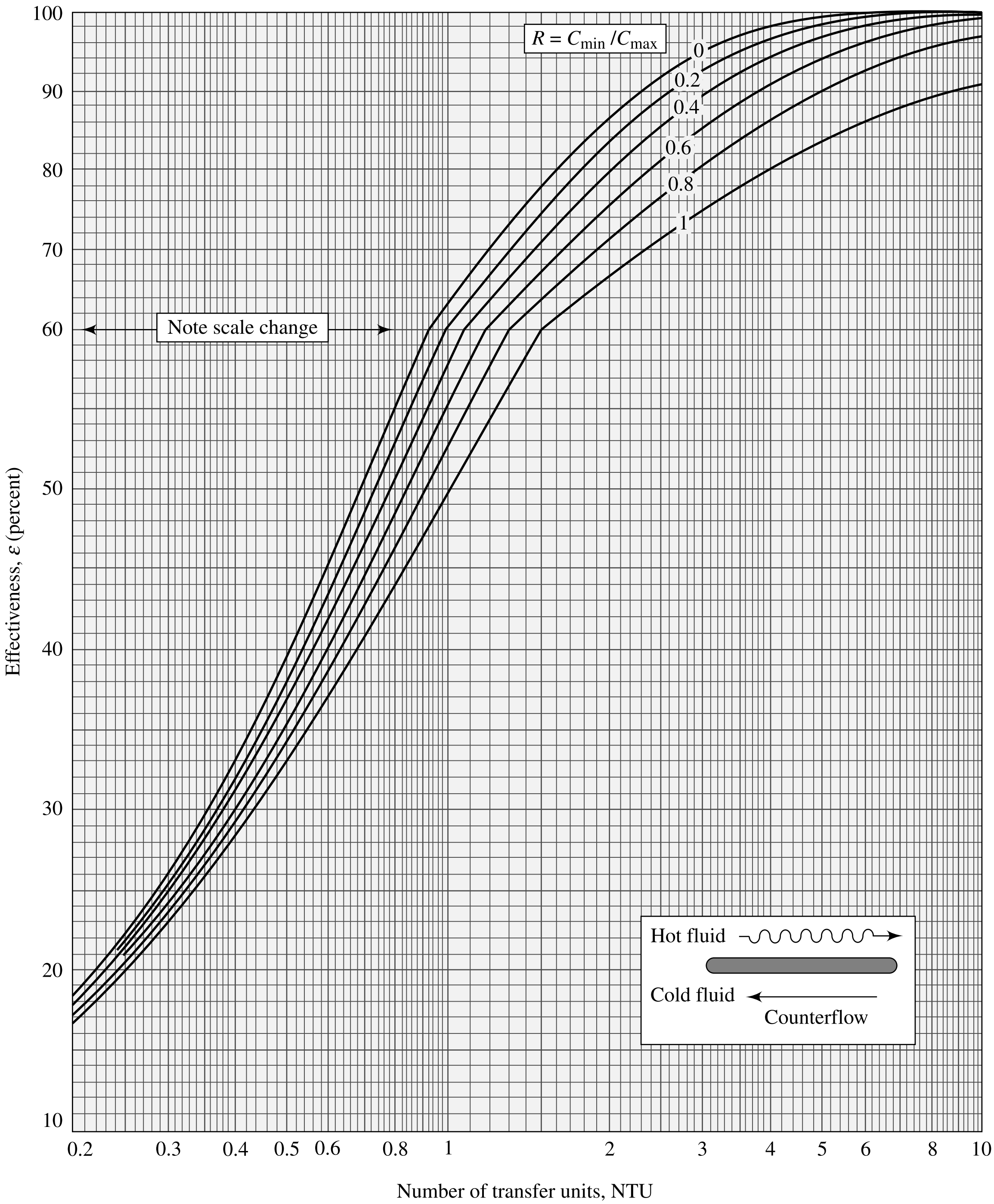
contd.

contd.

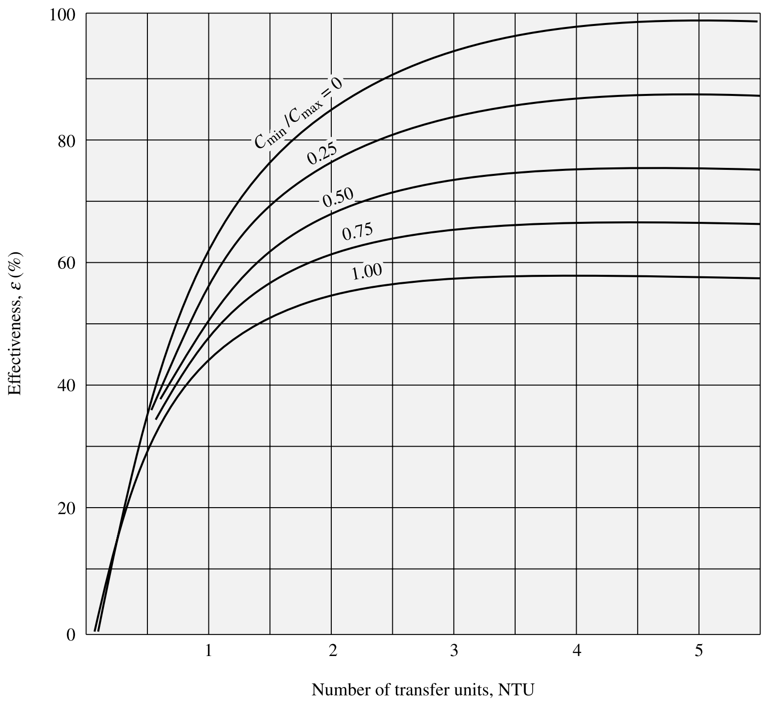
Special case for $R = 1$	$\varepsilon = \dfrac{\varepsilon_1 n}{1+(n-1)\varepsilon_1}$	—
■ ***Cross-flow*** **(single pass):** Both fluids unmixed	$\varepsilon = 1 - \exp\left[\dfrac{\text{NTU}^{0.22}}{R}\{\exp[-R(\text{NTU}^{0.78})] - 1\}\right]$	—
Both fluids *mixed*	$\varepsilon = \left[\dfrac{1}{1-\exp(-\text{NTU})} + \dfrac{R}{1-\exp(-R\,\text{NTU})} - \dfrac{1}{\text{NTU}}\right]^{-1}$	—
$C_{\min}$ *mixed*, $C_{\max}$ *unmixed*	$\varepsilon = 1 - \exp\left[-\dfrac{1}{R}\{1-\exp(-R\cdot\text{NTU})\}\right]$	$\text{NTU} = \dfrac{-\ln[R\ln(1-\varepsilon)+1]}{R}$
$C_{\max}$ *mixed*, $C_{\min}$ *unmixed*	$\varepsilon = \dfrac{1}{R}[1-\exp\{-R(1-\exp(-\text{NTU}))\}]$	$\text{NTU} = -\ln\left(1+\dfrac{\ln(1-\varepsilon R)}{R}\right)$



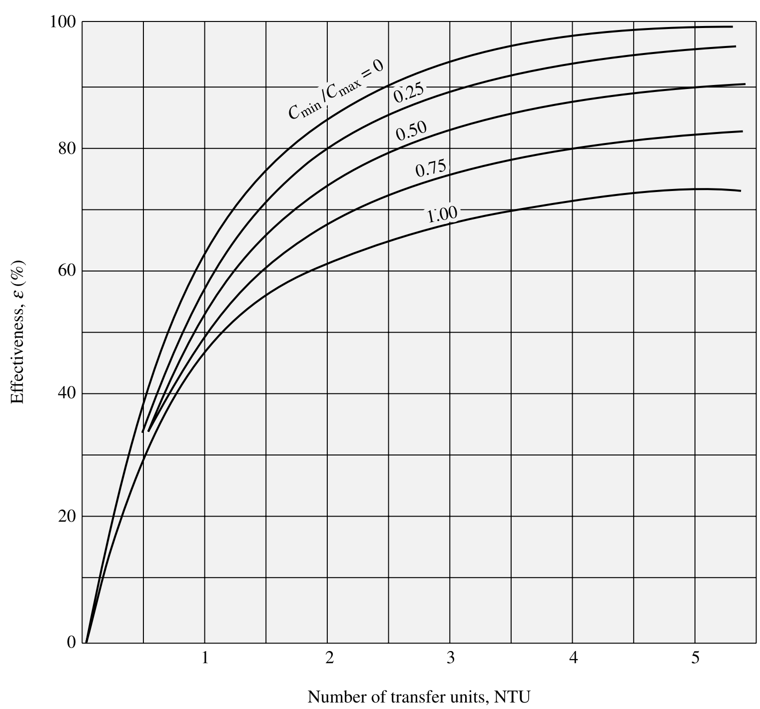
(a) Effectiveness as a function of *NTU* for the simple concentric-tube heat exchanger in parallel flow



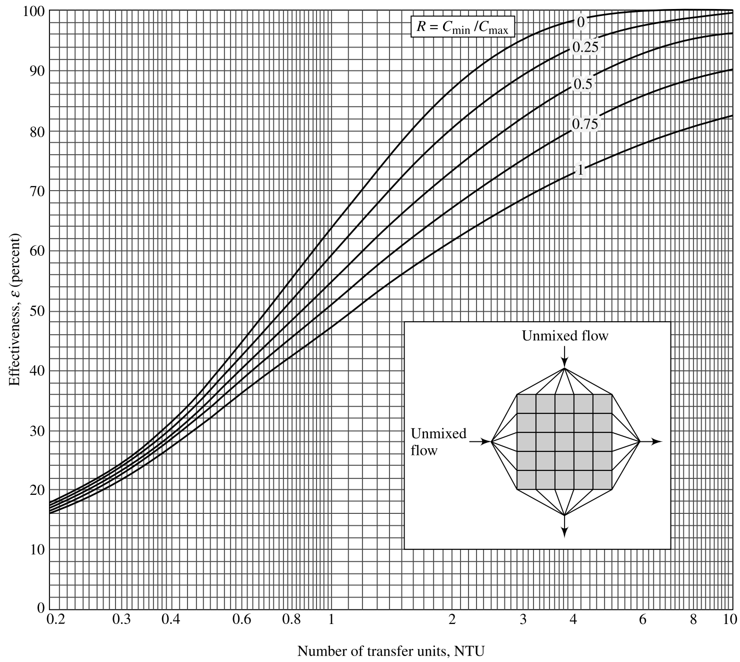
(b) Effectiveness as a function of *NTU* for the simple concentric-tube heat exchanger in counterflow



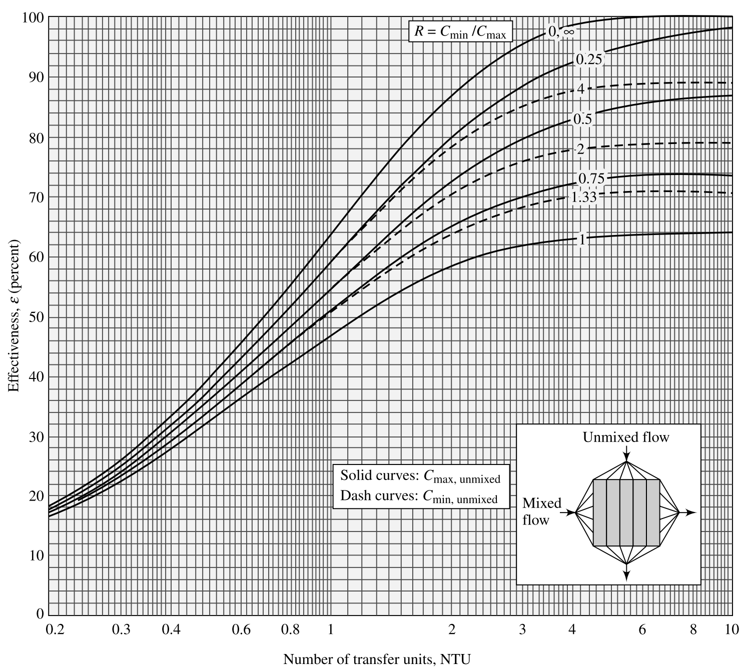
(c) Effectiveness of a shell-and-tube heat exchanger with one-shell pass and any multiple of two tube passes



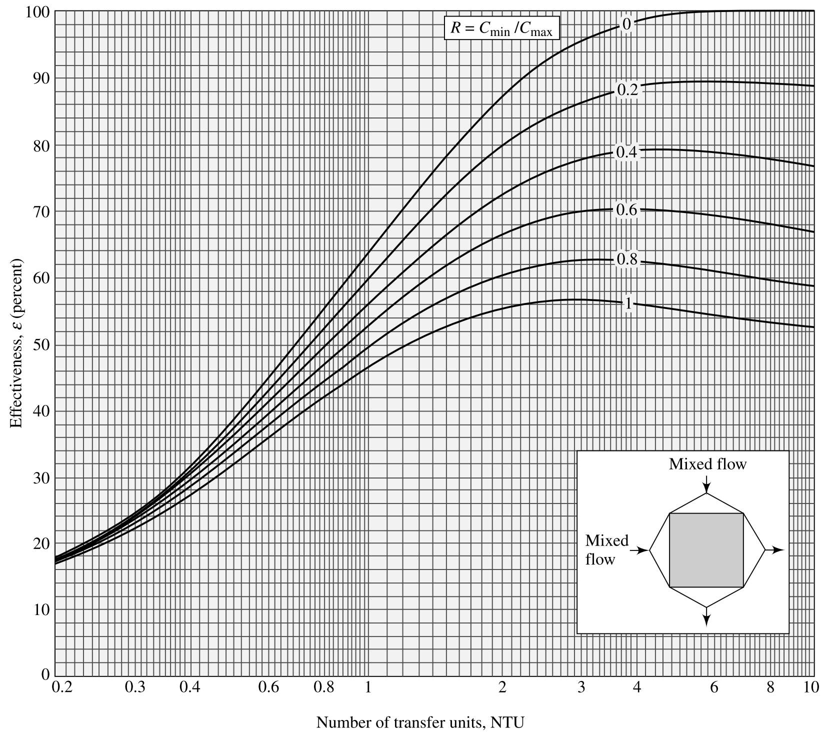
(d) Effectiveness of a shell-and-tube heat exchanger with two-shell-passes and any multiple of four tube passes.



(e) Effectiveness as a function of NTU for the cross flow heat exchanger with both fluids unmixed.



(f) Effectiveness as a function of *NTU* for the cross flow heat exchanger with the minimum heat capacity (C_{min}) fluid mixed (C_{max} fluid mixed for dashed curves)



(g) Effectiveness as a function of NTU for the cross flow heat exchanger with both fluids mixed.

Fig. 11.33 *Effectiveness curves for several heat exchanger types. (a) Parallel flow. (b) Counterflow. (c) Shell-and-tube with one shell pass and any multiple of two tube passes (two, four, etc., tube passes). (d) Shell-and-tube with two shell passes and any multiple of four tube passes (four, eight, etc. tube passes). (e) Single-pass, cross flow with both fluids unmixed. (f) Single-pass, cross flow with one fluid mixed and the other unmixed. (g) Single-pass, cross flow with both fluids mixed.*

11.8.3 • Effectiveness-NTU Relations and Charts

Note that the choice of a particular method is essentially governed by the designer's familiarity with it. However, the LMTD method is widely practised by designers of heat exchangers in *process industries* and *power plants* and the ε-NTU method has found favour with designers of compact heat exchangers. ε-NTU formulation is also preferred because of its suitability for computer-aided design.

Some common expressions for effectiveness are given in Table 11.2. Figure 11.35 shows the effect of varying parameters on the effectiveness for different heat-exchanger configurations. Note the exponential behaviour of the curves. When NTU is large, obtaining a small increase in the effectiveness may require a significant increase in area. For example, consider a simple counterflow heat exchanger with $R = 0.6$ and NTU = 4, with an effectiveness of 0.908. If we wanted to increase the effectiveness by 3% to 0.935 one would require an NTU of 4.78. This increase in effectiveness would require a 19.5 % increase in the surface area. Hence, for a heat exchanger that may have to be operated over a range of conditions, it would not be wise to design with an NTU near where the curve begins to flatten out. In striking contrast, a counterflow heat exchanger with $R = 0.6$ and NTU = 1.0 has an effectiveness of 0.56. To increase it by 3% one would need an increase in area of the order of only 5.65%.

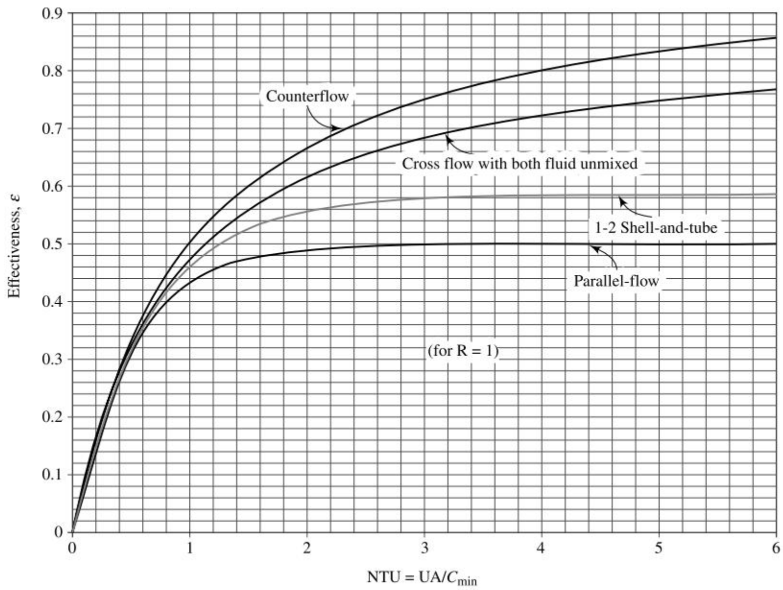


Fig. 11.34 *For a prescribed NTU and capacity ratio R, the counterflow heat exchanger has the highest effectiveness followed by cross flow, and the parallel flow with the lowest effectiveness.*

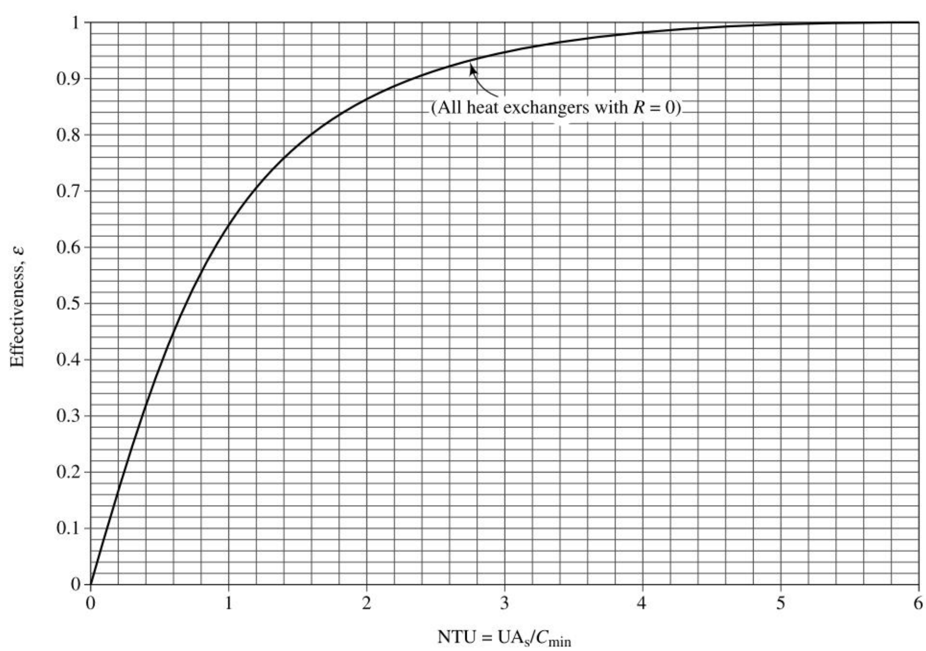


Fig. 11.35 *The effectiveness relation reduces to $\varepsilon = \varepsilon_{max} = 1 - [\exp(-NTU)]$ for all heat exchangers when the capacity ratio R = 0*

Note that for the given values of R and NTU, a *counterflow* heat exchanger has the *highest* effectiveness of any heat exchanger or flow arrangement, and a *parallel-flow* heat exchanger has the *lowest*. All others lie between these two extremes. This is illustrated in Fig. 11.34.

What is more, with a single-phase flow on one side of the heat exchanger and a constant wall or fluid temperature on the other side of the heat exchanger, $C_{max} \to \infty$, $R \to 0$, the geometry becomes irrelevant, and all heat exchangers have the same expression for effectiveness as shown in Fig. 11.35.

$$\boxed{\varepsilon = 1 - \exp(-\text{NTU}) \quad \text{for } R = 0} \tag{11.67}$$

Table 11.4 gives steps used in the ε-NTU method. Regardless of whether the ε-NTU or the LMTD method is used for either a *rating* or a *design* problem, identical results will be obtained (*within round off error*). It is generally not practical to design or use a heat exchanger which has an effectiveness less than 0.70.

Table 11.4 *Designing (Sizing) and Rating a Heat Exchanger using the ε–NTU Method*

Design (Sizing) Problem	Rating (Performance Prediction) Problem
Known: The type of heat exchanger and basic configuration (e.g., *diameter and wall thickness of tubes*); the two fluids and their flow rates; the inlet temperatures of the two fluids; the required heat duty or the two exit temperatures	**Known:** The geometry (*number, size, spacing, and layout of tubes, fin geometry, shell geometry, etc.*) and type of heat exchanger (*shell-and-tube, plate-fin, fin-tube, etc.*); the two fluids and their flow rates; the two inlet temperatures
Find: The surface area needed, A_s.	**Find:** The overall heat-transfer rate, $\dot{Q}$ or the two fluid exit temperatures, T_{ce} and T_{he}.
Procedure:	**Procedure:**
1. Calculate the heat transfer rate, Q.	**1.** Calculate the heat-transfer coefficients on each side of the heat exchanger using the prescribed geometry, fluid flow rates, and the relevant correlations.
2. Calculate the effectiveness from the specified data.	**2.** Calculate the wall resistance and estimate the fouling factor (s) if required.
3. Calculate the heat capacity rate ratio, R.	**3.** Determine the overall heat transfer coefficient.
4. Calculate C_{max}, C_{min}, and the wall resistance and estimate the fouling factor(s) if required.	**4.** Calculate C_{max}, C_{min} and the capacity rate ratio, $R = \left(\dfrac{C_{\min}}{C_{\max}}\right)$
5. Assume the number of tubes required in the heat exchanger.	**5.** Calculate NTU and $\dot{Q}_{\max}$.
6. Compute the heat-transfer coefficients on each side of the heat exchanger using the specified and assumed geometric parameters, fluid properties and flow rates and relevant correlations.	**6.** Evaluate the effectiveness ε, using the appropriate equation or figure.
7. Determine the overall heat transfer coefficient.	**7.** Calculate the actual heat duty, $\dot{Q}$ and the exit fluid temperatures. $T_{\text{out}} = T_{\text{in}} \pm (\dot{Q}/\dot{m}C_p)$
8. Evaluate the NTU for the heat exchanger geometry.	
9. Calculate the heat exchanger area, A = (NTU) C_{min}/U.	
10. Repeat steps 4 through 8 until the solution converges. (Use the area calculated in Step 8 to estimate the number tubes in Step 4.)	

11.9 ❑ SELECTION AND DESIGN ASPECTS OF HEAT EXCHANGERS

Heat exchangers are complex devices, and the results obtained with simplifying assumptions presented above should be used with caution. For example, the overall heat-transfer coefficient U was assumed constant throughout the heat exchanger and the convection heat transfer coefficients could be predicted using the empirical correlations. However, the uncertainty in the predicted value of U can sometimes exceed 30 percent. It is but natural to over-design the heat exchangers in order to avoid unpleasant surprises.

There are two aspects to the design of a heat exchanger-*thermo-hydraulic* and *mechanical*. The *thermo-hydraulic* design involves selecting the *flow arrangement* and *geometric configuration*, and determining the *surface area*. The second aspect—the *mechanical* design—deals with the *safety* considerations while taking note of the *standards* and *codes*. These two aspects are interdependent and cannot be carried out in isolation.

The range of heat-exchanger types is large, and to select a heat exchanger for a particular application can be a difficult decision. For a person with no experience, this could be a tall task. The task can be simplified by taking advantage of the experience of others. Unusual design requirements, and other considerations may however suggest a different approach. In such cases, it is important to consider several factors as follows:

- ***Heat-transfer Mechanisms*** This is the most important parameter in the selection of a heat exchanger. A heat exchanger should be capable of transferring heat at the specified rate in order to achieve the required change in the fluid temperature at the given mass-flow rate.

  Different flow paths and geometric configurations are used depending on the mode of heat transfer. Whether the flow is *laminar* or *turbulent*, *single-phase* (*gas* or *liquid*), or *two-phase* flow (boiling or condensation) will have an impact on the choice of surface and the type of heat exchanger.
- ***Cost*** Economy usually plays a significant role in the selection of heat exchangers, except in some specialized cases where money is not of much importance. An off-the-shelf heat exchanger is certainly less costly than those made to order. The operation and maintenance cost of a heat exchanger besides the capital cost should also be given due consideration.
- ***Pressure Drop and Pumping Power*** Greater heat transfer in heat exchangers is usually associated with *increased pressure drop,* and thus *higher pumping power*. Hence, any advantage from the increase in heat transfer has to be balanced against the cost of the accompanying pressure drop. Also, one must decide which fluid should pass through the *tube side* and which through the *shell side*. Generally, the *more viscous fluid is preferable for the shell side* (larger passage area and thus lower pressure drop) and *the fluid with the higher pressure for the tube side*.

  For an incompressible fluid flow: Pumping power (kW) = Pressure drop (kPa) × Volumetric flow rate (m^3/s). The operating (*running*) cost depends on this power consumption.

  Typically, fluid velocities in heat exchangers range between 0.7 and 7 m/s for *liquids* and between 3 *and* 30 m/s for *gases* to avoid long-term operational problems.
- ***Size and Weight*** *Smaller* and *lighter* heat exchangers are normally preferable. This is more so in the *automotive* and *aerospace* industries, where size and weight requirements are very stringent. In some applications like an oil refinery, a shell-and-tube heat exchanger may require a thick steel shell to contain a high-pressure petroleum product, and weight is not a critical consideration. But it also means higher running (*pumping*) cost.

  A heat exchanger is suitable to cool a liquid by a gas if the surface area on the gas side is several times that on the liquid side. *A plate* or *shell-and-tube* heat exchanger, on the other hand, is quite suitable for cooling a liquid by another liquid.

- ***Materials*** We should be careful in the selection of materials used in the construction of the heat exchanger. For example, one need not bother about the *thermal* and *structural stress effects* at pressures below 15 atm or temperatures less than 150°C. But, these effects become significant above 70 atm or 550°C.

  Another point to ponder is the high temperature differentials of 50°C or more between the tubes and the shell which may pose *differential thermal expansion* problems. Industrial heat exchangers using corrosive fluids, require expensive *corrosion-resistant* materials such as stainless steel or even titanium.

  Fabrication problems may also be important, since not all materials can be soldered, brazed, and/or welded.
- ***Heat-transfer Performance*** The main basis of selection of a heat exchanger is that it must satisfy the heat-transfer requirements for the specified application. This includes the required heat duty, the fluids used, their flow rates, inlet and exit temperatures, pressure levels, and permissible pressure drops. Since the exchanger design usually requires compromises between several competing factors, the relative importance of each factor should also be examined carefully.
- ***Pressure and Temperature*** Certain types of heat exchanger cannot sustain high pressures. For example, tubular heat exchangers can withstand high pressures, but heat exchangers with large flat areas and thinner materials (e.g., *plate-type* or *compact*) are limited in their maximum permissible pressures. Similarly, many heat exchangers can have restrictions on their permissible temperature level.
- ***Fouling Tendencies*** Fouling is difficult to predict, and fouling characteristics for a given application will depend on several parameters. Fluid velocity, flow distribution through the heat exchanger, channel dimensions, and fluid type are some major factors that may make fouling problematic. Experience with a particular fluid and heat exchanger type can come in handy to decide if fouling could be a serious problem. Periodic cleaning of the heat transfer surfaces while dealing with fluids that cause fouling is important. These surfaces should be easily accessible so as to reduce the shut-down time involved in servicing and maintenance.
- ***Safety, Reliability, and Environmental Aspects*** One must not forget issues related to safety, reliability and environmental considerations like the leakage of the process fluids into the environment, the use of toxic or inflammable fluids and waste disposal problems. Minimum noise and vibration, for example, is a key consideration in the selection of liquid-to-air heat exchangers used in heating and air-conditioning applications.

Illustrative Examples

(A) Overall Heat-Transfer Coefficient

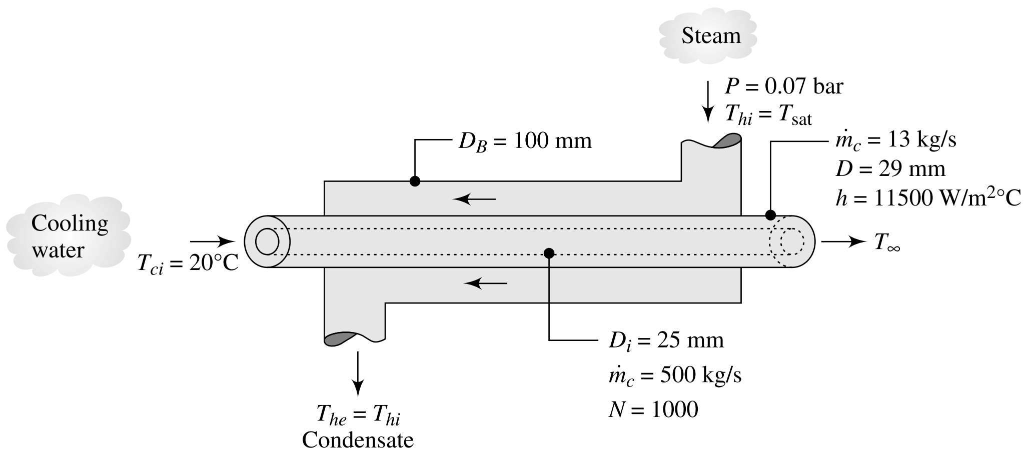
EXAMPLE 11.1 *The steam condenser of a thermal power plant operates at a pressure of 7 kPa. Cooling water is circulated with a mass-flow rate of 500 kg/s through 100 tubes of 25 mm ID and 29 mm OD made of brass (k = 110 W/m °C). The steam side heat-transfer coefficient is 11 500 W/m² °C. Assuming average properties of water: k = 0.6 W/m °C, μ = 9.6 E-4 kg/m s and Pr = 6.6, calculate the overall heat-transfer coefficient based on the outer diameter of the tubes, U_o. If the inside surfaces of the tubes are fouled due to scale formation over a period of time, and the overall heat-transfer coefficient is estimated to be 1968 W/m² °C, determine the corresponding fouling factor. Compute the exit temperature of water if the steam flow rate is 13 kg/s and the cooling-water inlet temperature is 20°C.*

Solution

Known Steam is condensed by cooling water in the condenser of a power plant.

Find Overall heat-transfer coefficient, U_o. Fouling factor, R_{fi}. Cooling water outlet temperature, T_{ce}.

Schematic



Assumptions (1) Steady operating conditions, (2) Constant properties. (3) Uniform convection coefficients. (4) No fouling on outer surface of the tubes.

Properties

$h_{fg_{@7\,kPa}}$ (*from the steam tables*) = 2409.1 kJ/kg

$C_{p_{water}}$ = 4.18 kJ/kg °C $\qquad$ k = 0.6 W/m °C

Pr = 6.6 $\qquad$ $\mu = 9.6 \times 10^{-4}$ kg /m s (*given in the problem statement*)

Analysis Overall heat-transfer coefficient based on the outside area, including fouling resistances,

$$U_{o,\text{dirty}} = \left[\frac{D_o}{D_i}\frac{1}{h_i} + \frac{D_o}{2k}\ln\frac{D_o}{D_i} + \frac{1}{h_o} + R_{fo} + \frac{D_o}{D_i}R_{fi}\right]^{-1}$$

In the absence of fouling resistance,

$$U_{o,\text{clean}} = \left[\frac{D_o}{D_i}\frac{1}{h_i} + \frac{D_o}{2k}\ln\frac{D_o}{D_i} + \frac{1}{h_o}\right]^{-1}$$

h_o = 11 500 W/m^2 °C $\qquad$ (*given*)

To determine h_i, let us first determine Re_D

$$Re_D = \frac{4\dot{m}/\text{tube}}{\pi D_i \mu} = \frac{4(500/1000)\text{kg/s}}{\pi(0.025\text{ m})(9.6\times 10^{-4}\text{ kg/m s})} = 26\,526$$

Using Dittus–Boelter correlation,

$$Nu_D = \frac{h_i D_i}{k} = 0.023\,(Re_D)^{0.8}(Pr)^{0.4}$$

$$\therefore \qquad h_i = \frac{0.023\times 0.6\text{ W/m°C}}{0.025\text{ m}}(26\,526)^{0.8}(6.6)^{0.4} = 4061.6\text{ W/m}^2\text{ °C}$$

Hence, $$U_{o,\text{clean}} = \left[\frac{29}{25}\frac{1}{4061.6} + \frac{0.029}{2\times 110}\ln\frac{29}{25} + \frac{1}{11500}\right]^{-1} = (3.921\times 10^{-4})^{-1}$$

$= \mathbf{2550\ W/m^2\ °C}$ **(Ans.)**

There is no fouling on the steam side of the tubes, i.e., $R_{fo} = 0$

$$\frac{1}{U_{o,\text{dirty}}} = \frac{1}{U_{o,\text{clean}}} + \frac{D_o}{D_i}R_{fi}$$

∴ Fouling factor on the water side.

$$R_{fi} = \frac{D_i}{D_o}\left[\frac{1}{U_{o,\text{dirty}}} - \frac{1}{U_{o,\text{clean}}}\right] = \frac{25}{29}\left[\frac{1}{1968} - \frac{1}{2550}\right]$$

$= \mathbf{10^{-4}\ m^2\ °C/W}$ **(Ans.)**

Heat-transfer rate,

$$\dot{Q} = \underbrace{\dot{m}_h h_{fg}}_{\textbf{steam}} = \underbrace{\dot{m}_c C_{p_c}(T_{ce} - T_{ci})}_{\textbf{cooling water}}$$

Therefore, the exit temperature of water is

$$T_{ce} = T_{ci} + \frac{\dot{m}_h h_{fg}}{\dot{m}_c C_{p_c}} = 20°\text{C} + \frac{(13\text{ kg/s})(2409.1\text{ kJ/kg})}{(500\text{ kg/s})(4.18\text{ kJ/kg°C})} = \mathbf{35°C}$$

(Ans.)

(B) LMTD Method

EXAMPLE 11.2 *For what value of end temperature difference ratio ($\Delta T_{max}/\Delta T_{min}$) is the arithmetic mean temperature difference 7 per cent greater than the log mean temperature difference?*

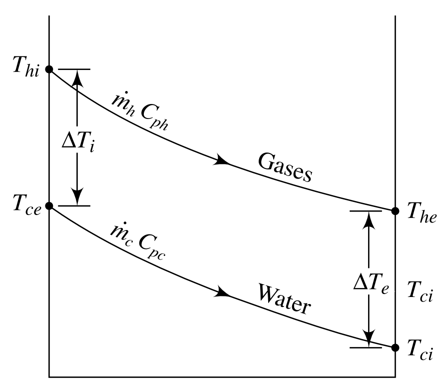
Solution

Known *AMTD* = 1.07 *LMTD*.

Find ($\Delta T_{\max}/\Delta T_{\min}$).

Assumptions (1) Steady-state conditions. (2) Constant properties.

Analysis AMTD (*arithmetic mean temperature difference*) $= \dfrac{\Delta T_{\max} + \Delta T_{\min}}{2}$



$$\text{LMTD (}\textit{log mean temperature difference}\text{)} = \frac{\Delta T_{\max} - \Delta T_{\min}}{\ln\left(\dfrac{\Delta T_{\max}}{\Delta T_{\min}}\right)}$$

The ratio

$$\frac{\text{AMTD}}{\text{LMTD}} = \frac{1}{2}\frac{\Delta T_{\max} + \Delta T_{\min}}{(\Delta T_{\max} - \Delta T_{\min})}\ln\left(\frac{\Delta T_{\max}}{\Delta T_{\min}}\right)$$

$$= \frac{1}{2}\frac{\Delta T_{\min}}{\Delta T_{\min}}\frac{[(\Delta T_{\max}/\Delta T_{\min}) + 1]}{[(\Delta T_{\max}/\Delta T_{\min}) - 1]}\ln\left(\frac{\Delta T_{\max}}{\Delta T_{\min}}\right)$$

Let $\dfrac{\Delta T_{\max}}{\Delta T_{\min}} = x$

Then $\dfrac{1.7\ \text{LMTD}}{\text{LMTD}} = \dfrac{1}{2}\dfrac{(x+1)}{(x-1)}\ln x$

or $\left(\dfrac{x+1}{x-1}\right)\ln x = 2.14$

By trial and error, for $x = 2.5$

$$\text{LHS} = \left(\frac{2.5+1}{2.5-1}\right)\ln 2.5 = 2.138$$

$$\text{RHS} = 2.14 \approx \text{LHS}$$

Hence, $x = \left(\dfrac{\Delta T_{\max}}{\Delta T_{\min}}\right) = \mathbf{2.5}$ **(Ans.)**

EXAMPLE 11.3 *In a counterflow steam superheater, steam enters at a pressure of 10 bar absolute as dry saturated vapour and leaves at 280°C. The hot flue gases (C_p = 1.05 kJ/kg °C) enter the superheater at 515°C. The mass-flow rates of steam and gases are 1200 kg/h and 2400 kg/h respectively. If the overall heat-transfer coefficient is 37 W/m² °C, determine the heat-transfer surface area required.*

Solution

Known A counterflow steam superheater operates under specified conditions.

Find Heat-exchanger area required, A(m^2).

Assumptions (1) Steady operating conditions prevail. (2) Heat exchanger is well insulated. (3) Changes in potential and kinetic energy are negligible. (4) Constant properties. (5) Uniform overall heat-transfer coefficient.

Analysis From steam tables:

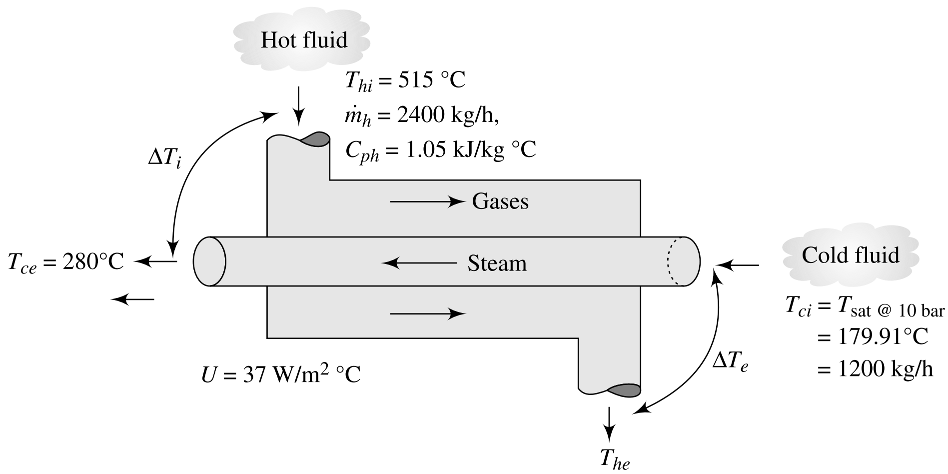
$$T_{ci} = T_{\text{sat}@P=10\,\text{bar}} = 179.91°\text{C}$$

The enthalpies of steam entering and leaving the superheater are:

$$h_i = h_{g\ @10\ \text{bar}} = 2778.1\ \text{kJ/kg}$$

$$h_e = h_{\text{sup}}\ (P = 10\ \text{bar},\ T = T_{ce} = 280°\text{C}) = 3008.2\ \text{kJ/kg}$$

Schematic



Heat transferred *from* hot combustion (*flue*) gases must equal the heat transferred *to* steam being superheated because the heat exchanger (*superheater*) is adiabatic with no loss to the surroundings. Hence,

$$\dot{Q}_{\text{out, gas}} = \dot{Q}_{\text{in, steam}} \quad \text{or} \quad \dot{m}_h C_{p_h}(T_{hi} - T_{he}) = \dot{m}_c(h_e - h_i)$$

Heat-transfer rate in the heat exchanger is

$$\dot{Q} = \dot{Q}_{\text{in, steam}} = \left(\frac{1200 \text{ kg}}{3600 \text{ s}}\right)[(3008.2 - 2778.1)\text{kJ/kg}]$$

$$= 76.7 \text{ kW} = \dot{Q}_{\text{out, gas}}$$

Temperature of gases leaving the superheater is

$$T_{he} = T_{hi} - \frac{\dot{Q}_{\text{out, gas}}}{\dot{m}_h C_{p_h}} = 515°\text{C} - \frac{76.7 \text{ kJ/s}}{(2400/3600)\text{kg/s} \times 1.05 \text{ kJ/kg°C}}$$

$$= 405.34°\text{C}$$

For counterflow arrangement:

$$\Delta T_i = T_{hi} - T_{ce} = 515 - 280 = 235°\text{C}$$

$$\Delta T_e = T_{he} - T_{ci} = 405.43 - 179.91 = 225.52°\text{C}$$

Log mean temperature difference,

$$\Delta T_{lm} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i / \Delta T_e)} = \frac{235 - 225.52}{\ln(235/225.52)} = 230.23°\text{C}$$

We note that, $\dot{Q} = UA\Delta T_{lm}$

Therefore, the heat-transfer surface area is determined to be

$$A = \frac{\dot{Q}}{U\Delta T_{lm}} = \frac{76.7 \times 10^3 \text{ W}}{37 \text{ W/m}^2 \text{°C} \times 230.23°\text{C}} = \mathbf{9.0 \ m^2} \qquad \textbf{(Ans.)}$$

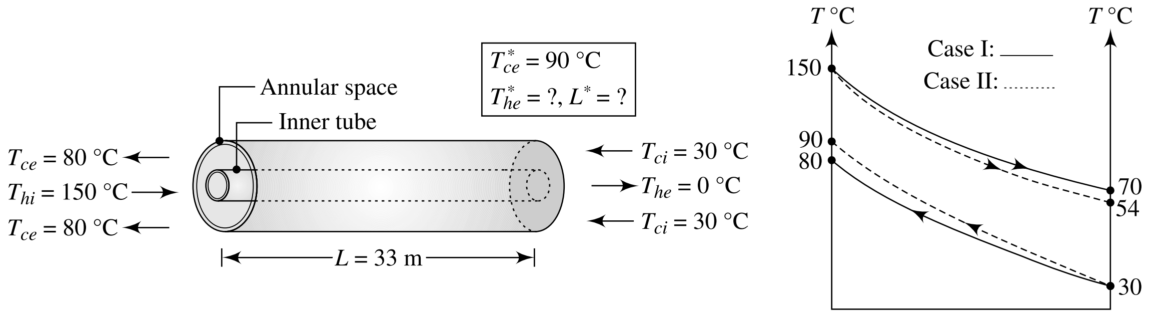
EXAMPLE 11.4 *A coaxial heat exchanger with counter-current arrangement of 33 m length is used to heat a cold fluid stream entering the annulus at 30°C and exiting at 80°C. The hot fluid stream passing through the inner tube enters at 150°C and leaves at 70°C. Determine the length of the heat exchanger if the cold fluid outlet temperature is to be increased by 10 K. The mass-flow rates, tube diameters and the fluid inlet temperatures are not to be altered for this purpose. Can parallel flow mode be used for the above two cases?*

Solution

Known A counterflow heat exchanger of prescribed length with all terminal temperatures specified.

Find New length of the exchanger if T_{ce} is to be raised from 80°C to 90°C.

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties and uniform heat-transfer coefficient. (3) No heat loss to the surroundings from the heat exchanger.

Analysis ***Case I***

Heat-transfer rate,

$$\dot{Q} = C_c(T_{ce} - T_{ci}) = C_h(T_{hi} - T_{he}) = UA\Delta T_{lm}$$

where $A = \pi DL, C_c = (\dot{m}C_p)_c$ and $C_h = (\dot{m}C_p)_h$

$$\Delta T_{lm} = \frac{(T_{hi} - T_{ce}) - (T_{he} - T_{ci})}{\ln\left(\dfrac{T_{hi} - T_{ce}}{T_{he} - T_{ci}}\right)} = \frac{(150 - 80) - (70 - 30)}{\ln\dfrac{70}{40}} = 53.6°\text{C}$$

$$\dot{Q} = U(\pi DL)\,\Delta T_{lm} = K \cdot L\Delta T_{lm} = K \times 33(\text{m}) \times 53.6(°\text{C})$$

Case II:

The cold fluid temperature at the outlet, T^*_{ce} = 80 + 10 = 90°C. There is no change in mass-flow rates, diameters, heat-transfer coefficients or inlet temperatures of the two fluids. The new heat rate is

$$\dot{Q}^* = C^*_c[T^*_{ce} - T_{ci}] = C^*_h[T_{hi} - T^*_{he}] = U(\pi DL^*)\Delta T^*_{lm} = K \times L^* \times \Delta T^*_{lm}$$

Heat-capacity rate ratio,

$$\frac{C^*_c}{C^*_h} = \frac{C_c}{C_h} = \frac{T_{hi} - T_{he}}{T_{ce} - T_{ci}} = \frac{150 - 70}{80 - 30} = \frac{80}{50} = 1.6$$

With $T^*_{ce} = 90°\text{C}, \dot{Q}^* = C^*_c(T^*_{ce} - T_{ci}) = C^*_h(T_{hi} - T^*_{he})$

or $$Q^* = 1.6C_h^*(90-30) = C_h^*(150 - T_{he}^*)$$

$\therefore$ $$T_{he}^* = 150 - (1.6)(60) = 54°C$$

$$\Delta T_{lm}^* = \frac{(150-90)-(54-30)}{\ln\frac{60}{24}} = 39.3°C$$

Ratio of heat rates,

$$\frac{Q^*}{Q} = \frac{C_c^*}{C_c}\frac{(90-30)}{(80-30)} = \frac{60}{50} = \frac{K \times L^* \times \Delta T_{lm}^*}{K \times L \times \Delta T_{lm}}$$

Hence, the new length of the heat exchanger is

$$L^* = L \times \frac{60}{50} \times \frac{\Delta T_{lm}}{\Delta T_{lm}^*} = 33(\text{m}) \times 1.2 \times \frac{53.6(°C)}{39.3(°C)} = \mathbf{54\ m} \qquad \text{(Ans.)}$$

In both cases, we note that cold fluid exit temperatures are *greater* than the hot fluid exit temperatures. This is just *not* possible in the case of a parallel-flow heat exchanger.

EXAMPLE 11.5 *A two-fluid, twin-tube, co-axial, counterflow heat exchanger is used to cool liquid Dowtherm* J *flowing through the outer annulus at a flow rate of 0.25 kg/s. The inside diameter (ID) of the outer tube is 25 mm. Water with a flow rate of 0.5 kg/s passes through the 15 mm ID and 1.5 mm thick inner copper tube (*k *= 393 W/m K). The length of the heat exchanger is 1 m. If the overall thermal resistance of the heat exchanger is 0.001 K/W, determine the length of the exchanger.*

Fluid properties:

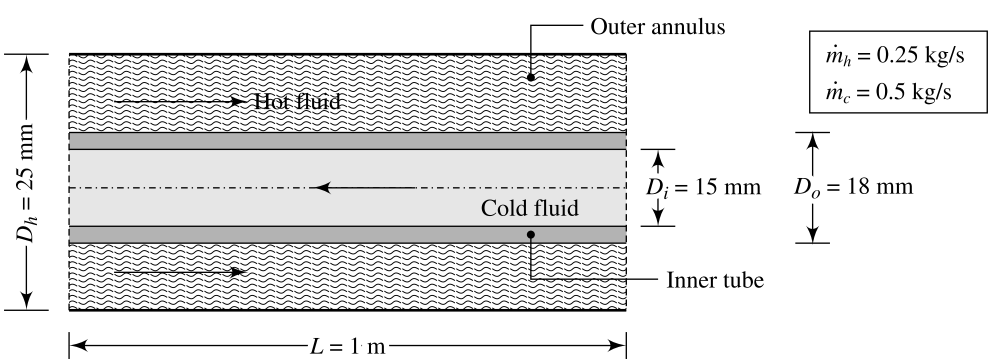
Dowtherm J:	k = *0.1185 W/m K*	μ = *1.72 × 10⁻⁴ kg/m s*	Pr = *3.48*
Water:	k = *0.680 W/m K*	μ = *2.79 × 10⁻⁴ kg/m s*	Pr = *1.76*

Solution

Known A concentric counterflow heat exchanger of given dimensions. Flow rates of the two fluids and their properties are specified.

Find Length of the heat exchanger for prescribed overall thermal resistance.

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties and uniform heat-transfer coefficient. (3) Fully developed fluid flows.

Analysis Overall thermal resistance is

$$\Sigma R_{\text{th}} = \frac{1}{UA} = \left[\frac{1}{h_i A_i} + \frac{1}{2\pi kL}\ln\frac{D_o}{D_i} + \frac{1}{h_o A_o}\right]\left(\frac{K}{W}\right)$$

Flow of water through inner tube:

$$D_i = 15 \times 10^{-3}\ \text{m},\ \dot{m}_c = 0.5\ \text{kg/s}$$

$$Re_D = \frac{4\dot{m}_c}{\pi D_i \mu} = \frac{4 \times 0.5(\text{kg/s})}{\pi \times 0.015(\text{m}) \times 2.79 \times 10^{-4}\ (\text{kg/m s})} = 1.52 \times 10^5\ (> 2300)$$

Accordingly, the flow is *turbulent: Pr* = 1.76

Using Dittus–Boelter equation:

$$Nu_D = 0.023(Re_D)^{0.8}(Pr)^{0.4}$$

Here, $n = 0.4$ as water is being heated.

$$\therefore \quad Nu_D = 0.023(1.52 \times 10^5)^{0.8}(1.76)^{0.4} = 403 = \frac{h_i D_i}{k}$$

Inside convective resistance is

$$\frac{1}{h_i A_i} = \frac{1}{h_i \pi D_i L} = \frac{1}{h_i D_i} \times \frac{1}{\pi \times 1} = \frac{1}{\pi k Nu_D} = \frac{1}{\pi \times 0.68 \times 403}$$

$$= \mathbf{1.161 \times 10^{-3}\ K/W}$$

The conduction resistance is

$$\frac{1}{2\pi kL}\ln\frac{D_o}{D_i} = \frac{1}{2\pi \times 393(\text{W/m K}) \times 1}\ln\left(\frac{18\ \text{mm}}{15\ \text{mm}}\right)$$

$$= \mathbf{7.836 \times 10^{-5}\ K/W}$$

Outside convective resistance is

$$\frac{1}{h_o A_o} = \frac{1}{h_o \pi D_o L}$$

Flow of liquid Dowtherm J through outer annulus:

Equivalent diameter,

$$D_e = \frac{4A_c}{P} = \frac{4 \times \frac{\pi}{4}[D_h^2 - D_o^2]}{\pi(D_h + D_o)}$$

where D_h is the ID of the outer tube.

$$\therefore \quad D_e = D_h - D_o = (25 - 18)\ (\text{mm}) = 7\ \text{mm} = 0.007\ \text{m}$$

$$Re_{D_e} = \frac{\rho V D_e}{\mu} = \frac{\rho \cdot \dfrac{\dot{m}_h}{\rho \frac{\pi}{4}(D_h^2 - D_o^2)}(D_h - D_o)}{\mu} = \frac{4\dot{m}_h}{\pi(D_h + D_o)\mu}$$

$$= \frac{4 \times 0.25(\text{kg/s})}{\pi(0.025 + 0.018)(\text{m}) \times (1.72 \times 10^{-4})(\text{kg/m s})}$$

$$= 43\ 038 \qquad (> 2300)$$

The annular flow is therefore *turbulent.*
As the liquid is getting cooled, $n = 0.3$

$$\therefore \quad Nu_{D_e} = 0.023(Re_{De})^{0.8}(Pr)^{0.3}$$

$$= 0.023\ (43038)^{0.8}\ (3.48)^{0.3} = 170.3 = \frac{h_o D_e}{k}$$

$$\frac{1}{h_o \pi D_o L} = \frac{D_e}{\pi D_o L Nu_D k} = \frac{0.007\ \text{m}}{\pi \times 0.018\ \text{m} \times 1\ \text{m} \times 170.3 \times 0.1185\ \text{W/m K}}$$

$$= 6.133 \times 10^{-3}\ \text{K/W}$$

Overall thermal resistance to heat transfer is

$$\Sigma R_{\text{th}} = \frac{1}{UA} = [(1.161 \times 10^{-3}) + (7.3836 \times 10^{-5}) + (6.133 \times 10^{-3})]\left(\frac{\text{K}}{\text{W}}\right)$$

$$= \mathbf{7.3678 \times 10^{-3}\ K/W}$$

This resistance corresponds to tube length, $L = 1$ m.
Hence, for an overall resistance of 0.0016 (K/W), the desired length of the tube will be

$$L = \frac{\Sigma R_{(L=1\text{m})} \times 1\ \text{m}}{\Sigma R_{(L)}} = \frac{7.3678 \times 10^{-3}\ (\text{K/W}) \times 1(\text{m})}{0.0016(\text{K/W})} = \mathbf{4.6\ m} \qquad \textbf{(Ans.)}$$

EXAMPLE 11.6 *A heat-exchanger design is required to cool 10 kg/s of gaseous methane at 10 bar, from 100°C to 20°C using 25 kg/s of sea water which is available at 5°C. The methane is to flow through 10 mm diameter tubes and the water flows over the outside.*

(a) Explain why this will not be able to operate in parallel flow.

(b) For a stack of 1000 tubes, evaluate the overall heat-transfer coefficient assuming that the tube side heat-transfer coefficient is the controlling heat-transfer coefficient.

(c) Find the heat exchanger surface area required.

(d) Estimate the length of tubing required.

For methane: $C_p = 2.226$ *kJ/kg K,* $\mu = 1.3 \times 10^{-5}$ *N s/m²,* $k = 0.03$ *W/m K*

For water: $C_p = 4.187$ *kJ/kg K*

Solution

Known Methane gas passing through the tubes is cooled in a heat exchanger by sea water flowing over the outside.

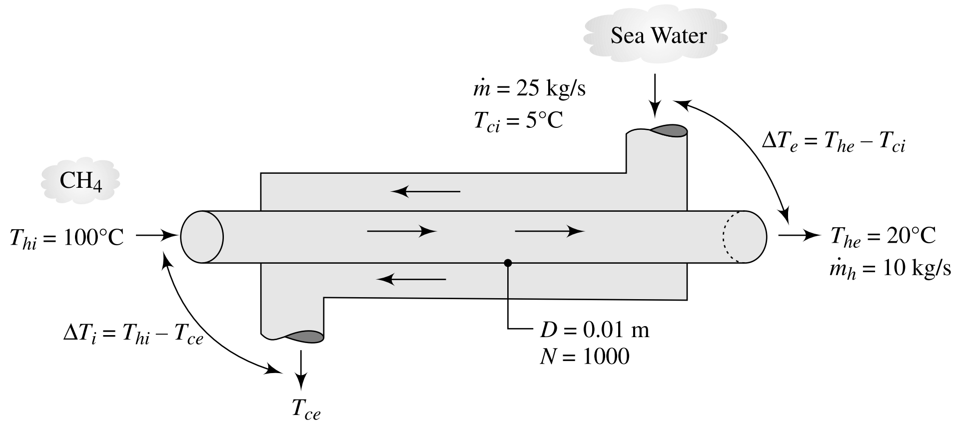
Find (a) Suitability for parallel-flow arrangement. (b) Overall heat-transfer coefficient. (c) Heat exchanger area. (d) Length of tubing required.

Assumptions (1) Steady-state conditions prevail. (2) Heat exchanger is adiabatic. (3) Kinetic and potential energy changes are negligible. (4) Gas-side heat-transfer coefficient is the controlling convection coefficient.

Analysis Heat-transfer rate in the heat exchanger,

$$\dot{Q} = UA_s\ \Delta T_{lm} = \underbrace{\dot{m}_c C_{pc}(T_{ce} - T_{ci})}_{\textbf{heat transfer to water}} = \underbrace{m_h C_{ph}(T_{hi} - T_{he})}_{\textbf{heat transfer from gas}}$$

Schematic



Cold fluid (*sea water*) outlet temperature,

$$T_{ce} = T_{ci} + \frac{\dot{m}_h C_{ph}}{\dot{m}_c C_{p_c}}(T_{hi} - T_{he})$$

$$= 5°\text{C} + \left(\frac{10 \text{ kg/s} \times 2.226 \text{ kJ/kg K}}{25 \text{ kg/s} \times 4.187 \text{ kJ/kg K}}\right)(100 - 20)°\text{C} = \mathbf{22°C}$$

For parallel-flow arrangement, T_{ce} is *always less* than T_{he}. However, in this case, T_{ce} is 22°C which is 2°C greater than T_{he}(= 20°C).

Hence, only counterflow mode is possible. **(Ans.) (a)**

To evaluate tube side (*gas side*) heat-transfer coefficient h_i, we first determine the Reynolds number.

$$Re = \frac{4\dot{m}(\text{per tube})}{\pi D \mu} = \frac{4 \times 10 \text{ kg/s}/1000}{\pi \times 0.01 \text{ m} \times 1.3 \times 10^{-5} \text{ kg/m s}}$$

$$= 97.94 \times 10^3 \quad \Rightarrow \quad \textit{Turbulent flow}$$

Prandtl number,

$$Pr = \frac{C_p \mu}{k} = \frac{(2226 \text{ J/kg K})(1.3 \times 10^{-5} \text{ N s/m}^2)}{0.03 \text{ W/m K}} \left|\frac{1 \text{ J}}{1 \text{ Nm}}\right| \left|\frac{1 \text{ W}}{1 \text{ J/s}}\right| = 0.9646$$

Using *Dittus Boelter* correlation, noting that n = 0.3 as methane (CH_4) is getting cooled, we have

$$Nu = \frac{h_i D}{k} = 0.023\ (Re)^{0.8}\ (Pr)^{0.3}$$

The heat-transfer coefficient (hot side) is

$$h_i = 0.023 \times \frac{0.03 \text{ W/m K}}{0.01 \text{ m}} \times (97.94 \times 10^3)^{0.8} \times (0.9646)^{0.3}$$

$$= 671.3 \text{ W/m}^2 \text{ K}$$

Assuming thin walled tubes without fouling, the overall heat-transfer coefficient, $U \approx h_i$ because h_i is the *controlling* heat-transfer coefficient.

Hence,

$$U = \mathbf{671.3\ W/m^2\ K} \qquad \textbf{(Ans.) (b)}$$

LMTD for counterflow configuration is

$$\Delta T_{lm} = \frac{(T_{hi} - T_{ce}) - (T_{he} - T_{ci})}{\ln[(T_{hi} - T_{ce})/(T_{he} - T_{ci})]} = \frac{(100 - 22) - (20 - 5)}{\ln[(100 - 22)/(20 - 5)]} = \mathbf{38.2°C}$$

Heat-flow rate,

$$\dot{Q} = \dot{m}_h C_{ph}(T_{hi} - T_{he})$$

$$= (10\ \text{kg/s})\ (2.226\ \text{kJ/kg K})\ (100 - 20)\ \text{K} = 1780.8\ \text{kW}$$

Heat-exchanger surface area,

$$A_s = \frac{\dot{Q}}{U\Delta T_{lm}} = \frac{1780.8 \times 10^3\ \text{W}}{671.3\ \text{W/m}^2\ \text{K} \times 38.2°\text{C}} = \mathbf{69.4\ m^2} \qquad \textbf{(Ans.) (c)}$$

Length of the tubing required,

$$L = \frac{A_s}{N(\pi D)} = \frac{69.4\ \text{m}^2}{(1000) \times (\pi \times 0.01\ \text{m})} = \mathbf{2.21\ m} \qquad \textbf{(Ans.) (d)}$$

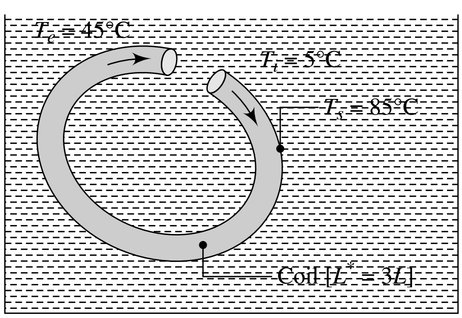
EXAMPLE 11.7 *A coil of single tubing is provided in a reaction vessel whose contents are at a uniform temperature of 85°C. The inlet and outlet temperatures of cooling water flowing through the tube are 5°C and 45°C respectively. If the tube length is increased three times the original, what would be the outlet temperature of the water. Assume the overall heat-transfer coefficient and the water flow rate to remain constant.*

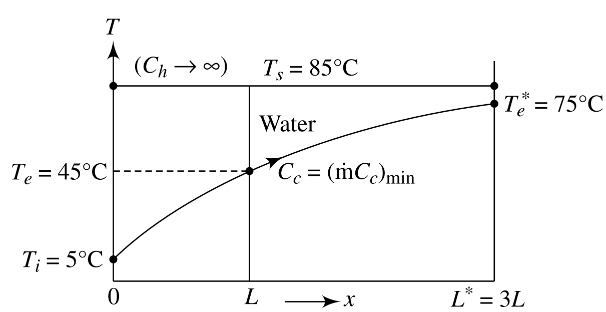
Solution

Known A single tube coil at constant surface temperature. Inlet and outlet temperatures of water flowing through the tube.

Find Water outlet temperature if length is increased threefold.

Schematic





Assumptions (1) Flow rate of water is constant. (2) Uniform and constant overall heat-transfer coefficient.

Analysis From energy balance:

$$(\dot{m}C_p)_c(T_e - T_i) = U\,A_s \Delta T_{lm}$$

As $\quad T_s = \text{const}, C_{h\to\infty}, \quad C_c = (\dot{m}C_p)_c = (\dot{m}C_p)_{\min}$

$$\Delta T_{lm} = \frac{(T_s - T_i) - (T_s - T_e)}{\ln(T_s - T_i)/(T_s - T_e)} = \frac{T_e - T_i}{\ln\left(\dfrac{T_s - T_i}{T_s - T_e}\right)}$$

and $\quad A_s$ = PL where P is the perimeter of the tube.

or $$\ln\left(\frac{T_s - T_i}{T_s - T_e}\right) = \frac{UA_s}{(\dot{m}C_p)_{\text{water}}} = \frac{\text{UPL}}{C_c}$$

$\therefore$ $$\frac{\text{UPL}}{C_c} = \ln\left(\frac{85-5}{85-45}\right) = 0.6931$$

When the length is increased (***threefold***), we have

$$\ln\left(\frac{T_s - T_i}{T_s - T_e^*}\right) = \frac{\text{UPL}^*}{C_c} \quad \text{or} \quad \frac{T_s - T_e^*}{T_s - T_i} = \exp\left[-3\frac{\text{UPL}}{C_c}\right]$$

Outlet temperature of water is then found to be

$$T_e^* = T_s - (T_s - T_i)\exp[-3 \times 0.6931]$$
$$= 85°\text{C} - (85-5)°\text{C}\exp(-2.0793) = \mathbf{75°C} \qquad \textbf{(Ans.)}$$

EXAMPLE 11.8 *A simple heat exchanger consisting of two concentric-flow passages is used for heating 1110 kg/h of oil (specific heat = 2.1 kJ/kg K) from a temperature of 27°C of 49°C. The oil flows through the inner pipe made of copper (*OD *= 2.86 cm,* ID *= 2.54 cm) and the surface heat-transfer coefficient on the oil side is 635 W/m² K. The oil is heated by hot water supplied at the rate of 390 kg/h and at an inlet temperature of 95°C. The water side heat-transfer coefficient is 1270 W/m² K. Take the thermal conductivity of copper to be 350 W/m K and the fouling factors on the oil and water sides to be 0.0001 and 0.0004 m² K/W. What is the length of heat exchanger for (a) parallel flow, and (b) counterflow?*

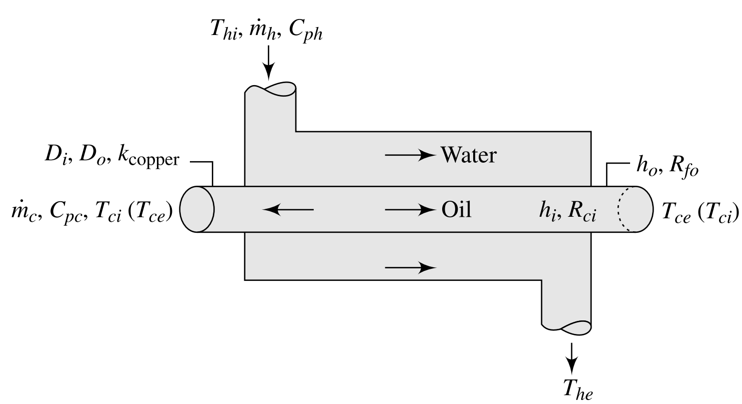
[NU: W 2002]

Solution

Known A double-pipe heat exchanger with cold oil in inner pipe being heated by hot water in the annulus.

Find L = ? for (a) parallel flow, and (b) counterflow

Schematic



T_{ci} = 27°C	T_{ce} = 49°C
$\dot{m}_{\text{oil}} = \dot{m}_c = 1110$ kg/h	$\dot{m}_{\text{water}} = \dot{m}_h = 390$ kg/h
C_{pc} = 2.1 kJ/kg K	C_{ph} = 4.2 kJ/kg K (*assumed*)
D_i = 2.54 cm	D_o = 2.86 cm
h_i = 635 W/m² K	k_{copper} = 350 W/m K
T_{hi} = 95°C	h_o = 1270 W/m² K
R_{fi} = 0.0001 m² KW	R_{fo} = 0.0004 m² K/W

Assumptions (1) Steady operating condition. (2) U is constant throughout the length of the exchanger.

Analysis Heat-transfer rate,

$$\dot{Q} = UA\Delta T_{lm} = C_c(T_{ce} - T_{ci}) = C_h(T_{hi} - T_{he})$$

$$C_c = \dot{m}_c C_{p_c} = \left(\frac{1110}{3600}\right)\frac{\text{kg}}{\text{s}} \times (2.1 \times 10^3)\frac{\text{J}}{\text{kg K}} = 647.5 \text{ W/K}$$

$$C_h = \dot{m}_h C_{p_h} = \left(\frac{390}{3600}\right) \times (4.2 \times 10^3) = 455 \text{ W/K}$$

Clearly, $C_h < C_c$ and, hence, $C_c = C_{\max}$ and $C_h = C_{\min}$

$\therefore$ $\dot{Q} = C_c(T_{ce} - T_{ci}) = 647.5$ W/K$(49 - 27)$K = 14 245 W

$$\frac{1}{UA} = \sum R_{\text{th}} = \frac{1}{h_i A_i} + \frac{1}{h_o A_o} + \frac{\ln D_o/D_i}{2\pi kL} + \frac{R_{fi}}{A_i} + \frac{R_{fo}}{A_o}$$

$$= \frac{1}{(635)(\pi \times 0.0254 \times L)} + \frac{1}{(1270)(\pi \times 0.0286 \times L)}$$

$$+ \frac{\ln(2.86/2.54)}{2\pi \times 350 \times L} + \frac{0.0001}{\pi \times 0.0254 \times L} + \frac{0.0004}{\pi \times 0.0286 \times L}$$

$$= \frac{1}{L}[(19.735 \times 10^{-3}) + (8.764 \times 10^{-3})$$

$$+ (0.054 \times 10^{-3}) + (1.253 \times 10^{-3}) + (4.452 \times 10^{-3})]$$

$$= \frac{34.258 \times 10^{-3}}{L}(\text{K/W})$$

$\therefore$ $$UA = \frac{L}{0.034258} = 29.19\, L(\text{W/K})$$

(a) ***Parallel flow***:

$$\Delta T_{lm} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}$$

where $\Delta T_i = T_{hi} - T_{ci}$

$\Delta T_e = T_{he} - T_{ce}$

We note that

$$\dot{Q} = C_h(T_{hi} - T_{he})$$

$$\therefore \quad T_{he} = T_{hi} - \frac{\dot{Q}}{C_h} = 95 - \left(\frac{14\,245}{455}\right) = \mathbf{63.7°C}$$

$$\therefore \quad \Delta T_i = 95 - 27 = 68°\text{C}$$

$$\Delta T_e = 63.7 - 49 = 14.7°\text{C}$$

$$\therefore \quad \Delta T_{lm} = \frac{68 - 14.7}{\ln(68/14.7)} = 34.8°\text{C}$$

As $\dot{Q} = UA\Delta T_{lm}$

$$14\,245 = (29.19\ L)\ (34.8°\text{C})$$

$\therefore$ Length of the heat exchanger,

$$L = \frac{14245}{29.19 \times 34.8} = \mathbf{14.0\ m} \qquad \textbf{(Ans.) (a)}$$

(b) ***Counterflow:***

$$\Delta T_i = T_{hi} - T_{ce} = 95 - 49 = 46°\text{C}$$

$$\Delta T_e = T_{he} - T_{ci} = 63.7 - 27 = 36.7°\text{C}$$

$$\therefore \quad \Delta T_{lm} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)} = \frac{46 - 36.7}{\ln(46/36.7)} = 41.18°\text{C}$$

Length of the heat exchanger,

$$L = \frac{14245}{29.19 \times 41.18} = \mathbf{11.85\ m} \qquad \textbf{(Ans.) (b)}$$

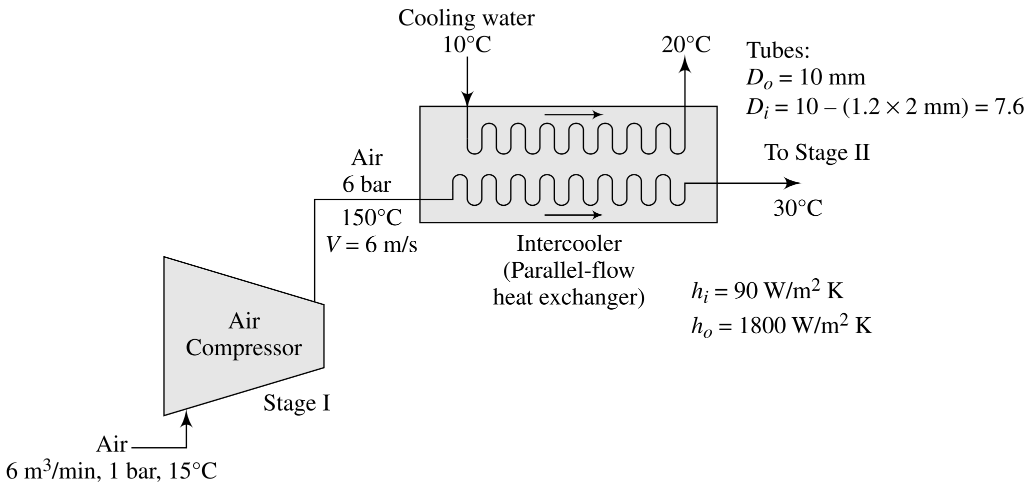
EXAMPLE 11.9 *An intercooler of air compressor takes in air at 6 bar and 150°C and passes it to the next stage at 30°C and at the equivalent rate of 6 m^3 of free air (15°C and 1 bar) per minute. The cooling water passes in parallel flow over the tubes which are 10 mm outside diameter and 1.2 mm thick. The inlet and outlet water temperatures are 10°C and 20°C, respectively and the air velocity at entry into the tubes in 6 m/s. Inside heat-transfer coefficient (air side) is 90 W/m^2 K and outside heat-transfer coefficient (water side) is 1800 W/m^2 K. Find (a) the number of tubes in the intercooler, and (b) the length of each tube. What will be the saving in the total tube length if the cooler is made counterflow with the inlet and outlet temperatures being maintained the same as before?* **[NU: S 2001]**

Solution

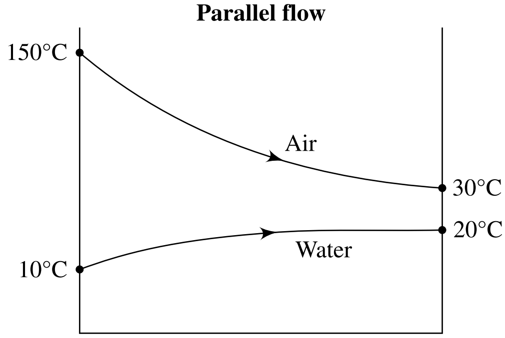
Known Intercooler of an air compressor with parallel flow configuration.

Find (a) Number of tubes, N; (b) Tube length, L, (c) Saving in tube length if the intercooler has counterflow arrangement.

Schematic



Parallel flow

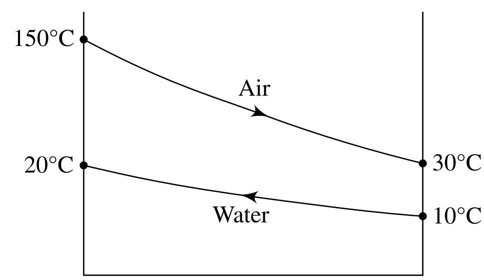


Counter flow: $\Delta T_i = 130°C$

$\Delta T_e = 20°C$

$$\Delta T_{\ell m,CF} = \frac{130-20}{ln(130/20)} = \mathbf{58.77°C}$$

Counter flow



Parallel flow: $\Delta T_i = 140°C$

$\Delta T_e = 10°C$

$$\Delta T_{\ell m,PF} = \frac{140-20}{ln(140/10)} = \mathbf{49.26°C}$$

Assumptions (1) Steady operating conditions. (2) U is constant along the exchanger.

Analysis Mass-flow rate of air.

$$\dot{m}_{\text{air}} = \frac{P_a \dot{V}_a}{R_a T_a} = \frac{100 \times (6/60)}{0.287 \times 288.15} = 0.1209 \text{ kg/s}$$

Heat-transfer rate,

$$\dot{Q} = \dot{m}_{\text{air}} C_{\text{pair}} (T_{hi} - T_{he}) = 0.1209 \times 1.005 \times (150 - 30)$$
$$= 14.583 \text{ kW}$$

Overall heat-transfer coefficient,

$$U_o = \left[\frac{1}{h_o} + \frac{D_o}{D_i}\frac{1}{h_i}\right]^{-1} = \left[\frac{1}{1800} + \frac{10}{7.6}\frac{1}{90}\right]^{-1} = 65.9 \text{ W/m}^2\text{K}$$

Area of heat exchanger,

$$A_o = N\pi D_o L$$

where N is the number of tubes

$$\text{Mass-flow rate, } \dot{m}_{\text{air}} = \rho A_c V = \rho \frac{\pi}{4} D_i^2 NV$$

Density of air at the average temperature, i.e., (150 + 30)/2 = 90°C is

$$\rho = \frac{P}{RT} = \frac{600}{0.287 \times (90 + 273.15)} = 5.757 \text{ kg/m}^3$$

$$\therefore \quad N = \frac{\dot{m}_{\text{air}}}{\rho(\pi/4)D_i^2 V} = \frac{0.1209}{5.757 \times (\pi/4) \times 0.0076^2 \times 6} = 77.2 \approx 78 \qquad \textbf{(Ans.) (a)}$$

Tube length with *parallel-flow* arrangement,

$$L_{\text{PF}} = \frac{A_{o,\text{PF}}}{N\pi D_o}$$

where $$A_{o,\text{PF}} = \frac{\dot{Q}}{U_o \Delta T_{lm,\text{PF}}} = \frac{14\ 583}{65.9 \times 49.26} = 4.492 \text{ m}^2$$

$$\therefore \quad L_{\text{PF}} = \frac{4.492}{78\pi \times 0.01} = \mathbf{1.83\ m} \qquad \textbf{(Ans.) (b)}$$

We note that $L \propto \dfrac{1}{\Delta T_{lm}}$

∴ Tube length with counterflow arrangement,

$$L_{\text{CF}} = L_{\text{PF}} \times \frac{\Delta T_{lm,\text{PF}}}{\Delta T_{lm,\text{CF}}} = 1.83 \times \frac{49.26}{58.77} = \mathbf{1.53\ m}$$

Percentage saving in tube length

$$= \left(\frac{1.83 - 1.53}{1.83}\right)(100) = \mathbf{16.4\%} \qquad \textbf{(Ans.) (c)}$$

(C) Multipass Heat Exchangers, Condensers and Evaporators

EXAMPLE 11.10 *In a solar power plant, 1 kg/s of saturated liquid Freon (refrigerant) is to be completely evaporated at 100°C, in a counterflow heat exchanger. Thermic fluid at 8 kg/s and 120°C is available for this purpose.*

(a) Determine the heat-transfer area

(b) What is the additional area required, if the Freon after evaporation is to be superheated to 110°C?

Neglect fouling in either case.
Data:
For thermic fluid:

C_p = *1.62 kJ/kg K* h = *450 W/m² K*

For Freon:

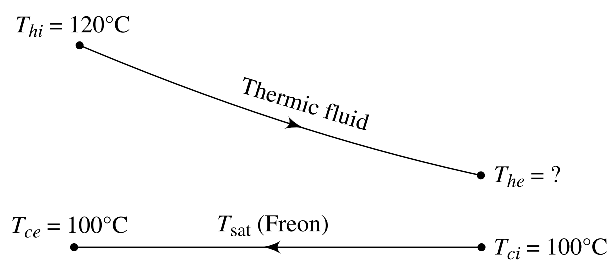
h_{fg} *at 100°C = 161.8 kJ/kg*
C_p *of superheated vapour = 0.63 kJ/ kg K*
h *during evaporation = 2500 W/m² K*
h *during superheating = 220 W/m² K* **[IIT, Bombay]**

Solution

Known In a solar power plant, liquid refrigerant undergoes evaporation by thermic fluid.

Find (a) Heat-exchanger area, A(m²). (b) Additional area, if the refrigerant is super heated.

Schematic



Hot side :	$\dot{m}_h = 8$ kg/s,	$C_{ph} = 1.62$ kJ/kg K
Cold side :	$\dot{m}_c = 1$ kg/s,	$h_{fg_c} = 161.8$ kJ/kg

Assumptions (1) Steady operating conditions. (2) Fouling is neglected on both cold and hot fluid sides. (3) Constant thermal properties of fluids.

Analysis (a) Evaporation ***(No superheating):***

By energy balance:

$$\begin{pmatrix}\text{Heat given up by} \\ \text{thermic fluid}\end{pmatrix} = \begin{pmatrix}\text{Heat received by saturated liquid Freon} \\ \text{for evaporation (phase change)}\end{pmatrix}$$

Heat-transfer rate,

$$\dot{Q} = \dot{m}_h C_{ph}(T_{hi} - T_{he}) = \dot{m}_c h_{fg_c}$$

$$= 8 \text{ kg/s} \times 1.62 \text{ kJ/kg K} \times (120 - T_{he}) \text{ °C} = 1 \text{ kg/s} \times 161.8 \text{ kJ/kg}$$

$$= \mathbf{161.8 \text{ kW}}$$

$$\therefore \qquad 120\text{°C} - T_{he} = \frac{161.8 \text{ kJ/s}}{8 \text{ kg/s} \times 1.62 \text{ kJ/kg K}} = 12.48\text{°C}$$

Thermic fluid exit temperature,

$$T_{he} = 120 - 12.48 = 107.52°C$$

$$\text{LMTD or } \Delta T_{lm} = \frac{(T_{hi} - T_{ce}) - (T_{he} - T_{ci})}{\ln\left(\dfrac{T_{hi} - T_{ce}}{T_{he} - T_{ci}}\right)} = \frac{(120 - 100) - (107.52 - 100)}{\ln\dfrac{120 - 100}{107.52 - 100}} = \frac{20 - 7.52}{\ln\dfrac{20}{7.52}}$$

$$= \mathbf{12.76°C \text{ or } K}$$

Heat rate, $\dot{Q} = UA\Delta T_{lm}$

Overall heat-transfer coefficient,

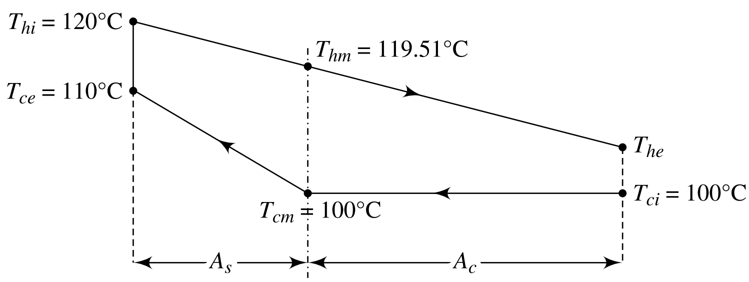
$$U = \left[\frac{1}{h_h} + \frac{1}{h_c}\right]^{-1} = \left[\frac{1}{450} + \frac{1}{2500}\right]^{-1} = \mathbf{381.36\ W/m^2\ K}$$

Heat-transfer area,

$$A = \frac{\dot{Q}}{U\Delta T_{lm}} = \frac{161.8 \times 10^3\ \text{W}}{381.36\ \text{W/m}^2\ \text{K} \times 12.76\ \text{K}} = \mathbf{33.25\ m^2} \qquad \textbf{(Ans.) (a)}$$

(b) ***Evaporation + Superheating to 110°C.***

Let suffix *e* stand for evaporation section and *s* for superheating section. Consider the superheater part.



$$\dot{Q}_s = \dot{m}_h C_{p_h}(T_{hi} - T_{hm}) = \dot{m}_c C_{pc}(T_{ce} - T_{cm})$$

$$= 8\ \text{kg/s} \times 1.62\ \text{kJ/kg K} \times (120 - T_{hm})°\text{C}$$

$$= 1\ \text{kg/s} \times 0.63\ \text{kJ/kg K} \times (110 - 100)°\text{C} = \mathbf{6.3\ kW}$$

$$\therefore \quad T_{hm} = 120°\text{C} - \frac{6.3\ \text{kJ/s}}{8\ \text{kg/s} \times 1.62\ \text{kJ/kg K}}$$

$$= \mathbf{119.51°C}$$

$$\therefore \quad \Delta T_{lm,s} = \frac{(T_{hi} - T_{ce}) - (T_{hm} - T_{cm})}{\ln\left(\dfrac{T_{hi} - T_{ce}}{T_{hm} - T_{cm}}\right)} = \frac{(120 - 110) - (119.51 - 100)}{\ln\left(\dfrac{120 - 100}{119.51 - 100}\right)} = \frac{10 - 19.51}{\ln\dfrac{10}{19.51}} = \mathbf{14.23°C}$$

Overall heat-transfer coefficient *(superheating section)* is

$$U_s = \left[\frac{1}{h_h} + \frac{1}{h_s}\right]^{-1} = \left[\frac{1}{450} + \frac{1}{220}\right]^{-1} = \mathbf{147.76\ W/m^2\ K}$$

Heat-exchanger area *(superheating section)* is

$$A_s = \frac{\dot{Q}_s}{U_s \Delta T_{lm,s}} = \frac{6.3 \times 10^3 \text{ W}}{147.76 \text{ W/m}^2 \text{ K} \times 14.23°\text{C}} = \mathbf{3.00 \ m^2}$$

In the evaporator section, $\dot{Q}_e$ and U_e do not change.

$\therefore$ $\dot{Q}_e$ = 161.8 kW; U_e = 381.36 W/m² K

$$\dot{Q}_e = \dot{m}_h C_{ph}(T_{hi} - T_{he})$$

$$161.8 \text{ kW} = 8 \text{ kg/s} \times 1.62 \text{ kJ/kg K} \times (119.51 - T_{he})°\text{C}$$

$\therefore$ T_{he} = **107.03°C**

$$\therefore \quad \Delta T_{lm,e} = \frac{(T_{hm} - T_{cm}) - (T_{he} - T_{ci})}{\ln\left(\dfrac{T_{hm} - T_{cm}}{T_{he} - T_{ci}}\right)} = \frac{(119.51 - 100) - (107.03 - 100)}{\ln\left(\dfrac{119.51 - 100}{107.03 - 100}\right)}$$

$$= \frac{19.51 - 7.03}{\ln \dfrac{19.51}{7.03}} = \mathbf{12.23°C}$$

Heat-exchanger area *(evaporating section)* is

$$A_e = \frac{\dot{Q}_e}{U_e \Delta T_{lm,e}} = \frac{161.8 \times 10^3 \text{ W}}{381.36 \text{ W/m}^2 \text{ K} \times 12.23°\text{C}} = \mathbf{34.70 \ m^2}$$

$\therefore$ Additional area required

$= A_s + A_e - A = (3.00 + 34.70 - 33.25)$ m² = **4.45 m²** **(Ans.) (b)**

EXAMPLE 11.11 *In an existing condenser of surface area 32 m², steam condenses at a pressure of 0.25 bar [*T_{sat} *= 65°C,* h_{fg} *= 2345.4 kJ/kg]. The overall heat-transfer coefficient based on previous experience is estimated to be 865 W/m² °C. Cooling water enters the condenser at 17°C with a flow rate of 14 kg/s. The mean specific heat of water may be taken as 4180 J/kg °C. Calculate the exit temperature of water and the rate of condensation of steam assuming that the steam entering the condenser is dry saturated vapour and at the exit, it is saturated liquid.*

How will the performance of the heat exchanger get affected if the overall heat-transfer coefficient is doubled?

Solution

Known Steam condenses in a condenser by circulating cooling water under specified conditions.

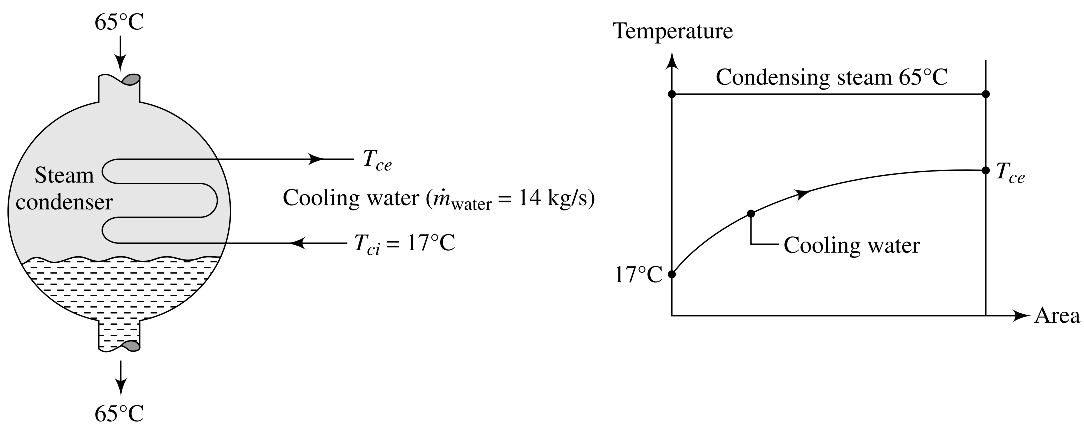
Find Cooling water exit temperature and steam-condensation rate. Effect of increasing the overall heat-transfer coefficient two fold on the exchanger performance.

Assumptions (1) Steady operating conditions exist. (2) Fluid properties are constant. (3) Uniform heat-transfer coefficient. (4) The condenser is effectively insulated.

Analysis Heat-transfer rate in the condenser,

$$\dot{Q} = \dot{m}_{\text{steam}} h_{fg} = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) = UA\Delta T_{lm}$$

Schematic



Log mean temperature difference,

$$\Delta T_{lm} = \frac{(T_{\text{sat}} - T_{ci}) - (T_{\text{sat}} - T_{ce})}{\ln\left(\dfrac{T_{\text{sat}} - T_{ci}}{T_{\text{sat}} - T_{ce}}\right)} = \frac{T_{ce} - T_{ci}}{\ln\left(\dfrac{\Delta T_i}{\Delta T_e}\right)}$$

Also, $$\Delta T_{lm} = \frac{\dot{Q}}{UA} = \frac{\dot{m}_c C_{pc}(T_{ce} - T_{ci})}{UA}$$

Hence $$\ln\frac{\Delta T_i}{\Delta T_e} = \frac{UA}{\dot{m}_c C_{pc}} = \frac{(865\ \text{W/m}^2\,°\text{C})(32\ \text{m}^2)}{(14\ \text{kg/s})(4180\ \text{J/kg}\,°\text{C})} = 0.473$$

where $\Delta T_i = T_{\text{sat}} - T_{ci} = 65 - 17 = 48°\text{C}$

and $\Delta T_e = T_{\text{sat}} - T_{ce} = 65 - T_{ce}$

∴ $$\frac{\Delta T_i}{\Delta T_e} = e^{0.473} = 1.6048$$

Hence, $$\Delta T_e = 65 - T_{ce} = \frac{48}{1.6048} = 29.9°\text{C}$$

Exit temperature of cooling water is

$T_{ce} = 65 - 29.9 =$ **35.1°C** **(Ans.)**

Rate of condensation of steam is

$$\dot{m}_{\text{steam}} = \frac{\dot{Q}}{h_{fg}} = \frac{\dot{m}_c C_{pc}(T_{ce} - T_{ci})}{h_{fg}} = \frac{(14\ \text{kg/s})(4180\ \text{J/kg}\,°\text{C})(35.1 - 17)°\text{C}}{2345.4 \times 10^3\ \text{J/kg}}$$

= **0.451 kg/s** **(Ans.)**

After the overall heat-transfer coefficient is *doubled*, the new value of U, i.e.,

$$U^* = 2U = 2 \times 865 = 1730\ \text{W/m}^2\,°\text{C}.$$

It follows that

$$\ln \frac{\Delta T_i}{\Delta T_e^*} = \frac{U^* A}{\dot{m}_c C_{pc}} = \frac{2UA}{\dot{m}_c C_{pc}} = 2 \times 0.473 = 0.946$$

$$\therefore \qquad \frac{\Delta T_i}{\Delta T_e^*} = 2.5754 \quad \text{and} \quad \Delta T_e^* = \frac{48}{2.5754} = 18.6°\text{C}$$

and $T_{ce}^* = 65 - 18.6 = 46.4$°C **(Ans.)**

Rate of condensation of steam,

$$\dot{m}_{\text{steam}}^* = \frac{m_c C_{pc}(T_{ce}^* - T_{ci})}{h_{fg}} = \frac{14 \times 4180 \times (46.4 - 17)}{2345.4 \times 10^3} = 0.733 \text{ kg/s}$$

Percentage increase of U = 100%

Percentage increase of steam condensation rate is

$$= \left(\frac{0.733 - 0.451}{0.451}\right)(100) = \mathbf{62.5\%}$$ **(Ans.)**

EXAMPLE 11.12 *Carbon dioxide from a gas-cooled reactor at 500°C and 4 bar pressure enters a shell and tube type steam generator at the flow rate of 90 × 10³ kg/h through the tubes and leaves the tubes at 330°C. The steam is generated dry and saturated at 250°C from the shell side. Using 2.5 cm ID copper tubes, 2 mm thick and designed for CO_2 mass-flow rate of 350 × 10³ kg/h m² calculate the length and number of tubes required. Neglect steam-side thermal resistance.*

Take the following properties of CO_2:

ρ = *3.26 kg/m³* $\qquad$ C_p = *1.172 kJ/kg K*

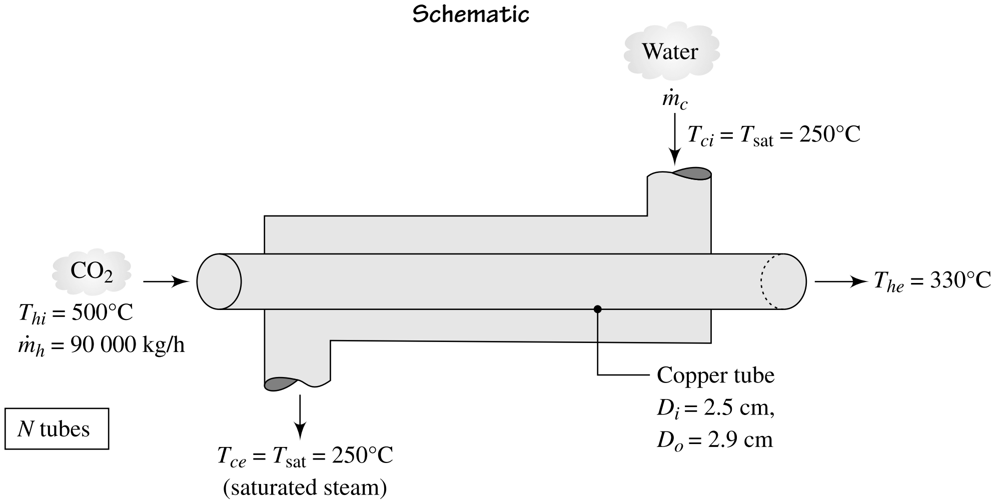
k = *0.043 W/m K* $\qquad$ μ = *0.0298 centipoise*

Solution

Known Shell-and-tube steam generator with steam on the shell side and CO_2 gas on the tube side operates under specified conditions.

Find Length of tubes, L(m). Number of tubes, N.

Schematic



Assumptions (1) Steady operating conditions exist. (2) Steam side thermal resistance is negligible. (3) Fully developed gas flow through the tube.

Analysis Log mean temperature difference,

$$\text{LMTD} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)} = \frac{(500-250)-(330-250)}{\ln(250/80)} = \frac{170}{\ln(25/8)} = 149.2°\text{C}$$

Mass-flow rate, $\dot{m} = (NA_c)\rho V$

$$\therefore \qquad \dot{m} = 90\times 10^3 \text{ kg/h and } \frac{\dot{m}}{(NA_c)} = 350\times 10^3 \text{ kg/h m}^2$$

$$NA_c = \frac{90\times 10^3}{350\times 10^3} = 0.257 \text{ m}^2$$

$\therefore$ Number of tubes required,

$$N = \frac{0.257}{(\pi/4)(0.025)^2} = \mathbf{523.85 = 524} \qquad \textbf{(Ans.)}$$

Overall heat-transfer coefficient, based on *the outer diameter*

$$U_o = \left[\frac{1}{h_o} + \frac{D_o}{D_i}\cdot\frac{1}{h_i}\right]^{-1}$$

As steam-side thermal resistance is negligible,

$$\frac{1}{h_o} = 0, \quad \text{and} \quad U_o = h_i \cdot \frac{D_i}{D_o} = h_i\left(\frac{2.5}{2.9}\right)$$

To find h_i, we use Dittus–Boelter equation:

$$\overline{Nu}_D = 0.023(Re_D)^{0.8}(Pr)^{0.3}$$

[*Note that here the index n is 0.3 as the fluid, i.e., CO_2 is getting cooled*]
Prandtl number,

$$Pr = \frac{C_p\mu}{k} = \frac{1.172\times 10^3\times 0.0298\times 10^{-3}}{0.043} = 0.8122 \quad [\textit{1 centipoise} = 10^{-3} \textit{ kg/m s}]$$

Reynolds number,

$$Re_D = \frac{\rho V D_i}{\mu} = \frac{4(\dot{m}/N)}{\pi D_i \mu} = \frac{4(90000/524\times 3600)\times 10^3}{\pi\times 0.025\times 0.0298} = 81\,540$$

$$Nu_D = 0.023(81540)^{0.8}(0.8122)^{0.3} = 183.53$$

$$\therefore \qquad h_i = \overline{Nu}_D\cdot\frac{k}{D_i} = 183.53\times\frac{0.043}{0.025} = 315.7 \text{ W/m}^2\text{ K}$$

$$U_o = 315.7 \times \frac{2.5}{2.9} = 272 \text{ W/m}^2\text{ K}$$

As $\qquad \dot{Q} = U_o A_o(\text{LMTD}) = \dot{m}_h C_{ph}(T_{hi} - T_{he})$

Outside surface area required,

$$A_o = \frac{90\,000 \times 1172 \times (500 - 330)}{272 \times 149.2 \times 3600} = 122.7 \text{ m}^2$$

Also, $A_o = N\pi D_o L$

Length of tubes, $L = \dfrac{122.7}{524 \times \pi \times 0.029}$ = **2.57 m** **(Ans.)**

EXAMPLE 11.13 *A single-pass steam condenser contains 100 thin-walled tubes of 25 mm nominal diameter and 2 m length. Cooling water enters at a temperature of 10°C, leaves at 50°C and flows through the tubes at a velocity of 2 m/s. The condenser pressure is 0.5 bar and the condensing heat-transfer coefficient is 5000 W/m² °C. Determine (a) the condensate-flow rate, and (b) the mean temperature of the tube metal work at the centre of the condenser length.*

The following properties of water may be used:

ρ = 995.8 kg/m³ μ = 801 × 10^{-6} kg/m s

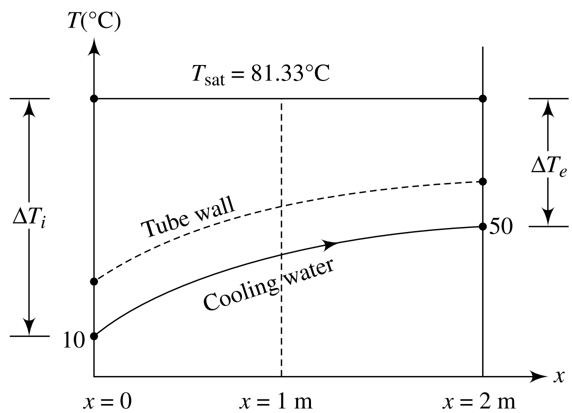
C_p = 4.178 kJ/kg K k = 0.617 W/m K

Solution

Known Steam condenser operates under the specified conditions.

Find (a) Condensate-flow rate. (b) Mean temperature of tube metal work at the midpoint.

Schematic



Assumptions (1) Steady operating conditions. (2) Uniform heat-transfer coefficient. (3) No fouling. (4) No subcooling of condensate.

Analysis The Dittus–Boelter equation can be used to find the heat-transfer coefficient on the cooling water side, h_i.

$$Re_D = \frac{\rho VD}{\mu} = \frac{(995.8 \text{ kg/m}^3)(2 \text{ m/s})(0.025 \text{ m})}{(801 \times 10^{-6} \text{ kg/m s})} = 62\,160$$

$$Pr = \frac{C_p \mu}{k} = \frac{(4178 \text{ J/kg K})(801 \times 10^{-6} \text{ kg/m s})}{0.617 \text{ W/m K}} = 5.42$$

$$Nu_D = 0.023(Re_D)^{0.8}(Pr)^{0.4} = 0.023(62\,160)^{0.8}(5.42)^{0.4} = 309 = \frac{h_i D}{k}$$

Water-side heat-transfer coefficient,

$$h_i = \frac{(309)(0.617 \text{ W/m K})}{0.025 \text{ m}} = 7629 \text{ W/m}^2 \text{ K}$$

The overall heat-transfer coefficient, U, is given by the following expression (*neglecting the thermal resistance of the tube*).

$$\frac{1}{U} = \frac{1}{h_i} + \frac{1}{h_o} = \frac{1}{7629} + \frac{1}{5000}$$

$\therefore$ $U = 3020 \text{ W/m}^2 \text{ K}$

At a pressure of 0.5 bar, the condensing temperature is 81.33°C (*from steam tables*). Neglecting the condensate subcooling,

$$\text{LMTD} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)} = \frac{(81.33 - 10) - (81.33 - 50)}{\ln\left[\dfrac{81.33 - 10}{81.33 - 50}\right]} = 48.6°\text{C}$$

and, the heat-transfer rate is

$$\dot{Q} = UA(\text{LMTD}) = U(N\pi DL)(\text{LMTD})$$

$$= (3020 \text{ W/m}^2 \text{ K})\ (100 \times \pi \times 0.025 \text{ m} \times 2 \text{ m})(48.6°\text{C}) \left|\frac{1 \text{ kW}}{10^3 \text{ W}}\right| = 2305.\ 5 \text{ kW}$$

Also, we note that, $\dot{Q} = \dot{m} h_{fg}$

Latent heat of condensation,

h_{fg} (at 0.5 bar) = 2305.4 kJ/kg

$\therefore$ Condensate mass-flow rate,

$$\dot{m} = \frac{\dot{Q}}{h_{fg}} = 2305.5 \text{ kJ/s}/2305.4 \text{ kJ/kg} = \mathbf{1.0\ kg/s} \qquad \textbf{(Ans.) (a)}$$

In order to find the tube metal work temperature at the centre of the exchanger, it is first necessary to calculate the cooling water temperature at this point.

The general equation for the temperature difference between the fluids may be expressed as $\log \Delta T = Ax + B$

where $\Delta T \equiv T_h - T_c = T_{sat} - T$, T is the cooling water temperature, and x is the distance along the heat exchanger as shown in the schematic.

At $x = 0$ m, $\Delta T_i = 71.33°\text{C}$

At $x = 2$ m, $\Delta T_e = 31.33°\text{C}$

$\log_{10} 71.33 = 0 + B$ and $\log_{10} 31.33 = 2A + B$

or $\log_{10} 71.33 + 2A = \log_{10} 31.33$ or $1.853 + 2A = 1.496$

$\therefore$ $A = -0.179$ and $B = 1.853$

Thus, the equation becomes

$$\boxed{\log_{10} \Delta T = -\ 0.179\ x + 1.853}$$

At the centre of the condenser length, i.e., at $x = 1$m

$$\log_{10} \Delta T_{\text{centre}} = -\ 0.179\ (1) + 1.853 = 1.674$$

$\therefore \qquad \Delta T_{\text{centre}} = 47.21°\text{C}$

and the cooling water temperature at $x = 1$ m,

$$T_{\text{centre}} = 81.33 - 47.21 = \mathbf{34.12°C}$$

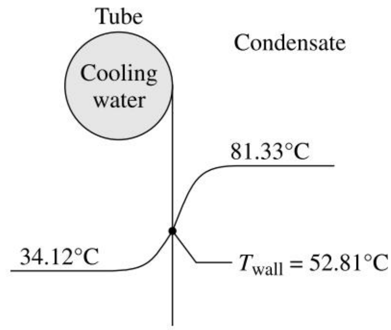
The radial conditions at the centre point are shown in the adjoining figure.

The radial heat-flow rate is given by

$$\dot{Q} = h_i A(T_{\text{wall}} - 34.12) = h_o A(81.33 - T_{\text{wall}})$$

where T_{wall} is metal work (wall) temperature

or $\qquad 7629(T_{\text{wall}} - 34.12) = 5000\ (81.33 - T_{\text{wall}})$



$\therefore \qquad T_{\text{wall}}$ = **52.81°C at midpoint** **(Ans.) (b)**

EXAMPLE 11.14 *A feedwater heater supplying hot water to a steam generator is a one-shell pass and two-tube pass heat exchanger with 100 thin-walled, 2.0 cm inside diameter tubes. Saturated steam at 1 atm condenses on the outside surface of the tubes with a heat-transfer coefficient of 10 000 W/m² °C. Due to space constraints the tube length per pass is restricted to 2.0 m. Water enters the tubes with a mass flow rate of 10 kg/s at 15°C. Calculate (a) the overall heat-transfer coefficient, and (b) the temperature of the water at the outlet.*

Properties of water:

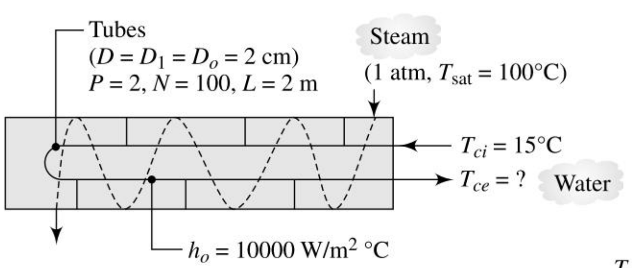
$\rho = 991$ kg/m³	$C_p = 4.179$ kJ/kg °C
$\mu = 0.631 \times 10^{-3}$ kg/m s	$k = 0.634$ W/m °C

Solution

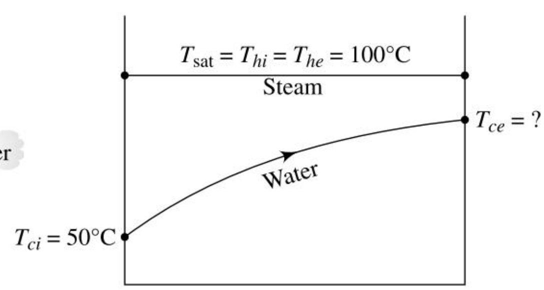
Known Feedwater heater (1-2 shell-and-tube heat exchanger) for heating water by condensing steam.

Find (a) U (W/m² °C); (b) T_{ce}(°C).

Schematic



One-shell-pass, two-tube-pass heat exchanger



Assumptions (1) Steady-state conditions. (2) Constant heat-transfer coefficient. (3) No fouling. (4) Tube-wall resistance is negligible. (5) Fully developed water flow.

Analysis Heat-transfer rate,

$$\dot{Q} = \dot{m}_{\text{steam}} h_{fg} = \dot{m}_c C_{p_c}(T_{ce} - T_{ci}) = UA_s \Delta T_{lm} \qquad \text{(A)}$$

where suffix c stands for cold fluid, i.e., water.

Overall heat-transfer coefficient, $U = \left[\frac{1}{h_i} + \frac{1}{h_o}\right]^{-1}$

where h_o = 10 000 W/m^2 °C

To evaluate h_i, we first determine the Reynolds number.

$$Re_D = \frac{\rho VD}{\mu} = \frac{4(\dot{m}/\text{tube})}{\pi D\mu} \qquad \textit{(for flow through a circular tube)}$$

$$= \frac{4 \times (10/100)\text{kg/s}}{\pi \times 0.02 \text{ m} \times (0.631 \times 10^{-3} \text{ kg/m s})} = 10\ 089 \quad \Rightarrow \quad \textit{turbulent flow}$$

$$Pr = \frac{C_p\mu}{k} = \frac{(4.179 \times 10^3 \text{ J/kg °C})(0.631 \times 10^{-3} \text{ kg/m s})}{0.634 \text{ W/m°C}}\left|\frac{1 \text{ W}}{1 \text{ J/s}}\right| = 4.16$$

For *fully developed turbulent flow*, using the Dittus–Boelter correlation with n = 0.4 (*water being heated*);

$$Nu_D = \frac{h_i D}{k} = 0.023(Re_D)^{0.8}(Pr)^{0.4}$$

$$= 0.023\ (10089)^{0.4}\ (4.16)^{0.4} = 64.93$$

Heat-transfer coefficient (*water side*),

$$h_i = Nu_D \frac{k}{D} = 64.93 \times \frac{0.634 \text{ W/m°C}}{0.02 \text{ m}} = 2058 \text{ W/m}^2\text{°C}$$

$\therefore$ $$U = \left[\frac{1}{2058} + \frac{1}{10000}\right] = \mathbf{1707\ W/m^2\ °C}$$ **(Ans.) (a)**

Heat-exchange surface area,

$$A_s = NP\pi DL = 100 \times 2 \times \pi \times 0.02 \text{ m} \times 2 \text{ m} = 25.13 \text{ m}^2$$

$$\Delta T_{lm} = [(T_{\text{sat}} - T_{ci}) - (T_{\text{sat}} - T_{ce})]/\ln\left(\frac{T_{\text{sat}} - T_{ci}}{T_{\text{sat}} - T_{ce}}\right)$$

$$= \frac{T_{ce} - T_{ci}}{\ln[(T_{\text{sat}} - T_{ci})/(T_{\text{sat}} - T_{ce})]} = \frac{T_{ce} - 15}{\ln\left(\frac{100 - 15}{100 - T_{ce}}\right)}$$

Substituting in Eq. (A):

$$(10 \text{ kg/s})\ (4179 \text{ J/kg °C})\ (T_{ce} - 15)\text{°C}$$

$$= 1707 \text{ W/m}^2\text{°C} \times 25.13 \text{ m}^2 \times \{(T_{ce} - 15)/\ln(85/100 - T_{ce})\}$$

or $$\ln\frac{85}{100 - T_{ce}} = \frac{1707 \times 25.13}{10 \times 4179} = 1.0266$$

or $$\frac{85}{100 - T_{ce}} = e^{1.0266} = 2.7916$$

Water exit temperature,

$$T_{ce} = 100 - \left(\frac{85}{2.7916}\right) = \mathbf{69.6°C}$$ **(Ans.) (b)**

EXAMPLE 11.15 *A single-pass multi-tube heat exchanger employed for the purpose of extracting heat from the exhaust gases of an internal combustion engine for a space heating system operates in counterflow. The exhaust gases (C_p = 1.1 kJ/kg °C and R = 0.3 kJ/kg K) flow through the tubes with a mean velocity of 20 m/s and are cooled from 300 to 100°C. The nominal tube diameter is 20 mm. Water flowing outside the tubes is heated from 20 to 90°C and the overall heat-transfer coefficient is 0.2 kW/m² °C.*

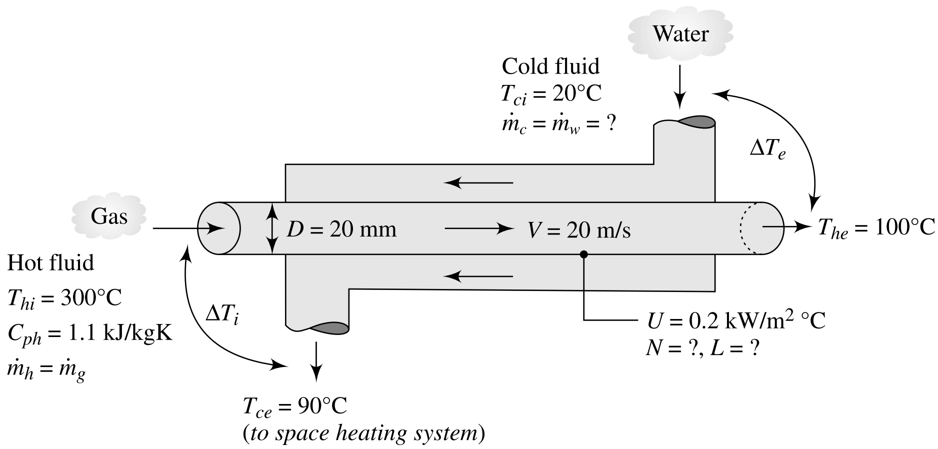
These conditions apply when the engine is producing a brake power (bp) of 100 kW. The brake specific fuel consumption (bsfc) is 0.2 kg/kWh, the air-fuel ratio (AFR) is 16 to 1, and the exhaust gas pressure is 100 mm of water. Estimate (a) the mass-flow rate of water, (b) the number of tubes required, and (c) the length of each tube.

Solution

Known Exhaust gases from an internal combustion (IC) engine are used to heat water in a counterflow heat exchanger under specified conditions.

Find (a) Mass-flow rate of water, $\dot{m}_w$ (kg/s). (b) Number of tubes, N. (c) Length of each tube, L(m).

Schematic



Assumptions (1) Steady operating conditions exist. (2) Gases are treated as an ideal gas. (3) No fouling. (4) Constant properties.

Analysis Rate of heat transfer,

$$\dot{Q} = UA\Delta T_{lm} = \dot{m}_h C_{p_h}(T_{hi} - T_{he}) = \dot{m}_c C_{p_c}(T_{ce} - T_{ci})$$

The mass-flow rate of the hot fluid (gases), $\dot{m}_g$ may be found from the engine data:

$$\dot{m}_f = \text{(bp) (bsfc)} = (100\text{ kW})\ (0.2\text{ kg/kWh}) = 20\text{ kg/h}$$

$$\dot{m}_g = \dot{m}_a + \dot{m}_f = \dot{m}_f\left[\frac{\dot{m}_a}{\dot{m}_f} + 1\right] = \dot{m}_f(\text{AFR} + 1)$$

$$= 20\text{ kg/h }(16 + 1) = 340\text{ kg/h} = \dot{m}_h$$

Heat rate,

$$\dot{Q} = \dot{m}_h C_{ph}(T_{hi} - T_{he}) = (340\text{ kg/h})\ (1.1\text{ kJ/kg °C})\ (300 - 100)\text{°C}$$

$= 74\ 800$ kJ/h

But $\quad \dot{Q} = \dot{m}_c C_{p_c}(T_{ce} - T_{ci})$

With $\quad C_{pc} = C_{pw} = 4.18$ kJ/kg °C,

Mass-flow rate of water,

$$\dot{m}_w = \dot{m}_c = \frac{\dot{Q}}{C_{pc}(T_{ce} - T_{ci})} = \frac{74800 \text{ kJ/h}}{(4.18 \text{ kJ/kg °C})(90 - 20)\text{°C}}$$

$=$ **255.6 kg/h** **(Ans.) (a)**

Gauge pressure, P_g = 100 mm water = 100 × 9.81 Pa = 0.981 kPa
Absolute pressure of exhaust gases,

P = Atmospheric pressure, P_a + Gauge pressure, P_g

= 101.3 kPa + 0.981 kPa = 102.281 kPa

From the characteristic gas equation:

$Pv = RT$

Specific volume of gases,

$$v = \frac{RT}{P} = \frac{(0.3 \text{ kJ/kg K})(573.15 \text{ K})}{102.281 \text{ kPa}} \left| \frac{1 \text{ kPa m}^3}{1 \text{ kJ}} \right| = 1.681 \text{ m}^3\text{/kg}$$

From the equation of continuity, the mass-flow rate of gases is $\dot{m}_g = \dfrac{A_c V}{v}$

Hence, the total cross-sectional area of tubes for gas flow, $A_c = \dot{m}_g v/V$

$$= \frac{(340 \text{ kg/h})(1.681 \text{ m}^3\text{/kg})}{(20 \times 3600 \text{ m/h})} = 0.007\,94 \text{ m}^2$$

Number of tubes required,

$$N = \frac{A_c}{A_c \text{ per tube}} = \frac{A_c}{(\pi/4)D^2} = \frac{0.00794 \text{ m}^2}{(\pi/4)(0.02 \text{ m})^2}$$

$=$ **25.27 ≅ 26** **(Ans.) (b)**

Log mean temperature difference,

$$\text{LMTD or } \Delta T_{lm} = \frac{\Delta T_i - \Delta T_e}{\ln \dfrac{\Delta T_i}{\Delta T_e}}$$

where $\quad \Delta T_i = T_{hi} - T_{ce} = 300 - 90 = 210\text{°C}$

$\Delta T_e = T_{he} - T_{ci} = 100 - 20 = 80\text{°C}$

$$\therefore \quad \Delta T_{lm} = \frac{210 - 80}{\ln(210/80)} = 134.7\text{°C}$$

As $\quad \dot{Q} = UA\Delta T_{lm}$, the total heat-exchanger surface area is

$$A = \frac{\dot{Q}}{U\Delta T_{lm}} = \frac{(74800/3600) \text{ kW}}{0.2 \text{ kW/m}^2 \text{°C} \times 134.7\text{°C}} = 0.7712 \text{ m}^2$$

For a single pass heat exchanger,

$$A = N\pi DL$$

Therefore, the length of each tube is

$$L = \frac{A}{N\pi D} = \frac{0.7712 \text{ m}^2}{26 \times \pi \times 0.02 \text{ m}} = \mathbf{0.472 \text{ m}} \qquad \textbf{(Ans.) (c)}$$

EXAMPLE 11.16 *A shell-and-tube heat exchanger with 2 shell passes and 8 tube passes is to be designed to heat water (C_p = 4.18 kJ/kg °C) with ethylene glycol (C_p = 2.68 kJ/kg °C).*

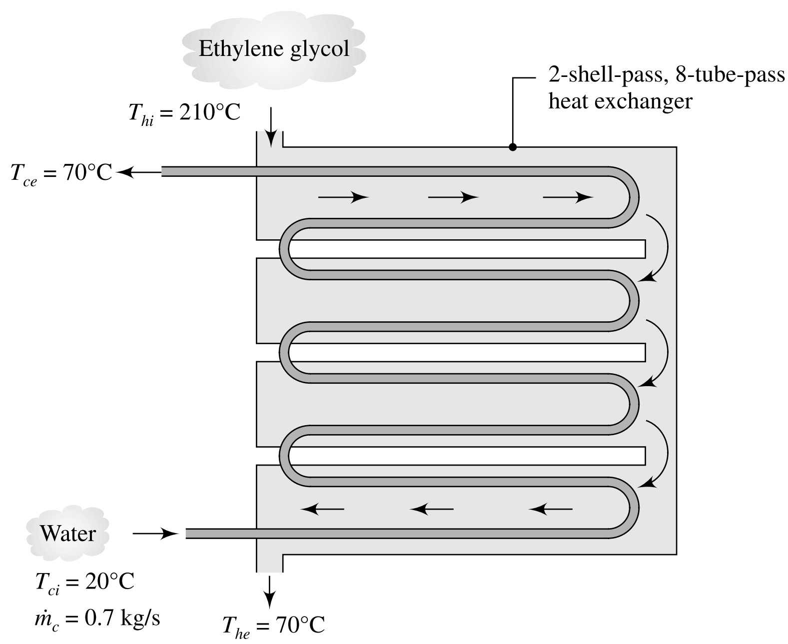
Water enters the tubes at 20°C at a flow rate of 0.7 kg/s and leaves at 70°C. Ethylene glycol enters the shell at 120°C and leaves at 70°C. If the convection heat-transfer coefficient on the tube side is 840 W/m² °C and that on the shell side is 420 W/m² °C, Calculate the heat-transfer area required.

Solution

Known A 2-8 shell-and-tube heat exchanger in used to heat water by ethylene glycol.

Find Heat-transfer area.

Schematic



Assumptions (1) Steady operating conditions. (2) Fouling is neglected. (3) Tube wall resistance is negligible. (4) Constant and uniform heat-transfer coefficient.

Analysis Heat-transfer rate is

$$\dot{Q} = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) = (0.7 \text{ kg/s})(4180 \text{ J/kg°C})(70 - 20)\text{°C} = 146.3 \times 10^3 \text{ W}$$

$$\Delta T_{lm,\text{CF}} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i / \Delta T_e)}$$

$$\Delta T_i = T_{hi} - T_{ce} = 120 - 70 = 50°\text{C}$$

where $\Delta T_e = T_{he} - T_{ci} = 70 - 20 = 50°\text{C} = 50°\text{C}$

Since $\Delta T_i = \Delta T_e,\ \Delta T_{lm,\text{CF}} = \Delta T_i = \Delta T_e = 50°\text{C}$

With
$$P = \frac{t_2 - t_1}{T_1 - t_1} = \frac{70 - 20}{120 - 20} = 0.50$$

$$R = \frac{T_1 - T_2}{t_2 - t_1} = \frac{120 - 70}{70 - 20} = 1.0$$

the LMTD correction factor, F = 0.95

Overall heat-transfer coefficient,

$$U = \left[\frac{1}{h_i} + \frac{1}{h_o}\right]^{-1} = \left[\frac{1}{840} + \frac{1}{420}\right]^{-1} = 280\ \text{W/m}^2\,°\text{C}$$

Heat-transfer rate,

$$\dot{Q} = UAF\Delta T_{lm,\text{CF}}$$

Heat exchanger surface area is

$$A = \frac{\dot{Q}}{UF\,\Delta T_{lm,\text{CF}}} = \frac{146.3 \times 10^3\ \text{W}}{280\ \text{W/m}^2\,°\text{C} \times 0.95 \times 50°\text{C}}$$

$= \mathbf{11.0\ m^2}$ **(Ans.)**

EXAMPLE 11.17 *In a shell-and-tube heat condenser steam is condensed at 0.1 bar and 0.9 dryness fraction. The cooling water enters at 30°C and leaves the condenser at 40°C. The condenser has 1000 tubes of 20 mm ID and 24 mm OD. One-shell pass is used along with two-tube passes. The heat-transfer coefficient on the outside of the tubes where the steam condenses may be assumed to be 8600 W/m² K. The rate of flow of cooling water is 1000 m³/h. Determine:*

(a) the quantity of steam condensed per hour, (b) the inside heat-transfer coefficient based on the correlation: $\text{Nu} = 0.023(\text{Re})^{0.8}(\text{Pr})^{0.4}$, *(c) the overall heat-transfer coefficient, (d) the length of the tube bundle*

Assume that the condensate is not subcooled and that the effect of fouling on both sides of the tubes may be neglected.

The properties of water at the average bulk temperature of 35°C may be taken as follows:

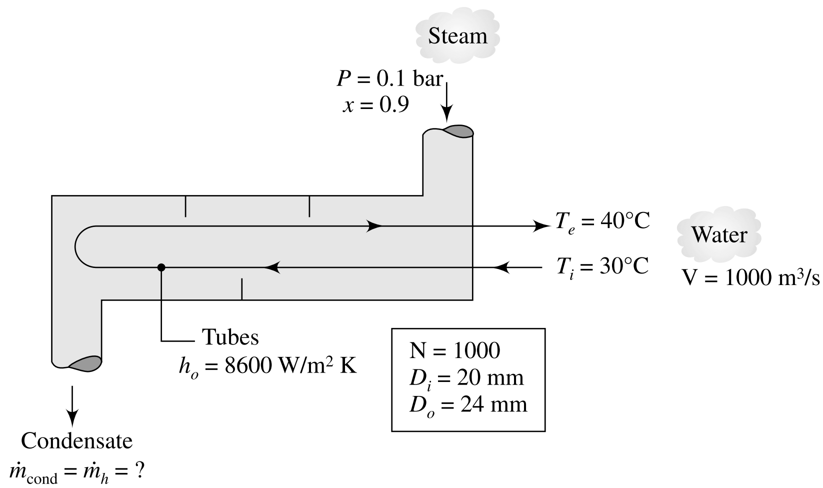
Dynamic viscosity = 0.72 × 10⁻³ N s/m²
Specific heat = 4.178 kJ/kg K
Thermal conductivity = 0.623 W/m K
Density of water = 994 kg/m³

Solution

Known Steam is condensed in a surface condenser by cooling water.

Find (a) $\dot{m}_{\text{cond}}$(kg/h), (b) h_i(W/m² K), (c) U(W/m² K), (d) L(m).

Schematic



Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Uniform overall heat-transfer coefficient. (4) No subcooling or fouling. (5) Tube side resistance negligible.

Analysis From steam tables:

At P = 0.1 bar, T_{sat} = 45.81°C, h_{fg} = 2392.1 kJ/kg, x = 0.90

Heat-transfer rate,

$$\dot{Q} = \dot{m}_{\text{steam}}[(h_f + x h_{fg})_{@0.1\,\text{bar}} - h_{f@0.1\,\text{bar}}] = \dot{m}_{\text{cond}} \cdot x h_{fg}$$

Also $\quad \dot{Q} = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) = (\rho\dot{\forall})_{\text{water}}\, C_{p_w}(T_e - T_i)$

$$= (994\ \text{kg/m}^3)\left(\frac{1000}{3600\ \text{m}^3/\text{s}}\right)(4.178\ \text{kJ/kg K})(40 - 30)\text{K} = 11535.9\ \text{kW}$$

Hence, the mass of steam condensed is

$$\dot{m}_{\text{cond}} = \frac{\dot{Q}}{x h_{fg}} = \frac{11535.9\ \text{kW}}{0.9 \times 2392.1\ \text{kJ/kg}} = \mathbf{5.36\ kg/s} \qquad \textbf{(Ans.) (a)}$$

Furthermore,

$$\dot{Q} = UA_s\Delta T_{lm}$$ (*For condensers, correction factor F is always 1*)

where $\quad A_s = NP\pi DL$

and $\quad \Delta T_{lm} = \dfrac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}$

with $\quad \Delta T_i = T_{\text{sat}} - T_i = 45.81 - 30 = 15.81°\text{C}$

$$\Delta T_e = T_{\text{sat}} - T_e = 45.81 - 40 = 5.81°\text{C}$$

$\therefore \quad \Delta T_{lm} = \dfrac{15.81 - 5.81}{\ln(15.81/5.81)} = 10°\text{C}$

If the overall heat-transfer coefficient is based on the *inner surface area* of the tubes,

$$UA_s = U_i A_i$$

Then $$\frac{1}{U_i} = \left[\frac{1}{h_i} + \frac{D_i}{D_o}\frac{1}{h_o}\right]$$

where h_i is the inside heat-transfer coefficient determined from

$$h_i = Nu\frac{k}{D_i}$$

Nusselt number,

$$Nu = 0.023(Re)^{0.8}(Pr)^{0.4}$$

where $$Re = \frac{\rho VD}{\mu} = \frac{4(\dot{m}/\text{tube})}{\pi D_i \mu} = \frac{4(994 \times 1000/1000 \times 3600)\text{kg/s}}{\pi(0.02\text{ m})(0.00072\text{ kg/m s})} = 24413.6$$

$$Pr = \frac{C_p \mu}{k} = \frac{4178\text{ J/kg K} \times 0.00072\text{ kg/m s}}{0.623\text{ W/m K}} = 4.83$$

Hence, $$Nu = 0.023(24413.6)^{0.8}(4.83)^{0.4} = 139.77$$

$\therefore$ $$h_i = (139.77 \times 0.623\text{ W/m K}/0.02\text{ m}) = \mathbf{4354\ W/m^2\ K}$$ **(Ans.) (b)**

Then, $$U_i = \left[\frac{1}{4354} + \frac{20}{24}\frac{1}{8600}\right]^{-1} = \mathbf{3062\ W/m^2\ K}$$ **(Ans.) (c)**

Length of the tube bundle,

$$L = \frac{\dot{Q}}{U_i NP\pi D_i \Delta T_{lm}} = \frac{11535.9 \times 10^3\text{ W}}{3062\text{ W/m}^2\text{ K} \times 1000 \times 2\text{ m} \times \pi \times 0.02\text{ m} \times 10}$$

$$= \mathbf{3.0\ m}$$ **(Ans.) (d)**

(D) Cross-Flow Heat Exchangers (LMTD Method)

EXAMPLE 11.18 *A single-pass cross-flow heat exchanger (both fluids unmixed) has the following operating data:*

	Hot fluid	*Cold fluid*
Inlet temperature (°C) ...	*70.2*	*35.1*
Exit temperature (°C) ...	*51.3*	*54.1*
Mass flow rate (kg/h) ...	*100.0*	–
Specific heat (kJ/kg °C) ...	*1.005*	–
Overall heat-transfer coefficient ...	*43 W/m² °C*	

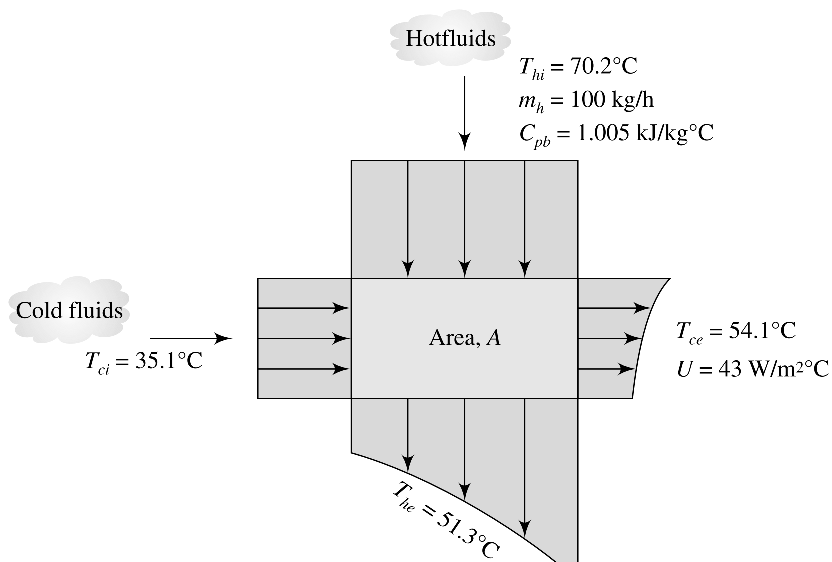
Find the area of the heat exchanger.

Solution

Known A cross-flow heat exchanger (*both fluids unmixed*) operates under specified conditions.

Find Heat exchanger surface area, A(m²).

Schematic



Single-pass cross-flow heat exchanger (both fluids unmixed)

Assumptions (1) Steady operating conditions. (2) Uniform overall heat-transfer coefficient. (3) Constant properties.

Analysis

$$(\text{LMTD})_{\text{counterflow}} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i / \Delta T_e)}$$

or

$$\Delta T_{lm,\text{CF}} = \frac{(T_{hi} - T_{ce}) - (T_{he} - T_{ci})}{\ln\left(\dfrac{T_{hi} - T_{ce}}{T_{he} - T_{ci}}\right)} = \frac{(70.2 - 54.1) - (51.3 - 35.1)}{\ln\dfrac{16.1}{16.2}} = \mathbf{16.15°C}$$

To find the LMTD correction factor, F, refer to the chart, for single pass cross flow heat exchanger *(both fluids unmixed),* where

$$P = \frac{T_{ce} - T_{ci}}{T_{hi} - T_{ci}} = \frac{(54.1 - 35.1)°\text{C}}{(70.2 - 25.1)°\text{C}} = \frac{19.0}{35.1} = 0.541$$

$$R = \frac{T_{hi} - T_{he}}{T_{ce} - T_{ci}} = \frac{(70.2 - 51.3)°\text{C}}{(54.1 - 35.1)°\text{C}} = \frac{18.9}{19.0} \simeq 1.0$$

From the appropriate chart, we read $\boldsymbol{F}$ = 0.71

Heat-transfer rate,

$$\dot{Q} = UA[F \Delta T_{lm,\text{CF}}] = \dot{m}_h C_{ph} (T_{hi} - T_{he})$$

Area of the heat exchanger,

$$A = \frac{\dot{m}_h C_{ph} (T_{hi} - T_{he})}{U(F \Delta T_{lm,\text{CF}})} = \frac{(100/3600 \text{ kg/s})(1005 \text{ J/kg°C})(18.9°\text{C})}{(43 \text{ W/m}^2\text{°C})(0.71 \times 16.15°\text{C})} \left| \frac{1 \text{ W}}{1 \text{ J/s}} \right|$$

$= \mathbf{1.07\ m^2}$ **(Ans.)**

(E) ε NTU Method

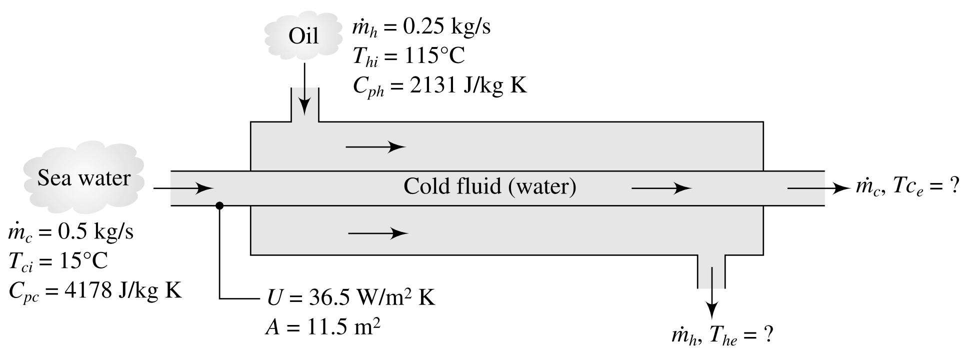
EXAMPLE 11.19 *A double-pipe, parallel-flow heat exchanger is to be used to cool oil [0.25 kg/s at 115°C], using sea water [0.5 kg/s at 15°C]. The area of the heat exchanger is 11.5 m^2 and the overall heat-transfer coefficient is 36.5 W/m^2 K. What are the exit states of the oil and the sea water from the heat exchanger? For oil, take* C_p *= 2131 J/kg K and for sea water, take* C_p *= 4178 J/kg K. Use the effectiveness* NTU *method.*

Solution

Known A parallel-flow heat exchanger with given (UA) value, hot and cold stream inlet temperatures, and flow rates.

Find Exit temperatures of hot and cold fluid streams, T_{he} and T_{ce}.

Schematic



Assumptions (1) Steady operating conditions. (2) Heat-transfer to the surroundings is negligible. (3) Negligible changes in kinetic and potential energy.

Analysis Since only inlet temperatures of the hot and cold fluids are known, we choose to use the ε NTU method.

Step 1: Identify C_{min} and R:

$$C_h = \dot{m}_h C_{p_h} = \dot{m}_{\text{oil}} C_{p_{\text{oil}}} = 0.25 \text{ kg/s} \times 2131 \text{ J/kg K} = 532.75 \text{ W/K}$$

$$C_c = \dot{m}_c C_{p_c} = \dot{m}_{\text{water}} C_{p_{\text{water}}} = 0.5 \text{ kg/s} \times 4178 \text{ J/kg K} = 2089.0 \text{ W/K}$$

Hence, $\quad C_{\min} = C_h, C_{\max} = C_c$

Heat-capacity rate ratio, $R = \dfrac{532.75 \text{ W/K}}{2089.0 \text{ W/K}} = 0.255$

Step 2: Calculate NTU:

Number of transfer units, $\text{NTU} = \dfrac{UA}{C_{\min}} = \dfrac{36.5 \times 11.5}{532.75} = 0.788$

Step 3: Compute effectiveness, ε

For a parallel-flow heat exchanger:

$$\varepsilon = \frac{1 - \exp[-\text{NTU}(1 + R)]}{1 + R} = \frac{1 - \exp[-(0.788)(1 + 0.255)]}{1 + 0.255} = 0.50$$

Step 4: ***Evaluate*** $\dot{Q}$ **:**

Heat-transfer rate,

$$\dot{Q} = \varepsilon \dot{Q}_{\max} = \varepsilon C_{\min} (T_{hi} - T_{ci})$$

$$= (0.5)\ (532.75\ \text{J/kg K})\ (115 - 15)°\text{C or K} = 26637.5\ \text{W}$$

Step 5: ***Apply energy balance and find exit temperatures:***

$$\dot{Q} = C_h (T_{hi} - T_{he}) = C_c (T_{ce} - T_{ci})$$

Oil exit temperature,

$$T_{he} = T_{hi} - \left(\frac{\dot{Q}}{C_h}\right) = 115 - \left(\frac{26637.5}{532.75}\right) = \mathbf{65°C} \qquad \textbf{(Ans.)}$$

Similarly,

Sea water exit temperature,

$$T_{ce} = T_{ci} + \left(\frac{\dot{Q}}{C_c}\right) = 15 + \left(\frac{26637.5}{2089.0}\right) = \mathbf{27.75°C} \qquad \textbf{(Ans.)}$$

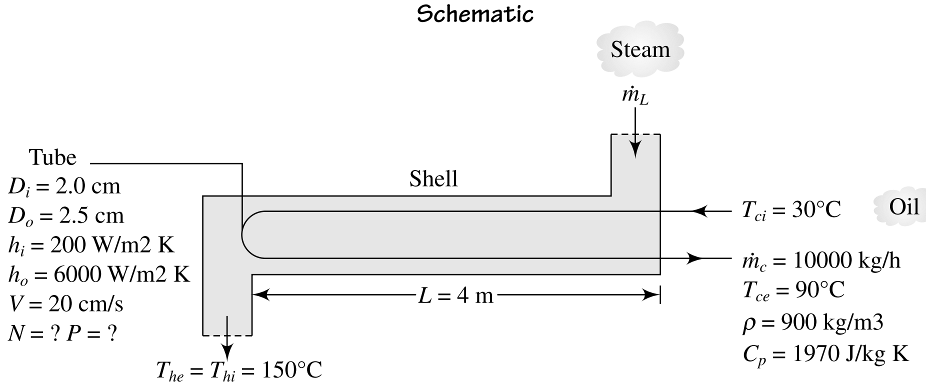
EXAMPLE 11.20 *10 000 kg/h of furnace oil is heated from 30°C to 90°C in a shell and tube type heat exchanger. The oil is flowing through the tube while steam at 150°C is to flow through the shell. The tubes are 2 cm ID and 2.5 cm OD in size. The heat-transfer coefficient on oil side and steam side are 200 and 6000 W/m² K. If the length of each tube is limited to 4 m, (a) calculate the number of tubes required in each pass, (b) the number of tube passes and (c) the tube length if the velocity of oil is limited to 20 cm/s. The properties of oil are the following: ρ = 900 kg/m³,* C_p *= 1970 J/kg K [NU: W 2001]*

Solution

Known Shell (single-pass) and tube (multiple-pass) heat exchanger. Tube dimeters, oil-flow rate and oil inlet and outlet temperatures. Convection coefficients (inside and outside). Tube length and oil velocity. Saturation temperature of steam and oil properties.

Find Number of tubes per pass, *N*; Number of tube pass, *P*; and length of the tube, *L*.

Schematic



Analysis Heat-transfer rate,

$$\dot{Q} = \dot{m}_c C_{pc} (T_{ce} - T_{ci}) = \left(\frac{10000}{3600}\right) \times (1970) \times (90 - 30)$$

$$= (5472.22)\ (60) = 328.33 \times 10^3\ \text{W}$$

Maximum heat-transfer rate,

$$\dot{Q}_{\max} = (\dot{m}C_p)_{\min}(T_{hi} - T_{ci}) = \left(\frac{10000}{3600}\right) \times (1970) \times (150 - 30)$$

$$= 656.66 \times 10^3 \text{ W}$$

Effectiveness of the heat exchanger is

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{328.33}{656.66} = 0.50$$

But $\quad \varepsilon = 1 - e^{-\text{NTU}} \quad \text{or} \quad e^{-\text{NTU}} = 1 - \varepsilon \quad \text{or} \quad \ln(1 - \varepsilon) = -\text{NTU}$

$\therefore$ Number of transfer units, NTU $= -\ln(1 - \varepsilon)$

$= -\ln(1 - 0.5) = 0.693$

But $\quad \text{NTU} = \dfrac{UA}{(\dot{m}C_p)_{\min}}$

$\therefore \quad UA = (0.693)(5472.22) = 3793 \text{ W/K} = U_oA_o = U_iA_i$

Overall heat-transfer coefficient,

$$U_o = \left[\frac{D_o}{D_i}\frac{1}{h_i} + \frac{1}{h_o}\right]^{-1} = \left[\frac{2.5}{2}\frac{1}{200} + \frac{1}{6000}\right] = 155.84 \text{ W/m}^2\text{ K}$$

Hence, $\quad A_o = \dfrac{3793}{155.84} = 24.34 \text{ m}^2$

But $\quad A_o = NP\pi D_oL$

Number of tubes/pass, $N = \dfrac{\dot{m}_{c,\text{total}}}{\dot{m}_c \text{ per tube}}$

$$\dot{m}_c \text{ per tube} = \rho A_cV = \rho \times \frac{\pi}{4} \times D_i^2V = (900 \text{ kg/m}^3)\left(\frac{\pi}{4} \times 0.02^2 \text{ m}^2\right)(0.20 \text{ m}^2)$$

$$= 0.05655\left(\frac{\text{kg}}{\text{s}}\right)\left(\frac{3600 \text{ s}}{1 \text{ h}}\right) = 203.58 \text{ kg/h}$$

Thus, $\quad N = \dfrac{10000 \text{ kg/h}}{203.58 \text{ kg/h}} = 49.12 \approx \mathbf{50}$ **(Ans.) (a)**

Number of tube passes,

$$P = \frac{A_o}{N\pi D_oL} = \frac{24.34 \text{ m}^2}{50 \times \pi \times 0.025 \text{ m} \times 4 \text{ m}} = 1.55 \approx \mathbf{2}$$ **(Ans.) (b)**

Length, $\quad L = \dfrac{A_o}{NP\pi d_O} = \dfrac{24.34 \text{ m}^2}{50 \times 2 \times \pi \times 0.025 \text{ m}} = \mathbf{3.1 \text{ m}}$ **(Ans.) (c)**

This is less than the permissible length of 4 m.

Comments Note that $\Delta T(°\text{C}) = \Delta T(\text{K})$ and $UA = U_iA_i = U_oA_o$.

For steam condensing on the outer tube surface, $(\dot{m}C_p)_{\text{steam}} = (\dot{m}C_p)_h \approx \infty$ as $T_{hi} = T_{he} = T_{\text{sat}}$. Hence, $(\dot{m}C_p)_c = (\dot{m}C_p)_{\text{oil}} = (\dot{m}C_p)_{\min}$. For any heat exchanger involving phase change (evaporator/condenser),

$\varepsilon = 1 - \exp(-\text{NTU})$.

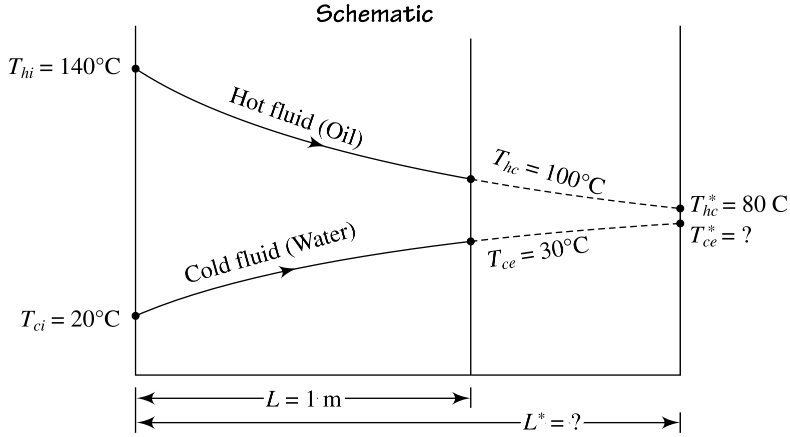
EXAMPLE 11.21 *A parallel-flow heat exchanger of one metre length cools an oil stream from 140°C to 100°C by a cooling water stream which enters the exchanger at 20°C and leaves at 30°C. Later, under the modified operating conditions, the oil is required to be cooled to 80°C and this is to be accomplished by increasing the length of the oil cooler. Assuming the oil and water mass-flow rates, their inlet temperatures and physical dimensions remain unchanged, calculate (a) the exit temperature of the cooling water in the modified cooler, and (b) the length of the new exchanger.*

Solution

Known A parallel-flow heat exchanger for cooling oil with water. Inlet and exit temperatures of oil and water.

Find (a) Water outlet temperature, T^*_{ce}, to cool the oil further to 80°C, and (b) Length of the modified oil cooler, L^*.

Schematic



Assumptions (1) Steady operating conditions. (2) Oil is the hot fluid with smaller heat-capacity rate. (3) No fouling.

Analysis ***Before increasing the length of the cooler:***

The temperature difference between the oil *(hot fluid)* and the water *(cold fluid)* at the two ends of the heat exchanger are

$$\Delta T_i = T_{hi} - T_{ci} = (140 - 20)°\text{C} = 120°\text{C}$$

and
$$\Delta T_e = T_{he} - T_{ce} = (100 - 30)°\text{C} = 70°\text{C}$$

The specific heats of oil and water are typically ***2.13*** and ***4.18*** kJ/kg °C. It is reasonable to identify the oil as the fluid with smaller heat-capacity rate. Then

$$R = \frac{C_{\min}}{C_{\max}} = \frac{C_h}{C_c} = \frac{\dot{m}_h C_{p_h}}{\dot{m}_c C_{p_c}}$$

From the energy balance:

$$\dot{Q} = \dot{m}_h C_{p_h}(T_{hi} - T_{he}) = \dot{m}_c C_{p_c}(T_{ce} - T_{ci}) = UA_s \Delta T_{lm}$$

Heat-capacity rate ratio then becomes

$$R = \frac{T_{ce} - T_{ci}}{T_{hi} - T_{he}} = \frac{30 - 20}{140 - 100} = 0.25$$

One can also write

$$C_{\min}(T_{hi} - T_{he}) = UA_s \Delta T_{lm}$$

or
$$\frac{UA_s}{C_{\min}} = \frac{T_{hi} - T_{he}}{\Delta T_{lm}} = \frac{(T_{hi} - T_{he})\ln(\Delta T_i/\Delta T_e)}{\Delta T_i - \Delta T_e}$$

$\therefore$
$$\text{NTU} \equiv \frac{UA_s}{C_{\min}} = \frac{(140 - 100)\ln(120/70)}{120 - 70} = 0.4312$$

With $A_s = \pi DL = \pi D$ since $L = 1$ m,

$U\pi D/C_{\min} = 0.4312$

After increasing the length of the cooler:

$$\text{NTU}^* = \frac{UA_s^*}{C_{\min}} = \frac{U\pi DL^*}{C_{\min}} = 0.4312\ L^*$$

Effectiveness of a parallel-flow heat exchanger in terms of NTU and R can be expressed as

$$\varepsilon = \frac{1 - \exp[-\text{NTU}(1 + R)]}{1 + R}$$

For the modified operating conditions, R remains same but NTU is changed. Then

$$\varepsilon^* = \{1 - \exp[-\text{NTU}^*(1 + R)]\}/(1 + R)$$

$$= \{1 - \exp\,[-0.4312\ L^* \times 1.25]\}/1.25 \tag{A}$$

Furthermore,

$$\varepsilon^* = \frac{C_h(T_{hi} - T_{he}^*)}{C_{\min}(T_{hi} - T_{ci})} = \frac{C_c(T_{ce}^* - T_{ci})}{C_{\min}(T_{hi} - T_{ci})} \tag{B}$$

Since $C_h = C_{\min}$, from Eq. (B):

$$\varepsilon^* = \frac{140 - 80}{140 - 20} = 0.5 \quad \text{and} \quad \frac{C_h}{C_c} = R = \frac{T_{ce}^* - T_{ci}}{T_{hi} - T_{he}^*}$$

Therefore,

$T_{ce}^* = T_{ci} + R(T_{hi} - T_{he}^*)$ = 20 + 0.25 (140 – 80) = **35°C** **(Ans.) (a)**

From Eq. (A):

$0.5 \times 1.25 = 1 - \exp\,(-\,0.4312 \times 1.25\ L^*)$

or $-0.4312 \times 1.25\ L^* = \ln\,(1 - 0.625) = \ln 0.375$

New length, $L^* = \dfrac{\ln 0.375}{0.539}$ = **1.82 m** **(Ans.) (b)**

EXAMPLE 11.22 *A heat exchanger used in a process industry is designed to operate under the following conditions:*

Parameter	Hot fluid	Cold fluid
Mass-flow rate (kg/s)	2.0	2.5
Specific heat capacity (kJ/kg °C)	3.5	4.18
Inlet temperature (°C)	80	15
Exit temperature (°C)	50	–

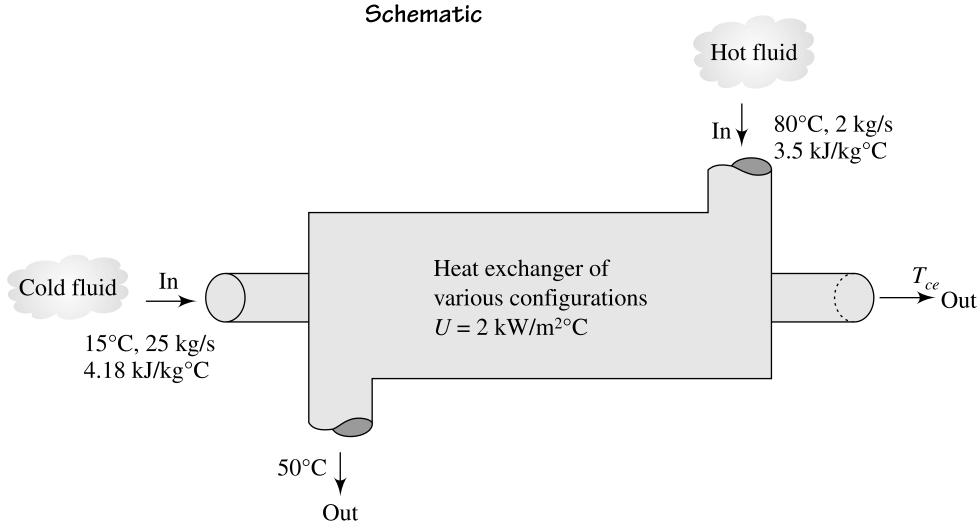
The overall heat-transfer coefficient is 2 kW/m² °C.
Determine the required heat-transfer area (m²) for the following exchanger configurations.
(a) Co-current flow, (b) Counter-current flow, (c) Cross-flow, single-pass, both fluids unmixed, (d) Shell and tube, one-shell pass and two-tube passes.

Solution

Known A heat exchanger operates under the specified conditions.

Find Heat-exchanger surface area required, A(m²) for different configurations: (a) Parallel flow, (b) Counterflow, (c) Cross flow, both fluids unmixed, (d) 1-2 shell and tube.

Schematic



Assumptions (1) Steady operating conditions. (2) Heat exchanger is adiabatic (no losses to surroundings). (3) Constant properties. (4) Constant and uniform overall heat-transfer coefficient.

Analysis ***Overall energy balance:***

$$\dot{Q} = \dot{m}_h C_{ph}(T_{hi} - T_{he}) = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) = \text{UAF(LMTD)}$$

Heat rate, $\dot{Q} = (2.0 \text{ kg/s})(3.5 \text{ kJ/kg°C})(80 - 50)\text{°C} = 210 \text{ kW}$

Cold fluid outlet temperature,

$$T_{ce} = T_{ci} + (\dot{Q}/\dot{m}_c C_{pc}) = 15\text{°C} + \left(\frac{210 \text{ kW}}{2.5 \text{ kg/s} \times 4.18 \text{ kJ/kg°C}}\right) = 35\text{°C}$$

LMTD = log mean temperature difference *in counterflow mode*

$$= \frac{\Delta T_i - \Delta T_e}{\ln \Delta T_i / \Delta T_e} = \frac{(T_{hi} - T_{ce}) - (T_{he} - T_{ci})}{\ln\left(\dfrac{T_{hi} - T_{ce}}{T_{he} - T_{ci}}\right)} = \frac{(80 - 35) - (50 - 15)}{\ln\left(\dfrac{80 - 35}{50 - 15}\right)} = \mathbf{39.8°C}$$

F is the LMTD correction factor (< 1) for different configurations to be found from the appropriate chart.

$$F = f(P, R)$$

where $$P = \frac{T_{hi} - T_{he}}{T_{hi} - T_{ci}} = \frac{80 - 50}{80 - 15} = 0.4615$$

$$R = \frac{T_{ce} - T_{ci}}{T_{hi} - T_{he}} = \frac{35 - 15}{80 - 50} = 0.667$$

(a) ***Co-current (parallel) flow:*** $\Delta T_i = T_{hi} - T_{ci} = 80 - 15 = 65°C$

$$\Delta T_e = T_{he} - T_{ce} = 50 - 35 = 15°C$$

$$(\text{LMTD})_{\text{PF}} = \frac{65 - 15}{\ln 65/15} = 34.1°C$$

Area required, $$A = \frac{\dot{Q}}{U(\text{LMTD})_{\text{PF}}} = \frac{210 \text{ kW}}{(2 \text{ kW/m}^2 °C)(34.1°C)}$$

$= \mathbf{3.08\ m^2}$ **(Ans.) (a)**

(b) ***Counter-current (Counter) flow:***

Area, $$A = \frac{\dot{Q}}{U(\text{LMTD})} = \frac{210}{2 \times 39.8} = \mathbf{2.64\ m^2}$$ **(Ans.) (b)**

(c) ***Cross flow: Single pass, Both fluids unmixed:***

From the chart, with $P = 0.4615$ and $R = 0.667$, $F = 0.951$

Area, $$A = \frac{\dot{Q}}{UF(\text{LMTD})} = \frac{210}{2 \times 0.951 \times 39.8} = \mathbf{2.78\ m^2}$$ **(Ans.) (c)**

(d) ***One-shell pass, two-tube pass heat exchanger:***

From the chart, with $P = 0.4615$ and $R = 0.667$, $F = 0.933$

Area, $$A = \frac{\dot{Q}}{UF(\text{LMTD})} = \frac{210}{2 \times 0.933 \times 39.8} = \mathbf{2.83\ m^2}$$ **(Ans.) (d)**

The lower the surface area, the better it is. Results are summarized and tabulated below, with the heat exchanger area *in increasing order*.

#	Configuration	Area required, A(m²)
1	**Counterflow (b)**	**2.64**
2	**Cross flow *(both fluids unmixed)* (c)**	**2.78**
3	**1-2 shell and tube (d)**	**2.83**
4	**Parallel flow (a)**	**3.08**

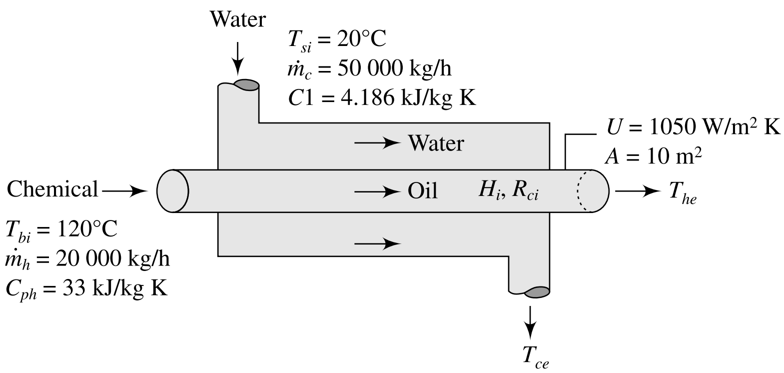
EXAMPLE 11.23 *A chemical (C_p = 3.3 kJ/kg K) following at the rate of 20 000 kg/hr enters a parallel flow heat exchanger at 120°C. The flow rate of cooling water (C_p = 4.186 kJ/kg K) is 50 000 kg/h with an inlet temperature of 20°C. The heat-transfer surface area is 10 m² and the overall heat-transfer coefficient is 1050 W/m² K. Calculate the (a) effectiveness of the heat exchanger, and (b) outlet temperatures of water and chemical*

Solution

Known A parallel-flow heat exchanger with specified hot and cold fluid inlet temperatures.

Find (a) Effectiveness, ε, (b) Hot and cold fluid exit temperatures, T_{he} and T_{ce}.

Schematic



Assumptions (1) Steady operating conditions. (2) U is constant.

Analysis This is a rating problem since only T_{hi} and T_{ci} are given.

Step 1: Identify $C_{\min}$

$$C_h = \dot{m}_h C_{p_h} = (20000/3600)(3.3) = 18.33 \text{ kW/K}$$

$$C_c = \dot{m}_c C_{p_h} = (50000/3600)(4.186) = 58.14 \text{ kW/K}$$

$\therefore$ $C_h = C_{\min}$ and $C_c = C_{\max}$

$$R = C_{\min}/C_{\max} = 18.33/58.14 = 0.315$$

Step 2: Determine NTU

$$\text{NTU} = UA/C_{\min} = (1050)(10)/18.33 \times 10^3 = 0.573$$

Step 3: Find ε

$$\varepsilon_{\text{parallel flow}} = \frac{1 - e^{-\text{NTU}(1+R)}}{1+R} = \frac{1 - e^{-(0.573)(1.315)}}{1.315}$$

$$= 0.402$$ **(Ans.)**

Step 4: Calculate $\dot{Q}$

$$\dot{Q} = \varepsilon \dot{Q}_{\max} = \varepsilon C_{\min}(T_{hi} - T_{ci})$$

$$= (0.402)(18.33)(120 - 20) = 737.0 \text{ kW}$$

Step 5: Calculate exit temperatures from energy balance

$$\dot{Q} = C_c(T_{ce} - T_{ci}) = C_h(T_{hi} - T_{he})$$

Hence, $$T_{ce} = T_{ci} + \frac{\dot{Q}}{C_c} \quad \text{and} \quad T_{he} = T_{hi} - \frac{\dot{Q}}{C_h}$$

$$T_{ce} = 20 + \frac{737.0}{58.14} = \mathbf{32.7°C} \qquad \textbf{(Ans.) (b)}$$

$$T_{he} = 120 - \frac{737.0}{18.33} = \mathbf{79.8°C} \qquad \textbf{(Ans.) (b)}$$

EXAMPLE 11.24 *A steam condenser is 4 m long and contains 2000 brass tubes (1.59 cm OD). In a test 125 kg/s cooling water at 22°C is supplied to the condenser, and when the steam pressure in the shell is 61 mm Hg(abs) the condensate is produced at a rate of 3.05 kg/s. Determine (a) the effectiveness of the exchanger, and (b) the overall heat-transfer coefficient. Take the specific heat of the water to be 4.178 kJ/kg K.*

Solution

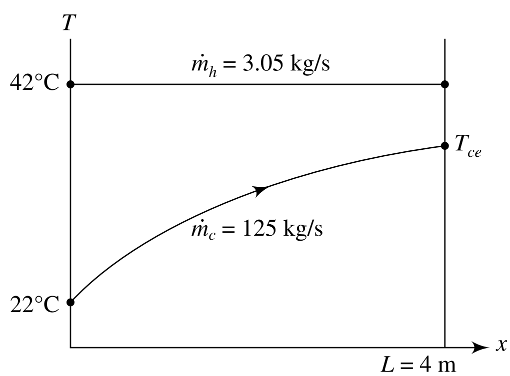
Known A shell and tube steam condenser.

Find (a) Effectiveness, ε, (b) Overall heat-transfer coefficient, U.

Assumptions U is constant along the exchanger.

Analysis The hot fluid temperature T_h is the saturation temperature corresponding to the given steam pressure of 61 mmHg which is equal to (61/750) bar = 0.0813 bar. $T_{\text{sat@0.0813}}$ = 42°C. The cooling water exit temperature can be found from the exchanger energy balance:

Schematic



$$\dot{m}_c C_{pc}(T_{ce} - T_{ci}) = \dot{m}_h h_{fg} \qquad \text{(A)}$$

From steam tables, the enthalpy of vaporization at

$$T_{\text{sat}} = 42°\text{C is } h_{fg} = 2402 \text{ kJ/kg}$$

Substituting values in Eq. (A), we have

$$(125 \text{ kg/s})(4.178 \text{ kJ/kg K})(T_{ce} - 22°\text{C}) = (3.05 \text{ kg/s})(2402 \text{ kJ/kg})$$

$$\Rightarrow \qquad T_{ce} = 36°\text{C}$$

The effectiveness, ε, is then obtained from

$$\varepsilon = \frac{T_{ce} - T_{ci}}{T_{\text{sat}} - T_{ci}} = \frac{36-22}{42-22} = \frac{14}{20} = \mathbf{0.7} \qquad \textbf{(Ans.) (a)}$$

For a condenser, $\varepsilon = 1 - \exp(-\text{NTU})$
Hence, the number of transfer units is

$$\text{NTU} = \ln\frac{1}{1-\varepsilon} = \ln\frac{1}{1-0.7} = 1.2 = \frac{UA}{\dot{m}_c C_{pc}}$$

It follows that

$$UA = 1.2\, \dot{m}_c C_{pc} = (1.2)(125 \text{ kg/s})(4.178 \text{ kJ/kg K}) = 627 \text{ kW/K}$$

For N tubes, the heat-transfer area is

$$A = N\pi DL = (2000)(\pi)(1.59 \times 10^{-2} \text{ m})(4 \text{ m}) = 399.6 \text{ m}^2$$

Hence, the overall heat-transfer coefficient based on the outside area of the tubes, is

$$U = UA/A = 627 \times 10^3 \text{ W/K}/399.6 \text{ m}^2 = \mathbf{1569\ W/m^2\ K} \qquad \textbf{(Ans.) (b)}$$

(F) Both LMTD and ε NTU Methods of Analysis

EXAMPLE 11.25 *Air enters a 5 m long, 15 mm ID and 25 mm OD tube (k = 50 W/m K) with a mass-flow rate of 50 kg/h and an inlet temperature of 45°C. The required outlet air temperature is 55°C. Air heating is accomplished by condensing steam on the outside surface of the tube. The condensation heat-transfer coefficient is estimated to be 915 W/m² K over a range of steam pressures.*

Determine: (a) the steam temperature and pressure required to obtain the specified outlet air temperature. (b) What is the steam-flow rate? Use both LMTD method and ε-NTU method.
The air properties at the average air temperature of 50°C are the following:

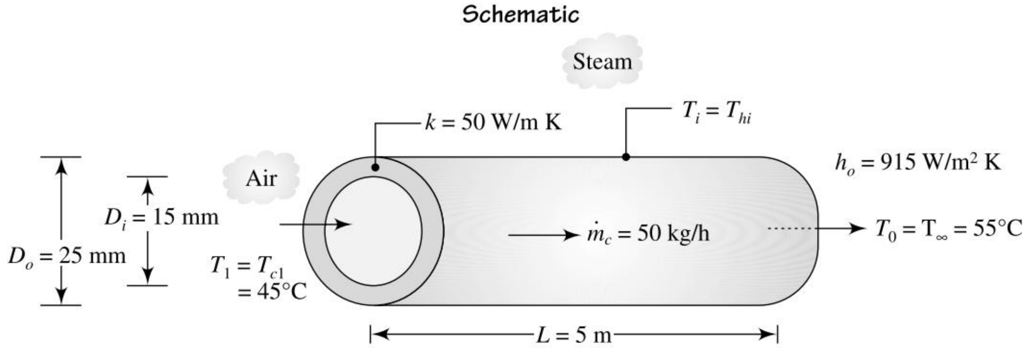
$\rho = 1.092$ kg/m³	$\mu = 1.963 \times 10^{-5}$ kg/m s	$Pr = 0.7228$
$Cp = 1.007$ kJ/kg K	$k = 0.2735$ W/m K	

Solution

Known A heat exchanger is used to raise the air temperature by means of condensing steam, thereby maintaining the constant wall temperature.

Find Saturation temperature of steam, T_{sat} (°C) and its pressure (bar). Steam-flow rate.

Schematic



Assumptions (1) Steady operating conditions exist. (2) Potential and kinetic energy effects are negligible. (3) No work occurs in the heat exchanger. (4) There is no fouling.

Analysis ***LMTD method*:**

For constant wall temperature, the energy balance gives:

$$\dot{Q} = UA\Delta T_{lm} = (\dot{m}C_p)_{\text{air}}(T_e - T_i) = \dot{m}_{\text{steam}}\, h_{fg} \tag{A}$$

where $$\Delta T_{lm} = \frac{(T_{\text{sat}} - T_i) - (T_{\text{sat}} - T_e)}{\ln\left(\dfrac{T_{\text{sat}} - T_i}{T_{\text{sat}} - T_e}\right)} = \frac{T_e - T_i}{\ln\{(T_{\text{sat}} - T_i)/(T_{\text{sat}} - T_e)\}} \tag{B}$$

From Eq. (A): $\Delta T_{lm} = \dfrac{\dot{Q}}{UA}$ and $\Delta T_{lm} = \dot{m}C_p(T_e - T_i)/UA$

or $$T_e - T_i = (\Delta T_{lm})(UA/\dot{m}C_p)$$

or $$T_e - T_i = \frac{(T_e - T_i)}{\ln\left(\dfrac{T_{\text{sat}} - T_i}{T_{\text{sat}} - T_e}\right)}(UA/\dot{m}C_p) \qquad \textit{from Eq. (B)}$$

or $$\ln\left(\frac{T_{\text{sat}} - T_e}{T_{\text{sat}} - T_i}\right) = -UA/\dot{m}C_p$$

∴ $$T_{\text{sat}} - T_e = (T_{\text{sat}} - T_i)\exp(-UA/\dot{m}C_p)$$

or $$T_{\text{sat}}\left[1 - \exp\left(-\frac{UA}{\dot{m}C_p}\right)\right] = T_e - T_i \exp\left(-\frac{UA}{\dot{m}C_p}\right)$$

∴ $$\boxed{T_{\text{sat}} = \frac{T_e - T_i \exp(-UA/\dot{m}C_p)}{1 - \exp(-UA/\dot{m}C_p)}}$$

The Reynolds number is $Re = \rho VD/\mu$. For a straight circular tube, we rewrite the Reynolds number as $Re_D = \dfrac{4\dot{m}}{\pi D_i \mu}$

Therefore, $$Re = \frac{4\dot{m}}{\pi D_i \mu} = \frac{4(50\text{ kg/h})\left(\dfrac{1\text{ h}}{3600\text{ s}}\right)}{\pi(1.963 \times 10^{-5}\text{ kg/m s})(0.015\text{ m})} = 60\ 057 \qquad (> 10\ 000)$$

This is *turbulent* flow. Using the Dittus–Böelter equation,

$$Nu = \frac{hD}{k} = 0.023\, Re^{0.8}\, Pr^{0.4} = 0.023(60\,057)^{0.8}\,(0.7228)^{0.4} = 134$$

Inside heat-transfer coefficient,

$$h_i = \frac{134(0.02735\text{ W/m K})}{0.015\text{ m}} = 248.94\text{ W/m}^2\text{ K}$$

Now, we can calculate the overall heat-transfer coefficient.

$$\frac{1}{UA}=\frac{1}{h_iA_i}+\frac{\ln(r_o/r_i)}{2\pi kL}+\frac{1}{h_oA_o}=\left\{\frac{1}{(244.94\ \mathrm{W/m^2})\pi(0.015\ \mathrm{m})(5\ \mathrm{m})}\right.$$

$$\left.+\frac{\ln(2.5/1.5)}{2\pi(50\ \mathrm{W/m\,K})(5\ \mathrm{m})}+\frac{1}{(915\ \mathrm{W/m^2\,K})\pi(0.025\ \mathrm{m})(5\ \mathrm{m})}\right\}$$

$$= [0.017\ 33\ \mathrm{K/W} + 0.000\ 325\ \mathrm{K/W} + 0.002\ 78\ \mathrm{K/W}] = 0.020\ 435\ \mathrm{K/W}$$

$\therefore$ $$UA = 48.94\ \mathrm{W/K} \Rightarrow \frac{UA}{(\dot{m}C_p)}=\frac{48.94\ \mathrm{W/K}}{(50\ \mathrm{kg/h})(1\ \mathrm{h}/3600\ \mathrm{s})(1007\ \mathrm{J/kg\,K})}=3.5$$

Incorporating these into our main equation,

$$T_{\text{sat}}=\frac{T_e-T_i\exp(-UA/\dot{m}C_p)}{1-\exp(-UA/\dot{m}C_p)}=\frac{55°\mathrm{C}-(45°\mathrm{C})\exp(-3.5)}{1-\exp(-3.5)}=\mathbf{55.3°C} \qquad \textbf{(Ans.) (a)}$$

At this temperature, from steam tables, the saturation pressure is

***P* = 0.16 bar** **(Ans.) (a)**

Latent heat of steam at 0.16 bar = h_{fg} = 2369 kJ/kg
Mass-flow rate of steam is then

$$\dot{m}_s=\frac{\dot{Q}}{h_{fg}}=\frac{(\dot{m}C_p)_{\text{air}}(T_e-T_i)}{h_{fg}}=\frac{(50\ \mathrm{kg/h})(1.007\ \mathrm{kJ/kg\,K})(55-45)°\mathrm{C}}{2369\ \mathrm{kJ/kg}}$$

= **0.213 kg/h** **(Ans.) (b)**

ε NTU method

Heat-transfer rate, $\dot{Q}=\varepsilon\dot{Q}_{\max}=\varepsilon(\dot{m}C_p)_{\min}(T_{hi}-T_{ci})$

For condensing steam, $T_{hi} = T_{he} = T_{\text{sat}}$ and $(\dot{m}C_p)_{\text{steam}}$ approaches infinity.

Obviously, $(\dot{m}C_p)_{\min}=(\dot{m}C_p)_{\text{air}}$.

Then $$\dot{Q}=(\dot{m}C_p)_{\text{air}}(T_{ce}-T_{ci})=\varepsilon(\dot{m}C_p)_{\text{air}}(T_{\text{sat}}-T_{ci})$$

$\therefore$ $$T_{\text{sat}}=T_{ci}+\frac{(T_{ce}-T_{ci})}{\varepsilon}$$

and, $$\text{NTU}=\frac{UA}{(\dot{m}C_p)}=\frac{48.94\ \mathrm{W/m^2\,K}}{(50/3600\ \mathrm{kg/s})(1007\ \mathrm{J/kg\,K})}=3.5$$

$\therefore$ Effectiveness of the heat exchanger,

$$\varepsilon=1-e^{-\text{NTU}}=1-e^{-3.5}=0.97$$

Hence, saturation temperature of steam,

$$T_{\text{sat}}=T_{ci}+\left(\frac{T_{ce}-T_{ci}}{\varepsilon}\right)=45°\mathrm{C}+\frac{(55°\mathrm{C}-45°\mathrm{C})}{0.97}=\mathbf{55.3°C} \qquad \textbf{(Ans.) (a)}$$

The corresponding saturation pressure from steam tables is found to be

0.16 bar or 16 kPa **(Ans.) (a)**

Latent heat of vaporization at 0.16 bar = 2369 kJ/kg from the steam table.

Heat-transfer rate,

$$\dot{Q} = (\dot{m}C_p)_{air}(T_{ce} - T_{ci}) = 50 \text{ kg/h} \times 1.007 \text{ kJ/kg K} \times (55 - 45) \text{ K} = 503.5 \text{ kJ/h}$$

With $\dot{Q} = \dot{m}_s h_{fg}$

Mass flow of steam or steam consumption is

$$\dot{m}_s = \frac{\dot{Q}}{h_{fg}} = \frac{503.5 \text{ kJ/h}}{2369 \text{ kJ/kg}} = \mathbf{0.213 \text{ kg/h}} \qquad \textbf{(Ans.) (b)}$$

EXAMPLE 11.26 *A counterflow heat exchanger is designed for the following specifications:*

$\dot{m}_h$ = *10 kg/s*	$\dot{m}_c$ = *20 kg/s*
C_{ph} = *1.6 kJ/kg K*	C_{pc} = *1.0 kJ/kg K*
h_h = *80 W/m² K*	h_c = *100 W/m² K*
R_{fh} = *0.005 m² K/W*	R_{fc} = *0.005 m² K/W*
T_{hi} = *200°C*	T_{ci} = *40°C*
T_{he} = *100°C*	

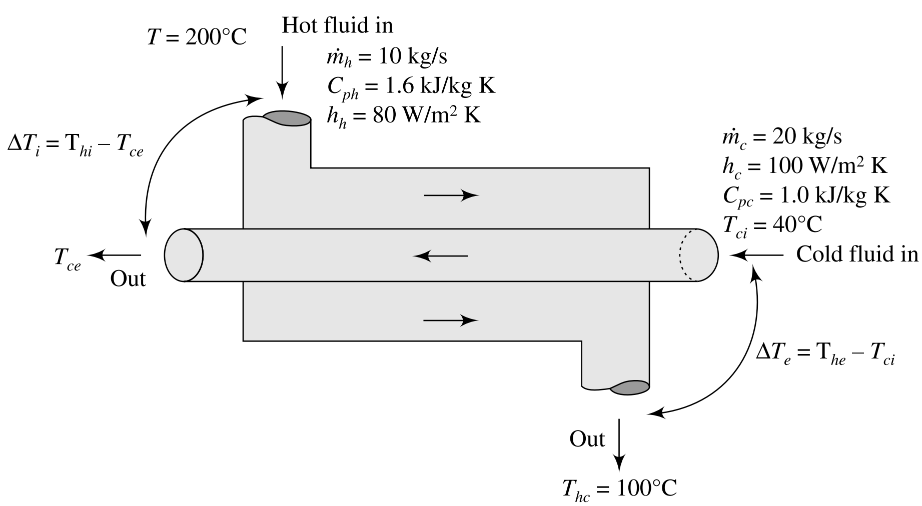
(a) Determine the heat-transfer area for the heat exchanger.
(b) When the heat exchanger is new (i.e., unfouled), the heat-transfer rate is higher. What will be the exit temperatures of the hot and cold fluids in this case?
(c) It is required that even for the new heat exchanger, the hot fluid exit temperature be 100°C. It is proposed to do this by adjusting the flow rate of the cold side fluid. Determine the mass flow rate of the cold side fluid (to the nearest 1 kg/s) which will achieve this. Assume that h_c *does not depend on* $\dot{m}_c$.

Solution

Known A counterflow heat exchanger operates under specified conditions.

Find (a) Heat-transfer area, A(m²). (b) Hot fluid exit temperature, T_{he}(*without fouling resistances*). (c) Mass-flow rate of cold fluid, $\dot{m}_c$ (kg/s) required for clean HX with T_{he} same as that for fouled HX.

Schematic



Assumptions (1) Steady operating conditions exist. (2) Constant fluid properties. (3) Tube wall conduction resistance is negligible. (4) Uniform heat-transfer coefficient. (5) Cold fluid convection coefficient, h_c is independent of its mass flow rate, $\dot{m}_c$.

Analysis **(a)** ***Fouled Heat Exchanger (HX):***

The overall heat-transfer coefficient is given by

$$\frac{1}{U} = \frac{1}{h_h} + \frac{1}{h_c} + R_{fh} + R_{fc} = \left[\frac{1}{80} + \frac{1}{100} + 0.005 + 0.005\right] \text{m}^2\,\text{K/W}$$

$$= 0.0325 \ \text{m}^2\,\text{K/W}$$

$$\therefore \qquad U = 30.77 \ \text{W/m}^2\,\text{K}$$

Energy balance:

Heat rate, $\dot{Q} = \dot{m}_h C_{p_h}(T_{hi} - T_{he})$

$$= (10 \text{ kg/s})(1600 \text{ J/kg K}) \, [(200 - 100) \text{ K}] \text{ W} = 1.6 \times 10^6 \text{ W}$$

$$= \dot{m}_c C_{pc}(T_{ce} - T_{ci})$$

$$\therefore \qquad T_{ce} = \frac{1.6 \times 10^6 \text{ W}}{20 \text{ kg/s} \times 1.0 \times 10^3 \text{ J/kg K}} + 40 = \mathbf{120°C}$$

$$\Delta T_{lm} = \frac{80 - 60}{\ln \dfrac{80}{60}} = 69.52°\text{C}$$

$\therefore$ Heat-exchanger surface area,

$$\therefore \qquad A = \frac{\dot{Q}}{U \Delta T_{lm}} = \frac{1.6 \times 10^6 \text{ W}}{30.77 \text{ W/m}^2\,\text{K} \times 69.52 \text{ K}} = \mathbf{748 \ m^2} \qquad \textbf{(Ans.) (a)}$$

(b) ***Unfouled HX:***

Overall heat-transfer coefficient,

$$U = \left[\frac{1}{h_h} + \frac{1}{h_c}\right]^{-1} = \left[\frac{1}{80} + \frac{1}{100}\right]^{-1} = 44.44 \text{ W/m}^2\,\text{K}$$

Heat-capacity rate ratio,

$$R = \frac{C_{\min}}{C_{\max}} = \frac{(\dot{m}C_p)_h}{(\dot{m}C_p)_c} = \mathbf{0.8} \qquad \{\because (\dot{m}C_p)_h < (\dot{m}C_p)_c\}$$

Number of transfer units,

$$\text{NTU} = \frac{UA}{(\dot{m}C_p)_h} = \frac{44.44 \text{ W/m}^2\,\text{K} \times 748 \text{ m}^2}{10 \text{ kg/s} \times 1600 \text{ J/kg K}} = \mathbf{2.0775}$$

Effectiveness,

$$\varepsilon = \frac{1 - \exp(-(1-R)\text{NTU})}{1 - R\exp(-(1-R)\text{NTU})} = \frac{1 - e^{-(1-0.8)(2.0775)}}{1 - 0.8\,e^{-(1-0.8)(2.0775)}} = \mathbf{0.7204}$$

Heat-transfer rate,

$$\dot{Q} = \varepsilon(\dot{m}C_p)_h(T_{hi} - T_{ci}) = 0.7204 \times 10 \text{ kg/s} \times 1.6 \text{ kJ/kg K} \times (200 - 40)°\text{C}$$

$= \mathbf{1844\ kW}$

Exit fluid temperatures are

T_{he} = 200°C – (1844 kW/10 kg/s × 1.6 kJ/kg K) = **84.74°C** **(Ans.) (b)**

T_{ce} = 40°C + (1844 kW/20 kg/s × 1.0 kJ/kg K) = **132.2°C** **(Ans.) (b)**

We must now change $\dot{m}_c$ to obtain T_{ce} = 100°C,

with $\dot{Q} = (\dot{m}C_p)_h(T_{hi} - T_{he})$ = 1600 kW (as before)

Let us use the ΔT_{lm} method with U = 44.44 W/m² K to find the desired value of $\dot{m}_c$. This involves an *iterative* procedure. A sample calculation and the results are tabulated below.

Let $\dot{m}_c$ = 13 kg/s.

With $\dot{Q}$ = 1600 kW,

$$T_{ce} = T_{ci} + (\dot{Q}/\dot{m}_c C_{pc}) = 40°\text{C} + \left[\frac{1600 \text{ kJ/s}}{13 \text{ kg/s} \times 1 \text{ kJ/kg K}}\right] = 163.1°\text{C}$$

$\Delta T_i = T_{hi} - T_{ce}$ = 200°C – 163.1°C = 36.9°C

$\Delta T_e = T_{he} - T_{ci}$ = 100°C – 40°C = 60°C

$$\therefore \quad \Delta T_{lm} = \frac{\Delta T_i - \Delta T_e}{\ln \Delta T_i/\Delta T_e} = \frac{36.9 - 60}{(36.9/60)} = 47.52°\text{C}$$

$\therefore \quad \dot{Q}_{\text{calculated}} = UA\Delta T_{lm}$ = (44.44 × 10⁻³ kW/m² K)(748 m²)(47.52°C)

= 1580 kW

$\dot{m}_c$ (kg/s)	Desired $\dot{Q}$ (kW)	T_{ce}(°C)	ΔT_{lm} (°C)	Calculated $\dot{Q}$ (kW)
20	1600	120	69.52	2311
15	1600	146.6	56.6	1882
14	1600	154.2	52.53	1746
13	**1600**	**163.1**	**47.52**	**1580 (≈ 1600)**
12	1600	173.3	41.11	1366

By linear interpolation, the cold-side fluid-flow rate,

$$\dot{m}_c = \frac{(1580 - 1746)}{(1366 - 1746)}(12 - 14) + 14 = 13.13 \text{ kg/s}$$

≈ **13 kg/s** ***(to the nearest 1 kg/s)*** **(Ans.) (c)**

EXAMPLE 11.27 *A shell-and-tube type steam condenser employed in a large steam power plant, effects a heat exchange rate of 2200 MW. The condenser consists of a single pass shell and 32 000 tubes, each executing two passes. The water at the rate of 3.2 × 10⁴ kg/s passes through the tubes which are* *thin walled and of diameter 30 mm. The steam condenses on the outer surface of the tubes. The heat-transfer coefficient on the steam side may be taken as 11 500 W/m² K. Steam condenses at 50°C while water enters the condenser at 20°C.*

Using the LMTD method and the ε–NTU method, calculate (a) the outlet temperature of the cooling water, and (b) the length of the tubes per pass.

Properties of water at 300 K:

$C_p = 4.179$ kJ/kg K	$k = 0.613$ W/m °C
$\mu = 855 \times 10^{-6}$ N s /m²	$Pr = 5.83$

[NU: S 2005]

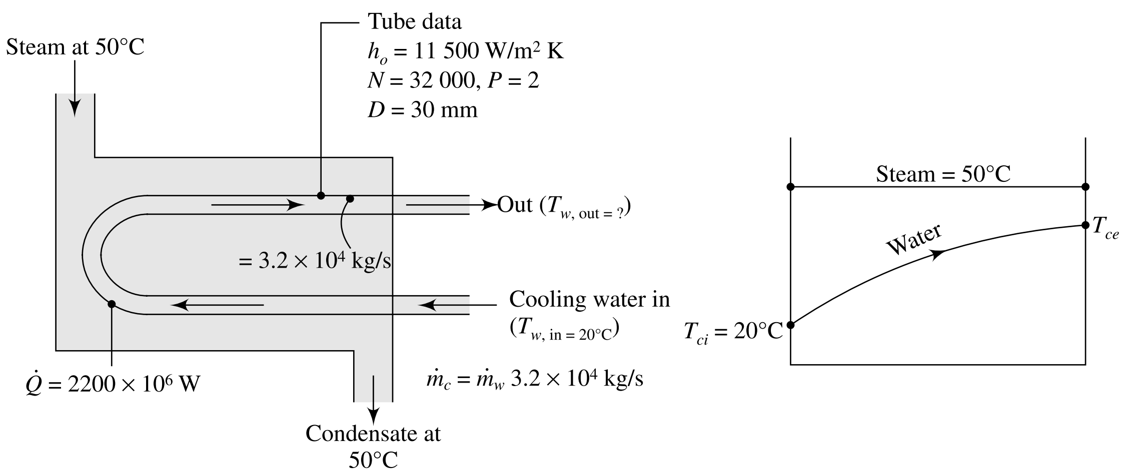
Solution

Analysis Heat-exchange rate,

$$\dot{Q} = 2200 \times 10^6 \text{ J/s} = \dot{m}_c C_{p_c} (T_{ce} - T_{ci})$$

The cold fluid is cooling water passing through the tubes.

Schematic



∴ Outlet temperature of cooling water,

$$T_{w,\text{out}} = T_{w,\text{in}} + \frac{\dot{Q}}{\dot{m}_w C_{pW}} = 20°\text{C} + \frac{2200 \times 10^6 \text{ J/s}}{3.2 \times 10^4 \text{ kg/s} \times 4179 \text{ J/kg K}}$$

$$= 36.45°\text{C}$$ **(Ans.) (a)**

LMTD Method:

Tubes are thin walled ($D_o \approx D_i$).

Hence, $R_{\text{wall}} = \dfrac{1}{2\pi k L} \ln \dfrac{D_o}{D_i} = 0$

Hence, $\dfrac{1}{U} = \dfrac{1}{h_i} + \dfrac{1}{h_o}$

To find inside heat-transfer coefficient, h_i, we use Dittus–Boelter correlation, provided the flow is *turbulent*.
Reynolds number,

$$Re_D = \frac{4\dot{m}(\text{per tube})}{\pi D\mu} = \frac{4(3.2\times10^4/32\,000)\text{kg/s}}{\pi(30\times10^{-3}\text{ m})(855\times10^{-6}\text{ N s/m}^2\text{ or kg/m s})}$$

$= 49\ 639 \quad \Rightarrow \quad$ *Turbulent flow*

$$Nu_D = 0.023(Re_D)^{0.8}(Pr)^{0.4}$$

or $$h_i = Nu_D\frac{k}{D} = \frac{0.613\text{ W/m K}}{0.03\text{ m}}\times0.023(49639)^{0.8}\times(5.83)^{0.4}$$

$= 5432.4$ W/m^2 K

Overall heat-transfer coefficient,

$$U = \left[\frac{1}{5432.4}+\frac{1}{11500}\right]^{-1} = 3689.54\text{ W/m}^2\text{ K}$$

$$\text{LMTD} = \frac{\Delta T_i - \Delta T_e}{\ln\left(\dfrac{\Delta T_i}{\Delta T_e}\right)} = \frac{(50-20)-(50-36.45)}{\ln\dfrac{30}{13.55}} = 20.7°\text{C}$$

$\therefore$ Area of heat exchanger,

$$A = NP\pi DL = \frac{\dot{Q}}{U(\text{LMTD})(F)}$$

Length of the heat exchanger required,

$$L = \frac{\dot{Q}}{(NP\pi D)U(\text{LMTD})} = \frac{2200\times10^6\text{ W}}{(32\,000\times2\times\pi\times0.03\text{ m})(3689.54\text{ W/m}^2\text{°C})(20.7\text{°C})}$$

$= 4.78$ m **(Ans.) (b)**

Note that LMTD correlation factor, $F = 1$ for condensers and evaporators.

Effectiveness–NTU Method:

$$\varepsilon = 1 - \exp[-\text{NTU}]$$

where $$\text{NTU} = \frac{UA}{C_{\min}} = \frac{UA}{\dot{m}_c C_{pc}}$$

Now $(1-\varepsilon) = e^{-\text{NTU}}$ or $\ln(1-\varepsilon) = -\text{NTU}$

Effectiveness,

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{T_{ce}-T_{ci}}{T_{hi}-T_{ci}} = \frac{(36.45-20)\text{°C}}{(50-20)\text{°C}} = 0.5483$$

$\therefore$ NTU = 0.7948

And, $$A = \frac{\dot{m}_c C_{p_c}\text{NTU}}{U} = \frac{3.2\times10^4\text{ kg/s}\times4180\text{ J/kg K}\times0.7948}{3689.54\text{ W/m}^2\text{ K}} = 28815\text{ m}^2$$

But $A = NP\pi DL$

$\therefore$ Length of the tubes required, per pass is

$$L = \frac{A}{NP\pi D} = \frac{28815\ \text{m}^2}{32000 \times 2 \times \pi \times 0.03\ \text{m}} = \mathbf{4.78\ m} \qquad \textbf{(Ans.) (b)}$$

EXAMPLE 11.28 *A plate-type heat exchanger cross flow, both fluids unmixed is to be designed to heat 1.5 kg/s of air (C_p = 1.046 kJ/kg K) from 50°C to 200°C. For this purpose, hot gases (C_p = 1.255 kJ/kg K) are available at 350°C and 1.8 kg/s.*

Assume h_{air} = 350 W/m² K, and h_{gas} = 465 W/m² K.

(a) Determine the heat-exchanger surface area. Neglect fouling.

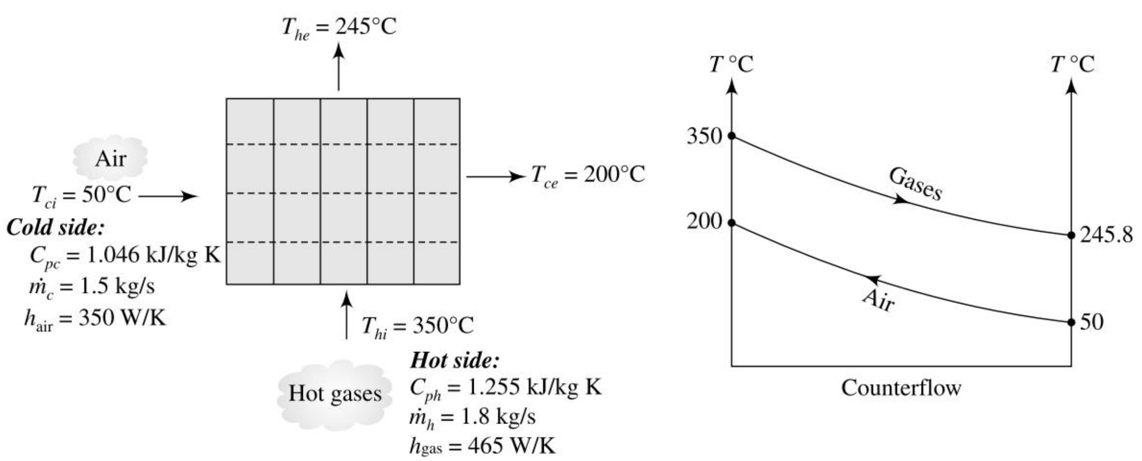
(b) After one year of use, the heat exchanger is fouled, and the fouling factors are estimated to be 0.001 m² K/W on the air side and 0.0012 m² K/W on the gas side. Determine the exit temperature of air in the fouled condition. [IIT, Bombay]

Solution

Known A plate-type cross-flow heat exchanger (*both fluids unmixed*) operates under specified conditions.

Find (a) Area of heat exchanger, A(m²) (b) Exit air temperature in the fouled condition, T_{ce}(°C).

Schematic



Assumptions (1) Steady-state conditions. (2) Heat-transfer coefficients are constant and uniform in both fouled and unfouled conditions. (3) Constant fluid properties.

Analysis (a) Heat-transfer rate,

$$\dot{Q} = \dot{m}_h C_{p_h}(T_{hi} - T_{he}) = \dot{m}_c C_{p_c}(T_{ce} - T_{ci})$$
$$= 1.8\ \text{kg/s} \times 1.255\ \text{kJ/kg K} \times (350 - T_{he})\text{°C}$$
$$= 1.5\ \text{kg/s} \times 1.046\ \text{kJ/kg K} \times (200 - 50)\text{°C} = \mathbf{235.4\ kW}$$

$\therefore$ T_{he} = **245.8°C**

Calculation of ΔT_{lm}:

$$\Delta T_{lm,\text{CF}} = \frac{195.8 - 150}{\ln\dfrac{195.8}{150}} = \mathbf{171.9°C}$$

From the schematic with

$$R = \frac{T_{hi} - T_{he}}{T_{ce} - T_{ci}} = \frac{350 - 245.8}{200 - 50} = 0.695$$

$$P = \frac{T_{ce} - T_{ci}}{T_{hi} - T_{ci}} = \frac{200 - 50}{350 - 50} = 0.50$$

LMTD correction factor, F is found to be

$$F = 0.94$$

$\therefore$ $\Delta T_m = F \cdot \Delta T_{lm} = 0.94 \times 171.9 = 161.6°C$

Overall heat-transfer coefficient,

$$U = \left[\frac{1}{h_{\text{air}}} + \frac{1}{h_{\text{gas}}}\right]^{-1} = \left[\frac{1}{350} + \frac{1}{465}\right]^{-1} = \mathbf{199.7\ W/m^2\ K}$$

Heat-exchanger surface area is

$\therefore$ $$A = \frac{\dot{Q}}{U \Delta T_m} = \frac{235.4 \times 10^3\ \text{W}}{199.7\ \text{W/m}^2\ \text{K} \times 161.6} = \mathbf{7.295\ m^2}$$ **(Ans.) (a)**

(b) Overall heat-transfer coefficient,

$$U = \left[\frac{1}{h_{\text{air}}} + R_{f,\text{air}} + R_{f,\text{gas}} + \frac{1}{h_{\text{gas}}}\right]^{-1} = \left[\frac{1}{350} + 0.001 + 0.0012 + \frac{1}{465}\right]^{-1} \text{W/m}^2\ \text{K}$$

$$= \mathbf{138.7\ W/m^2\ K}$$

$$\text{NTU} = \frac{UA}{(\dot{m}C_p)_{\min}} = \frac{138.7\ \text{W/m}^2\ \text{K} \times 7.295\ \text{m}^2}{1.5\ \text{kg/s} \times (1.046 \times 1000)\text{J/kg K}} = 0.645$$

$$R = \frac{C_{\min}}{C_{\max}} = \frac{(\dot{m}C_p)_{\min}}{(\dot{m}C_p)_{\max}} = \frac{1.5\ \text{kg/s} \times 1.046\ \text{kJ/kg K}}{1.8\ \text{kg/s} \times 1.255\ \text{kJ/kg K}} = 0.695$$

$\therefore$

$\therefore$ Effectiveness,

$$\varepsilon = 0.4 = \frac{T_{ce} - T_{ci}}{T_{hi} - T_{ci}} = \frac{T_{ce} - 50}{350 - 50} = \frac{T_{ce} - 50}{300}$$

Exit temperature of cold side fluid (*air*) is

$T_{ce} = 50 + 0.4\ (300) = \mathbf{170°C}$ **(Ans.) (b)**

EXAMPLE 11.29 *A cross-flow heat exchanger is to be designed to heat hydrogen gas with hot water. The water is on the tube side and enters at 150°C at a flow rate of 3 kg/s with a heat-transfer coefficient of 1250 W/m² °C. The hydrogen (*C_p *= 14.4 kJ/kg °C) is on the shell side and enters at 30°C at a flow rate of 120 kg/min with a heat-transfer coefficient of 1800 W/m² °C. The required hydrogen exit temperature is 60°C. The heat exchanger has 100, 2.5 mm thick tubes of 15 mm ID, made of stainless steel (*k *= 14.2 W/m °C).*

Determine (a) the overall heat-transfer coefficient based on the inner area, and (b) the required length of the tubes.

Solution

Known Cross-flow heat exchanger with hot fluid (water) on tube side and cold fluid (hydrogen) on shell side. Operating data are specified.

Find (a) U_i(W/m^2 °C); (b) L(m).

Assumptions (1) Steady operating conditions. (2) Constant properties. (3) Uniform heat-transfer coefficient. (4) No fouling.

Analysis

LMTD approach:

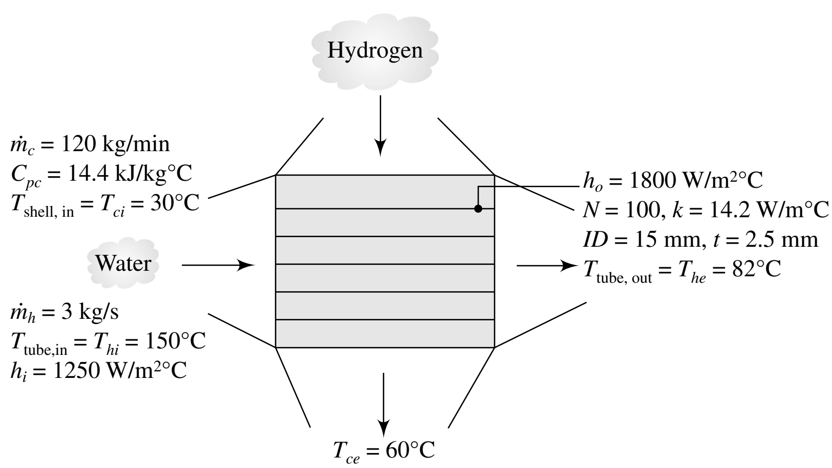
Heat-transferred by water to hydrogen,

$$\dot{Q} = \dot{m}_h C_{p_h}(T_{hi} - T_{he})$$

Heat received by hydrogen from water,

$$\dot{Q} = \dot{m}_c C_{pc}(T_{ce} - T_{ci})$$

Schematic



Cross-flow heat exchanger [Hydrogen: mixed; Water: unmixed]

Heat-capacity rates:

$$C_h = \dot{m}_h C_{p_h} = \dot{m}_{\text{water}} C_{p_{\text{water}}}$$

$C_{p_{\text{water}}}$ is to be evaluated at the mean temperature, $T_{bm} = \frac{1}{2}(T_{hi} + T_{he})$

But T_{he} is not known.

At an assumed mean water temperature of 120°C, $C_{p_{\text{water}}}$ = 4.232 kJ/kg °C

$\therefore$ C_h = (3 kg/s) (4.232 kJ/kg °C) = 12.7 kW/ °C

$$C_c = \dot{m}_c C_{pc} = \dot{m}_{H_2} C_{p_{H_2}} = \left(\frac{120}{60}\text{kg/s}\right) \text{ (14.4 kJ/kg °C) = 28.8 kW/°C}$$

Clearly, $C_c = C_{\text{mixed}} = C_{H_2} = C_{\max}$

And, $\quad C_h = C_{\text{unmixed}} = C_{\text{H}_2\text{O}} = C_{\min}$

Heat rate,

$$\dot{Q} = C_c(T_{ce} - T_{ci}) = (28.8 \text{ kW/°C})(60 - 30)\text{°C} = 864 \text{ kW}$$

Water outlet temperature, T_{he} can be found from an energy balance.

$$\dot{Q} = 864 \text{ kW} = C_h(T_{hi} - T_{he})$$

$$\therefore \quad T_{he} = T_{hi} - \frac{\dot{Q}}{C_h} = 150\text{°C} - \frac{864 \text{ kW}}{12.7 \text{ kW/°C}} = 82\text{°C}$$

For counter-current flow:

$$\Delta T_i = T_{hi} - T_{ce} = 150 - 60 = 90\text{°C}$$

$$\Delta T_e = T_{he} - T_{ci} = 82 - 30 = 52\text{°C}$$

$$\therefore \quad (\text{LMTD})_{\text{CF}} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)} = \frac{90 - 52}{\ln(90/52)} = 69.27\text{°C}$$

Now, let us calculate the parameters P and R to find the LMTD correction factor for cross flow arrangement (one fluid mixed, the other unmixed).

$$P = \frac{T_{\text{tube,out}} - T_{\text{tube,in}}}{T_{\text{shell,in}} - T_{\text{tube,in}}} = \frac{82 - 150}{30 - 150} = 0.566$$

$$R = \frac{T_{\text{shell,in}} - T_{\text{shell,out}}}{T_{\text{tube,out}} - T_{\text{tube,in}}} = \frac{30 - 60}{82 - 150} = 0.441$$

From the chart, LMTD correction factor, $F = 0.93$

Mean temperature difference,

$$\Delta T_m = F(\text{LMTD})_{\text{CF}} = 0.93 \times 69.27 = \mathbf{64.4\text{°C}}$$

For the tubes, $r_i = \frac{1}{2}D_i = 7.5$ mm

$$r_o = r_i + t = 7.5 + 2.5 = 10 \text{ mm}$$

Overall heat-transfer coefficient based on inner area,

$$U_i = \left[\frac{1}{h_i} + \frac{r_i}{r_o}\frac{1}{h_o} + \frac{r_i}{k}\ln\frac{r_o}{r_i}\right]^{-1} = \left[\frac{1}{1250} + \frac{7.5}{10}\frac{1}{1800} + \frac{7.5 \times 10^{-3}}{14.2}\ln\frac{10}{7.5}\right]^{-1}$$

$= \mathbf{730.7\ W/m^2\ °C}$ **(Ans.) (a)**

With $\quad \dot{Q} = U_i A_i \Delta T_m$

Area of heat exchanger, $A_i = NP\pi D_i L$

where P is the number of passes = 1

N is the number of tubes = 100

D_i is the ID of tubes = 0.015 m

$\therefore$ Required tube length is

$$L = \frac{\dot{Q}}{U_i(N\pi D_i)\Delta T_m} = \frac{864 \times 10^3 \text{ W}}{(730.7 \text{ W/m}^2\text{°C})(100 \times \pi \times 0.015 \text{ m})(64.4\text{°C})}$$

$= \mathbf{3.9\ m}$ **(Ans.) (b)**

Effectiveness–NTU approach:

Capacity-rate ratio,

$$R = \frac{C_{\min}}{C_{\max}} = \frac{C_h}{C_c} = \frac{C_{\text{unmixed}}}{C_{\text{mixed}}} = \frac{12.7}{28.8} = 0.441$$

Effectiveness,

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{C_h(T_{hi} - T_{he})}{C_{\min}(T_{hi} - T_{ci})} = \frac{T_{hi} - T_{he}}{T_{hi} - T_{ci}} = \frac{150 - 82}{150 - 30} = 0.566$$

For $C_{\max} = C_{\text{mixed}}$

$$\text{NTU} = -\ln\left[1 + \left(\frac{1}{R}\right)\ln(1 - \varepsilon R)\right] = -\ln\left[1 + \frac{1}{0.441} \times \ln(1 - 0.566 \times 0.441)\right]$$

$$= 1.056$$

But $\text{NTU} = \dfrac{UA}{C_{\min}} = \dfrac{U_i A_i}{C_h}$

$\therefore$

$$A_i = \frac{C_h \text{NTU}}{U_i} = \frac{(12.7 \times 10^3 \text{ W/°C})(1.056)}{730.7 \text{ W/m}^2 \text{ °C}} = 18.354 \text{ m}^2$$

As $A_i = N\pi D_i L$

$$L = \frac{A_i}{N\pi D_i} = \frac{18.354 \text{ m}^2}{(100)(\pi \times 0.015 \text{ m})} = \mathbf{3.9 \text{ m}} \quad \textbf{(Ans.)}$$

Comment One can also find NTU from the chart, corresponding to $R = 0.441$ and $\varepsilon = 0.566$ as 1.06. Then $A_i = 18.4 \text{ m}^2$ and $L = 3.9$ m

EXAMPLE 11.30 *A shell-and-tube heat exchanger (1-shell pass and 2-tube passes) is to be used to condense 2.73 kg/s of saturated steam at 67°C. Condensation takes place on the outer surface of the tubes and the steam side heat-transfer coefficient is 10 kW/m² °C. The temperature of the cooling water at inlet to the tubes is 15°C and the exit water temperature must not exceed 30°C. Thin-walled tubes of 19 mm diameter are prescribed and the average velocity of water flow through the tubes is to be maintained at 0.5 m/s. Determine (a) the minimum number of tubes that can be used and the corresponding length of the tubes per pass. (b) If the water-side heat-transfer coefficient is doubled by using a heat-transfer augmentation technique like inserting a wire mesh in the tubes, what would be the required tube length per pass? Use the following properties of water:*

Properties of saturated water:

$\rho_{@15°C} = 999$ kg/m³ $\quad h_{fg@67°C} = 2341.3$ kJ/kg

At $T_{bm} = (15 + 30)/2 = 22.5$°C:

$\rho = 998$ kg/m³ $\quad C_p = 4.181$ kJ/kg °C

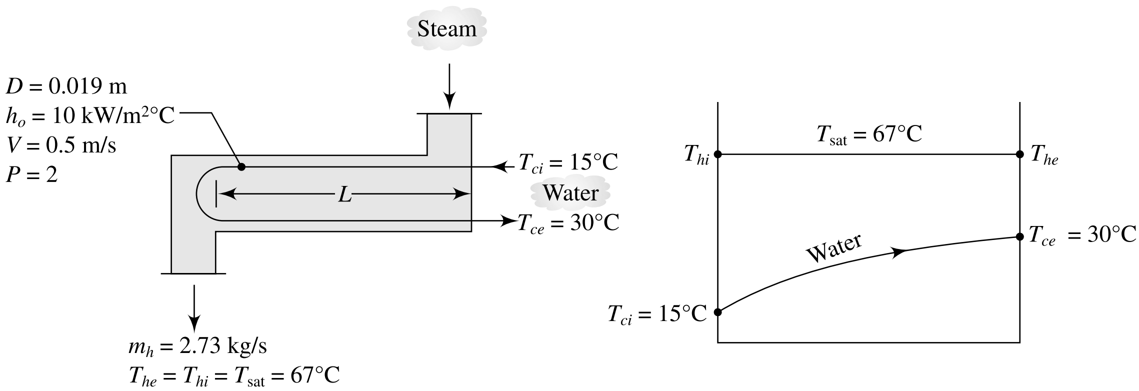
$\mu = 9.46 \times 10^{-4}$ kg/ms $\quad k = 0.607$ W/m °C $\quad \text{Pr} = 6.52$

Solution

Known Shell-and-tube heat exchanger for condensing steam with water as coolant operates under the specified conditions.

Find (a) Maximum number of tubes, N, and tube length, L(m). (b) Tube length, L(m) if h_{water} is increased by a factor of two.

Schematic



Assumptions (1) Steady operating conditions exist. (2) The thickness of the tubes is negligible because the tubes are thin walled. (3) Kinetic and potential energy changes of fluid streams are negligible. (4) The heat-transfer coefficients are constant and uniform. (5) The heat exchanger is effectively insulated. (6) Tube internal flow and thermal conditions are fully developed.

Analysis The heat-transfer rate is

$$\dot{Q} = \dot{m}_c C_{pc}(T_{ce} - T_{ci}) = \dot{m}_h h_{fg}$$

Heat rate, $\dot{Q} = (2.73 \text{ kg/s})(2341.3 \times 10^3 \text{ J/kg}) = 6.39 \times 10^6 \text{ W}$

Mass-flow rate of cooling water is

$$\dot{m}_c = \frac{\dot{Q}}{C_{pc}(T_{ce} - T_{ci})} = \frac{6.39 \times 10^6 \text{ J/s}}{4181 \text{ J/kg°C} \times (30 - 15)\text{°C}} = 101.9 \text{ kg/s}$$

Per tube, mass-flow rate,

$$\dot{m}_c/\text{tube} = \rho \frac{\pi}{4} D^2 V = (999 \text{ kg/m}^3)\left(\frac{\pi}{4} \times 0.019^2 \text{ m}^2\right)(0.5 \text{ m/s}) = 0.1416 \text{ kg/s}$$

Hence, the minimum number of tubes is

$$N = \frac{\dot{m}_c}{\dot{m}_c/\text{tube}} = \frac{101.9 \text{ kg/s}}{0.1416 \text{ kg/s}} \approx \mathbf{720} \qquad \textbf{(Ans.) (a)}$$

Overall heat-transfer coefficient is determined to be

$$U = \left[\frac{1}{h_i} + \frac{1}{h_o}\right]^{-1}$$

where h_o = 10 000 W/m² °C

To calculate h_i, let us first determine R_{eD}.

$$Re_D = \frac{\rho V D}{\mu} = \frac{(998 \text{ kg/m}^3)(0.5 \text{ m/s})(0.019 \text{ m})}{9.46 \times 10^{-4} \text{ kg/m s}} = 10\,022$$

Using Dittus–Boelter equation:

$$Nu_D = \frac{h_i D}{k} = 0.023(Re_D)^{0.8}(Pr)^{0.4}$$

It follows that

$$h_i = \frac{0.023 \times 607 \text{ W/m°C}}{0.019 \text{ m}} \times (10022)^{0.8}(6.52)^{0.4} = 2470 \text{ W/m}^2\text{°C}$$

Hence, $$U = \left[\frac{1}{2470} + \frac{1}{10000}\right]^{-1} = 1980 \text{ W/m}^2\text{°C}$$

ε–NTU Method:

Heat-capacity rates:

Hot fluid (steam): $C_h = \dot{m}_h C_{ph} \approx \infty = C_{\max}$ (*Phase change*)

Cold fluid (water) $= C_c = \dot{m}_c C_{pc} = (101.9 \text{ kg/s})(4181 \text{ J/kg°C})$

$$= 426 \times 10^3 \text{ W/°C} = C_{\min}$$

Heat-capacity rate ratio,

$$R = \frac{C_{\min}}{C_{\max}} = \frac{C_c}{\infty} = 0$$

Effectiveness of any heat exchanger with $R = 0$ is

$$\varepsilon = 1 - \exp(-\text{NTU}) \quad \text{or} \quad -\ln(1-\varepsilon) = \text{NTU}$$

where $$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{C_c(T_{ce} - T_{ci})}{C_{\min}(T_{hi} - T_{ci})} = \frac{T_{ce} - T_{ci}}{T_{hi} - T_{ci}} = \frac{30 - 15}{67 - 15} = 0.289$$

Number of transfer units, NTU = $UA/C_{\min}$

= – ln (1 – 0.289) = 0.341

Area of the heat exchanger is

$$A = \frac{(0.341)(426 \times 10^3 \text{ W/°C})}{1980 \text{ W/m}^2\text{°C}} = 73.4 \text{ m}^2$$

But $A = NP\pi DL$

Hence, the tube length required,

$$L = \frac{73.4 \text{ m}^2}{(720)(2)(\pi \times 0.019 \text{ m})} = \mathbf{0.85 \text{ m}}$$ **(Ans.) (a)**

If h_i increases by a factor of two then

$$U^* = \left[\frac{1}{2 \times 2470} + \frac{1}{10000}\right]^{-1} = 3307 \text{ W/m}^2\text{°C}$$

$$\text{NTU} = \frac{U^* A^*}{C_{\min}} = \frac{U^* NP\pi DL^*}{C_c} = 0.341$$

New tube length,

$$L^* = \frac{0.341 \times 426 \times 10^3 \text{ W/°C}}{3307 \text{ W/m}^2\text{°C} \times 2 \times \pi \times 0.019 \text{ m} \times 720} = \mathbf{0.51 \text{ m}}$$ **(Ans.) (b)**

LMTD Method:

$$(\text{LMTD})_{\text{counterflow}} = \frac{\Delta T_i - \Delta T_e}{\ln \dfrac{\Delta T_i}{\Delta T_e}} = \frac{(67-15)-(67-30)}{\ln \dfrac{52}{37}} = 44°\text{C}$$

Heat rate,

$$\dot{Q} = UA(\text{LMTD})_{\text{CF}}\,F$$

LMTD correction factor, $F = 1$ for a heat exchanger with $R = 0$.

$$A = NP\pi DL = \frac{\dot{Q}}{U \cdot \text{LMTD}}$$

Tube length is

$$L = \frac{6.39 \times 10^6 \text{ W}}{1980 \text{ W/m}^2\,°\text{C} \times 44°\text{C} \times 720 \times \pi \times 0.019 \text{ m}} = \mathbf{0.85\ m} \qquad \textbf{(Ans.) (a)}$$

For $U^* = 3307$ W/m^2 °C the new tube length will be

$$L^* = \frac{6.39 \times 10^6 \text{ W}}{3307 \text{ W/m}^2\,°\text{C} \times 44°\text{C} \times 720 \times 2 \times \pi \times 0.019 \text{ m}} = \mathbf{0.51\ m} \qquad \textbf{(Ans.) (b)}$$

Comment The tube length required is significantly reduced by enhancing the water side heat-transfer coefficient. It is noteworthy that it is the smaller convection coefficient (in this case h_i) which is decisive in evaluating the overall heat-transfer coefficient.

EXAMPLE 11.31 *In a regenerative gas turbine cycle, the pressure ratio is 11.2, the inlet air is at 30°C, 1 atm, and the turbine inlet temperature is 1150°C. Compressor and turbine are isentropic. The regenerator is a single pass, crossflow heat exchanger. One hundred silicon carbide ceramic tubes (*k *= 24 W/m K, 50 mm inner diameter, 75 mm outer diameter,* L *= 9.0 m) are arranged such that the outside heat-transfer coefficient is 35 W/m*^{2} *K. The total air-flow rate inside the tubes is 1.3 kg/s; the fuel flow rate is 5% of that of the air. The outer tube surface is clean and the inner tube surface has a fouling factor of 2 ×* 10^{-4} *m*^{2} *K/W. Determine the net power output and the cycle thermal efficiency. The following properties of air may be used:*

k = *0.0261 W/m K,* μ = *1.85 ×* 10^{-5} *kg/m s, and* Pr = *0.712*

Solution

Known Regenerator (*cross-flow heat exchanger with one fluid mixed*) operates under specified conditions.

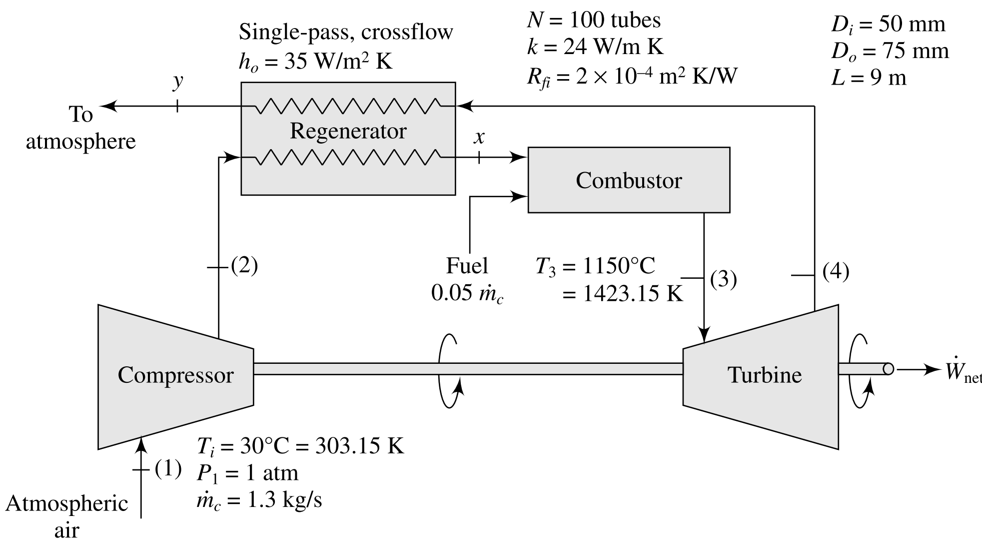
Find Net power output. Thermal efficiency.

Assumptions (1) The system operates under steady conditions. (2) Potential and kinetic energy effects are negligible. (3) Air is the working fluid and is an ideal gas with constant specific heats. (4) Specific heat is constant. (5) The turbine and compressor are isentropic.

Analysis The net power output is $\dot{W}_{\text{net}} = \dot{W}_t - \dot{W}_c$.

$$\dot{W}_c = \dot{m}_c C_p (T_2 - T_1) \quad \text{and} \quad \dot{W}_t = \dot{m}_t C_p (T_3 - T_4)$$

Schematic



Since fuel mass-flow rate is 5% of that of the air, $\dot{m}_t = 1.05\dot{m}_c$. The inlet temperatures (T_1 and T_3) are given. The pressure ratio is known for an isentropic turbine and compressor. Therefore, the outlet temperatures can be determined.

$$T_2 = T_1\left(\frac{P_2}{P_1}\right)^{(k-1/k)} = 303.15\text{ K }(11.2)^{(1.40-1)/(1.4)} = 604.55\text{ K}$$

$$T_4 = T_3\left(\frac{P_4}{P_3}\right)^{(k-1/k)} = 1423.15\text{ K}(1/11.2)^{(1.4-1)/(1.4)} = 713.63\text{ K}$$

The net power output is

$$\dot{W}_{\text{net}} = \dot{m}_t C_p(T_3 - T_4) - \dot{m}_c C_p(T_2 - T_1) = (1.3\text{ kg/s})(1.005\text{ kJ/kg K})(1.05)$$

$$\times[(1423.15 - 713.63) - (604.55 - 303.15)]\text{K}\left(\frac{1\text{ kW}}{1\text{ kJ/s}}\right) = \mathbf{580\ kW}\quad \textbf{(Ans.)}$$

The rate of heat supplied,

$$\dot{Q}_{\text{in}} = \dot{m}_t C_p(T_3 - T_x)$$

We note that the heat-exchanger effectiveness,

$$\varepsilon = \frac{\dot{Q}_{\text{act}}}{\dot{Q}_{\max}} = \frac{\dot{m}_c C_p(T_x - T_2)}{C_{\min}(T_4 - T_2)}$$

Since we have assumed constant specific heats and the mass flow rate through the compressor is smaller than that through the turbine, $C_{\min} = \dot{m}_c C_p$.

Hence, $\varepsilon = (T_x - T_2)/(T_4 - T_2)$

Solving for T_x, we have

$$\boxed{T_x = T_2 + \varepsilon(T_4 - T_2)}$$

The only unknown in this expression is the heat-exchanger effectiveness.

The heat-exchanger geometry, the flow rates, and the inlet temperatures are specified. We want to determine the heat-transfer rate. Hence, this is a *rating* problem, and the ε–NTU approach is preferred. This is a cross-flow heat exchanger, with one fluid *mixed* (*outside the tubes*) and one fluid *unmixed* (*inside the tubes*). We can obtain the effectiveness from the chart, once we have the heat-capacity ratio and the NTU. The heat capacity ratio is $C_{\text{mixed}}/C_{\text{unmixed}} = \dot{m}_t C_p / \dot{m}_c C_p = 1.05/1 = 1.05$.

As NTU = $UA/C_{\min}$, we need to evaluate the overall heat-transfer coefficient and the total surface area. Based on the inside surface area, $A_i = N\pi D_i L$, the overall heat-transfer coefficient is defined as

$$\frac{1}{U_i A_i} = \frac{1}{h_i A_i} + \frac{R_{fi}}{A_i} + R_{\text{wall}} + \frac{R_{fo}}{A_o} + \frac{1}{h_o A_o}$$

We are given the outside heat-transfer coefficient, the fouling resistance, and enough information to calculate the wall resistance.

To evaluate the inside heat-transfer coefficient, we need the Reynolds number, $Re = \rho V D_i/\mu$. For a straight circular tube, we rewrite the Reynolds number as $Re = \dfrac{4\dot{m}}{\pi D_i \mu}$.

Hence, $$Re = \frac{4\dot{m}_{\text{total}}/N}{\pi D_i \mu} = \frac{4(1.3\ \text{kg/s}/100)}{\pi(0.05\ \text{m})(1.85\times 10^{-5}\ \text{kg/m s})(0.05\ \text{m})} = 17894$$

This is a *turbulent* flow, and the Gnielinski correlation is appropriate:

$$Nu = \frac{hD}{k} = \frac{(f/8)(Re_D - 1000)Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)}$$

The friction factor is

$$f = (0.79 \ln Re - 1.64)^{-2} = [0.79 \ln 17894 - 1.64]^{-2} = 0.027$$

and the Nusselt number is

$$\text{Nu} = \frac{(0.027/8)(17894 - 1000)(0.712)}{1 + 12.7(0.027/8)^{1/2}(0.712^{2/3} - 1)} = 47.7$$

$$h_i = \frac{Nu\,k}{D_i} = \frac{47.7(0.0261\ \text{W/m K})}{0.05\ \text{m}} = 24.9\ \text{W/m}^2\,\text{K}$$

The overall heat-transfer coefficient is determined to be

$$U_i = \left[\frac{1}{h_i} + R_{fi} + \frac{D_i \ln(r_o/r_i)}{2k} + \frac{1}{h_o}\left(\frac{D_i}{D_o}\right)\right]^{-1}$$

$$= \left[\frac{1}{24.9\ \text{W/m}^2\,\text{K}} + 2\times 10^{-4}\ \text{m}^2\text{K/W} + \frac{(0.05\ \text{m})\ln(0.075/0.05)}{2(24\ \text{W/m K})} + \frac{0.05\ \text{m}}{(35\ \text{W/m}^2\,\text{K})(0.075\ \text{m})}\right]^{-1} = \mathbf{16.7\ W/m^2\,K}$$

$$\text{NTU} = \frac{U_i A_i}{C_{\min}} = \frac{U_i(N\pi D_i L)}{(\dot{m}C_p)_{\text{air}}}1 = \frac{(16.7\ \text{W/m}^2\,\text{K})\pi(100\times 0.05\ \text{m}\times 9\ \text{m})(1\ \text{J/1Ws})}{(1.3\ \text{kg/s})(1.005\ \text{kJ/kg K})(1000\ \text{J/1 kJ})} = 1.8$$

The heat-capacity ratio is 1.05. Therefore, from the chart, $\varepsilon \approx$ **0.58**.

We use T_2, T_4, and the heat exchanger effectiveness to determine the outlet temperature from the regenerator:

$$T_x = T_2 + \varepsilon(T_4 - T_2) = 604.55\text{ K} + 0.58(713.63 - 604.55)\text{K} = 667.8\text{ K}$$

The input heat-transfer rate is

$$\dot{Q}_{\text{in}} = \dot{m}_t C_p (T_3 - T_x)$$

$$= 1.05(1.3\text{ kg/s})(1.005\text{ kJ/kg K})(1423.15 - 667.8)\text{K}\left(\frac{1\text{ kW}}{1\text{ kJ/s}}\right)$$

$$= \mathbf{1036.2\ kW}$$

Hence, the cycle thermal efficiency is

$$\eta_{\text{cycle}} = \frac{\dot{W}_{\text{net}}}{\dot{Q}_{\text{in}}} = \frac{580\text{ kW}}{1036.2\text{ kW}} = \mathbf{0.56} \quad \text{or} \quad \mathbf{56\%} \qquad \textbf{(Ans.)}$$

EXAMPLE 11.32 *Cooling water at a steady rate of 0.5 kg/s flows through an inner tube having inner diameter of 25 mm and length of 10 m of a tube in tube condenser. The mean inlet temperature of cooling water is 10°C. Saturated steam condenses in the annulus at a uniform rate such that the inner surface temperature of the tube is constant throughout the length of the tube at 40°C. The average condensing side heat-transfer coefficient is 10000 W/m² K. Neglect the thickness of the heat exchanger tube. Calculate the effectiveness of the heat exchanger and the exit water temperature.*

Properties of water are given below:

Specific heat = 4180 J/kg K

Density = 90 kg/m³

Dynamic viscosity = 0.8 × 10⁻³ Pa s

Thermal conductivity = 0.57 W/m K

You may use the relation

$$\text{Nu} = 0.023\,\text{Re}_\text{D}^{0.8}\,\text{Pr}^{0.4} \qquad \textbf{[IES 2012]}$$

Solution

Known A co-axial, single stream (cooling water) heat exchanger with condensing steam in the annulus.

Find Heat-exchanger effectiveness, ε; cooling water exit temperature T_{ce}.

Assumptions (1) Thin-walled, inner tube with negligible tube-wall resistance. (2) The effect of fouling is ignored. (3) Constant U along the exchanger.

Analysis Reynolds number,

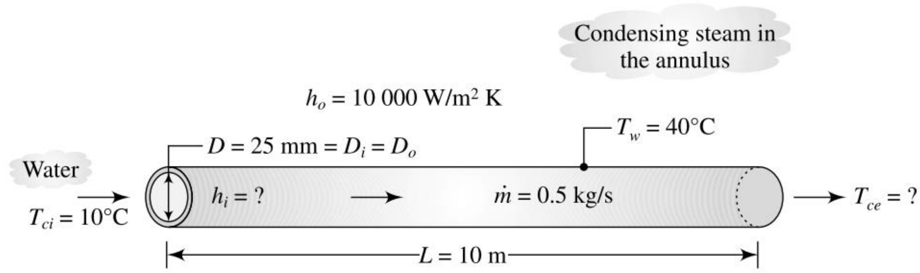
$$Re_D = \frac{\rho V D}{\mu} = \frac{4\dot{m}}{\pi D \mu} = \frac{4 \times 0.5}{\pi \times 0.025 \times 0.8 \times 10^{-3}} = 31\,831$$

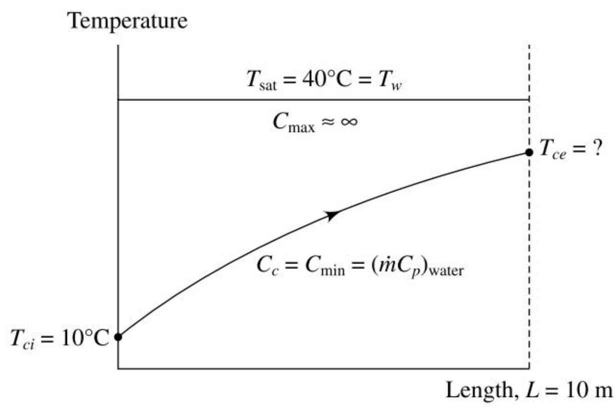
Prandtl number, $Pr = \dfrac{C_p \mu}{k} = \dfrac{4180 \times 0.8 \times 10^{-3}}{0.57} = 5.867$

Using the given relation:

$$Nu = \frac{h_i D}{k} = 0.023(Re_D)^{0.8}(Pr)^{0.4}$$

Schematic





∴ Inside (*water side*) heat-transfer coefficient

$$h_i = \frac{0.023 \times 0.57}{0.025} \times (31831)^{0.8} \times (5.867)^{0.4} = 4258.9 \text{ W/m}^2\text{ K}$$

Outside (*steam side*) heat-transfer coefficient, h_o = 10 000 W/m² K
Overall heat-transfer coefficient,

$$U = \left(\frac{1}{h_i} + \frac{1}{h_o}\right)^{-1} = \left[\frac{1}{4258.9} + \frac{1}{10000}\right]^{-1} = 2986.85 \text{ W/m}^2\text{ K}$$

Heat-exchanger surface area,

$$A = \pi DL = \pi(0.025)(10) = 0.7854 \text{ m}^2$$

$$(UA) \text{ value} = (2986.85)(0.7854) = 2345.86 \text{ W/K}$$

$$C_{\min} = \dot{m}_c C_{p_c} = (0.5)(4180) = 2090 \text{ W/K}$$

∴ Number of transfer units,

$$\text{NTU} = \frac{UA}{C_{\min}} = \frac{2345.86}{2090} = 1.122$$

Effectiveness of the heat exchanger,

$$\varepsilon = 1 - \exp(-\text{NTU}) = 1 - e^{-1.122} = \mathbf{0.6745} \qquad \textbf{(Ans.)}$$

Heat-transfer rate,

$$\dot{Q} = \varepsilon \dot{Q}_{\max} = \varepsilon (C_{\min})(T_{hi} - T_{ci}) = (0.6745)\ (2090)\ (40 - 10)$$

$$= 42.3 \times 10^3 \text{ W} \quad \text{or} \quad 42.3 \text{ kW}$$

Also, $\dot{Q} = C_c (T_{ce} - T_{ci})$

∴ Exit water temperature,

$$T_{ce} = T_{ci} + (\dot{Q}/C_c) = 10 + \left(\frac{42.3 \times 10^3}{2090}\right) = \mathbf{30.24°C} \qquad \textbf{(Ans.)}$$

Points to Ponder

- A heat exchanger is a process component which has the primary purpose of transferring thermal energy stored in one fluid to the other fluid.
- Heat exchangers can be classified as recuperators (*without storage*), regenerators (*with storage*) and direct contact type.
- According to flow arrangement, heat exchangers can be *parallel flow*, *counterflow* or *cross flow*.
- While evaluating the overall heat-transfer coefficient, we should be more particular about the controlling heat-transfer coefficient which the lower heat-transfer coefficient (*with larger thermal resistance*), and the use of fins, extended surfaces or turbulent promoters to increase the higher heat-transfer coefficient is wasteful.
- $\dfrac{1}{U_{\text{dirty}}} = \dfrac{1}{U_{\text{clean}}} + R_f$ where R_f is the fouling factor or resistance
- The expression for LMTD is the same for both *parallel flow* and *counterflow* heat exchangers.

  $$\text{LMTD} = \frac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}$$

  where
  $\Delta T_i \equiv T_{hi} - T_{ci}$ (*for parallel flow*)
  $\equiv T_{hi} - T_{ce}$ (*for counterflow*)
  $\Delta T_e \equiv T_{he} - T_{ce}$ (*for parallel flow*)
  $\equiv T_{he} - T_{ci}$ (*for counterflow*)
- In the case of a counterflow heat exchanger with $\Delta T_i = \Delta T_e$, the fluid temperature distributions are parallel and LMTD = $\Delta T_i = \Delta T_e$.
- $(\text{LMTD})_{\text{counterflow}}$ is *greater* than $(\text{LMTD})_{\text{parallel flow}}$
  Hence, $A_{\text{counterflow}}$ is *less* than $A_{\text{parallel flow}}$
- T_{ce} can be *greater* than T_{he} only in a counterflow heat exchanger.
- Baffles are used in shell-and-tube heat exchanger
  - to provide support to tubes
  - to increase the overall heat-transfer coefficient
  - to prevent stagnation of shell side fluid
- **Known:** U, T_{hi}, T_{he}, T_{ci}, T_{ce} **Known:** (UA), T_{hi}, T_{ci}
  Find: Heat exchanger area, A **Find:** T_{he}, T_{ce}
  Use: LMTD approach **Use:** ε–NTU approach
- Effectiveness–NTU approach facilitates comparison between the various types of exchangers which may be used for a particular application.

- The maximum possible energy transfer is $\dot{Q}_{\max} = (\dot{m}C_p)_{\text{xamller}}[T_{h,\text{inlet}} = T_{c,\text{inlet}}]$.
- For calculating the maximum possible heat-transfer, the maximum temperature difference ($T_{h,\text{in}}$ – $T_{c,\text{in}}$) is to be multiplied by the smaller heat capacity rate $(\dot{m}C_p)_{\min}$, i.e., $(\dot{m}C_p)_{\text{hot}}$ or $(\dot{m}C_p)_{\text{cold}}$, whichever is smaller.
- The Number of Transfer Units (NTU) provides some indication of the physical size of a heat exchanger.
- All heat exchangers (*concentric tube, crossflow, multipass exchangers or any other geometry*) in which the capacity-rate ratio R is found to be zero, that is, when one of the fluids either evaporates or condenses (*phase change*), the effectiveness will always be given by

  $$\varepsilon = 1 - \exp(-\text{NTU})$$
- In a multipass, shell-and-tube heat exchanger, the mass flow rate of the fluid flowing through a number of tubes is $\dot{m} = \rho\left[N\frac{\pi}{4}D_i^2\right]V$

  where N = Number of tubes per pass

  and, heat exchanger surface area,

  $$A = NP\pi D_o L$$

  where P = Number of tube passes

  L = Length of the tubes per pass
- No matter which fluid has smaller mass-flow specific heat product $(\dot{m}C_p)$, the maximum effectiveness of a parallel-flow heat exchanger is $\varepsilon_{\max} = \dfrac{1}{1+R}$ (as NTU $\to \infty$)
- Irrespective of which fluid has smaller capacity ratio $(\dot{m}C_p)$, the maximum effectiveness of a counterflow heat exchanger is $\varepsilon_{\max}$ = 1.0
  (as NTU $\to \infty$)
- For $R = \dfrac{C_{\min}}{C_{\max}} = 1, \varepsilon_{\text{counter flow}} = \dfrac{\text{NTU}}{1+\text{NTU}}$, and $\varepsilon_{\text{parallel flow}} = \dfrac{1}{2}[1 - \exp(-2\ \text{NTU})]$

GLOSSARY of Key Terms

• **Balanced heat exchanger**	A heat exchanger with equivalent fluid heat capacity rates.
• **Capacity-rate ratio**	Ratio of the smaller heat-capacity rate to the larger heat capacity rate.
• **Compact heat exchanger**	A device with high surface area density, usually greater than 700 m^2/m^3.
• **Condenser**	The temperature of the hot fluid remains essentially constant (due to change of phase from vapour to liquid).
• **Correction factor**	A multiplying factor (< 1) applied to the LMTD of a counterflow double-pipe heat exchanger with the same hot and cold fluid temperatures to take into account the departure from counter flow behaviour in the case of multi pass and cross-flow heat exchangers.
• **Counterflow:**	The two fluid flows are in the opposite direction.
• **Cross flow:**	One fluid stream flows across the direction of the other.

• **Direct-contact exchanger:**	Heat is transferred by direct contact between hot and cold fluids and their mixing.
• **Effectiveness:**	The ratio of the actual heat-transfer rate to the maximum possible heat-transfer rate.
• **Evaporator:**	The temperature of the cold fluid remains essentially constant (due to change of phase from liquid to vapour).
• **Fouling:**	Deposits on the heat exchanger passages formed from accumulation of dissolved salts, corrosion, chemical reactions, biological organisms, etc., which increase additional thermal resistance.
• **Heat exchanger:**	A device which facilitates transfer of heat from one fluid stream to another.
• **Log Mean Temperature Difference (LMTD):**	Effective mean temperature difference.
• **Mixed flow:**	The temperature variation is both in the flow and normal directions.
• **Multi-pass exchanger:**	A fluid (*shell side or tube side*) moving from one end to the other end of the exchanger many times.
• **Number of Transfer Units (NTU):**	The ratio of the product of overall heat-transfer coefficient and the exchanger surface area to the smaller heat capacity rate.
• **Parallel flow:**	The two fluid streams are in the same direction.
• **Rating:**	To determine the fluid outlet temperatures in a performance problem.
• **Recuperator:**	Heat exchangers without storage, with the two fluids separated by a solid partition.
• **Regenerator:**	Heat exchangers with storage in which a surface is alternately exposed to a hot fluid and then to a cold one.
• **Single stream exchanger:**	The temperature of only one stream changes, the temperature of the other stream being constant (due to phase change).
• **Sizing:**	To determine the heat exchange surface area required in a design problem
• **Two-stream exchanger:**	The temperatures of both streams change in the exchanger.
• **Unmixed flow:**	The temperature variation is only in the flow direction.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

11.1 The log mean temperature difference, ΔT_{lm} is given by

(a) $\Delta T_{lm} = \dfrac{\Delta T_i - \Delta T_e}{\ln(\Delta T_i/\Delta T_e)}$ (b) $\Delta T_{lm} = \dfrac{\ln(\Delta T_i - \Delta T_e)}{\ln \Delta T_i/\Delta T_e}$

(c) $\Delta T_{lm} = (\Delta T_i - \Delta T_e)\ln\left(\dfrac{\Delta T_i}{\Delta T_e}\right)$ (d) $\Delta T_{lm} = \dfrac{\ln(\Delta T_i/\Delta T_e)}{\Delta T_i/\Delta T_e}$

where ΔT_i and ΔT_e are temperature differences between hot and cold fluids at the entrance and exit respectively.

11.2 Which one of the following heat exchangers gives parallel straight-line pattern of temperature distribution for both cold and hot fluids?
(a) Parallel flow with unequal heat capacities
(b) Counterflow with equal heat capacities
(c) Parallel flow with equal heat capacities
(d) Counterflow with unequal heat capacities

11.3 In a heat exchanger with one fluid evaporating or condensing, the surface area required is least in
(a) parallel flow (b) counterflow (c) Cross flow (d) Same in all the above

11.4 The directions of fluid flow are immaterial in the case of heat exchange from
(a) wet or saturated steam to water (b) oil to gas
(c) oil to water (d) water to air

11.5 The steam condenser in a thermal power plant is a heat exchanger of the type
(a) recuperator (b) direct contact (c) regenerator (d) none of the above

11.6 A heat exchanger in which hot and cold fluids flow over the heat-transfer surface alternately is a
(a) direct contact exchanger (b) parallel-flow exchanger
(c) regenerator (d) all the above

11.7 Mark the correct answer in respect of heat exchanger area for the same heat duty:
(a) $A_{\text{counter flow}} > A_{\text{cross flow}} > A_{\text{parallel flow}}$ (b) $A_{\text{counter flow}} < A_{\text{parallel flow}} < A_{\text{cross flow}}$
(c) $A_{\text{counter flow}} < A_{\text{cross flow}} < A_{\text{parallel flow}}$ (d) $A_{\text{cross flow}} < A_{\text{parallel flow}} < A_{\text{counter flow}}$

11.8 Fins are usually provided to a heat-exchanger surface
(a) to augment heat transfer by increasing the heat-transfer coefficient
(b) to augment heat transfer by increasing the surface area
(c) to augment heat transfer by increasing the temperature difference
(d) to augment heat transfer by increasing turbulence ***[CSE: 2001]***

11.9 Match ***List I*** with ***List II*** and ***List III*** according to the code given below:

List I (Heat exchangers)	**List II (Hydraulic diameter, mm)**	**List III (Surface area density, mm^{-1})**
A. Human lungs	**1.** 0.5–1.2	**5.** 700–1700
B. Automotive radiators	**2.** 0.15–0.20	**6.** 60–600
C. Shell-and-tube heat exchanger	**3.** 2.5–5.0	**7.** 3000–7000
D. Gas turbine rotary regenerators	**4.** 5.5–60.0	**8.** 15000–20000

Codes:

	A	**B**	**C**	**D**
(a)	2,8	1,5	3,7	4,6
(b)	2,8	3,5	4,6	1,7
(c)	1,7	3,5	2,6	4,8

11.10 Fouling is considered in the design and selection of heat exchangers. Consider the following statements pertaining to fouling:
1. Fouling factor increases with increasing velocity and decreasing temperature.
2. Fouling factor is generally of the order of 10^{-4} m^2 K/W
3. Fouling is essentially the precipitation and accumulation of solid deposits on the heat transfer surfaces of a heat exchanger.

Of these statements:

(a) 1, 2, and 3 are correct. (b) 2 and 3 are correct
(c) 1 and 2 are correct (d) 1 and 3 are correct.

11.11 For multi-pass heat exchangers, the correction factor
(a) is less than unity
(b) is equal to one for a condenser or boiler
(c) depends on the geometry of the exit temperatures of the hot and cold fluid streams
(d) all of the above

11.12 The heat capacity of hot fluid is greater than the heat capacity of the cold fluid. The effectiveness of the heat exchanger would be
(a) $(T_{hi} - T_{he})/(T_{hi} - T_{ci})$ (b) $(T_{ce} - T_{ci})/(T_{hi} - T_{ci})$
(c) $(T_{hi} - T_{he})/(T_{hi} - T_{ce})$ (d) $(T_{hi} - T_{ci})/(T_{ce} - T_{ci})$

11.13 A heat exchanger with heat-transfer surface area A and overall heat-transfer coefficient U handles two fluids of heat capacities C_{max} and C_{min}. The parameter NTU (number of transfer units) used in the analysis of heat exchanger is specified as
(a) $\dfrac{AC_{\min}}{U}$ (b) $\dfrac{U}{AC_{\min}}$ (c) $AU\,C_{\min}$ (d) $\dfrac{AU}{C_{\min}}$

11.14 The equation of effectiveness of a heat exchanger $\varepsilon = 1 - \exp(-\text{NTU})$ is valid [(NTU is number of transfer units] in the case of
(a) boiler and condenser for parallel flow
(b) boiler and condenser for counterflow
(c) boiler and condenser for both parallel flow and counterflow
(d) gas turbine for both parallel flow and counterflow

11.15 From among the (A) parallel flow, (B) cross flow (both fluids *unmixed*), (C) counterflow, and (D) one-shell pass, two-tube pass heat exchangers, the correct sequence in the order of increasing effectiveness for $C_{min}/C_{max} = 1$ is
(a) A B C D (b) A D B C (c) B C D A (d) A C B D

11.16 The governing rate equation for a heat exchanger is
(a) $\dot{Q} = \varepsilon C_{\min}(T_{h,\text{in}} - T_{c,\text{in}})$ (b) $\dot{Q} = C_c(T_{c,\text{out}} - T_{c,\text{in}})$
(c) $\dot{Q} = C_h(T_{h,\text{in}} - T_{h,\text{out}})$ (d) all of the above

11.17 The effectiveness of a heat exchanger is defined as the ratio of the actual heat-transfer rate to
(a) the maximum possible heat-transfer rate
(b) the minimum heat-transfer rate
(c) the area of the heat exchanger
(d) the overall heat-transfer coefficient.

11.18 In a parallel-flow heat exchanger operating under steady state, the heat-capacity rates (product of specific heats at constant pressure and mass flow rate) of the hot and cold fluid are equal. The hot fluid, flowing at 1 kg/s with C_p = 4 kJ/kg K, enters the heat exchanger at 102°C while the cold fluid has an inlet temperature of 15°C. The overall heat-transfer coefficient for the heat exchanger is estimated to be 1 kW/m^2 K and the corresponding heat-transfer surface area is 5 m^2. Neglect heat transfer between the heat exchanger and the ambient. The heat exchanger is characterized by the following relation.
$2\varepsilon = 1 - \exp(-2\text{NTU})$. The exit temperature (in °C) for the cold fluid is
(a) 45 (b) 56 (c) 65 (d) 75

11.19 In a condenser of a power plant, the steam condenses at a temperature of 60°C. The cooling water enters at 30°C and leaves at 45°C. The logarithmic mean temperature difference (LMTD) of the condenser is

(a) 16.2°C (b) 21.6°C (c) 30°C (d) 37.5°C

11.20 The fouling factor in heat exchanger is defined as

(a) $R_f = U_{\text{dirty}} - U_{\text{clean}}$ (b) $R_f = \dfrac{1}{U_{\text{dirty}}} - \dfrac{1}{U_{\text{clean}}}$

(c) $\dfrac{1}{R_f} = \dfrac{1}{U_{\text{dirty}}} - \dfrac{1}{U_{\text{clean}}}$ (d) $\dfrac{1}{R_f} = U_{\text{dirty}} - U_{\text{clean}}$

11.21 What does NTU indicate?

(a) Effectiveness of a heat exchanger (b) Efficiency of a heat exchanger
(c) Size of a heat exchanger (d) Temperature drop in a heat exchanger

11.22 In a heat exchanger, the hot gases enter with a temperature of 150°C and leave at 75°C and 125°C. The capacity ratio of the exchanger

(a) 0.65 (b) 0.75 (c) 0.85 (d) 0.95

Answers

Multiple-Choice Questions

11.1 (a)	**11.2** (b)	**11.3** (d)	**11.4** (a)	**11.5** (a)	**11.6** (c)
11.7 (c)	**11.8** (b)	**11.9** (b)	**11.10** (b)	**11.11** (d)	**11.12** (b)
11.13 (d)	**11.14** (c)	**11.15** (b)	**11.16** (d)	**11.17** (a)	**11.18** (b)
11.19 (b)	**11.20** (b)	**11.21** (c)	**11.22** (b)		

REVIEW QUESTIONS

11.1 What is a heat exchanger? Enumerate various types of heat exchangers and describe briefly any one of them giving its advantages and disadvantages.

11.2 How are heat exchangers classified? Outline the salient characteristics of each type.

11.3 What are the various applications of heat exchangers? Explain the terms: single-stream and two-stream heat exchangers, balanced heat exchanger.

11.4 What is meant by a shell-and-tube heat exchanger? Why are baffles used in these heat exchangers? Distinguish between a *shell-and-tube* heat exchanger and a *cross-flow* heat exchanger.

11.5 Draw the schematic flow arrangement of a two-shell-pass and four-tube-pass shell-and-tube heat exchanger.

11.6 What is the main distinguishing feature of *compact heat exchangers* and where are they used?

11.7 Distinguish between *recuperative* and *regenerative* types of heat exchangers.

11.8 What is a regenerative heat exchanger? How does a stationary matrix type of regenerative heat exchanger differ from a rotating-matrix type?

11.9 How is a cross-flow heat exchanger different from a counterflow one? What is meant by the terms *mixed* and *unmixed* fluids in cross flow?

11.10 What restrictions are associated with the fluid exit temperatures in the *co-current* and *counter-current* double-pipe heat exchangers?

11.11 Sketch the temperature variations in parallel-flow and counterflow exchangers when

$\dot{m}_c C_{pc} < \dot{m}_h C_{ph}$ and when $\dot{m}_c C_{pc} > \dot{m}_h C_{ph}$, and when $\dot{m}_c C_{pc} = \dot{m}_h C_{ph}$

11.12 Enumerate the heat-transfer mechanisms and thermal resistances during heat transfer from the hot to the cold fluid? When can one neglect the thermal resistance of the pipe wall in a heat exchanger?

11.13 When can one reasonably calculate the overall heat-transfer coefficient from $\frac{1}{U} = \left[\frac{1}{h_i} + \frac{1}{h_o}\right]$? Under what conditions can one write $U \approx h_i$?

11.14 Define and explain the overall heat-transfer coefficient of a heat exchanger. Explain the significance of various terms in the expression.

11.15 Discuss the common causes of fouling in a heat exchanger. How does fouling affect its performance? What is the effect of fluid velocity and temperature on fouling?

11.16 What is meant by *fouling factor*? How does it affect the performance of a heat exchanger?

11.17 List the common assumptions made in the analysis of double-pipe heat exchangers.

11.18 Derive an expression for LMTD for a *parallel-flow* and for a *counter-flow* heat exchanger.

11.19 Why are the *counterflow* heat exchangers more effective than *parallel-flow* heat exchangers?

11.20 Define the log mean temperature difference (LMTD). When is the LMTD approximately equal to the arithmetic mean temperature difference (AMTD)? Can LMTD be ever negative?

11.21 In the analysis of multipass and crossflow heat exchangers, what is the LMTD correction factor F and how is it evaluated? Can F be greater than one?

11.22 A heat exchanger has both fluids with the same specific heats and different mass-flow rates. Which fluid will experience a larger temperature change: the one with the lower or higher mass flow rate?

11.23 Explain the effectiveness-NTU method of heat exchanger analysis. Discuss the conditions under which you would prefer the effectiveness-NTU method over the LMTD method for performing a heat exchanger analysis?

11.24 Why is the LMTD method preferred in sizing of a heat exchanger and is not usually applicable in the rating of a heat exchanger? What advantage does the effectiveness-NTU approach have over the LMTD approach?

11.25 What does the effectiveness of a heat exchanger represent? Can effectiveness be greater than one? What factors determine the effectiveness of a heat exchanger?

11.26 When can a parallel flow and a counterflow heat exchanger have an effectiveness of one? Comment.

11.27 Derive an expression for the effectiveness of a counterflow heat exchanger. Show that for the capacity rate ratio of one, the effectiveness, $\varepsilon = \text{NTU}/(1 + \text{NTU})$.

11.28 Derive an expression for the effectiveness of a parallel-flow heat exchanger. Show that $\varepsilon = \frac{1}{2}(1 - e^{-2\,\text{NTU}})$ if the capacity rates of the hot and cold fluids are equal?

11.29 Derive the expression for effectiveness of a counterflow heat exchanger and discuss the following cases?

(a) $C_{\min}/C_{\max} \to 0$ (b) $C_{\min}/C_{\max} \to 1$

11.30 Discuss the effect of *finned surfaces* on the overall heat-transfer coefficient in a heat exchanger. When is the use of fins most appropriate?

11.31 Explain the following terms as applied to heat exchangers:

(a) Capacity-rate ratio (b) Number of transfer units (c) Effectiveness

11.32 Describe the salient aspects in the selection and design of heat exchangers.

PRACTICE PROBLEMS

(A) Overall Heat-Transfer Coefficient

11.1 Lubricating oil is expected to be cooled in a double pipe heat exchanger from about 70°C to 30°C by passing cold water in the outer pipe (annulus). The inner pipe diameter is 2-cm and the outer pipe diameter is 4-cm. The thickness of both tubes is 2-mm. The heat transfer coefficient on the water side is 140 W/m^2 K and the oil side is 150 W/m^2 K. The pipes are made of galvanized iron (GI) whose thermal conductivity is 30 W/m K. Unfortunately the rate of heat transfer is grossly inadequate and the oil is not cooled sufficiently. The maintenance engineer decides to replace the GI pipes with copper pipes of the same dimensions (whose thermal conductivity is 385 W/m K) to increase the heat transfer rate. Do you agree with the decision of the maintenance engineer? Give justification for your answer. **[The decision of the maintenance engineer is therefore not correct.]**

(B) LMTD-Method

11.2 Hot fluid with a heat capacity rate of 2.4 kW/K flows through a double pipe heat exchanger. It enters at 380°C and leaves at 300 °C. Cold fluid enters at 35 °C and leaves at 200 °C. If the overall heat transfer coefficient is 750 W/m^2 K, determine the heat exchanger area required for: (a) Parallel flow, (b) Counter flow **[(a) 1.294 m^2 (b) 1.165 m^2]**

11.3 For a balanced counterflow heat exchanger where $\dot{m}_h C_{p_h} = \dot{m}_c C_{p_c}$ show that the temperature profiles of the two fluids along the heat exchanger length are linear and parallel.

11.4 In a concentric tube, counter-current heat exchanger, the hot fluid enters the inner tube at 150°C and leaves at 95°C. The cold fluid gets heated up from 30°C to 85°C while passing through the annular space. The length of the heat exchanger is 22 m. Determine the new length of the exchanger if the exit temperature of the cold fluid is required to be 100°C. The mass flow rates of the two fluids, the tube diameters and the fluid's inlet temperatures remain unchanged. **[36.4 m]**

11.5 In a counterflow double-pipe heat exchanger water flows through a copper tube (19-mm-OD and 16-mm-ID), at a flow rate of 1.48 m/s. The oil flows through the annulus formed by inner copper tube and outer steel tube (30-mm-OD and 26-mm-ID). The steel tube is insulated from outside. The oil enters at 0.4 kg/s and is cooled from 65°C to 50°C whereas water enters at 32°C. Neglecting the resistance of the copper tube, calculate the length of the tube required.
Fouling factor, water side = 0.0005 m^2 K/W
Fouling factor, oil side = 0.0008 m^2 K/W
Water and oil properties:

Property	**Oil (330 K)**	**Water (310 K)**
ρ (kg/m^3)	865.8	993
C_p (kJ/kg K)	2.035	4.178
k (W/m K)	0.141	0.628
ν (m^2/s)	9.66×10^{-5}	7.0×10^{-7}

[109 m]

11.6 Exhaust gases (C_p = 1.12 kJ/kg °C) flowing through a tubular heat exchanger at the rate of 1200 kg/h are cooled from 400 °C to 120 °C. The cooling is effected by water (C_p = 4.18 kJ/kg °C) that enters the system at 10 °C at the rate of 1500 kg/h. If the overall heat transfer coefficient is 500 kJ/m^2 h °C, what heat exchange area is required to handle the load for (a) parallel flow and (b) counterflow arrangement? **[(a) 4.55 m^2 (b) 3.76 m^2]**

11.7 Show that in a double-pipe heat exchanger if the overall heat transfer coefficient is a linear function of the temperature difference, the heat transfer rate is given by $Q = A_s \left[\dfrac{U_e \Delta T_i - U_i \Delta T_e}{\ln\left(\dfrac{U_e\,\Delta T_i}{Ui\,\Delta T_e}\right)} \right]$ where the suffixes i and e denote inlet and exit of the heat exchanger and A_s is the heat transfer surface area.

(C) Multipass Heat Exchangers, Condensers and Evaporators

11.8 A shell-and-tube heat exchanger is to be designed to heat 2.5 kg/s of water from 15°C to 85°C. Heating is done by passing hot oil available at 160°C through the shell side of the counter flow heat exchanger. The oil has an average convective coefficient of 400 W/m^2 K on the outer side of the tubes. Ten thin-walled tubes, each of 25-mm diameter and making 8 number of passes through the shell carry water to be heated. If the oil leaves the exchanger at 100°C, what is its flow rate? What should be the length of tubes in each pass? Take correction factor for LMTD as 0.85. Use the equation: $Nu = 0.023\,(Re)^{0.8}\,(Pr)^{0.4}$ to estimate the heat transfer coefficient for turbulent flow of fluid through tubes. Assume following properties of fluids:

Oil : $C_p = 2.35$ kJ/kg K

Water : $C_p = 4.181$ kJ/kg K, $Pr = 3.56$, $k = 0.643$ W/m K, $\mu = 5.48 \times 10^{-4}$ Pa s **[4.85 m]**

11.9 Lubricating oil at 17 °C is to be heated by saturated steam at 115°C in a double pipe heat exchanger to a temperature of 37 °C. The inner and outer diameters of the annular space are 30 mm and 50 mm respectively, and the oil enters the annulus with an average velocity of 0.8 m/s. The inner tube may be assumed to be isothermal at 115°C, and the outer tube is effectively insulated. Assuming fully developed flow for oil, calculate the length of the tube required to heat the oil to the specified temperature. Thermophysical properties of oil at 27°C are:

$\rho = 884.1$ kg/m^3 $\quad$ $\nu = 550 \times 10^{-6}$ m^2/s $\quad$ $\mu = 0.486$ kg/m s

$C_p = 1909$ J/kg K $\quad$ $Pr = 6400$ $\quad$ $k = 0.145$ W/m K **[38.2 m]**

11.10 A parallel-flow intercooler on a two-stage air compressor takes in air at 6 bar and 180°C and passes to the next stage at 30°C and at the equivalent rate of 6 m^3/min of free air (at atmospheric conditions). The cooling water passes over tubes of 10-mm outside diameter and 1.2-mm thick each. The inlet and outlet water temperatures are 12°C and 28°C respectively and the air velocity through the tubes is limited to 6 m/s. Atmospheric pressure and temperature are 1.013 bar and 15°C respectively. Find (a) the number of tubes required, and (b) the length of each tube.

Take $C_{p_{\text{air}}} = 1.005$ kJ/kg K, $\quad$ $R = 0.287$ kJ/kg K

$h_{a(\text{air-side})} = 105$ W/m^2 K, $\quad$ $h_{w(\text{water-side})} = 2100$ W/m^2 K

(c) What will be the percentage saving in the tube length if the intercooler is made counterflow with the same terminal temperatures as before? **[(a) 82 (b) 2.5 m (c) 40.6%]**

(D) Cross Flow Heat Exchangers (LMTD Method)

11.11 Determine the heat transfer surface area required for a heat exchanger constructed from a 2.5-cm-OD tube to cool 25 300 kg/h of alcohol ($C_p = 3.81$ kJ/kg °C) from 66°C to 40°C, using 23 000 kg/h of water ($C_p = 4.18$ kJ/kg °C) available at 10°C. Assume the overall heat transfer coefficient based on the outer tube area to be 580 W/m^2°C. Consider each of the following arrangements: (a) Parallel flow-tube and shell, (b) Counterflow-tube and shell, (c) Reversed-current exchanger with 2 shell passes and 72 tube passes, the alcohol flowing through the shell and the water flowing through the tubes, (d) Cross flow, with one tube pass and one shell pass, the shell side fluid mixed.

Flow arrangement	**Heat transfer surface area, A_o (m^2)**
(a) Parallel flow	**61.2**
(b) Counter flow	**46.0**
(c) Cross flow (one fluid) mixed, one fluid unmixed	**41.2**
(d) 2-shell pass and 72-tube pass Counter flow	**40.0**

(E) ε-NTU METHOD

11.12 A counterflow heat exchanger is to heat air entering at 400°C with a flow rate of 6 kg/s by the exhaust gas entering at 800°C with a flow rate of 4 kg/s. The overall heat transfer coefficient is 100 W/m^2 K and the outlet temperature of the air is 551.5 °C. Specific heat at constant pressure for both air and exhaust gas can be taken as 1100 kJ/kg K. Calculate the heat transfer area needed and the number of transfer units. **[48 m^2, 1.09]**

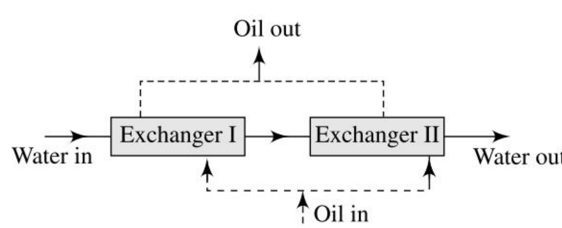
11.13 A chemical having specific heat of 3.3 kJ/kg K flowing at the rate of 20 000 kg/h enters a parallel flow heat exchanger at 120 °C. The flow rate of cooling water is 50 000 kg/h with an inlet temperature of 20°C. The heat transfer area is 10 m^2 and the overall heat transfer coefficient is 1050 W/m^2 K. Find the outlet temperatures of water and chemical, and the effectiveness of the heat exchanger. Take for water, specific heat = 4.186 kJ/kg K. **[0.402, 79.8, 32.7°C]**

11.14 A hot fluid at 200°C enters a heat exchanger at a mass flow rate of 10^4 kg/h. Its specific heat is 2000 J/kg K. It is to be cooled by another fluid entering at 25°C with a mass flow rate of 2500 kg/h and specific heat of 400 J/kg K. The overall heat transfer coefficient based on outside area of 20 m^2 is 250 W/m^2K. Find the exit temperature of the hot fluid when the fluids are in parallel flow. **[191.67°C]**

11.15 (a) Consider a parallel flow, concentric tube heat exchanger used to cool engine oil from 160°C to 70°C. Water, which is available at 20°C, is to be used as a coolant. The oil and water flow rates are each 2 kg/s. The specific heats of oil and water may be assumed to be 2.1 and 4.2 kJ/kg K, respectively. Calculate the number of transfer units (NTU) of the exchanger. (b) If the same heat exchanger were to operate in the counterflow mode, calculate the exit temperatures of the oil and the water for the same inlet fluid temperatures. (c) What will be the maximum temperature to which the water can be heated by increasing the tube length in parallel flow configuration? (d) Determine the maximum possible effectiveness of the exchanger in both parallel flow and counterflow operations.
[(a) 2.22, 47.6°C (b) 47.6°C 76.2°C (c) 66.7°C (d) 0.667, 1.0]

11.16 The feed water heater for a boiler is a shell-and-tube heat exchanger (single-pass on the shell side and two-pass on the tube side). 10 000 kg/h of water (C_p = 4.18 kJ/kg °C) are heated from 20 to 65°C by condensing steam at 1.3 bar (T_{sat} =107.3°C and h_{fg} = 2238 kJ/kg). Determine (a) the surface area of the heat exchanger if the overall heat transfer coefficient is 2 kW/m^2 °C. (b) What will be the steam mass flow rate in kg/min? **[(a) 4.2 m^2 (b) 14.0 kg/min]**

11.17 Two identical single-pass counterflow heat exchangers are used for heating water at 30°C with the help of hot oil (C_p = 2.09 kJ/kg °C) at 100 °C. The mass flow rates of the water and oil are 50 and 100 kg/min respectively. The heat exchangers are connected in series on the water side and in parallel on the oil side. The oil flow is split equally between the



two heat exchangers at the inlet and joined up later at the exit. Assuming that for each exchanger U = 350 W/m^2 °C and A = 10 m^2, calculate the exit temperatures of the water and the oil. **[56.2°C]**

11.18 In a gas turbine power plant, heat is being transferred in an exchanger from the hot gases leaving the turbine to the air leaving the compressor. The air flow rate is 5000 kg/h and the fuel-air ratio is 0.015 kg/kg. The inlet temperatures on the air and gas sides are 170 and 450°C, respectively. The overall heat transfer coefficient for the exchanger is 52 W/m^2 K, the surface area is 50 m^2 and the arrangement is cross flow (both fluids unmixed). Find the exit temperatures on the air and gas side and the rate of heat transfer in the exchanger (Take the specific heat on both sides to be 1.05 kJ/kg K). **[338°C, 284.5°C, 245 kW]**

11.19 A cross-flow heat exchanger used in a cardio pulmonary bypass procedure cools blood flowing at 5 L/min from a normal body temperature of 37°C to 25°C in order to induce body hypothermia, which reduces metabolic and oxygen requirements. The coolant is ice water at 0°C and its flow rate is adjusted to provide an exit temperature of 15°C. The heat exchanger operates with both fluids unmixed, and the overall heat transfer coefficient is 750 W/m^2 K. The density and specific heat of the blood are 1050 kg/m^3 and 3740 J/kg K respectively. Determine (a) the heat transfer rate for the heat exchanger, (b) the water mass flow rate, and (c) the surface area of the heat exchanger. **[(a) 3927 W (b) 3.76 L/min (c) 0.24 m^2]**

11.20 Two fluids, A and B, exchange heat in a counter current heat exchanger. Fluid A enters at 420 °C and has a mass flow rate of 1 kg/s. Fluid B enters at 20 °C and also has mass flow rate of 1 kg/s. Effectiveness of heat exchanger is 75%. Determine the heat transfer rate and exit temperature of Fluid B. (Specific heat of Fluid A is 1 kJ/kg K and that of Fluid B is 4 kJ/kg K). **[95°C, 300 kW]**

11.21 The hot and cold water inlet temperatures in a very long double-pipe heat exchanger are 85 °C and 25°C. The mass flow rate of hot water is twice that of cold water. Assuming equivalent hot and cold water specific heats, calculate the hot water exit temperature and effectiveness for (a) parallel flow and (b) counterflow arrangement. **[(a) 0.667 (b) 55°C]**

(F) Both LMTD and e-NTU Methods of Analysis

11.22 A double-pipe heat exchanger is made up of inner tube 37.5-mm-ID, 44.8-mm-OD and outer tube 72.7-mm-OD and 5.1-mm thick wall steel pipe, has an effective heating surface of 2.4 m^2 based on the outer surface of the inner pipe. This exchanger has a scale deposit on the heating surface with a fouling factor of 4.13 × 10^{-4} m^2 K/W. It is proposed to use this exchanger to preheat benzene from an initial temperature of 20°C by means of hot water which will enter the exchanger at 88°C. Benzene will flow through the annulus at the rate of 5500 kg/ h and the hot water will flow through the tube at 6250 kg/h. Determine the exit temperature of benzene if counter current flow is used. Thermal conductivity of steel = 45.7 W/m K.

Data:

	Viscosity (C_p)	Thermal conductivity (W/m K)	Specific heat capacity (kJ/kg °C)	Density (kg/m^3)
Water	0.8	0.640	4.187	970
Benzene	0.57	0.160	1.675	860
$\frac{hD}{k} = 0.023\,(Re)^{0.8}\,(Pr)^n$				
n = 0.3 for cooling, n = 0.4 for heating				

[52.17°C]

11.23 In a certain double-pipe heat exchanger hot water flows at a rate of 50 000 kg/h and gets cooled from 95°C to 65°C. At the same time 50 000 kg/h of cooling water at 30°C enters the heat exchanger. The flow conditions are such that the overall heat transfer coefficient remains constant at 2270 W/m^2 K. Determine the heat transfer area required and the effectiveness, assuming the two streams are in parallel flow. Assume for both streams C_p = 4.2 kJ/kg K. **[32.96 m^2, 0.4615]**

11.24 In a large steam power plant, the condenser, a shell-and-tube heat exchanger, consists of a single shell and 31 500 tubes, each executing two passes. Steam flows through the shell, while cooling water flows through the tubes. The tubes are of thin wall construction with diameter equal to 25-mm and steam condenses on their outer surface with an associated convection coefficient of h_o = 11 500 W/m^2 K. The heat transfer rate in the exchanger is 2200 MW. Cooling water passes through the tubes at a rate of 31 500 kg/s. Water enters at 18 °C, while the steam enters as saturated vapour at 60°C. What is the temperature of the cooling water leaving the condenser? What is the required tube length per pass? Use the following average properties of water:

C_p = 4.18 kJ/kg K, $\mu = 866 \times 10^{-6}$ N s/m^2, k = 0.609 W/m K

The correlation to be used is $Nu_D = \dfrac{(f/8)\,(Re_D - 1000)\,Pr}{1 + 12.7\sqrt{f/8}\,(Pr^{2/3} - 1)}$

where $f = [0.79 \ln Re_D - 1.64]^{-2}$ **[34.7°C, 2.74 m]**

11.25 In a tubular heat exchanger with two shell passes and eight tube passes, 12.6 kg/s of water are heated in the shell from 80 to 150°C. Hot exhaust gases (with almost the same thermophysical properties of air) enter the tubes at 350°C and leave at 180°C. The total outer surface area of the tubes is 930 m^2. Calculate (a) the LMTD if the exchanger were of counterflow type, (b) the correction factor for the actual configuration, (c) the heat exchanger effectiveness, and (d) the mean overall heat transfer coefficient. **[(a) 144.3°C (b) 0.975 (c) 0.59 (d) 28.2 W/m^2 K]**

12

Radiation Heat Transfer: Properties and Processes

12.1 ❑ INTRODUCTION

Radiation is one of the basic mechanisms by which energy is transferred between regions at different temperatures. It is essentially the transport of energy by electromagnetic waves. All materials continuously emit thermal radiation as long as they are above absolute zero temperature. It can occur irrespective of the material medium between the surfaces exchanging radiation. For example, radiation coming from the sun travels through the vacuum of outer space and penetrates the glass of a window pane before it is absorbed in a room. Radiation heat transfer can even occur between two bodies separated by a medium which may be colder than both bodies. Solar radiation, for instance, reaches the earth's surface after passing through the cold atmosphere at high altitudes. Inside a greenhouse, objects that absorb radiation attain high temperatures even though the glass cover remains colder.

If a heated solid is placed in a vacuum chamber whose surfaces are at the ambient temperature, it will lose heat until its temperature reaches the temperature of the chamber walls. It is noteworthy that heat transfer between the object and the chamber by conduction or by convection is impossible in an evacuated space simply because both require a physical medium. In fact, heat transfer by radiation is the fastest (*at the speed of light*) mode and there is no reduction in it in a *vacuum*. Radiation heat transfer takes place in solids as well as liquids and gases. In many engineering applications, all *three* modes of heat transfer can occur simultaneously but radiation is the only significant mode of heat transfer in the evacuated space.

Some familiar examples of thermal radiation are *the heat dissipation from the filament of a* light bulb or the *heat leakage through the evacuated walls of a thermos flask*. Thermal radiation also finds applications in many energy-conversion systems *like powers plants* that involve combustion and solar radiation. It is also of use in several *industrial heating, cooling and drying processes, laser cutting and welding, climate control of buildings, automobiles, combustion chambers, cryogenic containers, high-performance thermal insulation, solar collectors, spacecraft*, etc.

While radiation is usually important in high-temperature applications, it can also be significant in many moderate and low-temperature applications. When other modes of heat transfer are relatively weak, radiation needs to be considered.

12.2 ❑ NATURE OF THERMAL RADIATION

The basic nature, composition, and velocity of propagation of all radiation is the same. However, the significant feature is frequency (*or wavelength*). Propagation of radiation takes place at the velocity of light in a vacuum, c_o. The wavelength λ, the velocity c_o, and the frequency (number of oscillations per second of radiation), ν are related by

$$\boxed{\lambda \nu = c_o = 2.998 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s}} \tag{12.1}$$

The wavelength and frequency of electromagnetic radiation are inversely proportional to each other.

The frequency of radiation ν is independent of the medium through which the radiation propagates. The speed of propagation and wavelength vary alike so that ν remains constant. If the speed of propagation of radiation is c in a medium, the refractive index n of the medium equals c_o/c. By definition, $n \geq 1$, because the speed of light in a medium cannot exceed c_o.

The frequency of electromagnetic radiation depends only on the nature of the source and does not depend on the medium through which it travels.

The unit of wavelength is the *micron* (μm) or the *angstrom* (Å).

$$\boxed{\begin{array}{l} 1\ \mu\text{m (micron)} \equiv 10^{-6}\ \text{m} \equiv 10^{-4}\ \text{cm} \equiv 10^{-3}\ \text{mm} \equiv 10^{4}\ \text{Å} \\ 1\text{Å(angstrom)} \equiv 10^{-4}\ \mu\text{m} \equiv 10^{-10}\ \text{m} \end{array}}$$

A body emits radiation in the form of *continuous* or *discontinuous* spectrum with respect to wavelength.

The nature of radiation and its transport are not fully understood but they can be described satisfactorily either by *wave* or *particle* theory.

12.2.1 • Maxwell's Wave Theory

Radiation has a *dual* character since it possesses the continuity properties of electromagnetic waves and the properties of discreteness of photons. The wave properties are distinctly observed in *radio waves* while the quantum properties are most pronounced in *short-wave radiation*. According to Maxwell's classical electromagnetic theory, the energy transfer may be considered as being transported by electromagnetic *waves*.

For example, a metal bombarded by high-energy electrons emits *X-rays*, high-frequency electric currents generate *radio waves* and a body emits *thermal radiation* by virtue of its temperature. This *wave* model is useful in studies involving prediction of the radiation properties of the surfaces and materials.

12.2.2 • Max Planck's Quantum Theory

According to the quantum theory, when a solid body is heated, its atoms and molecules are raised to excited states of higher energy. These atoms tend to return spontaneously to lower energy states. The energy which is released is *not continuous* but is in the form of a collection of successive and separate *discrete* packets or quanta of energy called *photons*. The photons are propagated through space as rays, the movement of swarm of photons is known as *electromagnetic waves*. These electromagnetic waves travel with the speed of light in a straight path with unchanged frequency.

The *particle model*, in which radiation energy is carried by *photons*, helps in understanding *emission* and *absorption*. At an atomic level when an electron falls from a high-energy level to a low-energy level, a photon is emitted. The photon travels at the speed of light through the space until it is absorbed by another atom. When a photon is absorbed, an electron rises from a low energy level to a high energy level.

The energy carried by a photon is proportional to the frequency of the electromagnetic waves. The energy associated with the photon is given by

$$\boxed{E = h\nu = \frac{hc}{\lambda}} \tag{12.2}$$

and each photon has a momentum equal to $mc = \dfrac{mc^2}{c} = \dfrac{E}{c} = \dfrac{h\nu}{c}$

where $h = 6.626\ 069 \times 10^{-34}$ J s is *Planck's constant* and ν is the frequency of emitted photons. The energy of a photon is, therefore, inversely proportional to its wavelength. The shorter the wavelength of radiation, the larger the energy of the photon. *Gamma rays* and *X-rays* with their *very short* wavelengths are considered *highly destructive* and must be avoided. This *particle* model is used to predict the magnitude of energy emitted by a body at a given temperature under ideal conditions.

Prevost's Principle of Exchange If a body is placed in the surroundings at the same temperature as itself, its temperature does not change. Nevertheless, it continues to radiate energy and, simultaneously, receive energy at the same rate from its surroundings.

12.2.3 • Electromagnetic Spectrum

The electromagnetic radiation covers a wide range of wavelengths, varying from less than 10^{-10} μm for *cosmic rays* to greater than 10^{10} μm for *electrical power waves. The electromagnetic spectrum* shown in Fig. 12.1 includes *gamma rays, X-rays, ultraviolet radiation, visible light, infrared radiation, thermal radiation,* and *microwaves*.

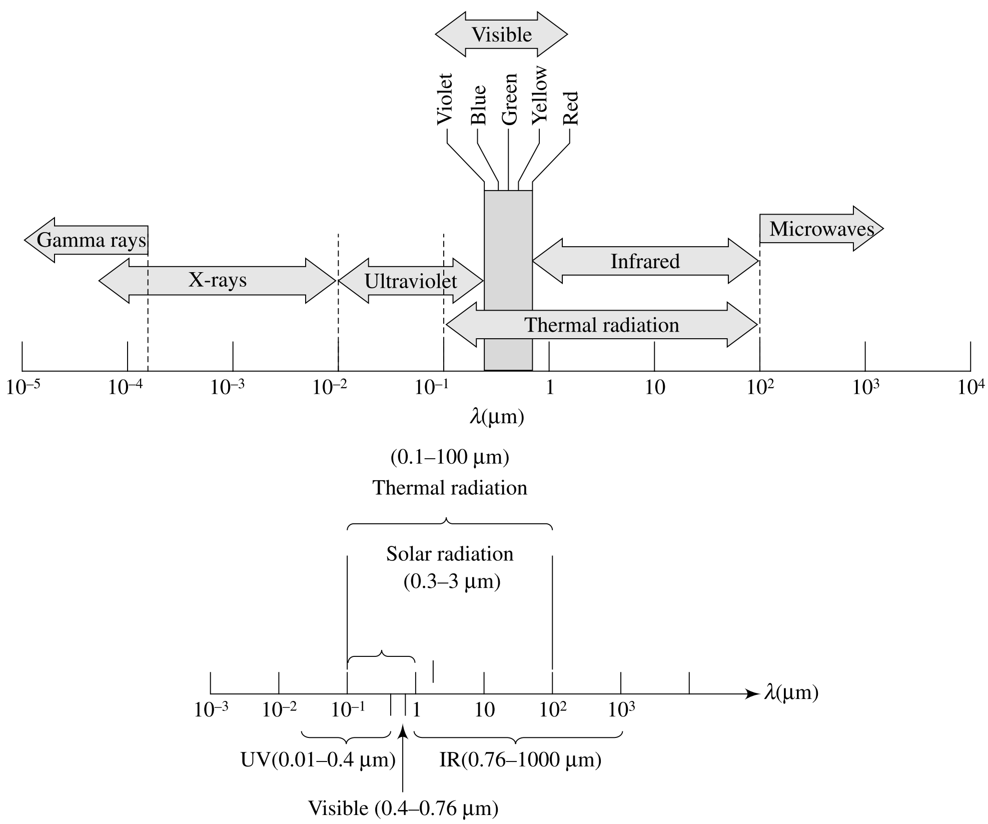


Fig. 12.1 *Electromagnetic radiation spectrum*

Nuclear engineers are more interested in the short-wavelength *gamma rays* with powers of penetration while the electrical engineers find the long wavelength *microwaves and radio waves* more useful.

We are primarily concerned with the *thermal radiation* which is that portion of the electromagnetic spectrum which ranges from about 0.1 to 100 μm, and includes the entire *visible* and *infrared* (IR) radiation as well as a part of the *ultraviolet* (UV) radiation. Thermal radiation, like the other forms of electromagnetic radiation, can often be considered to travel in straight lines in a uniform medium. Consequently, opaque bodies cast shadows when placed in the path of thermal radiation and one body cannot receive radiation directly from another unless it can *see* it.

The visible part of the spectrum (*radiation to which the human eye is sensitive*) falls in the thermal range. Light is nothing but the *visible* range of the electromagnetic spectrum lying between 0.40 and 0.76 μm. The visible spectrum, consists of narrow bands of colour from violet (0.40 0.44 μm) to red (0.63 0.76 μm), as shown in Table 12.1.

Table 12.1 *The wavelength spectrum of different colours*

Colour	Violet	Blue	Green	Yellow	Orange	Red
Wavelength band (μm)	0.40 to 0.44	0.44 to 0.49	0.49 to 0.54	0.54 to 0.60	0.60 to 0.63	0.63 to 0.76

The electromagnetic radiation emitted by the sun is *solar radiation*, with a wavelength range of 0.3–3 μm. Almost half of solar radiation is *light* and the rest is *ultraviolet* and *infrared* radiation. Also radiation emitted by earth is in the IR region and has a peak at around 10 μm.

The radiation emitted by bodies at the room temperature is in the *infrared region* of the spectrum, which ranges from 0.76 to 100 μm. Bodies start emitting visible radiation at temperatures above 500°C. The tungsten filament of a light bulb is to be heated to temperatures above 1700°C before it emits some radiation in the visible range.

The *ultraviolet* radiation is in the wavelength band of 0.01 to 0.40 μm. Ultraviolet rays can destroy microorganisms and are very harmful to human beings and other living creatures. About 12.5 percent of solar radiation lies in the ultraviolet range. The ozone (O_3) layer in the stratosphere acts as a protective umbrella and absorbs most of it.

12.2.4 • Volumetric vs Surface Phenomenon

Radiation is constantly *emitted*, as well as *absorbed* or *transmitted* throughout the whole volume of matter above absolute zero temperature. Radiation can, therefore, be viewed as a *volumetric phenomenon* especially at high temperatures, for gases and semi-transparent solids like glass as depicted in Fig. 12.2. However, for opaque (*non transparent*) solids and liquids, radiation is essentially a *surface phenomenon*, because the radiation emitted by the inner regions hardly reaches the surface, and the radiation incident on such bodies is usually absorbed within a few microns from the surface. It is, therefore, mostly the surface finish rather than the material itself which governs the radiation properties. Thin layers of coatings on the surfaces can significantly change the radiation characteristics.

12.2.5 • Participating vs Non-participating Medium

Radiation transport can involve either a *non-participating* or *participating* medium. We are basically interested in a solid surface surrounded by a transparent gas or a vacuum. Air is transparent to thermal radiation, except when distances are very large. Thus, a non-participating medium includes not only vacuum and outer space but also atmospheric air over short distances. Photons can travel from one surface to another without any restriction or resistance by the intervening medium. Radiation heat exchange between such bodies is dependent only on the exchanging surfaces, their radiative properties and the

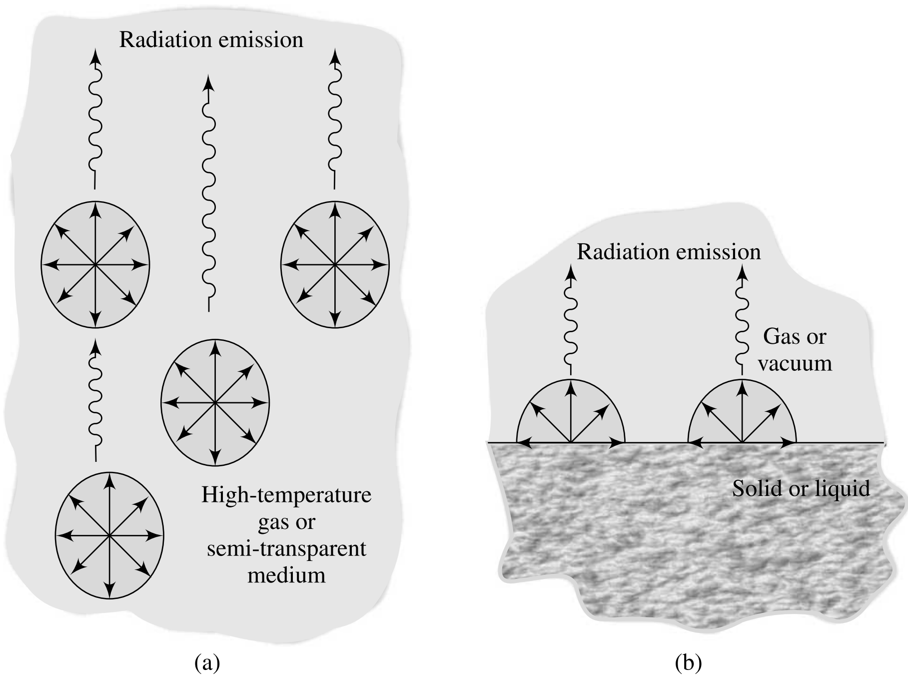


Fig. 12.2 *The emission process as a (a) volumetric phenomenon, and (b) surface phenomenon.*

geometrical configuration. On the other hand, radiation heat exchange between surfaces separated by participating media like water vapour, carbon dioxide, etc., also depend on the radiation properties of medium like emissivity and absorptivity which in turn are a strong function of temperature.

12.2.6 • Spectral vs Directional Characteristics

Thermal radiation depends on two factors, viz., wavelength and direction. The wavelength dependence is illustrated in Fig. 12.3(a) which shows how the monochromatic or spectral (at a particular wavelength) radiation emission per unit area (W/m^2 μm) varies with wavelength. The nature and temperature of the emitting surface govern the magnitude of radiation at a single wavelength (monochromatic) and the spectral distribution. The area under the graph is a measure of radiation emission per unit area (W/m^2) over the entire range of wavelengths. Figure 12.3(b) shows the directional nature of radiation. A surface emits radiation preferentially in different directions and magnitude depends on the nature of the surface. In the analysis of thermal radiation problems, we should address both wavelength and direction related issues. We can, thus, define radiation properties in four different ways: (a) *Monochromatic directional* (for a particular wavelength in a particular direction, (b) *Total directional* (of all wavelengths in a specific direction), (c) *Monochromatic hemispherical* (at a given wavelength in all directions, and (d) *Total hemispherical* (including all wavelength and all directions).

For the purpose of engineering analysis, two simplifying approximations and assumptions are generally made:

- The surface emits uniformly in all directions as a *diffuse* emitter.
- The total emission is integrated over all wavelengths.

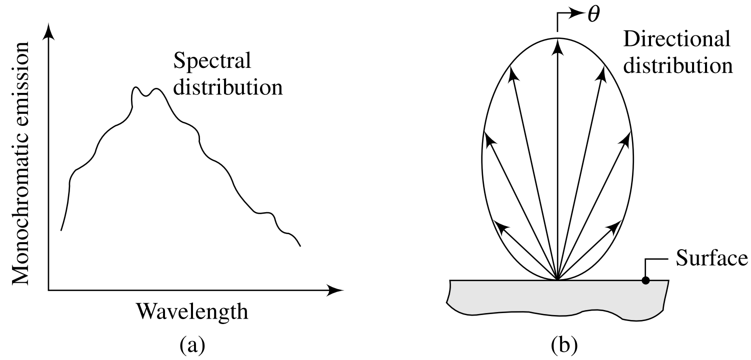


Fig. 12.3 *Radiation emission from a surface: (a) Wavelength dependence (b) Directional nature*

12.3 ❑ RADIATION DEFINITIONS

Emissive Power (E) It is defined as the radiant energy leaving the surface, in all directions, due only to the absolute temperature of the surface. The quantity E or $\dot{Q}_{\text{rad}}/A$ is the total energy summed up over all wavelengths. The *monochromatic* emissive power E_λ (W/m^2 μm) is the energy contained in an infinitesimally small wavelength band centred about the particular wavelength λ being considered. It follows that the *total*, *hemispherical* emissive power is

$$\boxed{E = \int_0^\infty E_\lambda d\lambda} \qquad (\text{W/m}^2) \tag{12.3}$$

The term *total* means the quantity in question is the summation of radiation over all wavelengths and the term *hemispherical* indicates that the summation is in all directions over a hemisphere.

Black-body Emissive Power Different bodies may emit different amounts of radiation per unit surface area, even when they are at the same temperature. The *maximum* amount of radiation is emitted by a *black body* at a given temperature.

The total radiation emitted by a black body at all wavelengths from $\lambda = 0$ to $\lambda = \infty$ per unit time and per unit area is called black body emissive power and denoted by E_b.

That is,

$$\boxed{E_b = \int_0^\infty E_{b\lambda} d\lambda \; (\text{W/m}^2)} \tag{12.4}$$

where $E_{b\lambda}$ is the monochromatic black-body emissive power (W/m^2 μm)

Since the emissive power depends on the fourth power of the absolute temperature, high-temperature bodies emit much more than the low temperature ones.

Emissivity *Emissivity*, of a surface is defined as

$$\boxed{\varepsilon = \frac{\text{Actual emissive power, } E}{\text{Blackbody emissive power, } E_b}}$$

The value of emissivity ranges between *zero* and *unity*. For a black surface, $\varepsilon = 1$. In general, emissivity depends on *temperature, wavelength,* and *direction of emission*.

Irradiation Irradiation (G) is defined as the combination, *from all sources and directions*, of radiant energy that strikes the surface per unit time and per unit area. Note that G does not depend on the temperature of the surface concerned. There may, however, be an indirect dependence. For instance, the radiation received by a surface is partly its only reflected radiation. Irradiation is not a surface property and may not be constant unless the surface is receiving radiation from a very distant source. The solar radiation flux incident on the earth's surface can therefore be taken as uniform. Total irradiation G is expressed as

$$G = \int_0^\infty G_\lambda d_\lambda \quad (\text{W/m}^2)$$

where G_y(W/m^2 μm) denotes spectral irradiation

Radiosity Radiosity (J) is defined as the rate at which the total is radiant energy leaves the surface per unit time and per unit area in all directions. Surfaces both emit as well as reflect radiation. The radiation leaving a surface thus has two separating components: one which is *reflected* by the surface, and the other which is *emitted* by the surface by virtue of its temperature (see Figure 12.4).

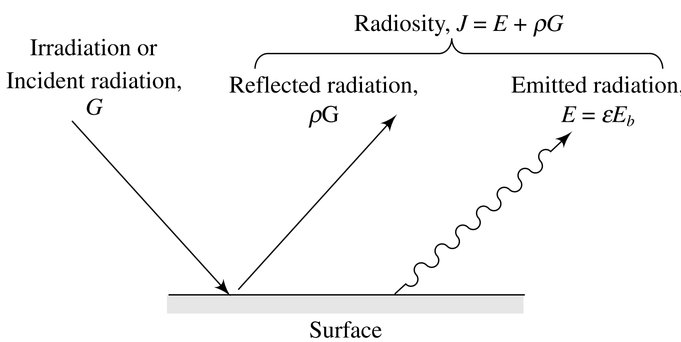


Fig. 12.4 *Radiosity represents the sum of the emitted energy and reflected energy*

If the spectral radiation J_λ (W/m μm) represents the radiation flux at wavelength λ leaving the follows wavelength λ

$$\boxed{J = \int_0^\infty J_\lambda d\lambda}$$

The emissive power E and radiosity J will be equal only when there is no reflection from the surface. This is possible when either the irradiation G is zero, or when the surface absorbs (or transmits) all the energy incident on it (i.e., when the surface is a black body or is completely transparent). Radiosity is constant only when there is uniform irradiation.

12.4 ❑ SURFACE CHARACTERISTICS: ABSORPTIVITY, REFLECTIVITY AND TRANSMISSIVITY

When radiation energy per unit time, per unit surface area, called irradiation, G, strikes a surface, part of it is absorbed, part of it reflected, and the remaining part, if any, is transmitted as shown in Fig. 12.5

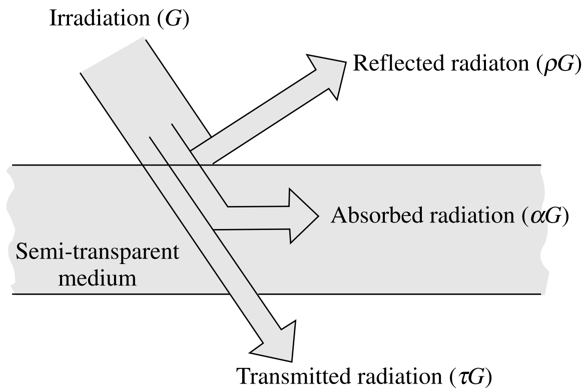


Fig. 12.5 *Surface absorption, reflection, and transmission of incident radiation*

The fraction of incident radiation absorbed by a surface is called absorptivity, the fraction reflected by the surface is known as reflectivity, and the fraction transmitted is referred to as transmissivity.

Absorptivity, reflectivity and transmissivity for a medium can be expressed as follows:

Absorptivity $$\boxed{\alpha = \frac{\text{Absorbed radiation}}{\text{Incident radiation}} = \frac{G_{\text{abs}}}{G}, \quad 0 \le \alpha \le 1}$$

Reflectivity: $$\boxed{\rho = \frac{\text{Reflected radiation}}{\text{Incident radiation}} = \frac{G_{\text{ref}}}{G}, \quad 0 \le \rho \le 1} \tag{12.5}$$

Transmissivity: $$\boxed{\tau = \frac{\text{Transmitted radiation}}{\text{Incident radiation}} = \frac{G_{\text{trans}}}{G}, \quad 0 \le \tau \le 1} \tag{12.6}$$

If G is the radiation flux (*irradiation*) incident on the surface (*solid, liquid or gas*) and G_{abs}, G_{ref}, and G_{trans} are the *absorbed*, *reflected*, and *transmitted* components of it, respectively, the sum total of the absorbed, reflected, and transmitted radiation is equal to incident radiation. That is,

$$\boxed{G_{\text{abs}} + G_{\text{ref}} + G_{\text{trans}} = G} \tag{12.7}$$

Dividing each term by G gives

$$\boxed{\alpha + \rho + \tau = 1} \tag{12.8}$$

The above definitions are for *total hemispherical* properties, since G represents the irradiation incident on the surface from *all directions over the hemispherical space* and *over all wavelengths*. Thus, α, ρ, and τ are the *average* properties of a medium for all directions and all wavelengths.

Opaque Body A material with a transmissivity of zero is *opaque* ($\tau = 0$). Most solids are opaque in the visible wavelength range. Even *glass* and *water*, which transmit visible light, are opaque to infrared radiation. For an opaque body, the transmissivity, $\tau = 0$ and $\alpha + \rho = 1$. Most of our discussion will be concentrated on solids which are treated as opaque. Glass and rock salt and other inorganic crystals are some exceptions among the solids, because, unless very thick, they are to a certain degree transparent to radiation of certain wavelengths. Many surfaces of engineering importance are opaque.

Black Body When a body is such that no incident radiation is *reflected* or *transmitted*, all the radiant energy must be absorbed. For a *black body*, therefore,

$$\boxed{\rho = 0, \tau = 0, \text{and } \alpha = 1}$$

Black surfaces are perfect absorbers as well as perfect emitters, emitting the maximum possible energy that any surface can emit at a given temperature.

In practice, there is no perfectly black body but many surfaces can be made to approach it. For example, a body may be coated with carbon black to produce a near black surface from the point of view of thermal radiation.

The concept of a black body is very useful in the study of radiation heat transfer much like the concept of an ideal gas in the study of thermodynamics.

White Body It reflects all the incident thermal radiation and neither transmits nor absorbs any part of it. For a white body, α = 0, τ = 0, and $\boxed{\rho = 1}$.

Gray Body A gray body is one whose absorptivity of a surface does not vary with temperature and wavelength of the incident radiation. For a gray body,

$$\boxed{\alpha = \alpha_\lambda = \text{constant}}$$

A surface whose α_λ is independent of λ would absorb equal fraction of each wavelength. Absorptivity of a gray body is independent of the spectrum of the irradiation.

Coloured Body A coloured body is one whose absorptivity of a surface varies with the wavelength of radiation, $\alpha \neq \alpha_\lambda$

It is important recognize that the dependence of surface-radiation properties on the *wavelength* and *direction*, makes the analysis more complicated. Hence, very often *gray* and *diffuse* approximations are used in radiation calculations. A surface is called *diffuse* if its properties are *independent of direction*, and *gray* if its properties are *independent of wavelength*.

12.5 ❑ BLACK-BODY RADIATION

A black body is defined as a perfect absorber and a perfect emitter. All real (*non-ideal*) surfaces emit *less* than a black surface. Real surfaces are characterized by how closely they resemble black surfaces. By definition:

- *A black body absorbs all the radiation incident upon it,*
- *The black body is a diffuse emitter.* The radiation emitted by a black body irrespective of wavelength and directon is independent of direction, though it does depend on the wave length and temperature.
- *For a given wavelength and temperature, no surface can emit more energy as thermal radiation than a black body.*
- A black body does not *reflect* or *transmit* any incident radiation.

Materials such as *carbon black*, *carborundum*, and *platinum black* are good approximations of black surfaces in their ability to absorb incident radiation.

We must recognize that surfaces which are nearly black, for radiation purposes, are not necessarily black to visible light, because the visible-light wavelength range is only a small part of the overall thermal radiation range. White paper, for example, is nearly *radiation black* with an absorptivity of 0.97.

• Concept of a Black Body

It is possible to artificially create an almost perfect black body by forming a cavity in a material. A *large cavity with a small opening* called Hohlraum, closely approximates a black body as shown in Fig. 12.6. Radiation enters the cavity through the opening (a narrow aperture) and the incident radiation undergoes repeated reflections on the wall of the cavity. Thus, there are many chances for it be absorbed by the interior surfaces of the cavity before any part of it can possibly escape. Also, if the surface of the cavity is at a unform temperature, the radiation emitted by the interior surfaces comes out through the opening after undergoing multiple internal reflections, and thus it has a diffuse nature. Therefore, the cavity acts as a perfect absorber and perfect emitter, and the opening will behave as an idealized black body regardless of whether the surface the cavity of highly absorbing or highly reflecting.

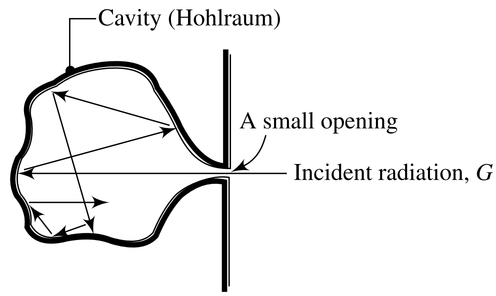


Fig. 12.6 *Cavity as a black body enclosure*

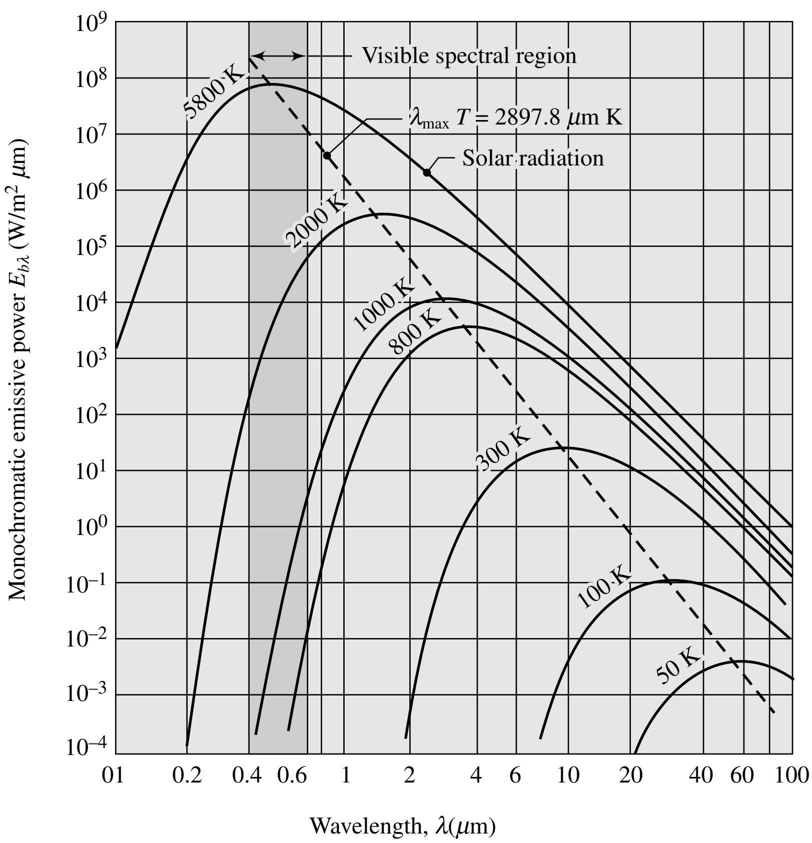


Fig. 12.7 *Monochromatic emissive power as function of wavelength with the absolute temperature as the parameter according to Planck's law*

The opening of a peephole on the side of a large boiler is another fairly good approximation of a black surface. The beam leaving the boiler through the peephole is weekened by multiple absorption reflection processes to the point where the escaped radiation is negligible. Thus, all the energy of the radiant beam is absorbed when it passes through the opening.

12.6 ❑ LAWS OF BLACK-BODY RADIATION

12.6.1 • Stefan–Boltzmann Law

The radiation energy emitted by a black body per unit time and per unit surface area is expressed as

$$\boxed{\frac{\dot{Q}_{\max}}{A} = E_b(T) = \sigma T^4} \quad (\mathrm{W/m^2}) \tag{12.9}$$

where $\sigma = 5.67 \times 10^{-8}$ W/m^2 K^4 is called the *Stefan–Boltzmann constant* and T is the absolute temperature of the surface in K. *Equation* (*12.9*) is known as the **Stefan–Boltzmann law** and E_b is called the *black-body emissive power*. The emission of thermal radiation is thus proportional to the *fourth power* of the absolute temperature. For calculations, it is convenient to use this expression in the form: $E_b = 5.67 \times (T/100)^4$ (W/m^2).

12.6.2 • Planck's Law of Distribution

Thermal energy is not emitted at a single wavelength but rather over a range of wavelengths. In 1900, *Max Planck* derived an equation for the energy emitted by a black body into vacuum as a function of wavelength.

A black body absorbs all radiant energy incident on it and at a prescribed temperature it possesses the *maximum* emissive power of any body. Planck's law of distribution gives the energy emitted by a black body as a function of wavelength and absolute temperature.

$$\boxed{E_{b\lambda} = \frac{C_1 \lambda^{-5}}{\{\exp[C_2/\lambda T] - 1\}}} \tag{12.10}$$

where $E_{b\lambda}$ is the maximum radiant energy emitted by a black body per unit area per unit wavelength (W/m^2 μm), called the *monochromatic or spectral hemispherical emissive power of a black body*.
where $E_{b\lambda}$ is emissive power per unit area, per unit wavelength, and λ is wavelength. The constants C_1 and C_2 are

$$\boxed{C_1 = 2\pi h c_0^2 = 3.741\,77 \times 10^8} \quad (\mathrm{W\,\mu m^4/m^2})$$

$$\boxed{C_2 = \frac{hc_o}{k} = 1.438\,78 \times 10^4} \quad (\mathrm{\mu m\,K})$$

where h is Planck's constant, k is Boltzmann's constant, and C_o is the speed of light in vacuum. Wavelength is typically measured in microns, with 1 μm = 10^{-6} m.

12.6.3 • Maximum Monochromatic Emissive Power

The value of maximum monochromatic emissive power of a black body at a given temperature can be obtained by substituting this value of $\lambda_{\max} T$(= 2897.8 μm K) in the Planck's equation.

$$E_{b\lambda,\max} = \frac{C_1\lambda^{-5}}{\exp\left(\dfrac{C_2}{\lambda T}\right)-1} = \frac{(3.74177\times10^{8}\ \mathrm{W\,\mu m^4/m^2})\left(\dfrac{2897.8\ \mu\mathrm{m\,K}}{T(K)}\right)^{-5}}{\exp\left(\dfrac{1.43878\times10^{4}\ \mu\mathrm{m\,K}}{2897.8\ \mu\mathrm{m\,K}}\right)-1}$$

$$= \frac{1.8312\times10^{-9}\ \mathrm{W/\mu m\,m^2}\times T^5(K^5)K^{-5}}{e^{4.965}-1}$$

$$\boxed{E_{b\lambda,\max} = 1.2868\times10^{-11}\,T^5(\mathrm{W/m^2\,\mu m}) \approx 1.287\times10^{-5}\,T^5(\mathrm{W/m^3})} \tag{12.11}$$

Note: The maximum monochromatic emissive power of a black body varies as the *fifth power* of the *absolute temperature* of the body.

This expression can be used to predict very high temperatures simply by measuring the wavelength of the radiation emitted.

Figure 12.7 is a plot of Planck's law and shows the spectral energy distribution from a black body at different temperatures (*spectral refers to any quantity which varies with wavelength*). The following important features of black-body radiation are quite obvious from this figure:

1. The monochromatic emissive power varies *continuously* with wavelength.
2. The amount of emitted radiation at any wavelength *increases* with increasing temperature.
3. As temperature increases, the total amount of radiation emitted also increases. This is consistent with the *Stefan–Boltzmann law*.
4. The wavelength corresponding to the maximum monochromatic emissive power depends on temperature.
5. At higher temperatures, the peak of the distribution shifts to the left, and comparatively the more radiation is emitted at short wavelengths.
6. A significant portion of the radiation emitted by the sun (approximated as a black body at a surface temperature of 5779 K or about 5800 K) lies in the visible region of the spectrum. However, at low temperatures (≤ 800 K), most of the radiation emitted by surfaces is in the infrared range of the spectrum and is hardly visible to the eye, unless the surfaces reflect light coming from other sources.

It is important to note that if the temperature of a hot body is less than 500°C (≈ 800 K), virtually none of the radiation will fall within the band of wavelengths corresponding to visible light. If the temperature of the body is increased, some radiation will fall within the *visible range* and, at about 700°C, the surface glows *dull red*. With further increase in temperature, the colour changes to *cherry red* at 900°C, *orange red* at 1100°C and, finally, at temperatures greater than about 1400°C, when sufficient energy is emitted in the visible range, the body becomes *white hot*. At the same time, the total quantity of heat radiated, which is proportional to T^4, increases rapidly. Even at 2500°C, the average temperature of the tungsten filament of the incandescent lamp, only about 10 percent of the energy is emitted in the visible range, which illustrates that *the incandescent lamp is a more efficient source of heat than light.*

12.6.4 • Derivation of Stefan–Boltzmann Law from Planck's Law

The emissive power of a black body given by the *Stefan–Boltzmann law* can be found by integrating the expression for *Planck's law* over all wavelengths. The total amount of radiation emitted at a given temperature is the area under the curve in Fig. 12.8.

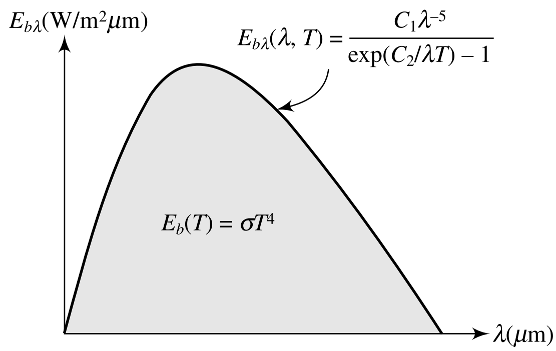


Fig. 12.8 *Representation of the total emissive power of a black surface at temperature T*

Planck's distribution law for the monochromatic emissive power of a black body is expressed as

$$E_{b\lambda}(\lambda T) = C_1/\lambda^5[\exp(C_2/\lambda T) - 1] \quad \text{(A)}$$

The total emissive power is then given by, $E_b(T) = \int_0^\infty E_{b\lambda}(\lambda, T)d\lambda$ (B)

Substituting from Eq. (A), one has, $E_b(T) = \int_0^\infty \dfrac{C_1}{\lambda^5[\exp(C_2/\lambda T) - 1]}d\lambda$ (C)

Defining the variable x as

$$x = \frac{C_2}{\lambda T} \quad dx = -\frac{C_2}{\lambda^2 T}d\lambda \quad \text{or} \quad d\lambda = -\frac{\lambda^2 T}{C_2}dx = -\frac{(C_2/xT)^2 T\,dx}{C_2}$$

The limits of integration are $\lambda = 0$, $x = \infty$, and $\lambda = \infty$, $x = 0$

Substituting for the new variable x with its limits, Eq. (C) becomes

$$E_b(T) = \int_\infty^0 \frac{C_1}{(C_2/xT)^5[e^x - 1]}\left(-\frac{(C_2/xT)^2 T}{C_2}dx\right) = \frac{C_1}{C_2^4}T^4\int_0^\infty \frac{x^3}{e^x - 1}dx \quad \text{(D)}$$

We note that $(e^x - 1)^{-1} = e^{-x} + e^{-2x} + \ldots = e^{-nx}$

The definite integral $\int_0^\infty x^3 e^{-nx}\,dx = \dfrac{3!}{(3+1)} = \dfrac{3!}{n^4} = \dfrac{3\times 2}{n^4} = \dfrac{6}{n^4}$

$$\therefore \qquad E_b(T) = \frac{6C_1}{C_2^4}T^4\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right] = \frac{6C_1T^4}{C_2^4} \times \frac{\pi^4}{90} = \left(\frac{\pi}{C_2}\right)^4 \frac{C_1}{15}T^4$$

Thus $\qquad \boxed{E_b = \sigma T^4}$

where $\qquad \sigma = \left(\dfrac{\pi}{C_2}\right)^4 \dfrac{C_1}{15}$

Substituting numerical values for C_1 and C_2, the Stefan–Boltzmann constant is

$$\sigma = \left(\frac{\pi}{14\,387.8\ \mu\text{m K}}\right)^4 \frac{3.74177 \times 10^8\ \text{W}\,\mu\text{m}^4/\text{m}^2}{15} = 5.670 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4$$

12.6.5 • Wien's Displacement Law

Figure 12.6 shows the variation of $E_{b\lambda}$ with wavelength at different temperatures. The curve of monochromatic emissive power versus wavelength has the same form for every temperature but as the temperature is increased, the height of the curve increases and the maximum moves towards the shorter wavelengths. At each temperature, there is a maximum in the spectral energy distribution. The dashed line in the figure represents the locus of the maximum emissive power and we note the maximum shifts towards lower wavelengths as the surface temperature increases. The location of the maximum emissive power at a given temperature can be calculated from Wien's law, which states that

$$\boxed{\lambda_{\max} T = C_3 = 2897.8 \approx 2898\ (\mu\text{m K})} \tag{12.12}$$

where $\lambda_{\max}$ is the wavelength at which the spectral black-body emissive power is a *maximum* for a specified temperature and is inversely proportional to the absolute temperature, T. The *greater* the *temperature*, the *shorter* the *wavelength* at which the peak occurs. The peak of solar radiation, for instance, occurs at $\lambda \approx 2898/5880 \approx 0.5$ μm which is almost at the middle of the visible band while the peak of the radiation emitted at the room temperature of 300 K occurs at $\lambda \approx 2898/300 = 9.66$ μm which is in the infrared (IR) range of the spectrum. Infrared radiation caanot be sensed by our eyes.

Derivation of Wien's Displacement Law from Planck's Law To find the wavelength at a specified temperature for which the black-body spectral (*monochromatic*) emissive power is a maximum, we can differentiate $E_{b\lambda}$ with respect to λ and equate the resulting derivative to zero.

$$E_{b\lambda} = \frac{C_1}{\lambda^5 (e^{C_2/\lambda T} - 1)}$$

$$\frac{d}{d\lambda}(E_{b\lambda}) = 0 \quad \text{for} \quad E_{b\lambda,\max} \quad \text{or} \quad \frac{d}{d\lambda}[\lambda^5 (e^{C_2/\lambda T} - 1)] = 0 \qquad \text{(since } C_1 \text{ is a constant)}$$

or
$$\lambda^5 \left[e^{C_2/\lambda T} \cdot \frac{d}{d\lambda}\left(\frac{C_2}{\lambda T}\right) - 0 \right] + (e^{C_2/\lambda T} - 1)(5\lambda^4) = 0$$

or
$$\lambda \left[e^{C_2/\lambda T} \left(-\frac{C_2}{\lambda^2 T} \right) \right] + 5(e^{C_2/\lambda T} - 1) = 0$$

Let
$$\frac{C_2}{\lambda T} \equiv x$$

Then, $\quad xe^x + 5(e^x - 1) = 0$ or $e^x[5 - x] = 5$

To solve this equation, a *trial-and-error* solution is necessary.

x	LHS	RHS
4.9	13.43	5.0
4.95	7.06	5.0
4.97	4.32	5.0
4.965	5.0158	5.0
4.9651	**5.0**	**5.0**

$$\therefore \quad x = \frac{C_2}{\lambda T} = 4.9651$$

$$\therefore \qquad (\lambda T)_{\text{max power}} = \frac{1.43878 \times 10^4\ \mu\text{m K}}{4.9651} = \mathbf{2898\ \mu m\ K}$$

$$\boxed{\boldsymbol{\lambda_{\max} T = 2898\ \text{mm K}}}$$

Clearly, the wavelength at which the spectral emissive power is maximum goes on decreasing as the temperature goes on increasing as illustrated in Fig. 12.7.

12.6.6 • Corollaries of Planck's Law

Approximations to Planck's law of distribution are the *Wien* and *Rayleigh–Jeans* laws which can be used to determine the monochromatic emissive power fairly easily for the *extremely low* and *high* limits of the product λT, respectively.

Now consider these two extreme cases when $C_2/\lambda T >> 1$ (*very low* λT) and $C_2/\lambda T << 1$ (*very high* λT).

Planck's law has two limiting cases, one of which is that when the product λT is small compared with the constant C_2. The second extreme case corresponds to a large value of the product λT as compared to the constant C_2.

Wien's Law When $C_2/\lambda T >> 1$ (or $\lambda T << C_2$), it follows that exp $(C_2/\lambda T) >> 1$. Hence, the –1 term in the denominator of the Planck's law is insignificant and can be dropped giving

$$\boxed{E_{b\lambda}(\lambda, T) \approx C_1/\lambda^5 \exp(-C_2/\lambda T)} \tag{12.13}$$

This relationship is known as *Wien's law*. The ratio of the emissive power by Wien's law to that by the Planck's law is,

$$\frac{E_{b\lambda}|_{\text{Wien}}}{E_{b\lambda}|_{\text{Planck}}} = \frac{1/\exp(C_2/\lambda T)}{1/[\exp(C_2/\lambda T) - 1]}$$

For the condition $\lambda T = \lambda_{\max} T = 2897.8\ \mu\text{m K}, C_2/\lambda T = \dfrac{14387.8\ \mu\text{m K}}{2897.8\ \mu\text{m K}} = 4.965$

and $$\frac{E_{b\lambda}|_{\text{Wien}}}{E_{b\lambda}|_{\text{Planck}}} = \frac{1/\exp(4.965)}{1/[\exp(4.965) - 1]} = \frac{[\exp(4.965) - 1]}{\exp(4.965)} = \mathbf{0.9930}$$

That is, for $\lambda T \le 2898\ \mu$m K, Wien's law is a good approximation (*accurate within 1%*) to the Planck distribution.

Rayleigh–Jean's Law This law is useful in analyzing *long-wavelength radiation* such as *radio waves*.

When the product λT is large compared with the constant C_2, i.e., $C_2/\lambda T << 1$ (or $\lambda T >> C_2$), the exponential term expressed as a series can be approximated by the first two terms. That is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \approx 1 + x \quad \text{when} \quad x << 1$$

The *Rayleigh–Jean's* (RJ) approximation is then

$$E_{b\lambda}(\lambda T) \approx C_1/\lambda^5 [1 + (C_2/\lambda T) - 1]$$

or $$\boxed{E_{b\lambda}(\lambda T) \approx C_1 T / C_2 \lambda^4} \tag{12.14}$$

For the condition $\lambda T \geq 10^5$ μm K, $C_2/\lambda T = 0.143\ 878$

$$\frac{E_{b\lambda}|_{R-J}}{E_{b\lambda}|_{\text{Planck}}} = \frac{C_1 T / C_2 \lambda^4}{C_1/\lambda^5}[\exp(C_2/\lambda T) - 1] = (\lambda T / C_2)[\exp(C_2/\lambda T) - 1]$$

$$= \frac{\exp(0.143\ 878) - 1}{0.143\ 878} = \mathbf{1.0755}$$

That is, for $\lambda T \geq 10^5$ μm K, the *Rayleigh–Jean's law* is a good approximation better than 10% to the Planck distribution.

12.6.7 • Black-body Radiation Functions

Many problems involve an estimate of the energy radiated at a specified wavelength or within a finite band of wavelengths.

The radiation energy emitted by a black body per unit area over a wavelength band from $\lambda = 0$ to λ can be determined from

$$\boxed{E_{b,0-\lambda}(T) = \int_0^\lambda E_{b\lambda}(\lambda, T) d\lambda} \quad (\text{W/m}^2) \tag{12.15}$$

Figure 12.9 shows graphically (*the shaded area*) the representation of $E_b((0 - \lambda)\ (T))$. The total area under the curve obviously represents $E_{b(0 \to \infty)}(T)$ which is the black-body emissive power at temperature T given by σT^4.

A dimensionless quantity f_λ called the *black-body radiation function,* is defined as

$$\boxed{f_\lambda(T) = \frac{\int_0^\lambda E_{b\lambda}(\lambda, T) d\lambda}{\sigma T^4}} \tag{12.16}$$

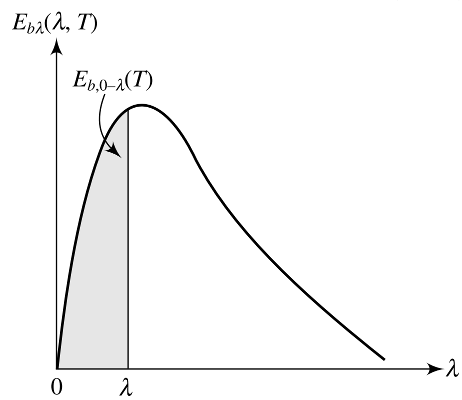


Fig. 12.9 *Graphical representation of $E_{b,\,0-\lambda}(T)$*

Substituting the expression for Planck's law and rearranging terms, we get

$$f_\lambda(T) = f_{0-\lambda} = \int_0^\lambda \frac{C_1}{\lambda^5[\exp(C_2/\lambda T) - 1]} \frac{1}{\sigma T^4} d\lambda$$

$$= \int_0^\lambda \frac{C_1}{\sigma(\lambda T)^5 [\exp(C_2/\lambda T) - 1]} T d\lambda$$

Changing the variable of integration from λ to λT, we get

$$\boxed{f_\lambda(T) = \int_0^{\lambda T} \frac{C_1}{\sigma(\lambda T)^5[\exp(C_2/\lambda T) - 1]} d(\lambda T)} \tag{12.17}$$

The function f_λ represents the *fraction of radiation emitted from a black body at temperature T in the wavelength band from λ = 0 to λ*, where λ is in μm and T is in K.

Table 12.2 lists the radiation functions $f_{(0-\lambda)}$(T) as functions of λT.

Table 12.2 *Black-body radiation functions*

λT (μm K)	$f_{0-\lambda}(T)$	λT (μm K)	$f_{0-\lambda}(T)$	λT (μm K)	$f_{0-\lambda}(T)$	λT (μm K)	$f_{0-\lambda}(T)$
500	0	3600	0.40360	6800	0.79610	10 000	0.91416
600	0	3700	0.42376	6900	0.80220	10 500	0.92367
700	0	3800	0.44337	7000	0.80808	11 000	0.93185
800	0.000016	3900	0.46241	7100	0.81373	11 500	0.93892
900	0.000087	4000	0.48087	7200	0.81918	12 000	0.94505
1000	0.000321	4100	0.49873	7300	0.82443	12 500	0.95041
1100	0.000911	4200	0.51600	7400	0.82949	13 000	0.95509
1200	0.00213	4300	0.53268	7500	0.83437	13 500	0.95921
1300	0.00432	4400	0.54878	7600	0.83906	14 000	0.96285
1400	0.00779	4500	0.5643	7700	0.84359	14 500	0.96607
1500	0.01285	4600	0.57926	7800	0.84796	15 000	0.96893
1600	0.01972	4700	0.59367	7900	0.85218	15 500	0.97149
1700	0.02853	4800	0.60754	8000	0.85625	16 000	0.97377
1800	0.03934	4900	0.62089	8100	0.86017	16 500	0.97581
1900	0.05211	5000	0.63373	8200	0.86396	17 000	0.97765
2000	0.06673	5100	0.64608	8300	0.86762	18 000	0.98081
2100	0.08305	5200	0.65795	8400	0.87115	19 000	0.98341
2200	0.10089	5300	0.66936	8500	0.87457	20 000	0.98555
2300	0.12003	5400	0.68034	8600	0.87786	25 000	0.99217
2400	0.14026	5500	0.69088	8700	0.88105	30 000	0.99529
2500	0.16136	5600	0.70102	8800	0.88413	35 000	0.99695
2600	0.18312	5700	0.71077	8900	0.88711	40 000	0.99792
2700	0.20536	5800	0.72013	9000	0.88999	45 000	0.99852
2800	0.22789	5900	0.72914	9100	0.89277	50 000	0.99890
2897.8	**0.25011**	6000	0.73779	9200	0.89547	55 000	0.99917
2900	0.25056	6100	0.74611	9300	0.89807	60 000	0.99935
3000	0.27323	6200	0.75411	9400	0.90060	65 000	0.99949
3100	0.29578	6300	0.76181	9500	0.90304	70 000	0.99959
3200	0.3181	6400	0.7692	9600	0.90541	75 000	0.99966
3300	0.34011	6500	0.77632	9700	0.90770	85 000	0.99977
3400	0.36173	6600	0.78317	9800	0.90992	100 000	0.99985
3500	0.38291	6700	0.78976	9900	0.91207	∞	1.0000

12.6.8 • Band Emission

We are often interested in knowing the fraction of the total radiation emission from a black body over a certain wavelength interval or band.

The energy emitted by a black body in the wavelength band defined by λ_1 and λ_2 is $\int_{\lambda_1}^{\lambda_2} E_{b\lambda} d\lambda$, or the area under the curve in Fig. 12.10 between λ_1 and λ_2.

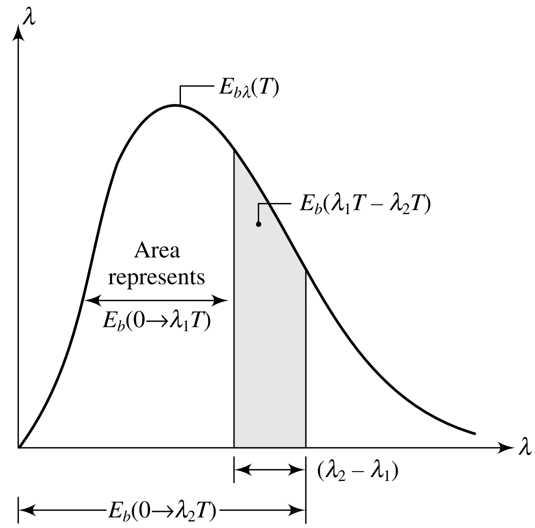


Fig. 12.10 *Graphical representation of the emitted radiation in the wavelength band from λ_1 to λ_2 at temperature T*

The fraction of radiation energy emitted by a black body at temperature T over a finite wavelength band from $\lambda = \lambda_1$ to $\lambda = \lambda_2$ can be obtained from

$$f_{\lambda_1 - \lambda_2} = \frac{\int_{\lambda_1}^{\lambda_2} E_{b\lambda} d\lambda}{\int_0^{\infty} E_{b\lambda} d\lambda} = \frac{\int_0^{\lambda_2} E_{b\lambda} d\lambda - \int_0^{\lambda_1} E_{b\lambda} d\lambda}{\sigma T^4} = f_{\lambda_2} - f_{\lambda_1}$$

or
$$\boxed{f_{\lambda_1 - \lambda_2}(T) = f_{\lambda_2}(T) - f_{\lambda_1}(T)} \tag{12.18}$$

where $f_{\lambda_1}(T)$ and $f_{\lambda_2}(T)$ are black-body radiation functions corresponding to $\lambda_1 T$ and $\lambda_2 T$, respectively.

12.7 ❑ INTENSITY OF RADIATION

Radiation is emitted by all parts of a body in all directions into the surrounding hemispherical space above it. However, the directional distribution of emitted (*or incident*) radiation is generally *not* uniform. It is of great importance to know the amount of radiation emitted *or* streaming into a given direction. This quantity is *radiation intensity*, denoted by I. The direction of radiation passing through a point is best described in spherical coordinates in terms of zenith (polar) angle θ and the azimuth angle ϕ,

Fig. 12.11. The intensity of radiation is used to describe how the emitted radiation varies with the zenith and azimuth angles. One must note that the radiation emission by a black body per unit normal area is uniform in all directions, and is independent of direction. But for non-black surfaces, this is not the case.

We should also note that *intensity* could be due to some other aspect than *emission*, such as *reflection* or *transmission*.

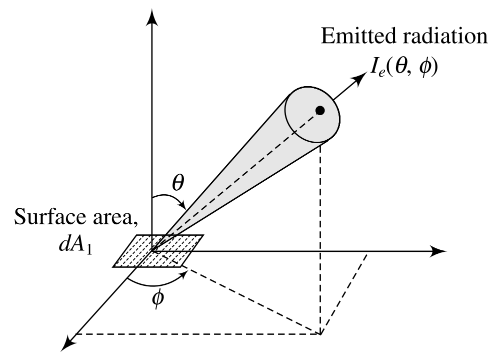


Fig. 12.11 *Intensity of radiation emitted from a surface*

12.7.1 • Plane Angle and Solid Angle

Radiation is emitted in *three dimensional* space and propagates in all directions from a given surface area. To describe such a propagation process, we need to define the geometric concept of the *solid angle*, which is analogous to the plane angle. Figure 12.12 indicates how a plane angle and a solid angle are defined. Solid angle is defined as the region in sphere which is enclosed by a conical surface whose vertex is the centre of the sphere. The differential plane angle $d\alpha$ is the arc length ds on a circle divided by the circle radius. That is, $d\alpha = ds/r$. Similarly, the differential solid angle is the area element dA on a sphere divided by the square of the sphere radius. That is, $d\omega = \frac{dA}{r^2}$. While the plane angle (*two-dimensional*) has the unit of *radian* (rad), the solid angle (*three-dimensional*) is measured in *steradian* (sr).

The total solid angle subtended by the sphere at its centre is $\omega = \frac{\text{sphere surface area}}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi(\text{sr})$. The solid angle subtended by a complete hemisphere is $\frac{2\pi r^2}{r^2} = 2\pi$.

Note that the area dA_n is normal to the direction of viewing since dA_n is viewed from the centre of the sphere.

When the solid angle subtended by an area element at an arbitrarily located point with respect to it is required, the solid angle is defined as

$$\boxed{d\omega = \frac{dA_n}{r^2}} \tag{12.19}$$

where the subscript n is used to denote that the receiving area is normal to the radiation emitted from dA.

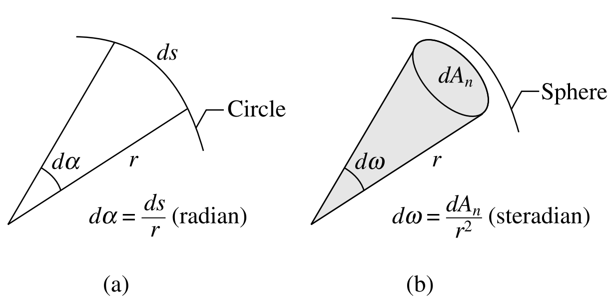


Fig. 12.12 *Definition of (a) plane and (b) solid angles*

12.7.2 • Relation to Emission

The **intensity** of emitted radiation $I_e(\theta, \phi)$ is defined as *the rate at which the radiation energy $d\dot{Q}_e$ is emitted ina given direction per unit area of the emitting surface perpendicular to this direction and per unit solid angle about this direction.*

We should note that the area used to define the radiation intensity is the component of area dA_1 normal to the direction of radiation. From Fig. 12.13, it can be seen that this projected area is $dA_1 \cos \theta$. What it means is that this is how the area dA_1 would appear to an observer at dA_n.

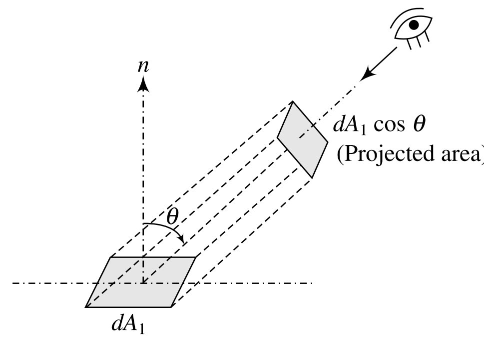


Fig. 12.13 *Projection of area dA_1 perpendicular to the direction of radiation*

$$\boxed{I_e(\theta, \phi) = \frac{d\dot{Q}_e}{dA_1 \cos\theta \, d\omega} = \frac{d\dot{Q}_e}{dA \cos\theta \sin\theta \, d\theta \, d\phi}} \quad (\text{W/m}^2\text{sr}) \tag{12.20}$$

We know that the rate at which the radinat energy is emitted per unit area of the emitting surface is the emissive power which can be expressed in differential form,

$$\boxed{dE = \frac{d\dot{Q}_e}{dA_1} = I_e(\theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi} \tag{12.21}$$

By integrating the above expression, we can get the emissive power from the surface into the hemisphere. Thus,

$$\boxed{E = \int_{\text{hemisphere}} dE = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_e(\theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi} \quad (\text{W/m}^2) \tag{12.22}$$

12.7.3 • Relation between Emissive Power and Intensity of Radiation

Consider the radiation emitted from the differential area dA at the centre of a sphere towards the area dA_n. Suppose this radiation is absorbed by a second elemental area dA_n, a portion of the hemispherical surface.

The amount of radiation falling on area dA_n is given by

$d\dot{Q}$ = (Intensity of radiation of the area dA) × (Projected area of dA on the plane perpendicular to the line joining dA and dA_n) × (Solid angle made by the area dA_n with the centre of the base circle of the hemisphere)

$$d\dot{Q} = I_e (dA \cos\theta)\,(d\omega)$$

where I_e = intensity of emitted radiation

The differential solid angle $d\omega$ can be related to the polar angle θ and the azimuth angle ϕ. By definition, $d\omega = \dfrac{dA_n}{r^2}$.

From Figure 12.14,

$$dA_n = (r d\theta)(r \sin\theta)(r \sin\theta \, d\phi) = r^2 \sin\theta \, d\theta \, d\phi$$

or $$d\omega = \frac{r^2 \sin\theta \, d\theta \, d\phi}{r^2} = \sin\theta \, d\theta \, d\phi$$

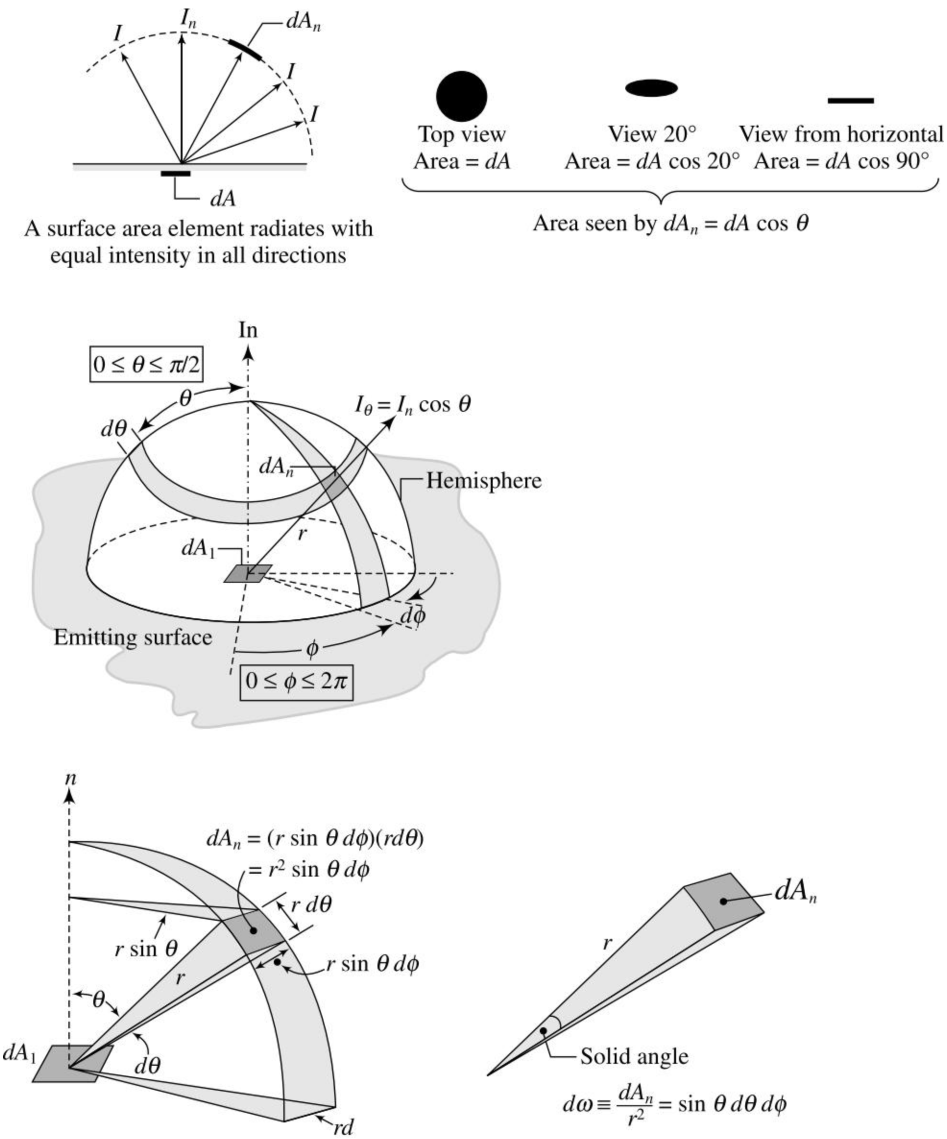


Fig. 12.14 *Radiation emission from a surface area element into the surrounding hemispherical space through a differential solid angle.*

Total radiation emission EdA from the surface can be obtained by integrating $d\dot{Q}$ over the hemispherical surface

$$EdA = \int_{\text{hemisphere}} d\dot{Q} = dA \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=2\pi} I_e \sin\theta \cos\theta \, d\theta \, d\phi$$

The intensity of radiation emitted by a surface, usually varies with direction (*especially with the polar (zenith) angle* θ). But many surfaces in practice can be approximated as being *diffuse*. For a *diffusely* *emitting* (isotropic) surface, the intensity of the emitted radiation is independent of direction and I_e = constant.

Emissive power,

$$E = I_e \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=2\pi} \cos\theta \sin\theta \, d\theta \, d\phi$$

or
$$E = I_e \left[\int_{\theta=0}^{\theta=\pi/2} \cos\theta \sin\theta \, d\theta \right] [\phi]_0^{2\pi} = 2\pi I_e \int_{\theta=0}^{\theta=\pi/2} \cos\theta \sin\theta \, d\theta$$

or
$$E = \pi I_e \int_{\theta=0}^{\theta=\pi/2} 2\cos\theta \sin\theta \, d\theta = \pi I_e \int_{\theta=0}^{\theta=\pi/2} \sin\theta \, d\theta$$

or
$$E = \pi I_e \left[\frac{-\cos 2\theta}{2} \right]_0^{\pi/2} = \frac{\pi I_e}{2} [\cos 0 - \cos \pi] = \frac{\pi I_e}{2} \left[\frac{-(-1-1)}{2} \right] = \pi I_e$$

Thus, the emissive power relation reduces to

$$\therefore \quad \boxed{E = \pi I_e} \quad (\text{W/m}^2) \tag{12.23}$$

The total emissive power of a diffuse surface is equal to π times its intensity of radiation.

Note that the *emissive power is based on the actual surface area whereas the intensity is based on the projected area*. Also note that the constant appearing in the two expressions is π and *not* 2π and has the unit steradian (sr).

For a black body, which is a diffuse emitter, one can write

$$\boxed{E_b = \pi I_b} \tag{12.24}$$

where $E_b = \sigma T^4$ is the black-body emissive power.

The intensity of the radiation emitted by a black body at the absolute temperature T is

$$\boxed{I_b(T) = \frac{E_b(T)}{\pi} = \frac{\sigma T^4}{\pi}} \quad (\text{W/m}^2\,\text{sr}) \tag{12.25}$$

12.7.4 • Relation to Irradiation

Apart from emitting radiation, all surfaces also receive radiation emitted or reflected by other surfaces.

The intensity of radiation incident on a surface I_i, (Fig. 12.15) is related to the total irradiation from all directions. The intensity of *incident* radiation $I_i(\theta, \phi)$ is defined as *the rate at which the energy is incident from a given direction per unit area of the receiving surface normal to this direction and per unit solid angle about this direction.* Here, θ is the angle between the direction of incident radiation and the normal to the surface. Total irradiation G can then be expressed as

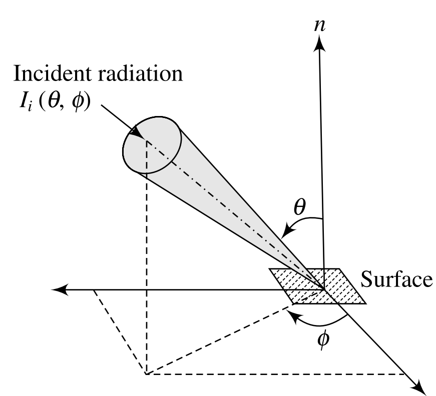


Fig. 12.15 *Intensity of radiation incident on a surface*

$$\boxed{G = \int_0^\infty G_\lambda d\lambda \quad \text{and} \quad G = \int_0^{2\pi}\int_0^{\pi/2} I_i(\theta, \phi)\cos\theta \sin\theta d\theta\, d\phi} \cdot (\text{W/m}^2) \tag{12.26}$$

If the intensity of incident radiation is independent of θ and ϕ, i.e., *diffuse* then we will get

$$\boxed{G = \pi I_i} \quad (\text{W/m}^2) \tag{12.27}$$

Note that *irradiation* is based on the *actual* surface area, while the *intensity of incident radiation* is based on the *projected* area.

12.7.5 • Relation to Radiosity

Radiosity J accounts for both the *emitted* radiation as well as the *reflected* component of irradiation. Hence, integrating over the hemisphere, we obtain the radiosity J.

$$\boxed{J = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_{(e+r)}(\theta, \phi)\cos\theta \sin\theta\, d\theta\, d\phi} \quad (\text{W/m}^2) \tag{12.28}$$

where $I_{(e+r)}$ is the sum of the emitted and reflected intensities. For a surface that is both a *diffuse emitter* and a *diffuse reflector*, $I_{(e+r)}$ = constant, and the radiosity can be expressed as

$$\boxed{J = \pi I_{(e+r)}} \tag{12.29}$$

A black surface radiates diffusely with a radiosity $J = E_b$. Then $J = E_b = \pi I_e$

12.8 ❑ LAMBERT'S COSINE LAW

Lambert's cosine law states that *the total emissive power E_θ from a radiating plane surface in any particular direction is directly proportional to the cosine of the angle between the direction under consideration and the normal to the surface.* If the total emissive power of the radiating surface in the direction of its normal is E_n, then

$$\boxed{E_\theta = E_n \cos\theta} \tag{12.30}$$

Clearly, it decreases with an increase in the angle θ and is zero when θ is 90°.
Consider an elemental area dA of a *diffuse* radiating surface. The total amount of energy radiating from this surface will be

$$dA\, E_\theta = 1_\theta\, dA \cos\theta\, d\omega$$

or

$$\boxed{E_\theta = 1_\theta \cos\theta\, d\omega} \tag{12.31}$$

where $d\omega$ = differential solid angle

and $\quad dA\ E_n = 1_n\ dAd\omega \quad$ (*since* θ = 0°) $\quad$ or $\quad \boxed{E_n = I_n\, d\omega}$ (12.32)

Dividing one by the other, we get

$$\frac{E_\theta}{E_n} = \frac{1_\theta \cos\theta}{I_n} \tag{12.33}$$

But from the *Lambert's cosine law*, $E_\theta = E_n \cos\theta$ (Fig. 12.16).

Equation (12.33) then becomes,

$$\frac{E_n \cos\theta}{E_n} = \frac{I_\theta \cos\theta}{I_n}$$

i.e. $$\boxed{I_n = 1_\theta} \tag{12.34}$$

This shows that if the surface is *diffuse*, the intensity of radiation is uniform in all directions. Such surfaces which behave in this way are often known as *Lambertonian surfaces*.

A *black body* is also a *Lambertonian* surface. The surfaces which obey Lambert's law have radiation intensities independent of direction.

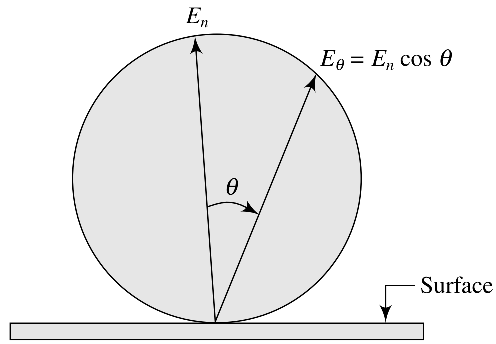


Fig. 12.16 *Lambert's cosine law*

12.9 ❑ KIRCHHOFF'S LAW

Consider a large evacuated isothermal enclosure (*at uniform absolute temperature T*) in which several small bodies are placed. The irradiation experienced by any body in the enclosure, regardless of its orientation, is diffuse and equal to emission from the walls of the enclosure acting as a black body at temperature T. Thus, $G = E_b(T)$

Under steady-state conditions, thermal equilibrium must exist between the bodies and the enclosure. Hence, $T_1 = T_2 = T_3 \ldots = T$, and the net rate of energy transfer to each surface has to be zero. Applying control surface energy balance about body of the area A_1

Energy absorbed $(\dot{E}_{\text{in}})$ – Energy emitted $(\dot{E}_{\text{out}}) = 0$

i.e., $$\boxed{\alpha_1 G A_1 - E_1(T) A_1 = 0} \tag{12.35}$$

where α_1 is the absorptivity of body 1 and A_1 its area.

or $$\boxed{\alpha_1 E_b(T) = E_1(T)} \tag{12.36}$$

Since this result must apply to each of the enclosed bodies, it follows that under thermal equilibrium conditions,

$$\boxed{\frac{E_1(T)}{\alpha_1} = \frac{E_2(T)}{\alpha_2} = \cdots = E_b(T)} \tag{12.37}$$

This relation is known as *Kirchhoff's law* or Kirchhoff's identity. Since $\alpha \le 1$, $E(T) \le E_b(T)$.

Thus, no real surface can have an emissive power greater than that of a black body at the same temperature.

Total, hemispherical emissivity is defined as the ratio of emissive power of a real surface to that of a black surface at the same temperature.

Thus, $$\boxed{\frac{E_1}{E_b} = \varepsilon_1, \frac{E_2}{E_b} = \varepsilon_2 \quad \text{and} \quad \text{so on}} \tag{12.38}$$

Hence, for any surface in the enclosure, $\boxed{\varepsilon = \alpha}$

Thus, the total, hemispherical emissivity of the surface is equal to its total hemispherical absorptivity.

It is important to recognize the *restrictive conditions* because of dependence of emissivity on temperature, wavelength, and direction.

The monochromatic emissivity, like monochromatic absorptivity, is generally found to be a function of surface temperature as well as wavelength and angle of emission.

There are no restrictions if the Kirchhoff's law is expressed in the form

$$\boxed{\varepsilon_{\lambda,\theta} = \alpha_{\lambda,\theta}} \tag{12.39}$$

This involves *spectral* as well as *spatial* (directional) properties.
In practice, it is usually assumed that there is little variation of emissivity with angle of emission.
The *spectral* form of the Kirchhoff's law is given by

$$\boxed{\varepsilon_{\lambda}(T) = \alpha_{\lambda}(T)} \tag{12.40}$$

Kirchhoff's law may thus be stated as: The monochromatic emissivity of a surface (ε_λ) is equal to its monochromatic absorptivity (α_λ) regardless of the difference in temperatures corresponding to the emitted ($T_{surface}$) and incident (T_{source}) radiation.

i.e.,

$$\boxed{\varepsilon_{\lambda} = \alpha_{\lambda}} \tag{12.41}$$

It must be emphasized that the absorptivity of a surface equals the emissivity of the surface only when the source temperature of the irradiation is the same as the surface temperature. Fortunately, most substances have α that is relatively insensitive to wavelength so that a change in the temperature of the irradiating source (or irradiation from several different source temperatures) does not change the absorptivity. Consequently, the emissivity remains the same.

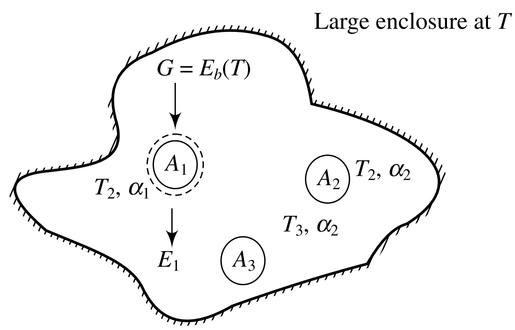


Fig. 12.17 *Large enclosure with inside walls at uniform temperature T*

In the case of black or gray bodies $T_{source} = T_{surface}$ (whose emissivities do not vary with wavelength), the total absorptivity and emissivity are the same even when $T_{source} \neq T_{surface}$. But, for *real surfaces*, when $T_{source} \neq T_{surface}$, the deviation from this law increases as ($T_{source} - T_{surface}$) increases. For example, in a tropical climate, white clothing is preferable because *white* has a *low absorptivity* for the high temperature, *short wavelength*, radiation from the sun (≈ 5800 K) but a *high emissivity* for its own low-temperature radiation (≈ 300 K).

If the temperature difference between the surfaces is well within a few hundred degrees, Kirchhoff's law can be applied as an approximation. Kirchhoff's law is not applicable for example, to solar radiation exchanging radiant energy with surfaces at room or moderate temperatures. The sun is thousands of degrees hotter than the surface of a solar collector, and the absorptivity of the solar collector is not generally same as its emissivity.

The emissivity of a surface at a particular *wavelength*, *direction*, and *temperature* is thus always equal to the *absorptivity* at the same *wavelength*, *direction* and *temperature*.

For gray diffuse surfaces an idealization the properties like emissivity are independent of wavelength as well as direction.

Kirchhoff's law is very useful in radiation analysis because by using the relation $\varepsilon = \alpha$ along with $\rho = 1 - \alpha$, we can find all three properties of an opaque surface from a knowledge of just *one* property.

12.10 ❑ RADIATION FROM REAL SURFACES

12.10.1 • Real Surfaces and Band Approximation

If the variation of monochromatic emissivity of an engineering (*non-gray*) surface with wavelength at the specified temperature is known, we can divide the spectrum into a sufficient number of *wavelength bands* and take the emissivity to be constant over each band by expressing the function $\varepsilon_\lambda(\lambda, T)$ as a *step function*. Let the *three-band* approximation be an adequate representation of the variation of emissivity of a real engineering surface as shown in Fig. 12.18. The constant band emissivities ε_1, ε_2, and ε_3 and the corresponding wavelength bands are

$$\boxed{\varepsilon_\lambda = \begin{cases} \varepsilon_1 = \text{constant}, & 0 \le \lambda < \lambda_1 \\ \varepsilon_2 = \text{constant}, & \lambda_1 \le \lambda < \lambda_2 \\ \varepsilon_3 = \text{constant}, & \lambda_2 \le \lambda < \infty \end{cases}} \tag{12.42}$$

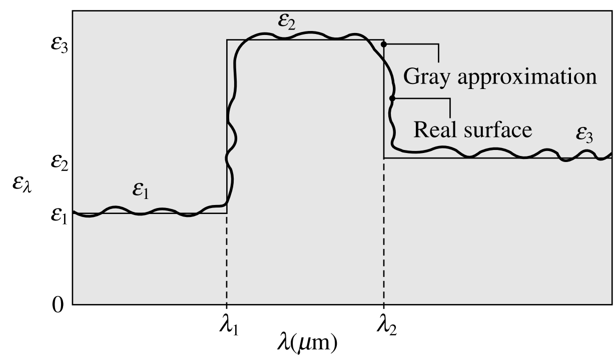


Fig. 12.18 *Approximation of the emissivity variation with wavelength of a real surface as step function in three bands (intervals)*

The average emissivity can now be found by breaking the integral into *three* parts and using the definition of the black-body radiation function as follows:

$$\boxed{\begin{aligned} \varepsilon(T) &= \frac{\varepsilon_1 \int_0^{\lambda_1} E_{b\lambda} d\lambda}{E_b} + \frac{\varepsilon_2 \int_{\lambda_1}^{\lambda_2} E_{b\lambda} d\lambda}{E_b} + \frac{\varepsilon_3 \int_{\lambda_2}^{\infty} E_{b\lambda} d\lambda}{E_b} \\ &= \varepsilon_1 f_{0-\lambda_1}(T) + \varepsilon_2 f_{\lambda_1-\lambda_2}(T) + \varepsilon_3 f_{\lambda_2-\infty}(T) \end{aligned}} \tag{12.43}$$

12.10.2 • Monochromatic Directional Surface Radiation Properties

In the earlier section, we had defined total hemispherical absorptivity, reflectivity, transmissivity.

The properties that characterize absorption reflection and transmission processes depend upon surface material and finish, surface temperature, and the direction and wavelength of the incident radiation.

Reflectivity Like emissivity, reflectivity of real surfaces depends on *temperature, wavelength*, and *direction*. *Reflectivity* is actually more complex than emissivity, because it varies, in general, with both angle of *incidence* and angle of *reflection*. The reflection of radiation can be either regular or diffuse. In the case of highly polished and smooth surface, the reflection of radiation is similar to the reflection of a light beam, i.e., the angle of incidence equals the angle of reflection. This is known as *specular* reflection. Most materials that we come across are *rough* because their surfaces have asperities which are large compared with the wavelength. The reflection of radiation from a rough surface occurs in almost all directions and is known as *diffuse*. There are two limiting cases of reflective behaviour that are used to simplify analysis the *diffuse* surface and the *specular surface.* In a diffuse (*optically rough*) reflector, the angle of reflection is independent of the angle of incidence. In a *specular* (*optically smooth*) reflector, the angle of reflection equals the angle of incidence. Real surfaces often fall somewhere between these two behaviours, as shown in Fig. 12.8.

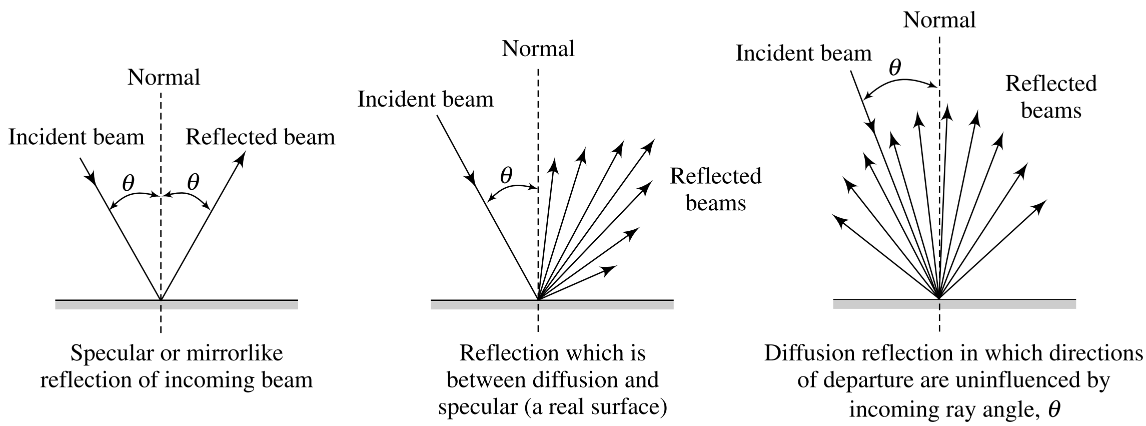


Fig. 12.19 *Specular and diffuse reflection of radiation*

Specular reflectors in the visible range include *mirrors, shiny metal surfaces*, *glass sheets*, and *still water*. The near-perfect reflection behaviour allows us to see images in these surfaces. Surfaces that are specular in the visible range are generally specular in the infrared range as well, since infrared radiation has longer wavelengths than visible light. It is possible to perform straightforward radiation analyses assuming either perfectly diffuse reflectors or perfectly specular reflectors. Most common surfaces are more nearly diffuse than specular.

Absorptivity The *total or average absorptivity* depends on the spectral distribution of the *irradiation* itself. Hence, if a *red* surface is irradiated by light dominated by *red* colour ($\lambda \approx 0.65$ μm), its absorptivity would be *much less* than that when the irradiation is dominated by *blue light* ($\lambda \approx 0.47$ μm).

We must realize that while freshly fallen snow with its almost blinding whiteness would appear to possess very large reflectivity, this high reflectivity is only in the visible part of the spectrum. Its absorptivity for the wider band of infrared radiation is quite large and, thus, its total absorptivity is fairly high. In fact, snow can be idealized as a black body for longer wavelength (*infrared*) radiation.

Depending on different radiation properties, we can broadly classify various bodies as follows:

Transparent Body It transmits the entire incident radiation falling on its surface and allows it to pass through $\boxed{\tau = 1, \alpha = \rho = 0}$.

Transmissivity Transmissivity like the other radiative properties depends on *temperature, wavelength*, and *direction*. It also depends on the thickness of the layer through which radiation travels. A material with a transmissivity of unity (τ = 1) is perfectly *transparent*. Thin layers of some gases, like air and oxygen, are virtually transparent to thermal radiation. A material with a transmissivity between zero and unity is called *semi-transparent*. Ordinary glass is *semi-transparent* in the visible wavelength range, and so is liquid water.

Solids generally transmit no radiation unless the material is of very thin section. *Metals* absorb radiation within a fraction of a micrometre (μm) and electrical insulators within a fraction of a millimetre (mm).

$$\boxed{\alpha + \rho = 1}$$

On the other hand, most elementary *gases* such as *hydrogen*, *oxygen*, and *nitrogen* (*and mixtures of these such as air*) have a transmissivity of practically *unity*; i.e., their *reflectivity* and *absorptivity* are nearly *zero*. For this reason, *radiation* heat transfer through air is generally estimated using the relationships for radiation through a *vacuum*. Gases with a more complicated structure, such as *steam* (H_2O) and *carbon dioxide*, (CO_2) generally absorb and emit as well as transmit radiation.

Certain solids and liquids transmit radiation at specific wavelengths unless they are very thick. These materials (*glass, inorganic crystals, etc*.) are transparent to radiation only at these wavelengths; at other wavelengths they are opaque to radiation. Thus, ordinary clear glass is transparent in the visible (*short wavelength*) range and in the infrared (*long wavelength*) range up to 2.5 μm, whereas very thin slices of semi conductor materials such as silicon and germanium are opaque in the visible region but transparent over parts of the infra redregion beyond 1.0 and 1.8 μm respectively.

Monochromatic (or Spectral) Emissivity It is the ratio of the monochromatic emissive power of a surface to the monochromatic emissive power of a black body at the same wavelength and temperature. Emissivity can be defined in three ways.
The spectral hemispherical emissivity is given by

$$\boxed{\varepsilon_\lambda(\lambda, T) = \frac{E_\lambda(\lambda, T)}{E_{b\lambda}(\lambda, T)}} \tag{12.44}$$

The emissivity of a surface at a given wavelength changes with temperature.

Total Hemispherical Emissivity It can be expressed in terms of the radiant energy emitted over all wavelengths and in all directions as

$$\boxed{\varepsilon(T) = \frac{E(T)}{E_b(T)}} \tag{12.45}$$

Since $E = \int_0^\infty E_\lambda d\lambda$ and $E_\lambda(\lambda, T) = \varepsilon_\lambda(\lambda, T) E_{b\lambda}(\lambda, T)$, the total hemispherical emissivity can also be expressed as

$$\boxed{\varepsilon(T) = \frac{E(T)}{E_b(T)} = \frac{\int_0^\infty \varepsilon_\lambda(\lambda, T) E_{b\lambda}(\lambda, T) d\lambda}{E_b(T)}} \tag{12.46}$$

Normal Total Emissivity ε_n Is the ratio of the normal component of the total emissive power of a surface (E_n) to the *normal component* of the total emissive power of a black body E_{bn} at the same temperature

$$\boxed{\varepsilon_n = \frac{E_n}{E_{bn}}} \tag{12.47}$$

Clearly, for a black body,

$$\boxed{\varepsilon_{b\lambda} = 1, \quad \varepsilon_b = 1, \quad \varepsilon_{bn} = 1} \tag{12.48}$$

Now, considering the wavelength and direction dependence, *the monochromatic directional absorptivity* and the *monochromatic directional reflectivity* of a surface are defined, respectively, as the absorbed and reflected fractions of the intensity of radiation incident at a particular wavelength in a specified direction as

$$\boxed{\alpha_{\lambda,\theta}(\lambda,\theta,\phi) = \frac{I_{\lambda,\text{abs}}(\lambda,\theta,\phi)}{I_{\lambda,i}(\lambda,\theta,\phi)}} \quad \text{and} \quad \boxed{\rho_{\lambda,\theta}(\lambda,\theta,\phi) = \frac{I_{\lambda,\text{ref}}(\lambda,\theta,\phi)}{I_{\lambda,i}(\lambda,\theta,\phi)}} \tag{12.49}$$

The *monochromatic hemispherical absorptivity* and *monochromatic hemispherical reflectivity of a surface* can be expressed as

$$\boxed{\alpha_\lambda(\lambda) = \frac{G_{\lambda,\text{abs}}(\lambda)}{G_\lambda(\lambda)} \quad \text{and} \quad \rho_\lambda(\lambda) = \frac{G_{\lambda,\text{ref}}(\lambda)}{G_\lambda(\lambda)}} \tag{12.50}$$

where G_λ is the *monochromatic* irradiation (in W/m^2 μm) incident on the surface, and $G_{\lambda,\text{ abs}}$ and $G_{\lambda,\text{ ref}}$ are its reflected and absorbed components, respectively. In a similar fashion, for the transmissivity of *semi-transparent* materials, *the monochromatic hemispherical transmissivity* of a medium can be expressed as

$$\boxed{\tau_\lambda(\lambda) = \frac{G_{\lambda,\text{trans}}(\lambda)}{G_\lambda(\lambda)}} \tag{12.51}$$

The average (*total, hemispherical*) *absorptivity, reflectivity*, and *transmissivity* of a surface are the integrated effect of their monochromatic values and can be expressed as

$$\boxed{\alpha = \frac{\int_0^\infty \alpha_\lambda G_\lambda d\lambda}{\int_0^\infty G_\lambda d\lambda} \qquad \rho = \frac{\int_0^\infty \rho_\lambda G_\lambda d\lambda}{\int_0^\infty G_\lambda d\lambda} \qquad \tau = \frac{\int_0^\infty \tau_\lambda G_\lambda d\lambda}{\int_0^\infty G_\lambda d\lambda}} \tag{12.52}$$

The absorptivity of *aluminium increases* with the source temperature, a characteristic for metals, and the absorptivity of *non-conductors*, in general, decreases with temperature. This decrease is most obvious for surfaces which appear white to the eye. For example, the *absorptivity* of a white painted surface is *low* for *solar* radiation, but it is quite *high* for *infrared* radiation.

Emissivity depends also, on whether or not a material is an electrical conductor. Insulators usually have high values of emissivity, typically between 0.8 to 0.99. Metals, on the contrary, have low values of emissivity, varying from 0.001 to 0.7. Surface condition also affects significantly the emissivity, particularly in metals.

12.10.3 • Gray Surface

The effect of the gray approximation on emissivity and emissive power of a real surface is shown in Fig. 12.20 and 12.21. The radiation emission from a real surface, differs by and large, from the Planck distribution, and the emission curve may have several peaks and valleys. A gray surface must emit as much radiation as the real surface it represents at the same temperature. The areas under the emission curves of the real and gray surfaces must, therefore, be equal.

$$\boxed{\frac{\dot{Q}_{\text{emitted}}}{\text{A}} = E = \varepsilon\sigma T^4} \quad (\text{W/m}^2) \tag{12.53}$$

where E is the emissive power of a real surface.

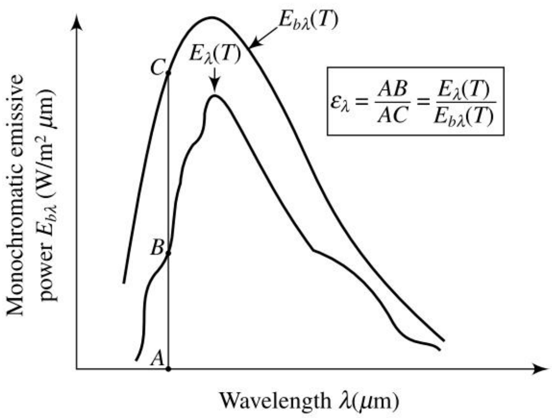


Fig. 12.20 *Definition of monochromatic emissivity ε_λ (The variations of E_λ and $E_{b\lambda}$ shown are for the same temperature)*

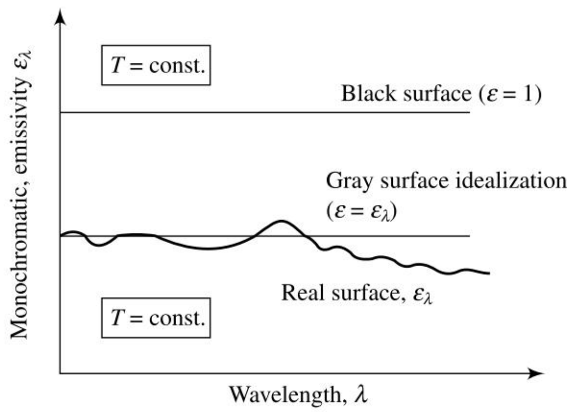


Fig. 12.21 *Typical variation of monochromatic emissivity with wavelength for a real surface and the 'gray surface' idealization*

Emissive power is a constant proportion of the black-body emissive power at any wavelength, i.e., its emissivity at every wavelength is the same. A surface which has this type of spectral distribution is called a *gray surface*. This is a useful concept, because it enables *non-black* surfaces to be simplified and analyzed without bothering about the details of their spectral emission bands. Many real surfaces do approximate to gray surfaces and the error involved in the simplification is often very small. It should be noted that the relationships derived for non-black surfaces are strictly applicable only to *gray* surfaces or, alternatively, to *monochromatic* radiation. Thus, a body whose emissivity is constant with wavelength is known as a gray body and many materials approximate to this ideal case.

A material whose monochromatic emissivity is not constant with wavelength, angle of emission, and surface temperature is called a *selective emitter*.

Analysis of radiation heat transfer is greatly simplified by assuming that surfaces are both gray and diffuse.

If ε_λ and α_λ do not vary with λ, the surface is said to be *gray*. When an opaque surface absorbs all incident radiation, nothing is reflected for the eye to see. Such surfaces are, therefore, called *black* surfaces or perfect absorbers. Perfect absorbers are naturally perfect emitters. An opaque surface that does not absorb any radiation will reflect all radiation. When our eye sees all radiation to which it is sensitive (*in relatively equal energy amounts*); the visual sensation is described as *white*. Hence, surfaces which absorb and reflect between these limits and are insensitive to wavelength are known as *gray surfaces*, since a

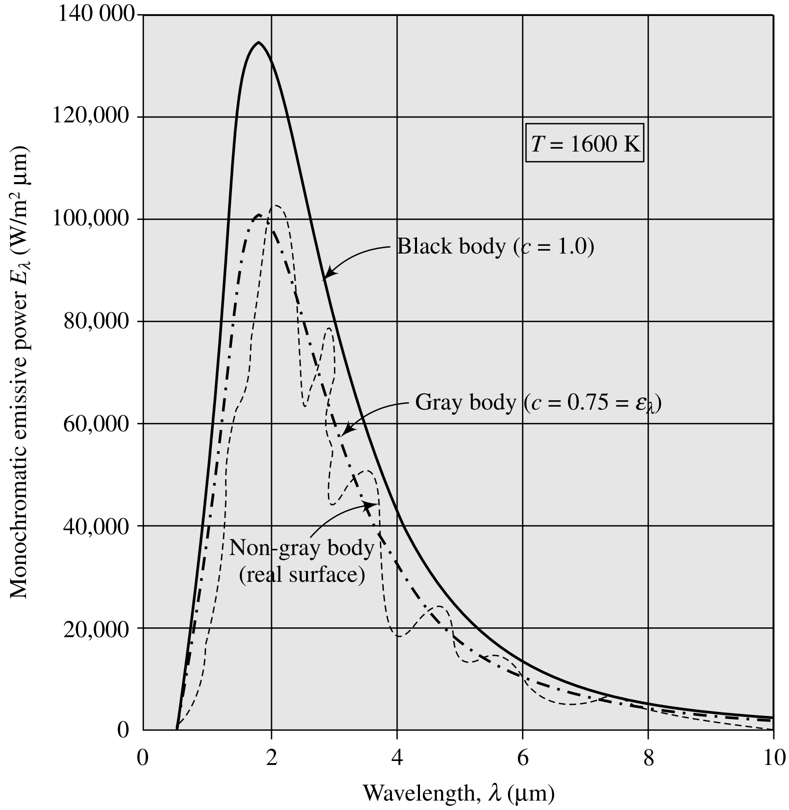


Fig. 12.22 *Monochromatic emissive power of a black body, a gray body, and a real (non-gray) body at constant temperature*

combination of black and white gives *gray!* The spectral behaviour of block, gray and real surfaces at 1600 K is shown in Fig. 12.22.

12.10.4 • Specular and Diffuse Surfaces

A body that reflects all the incident thermal radiation is known as a *specular surface*. A *specular* surface is a *highly polished* and *smooth* mirrorlike surface in which the angle of incidence is equal to the angle of reflection. On the other hand, a surface on which an incident beam is distributed uniformly in all directions after reflection, is called a *diffuse surface*.

Non-conductors show a relative insensitivity to polar angle, whereas *conductors* (*metals*) show a strong dependence on polar angles.

Surfaces having a roughness whose dimensions (for example mean pit depth) are *large* compared to the wavelength of the incident radiation reflect diffusively. On the other hand, surfaces whose roughness dimensions are considerably *smaller* than the wavelength of the radiation behave as specular reflectors. A diffuse surface can thus be defined as one whose intensity of emitted and reflected radiation is neither a function of polar (or zenith) angle θ nor of azimuthal angle ϕ. A diffuse gray surface obeys Lambert's law, i.e., the intensity of radiation leaving the surface does not depend on direction. This means that the emissivity and reflectivity are independent of angle. Fortunately, most of the real surfaces behave in an almost *diffuse* manner.

Emissivity and Absorptivity The *emissivity* of a surface (ε) is defined as the ratio of the emissive power of the surface to the emissive power of a hypothetical black body *at the same temperature.* According to Kirchhoff's identity, this ratio (E/E_b) *at the same temperature* also equals the absorptivity (α).

Emissivity is a property of the surface. It depends only on the nature or characteristics of the surface and is *independent of the nature or wavelength of the incident radiation*.

The fraction of incident radiation absorbed by a surface is called *absorptivity* (α).

It may be noted that unlike emissivity, the absorptivity of a material is *almost independent of the surface temperature*. However, the absorptivity does *depend strongly on the temperature of the source* of the incident radiation. For example, the absorptivity of the roof of a house is nearly 0.6 for solar radiation (*source temperature: 5779 K*) and 0.9 for radiation originating from the surrounding trees and buildings (*source temperature: 300 K*).

$$\boxed{\text{Emissivity, } \varepsilon = \frac{E}{E_b} = \frac{\int_0^\infty \varepsilon_\lambda(T) E_{b\lambda}(T)\, d\lambda}{\sigma T^4} = \varepsilon(T)} \tag{12.54}$$

$$\boxed{\text{Absorptivity, } \alpha = \frac{G_{\text{abs}}}{G} = \frac{\int_0^\infty \alpha_\lambda(T) G_\lambda(T^*)\, d\lambda}{G(T^*)} = \alpha(T, T^*)} \tag{12.55}$$

It is noteworthy that α depends on the *source temperature* T^* while ε is independent of T^* but a function of the *surface temperature T*. Hence, α and ε will usually be unequal.

Even when the emissivity does vary with wavelength, an average emissivity or absorptivity for the wavelength band in which the bulk of the radiation is emitted or absorbed is often sufficiently accurate for calculation.

The emissivity and absorptivity for a *non-gray* surface are not equal.

Selective Surfaces A selective surface is one in which radiation properties are manipulated to meet the specific requirement. *Selective surfaces* are non-gray surfaces which play an important role in *solar energy devices* and *thermal control of spacecraft systems*. In the case of solar collectors, the aim is to gather solar heat while in the spacecraft systems, the purpose is to keep the solar heat away. Hence, the name *selective surface*.

Higher emissivity (*and, hence, absorptivity*) up to 4 μm and low emissivity beyond 4 μm in the infrared (IR) region will lead to higher equilibrium surface temperature. Thus, typically for a selective surface with $\alpha_{\text{solar}} \approx 0.9$ and $\varepsilon_{\text{IR}} \approx 0.2$, $T_{\text{equil}} \approx 300°\text{C}$ for an incident solar flux of 1300 W/m^2. Hence, it is desirable to have a selective surface for solar heating application with a high (α_s/ε) ratio.

Lower emissivity up to 4 μm and higher emissivity beyond 4 μm ($T_{\text{equil}} \approx 140°\text{C}$ with $\alpha_{\text{solar}} \approx 0.11$ and $\varepsilon_{\text{IR}} = 0.9$ for the same solar flux of 1300 W/m^2) will make the surface cool while dissipating heat. Thus, a selective surface with lower (α_s/ε) ratio is desirable for spacecraft systems.

At very large differences (*like solar irradiation*) in temperatures compared to ambient temperatures, the absorptivity is vastly different from the emissivity. For example, for clean, smooth aluminium the absorptivity at $T = 24°\text{C}$ to solar radiation is ($T^* = 5500°\text{C}$) $\alpha = 0.45$ and the emissivity at 24°C is ($T = 24°\text{C}$) $\varepsilon = 0.09$ so that in this case, α is *five* times as large as ε !

The ratio (α_s/ε) is an important engineering parameter. Kirchhoff's law ($\alpha = \varepsilon$) is not applicable in this case.

12.10.5 • Real Surfaces: Directional Dependence

Real surfaces do not emit radiation in a perfectly diffuse manner and the variation of emissivity with direction for both *electrical conductors* and *non-conductors* is given in Fig. 12.23. Here, θ is the angle measured from the normal of the surface, and thus $\theta = 0°$ is for radiation emitted in a direction normal to the surface ε_θ remains almost constant for about $\theta < 40°$ for *conductors* such as *metals* and for $\theta < 70°$ for *non-conductors* such as *plastics*. The *directional emissivity* of a surface in the normal direction is representative of the hemispherical emissivity of the surface. The surfaces are generally assumed to be diffuse emitters with an emissivity equal to that in the normal ($\theta = 0$) direction.

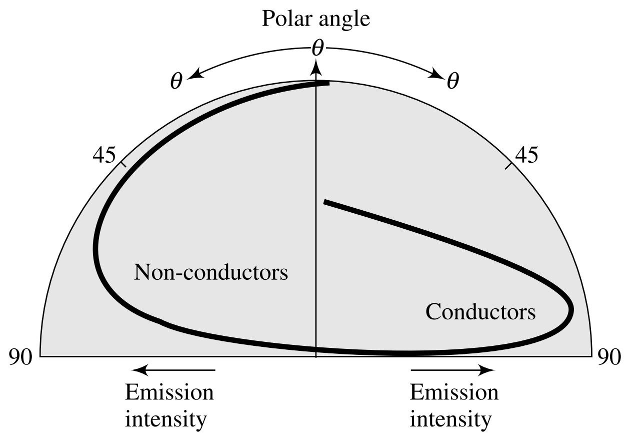


Fig. 12.23 *Relationship of radiation intensity of emission with polar angle for conductors and non-conductors*

The emissivities of some common materials are given in Table 12.3. The variation of emissivity with wavelength and temperature is indicated in Fig. 12.25. Typical ranges of emissivity of various materials are presented in Fig. 12.26. It may be noted that *metals* generally have *low emissivities*, as low as 0.02 for *polished surfaces*. *Non-metals* such as *ceramics* and *organic materials* have *high emissivities*. The emissivity of metals *increases* with temperature. Also, oxidation results in significant *increase* in the emissivity of metals. Heavily oxidized metals can have emissivities almost equal to those of non-metals. Most non-metallic substances have high emissivities and are usually considered gray. Radiation from electrical conductors, especially polished metals, is markedly different. Emissivities are much lower and vary considerably with wavelength (Fig. 12.23).

The radiation properties depend strongly on the surface conditions like *oxidation*, *roughness*, *type of finish*, and *cleanliness*. In general, dirt and oxidation considerably increase the emissivities of most surfaces, both dirt and oxides being poor conductors.

Over small temperature ranges, emissivities do not vary much. When considering the absorption of radiation from a high-temperature source to a much cooler surface, it is usually necessary to use an emissivity corresponding to the lower temperature and an absorptivity corresponding to the high temperature.

Table 12.3 *Emissivity values of some common materials*

Material	Emissivity	Material	Emissivity
Aluminium, polished oxidized	0.0095 0.20	White epoxy paint	0.88
Aluminium foil	0.04	White acrylic paint	0.92
Brass, polished dull plate	0.03 0.60	Snow	0.82
Copper, polished oxidized	0.02 0.80	Water	0.96
Iron, polished oxidized	0.20 0.70	Concrete	0.94
Steel, polished galvanized oxidized	0.07 0.30 0.80	White marble	0.95
Black enamel paint	0.81	Rubber	0.92
Stainless steels	0.2 to 0.7	Silver, polished	0.01 0.03
Alumina	0.40	Gold, polished	0.018
Asbestos	0.95	Tin, polished	0.05
Brick, red building fireclay	0.93 0.45 0.75	White paper	0.97
Glass	0.95	Wood	0.94
Silica	0.4		

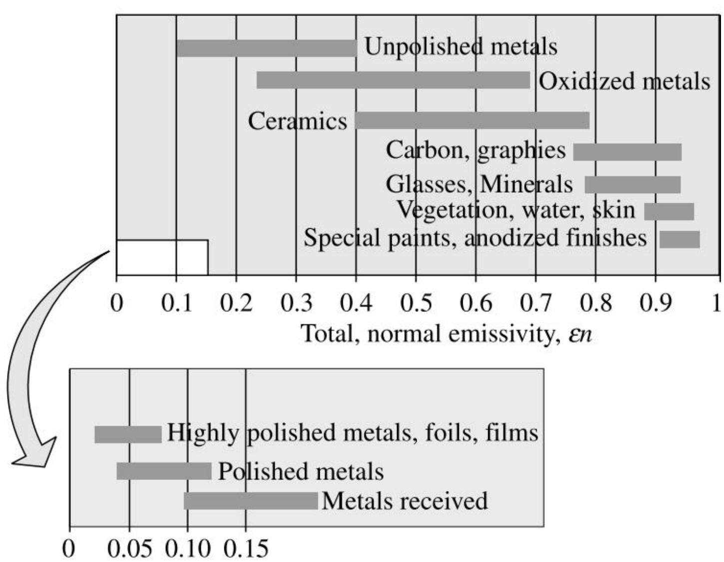


Fig. 12.24 *Representative ranges of emissivity of selected materials*

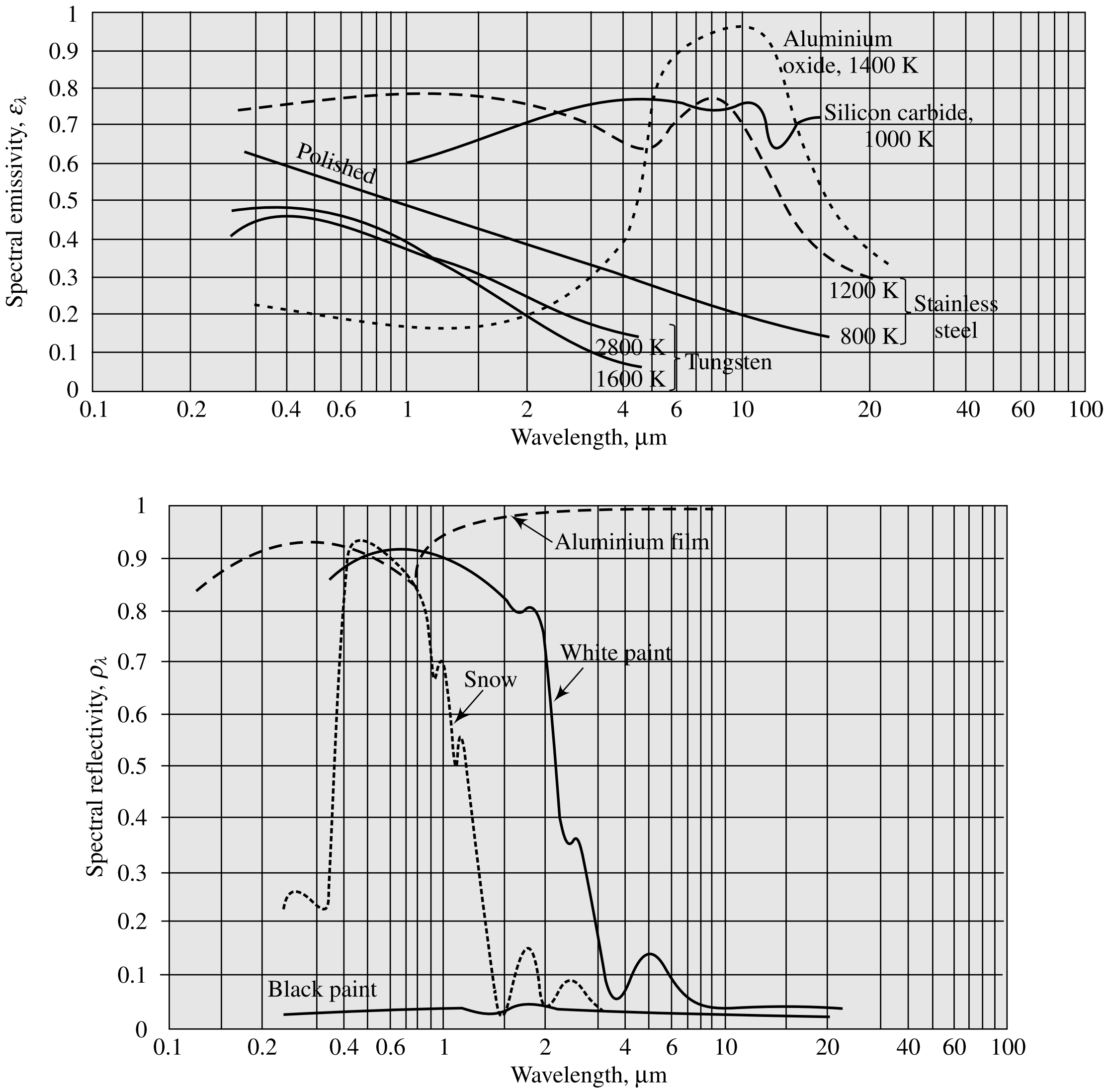


Fig. 12.25 *Spectral emissivity of some selected materials*

Furthermore, the emissivity of many surfaces varies with the angle of emission. Not much experimental data on the directional variation of emissivity is available. Generally, we take $\varepsilon/\varepsilon_n = 1.2$ for low-emissivity polished *metallic surfaces* and $\varepsilon/\varepsilon_n = 0.96$ for high-emissivity *non-metallic surfaces*, where ε is the average emissivity throughout the hemispherical solid angle of 2π steradians and ε_n is the emissivity in the direction normal to the surface.

Figure 12.24 shows some simplified spectral *reflectivity* curves for some selected opaque surfaces. Note that many surfaces such as *white paint* or *aluminium paint* demonstrate snowlike behaviour, but not as marked. In these cases, high reflectivity is spread over a *wider* band of wavelengths.

Figure 12.25 shows the *spectral transmissivity* of some types of glasses. Note that in each case, the high value of transmissivity occurs for low wavelengths, and in each case, τ_λ drops to almost zero very sharply at a specific wavelength. This property is responsible for the *greenhouse effect*, where the temperature within a glass enclosure for plants is higher than the ambient. The solar radiation at the higher radiation temperature is dominated by shorter wavelength, which pass into the enclosure across the glass walls. But the radiation from the plants, being at a lower *colour temperature*, is dominated by higher wavelengths, and is blocked by the glass. This results in retention of heat within the enclosure, giving rise to the hot house effect.

A layer of carbon dioxide has similar properties, and hence the ever-increasing concentration of CO_2 in the atmosphere due to the increased energy consumption in the world is producing the so-called *greenhouse effect*. This is raising the atmospheric temperature, and is threatening to *melt the polar ice caps, raising the level of oceans and submerging* vast tracts of lands.

Figure 12.26 shows the variation of spectral emissivity with wavelength for same selected materials.

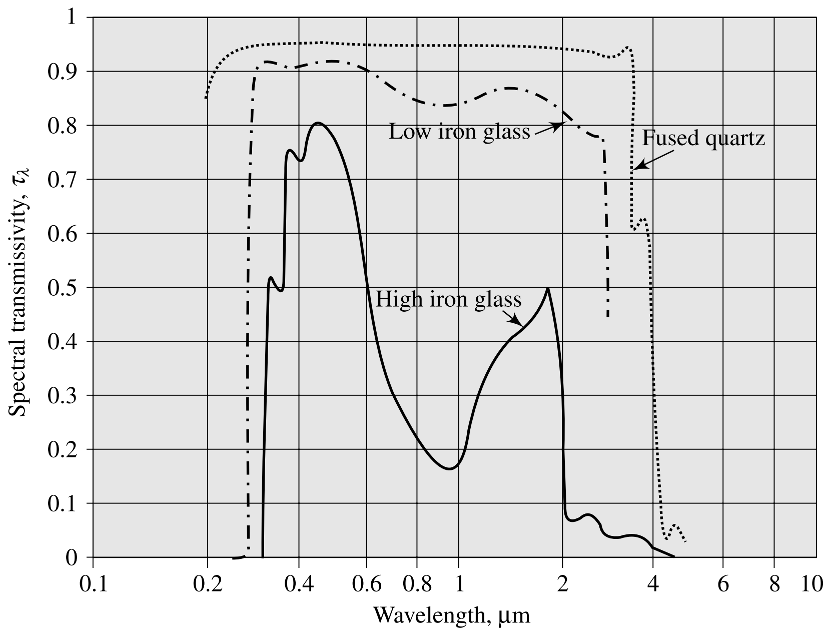


Fig. 12.26 *Spectral transmissivity of some selected materials*

12.11 ❑ SOLAR AND ENVIRONMENTAL RADIATION

The energy emitted by the sun and incident on the earth's surface is known as *solar radiation*. The sun is an almost spherical body of diameter $D = 1.39 \times 10^6$ km and is located at a mean distance of $L = 1.495 \times 10^8$ km from the earth. Even though the sun radiates an enormous amount of energy, ($E_{sun} \approx 3.81 \times 10^{26}$ W) only less than a billionth of this energy ($\approx 1.7 \times 10^{17}$ W) reaches the earth's surface. Solar radiation travels through the vacuum of space till it encounters earth's atmosphere. The average value of solar energy reaching the upper surface of the earth's atmosphere is about 1367 W/m^2. This value is known as ***solar constant***, $\boldsymbol{G_s}$ which is defined as the rate at which solar energy is incident on a surface normal to the sun's rays at the outer edge of the atmosphere when the earth is at its mean distance from the sun.

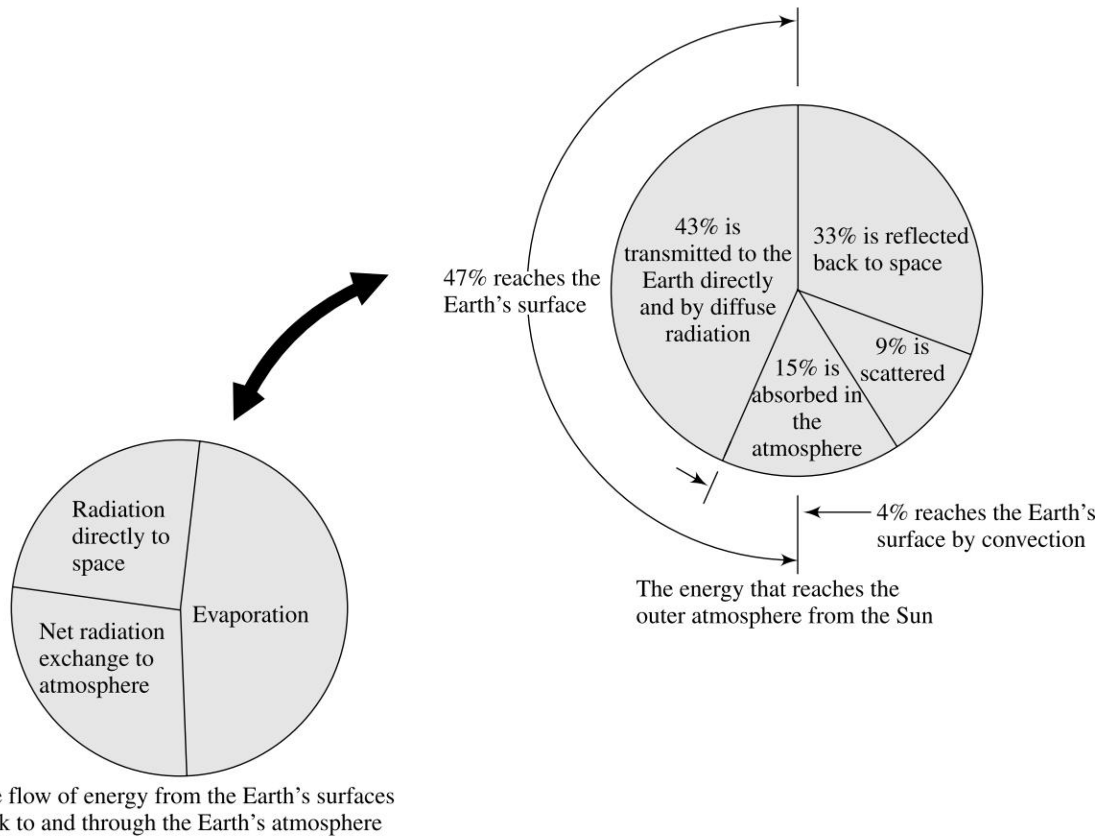


Fig. 12.27 *Approximate distribution of the flow of solar energy to and from the earth's surface*

As the earth moves in an elliptical orbit around the sun, the mean sun–earth distance (L) varies with the position of the earth. The value of G_S also varies, as a result. Constituents of the atmosphere absorb and/or scatter radiation of different wavelengths contained in solar radiation. As a result, the amount of solar energy actually reaching the earth's surface is consideratly reduced.

The approximate break-up of solar energy to and from the earth's surface is given as follows (Fig. 12.27).

- 9% is scattered.
- 15% is absorbed in the atmosphere and out of it, 4% reaches the earth's surface by convection.
- 43% is transmitted to the earth directly and by diffuse radiation.
- 33% is reflected back to space.

With so much clean, safe and reliable solar energy falling on all parts of the world today, the spotlight today is on sunlight.

The spectral distribution of solar radiation shows that the sun behaves almost like a black body. Hence, for engineering calculations, the sun is assumed to be a black body at an effective surface temperature of 5779 K.

The solar radiation has to first penetrate the earth's atmospheric layer in order to reach the earth's surface. For a horizontal surface outside the earth's atmosphere, the solar radiation appears as a beam

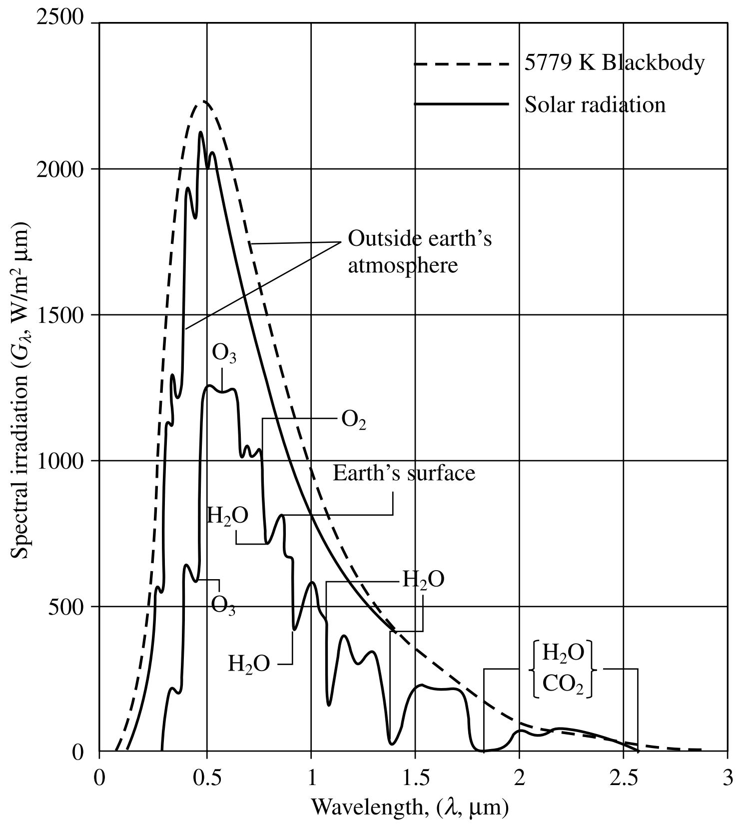


Fig. 12.28 *Spectral distribution of solar radiation outside the earth's atmosphere and on ground under a clear atmosphere.*

of nearly parallel rays. The spectral distribution of solar radiation on the ground as compared to that of extraterrestrial solar radiation is shown in Fig. 12.28. Clearly, there is a significant reduction of solar radiation as it passes through the earth's atmosphere due to absorption and scattering as illustrated in Fig. 12.29.

Usually, about 950 W/m^2 of solar energy reaches the earth's surface on a sunny day and much less on a cloudy day. Almost all the solar radiation reaching the earth falls in the wavelength range of 0.3 to 2.5 µm.

The *absorption* of solar radiation is mainly by O_3 (ozone), H_2O, O_2, and CO_2.

The *scattering* of solar radiation occurs because of scattering by the gas molecules (known as *Rayleigh scattering*) and scattering by the dust and aerosol particles of the atmosphere (known as *Mie scattering*). The portion of radiation that has penetrated the atmosphere without having been scattered is termed *direct radiation*. The remaining radiation (*i.e., scattered*) is known as *diffuse radiation*. Hence, the total solar radiation reaching the earth's surface is the sum of *direct* and *diffuse* radiation (Fig. 12.29). The diffuse radiation may vary from 10% of total radiation on a clear day to nearly 100% on a cloudy day.

Solar energy incident on the earth's surface consists of two parts: *direct or beam solar radiation,* G_D (*which reaches the surface without any attenuation in the atmosphere*) and *diffuse solar radiation* G_d (*scattered radiation reaching the earth uniformly from all directions*).

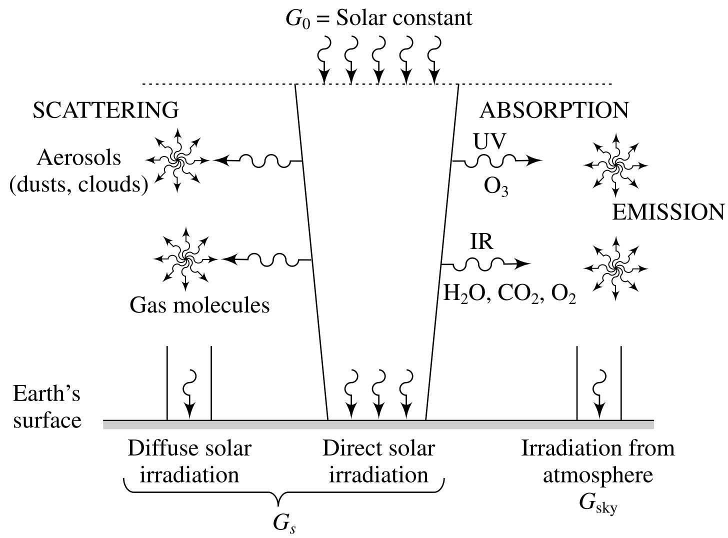


Fig. 12.29 *Absorption and scattering of solar radiation in the atmosphere*

Then, the *total* solar energy incident on a *horizontal surface* per unit area on the ground is

$$\boxed{G_{solar} = G_D \cos\theta + G_d} \quad (\mathrm{W/m^2}) \tag{12.56}$$

where θ is the angle between the sun's rays and the normal to the surface.
The extraterrestrial solar irradiation $G_{S,0}$ depends on the angle θ between the incident rays from the sun and a normal to the surface of the earth. Hence,

$$\boxed{G_{s,0} = f G_S \cos\theta} \tag{12.57}$$

where f is a correction factor to account for the eccentricity of the earth's orbit around the sun ($0.97 \le f \le 1.03$). G_S is the solar constant which can be calculated from $E_{sun} = \sigma T_{sun}^4$, and since the intensity of radiation obeys the inverse square law

$$G_S = \left(\frac{r_{sun}}{r_{orbit}}\right)^2 E_{sun} = \left(\frac{r}{L}\right)^2 (\sigma T_{sun}^4)$$

$$= \left(\frac{0.695 \times 10^9}{1.495 \times 10^{11}}\right)^2 (5.67 \times 10^{-8}\ \mathrm{W/m^2\,K^4})(5779^4\ \mathrm{K^4}) = 1367\ \mathrm{W/m^2}$$

An important consideration when dealing with solar radiation to and from surfaces at temperatures significantly different from the source of radiation is that $\varepsilon \neq \alpha$. Consider a surface absorbing radiation from the sun which is known to approximate a black body at 5779 K. The surface also emits radiant energy but at a much lower temperature. This difference in temperatures is large enough for there to be a significant difference in the respective wavelengths of the absorbed and emitted radiation: $\lambda_{absorbed} << \lambda_{emitted}$ and, hence, $\alpha \neq \varepsilon$. This may be summarized by the rule that α *sees* T_{source}, while ε *sees* T_{sink}. The absorptivity of a surface designed to receive radiation from the sun is usually referred to as the *solar absorptivity* α_S.

Table 12.4 lists the values of solar absorptivity, α_S and emissivity ε (at 300 K) for some selective materials. Obviously, solar collectors, widely used in solar energy applications, must be made of materials having higher value of α_S/ε.

Table 12.4 *Solar absorptivity (α_s) and emissivity (ε) of some selective surfaces*

Surface	α_s	ε(300 K)	α_s/ε
Aluminium			
Polished	0.09	0.03	3.0
Anodized	0.14	0.84	0.17
Foil	0.15	0.05	3.0
Copper			
Polished	0.18	0.03	6.0
Tarnished	0.65	0.75	0.87
Stainless steel			
Polished	0.37	0.60	0.62
Dull	0.50	0.21	2.4
Concrete	0.60	0.88	0.68
White marble	0.46	0.95	0.48
Red brick	0.63	0.93	0.68
Asphalt	0.90	0.90	1.0
Black paint	0.97	0.97	1.0
White paint	0.21	0.96	0.22
Snow	0.28	0.97	0.29
Human skin	0.62	0.97	0.64

12.11.1 • Environmental Radiation

The environmental radiation includes both the emission from the earth's surface, and the emission from the atmosphere.

Atmospheric molecules, especially CO_2 and H_2O vapour, not only absorb solar radiation but also absorb radiation from the earth's surface. The radiation emission from ground is usually from sources at 250 to 320 K which produce radiation of wavelengths from nearly 4 to 40 μm with a maximum of about 10 μm. Environmental emission and absorption is in the range of wavelengths of 5 to 8 μm and above 13 μm, and this radiation is not distributed like the black-body radiation. However, it is possible to approximate the emission from the atmosphere or sky as a fraction of black-body radiation corresponding to the air temperature on the ground.

$$\boxed{J_{\text{sky}} = \varepsilon_{\text{sky}}\, \sigma T^4} \tag{12.58}$$

Emissive power associated with the earth's surface is given by

$$\boxed{E_{\text{earth}} = \varepsilon \sigma T_s^4} \tag{12.59}$$

where ε = surface emissivity (≈ 1)

T_s = surface temperature ($250\text{ K} \le T < 320\text{ K}$)

The main constituents contributing to the *atmospheric radiation* are CO_2 and H_2O molecules.

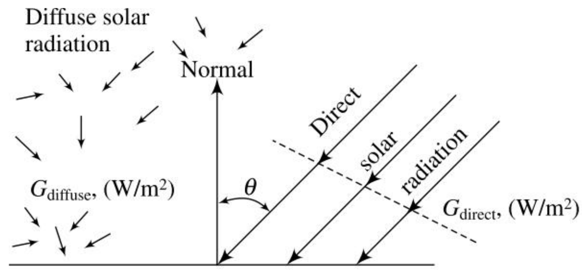


Fig. 12.30 *Direct and diffuse solar radiation incident on the horizontal earth's surface*

Effective sky temperature, T_{sky} is calculated assuming the atmosphere to be a black body, i.e.,

$$\boxed{G_{sky} = \sigma T_{sky}^4} \quad (\mathrm{W/m^2}) \tag{12.60}$$

The value of T_{sky} varies from 230 K to 285 K, depending on the atmospheric conditions.
Sky radiation absorbed by the earth's surface can be expressed as:

$$E_{sky,absorbed} = \alpha G_{sky} = \alpha \sigma T_{sky}^4 = \varepsilon \sigma T_{sky}^4 \quad (\mathrm{W/m^2})$$

When the surface at a temperature T_s is exposed to both solar and atmospheric radiation, the net rate of heat transfer to the surface is found from the energy balance as follows:

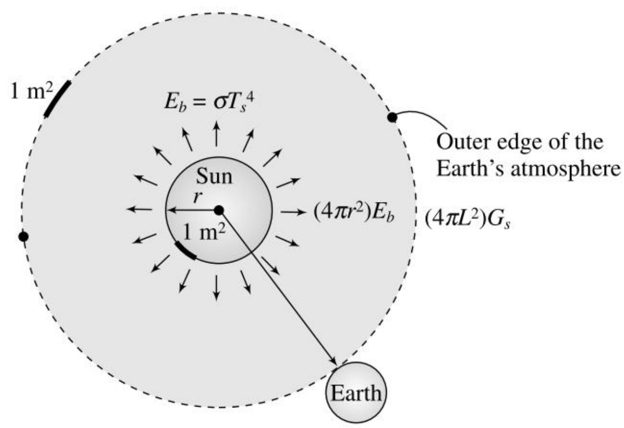


Figure 12.31 *The total solar energy passing through concentric spheres remains constant, but the energy falling per unit area decreases with increasing radius.*

$$\boxed{\begin{aligned} q_{net,rad} &= \Sigma E_{absorbed} - \Sigma E_{emitted} \\ &= (\alpha_s G_{solar} + \varepsilon\sigma T_{sky}^4) - \varepsilon\sigma T_s^4 = \alpha_s G_{solar} + \varepsilon\sigma(T_{sky}^4 - T_s^4) \end{aligned}} \quad (\mathrm{W/m^2}) \tag{12.61}$$

12.11.2 • Greenhouse Effect

Typically, a *greenhouse* is a closed chamber made up of glass or plastic sheets.
In a greenhouse, window glass transmits radiation in the range of wavelengths from about 0.15 to 3 μm as shown in Fig. 12.31. It is almost opaque to infrared radiation of longer wavelengths; i.e., above 3 μm which is absorbed or reflected. Most of the radiation which reaches the earth from the sun is within this

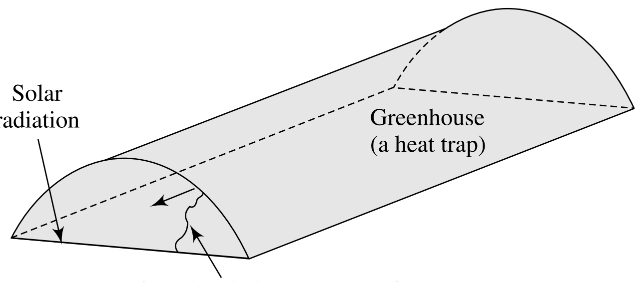


Figure 12.32 *A greenhouse traps energy by permitting the solar radiation to enter but it is practically opaque to infrared radiation*

range and solar radiation is, therefore, passed through the glass to the plants and soil in the greenhouse. On the other hand, the earth's surface radiates mainly in the longer wavelengths and re-radiation from it to the surroundings is unable to pass through the glass. The heat transferred by solar radiation is, therefore, trapped in the greenhouse and causes the temperature to be higher than the external surroundings. The steady-state temperature in the greenhouse is dictated by an equilibrium between the radiant heat entering and the heat loss to the surroundings. You might have experienced the heating effect due to accumulation of energy inside a closed compartment of a car, which is exposed under direct sunlight. The ozone layer along with the CO_2 and H_2O vapour surrounding the earth also exhibits a similar behaviour as that of the glass. This results in global warming of the earth's atmosphere.

The average global temperature has risen by about 0.3 to 0.6°C over the past 100 years, and that the rate of rise is increasing alarmingly. Even a small increase in the average global temperature may lead to catastrophic consequences. Global warming (*or a global warning*!) is due to what is popularly called *greenhouse effect*. Reduced consumption of energy and enhanced efficiency of energy utilization have been identified as the feasible strategies for the reduction of emission of greenhouse gases, particularly of carbon dioxide, thereby mitigating the adverse effects of global warming.

We must, therefore, recognize that in solar energy applications, the radiation properties of surfaces are appreciably different for incident solar radiation and surface emission. Solar radiation is concentrated in the short wavelength region but the emitted radiation is in the longer wavelength infra red region. Clearly, such surfaces cannot be approximated as gray. We, therefore, require two sets of properties: solar absorptivity α_S and emissivity ε at moderate temperatures. The ratio α_s/ε is a significant parameter. Large values of α_S/ε are desirable if we intend to collect solar energy (as in solar collectors) while smaller values are preferred if our intention is to reject heat to keep the surface cool under the sun.

Illustrative Examples

(A) Black-body Radiation

EXAMPLE 12.1

(a) Calculate the wavelength of the telephone waves if a cordless telephone is designed for a frequency of 850 MHz.

(b) Determine the frequency of the radio waves which are broadcast from a radio station at a wavelength of 200 m.

(c) A microwave oven is designed to operate at a frequency of 2.2 GHz. Find the wavelength of these microwaves and the energy of each microwave.

(d) Generation and transmission of electricity in power lines take place at a frequency of 50 hertz (1 hertz is 1 cycle per second) in India. What is the wavelength of the electromagnetic waves generated by the passage of electricity in the power lines?

Solution

Known Telephone waves, radio waves, microwaves and electromagnetic waves.

Find (a) $\lambda(\mu m)$ for $\nu = 850$ MHz; (b) ν for $\lambda = 200$ m; (c) λ and E for $\nu = 2.2$ GHz; (d) λ for $\nu = 50$ Hz.

Analysis (a) Wavelength of telephone waves is

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8 \text{ m/s}}{850 \times 10^6 \text{ Hz}} = \mathbf{0.353\ m} \quad (\textit{1 Hz = 1 cycle/s}) \qquad \textbf{(Ans.) (a)}$$

where c is the velocity of light in vacuum.

(b) Frequency of radio waves,

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^8 \text{ m/s}}{200 \text{ m}} = 1.5 \times 10^6 \text{ Hz} = \mathbf{1.5\ MHz} \qquad \textbf{(Ans.) (b)}$$

(c) Wavelength of the microwaves,

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8 \text{ m/s}}{2.2 \times 10^9 \text{ cycles/s}} = \mathbf{0.136\ m} \qquad \textbf{(Ans.) (c)}$$

Energy of each microwave, $E = h\nu$

where $h = 6.626069 \times 10^{-34}$ (J s) is *Planck's constant.*

$$\therefore \quad E = (6.626069 \times 10^{-34} (\text{J s}))(2.2 \times 10^9 \text{ cycles/s}) = \mathbf{1.458 \times 10^{-24}\ J} \qquad \textbf{(Ans.) (c)}$$

(d) Wavelength of electromagnetic waves,

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8 \text{ m/s}}{50 \text{ Hz}} = \mathbf{6 \times 10^6\ m} \qquad \textbf{(Ans.) (d)}$$

EXAMPLE 12.2 *A large industrial furnace can be approximated as a black body with a uniform surface temperature of 2300 K. Determine (a) the monochromatic emissive power at a wavelength of 1 μm, (b) the total (hemispherical) emissive power, (c) the wavelength and magnitude of the maximum monochormatic emissive power, (d) the fraction of the total emission which occurs between the wavelength of 2.0 and 6.0 μm, (e) the percentage reduction in the total emissive power when the temperature falls to 1800 K, and (f) the wavelength λ such that the emission from 0 to λ equals the emission from λ to ∞.*

Solution

Known Black body at a specified temperature.

Find (a) $E_{b\lambda}$ at $\lambda = 1$ μm, (b) E_b, (c) $E_{b\lambda,\max}$, and $\lambda_{\max}$, (d) $f_{(2 \to 6\ \mu m)}$, (e) Percent change in E_b if $T = 1800$ K, (f) the value of λ such that the emission from 0 to λ equals that from λ to ∞.

Assumption Diffuse black surface.

Analysis (a) Monochromatic emissive power at 1 μm is

$$E_{b\lambda} = \frac{C_1\lambda^{-5}}{\exp(C_2/\lambda T)-1} = \frac{3.74177\times10^8\times(1)^{-5}}{\exp\left[\dfrac{14387.8}{1\times2300}\right]-1} = \mathbf{719.7\ kW/m^2\ \mu m} \qquad \textbf{(Ans.) (a)}$$

(b) Total emissive power

$$E_b = \sigma T^4 = (5.67\times10^{-8}\ \text{W/m}^2\ \text{K}^4)(2300\ \text{K})^4$$

$$= 5.67(23)^4 = 1.587\times10^6\ \text{W/m}^2 \quad \text{or} \quad \mathbf{1587\ kW/m^2} \qquad \textbf{(Ans.) (b)}$$

(c) Wavelength of maximum monochromatic emissive power is

$$\lambda_{\max} = \frac{2897.8}{2300} = \mathbf{1.26\ \mu m} \qquad \textbf{(Ans.) (c)}$$

We note that

$$\frac{E_{b\lambda}}{T^5} = 1.287\times10^{-11}\ \text{W/m}^2\ \mu\text{m}$$

$$E_{b\lambda,\max} = (1.287\times10^{-11})(2300)^5 = \mathbf{828.36\ kW/m^2\ \mu m} \qquad \textbf{(Ans.) (c)}$$

(d) From Table 12.1, $\lambda_1 T = 2\times2300 = 4600\ \mu\text{m K} \Rightarrow f_{(0-\lambda_1)} = 0.57926$

$$\lambda_1 T = 6\times2300 = 13800 \quad\Rightarrow\quad f_{(0-\lambda_1)} = 0.96144$$

The fraction of total emission occurring between λ_1 = 2 μm and λ_2 = 6 μm is

$$f_{(\lambda_1\to\lambda_2)} = f_{(0\to\lambda_2)} - f_{(\lambda\to\lambda_1)} = 0.9614 - 0.57926 = \mathbf{0.382} \qquad \textbf{(Ans.) (d)}$$

(e) Percentage reduction in total emissive power when the temperature falls to 1800 K

$$= \frac{E_b(2300\ \text{K}) - E_b(1800\ \text{K})}{E_b(2300\ \text{K})}\times100 = 1 - \frac{E_b(1800)}{E_b(2300)} = \left[1-\left(\frac{1800}{2300}\right)^4\right]\times100$$

$$= \mathbf{62.5\%} \qquad \textbf{(Ans.) (e)}$$

(f) Since emissive power from 0 to λ is same as that from λ to ∞ at the given temperature T = 2300 K, $f_{0\to\lambda}$ = 0.5. From Table 12.1, λT (corresponding to f = 0.5 at 2300 K) =

$$\left(\frac{0.50000-0.49873}{0.51600-0.49873}\right)(4200-4100)+4100 = 4107.35\ \mu\text{m K}.$$

Hence, λ = 4107.35/2300 = **1.786 μm** **(Ans.) (f)**

EXAMPLE 12.3 *What is the rate of emission at which a surface idealized as a black body of area A = 4 cm² at 2000 K emits radiation in directions corresponding to 0° ≤ θ ≤ 60° and in the wavelength band of 2.5 μm ≤ λ ≤ 5 μm?*

Solution

Known Temperature of a surface which emits as a black body.

Find Rate of emitted energy for $0° < \theta < 60°$ and $2.5\ \mu m \le \lambda \le 5\ \mu m$.

Assumptions Surface is a black body that emits diffusely.

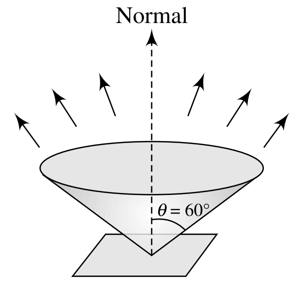
Analysis The desired emission by the black body with the limits of integration imposed as follows is given by

$$\Delta E = \int_{2.5}^{5}\int_{0}^{2\pi}\int_{0}^{\pi/3} I_{b\lambda}\cos\theta\sin\theta\, d\theta\, d\phi\, d\lambda$$

$$= \int_{2.5}^{5} I_{b\lambda}\left\{\int_{0}^{2\pi}\int_{0}^{\pi/3}\cos\theta\sin\theta\, d\theta\, d\phi\right\}d\lambda$$

$$= \int_{2.5}^{5} I_{b\lambda}\left(2\pi\frac{\sin^2\theta}{2}\Bigg|_{0}^{\pi/3}\right)d\lambda$$

Schematic



Black body at 2000 K, $A = 4\ cm^2$

Directional restriction:
$0° \le \theta \le 60°$ or $\pi/3$ rad
Spectral restriction:
$2.5\ \mu m \le \lambda \le 5\ \mu m$

Noting that $\int_{0}^{2\pi} d\phi = 2\pi$, $\int 2\sin\theta\cos\theta\, d\theta = \sin^2\theta$, and $E_{b\lambda} = I_{b\lambda}\pi$, we have

$$\Delta E = \int_{2.5}^{5}(\pi I_{b\lambda})\{(\sin^2 60° - 0)\}d\lambda = 0.75\int_{2.5}^{5} E_{b\lambda}d\lambda$$

$$= 0.75\, E_b\left[\frac{\int_{2.5}^{5} E_{b\lambda}d\lambda}{E_b}\right] = 0.75\, E_b\,\frac{\int_{0}^{5} E_{b\lambda}d\lambda - \int_{0}^{2.5} E_{b\lambda}d\lambda}{\sigma T^4}$$

$$= 0.75\, E_b[f_{(0\to5)} - f_{(0\to2.5)}] = 0.75(\sigma T^4)[f_{(0\to5)} - f_{(0\to2.5)}]$$

From ***Table 12.1***, for

$$\lambda_1 T = 2.5\ \mu m \times 2000\ K = 5000\ \mu m\,K \quad f_{(0\to2.5)} = 0.634$$

$$\lambda_2 T = 5\ \mu m \times 2000\ K = 10\,000\ \mu m\ K \quad f_{(0\to5)} = 0.914$$

Hence, substituting the values, we get

$$\Delta E = 0.75 \times 5.67 \times 10^{-8}\ W/m^2\,K^4 \times (2000\ K)^4\,(0.914 - 0.634)$$

$$= \mathbf{190.8 \times 10^3\ W/m^2} \qquad \textbf{(Ans.)}$$

$\therefore$ Rate of emission

$$= (190.8 \times 10^3\ W/m^2)(4 \times 10^{-4}\ m^2) = \mathbf{76.3\ W} \qquad \textbf{(Ans.)}$$

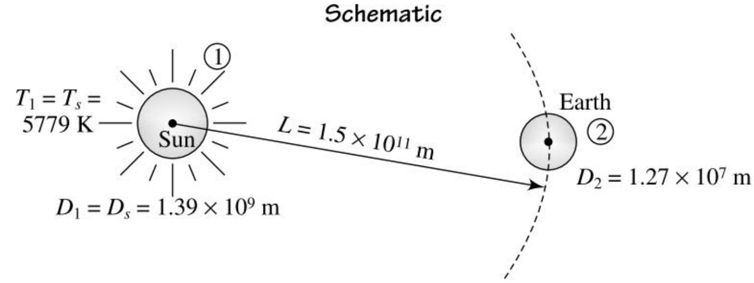
EXAMPLE 12.4 *Approximating the sun's surface as black and an equivalent black body temperature of 5779 K, (a) estimate the rate at which the sun emits radiant energy. (b) What fraction of this energy is intercepted by the earth, and (c) what is the amount intercepted? Given: Diameter of the sun = 1.39 × 10⁹ m, Diameter of the earth = 1.27 × 10⁷ m, Distance between the sun and the earth = 1.5 × 10¹¹ m.*

Solution

Known Radiation emitted by the sun is intercepted by the earth.

Find (a) Solar radiation emitted. (b) Fraction intercepted by the earth. (c) Intercepted radiation by the earth.

Schematic



Assumptions (1) The sun is a black body. (2) Both the sun and the earth are spheres.

Analysis (a) Rate at which the sun emits energy is

$$\dot{Q}_{\text{emit,sun}} = A_s \sigma T_s^4 = (\pi D_s^2)\sigma T_s^4 = \pi \times (1.39 \times 10^9 \text{ m})^2 (5.67 \times 10^{-8} \text{ W/m}^2\text{ K}^4)(5779 \text{ K})^4$$

$$= \mathbf{3.84 \times 10^{26}\ W} \qquad \textbf{(Ans.) (a)}$$

(b) Radiation from the sun (1) spreads out evenly in all directions and we imagine that it falls on the inside of a hollow sphere of radius L (the earth–sun distance). It follows that the proportion of the total radiation falling on the earth (2) from the sun (1) is the ratio of the *projected* area of the earth (2) (which will be a circle of diameter D_2) to the *total* area of the sphere (of radius L). Then, the fraction of radiation emitted by the sun and intercepted by the earth is

$$F_{S-E} = F_{1-2} = \frac{(\pi D_2^2/4)}{4\pi L^2} = \frac{D_2^2}{16L^2} = \left(\frac{D_2}{4L}\right)^2 = \left[\frac{1.27 \times 10^7 \text{ m}}{4 \times 1.5 \times 10^{11} \text{ m}}\right]^2 = \left(\frac{1.27}{6 \times 10^4}\right)^2$$

$$= \mathbf{4.48 \times 10^{-10}} \qquad \textbf{(Ans.) (b)}$$

(c) The amount of energy from the sun intercepted by the earth is

$$\dot{Q}_{\text{recd,earth}} = \dot{Q}_{\text{emit,sun}} \times F_{12} = (3.84 \times 10^{26} \text{ W})(4.48 \times 10^{-10}) = \mathbf{1.72 \times 10^{17}\ W} \quad \textbf{(Ans.) (c)}$$

EXAMPLE 12.5 *The directional distribution of the solar radiation incident on the earth's surface is approximately given by* $I_{inc} = I_n \cos\theta$, *where* θ *is the zenith angle and* $I_n = 85$ *W/m² sr is the total intensity of radiation normal to the surface. Determine the solar irradiation on the earth's surface.*

Solution

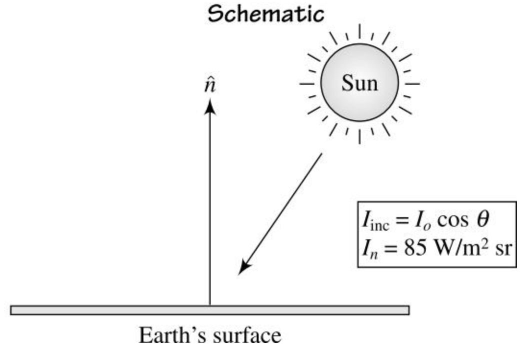
Known Directional distribution of solar radiation incident on earth surface.

Find Solar irradiation on the earth surface.

Assumptions Intensity is independent of azimuth angle ϕ.

Analysis The solar radiation flux incident on the earth's surface from all directions, that is, solar irradiation can be expressed as

Schematic



$$G = \int_{\text{hemisphere}} dG = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{\text{inc}}(\theta)\cos\theta\, d\theta\, d\phi (\text{W/m}^2)$$

$$= \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi/2} I_n \cos\theta \cdot \cos\theta \sin\theta\, d\theta = 2\pi I_n \int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta$$

Noting that $\frac{d}{d\theta}(\cos^3\theta) = 3\cos^2\theta(-\sin\theta)$,

$$G = 2\pi I_n \left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{2}{3}\pi I_n[\cos^3 0 - \cos^3 \pi/2]$$

$$= \frac{2}{3}\pi I_n(1-0) = \frac{2}{3}\pi \times 85 = \mathbf{178.0\ W/m^2}$$ **(Ans.)**

(B) Radiation Properties

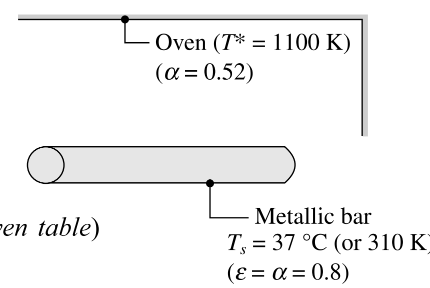
EXAMPLE 12.6 *A metallic bar at 37°C is placed inside an oven whose interior is maintained at a temperature of 1100 K. The absorptivity of the bar (at 37°C) is a function of the temperature of incident radiation and a few representative values are given below:*

Temperature (K)	*310*	*700*	*1100*
α	*0.8*	*0.68*	*0.52*

Estimate the rate of absorption and emission by the metallic bar. *[AMIE S 2007]*

Solution

Schematic



Known A metallic bar is placed inside an oven.

Find Rate of absorption and emission.

Assumptions (1) Constant surface temperature. (2) Diffuse gray surface.

Analysis We note that the emissivity of the bar is at its surface temperature,

T_s = 37°C or 310 K (*from the given table*)

Hence, $\varepsilon = \alpha$(at 310 K) = 0.8

The rate of emission from the metallic bar,

$$\frac{\dot{Q}_{\text{emit}}}{A} = \varepsilon\sigma T^4 = 0.8 \times 5.67 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4 \times (310\ \text{K})^4$$

$$= \mathbf{418.9\ W/m^2 \approx 0.42\ kW/m^2}$$ **(Ans.)**

Significantly, the absorptivity is always taken at the radiation source (*oven*) temperature (T^* = 1100 K). Hence,

α (at 1100 K) = 0.52 (*from the given table*)

The rate of absorption by the metallic bar,

$$\frac{\dot{Q}_{\text{abs}}}{A} = \alpha\sigma T^{*4} = 0.52 \times 5.67 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4 \times (1100\ \text{K})^4$$

$$= 43.17 \times 10^3\ \text{W/m}^2 \approx \mathbf{43.2\ kW/m^2}$$ **(Ans.)**

EXAMPLE 12.7 *Determine the ratio of the total hemispherical emissivity to the normal emissivity for a non-diffuse surface if the intensity of emission varies as the cosine of the angle measured from the normal.*

Solution

Known In a non-diffuse surface, the intensity of emission varies as the cosine of angle measured from normal.

Find Ratio of total emissivity to normal emissivity.

Assumption The surface is opaque, isothermal, and gray.

Analysis Total hemispherical emittance, $E_N = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I \cos\theta \sin\theta \, d\theta \, d\phi$

where θ is the *zenith angle* and ϕ is the *azimuth angle*

Here, $I = I_0 \cos \theta$(*Lambert's cosine law*) where I_0 is the normal intensity of radiation.

We also note that $d(\cos \theta) = -\sin \theta \, d\theta$

$$E_N = 2\pi I_o \int_{\theta=0}^{\pi/2} -\cos^2 \theta d(\cos\theta) = 2\pi I_0 \left[-\frac{\cos^3\theta}{3} \right]_0^{\pi/2} = \frac{2}{3}\pi I_0 [\cos^3 0 - \cos^3(\pi/2)]$$

$$\therefore \qquad = (2/3)\pi I_0 (1-0) = \frac{2}{3}\pi I_0$$

For a diffuse surface, $E_D = \pi I_0$

Hence, the ratio of total directional emissivity to total hemispherical emissivity is

$$\frac{\varepsilon_N}{\varepsilon_D} = \frac{E_N/E_b}{E_D/E_b} = \frac{E_N}{E_D} = \frac{(2/3)\pi I_0}{\pi I_0} = \mathbf{\frac{2}{3}}$$

or $\varepsilon(T)/\varepsilon(\theta, T) = 3/2 = \mathbf{1.5}$ **(Ans.)**

EXAMPLE 12.8 *The spectral distribution of monochromatic emissive power for a diffuse emitting surface at 1100 K can be expressed as*

$$E_\lambda \begin{cases} E_1 = 0 & \text{for } 0 \le \lambda \le 1 \\ E_2 = 1500 & \text{for } 1 \le \lambda \le 3 \\ E_3 = 4000 & \text{for } 3 \le \lambda \le 5 \\ E_4 = 2700 & \text{for } 5 \le \lambda \le 7 \\ E_5 = 0 & \text{for } \lambda > 7 \end{cases}$$

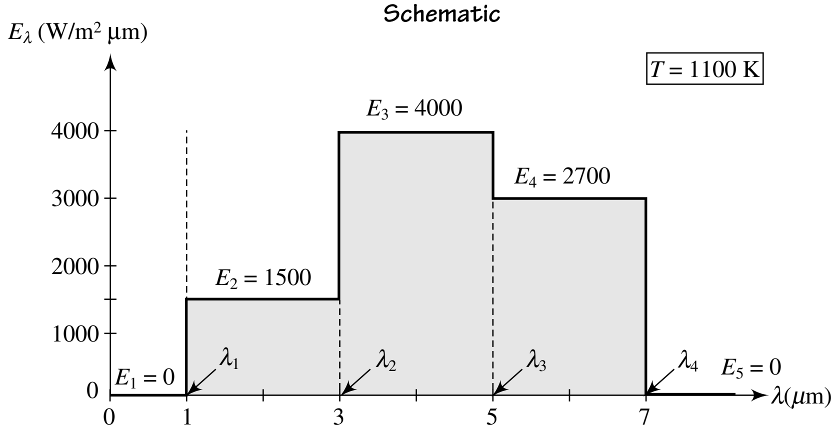
where E_λ is in W/m² μm and λ is in μm. Determine (a) the total emissive power, (b) the total intensity of radiation associated with $\theta = 0°$ and $\theta = 40°$, (c) the total emissivity of the surface, (d) the spectral emissivity at $\lambda = 6$ μm, and (e) the spectral intensity of radiation in the normal direction at $\lambda = 4.5$ μm.

Solution

Known Spectral distribution of E_λ for a diffuse emitting surface.

Find (a) E(W/m²); (b) I_e at $\theta = 0°$ and $\theta = 40°$(W/m² sr); (c) ε; (d) ε_λ at $\lambda = 6$ μm; (e) $I_{e,\lambda}$(W/m² μm sr) at $\lambda = 4.5$ μm.

Schematic



Assumption Diffusely emitting surface.

Analysis (a) Total emissive power,

$$E = \int_0^\infty E_\lambda d\lambda = \int_0^{\lambda_1} E_1 d\lambda + \int_{\lambda_1}^{\lambda_2} E_2 d\lambda + \int_{\lambda_2}^{\lambda_3} E_3 d\lambda + \int_{\lambda_3}^{\lambda_4} E_4 d\lambda + \int_{\lambda_4}^{\infty} E_5 d\lambda$$

$$= \int_0^1 (0) d\lambda + \int_1^3 (1500) d\lambda + \int_3^5 (4000) d\lambda + \int_5^7 (2700) d\lambda + \int_7^\infty (0) d\lambda$$

$$= 0 + 1500(3-1) + 4000(5-3) + 2700(7-5) + 0 = \mathbf{16\,400\ W/m^2} \quad \textbf{(Ans.) (a)}$$

(b) For a diffuse emitter, I_e is *independent* of the angle θ.
Hence, for both $\theta = 0°$ and $\theta = 40°$, the total intensity of radiation,

$$I_e = \frac{E}{\pi} = \frac{16\,400\ \text{W/m}^2}{\pi\ \text{sr}} = \mathbf{5220\ W/m^2\ sr} \quad \textbf{(Ans.) (b)}$$

(c) Total emissivity of surface, $\varepsilon = \dfrac{E}{E_b}$

where the black-body emissive power,

$$E_b = \sigma T^4 = (5.67 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4)(1100\ \text{K})^4 = 83\,015\ \text{W/m}^2$$

Hence,
$$\varepsilon = \frac{16\,400\ \text{W/m}^2}{83\,015\ \text{W/m}^2} = \mathbf{0.198} \quad \textbf{(Ans.) (c)}$$

(d) Spectral emissivity at $\lambda = 6$ μm, $\varepsilon_\lambda = \dfrac{E_\lambda}{E_{b\lambda}}$

At 1100 K and $\lambda = 6$ μm, the spectral emissive power is known to be E_λ = 2700 W/m² μm

And, the spectral emissive power of a black surface at $\lambda = 6$ μm and T = 1100 K is

$$E_{b\lambda} = \frac{C_1}{\lambda^5\left[\exp\left(\dfrac{C_2}{\lambda T}\right) - 1\right]} = \frac{3.742 \times 10^8\ \text{W}\,\mu m^4/\text{m}^2}{(6\ \mu\text{m})^5\left[\exp\left(\dfrac{1.4387 \times 10^4\ \mu\text{m K}}{6\ \mu\text{m} \times 1100\ \text{K}}\right) - 1\right]}$$

$$= 6134\ \text{W/m}^2\ \mu\text{m}$$

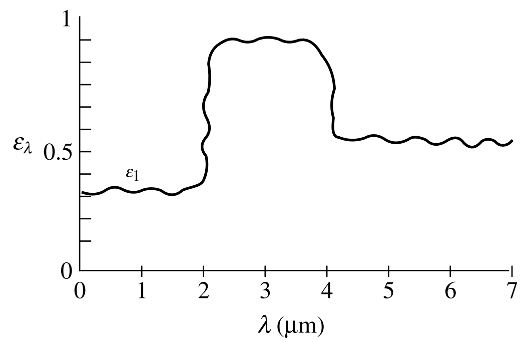
Therefore,

$$\varepsilon_\lambda = \frac{2700 \text{ W/m}^2\ \mu\text{m}}{6134 \text{ W/m}^2\ \mu\text{m}} = \mathbf{0.44} \qquad \textbf{(Ans.) (d)}$$

(e) Spectral intensity of radiation in the normal direction at λ = 4.5 μm is

$$I_{e,\lambda} = \frac{E_\lambda}{\pi} = \frac{4000 \text{ W/m}^2\ \mu\text{m}}{\pi \text{ sr}} = \mathbf{1273\ W/m^2\ \mu m\ sr} \qquad \textbf{(Ans.) (e)}$$

EXAMPLE 12.9 *The typical variation of monochromatic emissivity of a real surface is shown below:*



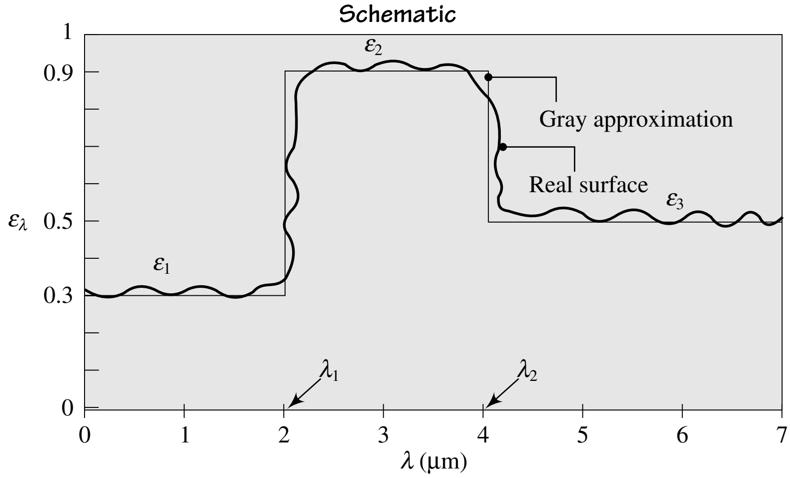
Determine using the band approximation: (a) the effective emissivity over the whole spectrum, (b) the emissive power at T = *1200 K, and (c) the solar absorptivity based on black-body distribution at* T_S = *5800 K.*

Solution

Known Spectral emissivity distribution of a non-gray engineering surface at a specified temperature.

Find (a) Effective emissivity, $\varepsilon(T)$, (b) Emissive power, $E(T)$, and (c) Solar absorptivity, $\alpha_S(T_S)$.

Schematic



Assumptions (1) Isothermal surface. (2) A piecewise gray model approximates the real non-gray surface.

Analysis The three-band approximation is a satisfactory representation of the monochromatic variation of emissivity of the real surface as illustrated in the schematic.

The equivalent gray emissivity of a non-gray body is determined from

$$\boxed{\varepsilon(T) = \sum_{i=1}^{N} \varepsilon_i f_{\lambda_{i-1}\to\lambda_i}(T)}$$

where there are N bands with constant band emissivities given by ε_i.

(a) In this case, λ_1 = 2 μm, λ_2 = 4 μm, and $\lambda_3 = \infty$.
ε_1 = 0.3, ε_2 = 0.9, and ε_3 = 0.5

$$\therefore \quad \varepsilon(T) = \varepsilon_1 f_{0-\lambda_1} + \varepsilon_2 f_{\lambda_1-\lambda_2} + \varepsilon_3 f_{\lambda_2-\infty} = \varepsilon_1 f_{\lambda_1} + \varepsilon_2 (f_{\lambda_1-\lambda_2}) + \varepsilon_3 (1 - f_{\lambda_2})$$

From Table 12.1, with T = 1200 K:

$$\lambda_1 T = (2\ \mu\text{m})(1200\ \text{K}) = 2400\ \mu\text{m K} \Rightarrow f_{\lambda_1} = 0.14026$$

$$\lambda_2 T = (4\ \mu\text{m})(1200\ \text{K}) = 4800\ \mu\text{m K} \Rightarrow f_{\lambda_2} = 0.60754$$

Substituting the values of f_{λ_1} and f_{λ_2} in the expression for equivalent (*effective*) emissivity, we have

$$\varepsilon(T) = (0.3)(0.14026) + 0.9(0.60754 - 0.14026) + 0.5(1 - 0.60754)$$
$$= \mathbf{0.659} \qquad \textbf{(Ans.) (a)}$$

(b) Emissive power,

$$E(T) = \varepsilon\sigma T^4 = (0.659)(5.67 \times 10^{-8}\ \text{W/m}^2\,\text{K}^4)(1200\ \text{K})^4$$
$$= \mathbf{77.5 \times 10^3\ W/m^2} \quad \text{or} \quad \mathbf{77.5\ kW/m^2} \qquad \textbf{(Ans.) (b)}$$

(c) Solar absorptivity, $\alpha_S(T_S) = \dfrac{\int_0^\infty \alpha_\lambda G_\lambda(T_S)\,d\lambda}{\int_0^\infty G_\lambda(T_S)\,d\lambda}$

Since $\quad \alpha_\lambda = \varepsilon_\lambda, \alpha_S(T_S) = \varepsilon_1 f_{\lambda_1}(T_S) + \varepsilon_2 f_{\lambda_1-\lambda_2}(T_S) + \varepsilon_3 f_{\lambda_2-\infty}(T_S)$

$$= \varepsilon_1 f_{\lambda_1}(T_S) + \varepsilon_2 (f_{\lambda_2} - f_{\lambda_1})(T_S) + \varepsilon_3 (1 - f_{\lambda_2})(T_S)$$

With T_S = 5800 K, from Table 12.1, by interpolation

$$\lambda_1 T_S = 2(5800) = 11600\ \mu\text{m K} \Rightarrow f_{\lambda_1} = 0.94015$$

$$\lambda_2 T_S = 4(5800) = 23200\ \mu\text{m K} \Rightarrow f_{\lambda 2} = 0.9898$$

$$\therefore \quad \alpha_S(T_S) = (0.3)(0.94015) + 0.9(0.9898 - 0.94015) + 0.5(1 - 0.9898)$$
$$= \mathbf{0.332} \qquad \textbf{(Ans.) (c)}$$

EXAMPLE 12.10 *An opaque surface has the spectral reflectivity given by (λ in μm):*

$$\rho_\lambda = \begin{cases} 0 & \lambda < 5 \\ 0.5 & 5 < \lambda < 10 \\ 0.9 & \lambda > 10 \end{cases}$$

The surface is subjected to the spectral irradiation given by $G_\lambda = \begin{cases} 120\,\lambda & \lambda < 5 \\ 600 & 5 < \lambda < 10 \\ 1200 - 60\,\lambda & 10 < \lambda < 20 \\ 0 & \lambda > 20 \end{cases}$

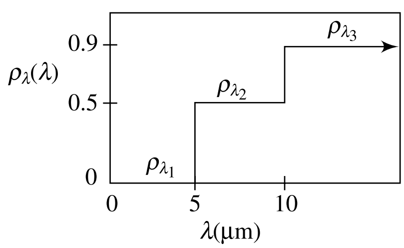
Calculate (a) the total irradiation of the surface, (b) the total absorptivity of the surface, (c) the total emissivity of the surface if the surface is at 300 K, and (d) the total radiosity of the surface

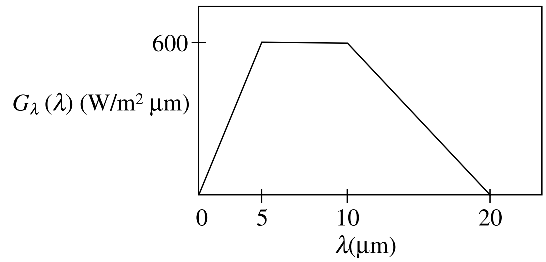
Solution

Known Spectral reflectivity and spectral irradiation for an opaque surface at 300 K.

Find (a) G, (b) G_{abs}, (c) ε, (d) J.

Schematic





Assumptions (1) Opaque and diffuse surface. (2) $\varepsilon_\lambda = \alpha_\lambda$.

Analysis (a) Total irradiation of the surface,

$$G = \frac{1}{2}(600\ \text{W/m}^2\ \mu\text{m})(5-0)\mu\text{m} + (600\ \text{W/m}^2\ \mu\text{m})(10-5)\mu\text{m} + \frac{1}{2}(600\ \text{W/m}^2\ \mu\text{m})(20-10)\mu\text{m}$$

$$= (1500 + 3000 + 3000)\text{W/m}^2 = \mathbf{7500\ W/m^2} \qquad \textbf{(Ans.) (a)}$$

(b) For an opaque surface, $\tau_\lambda = 0$ and $\alpha_\lambda = 1 - \rho_\lambda$

Total absorptivity of the surface is $\alpha = \int_0^\infty \alpha_\lambda G_\lambda d_\lambda \Big/ \int_0^\infty G_\lambda d\lambda = \frac{G_{abs}}{G}$

Subdividing the integral into parts,

$$G_{abs} = (1-\rho_{\lambda_1})\int_0^5 G_\lambda d_\lambda + (1-\rho_{\lambda_2})\int_5^{10} G_\lambda d\lambda + (1-\rho_{\lambda_3})\int_{10}^{20} G_\lambda d\lambda$$

$$= (1-0)[0.5\times 600\ \text{W/m}^2\ \mu\text{m}(5-0)\mu\text{m}] + (1-0.5)[600\ \text{W/m}^2\ \mu\text{m}(10-5)\mu\text{m}] + (1-0.9)[0.5\times 600\ \text{W/m}^2\ \mu\text{m}(20-10)\mu\text{m}]$$

$$= [(1\times 1500) + (0.5\times 3000) + (0.1\times 3000)]\text{W/m}^2 = 3300\ \text{W/m}^2$$

$$\therefore \qquad \alpha = \frac{G_{abs}}{G} = \frac{3300\ \text{W/m}^2}{7500\ \text{W/m}^2} = \mathbf{0.44} \qquad \textbf{(Ans.) (b)}$$

(c) Total emissivity of the surface is, $\varepsilon = \int_0^\infty \varepsilon_\lambda E_{b\lambda}(\lambda, T)d\lambda / E_b(T)$

Since $\qquad \varepsilon_\lambda = \alpha_\lambda = 1 - \rho_\lambda,\ \varepsilon = \int_0^\infty (1-\rho_\lambda)E_{b\lambda}\, d\lambda / E_b$

$$= (1-\rho_{\lambda_1})\int_0^5 E_{b\lambda} d\lambda / E_b + (1-\rho_{\lambda_2})\int_5^{10} E_{b\lambda} d\lambda / E_b$$

$$+ (1-\rho_{\lambda_3})\int_{10}^{\infty} E_{b\lambda} d\lambda / E_b = (1-\rho_{\lambda_1}) f_{(0\to 5\,\mu\text{m})}$$

$$+ (1-\rho_{\lambda_2}) f_{(5\to 10\,\mu\text{m})} + (1-\rho_{\lambda_3}) f_{(10\to\infty\,\mu\text{m})}$$

From Table 12.1:

With $\lambda_1 T = 5 \times 300 = 1500\ \mu\text{m K}$, $f_{(0\to 5\,\mu\text{m})} = 0.013754$

and, with $\lambda_2 T = 10 \times 300 = 3000\ \mu\text{m K}$, $f_{(0\to 10\,\mu\text{m})} = 0.273232$

Then, $\varepsilon = (1-\rho_{\lambda_1}) f_{(0\to 5\,\mu\text{m})} + (1-\rho_{\lambda_2})(f_{(0\to 10\,\mu\text{m})} - f_{(0\to 5\,\mu\text{m})})$

$$+ (1-\rho_{\lambda_3})(f_{(0\to\infty)} - f_{(0\to 10\,\mu\text{m})})$$

$$= (1 \times 0.013754) + (1-0.5)(0.273232 - 0.013754)$$

$$+ (1-0.9)(1-0.273232)$$

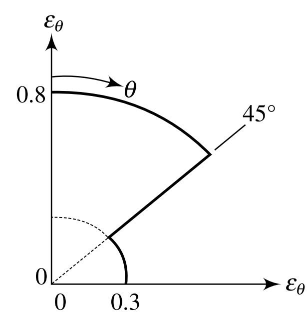
$$= \mathbf{0.216}$$ **(Ans.) (c)**

(d) Radiosity of the surface, $J = \varepsilon E_b + \rho G = \varepsilon\sigma T^4 + (1-\alpha)G$

$$= (0.216)(5.67 \times 10^{-8}\ \text{W/m}^2\,\text{K}^4)(300\ \text{K})^4 + (1-0.44)(7500\ \text{W/m}^2)$$

$$= \mathbf{4299\ W/m^2}$$ **(Ans.) (d)**

EXAMPLE 12.11 *Consider the directionally selective surface having the directional emissivity ε_θ as shown.*
Calculate the ratio of the normal emissivity to the hemispherical emissivity.



Solution

Known Directional emissivity, ε_θ, of a selective surface.

Find Ratio of normal emissivity to hemispherical emissivity.

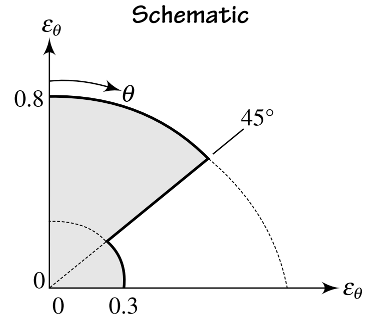
Assumptions Surface is isotropic in the ϕ-direction.

Analysis Hemispherical emissivity, $\varepsilon = \dfrac{\int_0^{2\pi}\int_0^{\pi/2} \varepsilon_\theta \cos\theta \sin\theta\, d\theta\, d\phi}{\int_0^{2\pi}\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta\, d\phi}$

Assuming ε to be independent of ϕ, we have

$$\varepsilon = \frac{2\pi\int_0^{\pi/2} \varepsilon_\theta \cos\theta\sin\theta\, d\theta}{2\pi\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta} = \frac{\int_0^{\pi/2} \varepsilon_\theta\, 2\sin\theta\cos\theta\, d\theta/2}{\left[\dfrac{\sin^2\theta}{2}\right]_0^{\pi/2}}$$

Schematic



$$= \frac{\frac{1}{2} \times 0.8 \int_0^{\pi/4} 2 \sin\theta \cos\theta \, d\theta + \frac{1}{2} \times 0.3 \int_{\pi/4}^{\pi/2} 2 \sin\theta \cos\theta \, d\theta}{\frac{1}{2}\left[\sin^2 \frac{\pi}{2} - \sin^{-2} 0\right]}$$

$$= \frac{\frac{1}{2} \times 0.8 [\sin^2 \theta]_0^{\pi/4} + \frac{1}{2} \times 0.3 [\sin^2 \theta]_{\pi/4}^{\pi/2}}{\frac{1}{2}(1-0)}$$

$$= \left[0.8 \times \left(\sin^2 \frac{\pi}{4} - 0\right)\right] + \left[0.3 \times \left(\sin^2 \frac{\pi}{2} - \sin^2 \frac{\pi}{4}\right)\right] = (0.8)(0.5-0) + (0.3)(1-0.5) = 0.55$$

Ratio of normal emissivity, ε_n to hemispherical emissivity is, $\dfrac{\varepsilon_n}{\varepsilon} = \dfrac{0.8}{0.55} = \mathbf{1.455}$ **(Ans.)**

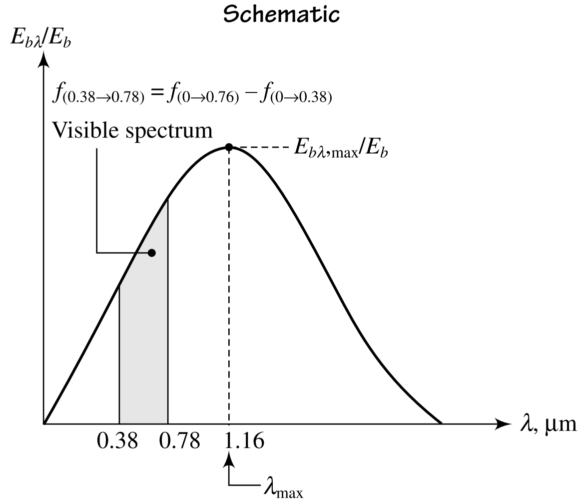
EXAMPLE 12.12 *Consider an incandescent tungsten filament light bulb whose filament is at a temperature of 2500 K. Assuming the filament to be a black body, determine the fraction of the total radiation energy emitted by the bulb in the visible wavelength spectrum from 0.38 μm to 0.78 μm. Also find the wavelength at which the emission of radiation from the filament is maximum.*

Solution

Known The filament of a light bulb is at 2500 K. The visible wavelength band is 0.38 to 0.78 micron.

Find $f_{(\lambda_1 \to \lambda_2)}$, $\lambda_{\max}$.

Schematic



Assumptions The filament closely approximates a black body.

Analysis The black-body radiation fractions corresponding to $\lambda_1 T$ and $\lambda_2 T$ are determined from Table 12.1.

$$\lambda_1 T = (0.38 \ \mu\text{m})\ (2500 \ \text{K}) = 950 \ \mu\text{m K}$$

$$\Rightarrow \qquad f_{(0\to\lambda_1)} = f_{\lambda_1} = 0.000\ 204$$

$$\lambda_2 T = (0.78\ \mu\text{m})\ (2500\ \text{K}) = 1950\ \mu\text{m K}$$

$$\Rightarrow \qquad f_{(0\to\lambda_2)} = f_{\lambda_2} = 0.059\ 42$$

The fraction of radiation emitted between these two wavelengths is

$$f_{\lambda_1-\lambda_2} = f_{\lambda_2} - f_{\lambda_1} = 0.059\ 42 - 0.000\ 204 = 0.0592 \approx \mathbf{0.06} \qquad \textbf{(Ans.)}$$

According to Wien's displacement law,

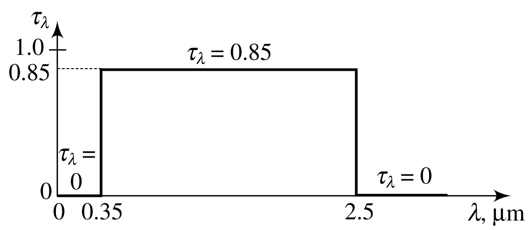
$$\lambda_{max}\ T = 2898\ \mu\text{m K}$$

The wavelength at which the emission of radiation from the filament is maximum, is

$$\lambda_{max} = \frac{2898\ \mu\text{m K}}{2500\ \text{K}} = \mathbf{1.16\ \mu m} \qquad \textbf{(Ans.)}$$

Comment Only about 6% of the radiation emitted by the filament of the bulb falls in the visible range. The remaining 94% of the radiation appears in the infrared region as radiant heat and goes into heating the surrounding room. Incandescent bulbs are, therefore, considered very inefficient as sources of light. Fluorescent tubes are a better option.

EXAMPLE 12.13 *The spectral hemispherical transmissivity of a 3 mm thick glass varies with wavelength in the following manner:*



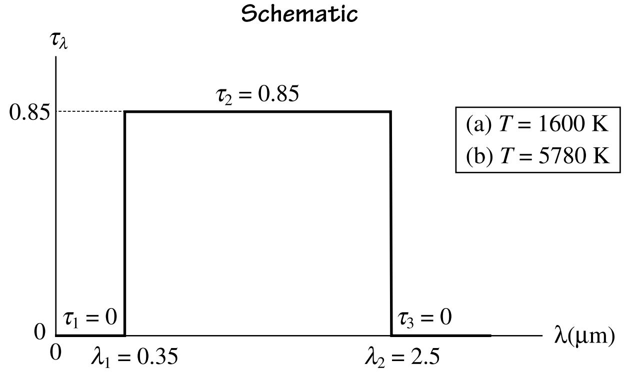
(a) Calculate the average transmissivity and the rate of radiation transmitted through a 2 m by 2 m glass window if irradiation from a black surface at 1600 K is incident on it. (b) What will be the amount of solar radiation transmitted through the window assuming the sun to be a black-body source at 5780 K?

Solution

Known Spectral transmissivity of glass.

Find Amount of heat transmitted through a glass window for, $T_{source} = 1600$ K, and (b) $T_{source} = 5780$ K.

Schematic



Assumption Spectral distribution is proportional to that of black-body emission at the source temperature.

Analysis (a) Average transmissivity, T = 1600 K: $\tau = \int_0^\infty \tau_\lambda G_\lambda d\lambda / \int_0^\infty G_\lambda d\lambda$

We note that, $G_\lambda \propto E_{b\lambda}$ (T = 1600 K)

With the proportionality constant cancelling out in the numerator and denominator of the expression for τ, we have

$$\overline{\tau} = \frac{\int_0^\infty \tau_\lambda E_{b\lambda}(1600\text{ K})d\lambda}{\int_0^\infty E_{b\lambda}(1600\text{ K})d\lambda} = \frac{\tau_1 \int_0^{\lambda_1} E_{b\lambda} d\lambda}{E_b} + \frac{\tau_2 \int_{\lambda_1}^{\lambda_2} E_{b\lambda} d\lambda}{E_b} + \frac{\tau_3 \int_{\lambda_2}^{\infty} E_{b\lambda} d\lambda}{E_b}$$

$$= 0 \times f_{(0-\lambda_1)} + 0.85 f_{(\lambda_1-\lambda_2)} + 0 \times f_{(\lambda_2 \to \infty)} = 0.85[f_{(0-\lambda_2)} - f_{(0-\lambda_1)}]$$

Rate of radiation transmitted,

$$\dot{Q}_{\text{tr}} = A G_{\text{tr}} = A \overline{\tau} E_b(1600\text{ K})$$

$$= \overline{\tau} \times (2\text{ m} \times 2\text{ m})(5.67 \times 10^{-8}\text{ W/m}^2\text{ K}^4)(T, K^4) = 22.68\left(\frac{T}{100}\right)^4 \times \overline{\tau}(T)$$

With $\lambda_2 T = 2.5 \times 1600 = 4000\ \mu\text{m K} \Rightarrow f_{\lambda_2} = 0.48087$

and $\lambda_1 T = 0.35 \times 1600 = 560\ \mu\text{m K} \Rightarrow f_{\lambda_1} = 0.0000$ [*from Table 12.1*]

$\therefore$ $\overline{\tau}(1600\text{ K}) = 0.85[0.48087 - 0.0000] = \mathbf{0.409}$ **(Ans.) (a)**

Hence, $\dot{Q}_{\text{tr}} = 22.68\left(\frac{1600}{100}\right)^4 \times 0.409 = \mathbf{607.5 \times 10^3\ W}$ or **607.5 kW** **(Ans.) (a)**

(b) With T = 5780 K,

$$\lambda_1 T = 0.35 \times 5780 = 2023\ \mu\text{m K} \Rightarrow f_{\lambda_1} = 0.07066$$

$$\lambda_2 T = 2.5 \times 5780 = 14450\ \mu\text{m K} \Rightarrow f_{\lambda_2} = 0.966$$

$\therefore$ $\overline{\tau}(5780\text{ K}) = 0.85(0.966 - 0.07066) = 0.761$

and $\dot{Q}_{\text{tr}} = 22.68\left(\frac{5780}{100}\right)^4 \times 0.761 = \mathbf{192.6 \times 10^6\ W}$ or **192.6 MW** **(Ans.) (b)**

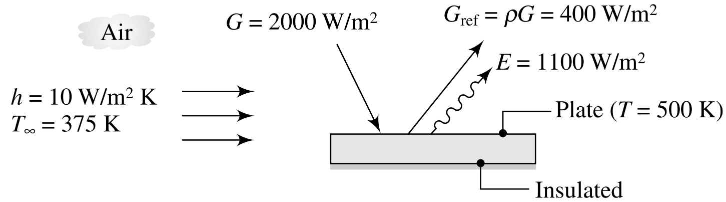
EXAMPLE 12.14 *An opaque horizontal plate, well insulated on its back is at 500 K and has an emissive power of 1100 W/m². The irradiation on the plate is 2000 W/m², of which 400 W/m² is reflected. Air at 375 K flows over the plate, the convective heat-transfer coefficient being 10 W/m² K. Determine the emissivity, absorptivity, and radiosity of the plate. Also calculate the net heat flux.*

Solution

Known An isothermal plate loses heat by both radiation and convection.

Find Emissivity (ε), Absorptivity (α), Radiosity (J), and net heat flux (q_{net}).

Schematic



Assumptions (1) Opaque, isothermal, diffuse surface. (2) The back of the plate is effectively insulated.

Analysis The total hemispherical emissivity of the plate,

$$\varepsilon = \frac{E(T)}{E_b(T)} = \frac{1100\ \text{W/m}^2}{(5.67 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4)(500\ \text{K})^4} = \mathbf{0.31} \quad \textbf{(Ans.)}$$

The total hemispherical absorptivity for an opaque surface is related to its reflectivity.

$$\alpha = 1 - \rho$$

The reflectivity is the fraction of the irradiation reflected.

Thus, $\rho = \dfrac{G_{ref}}{G} = \dfrac{400\ \text{W/m}^2}{2000\ \text{W/m}^2} = 0.20$

Hence, $\alpha = 1 - \rho = 1 - 0.20 = \mathbf{0.80}$ **(Ans.)**

The radiosity, J, is the radiant flux leaving the plate surface by both emission and reflection.

$$J = \varepsilon E_b + \rho G = E + G_{ref} = (1100 + 400)\ \text{W/m}^2 = \mathbf{1500\ W/m^2} \quad \textbf{(Ans.)}$$

From the energy balance,

Net heat flux, $q_{net} = q_{in} - q_{out} = G - J - q_{conv}$

Convective heat flux, $q_{conv} = h(T_s - T_\infty) = (10\ \text{W/m}^2\ \text{K})\ (500 - 375)\ \text{K} = 1250\ \text{W/m}^2$

Hence, $q_{net} = (2000 - 1500 - 1250)\ \text{W/m}^2 = \mathbf{-\ 750\ W/m^2}$ **(Ans.)**

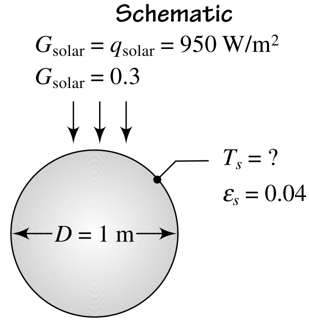
Comment The *negative* sign indicates that energy must be added to the plate to enable it to maintain the temperature of 500 K. Note that $\alpha \neq \varepsilon$. The plate is thus *not* a gray body.

(C) Solar and Environmental Radiation

EXAMPLE 12.15 *A deep space probe is constructed as a 1 m diameter polished aluminium sphere. Determine the equilibrium temperature that the probe reaches if the incident solar radiation is 950 W/m². For solar radiation, the absorptivity of aluminium is 0.3 and the emissivity of aluminium is 0.04.*

Solution

Schematic



Known Spherical space probe is exposed to solar radiation under specified conditions.

Find Equilibrium temperature, T_s(K).

Assumptions (1) Steady-state conditions. (2) Surface is diffuse gray. (3) Sky temperature is too low to be negligible.

Analysis Under equilibrium conditions,

or $\qquad \dot{E}_{in} = \dot{E}_{out}$

$$\begin{pmatrix}\text{Rate of radiation heat transfer to the}\\ \text{probe surface exposed to solar energy}\end{pmatrix} = \begin{pmatrix}\text{Rate at which radiation is emitted by the}\\ \text{probe surface at its temperature}\end{pmatrix}$$

$$\alpha_{\text{solar}} G_{\text{solar}} A_{\text{projected}} = \varepsilon_s \sigma A_s T_s^4$$

The projection of a sphere is a *circle*. Hence, the projected area = $\frac{\pi}{4}D^2$ while the surface area of the spherical probe, $A_s = \pi D^2$

Inserting the appropriate values, we have

$$(0.3)(950 \text{ W/m}^2)\left(\frac{\pi}{4} \times 1^2 \text{ m}^2\right) = 0.04 \times 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4 \times (\pi \times 1^2 \text{ m}^2) \times T_s^4 (K^4)$$

Equilibrium surface temperature of the probe is

$$T_s = \left[\frac{0.3 \times 950 \times 10^8}{0.04 \times 4 \times 5.67}\right]^{1/4} = \mathbf{421\ K (\approx 148°C)} \qquad \textbf{(Ans.)}$$

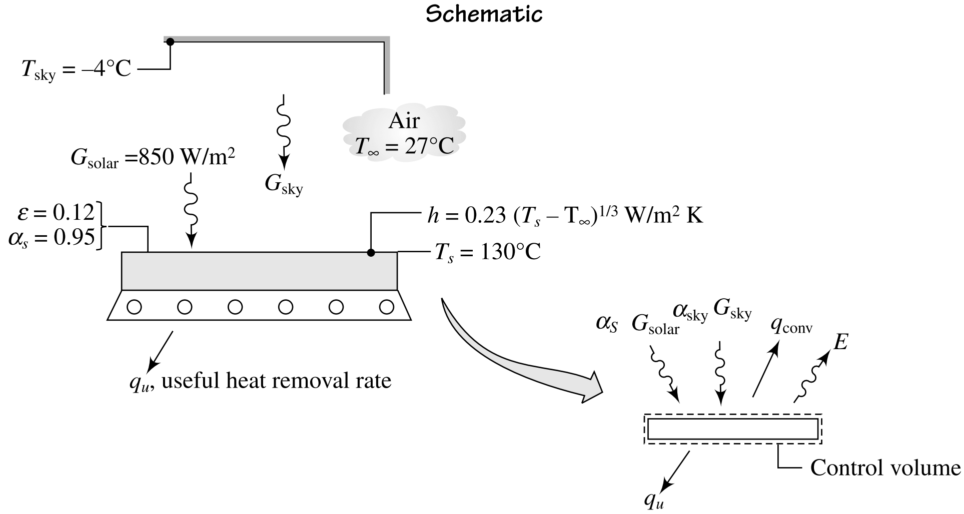
EXAMPLE 12.16 *A flat-plate solar collector with no cover plate is used to collect the solar radiation to heat water in a commercial installation. The surface emissivity of the absorber is 0.12 while its solar absorptivity is 0.95. At a given time of the day, the absorber surface temperature is 130°C when the solar irradiation is 850 W/m², the effective sky temperature is –4°C and the ambient air temperature is 27°C. Assume that the heat-transfer convection coefficient for the calm day condition is given by,* $h = 0.23(T_s - T_\infty)^{1/3}$ *W/m² K where* T_s *is the surface temperature and* T_∞ *is free stream ambient temperature. Assume steady-state conditions, the bottom surface is well insulated and the absorber surface is diffuse. Work out the following: (a) Sketch the system and show the control volume. (b) The useful heat removal rate in W/m² from the collector. (c) The efficiency of collector. (d) Comment on the results.* **[IES 2007]**

Solution

Known Operating conditions for a flat-plate solar collector.

Find (a) Sketch of the system. (b) Useful heat-removal rate (W/m²). (c) Collector efficiency. (d) Comment.

Schematic



Assumptions (1) Steady-state conditions. (2) The bottom of the collector is well insulated. (3) Absorber surface is diffuse.

Analysis (b) Performing an energy balance on the absorber, $\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0$

Per unit surface area

$$\alpha_S G_{\text{solar}} + \alpha_{\text{sky}} G_{\text{sky}} - q_{\text{conv}} - E - q_u = 0$$

where $G_{\text{sky}} = \sigma T_{\text{sky}}^4$

Since the sky radiation is concentrated in approximately the same spectral region as that of surface emission, it is reasonable to assume that, $\alpha_{\text{sky}} \approx \varepsilon = 0.12$

Now, $q_{\text{conv}} = h(T_s - T_\infty) = 0.23(T_s - T_\infty)^{1/3}(T_s - T_\infty) = 0.23(T_s - T_\infty)^{4/3}$

Also, $E = \varepsilon\sigma T_s^4$

Useful heat-removal rate,

$$\begin{aligned} q_u &= \alpha_S G_{\text{solar}} + \varepsilon\sigma T_{\text{sky}}^4 - 0.23(T_S - T_\infty)^{4/3} - \varepsilon\sigma T_S^4 \\ &= \alpha_S G_{\text{solar}} - 0.23(T_S - T_\infty)^{7/3} - \varepsilon\sigma(T_S^4 - T_{\text{sky}}^4) \\ &= (0.95 \times 850) - 0.23(130 - 27)^{4/3} - 0.12 \times 5.67 \times 10^{-8}(403.15^4 - 269.15^4) \\ &= \mathbf{552.6\ W/m^2} \end{aligned}$$

(Ans.) (b)

(c) Efficiency of collector

$$\eta = \frac{\text{Useful energy extracted, } q_u}{\text{Solar irradiation, } G_{\text{solar}}} = \frac{552.6}{850} = 0.65 \quad \text{or} \quad \mathbf{65.0\%}$$

(Ans.) (c)

(d) **Comments:**

Since the spectral range of G_{sky} is entirely different from that of G_S it would be incorrect to assume that $\alpha_{\text{sky}} = \alpha_S$

The convection heat transfer coefficient is very low ($h \approx 0.23\ (130 - 37^{1/3} \approx 1$ Wm^2 K). With a slight increase in h, both the collector efficiency η and useful heat flux q_u would reduce substantially. Hence, it is advisable to use the cover plate to reduce convection (and radiation) heat loss from the absorber plate. **(Ans.) (d)**

EXAMPLE 12.17 *The irradiation received by a stratospheric balloon was found to be 1250 W/m². If the earth's atmosphere allows only 82% of the emitted radiation reaching the receiver, estimate the temperature of the sun. The sun may be considered a black body. The distance from the earth to the sun is 1.49 × 10⁸ km and the diameter of the sun is 1.39 × 10⁶ km.*

Solution

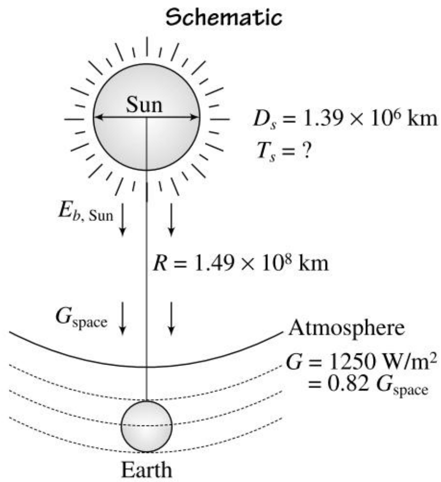
Known Solar irradiance. Distance between sun and earth. Diameters of sun and earth.

Find Temperature of the sun, T_S(K).

Assumption (1) The sun is a black body. (2) 18% attenuation of solar irradiation *en route* to earth.

Analysis Solar heat flux,

$$G_{\text{space}} = \frac{1250}{0.82}\ \text{W/m}^2 = 1524.39\ \text{W/m}^2$$



If the sun is black,

$$G_{\text{space}} = E_{b,\text{Sun}} \times \frac{\pi D_S^2}{4\pi R^2} = \sigma T_S^4 \left(\frac{D_S}{2R}\right)^2$$

Here, $\dfrac{\pi D_S^2}{4\pi R^2}$ is the solid angle ratio.

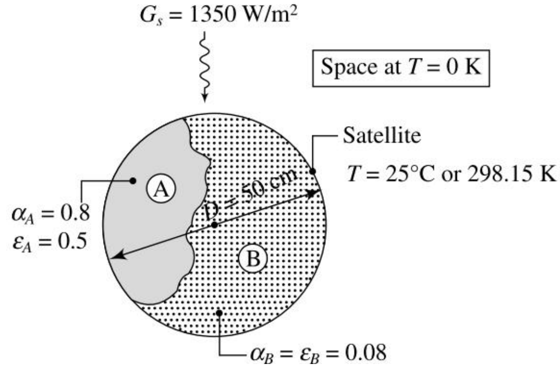
where D_S = Diameter of the sun
R = Distance between the earth and the sun.

Then $T_S^4 = \dfrac{G_{\text{space}}}{\sigma}\left(\dfrac{2R}{D_S}\right)^2 = \dfrac{1524.39\ \text{W/m}^2}{5.67 \times 10^{-8}\ \text{W/m}^2\,\text{K}^4}\left(\dfrac{2 \times 1.49 \times 10^8\ \text{km}}{1.39 \times 10^6\ \text{km}}\right)^2 = 1235.93 \times 10^{12}\ \text{K}^4$

$\therefore$ Surface temperature of the sun is

$$T_S = [1235.93 \times 10^{12}\ \text{K}^4]^{1/4} = \mathbf{5929\ K} \qquad \textbf{(Ans.)}$$

EXAMPLE 12.18 *The main body of a communications satellite, modelled as a 50 cm diameter sphere, rotates continuously and is exposed to solar radiation with an energy flux of 1350 W/m². Determine how much of its surface should be covered with each of the following materials to maintain the temperature of the surface material at 25°C.*



Material A: $\alpha_A = 0.8$ *for solar radiation*

$\varepsilon_A = 0.5$ *for all wavelengths associated with emitted radiation*

Material B: $\alpha_B = \varepsilon_B = 0.08$ *for all wavelengths*

Solution

Known A spherical communications satellite with the surface covered with two different materials continuously rotates while exposed to solar radiation.

Find Surface area covered with each material to maintain constant surface temperature.

Assumption (1) Outer space is a black body at 0 K. (2) Let x equal the fraction of area to be covered by material A.

Analysis Energy absorbed = $[G_s x\alpha_A + G_s(1-x)\alpha_B]$ × Projected area, A_p.

$$= [(1350 \text{ W/m}^2)(x)(0.8) + (1350 \text{ W/m}^2)(1-x)(0.08)] \times \frac{\pi D^2 (m^2)}{4}$$

Energy emitted = $[\varepsilon_A xE_b(T) + \varepsilon_B(1-x)E_b(T)]$ × Surface area, A_s = [0.5) (x) σT^4 + (0.08) $(1-x)$ $(\sigma T^4)] \times \pi D^2$

[*Note that the projected area of a sphere of diameter D is a circle of area, $\pi D^2/4$ and the surface area of a sphere is πD^2.*]

Since no energy is supplied from within the satellite, the energy absorbed must equal the energy emitted,

Hence, from energy balance: Energy absorbed = Energy emitted

$$1080x + 108 - 108x = 4\left[(0.5)(x)(5.67)\left(\frac{298.15}{100}\right)^4 + (0.08)(1-x)(5.67)\left(\frac{298.15}{100}\right)^4\right]$$

or $\quad 972x + 108 = 4[224x + 35.84 - 35.84x]$

or $\quad x[972 - 752.64] = 143.375 - 108 \quad \text{or} \quad 219.36x = 35.375$

$\therefore \quad x = 35.375/219.36 = 0.16 \text{ or } 16\% \text{ and } (1-x) = (1-0.16) = \mathbf{0.84 \text{ or } 84\%}$

$\therefore \quad A_A = x(\pi D^2) = (0.16)(\pi)(0.5 \text{ m})^2 = \mathbf{0.126 \text{ m}^2}$ **(Ans.)**

$A_B = (1-x)(\pi D^2) = (0.84)(\pi)(0.5 \text{ m})^2 = \mathbf{0.66 \text{ m}^2}$ **(Ans.)**

Comment 16 percent (0.126 m^2) of the surface must be covered with material 'A' and 84 percent (0.66 m^2) must be covered with material 'B' to maintain a surface temperature of 25°C.

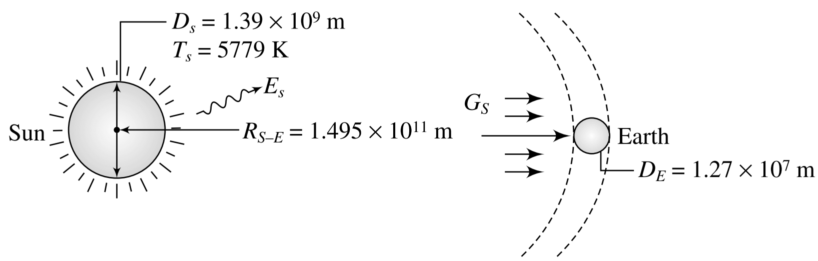
EXAMPLE 12.19 *The sun may be idealized as a black body with a surface temperature of 5779 K at a mean distance of 1.495×10^{11} m from the earth. The diameters of the sun and earth are 1.39×10^9 m and 1.27×10^7 m, respectively. Evaluate (a) the total radiation energy emitted by the sun, (b) the solar constant (the energy received per unit time per unit area normal to sun's rays in the extraterrestrial atmosphere at the mean earth–sun distance, (c) the total radiant energy intercepted by the earth in the absence of the earth's atmosphere, and (d) the energy received by a 1.5 m × 1.5 m flat-plate solar collector whose normal is inclined at 40° to the sun. The energy lost in the earth's atmosphere is estimated to be 30% and the diffuse radiation is 12% of the total energy received by the earth.*

Solution

Known Diameter and surface temperature of sun, diameter of earth and mean earth–sun distance.

Find (a) Energy emitted by the sun; (b) Solar constant; (c) Total radiant energy received by the earth; (d) Energy received by a solar collector.

Schematic



Assumptions (1) Sun is a black body. (2) Sun and earth are spherical bodies.

Analysis (a) Total radiant energy emitted by the sun,

$$\dot{Q}_{\text{sun}} = E_S A_S = \sigma T_s^4 (\pi D_s^2) = (5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4)(5779 \text{ K})^4 [\pi \times (1.39 \times 10^9 \text{ m})^2]$$

$$= \mathbf{3.84 \times 10^{26} \text{ W}} \qquad \textbf{(Ans.) (a)}$$

(b) Solar constant or solar flux at the outer edge of the earth's atmosphere is

$$G_S = \frac{\dot{Q}_{\text{sun}}}{4\pi\left[R_{S-E} - \dfrac{D_E}{2}\right]^2} = \frac{3.84 \times 10^{26} \text{ W}}{4\pi[1.495 \times 10^{11} - (0.5 \times 1.27 \times 10^7)]m^2}$$

$$= \mathbf{1367 \text{ W/m}^2} \qquad \textbf{(Ans.) (b)}$$

(c) Total radiant energy intercepted by the earth

= G_S × Projected area of the earth (*as a circle of diameter* D_E)

$$= G_S \times \frac{\pi}{4} D_E^2 = (1367 \text{ W/m}^2)\left[\frac{\pi}{4} \times (1.27 \times 10^7 \text{ m})^2\right] = \mathbf{1.73 \times 10^{17} \text{ W}} \qquad \textbf{(Ans.) (c)}$$

(d) Total energy received by the solar collector

$$= A_{\text{collector}} \times G_S = A_{\text{collector}} \times \left[\underbrace{G_D \cos\theta}_{\text{direct radiation}} + \underbrace{G_d}_{\text{diffuse radiation}}\right]$$

where $G_D = (1 - 0.30)G_S = 0.7 \times 1367 \text{ W/m}^2$

$\theta = 40°$ (*angle of incidence*)

$G_d = 0.12 \times G_s$

∴ $G_S = (0.7 \times 1367) \cos 40° + 0.12 \times G_S$

or $G_S = \dfrac{0.7}{[1 - 0.12]} \times 1367 \times \cos 40° = 833 \text{ W/m}^2$

Energy received = (1.5 m × 1.5 m)(833 W/m^2) = **1874 W** **(Ans.) (d)**

Points to Ponder

- The surface temperature of the sun is nearly 5800 K.
- The black body emissive power, E_b and wavelength λ_{max} at 5800 K are 64.16×10^6 W/m^2 and 0.5 respectively.
- The Planck's spectral distribution function is given by $E_{b\lambda} = \dfrac{C_1}{\lambda^5}[\exp(C_2/\lambda T) - 1]^{-1}$
- Selective surfaces exhibit non-gray surface properties.
- According to Wien's displacement law: $\lambda_{max}\, T = 2898$ μm K.
- Gray body assumption for white paint is not valid.
- For a diffusely emitting surface, intensity is independent of direction.
- The emissive power distribution for a gray surface is a scaled version of the black-body spectral emissive power distribution with a constant scale factor less than or equal to one.
- The emissive power of a surface also depends on the surface material and roughness.
- White paper is nearly radiation black with an absorptivity of 0.97.
- Window glass transmits radiation in the range of wavelengths from about 0.15 to 3 μm.
- Window glass does not transmit radiation in the range of wavelengths from about 3 μm to 10 μm.
- A highly polished surface can be called a specular surface.
- The albedo of a surface is defined as the ratio of the reflected energy to the incident energy.
- The blue colour of the sky does not result from particulate scattering in the atmosphere.
- The value of the solar constant is 1376 W/m^2.
- No medium is necessary for radiative heat transfer.
- All bodies at above the absolute zero temperature emit thermal radiation. This statement is based on the Prevost theory of heat exchange.
- Gases have poor reflectivity.
- Emissivity and absorptivity of a surface are equal for radiation with equal temperature and wavelength.
- The Stefan–Boltzman constant has a value of 5.67×10^{-8} W/m^2 K^4.
- Thermal radiation refers to the portion of the wavelength spectrum between 0.1 μm and 100 μm.
- Visible radiation range is between the wavelengths of 0.4 and 0.76 micron.
- The emission of radiation from a semi-transparent material is a volumetric phenomenon while that from an opaque material is regarded as a surface phenomenon.
- Spectral radiation intensity has the units of W/m^2 μm sr.
- The ratio of $I_b(T)$ and $E_b(T)$ is $1/\pi$.
- Solar radiation ranges between the wavelengths of 0.1 and 3 μm.
- Solid angle is defined as $d\omega = \dfrac{dA_{\text{normal}}}{r^2}$.
- The fractions of ultraviolet, visible, and infrared radiation at T = 5800 K are 0.112, 0.456, and 0.432 respectively.
- If r_1 and T_1 denote the radius and absolute temperature of the sun, and x is the mean earth-sun distance, then the mean earth temperature, T_2 is estimated to be $T_2 = T_1\sqrt{r_1/2x}$.
- The expression for solar constant G_S in terms of earth–sun distance L, sun's surface temperature T, its radius r and Stefan–Boltzmann constant σ is given by $G_S = \left(\dfrac{r}{L}\right)^2 \sigma T^4$.
- The intensity of radiation at an angle to the diffuse surface, I_θ is related to the normal intensity I_n by the relation $I_\theta/I_n = \cos\,\theta$.

- Solar constant is the amount of solar radiation incident upon a surface of unit area that is normal to the sun rays and situated just outside the earth's atmosphere.
- The intensity of radiation leaving the sun is 20.354×10^6 W/m^2 sr.
- A diffuse gray surface at 0°C and of emissivity 0.9 radiates to the environment at absolute zero temperature. The radiation heat loss per unit area is 284 W.
- Spectral concentration of surface radiation depends strongly on surface temperature.
- Much of the UV solar radiation is absorbed in the earth's atmosphere.
- Less than 6% of the energy consumed by the lamp is converted to visible light.
- The assumption $\varepsilon_\lambda = \alpha_\lambda$ can be satisfied if this surface were irradiated diffusely or if the surface itself were diffuse. Under the specified conditions of solar irradiation and surface temperature, $\alpha_S \neq \varepsilon$. Such a surface is referred to as a spectrally selective surface.
- A large value of α/ε is desirable for solar absorbers.

GLOSSARY of Key Terms

• **Absorption**	The process of converting radiation intercepted by matter to internal thermal energy.
• **Absorptivity**	Fraction of the incident radiation absorbed by a body. Ratio of the radiation absorbed by a body to that incident upon it.
• **Albedo**	Ratio of the radiation reflected from the earth to the incident radiation.
• **Black body or black surface**	A body (*or surface*) which (a) absorbs all radiation incident upon it (*and reflects or transmits none*), or (b) emits, at any particular temperature, the maximum possible amount of thermal radiation.
• **Black-body source or enclosure**	A source (*or enclosure*) whose absorption and emission characteristics closely approach that of a black body.
• **Diffuse reflection**	Uniform reflection of radiation over all angles, independent of the angle of incidence.
• **Effective black-body temperature (brightness temperature)**	The temperature of a black body which emits the same amount of radiation as the body being considered.
• **Emission**	The process of radiation production by matter at a finite temperature.
• **Emissive power**	Rate of radiant energy emitted by a surface in all directions per unit area of the surface, E (*W/m*^{2}).
• **Emissivity**	The ratio of the radiant energy emitted by the body under consideration to that emitted by a black body at the same temperature.
• **Gray surface**	Surface for which the spectral absorptivity and the emissivity are independent of wavelength over the spectral regions of surface irradiation and emission.
• **Global radiation**	The sum of the vertical component of the direct solar radiation on the earth and the diffuse (*or scattered*) solar radiation.

• **Greenhouse effect**	Solar heating of objects shielded by glass which transmit solar radiation but absorb most of the radiation emitted by the bodies themselves.
• **Intensity of radiation**	Radiant energy per unit area per unit time per unit solid angle.
• **Irradiation**	Rate at which radiation is incident on a surface from all directions per unit area of the surface, $G(W/m^2)$.
• **Kirchhoff's law**	Relation between emission and absorption properties for surfaces irradiated by a black body at the same temperature.
• **Monochromatic radiation**	Radiation of one particular wavelength.
• **Planck's law**	Spectral distribution of emission from a black body.
• **Photon**	Quantum or bundle of radiation.
• **Pyrometer**	A radiometer calibrated to read temperature.
• **Radiation**	(1) Transmission of energy by electromagnetic waves; (2) Radiant energy.
• **Radiosity**	The sum of the incident and reflected radiation fluxes from a surface.
• **Reflectivity**	The ratio of the radiation reflected by a body to that incident upon it.
• **Reflection**	The process of redirection of radiation incident on a surface.
• **Semi-transparent**	Refers to a medium in which radiation absorption is a volumetric process.
• **Solid angle**	Region subtended by an element of area on the surface of a sphere with respect to the centre of the sphere, ω (sr).
• **Specular**	Refers to a surface for which the angle of reflected radiation is equal to the angle of incident radiation.
• **Scattering**	Attenuation of radiation passing through a medium by means other than absorption.
• **Selective emitter**	A material whose monochromatic emissivity is a function of wavelength, angle of emission, or surface temperature.
• **Solar constant**	The amount of solar radiation incident upon a surface normal to the radiation and situated just outside the earth's atmosphere.
• **Thermal radiation**	Electromagnetic energy emitted by matter at a finite temperature and concentrated in the spectral region from approximately 0.1 to 100 μm.
• **Transmission**	The process of thermal radiation passing through matter.
• **Transmissivity**	Fraction of the incident radiation transmitted by matter.
• **Ultraviolet**	Radiation characterized by wavelengths in the 0.1 to 0.4 μm range. Radiation emitted by one surface which is directly incident upon a second surface.
• **Wien's law**	Locus of the wavelength corresponding to peak emission by a black body.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

12.1 A diffuse radiating surface has
(a) radiation intensity independent of angle
(b) emissive power independent of angle
(c) emissive power independent of wavelength
(d) radiation intensity independent of both angle and wavelength.

12.2 In radiation heat transfer, a gray surface is one
(a) which appears gray to the eye (b) whose emissivity is independent of wavelength
(c) which has reflectivity equal to zero. (d) which appears equally bright from all directions.

12.3 What is the basic equation of thermal radiation from which all other equations of radiation can be derived?
(a) Stefan–Boltzmann equation (b) Planck's equation
(c) Wien's equation (d) Rayleigh–Jeans formula

12.4 What is the radiation intensity in a particular direction?
(a) Radiant energy per unit time per unit area of the radiating surface
(b) Radiant energy per unit time per unit solid angle per unit area of the radiating surface
(c) Radiant energy per unit time per unit solid angle per unit projected area of the radiating surface in the given direction
(d) Radiant energy per unit time per unit projected area of the radiating surface in the given direction.

12.5 *All bodies emit thermal radiation, unless the body is at absolute zero temperature*. This statement is based upon
(a) Stefan–Boltzmann law (b) Prevost theory
(c) Wien's law (d) Planck's law

12.6 The following figure was generated from experimental data relating spectral black-body emissive power to wavelength at the three temperatures T_1, T_2 and $T_3(T_1 > T_2 > T_3)$.
The conclusion is that the measurements are
(a) correct because the maxima in E_{b_λ} show the correct trend
(b) correct because Planck's law is satisfied
(c) wrong because the Stefan–Boltzmann law is not satisfied
(d) wrong because Wien's displacement law is not satisfied

12.7 A source of radiation has an emissive power of 1500 W/m^2. How many photons per second per square metre does this intensity represent if the wavelength is 530 nm?
(Speed of light = 2.998×10^8 m/s and Planck's constant = 6.6256×10^{-34} J s)
(a) 4.0×10^{21} (b) 12.8×10^{21} (c) 7.55×10^{21} (d) 1.0×10^{21}

12.8 A small body at 40°C is placed in a large heating oven whose walls are maintained at 1100°C. The average absorptivity of the body varies with the temperature of the emitter as follows:

Temperature (°C)	40	540	1100
Absorptivity, α	0.8	0.6	0.5

What is the rate at which radiant energy is absorbed by the body per unit surface area? (Stefan–Boltzmann constant, $\sigma = 5.67 \times 10^{-8}$ W/m^2 K^4)
(a) 6.65×10^6 W/m^2 (b) 43.3 W/m^2 (c) 10.66×10^5 W/m^2 (d) 1.0×10^5 W/m^2

12.9 A 10 cm diameter spherical ball is known to emit radiation at a rate of 57 W when its surface temperature is 550 K. The average emissivity of the ball at this temperature is

(a) 0.173 (b) 0.35 (c) 0.50 (d) 0.75

12.10 A large spherical enclosure has a small opening. The rate of emission of radiative flux through this opening is 7.35 kW/m^2. The temperature at the inner surface of the sphere will be about (assume Stefan–Boltzmann constant $\sigma = 5.67 \times 10^8$ W/m^2 K^4)

(a) 600 K (b) 330 K (c) 373 K (d) 1000 K

12.11 The maximum rate of heat that can be emitted by radiation per unit area (W/m^2) from a surface at –3°C kept in an environment at 40°C is

(a) 302 (b) 175 (c) 590 (d) 0

12.12 Ten percent of radiation at wavelengths less that $\lambda = 3$ μm and 50 per cent of radiation at wavelengths more than 3 μm is absorbed by a surface. If the radiation emitted by the sun is at $T = 5800$ K, the average solar absorptivity of the surface [$f_{0-\lambda}$(5800 K) = 0.98] is

(a) 0.272 (b) 0.108 (c) 0.425 (d) 0.306

12.13 Match ***List I*** with ***List II*** according to the code given below:

List I (***Body temperature***)	**List II** (***Body colour***)
A. 700°C	**1.** While hot
B. 900°C	**2.** Cherry red
C. 1100°C	**3.** Dull red
D. 1400°C	**4.** Orange red

Codes:

	A	**B**	**C**	**D**
(a)	1	3	4	2
(b)	4	1	2	3
(c)	3	2	1	4
(d)	3	2	4	1

Answers

Multiple-Choice Questions

12.1 (b) **12.2** (b) **12.3** (b) **12.4** (c) **12.5** (b) **12.6** (d)
12.7 (a) **12.8** (d) **12.9** (b) **12.10** (a) **12.11** (a) **12.12** (b)
12.13 (d)

REVIEW QUESTIONS

12.1 Explain the nature of thermal radiation. Which important features characterize radiation?

12.2 What is an *electromagnetic wave*? How is it different from an *acoustic wave*? What is the relation between wavelength and frequency of radiation propagating in a medium?

12.3 What is *visible light*? How does it differ from the other forms of electromagnetic radiation? What is the range of wavelength for visible radiation?

12.4 What are the wavelength bands covering *ultraviolet* (UV), *visible*, and *infrared* (IR) radiation?

12.5 What is *thermal radiation*? In what region of the electromagnetic spectrum is it concentrated?

12.6 Explain why freshly fallen snow gives high glare but melts quickly when the sun shines.

12.7 When a body is said to be black, what is the range of wavelengths which it will absorb?

12.8 What do you mean by a *black body*? Does a black body actually exist? Describe an arrangement by which a black body can be approximated in a laboratory.

12.9 Distinguish between the *total* and *monochromatic* emissive powers. How are they related to each other and what are their units?

12.10 Distinguish between *spectral* and *total* radiation. What is the distinction between *directional* and *hemispherical* radiation?

12.11 Define *solid angle* and *intensity of radiation*.

12.12 What is the *black-body radiation function*? What does it represent? For what is it used?

12.13 Describe the phenomenon of radiation from *real surfaces*.

12.14 Does *emissivity* or *absorptivity* vary with temperature? Would their change in turn affect *transmissivity*?

12.15 Differentiate between *specular* and *diffuse* surfaces.

12.16 Show that emissivity ε and absorptivity α are not necessarily equal for irradiation of a source at temperature T by a source at a different temperature T^*.

12.17 What is the *Stefan–Boltzmann law*?

12.18 Distinguish between *black-body radiation* and *gray-body radiation*. Can the *Wiens' displacement law of radiation* be applied to *gray-body radiation*?

12.19 Define *Planck's distribution law* and derive *Stefan–Boltzmann law* from *Planck's distribution equation*.

12.20 State *Planck's distribution law*. For extremely low and high limits of the *wavelength absolute temperature product* λT, derive the *Wien's law* and *Rayleigh–Jeans law* from the *Planck's law*.

12.21 Enunciate and explain the *Wien's displacement law*. Show that the spectral emissive power of a black body will be maximum when $\lambda_{max} T = 2897.8$ μm K.

12.22 Prove that for a diffusely emitted surface, the emissive power is π times the intensity of emitted radiation.

12.23 What is *Kirchhoff's law*? What are the conditions under which it is applicable?

12.24 What is meant by a *black body* and a *gray body*? Is the black body always black?

12.25 What is *Lambert's cosine law*?

12.26 Define *absorptivity*, *emissivity*, *emissive power, reflectivity,* and *transmissivity*.

12.27 What is the *solar constant*? How is it used to determine the *effective surface temperature* of the sun?

12.28 What is meant by *effective sky temperature*?

12.29 Explain what is meant by *greenhouse effect*?

12.30 What physical ratio determines whether a real surface is an almost specular reflector or an almost diffuse reflector?

PRACTICE PROBLEMS

(A) Black-Body Radiation

12.1 A large isothermal enclosure is maintained at a uniform temperature of 2000 K. (a) Calculate the emissive power of a small aperture on the enclosure. (b) What are the wavelengths below which and above which 10% of the emission is concentrated? (c) Find the monochromatic emissive power and the wavelength associated with maximum emission. (d) Determine the irradiation on a small object placed inside the enclosure.

[(a) 907.2 kW/m^2 (b) 1.10 μm, 4.69 μm (c) 411.7 W/m^2 μm (d) 907.2 kW/m^2]

12.2 A spherical ball of 15 cm diameter is known to emit radiant energy at a rate of 650 kJ/h when its surface temperature is 250°C. What is the average emissivity of the ball at this temperature? **[0.60]**

12.3 The intensity of radiation leaving a surface varies as follows: $I(\theta) = I_0(\cos\theta)^{0.2}$ where θ is the angle to the normal and I_o is a constant that corresponds to the intensity of a blackbody at 2350 K. Calculate the emitted radiant energy flux and compare it with that from a blackbody at 2350 K.
[1.729×10^6 W/m²]

12.4 A mildly polished steel surface with an emissivity of 0.07 and held at a temperature of 760°C emits radiation diffusely. Compute (a) the total radiation intensity in the normal direction, and (b) the fraction of the total hemispherical emissive power irradiated into a solid angle subtended by $0° \le \theta \le 55°$ and $0 \le \varphi \le 2\pi$? **[0.671]**

12.5 An industrial furnace in the form of a black body emits radiation at 3000 K. Calculate the following: (a) Monochromatic emissive power at 1 micrometre wavelength (b) Wavelength at which the emission is maximum (c) Total and maximum emissive power (d) Wavelength λ such that the emission in the wavelength band 0 to λ equals that from λ to ∞.
[(a) 3.12×10^6 W/m² μm (b) 0.966 μm (c) 4592.7 kW/m² (d) 1.37 μm]

12.6 A hole of area dA = 2 cm² is opened on the surface of a large spherical cavity whose inside is maintained at T = 1000 K. (a) Calculate the radiant energy streaming through the hole in all directions into space. (b) Find the radiant energy streaming per unit solid angle in the direction making an angle of 50° with the normal to the surface of the opening. **[(a) 11.34 W (b) 2.32 W]**

12.7 A body having an area of 1000 cm² has an effective temperature of 900 K. Determine (a) the total rate of energy emission, (b) the intensity of normal radiation, (c) the intensity of radiation along a direction at 60° to the normal, and (d) the wavelength of maximum monochromatic emissive power.
[(a) 3.72 kW (b) 11.84 kW/m² μm (c) 5.92 kW/m² (d) 3.22 μm]

12.8 A heat flux gauge of sensitive area 1.4×10^{-5} m² is calibrated by irradiating it with 1 kW/m² from an aperture in the furnace. The aperture diameter is 20 mm and its emissive power is 6.25×10^5 W/m². (a) Find the distance (*along the normal to the aperture*) at which the gauge must be held to receive the requisite irradiation. (b) If the gauge is tilted at 40° to the normal, what will be the irradiation?
[(a) 250 mm (b) 766 W/m²]

12.9 Show that the ratio of the fractional change in the monochromatic intensity of radiation of a blackbody to the fractional change in its absolute temperature is given by $\frac{dI_\lambda/I_\lambda}{dT/T} = \frac{C_2}{\lambda T}[1 - \exp(-C_2/\lambda T)]^{-1}$. Using the above expression, calculate the permissible variation in the temperature of a furnace at 2200 K if the variation in the monochromatic intensity of radiation at 0.65 μm is restricted to 1%. What would this variation be if the wavelength were 10 μm. Also find the wavelength corresponding to the peak of the blackbody curve at 2200 K. **[2.2 K, 16.2 K, 1.32 μm]**

(B) Radiation Properties

12.10 Calculate the absorptivity of a surface having the following characteristics:

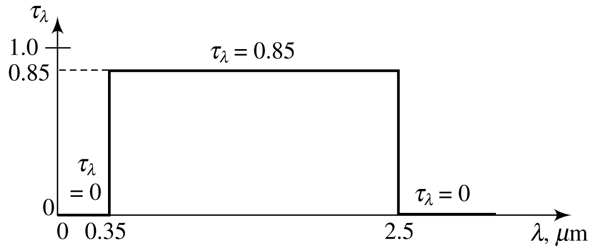
$\alpha_\lambda = 0.92 \qquad 0 \le \lambda \le 4$ μm

$\alpha_\lambda = 0.15 \qquad 4 \text{ μm} \le \lambda \le \infty$

Assume that blackbody radiation at 5770 K is incident on the surface.
If $\varepsilon_\lambda = \alpha_\lambda$, calculate the emissivity of the surface if it is at 400 K. **[0.912, 0.165]**

12.11 The spectral hemispherical transmissivity of a 3 mm thick glass varies with wavelength in the following manner:

(a) Calculate the average transmissivity and the rate of radiation transmitted through a 2 m by 2 m glass window if irradiation from a black surface at 1600 K is incident on it. (b) What will be the amount of solar radiation transmitted through the window assuming the sun to be a blackbody source at 5780 K. **[(a) 0.409, 607.5 kW (b) 0.761, 192.6 kW]**



12.12 A diffuse, fire brick wall of temperature T_b = 600 K has the spectral emissivity distribution approximated by the following stair step function:

ε_λ	0.1	0.5	0.8
λ (μm)	< 2.0	2.0–15.0	> 15.0

The brick wall is exposed to a bed of coals at T_c = 2000 K. Calculate (a) the total hemispherical emissivity, and (b) the total emissive power of the brick wall. (c) Determine the total absorptivity of the wall to irradiation resulting from emission by the coals. **[(a) 0.532 (b) 3910 W/m^2 (c) 0.309]**

12.13 (a) Determine the hemispherical absorptivity of a gray but directionally selective material with $\alpha(\theta, \phi) = 0.6(1 - \cos \phi)$ when the solar irradiation on the surface is such that the zenith angle is 45° and the azimuth angle is 0°. (b) Also calculate the material's hemispherical emissivity. **[0.6]**

(C) Solar and Environmental Radiation

12.14 A sheet of glass is placed over a number of black objects in direct sunlight. The glass transmits the short wavelength radiation from the sun but absorbs 90 percent of the longer wavelength radiation from the objects themselves. Estimate the equilibrium temperature of the objects for a glass temperature of 20°C. **[75.5°C]**

12.15 A thin metal plate is exposed to solar radiation. The air and the surroundings are at 30°C. The heat transfer coefficient by free convection from the upper surface of the plate is 17.4 W/m^2 K. The plate has an absorptivity of 0.9 at solar wavelength and an emissivity of 0.1 at the long wavelength. Neglecting any heat loss from the lower surface, determine the incident solar radiation intensity in kW/m^2, if the measured equilibrium temperature of the plate is 50°C. Stefan–Boltzmann constant is 5.67×10^{-8} W/m^2 K^4. **[0.402 kW/m^2]**

12.16 Droplets are injected into open space in extraterrestrial applications for radiation cooling. After a short interval, a liquid collector collects them under microgravity conditions. The initial temperature of a spatially isothermal droplet is 50°C and the temperature of the outer space, assumed as blackbody is 3 K. Determine the time elapsed during cooling of the droplets of 2 mm diameter to 30°C. The emissivity of the droplet may be taken as 0.8. **[63.6 s]**

12.17 The distance of the sun from the earth is 150×10^6 km. If the radius of the sun is 0.7×10^6 km and its temperature is 6200 K, estimate approximately the mean temperature of the earth. Assume that the rate of radiative transfer from the sun to the earth is equal to the radiant transfer from the earth to the outer space which is at 0 K. Consider the earth and the sun as black bodies. **[299.5 K]**

12.18 Calculate the equilibrium temperature for plate, exposed to a solar flux of 700 W/m^2 and convection environment at 25°C, with convection coefficient of 10 W/m^2 K. If the plate is coated with

(a) White paint: $\alpha_{\text{sun}} = 0.12$; $\alpha_{\text{low temp.}} = 0.9$.

(b) Flat black lacquer: $\alpha_{\text{sun}} = 0.96$; $\alpha_{\text{low temp.}} = 0.95$.

[64.6°C]

12.19 Solar irradiation of 850 W/m^2 is incident on a large, flat, horizontal metal roof on a day when the wind blowing over the roof causes the heat transfer coefficient of 30 W/m^2 K. The ambient temperature is 25°C, the metal surface emissivity is 0.30, the metal surface absorptivity for the incident solar radiation is 0.70, and the roof is essentially insulated from below. Estimate the roof surface temperature under steady-state conditions. **[39.4°C]**

13 Radiation Heat Exchange Between Surfaces

13.1 ❑ INTRODUCTION

In the previous chapter, we discussed the radiation properties of a single surface. In engineering practice, however, we are mostly interested in the radiation heat exchange between two or more surfaces. This exchange among surfaces depends not only on the surface characteristics like *temperature*, *emissivity*, *absorptivity*, *reflectivity*, etc, but also the *surface geometry*, *relative orientation,* and the *distance between* the *exchanging surfaces*.

In this chapter, the concept of shape factor will be introduced first to be followed by the methods to determine it for different configurations. The net radiation heat exchange in a *two-surface* enclosure for black and non-black bodies will then be dealt with. *Three-surface* enclosure, assuming diffuse-gray surfaces, will be analyzed. Refractory (*reradiating*) surfaces and radiation shields will then be discussed. The main assumption will be the non-participating medium (ideally vacuum but practically most of the gases) between surfaces.

Towards the end, we will also deal with radiation exchange involving a participating medium like some gases, primarily carbon dioxide and water vapour.

13.2 ❑ THE CONCEPT OF SHAPE FACTOR

Radiation heat exchange between surfaces depends on their radiation properties and temperatures as well as the orientation of the surfaces relative to each other.

The fraction of radiation leaving a surface that strikes another surface can be expressed in terms of the orientation of these two surfaces with respect to each other. The surfaces are assumed to be isothermal and diffuse emitters and reflectors and the surfaces are *separated* by a non-participating medium such as vacuum or air.

To take into account the effects of orientation on radiation exchange between two bodies (surfaces), we define a new quantity called the *shape factor*, which is a purely geometric parameter and is independent of the surface properties and temperature. The shape factor depends only on the *shape*, *size, orientation,* and *spacing* of the surfaces involved. The shape factor from an emitting surface i to a receiving surface j is denoted by $F_{i\to j}$ or just F_{ij} and is defined as

F_{ij} = fraction of the radiation leaving the surface i that strikes the surface j directly, i.e., by a straight-line route.

The notation $F_{i\to j}$ or F_{ij} emphasizes that the shape factor is for radiation that travels from the surface *i* to the surface *j*. Radiation leaving the surface *i* may either arrive at or miss the surface *j*. The shape factor is the ratio of all the radiative contributions that strike the surface *j* to all the energy leaving the surface *i*. It is a *dimensionless* quantity with a value varying between *zero* and *unity*. Shape factors are also called *configuration factors, view factors, or angle factors.* The shape factor F_{12}, for example, represents the fraction of radiation leaving the surface 1 heading towards the surface 2, and F_{21}, represents the fraction of radiation leaving the surface 2 heading towards the surface 1. It is worth noting that the radiation that strikes a surface need not necessarily be absorbed by that surface. Furthermore, radiation that strikes a surface after being reflected by other surfaces is not included in the evaluation of shape factors.

13.3 ❑ SHAPE-FACTOR DETERMINATION

1. Shape Factor $F_{dA_1\to dA_2}$ To derive an expression for the shape factor, we first consider two elemental surfaces dA_1 and dA_2 on two arbitrarily oriented surfaces A_1 and A_2, respectively, as shown in Fig. 13.1. The distance between dA_1 and dA_2 is *r*, and the angles between the normals of the surfaces and the line joining dA_1 and dA_2 are θ_1 and θ_2, respectively. Surface 1 emits and reflects radiation diffusely in all directions with a constant intensity of I_1, and the solid angle subtended by dA_2 when viewed from dA_1 is $d\omega_{21}$.

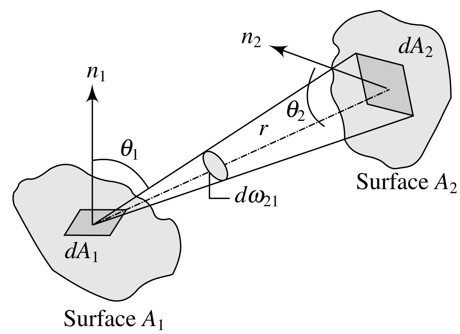


Fig. 13.1 *Shape-factor geometry: Radiant heat exchange between area elements*

The rate at which radiation leaves dA_1 in the direction of θ_1 is $I_1 \cos\theta_1\, dA_1$. Also, $d\omega_{21} = dA_2 \cos\theta_2/r^2$. Then the portion of this radiation that is intercepted by dA_2 is

$$\boxed{\dot{Q}_{dA_1\to dA_2} = I_1 \cos\theta_1\, dA_1\, d\omega_{21} = I_1 \cos\theta_1\, dA_1 \frac{dA_2 \cos\theta_2}{r^2}} \tag{13.1}$$

The total radiation leaving dA_1 (*through emission and reflection*) in all directions is

$$\boxed{\dot{Q}_{dA_1} = J_1 dA_1 = \pi I_1 dA_1} \tag{13.2}$$

where $J_1 = \pi I_1$ is the radiosity and dA_1 is the surface area.

The differential shape factor $dF_{dA_1\to dA_2}$, which is the fraction of the radiation leaving dA_1, intercepted by dA_2 is

$$\boxed{dF_{dA_1\to dA_2} = \frac{\dot{Q}_{dA_1\to dA_2}}{\dot{Q}_{dA_1}} = \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_2} \tag{13.3}$$

From the symmetry of the problem, the differential shape factor $dF_{dA_2\to dA_1}$ can be simply written by interchanging the subscripts 1 and 2. Thus,

$$dF_{dA_2\to dA_1} = \frac{\cos\theta_2 \cos\theta_1 dA_1}{\pi r^2}$$

Hence, it follows that

$$\boxed{dA_1 dF_{dA_1\to dA_2} = dA_2 dF_{dA_2\to dA_1}} \tag{13.4}$$

This is the reciprocity relationship for infinitesimal or differential shape factors.

2. Shape Factor $\boldsymbol{F_{dA_1 \to A_2}}$ The shape factor from the element dA_1 to a finite area A_2 can be found from the fact that the fraction of radiation leaving dA_1 that is intercepted by A_2 is the sum of the fractions of radiation striking the elemental areas dA_2. Hence, the shape factor $F_{dA_1 \to A_2}$ is obtained by integrating $dF_{dA_1 \to dA_2}$ over the area A_2.

$$\boxed{F_{dA_1 \to A_2} = \int_{A_2} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_2} \tag{13.5}$$

3. Shape Factor $\boldsymbol{F_{A_1 \to A_2}}$ The total rate at which radiation leaves the entire surface A_1 (*through emission and reflection*) in all directions is

$$\dot{Q}_{A_1} = J_1 A_1 = \pi I_1 A_1$$

The part of this radiation intercepted by dA_2 is found by considering the radiation that leaves dA_1 and strikes dA_2, and integrating it over A_1,

$$\boxed{\dot{Q}_{A_1 \to dA_2} = \int_{A_1} \dot{Q}_{dA_1 \to dA_2} = \int_{A_1} \frac{I_1 \cos\theta_1 \cos\theta_2}{r^2} dA_2 dA_1} \tag{13.6a}$$

Integration of this relation over A_2 yields the radiation that strikes the entire surface A_2,

$$\boxed{\dot{Q}_{A_1 \to A_2} = \int_{A_2} \dot{Q}_{A_1 \to dA_2} = \int_{A_2} \int_{A_1} \frac{I_1 \cos\theta_1 \cos\theta_2}{r^2} dA_1 dA_2} \tag{13.6b}$$

Dividing this by the total radiation leaving A_1, we obtain the fraction of radiation leaving A_1 that strikes A_2, that is the shape factor $F_{A_1 \to A_2}$ or F_{12}.

$$\boxed{F_{12} = F_{A_1 \to A_2} = \frac{\dot{Q}_{A_1 \to A_2}}{\dot{Q}_{A_1}} = \frac{1}{A_1} \int_{A_2} \int_{A_1} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_1 dA_2} \tag{13.7}$$

The shape factor $F_{A_2 \to A_1}$ is readily obtained by simply interchanging the subscripts 1 and 2 in the above equation.

$$\boxed{F_{21} = F_{A_2 \to A_1} = \frac{\dot{Q}_{A_2 \to A_1}}{\dot{Q}_{A_2}} = \frac{1}{A_2} \int_{A_2} \int_{A_1} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_1 dA_2} \tag{13.8}$$

It is worth mentioning that the order of integration does not matter since the integration limits are constants. Also, the shape factor between two surfaces depends on their relative orientation and the distance between them.

Combining Eqs. (13.7) and (13.8) after multiplying the former by A_1 and the latter by A_2 gives

$$\boxed{A_1 F_{12} = A_2 F_{21} = \int_{A_2} \int_{A_1} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_1 dA_2} \tag{13.9}$$

It is worth noting that the radiation heat exchange between two surfaces is dependent not only on the areas but also on their distance apart. More precisely, it is inversely proportional to the square of the distance between them.

This is both an expression for the shape factor and a statement of the *reciprocal relation* for shape factors.

Shape factor (view factor) $F_{1\text{-}2}$ between two surfaces A_1 and A_1 can be determined directly by performing integration as explained above. But this method is quite complicated even for simple geometries. Hence, usually the analytical, tabular or graphical methods are preferred. Shape factors for some common geometries are given in analgitical form in **Tables 13.1** to **13.3** and in **Figs 13.2** to **13.5** in the graphical form (as charts).

Table 13.1 *Shape factors for some common three-dimensional configurations*

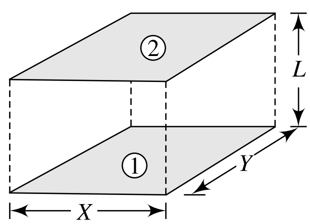
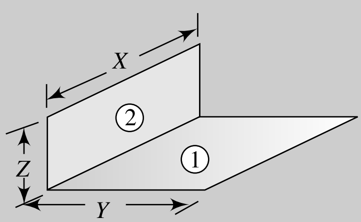
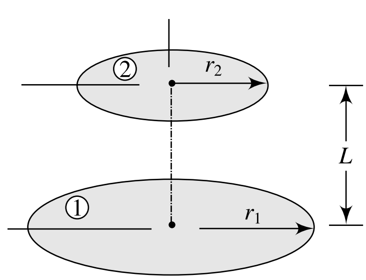
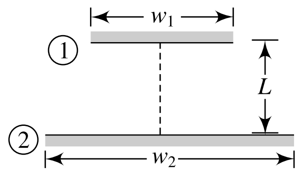
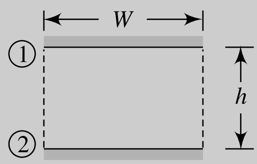
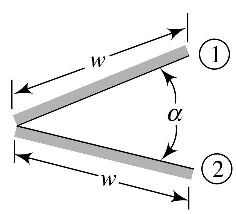
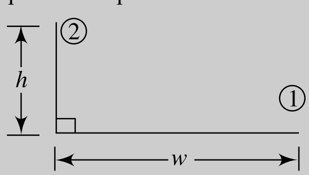
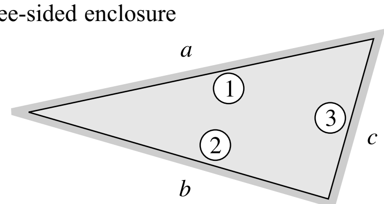
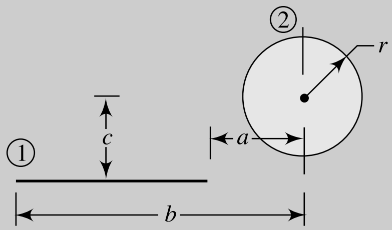
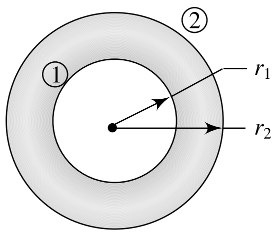
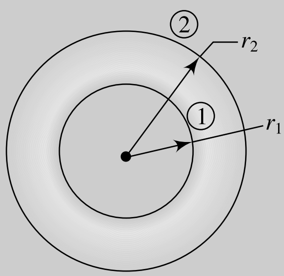
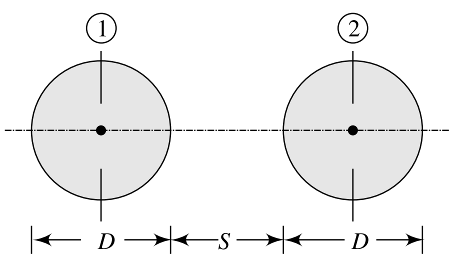
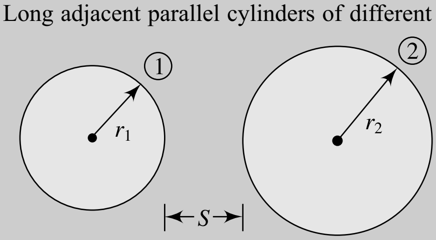
Configuration	Equation
1. Aligned parallel rectangles 	Let $\bar{X} = X/L$ and $\bar{Y} = Y/L$. Then $F_{12} = \frac{2}{\pi \bar{X}\bar{Y}}\left\{ \ln\left[\frac{(1+\bar{X}^2)(1+\bar{Y}^2)}{1+\bar{X}^2+\bar{Y}^2}\right]^{1/2} + \bar{X}\sqrt{1+\bar{Y}^2}\tan^{-1}\frac{\bar{X}}{\sqrt{1+\bar{Y}^2}} + \bar{Y}\sqrt{1+\bar{X}^2}\tan^{-1}\frac{\bar{Y}}{\sqrt{1+\bar{X}^2}} - \bar{X}\tan^{-1}\bar{X} - \bar{Y}\tan^{-1}\bar{Y}\right\}$
2. Perpendicular rectangles with a common edge 	Let $H = Z/X$ and $W = Y/X$. Then $F_{12} = \frac{1}{\pi W}\left[W\tan^{-1}\frac{1}{W} + H\tan^{-1}\frac{1}{H} - \sqrt{(H^2+W^2)}\tan^{-1}(H^2+W^2)^{-1/2} + \frac{1}{4}\ln\left\{\frac{(1+W^2)(1+H^2)}{1+W^2+H^2}\times\left[\frac{W^2(1+W^2+H^2)}{(1+W^2)(W^2+H^2)}\right]^{W^2}\times\left[\frac{H^2(1+H^2+W^2)}{(1+H^2)(W^2+H^2)}\right]^{H^2}\right\}\right]$
3. Coaxial parallel disks 	Let $R_1 = r_1/L$, $R_2 = r_2/L$, and $S = 1 + \frac{1+R_2^2}{R_1^2}$ Then $F_{12} = \frac{1}{2}[S - \sqrt{S^2 - 4(R_2/R_1)^2}]$

Table 13.2 *Shape factors for a variety of two-dimensional configurations (infinite in extent normal to the paper)*

Configuration	Equation
1. Parallel plates with centrelines connected by perpendicular line 	$F_{1,2} = \dfrac{[(W_1+W_2)^2+4]^{1/2}-[(W_2-W_1)^2+4]^{1/2}}{2W_1}$ $W_1 = w_1/L,\ W_2 = w_2/L$
2. Long parallel plates of equal width 	$F_{12} = F_{21} = \sqrt{1+\left(\dfrac{h}{w}\right)^2} - \left(\dfrac{h}{w}\right)$
3. Inclined plates of equal width and with a common edge 	$F_{12} = F_{21} = 1 - \sin(\alpha/2)$
4. Long perpendicular plates with a common edge 	$F_{12} = \dfrac{1}{2}[1+H-\sqrt{1+H^2}]$ $H \equiv \dfrac{h}{w}$
5. Three-sided enclosure 	$F_{12} = (A_1 + A_2 - A_3)/2\,A_1$ $= (a + b - c)/2a$

Contd.

Contd.

6. Long cylinder and parallel rectangle 	$F_{12} = \frac{r}{b-a}\left[\tan^{-1}\frac{b}{c} - \tan^{-1}\frac{a}{c}\right]$
7. Concentric cylinders 	$F_{12} = 1$, $F_{21} = r_1/r_2$ $F_{22} = 1 - (r_1/r_2)$
8. Concentric spheres 	$F_{12} = 1$, $F_{21} = (r_1/r_2)^2$ $F_{22} = 1 - (r_1/r_2)^2$
9. Long adjacent parallel cylinders of equal diameters 	Let $X = 1 + (s / D)$. Then $F_{12} = F_{21} = \frac{1}{\pi}\left[\sqrt{X^2-1} + \sin^{-1}\frac{1}{X} - X\right]$
10. Long adjacent parallel cylinders of different radii 	$F_{12} = \frac{1}{2\pi}\Big\{\pi + [C^2-(R+1)^2]^{1/2} - [C^2-(R-1)^2]^{1/2}$ $+ (R-1)\cos^{-1}\left[\left(\frac{R}{C}\right)-\left(\frac{1}{C}\right)\right] - (R+1)\cos^{-1}\left[\left(\frac{R}{C}\right)+\left(\frac{1}{C}\right)\right]\Big\}$ $R = r_2/r_1$, $S = s/r_1$ $C = 1 + R + S$

Contd.

Contd.

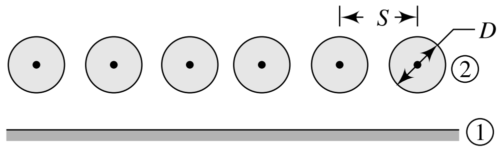
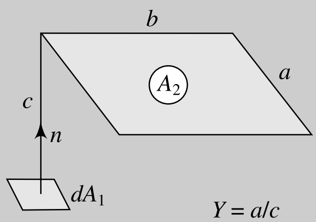
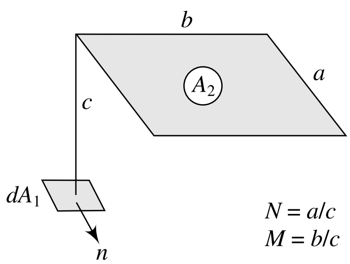
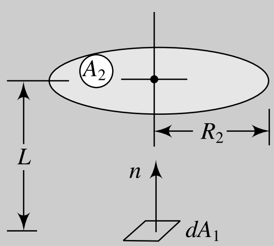
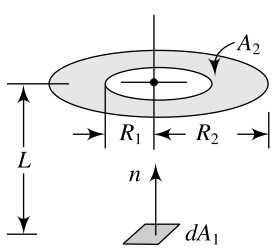
11. Infinite plane and row of cylinders 	$F_{12} = 1 - \left[1 - \left(\frac{D}{s}\right)^2\right]^{1/2} + \left(\frac{D}{s}\right)\tan^{-1}\left(\frac{s^2 - D^2}{D^2}\right)^{1/2}$

Table 13.3 *Shape factors for rectangular and circular areas*

1.  $Y = a/c$ $X = b/c$	Area dA_1 to rectangle; normal to dA_1 perpendicular to plane of rectangle as shown. $F_{12} = \frac{1}{2\pi}\left[\frac{Y}{\sqrt{1+Y^2}}\tan^{-1}\left(\frac{X}{\sqrt{1+Y^2}}\right) + \frac{X}{\sqrt{1+X^2}}\tan^{-1}\left(\frac{Y}{\sqrt{1+X^2}}\right)\right]$
2.  $N = a/c$ $M = b/c$	Area dA_1 to rectangle; normal to dA_1 parallel to plane of rectangle as shown. $F_{12} = \frac{1}{2\pi}\left[\tan^{-1}\frac{1}{M} - \frac{M}{\sqrt{M^2+N^2}}\tan^{-1}\left(\frac{1}{\sqrt{M^2+N^2}}\right)\right]$
3. 	Area dA_1 to disk; normal to dA_1 perpendicular to and passing through the centre of disk. $F_{12} = \frac{R_2^2}{R_2^2 + L^2}$
4. 	Area dA_1 to washer; normal to dA_1 perpendicular to and passing through the centre of hole. $F_{12} = \frac{R_2^2}{R_2^2 + L^2} - \frac{R_1^2}{R_1^2 + L^2}$

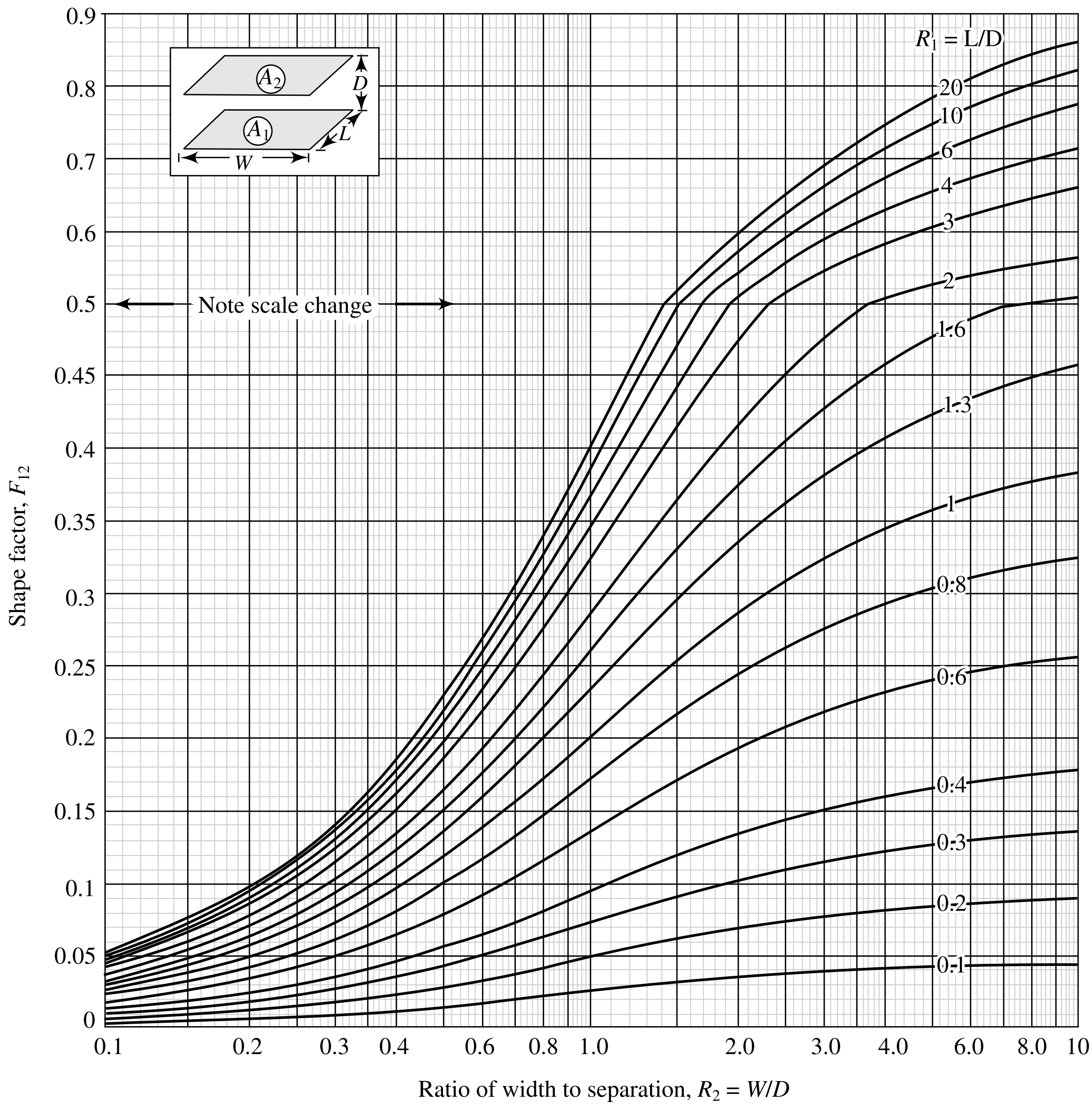


Fig. 13.2 *Shape factor between two aligned parallel rectangles of equal size*

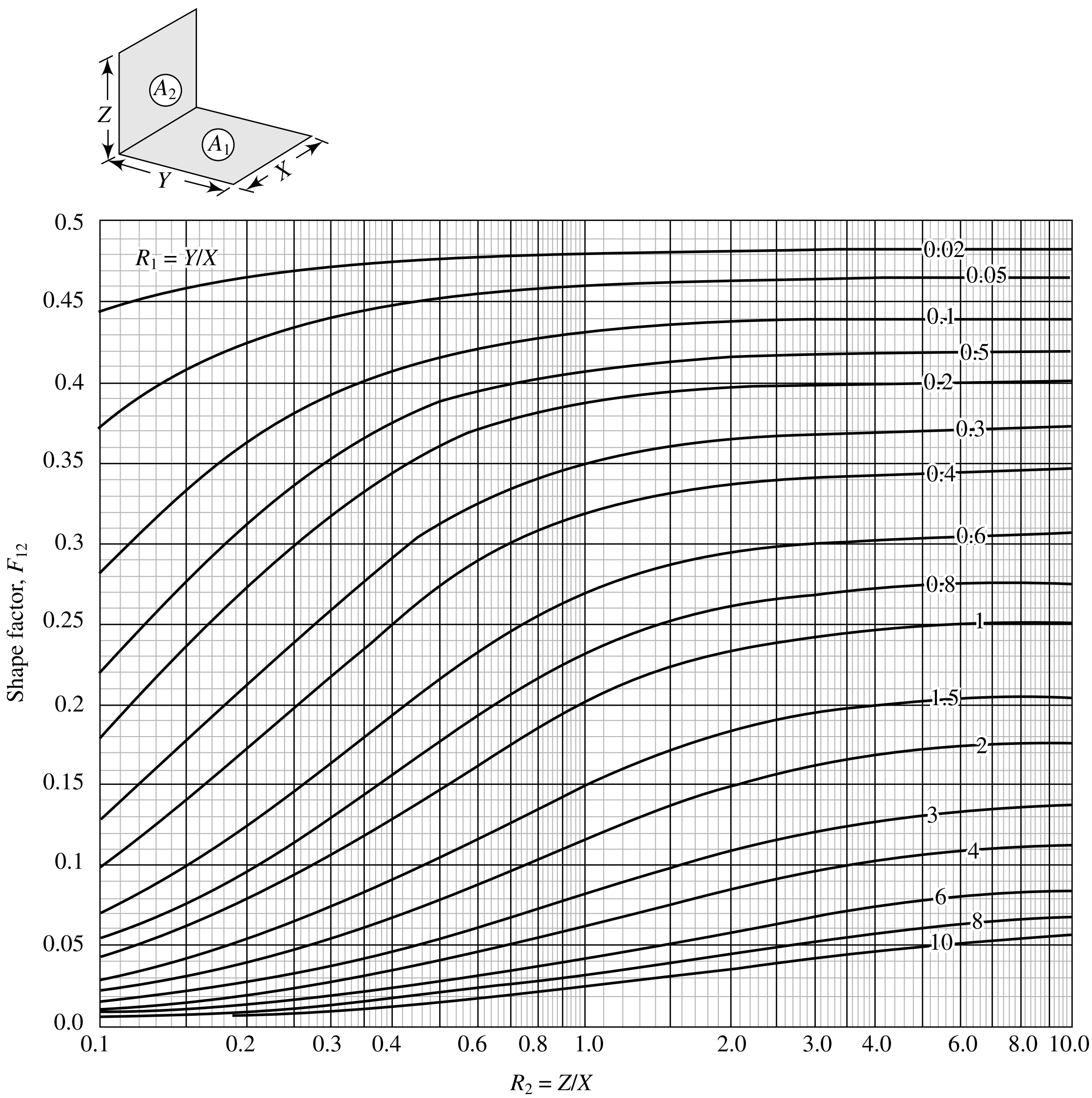


Fig. 13.3 *Shape factor between two perpendicular rectangles with a common edge*

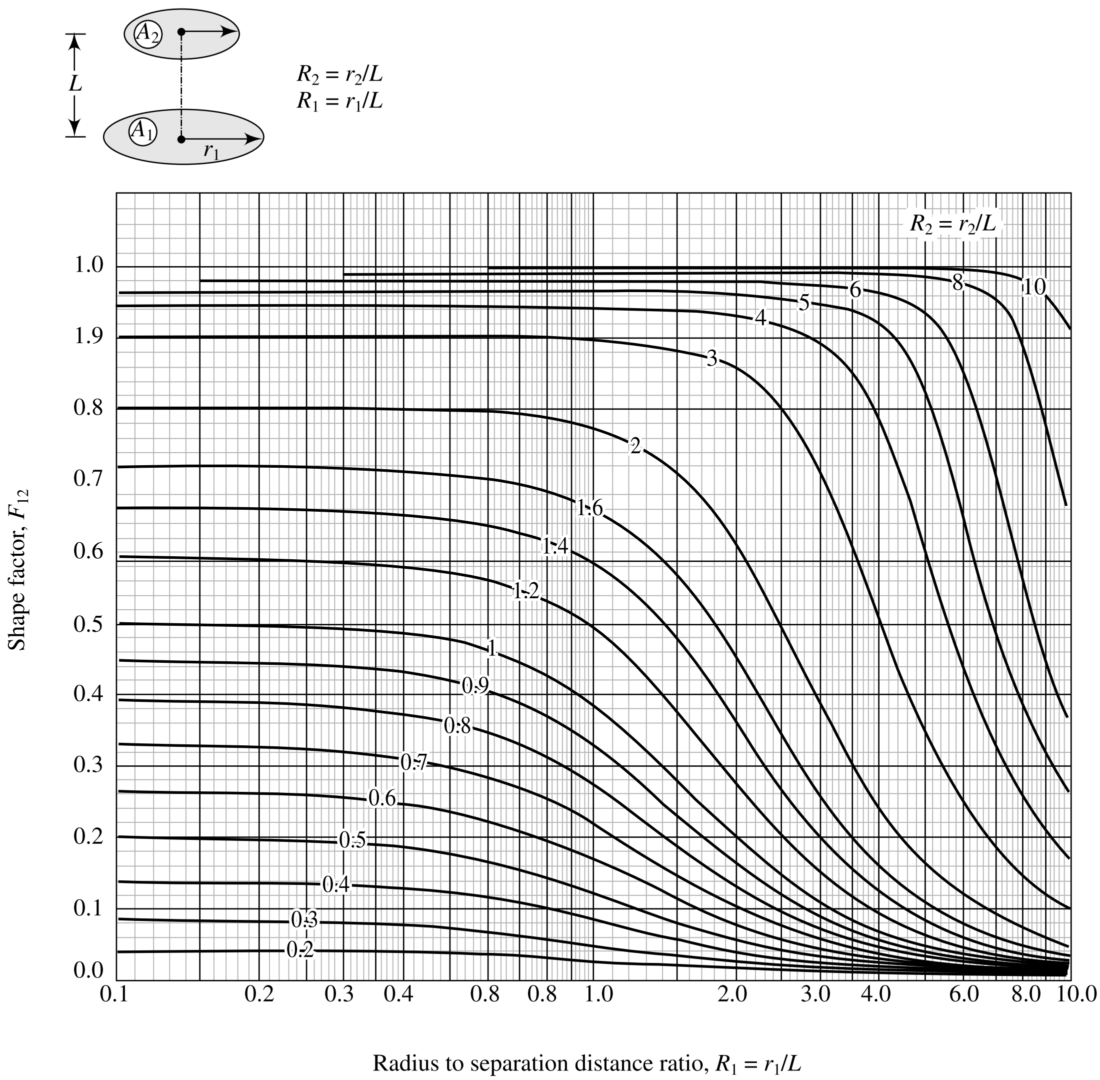


Fig. 13.4 *Shape factor between two aligned parallel discs*

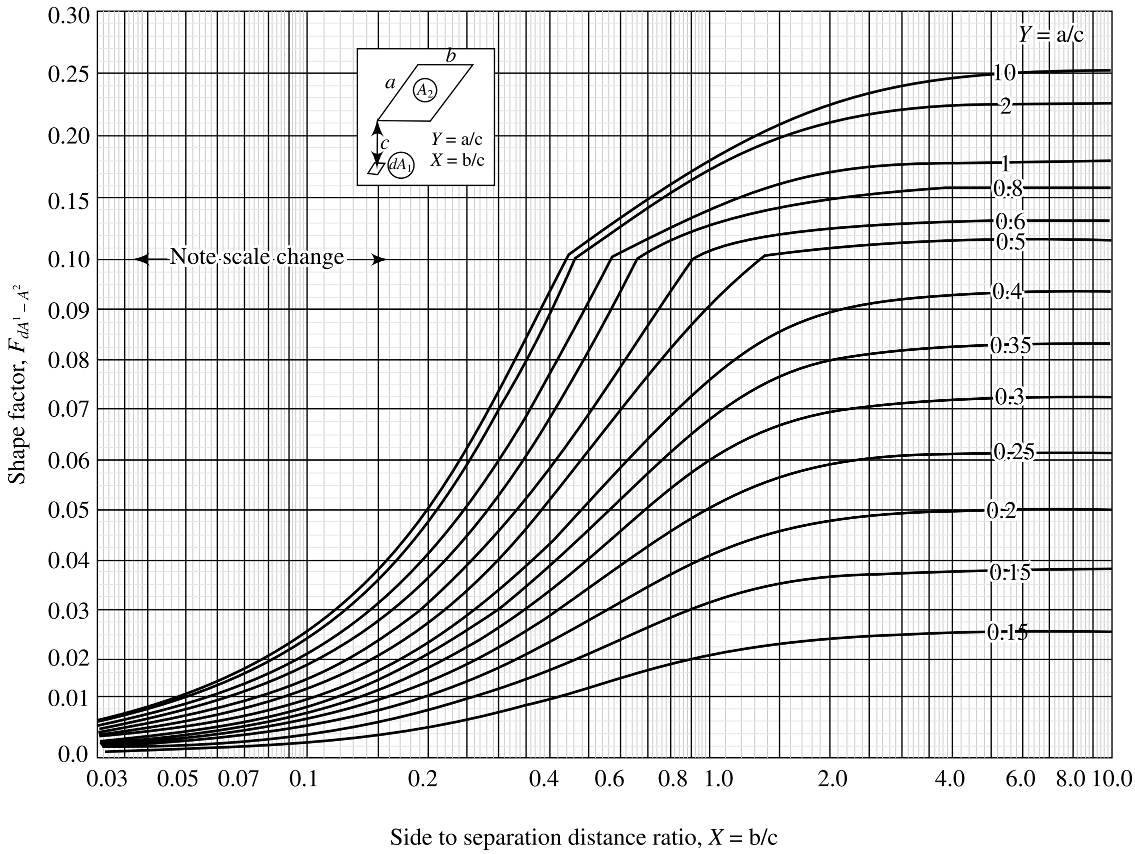


Fig. 13.5 *Shape factor between a surface element dA_1 and a rectangular surface A_2 parallel to it*

13.4 ❑ SHAPE-FACTOR PROPERTIES AND SHAPE-FACTOR ALGEBRA

Consider an enclosure with N surfaces. The number of shape factors involved is N^2 (all F_{ij}'s for $i = 1\ldots N$ and $j = 1 \ldots N$) as represented in the matrix form given below.

$$\begin{bmatrix} F_{11} & F_{12} \ldots F_{1N} \\ F_{21} & F_{22} \ldots F_{2N} \\ F_{N1} & F_{N2} \ldots F_{NN} \end{bmatrix} \tag{13.10}$$

However, it is tedious and time consuming to evaluate all the shape factors directly. Once a sufficient number of shape factors are available, the rest of them can be determined by utilizing some fundamental relations for shape factors.

Summation rule applied to every surface yields N relations among the shape factors.

Considering the reciprocity relation, for the N-walled enclosures, we have ${}^NC_2$ or $N(N-1)/2$ equations in the shape factors. For a given i, there are $(N-1)$ values of j. Now, i may be varied from 1 to N. Thus, the total number of equations available is

$$\boxed{N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}} \tag{13.11}$$

The independent shape factors to be determined are

$$\boxed{N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}} \tag{13.12}$$

If all the surfaces are *convex* or *flat*, $F_{ii} = 0$ for $i = 1 \ldots N$. Thus, the number of independent shape factors to be determined is $\boxed{\frac{N(N-1)}{2} - N = \frac{N(N-3)}{2}}$. For instance, if $N = 3$, the shape factor algebra provides all the shape factors.

Determination of shape factor often becomes difficult for bodies having complex geometries. Sometimes the problem can be tackled by making use of the definition of shape factor and the reciprocity relation. *Shape-factor algebra* is a methodology to determine the shape factor for a pair of surfaces from known shape factors of surfaces of other geometries and orientations.

13.4.1 • Summation Rule

An enclosure is a *three-dimensional* region in space completely encased by bounding surfaces, as shown schematically in Fig. 13.6. Generally, all the enclosure surfaces are *plane* or *flat* so that none of the radiation that leaves the surface strikes the surface itself directly (*non-reentrant*). For surfaces that have a concave curvature, a fraction of the radiant energy that leaves the surface strikes the surface directly i.e. the surface *sees* itself. For example, the inside of a hemisphere can *see* itself.

This fraction is called *self-view factor* and is designated as F_{11} or F_{ii} in the general sense. Figure 13.7 illustrates this concept.

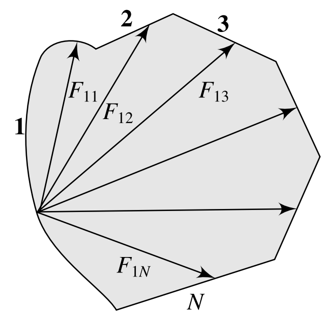
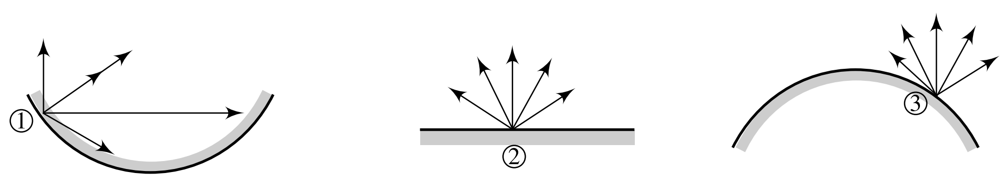


Fig. 13.6 *A black-body enclosure with N surfaces*



$F_{11} \neq 0$ (a) Concave surface

$F_{22} = 0$ (b) Flat surface

$F_{33} = 0$ (c) Convex surface

Fig. 13.7 *Self-view factor is non-zero for concave surface, and zero for flat and convex surfaces*

Consider radiation leaving the surface 1 in Fig. 13.6. Conservation of energy states that we must account for all the energy leaving the surface 1. Since the enclosure is closed, all the radiation leaving the surface 1 must arrive at either 2, 3, or 4.

More generally, the principle of conservation of energy requires that the entire radiation leaving any surface i of an enclosure be intercepted by the surface of the enclosure. The sum of the shape factors from the surface i to all the surfaces of the enclosure, *including to itself*, must, therefore, equal unity. This is known as the *summation rule* for an enclosure and can be expressed as.

$$\boxed{\sum_{j=1}^{N} F_{i\to j} = 1}$$

where N is the number of surfaces of the enclosure, i.e., all the radiation leaving a surface must be accounted for by the radiation received by all the surfaces in an enclosure.

If the surface A_1 is re-entrant (*not flat*), the summation should include surface A_1 and the index j should range from 1 to the N^{th} surface. Thus, for *re-entrant* surfaces,

$$\boxed{F_{11} + F_{12} + F_{13} + F_{14} + \cdots + F_N = 1}$$

For *non-re-entrant* surfaces.

$$\boxed{F_{12} + F_{13} + F_{14} + \cdots + F_N = 1}$$

13.4.2 • Reciprocity Rule

We have shown earlier that the pair of shape factors F_{i-j} and F_{j-i} are related to each other by

$$\boxed{A_i F_{i-j} = A_j F_{j-i}}$$

- This relation is referred to as the *reciprocity relation* or the *reciprocity rule*. The shape factors $F_{i\to j}$ and $F_{j\to i}$ are *not* equal to each other unless the areas of the two surfaces are equal. That is,

$$F_{j-i} = F_{i-j} \quad \text{when} \quad A_i = A_j$$

$$F_{j-i} \neq F_{i-j} \quad \text{when} \quad A_i \neq A_j$$

When determining the pair of shape factors F_{i-j} and F_{j-i}, it makes sense to evaluate first the easier one directly and then the more difficult one by applying the reciprocity relation. If a surface A_1 is completely enclosed by a second surface A_2, and if A_1 does not *see* itself ($F_{1-1} = 0$) then $F_{1-2} = 1$.

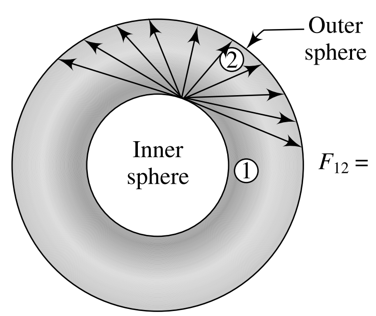


Fig. 13.8 *In the case of two concentric spheres, the shape factor F_{12} = 1 as the entire radiation emitted by the surface 1 is intercepted by the surface 2*

Let us now consider a simple example of radiation exchange in an enclosure formed by two concentric spherical surfaces (Fig. 13.8). The outer surface has an area A_2, and the inner one has an area A_1. The inner surface cannot *see* itself and all the radiation it emits is intercepted by the area A_2 of the outer sphere. Hence, $F_{12} = 1$. But the outer spherical surface, being concave inwards, can see itself partly ('partly', because its view is partially obstructed by the surface A_1). Using the reciprocity relation, we have

$$\boxed{A_1 F_{12} = A_2 F_{21}} \quad \text{or} \quad \boxed{F_{21} = \frac{A_1}{A_2} F_{12} = \frac{A_1}{A_2}}$$

13.4.3 • Symmetry Rule

The presence of symmetry can be determined by inspection, keeping the definition of the shape factor in mind. Identical surfaces that are oriented in an identical manner with respect to another surface will intercept identical amounts of radiation leaving that surface, Therefore, the symmetry rule can be expressed as *two (or more) surfaces that possess symmetry about a third surface will have identical shape factors from that surface*.

Consider a *right circular cylinder* with the cylindrical (*lateral*) surface A_3, one flat end surface A_1, the other flat end surface A_2 and we can determine the shape factor F_{33} from $F_{31} + F_{32} + F_{33} = 1$. We note that $F_{31} = F_{32}$ by *symmetry* (from the observation) so that $2F_{31} + F_{33} = 1$. Together with *reciprocity rule*, $A_3F_{31} = A_1F_{13}$, we can determine F_{33} because F_{12} can be obtained from the chart or Table and $F_{13} = 1 - F_{12}$.

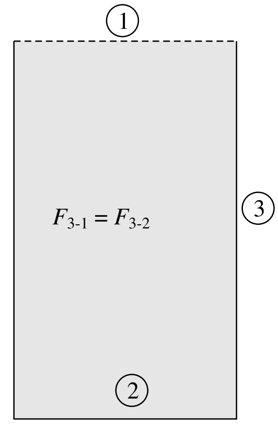


Fig. 13.9 *Symmetry rule*

13.4.4 • Law of Corresponding Corners

As shown in Fig. 13.10, by simple observation, one can write

$$\boxed{A_1F_{12} = A_3F_{34}}$$

This is often referred to as the *law of corresponding corners*.

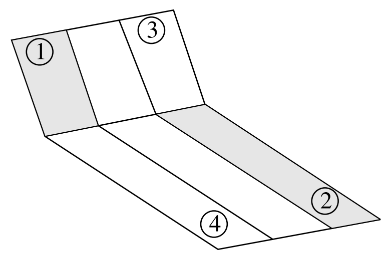


Fig. 13.10 *Law of corresponding corners*

13.4.5 • Shape Decomposition and Superposition Rule

Sometimes the shape factor associated with a given geometry is not available in standard tables and charts. In such a case, it is desirable to use the method of shape decomposition. A major advantage of shape decomposition is that it allows the calculation of shape factors without resorting to integration. In this case, we express the given geometry as the sum or difference of some geometries with known shape factors, and then apply the *superposition rule,* which can be expressed as *the shape factor from a surface i to a surface j is equal to the sum of the shape factors from surface i to the parts of surface j*.

Referring to Fig. 13.10, the arbitrary surface 1 radiates to the arbitrary surface 2, which is composed of two subsurfaces, 2*a* and 2*b*. All the radiation striking 2 must strike either 2*a* or 2*b*; therefore,

$$\boxed{F_{1\to2} = F_{1\to2a} + F_{1\to2b}} \tag{13.13}$$

This equation is useful if two of the shape factors are known and third one is to be evaluated. Shape decomposition is thus a simple and effective technique for finding this shape factor.

The receiving surface 2 can be divided into any arbitrary number of subsurfaces and is not confined to two. If the surface 2 is subdivided into N subsurfaces then

$$\boxed{F_{1\to2} = F_{1\to2a} + F_{1\to2b} + F_{1\to2c} + \cdots + F_{1\to2N} = \sum_{i=1}^{N} F_{1-2i}}$$

Shape decomposition is specially useful when used along with *reciprocity relation* and *principle of symmetry*.

Note that the reverse of this is not true. That is, the shape factor from a surface j to a surface i is not equal to the sum of the shape factors from the parts of surface j to surface i.

Consider the geometry in Fig. 13.11, which is infinitely long in the direction perpendicular to the plane of the paper. The radiation that leaves the surface 1 and strikes the combined surfaces 2 and 3 is equal to the sum of the radiation that strikes surfaces 2 and 3. Therefore, the shape factor from surface 1 to the combined surfaces of 2 and 3 is

$$\boxed{F_{1-(2,3)} = F_{1-2} + F_{1-3}} \tag{13.14}$$

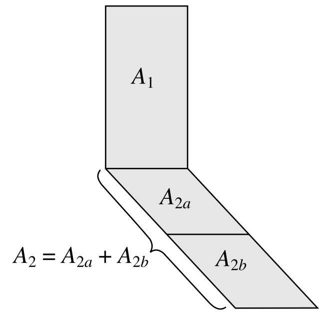


Fig. 13.11 *Shape decomposition rule*

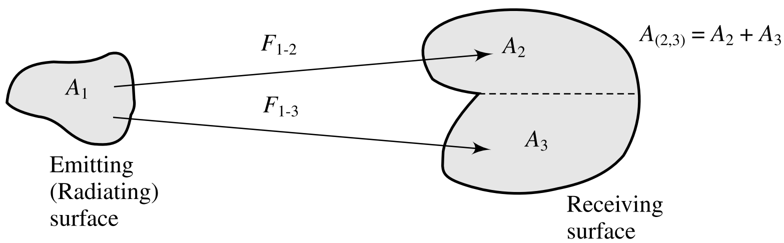


Fig. 13.12 *Summation rule*

To obtain a relation for the shape factor $F_{(2,\,3)-1}$, we multiply Eq. (13.14) by A_1,

$$\boxed{A_1 F_{1-(2,3)} = A_1 F_{1-2} + A_1 F_{1-3}}$$

and apply the reciprocity relation to each term to get

$$(A_2 + A_3) F_{(2,3)-1} = A_2 F_{2-1} + A_3 F_{3-1}$$

or

$$\boxed{F_{(2,3)-1} = \frac{A_2 F_{2-1} + A_3 F_{3-1}}{A_2 + A_3}}$$

Thus, if the transmitting (*radiating* or *emitting*) surface is subdivided, the shape factor for that surface with respect to a receiving surface is *not* simply the sum of the individual shape factors, although the *AF* product is expressed by such a sum. On the other hand, the shape factor from a radiating surface to a subdivided (*decomposed*), receiving surface is simply the sum of the individual shape factors.

In some cases, the geometry under consideration is *not* totally enclosed. For example, consider a small, cylindrical tube open at both ends suspended in a large room. It is usually possible to *plug* the open ends by using black hypothetical surfaces.

Take another case of a cavity. Radiation incident on the opening in the cavity is effectively trapped inside. One can cover the opening with an imaginary black surface at the temperature of the interior walls of the cavity.

13.5 ❑ HOTTEL'S CROSSED-STRINGS METHOD

This is a simple method to determine shape factors of two-dimensional configurations. We come across many problems in practice involving geometries of constant cross section such as ducts and channels which are *very long* in one direction relative to the other directions. Such geometries are considered *two-*

dimensional, because any radiation exchange through their end surfaces is negligible. The shape factor between the surfaces of such geometries can be determined by the crossed-strings method developed by H C Hottel in the 1950s.

The crossed-string rule is applicable when the surfaces have

- lengths much greater than their widths and separation,
- constant cross sections normal to their lengths, and
- constant separation along their lengths.

The surfaces of these geometrical arrangements need not be *flat*. They can be *convex*, *concave*, or of any *irregular* shape.

To find the shape factor F_{1-2} between surfaces 1 and 2 (Fig. 13.13), first we identify the end points of the surfaces *A*, *B*, *C*, and *D* and connect them to each other with tightly stretched strings. The shape factor F_{1-2} can be expressed in terms of the lengths of these stretched strings, as follows:

$$\boxed{F_{1-2} = \frac{(AD + BC) - (AC + BD)}{2AB}}$$

Note that (*AD* + *BC*) is the sum of the lengths of the *crossed strings*, and (*AC* + *BD*) is the sum of the lengths of the *uncrossed strings* attached to the end points. Therefore, Hottel's crossed-strings method can be expressed as

$$\boxed{F_{12} = \frac{(\text{Sum of the crossed strings}) - (\text{Sum of the uncrossed or straight strings})}{2 \times \text{String on Surface 1}}} \tag{13.15}$$

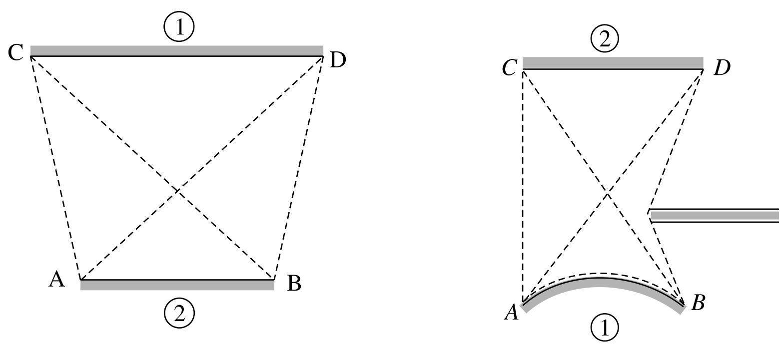


Fig. 13.13 *The 'string rule' for shape factors of two-dimensional configurations*

The crossed-strings method is also useful even when the two surfaces of interest share a common edge, as in a triangle. The common edge in this case can be looked upon as an imaginary string of zero length. One can also apply this method to surfaces partially blocked by other surfaces by allowing the strings to bend around the blocking surfaces.

Let me illustrate the use of the Hottel's string rule for the determination of shape factor for the configuration shown in Fig. 13.14.

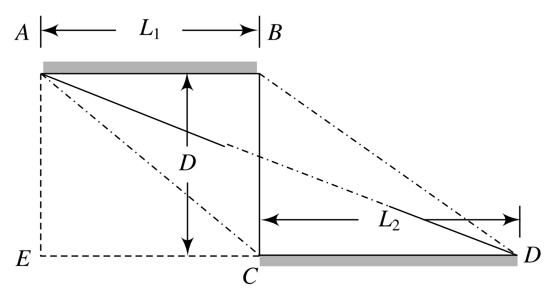


Fig. 13.14 *Hottel's crossed-strings rule*

The shape factor is given by

$$2L_1F_{12} = \begin{pmatrix}\text{Sum of lengths} \\ \text{of two crossed strings}\end{pmatrix} - \begin{pmatrix}\text{Sum of the lengths} \\ \text{of uncrossed strings}\end{pmatrix}$$

$$= (AD + BC) - (AC + BD)$$

Now, $$AC = \sqrt{AB^2 + BC^2} = \sqrt{L_1^2 + D^2}$$

$$BD = \sqrt{BC^2 + CD^2} = \sqrt{L_2^2 + D^2}$$

$$AD = \sqrt{AE^2 + DE^2} = \sqrt{AE^2 + (EC + CD)^2} = \sqrt{D^2 + (L_1 + L_2)^2}$$

Therefore, $$F_{12} = [\sqrt{D^2 + (L_1 + L_2)^2} + D - (\sqrt{L_1^2 + D^2} + \sqrt{L_2^2 + D^2})/2L_1]$$

For $D = L_1 = L_2$, the shape factor will be

$$F_{12} = [\sqrt{1+4} + 1 - 2\sqrt{2}]/2 = \mathbf{0.204}$$

13.6 ❑ RADIANT HEAT EXCHANGE BETWEEN NON-BLACK SURFACES

13.6.1 • (a) Radiation Heat Exchange Between Small Grey Bodies

Consider two small grey bodies 1 and 2 of areas A_1 and A_2 having emissivities ε_1 and ε_2 or absorptivities α_1 and α_2. The radiation emitted by the surface 1 is partly absorbed by the surface 2. The part of radiation not absorbed and thus reflected on the first incidence is considered lost in space (because of the assumption of small surfaces compared to the distance between them). This means that nothing returns again to the surface 1. The same can be said about the surface 2 as well.

The radiant energy emitted by the body $1 = A_1\varepsilon_1\sigma T_1^4$

The fraction of this energy striking the body $2 = F_{12}A_1\varepsilon_1\sigma T_1^4$

The energy absorbed by the body $2 = \alpha_2 F_{12}A_1\varepsilon_1\sigma T_1^4$

The radiant energy transfer from the surface 1 to the surface 2 is

$$\dot{Q}_{1-2} = \varepsilon_1\varepsilon_2 A_1 F_{12}\sigma T_1^4 \qquad (\because\ \alpha_2 = \varepsilon_2 \text{ by Kirchhoff's law})$$

Similarly, energy transfer from the surface 2 to the surface 1 is

$$\dot{Q}_{2-1} = \varepsilon_1\varepsilon_2 A_2 F_{21}\sigma T_2^4$$

By reciprocity relation: $A_1F_{12} = A_2F_{21}$

or $$\dot{Q}_{12} = \bar{\varepsilon} A_1 F_{1-2}\,\sigma(T_1^4 - T_2^4)$$

Net radiant interchange between the two bodies is

$$\boxed{\dot{Q}_{12} = \dot{Q}_{1-2} - \dot{Q}_{2-1} = \varepsilon_1\varepsilon_2 A_1 F_{12}\,\sigma(T_1^4 - T_2^4)} \tag{13.16}$$

where $\bar{\varepsilon} = \varepsilon_1\varepsilon_2$ is called the *equivalent emissivity* of the system comprising two small grey bodies.

13.6.2 • Radiation Heat Exchange Between a Small Grey Body in a Large Grey Enclosure

When a small grey body 1 is placed in a large grey enclosure 2 ($A_2 >> A_1$), the enclosure acts like a black body. The grey surroundings are effectively black because only a negligible amount of energy is reradiated to the small grey body. Thus, if the small body 1 emits a radiation of $\varepsilon_1 A_1 \sigma T_1^4$, all of it will be absorbed by the enclosure. The enclosure emits $A_2 \sigma T_2^4$, of which $F_{21} A_2 \sigma T_2^4$ will be intercepted by the body 1. Out of this, $\alpha_1 F_{21} A_2 \sigma T_2^4$ will be absorbed by the body 1.

The net radiant exchange is given by

$$\dot{Q}_{12} = \varepsilon_1 A_1 \sigma T_1^4 - \varepsilon_1 A_2 F_{21} \sigma T_2^4 \qquad \text{(since } \alpha_1 = \varepsilon_1 \text{ by Kirchhaff's law)}$$

By reciprocity,

$$A_2 F_{21} = A_1 F_{12} = A \qquad \text{since } F_{12} = 1.$$

It follows that

$$\boxed{\dot{Q}_{12} = \varepsilon_1 A_1 \sigma (T_1^4 - T_2^4)} \tag{13.17}$$

13.6.3 • Radiation Heat Exchange Between Two Infinite Parallel Surfaces

Consider two infinite (*very large*) flat parallel planes 1 and 2 which are maintained at absolute temperatures T_1 and T_2 and have emissivities ε_1 and ε_2, respectively, as shown in Fig. 13.15. Together they constitute a two-surface enclosure. Our objective is to determine the net exchange of radiant heat transfer between the two surfaces.

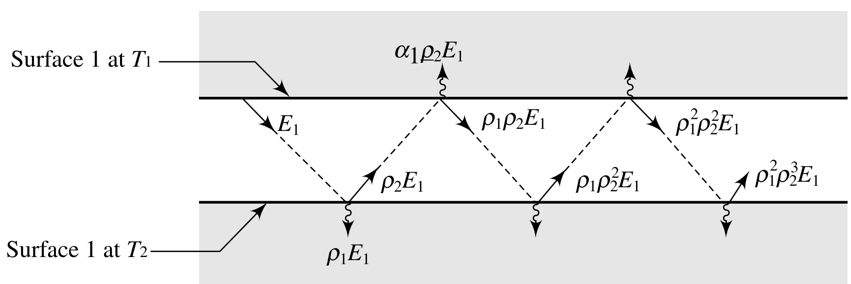


Fig. 13.15 *Radiant heat exchange between two infinite parallel plates*

Assumptions

- The two surfaces are *opaque (no transmissivity),* isothermal (*constant and uniform temperature*) and *diffuse grey* (*independent of direction and wavelength*).
- There exists a *vacuum* or *non-participating medium* between the two surfaces.
- Since the surfaces are very large, the areas are *equal* ($A_1 = A_2 \approx A$).
- Kirchhoff's law is applicable ($\alpha = \varepsilon$).
- Surfaces being parallel and large, the entire radiation leaving one is intercepted completely by the other. Hence, the shape factor, $F_{12} = F_{21} = 1$.
- All emission and reflection properties are same over the whole surface.

The emissive powers of the surfaces 1 and 2 are

$$E_1 = \varepsilon_1 E_{b1} = \varepsilon_1 \sigma T_1^4 \quad \text{and} \quad E_2 = \varepsilon_2 E_{b2} = \varepsilon_2 \sigma T_2^4$$

The radiant heat flux emitted by the surface 1 that strikes the surface 2 is E_1. A fraction of it ($\alpha_2 E_1$) is absorbed by the surface 2 and the rest ($\rho_2 E_1$) is reflected back towards the surface 2. This process of successive absorption and reflection back and forth goes on indefinitely and the amounts involved become progressively smaller. The radiant heat flux emitted by the surface 1 and absorbed by the surface 2 is

$$q_{1-2} = \alpha_2 E_1 + \alpha_2 \rho_1 \rho_2 E_1 + \alpha_2 \rho_1^2 \rho_2^2 E_1 + \cdots$$

$$= \alpha_2 E_1 [1 + \rho_1 \rho_2 + (\rho_1 \rho_2)^2 + \cdots] = \alpha_2 E_1 \times \frac{1}{1 - \rho_1 \rho_2}$$

Since for $\rho_1 \rho_2 < 1$, the infinite series $[1 + \rho_1 \rho_2 + \rho_1^2 \rho_2^2 + \cdots \infty] = \left(\dfrac{1}{1 - \rho_1 \rho_2} \right)$

In a similar way, the radiant heat flux emitted by the surface 2 and absorbed by the surface 1 is

$$q_{2-1} = \alpha_1 E_2 + \alpha_1 \rho_1 \rho_2 E_2 + \alpha_1 \rho_1^2 \rho_2^2 E_2$$

$$= \alpha_1 E_2 [1 + \rho_1 \rho_2 + (\rho_1 \rho_2)^2 + \cdots] = \alpha_1 E_2 \times \frac{1}{1 - \rho_1 \rho_2}$$

Thus, the net radiant heat flux is

$$q_{12} = \frac{\dot{Q}_{12}}{A} = q_{1-2} - q_{2-1} = \frac{\alpha_2 E_1}{1 - \rho_1 \rho_2} - \frac{\alpha_1 E_2}{1 - \rho_1 \rho_2}$$

From *Kirchhoff's law:* $\alpha_1 = \varepsilon_1$, and $\alpha_2 = \varepsilon_2$. For an opaque surface, the reflectivities are $\rho_1 = 1 - \alpha_1 = 1 - \varepsilon_1$ and $\rho_2 = 1 - \alpha_2 = 1 - \varepsilon_2$.
It follows that

$$q_{12} = \frac{\varepsilon_2 E_1 - \varepsilon_1 E_2}{1 - (1 - \varepsilon_1)(1 - \varepsilon_2)} = \frac{\varepsilon_2 \sigma \varepsilon_1 T_1^4 - \varepsilon_1 \sigma \varepsilon_2 T_2^4}{1 - [1 - \varepsilon_1 - \varepsilon_2 + \varepsilon_1 \varepsilon_2]}$$

$$= \frac{\sigma (T_1^4 - T_2^4) \varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2 - \varepsilon_1 \varepsilon_2} = \frac{\sigma (T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1} \; (\text{W/m}^2)$$

or

$$\boxed{\dot{Q}_{12} = \frac{\sigma A (T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}} \quad (\text{W}) \tag{13.18}$$

The equivalent emissivity, $\bar{\varepsilon}$ for this case is $\left[\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1 \right]^{-1} = \dfrac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2 - \varepsilon_1 \varepsilon_2}$

13.6.4 • Radiation Heat Exchange between Two Concentric Cylinders/Spheres

Consider two large concentric cylinders of areas, $A_1 = \pi D_1 L$ and $A_2 = \pi D_2 L$, as shown in Fig. 13.16. The inner and outer surfaces of the concentric cylinders facing each other across the annular space are maintained at temperatures T_1 and T_2. The surfaces are assumed to be diffuse-grey and opaque, and their emissivities are ε_1 and ε_2, respectively. As the inner cylinder is entirely enclosed by the outer cylinder, $F_{12} = 1$, and since by reciprocity relation: $A_1 F_{12} = A_2 F_{21}$, $F_{21} = \dfrac{A_1}{A_2}$. From Kirchhoff's law, the absorptivities

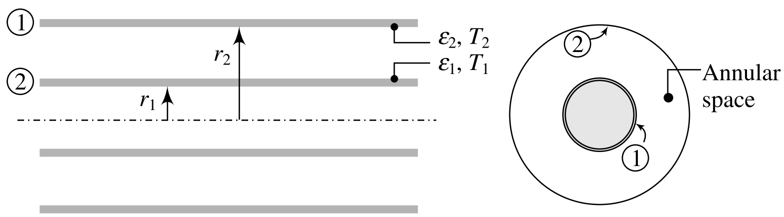


Fig. 13.16 *Radiant heat exchange in the annular space between two concentric cylinders (tubes)*

of the two surfaces are $\alpha_1 = \varepsilon_1$ and $\alpha_2 = \varepsilon_2$. The surfaces being opaque, $\tau = 0$ and $(\alpha + \rho) = 1$ and the reflectivities are $\rho_1 = (1 - \varepsilon_1)$ and $\rho_2 = (1 - \varepsilon_2)$, respectively.

The radiant energy emitted (per unit area) from the outer surface of the inner cylinder (*Surface 1*) towards the inner surface of the outer cylinder (*Surface 2*) is E_1. As $F_{1-2} = 1$, the whole emissive power (E_1) is intercepted by the outer cylinder (Surface 2). Out of this, $\alpha_2 E_1$ or $\varepsilon_2 E_1$ is absorbed by the outer cylinder. The remaining energy $\rho_2 E_1$ or $(1 - \varepsilon_2) E_1$ is diffusely *reflected* back towards the inner cylinder (Surface 1).

By reciprocity: $A_1F_{12} = A_1F_{21}$

Hence, $\qquad F_{21} = A_1/A_2 \qquad$ (as $F_{12} = 1$)

The energy intercepted by the inner cylinder will then be $F_{21}(1- \varepsilon_2)\ \varepsilon_1 = (A_1/A_2)\ (1 - \varepsilon_2)\ E_1$. Out of this, the energy absorbed by the inner cylinder is $\varepsilon_1(A_1/A_2)\ (1 - \varepsilon_2)E_1$ (since $\alpha_1 = \varepsilon_1$).

$$\begin{pmatrix}\text{The energy now reflected}\\ \text{from the inner cylinder}\end{pmatrix} = \begin{pmatrix}\text{Energy reflected by}\\ \text{the outer cylinder}\end{pmatrix} - \begin{pmatrix}\text{Energy absorbed by}\\ \text{the inner cylinder}\end{pmatrix}$$

$$= E_1(1-\varepsilon_2) - \varepsilon_1(A_1/A_2)(1-\varepsilon_2)E_1$$

$$= E_1(1-\varepsilon_2)[1-\varepsilon_1(A_1/A_2)]$$

And the energy absorbed by the outer cylinder

$$= \varepsilon_2\ E_1(1 - \varepsilon_2)[1 - \varepsilon_1(A_1/A_2)] \qquad (\textit{since } \alpha_2 = \varepsilon_2)$$

This process of partial absorption and reflection back and forth continues endlessly and the quantities involved become progressively smaller.

Finally, the rate at which radiation is emitted by the inner cylinder and absorbed by the outer cylinder is

$$\dot{Q}_{1-2} = A_1\left[\varepsilon_2 E_1 + \left(1-\varepsilon_1\frac{A_1}{A_2}\right)\right](1-\varepsilon_2)\varepsilon_2 E_1 + \left(1-\varepsilon_1\frac{A_1}{A_2}\right)^2(1-\varepsilon_2)^2\varepsilon_2 E_1 + \cdots$$

$$= A_1\varepsilon_2 E_1\left[1 + \underbrace{\left(1-\varepsilon_1\frac{A_1}{A_2}\right)(1-\varepsilon_2)}_{x} + \underbrace{\left(1-\varepsilon_1\frac{A_1}{A_2}\right)^2(1-\varepsilon_2)^2}_{x^2} + \cdots\right]$$

$$= \frac{A_1\varepsilon_2 E_1}{1-(1-\varepsilon_2)\left(1-\varepsilon_1\dfrac{A_1}{A_2}\right)} \qquad \left(\text{since} \quad 1 + x + x^2 + \cdots = \frac{1}{1-x}\right)$$

$$= \frac{A_1\varepsilon_2 E_1}{1-1+\varepsilon_2+\varepsilon_1\dfrac{A_1}{A_2}-\varepsilon_1\varepsilon_2\dfrac{A_1}{A_2}} = \frac{A_1\varepsilon_2 E_1}{(A_1/A_2)\varepsilon_1+\varepsilon_2-(A_1/A_2)\varepsilon_1\varepsilon_2}$$

Likewise, the rate at which radiation is emitted by the outer cylinder (Surface 2) and absorbed by the inner cylinder (Surface 1) is

$$\dot{Q}_{2-1} = \frac{A_1\varepsilon_1 E_2}{(A_1/A_2)\varepsilon_1 + \varepsilon_2 - (A_1/A_2)\varepsilon_1\varepsilon_2}$$

The net radiant heat exchange between the inner and outer surfaces is then given by

$$\begin{aligned}\dot{Q}_{12} &= \frac{A_1\varepsilon_2 E_1}{(A_1/A_2)\varepsilon_1 + \varepsilon_2 - (A_1/A_2)\varepsilon_1\varepsilon_2} - \frac{A_1\varepsilon_1 E_2}{(A_1/A_2)\varepsilon_1 + \varepsilon_2 - (A_1/A_2)\varepsilon_1\varepsilon_2}\\ &= \frac{A_1\varepsilon_1\varepsilon_2 E_{b1} - A_1\varepsilon_1\varepsilon_2 E_{b2}}{\dfrac{A_1}{A_2}(\varepsilon_1 - \varepsilon_1\varepsilon_2) + \varepsilon_2} = \frac{A_1\varepsilon_1\varepsilon_2(E_{b1} - E_{b2})}{\left(\dfrac{A_1}{A_2}\right)\varepsilon_1(1-\varepsilon_2) + \varepsilon_2}\\ &= \frac{\sigma A_1(T_1^4 - T_2^4)}{(1/\varepsilon_1) + (A_1/A_2)[(1-\varepsilon_2)/\varepsilon_2]}\end{aligned}$$

$$\therefore \qquad \boxed{\dot{Q}_{12} = \frac{\sigma A_1(T_1^4 - T_2^4)}{(1/\varepsilon_1) + (A_1/A_2)[(1/\varepsilon_2) - 1]}} \tag{13.19}$$

As $\qquad A_1/A_2 = \pi D_1 L/\pi D_2 L = D_1/D_2 \quad$ or $\quad r_1/r_2$,

we have for *concentric cylinders*,

$$\boxed{\dot{Q}_{12} = \frac{\sigma(\pi D_1 L)(T_1^4 - T_2^4)}{(1/\varepsilon_1) + (D_1/D_2)\{(1/\varepsilon_2) - 1\}}} \tag{13.20}$$

Similarly, for *concentric spheres*, the only difference will be the areas: $A_1 = \pi D_1^2$ and $A_2 = \pi D_2^2$, so that

$$\frac{A_1}{A_2} = \frac{\pi D_1^2}{\pi D_2^2} = \left(\frac{D_1}{D_2}\right)^2 \text{ or } (r_1/r_2)^2$$

and

$$\boxed{\dot{Q}_{12} = \frac{\sigma(\pi D_1^2)(T_1^4 - T_2^4)}{(1/\varepsilon_1) + (D_1/D_2)^2[(1/\varepsilon_2) - 1]}} \tag{13.21}$$

13.7 ❑ RADIANT ENERGY EXCHANGE BETWEEN BLACK SURFACES

By and large, radiation leaving a surface is by both emission and reflection and after being intercepted by another surface, it is partly absorbed and partly reflected. The analysis of radiation exchange between surfaces can be simplified considerably by assuming black-body behaviour. Radiation leaving a black surface is then by emission only and the entire radiation is absorbed by the other black surface without any reflection. Hence, radiosity J equals black-body emissive power E_b.

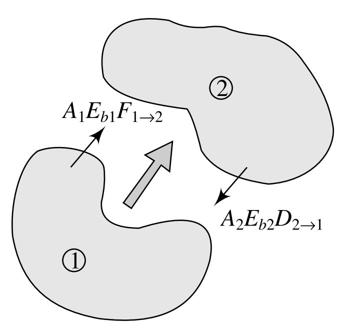


Fig. 13.17 *Two isothermal black surfaces exchanging heat by radiation*

Consider two black surfaces of arbitrary shape at specified uniform temperatures T_1 and T_2 that exchange heat by radiation, as shown in Fig. 13.17. Surface 1 emits radiation, and some fraction of this radiation strikes the surface 2. In addition, the surface 2 emits radiation, some of which strikes the surface 1. The net rate of radiation heat transfer between them is given by

$$\begin{pmatrix}\text{Net radiation heat transfer}\\ \text{between surfaces 1 and 2}\end{pmatrix} = \begin{pmatrix}\text{Radiation } \textit{leaving} \text{ the entire}\\ \text{surface 1 and the striking surface 2}\end{pmatrix} - \begin{pmatrix}\text{Radiation } \textit{leaving} \text{ the entire}\\ \text{surface 2 and the striking surface 1}\end{pmatrix}$$

i.e.,
$$\boxed{\dot{Q}_{12} = \dot{Q}_{1\to 2} - \dot{Q}_{2\to 1}}$$

We note that the radiant energy emitted by a black surface per unit surface area is the emissive power, $E_b = \sigma T^4$, and the rate of energy emitted by it is $E_b A$, where A is the surface area. The fraction of radiation leaving the surface 1 that strikes the surface 2 is the shape factor $F_{1\to 2}$ and that leaving the surface 2 and striking the surface 1 is $F_{2\to 1}$.

Net radiant heat exchange between the surfaces 1 and 2 can then be expressed as

$$\dot{Q}_{12} = E_{b1}A_1F_{1-2} - E_{b2}A_2F_{2-1}$$

Since $A_1F_{1\to 2} = A_2F_{2\to 1}$, (*reciprocity relation*)

$$\dot{Q}_{12} = E_{b1}A_1F_{1\to 2} - E_{b2}A_1F_{1\to 2} = A_1F_{1\to 2}(E_{b1} - E_{b2})$$

With $E_b = \sigma T^4$, we have

$$\boxed{\dot{Q}_{12} = A_1F_{1\to 2}\,\sigma(T_1^4 - T_2^4)} \quad \text{(W)} \tag{13.22}$$

If $T_1 > T_2$, $\dot{Q}_{12}$ is *positive*. Conversely, if $T_2 > T_1$, $\dot{Q}_{12}$ is *negative*.

This result can also be used to evaluate the net radiation transfer from any surface in an enclosure of surfaces.

Consider an enclosure with *four* black surfaces as shown in Fig 13.18. The net heat leaving the surface 1 by radiation is given by

$$\boxed{\dot{Q}_1 = \dot{Q}_{1\to 2} + \dot{Q}_{1\to 3} + \dot{Q}_{1\to 4}}$$

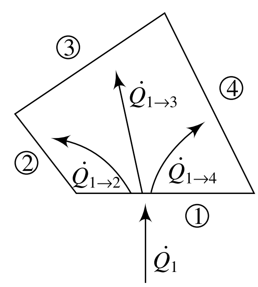


Fig. 13.18 *An enclosure consisting of four isothermal black bodies surfaces*

In general, for an enclosure comprising N black surfaces maintained at different temperatures, the net radiation heat transfer from the surface i of this enclosure can be expressed as

$$\boxed{\dot{Q}_i = \sum_{j=1}^{N} Q_{i\to j} = \sum_{j=1}^{N} A_iF_{i\to j}\,\sigma(T_i^4 - \mathrm{T}_j^4)} \quad \text{(W)} \tag{13.23}$$

The net radiant heat that leaves one surface must arrive at one or more other surfaces in the enclosure. The net radiant heat *leaving* is a *positive* quantity and the net radiant heat *arriving* is a *negative* quantity. Energy conservation requires that, in the steady state, for a complete enclosure, the net radiation heat transfer from a surface to itself is zero, irrespective of the shape of the surface. That is,

$$\boxed{\sum_{i=1}^{N} \dot{Q}_i = 0} \tag{13.24}$$

It is worth noting that if a system of black surfaces does not form a complete enclosure, it may be treated as one, by enclosing the system with a fictitious surface with $E_b = 0$ and $\varepsilon = 1$.

13.8 ❑ SIMPLE ARRANGEMENT INVOLVING BLACK SURFACES

Consider two spheres, the sun (1) and the earth (2), situated a large distance apart in surroundings which are essentially black. This includes surroundings which are non-black but at a large distance from the bodies so that hardly any radiation from the bodies is reflected back to them from the surroundings.

The radiation from the sun (Body 1) spreads out evenly in all directions and that it falls on the inside of a hollow sphere of radius L which is the earth–sun distance which is the earth-sun distance. Since the radiation is spread evenly on the hollow sphere it follows that the proportion of the total radiation falling on the Body 2 (the earth) from the body 1 (the sun) is the ratio of the projected area of the body 2 to the total area of the sphere. Therefore,

$$\boxed{F_{12} = \frac{\pi r_2^2}{4\pi L^2} = \frac{1}{4}\left(\frac{r}{L}\right)^2}$$

where r_2 is the radius of the body 2. The net radiant heat exchange between two black surfaces is given by

$$\boxed{\dot{Q}_{12} = A_1 F_{12} \sigma (T_1^4 - T_2^4)} \tag{13.25}$$

Substitution in Eq. (13.25) yields

$$\boxed{\dot{Q}_{12} = (4\pi r_1^2) \times \frac{1}{4}\left(\frac{r_2}{L}\right)^2 \times \sigma(T_1^4 - T_2^4) = \frac{\pi r_1^2 r_2^2}{L^2}\sigma(T_1^4 - T_2^4)}$$

The radiation emitted from the earth to the outer space (assumed at 0) is given by

$$\boxed{\dot{Q}_{20} = 4\pi r_2^2 \times 1 \times \sigma(T_2^4 - 0) = (4\pi r_2^2)\sigma T_2^4}$$

Energy Balance (Heat received from the sun) = (Heat radiated from the earth to the outer space)

i.e., $$\dot{Q}_{12} = \dot{Q}_{20} \quad \text{or} \quad \frac{\pi r_1^2 r_2^2}{L^2}\sigma(T_1^4 - T_2^4) = 4\pi r_2^2 \sigma T_2^4$$

or $$\frac{r_1^2}{4L^2} = \frac{T_2^4}{T_1^4 - T_2^4}$$

Since $$T_2 \text{ or } T_{\text{earth}} <<< T_1 \text{ or } T_{\text{sun}}$$

$$T_1^4 - T_2^4 \approx T_1^4$$

∴ $$T_2^4 = T_1^4\left(\frac{r_1}{2L}\right)^2 \quad \text{or} \quad \boxed{T_2 = T_1\sqrt{\frac{r_1}{2L}}} \tag{13.26}$$

13.9 ❑ RADIATION HEAT EXCHANGE BETWEEN DIFFUSE-GREY SURFACES—RADIOSITY-IRRADIATION METHOD

13.9.1 • Net Radiation Exchange at a Surface

When incident radiation, G, strikes a diffuse-grey surface, it is partially (ρG) reflected. Additionally, the surface emits radiation εE_b. Then, the *radiosity*, J, is defined as all the radiation leaving a surface and includes both emitted and reflected components.

$$\boxed{J = \varepsilon E_b + \rho G} \qquad (\text{W/m}^2) \tag{13.27}$$

For an opaque surface, $\alpha + \rho = 1$

or $$\boxed{\rho = 1 - \alpha}$$

Using Kirchhoff's law, $\alpha = \varepsilon$

Hence, $$J = \varepsilon E_b + (1 - \varepsilon)G$$

or $$\boxed{G = \frac{J - \varepsilon E_b}{1 - \varepsilon}} \tag{13.28}$$

The net heat flux leaving a surface by radiation is then the difference between outgoing and incoming radiation. It follows that

$$\frac{\dot{Q}}{A} = J - G \tag{13.29}$$

Substituting for G, we have

$$\frac{\dot{Q}}{A} = J - \left[\frac{J - \varepsilon E_b}{(1 - \varepsilon)}\right] = \frac{J - J\varepsilon - J + \varepsilon E_b}{1 - \varepsilon} = \frac{\varepsilon(E_b - J)}{1 - \varepsilon}$$

Net radiation heat-transfer rate from a surface,

$$\boxed{\dot{Q} = \frac{\varepsilon A}{1 - \varepsilon}(E_b - J) \qquad (\text{W}) \quad \textit{diffuse-grey surface}} \tag{13.30}$$

For a *black surface*, the only radiant energy that leaves the black surface is the black-body emissive power and there is no reflected component.

$$\boxed{J = E_b = \sigma T^4 \ (\text{W/m}^2) \qquad \textit{black surface}} \tag{13.31}$$

where the only radiant energy that leaves the black surface is the black-body emissive power and there is no reflected component.

One can also write using electrical analogy:

$$\dot{Q} = \frac{E_b - J}{(1 - \varepsilon)/\varepsilon A}$$

where the *driving potential* is (E_b – J) and the *surface resistance*, R is $(1 - \varepsilon)/\varepsilon A$. The surface resistance depends only on the area and the emissivity of the surface and does not depend on the *placement*, *size*, or *properties* of any other surface in the enclosure. For a *black* surface, $R = 0$ and $J = E_b$.

13.9.2 • Radiation Exchange Between Surfaces

Consider radiation in an enclosure of diffuse-grey surfaces. If there are two, diffuse-grey surfaces that exchange heat by radiation, the net radiant heat transfer between the two surfaces is

$$\left\{\begin{matrix}\text{Net radiation heat transfer} \\ \text{between 1 and 2}\end{matrix}\right\} = \left\{\begin{matrix}\text{Radiation leaving 1} \\ \text{and striking at 2}\end{matrix}\right\} - \left\{\begin{matrix}\text{Radiation leaving 2} \\ \text{and striking at 1}\end{matrix}\right\} \tag{13.32}$$

The total radiant energy leaving the surface 1 per unit area is radiosity J_1 and the total energy leaving the surface is J_1A_1, where A_1 is the surface area. The fraction of this energy that strikes 2 is $F_{1\to2}$.

It is follows that

$$\boxed{\dot{Q}_{1\to2} = J_1A_1F_{1\to2} - J_2A_2F_{2\to1}} \tag{13.33}$$

From reciprocity, $A_1F_{1\to2} = A_2F_{2\to1}$.

Hence,
$$\boxed{\dot{Q}_{1\to2} = A_1F_{1\to2}(J_1 - J_2) = \frac{J_1 - J_2}{1/A_1F_{12}}} \quad \text{(W)} \tag{13.34}$$

We can look upon $(J_1 - J_2)$ as the driving potential and $(1/A_1F_{12})$ as the space resistance, R_{12} which involves the shape factor.

Likewise,
$$\begin{aligned}\dot{Q}_{2\to1} &= J_2A_2F_{2\to1} - J_1A_1F_{1\to2} \\ &= (J_2 - J_1) = A_1F_{1\to2} = -\dot{Q}_{1\to2}\end{aligned}$$

In an enclosure of N surfaces, the net heat leaving the surface i by radiation is the sum of the net radiation heat transfers between the surface i and each of the other surfaces in the enclosure. In the equation form,

$$\boxed{\dot{Q}_i = \sum_{j=1}^{N} \dot{Q}_{i\to j}} \tag{13.35}$$

With $i = 1$ and $N = 3$, $\dot{Q}_1 = \dot{Q}_{1\to2} + \dot{Q}_{1\to2} + \dot{Q}_{1\to3}$

since $\dot{Q}_{1\to3}$ is identically zero.

As in the case of an enclosure with black surfaces, energy conservation requires that

$$\boxed{\sum_{i=1}^{N} \dot{Q}_i = 0}$$

The assumptions in this analysis are: *The radiosity is uniform over the surface*. In other words, *the surface must be isothermal* and *must reflect the same amount of energy at every location*.

13.10 ❑ METHODS TO DETERMINE RADIANT HEAT EXCHANGE IN ENCLOSURES OF DIFFUSE-GREY SURFACES

Let us now consider the general case of radiation heat exchange in which the surfaces involved are of finite size so that the shape factor $F_{i-j} \neq 1$. We assume that the surfaces constitute a complete enclosure. In cases where the surfaces do not form an enclosure, but radiate at each another in an otherwise radiation-free environment, for instance, between parts of a spacecraft in deep space. In such cases, the system may be made an enclosure by adding an enclosing surface with $E_b = 0$ and $\alpha = \varepsilon = 1$.

Assumptions (1) All surfaces are opaque and grey-diffuse emitters and reflectors. (2) Emissive power, irradiation, and radiosity of each isothermal surface are uniform. (3) Steady state exists so that all quantities are time independent. (4) The medium within the enclosure is non-participating.

Two methods are commonly used in the radiation analysis of such an enclosure.

1. Direct Method: Surfaces with Known Surface Temperature, T_i

$$E_{b1} = \sigma T_1^4 = J_1 + R_1 \left\{ \frac{J_1 - J_2}{R_{12}} + \frac{J_1 - J_3}{R_{13}} \right\}$$

In general, for the N-surface enclosure,

$$\boxed{\sigma T_i^4 = J_i + R_i \sum_{j=1}^{N} \left\{ \frac{J_i - J_j}{R_{ij}} \right\}}$$

The network representation of net radiation heat transfer from the surface i to the remaining surfaces of an N-surface enclosure is given in Fig. 13.19. Note that $\dot{Q}_{i\to i}$ (the net rate of heat transfer from a surface to itself) is zero regardless of the shape of the surface.

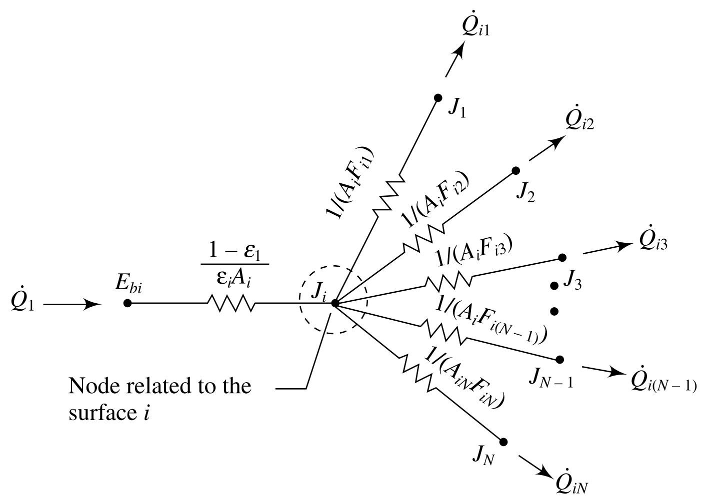


Fig. 13.19 *Network representation of net radiation heat transfer from surface to the remaining surfaces of an N-surface enclosure*

$$\boxed{\frac{E_{bi}-J_i}{R_i}=\sum_{j=1}^{N}\frac{J_i-J_j}{R_{i\to j}}} \qquad \text{(W)} \tag{13.36}$$

which has the electrical analogy interpretation that the net radiation flow from a surface through its *surface resistance* is equal to the sum of the radiation flows from that surface to all other surfaces through the corresponding *space resistances*.

In terms of shape factors, $R_{ij}=\dfrac{1}{A_iF_{ij}}$

and $$R_i=(1-\varepsilon_i)/\varepsilon_i A_i$$

$$\sigma T_i^4=J_i+\left(\frac{1-\varepsilon_i}{\varepsilon_i A_i}\right)\sum_{j=1}^{N}A_iF_{ij}(J_i-J_j)$$

or $$\boxed{\sigma T_i^4=J_i+\left(\frac{1-\varepsilon_i}{\varepsilon_i}\right)\sum_{j=1}^{N}F_{ij}(J_i-J_j)} \qquad \text{(W)} \tag{13.37}$$

Surfaces with Known Net Heat-Transfer Rate $\dot{Q}_i$ For a three-surface enclosure,

$$\dot{Q}_1=\frac{E_{b_1}-J_1}{R_1}=\dot{Q}_{12}+\dot{Q}_{13}=\frac{J_1-J_2}{R_{12}}+\frac{J_1-J_3}{R_{13}}$$

or $$\dot{Q}_1=A_1F_{12}(J_1-J_2)+A_1F_{13}(J_1-J_3)=A_1[F_{12}(J_1-J_2)+F_{13}(J_1-J_3)]$$

If the net heat-transfer rate is known for each surface in the N-surface enclosure,

$$\boxed{\dot{Q}_i=\sum_{j=1}^{N}\dot{Q}_{i\to j}=A_i\sum_{j=1}^{N}F_{ij}(J_i-J_j)=\sum_{j=1}^{N}\frac{J_i-J_j}{R_{ij}}} \tag{13.38}$$

It is noteworthy that

- For a *black surface*, $E_{bi}=J_i=\sigma T_i^4$ and R_i = 0 since ε_i = 1.
- For an *insulated* (*or re-radiating*) surface, $\dot{Q}=0$ and E_{bi} = J_i.

It is well known that the number of unknowns must be equal to the number of equations to be solved. There are N linear algebraic equations and N unknown radiosities for an N-surface enclosure. Now, the radiosities J_1, J_2,..., J_N can be evaluated. The unknown heat-transfer rates can then be determined from Eq. 13.37. To determine the unknown surface temperatures Eq. (13.38) can be used. The temperatures of insulated or reradiating surfaces can be calculated from J_i = σT_i^4 . A *positive* value for $\dot{Q}_i$ implies net radiation heat-transfer rate *from* surface *i* to other surfaces in the enclosure whereas a *negative* value of $\dot{Q}_i$ indicates net radiation heat-transfer rate *to* the surface.

2. Electrical Network Analogy Method Draw a *surface resistance* associated with each surface of an enclosure and connect them with *space resistances*. Then solve the radiation problem by treating it as an electrical network problem where the radiation heat transfer replaces the current and radiosity replaces the potential.

The network method is *not* preferable for enclosures with more than *three* or *four* surfaces since the analysis becomes quite involved.

13.11 ❑ RADIATION HEAT TRANSFER IN TWO-SURFACE ENCLOSURES

A two-surface enclosure exchanging radiation only with each other is shown schematically in Fig. 13.20. Since there are only two surfaces, the net rate of radiation transfer from surface 1, $\dot{Q}_1$, must equal the net rate of radiation transfer to surface 2, $-\dot{Q}_2$, and both quantities must equal the net rate at which radiation is exchanged between 1 and 2. Accordingly,

$$\boxed{\dot{Q}_1 = -\dot{Q}_2 = \dot{Q}_{12}} \tag{13.39}$$

The radiation network of this *two-surface enclosure* consists of *two surface resistances* and *one space resistance*, as shown in Fig. 13.20. Using the electrical analogy the net rate of radiation heat transfer can be expressed as

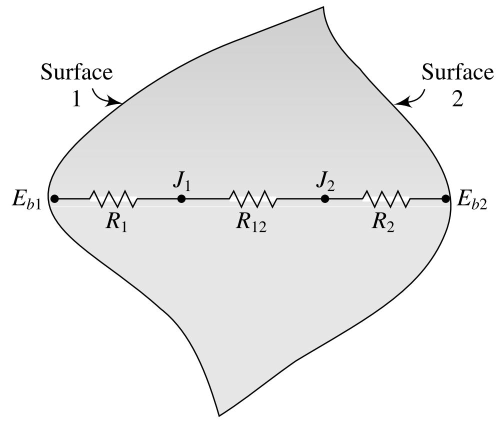


Fig. 13.20 *The resistance analogy for two surfaces*

$$\dot{Q}_{12} = \frac{\text{Potential difference between points 1 and 2}}{\text{Total resistance between the same two points}}$$

$$= \frac{E_{b1} - E_{b2}}{R_1 + R_{12} + R_2} = \dot{Q}_1 = -\dot{Q}_2 \tag{13.40a}$$

Noting that, $E_{b1} = \sigma T_1^4 \quad E_{b2} = \sigma T_2^4$

$R_1 = \dfrac{1-\varepsilon_1}{\varepsilon_1 A_1}$, $R_{12} = \dfrac{1}{A_1 F_{12}}$, and $R_2 = \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}$, we can write

$$\boxed{\dot{Q}_{12} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}} \quad \text{(W)} \tag{13.40b}$$

This important result is applicable to any two *gray, diffuse*, and *opaque* surfaces that make up an enclosure. The shape factor F_{12} depends on the geometry and must be determined first. *Note that F_{12} = 1 for all of these special cases.*

To solve problems with two surface enclosures, the surface temperature must be known at one or both surfaces. If the surface temperature is not known at one of the surfaces, the net radiation leaving the surface, $\dot{Q}_1$ must be known.

If the surrounding surfaces are large compared to the surface 1 and all radiation from the surface 1 reaches the surroundings, then $A_2 >> A_1$ and $F_{12} = 1$, and Eq. (13.40b) reduces to

$$\dot{Q}_{12} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1}}$$

which simplifies to $\boxed{\dot{Q}_{12} = \sigma A_1 \varepsilon_1 (T_1^4 - T_2^4)}$ (13.41)

The direction of the net radiation heat transfer between two surfaces i and j depends on the relative magnitudes of J_i and J_j. A *positive* value for $\dot{Q}_{i \to j}$ indicates that the net heat transfer is from the surface i to the surface j. A negative value indicates the opposite.

Situations may arise for which the net radiation transfer rate at the surface $\dot{Q}_i$, rather than the temperature T_i, is known.

13.12 ❑ TWO-SURFACE ENCLOSURE: ENERGY RADIATED FROM A GREY CAVITY

Consider a grey cavity as shown in Fig. 13.21. Let ε_1, A_1 and T_1 be its emissivity, surface area, and temperature (in kelvin), respectively. The radiant energy will stream out of the cavity into the surrounding space through the opening of the cavity. Let the opening be covered by a hypothetical surface A_2. This is a *two-surface* enclosure problem. As the cavity is very small compared to the space outside, almost all the energy emitted by the cavity will be absorbed by space. The radiation entering the cavity from space can be assumed to be negligibly small. Hence, the space can be idealized to behave like a black body at the temperature of the cavity. Thus, the surface 2 is a black body at 0 K for the purpose of our analysis. It implies that the surface resistance of the surface 2 is zero, and radiosity of the surface 2 equals its emissive power, which is equal to zero since the temperature is 0 K. The radiation network for this configuration is also shown in Fig. 13.21. Net energy radiated from a grey cavity is then expressed as

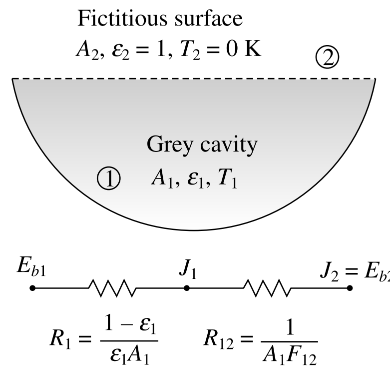


Fig. 13.21 *Radiation from a grey cavity*

$$\dot{Q}_{net} = \frac{E_{b1} - E_{b2}}{R_1 + R_{12} + R_2}$$ (where R_1 *is surface resistance for the cavity and* R_{12} *is the space resistance*)

or
$$\dot{Q}_{net} = \frac{\sigma T_1^4}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}}}$$

Surface 1: $F_{11} + F_{12} = 1$ (*summation rule*)

or $F_{12} = 1 - F_{11}$

Then,
$$\dot{Q}_{net} = \frac{\sigma T_1^4}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1(1-F_{11})}} = \frac{A_1 \varepsilon_1 \sigma T_1^4 (1-F_{11})}{(1-\varepsilon_1)(1-F_{11}) + \varepsilon_1} = \frac{A_1 \varepsilon_1 \sigma T_1^4 (1-F_{11})}{1-\varepsilon_1 - F_{11} + \varepsilon_1 F_{11} + \varepsilon_1}$$

Simplifying,

$$\boxed{\dot{Q}_{net} = A_1 \varepsilon_1 \sigma T_1^4 \left[\frac{1-F_{11}}{1-(1-\varepsilon_1)F_{11}}\right]} \quad \text{(W)}$$

• Radiation Heat Transfer in Three-Surface Enclosures

Consider an enclosure comprising three *opaque*, *diffuse*, and *grey* surfaces, as shown in Fig. 13.22. Surfaces 1, 2, and 3 have areas A_1, A_2, and A_3; emissivities ε_1, ε_2, and ε_3; and uniform temperatures T_1, T_2, and T_3, respectively. The radiation network of this geometry is constructed with each of the three surfaces and connects these surface resistances with space resistances.

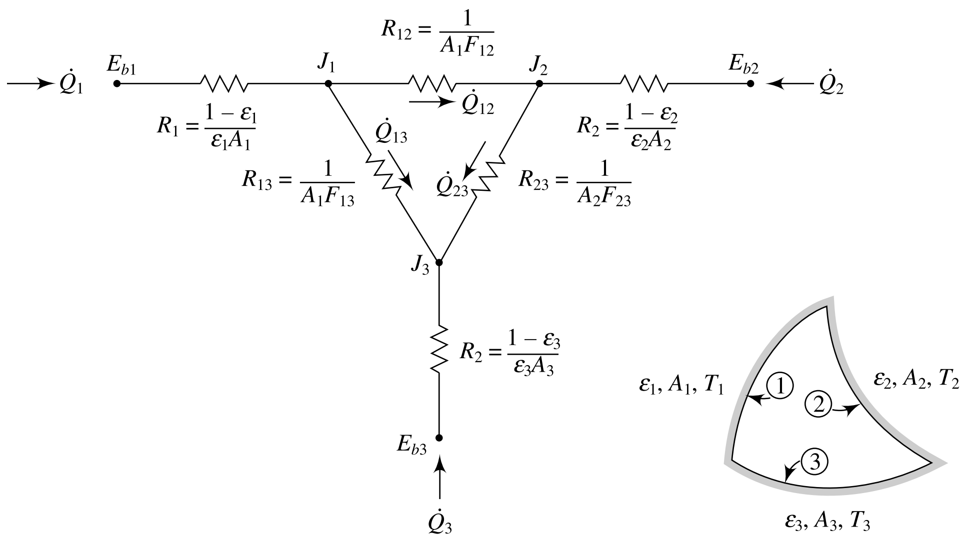


Fig. 13.22 *Network representation of a three-surface enclosure*

Relations for the surface and space resistances are given by Eqs. (13.28) and (13.33). The three end-point potentials E_{b1}, E_{b2}, and E_{b3} are considered known, since the surface temperatures are specified. Then all we need to find are the radiosities J_1, J_2, and J_3. The *three equations* for the determination of these *three unknowns* are obtained from the requirement that the algebraic sum of the currents (*net radiation heat transfer*) at each node must equal zero. That is,

$$\boxed{\frac{E_{b1}-J_1}{R_1}+\frac{J_2-J_1}{R_{12}}+\frac{J_3-J_1}{R_{13}}=0} \tag{13.42a}$$

$$\boxed{\frac{J_1-J_2}{R_{12}}+\frac{E_{b2}-J_2}{R_2}+\frac{J_3-J_2}{R_{23}}=0} \tag{13.42b}$$

$$\boxed{\frac{J_1-J_3}{R_{13}}+\frac{J_2-J_3}{R_{23}}+\frac{E_{b3}-J_3}{R_3}=0} \tag{13.42c}$$

Once the radiosities J_1, J_2 and J_3 are available, the net rate of radiation heat transfers at each surface can be determined from Eq. (13.34).

The set of equations above can be simplified further if one or more surfaces are *special* in some way. For example, $J_i = E_{bi} = \sigma T_i^4$ for a *black* or *reradiating* surface. Also, $\dot{Q}_i = 0$ for a *reradiating* surface.

Finally, when the net rate of radiation heat transfer $\dot{Q}_i$ is specified at surface *i* instead of the temperature, the term $(E_{bi} - J_i)/R_i$ should be replaced by the specified $\dot{Q}_i$.

In Fig. 13.22, the resistance analogy for an enclosure formed from three diffuse-grey surfaces is illustrated. Each surface is at a uniform temperature and radiosity. At each of the three node points, J_1, and J_2, and J_3, the sum of the incoming heat fluxes must equal the sum of the outgoing heat fluxes (see Eq. 13.35). This leads to the following *three* equations:

$$\boxed{\begin{aligned} \dot{Q}_1 &= \dot{Q}_{12} + \dot{Q}_{13} = \frac{J_1 - J_2}{R_{1-2}} + \frac{J_1 - J_3}{R_{1-3}} \\ \dot{Q}_2 &= \dot{Q}_{23} + \dot{Q}_{21} = \frac{J_2 - J_3}{R_{2-3}} + \frac{J_2 - J_1}{R_{2-1}} \\ \dot{Q}_3 &= \dot{Q}_{31} + \dot{Q}_{32} = \frac{J_3 - J_1}{R_{3-1}} + \frac{J_3 - J_2}{R_{3-2}} \end{aligned}} \tag{13.43}$$

The net heat transfer at each surface is related to the surface resistance by

$$\boxed{\begin{aligned} \dot{Q}_1 &= \frac{E_{b1} - J_1}{R_1} = \frac{\sigma T_1^4 - J_1}{R_1} \\ \dot{Q}_2 &= \frac{E_{b2} - J_2}{R_2} = \frac{\sigma T_2^4 - J_2}{R_2} \\ \dot{Q}_3 &= \frac{E_{b3} - J_3}{R_3} = \frac{\sigma T_3^4 - J_3}{R_3} \end{aligned}} \tag{13.44}$$

Rearranging, we have

$$\boxed{\begin{aligned} J_1 &= \sigma T_1^4 - \dot{Q}_1 R_1 \\ J_2 &= \sigma T_2^4 - \dot{Q}_2 R_2 \\ J_3 &= \sigma T_3^4 - \dot{Q}_3 R_3 \end{aligned}} \tag{13.45}$$

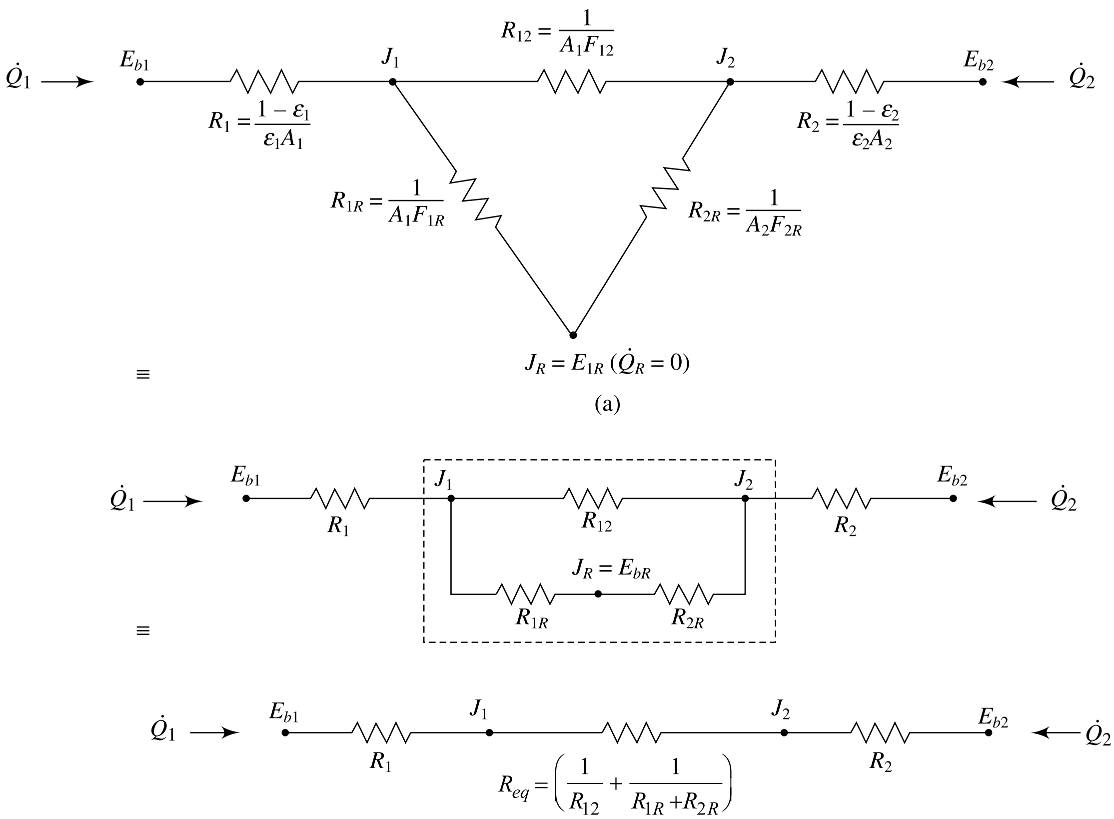
Equations (13.43) and (13.45) are *six* equations with *six* unknowns. There are *there* unknown quantities: the radiosities, J_1, J_2, and J_3. The other *three* are temperatures or net heat-transfer rates at a surface. At each surface, either T_i or $\dot{Q}_i$ must be known. Furthermore, the temperature must be known for at least *one* surface. For example, the *six* unknowns might be $J_1, J_2, J_3, T_1, \dot{Q}_2, \dot{Q}_3$. Alternatively, the six unknowns might be J_1, J_2, J_3, T_1, T_2, T_3. However, the six unknowns *cannot be* $J_1, J_2, J_3, \dot{Q}_1, \dot{Q}_2, \dot{Q}_3$. In practical applications, the basic quantities of interest are *surface temperature* and *net heat-transfer rate* at a surface. The radiosity is only an intermediate variable required to determine radiation heat exchange.

Equations (13.43) and (13.45) have been written for *three* surfaces. The analysis can easily be extended to any finite number of surfaces.

In addition to the *six* equations in Eq. (13.43) and Eq. (13.45), we may also write (see Eq. 13.24)

$$\boxed{\dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 = 0} \tag{13.46}$$

This equation is not linearly independent of the other six. It may be used in place of one of the six equations in the system. Note that all of the above equations apply to only steady-state systems.



(b)

Fig. 13.23 *Equivalent electrical network for an enclosure formed by the diffuse-grey surfaces; surface R is a refractory (adiabatic) surface.*

Special Cases

1. If the surface 3 is *black* then the surface resistance R_3, is zero, and the radiosity is equal to the emissive power

$$\boxed{J_3 = E_{b_3}}$$

2. If the third surface is of large surface area ($A_3 \to \infty$) then $R_3 = \dfrac{1-\varepsilon_3}{\varepsilon_3 A_3}$ approaches zero and again

$$\boxed{J_3 = E_{b_3}}$$

3. If the third surface is insulated, $\dot{Q}_3 = A_3(J_3 - G_3) = 0$
   Then $J_3 = G_3$.
   Also, since $J_3 = (1-\varepsilon_3)G_3 + \varepsilon_3 E_{b_3}$

$$J_3 = (1-\varepsilon_3)J_3 + \varepsilon_3 E_{b_3}$$

   It follows that $\boxed{J_3 = E_{b_3}}$

13.13 ❑ REFRACTORY SURFACES

A refractory surface is that surface which receives radiant energy and reaches a temperature that is just high enough to reradiate all the energy received. Consequently, the surface has no net radiant energy associated with it. Such surfaces are also sometimes called *reradiating surfaces*, but since the application is mostly in furnace calculations where such surfaces are made of *refractory materials*, the name *refractory surface* is usually used.

Some surfaces encountered in numerous practical heat-transfer applications are modelled as being *adiabatic* since their back sides are *well insulated* and the net heat transfer through them is zero. When the convection effects on the front (*heat transfer*) side of such a surface is negligible and steady-state conditions are reached, the surface must lose as much radiation energy as it gains, and thus $\dot{Q}_i = 0$. In such cases, the surface is said to *reradiate* all the radiation energy it receives, and such a surface is called a *reradiating surface*. Setting $\dot{Q}_i = 0$ in Eq. (13.27) yields

$$\boxed{J_i = E_{bi} = \sigma T_i^4} \quad (\text{W/m}^2)$$

Therefore, the temperature of a re-radiating surface under steady conditions can easily be determined from the equation above, once its radiosity is known. Note that the temperature of a reradiating surface is *independent* of its emissivity. In radiation analysis, the surface resistance of a reradiating surface is disregarded since there is no net heat transfer through it. (This is like the fact that there is no need to consider a resistance in an electrical network if no current is flowing through it.).

In an enclosure, *reradiating* surface is defined as one which is thermally *isolated* so that the net heat flow away from the surface is zero. Such a surface interacts with the other surfaces of the enclosure, absorbing and reflecting incident irradiation and re-emitting the absorbed energy. Through this interaction it is allowed to come to an equilibrium emissive power (*or temperature*) compatible with its radiative environment so that its net heat flow is zero. *The refractory walls in a furnace which serve to reflect or absorb and re-radiate energy from the fire are examples of such surfaces.*

Consider two grey surfaces connected by a *third* surface which is *non-conducting*. This three-surface enclosure, for which the third surface *R*, is reradiating, is shown in Fig. 13.23a, along with the radiation network. Surface *R* is presumed to be well insulated, and convection effects are assumed to be negligible. Hence, with $\dot{Q}_R$ = 0, the net radiation transfer *from* the surface 1 must equal the net radiation transfer *to* the surface 2. The network is a simple *series–parallel arrangement*.

Since the refractory wall does not exchange energy, its surface resistance will be zero and hence, the node J_R is not connected to a *surface resistance*.

As the reradiating surface is characterized by *zero* net radiation heat transfer ($\dot{Q}_R$ = 0), $\boxed{G_R = J_R = E_{bR} = \sigma T_R^4}$. Therefore, if one knows the radiosity of a reradiating surface, one can easily find its temperature. In an enclosure, the equilibrium temperature of a reradiating surface is obtained by its interaction with the other surfaces, and it is *independent of the emissivity of the re-radiating surface*.

The network shown in Fig. 13.23(a) is further simplified as shown in Fig. 13.23(b).

We have

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{1/A_1F_{12}} + \frac{1}{1/A_1F_{1R} + 1/A_2F_{2R}}}$$

Since the two active surfaces 1 and 2 are *plane* or *convex*, $F_{11} = F_{22} = 0$,

$$F_{12} + F_{1R} = 1 \quad \Rightarrow \quad F_{1R} = 1 - F_{12}$$

$$F_{21} + F_{2R} = 1 \quad \Rightarrow \quad F_{2R} = 1 - F_{21}$$

Substituting these values, we get

$$R_{\text{eq}} = \frac{1}{A_1 F_{1-2} + \dfrac{1}{\dfrac{1}{A_1(1-F_{12})} + \dfrac{1}{A_2(1-F_{21})}}} = \frac{1}{A_1 F_{12} + \dfrac{1}{\left[\dfrac{A_2(1-F_{21}) + A_1(1-F_{12})}{A_1 A_2 (1-F_{12})(1-F_{21})}\right]}}$$

$$= \frac{1}{A_1 F_{12} + \dfrac{A_1 A_2 (1-F_{12})(1-F_{21})}{A_2(1-F_{21}) + A_1(1-F_{12})}}$$

$$= \frac{A_2(1-F_{21}) + A_1(1-F_{12})}{A_1 F_{12} A_2 (1-F_{21}) + A_1 F_{12} A_1 (1-F_{12}) + A_1 A_2 (1-F_{12})(1-F_{21})}$$

$$= \frac{A_1 + A_2 - A_2 F_{21} - A_1 F_{12}}{A_1 A_2 (F_{12} - F_{12} F_{21}) + A_1^2 (F_{12} - F_{12}^2) + A_1 A_2 (1 - F_{12} - F_{21} + F_{12} F_{21})}$$

or

$$R_{\text{eq}} = \frac{A_1 + A_2 - 2A_1 F_{12}}{A_1 A_2 (F_{12} - F_{12}F_{21} + 1 - F_{12} - F_{21} + F_{12}F_{21}) + A_1^2 (F_{12} - F_{12}^2)} \quad (\because A_1 F_{12} = A_2 F_{21})$$

$$= \frac{A_1 + A_2 - 2A_1 F_{12}}{A_1 A_2 (1 - F_{21}) + A_1^2 (F_{12} - F_{12}^2)} = \frac{A_1 + A_2 - 2A_1 F_{1-2}}{A_1 (A_2 - A_2 F_{21} + A_1 F_{12} - A_1 F_{12}^2)}$$

As $A_2 F_{21} = A_1 F_{12}$, one gets

$$R_{\text{eq}} = \frac{A_1 + A_2 - 2A_1 F_{12}}{A_1 (A_2 - A_1 F_{12}^2)}$$

Therefore,

$$\dot{Q}_1 = \frac{E_{b_1} - E_{b_2}}{\dfrac{1-\varepsilon_1}{A_1 \varepsilon_1} + \dfrac{1-\varepsilon_2}{A_2 \varepsilon_2} + \dfrac{A_1 + A_2 - 2A_1 F_{12}}{A_1 (A_2 - A_1 F_{12}^2)}}$$

i.e.,

$$\boxed{\dot{Q}_1 = \frac{\sigma A_1 (T_1^4 - T_2^4)}{\left(\dfrac{1}{\varepsilon_1} - 1\right) + \dfrac{A_1}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right) + \left(\dfrac{A_1 + A_2 - 2A_1 F_{12}}{A_2 - A_1 F_{12}^2}\right)}} \tag{13.47}$$

If the two grey bodies are replaced by *black bodies* then we have $\varepsilon_1 = 1$ and $\varepsilon_2 = 1$. Then, $\dot{Q}_1 = \dot{Q}_2 = \dot{Q}_{12}$ is given by

$$\boxed{\dot{Q}_1 = \frac{\sigma A_1 (T_1^4 - T_2^4)}{\dfrac{A_1 + A_2 - 2A_1 F_{12}}{A_2 - A_1 F_{12}^2}} = \sigma A_1 (T_1^4 - T_2^4)\left[\frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{13}}\right]} \tag{13.48}$$

The term $\left\{\dfrac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}}\right\}$ represents the *shape factor* of the surface 1 with respect to the surface 2, when the two surfaces are connected by a *reradiating* surface.

If $A_1 = A_2$ then the shape factor simply becomes

$$\frac{A_1(1-F_{12}^2)}{2A_1 - 2A_1F_{12}} = \frac{A_1(1-F_{12})(1+F_{12})}{2A_1(1-F_{12})} = (1+F_{12})/2$$

Hence,
$$\boxed{\dot{Q}_1 = \sigma A_1(T_1^4 - T_2^4) \times \left(\frac{1+F_{12}}{2}\right)} \tag{13.49}$$

To find the temperature of the adiabatic (reradiating) surface, we have $\dot{Q}_R = 0 = A_R F_{R2}(J_R - J_2)$
By symmetry: $F_{R1} = F_{R2}$ and for black bodies,

$$J_1 = E_{b_1}, J_2 = E_{b_2}$$

Also, $R_R = E_{bR}$

Hence, $(E_{bR} - E_{b_1}) + (E_b - E_{b_2}) = 0$

or $2E_{bR} = E_{b_1} + E_{b_2} \quad \Rightarrow \quad 2\sigma T_R^4 = \sigma T_1^4 + \sigma T_2^4$

The radiosity of the reradiating surface J_R can be found from the radiation balance:

$$\boxed{\frac{J_1 - J_R}{(1/A_1F_{1R})} - \frac{J_R - J_2}{(1/A_2F_{2R})} = 0} \tag{13.50}$$

The temperature of the reradiating surface may then be readily determined from

$$\boxed{\sigma T_R^4 = J_R}$$

It may be noted that in the above analysis, only the *single shape factor* between the two active surfaces, F_{1-2}, is needed, and the *configuration of the adiabatic surface is immaterial as long as it does not obstruct the view of A_1 and A_2*. Again, the reduction to the case when the active surfaces are black is obvious.

Furthermore, the expressions derived above are applicable only to those surfaces which do not 'see' themselves, that is, flat or convex surfaces for which $F_{11} = F_{22} = 0$.

13.14 ❑ RADIATION HEAT EXCHANGE IN A FOUR-ZONE ENCLOSURE

1. All Surfaces are Diffuse-Grey (Fig. 13.24)

Net radiant heat flow rate from the surface 1 is

$$\dot{Q}_1 = \frac{E_{b1} - J_1}{R_1} = \frac{J_1 - J_2}{R_{12}} = \frac{J_1 - J_3}{R_{13}} = \frac{J_1 - J_4}{R_{14}}$$

Similarly,

$$\dot{Q}_2 = \frac{E_{b2} - J_2}{R_2} = \frac{J_2 - J_3}{R_{23}} = \frac{J_2 - J_4}{R_{24}} = \frac{J_2 - J_1}{R_{21}}$$

$$\dot{Q}_3 = \frac{E_{b3} - J_3}{R_3} = \frac{J_3 - J_4}{R_{34}} = \frac{J_3 - J_1}{R_{31}} = \frac{J_3 - J_2}{R_{32}}$$

$$\dot{Q}_4 = \frac{E_{b4} - J_4}{R_4} = \frac{J_4 - J_1}{R_{41}} = \frac{J_4 - J_2}{R_{42}} = \frac{J_4 - J_3}{R_{43}}$$

Note that $R_{12} = R_{21}, \quad R_{13} = R_{31}, \quad R_{14} = R_{41}$

$R_{23} = R_{32}, \; R_{24} = R_{42}, \quad R_{34} = R_{43}$

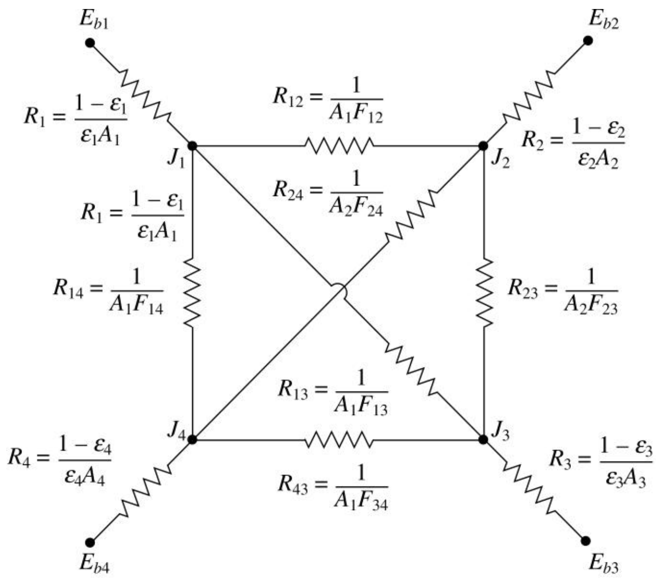


Fig. 13.24 *Network representation of four diffuse-grey surfaces*

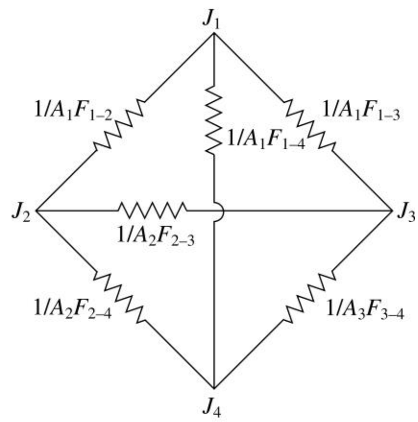


Fig. 13.25 *Network representation of four black surface enclosures*

2. All Surfaces are Black Refer Fig. 13.25 showing the radiation network for four black surface enclosures.

In this case, $E_{b1} = J_1$, $E_{b2} = J_2$, $E_{b3} = J_3$, and $E_{b4} = J_4$

3. Four Surface Enclosures with One Surface Black and One Surface Adiabatic Refer Fig. 13.26. Clearly, $\dot{Q}_4 = 0$ and $E_{b4} = J_4$ and $E_{b1} = J_1$ but $\dot{Q}_1 \neq 0$

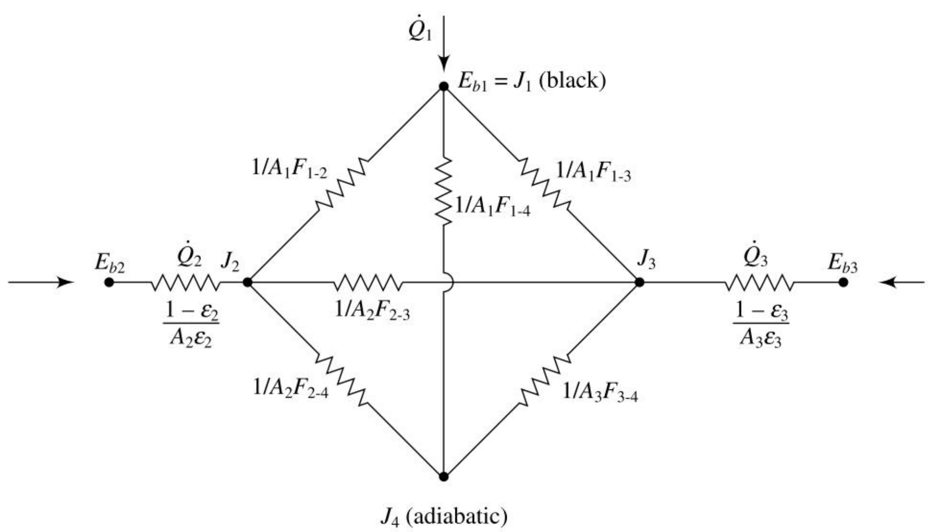


Fig. 13.26 *Network representation of an enclosure comprising two grey surfaces, one black surface, and one adiabatic surface*

13.15 ❑ RADIATION SHIELDING

It is possible to reduce the amount of radiant energy exchange between two surfaces by placing a thin, opaque, high-reflectivity (*low emissivity*) barrier (*thin plate or shell*) between the surfaces. This introduces additional surface and space resistances into the thermal circuit which reduces considerably the net radiation heat transfer. The lower the emissivity of the shield, the higher the surface resistance $\frac{1}{A}\left(\frac{1}{\varepsilon}-1\right)$. *Remember that the radiation shield neither adds nor removes any energy from the system.*

Multiple radiation shields are extensively used in *cryogenic* and *space* applications. In temperature measurement of fluids, the use of shield is common to reduce the error caused by the radiation effect.

Let us examine the benefit derived in terms of the reduction in radiation heat exchange by using a radiation shield.

The emissivities of the two surfaces of the shield may be same or different depending on the surface characteristics. Let the emissivity of one side of the shield facing plate 1 is $\varepsilon_{3,1}$ and that on the opposite side facing the plate 2 is ($\varepsilon_{3,2}$). If the emissivity on both sides of the shield is same then $\varepsilon_{3,1} = \varepsilon_{3,2} = \varepsilon_3$. The two large parallel plates are maintained at temperatures T_1 and T_2 ($T_1 > T_2$). The equilibrium (*steady-state*) temperature of the shield, T_3 will be between T_1 and T_2.

The resistances are connected in series, and the rate of radiation heat transfer is given by

$$\boxed{\dot{Q}_{12,\text{one shield}} = \frac{E_{b1}-E_{b2}}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1}+\dfrac{1}{A_1F_{12}}+\dfrac{1-\varepsilon_{3,1}}{\varepsilon_{3,1}A_3}+\dfrac{1-\varepsilon_{3,2}}{\varepsilon_{3,2}A_3}+\dfrac{1}{A_3F_{32}}+\dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}} \quad (13.51)$$

Clearly, $F_{13} = F_{32} = 1$ and $A_1 = A_2 = A_3 = A$ for large parallel plates.
It follows that

$$\dot{Q}_{12,\text{one shield}} = \frac{\sigma(T_1^4-T_2^4)}{\dfrac{1}{A}\left[\dfrac{1-\varepsilon_1}{\varepsilon_1}+1+\dfrac{1-\varepsilon_{3,1}}{\varepsilon_{3,1}}+\dfrac{1-\varepsilon_{3,2}}{\varepsilon_{3,2}}+1+\dfrac{1-\varepsilon_2}{\varepsilon_2}\right]}$$

$$= \frac{\sigma A(T_1^4-T_2^4)}{\left[\dfrac{1-\varepsilon_1}{\varepsilon_1}+1+\dfrac{1-\varepsilon_{3,1}}{\varepsilon_{3,1}}+\dfrac{1-\varepsilon_{3,2}}{\varepsilon_{3,2}}+1+\dfrac{1-\varepsilon_2}{\varepsilon_2}\right]}$$

$$\boxed{\dot{Q}_{12,\text{one shield}} = \frac{\sigma A(T_1^4-T_2^4)}{\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}-1\right)+\underbrace{\left(\dfrac{1}{\varepsilon_{3,1}}+\dfrac{1}{\varepsilon_{3,2}}-1\right)}_{\text{additional resistance for one shield}}}} \quad (13.52)$$

The radiation heat transfer through large parallel plates separated by N radiation shields can now be expressed as

$$\boxed{\dot{Q}_{12,N\,\text{shields}} = \frac{\sigma A(T_1^4-T_2^4)}{\left(\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}-1\right)+\underbrace{\left(\dfrac{1}{\varepsilon_{3,1}}+\dfrac{1}{\varepsilon_{3,2}}-1\right)+\cdots\left(\dfrac{1}{\varepsilon_{N,1}}+\dfrac{1}{\varepsilon_{N,2}}-1\right)}_{\text{additional resistance for }N\text{ shields}}}} \quad (13.53)$$

It is important to recognize that the resistances associated with the radiation shield become very large when the emissivities $\varepsilon_{3,1}$ and $\varepsilon_{3,2}$ are very small.

In the absence of any radiation shield or screen, the network representation is shown in Fig. 13.27.

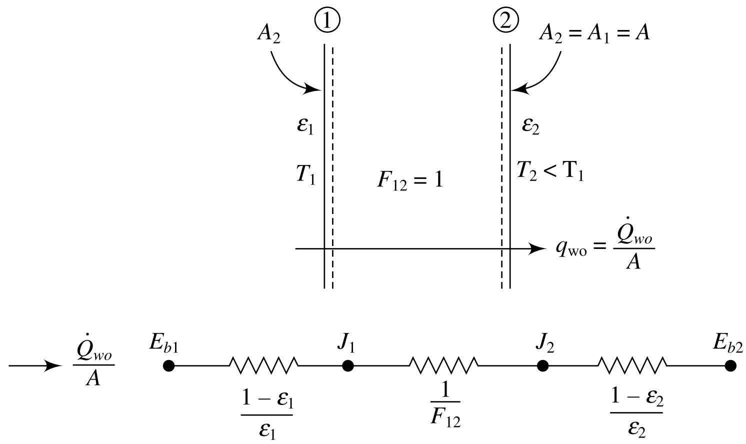


Fig. 13.27 *Radiation network for a pair of two large parallel plane surfaces without a radiation shield*

The net rate of radiation heat transfer between surfaces 1 and 2 *without* a radiation shield is given by

$$\boxed{\dot{Q}_{12,\text{without}} = \frac{\sigma A(T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}} \quad (\text{W/m}^2) \tag{13.54}$$

If the emissivities of all surfaces are equal, including the N shields and the two large parallel planes, there will be $(N + 1)$ spaces created by $(N + 2)$ parallel planes. The space resistance of each space will be $\frac{1}{A}$. Also there will be *two* surface resistances for the original two surfaces *plus* $2N$ surface resistances for the N shields. Each surface resistance will be equal to $\frac{(1-\varepsilon)}{\varepsilon A}$.

Hence, the total resistance with N shields will be

$$R_{\text{tot, with}} = \underbrace{(N+1)\left(\frac{1}{A}\right)}_{\text{space resistance}} + \underbrace{(2N+2)\left(\frac{1-\varepsilon}{\varepsilon A}\right)}_{\text{surface resistance}} = \frac{1}{A}\left[(N+1) + 2(N+1)\left(\frac{1}{\varepsilon} - 1\right)\right]$$

$$\boxed{R_{\text{tot, with}} = \frac{1}{A}(N+1)\left[1 + \frac{2}{\varepsilon} - 2\right] = (N+1)\left(\frac{2}{\varepsilon} - 1\right)\frac{1}{A}}$$

Without shields, the total resistance is

$$\boxed{R_{\text{tot, without}} = 2\left(\frac{1-\varepsilon}{\varepsilon A}\right) + \frac{1}{A} = \left(\frac{2}{\varepsilon} - 1\right)\frac{1}{A}}$$

$$\therefore \qquad \frac{\dot{Q}_{\text{with shields}}}{\dot{Q}_{\text{without shields}}} = \frac{R_{\text{without}}}{R_{\text{with}}} = \frac{\left(\frac{2}{\varepsilon} - 1\right)\frac{1}{A}}{(N+1)\left(\frac{2}{\varepsilon} - 1\right)\frac{1}{A}} = \frac{1}{N+1}$$

This results in the reduction of radiation heat transfer by a factor of $\left(\dfrac{1}{N+1}\right)$ when there are N shields.

$$\boxed{\dot{Q}_{12,N\,\text{shields}} = \frac{\sigma A(T_1^4 - T_2^4)}{(N+1)\left(\dfrac{1}{\varepsilon} + \dfrac{1}{\varepsilon} - 1\right)} = \frac{1}{N+1}\dot{Q}_{12,\,\text{no shield}}} \tag{13.55}$$

Therefore, when all emissivities are equal, one shield reduces the rate of radiation heat transfer to *one-half*, 9 shields reduce it to *one-tenth*, and 19 shields reduce it to *one-twentieth* (or *5 percent*) of what it was without any shield. Note that the exact position of the shield between the plates does not affect the result. Also, note the assumption that the reduction in heat transfer does not affect the plate surface temperatures.

If the emissivity of the shield (3) is much less than that of the original surfaces (i.e., if $\varepsilon_3 << \varepsilon_1$ or ε_2) then the net radiative heat transfer will be reduced by a considerably larger amount than would be the case when all emissivities are same. This principle is utilized in multi-layer insulation in which each layer is an aluminium foil or mylar (plastic) with $\varepsilon \approx 0.05$. Several such layers can be prepared in the form of a blanket.

The equilibrium temperature of the radiation shield T_3 in Fig. 13.28 can be determined by expressing Eq. (13.52) for $\dot{Q}_{13}$ or $\dot{Q}_{32}$ (which involves T_3) after $\dot{Q}_{12}$ from Eq. (13.52) and noting that $\dot{Q}_{12} = \dot{Q}_{13} = \dot{Q}_{32}$ under steady state conditions.

Energy balance: $\dot{Q}_{13} = \dot{Q}_{32}$

$$\boxed{\frac{\sigma A(T_1^4 - T_3^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_{3,1}} - 1} = \frac{\sigma A(T_3^2 - T_2^4)}{\dfrac{1}{\varepsilon_{3,2}} + \dfrac{1}{\varepsilon_2} - 1}}$$

If $\quad \varepsilon_{3,1} = \varepsilon_{3,2} = \varepsilon_3$

$$(T_1^4 - T_3^4)\underbrace{\left[\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} - 1\right]}_{L} = (T_3^2 - T_2^4)\underbrace{\left[\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} - 1\right]}_{M}$$

or $\quad LT_1^4 - LT_3^4 = MT_3^4 - MT_2^4$

or $\quad (L+M)T_3^4 = LT_1^4 + MT_2^4$

$$\boxed{T_3 = \left[\frac{LT_1^4 + MT_2^4}{L+M}\right]^{1/4}}$$

If $\quad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$

Then $\quad L = M = \dfrac{2}{\varepsilon} - 1$

Radiation shield temperature,

$$T_3 = \left\{\frac{L(T_1^4 + T_2^4)}{2L}\right\}^{1/4}$$

or

$$\boxed{T_3 = \left\{\frac{1}{2}(T_1^4 + T_2^4)\right\}^{1/4}} \tag{13.56}$$

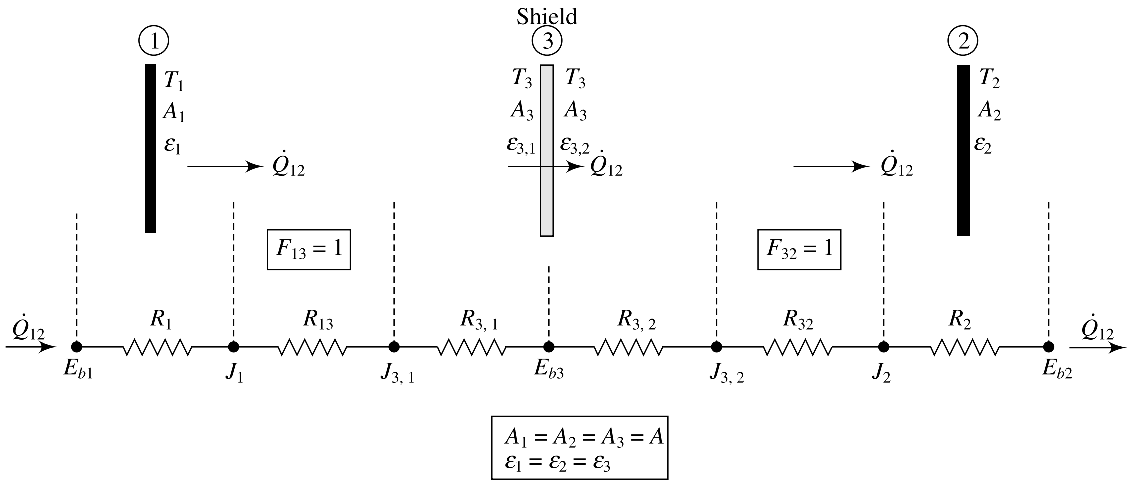


Fig. 13.28 *Radiation network for two parallel plane surfaces with a radiation shield between them*

$$R_1 = \frac{1-\varepsilon_1}{\varepsilon_1 - A_1} \quad R_{3,1} = \frac{1-\varepsilon_{3,1}}{\varepsilon_{3,1}A_3} \quad R_{32} = \frac{1}{A_3F_{32}}$$

$$R_{13} = \frac{1}{A_1F_{13}} \quad R_{3,2} = \frac{1-\varepsilon_{3,2}}{\varepsilon_{3,2}A_3} \quad R_2 = \frac{1-\varepsilon_2}{\varepsilon_2A_2}$$

Radiation Shield in Concentric Long Cylinders and Concentric Spheres The effect of radiation shields between *concentric cylinders* or *concentric spheres* can be determined in a similar fashion: With one shield, Eq. (13.51) can be used by noting $F_{13} = F_{32} = 1$ for both cases and replacing with respective areas A_1, A_2, and A_3.

In the case of concentric tubes or coaxial cylinders, $A = \pi DL$ and for concentric spheres, $A = \pi D^2$. To find the temperature of the shield, the energy balance becomes: $\dot{Q}_{13} = \dot{Q}_{32}$.

$$\frac{\sigma A_1(T_1^4 - T_3^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_3}\left(\dfrac{1}{\varepsilon_{3,1}} - 1\right)} = \frac{\sigma A_3(T_3^4 - T_2^4)}{\dfrac{1}{\varepsilon_{3,2}} + \dfrac{A_3}{A_2}\left(\dfrac{1}{\varepsilon_2} - 1\right)}$$

13.16 ❑ RADIATION EFFECT ON TEMPERATURE MEASUREMENT

Gas temperature measurement by means of a thermocouple involves a radiation error so that the temperature measured is less than the true gas temperature. When a thermocouple is placed in a fluid stream, heat transfer takes place between the couple and the fluid by convection until the couple reaches the fluid temperature. But when the couple is surrounded by surfaces that are at a temperature different from the fluid, there is radiation exchange between the couple and the surrounding surfaces. When the convection and radiation balance each other, the thermocouple indicates a temperature which is less than the true fluid temperature. To account for the radiation effect and to determine the error in temperature measurement, consider a thermocouple placed in a fluid flowing through a large duct whose walls are at a lower temperature than that of the fluid (Fig. 13.29).

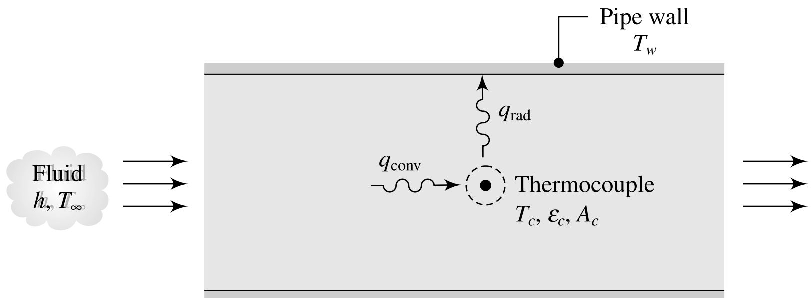


Fig. 13.29 *Error in measurement of temperature caused by radiation effect*

Under equilibrium conditions:

(Heat *gain* by convection) = (Heat *loss* by radiation)

i.e., $$\dot{Q}_{\text{conv}} = \dot{Q}_{\text{rad}} \quad \text{or} \quad hA_c(T_\infty - T_c) = \varepsilon_c \sigma A_c (T_c^4 - T_w^4)$$

The error in temperature measurement is

or $$\boxed{T_\infty - T_c = \frac{\varepsilon_c \sigma (T_c^4 - T_w^4)}{h}} \tag{13.57}$$

where

T_∞ = true temperature of the fluid (K)
T_c = temperature measured by the thermocouple (K)
T_w = temperature of the surrounding duct wall (K)
h = convection heat transfer coefficient (W/m^2 K)
ε_c = emissivity of the thermocouple

It is obvious that to reduce this error, the convection heat-transfer coefficient h should be small and the emissivity of the surface of the thermocouple should be large. Hence, the thermocouple is usually coated with a material of high reflectivity (*low emissivity*) to reduce the error. This error can also be considerably reduced by placing a cylindrical shield over the thermocouple (*or other temperature-measuring device*) in such a manner that the gas flow over the thermocouple is not hampered but the radiation from the thermocouple to the wall is partially blocked.

The above equation is valid only under the assumption of very large or black surroundings, and large errors in temperature measurement may result in actual practice. The thermocouple and a low emissivity radiation shield are arranged as shown in Fig. 13.30.

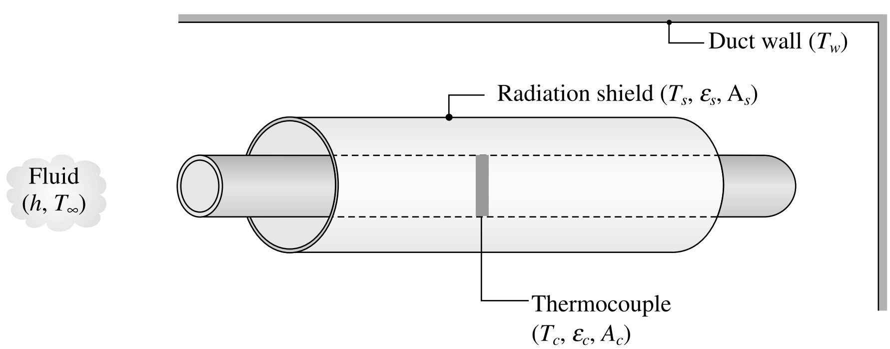


Fig. 13.30 *Radiation shield to reduce the error in temperature measurement*

With the radiation shield in place, radiation from the thermocouple is determined by the temperature of the shield. If the shield is long compared to its diameter, the following approximations can be made:

- Radiation from the shield to the surroundings is totally from the outer surface of the shield.
- Radiation from the thermocouple bead is entirely to the inner surface of the shield, and is negligible compared to the radiation from the outer surface of the shield to the surroundings (due to the surface areas and temperature differences involved).
- Convection occurs both inside and outside the shield.
- The internal resistance of the shield is negligible (the temperature on both sides of the shield is the same).

Under these assumptions, an energy balance yields

$$\boxed{2hA_s(T_\infty - T_s) = \varepsilon_s A_s \sigma (T_s^4 - T_w^4)} \tag{13.58}$$

where A_s is the area of the shield and ε_s is its emissivity. The factor 2 arises since convection occurs both outside and inside the shield. The temperature of the shield, T_s, can be found from this equation by trial and error.

An energy balance between convection to the thermocouple bead and radiation from the bead to the shield (assuming it to be a small body in a large gray enclosure) gives

$$\boxed{hA_c(T_\infty - T_c) = \varepsilon_c A_c \sigma (T_c^4 - T_s^4)} \tag{13.59}$$

The temperature, T_c, can be calculated from this equation. The temperature error with shield will be found to be considerably less than tha without shield. The lower the value of ε_s, the greater will be the reduction in error.

13.17 ❑ RADIATION HEAT-TRANSFER COEFFICIENT

Heat transfer from any surface usually takes place by *convection* as well as *radiation*. Typical examples of such practical situations include heat loss from a steam pipe passing through a room, heat loss from the door and walls of a furnace, and hot combustion gases passing through a duct. In such cases, the heat-transfer calculations become complicated due to the fact that

- The rate of convection heat transfer is proportional to $(T - T_\infty)$, while
- The rate of radiation heat transfer is proportional to $(T^4 - T_{\text{sur}}^4)$ or $(T^4 - T_\infty^4)$ if $T_{\text{sur}} \approx T_\infty$.

The analysis becomes simpler by expressing the radiation heat transfer $\dot{Q}_{\text{rad}}$ in the same manner as the convection heat transfer, $\dot{Q}_{\text{conv}}$.

One can then write: $\dot{Q}_{\text{rad}} = h_{\text{rad}} A_s (T - T_\infty)$ (A)

where $\dot{Q}_{\text{rad}}$ = rate of heat flow by radiation [W]

A_s = surface area (m^2)

h_{rad} = radiation heat transfer coefficient [W/m^2 K]

The heat-transfer rate by radiation from the surface to the large surroundings is expressed as:

$$\dot{Q}_{\text{rad}} = \sigma A_s \varepsilon (T^4 - T_\infty^4)$$

Rearranging, one obtains

$$\dot{Q}_{\text{rad}} = \sigma A_s \varepsilon [(T + T_\infty)(T^2 + T_\infty^2)](T - T_\infty) \tag{B}$$

Comparing Eq. (A) with Eq. (B), one can express the radiation heat-transfer coefficient as

$$\boxed{h_{\text{rad}} = \sigma\varepsilon(T + T_\infty)(T^2 + T_\infty^2)} \quad (\text{W/m}^2\text{K}) \tag{13.60}$$

But in many cases such as *composite slabs, composite cylinders, composite spheres,* etc., one often does not know the surface temperature and, hence, it becomes very difficult to estimate the value of h_{rad} since the analysis involves *trial and error*. However, with the help of a little approximation, the *trial-and-error* method can be avoided as follows.

$$\dot{Q}_{\text{rad}} = \sigma A_s \varepsilon (T^4 - T_\infty^4) \quad \text{or} \quad \dot{Q}_{\text{rad}} = \sigma A_s \varepsilon T_\infty^4 \left\{ \left(\frac{T}{T_\infty} \right)^4 - 1 \right\}$$

Let $\quad (T - T_\infty) \equiv \theta \quad \text{or} \quad T = T_\infty + \theta$

Then, $$\left[\left(\frac{T}{T_\infty} \right)^4 - 1 \right] = \left(\frac{T_\infty + \theta}{T_\infty} \right)^4 - 1 = \left(1 + \frac{\theta}{T_\infty} \right)^4 - 1 = \left\{ \left[1 + \frac{4\theta}{T_\infty} + \cdots \right] - 1 \right\}$$

[*As* $\theta <<< T_\infty$, *the terms with higher powers of* $\dfrac{\theta}{T_\infty}$ *are neglected.*]

It follows that

$$\left[\left(\frac{T}{T_\infty} \right)^4 - 1 \right] \cong 1 + \frac{4\theta}{T_\infty} - 1 \quad \text{or} \quad \left[\left(\frac{T}{T_\infty} \right)^4 - 1 \right] \cong \frac{4\theta}{T_\infty}$$

The rate of radiation heat transfer is then given by

$$\dot{Q}_{\text{rad}} \cong \sigma A_s \varepsilon T_\infty^4 \left[\frac{4\theta}{T_\infty} \right] \cong 4\sigma A_s \varepsilon T_\infty^3 \theta$$

$$\therefore \quad \dot{Q}_{\text{rad}} \cong 4\sigma A_s \varepsilon T_\infty^3 (T - T_\infty)$$

Comparing the above equation with Eq. (A), one gets,

$$\boxed{h_{\text{rad}} = 4\sigma\varepsilon T_\infty^3 \quad (\text{W/m}^2\text{K})} \tag{13.61}$$

It is worth noting that unlike convection heat-transfer coefficient, the radiation heat-transfer coefficient is *strongly dependent on temperature*.

The total heat transfer rate, $\dot{Q} = \dot{Q}_{\text{conv}} + \dot{Q}_{\text{rad}} = (h_{\text{conv}} + h_{\text{rad}}) A_s (T - T_\infty) = h A_s (T - T_\infty)$ where h is the combined heat-transfer coefficient.

13.18 ❑ RADIATION PROPERTIES OF A PARTICIPATING MEDIUM: VOLUMETRIC ABSORPTIVITY AND EMISSIVITY

Consider a gas layer (*participating medium*) of thickness L. A monochromatic beam of intensity $I_{\lambda,0}$ is incident on the medium, which is attenuated (reduced) as it propagates due to absorption. The decrease in the intensity of radiation as it passes through a layer of thickness dx is proportional to the intensity itself and the thickness dx.

This decay in radiation intensity expressed as the local intensity where $I_{\lambda, x}$ and the proportionality constant called *the monochromatic absorption or extinction coefficient*.

Figure 13.31 shows a monochromatic beam of intensity $1_{\lambda,0}$ incident on the gas layer at $x = 0$. Its intensity reduces as a result of absorption. At $x = L$, intensity $I_{\lambda,L}$ is less than $I_{\lambda,0}$.

$$\boxed{dI_{\lambda,x} = -k_\lambda I_{\lambda,x} dx}$$

The absorption coefficient k_λ has the units of m^{-1}.

Separating variables in the above equation and integrating over the limits $x = 0$ where $I_{\lambda x} = I_{\lambda,0}$ and $x = L$ where $I_{\lambda x} = I_{\lambda,L}$ yields $\boxed{\ln\frac{I_{\lambda,L}}{I_{\lambda,0}} = -k_\lambda L}$ where k_λ is assumed to be independent of x.

The *monochromatic transmissivity* of the gas can be defined as the ratio of the intensity of radiation leaving the medium to that entering the medium. That is,

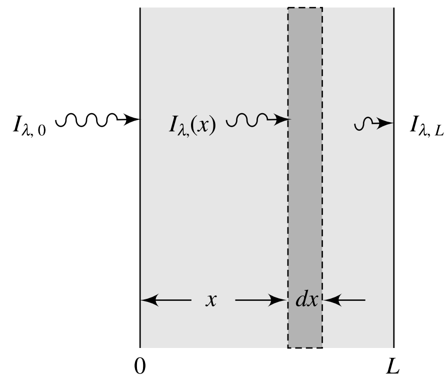


Fig. 13.31 *Absorption of a radiation beam through a gas layer (an absorbing medium) of thickness L*

$$\boxed{\tau_{g\lambda} = \frac{I_{\lambda,L}}{I_{\lambda,0}} = e^{-k_\lambda L}}$$

This exponential decay, known as *Beer's law*, is a valuable tool in radiation analysis.
Note that $\tau_\lambda = 1$ when no radiation is absorbed and thus radiation intensity remains constant. Also, the monochromatic transmissivity of a medium represents the fraction of radiation transmitted by the medium at a given wavelength.

The amount of radiant energy absorbed in the length L is $(I_{\lambda,0} - I_{\lambda,L})$ and the ratio of energy absorbed to incident energy gives the monochromatic absorptivity of the gas, $\alpha_{g \cdot \lambda}$. Thus,

$$\boxed{(I_{\lambda,0} - I_{\lambda,L})/I_{\lambda,0} = 1 - e^{-k_\lambda \mathrm{L}} = 1 - \tau_{g\lambda} = \alpha_{g\lambda}} \tag{13.62}$$

Then, from the Kirchhoff's law, since absorptivity is equal to emissivity, we have

$$\boxed{\varepsilon_{g\lambda} = 1 - \exp(-k_\lambda L)} \qquad \textit{(monochromatic emissivity of gas)}$$

For an *optically thick* medium (a medium with a large value of $k_\lambda L$), $\varepsilon_{g\lambda} \approx \alpha_{g\lambda} \approx 1$. For $k_\lambda L = 5$, for example, $\varepsilon_{g\lambda} = \alpha_{g\lambda} = 0.993$. Therefore, an optically thick medium emits like a blackbody at the given wavelength. As a result, an optically thick *absorbing-emitting* gas with no significant scattering at a given temperature T_g can be viewed as a *black surface* at T_g since it will absorb essentially all the radiation passing through it, and it will emit the maximum possible radiation that can be emitted by a surface at T_g, which is $E_{b\lambda}(T_g)$.

i.e., $\boxed{\alpha_{g\lambda} = \varepsilon_{g\lambda} = 1}$

If the radiation is not parallel, the various beams will travel different distances through the medium and will be attenuated by different amounts. The overall attenuation can still be described by Eq. (13.62) if an appropriate average length (the *mean* or *effective beam length, L*) is chosen. In general, *the mean beam length is a function of the geometry and the attenuation coefficient.*

Note that the monochromatic absorptivity, transmissivity, and emissivity of a medium are dimensionless quantities, with values *less than* or *equal to* 1. The monochromatic absorption coefficient of a medium and thus (ε_λ, α_λ, and τ_λ), in general, vary with wavelength, temperature, pressure, and composition.

13.19 ❑ EMISSION AND ABSORPTION OF RADIATION BY AN ABSORBING MEDIUM

We restricted our attention so far to radiation heat transfer between surfaces that are separated by a non-participating medium which does not emit, absorb, or scatter radiation and is completely transparent to thermal radiation. A vacuum is ideally suited and air at ordinary temperatures and pressures is considered a non-participating medium. Radiation calculations involve complexity if a participating medium is present.

A participating medium emits and absorbs radiation throughout its entire volume. That is, gaseous radiation is a *volumetric phenomenon*, and thus it depends on the size and shape of the body. This is the case even if the temperature is uniform throughout the medium.

- Gases emit and absorb radiation at a number of narrow wavelength bands. This is in contrast to solids, which emit and absorb radiation over the entire spectrum. Therefore, the grey assumption is usually not appropriate for a gas.
- The emission and absorption characteristics of the components of a gas mixture also depend on the temperature, pressure, and composition of the gas mixture. Hence the presence of other participating gases also affects the radiation characteristics of a particular gas.

We will consider only those gases which emit and absorb radiation, specially H_2O and CO_2 since they are most commonly encountered in practice

We have discussed so far the radiant heat exchange between surfaces in an enclosure, separated by perfectly transparent or a non-participating medium. This assumption is justified for monatomic gases like *argon* and *helium*, and for *diatomic* gases such as oxygen and nitrogen. These gases are inert to thermal radiation. However, for polyatomic gases like CO_2, H_2O (vapour), NH_3, and hydrocarbon gases which absorb and emit radiation the assumption is not valid. Moreover, radiation from solids and liquids covers the entire wavelength range while radiation from gases cover selected wavelength *bands*. Gaseous radiation is concentrated in specific wavelength intervals (called bands). As mentioned earlier, radiation from solids is a surface phenomenon but that from gases is a volumetric phenomenon.

The absorption (or emissiion) does not take place continuously over the entire spectrum. Clearly the wide radiation bands of relatively strong absorption properties are not grey.

The shape and size of a gas volume affects the gas emissivity. This dependence is reflected through the mean beam length. The emissivity of a gas also depends upon its concentration which is related to the partial pressure of the gas. When a gas mixture is considered, the total pressure of the gas (if it is a mixture of several gases) and the concentration of other gases also influence absorptivity and emissivity since the absorption bands of these constituents may interfere with the absorption bands of the gas.

A common engineering calculation is one that requires determination of the radiant heat flux from a gas to an adjoining surface. Despite the complicated spectral and directional effects inherent in such calculations, a simplified procedure may be used.

We must note that the emissivity of a gas also depends on the mean length an emitted radiation beam travels in the gas before reaching a bounding surface, and thus on the shape and the size of the gas body involved.

Results for the emissivity of water vapour in a mixture of non-participating gases are plotted in Fig. 13.33 as a function of the gas temperature, for a total pressure of 1 atm, and for different values of the product of the partial pressure of H_2O vapour and the mean path length of the beam, L_1 which may be interpreted as the radius of a hemi-spherical gas volume radiating to the centre of the base.

Emissivity at a total pressure P other than P = 1 atm is determined by multiplying the emissivity value at 1 atm by a *pressure correction factor* C_w obtained from Fig. 13.34 for water vapour. That is,

$$\boxed{\varepsilon_w = C_w \varepsilon_{w,1\,\text{atm}}} \tag{13.63}$$

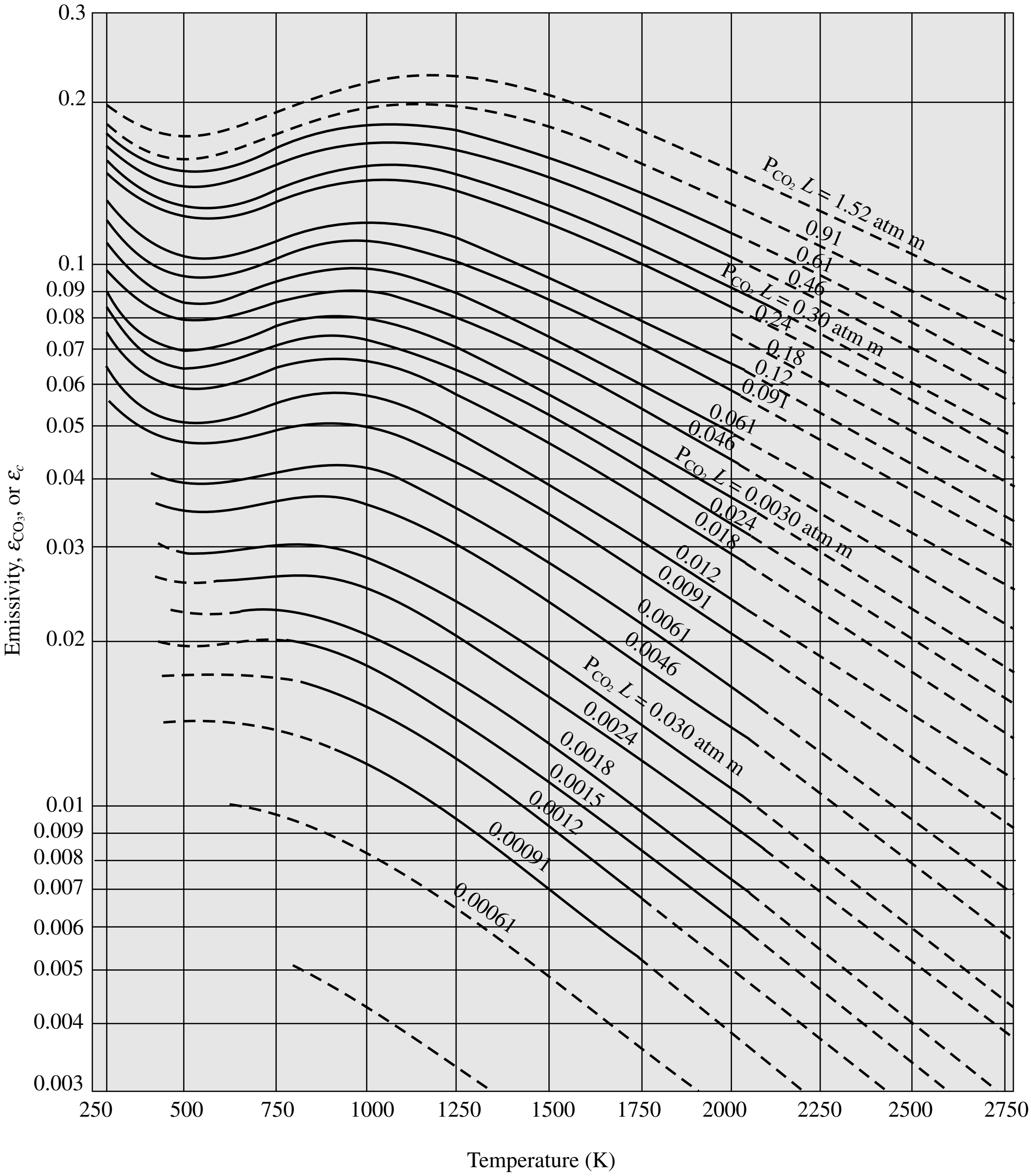


Fig. 13.32 *Emissivity of carbon dioxide at a total pressure of 1 atm.*

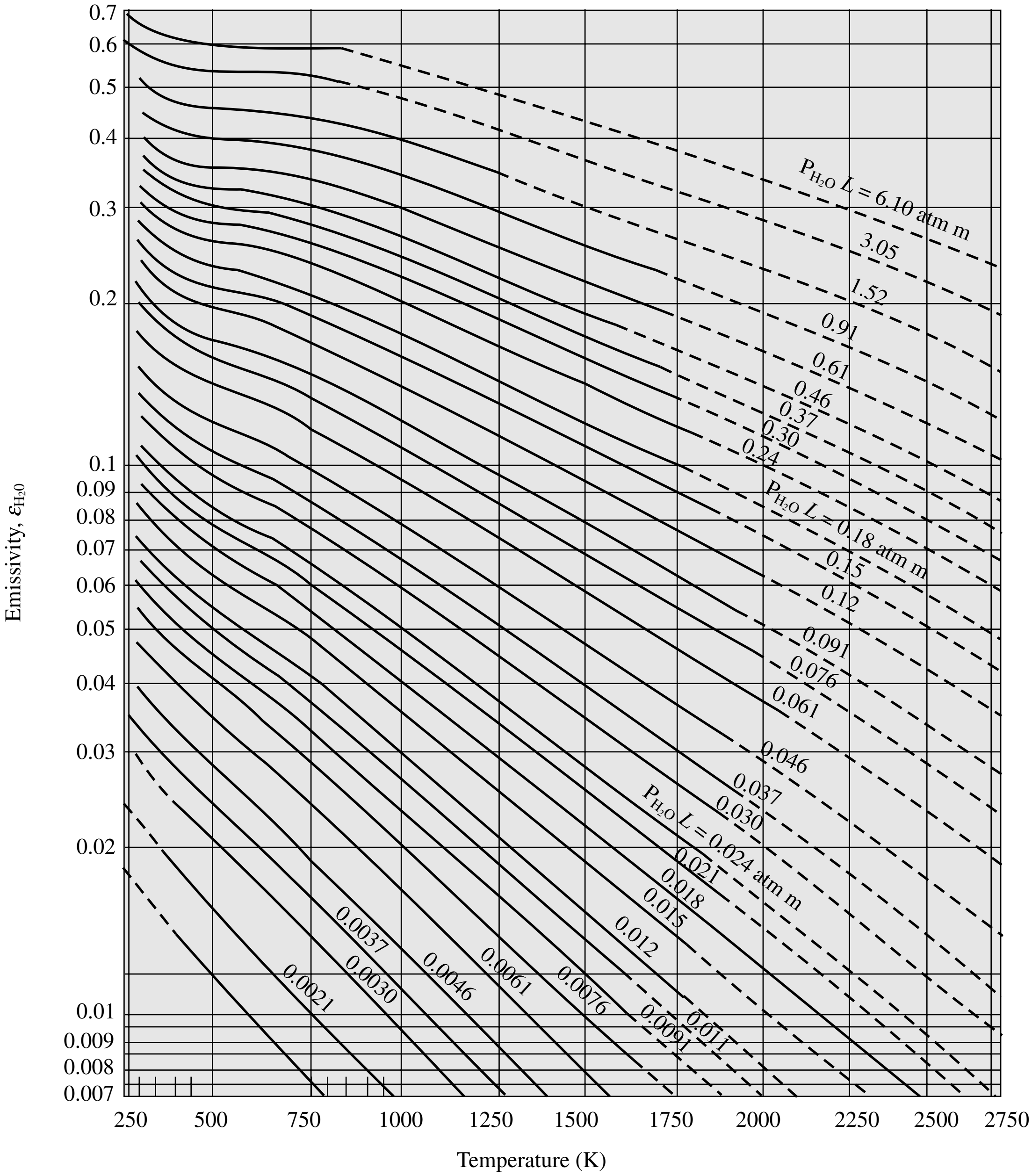


Fig. 13.33 *Emissivity of water vapour at a total presure of 1 atm.*

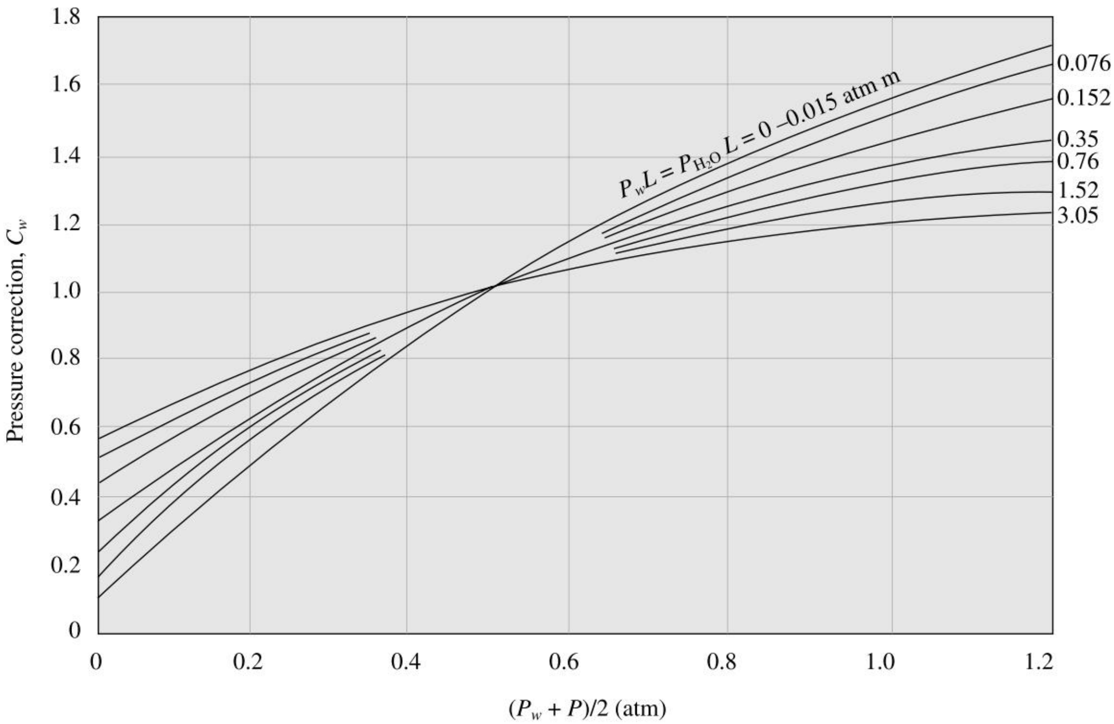


Fig. 13.34 *Correction factor, C_w for the emissivity of water vapour at pressures other than 1 atm*

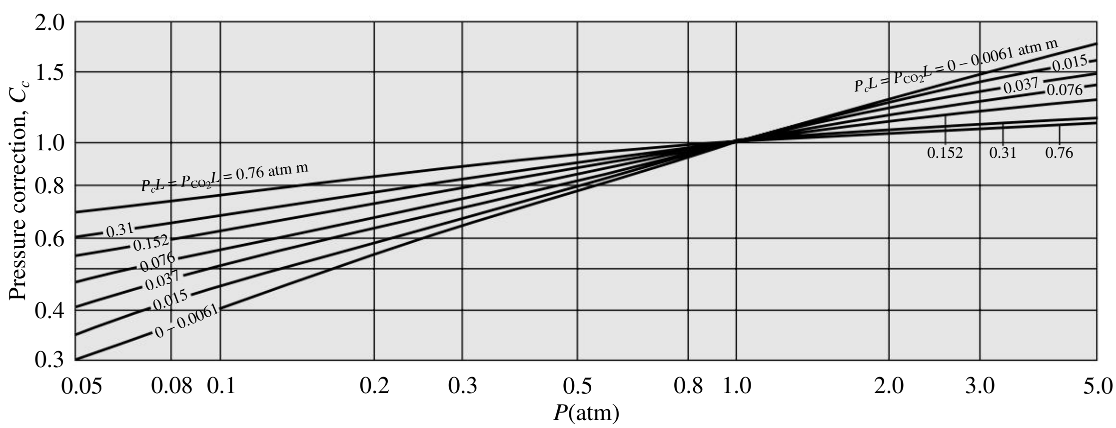


Fig. 13.35 *Correction factor: Total pressure, C_c, for the emissivity of carbon dioxide at pressures other than 1 atm.*

Note that $C_w = 1$ for $P = 1$ atm and thus $(P_w + P)/2 \cong 0.5$. Emissivity values are presented in a similar manner for a mixture of CO_2 and non-participating gases in Fig. 13.32 and 13.35.

If the CO_2 and H_2O gases appear *together* in a mixture with other non-radiating gases, the emissivity of each participating gas can still be determined using its partial pressure, but the *effective emissivity* of the mixture cannot be determined by simply adding the emissivities of individual gases. In such a case, the total gas emissivity can be expressed as

$$\boxed{\varepsilon_g = \varepsilon_c + \varepsilon_w - \Delta\varepsilon = C_c \varepsilon_{c,1\,\text{atm}} + C_w \varepsilon_{w,1\,\text{atm}} - \Delta\varepsilon} \tag{13.64}$$

where $\Delta\varepsilon$ is the emission correction factor to account for the overlapping of the carbon dioxide and water vapour emission bands and can be obtained form Fig. 13.36.

Similarly, in the presence of both water vapour and carbon dioxide, the total gas absorptivity can be expressed as

$$\boxed{\alpha_g = \alpha_c + \alpha_w - \Delta\alpha} \tag{13.65}$$

The pressure correction factors C_c and C_w are evaluated using $P_c L$ and $P_w L$, as in emissivity calculations.

The absorptivity of the gas depends not only on the *gas temperature*, but also on the source temperature of the radiation being absorbed, T_s.

For *water vapour* and *carbon dioxide,* the required gas absorptivity α_g may be evaluated from the emissivity by expressions of the following form:

Water vapour:

$$\boxed{\alpha_w = C_w \left(\frac{T_g}{T_s}\right)^{0.45} \times \left(T_s, P_w L \frac{T_s}{T_g}\right)} \tag{13.66a}$$

Carbon dioxide:

$$\boxed{\alpha_c = C_c \left(\frac{T_g}{T_s}\right)^{0.65} \times \varepsilon_c \left(T_s, P_c L \frac{T_s}{T_g}\right)} \tag{13.66b}$$

where $\Delta\alpha = \Delta\varepsilon$ and is determined from Fig. 13.36 at the source temperature T_s.

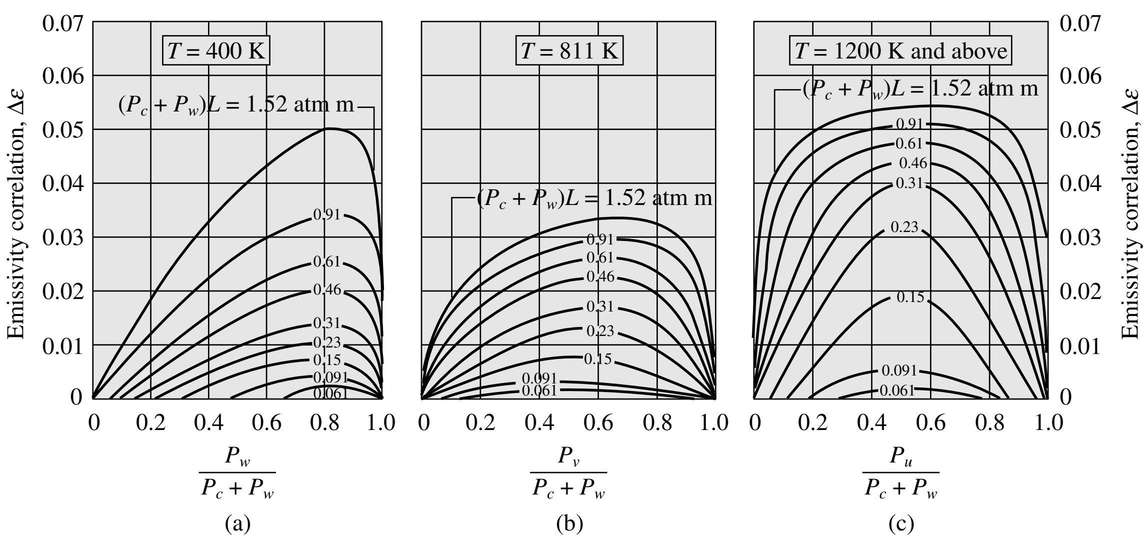


Fig. 13.36 *Emissivity correction factor, $\Delta\varepsilon$ for mutual absorption when both CO_2 and H_2O(v) are present in a gas mixture*

Table 13.4 gives a few mean beam length values, L for certain geometries of a gas body used in Figs. 13.32 to 13.36.

Table 13.4 *Effective mean beam lengths*

S.No	Geometry of gas body	Mean beam length, L
1.	Sphere of diameter D radiating to its inner surface	0.65 D
2.	Hemisphere of radius R radiating to the centre of its base	R
3.	Infinitely long circular cylinder of diameter D radiating to inner cylindrical (curved) surface	0.95 D
4.	Semi-infinite circular cylinder of diameter D radiating to (a) its entire base, and (b) the element at the centre of its base	0.65 D 0.90 D
5.	Infinite semi-circular cylinder of radius R radiating to the centre of its base	1.26 R
6.	Circular cylinder of height equal to diameter D radiating to (a) the entire surface,and (b) the centre of its base	0.60 D 0.70 D
7.	Cube of edge (side length) L radiating to any of its six faces	0.6 L
8.	Arbitrary shape of volume $\forall$ and surface area A_s radiating to surface	3.6 ($\forall/A_s$)

When the total emissivity of a gas ε_g at a temperature T_g is known, the emissive power of the gas can be expressed as $\boxed{E_g = \varepsilon_g \sigma T_g^4}$.

Then the rate of radiation energy emitted by a gas to a bounding surface of area A_s becomes

$$\boxed{\dot{Q}_{g,e} = \varepsilon_g A_s \sigma T_g^4} \tag{13.67}$$

If the bounding surface is *black* at temperature T_s, the surface will emit radiation to the gas at a rate of $A_s \sigma T_g^4$ and the gas will absorb all this radiation at a rate of $\alpha_g A_s \sigma T_s^4$, where α_g is the absorptivity of the gas. Then the net rate of radiation heat transfer between the gas and surrounding black surface is given by

$$\boxed{\dot{Q}_{\text{net}} = \sigma A_s (\varepsilon_g T_g^4 - \alpha_g T_s^4)} \qquad (\textit{Black enclosure}) \tag{13.68}$$

If the surface is *not* black, the analysis becomes quite complex because of the radiation reflected by the surface. However, for nearly black surfaces with an emissivity $\varepsilon_s > 0.7$, we have

$$\boxed{\dot{Q}_{\text{net, gray}} = \varepsilon_{s,\text{eff}}\, \dot{Q}_{\text{net, black}} = \left(\frac{\varepsilon_s + 1}{2}\right) \sigma A_s (\varepsilon_g T_g^4 - \alpha_g T_s^4)} \qquad (\textit{Grey enclosure}) \tag{13.69}$$

The functional notation means that the emissivity of CO_2 is evaluated at the temperature of the source T_s and an adjusted optical depth $P_c L(T_s/T_g)$ and then multiplied by the ratio T_g/T_s raised to the power 0.65. Remember that the temperature ratios must always be calculated with absolute temperatures in kelvin.

For *water vapour* the procedure is similar to that for CO_2 except that the ratio of absolute temperatures T_g/T_s is raised to a different power as follows:

$$\boxed{\alpha_{g\,H_2O}(T_g, T_s, P_{wL}) = \left(\frac{T_g}{T_s}\right)^{045} \times E_{gw}\left(T_s, P_{wL}\frac{T_s}{T_s}\right)}$$

Illustrative Examples

(A) Shape Factor

EXAMPLE 13.1 *Calculate the solar constant from the following data:*

Diameter of the sun = 1.39 × 10⁹ m

Diameter of the earth = 1.27 ×10⁷ m

Distance between the sun and the earth = 1.495 × 10¹¹ m

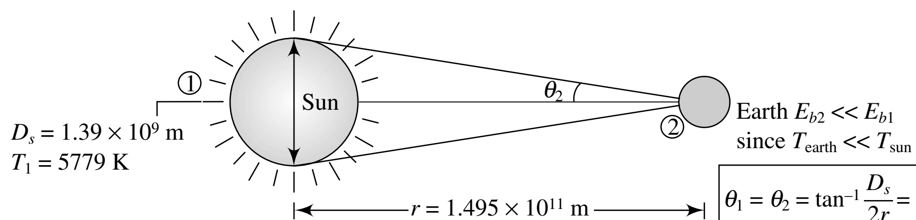
Effective black body temperature of the sun = 5779 K.

Solution

Known Diameters of sun and earth. Distance between sun and earth and the surface temperature of the sun.

Find Solar constant, G_S.

Schematic



$$\theta_1 = \theta_2 = \tan^{-1}\frac{D_s}{2r} = \sin^{-1}\frac{D_s}{2r} \approx \frac{D_s}{2r}$$

$$= \frac{1.39 \times 10^9 \text{ m}}{2 \times 1.495 \times 10^{11} \text{ m}} = 0.00465 \text{ rad}$$

$$\cos\theta_1 = \cos\theta_2 \approx \cos(0.00465 \text{ rad}) \approx 1$$

Assumptions (1) The sun is a black body. (2) $\theta_1 = \theta_2 \approx 0$. (3) $T_2 << T_1$ and $E_{b_2} \approx 0$.

Analysis The *solar constant, G_S*, is the energy from the sun, per unit time, received on a unit area of surface perpendicular to the direction of propagation of the radiation, at mean earth sun distance, outside the earth's atmosphere.

The sun appears as a circular disc of area $\pi D_s^2/4$. The net radiant heat exchange between the differential areas dA_1 and dA_2 on the sun (1) and the earth (2) is given by

$$d\dot{Q}_{12} = (E_{b_1} - E_{b_2})(\cos\theta_1 \cos\theta_2 dA_1 dA_2/\pi r^2)$$

The earth sun distance (r) being very large, θ_1 and θ_2 may be considered zero (the sun rays fall normal on the earth surface). The emissive power of the earth is also vanishingly small as compared to that of the sun.

Thus, solar constant, G_s

$$= \frac{d\dot{Q}_{12}}{dA_2} = \frac{\sigma T_1^4 dA_1}{\pi r^2} = \frac{\sigma T_s^4 (\pi D_s^2/4)}{\pi r^2}$$

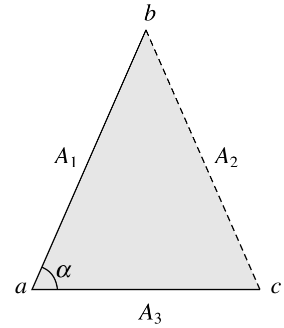
$$= \frac{(5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4)(5779 \text{ K})^4 (1.39 \times 10^9 \text{ m})^2/4}{(1.495 \times 10^{11} \text{ m})^2}$$

$$= \frac{5.67(57.79)^4}{4}\left(\frac{1.39}{149.5}\right)^2 \frac{\text{W}}{\text{m}^2} = \mathbf{1367\ W/m^2} \qquad \textbf{(Ans.)}$$

EXAMPLE 13.2 A_1 *and* A_3 *are two rectangular flat surfaces having a common edge and inclined at an arbitrary angle* α *to each other. They are very long along the common edge and have lengths of* ab *and* ac *respectively in the other directions. Show that*

$$F_{1-3} = \frac{(ab)+(ac)-(bc)}{2ab}$$

Schematic



Solution

Known Two inclined rectangular flat surfaces with a common edge.

Find Shape factor, F_{1-3}.

Assumption Diffuse grey surfaces.

Analysis Let us introduce and define a hypothetical surface A_2 of length bc joining lengths ab and ac of surfaces A_1 and A_3 to form an enclosure.

We note that A_1, A_2, and A_3 are flat surfaces and form an enclosure.

Surface A_1: $\underbrace{\cancel{F_{11}}^{\,0}}_{\text{flat surface}} + F_{12} + F_{13} = 1$ *(summation rule)*

Also, $A_1F_{1-2} = A_2F_{2-1}$ *(reciprocity rule)*

Hence, $F_{1-3} = 1 - F_{1-2} = 1 - \dfrac{A_2}{A_1}F_{2-1}$

Surface A_2: $F_{2-1} + \underbrace{\cancel{F_{2-2}}^{\,0}}_{\text{flat surface}} + F_{2-3} = 1$ *(summation rule)*

$\therefore$ $\boxed{F_{2-1} = 1 - F_{2-3}}$

Hence, $F_{1-3} = 1 - \dfrac{A_2}{A_1}[1 - F_{2-3}] = 1 - \dfrac{A_2}{A_1} + \dfrac{A_2}{A_1}F_{2-3}$

But, $A_2F_{2-3} = A_3F_{3-2}$ *(reciprocity rule)*

$\therefore$ $\boxed{F_{2-3} = \dfrac{A_3}{A_2}F_{3-2}}$

Surface A_3: $F_{3-1} + F_{3-2} + \underbrace{\cancel{F_{3-3}}^{\,0}}_{\text{flat surface}} = 1$ *(summation rule)*

$\therefore$ $\boxed{F_{3-2} = 1 - F_{3-1}}$

$\therefore$ $F_{1-3} = 1 - \dfrac{A_2}{A_1} + \dfrac{A_2}{A_1}\cdot\dfrac{A_3}{A_2}F_{3-2}$ or $F_{1-3} = 1 - \dfrac{A_2}{A_1} + \dfrac{A_3}{A_1}(1 - F_{3-1})$

Since $A_1F_{1-3} = A_3F_{3-1}$, *(reciprocity rule)*

$\boxed{F_{3-1} = A_1F_{1-3}/A_3}$

Hence, $F_{1-3} = 1 - \dfrac{A_2}{A_1} + \dfrac{A_3}{A_1} - \dfrac{A_3}{A_1}\cdot\dfrac{A_1}{A_3}F_{1-3}$ or $2F_{1-3} = 1 - \dfrac{A_2}{A_1} + \dfrac{A_3}{A_1} = \dfrac{A_1 - A_2 + A_3}{A_1}$

$\therefore$ $F_{1-3} = \dfrac{A_1 - A_2 + A_3}{2A_1}$

It follows that

$$\boxed{F_{1-3} = \frac{(ab)+(ac)-(bc)}{2(ab)}}$$ **QED**

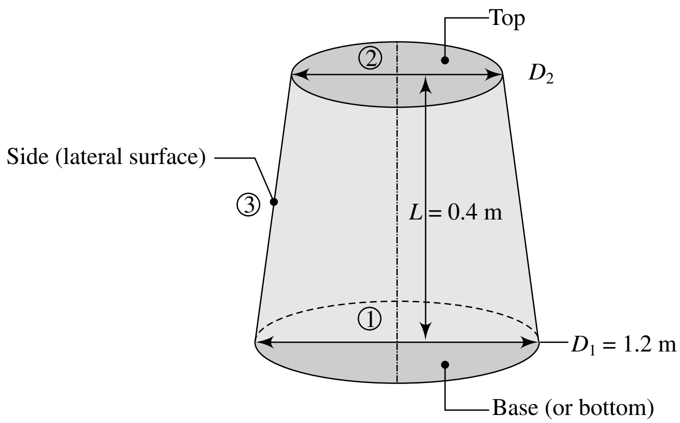
EXAMPLE 13.3 *The frustum of a cone has a base diameter of 1.2 m and a height of 0.4. The semicone angle is 30°. Determine all the view factors.*

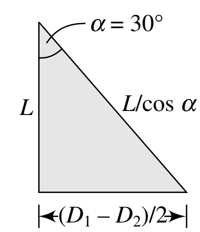
Solution

Known Dimensions of the frustum of a cone.

Find All view factors.

Schematic





Assumptions (1) Diffuse surfaces.

Analysis As there are three surfaces involved, namely, top, bottom, and side, $N = 3$. The number of view factors available is $N^2 = 3^2 = 9$. The number of independent view factors to be calculated $= \dfrac{N(N-1)}{2} = 3$

For the given geometric configuration, $\tan\alpha = (D_1 - D_2)/2$

Diameter of the top surface, $D_2 = D_1 - 2\,L\tan\alpha$

$$= 1.2\text{ m} - (2)(0.4\text{ m})\tan 30° = 0.738\text{ m}$$

Area calculations:

Area of the base surface, $A_1 = \pi D_1^2/4 = \dfrac{\pi}{4} \times 1.2^2 = 1.131\text{ m}^2$

Area of the top surface, $A_2 = \pi D_2^2/4 = \dfrac{\pi}{4} \times 0.738^2 = 0.428\text{ m}^2$

Area of the lateral (*curved*) surface,

$$A_3 = \pi\left(\frac{D_1 + D_2}{2}\right)\frac{L}{\cos\alpha} = \frac{\pi}{2}(1.131 + 0.428)\text{m} \times \frac{0.4\text{ m}}{\cos 30°} = 1.131\text{ m}^2$$

View factor calculations:

F_{12} is base to top view factor. The geometry corresponds to two directly opposed, parallel circular discs with

$$R_1 = \frac{r_1}{L} = \frac{1.2\text{ m}}{(2\times 0.4)\text{m}} = 1.5 \quad \text{and} \quad R_2 = \frac{r_2}{L} = \frac{0.738\text{ m}}{2\times 0.4} = 0.923$$

From the appropriate chart,

$$\boxed{F_{12} = 0.24}$$

Since $\quad A_1F_{12} = A_2F_{21} \quad$ (*reciprocity rule*)

$$F_{21} = \left(\frac{A_1}{A_2}\right)F_{12} = \left(\frac{1.131}{0.428}\right)(0.24) \quad \Rightarrow \quad F_{21} = 0.63$$

$$\underbrace{\cancelto{0}{F_{11}}}_{\text{flat surface}} + F_{12} + F_{13} = 1 \qquad \text{(summation rule)}$$

$$F_{13} = 1 - F_{12} = 1 - 0.24 \quad \Rightarrow \quad F_{13} = 0.76$$

$$F_{21} + \underbrace{\cancelto{0}{F_{22}}}_{\text{flat surface}} + F_{23} = 1 \qquad \text{(summation rule)}$$

$$F_{23} = 1 - F_{21} = 1 - 0.63 \quad \Rightarrow \quad F_{23} = 0.37$$

Now, $\quad A_1F_{13} = A_3F_{31} \quad$ (*reciprocity rule*)

$$F_{31} = F_{13}(A_1/A_3) = (0.76)(1.131/1.131) \quad \Rightarrow \quad F_{31} = 0.76$$

Also, $\quad A_2F_{23} = A_3F_{32} \quad$ (*reciprocity rule*)

$$F_{32} = (A_2/A_3)F_{23} = (0.428/1.131)(0.37) \quad \Rightarrow \quad F_{32} = 0.14$$

Furthermore, $F_{31} + F_{32} + F_{33} = 1 \quad$ (*summation rule*)

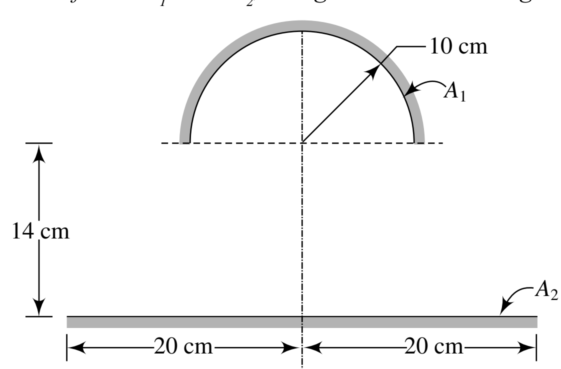
As the curved surface is *concave*, the self-view factor, $F_{33} \neq 0$.

Hence, $\quad F_{33} = 1 - F_{31} - F_{32} = 1 - 0.76 - 0.14 \quad \Rightarrow \quad F_{33} = 0.10$

All the possible 9 view factors are arranged in the form of a matrix below:

$$\boldsymbol{F} = \begin{vmatrix} \boldsymbol{F_{11}} & \boldsymbol{F_{12}} & \boldsymbol{F_{13}} \\ \boldsymbol{F_{21}} & \boldsymbol{F_{22}} & \boldsymbol{F_{23}} \\ \boldsymbol{F_{31}} & \boldsymbol{F_{32}} & \boldsymbol{F_{33}} \end{vmatrix} = \begin{vmatrix} \mathbf{0} & \mathbf{0.24} & \mathbf{0.76} \\ \mathbf{0.63} & \mathbf{0} & \mathbf{0.37} \\ \mathbf{0.76} & \mathbf{0.14} & \mathbf{0.10} \end{vmatrix} \qquad \textbf{(Ans.)}$$

EXAMPLE 13.4 *An infinitely long semi-cylindrical surface* A_1 *of radius 10 cm and an infinitely long flat plate* A_2 *of half width 20 cm are located a 14 cm distance apart as shown in the figure. Determine the shape factor* $F_{1\,2}$ *between surfaces* A_1 *and* A_2*, using the crossed string method.*

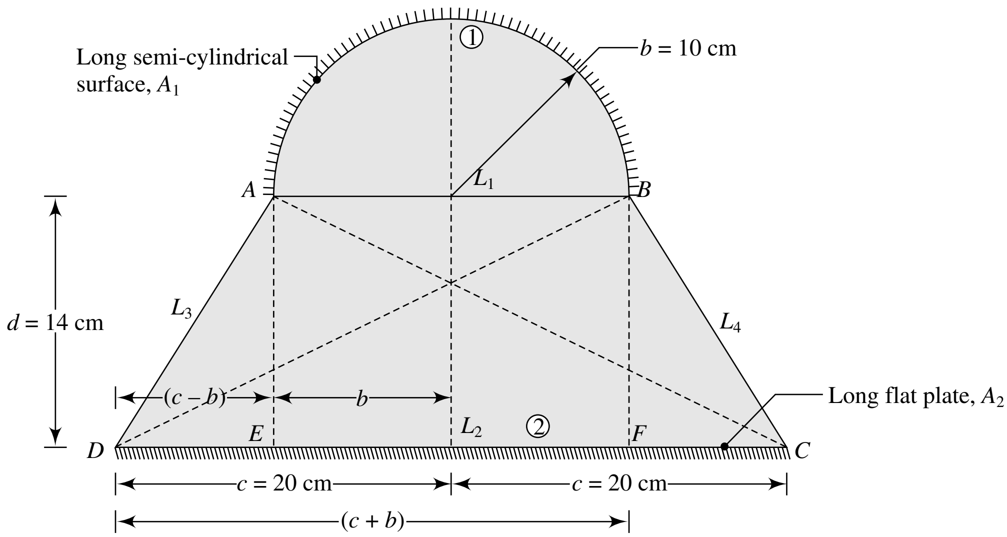


Solution

Known A long semi-cylindrical surface and a long flat plate a distance apart with prescribed dimensions.

Find Shape factor, F_{12}.

Schematic



Assumptions The surfaces are diffuse emitters and reflectors.

Analysis The end points of both surfaces A_1 and A_2 are located as A, B, C, and D. We draw dashed straight lines between the end points and identify the crossed and uncrossed strings. Hottel's crossed strings method can be stated as

$$F_{i-j} = \frac{\Sigma \text{Crossed strings} - \Sigma \text{Uncrossed strings}}{2 \times \text{String on surface } i}$$

$$\therefore \qquad F_{12} = \frac{(L_5 + L_6) - (L_3 + L_4)}{2L_1}$$

where $L_1 = 2b = 2 \times 10 \text{ cm} = 20 \text{ cm}$

$$L_2 = 2c = 2 \times 20 \text{ cm} = 40 \text{ cm}$$

$$L_3 = L_4 = AD = \sqrt{DE^2 + AE^2} = [(c-b)^2 + d^2]^{1/2}$$

$$= [(20-10)^2 + 14^2]^{1/2} = \mathbf{17.2\ cm}$$

$$L_5 = L_6 = BD = \sqrt{DF^2 + BF^2} = [(c+b)^2 + d^2]^{1/2}$$

$$= [(20+10)^2 + 14^2]^{1/2} = \mathbf{33.1\ cm}$$

Shape factor,

$$F_{12} = \frac{2L_5 - 2L_3}{2L_1} = \frac{L_5 - L_3}{L_1} = \frac{(33.1 - 17.2)\text{cm}}{20 \text{ cm}} = \mathbf{0.795} \qquad \textbf{(Ans.)}$$

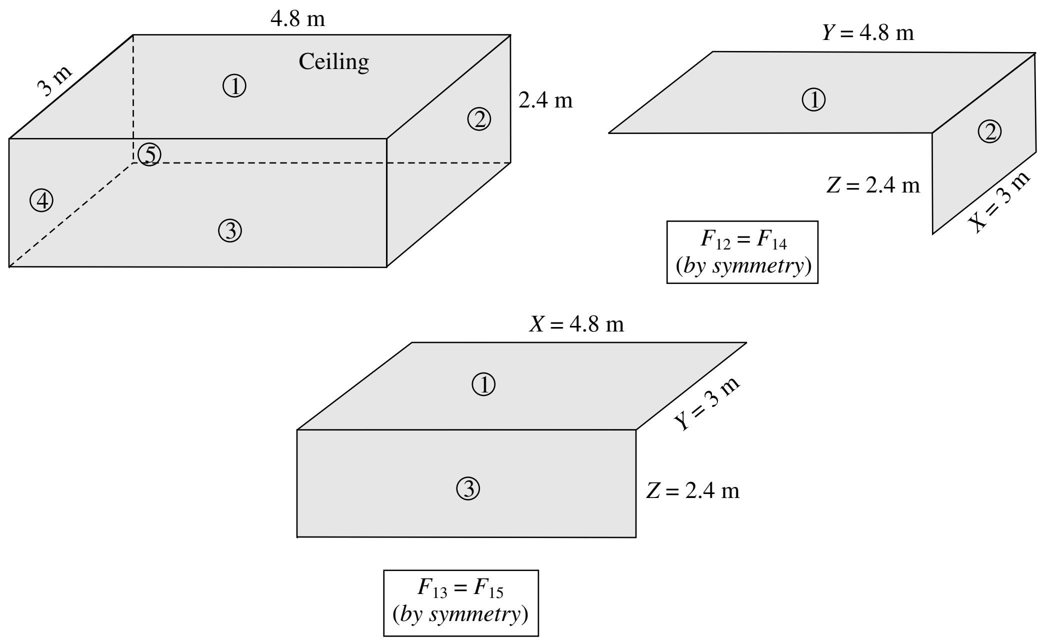
EXAMPLE 13.5 *A room is 3 m by 4.8 m with a 2.4 m high ceiling. The ceiling contains heating elements and is textured so that it acts essentially as a black diffuse surface. What percentage of the radiant energy leaving the ceiling strikes all four walls?*

Solution

Known Dimensions of a room with its ceiling containing heating elements.

Find Percentage of radiation energy leaving ceiling and striking the four walls.

Schematic



Assumption Black diffuse surfaces.

Analysis Fraction of energy emitted by surface 1 (*ceiling*) and striking the four walls 2, 3, 4, and 5 is the sum total of shape factors F_{12}, F_{13}, F_{14} and F_{15}.

From symmetry: $F_{12} = F_{14}$ and $F_{13} = F_{15}$

$\therefore$ Percentage of radiation intercepted by *four* walls is

$$100[(2 \times F_{12}) + (2 \times F_{13})]$$

To determine F_{12}: $\dfrac{Y}{X} = \dfrac{4.8\text{ m}}{3\text{ m}} = 1.6\text{ m} \quad \dfrac{Z}{X} = \dfrac{2.4\text{ m}}{3\text{ m}} = 0.8$

From the chart for perpendicular rectangles with a common edge:

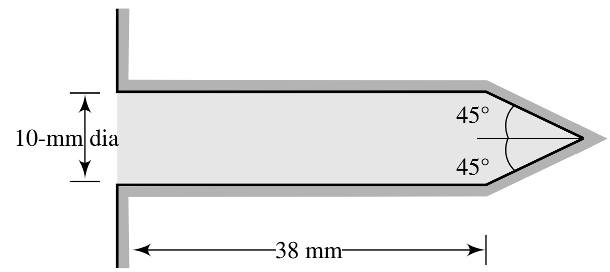
$$F_{12} = \mathbf{0.125}$$

To determine F_{13}: $\dfrac{Y}{X} = \dfrac{3\text{ m}}{4.8\text{ m}} = 0.625 \quad \dfrac{Z}{X} = \dfrac{2.4\text{ m}}{4.8\text{ m}} = 0.5 \quad F_{13} = \mathbf{0.21}$

Hence, percentage of radiant energy striking the walls is

$$100[(2 \times 0.125) + (2 \times 0.21)] = \mathbf{67\ \%}$$ **(Ans.)**

EXAMPLE 13.6 *A 10 mm diameter hole is drilled into a metal slab as shown. If the metal is at a uniform temperature and has a black body behaviour, determine the percentage of the emission from the cavity surface that will escape to the surroundings.*



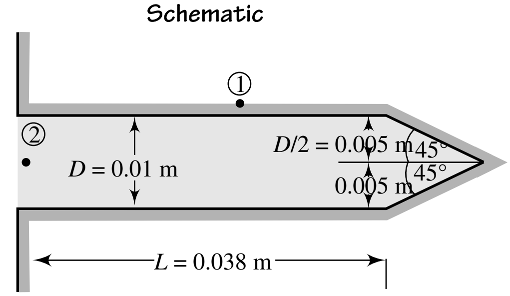
Solution

Known A hole is drilled into a slab of metal.

Find Emission from the cavity surface lost through the opening, i.e., shape factor F_{12}.

Assumptions The metal slab is isothermal and approximated as a black surface.

Schematic



Analysis The walls in the cavity are represented as the surface 1 and the circular hole is designated as the surface 2. Let us first calculate the surface areas of 1 and 2.

$$A_1 = \pi DL + \frac{1}{2}\pi D\left(\frac{D/2}{\sin 45°}\right)$$

$$= \pi(0.01\text{ m})(0.038\text{ m}) + \frac{1}{2}\pi(0.01\text{ m}) \times (0.005\text{ m}/\sin 45°) = 0.0013\text{ m}^2$$

$$A_2 = \frac{\pi}{4}D^2 = \left(\frac{\pi}{4}\right)(0.01\text{ m})^2 = 78.54 \times 10^{-6}\text{ m}^2$$

The shape factor between 2 and 1 is

$$F_{21} = 1$$

$$A_1F_{12} = A_2F_{21} = A_2 \qquad \textit{(Reciprocity theorem)}$$

$$\therefore \qquad F_{12} = \frac{A_2}{A_1} = \frac{78.54 \times 10^{-6}\text{ m}^2}{0.0013\text{ m}^2} = \mathbf{0.06}$$

Hence, 6% of the emission from the surface of the cavity escapes to the surroundings through the opening. **(Ans.)**

(B) Black Bodies: Radiation Heat Exchange

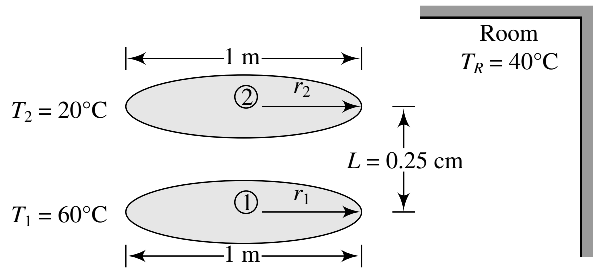
EXAMPLE 13.7 *Two perfectly black, parallel disks, 1 m in diameter are separated by a distance of 0.25 m. One disk is kept at 60°C while the other is held at 20°C. The discs are placed in a large room whose walls are maintained at 40°C. Determine the net radiation heat exchange (a) between the disks, and (b) between the discs and the room.*

Solution

Known Two black parallel disks opposite each other are placed in a large room.

Find Net radiation heat exchange (a) between the disks, and (b) between discs and room.

Schematic



Assumptions (1) All surfaces are black and isothermal. (2) Exterior surfaces (the surfaces which do not face each other) are well insulated.

Analysis (a) Net rate of radiation heat transfer from the surface 1 to the surface 2 is

$$\dot{Q}_{12} = A_1 F_{12} \sigma (T_1^4 - T_2^4)$$

where $A_1 = \frac{\pi}{4} D^2$

From the relevant shape-factor chart with $\frac{r_1}{L} = \frac{r_2}{L} = \frac{0.50 \text{ m}}{0.25 \text{ m}} = 2$, we get $F_{12} = 0.62$

$$\therefore \quad \dot{Q}_{12} = \left(\frac{\pi}{4} \times 1^2\right) \text{m}^2 (0.62)(5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4) \left[\left(\frac{60 + 273.15}{100}\right)^4 - \left(\frac{20 + 273.15}{100}\right)^4 \right] (10^8) \text{K}^4$$

$$= 2.76[3.3315^4 - 2.9315^4]\text{W} = \mathbf{136.2 \text{ W}} \qquad \textbf{(Ans.) (a)}$$

(b) ***Disk 1 to room:*** $\dot{Q}_{1R} = A_1 F_{1R} \sigma (T_1^4 - T_R^4)$

where $F_{1R} = 1 - F_{12} = 1 - 0.62 = 0.38$

Therefore, $\dot{Q}_{1R} = \left(\frac{\pi}{4} \times 1^2\right) \text{m}^2 (0.38)(5.67 \times 10^{-8}) \text{W/m}^2 \text{ K}^4 [3.3315^4 - 3.1315^4](10^8)\text{K}^4$

$= 45.72$ W

Disk 2 to room: $\dot{Q}_{2R} = A_2 F_{2R} \sigma (T_2^4 - T_R^4)$

where $F_{2R} = 1 - F_{12} = 1 - 0.62 = 0.38$

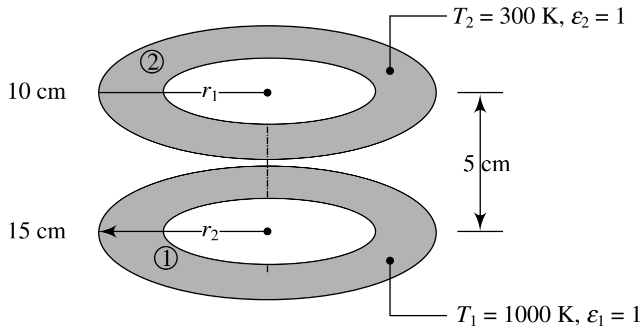
Therefore, $\dot{Q}_{2R} = \left(\frac{\pi}{4} \times 1^2\right) \text{m}^2 (0.38)(5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4)[2.9315^4 - 3.1315^4](10^8)\text{K}^4$

$= -37.76$ W

The *net* radiative heat exchange, then, between the two discs and the room is

$$\dot{Q}_{(12)R} = (45.72 - 37.76) \text{ W} = \mathbf{7.96 \text{ W}} \qquad \textbf{(Ans.) (b)}$$

EXAMPLE 13.8 *Determine the view factor F_{12} for the areas A_1 and A_2 oriented as shown in the accompanying figure and calculate the net radiative heat exchange between the two surfaces.*

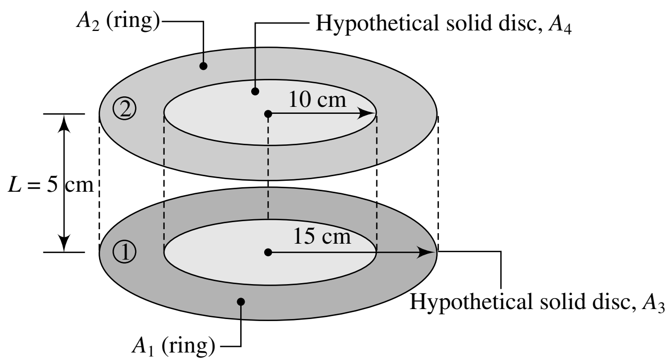


Solution

Known Two parallel, black, ring shaped, coaxial and identical disks at a specified distance apart with prescribed temperatures.

Find View factor, F_{12} and $\dot{Q}_{\text{net}}$.

Schematic



Assumptions (1) The two ring shaped disks are black and diffuse surfaces. (2) Convection is neglected.

Analysis Let A_3 and A_4 represent the hypothetical co planar central solid disks of radius, 10 cm. (Area A_3 + Area A_1) = Area, $A_{1,3}$ of the solid disks of 15 cm radius. Similarly, (Area A_2 + Area A_4) = Area $A_{2,4}$ of the solid disk of radius 15 cm.

Then, we have, using the view-factor algebra.

$$A_1F_{12} = A_2F_{21} \qquad \textit{(reciprocity relation)}$$

$$A_2F_{21} = A_2[F_{2(1,3)} - F_{23}] = A_2F_{2(1,3)} - A_2F_{23}$$

Using the reciprocity relation, $A_2F_{2(1,3)} = A_{(1,3)}F_{(1,3)2}$ and $A_2F_{23} = A_3F_{32}$

Now, $\quad F_{(1,3)2} = F_{(1,3)(2,4)} - F_{(1,3)4} \quad$ and $\quad F_{32} = F_{3(2,4)} - F_{34}$

Hence, $\quad A_2F_{21} = A_{1,3}[F_{(1,3)(2,4)} - F_{(1,3)4}] - A_3[F_{3(2,4)} - F_{34}]$

Reciprocity relation: $A_2F_{21} = A_1F_{12}$

$\therefore \quad A_1F_{12} = A_{1,3}[F_{(1,3)(2,4)} - F_{(1,3)4}] - A_3[F_{3(2,4)} - F_{34}]$

The view factor,

$$\boxed{\boldsymbol{F_{12} = \frac{A_{1,3}}{A_1}[F_{(1,3)(2,4)} - F_{(1,3)4}] - \frac{A_3}{A_1}[F_{3(2,4)} - F_{34}]}}$$

With $\quad A_{1,3} = \pi(15)^2\,\text{cm}^2,\ A_3 = \pi(10)^2\,\text{cm}^2 \quad$ and $\quad A_1 = A_{1,3} - A_3 = \pi[15^2 - 10^2]\text{cm}^2$

$$\frac{A_{1,3}}{A_1} = \frac{\pi(15)^2}{\pi(15^2 - 10^2)} = 1.8 \quad \text{and} \quad \frac{A_3}{A_1} = \frac{\pi(10)^2}{\pi(15^2 - 10^2)} = 0.8$$

Each term in the expression for F_{12} can be evaluated from the appropriate chart.

Calculation of $\boldsymbol{F_{(1,3)(2,4)}}$: $\dfrac{r_i}{L} = \dfrac{r_j}{L} = \dfrac{15}{5} = 3$

$$\boxed{F_{(1,3)(2,4)} = 0.72}$$

Calculation of $\boldsymbol{F_{(1,3)4}}$: $\dfrac{r_i}{L} = \dfrac{15}{5} = 3,\ \dfrac{r_j}{L} = \dfrac{10}{5} = 2$

$$\boxed{F_{(1,3)4} = 0.38}$$

Calculation of $\boldsymbol{F_{3(2,4)}}$: $\dfrac{r_i}{L} = \dfrac{10}{5} = 2,\ \dfrac{r_j}{L} = \dfrac{15}{5} = 3$

$$\boxed{F_{3(2,4)} = 0.85}$$

Calculation of $\boldsymbol{F_{34}}$: $\dfrac{r_i}{L} = \dfrac{10}{5} = 2,\ \dfrac{r_j}{L} = \dfrac{10}{5} = 2$

$$\boxed{F_{34} = 0.61}$$

Substituting the relevant values, the view factor F_{12} is determined to be

$$F_{12} = 1.8[0.72 - 0.38] - 0.8[0.85 - 0.61] = \mathbf{0.42}$$

Net radiation heat exchange between the two surfaces is

$$\begin{aligned}\dot{Q}_{\text{net}} &= A_1 F_{12} \sigma [T_1^4 - T_2^4] \\ &= \pi[0.15^2 - 0.10^2]\text{m}^2 \times 0.42 \times 5.67 \times 10^{-8}\ \text{W/m}^2\,\text{K}^4 \times [(1000\ \text{K})^4 - (300\ \text{K})^4] \\ &= \mathbf{928\ W}\end{aligned}$$

(Ans.)

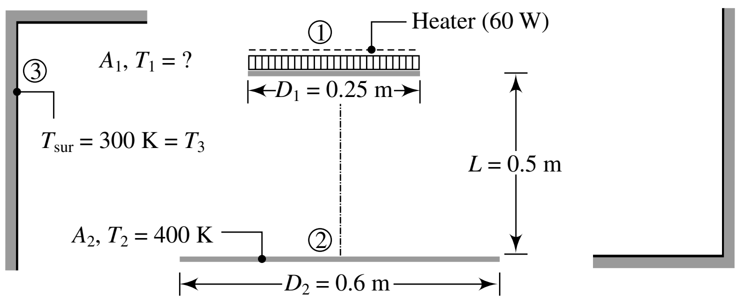
EXAMPLE 13.9 *Two directly opposed, parallel black disks are 50 cm apart. The diameter of the top disk is 25 cm. The bottom disk of 60 cm diameter is maintained at a uniform temperature of 400 K. An electric power of 60 W is supplied to the heater on the back side of the 25 cm diameter top disk. If the surroundings are at a temperature of 300 K, determine the temperature of the top disk.*

Solution

Known Two parallel, black, coaxial disks with surroundings around. Heater input to upper disk, disk diameters and lower disk temperature are specified.

Find Upper disk temperature (K).

Schematic



Assumptions (1) Black disks at uniform temperatures. (2) Backside of heater is insulated.

Analysis $\begin{pmatrix}\text{Electrical power input} \\ \text{to the heater}\end{pmatrix} = \begin{pmatrix}\text{Net radiation heat transfer from the} \\ \text{top disk 1, } \dot{Q} = 60 \text{ W}\end{pmatrix}$

$$\dot{Q}_1 = A_1 F_{12}\sigma(T_1^4 - T_2^4) + A_1 F_{13}\sigma(T_1^4 - T_3^4)$$
$$= A_1\sigma[F_{12}(T_1^4 - T_2^4) + F_{13}(T_1^4 - T_{sur}^4)] \qquad \text{(A)}$$

where $A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}(0.25 \text{ m})^2 = 0.0491 \text{ m}^2, T_2 = 400 \text{ K}, T_3 = T_{sur} = 300 \text{ K}$

Calculation of shape factors: $R_1 = \frac{r_1}{L} = \frac{0.125 \text{ m}}{0.5 \text{ m}} = 0.25, R_2 = \frac{r_2}{L} = \frac{0.3 \text{ m}}{0.5 \text{ m}} = 0.6$

$$S = 1 + \frac{1 + R_2^2}{R_1^2} = 1 + \left(\frac{1 + 0.6^2}{0.25^2}\right) = 22.76$$

$\therefore$ $$F_{12} = \frac{1}{2}[S - \sqrt{S^2 - 4(R_2/R_1)^2}] = \frac{1}{2}[22.76 - \sqrt{22.76^2 - 4(0.6/0.25)^2}] = \mathbf{0.256}$$

$$\underbrace{F_{11}}_{\text{flat surface}}{}^{0} + F_{12} + F_{13} = 1 \qquad (\textit{summation rule})$$

$\therefore$ $$F_{13} = 1 - F_{12} = 1 - 0.256 = \mathbf{0.744}$$

Hence, substituting the known quantities in Eq. (A), one gets

$$60 \text{ W} = 0.0491 \text{m}^2 \times 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4[0.256(T_1^4 - 400^4) + 0.744(T_1^4 - 300^4)]\text{K}^4$$

or $$215.6 = 0.256[x^4 - 256] + 0.744[x^4 - 81] \text{ where } x \equiv \frac{T_1}{100}$$

By *trial* and *error*, one finds, x = 4.3 and, the top-disk temperature,

$\mathbf{T_1 = 430 \text{ K}}$ **(Ans.)**

(C) Grey Bodies: Two Surface Enclosures

EXAMPLE 13.10 *Radiative heat transfer is intended between the inner surfaces of two very large isothermal parallel metal plates. While the upper plate (designated as Plate 1) is a black surface and is the warmer one being maintained at 1000 K, the lower plate (Plate 2) is a diffuse and grey surface with an emissivity of 0.8 and is kept at 500 K. Assume that the surfaces are sufficiently large to form a two-surface enclosure and steady-state conditions exist.*

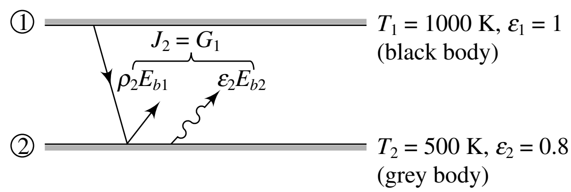
Determine (a) irradiation to the top plate, (b) radiosity of the top plate, (c) radiosity of the lower plate, and (d) net radiative heat exchange between the plates per unit area of the plates if the plate 1 is a diffused grey surface with an emissivity of 0.5.

Solution

Known Two large horizontal, parallel plates with given surface conditions and temperatures.

Find (a) Irradiation to the top plate, G_1. (b) Radiosity of the top plate, J_1. (c) Radiosity of the lower plate, J_2. (d) Net radiative exchange between the plates per unit area of the plates if ε_1 is 0.7.

Schematic



Assumptions (1) The plates make up a two-surface enclosure (2) Diffuse grey surfaces.

Analysis (a) The irradiation to the upper plate is defined as the radiant flux incident on that surface. The irradiation to the upper plate, G_1 is comprises the radiant flux emitted by the surface 2 and the reflected flux emitted by the surface 1.

$$G_1 = \varepsilon_2 E_{b2} + \rho_2 E_{b1} = \varepsilon_2 \sigma T_2^4 + (1 - \varepsilon_2)\sigma T_1^4$$

or $$G_1 = 0.8 \times 5.67 \times 10^{-8} \text{ W/m}^2\text{ K}^4 (500 \text{ K})^4 + (1 - 0.8) \times 5.67 \times 10^{-8} \text{ W/m}^2\text{ K}^4 (1000 \text{ K})^4$$

$$G_1 = 2835 \text{ W/m}^2 + 11\,340 \text{ W/m}^2 = \mathbf{14\,175\ W/m^2} \qquad \textbf{(Ans.) (a)}$$

(b) The radiosity is the radiant flux leaving the surface by emission and reflection. For the black body surface 1,

$$J_1 = E_{b1} = \sigma T_1^4 = 5.67 \times 10^{-8} \text{ W/m}^2\text{ K}^4 (1000 \text{ K})^4 = \mathbf{56\,700\ W/m^2} \qquad \textbf{(Ans.) (b)}$$

(c) The radiosity of the surface 2 is,

$$J_2 = \varepsilon_2 E_{b1} + \rho_2 G_2$$

Since the upper plate is a black body, we note that $G_2 = E_{b1}$ and

$$J_2 = \varepsilon_2 E_{b1} + \rho_2 E_{b1} = \varepsilon_2 \sigma T_2^4 + (1 - \varepsilon_2)\sigma T_1^4 = \mathbf{14\,175\ W/m^2} \qquad \textbf{(Ans.) (c)}$$

Note that $J_2 = G_1$. That is, the radiant flux leaving the surface 2 (J_2) is incident upon the surface 1 (G_1).

(d) The net radiation heat exchange per unit area is

$$q_{\text{net}} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1} = \frac{5.67 \times 10^{-8}(1000^4 - 500^4)}{\dfrac{1}{0.5} + \dfrac{1}{0.8} - 1} = \mathbf{23\,625\ W/m^2} \qquad \textbf{(Ans.) (d)}$$

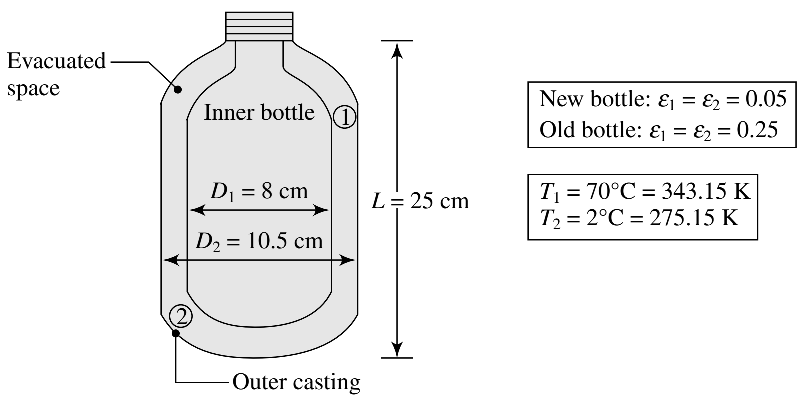
EXAMPLE 13.11 *Hot coffee at 70°C is contained in a cylindrical vacuum bottle with a height of 25 cm. The coffee container consists of an inner bottle centred within an outer casing that is at 2°C. The space between the inner bottle and the casing is evacuated, and the walls are coated with aluminium to minimize radiative heat loss. There is negligible heat transfer at the ends of the container. In a new vacuum bottle, the emissivity of all surfaces is 0.05, but in an older container, the finish becomes dull and the emissivity rises to 0.25. Calculate the rate of heat loss from the coffee for both a new and an old vacuum bottle.*

Solution

Known A vacuum bottle comprising two concentric cylinders with evacuated space in between contains coffee. Surface emissivities for old and new bottle are specified.

Find Heat loss from the coffee for both *new* and *old* bottle.

Schematic



Assumptions (1) The walls are grey and diffuse. (2) The walls are isothermal. (3) Radiosity on each surface is constant. (4) There is no net radiation in the axial direction outside the ends of the evacuated space.

Analysis The net rate of heat transfer between the inner bottle (*Surface 1*) and the casing (*Surface 2*) is

$$\dot{Q}_{1\to2} = \frac{E_{b1} - E_{b2}}{R_{\text{total}}} = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{A_1\varepsilon_1} + \dfrac{1}{A_1F_{1-2}} + \dfrac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

If we neglect the ends of the container, all the radiation leaving the bottle arrives at the casing. Therefore, $F_{1\to2} = 1$. The areas of the two cylinders are

$$A_1 = \pi D_1 L = \pi(8\text{ cm})(25\text{ cm}) = 628.32\text{ cm}^2 \qquad \frac{A_1}{A_2} = \frac{D_1}{D_2} = \frac{8\text{ cm}}{10.5\text{ cm}}$$

$$A_2 = \pi D_2 L = \pi(10.5\text{ cm})(25\text{ cm}) = 324.67\text{ cm}^2$$

- When the vacuum bottle is *new*, the emissivity of both surfaces is 0.05 and the rate of heat loss is calculated as

$$\dot{Q}_{1\to 2} = \frac{\sigma A_1(T_1^4 - T_2^4)}{\left(\frac{1-\varepsilon_1}{\varepsilon_1}\right) + 1 + \frac{A_1}{A_2}\left(\frac{1-\varepsilon_2}{\varepsilon_2}\right)}$$

$$= \frac{(5.67\times 10^{-8}\ \text{W/m}^2\,\text{K}^4)(628.32\times 10^{-4}\text{m}^2)[343.15^4 - 275.15^4]\text{K}^4}{\frac{0.95}{0.05} + 1 + \frac{8}{10.5}\left(\frac{0.95}{0.05}\right)}$$

$= \mathbf{0.84\ W}$ **(Ans.)**

- For an *old* vacuum bottle, the calculation is repeated with $\varepsilon_1 = \varepsilon_2 = 0.25$
  The heat loss rate from coffee is determined to be

$$\dot{Q}_{1\to 2} = \frac{(5.67\times 10^{-8}\ \text{W/m}^2\,\text{K}^4)(638.32\times 10^{-4}\ \text{m}^2)[343.15^4 - 275.15^4]\text{K}^4}{\left(\frac{1-0.25}{0.25}\right) + 1 + \frac{8}{10.5}\left(\frac{1-0.25}{0.25}\right)}$$

$= \mathbf{4.61\ W}$ **(Ans.)**

Comments Aging of the coating on a vacuum bottle can adversely affect its performance. Coffee will cool faster in the older container.

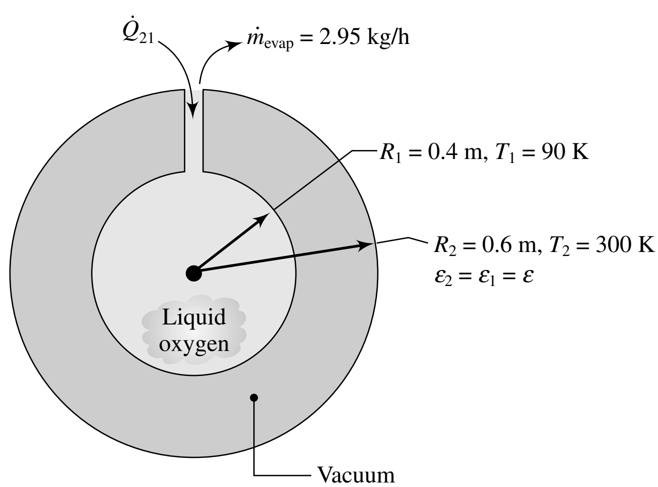
EXAMPLE 13.12 *Two concentric spheres of radii R_1 = 0.4 m and R_2 = 0.6 m are separated by a vacuum and have surface temperatures of T_1 = 90 K and T_2 = 300 K. The surfaces of the sphere have the same emissivity and the system is used for storing liquid oxygen. The latent heat of vaporization of liquid oxygen is 210 kJ/kg. The boil-off rate is 2.95 kg/h. Calculate the emissivity of the surfaces. If the rate of evaporation is to be reduced by 90.7%, what should be the surface emissivity?*

Solution

Known Concentric sphere system for storage of liquid oxygen. Diameters and temperatures. Rate of evaporation.

Find Emissivity $\varepsilon = \varepsilon_1 = \varepsilon_2$ for two specified boil-off rates.

Schematic



Assumptions (1) Steady operating conditions. (2) Diffuse grey surfaces.

Analysis Rate of heat transfer by radiation,

$\dot{Q}_{21} =$ (Rate of evaporation, $\dot{m}_{\text{evap}}$) $\times$ (Latent heat of vaporization, h_{fg}).

$$= \left(\frac{2.95}{3600}\frac{\text{kg}}{\text{s}}\right)\left(210 \times 10^3 \frac{\text{J}}{\text{kg}}\right) = 172.08 \text{ W}$$

For two concentric spheres,

$$\dot{Q}_{21} = \frac{\sigma A_1 (T_2^4 - T_1^4)}{\dfrac{1}{\varepsilon_1} + \left(\dfrac{1}{\varepsilon_2} - 1\right)\dfrac{A_1}{A_2}} = \frac{\sigma (4\pi R_1^2)(T_2^4 - T_1^4)}{\dfrac{1}{\varepsilon_1} + \left(\dfrac{R_1}{R_2}\right)^2 \left(\dfrac{1}{\varepsilon_2} - 1\right)}$$

$$= \frac{(5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4)(4\pi \times 0.4^2 \text{ m}^2)(300^4 - 90^4)\text{K}^4}{(1/\varepsilon) + (0.4/0.6)^2 \{(1/\varepsilon) - 1\}}$$

$$= \frac{915.94 \text{ W}}{\dfrac{1}{\varepsilon}(1 + 0.4444) - 0.4444} = 172.08 \text{ W}$$

Hence, the emissivity of both surfaces is determined to be

$$\varepsilon = \frac{1.4444}{0.4444 + (915.94/172.08)} = \mathbf{0.25} \qquad \textbf{(Ans.)}$$

If the evaporation rate (*and hence the heat transfer rate*) is to be reduced by 90.7 percent i.e. $\dot{Q}_{21} = (1 - 0.907)(172.08) = 16$ W, then

$$\varepsilon = \frac{1.4444}{1 + 0.4444 + (915.94/16)} = \mathbf{0.025} \qquad \textbf{(Ans.)}$$

EXAMPLE 13.13 *A spherical tank of 2 m diameter that is filled with liquid oxygen at–183°C is kept in an evacuated cubic enclosure whose sides are 3 m long. The emissivities of the spherical tank and the enclosure are 0.1 and 0.9, respectively. If the temperature of the cubic enclosure is measured to be –1°C, determine the net rate of radiation heat transfer to the liquid oxygen.*

Solution

Known A spherical tank filled with liquid oxygen is kept in a cubic enclosure.

Find Net radiation heat transfer, $\dot{Q}_{\text{net}}$ (W).

Assumptions (1) All surfaces are opaque, grey, and diffuse. (2) Steady operating conditions exist.

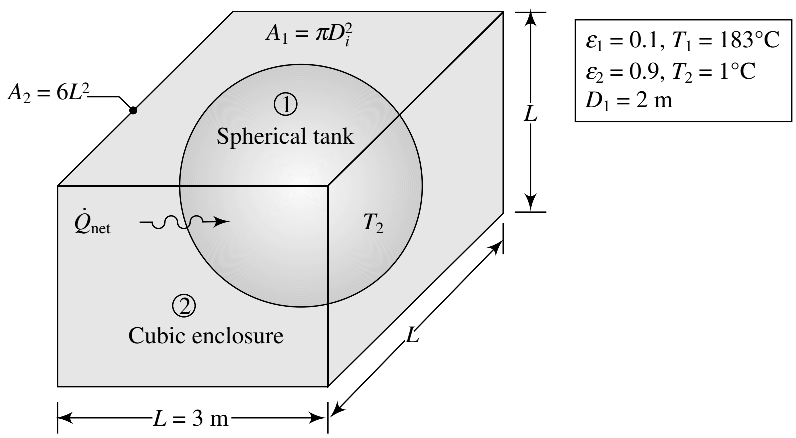
Analysis The net radiant heat exchange is given by

$$\dot{Q}_{\text{net}(21)} = \frac{\sigma A_1 [T_2^4 - T_1^4]}{(1/\varepsilon_1) + (A_1/A_2)\{(1/\varepsilon_2) - 1\}}$$

where $\sigma = 5.67 \times 10^{-8}$ W/m² K⁴ (*Stefan–Boltzmann constant*)

$A_1 = \pi D_1^2 = \pi (2 \text{ m})^2 = 4\pi \; m^2 \qquad A_2 = 6L^2 = 6(3 \text{ m})^2 = 54 \text{ m}^2$

Schematic



$T_1 = -183 + 273.15 = 90.15$ K $\quad$ $T_2 = -1 + 273.15 = 272.15$ K

$\varepsilon_1 = 0.1$ $\quad$ $\varepsilon_2 = 0.9$

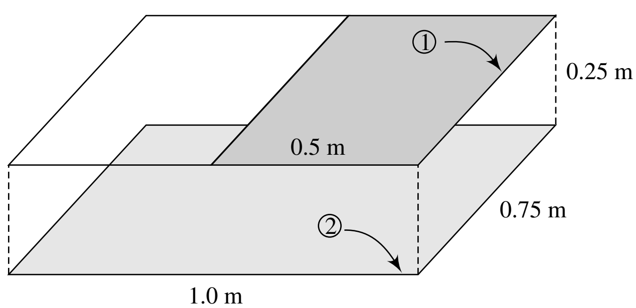
Substituting the numerical values, we have

$$\dot{Q}_{net} = \frac{(5.67 \times 10^{-8}\ \text{W/m}^2\,\text{K}^4)(4\pi\ m^2)[272.15^4 - 90.15^4]\text{K}^4}{\dfrac{1}{0.1} + \dfrac{4\pi}{54}\left\{\dfrac{1}{0.9} - 1\right\}} = \mathbf{385\ W} \qquad \textbf{(Ans.)}$$

EXAMPLE 13.14 *Calculate the net radiation heat exchange between the surfaces 1 and 2 as shown in the adjoining figure:*

Given: $T_1 = 300$ K $\quad$ $T_2 = 1000$ K

$\varepsilon_1 = 0.7$ $\quad$ $\varepsilon_2 = 0.9$

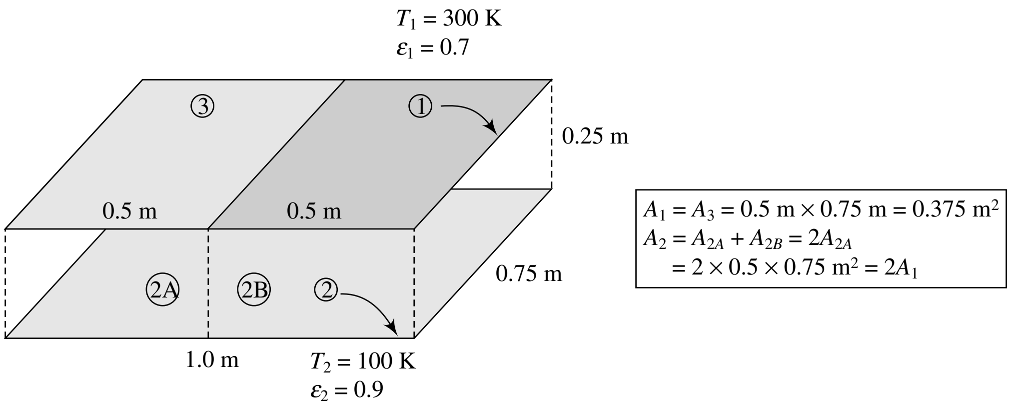


Solution

Known Arrangement of rectangles.

Find Net radiation heat exchange, $\dot{Q}_{net}$.

Schematic



Assumption Diffuse grey surface behaviour.

Analysis Let us define the hypothetical surface A_3 and divide A_2 into two sections A_{2A} and A_{2B}.

$$A_{(1,3)}F_{(1,3)2} = A_1F_{12} + A_3F_{32} \qquad (\textit{Summation rule})$$

$$A_3F_{32} = A_3F_{3(2A,2B)} = A_3F_{3(2A)} + A_3F_{3(2B)}$$

$$\boldsymbol{A_{(1,3)}F_{(1,3)2}} = A_1F_{12} + A_3F_{3(2A)} + A_3F_{3(2B)} \qquad \text{(A)}$$

The boldface type shape factors can be evaluated *directly* from the chart 13.9 for aligned, parallel rectangles.

To evaluate $F_{3(2B)}$, we will ultimately need a relationship involving $F_{(2B)1}$.

Summation rule: $\quad A_{2B}F_{(2B)(1,3)} = \boldsymbol{A_{2B}F_{(2B)1}} + A_{2B}F_{(2B)3} \qquad$ (B)

By symmetry: $\quad A_{2B}F_{(2B)(1,3)} = A_1F_{1(2A,2B)} \quad \text{or} \quad A_1F_{12} \qquad$ (C)

Reciprocity rule: $\quad A_3F_{3(2B)} = A_{2B}F_{(2B)3} \qquad$ (D)

Substituting for $A_3F_{3(2B)}$ from Eq. (D), Eq. (A) becomes

$$\boldsymbol{A_{(1,3)}F_{(1,3)2}} = A_1F_{12} + \boldsymbol{A_3F_{3(2A)}} + A_{2B}F_{(2B)3}$$

From Eq. (B) $\quad A_{2B}F_{(2B)3} = A_{2B}F_{(2B)(1,3)} - \boldsymbol{A_{2B}F_{(2B)1}}$

From Eq. (C), $\quad A_{2B}F_{(2B)(1,3)} = A_1F_{12}$

Substituting these values, Eq. (A) finally becomes

$$\boldsymbol{A_{(1,3)}F_{(1,3)2}} = A_1F_{12} + \boldsymbol{A_3F_3(2A)} + A_1F_{12} - \boldsymbol{A_{2B}F_{(2B)1}}$$

or $\quad 2A_1F_{12} = A_{(1,3)}F_{(1,3)2} + A_{2B}F_{(2B)1} - A_3F_{3(2A)}$

Noting that $A_{(1,3)} = A_2 = 2\,A_1$, and $A_{2B} = A_1 = A_3$, we have

Shape factor, $\quad F_{12} = \dfrac{1}{2A_1}[2A_1F_{(1,3)2} + A_1F_{(2B)1} - A_1F_{3(2A)}]$

or $\quad \boxed{\boldsymbol{F_{12} = \frac{1}{2}[2F_{(1,3)2} + F_{(2B)1} - F_{3(2A)}]}} \qquad$ (E)

Using the relevant chart, we evaluate the shape factors which are tabulated below:

$\mathbf{i_j}$	X/L	Y/L	$\mathbf{F_{ij}}$
(1, 3) 2	$\dfrac{1}{0.25} = 4$	$\dfrac{0.75}{0.25} = 3$	0.588
(2 B) 1	$\dfrac{0.5}{0.25} = 2$	$\dfrac{0.75}{0.25} = 3$	0.48
3 (2 A)	$\dfrac{0.5}{0.25} = 2$	$\dfrac{0.75}{0.25} = 3$	0.48

Substituting numerical values in Eq. (E), $F_{12} = \dfrac{1}{2}[(2 \times 0.588) + 0.48 - 0.48] = 0.588$

Not radiant heat exchange between surfaces 1 and 2 is

$$\dot{Q}_{\text{net}} = \dot{Q}_{21} = \frac{E_{b_2} - E_{b_1}}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}} = \frac{\sigma A_1 (T_2^4 - T_1^4)}{\left(\dfrac{1-\varepsilon_1}{\varepsilon_1}\right) + \dfrac{1}{F_{12}} + \left(\dfrac{1-\varepsilon_2}{\varepsilon_2}\right)\dfrac{A_1}{A_2}}$$

$$= \frac{(5.67 \times 10^{-8}\ \text{W/m}^2\,\text{K}^4)(0.375\ \text{m}^2)[1000^4 - 300^4]\text{K}^4}{\left(\dfrac{1-0.7}{0.7}\right) + \dfrac{1}{0.588} + \left(\dfrac{1-0.9}{0.9}\right)\left(\dfrac{0.375\ \text{m}^2}{0.75\ \text{m}^2}\right)}$$

$$= \mathbf{9.65 \times 10^3\ W \quad or \quad 9.65\ kW} \qquad \textbf{(Ans.)}$$

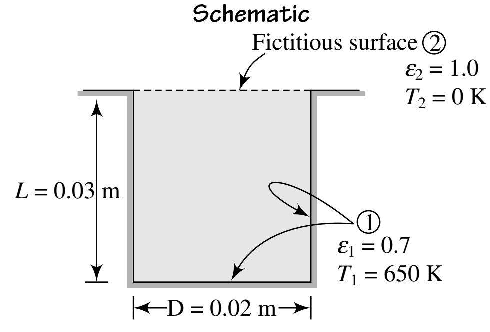
EXAMPLE 13.15 *A blind cylindrical hole of 20 mm diameter and 30 mm length is drilled into a metal block having an emissivity of 0.7. If the metal slab temperature is maintained at 650 K, determine the rate at which radiant heat escapes the hole. Also calculate the effective emissivity of the hole described above.*

Solution

Known Heat (*radiant energy*) streams out of a cylindrical hole (*cavity*) of given dimensions at specified temperature and emissivity to the surrounding space.

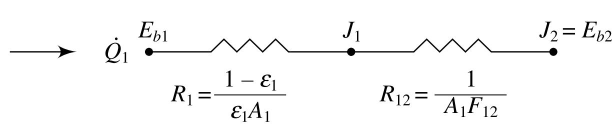
Find Heat transfer rate by radiation leaving the opening of the cavity, $\dot{Q}_1$; Effective emissivity, ε_e.

Schematic



Assumptions (1) Grey diffuse surface of the hole (cavity). (2) The fictitious closing surface is black at zero kelvin.

Analysis The equivalent radiation network of the prescribed two surface enclosure is



Cavity surface 1: $F_{11} + F_{12} = 1$ (*Summation rule*)

Closing imaginary surface 2: $F_{21} + \underbrace{\cancelto{0}{F_{22}}}_{\text{flat surface}} = 1 \Rightarrow F_{21} = 1$

$$A_1F_{12} = A_2F_{21} = A_2 \qquad (\textit{Reciprocity rule}) \qquad (\because F_{21} = 1)$$

$$\therefore \qquad F_{12} = \frac{A_2}{A_1}$$

Hence, $F_{11} = 1 - F_{12} = 1 - \dfrac{A_2}{A_1}$

Area of the opening, $A_2 = \dfrac{\pi}{4}D^2$

Area of the inner surfaces of the cavity, $A_1 = \dfrac{\pi}{4}D^2 + \pi DL$

$$\frac{A_1}{A_2} = \frac{(\pi/4)D^2 + \pi DL}{(\pi/4)D^2} = \frac{\left[\dfrac{D}{4} + L\right]}{D/4} = \frac{D+4L}{D}$$

$$\therefore \qquad F_{11} = 1 - \frac{A_2}{A_1} = 1 - \frac{D}{D+4L} = 1 - \frac{20}{20 + (4 \times 30)} = 0.857$$

$$A_1 = \pi D\left[\frac{D}{4} + L\right] = \pi \times 0.02\left[\frac{0.02}{4} + 0.03\right] = 0.0022 \text{ m}^2$$

Radiant heat exchange between the two surfaces 1 and 2 is, $\dot{Q}_{12} = \dfrac{E_{b_1} - E_{b_2}}{R_1 + R_{12}}$

Surface resistances: $R_1 = \dfrac{1-\varepsilon_1}{\varepsilon_1 A_1}$, $R_2 = \cancelto{0}{\dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}$ $(\because \varepsilon_2 = 1)$

Space resistance: $R_{12} = \dfrac{1}{A_1F_{12}}$

Black-body emissive powers:

$$E_{b_1} = \sigma T_1^4 \quad E_{b_2} = \cancelto{0}{\sigma T_2^4} \quad (\because T_2 = 0 \text{ K})$$

$$\therefore \qquad \dot{Q}_1 = \frac{\sigma T_1^4}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1F_{12}}} = \frac{A_1\sigma T_1^4}{\dfrac{1-\varepsilon_1}{\varepsilon_1} + \dfrac{1}{(1-F_{11})}} = \frac{A_1\sigma T_1^4\varepsilon_1(1-F_{11})}{(1-\varepsilon_1)(1-F_{11}) + \varepsilon_1}$$

$$= \frac{A_1\varepsilon_1\sigma T_1^4(1-F_{11})}{(1-\varepsilon_1) - (1-\varepsilon_1)F_{11} + \varepsilon_1} = \frac{A_1\varepsilon_1\sigma T_1^4(1-F_{11})}{1-(1-\varepsilon_1)F_{11}}$$

Radiant power leaving the hole is

$$\dot{Q}_1 = A_1 \varepsilon_1 \sigma T_1^4 \left[\frac{1 - F_{11}}{1 - (1 - \varepsilon_1)F_{11}}\right]$$

$$= 0.0022 \text{ m}^2 \times 0.7 \times 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4 \times 650^4 \text{ K}^4 \times \left[\frac{1 - 0.857}{1 - (1 - 0.7)(0.857)}\right]$$

$$= \mathbf{3.0 \text{ W}} \qquad \textbf{(Ans.)}$$

$$\text{Effective emissivity } (\varepsilon_e) = \frac{\text{Radiant power streaming out of the cavity}}{\begin{pmatrix}\text{Radiant power from a black body with the area of the cavity} \\ \text{opening and the temperature of inner surfaces of the cavity}\end{pmatrix}}$$

or
$$\varepsilon_e = \frac{\dot{Q}_1}{E_{b_1} A_2} = \frac{\dot{Q}_1}{\sigma T_1^4 A_2} = \frac{\dot{Q}_1}{\sigma T_1^4 (\pi D^2/4)}$$

$$= \frac{4 \times 3.0}{5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4 \times 650^4 \text{K}^4 \times \pi \times 0.02^2 \text{m}^2} = \mathbf{0.9435} \qquad \textbf{(Ans.)}$$

(D) Three Surface Enclosures

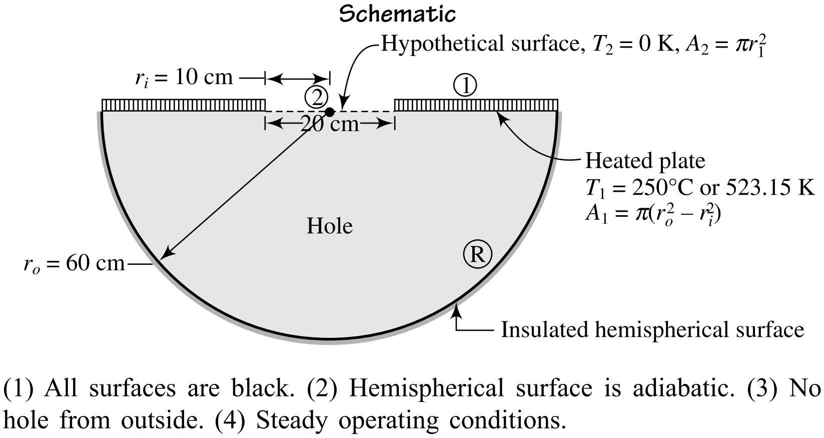
EXAMPLE 13.16 *A hemispherical cavity of 60 cm radius is covered by a plate with a hole of 20 cm diameter drilled in its centre. The inner surface of the plate is maintained at 250°C by a heater embedded in the surface. The surfaces may be assumed to be black and the hemisphere is well insulated. Assuming that the energy entering the hole from outside is negligible, calculate the temperature of the surface of the hemisphere and the power input to the heater.*

Solution

Known A hemispherical insulated cavity is covered with a heated plate having a hole at the centre and held at a specified temperature. All surfaces are black.

Find Hemisphere surface temperature (°C) and power input to heater (W).

Schematic



Assumptions (1) All surfaces are black. (2) Hemispherical surface is adiabatic. (3) No heat enters the hole from outside. (4) Steady operating conditions.

Analysis We can treat this problem as a *three-surface enclosure:* Plate 1, Hole (covered with an imaginary surface) 2, and hemispherical cavity as a re-radiating surface R. Since no energy enters the hole from the surroundings, $T_2 = T_{sur} = 0$ K. The hemispherical surface is adiabatic (*insulated*). Hence, $\dot{Q}_1 = -\dot{Q}_2$ where $\dot{Q}_1$ is the power supplied to the heater embedded in the plate.

For a three surface enclosure involving a re radiating surface, we have

$$\dot{Q}_1 = -\dot{Q}_2 = \frac{E_{b_1} - E_{b_2}}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{\left[A_1F_{12} + \left\{\dfrac{1}{A_1F_{1R}} + \dfrac{1}{A_2F_{2R}}\right\}^{-1}\right]} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

We note that since $T_2 = 0$ K, $E_{b_2} = \sigma T_2^4 = 0$

Also, $\varepsilon_1 = \varepsilon_2 = 1$ because the surfaces are black.

Furthermore, $F_{12} = 0$, $F_{1R} = F_{2R} = 1$

One can therefore write $\dot{Q}_1 = \dfrac{\sigma T_1^4}{0 + \dfrac{1}{\left[0 + \left\{\dfrac{1}{A_1} + \dfrac{1}{A_2}\right\}^{-1}\right]} + 0} = \dfrac{\sigma T_1^4}{\left(\dfrac{1}{A_1}\right) + \left(\dfrac{1}{A_2}\right)} = \dfrac{\sigma A_1 T_1^4}{\left[1 + \left(\dfrac{A_1}{A_2}\right)\right]}$

With $\quad A_1 = \pi(r_o^2 - r_i^2) = \pi[60^2 - 10^2]\text{cm}^2 = 3500\pi(\text{cm}^2)$

and $\quad A_2 = \pi r_i^2 = \pi(10^2)\text{cm}^2 = 100\pi(\text{cm}^2), \quad \dfrac{A_1}{A_2} = \dfrac{3500\pi}{100\pi} = 35$

Power input to the heater is then determined from

$$\dot{Q}_1 = \frac{(5.67 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4)(3500\pi \times 10^{-4}\ \text{m}^2)(523.15^4)\text{K}^4}{[1+35]} = \mathbf{129.7\ W} \quad \textbf{(Ans.)}$$

We note that for black surfaces forming an enclosure:

$$\dot{Q}_i = \sum_{j=1}^{N} \dot{Q}_{i-j} = \sum_{j=1}^{N} A_i F_{ij} \sigma (T_i^4 - T_j^4)$$

or $\quad \dot{Q}_1 = A_1 \overset{0}{\cancel{F_{12}}} \sigma (T_1^4 - T_2^4) + A_1 F_{1R} \sigma (T_1^4 - T_R^4)$ [*The third term is zero because* $F_{11} = 0$]

or $\quad \dot{Q}_1 = A_1 \sigma (T_1^4 - T_R^4) \qquad$ since $F_{1R} = 1$

$\therefore \quad T_R = \left[T_1^4 - \dfrac{\dot{Q}_1}{A_1\sigma}\right]^{1/4} = [523.15^4 - \{129.7/(0.35\pi \times 5.67 \times 10^{-8})\}]^{1/4} = 519.48\ \text{K}$

Hemispherical surface temperature,

$T_R = (519.48 - 273.15)$°C $=$ **246.3°C** **(Ans.)**

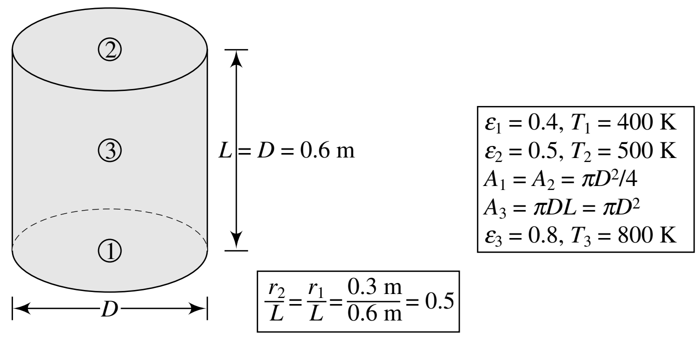
EXAMPLE 13.17 *A furnace of cylindrical shape has both diameter and length of 0.6 m. The top (surface 1) and the base (surface 2) of the furnace has emissivities of 0.4 and 0.5 respectively, and are maintained at uniform temperatures of 400 K and 500 K respectively. The lateral surface has emissivity of 0.8 and is maintained at 800 K. Determine the net rate of radiation heat transfer at each surface during steady operation. How these surfaces can be maintained at prescribed temperatures?*

Solution

Known A circular furnace with specified dimensions, emissivities and temperatures.

Find Radiation heat transfer rate at each of the three surfaces.

Schematic



Assumptions (1) Steady operating conditions exist. (2) All surfaces are opaque, isothermal, diffuse, and grey.

Analysis Shape factor, $F_{12} = 0.17$ for $\dfrac{r_1}{L} = \dfrac{r_2}{L} = 0.5$ (*from the chart*)

$$\underbrace{\cancel{F_{11}}^{\,0}}_{\text{flat surface}} + F_{12} + F_{13} = 1 \qquad (\textit{Summation rule})$$

$\therefore$ $F_{13} = 1 - F_{12} = 1 - 0.17 = 0.83$

$A_1F_{12} = A_2F_{21}$ (*Reciprocity rule*)

$\therefore$ $F_{21} = F_{12} = 0.83$ (*because* $A_2 = A_1$)

$F_{32} = F_{31}$ (*By symmetry*)

$$F_{21} + \underbrace{\cancel{F_{22}}^{\,0}}_{\text{flat surface}} + F_{23} = 1 \qquad (\textit{Summation rule})$$

$\therefore$ $F_{23} = 1 - F_{21} = 1 - 0.17 = 0.83$

$A_2F_{23} = A_3F_{32}$ (*Reciprocity rule*)

$$\therefore \quad F_{32} = F_{23} \times \frac{A_2}{A_3} = 0.83 \times \frac{(\pi/4)D^2}{\pi D^2} = 0.2075 = F_{31}$$

Black-body emissive powers:

$$E_{b_1} = \sigma T_1^4 = (5.67 \times 10^{-8}\ \text{W/m}^2\ \text{K}^4)(400\ \text{K})^4 = 1452\ \text{W/m}^2$$

$$E_{b_2} = \sigma T_2^4 = (5.67 \times 10^{-8})(500)^4 = 3\,544\ \text{W/m}^2$$

$$E_{b_3} = \sigma T_3^4 = (5.67 \times 10^{-8})(800)^4 = 23\,224\ \text{W/m}^2$$

Also,
$$\frac{1-\varepsilon_1}{\varepsilon_1} = \frac{1-0.4}{0.4} = 1.5, \quad \frac{1-\varepsilon_2}{\varepsilon_2} = \frac{1-0.5}{0.5} = 1$$

$$\frac{1-\varepsilon_3}{\varepsilon_3} = \frac{1-0.8}{0.8} = 0.25$$

Using the direct method, we can write for the three surfaces:

$$\boldsymbol{A_1}: \quad E_{b_1} = J_1 + \left(\frac{1-\varepsilon_1}{\varepsilon_1}\right)[F_{12}(J_1 - J_2) + F_{13}(J_1 - J_3)$$

$$\boldsymbol{A_2}: \quad E_{b_2} = J_2 + \left(\frac{1-\varepsilon_2}{\varepsilon_2}\right)[F_{21}(J_2 - J_1) + F_{23}(J_2 - J_3)]$$

$$\boldsymbol{A_3}: \quad E_{b_3} = J_3 + \left(\frac{1-\varepsilon_3}{\varepsilon_3}\right)[F_{31}(J_3 - J_1) + F_{32}(J_3 - J_2)]$$

Substituting the values, we get

$$\boldsymbol{A_1}: \quad J_1\left[1 + (F_{12} + F_{13})\left(\frac{1-\varepsilon_1}{\varepsilon_1}\right)\right] - F_{12}\left(\frac{1-\varepsilon_1}{\varepsilon_1}\right)J_2 - F_{13}\left(\frac{1-\varepsilon_1}{\varepsilon_1}\right) = E_{b_1}$$

or $$J_1[1 + (0.17 + 0.83)(1.5)] - (0.17)(1.5)J_2 - (0.83)(1.5) = 1452$$

$$\mathbf{A_2}: \quad J_1\left[-F_{21}\left(\frac{1-\varepsilon_2}{\varepsilon_2}\right)\right] + J_2\left[1 + (F_{21} + F_{23})\left(\frac{1-\varepsilon_2}{\varepsilon_2}\right)\right] - F_{23}\left(\frac{1-\varepsilon_2}{\varepsilon_2}\right)J_3 = E_{b_2}$$

or $$J_1[-(0.17)(1)] + J_2[1 + (0.17 + 0.83)(1)] - (0.83)(1)J_3 = 3544$$

$$\boldsymbol{A_3}: \quad J_1\left[-F_{31}\left(\frac{1-\varepsilon_3}{\varepsilon_3}\right)\right] + J_2\left[-F_{32}\left(\frac{1-\varepsilon_3}{\varepsilon_3}\right)\right] + J_3\left[1 + (F_{31} + F_{32}) \times \left(\frac{1-\varepsilon_3}{\varepsilon_3}\right)\right] = E_{b_3}$$

or $$J_1[-(0.2075)(0.25)] + J_2[-(0.2075)(0.25)] + J_3[1 + (0.2075 + 0.2075)(0.25)] = 23\,224$$

Simplifying, we obtain

$$\boxed{\begin{array}{l} 2.5J_1 - 0.255J_2 - 1.245J_3 = 1452 \\ -0.17J_1 + 2.0J_2 - 0.83J_3 = 3544 \\ -0.052J_1 - 0.052J_2 + 1.104J_3 = 23\,224 \end{array}} \qquad \begin{array}{r} \text{(A)} \\ \text{(B)} \\ \text{(C)} \end{array}$$

Solving these simultaneous equations with three unknowns, we have

J_1 = 12 875 W/m^2, J_2 = 12 084 W/m^2, J_3 = 22 212 W/m^2

Net radiation heat transfer rates at each surface:

$$\dot{Q}_1 = A_1[F_{12}(J_1 - J_2) + F_{13}(J_1 - J_3)]$$

$$= \frac{\pi}{4}(0.6\text{ m})^2\,[0.17(12875 - 12084) + 0.83(12875 - 22212)]\text{W/m}^2$$

$$= -\mathbf{2153\ W} \qquad \textbf{(Ans.)}$$

$$\dot{Q}_2 = A_2[F_{21}(J_2 - J_1) + F_{23}(J_2 - J_3)]$$

$$= \frac{\pi}{4}(0.6\text{ m})^2[0.17(12084 - 12875) + 0.83(12\,084 - 22\,212)]\text{W/m}^2$$

$$= -\mathbf{2415\ W} \qquad \textbf{(Ans.)}$$

$$\dot{Q}_3 = A_3[F_{31}(J_3 - J_1) + F_{32}(J_3 - J_2)]$$

$$= \pi\,(0.6\text{ m})\,(0.6\text{ m})\,[0.2075\,(22\,212 - 12\,875) + (0.2075)\,(22\,212 - 12\,084)]\text{ W/m}^2$$

$$= +\ \mathbf{4568\ W} \qquad \textbf{(Ans.)}$$

***Check*:** The algebraic sum of all heat-transfer rates should be zero.

i.e., $$\sum_{i=1}^{3}\dot{Q}_i = 0 \quad \text{or} \quad \dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 = 0$$

or $$(-2153 \text{ W}) + (-2415 \text{ W}) + (4568 \text{ W}) = 0$$

The *negative* sign implies that the heat is received by surfaces 1 and 2, and must be extracted out non radiatively and continuously for equilibrium at the rates of 2153 W and 2415 W respectively. Similarly, to ensure steady operation (to maintain the three surfaces at constant temperatures), we should also supply heat to the side surface continuously at a rate of 4568 W. **(Ans.)**

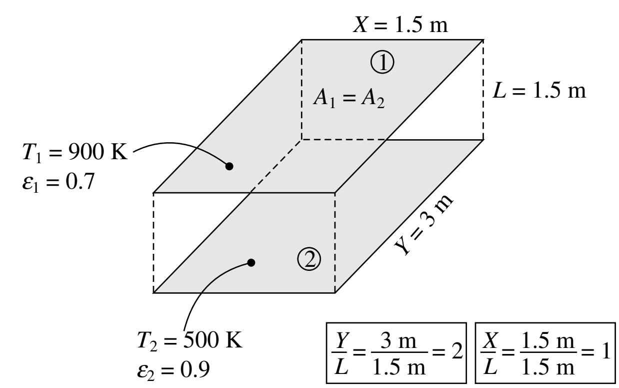
EXAMPLE 13.18 *Two parallel and directly opposed rectangles of the same size (1.5 m × 3 m) are separated by a distance of 1.5 m. One rectangle is maintained at* T_1 *= 900 K and has an emissivity of* ε_1 *= 0.7. The second rectangle has a temperature of* T_2 *= 500 K and emissivity* ε_2 *= 0.9. Determine the net radiative heat transfer at each of the two surfaces (considering only the opposed faces) if (a) they are located in a radiation free environment, (b) they are connected by a single re radiating or adiabatic surface. (c) Find the temperature of the adiabatic surface,* T_3*. Sketch the radiation network.*

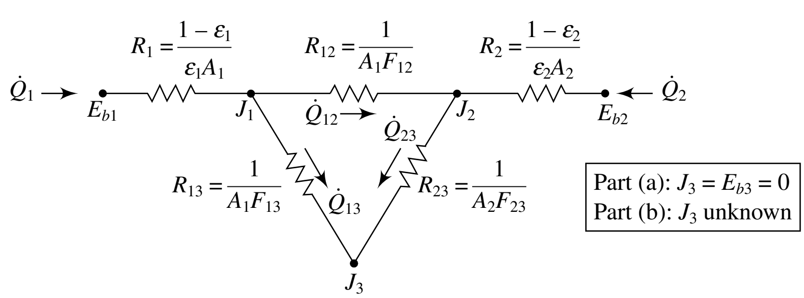
Solution

Known Two aligned parallel rectangles of equal size. Dimensions, temperatures, and emissivities.

Find Net rate of radiation heat transfer at each surface if (a) the surroundings are radiation free, and (b) the two surfaces are connected by an adiabatic surface. (c) Adiabatic surface temperature.

Schematic





Assumptions (1) Steady operating conditions exist. (2) The surfaces are opaque, diffuse and grey. (3) Convection heat transfer is ignored.

Analysis The shape factor, $F_{12}\left\{\text{for } \dfrac{Y}{L}=2 \text{ and } \dfrac{X}{L}=1\right\} = 0.286$ (*from the chart*)

Reciprocity theorem: $A_1F_{12} = A_2F_{21}$

As $A_1 = A_2 = 1.5 \times 3 = 4.5\ \text{m}^2$, $F_{21} = F_{12} = 0.286$

If A_3 is used to denote the radiation free space in part (a) or the adiabatic surface in part (b), one has

Summation rule: $F_{11} + F_{12} + F_{13} = 1$ and $F_{21} + F_{22} + F_{23} = 1$

Since $F_{11} = F_{22} = 0$ *(flat surface),*

$\therefore$ $F_{13} = 1 - F_{12} = 1 - 0.286 = 0.714$ and $F_{23} = 1 - F_{21} = 1 - 0.286 = 0.714$

Now,
$$\frac{\varepsilon_1 A_1}{1-\varepsilon_1} = \left(\frac{0.7}{1-0.7}\right)(4.5\ \text{m}^2) = 10.5\ \text{m}^2$$
$$\frac{\varepsilon_2 A_2}{1-\varepsilon_2} = \left(\frac{0.9}{1-0.9}\right)(4.5\ \text{m}^2) = 40.5\ \text{m}^2$$
$$A_1F_{12} = A_2F_{21} = (4.5\ \text{m}^2)(0.286) = 1.287\ \text{m}^2$$
$$A_1F_{13} = A_2F_{23} = A_3F_{31} = A_3F_{32} = (4.5\ \text{m}^2)(0.714) = 3.213\ \text{m}^2$$
$$T_1 = 900\ \text{K} \quad E_{b1} = \sigma T_1^4 = (5.67\times10^{-8}\ \text{W/m}^2\ \text{K}^4)(900^4)\text{K}^4 = 37\ 201\ \text{W/m}^2$$
$$T_2 = 500\ \text{K} \quad E_{b2} = \sigma T_2^4 = (5.67\times10^{-8}\ \text{W/m}^2\ \text{K}^4)(500^4)\text{K}^4 = 3543.75\ \text{W/m}^2$$

(a) In this case, the surfaces are located in a radiation free space. We complete the enclosure by representing the space as a third active *black* surface at $T_3 = 0$ K.

$\therefore$ $E_{b3} = \sigma T_3^4 = 0$. No adiabatic surfaces are present. Hence, only energy balances need to be made at the three active ones.

Node J$_1$:
$$\frac{E_{b1}-J_1}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1}} + \frac{J_2-J_1}{\dfrac{1}{A_1F_{12}}} + \frac{J_3-J_1}{\dfrac{1}{A_1F_{13}}} = 0 \tag{A}$$

Node J$_2$:
$$\frac{E_{b2}-J_2}{\dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}} + \frac{J_1-J_2}{\dfrac{1}{A_1F_{12}}} + \frac{J_3-J_2}{\dfrac{1}{A_2F_{23}}} = 0 \tag{B}$$

Node J$_3$: $J_3 = E_{b3} = 0$

On rearrangement: $\dfrac{\varepsilon_1 A_1}{1-\varepsilon_1}(E_{b1}-J_1) + A_1F_{12}(J_2-J_1) + A_1F_{13}(0-J_1) = 0$

or $10.5\ \text{m}^2(37201 - J_1)\text{W/m}^2 + 1.287\ \text{m}^2(J_2 - J_1)\text{W/m}^2 - 3.213\ \text{m}^2\,(J_1)\text{W/m}^2 = 0$

or
$$\boxed{15J_1 - 1.287J_2 = 390.6\times10^3} \tag{C}$$

Also, $\dfrac{\varepsilon_2 A_2}{1-\varepsilon_2}(E_{b2}-J_2) + A_1F_{12}(J_1-J_2) + A_2F_{23}(0-J_2) = 0$

or $40.5(3543.75 - J_2) + 1.287\,(J_1 - J_2) - 3.213\,J_2 = 0$

or $\boxed{45J_2 - 1.287J_1 = 143522}$ (D)

Solving equations (C) and (D), we have

$J_1 = 26378.4$ W/m^2 and $J_2 = 3943.8$ W/m^2

The radiation heat transfers at each of the two surfaces are

$$\dot{Q}_1 = \frac{\varepsilon_1 A_1}{1-\varepsilon_1}(E_{b1} - J_1) = (10.5\text{ m}^2)(37\,201 - 26\,378.4)\text{W/m}^2$$

$= \mathbf{113.637 \times 10^3\ W \cong 113.64\ kW}$ **(Ans.) (a)**

$$\dot{Q}_2 = \frac{\varepsilon_2 A_2}{1-\varepsilon_2}(E_{b2} - T_2) = (40.5\text{ m}^2)(3543.75 - 3943.8)\text{W/m}^2$$

$= \mathbf{-16.20 \times 10^3\ W = -16.20\ kW}$ **(Ans.) (a)**

The heat flows $\dot{Q}_1$ and $\dot{Q}_2$ are *not* equal in magnitude because some energy is *lost* to the radiation free space.

(b) In this case, the surfaces are enclosed by a single adiabatic surface $(\dot{Q}_3 = 0)$

There are three unknowns, viz., J_1, J_2, and J_3.

Hence three equations are required to evaluate them.

Node J_1: $\frac{\varepsilon_1 A_1}{1-\varepsilon_1}(E_{b1} - J_1) + A_1F_{12}(J_2 - J_1) + A_1F_{13}(J_3 - J_1) = 0$

Node J_2: $\frac{\varepsilon_2 A_2}{1-\varepsilon_2}(E_{b2} - J_2) + A_2F_{21}(J_1 - J_2) + A_2F_{23}(J_3 - J_2) = 0$

Node J_3: $A_1F_{13}(J_1 - J_3) + A_2F_{23}(J_1 - J_3) = 0$

$10.5\,(37201 - J_1) + 1.287\,(J_2 - J_1) + 3.213\,(J_3 - J_1) = 0$ (E)

$40.5\,(3543.75 - J_2) + 1.287\,(J_1 - J_2) + 3.213\,(J_3 - J_2) = 0$ (F)

$3.213\,(J_1 - J_3) + 3.213\,(J_2 - J_3) = 0$ (G)

On rearrangement,

$$\boxed{\begin{aligned} 15J_1 - 1.287J_2 - 3.213J_3 &= 390\,610.5 \\ -1.287J_1 + 45J_2 - 3.213J_3 &= 143\,521.9 \\ -3.213J_1 - 3.213J_2 + 6.426J_3 &= 0 \end{aligned}}$$

Solving the above three simultaneous equations, we have

$J_1 = 30\,315.5$ W/m^2

$J_2 = 5328.88$ W/m^2 and $J_3 = 17\,822.1$ W/m^2

For the active surfaces:

$$\dot{Q}_1 = \frac{\varepsilon_1 A_1}{1-\varepsilon_1}(E_{b_1} - J_1) = 10.5\text{ m}^2\,(37\,201 - 30\,315.4)\text{ W/m}^2$$

$= 72.3 \times 10^3$ W $=$ **72.3 kW** **(Ans.) (b)**

$$\dot{Q}_2 = \frac{\varepsilon_2 A_2}{1-\varepsilon_2}(E_{b_2} - J_2) = 40.5 \text{ m}^2 (3543.75 - 5328.88) \text{ W/m}^2$$

$$= -72.3 \times 10^3 \text{ W} = \mathbf{-72.3 \text{ kW}}$$ **(Ans.) (b)**

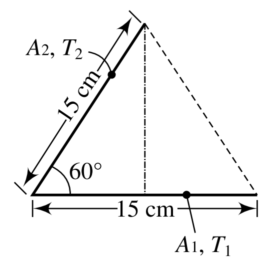
The heat flows $\dot{Q}_1$ and $\dot{Q}_2$ are equal in magnitude since there are just two active surfaces and no energy is lost to the surroundings.

As $J_3 = E_{b_3} = \sigma T_3^4 = 17\ 822.1 \text{ W/m}^2$, the adiabatic surface temperature is

$$T_3 = \left[\frac{J_3}{\sigma}\right]^{1/4} = \left(\frac{17822.1 \text{ W/m}^2}{5.67\times10^{-8} \text{ W/m}^2\text{ K}^4}\right)^{1/4}$$

$$= \mathbf{748.76 \text{ K} \quad or \quad 475.6°C}$$ **(Ans.) (c)**

EXAMPLE 13.19 *Two very long, inclined black surfaces* A_1 *and* A_2 *are maintained at uniform temperatures of* T_1 *= 1000 K and* T_2 *= 600 K as indicated below:*



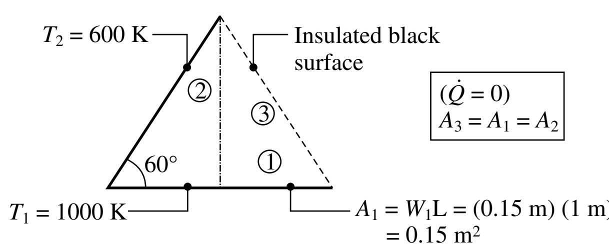
(a) Calculate the net radiative heat exchange between the two surfaces per unit length. (b) If a black surface A_3 *with its backside insulated is placed along the third side of the triangle shown above (dashed line) to form a three surface enclosure, determine the net radiation heat transfer to surface* A_2 *per unit length. (c) Compute the equilibrium temperature of the third insulated surface* A_3*.*

Solution

Known Long, inclined black surfaces at specified temperatures.

Find (a) Net radiation heat exchange, $\dot{Q}_{12}$ per metre length. (b) Net radiation heat transfer to surface 2 if insulated black surface 3 is positioned along the dashed line. (c) Insulated surface temperature T_3.

Schematic



Assumptions (1) Black, diffuse surfaces. (2) Surfaces are very long in a direction perpendicular to the plane of the paper.

Analysis Net radiation heat exchange between two black surfaces is determined from

$$\dot{Q}_{12} = A_1 F_{12} \sigma (T_1^4 - T_2^4)$$

For inclined parallel plates of equal width and a common edge with the angle of inclination α, the shape factor is given by

$$F_{12} = 1 - \sin(\alpha/2)$$

With α = 60°C,

$$F_{12} = 1 - \sin (60/2)° = 1 - 0.5 = 0.5$$

Hence, per unit length,

$$\dot{Q}_{12} = (0.15\text{ m})^2\,(0.5)(5.67\times10^{-8}\text{ W/m}^2\text{ K}^4)$$

$$\times[(1000\text{ K})^4-(600\text{ K})^4]\left|\frac{1\text{ kW}}{10^3\text{ W}}\right| = \mathbf{3.7\ kW} \qquad \textbf{(Ans.) (a)}$$

With insulated surface A_3 put in place as prescribed in the problem, we end up with a three-black surface enclosure whose radiation network is shown in the figure above. Consider energy balance on the node representing surface A_2.

$$-\dot{Q}_2 = \dot{Q}_{12}+\dot{Q}_{32} = A_1F_{12}\sigma(T_1^4-T_2^4)+A_3F_{32}\sigma(T_3^4-T_2^4)$$

Noting that $F_{32} = F_{12} = 0.5$ (*from symmetry*), radiation heat transfer to surface A_2 is

$$-\dot{Q}_2 = 3.7\text{ kW}+(0.15\text{ m}^2)(0.5)(5.67\times10^{-8}\text{ W/m}^2\text{ K}^4)(T_3^4-600^4)\text{K}^4\left|\frac{1\text{ kW}}{10^3\text{ W}}\right| \qquad \text{(A)}$$

To find T_3, concentrate on the node 3.

As $\quad \dot{Q}_3 = 0,\ \dot{Q}_{13} = \dot{Q}_{32}$

i.e., $\quad A_1F_{12}\sigma(T_1^4-T_3^4) = A_3F_{32}\sigma(T_3^2-T_2^4)$

With $\quad A_1 = A_3$ and $F_{12} = F_{32}$, we have

$$T_1^4-T_3^4 = T_3^4-T_2^4 \quad\text{or}\quad T_3^4 = \frac{1}{2}(T_1^4+T_2^4) = \frac{1}{2}(1000^4+600^4)\text{K}^4 = 5648\times10^8\text{ K}^4$$

$\therefore$ Equilibrium surface temperature, $T_3 = [5648\times10^8]^{0.25} = \mathbf{866.9\ K}$ **(Ans.) (c)**

Substituting this value of T_3 in Eq. (A), the net radiative heat transfer to the surface A_2 is

$$-\dot{Q}_2 = \dot{Q}_1 = 3.7\text{ kW}+\{(0.15)(0.5)(5.67\times10^{-8})(866.9^4-600^4)\}(10^{-3})\text{kW}$$

$$= 3.7\text{ kW} + 1.85\text{ kW} = \mathbf{5.55\ kW} \qquad \textbf{(Ans.) (b)}$$

Comment Overall energy balance: $\dot{Q}_1+\dot{Q}_2+\underbrace{\cancelto{0}{\dot{Q}_3}}_{\text{Insulated}}=0$

Hence, $\dot{Q}_1 = -\dot{Q}_2$. The net radiation heat transfer *from* the surface 1 is equal to that *of* the surface 2.

EXAMPLE 13.20 *A thermocouple sensor of 1 mm diameter and emissivity of 0.4 measures the temperature of air stream in a large channel having a wall temperature of 640°C. The temperature indicated by the thermocouple is 760°C. The air velocity is 3 m/s. Calculate the true temperature of air. Use the following correlation and properties of air:* Nu = *0.8 (*Re*)*$^{0.38}$

k = *0.06581 W/m °C and* ν = *1.133 × 10*$^{-4}$ *m*^{2}*/s*

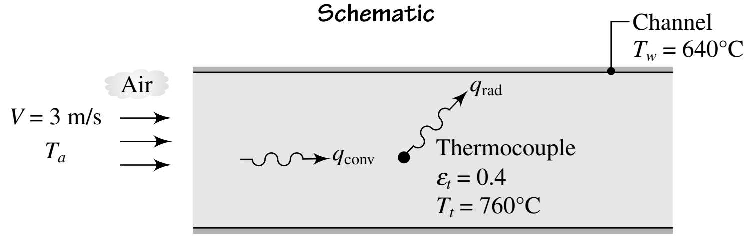
What should be the emissivity of the sensor in order to reduce the error by 30%?

Solution

Known A thermocouple is placed in an air stream passing through a duct for temperature measurement.

Find True air temperature. Emissivity to reduce the error by 30%.

Schematic



Assumptions (1) Diffuse grey surfaces. (2) Steady-state conditions.

Analysis To find convective heat transfer coefficient, *h*:

$$Nu = \frac{hD}{k} = 0.8(Re)^{0.38} = 0.8\left(\frac{VD}{\nu}\right)^{0.38}$$

$$h = \frac{k}{D}Nu = \frac{k}{D} \times 0.8\left[\frac{VD}{\nu}\right]^{0.38} = \frac{0.06581 \text{ W/m °C}}{0.001 \text{ m}}$$

$$\times 0.8\left[\frac{3 \text{ m/s} \times 0.001 \text{ m}}{1.133 \times 10^{-4} \text{ m}^2/\text{s}}\right]^{0.38} = 182.84 \text{ W/m}^2 \text{ °C}$$

Energy balance: $\dot{Q}_{\text{conv, to sensor}} = \dot{Q}_{\text{rad, from sensor}}$

i.e., $hA_t(T_a - T_t) = \sigma A_t \varepsilon_t (T_t^4 - T_w^4)$

Hence, the true temperature of the air stream,

$$T_a = T_t + \frac{\sigma\varepsilon_t}{h}(T_t^4 - T_w^4)$$

$$= 760°\text{C} + \frac{(5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4)(0.4)}{182.84 \text{ W/m}^2 \text{ °C}}\left[\left(\frac{1033.15}{100}\right)^4 - \left(\frac{913.15}{100}\right)^4\right]10^8 \text{ K}^4$$

$$= (760 + 55)°\text{C} = \mathbf{815°C} \qquad \textbf{(Ans.)}$$

If the error is reduced by 30%,

$(T_a - T_t^*)$ will be $55(1 - 0.3) = 38.5°\text{C}$ or K

The sensor will then indicate a temperature, T_t^* equal to (815 – 38.5)°C

i.e., 776.5°C or 1049.65 K

The new emissivity is

$$\varepsilon_t^* = (T_a - T_t^*)/\sigma(T_t^{*4} - T_w^4) = \frac{(38.5)(182.84)}{5.67[10.4965^4 - 9.1315^4]} = \mathbf{0.24} \qquad \textbf{(Ans.)}$$

EXAMPLE 13.21 *A thermocouple of emissivity 0.8 is used to measure the temperature of non-absorbing gas flowing in a large duct. The temperature of the duct wall is 227°C. The temperature indicated by the thermocouple is 505°C. Determine the true temperature of the gas if the value of convection heat transfer coefficient is 142.5 W/m² K.*

To measure the gas temperature more accurately, a thin cylindrical radiation shield (ε = 0.3) with inside diameter 4 times the outer diameter of the thermocouple is placed around the thermocouple.

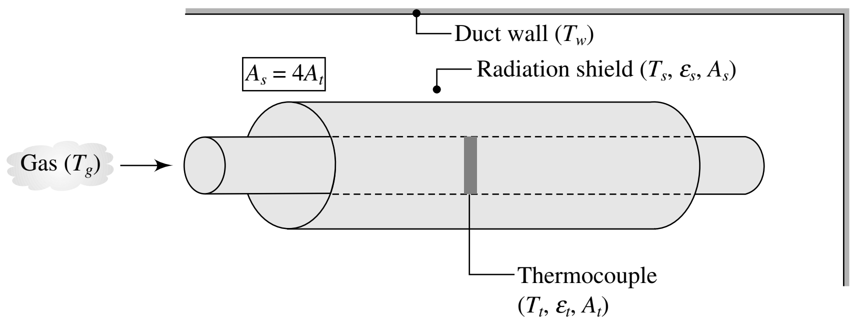
The temperature indicated by the thermocouple is 505°C. Determine the true temperature of the gas if the convection heat transfer coefficient for the shielded thermocouple remains 142.5 W/m² K.

Solution

Known A thermocouple placed in a gas stream inside a duct measures the gas temperature. A shield is inserted to reduce the error in temperature measurement.

Find True temperature of the gas for unshielded and shielded thermocouple.

Schematic



Assumptions (1) Steady operating conditions exist. (2) Radiation surfaces are diffuse grey. (3) The gas is non-absorbing. (4) The shield is too small relative to the large duct.

Analysis Let the suffixes g, t, and w relate to gas, thermocouple, and wall of the duct, respectively.

(a) ***Without radiation shield:*** Under steady-state conditions, applying energy balance, we have:

$$\begin{pmatrix}\text{Heat transfer from gas to}\\ \text{thermocouple by convection}\end{pmatrix} = \begin{pmatrix}\text{Heat transfer from thermocouple to}\\ \text{duct wall by radiation}\end{pmatrix}$$

i.e., $$\dot{Q}_{\text{conv}} = \dot{Q}_{\text{rad}} \quad \text{or} \quad \bar{h}_c A_t (T_g - T_t) = \sigma A_t \varepsilon_t (T_t^4 - T_w^4)$$

or $$T_g - T_t = \frac{\sigma \varepsilon_t}{\bar{h}_c}(T_t^4 - T_w^4)$$

$$= \frac{(5.67)\times 10^{-8}\ \text{W/m}^2\,\text{K}^4 (0.8)}{142.5\ \text{W/m}^2\,\text{K}} \left[\left(\frac{505 + 273.15}{100}\right)^4 - \left(\frac{227 + 273.15}{100}\right)^4\right] \text{K}^4 (10^8) = 96.8\ \text{K}$$

True temperature of the gas is

$$T_g = (778.15 + 96.8)\ \text{K} = 874.95\ \text{K} \quad \text{or} \quad \mathbf{601.8°C}$$ **(Ans.) (a)**

(b) ***With radiation shield:*** An energy balance on the shield gives:

$$\begin{pmatrix}\text{Convective heat transfer from}\\ \text{the gas to the shield}\end{pmatrix} + \begin{pmatrix}\text{Radiative heat transfer from the}\\ \text{thermocouple to the shield}\end{pmatrix} = \begin{pmatrix}\text{Radiative heat transfer from}\\ \text{the shield to the duct wall}\end{pmatrix}$$

or $$\bar{h}_{cs}\, 2A_s(T_g - T_s) + \frac{\sigma(T_t^4 - T_s^4)A_t}{\dfrac{1}{\varepsilon_t} + \dfrac{A_t}{A_s}\left(\dfrac{1}{\varepsilon_s} - 1\right)} = \sigma A_s \varepsilon_s (T_s^4 - T_w^4)$$

or $$2\bar{h}_{cs}(T_g - T_s) + \frac{\sigma(T_t^4 - T_s^4)\left(\dfrac{A_t}{A_s}\right)}{\dfrac{1}{\varepsilon_t} + \dfrac{A_t}{A_s}\left(\dfrac{1}{\varepsilon_s} - 1\right)} = \sigma\varepsilon_s(T_s^4 - T_w^4)$$

or $$2(142.5)(T_g - T_s) + \frac{(5.67/4)\left[\left(\dfrac{778.15}{100}\right)^4 - \left(\dfrac{T_s}{100}\right)^4\right]}{\dfrac{1}{0.8} + \dfrac{1}{4}\left(\dfrac{1}{0.3} - 1\right)} = (5.67)(0.3)\left[\left(\dfrac{T_s}{100}\right)^4 - \left(\dfrac{500.15}{100}\right)^4\right]$$

or $$285(T_g - T_s) + 0.7732[(7.7815)^4 - (T_s/100)^4] = 1.701[(T_s/100)^4 - (5.0015)^4] \quad \text{(A)}$$

An energy balance on the thermocouple gives

$$\bar{h}_{ct} A_t (T_g - T_t) = \sigma A_t \varepsilon_t (T_t^4 - T_s^4)$$

or $$(142.5)\ (T_g - 778.15) = (5.67)(0.8)[7.7815^4 - (T_s/100)^4]$$

or $$T_g = 778.15 + 0.03183[3666.5 - (T_s/100)^4] \quad \text{(B)}$$

Substituting the value of T_g from Eq. (B) into Eq. (A),

$$285\ [778.15 + 0.03183\ \{3666.5 - (T_s/100)^4\} - T_s] + 0.7732\ [3666.5 - (T_s/100)^4]$$
$$= 1.701\ [(T_s/100)^4 - 625.75] \quad \text{(C)}$$

The solution of Eq. (C) for finding T_s is by *trial and error.*

T_s K	LHS	RHS
770	3694.33 + 116.90 = 3811.23	4915.13
768	4594.36 + 145.0 = 4739.36	4853.25
767.8	4679.29 + 147.42 = **4826.71**	**4847.08**

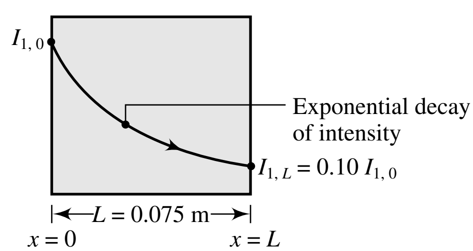
Thus, T_s = 767.8 K and T_g = **784.2 K** or 511°C **(Ans.) (b)**

Error = 511 – 505 = 6°C or 6 K.

The error is reduced from **96.8 K to 6 K.**

(G) Gas Radiation

EXAMPLE 13.22 *Determine the spectral absorption (extinction) coefficient for radiation passing through a 7.5 cm thick gas layer if the monochromatic intensity of radiation is reduced by 90 percent. What is the spectral transmissivity of the gas?*



Solution

Known Extinction coefficient, K_λ. Transmissivity of the medium.

Assumptions The absorption coefficient is independent of x.

Analysis According to *Beer's law:*

$$\frac{I_\lambda, L}{I_{\lambda,0}} = \exp(-K_\lambda, L) \tag{A}$$

where K_λ is the extinction coefficient for a specific wavelength λ, also called spectral absorption coefficient whose unit is m^{-1}. L is the thickness of the participating medium (the gas layer) and I_λ is the spectral (monochromatic) intensity of radiation.

As $I_{\lambda,L} = 0.101_{\lambda,0}$ we have from Eq. (A):

$$\ln\left(\frac{I_{\lambda,L}}{I_{\lambda,o}}\right) = -K_\lambda L \quad \Rightarrow \quad K_\lambda = -\frac{1}{L}\ln\left(\frac{I_\lambda, L}{I_{\lambda,0}}\right) = -\frac{1}{0.075}\ln(0.10) = \mathbf{30.7\ m^{-1}} \qquad \textbf{(Ans.)}$$

Spectral transmissivity of the medium is, $\tau_\lambda = \dfrac{I_{\lambda,}\ L}{I_{\lambda,o}} = \mathbf{0.10}$

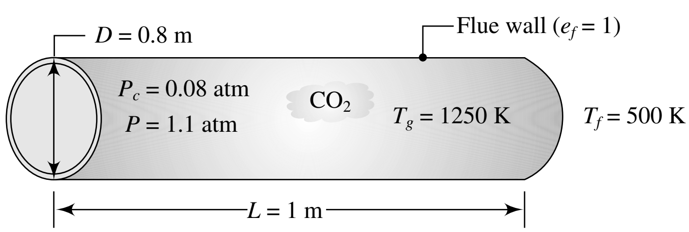
EXAMPLE 13.23 *Combustion products at 1250 K leave a very long cylindrical flue, which is 0.8 m in diameter. The partial pressure of carbon dioxide (CO_2) is 0.08 atm, and its total pressure is 1.1 atm. Calculate the radiation heat transfer per metre length from CO_2 to the flue wall if the wall emissivity is unity and its temperature is 500 K.*

Solution

Known Combustion products pass through a long cylindrical flue under specified conditions.

Find Radiation heat transfer from CO_2 to flue wall.

Schematic



Assumptions (1) Steady operating conditions. (2) Interior of flue wall is essentially black. (3) CO_2 is an ideal gas.

Analysis Since the gas (CO_2) is enclosed, it can be looked upon as a small body in a large enclosure (*long cylindrical flue*).

Radiation heat transfer, $\dot{Q}_{c-f} = \sigma A_c \varepsilon_c (T_c^4 - T_f^4)$

where the suffix c stands for CO_2, and f for flue wall.

Effective area of CO_2 is the inner area of the flue, i.e., $A_c = \pi DL$

Per unit length, $A_c = \pi \times 0.8$ m

To determine the effective emissivity $\varepsilon_{c:}$

$$\begin{pmatrix}\text{Optical thickness}\\ P_c L\end{pmatrix} = \begin{pmatrix}\text{Partial pressure of } CO_2\\ \text{in the combustion products, } P_c\end{pmatrix} \times \begin{pmatrix}\text{Mean beam length}\\ L\end{pmatrix}$$

Mean beam length for an infinitely long circular cylinder of diameter D radiating to curved surface is

$$L = 0.95\,D = 0.95 \times 0.8 \text{ m} = 0.76 \text{ m} \quad \Rightarrow \quad P_c L = (0.08 \text{ atm})(0.76 \text{ m}) = 0.061 \text{ m atm}$$

With $\quad P_c L = 0.061$ m atm and $T = 1250$ K $\qquad \varepsilon_{c,1\text{atm}} = 0.09$

Correction factor for CO_2 at $P_c L$ = 0.061 m atm and 1.1 atm is, $C_c \approx 1$

Hence, $\quad \varepsilon_c = (0.09)(1) = \mathbf{0.09}$

Radiation heat transfer per m length is

$$\dot{Q}_{c-f} = (\pi \times 0.8 \text{ m} \times 1 \text{ m})(5.67 \times 10^{-8} \text{ W/m}^2\text{ K}^4)(0.09)[1250^4 - 500^4]\text{K}^4$$

$$= \mathbf{30510\ W} \quad \text{or} \quad \mathbf{30.5\ kW} \qquad \textbf{(Ans.)}$$

Points to Ponder

- The fraction of the radiant energy leaving one surface that strikes the other surface directly is called shape factor.
- The shape factors with themselves of two infinitely long black body concentric spheres with a diameter ratio of 3 are 1 for the inner and 8/9 for the outer.
- The shape factor from the base of a tetragon to each of its four side surfaces is 0.25.
- The shape factor from a surface of a long equiangular triangular duct to another surface is 0.5.
- In a cubical enclosure, there are 36 individual view factors between the 6 surfaces that constitute the enclosure.
- The view factor of a small enclosed body with respect to the enclosing surface is one.
- Radiation need not always be considered in natural convection systems with polished surfaces.
- The surface resistance to radiation for a black body is zero.
- The radiosity of a black body is equal to its emissive power.
- Crossed string method is a technique for determining exchange areas between long, parallel bodies of uniform cross section.
- The temperature of a reradiating surface is not dependent on its emissivity.
- The shape factor between two black parallel planes of equal area placed symmetrically with re radiating enclosure is $(1 + F_{12})/2$ if the shape factor without re radiation is F_{12}.
- Radiation heat transfer between two surfaces can be decreased by introducing radiation shields between the surfaces.
- A radiation shield should have high reflectivity.
- The ratio of heat transfer rates for parallel plate systems having three radiation shields and no shield, when all emissivities are equal is 0.25.
- Increasing the heat transfer coefficient and reducing the emissivity of the thermocouple placed in a duct reduces the error in temperature measurement.
- Gases such as carbon dioxide and water vapour are opaque to thermal radiation.
- The mean beam length for radiation of an arbitrary volume $\forall$ of surface area A is 3.6 ($\forall/A$).
- Since gases are inherently non-grey, the emissivity, and absorptivity are not the same.

- The rate of decrease of radiation with distance as it passes through an absorbing medium is known as extinction coefficient.
- The monochromatic absorptivity α_λ of a gas is related to the gas thickness L and the monochromatic absorption coefficient κ_λ by the expression $\alpha_\lambda = 1 - e^{-\kappa_\lambda L}$.
- For a grey gas has an emissivity ε its transmissivity is $(1 - \varepsilon)$.
- The absorptivity of CO_2 gas depends, in addition to gas temperature, on the surface temperature.
- Note that the summation rule is applied to an enclosure. To complete the enclosure, it to necessary in several cases to define a third surface which is shown by dashed lines.
- Remember that Hottel's crossed string method is applicable only to surfaces that are of infinite extent in one direction and have unobstructed views of one another.
- With the re radiating walls,

$$J_R = E_{bR} = (J_1 + J_2)/2 = (E_{b1} + E_{b2})/2$$

GLOSSARY of Key Terms

• **Non-participating medium**	A medium which does not emit, absorb, or scatter radiation and has no effect on the radiation exchange between surfaces.
• **Self-view factor**	The fraction of the radiation which leaves a surface and is directly intercepted by it. If the surface is concave, it sees itself and is non zero. For a flat or convex surface, it is zero.
• **Shape factor**	The fraction of the radiant energy leaving one surface which strikes the other surface directly. (Energy transferred by reflection or re radiation from other surfaces that may be present is not to be considered.)
• **Adiabatic, reradiating surface**	A surface which is thermally insulated so that the net heat flow away from the surface is zero. Such surface interacts radiatively with the other surfaces of the enclosure, absorbing and reflecting incident irradiation and re emitting the absorbed energy. Radiation shields and refractory walls in a furnace are examples of adiabatic surface.
• **Radiation-free space**	When two surfaces are located in a radiation-free space, the space is treated as a third active black surface to complete the enclosure at $J_3 = E_{b3} = 0$.
• **Shape-factor algebra**	The interrelation between various shape factors.
• **Radiation shield**	A low-emissivity surface inserted between two radiation exchanging surfaces to effectively reduce the net radiation heat transfer without actually adding to or removing any heat from the overall system.
• **Space resistance**	In electrical network analogy, the resistance across which the potential difference is the difference between radiosities.
• **Surface resistance**	The resistance across which the potential difference is the black body emissive power minus radiosity.

• **Solar constant**	The average solar-energy flux incident on the outer fringes of the earth's atmosphere when the earth is at its mean distance from the sun.
• **Summation rule**	The rule that follows from the conservation requirement that all thermal radiation leaving a surface must be intercepted by the enclosure surfaces.
• **Reciprocity rule**	$A_i F_{ij} = A_j F_{ji}$ is useful in determining one shape factor form the knowledge of the other.
• **Volumetric absorption**	Spectral absorption of radiation in a gas (or semi-transparent liquid or solid) follows exponential decay.
• **Mean beam length**	Interpreted as the radius of a hemispherical gas mass whose emissivity is equivalent to that for the geometry of interest.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

13.1 Which of the following is a wrong statement?
The shape factor is equal to one
(a) for any surface completely enclosed by another surface
(b) for infinite parallel planes radiating only to each other
(c) for a flat or convex surface with respect to itself
(d) inner cylinder to outer cylinder of a long coaxial cylinder

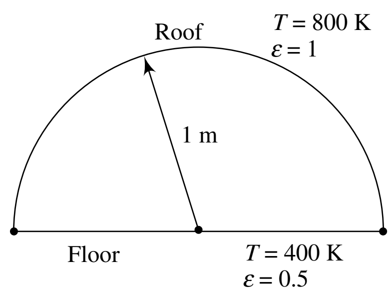
13.2 The number of shape factors that need to be determined directly for a four-surface enclosure is
(a) 6 (b) 1 (c) 10 (d) 16

13.3 The reciprocity theorem states that
(a) $F_{12} = F_{21}$ (b) $A_1F_{12} = A_2F_{21}$ (c) $A_1/F_{12} = A_2/F_{21}$ (d) $A_2F_{12} = A_1F_{21}$
where the symbols have their usual meanings

13.4 A solid cylinder (Surface 2) is located at the centre of a hollow sphere (Surface 1). The diameter of the sphere is 1 m, while the cylinder has a diameter and length of 0.5 m each. The radiation configuration factor F_{11} is
(a) 0.375 (b) 0.625 (c) 0.75 (d) 1

13.5 The value of the shape factor for two inclined flat plates having a common edge of equal width and with an angle of 35 degrees is
(a) 0.70 (b) 1.17 (c) 0.66 (d) 1.34

13.6 Refer to the figure of a hemispherical oven shown below:



The net radiation heat transfer is

(a) 34.2 kW (b) 68.4 kW (c) 136.8 kW (d) 17.1 kW

13.7 The energy radiated from a cavity of area A_1, emissivity ε_1 and absolute temperature T_1 to the surrounding space is given by

(a) $\sigma A_1 \varepsilon_1 T_1^4 \left\{ \dfrac{1 - F_{11}}{1 - (1 - \varepsilon_1) F_{11}} \right\}$ (b) $\sigma A_1 \varepsilon_1 T_1^4 \{1 - F_{11}\}$

(c) $\sigma A_1 \varepsilon_1 T_1^4 F_{11}$ (d) $\sigma A_1 \varepsilon_1 T_1^4 [1 - (1 - \varepsilon_1) F_{11}]$

13.8 A spherical tank of 10 cm diameter is placed in a cubic enclosure whose sides are 10 cm long. The shape factor from any of the square cube surfaces to the sphere is

(a) 0.52 (b) 0.08 (c) 1.0 (d) 0.26

13.9 Consider a two-surface enclosure.

Surface	Temperature (K)	Emissivity	Area (m²)
1	300	0.3	0.3
2	500	1	04

Shape factor $F_{12} = 0.25$

The net radiant heat exchange between the two surfaces is

(a) 85 W (b) 467 W (c) 146 W (d) 737 W

13.10 Two infinite parallel plates are placed at a certain distance apart. An infinite radiation shield in inserted between the plates without touching any of them to reduce heat exchange between the plates. Assume that the emissivities of plates and radiation shield are equal. The ratio of the net heat exchange between the plates with and without the shield is

(a) 1 / 2 (b) 1 / 3 (c) 1 / 4 (d) 1 / 8

13.11 Which of these gases absorb and emit thermal radiation?

(1) Oxygen (2) Water vapour

(3) Nitrogen (4) Carbon dioxide

Codes:

(a) 1 and 2 (b) 1 and 3 (c) 2 and 4 (d) 1, 3 and 4

13.12 The mean beam length in gas radiation is defined as

(a) volume, $\forall$ / surface area, A (b) (volume, $\forall$/3.6) (surface area, A)

(c) 3.6 $\forall / A$ (d) 4 $\forall / A$

13.13 The error in temperature measurement by a thermocouple placed in a gas stream due to radiation can be reduced by

(a) shielding the thermocouple junction with one or more concentric radiation shields
(b) increasing the gas velocity over the junction
(c) using finer wire of thermocouple
(d) all of the above

Answers

Multiple-Choice Questions

13.1 (c) **13.2** (a) **13.3** (b) **13.4** (c) **13.5** (a) **13.6** (a)
13.7 (a) **13.8** (a) **13.9** (c) **13.10** (a) **13.11** (c) **13.12** (c)
13.13 (d)

REVIEW QUESTIONS

13.1 What is the physical significance of the shape factor? When is the shape factor from a surface to itself not zero?

13.2 What assumptions are associated with determining shape factor between two bodies?

13.3 How would you determine the shape factor F_{12} when the shape factor F_{21} and the surface areas are specified?

13.4 Explain the *summation rule, reciprocity rule,* and the *superposition rule* for shape factors?

13.5 Explain the *Hottel's crossed-strings method.* For what kind of geometries is the crossed-strings method appropriate?

13.6 Discuss the radiation analysis of enclosures which comprise black surfaces only. How is the rate of radiation heat transfer between two surfaces expressed?

13.7 Discuss the usefulness of shape factor charts.

13.8 Distinguish between *radiosity* and *radiation emission* for a surface. When will these two quantities be identical?

13.9 Define the *surface* and *space* resistances in radiant heat exchange between two diffuse grey surfaces. In which type of surface will the *surface resistance* be zero?

13.10 Explain the two methods used in solving radiation problems, viz., *direct method* and *network method.*

13.11 Obtain an expression for radiative heat transfer between two grey surfaces connected by a single refractory surface using the electrical network method.

13.12 Explain the meaning of the term *geometric factor* in relation to heat exchange by radiation. Derive an expression for the geometric factor F_{11} for the inside surface of a black hemispherical cavity of radius R with respect to itself.

13.13 Show that the shape factor F_{1R2} for two parallel black surfaces of equal area connected by the re radiating walls at constant temperature is given by
$F_{1R2} = (1 + F_{12})/2$
where F_{1R2} is the shape factor of surface 1 with respect to the surface 2, when connected by a reradiating surface R.

13.14 What is a *reradiating surface*? What simplifications does a reradiating surface offer in the radiation analysis? Does the temperature of such a surface depend on its radiative properties?

13.15 What is a *radiation shield*? What purpose does a radiation shield serve? Is it beneficial for a shield to have a large *reflectivity* or *absorptivity*?

13.16 What is the radiation effect? How does it cause an error in the temperature measurement?

13.17 What is a *non-participating* medium? How does radiation heat transfer through a participating medium differ from that through a non participating one?

13.18 Gaseous radiation is a volumetric phenomenon. Explain. What is Beer's law?

13.19 Explain the concept of *mean beam length*.

13.20 How does the wavelength distribution of radiation emitted by a gas differ from that of a surface at the same temperature?

PRACTICE PROBLEMS

(A) Shape Factor

13.1 The sun and the earth are separated by a distance of 149.5×10^6 km on an average. The diameter of the sun is approximately 1.384×10^6 km and that of the earth is approximately 12 900 km. On a clear day solar irradiation has been measured at the earth's surface to be 1135 W/m^2, with an additional 284 W/m^2 absorbed by the earth's atmosphere. Assuming the sun emits as a black body, estimate its surface temperature from this information. The emissive power of the earth may be neglected.

[5846 K]

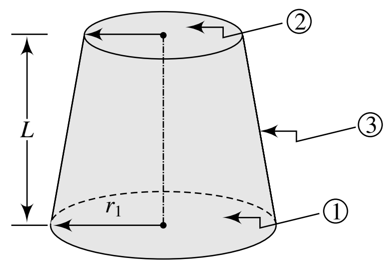
13.2 To construct a hohlraum, a small opening is made on the surface of a sphere of diameter 10 cm with its internal surface highly oxidized. Find the area of the opening if the desired absorptivity is 0.95. **[14.28 cm²]**

13.3 Find the shape factor between two areas 1 and 2 which are in the form of circular ring, are co-axial and in two parallel planes at a distance of 10 cm. Area 1 has inner radius of 5 cm and outer radius of 10 cm. Area 2 has inner radius of 8 cm and outer radius of 20 cm. Use the following formula for calculating the shape factor between the two circular areas located coaxially in two parallel planes:

$$F_{ij} = \frac{1}{2B^2}[X - \sqrt{X^2 - 4B^2C^2}] \text{ where } B = \frac{R_i}{H}, C = \frac{R_j}{H} \text{ and } X = (1 + B^2 + C^2)$$

where R_i and R_j are the radii of the circular planes and H is the distance between them. **[0.513]**

13.4 Calculate the radiation shape factor for the configuration shown below:



$F_{12} = 0.12$
$r_1 = 0.1$ m
$r_2 = 0.05$ m
$L = 0.1$ m
Calculate F_{13}, F_{32} and F_{33}. **[0.398]**

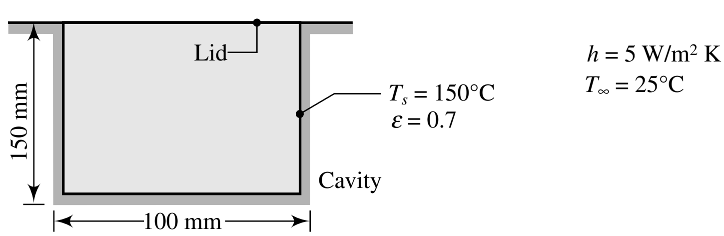
(B) Black Bodies: Radiation Heat Exchange

13.5 An electrically heated industrial furnace cavity is modelled in the form of a cylinder having diameter 10-cm and length 20-cm. It is opened at one end to surroundings that are at a temperature of 300 K. The electrically heated sides and the bottom of the cavity which are well insulated and may be approximated as blackbodies are maintained at a temperature of 1800 K and 2000 K, respectively. By showing the sketch of the furnace, find the power required to maintain the surface at this condition. Take shape factor from the bottom surface to surroundings as 0.06. **[4818 W]**

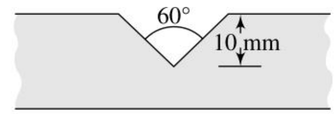
(C) Grey Bodies: Two-Surface Enclosures

13.6 The heat exchange between two walls coated with silver on sides facing each other is by radiation only. The contents of the bottle are at 100°C. The ambient temperature is 14°C. The emissivity of silver is 0.02. Calculate the heat loss. If the inner diameter of the thermos bottle is 10 cm, calculate approximately the outer diameter of the thermos if the thickness of the cork insulation is such as to achieve the same heat transfer. Thermal conductivity of cork is 0.042 W/m °C. **[1.1 m]**

13.7 An evacuated cylindrical cavity 100-mm-diameter and 150-mm-deep is covered with a black lid as shown. The cavity surface is maintained at 150°C and has an emissivity of 0.7. If the lid is exposed to a convective environment where the heat transfer coefficient is 5 W/m² K and the temperature equal to 25°C calculate the steady state lid temperature. **[118°C]**



13.8 Consider a long V-groove, 10-mm deep, machined on a block that is maintained at 1000°C. (a) If the emissivity of the groove surface is 0.6, determine the radiant heat flux leaving the groove to the surroundings. (b) Also find the effective emissivity of the groove



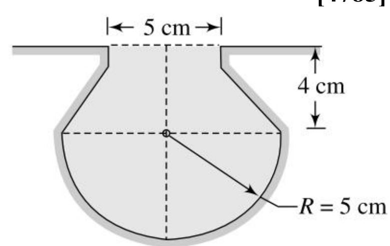
[117 W]

13.9 A hot steel billet, 3-m × 1-m × 1-m has 95% of its surface exposed. The billet has an emissivity (gray) of 0.30, its thermal conductivity is very high and the surroundings are at 30°C. The specific heat and density of steel are 0.50 kJ/kg K and 7800 kg / m^3 respectively.
Assuming that negligible heat is lost by conduction through the supports and that the convective heat losses are small, calculate the time for the billet to cool from 1000°C to 800°C. **[5595 s or 1.55 h]**

13.10 A spherical Dewar flask has inner and outer diameters of 30 cm and 40 cm respectively. The emissivity of both surfaces is 0.05. Liquid oxygen at 90 K is kept in the inner sphere while the temperature of outer sphere is 300 K. Assuming only radiation heat transfer, determine the evaporation rate of oxygen if latent heat of evaporation for oxygen is 214.2 kJ/kg and $\sigma = 5.67 \times 10^{-8}$ W/m^2 K^4. What should be the emissivity of the two surfaces if the rate of boil-off is to be reduced by 80 percent? **[0.01]**

13.11 An enclosure measures 1.5 m × 1.7 m with a height of 2 m. The walls and ceiling are maintained at 250°C and the floor at 130°C. The walls and ceiling have an emissivity of 0.82 and the floor 0.7. Determine the net radiation to the floor. **[4785]**

13.12 Figure shows a cavity having a surface temperature of 900°C and an emissivity of 0.6. Determine the rate of radiant heat loss from the cavity to the surroundings. **[201 W]**



13.13 A hot water radiator of overall dimensions 2 m × 1 m × 0.2 m is used to heat a room at 18°C. The surface temperature of the radiator is 60°C, and its surface is nearly black. The actual surface area of the radiator is 2.5 times the area of its envelope, and the convection heat transfer coefficient is $h_c = 1.31(\theta)^{1/13}$ W/m^2 K. Calculate the rate of heat loss from the radiator. **[4000 W]**

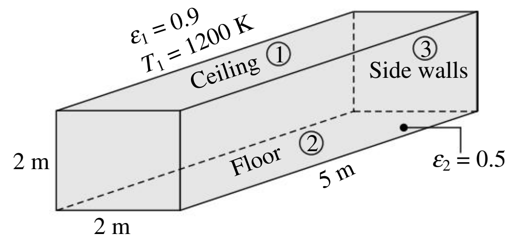
(D) Three-Surface Enclosures

13.14 Two rectangles 0.6 by 0.6 m are placed at right angles with a common edge. The temperature and emissivity of one surface are 1200 K and 0.4 respectively, while the other surface is effectively insulated and in radiant balance with a large surrounding room at 300 K. Calculate the temperature of the insulated surface and the heat lost by the surface at 1200 K. **[16.6 kW, 649 K]**

13.15 A heater of 1-m-diameter is covered by a hemisphere of 4-m-diameter. The surface of the hemisphere is maintained at 400 K. The emissivity of the hemisphere as well as the heater is 0.8. The heater surface is maintained at 1000 K. The remaining base area is open to surroundings at 300 K. The surroundings may be considered as black. Determine the net radiation heat exchange from the heater to the hemisphere and to the surroundings. **[–14 kW]**

13.16 A paint-baking oven consists of a long, triangular duct in which a heated surface is maintained at 1200 K and another surface is insulated. Painted panels, which are maintained at 500 K, occupy the third surface. The triangle is of width 1 m on a side, and the heated and insulated surfaces have an emissivity of 0.8. The emissivity of the panels is 0.4. During steady-state operation, at what rate must energy be supplied to the heated side per unit length of the duct to maintain its temperature at 1200 K? What is the temperature of the insulated surface? **[1102 K]**

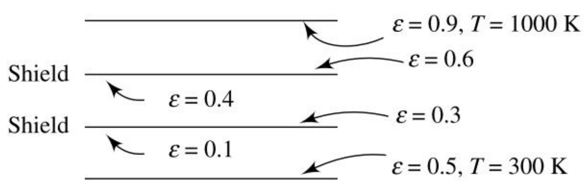
13.17 Figure shows a radiant furnace in the form of a rectangular box. The ceiling (surface 1) is at 1200 K. Determine the temperature of the floor (surface 2) if it receives 100 kW of heat. Also find the temperature of the side walls (surface 3). **[1177 K]**



(E) Radiation Shields

13.18 A large *pandal* is usually put up for big social events. Find the percentage reduction in the net radiative flux to the ground on account of the *pandal*. Assume the ground and the *pandal* to be infinite parallel plates and the solar flux to be 1 kW/m^2. Assume the ground and the sky to be blackbodies at 27°C and the *pandal* a gray-diffuse surface of emissivity 0.8. **[60%]**

13.19 Two very large parallel plates are maintained at uniform temperatures of 1000 K and 300 K. Two radiation shields are interposed between the two planes as shown in the adjoining figure. Calculate the steady-state temperatures of the two shields and the percentage reduction in heat transfer by placing the shields. **[3.2 kW/m^2]**



(F) Error in Temperature Measurement

13.20 A black thermocouple of 2-mm-diameter is inserted at the middle of a 30-cm-long pipe of 75-mm-diameter to measure the temperature of air flowing through it with a velocity of 10 m/s. If the temperature recorded by the thermocouple is 80°C when the pipe wall temperature is 50°C, determine the true temperature of air.

Properties of air: $k = 0.0291$ W/m K $\nu = 19.73 \times 10^{-6}$ m^2/s $Pr = 0.702$

[86.9°C]

13.21 Exhaust gas from a Diesel engine flows at a rate of 0.07 m^3/s along an exhaust duct with a diameter of 40 mm. The temperature of the gas is measured using a spherical probe of 3-mm-diameter and emissivity 0.5. If the duct walls are at 500 K and the thermocouple records a temperature of 785 K, what is the actual temperature of the exhaust gas? Gas properties may be approximated as those of atmospheric air. For a sphere of diameter D, the Nusselt number is given by

$$\overline{Nu_D} = 2 + (0.4\, Re_D^{1/2} + 0.06\, Re_D^{2/3})\, Pr^{0.4} \left(\frac{\mu_\infty}{\mu_w}\right)^{1/4}$$

Properties of air at 1 bar, 800 K: $k = 0.0573$ W/m K, $C_p = 1.099$ kJ/kg K, $\rho = 0.4354$ kg/m^3
$\mu_\infty = 369.8 \times 10^{-7}$ kg/m s, $\mu_w = 270.1 \times 10^{-7}$ kg/m s

[802 K]

14 Mass Transfer

14.1 ❑ INTRODUCTION

Mass-transfer phenomena are found everywhere in nature. The transport of one component in a mixture from a region of higher concentration to that of lower concentration is known as *mass transfer*. For example, if a pan with some water in the bottom is placed in a room, water vapour will diffuse into the air. There is a mass transfer of water from just above the liquid surface where its concentration is higher to the main portion of the air stream where the water concentration is lower.

Consider another example: When a crystal of potassium permanganate ($KMnO_4$) is dropped in a beaker containing clear, stagnant water, the $KMnO_4$ crystal starts dissolving in water and a dark purple colour is absorbed in the vicinity of the crystal. The local concentration of permanganate, indicated by the deepest purple colour will be in the neighbourhood of the crystal. The diffusion is always in the direction of decreasing concentration.

Even though molecular diffusion also takes place in *solids* and *liquids*, the mass-transfer rates *are lower* because the molecules are *more closely packed* in solids and liquids compared to gases.

It is worth noting that the term *mass transfer* refers to the movement of a component in a mixture in the presence of the concentration gradient and does not include gross or bulk motion of the fluid. For instance, air, a mixture of several gases, flowing down a tube is *not* the mass transfer we have in mind but the mixing of oxygen and nitrogen in a container—*whether stirred or not*—is certainly a mass-transfer process.

14.2 ❑ AREAS OF APPLICATION

Mass-transfer processes find vast industrial applications in varied fields such as *mechanical, chemical* and *aerospace engineering*, *physics, chemistry, biology*, etc.

Some of the important industrial and day-to-day applications involving mass transfer are

- Absorption and desorption (e.g., *ammonia-water absorption refrigeration systems*)
- Solvent extraction
- Humidification of air (e.g., *cooling towers, desert coolers, and air-conditioning applications*)
- Oxygenation of blood, food, and drug assimilation
- Transpiration cooling of jet engines and rocket motors
- Ablative cooling of space re-entry vehicles
- Sublimation
- Respiratory mechanism
- Desalination of water by reverse osmosis, ultra-filtration, etc.
- Evaporation of petrol in internal combustion engines
- Neutron diffusion in nuclear reactors
- Distillation columns to separate components in a mixture

- Evaporation of liquid ammonia in the atmosphere of hydrogen in a three-fluid vapour absorption refrigeration system (*electrolux refrigerator*)
- Evaporation of water vapour into dry air
- Diffusion of smoke from a tall chimney into the atmosphere
- Dissolution of sugar added to a cup of tea or coffee
- Diffusion of exhaust gases from an engine into the stagnant ambient air
- Spread of fragrance of perfumes or flowers in the surrounding atmosphere
- Drying of wood, clothes, and coal from mines
- Penetration of carbon in mild steel in the carburizing process (*case hardening*)

14.3 ❑ DIFFERENT MODES OF MASS TRANSFER

Mass transfer occurs whenever there is a *concentration gradient* between two fluids, just as heat transfer takes place whenever there exists a *temperature gradient*. There are essentially *three* modes of mass transfer.

14.3.1 • Mass Transfer by Diffusion

Whenever the transfer of mass of one substance or species through another occurs at a *microscopic* level, it is called *molecular diffusion*. Molecular diffusion occurs in a *gaseous* mixture as a result of *random motion of the molecules*. If such diffusion occurs through a layer of stagnant fluid, it may do so as a result of

- Concentration gradient (*ordinary diffusion*)
- Temperature gradient (*thermal diffusion*)
- Pressure gradient (*pressure diffusion*)

The mechanism of mass transfer by molecular diffusion is akin to heat transfer by conduction.

When one of the diffusing fluids is in *turbulent motion*, the mass transfer process is known as *eddy diffusion*. Dissipation of smoke from a smoke stack (*chimney*) involves an eddy diffusion process. Turbulence causes mixing and transfer of smoke to the ambient air. The mass transfer increases at a slower rate by *molecular diffusion* than by *eddy diffusion*.

14.3.2 • Mass Transfer by Convection

In most cases, diffusion proceeds simultaneously with convection. This occurs when the fluid is in motion. When water vaporizes in a lake, the mixing of water vapour with air is partly due to convection and partly due to diffusion. Mass transfer by convection occurs between a *moving fluid and a surface, or between two relatively immiscible moving fluids which do not mix with each other*. The evaporation of ether is an example of convection mass transfer.

In fact, convective mass transfer occurs by both *molecular diffusion* and *convective motion* of the fluid. This process is analogous to convective heat transfer. For *low concentrations* and *low mass-transfer rates*, many of the equations for convective mass transfer are analogous to those derived for convective heat transfer.

14.3.3 • Mass Transfer by Change of Phase

- Mass transfer also occurs in some cases that involve a change from one phase to another. This is characterized by the simultaneous action of *convection* and *diffusion*.
  The familiar example is the evaporation of a cryogenic fluid from its container.

- When water boils in open air, mass is transferred from liquid to vapour. There is further transfer of vapour from the liquid-vapour interface to the open air by convection as well as by diffusion.

14.4 ❑ CONCENTRATIONS, VELOCITIES, AND FLUXES

In diffusion mass transfer, the *concentration gradient* of a component in a mixture provides the *driving potential*. There are many ways of defining *concentrations, velocities,* and *fluxes* pertinent to mass transfer. We confine our discussion to the *non-reactive*, *two-component* systems only for the sake of simplicity.

14.4.1 • Concentrations

The concentration of species in a multi-component fluid mixture can be expressed in many ways. The *molar concentration*, C_A, of a component A (*or molar density*) is defined as the number of moles of component A per unit volume of the mixture. It is expressed in *kg mole/m*^{3} or *kmol/m*^{3}. The *mass concentration* of a component A, ρ_A (*or mass density*) is the mass of component A per unit volume of the mixture. It is expressed in *kg/m*^{3}.

Concentration, on *mass basis*, is more appropriate in the study of diffusion in *liquids* and *solids*, while molar concentration is more convenient to use in the case of *gases*. Mass and molar concentrations are related to each other as follows:

$$\boxed{C_A(\text{kmol/m}^3) = \frac{\rho_A(\text{kg/m}^3)}{\tilde{\text{M}}_A(\text{kg/kmol})}} \tag{14.1}$$

where $\tilde{M}_A$ is the molecular weight (*relative molar mass*) of the component A.

- The *mass fraction*, w_A, is defined as the ratio of the mass concentration of component A to the total mass density, ρ, of the mixture, and is given by.

$$\boxed{w_A = \frac{\rho_A}{\rho} = \frac{\rho_A/\tilde{M}_A}{r/\tilde{M}}} \tag{14.2}$$

- The *molar fraction*, y_A, of the component A is defined as the ratio of molar concentration of the species A to the total molar concentration of the mixture, and is expressed as

$$\boxed{y_A = \frac{C_A}{C}} \tag{14.3}$$

- For a binary mixture of two components A and B, the following summation rules are valid.

$$\boxed{w_A + w_B = 1} \tag{14.4}$$

$$\boxed{y_A + y_B = 1} \tag{14.5}$$

$$\boxed{\rho_A + \rho_B = \rho} \tag{14.6}$$

$$\boxed{C_A + C_B = C} \tag{14.7}$$

Ideal-Gas Mixtures At *low pressures*, a gas or gas mixture can be considered an ideal gas. The familiar example of such a case is the mixture of *dry air* and *water vapour* existing under atmospheric conditions.

Assuming the mixture of ideal gases, the *Dalton's law of partial pressures* states that the total pressure of the mixture equals the sum total of the partial pressures of the constituent components. That is, P =

$(P_A + P_B)$ for a binary (two-component) maxture of species A and B or, in general, $P = \sum P_i$, where P_i is the partial pressure of the component i and it is the pressure that would be exerted by the component i if it alone occupied the entire volume of the mixture. Then, using the ideal-gas relation $P\forall = n\bar{R}T$,

where $\bar{R}$ = universal gas constant = 8.3143 kJ/kmol K), one can write $\dfrac{P_i}{P} = \dfrac{\dfrac{n_i\bar{R}T}{\forall}}{\dfrac{n\bar{R}T}{\forall}} = \dfrac{n_i}{n} = y_i$.

Molar density of a multi-component mixture, and, mass density, $C = \sum C_i$, $\rho = \sum \rho_i$

Clearly, $\sum y_i = 1$ and $\sum w_i = 1$

We note that the *pressure fraction* is equal to the *mole fraction* for an ideal-gas mixture.

As stated earlier for an ideal gas mixture of the two components A and B.

$$\boxed{P = P_A + P_B}$$

From the equation of state (the perfect-gas equation),

we have
$$\boxed{P_A\forall = n_A\bar{R}T} \tag{14.8}$$

Also
$$\boxed{C_A = \frac{n_A}{\forall} = \frac{P_A}{\bar{R}T}} \tag{14.9}$$

Therefore
$$\boxed{y_A = \frac{C_A}{C} = \frac{P_A/\bar{R}T}{P/\bar{R}T} = \frac{P_A}{P}} \tag{14.10}$$

or
$$\boxed{C = C_A + C_B = \frac{P_A}{\bar{R}T} + \frac{P_B}{\bar{R}T} = \frac{P}{\bar{R}T}} \tag{14.11}$$

14.4.2 • Velocities: Diffusion in a Moving Medium

Let us now consider mass diffusion in a moving medium. In a multi-component mixture, various constituent components move at different velocities. Here, we consider only one-dimensional steady state problems. Assume that there is a mixture of two gases A and B moving at different velocities V_A and V_B in a direction x relative to a stationary frame of reference. We assume further that diffusion is taking place in the direction of the flow because of the presence of concentration gradients. Thus, the molar densities C_A and C_B and the molar fractions y_A and y_B also vary with the direction x.

Let V_A and V_B be the mean velocities of components A and B, respectively, in a mixture of two components, relative to a stationary frame of reference, One can define two types of average velocities, depending on whether one is interested in the *mass* movement or the *molar* movement.

The *molar-average velocity* of the gas mixture is defined in terms of the velocities and molar densities of the components by the equation

$$\boxed{\bar{V} = \frac{V_A C_A + V_B C_B}{C_A + C_B}} \tag{14.12}$$

or
$$\boxed{\bar{V} = \frac{V_A C_A + V_B C_B}{C}} \tag{14.13}$$

Noting that $C_A/C = y_A$, and $C_B/C = y_B$,

$$\boxed{\bar{V} = y_A V_A + y_B V_B} \tag{14.14}$$

where y_A and y_B are the mole-fractions of the components A and B, respectively.

Similar to the *molar-average velocity* of the mixture $(\bar{V})$, one can also define the *mass-average velocity* of the mixture (V) in terms of the velocities V_A and V_B and the mass densities ρ_A and ρ_B of the components. Thus, we have

$$\dot{m} = \dot{m}_A + \dot{m}_B \quad \text{i.e.,} \quad \rho AV = \rho A\ AV_A + \rho_B\ AV_B$$

or
$$\boxed{V = \frac{\rho_A V_A + \rho_B V_B}{\rho_A + \rho_B} = \frac{\rho_A V_A + \rho_B V_B}{\rho}} \tag{14.15}$$

Noting that $\rho_A/\rho = w_A$ and $\rho_B/\rho = w_B$

$$\boxed{V = w_A V_A + w_B V_B} \tag{14.16}$$

where w_A and w_B are the *mass fractions* of the components A and B, respectively.

In general, the values of the molar-average velocity and the mass-average velocity will be different. In the case of a *stationary* medium, *mass-average velocity* as well as *molar-average velocity* is equal to *zero*.

The molar diffusion velocities of components A and B with respect to the molar average velocity $\bar{V}_{\text{diff}}$ are as follows:

Molar diffusion velocity of the component A, $\boxed{\bar{V}_{\text{diff},A} = V_A - \bar{V}}$ (14.17)

Molar diffusion velocity of the component B, $\boxed{\bar{V}_{\text{diff},B} = V_B - \bar{V}}$ (14.18)

Mass transfer by diffusion can take place either in a *stationary* medium or a *moving* medium. The movement of components in a mixture in a stationary medium is due only to the concentration gradients and the velocity of each component is equal to the *diffusion velocity*. However, in a moving medium, the *absolute velocity* of a component will be equal to the *diffusion velocity* plus the *bulk flow velocity* as well.

When there is no concentration gradient, the velocity of all species is equal to mass-average velocity of flow.

14.4.3 • Fluxes

We will express here *four* types of fluxes: *two* with reference to *stationary surfaces* and *two* relative to the *molar average velocity* $\bar{V}$ or *mass average velocity* V.

The *molar flux* is defined as the amount of that component, in moles, which crosses a given area per unit time.

Let $\bar{j}_A$ and $\bar{j}_B$ be the *molar fluxes* of components A and B, respectively, relative to *stationary* coordinates. We then have

$$\bar{j}_A = C_A \bar{V}_A \tag{14.19}$$

and
$$\bar{j}_B = C_B \bar{V}_B \quad (\text{kmol/s m}^2) \tag{14.20}$$

These are known as total (*convection* + *diffusion*) fluxes on *molar* basis.

Let $\bar{j}_{\text{diff},A}$ and $\bar{j}_{\text{diff},B}$ be the *molar fluxes* of the components A and B, respectively, relative to the *molar average velocity*, $\bar{V}$. Therefore,

$$\bar{j}_{\text{diff},A} = C_A(V_A - \bar{V}) \quad (\text{kmol/s m}^2) \tag{14.21}$$

$$\bar{j}_{\text{diff},B} = C_B(V_B - \bar{V}) \tag{14.22}$$

These are defined as *diffusion* fluxes or *molar* basis.

The *mass flux* is defined as the amount of that component in mass units that passes a unit area perpendicular to the x-axis per unit time.

Let j_A and j_B be the *mass fluxes* of the components A and B, respectively, relative to *stationary* coordinates. Then, we have

$$\boxed{\begin{aligned} j_A &= \rho_A V_A \\ j_B &= \rho_B V_B \end{aligned}} \quad (\text{kg/s m}^2) \qquad \begin{aligned}&(14.23)\\&(14.24)\end{aligned}$$

These are called *total* fluxes on *mass* basis.

Let $j_{\text{diff},\,A}$ and $j_{\text{diff},\,B}$ be the *mass fluxes* of the components A and B, respectively, relative to *mass average velocity*, V. Then

$$\boxed{\begin{aligned} j_{\text{diff,A}} &= \rho_A(V_A - V) \\ j_{\text{diff},B} &= \rho_B(V_B - V) \end{aligned}} \quad (\text{kg/s m}^2) \qquad \begin{aligned}&(14.25)\\&(14.26)\end{aligned}$$

These are *diffusion* fluxes on *mass* basis.

Substituting for $\overline{V}$, one gets

$$\boxed{\begin{aligned} \overline{j}_{\text{diff},A} &= C_A V_A - \frac{C_A}{C}(V_A C_A + V_B C_B) \\ \overline{j}_{\text{diff},B} &= C_B V_B - \frac{C_B}{C}(V_A C_A + V_B C_B) \end{aligned}} \qquad \begin{aligned}&(14.27)\\&(14.28)\end{aligned}$$

Adding the two equations (14.27) and (14.28), one gets

$$\overline{j}_{\text{diff},A} + \overline{j}_{\text{diff},B} = (C_A V_A + C_B V_B) - \frac{C_A + C_B}{C}(C_A V_A + C_B V_B)$$

Therefore, $\overline{j}_{\text{diff},A} + \overline{j}_{\text{diff},B} = 0$ (since $C_A + C_B = C$) (14.29)

or $$\overline{j}_{\text{diff},A} + \overline{j}_{\text{diff},B} = 0 \qquad (14.30)$$

This shows that the *diffusion fluxes* $\boxed{\overline{j}_A = -\overline{j}_B}$ and $\overline{j}_{\text{diff},A}$ in a mixture of two components are *equal* in magnitude but in the *opposite* direction.

Table 14.1 presents the summary of definitions of basic quantities used in mass transfer.

Table 14.1 *Basic quantities in mass transfer*

	Mass quantities	Molar quantities	Relationships
Densities Species density	ρ_i(kg/m^3)	C_i(kmol/m^3)	$C_i = \rho_i/<M_i$ (for species i) $C = \dfrac{\rho}{\tilde{M}}$ (for the mixture)
Fractional density	$w_i = \rho_i/\rho$ (mass fraction)	$y_i = C_i/C$ (mole fraction)	$\dfrac{w_i}{y_i} = \dfrac{\tilde{M}_i}{\tilde{M}}$ with
Mixture density	$\rho = \Sigma\rho_i$ = constant	$C = \Sigma C_i$ = constant	$\tilde{M} = \sum y_i \tilde{M}_i$

contd.

contd.

Velocities			
Species velocity	V_i (m/s)	V_i (m/s)	$V_i = V_{\text{diff},i} + V$ $V_i = \bar{V}_{\text{diff},i} + \bar{V}$
Average (bulk) flow velocity	$V = \sum w_i V_i$ (mass-average velocity)	$\bar{V} = \sum y_i \bar{V}_i$ (molar-average velocity)	$V_i = V$ (no concentration gradient)
Diffusion velocity	$V_{\text{diff},i} = V_i - V$	$\bar{V}_{\text{diff},i} = V_i - \bar{V}$	$V = 0$ (stationary medium)
Fluxes			
Total flux (Convection flux + Diffusion flux)	$J_i = \rho_i V_i$(kg/s m^2)	$\bar{j}_i = C_i V_i$ (kmol/s m^2)	$j_i = \rho_i V_i = \rho_i V = \rho_i V_{\text{diff},i} = w_i(\Sigma j_i)$
Diffusion flux	$J_{\text{diff},i} = \rho_i V_{\text{diff},i} = \rho_i(V_i - V)$	$\bar{j}_{\text{diff},i} = C_i \bar{V}_{\text{diff},i} = C_i(V_i - \bar{V})$	$\bar{j}_i = C_i V_i = C_i \bar{V} + C_i \bar{V}_{\text{diff},i}$ $= y_i \Sigma(\bar{j}_i) + \bar{j}_{\text{diff},i}$

14.5 ❑ FICK'S LAW OF DIFFUSION

Fick's law is based on experimental evidence and cannot be derived from first principles. It is valid for all types of matter-*solid*, *liquid,* or *gas*. In general, the diffusion coefficient D_{AB} depends upon the *pressure*, the *temperature* and the *nature* of the constituents. However, for *ideal gases* and *dilute liquids*, the diffusion coefficient is presumed to be constant for a given range of pressure and temperature. The diffusion coefficient is a *property of the system* and is expressed in m^2/s.

Mass flux being a vector quantity, a more general statement of the Fick's law can be written as follows:

$$\left(\frac{\vec{\dot{m}}_A}{A}\right) = -\rho\, D_{AB}\, \nabla\, w_A \text{ where } \nabla \text{ is the } \textit{three-dimensional del operator.} \tag{14.31}$$

The concentration of the gas A can also be described in terms of a *molar fraction* or a *molar density*. In that case,

$$\left(\frac{\dot{N}_A}{A}\right)_n = -C\, D_{AB} \frac{\partial (C_A/C)}{\partial n} = -C\ D_{AB} \frac{\partial y_A}{\partial n} \tag{14.32}$$

where $(\dot{N}_A/A)_n =$ molar flux of the gas A (kmol/m^2 s) in the n-direction.

If C_A = molar density of the gas A, i.e., moles of the gas A per unit volume of mixture in (kmol/m^3), and C_B = molar density of gas B, i.e. moles of the gas B per unit volume of mixture in (kmol/m^3).

The molar density of the mixture is given by $\boxed{C = C_A + C_B}$ and y_A, the molar fraction of A is

$$\boxed{y_A = \frac{C_A}{C}}$$

If the molar density of the mixture C is constant then

$$\boxed{\left(\frac{\dot{N}_A}{A}\right)_n = -D_{AB} \frac{\partial C_A}{\partial n}} \tag{14.33}$$

If the gases constituting the mixture obey the perfect gas law, then in terms of the partial pressure,

$$\left(\frac{\dot{N}_A}{A}\right)_n = -\frac{P}{\bar{R}T} D_{AB} \frac{\partial(PA/P)}{\partial n} \quad \text{and} \quad \left(\frac{\dot{N}_A}{A}\right)_n = -D_{AB} \frac{\partial(PA/\bar{R}T)}{\partial n} \tag{14.34}$$

where P_A is the partial pressure of the gas A, P, the total absolute pressure, and T, the absolute temperature.

Consider a stationary mixture of two gases A and B in a chamber which is divided into two equal compartments by a partition, as shown in Figure 14.1. The concentration of the two gases is not uniform.

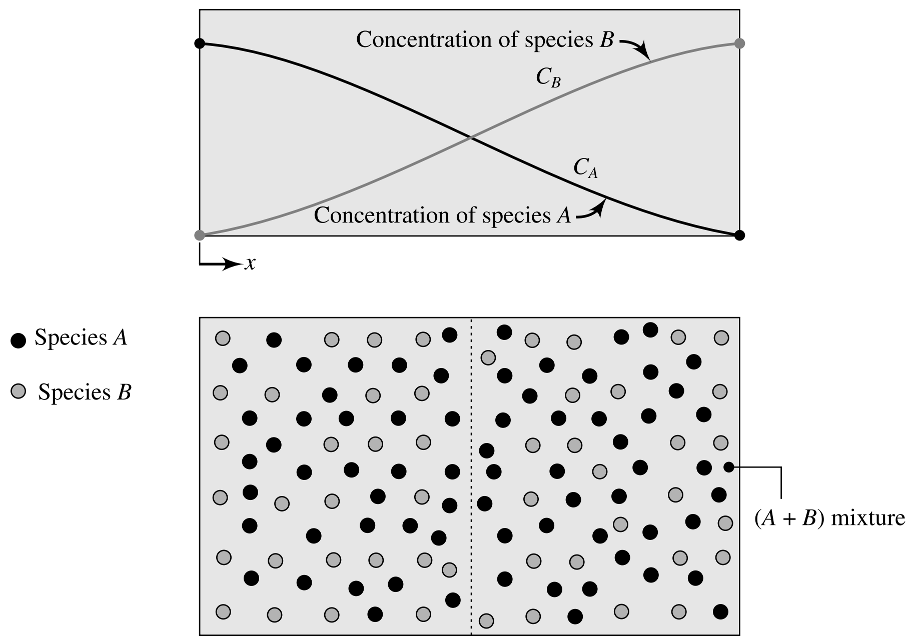


Fig. 14.1 *Diffusion mass transfer in a binary gas mixture (A + B)*

Let the gas mixture in the volume A be rich in the species A, and the volume B be rich in the species B. In such a situation, if the partition is removed, molecules of A would diffuse to the right, i.e., in the direction of decreasing concentration of A, and the molecules of B would diffuse to the left. The lower part of **Figure 14.1** also shows the concentration profiles of A and B shortly after the partition is removed. After sufficient time has elapsed, equilibrium conditions would be achieved, i.e., uniform concentrations of A and B would be attained and there would be no more mass diffusion.

Fick's law relates the mass flux by diffusion to the concentration gradient. Diffusion mass flux of a species through a medium is proportional to the concentration gradient, i.e.,

$$\boxed{\frac{\dot{m}_A}{A} \alpha \frac{d\rho_A}{dx}} \tag{14.35}$$

or

$$\boxed{j_A = \frac{\dot{m}_A}{A} = -D_{AB} \frac{d\rho_A}{dx}} \qquad (\text{kg/s m}^2) \tag{14.36}$$

or in general,

$$\boxed{\left(\frac{\dot{m}_A}{A}\right)_n = -\rho D_{AB} \frac{\partial w_A}{\partial n}} \tag{14.37}$$

where n = direction under consideration

$j_A = \dot{m}_A / A$ = mass flux (kg/s m^2)

A = area normal to the direction of propagation of mass (m^2)

ρ_A = concentration of the species A (kg/m^3)

$d\rho_A/dx$ = concentration gradient for species $A = \rho \dfrac{dw_A}{dx}$ where $w_A = \dfrac{\rho_A}{\rho}$

D_{AB} = constant of proportionality called binary diffusion coefficient or mass diffusivity for the binary mixture of A and B (m^2/s).

The *negative* sign in Eq. (14.18) indicates that diffusion takes place in the direction of *decreasing* concentration, so that mass flux is a *positive* quantity.

Molar flux can be obtained by simply dividing j_A by the molecular weight of the species A.

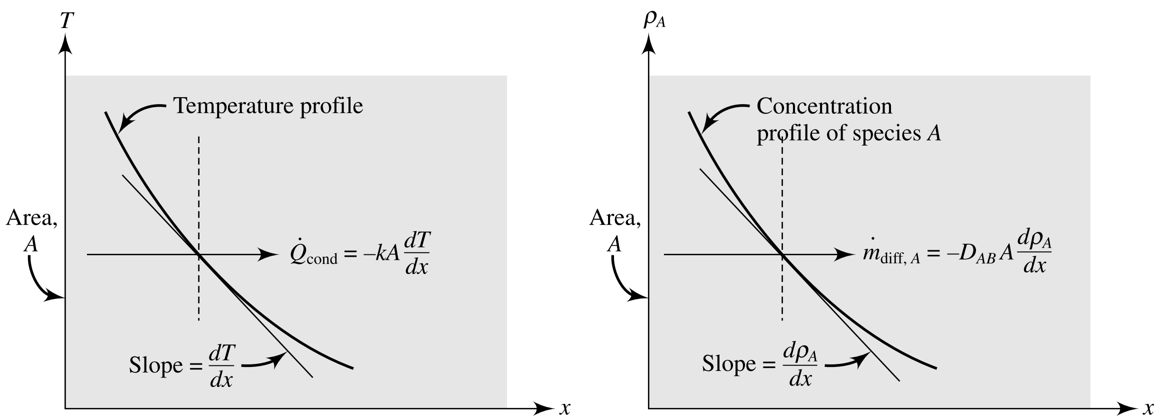


Fig. 14.2 *Analogy between conduction heat transfer and diffusion mass transfer*

$$\boxed{\bar{j}_A = \frac{j_A}{\tilde{M}_A}} \qquad \text{(kmole/s m}^2\text{)} \tag{14.38}$$

Fick's law of diffusion is analogous to the Fourier's law of heat conduction and to the Newton's law of viscosity, i.e.,

$$\boxed{q = \frac{\dot{Q}}{A} = -k\frac{dT}{dx} = \alpha\frac{d}{dx}(\rho C_p T)} \qquad \left(\because \alpha = \frac{k}{\rho C_p}\right)$$

(*Fourier's law of heat conduction*) (14.39)

$$\boxed{\tau = \mu\frac{du}{dy} = \nu\frac{d}{dy}(\rho u)} \qquad \left(\because \nu = \frac{\mu}{\rho}\right) \qquad \textit{(Newton's law of viscosity)} \quad (14.40)$$

The following table sums up the analogy of heat, mass and momentum transfer.

Table 14.2 *Analogy between heat, mass, and momentum transfer*

Heat transfer	**Fourier's law**	**Transport of *heat* due to *temperature gradient* (dT/dx)**	**Thermal conductivity (k)**
Momentum transfer	Newton's law	Transport of *momentum* due to *velocity gradient* (du/dy)	Dynamic viscosity (μ)
Mass transfer	Fick's law	Transport of *mass* due to *concentration gradient* ($d\rho_A/dx$)	Mass diffusivity (D_{AB})

We note that the units of *mass diffusivity* (D), *thermal diffusivity* (α), and *kinematic viscosity* (ν) are all same, i.e., m^2/s.

The analogy between heat transfer and diffusion mass transfer is illustrated in Figure 14.2.

For the species A: From the equation of state: $P_A = \rho_A R_A T = \rho_A \dfrac{\bar{R}T}{\tilde{M}_A}$ (kPa) (14.41)

where $\bar{R}$ = universal gas constant = 8.3143 kJ/kmol K

P_A = partial pressure of the species A (kPa)

ρ_A = density of species A (kg/m^3)

$\tilde{M}_A$ = molecular weight of species A (kg/kmol)

and T = absolute temperature (K)

Mass diffusion flux,
$$j_A = \frac{\dot{m}_A}{A} = -D_{AB}\frac{d\rho_A}{dx} = -D_{AB}\frac{d}{dx}\left(\frac{P_A\tilde{M}_A}{\bar{R}T}\right) \tag{14.42}$$

or
$$\boxed{j_A = -D_{AB}\frac{\tilde{M}_A}{\bar{R}T}\frac{dP_A}{dx}} \tag{14.43}$$

Similarly, ***for the species B***, one can write
$$\boxed{j_B = \frac{\dot{m}_B}{A} = -D_{AB}\frac{\tilde{M}_B}{\bar{R}T}\frac{dP_B}{dx}} \tag{14.44}$$

Note that the above equations are valid for *isothermal* conditions only.

- The statements of Fick's law given so far have been for the diffusion of the gas A in the gas B. Similar equations could also be written for the diffusion of the gas B in the gas A. For example, corresponding to Eq. (14.37), we have
$$\boxed{\left(\frac{\dot{m}_B}{A}\right)_n = -\rho D_{BA}\frac{\partial w_B}{\partial n}} \tag{14.45}$$
- The Fick's law is not only applicable to binary gas mixtures but is also valid for a mixture containing more than two gases.
- The applicability of Fick's law is not confined to mass diffusion in stationary gas mixtures. It can be used for mass diffusion in stationary liquid solutions and in solid solutions in which the concentrations of the components are not uniform.
- This law holds good for mass diffusion due only to concentration gradient. It cannot be used in the case of mass diffusion due to other reasons like pressure gradient, temperature gradient or other external influences.
- *Fick's law*, like *Fourier's law*, is developed based on experimental observation. One cannot analytically derive it from first principles.

- *Mass diffusion* takes place in the direction of decreasing concentration, in the same manner as *heat transfer* occurs in the direction of decreasing temperature.
- *Mass diffusivity or binary diffusion coefficient* (D) depends upon *pressure*, *temperature* and the *nature* of the component concerned. However, it can be assumed as constant for *ideal gases* and *dilute liquids* for a specified range of temperature and pressure.

• Diffusion Coefficient: Equivalence of Diffusivities

The molar flux of a species (*which occurs due to concentration gradients*) causes the species to have a velocity relative to the molar-average velocity of the mixture. The magnitude of this molar flux is given by *Fick's law*. Thus the total molar flux of a species at any cross-section in a moving medium equals the sum of the flux of the species due to the molar-average velocity of the mixture and the flux given by the Fick's law. We have the following equations:

For the species A: $$\left(\frac{\dot{N}_A}{A}\right) = C_A\bar{V} - CD_{AB}\frac{dy_A}{dx} \tag{14.46}$$

For the species B: $$\left(\frac{\dot{N}_B}{A}\right) = C_B\bar{V} - C\,D_{BA}\frac{dy_B}{dx} \tag{14.47}$$

If C, the *molar density* of the mixture, is a constant, one can also write

$$\left(\frac{\dot{N}_A}{A}\right) = C_A\bar{V} - D_{AB}\frac{dC_A}{dx} \qquad \left(\because \quad y_A = \frac{C_A}{C}\right)$$

and $$\left(\frac{\dot{N}_B}{A}\right) = C_B\bar{V} - D_{BA}\frac{dC_B}{dx}$$

The molar flux of the mixture at any cross-section must be equal to the sum of the molar fluxes of the two species. Thus,

$$\boxed{\left(\frac{\dot{N}}{A}\right) = \left(\frac{\dot{N}_A}{A}\right) + \left(\frac{\dot{N}_B}{A}\right)} \tag{14.48}$$

Substituting $(\dot{N}/A) = C\bar{V}$ and the expressions for $(\dot{N}_A/A)$ and $(\dot{N}_B/A)$ from equations (14.46) and (14.47) in Eq. (14.48), we have

$$\frac{\dot{N}}{A} = C\bar{V} = C_A\bar{V} - CD_{AB}\frac{dy_A}{dx} + C_B\bar{V} - CD_{BA}\frac{dy_B}{dx} \tag{14.49}$$

or $$C\bar{V} = (C_A + C_B)\bar{V} - CD_{AB}\frac{dy_A}{dx} - CD_{BA}\frac{dy_B}{dx} \tag{14.50}$$

As $C = C_A + C_B$, we have

$$-CD_{AB}\frac{dy_A}{dx} = CD_{BA}\frac{dy_B}{dx}$$

Since $(y_A + y_B) = 1$, $(dy_A/dx) = (-dy_B/dx)$
It follows that

$$\boxed{D_{AB} = D_{BA} = D} \tag{14.51}$$

Thus, the binary diffusion coefficient of the species A with respect to the species B, (D_{AB}) is equal to the binary diffusion coefficient of B with respect to A, (D_{BA}).

From the kinetic theory of gases, it can be shown that at ordinary pressures, the diffusion coefficient is independent of mixture composition, but *increases* with *temperature* and *decreases* with pressure, i.e., for a binary gas mixture of two components A and B, we have

$$\boxed{D_{AB} = 0.0043 \frac{T^{2/3}}{P(\forall_A^{1/3} + \forall_B^{1/3})^2} \left(\frac{1}{\tilde{M}_A} + \frac{1}{\tilde{M}_B} \right)^{1/2}} \quad [\text{cm}^2/\text{s}] \tag{14.52}$$

where P = Total pressure (atm)

T = Absolute temperature (K)

$\tilde{M}_A, \tilde{M}_B$ = Molecular weights of gas species (kg/kmol)

$\forall_A \forall_B$ = Molecular volumes of A and B at normal boiling points, (cm^3/gmol)

Molecular weights and molecular volumes of a few gases are presented in Table 14.3.

Table 14.3 *Molecular weights and molecular volumes of a few gases*

Gas	Molecular weight (*relative molecular mass*) (kg/kmol) or (g/gmol)	Molecular volume at normal boiling point (cm^3/gmol)
Air	29	29.89
Ammonia (NH_3)	17	25.81
Carbon dioxide (CO_2)	44	34.00
Carbon monoxide (CO)	28	30.71
Hydrogen (H_2)	2	14.28
Nitrogen (N_2)	28	31.20
Oxygen (O_2)	32	25.63
Sulphur dioxide (SO_2)	64	44.78

Another equation suggested for the diffusion coefficient for gas pairs of *non-polar, non-reacting* molecules is of the form

$$\boxed{D_{AB} = \frac{1.8583 \times 10^{-3} T^{3/2} \left[(1/M_A) + (1/M_B) \right]^{1/2}}{P(\sigma_{AB})^2 \, \Omega}} \tag{14.53}$$

where, D_{AB} = Mass diffusivity of the gas species A diffusing through another gas species B, cm^2/s

T = Absolute temperature, K

M_A, M_B = Molecular weights of the components A and B respectively

P = Absolute pressure in atmospheres

σ_{AB} = Collision diameter in ^{o}A (angstroms)

Ω = Collision integral or a dimensionless function depending on the temperature and of the intermolecular forces.

The values of collision integral Ω have been compiled as a function of kE/ε_{AB} where k is the Boltzman constant (1.38×10^{-6} ergs/K) and ε_{AB} is the energy of molecular interaction for the binary system.

For a binary system composed of non-polar molecule pairs,

$$\sigma_{AB} = \frac{\sigma_A + \sigma_B}{2} \quad \text{and} \quad \varepsilon_{AB} = \sqrt{\varepsilon_A \varepsilon_B} \tag{14.54}$$

Charts are available to determine the value of F for different values of solute molal volume and solvent factor (a ratio of the value of F in the solvent of the value F for diffusion in water at constant molal volume).

Table 14.4 gives the values of *diffusion coefficient* and *Schmidt number* for some important substances diffusing through air at 25°C and 1 atm.

Table 14.4 *Diffusion coefficient and Schmidt number for a few gases and vapours diffusing through air at 25°C and 1 atm.*

Substance	Diffusion coefficient, D_{AB}(m²/s)	Schmidt number, $Sc \equiv \nu/D$
Ammonia	2.80×10^{-5}	0.78
Carbon dioxide	1.64×10^{-5}	0.94
Hydrogen	4.10×10^{-5}	0.22
Oxygen	2.06×10^{-5}	0.75
Water	2.56×10^{-5}	0.60
Methanol	1.59×10^{-5}	0.97
Ethyl alcohol	1.19×10^{-5}	1.30
Acetic acid	1.33×10^{-5}	1.16
Benzene	0.88×10^{-5}	1.76
Toluene	0.84×10^{-5}	1.84
Ethyl benzene	0.77×10^{-5}	2.01
Propyl benzene	0.59×10^{-5}	2.62

It can be seen from Table 14.5 that the diffusion coefficient of water vapour in air increases with temperature. This statement is true in general for gases or vapours diffusing in a gaseous medium. Furthermore, the diffusion coefficient decreases with increase in pressure and can be accounted for in the following approximate relation:

$$\boxed{D_{AB} \propto \frac{T^n}{P}} \tag{14.55}$$

where T is the absolute temperature and P, the total pressure. The exponent n varies from 1.5 to 2. This equation can be used over a limited range to estimate the diffusion coefficient in a binary system at a desired temperature and pressure, if the value at a certain temperature and pressure is known.

With n = 1.5, Eq. (14.56) can be expressed as

$$\boxed{\frac{D_1}{D_2} = \left(\frac{T_1}{T_2}\right)^{3/2}\left(\frac{P_2}{P_1}\right)} \tag{14.56}$$

For values of diffusion coefficient of water vapour in air, the following formula has been suggested by *Marrero and Mason*:

$$\boxed{D_{H_2O-Air} = 1.87 \times 10^{-10} \frac{T^{2.072}}{P} \quad (m^2/s)} \tag{14.57}$$

where, P is the total pressure (*atm*) and T is the absolute temperature in (*kelvin*).

Variation of diffusion coefficient for water vapour in air at a total pressure of 1 atm with temperature can be seen from Table 14.5.

Table 14.5 *Binary diffusion coefficient for water at different temperatures*

T(°C)	D_{AB} (m^2/s)	T(°C)	D_{AB} (m^2/s)
0	2.09×10^{-5}	80	3.55×10^{-5}
5	2.17×10^{-5}	85	3.66×10^{-5}
10	2.25×10^{-5}	90	3.77×10^{-5}
15	2.33×10^{-5}	95	3.88×10^{-5}
20	2.42×10^{-5}	100	3.99×10^{-5}
25	2.5×10^{-5}	105	4.10×10^{-5}
30	2.59×10^{-5}	110	4.21×10^{-5}
35	2.68×10^{-5}	115	4.32×10^{-5}
40	2.77×10^{-5}	120	4.44×10^{-5}
45	2.86×10^{-5}	125	4.56×10^{-5}
50	2.96×10^{-5}	130	4.68×10^{-5}
55	3.05×10^{-5}	135	4.80×10^{-5}
60	3.15×10^{-5}	140	4.92×10^{-5}
65	3.25×10^{-5}	145	5.05×10^{-5}
70	3.35×10^{-5}	150	5.18×10^{-5}
75	3.45×10^{-5}		

In a binary ideal-gas mixture of species A and B, the diffusion coefficient of A in B is equal to the diffusion coefficient of B in A, and both *increase* with temperature but *decrease* with increase in pressure.

For steady-state diffusion through a *non-diffusing*, *multi-component* mixture, an *effective diffusivity* is defined as

$$\boxed{D = \frac{1}{\dfrac{y_B}{D_{AB}} + \dfrac{y_C}{D_{AC}} + \dfrac{y_D}{D_{AD}}}} \tag{14.58}$$

where y_B, y_C, y_D ... mole fractions of components of the mixture on a free basis.

D_{AB}, D_{AC}, D_{AD} ... diffusivities of the species A through B, C, D.

For the determination of the binary diffusion coefficient of liquids and solids, there are no established methods available and one has to depend on experimentally obtained values. The following *semi-empirical* relation is suggested to estimate the diffusion coefficient of dilute liquids.

$$\boxed{F = \frac{T}{D_{AB}\,\mu_B}} \tag{14.59}$$

where T = Absolute temperature (K)

D_{AB} = Diffusivity of the solute A through a solvent B (cm^2/s)

μ_B = Viscosity of the solvent B (centipoise), and

F = A function of molal volume of the solute A (K s/cm^2 centipoise)

The values of D_{AB} for liquids are much less then those for gases because of higher molecular density. These values also show an increase with increase in temperature. In the case of solids, the diffusion processe is far more complex and the values of diffusion coefficients are even significantly lower then those for liquids.

14.6 ❑ CONCENTRATION AT THE GAS-LIQUID AND GAS-SOLID INTERFACE

The mole fractions of a species A in the gas and liquid phases at the interface of a dilute mixture are proportional to each other and are expressed by *Henry's law* as

$$\boxed{y_{A,\text{liquid side}} = \frac{P_{A,\text{gas side}}}{H}} \quad \text{(at the interface)} \tag{14.60}$$

where H is *Henry's constant*.

When the mixture is *not* dilute, an approximate relation for the mole fraction of a species on the liquid and gas sides of the interface is expressed approximately by *Raoult's law* as

$$\boxed{P_{A,\text{gas side}} = y_{A,\text{gas side}}\, P = y_{A,\text{liquid side}}\, P_{i,\text{sat}}\,(T)} \tag{14.61}$$

where $P_{A,sat}$ (T) is the saturated pressure of the species A at the interface temperature and P is the *total pressure* on the gas-phase side.

The concentration of the gas species A in the solid at the interface $C_{A,\text{ solid side}}$ is proportional to the *partial* pressure of the species A in the gas $P_{i,\text{gas side}}$ on the gas side of the interface and is given by

$$\boxed{C_{A,\text{solid side}} = S \times P_{A,\text{gas side}}} \qquad (\text{kmol/m}^3) \tag{14.62}$$

where S is the property called *solubility*. The product of the *solubility* of a gas and the *diffusion coefficient* of the gas in a solid is defined as the *permeability P*, which is a measure of the ability of the gas to penetrate a solid. Table 14.6 presents the solubility of a few gases and solids.

Table 14.6 *Solubility of selected gases and solids*

Gas	Solid	T (K)	$S = C_{A,i}/P_{A,i}$ (kmol/m^3 bar)
O_2	Rubber	298	3.12×10^{-3}
N_2	Rubber	298	1.56×10^{-3}
CO_2	Rubber	298	40.15×10^{-3}
He	SiO_2	293	0.45×10^{-3}
H_2	Ni	358	9.01×10^{-3}

(**Note**: Permeability, $P = S.D_{AB}$ where D_{AB} = diffusivity of gas in a solid)

14.7 ❑ GENERAL DIFFERENTIAL EQUATION FOR DIFFUSION IN STATIONARY MEDIA

14.7.1 • Species Conservation Balance in a Control Volume

Similar to the control volume energy balance in heat transfer discussed in Chapter 1, in mass transfer toowe have a conservation balance for a species mass diffusing through a medium. Referring to Fig. 14.3, the rate at which the mass of some species enters a control volume *minus* the rate at which this species mass leaves the control volume *plus* the rate at which the mass of the species is generated (*or absorbed*), due to chemical reactions occurring within the system, must equal the rate at which the species mass is stored in the control volume.

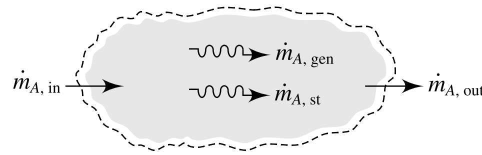


Figure 14.3 *Species conservation balance in a control volume*

Any species A may enter and leave the control volume due to both *fluid motion* and *mass diffusion* across the control surface. These processes are surface phenomena represented by $\dot{m}_{A,\text{in}}$ and $\dot{m}_{A,\text{out}}$. The same species A may also be generated, $\dot{m}_{A,\text{gen}}$, and accumulated or stored, $\dot{m}_{A,st}$, within the control volume. On a *rate basis* it follows that

$$\boxed{\dot{m}_{A,\text{in}} - \dot{m}_{A,\text{out}} + \dot{m}_{A,\text{gen}} = \frac{d\dot{m}_A}{dt} \quad \text{or} \quad \dot{m}_{A,st}} \tag{14.63}$$

14.7.2 • General Mass Diffusion Equation in Cartesian Coordinates

Let us consider a homogeneous and stationary medium which is a binary mixture of the species A and B. The *mass-average* or *molar-average* velocity of the mixture will be zero everywhere and mass transfer will occur only by diffusion because the medium is stationary. Fick's law can be used to determine the species diffusion rate at any point in the medium.

In Cartesian coordinates, there will, in general be the concentration gradients in each of the x, y, and z-directions. Consider a differential control volume, ($dxdydz$), within the medium as shown in Fig. 14.4. With these concentration gradients, mass diffusion of the species A will occur through the surfaces. Relative to stationary coordinates, the species diffusion rates at opposite surfaces take the following forms:

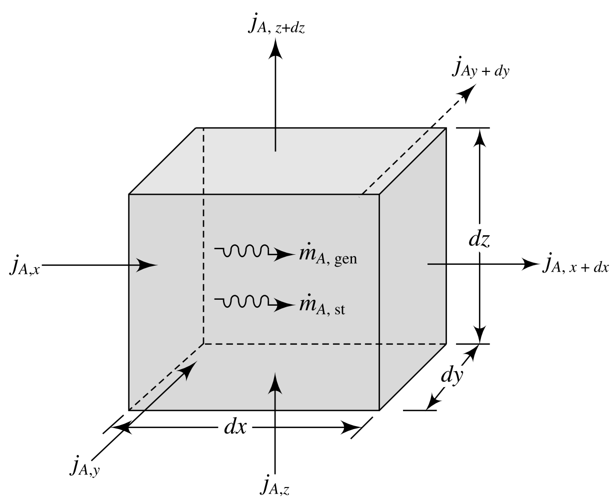


Figure 14.4 *Differential control volume for mass diffusion in Cartesian coordinates*

$$\dot{m}_{A,x+dx} = \dot{m}_{A,x} + \frac{\partial}{\partial x}(\dot{m}_{A,x})\,dx \quad \Rightarrow \quad j_{A,x+dx}dy\,dz = j_{A,x}dy\,dz + \frac{\partial(j_{A,x}dy\,dz)}{\partial x}dx \tag{14.64}$$

$$\dot{m}_{A,y+dy} = \dot{m}_{A,y} + \frac{\partial}{\partial y}(\dot{m}_{A,x})\,dy \quad \Rightarrow \quad j_{A,y+dy}dx\,dz = j_{A,y}dx\,dz + \frac{\partial(j_{A,y}dx\,dz)}{\partial y}dy \tag{14.65}$$

$$\dot{m}_{A,z+dz} = \dot{m}_{A,z} + \frac{\partial}{\partial z}(\dot{m}_{A,z})\,dz \quad \Rightarrow \quad j_{A,z+dz}\,dx\,dy = j_{A,z}dx\,dy + \frac{\partial (j_{A,z}dx\,dy)}{\partial z}dz \tag{14.66}$$

There may also be volumetric (*homogeneous*) chemical reactions taking place throughout the medium. The rate at which the species A is generated within the control volume due to these chemical reactions may be written as

$$\boxed{\dot{m}_{A,\text{gen}} = j_{A,\text{gen}}\,dx\,dy\,dz} \tag{14.67}$$

where $j_{A,\,B}$ is the rate of increase of the mass of the species A per unit volume of the mixture (kg/s m^3) (analogous to the rate of heat generation, W/m^3) The rate of change of the mass of the species A stored within the control volume, can be expressed as

$$\boxed{\dot{m}_{A,st} = \frac{\partial \rho_A}{\partial t}dx\,dy\,dz} \tag{14.68}$$

Now, $\dot{m}_{A,\text{in}} - \dot{m}_{A,\text{out}}$ = (Total mass influx – Total mass efflux) in all the three (x, y, z) directions

$$= -\left\{\frac{\partial}{\partial x}(j_{A,x}) + \frac{\partial}{\partial y}(j_{A,y}) + \frac{\partial}{\partial z}(j_{A,z})\right\}dx\;dy\;dz \tag{14.69}$$

Substituting Eqs. (14.67) to (14.69) into Eq. (14.63), we have

$$\boxed{-\frac{\partial j_{A,x}}{\partial x} - \frac{\partial j_{A,y}}{\partial y} - \frac{\partial j_{A,z}}{\partial z} + j_{A,\text{gen}} = \frac{\partial \rho_A}{\partial t}} \tag{14.70}$$

For a stationary medium, the mass-average velocity V is zero, and $\dfrac{\dot{m}_A}{A} = j_A$. From Fick's law of diffusion, we note that

$$j_{A,x} = -\rho D_{AB}\frac{\partial w_A}{\partial x},\; j_{A,y} = -\rho D_{AB}\frac{\partial w_A}{\partial y} \quad \text{and} \quad j_{A,z} = -\rho D_{AB}\frac{\partial w_A}{\partial z}$$

Substituting these values in Eq. (14.70), we obtain

$$\boxed{\frac{\partial}{\partial x}\left(\rho D_{AB}\frac{\partial w_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho D_{AB}\frac{\partial w_A}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho D_{AB}\frac{\partial w_A}{\partial z}\right) + j_{A,\text{gen}} = \frac{\partial \rho_A}{\partial t}} \tag{14.71}$$

On the basis of the molar concentration, a similar derivation will give

$$\boxed{\frac{\partial}{\partial x}\left(CD_{AB}\frac{\partial y_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(CD_{AB}\frac{\partial y_A}{\partial y}\right) + \frac{\partial}{\partial z}\left(CD_{AB}\frac{\partial y_A}{\partial z}\right) + \bar{j}_{A,\text{gen}} = \frac{\partial C_A}{\partial t}} \tag{14.72}$$

If D_{AB} and ρ are constant, Eq. (14.71) may be expressed as

$$\boxed{\frac{\partial^2 \rho_A}{\partial x^2} + \frac{\partial^2 \rho_A}{\partial y^2} + \frac{\partial^2 \rho_A}{\partial z^2} + j_{A,\text{gen}} = \frac{1}{D_{AB}}\frac{\partial \rho_A}{\partial t}} \tag{14.73}$$

Likewise, if D_{AB} and C are constant, Eq. (14.72) may be written as

$$\boxed{\frac{\partial^2 C_A}{\partial x^2} + \frac{\partial^2 C_A}{\partial y^2} + \frac{\partial^2 C_A}{\partial z^2} + \bar{j}_{A,\text{gen}} = \frac{1}{D_{AB}}\frac{\partial C_A}{\partial t}} \tag{14.74}$$

In the absence of chemical reactions $(j_{A,\text{gen}} = \bar{j}_{A,\text{gen}} = 0)$ and with one-dimensional, steady-state conditions, one can write

$$\boxed{\frac{d^2\rho_A}{dx^2}=0} \quad \text{and} \quad \boxed{\frac{d^2C_A}{dx^2}=0} \tag{14.75}$$

In *cylindrical* and *spherical* coordinates too, in terms of the molar concentration, we can write the general species diffusion equation analogous to conduction heat transfer, as follows:

Cylindrical Coordinates

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(CD_{AB}r\frac{\partial y_A}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(CD_{AB}\frac{\partial y_A}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(CD_{AB}\frac{\partial y_A}{\partial z}\right)+\overline{j}_{A,\text{gen}}=\frac{\partial C_A}{\partial t}} \tag{14.76}$$

Spherical Coordinates

$$\boxed{\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left(CD_{AB}r\frac{2\partial y_A}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \varphi}\left(CD_{AB}\frac{\partial y_A}{\partial \varphi}\right)\\&\qquad+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(CD_{AB}\sin\theta\frac{\partial y_A}{\partial \theta}\right)+\overline{j}_{A,\text{gen}}=\frac{\partial C_A}{\partial t}\end{aligned}} \tag{14.77}$$

14.7.3 • Boundary Conditions

Boundary conditions in mass transfer are very similar to those in heat transfer. Some of these boundary conditions generally encountered in engineering practice are the following:

- *Specified Concentrations at the Boundary* (*analogous to specified temperature, T_s at the boundary*)

$$\boxed{\rho_A=\rho_{A,0} \text{ at } x=0 \text{ and } \rho_A=\rho_{A,L} \text{ at } x=L} \tag{14.78}$$

- *Impermeable Surface at the Boundary* *(analogous to insulated surface in heat transfer)*

$$\boxed{(\partial\rho_A/\partial x)=0 \quad \text{at} \quad x=0,\ \dot{m}_A(0)=0} \tag{14.79}$$

- *Specified Mass Flux at the Surface* *(analogous to specified heat flux, q_w at the surface)*

$$\boxed{j_{A,s}=(\dot{m}_{A,s}/A)=-D_{AB}\left.\frac{\partial\rho_A}{\partial x}\right|_{x=0}} \tag{14.80}$$

- *Specified Mass Transfer Coefficient (Convective) at the Surface*

$$\boxed{\frac{\dot{m}_A}{A}=j_A=h_m(\rho_{A,s}-\rho_{A,\infty})} \tag{14.81}$$

where h_m = convective mass-transfer coefficient (*analogous to specified convection coefficient, h*),
$\rho_{A,s}$ = concentration in the fluid adjacent to the surface, and
$\rho_{A,\infty}$ = bulk concentration in the fluid stream.

14.8 ❑ STEADY-STATE MASS DIFFUSION IN A STATIONARY MEDIUM

Let us now obtain solutions to some simple one-dimensional steady state problems of mass diffusion in a stationary medium.

14.8.1 • Steady-state Diffusion Through a Plain Membrane (Wall)

Consider a plane membrane (wall) of thickness L with the mass fractions of the species A on the two faces as w_{A_1} and w_{A_2} .

Assumptions (1) Steady-state, one-dimensional mass diffusion. (2) Constant mass diffusivity. (3) No chemical rections (no internal mass generation).

For steady-state diffusion, according to Fick's law, we know that

$$\frac{\dot{m}_A}{A} = -\rho D_{AB} \frac{dw_A}{dx} \tag{14.82}$$

Assuming that ρ, the density of the binary mixture, is a constant, we have

$$\frac{\dot{m}_A}{A} = -D_{AB} \frac{d\rho_A}{dx} \tag{14.83}$$

Integrating Eq. (14.80) between the limits 0 and L, and assuming that the diffusion coefficient is a constant, we get

$$\frac{\dot{m}_A}{A} \int_0^L dx = -\int_{\rho_{A_1}}^{\rho_{A_2}} D_{AB} d\rho_A$$

$$\frac{\dot{m}_A}{A} = D_{AB} \frac{(\rho_{A_1} - \rho_{A_2})}{L} = \rho D_{AB} \frac{(w_{A1} - w_{A2})}{L} \quad \text{(on mass basis)} \tag{14.84}$$

Also, $$\frac{\dot{N}_A}{A} = D_{AB} \frac{(C_{A1} - C_{A2})}{L} = CD_{AB} \frac{(y_{A1} - y_{A2})}{L} \quad \text{(on molar basis)} \tag{14.85}$$

Similarly, integrating between 0 and x with corresponding values ρ_{A_1} and ρ_A, we have

$$\frac{\dot{m}_A}{A} = \frac{D_{AB}}{x} (\rho_{A_1} - \rho_A) \tag{14.86}$$

Equating the two, we get the linear concentration distribution

$$\boxed{\frac{\rho_{A_1} - \rho_A}{\rho_{A_1} - \rho_{A_2}} = \frac{w_{A1} - w_A}{w_{A1} - w_{A2}} = \frac{x}{L}} \tag{14.87}$$

or $$\boxed{\rho_A = \rho_{A1} - \frac{x}{L}(\rho_{A1} - \rho_{A2})} \tag{14.88}$$

We can also express the mass-diffusion rate as

$$\dot{m}_A = \frac{\rho_{A_1} - \rho_{A_2}}{\dfrac{L}{D_{AB} A}} = \frac{\text{Concentration potential}}{\text{Diffusion resistance}} \tag{14.89}$$

Note that the above equation gives diffusion mass-flow rate (kg/s), which can be expressed in a form analogous to Ohm's law, i.e., as a ratio of concentration potential to the diffusion resistance.

The diffusion resistance for a plane membrane is given by

$$\boxed{R_{\text{diff}} = \frac{L}{D_{AB} A} \qquad (\text{s/m}^3)} \tag{14.90}$$

14.8.2 • Steady-State Diffusion Through a Cylindrical Shell (a Long Hollow Cylinder)

Consider a cylindrical shell of length L and inner and outer radii equal to r_1 and r_2, respectively. Let the corresponding concentrations of the species A at these radii be ρ_{A_1} and ρ_{A_2} .

Assumptions

- Steady-state, one-dimensional diffusion in the *radial* direction
- Constant mass diffusivity
- No internal mass generation

For an elemental cylindrical shell at any radius r, with a thickness dr, one can write,

From Fick's law, $\dfrac{\dot{m}_A}{A(r)} = -D_{AB}\dfrac{d\rho_A}{dr}$

For this system, we write $\dfrac{\dot{m}_A}{2\pi r L} = -D_{AB}\dfrac{d\rho_A}{dr}$

Separating the variables, and integrating form r_1 to r_2,

$$\dot{m}_A \int_{r_1}^{r_2} \left(\frac{1}{r}\right) dr = -D_{AB}\, 2\pi L \int_{\rho_{A1}}^{\rho_{A2}} d\rho_A$$

or
$$\dot{m}_A \ln(r_2/r_1) = 2\pi D_{AB} L(\rho_{A_1} - \rho_{A_2})$$

or
$$\dot{m}_A = \frac{\rho_{A1} - \rho_{A2}}{\ln\dfrac{(r_2/r_1)}{2\pi D_{AB} L}} \quad \text{or} \quad \dot{m}_A = \frac{\rho_{A_1} - \rho_{A_2}}{R_{\text{diff,cyl}}} \quad \text{(kg/s)} \tag{14.91}$$

where,
$$R_{\text{diff,cyl}} = \frac{\ln(r_2/r_1)}{2\pi D_{AB} L} \quad (s/m^3) = \textit{is the diffusion resistance of cylindrical shell.} \tag{14.92}$$

Integrating from r_1 to r (and mass concentration from ρ_{A1} to ρ_A), we can obtain

$$\dot{m}_A = \frac{\rho_{A1} - \rho_A}{\dfrac{\ln(r/r_1)}{2\pi D_{AB} L}} \quad \text{(kg/s)} \tag{14.93}$$

Equating the two, we have

$$\frac{\rho_{A_1} - \rho_{A_2}}{\dfrac{\ln(r_2/r_1)}{2\pi D_{AB} L}} = \frac{\rho_{A_1} - \rho_A}{\dfrac{\ln(r/r_1)}{2\pi D_{AB} L}} \quad \text{or} \quad \boxed{\frac{\rho_{A_1} - \rho_A}{\rho_{A_1} - \rho_{A_2}} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}} \tag{14.94}$$

In terms of mass fractions, we can write

$$\boxed{\dot{m}_A = \frac{2\pi \rho L D_{AB}(w_{A1} - w_{A2})}{\ln(r_2/r_1)}} \quad \text{and} \quad \boxed{\frac{w_A - w_{A1}}{w_{A2} - w_{A1}} = \frac{\ln(r_2/r_1)}{\ln(r_2/r_1)}} \tag{14.95}$$

The concentration profile can be obtained, on similar lines, as this gives the concentration distribution in the cylindrical shell as a function of the radius, r. Note that the concentration distribution is *logarithmic*.

14.8.3 • Steady-state Diffusion through a Spherical Shell

Consider a spherical shell of inner and outer radii r_1 and r_2 with corresponding mass density concentrations of ρ_{A_1} and ρ_{A_2}.

Assumptions (1) Steady-state, one-dimensional (radial) diffusion. (2) Constant binary diffusion coeffcient. (3) No internal mass generation.
Then, according to

Fick's law, $$\frac{\dot{m}_A}{A(r)} = -D_{AB}\frac{d\rho_A}{dr} \quad \text{or} \quad \frac{\dot{m}_A}{4\pi r^2} = -\rho D_{AB}\frac{dw_A}{dr}$$

Separating the varratles and integrating, we have

$$\frac{\dot{m}_A}{4\pi\rho D_{AB}}\int_{r_1}^{r_2}\frac{dr}{r^2} = \int_{w_{A_1}}^{w_{A_2}} \quad \text{or} \quad \frac{\dot{m}_A}{4\pi\rho D_{AB}}\left[-\frac{1}{r}\right]_{r_1}^{r_2} = w_{A_1} - w_{A_2}$$

or $$\dot{m}_A\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = 4\pi\rho D_{AB}(w_{A_1} - w_{A_2})$$

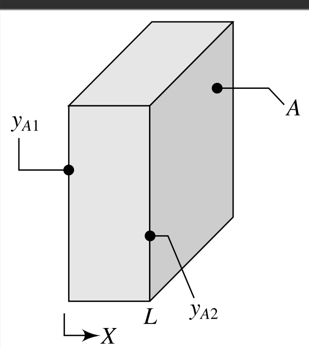
or $$\boxed{\dot{m}_A = 4\pi\rho D_{AB}(w_{A1} - w_{A2})\Big/\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \tag{14.96}$$

The concentration distribution can be obtained, on similar lines, as

$$\boxed{\frac{w_{A1} - w_A}{w_{A1} - w_{A2}} = \left(\frac{1}{r_1} - \frac{1}{r}\right)\Big/\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \tag{14.97}$$

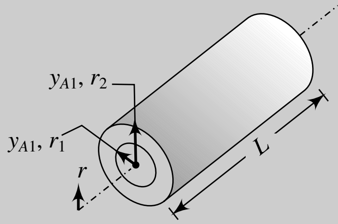
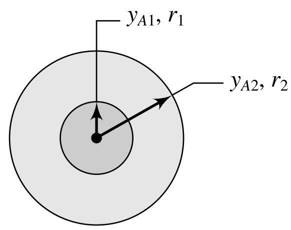
Table 14.7 summarizes the concentration variation and the *molar* diffusion resistance for the *three* geometries discussed above.

Table 14.7 *Concentration distribution and diffusion resistance for different geometries*

S. No.	Geometry	Concentration distribution $y_A(x)$ or $y_A(r)$	Molar diffusion resistance $\bar{R}_{\text{diff}}$ (s/kmol)
1.	 **Plane wall**	$y_A(x) = y_{A_1} - (y_{A_1} - y_{A_2})\frac{x}{L}$ $\frac{y_{A_1} - y_A}{y_{A_1} - y_{A_2}} = \frac{x}{L}$	$\bar{R}_{\text{diff}} = \frac{L}{CD_{AB}A}$

contd.

contd.

2.	 **Cylindrical shell**	$y_A(r) = y_{A_1} - \dfrac{y_{A_1} - y_{A_2}}{\ln(r_2/r_1)} \ln(r/r_1)$ $\dfrac{y_{A_1} - y_A}{y_{A_1} - y_{A_2}} = \dfrac{\ln(r/r_1)}{\ln(r_2/r_1)}$	$\bar{R}_{\text{diff}} = \dfrac{\ln(r_2/r_1)}{2\pi L C D_{AB}}$
3.	 **Spherical shell**	$y_A(r) = y_{A_1} - \dfrac{y_{A_1} - y_{A_2}}{(1/r_1 - 1/r2)} \left(\dfrac{1}{r_1} - \dfrac{1}{r} \right)$ $\dfrac{y_{A_1} - y_A}{y_{A_1} - y_{A_2}} = \dfrac{(1/r_1) - (1/r)}{(1/r_1) - (1/r_2)}$	$\bar{R}_{\text{diff}} = \dfrac{1}{4\pi C D_{AB}} \left(\dfrac{1}{r_1} - \dfrac{1}{r_2} \right)$

14.9 ❑ STEADY-STATE EQUIMOLAR COUNTER-DIFFUSION OF TWO GASES

Consider two large reservoirs (chambers) containing uniform binary mixtures of two gases A and B at different concentrations as shown in Figure 14.5. In the reservoir 1, the molar densities of A and B are C_{A1} and C_{B1}, while in the reservoir 2, they are C_{A2} and C_{B2} respectively. Let us assume that the reservoirs are at the same pressure P and at a uniform temperature T and that they are connected by a long pipe (passage) of length L having a small diameter. Thus, the mass transfer through the pipe is basically one-dimensional. Also, since the reservoirs are of large capacity, the molar densities of the gases inside them do not change with time and a steady-state condition exists in the connecting pipe.

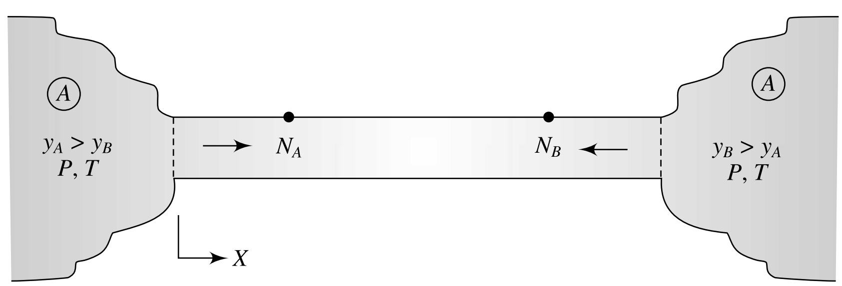


Fig. 14.5 *Equimolar counter diffusion of two gases A and B*

For the reservoir 1, the total pressure P which is the sum of the partial pressures of gases A and B, is constant in steady state and can be expressed as

$$P = P_{A1} + P_{B1} = \bar{R}\,T(C_{A1} + C_{B1})$$

Similarly, for the reservoir 2:

$$P = P_{A2} + P_{B2} = \bar{R}\,T(C_{A2} + C_{B2})$$

Hence, $C_{A1} + C_{B1} = C_{A2} + C_{B2} = \text{constant}$

This constant value is the molar density of the mixture in either reservoir.

Let $C_{A1} > C_{B1}$. Then $C_{A2} > C_{B2}$. Thus, the gas A will diffuse from *left* to *right*, while the gas B will diffuse from *right* to *left*. Since the molar density of the mixture is constant everywhere, the counter-diffusion of A and B occurs in such a manner that. This process is called *equimolar counter-diffusion*. In equimolar counter-diffusion, the molar-average velocity of the gas mixture, $\bar{V} = 0$. Therefore,

$$\left(\frac{\dot{N}_A}{A}\right)\left(\frac{\text{kmol}}{\text{m}^2}\right) = -D_{AB}\frac{dC_A}{dx}\left(\frac{m^2}{s}\right)\left(\frac{\text{kmol}}{\text{m}^3}\frac{1}{\text{m}}\right) \tag{14.98}$$

and

$$\left(\frac{\dot{N}_B}{A}\right) = -D_{BA}\frac{dC_B}{dx} \tag{14.99}$$

Integrating equations (14.96) and (14.97) between the limits $x = 0$ to $x = L$ and using the ideal-gas equation, we have

$$\left(\frac{\dot{N}_A}{A}\right) = -D_{AB}\frac{(C_{A2} - C_{A1})}{L} = -\frac{D_{AB}}{\bar{R}T}\frac{(P_{A2} - P_{A1})}{L}$$

$$\left(\frac{\dot{N}_B}{A}\right) = -D_{AB}\frac{(C_{B2} - C_{B1})}{L} = -\frac{D_{AB}}{\bar{R}T}\frac{(P_{B2} - P_{B1})}{L}$$

Obviously, $(\dot{N}_A) = -(\dot{N}_B)$

Integrating between the limits 0 and x, the molar density distribution can be expressed as

$$\boxed{\frac{C_A - C_{A1}}{C_{A2} - C_{A1}} = \frac{C_B - C_{B1}}{C_{B2} - C_{B1}} = \frac{x}{L}} \tag{14.100}$$

This shows that during the process of equimolal counter-diffusion, the *molar density* and *partial pressure* of each gas varies *linearly* in the pipe connecting the reservoirs.

It may be noted that although the molar-average velocity $\bar{V}$ *is zero during equimolal counter-diffusion, the mass-average velocity* V *will not be equal to zero.*

Since $P = P_A + P_B$, we get after differentiation

$$\frac{dP_A}{dx} + \frac{dP_B}{dx} = 0$$

or

$$\frac{dP_A}{dx} = -\frac{dP_B}{dx} \tag{14.101}$$

This shows that the partial-pressure gradients are equal.

Substitution of $\dfrac{dP_B}{dx}$ as $-\dfrac{dP_A}{dx}$ gives

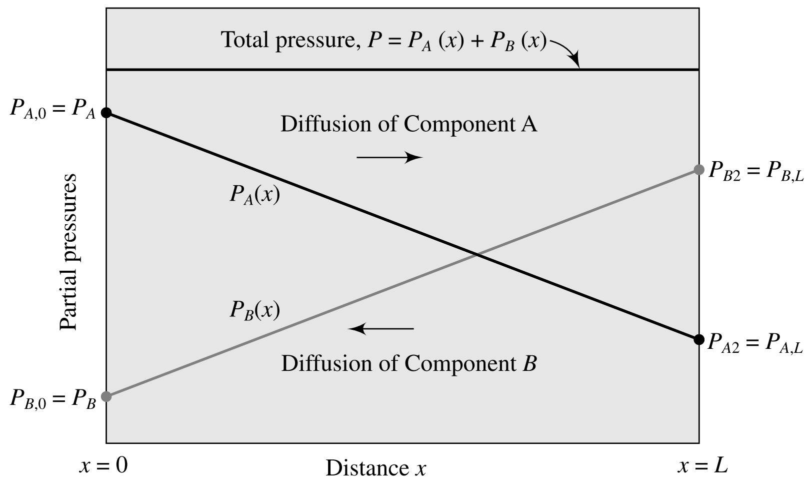


Fig. 14.6 *Distribution of partial pressures P_A and P_B in equimolar counter diffusion of a binary gas mixture.*

$$\dot{N}_B = +\frac{D_{BA}}{\bar{R}T} A \frac{dP_A}{dx} \tag{14.102}$$

Since $\dot{N}_A = -\dot{N}_B$, we have $-\frac{D_{AB}}{\bar{R}T} A \frac{dP_A}{dx} = -\frac{D_{BA}}{\bar{R}T} A \frac{dP_A}{dx}$

We thus find $D_{AB} = D_{BA}$ (14.103)

$$\dot{N}_A = -\frac{D_{AB}}{\bar{R}T} A \frac{dP_A}{dx}$$

Integration of the preceding equation between any two planes gives

$$\boxed{\dot{N}_A = -\frac{D_{AB}}{\bar{R}T} A \left(\frac{P_{A1} - P_{A2}}{L}\right) \text{(kmol/s)}} \tag{14.104}$$

where P_{A1} and P_{A2} are the partial pressures of the gas A at locations 0 and L of the system.

A similar relation can be written for $\dot{N}_B$. Figure 14.6 shows the distribution of partial pressures of the two components as a function of the distance x, which is linear. Also, $\dot{m}_{\text{diff},A} = \dot{N}_A \cdot \tilde{M}_A$ where $\tilde{M}_A$ is the molecular weight of the species A.

Distillation columns are good examples for equimolal counter-diffusion. Venting of a gas to atmosphere also involves equimolal counter-diffusion.

14.10 ❑ ISOTHERMAL EVAPORATION OF WATER INTO AIR FROM A SURFACE

Diffusion of one gas component through another non-diffusing (*stagrant*) gas component occurs in many mass-transfer opertions. Familiar examples of such a unidirectional diffusion through a stagnant gas layer are *absorption*, *humidification,* and the *diffusion of water vapour through a layer of air when evaporation of water occurs, say in a well or a test tube*.

Now consider the isothermal evaporation of water contained in a well and the subsequent diffusion of this water vapour through the stagnant air layer above the water as illustrated in Figure 14.7.

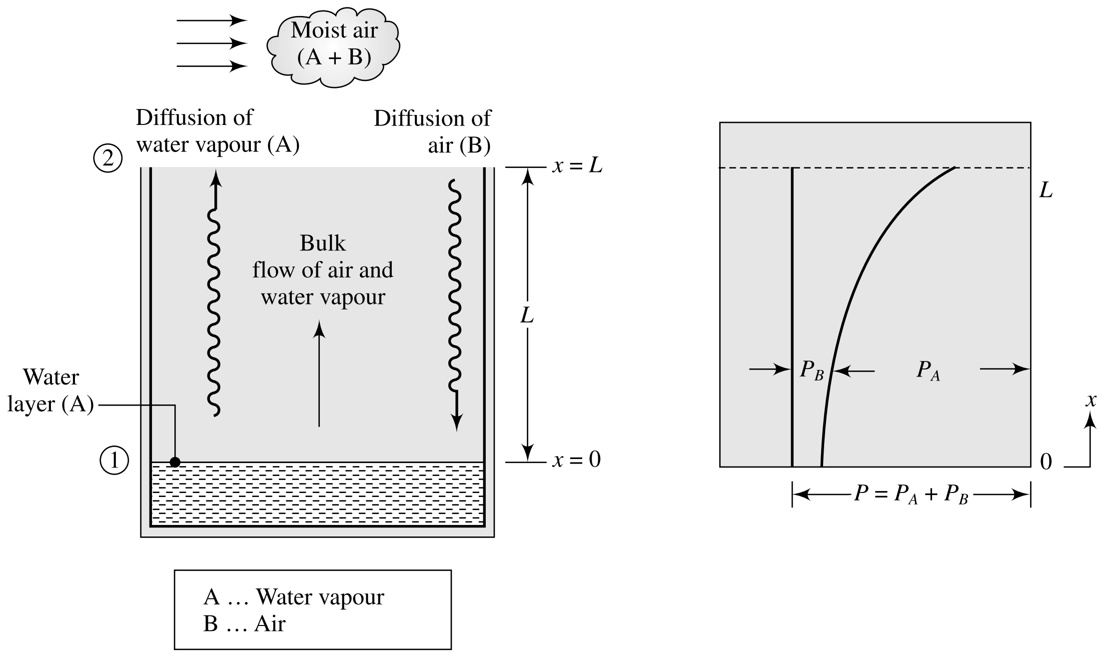


Fig. 14.7 *Evaporation of water in stagnant air*

Assumptions

- Steady-state and isothermal conditions exist.
- Total pressure within the system remains constant.
- Both air and water vapour behave like perfect gases.
- Air has negligible solubility in water.
- Air movement over the top of the liquid surface does not create turbulence and is just sufficient to carry away the evaporated water, but not large enough to cause any change in the concentration profile of air.

As the water (*Species A*) evaporates, it diffuses upwards through the air. The *upward* movement of *water* must be balanced by a *downward* diffusion of *air* so (*Species B*) that the concentration at any distance from the water surface remains constant.

Mass diffusion of air in the *downward* direction is given as follows:

$$\boxed{\dot{m}_B = -\frac{D_{AB}\tilde{M}_B}{\bar{R}T}\frac{dP_B}{dx}} \tag{14.105}$$

where A is the cross-sectional area of the tube. At the surface of water, since air is not soluble in water, the air cannot move downwards. Hence, there must be a bulk mass movement upwards with a velocity just large enough balance the diffusion of air *downwards*. Consequently, this will produce on additional mass flux of water vapour upwards.

Bulk mass transfer of air is given by $\rho_B AV$, i.e.,

$$\dot{m}_B = -\frac{P_B \tilde{M}_B}{\bar{R}T} AV \tag{14.106}$$

where V is the bulk mass velocity in the *upward* direction.

Combining the two expressions of $\dot{m}_B$, we have

$$-\frac{D_{AB}\tilde{M}_B}{\bar{R}T}\frac{dP_B}{dx} = -\frac{P_B \tilde{M}_B}{\bar{R}T} AV$$

or
$$\boxed{V = \frac{D_{AB}}{P_B}\frac{dP_B}{dx}} \tag{14.107}$$

The mass diffusion of water vapour upwards,

$$\dot{m}_A = -D_{AB} A \frac{\tilde{M}_A}{\bar{R}T}\frac{dP_A}{dx}$$

Clearly,

$$\begin{pmatrix}\text{Total mass transport}\\ \text{of water vapour}\end{pmatrix} = \begin{pmatrix}\text{Upward mass diffusion}\\ \text{of water vapour}\end{pmatrix} + \begin{pmatrix}\text{Water vapour carried upwards}\\ \text{along with bulk movement of air}\end{pmatrix}$$

Bulk transport of water vapour,

$$\dot{m}_A = \rho_A AV = \frac{P_A \tilde{M}_A}{\bar{R}T} AV$$

Hence,
$$\dot{m}_{A,\text{total}} = \frac{-D_{AB} A\tilde{M}_A}{\bar{R}T}\frac{dP_A}{dx} + \frac{P_A \tilde{M}_A}{\bar{R}T} AV$$

Substituting the value of V, we get

$$\dot{m}_{A,\text{total}} = -D_{AB} - \frac{A\tilde{M}_A}{\bar{R}T}\frac{dP_A}{dx} + \frac{P_A \tilde{M}_A}{\bar{R}T} \mathrm{A}\left(\frac{D_{AB}}{P_B}\frac{dP_B}{dx}\right) \tag{14.108}$$

or
$$\dot{m}_{A,\text{total}} = \frac{-D_{AB}\tilde{M}_A A}{\bar{R}T}\left(\frac{dP_A}{dx} - \frac{P_A}{P_B}\frac{dP_B}{dx}\right) \tag{14.109}$$

Making use of Dalton's law of partial pressures, we get $\boxed{P = P_A + P_B}$

where P = total pressure, P_A = partial pressure of water vapour, and P_B = partial pressure of air

Differentiating, we get $\dfrac{dP_A}{dx} + \dfrac{dP_B}{dx} = 0$

or
$$\boxed{\frac{dP_A}{dx} = -\frac{dP_B}{dx}} \tag{14.110}$$

Substituting Eq. (14.110) into Eq. (14.109), we get

$$\dot{m}_{A,\text{total}} = \frac{-D_{AB}\tilde{M}_A A}{\bar{R}T}\left[\frac{dP_A}{dx} + \frac{P_A}{P_B}\frac{dP_A}{dx}\right] = \frac{-D_{AB}\tilde{M}_w}{\bar{R}T} A\frac{dP_A}{dx}\left[\frac{P_A + P_B}{P_B}\right]$$

$$\boxed{\dot{m}_{A,\text{total}} = \frac{-D_{AB}\tilde{M}_A A}{\bar{R}T}\frac{P}{P - P_A}\frac{dP_A}{dx}} \tag{14.111}$$

This relation is called *Stefan's law* for an ideal gas diffusing through another stationary ideal gas in a binary gas mixture. Integration of this equation gives

$$\dot{m}_{A,\text{total}} \int_0^L dx = -\frac{D_{AB}\tilde{M}_A A}{\bar{R}T} P \int_{P_{A_1}}^{P_{A_2}} \frac{dP_A}{P - P_A}$$

$$\boxed{\dot{m}_{A,\text{total}} = \frac{D_{AB}\tilde{M}_A A}{L\bar{R}T} P \ln\left[\frac{P - P_{A2}}{P - P_{A1}}\right] = \frac{D_{AB}\, P\tilde{M}_A A}{L\bar{R}T} \ln\frac{P_{B_2}}{P_{B_1}}} \quad \text{(kg/s)} \qquad (14.112)$$

where $P_{B1} = P - P_{A_1}$ or $P - P_{A,0}$ and $P_{B2} = P - P_{A_2}$ or $P - P_{A,L}$

Total molar transport of water vapour is

$$\dot{N}_{A,\text{total}} = \frac{\dot{m}_{A,\text{total}}}{\tilde{M}} = \frac{CD_{AB}A}{L} \ln\left[\frac{1 - (P_{A2}/P)}{1 - (P_{A1}/P)}\right]$$

or

$$\boxed{\dot{N}_{A,\text{total}} = \frac{CD_{AB}A}{L} \ln\left[\frac{1 - y_{A2}}{1 - y_{A1}}\right]} \quad \text{(kmol/s)} \qquad (14.113)$$

where $C = \dfrac{P}{\bar{R}T}$, $y_{A_2} = y_A(L)$, and $y_{A_1} = y_A(0)$

Figure 14.8 illustrates the distribution of partial pressures of air and water vapour with distance x in the medium.

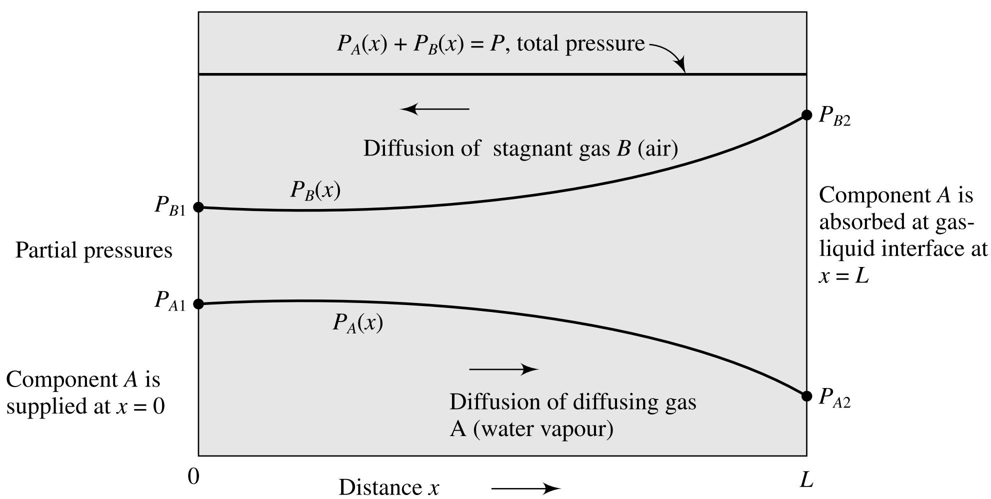


Figure 14.8 *Distribution of partial pressure P_A and P_B for unidirectional diffusion of gas A (water vapour) through gas B (air).*

Now, let us define *Log Mean Partial Pressure of Air* (LMPA) as $LMPA = \dfrac{P_{B2} - P_{B1}}{\ln(P_{B2}/P_{B1})}$

or

$$\ln\left(\frac{P_{B2}}{P_{B1}}\right) = \frac{P_{B2} - P_{B1}}{LMPA}$$

Total mass flow of water vapour can now be written as

$$\dot{m}_{A,\text{total}} = \frac{D_{AB}A}{\bar{R}T}\frac{\tilde{M}_A P}{L}\frac{(P_{B2}-P_{B1})}{LMPA} = \frac{D_{AB}A}{\bar{R}T}\frac{\tilde{M}P}{A}\frac{(P_{A1}-P_{A2})}{LMPA} \qquad \text{(kg/s)} \tag{14.114}$$

Note that, in Eq. (14.114), instead of LMPA we can use arithmetic mean pressure, [i.e., $(P_{B1} + P_{B2})/2$], if the partial pressure of water vapour does not change much as compared to the total pressure of the mixture.

Let the partial pressure of water be P_A at any plane x. Then, integrating the *Stefan's equation* between planes 0 and x, we get $\dot{m}_{A,\text{total}} = \frac{D_{AB}\,A}{\bar{R}T\cdot x}\tilde{M}_A P \ln\left(\frac{P-P_A}{P-P_{A1}}\right)$

$$\boxed{P_A(x) = P - (P-P_{A1})\exp\left[\frac{\dot{m}_{A,\text{total}}}{P\tilde{M}_A}\frac{\bar{R}T\cdot x}{D_{AB}\,A}\right]} \tag{14.115}$$

Equation (14.115) gives the variation of partial pressure of water vapour with the distance x along the tube. And, for the stagnant gas, i.e., air $P_B = P - P_B$. It follows that

$$\boxed{P_B(x) = (P-P_{A1})\exp\left[\frac{\dot{m}_{A,\text{total}}}{P\tilde{M}_A}\frac{\bar{R}T\cdot x}{D_{AB}A}\right] \qquad \text{(Air)}} \tag{14.116}$$

14.11 ❑ TRANSIENT MASS DIFFUSION IN SEMI-INFINITE, STATIONARY MEDIUM

Transient mass diffusion processes occur when the concentration at any location varies with time. Consider the diffusion of a dilute species A in a stationary medium with constant mass diffusivity D_{AB}.

Transient mass diffusion finds an important application in case hardening of mild steel by carburizing process in which the steel component is packed in a carbonaceous material and kept in a furnace at high temperature for a desired length of time.

The mathematical formulation and solution are exactly similar to that for transient heat conduction discussed in Chapter 5. The one-term approximation (analytical) and transient temperature chart (graphical) solutions presented for transient conduction problems are also applicable to transient mass diffusion problems, provided:

- The diffusion coefficient (D_{AB}) is constant (corresponding to *constant thermal diffusivity*, in transient heat conduction).
- There are no homogeneous reactions occurring (corresponding to *no heat generation*).
- Initial concentration of the species A is constant throughout the medium (corresponding to *uniform initial temperature*).

The correspondence between heat and mass transfer variables for transient diffusion is summarized in Table 14.8.

Let us now consider the problem of diffusion of the component A in a semi-infinite slab of material B with an initial concentration of A as $C_{A,i}$. At $t = 0$, the slab is suddenly exposed to a surface concentration, $C_{A,s}$. For $C_{A,s} > C_{A,i}$, the diffusion of component A into the surface of a semi-infinite medium will result in different concentration profiles at different times as shown in Figure 14.9.

Table 14.8 *Analogous quantities in transient heat conduction and transient mass diffusion in a stationary medium*

Heat conduction	Mass diffusion
$\theta(x,t)=\dfrac{T(x,t)-T_\infty}{T_i-T_\infty}$	$\theta_m=\dfrac{C_{A(x,t)}-C_{A,s}}{C_{A,i}-C_{A,s}}$ or $\dfrac{w_{A(x,t)}-w_{A,\infty}}{w_{A,i}-w_{A,\infty}}$
$\dfrac{x}{2\sqrt{\alpha t}}$	$\dfrac{x}{2\sqrt{D_{AB}\,t}}$
$Bi=\dfrac{hL}{k}$	$Bi_m=\dfrac{h_m L}{D_{AB}}$
$Fo=\dfrac{\alpha t}{L^2}$	$Fo_m=\dfrac{D_{AB}t}{L^2}$

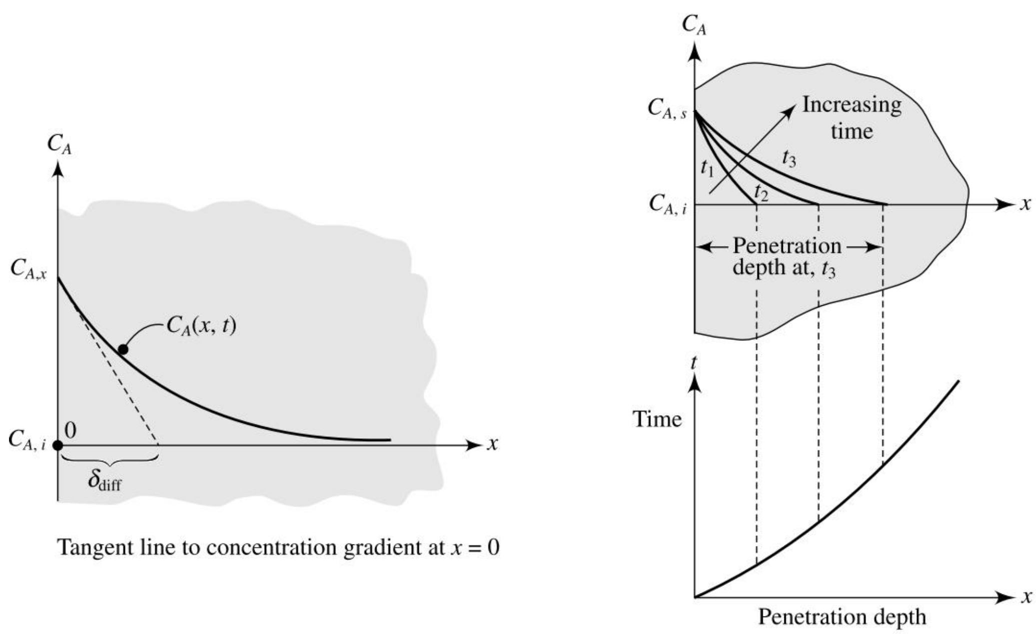


Figure 14.9 *Transient mass diffusion in a semi-infinite medium*

The appropriate differential equaiton for one-dimensional, transient mass diffusion in the x-direction in a stationary medium is given by

$$\boxed{\frac{\partial^2 C_A}{\partial x^2}=\frac{1}{D_{AB}}\left(\frac{\partial C_A}{\partial t}\right)} \tag{14.117}$$

The above equation can be solved with the following initial and boundary conditions:

$C_A(x, 0) = C_{A,i}$ at $t = 0$ for all values of x

$C_A(x, 0) = C_{A,s}$ at $x = 0$ for all values of t

$C_A(\infty, t) = C_{A,i}$ at $x \to \infty$, for all values of t

By analogy, the solution to the diffusion problem is

$$\frac{C_{A(x,t)} - C_{A,s}}{C_{Ai} - C_{A,s}} = \operatorname{erf}\left(\frac{x}{2\sqrt{D_{AB}\,t}}\right) = \operatorname{erf}(\eta)$$

or

$$\boxed{\frac{C_{A(x,t)} - C_{A,i}}{C_{A,s} - C_{A,i}} = \operatorname{erfc}\left(\frac{x}{2\sqrt{D_{AB}\,t}}\right)} \qquad (14.118)$$

where

$$\eta = \frac{x}{2\sqrt{D_{AB}\,t}}$$

Another quantity of interest in mass diffusion process is the *penetration depth* (δ_{diff}), usually defined as the location x where the tangent to the concentration profile at the surface ($x = 0$) intercepts the $C_A = C_{A,i}$ line.

The penetration depth is sometimes also defined as the depth from the surface where the effect of surface concentration change has been reduced to 1 per cent. Then, $\delta_{\text{diff}} = 3.6\sqrt{D_{AB}\,t}$.

14.12 ❑ MASS-TRANSFER COEFFICIENT

Mass-transfer coefficient is in many ways analogous to convective heat-transfer coefficient. You will recall that the rate of heat convection was expressed by Newton's law of cooling as

$$\dot{Q}_{\text{conv}} = hA(T_s - T_\infty)$$

where h_c is the average heat-transfer coefficient, A the surface area and $(T_s - T_\infty)$ is the temperature difference causing heat flow. Similarly, the rate of mass transfer is given by

$$\dot{m}_A = h_m A(\rho_{A,s} - \rho_{A,\infty})$$

where h_m is the average mass-transfer coefficient, A is the surface area and $(\rho_{A,s} - \rho_{A,\infty})$ is the mass concentration difference. If the mixture density ρ is constant,

$$\dot{m}_{\text{conv}}/A = h_m \rho(w_{A,s} - w_{A,\infty}) \qquad (\text{kg/s m}^2)$$

Note that h_m has the units of m/s while h_c has the units of W/m^2 K.

(a) Steady-state Diffusion of a Fluid Across a Plane Membrane (Solid Layer) of Thickness, L Mass-transfer rate for diffusion through a plane membrane is given by

$$\dot{m}_A = \frac{D_{AB}\,A(\rho_{A_1} - \rho_{A_2})}{L} \qquad (\text{kg/s})$$

$$\dot{m}_A = \frac{D_{AB}\,A(\rho_{A1} - \rho_{A2})}{L} = h_m\,A(\rho_{A_1} - \rho_{A_2})$$

Therefore, h_m, the mass-transfer coefficient based on *concentration difference*, can be written as

$$\boxed{h_m = \frac{D_{AB}}{L} \quad (\text{m/s})} \tag{14.119}$$

(b) Steady-state Equimolal Counter-Diffusion The mass diffusion rate in this case is

$$\dot{m}_{\text{diff},A} = D_{AB}\, A \frac{\tilde{M}_A}{\bar{R}T} \frac{(P_{A1} - P_{A2})}{L} = \frac{D_{AB}}{L} \frac{\tilde{M}_A}{\bar{R}T} A(P_{A1} - P_{A2})$$

$$= h_{mc} \frac{\tilde{M}_A}{\bar{R}T} A(P_{A1} - P_{A2}) \quad (\text{kg/s})$$

Defining a mass-transfer coefficient h_{mp}, based on *partial pressure difference*, we get:

Clearly, $$\dot{m}_{\text{diff},A} = h_{mc} \frac{\tilde{M}_A}{\bar{R}T} A(P_{A1} - P_{A2}) = h_{mp}\, A(P_{A1} - P_{A2}) \tag{14.120}$$

$$h_{mp} = h_{mc} \frac{\tilde{M}_A}{\bar{R}T} = \frac{h_{mc}}{RT}$$

where $R \equiv \bar{R}/\tilde{M}_A$

Thus the mass-transfer coefficient based osn *pressure difference* is obtained by simply dividing the mass-transfer coefficient based on *concentration differences* by (RT) where R = characteristic gas constant and T = absolute temperature in kelvin.

(c) Diffusion of Water Vapour Through a Layer of Stagnant Air In this case, the mass diffusion rate of water vapour

$$\dot{m}_{\text{diff},A} = \frac{D_{AB}A}{\bar{R}T} \frac{\tilde{M}_A P}{L} \ln\left(\frac{P_{A2}}{P_{B1}}\right) = \frac{D_{AB}A}{\bar{R}T} \frac{\tilde{M}_A P}{L} \ln\left(\frac{P - P_{A2}}{P - P_{A1}}\right) = h_{mp}\, A(P_{A1} - P_{A2})$$

Then, for this case, the mass-transfer coefficient based on *pressure difference* can be written as

$$h_{mp} = \frac{D_{AB}P}{L(P_{A1} - P_{A2})} \frac{\tilde{M}_A}{\bar{R}T} \ln\left(\frac{P - P_{A2}}{P - P_{A1}}\right) = \frac{D_{AB}P}{L(P_{A1} - P_{A2})RT} \ln\left(\frac{P - P_{A2}}{P - P_{A1}}\right)$$

Obviously, the mass-transfer coefficient based on *concentration difference* would be

$$\boxed{h_{mc} = h_{mp}(RT) = \frac{D_{AB}P}{L(P_{A1} - P_{A2})} \ln\left(\frac{P - P_{A2}}{P - P_{A1}}\right)} \tag{14.121}$$

14.13 ❑ MASS CONVECTION

So far, we have discussed mass diffusion due only to a concentration gradient. Now, we will focus on mass convection involving transfer of mass due to both mass diffusion as well as bulk fluid transport. This is analogous to convective heat transfer, just as molecular diffusion is analogous to conduction heat transfer. This analogy between heat and mass convection is valid for both *free* and *forced* convection, *external* and *internal* flow, as also *laminar* and *turbulent* flow.

Analytical treatment of convective mass transfer is complicated, because of the effects of *flow velocity*, *surface geometry, flow conditions* (i.e., *external* or *internal flow*), *composition,* and *variation of fluid properties*. Empirical relations, based on experimentation, are, therefore, usually relied upon. In convective heat transfer, the heat-transfer coefficient is defined by the relation

$$\frac{\dot{Q}}{A_s} = h(T_s - T_\infty)$$

On similar footing, the convective mass-transfer coefficient is defined as

$$\frac{\dot{m}_A}{A_s} = h_m(\rho_{A,s} - \rho_{A,\infty})$$

The units of h_m are those of velocity, i.e., m/s. The empirical correlations used in solving convective mass-transfer problems are in terms of dimensionless numbers which have a functional relationship determined by dimensional analysis as explained in the following sections.

14.14 ❑ DIMENSIONAL ANALYSIS: FORCED CONVECTION MASS TRANSFER

The general functional relationship between the convective mass transfer coefficient and the several variables that affect it can be expressed as

$$\boxed{h_m = f[D_{AB}, \mu, \rho, V, D]} \tag{14.122}$$

The dimensions of the variables using the MLt system are

Mass transfer coefficient, h_m (m/s)		…	$[L\,t^{-1}]$
Binary diffusion coefficient, D_{AB}(m^2/s)		…	$[L^2\,t^{-1}]$
Absolute viscosity, μ (kg/m s)		…	$[ML^{-1}t^{-1}]$
Density, ρ(kg/m^3)	…	…	$[M\,L^{-3}]$
Velocity, V (m/s)	…	…	$[L\,t^{-1}]$
Diameter, D (m)	…	…	$[L]$

There are in all *six* variables and *three* basic dimensions so that *three* pi-terms will be needed.

$$(p = n - m = 6 - 3 = 3)$$

We select *three* repeating variables such as D, D_{AB}, and ρ.

Starting with the *first* non-repeating (*dependent*) variable h_m, with the repeating variables, such that

$$\Pi_1 = h_m D^a D_{AB}^b \rho^c$$

and in terms of dimensions

$$[Lt^{-1}][L]^a\,[L^2\,t^{-1}]b\,[M\,L^{-3}]c = [M^0\,L^0\,t^0]$$

For Π_1 to be dimensionless, it follows that

$$c = 0 \qquad \text{(for M)}$$
$$1 + a + 2b - 3c = 0 \qquad \text{(for L)}$$
$$-1 - b = 0 \qquad \text{(for t)}$$

and, therefore, $c = 0$, $b = -1$, and $a = 1$

The *first* pi-term then becomes

$$\boxed{\Pi_1 = \frac{h_m D}{D_{AB}}}$$

The procedure is repeated with the *second* non-repeating variable, μ, so that

$$\Pi_2 = \mu D^a D_{AB}^b \rho^c$$

It follows that

$$[ML^{-1}\ t^{-1}][L]^a\ [L^2\ t^{-1}]^b\ [M\ L^{-3}]^c = [M^0\ L^0\ t^0]$$

Equating powers on both sides,

$$1 + c = 0 \qquad \text{(for M)}$$
$$-1 + a + 2b - 3c = 0 \qquad \text{(for L)}$$
$$-1 - b = 0 \qquad \text{(for t)}$$

so that $c = -1$, $b = -1$ and $a = 0$

The *second* pi-term is then

$$\boxed{\Pi_2 = \frac{\mu}{D_{AB}\rho}}$$

Finally, with the *third* non-repeating variable V, $\Pi_3 = V\ D^a\ D_{AB}^b \rho^c$

It follows that

$$[LT^{-1}][L]^a\ [L^2\ t^{-1}]^b\ [M\ L^{-3}]^c = [M^o\ L^o\ t^o]$$

Equating powers on both sides,

and $$c = 0 \qquad \text{(for M)}$$
$$1 + a + 2b - 3c = 0 \qquad \text{(for L)}$$
$$-1 - b = 0 \qquad \text{(for t)}$$

so that $c = 0$, $b = -1$, and $a = 1$

The *third* pi-term becomes

$$\boxed{\Pi_3 = \frac{VD}{D_{AB}}}$$

One can also define $\Pi_4 = \dfrac{\Pi_3}{\Pi_2} = \dfrac{VD}{D_{AB}} \times \dfrac{D_{AB}\rho}{\mu} = \dfrac{\rho VD}{\mu}$

$\therefore$ $$\boxed{\Pi_4 = \frac{\rho VD}{\mu}}$$

Then,

$$\boxed{\Pi_1 = \phi[\Pi_2, \Pi_4]}$$

where $\Pi_1 \equiv \dfrac{h_m D}{D_{AB}}$ is called *Sherwood number*, *Sh*

$\Pi_2 \equiv \dfrac{\mu}{D_{AB}\rho} = \dfrac{\nu}{D_{AB}}$ is known as *Schmidt number*, *Sc*

$\Pi_4 \equiv \dfrac{\rho VD}{\mu} = \dfrac{VD}{\nu}$ is identified as *Reynolds number*, *Re*

The non-dimensional relationship is given by

$$\boxed{\boldsymbol{Sh = \phi[Sc, Re]}} \tag{14.123}$$

14.15 ❑ DIMENSIONAL ANALYSIS: NATURAL (*FREE*) CONVECTION MASS TRANSFER

The variables involved in this case are functionally related as

$$\boxed{h_m = f[D_{AB}, L, \rho, \mu, (g\Delta\rho)]} \tag{14.124}$$

The dimensions of these parameters are listed in the following table:

h_m	D_{AB}	L	r	m	$(g\Delta\rho)$
(m/s)	(m^2/s)	(m)	(kg/m^3)	(kg/m s)	(kg/m^2 s^2)
$[L\, t^{-1}]$	$[L^2 t^{-1}]$	$[L]$	$[ML^{-3}]$	$[ML^{-1}t^{-1}]$	$[ML^{-2}\, t^{-2}]$

Number of Π-terms, p = number of variables, n = mumber of main dimensions, $m = 6 - 3 = 3$.

Using the *Buckingham–Π method*, *three* dimensionless parameters can be formed with *three* repeating variables, namely, ρ, Δ_{AB} and L where L is the characteristic length.
It follows that with the *first* non-repeating (*dependent*) variable, h_m:

$$\Pi_1 = h_m \rho^a D_{AB}^b L^c \quad \text{or} \quad [Lt^{-1}][ML^{-3}]^a [L^2 t^{-1}]^b [L]^c = [M^0 L^0 t^0]$$

Equating the exponents on both sides,

$$a = 0 \qquad \text{(for M)}$$
$$1 + 3a + 2b + c = 0 \qquad \text{(for L)}$$
$$-1 - b = 0 \qquad \text{(for t)}$$

Then, $a = 0$, $b = -1$, and $c = 1$
The *first* pi-term is given by

$\therefore$ $\boxed{\Pi_1 = h_m L/D_{AB}}$ This is the *Sherwood number*, *Sh*.

Also, with the *second* non-repeating variable μ,

$$\Pi_2 = \mu\rho^a D_{AB}^b L^c$$

or $$[ML^{-1} t^{-1}][ML^{-3}]^a [L^2 t^{-1}]^b [L]^c = [M^0 L^0 t^0]$$

Equating the powers on both sides,

$$1 + a = 0 \qquad \text{(for M)}$$
$$-1 - 3a + 2b + c = 0 \qquad \text{(for L)}$$
$$-1 - b = 0 \qquad \text{(for t)}$$

Then $a = -1$, $b = -1$, and $c = 0$
The *second* pi-term then becomes

$$\boxed{\Pi_2 \equiv \frac{\mu}{\rho D_{AB}} \quad \text{or} \quad \frac{\nu}{D_{AB}}}$$ This is the *Schmidt number*, *Sc*.

Similarly, with the *third* non-repeating variable ($g\Delta\rho$), the *third* pi-term will be

$$\Pi_3 = (g\Delta\rho)\rho^a D_{AB}^b L^c$$

or $$[ML^{-2}t^{-2}][ML^{-3}]^a [L^2 t^{-1}]^b [L]^c = [M^0 L^0 t^0]$$

Equating the exponents on both sides,

$$1 + a = 0 \qquad \text{(for M)}$$
$$-2 - 3a + 2b + c = 0 \qquad \text{(for L)}$$
$$-2 - b = 0 \qquad \text{(for t)}$$

Then $a = -1$, $b = -2$, and $c = 3$

The third pi-term is then

$$\boxed{\Pi_3 = \frac{g(\Delta\rho)L^3}{\rho D_{AB}^2}}$$

One can define Π_4 as Π_3/Π_2^2 to eliminate D_{AB}.

$$\Pi_4 = \frac{g(\Delta\rho)L^3}{\rho D_{AB}^2} \times \frac{\rho^2 D_{AB}^2}{\mu^2} = \frac{g(\Delta\rho)L^3/\rho}{\mu^2/\rho^2}$$

or $$\boxed{\Pi_4 = \frac{g(\Delta\rho/\rho)L^3}{\nu^2}}$$ This is the *Grashof number*, *Gr*.

The correlation in the dimensionless form is thus given by

$$\Pi_1 = f(\Pi_2, \Pi_4)$$

or $$\boxed{\boldsymbol{Sh = \phi(Sc, Gr)}} \qquad (14.125)$$

14.16 ❑ ANALOGIES BETWEEN HEAT, MASS, AND MOMENTUM TRANSFER

Recall that the phenomenological laws governing heat, mass, and momentum transfer are strikingly similar. The equations pertaining to the three transport phenomena for a laminar boundary layer over a flat plate are

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}} \qquad (\textit{Momentum Transfer})$$

$$\boxed{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}} \qquad (\textit{Heat Transfer})$$

$$\boxed{u\frac{\partial C_A}{\partial x} + v\frac{\partial C_A}{\partial y} = D_{AB}\frac{\partial^2 C_A}{\partial y^2}} \qquad (\textit{Mass Transfer})$$

where C_A is the molar concentration of the component A that diffuses through the boundary layer.

$$\left[C_A = \frac{\rho_A}{\tilde{M}} \text{ where } \tilde{M} \text{ is the molecular weight of the component } A\right]$$

- The dimensionless ratio ν/α defines the *Prandtl number* and it forms the connecting link between the velocity and temperature profiles. These profiles become identical when $Pr = \nu/\alpha = 1$.
  Prandtl number, $Pr \equiv$ (Momentum diffusivity)/(Thermal diffusivity) = ν/α.
- The dimensionless ratio ν/D_{AB} defines the *Schmidt number*, $Sc = \nu/D_{AB}$, and it forms the connecting link between the velocity and concentration profiles. These profiles will have the same shape when the Schmidt number equals unity.
  Schmidt number, Sc = (Momentum diffusivity) /(Mass diffusivity) = ν/D_{AB}
- The dimensionless ratio α/D_{AB} defines the *Lewis number*, $Le = \alpha/D_{AB}$, and it forms the connecting link between the temperature and concentration profiles. When Lewis number equals unity, these two profiles are identical.
  Lewis number, Le = (Thermal diffusivity)/(Mass diffusivity)

$$= \frac{\alpha}{D_{AB}} = \frac{\nu/D_{AB}}{\nu/\alpha} = \frac{Sc}{\Pr}$$

Clearly, the solution for velocity, temperature and concentration boundary layers will be same if, $\boxed{Pr = Sc = Le}$ and all the *three* boundary layers coincide with each other.

The empirical correlations for mass transfer coefficient h_m are similar to those for the heat-transfer coefficient, h. For example, corresponding to the correlation

$$\boxed{Nu = \frac{hL}{k} = f(Re, Pr)}$$

for heat-transfer coefficient, we have the following correlation for convective mass-transfer coefficient.

$$\boxed{Sh = \frac{h_m L}{D} = f(Re, Sc)}$$

where Sh is Sherwood number, which represents a non-dimensional mass-transfer coefficient.

14.17 ❑ CONCENTRATION BOUNDARY LAYER

14.17.1 • External Flow

Consider the flow of air over a water surface like a lake surface under isothermal conditions. Assume that in the free stream, the air is *not* saturated and has a mass density $\rho_{A,\infty}$. As a result, the concentration of the water vapour (Species A) in the mixture will vary from the water surface to the free stream. At the water surface, the concentration mass density $\rho_{A,s}$ or $\rho_{A,w}$ will be a maximum corresponding to saturation conditions. The concentration boundary layer and the concentration profile at a location are shown in Figure 14.10. The thickness of the concentration boundary layer, δ_c at any location is defined as the normal distance y above the flat surface (*fluid side*) such that $\dfrac{C_{A,w} - C_A}{C_{A,w} - C_{A,\infty}} = 0.99$ where $C_{A,w}$ and $C_{A,\infty}$ are the molar concentrations of the species A at the surface and free stream respectively.

The convective mass transfer is given by

$$\boxed{\frac{\dot{m}_A}{A} = h_m(\rho_{A,s} - \rho_{A,\infty})} \qquad \left(\frac{kg}{sm^2}\right) \tag{14.126}$$

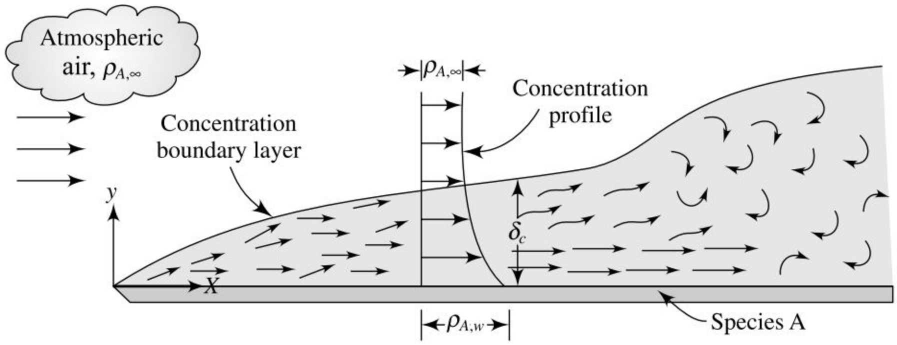


Figure 14.10 *Development of a concentration boundary layer for species A during external flow over a flat surface*

or $$\boxed{\frac{\dot{m}_A}{A} = h_m\,\rho(w_{A,s} - w_{A\infty})} \quad \text{(If the mixture density } \rho \text{ is a constant)} \tag{14.127}$$

where $(\dot{m}_A/A)$ = convective mass flux of the water vapour (Species A) in (kg/s m^2).
$\rho_{A,s}$ = mass density of the water vapour in the saturated air at the water surface in (kg/m^3).
$w_{A,s}$ = $(\rho_{A,s}/\rho)$ = mass fraction of the water vapour in the saturated air
$\rho_{A,\infty}$ = mass density of the water vapour in the free stream in (kg/m^3)
$w_{A,\infty}$ = $(\rho_{A,\infty}/\rho)$ = mass fraction of the water vapour in the free stream
and h_m = convective mass-transfer coefficient in (m/s).

14.17.2 • Internal Flow

Consider convective mass transfer taking place in a tube. A concentration boundary layer develops along with hydrodynamic and thermal boundary leyers in the flow direction. Beyond the concentration entry length, there is a fully developed region. The three boundary layers can be seen in Figure 14.11.

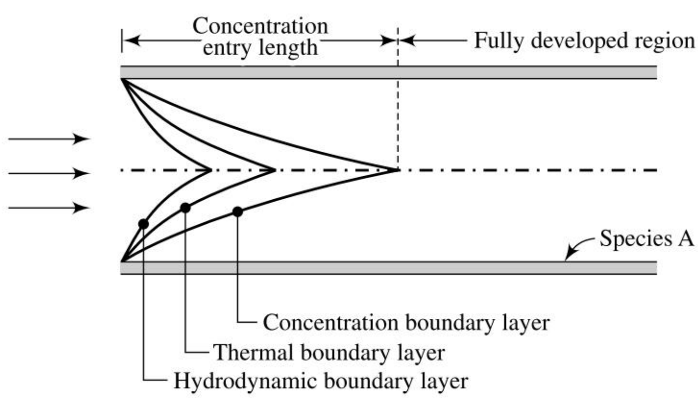


Figure 14.11 *Development of the hydrodynamic, thermal, and concentration boundary layers in internal flow.*

14.18 ❑ CONVECTIVE MASS-TRANSFER CORRELATIONS

Recall the case of laminar-flow heat transfer over a flat plate in which the expression for the local drag coefficient was determined to be

$$\boxed{C_{f,x} = 0.664\, Re_x^{-1/2}} \tag{14.128}$$

Also, the *local heat-transfer coefficient*, expressed as a Nusselt number, was derived as

$$\frac{hx}{k} = 0.332\left(\frac{\nu}{\alpha}\right)^{1/3} Re_x^{1/2} \quad \text{or} \quad \boxed{Nu_x = 0.332\, Pr^{1/3}\, Re_x^{1/2}} \tag{14.129}$$

The analogy between convection heat transfer and mass convection is a powerful tool which makes it possible to predict the heat-transfer coefficient from the knowledge of mass-transfer coefficient and *vice versa* provided the thermal and concentration boundary layers are similar. Table 14.9 shows the analogy between the quantities appearing in the formulation and solution of heat and mass convection.

Table 14.9 *Analogy between heat and mass transfer by convection*

Convective heat Transfer	Convective Mass Yransfer
T	r_A
h	h_m
δ_T	δ_c
$Re = \dfrac{\rho V L_c}{\mu}$	$Re = \dfrac{\rho V L_c}{\mu}$
$Gr = \dfrac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$	$Gr = \dfrac{g(\rho_\infty - \rho_s)L_c^3}{\rho \nu^2}$
$Pr = \dfrac{\nu}{\alpha}$	$Sc = \dfrac{\nu}{D_{AB}}$
$St = \dfrac{h}{\rho C_p V}$	$St_{\text{mass}} = \dfrac{h_m}{V}$
$Nu = \dfrac{hL_c}{k}$	$Sh = \dfrac{h_m L_c}{D_{AB}}$
$Nu = f\,(Re, Pr)$	$Sh = f\,(Re, Sc)$
$Nu = f\,(Gr, Pr)$	$Sh = f\,(Gr, Sc)$

Using the heat-mass transfer analogy, the following expression for the *local mass transfer coefficient* can be written as

$$\boxed{\frac{h_m x}{D_{AB}} = 0.332\left(\frac{\nu}{D_{AB}}\right)^{1/3} Re_x^{1/2}} \tag{14.130}$$

The dimensionless parameter ($h_m x/D_{AB}$) on the left-hand side is called the *local Sherwood number* (*Sh*). It is the mass transfer equivalent of the local Nusselt number (hx/k). The dimensionless number (n/D_{AB}) is the ratio of the kinematic viscosity to the diffusion coefficient. It is called the *Schmidt number* (*Sc*) and is analogous to the Prandtl number (*Pr*). Thus,

$$\boxed{Sh_x = 0.332\, Sc^{1/3}\, Re_x^{1/2}} \tag{14.131}$$

By integrating, we get the average mass-transfer coefficient for a plate of length L as

$$\boxed{\overline{Sh}_L = 0.664\, Sc^{1/3}\, Re_L^{1/2}} \tag{14.132}$$

where $\overline{Sh}_L = (\overline{h}_m L/D_{AB})$ is the *average Sherwood number* based on the length L. Table 14.12 summarizes some of the significant convective heat- and mass-transfer correlations.
It may be noted that for mass transfer, the *Grashof number* is defined as follows:

$$\boxed{Gr = \frac{g(\rho_\infty - \rho_s)L^3}{\rho v^2} = \frac{g\left(\dfrac{\Delta\rho}{\rho}\right)L^3}{v^2}} \tag{14.133}$$

This is applicable for both *homogeneous fluids* (fluids in which density differences are due only to temperature differences with no concentration gradients) and *non-homogeneous fluids* (in which density differences are the result of combined effects of temperature and concentration difference). $\Delta\rho/\rho$ can be replaced by $\beta\Delta T$ only when there is no mass transfer involved.

Table 14.10 *Convective heat- and mass-transfer correlations*

Convective Heat Transfer	Convective Mass Transfer
Forced convection over a flat plate	
Local heat-transfer coefficient: $N_{ux} = \dfrac{h_x x}{k} = 0.332\, Re_x^{1/2}\, Pr^{1/3}$ *Laminar flow* ($Re < 5 \times 10^5$) $Nu_x \dfrac{h_x x}{k} = 0.0296\, Re_x^{4/5}\, Pr^{1/3}$ *Turbulent flow* ($Re > 5 \times 10^5$)	Local mass-transfer coefficient: $Sh_x = \dfrac{h_m x}{D} = 0.332\, Re_x^{1/2}\, Sc^{1/3}$ $Sh_x = \dfrac{h_m x}{D} = 0.0296\, Re_x^{4/5}\, Sc^{1/3}$
Average heat-transfer coefficient: $\overline{Nu}_L = \dfrac{\overline{h} L}{k} = 0.664\, Re_x^{1/2}\, Pr^{1/3}$ *Laminar flow* $\overline{Nu} = \dfrac{hL}{k} = 0.037\, Re_L^{4/5}\, Pr^{1/3}$ *Turbulent flow* For *mixed* boundary layer conditions with $Re_c = 5 \times 10^5$: $\overline{Nu} = (0.037\, Re_L^{4/5} - 871)Pr^{1/3}$	Average mass-transfer coefficient: $\overline{Sh} = \dfrac{\overline{h}_m L}{D} = 0.664\, Re_L^{1/2}\, Sc^{1/3}, Sc > 0.5$ $\overline{Sh} = \dfrac{\overline{h}_m L}{D} = 0.037\, Re_L^{4/5}\, Sc^{1/3}, Sc > 0.5$ For *mixed* boundary layer conditions with $Re_c = 5 \times 10^5$: $\overline{Sh} = (0.037\, Re^{4/5} - 871)Sc^{1/3}$
Fully developed flow in smooth, circular pipes	
***Laminar flow* (*Re* < 2300):** $Nu_x = \dfrac{hD}{k} = 3.657$ (*for uniform wall temperature*) $Nu = 4.364$ (*for uniform wall heat flux*)	***Laminar flow* (*Re* < 2300):** $Sh = \dfrac{h_m D}{D_{AB}} = 3.657$ (*for uniform wall mass concentration*) $Sh = 4.364$ (*for uniform wall mass flux*).

contd.

contd.

***Turbulent flow*:** $Nu_D = 0.023\ Re^{0.8}\ Pr^{1/3}$ ($0.7 < Pr < 100$, $Re_D > 10\ 000$) $Nu_D = \dfrac{(f/8)(Re_D - 1000)Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)}$ ($3000 < Re_D < 5 \times 10^6$ and $0.5 < Pr < 2000$)	***Turbulent flow:*** Gilliland and Sherwood correlation $Sh_D = 0.023\ Re^{0.83}\ Sc^{0.44}$ ($2000 < Re < 35\ 000$, $0.6 < Sc < 2.5$) $Sh_D = \dfrac{(f/8)(Re_D - 1000)Sc}{1 + 12.7(f/8)^{1/2}(Sc^{2/3} - 1)}$ ($3000 < Re_D < 5 \times 10^6$ and $0.5 < Sc < 2000$)
Natural convection over surfaces	
(a) *Vertical plate*: $Nu = 0.59\ (Gr\ Pr)^{1/4}$ $10^5 < GrPr < 10^9$ $Nu = 0.10\ (Gr\ Pr)^{1/3}$ $10^9 < GrPr < 10^{13}$ **(b) *Upper surface of a horizontal plate:*** (Surface is hot, $T_s > T_\infty$): $Nu = 0.54\ (Gr\ Pr)^{1/4}$ $10^4 < GrPr < 10^7$ $Nu = 0.15\ (Gr\ Pr)^{1/3}$ $10^7 < GrPr < 10^{11}$ **(c) *Lower surface of a horizontal plate:*** (Surface is hot, $T_s > T_\infty$): $Nu = 0.27\ (Gr\ Pr)^{1/4}$ $10^5 < GrPr < 10^{11}$	**(a) *Vertical plate:*** $Sh = 0.59\ (GrSc)^{1/4}$ $10^5 < GrSc < 10^9$ $Sh = 0.10\ (GrSc)^{1/4}$ $10^9 < GrSc < 10^{13}$ **(b) *Fluid near the surface is light, ($\rho_s < \rho_\infty$):*** $Sh = 0.54\ (GrSc)^{1/4}$ $10^4 < GrPr < 10^7$ $Sh = 0.15\ (GrSc)^{1/3}$ $10^7 < GrPr < 10^{11}$ **(c) *Fluid near the surface is light ($\rho_s < \rho_\infty$):*** $Sh = 0.27\ (GrSc)^{1/4}$ $10^5 < GrSc < 10^{11}$
Chilton–Colburn analogy	
($C_f/2 = St\ Pr^{2/3} = j_H$) ($0.5 < Pr < 50$)	($C_f/2 = St_m\ Sc^{2/3} = j_M$) ($0.6 < Sc < 3000$)
j_H and j_M are Colburn factors for heat and mass transfer, respectively	
Forced convection mass transfer from a sphere	
$Nu = 2 + [0.4Re + 0.06Re^{2/3}]Pr^{0.4}$	$Sh = 2 + [0.4Re^{1/2} + 0.06Re^{2/3}]\ (Sc)^{0.4}$

14.19 ❑ REYNOLDS AND COLBURN ANALOGIES FOR MASS TRANSFER

Reynolds and *Colburn* analogies for heat transfer can be extended to the case of mass transfer to get a relation between the mass transfer coefficient and the friction factor.

Reynolds analogy for heat transfer over a flat plate can be expressed as

$$\boxed{\frac{Nu}{Re\,Pr} = St = \frac{h}{\rho C_p V} = \frac{C_f}{2}} \tag{14.134}$$

where St is the Stanton number for heat transfer and C_f is the skin-friction coefficient.

Similarly, for mass transfer, we write

$$\boxed{\frac{Sh}{Re\,Sc} = St_m = \frac{h_m}{V} = \frac{C_f}{2}} \tag{14.135}$$

where St_m is the *Stanton number* for mass transfer.

Remember that Reynolds analogy is valid only when $\boxed{Pr \approx Sc \approx 1}$.

When Pr (or Sc) is different form unity, we use the ***Chilton–Colburn analogy*:**

$$\boxed{j_H = St(Pr)^{2/3} = \frac{C_f}{2}} \qquad \text{(for } 0.6 < Pr < 60\text{)} \tag{14.136}$$

and $$\boxed{j_M = St_m (Sc)^{2/3} = \frac{C_f}{2}} \qquad (0.6 < Sc < 3000) \qquad (14.137)$$

where j_H and j_M are the *Colburn factors* for heat transfer and mass transfer, respectively.
From Eqs (14.136) and (14.137), we can write

$$\frac{St}{St_m} = \left(\frac{Sc}{Pr}\right)^{2/3}$$

Also, from Eqs (14.134) and (14.135),

$$\boxed{\frac{h}{h_m} = \rho C_p \frac{St}{St_m} = \rho C_p \left(\frac{Sc}{Pr}\right)^{2/3} = \rho C_p \left(\frac{\nu / D_{AB}}{\nu / \alpha}\right)^{2/3} = \rho C_p \left(\frac{\alpha}{D_{AB}}\right)^{2/3}}$$

We recognise the non-dimensional number (α/D_{AB}) as ***Lewis number*, *Le.***
Therefore, the analogy can be expressed as

$$\boxed{\frac{h}{h_m} = \rho C_p Le^{2/3}} \qquad (14.138)$$

The above relation is useful in cases of *simultaneous heat and mass transfer*.

Air-water vapour mixtures are of special interest in air conditioning applications. For air-water vapour mixtures, Le = 0.872 and $Le^{2/3}$ = nearly equal to unity. Therefore, *for air-water vapour mixtures*, the relation between heat- and mass-transfer coefficients can be conveniently expressed as

$$\boxed{h \approx \rho C_p h_m} \qquad \text{(air-water vapour mixture, i.e., moist air)} \qquad (14.139)$$

Equation (14.139) is known as ***Lewis relation*** and is normally used in air-conditioning applications.
Note: It should be remembered that the analogy between convection heat and mass transfer is valid *only for low mass flux conditions,* i.e., when the mass flux of the diffusing species is low compared to the mass flux existing in a tube or over a surface in the absence of the mass transfer process.

The results for laminar boundary-layer flow over a flat plate can also be expressed in terms of a *local* and *average* Stanton number for mass transfer as follows:

$$\boxed{St_m = 0.332\, Re_x^{-1/2}\, Sc^{-2/3}} \qquad (14.140)$$

$$\boxed{\overline{St}_m = 0.664\, Re_L^{-1/2}\, Sc^{-2/3}} \qquad (14.141)$$

where

St_m = local Stanton number for mass transfer = ($Sh_x/Re_x\ Sc$) = (h_m/u_∞)

$\overline{St}_m$ = average Stanton number for mass transfer = ($\overline{Sh}_L/Re_L\ Sc$) = ($\overline{h}_m/u_\infty$)

Combining equations (14.136) and (14.137) giving the Colburn analogy between heat and momentum transfer, we have

$$\boxed{St_m Sc^{2/3} = St\, Pr^{2/3} = C_{f,x}/2} \qquad (14.142)$$

$$\boxed{\overline{St}_m Sc^{2/3} = \overline{St}\ Pr^{2/3} = \overline{C}_f/2} \qquad (14.143)$$

The *Colburn analogy* for heat and momentum transfer has thus been extended to mass transfer as well.

The usefulness of the Colburn analogy lies in the fact that it can be shown to be reasonably applicable for *turbulent* flow cases.

For turbulent flow through a circular tube, we can evaluate the mass-transfer coefficient by replacing the Nusselt number by the Sherwood number and the Prandtl number by the Schmidt number in the *Dittus–Boelter* equation or in the *Gnielinski equation*. Thus, we have the equivalent mass-transfer correlations:

$$\boxed{Sh_D = 0.023\, Re_D^{0.83}\, Sc^{0.44}} \tag{14.144}$$

and
$$\boxed{Sh_D = \frac{(f/2)(Re_D - 1000)Sc}{1 + 12.7(f/2)(Sc^{2/3} - 1)}} \tag{14.145}$$

where $Sh_D = (h_m D/D_{AB})$

14.20 ❑ SIMULTANEOUS HEAT AND MASS TRANSFER

There are many engineering situations in which heat and mass transfer take place simultaneously, i.e., *humidifiers*, *dehumidifiers*, *absorbers*, *desert coolers*, *wet-bulb thermometer,* etc.

14.20.1 • Evaporation of Water into Air

When the evaporation of water into air or condensation of water vapour from moist air takes place on a surface, the heat-mass convection analogy can be used.

$$\frac{h}{h_m} = \rho C_p \left(\frac{Sc}{Pr}\right)^{2/3} = \rho C_p L_e^{2/3} \tag{14.146}$$

During evaporation of water by blowing air over the water surface (*evaporative cooling*) (Figure 14.12), the energy associated with the phase change is the latent heat of vaporisation of water. The energy required for evaporation must come from the internal energy of the water lowering its temperature. However, under steady-state conditions, the latent heat *lost* by water due to evaporation must be equal to the heat supplied to the water from the surrounding air which in turn gets cooled. Applying the energy balance at the air-water interface under steady-state conditions and neglecting radiation effects, we get

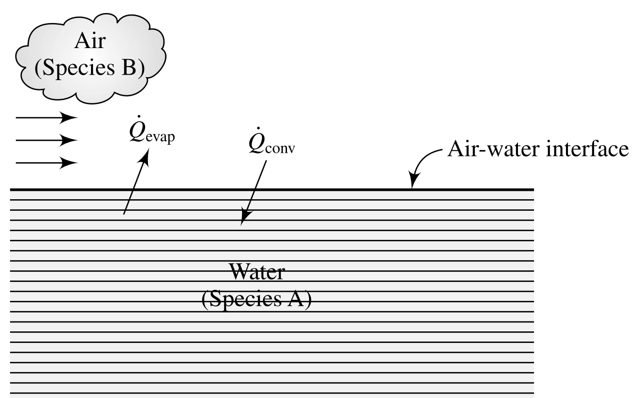


Figure 14.12 *Simultaneous heat and mass transfer*

$$\dot{Q}_{\text{conv}} = \dot{Q}_{\text{evap}}$$

Now
$$\dot{Q}_{\text{evap}} = \dot{m}_A h_{fg} = h_m A_s(\rho_{A,s} - \rho_{A,\infty}) = [\tilde{M}_A h_m A_s (C_{A,s} - C_{A,\infty})]$$

and
$$\dot{Q}_{\text{conv}} = hA_s(T_\infty - T_s)$$

Hence,
$$h(T_\infty - T_s) = [\tilde{M}_A h_m (C_{A,s} - C_{A,\infty})]h_{fg}$$

where $C_{A,s}$ is the water-vapour concentration corresponding to the saturation conditions at T_s. Assuming water vapour to be an ideal gas, one can write

$$(T_\infty - T_s) = h_{fg}\frac{\tilde{M}_A(h_m/h)}{\bar{R}}\left[\frac{P_{A,s}}{T_s} - \frac{P_{A,\infty}}{T_\infty}\right]$$

Since $h/h_m = \rho C_p Le^{2/3}$, we have

$$\boxed{(T_\infty - T_s) = \frac{h_{fg}\tilde{M}_A}{\bar{R}\rho C_p L_e^{2/3}}\left[\frac{P_{A,s}}{T_s} - \frac{P_{A,\infty}}{T_\infty}\right]} \tag{14.147}$$

Equation (14.147) represents the *cooling effect* in evaporative cooling. The properties of air in the above equation (ρ, C_p, L_e) are to be evaluated at the mean film temperature,

$$T = \frac{1}{2}(T_s + T_\infty)$$

If T_s and T_∞ are approximately equal to T then

$$\frac{1}{T_s} = \frac{1}{T_\infty} = \frac{1}{T}$$

$$(T_\infty - T_s) \approx \frac{h_{fg}\cdot\tilde{M}_A}{R\rho C_p Le^{2/3} T}[P_{A,s} - P_{A,\infty}]$$

Also, from the ideal-gas equation,

$$P = \rho RT = \rho\frac{\bar{R}}{\tilde{M}_B}T$$

or $\rho\bar{R}T = \tilde{M}_B P$ where $P = P_A + P_B$

It follows that

$$\boxed{(T_\infty - T_s) \approx \frac{(\tilde{M}_A/\tilde{M}_B)h_{fg}}{C_p Le^{2/3}}\left[\frac{P_{A,s} - P_{A,\infty}}{P}\right]} \tag{14.148}$$

14.20.2 • Wet-bulb Thermometer

Consider the measurement of wet-bulb temperature of an air stream flowing with a velocity u_∞ having a dry-bulb temperature T_∞ or T_{db} and water vapour concentration $\rho_{A,\infty}$ or $\rho_{v,\infty}$. The wet-bulb temperature is measured by wrapping the thermometer bulb with a damp cloth (Fig. 14.13). Let the wet bulb temperature be T_s or T_{wb} and the corresponding water vapour mass density for saturated air at that temperature be $\rho_{A,s}$ or $\rho_{v,s}$. Heat is transferred from the air stream to the surface of the wet bulb because of the temperature difference $(T_\infty - T_s)$, while water vapour is being transferred from the wet bulb surface to the air stream because of the mass concentration difference $(\rho_{A,s} - \rho_{A,\infty})$.

The heat-transfer rate is given by

$$\boxed{\dot{Q} = hA(T_s - T_\infty)} \tag{14.149}$$

while the mass-transfer rate is given by

$$\dot{m}_A = \bar{h}_m A(\rho_{A,s} - \rho_{A,\infty}) \tag{14.150}$$

Under steady operating conditions,

$$\dot{Q}_{\text{sensible, lost}} = \dot{Q}_{\text{latent, gained}}$$

or $$\dot{Q} = \dot{m}_{\text{evap}}\, h_{fg}$$

where $$\dot{m}_{\text{evap}} = h_m A(\rho_{v,s} - \rho_{v,\infty})$$

$$P_v = \rho_v R_v T$$

Hence, $$\dot{m}_{\text{evap}} = \frac{h_m A}{R_v}\left[\frac{P_{v,s}}{T_s} - \frac{P_{v,\infty}}{T_\infty}\right]$$

Using heat-mass transfer analogy,

$$h/h_m = \rho C_p (Le)^{2/3}$$

Therefore, $$\dot{m}_{\text{evap}} = \frac{hA}{\rho C_p Le^{2/3} R_v}\left[\frac{P_{v,s}}{T_{wb}} - \frac{P_{v,\infty}}{T_{db}}\right]$$

Also, $$\dot{Q} = hA(T_\infty - T_s) = hA(T_{db} - T_{wb}) = \dot{m}_{\text{evap}}\, h_{fg}$$

Hence, $$hA(T_{db} - T_{wb}) = \frac{hAh_{fg}\tilde{M}_v}{\rho C_p^{2/3}\bar{R}}\left[\frac{P_{v,s}}{T_{wb}} - \frac{P_{v,\infty}}{T_{db}}\right]$$

Noting that $$\frac{1}{T_{wb}} \simeq \frac{1}{T_{db}} \approx \frac{1}{T},$$

$$T_{db} - T_{wb} = \frac{h_{fg}\tilde{M}_v}{C_p Le^{2/3}}\frac{1}{\rho\bar{R}T}[P_{v,s} - P_{v,\infty}]$$

Since $$P = \rho R T = \rho\frac{\bar{R}}{\tilde{M}_a}T, \quad \rho\bar{R}T = P\tilde{M}_a$$

$$\boxed{T_{db} - T_{wb} = \frac{h_{fg}}{C_p Le^{2/3}}\frac{\tilde{M}_v}{\tilde{M}_a}\left(\frac{P_{v,s} - P_{v,\infty}}{P}\right)} \tag{14.151}$$

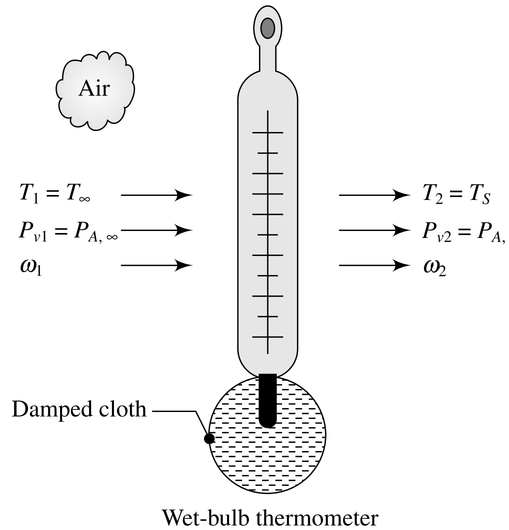


Figure 14.13 *Convective mass transfer in a wet-bulb thermometer*

Defining the specific humidity, ω (*also called absolute humidity or humidity ratio*) as

$$\omega = \frac{\text{Mass of water vapour, } m_v}{\text{Mass of dry air, } m_a}$$

According to Dalton's law of partial pressures, $P = P_a + P_v$
Assuming air and water vapour to be an ideal gas, we get

$$P_a \forall = m_v R_v T = m_v \frac{\bar{R}}{\tilde{M}_v} T \text{ and } P_a \forall = m_a R_a T = m_a \frac{\bar{R}}{\tilde{M}_a} T$$

Therefore, $$\omega = \frac{m_v}{m_a} = \frac{\tilde{M}_v}{\tilde{M}_a}\frac{P_v}{P_a}$$

Taking $T_{ab} = T_1$ and $T_{wb} = T_2$, $P_{v,s} = P_{v_2}$ and $P_{v,\infty} = P_{v_1}$,

$$\omega_1 = \frac{\tilde{M}_v}{\tilde{M}_a} \cdot \frac{P_{v1}}{P_{a1}} \approx \frac{\tilde{M}_v}{\tilde{M}_a} \cdot \frac{P_{v1}}{P} = \frac{\tilde{M}_v}{\tilde{M}_a} \cdot \frac{P_{v,\infty}}{P} \qquad \text{(since } P_a \approx P \text{ as } P_v << P_a\text{)}$$

and
$$\omega_2 \approx \frac{\tilde{M}_v}{\tilde{M}_a} \cdot \frac{P_{v2}}{P} = \frac{\tilde{M}_v}{\tilde{M}_a} \cdot \frac{P_{v,s}}{P}$$

Substituting these values in Eq. (14.149), we get

$$T_1 - T_2 = \frac{h_{fg}}{C_p L_e^{2/3}} \left[\frac{M_v}{M_a} - \left(\frac{P_{v,s}}{P} - \frac{P_{v,\infty}}{P} \right) \right] = \frac{h_{fg}}{C_p L_e^{2/3}} [\omega_1 - \omega_2]$$

$$T_1 - T_2 = \frac{h_{fg}}{C_p L_a^{2/3}}$$

$$\boxed{\frac{\omega_2 - \omega_1}{T_1 - T_2} = \frac{C_p}{h_{fg}} (Le)^{2/3}} \tag{14.152}$$

In this case, air is dry, i.e., relative humidity, $\phi_\infty = 0$.

As
$$\phi_\infty = \frac{P_{v,\infty}}{P_{\text{sat},\infty}} = 0, \quad P_{v,\infty} = 0$$

$\therefore$
$$T_{wb}(°\text{C}) = T_{db}(°\text{C}) - \left\{ \frac{h_{fg}}{C_p Le^{2/3}} \frac{\tilde{M}_v}{\tilde{M}} \frac{P_{v,s}}{P} \left| \frac{\text{kJ/kg}}{\text{kJ/kg°C}} \frac{\text{kg/kmol}}{\text{kg/kmol}} \times \frac{\text{kPa}}{\text{kPa}} \right| \right\}$$

or
$$\boxed{T_{wb} = T_{db} - \left(\frac{\tilde{M}_v}{\tilde{M} P} \right) \frac{(P_{v,s} \times h_{fg})_{@T_{wb}}}{(C_p Le^{2/3})_{@T_f}}} \tag{14.153}$$

Illustrative Examples

(A) Mixture Composition, Fick's Law, Diffusivity

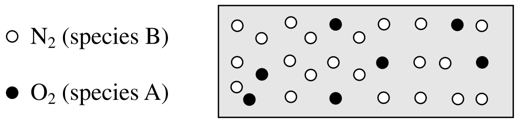
EXAMPLE 14.1 *A binary mixture of oxygen and nitrogen with their partial pressures in the ratio of 0.21 to 0.79 is contained in a vessel at 300 K. Determine (a) the molar concentration, (b) the mass density, (c) the mole fraction of each component for a total pressure of 1 atm. Also calculate (e) the average relative molecular mass (molecular weight) of the mixture.*

Solution

Known A mixture of O_2 and N_2 at the prescribed pressure and temperature, and the proportion of their pressures.

Find (a) Molar concentration, (b) Mass density, (c) Mole fraction, (d) Mass fraction, (e) Relative molecular mass (*average*).

Schematic



$(A + B)$ mixture
$P = 1$ atm
$T = 300$ K

$\frac{P_A}{P} = 0.21$

$\frac{P_a}{P} = 0.79$

Assumptions (1) Ideal-gas behaviour. (2) Constant total pressure and temperature.

Analysis (a) Molar concentration of a species or constituent, $C_i = P_i / \bar{R}\, T$.

where P_i is the partial pressure of a species or constituent, $\bar{R}$ is the universal gas constant, and T is the absolute temperature.

$$C_A = C_{O_2} = \frac{P_A}{\bar{R}T} = \frac{(P_A/P)P}{\bar{R}T} = \frac{0.21 \times 1 \text{ atm}}{8.205 \times 10^{-2} \text{ atm m}^3/\text{kmol K} \times 300 \text{ K}}$$

$$= \mathbf{0.008\ 53\ kmol/m^3} \qquad \textbf{(Ans.) (a)}$$

Else $$C_A = C_{O_2} = \frac{0.21 \times 101.325 \text{ kPa}}{8.3143 \text{ kJ/kmol K} \times 300 \text{ K}} \left| \frac{1 \text{ kJ}}{1 \text{ kPa m}^3} \right| = \mathbf{0.008\ 53\ kmol/m^3}$$

$$C_B = C_{N_2} = \frac{P_B}{\bar{R}T} = \frac{(P_B/P)P}{\bar{R}T} = \frac{0.79 \times 1 \text{ atm}}{8.205 \times 10^{-2} \text{ atm m}^3/\text{kmol K} \times 300 \text{ K}}$$

$$= \mathbf{0.032\ 09\ kmol/m^3} \qquad \textbf{(Ans.) (a)}$$

(b) Mass density can be obtained from, $\rho_i = \tilde{M}_i C_i$

Hence, $$\rho_A = \tilde{M}_A C_A = \tilde{M}_{O_2} C_{O_2} = \left(32 \frac{\text{kg}}{\text{kmol}}\right)\left(0.008\ 53 \frac{\text{kmol}}{\text{m}^3}\right) = \mathbf{0.273\ kg/m^3} \qquad \textbf{(Ans.) (b)}$$

and $$\rho_B = \tilde{M}_B C_B = \tilde{M}_{N_2} C_{N_2} = \left(28 \frac{\text{kg}}{\text{kmol}}\right)\left(0.032\ 09 \frac{\text{kmol}}{\text{m}^3}\right) = \mathbf{0.899\ kg/m^3} \qquad \textbf{(Ans.) (b)}$$

The total mass density of the mixture,

$$\rho = \Sigma \rho_i = \rho_A + \rho_B = 0.273 + 0.899 = 1.172 \text{ kg/m}^3$$

(c) Mole fraction of a component of an ideal-gas mixture is equal to its pressure fraction.

Hence, $y_i = \frac{P_i}{P}$

Then, $y_A = y_{O_2} = \mathbf{0.21}$, and $y_B = y_{N_2} = \mathbf{0.79}$ **(Ans.) (c)**

(d) Mass fraction of a species, $w_i = \frac{\rho_i}{\rho} = y_i \frac{\tilde{M}_i}{\tilde{M}}$

Therefore, $$w_A = w_{O_2} = \frac{\rho_A}{\rho} = \frac{0.273 \text{ kg/m}^3}{1.172 \text{ kg/m}^3} = \mathbf{0.233} \qquad \textbf{(Ans.) (d)}$$

$$w_B = w_{N_2} = \frac{\rho_B}{\rho} = \frac{0.899 \text{ kg/m}^3}{1.172 \text{ kg/m}^3} = \mathbf{0.767} \qquad \textbf{(Ans.) (d)}$$

Check: $\Sigma w_i = 1$ i.e. $w_A + w_B = 0.233 + 0.767 = 1.0$

(e) Average relative molecular mass or molecular weight of the mixture,

$$\tilde{M} = \Sigma y_i \tilde{M}_i = y_A \tilde{M}_A + y_B \tilde{M}_B$$

$= (0.21)\ (32\ \text{kg/kmol}) + (0.79)\ (28\ \text{kg/kmol}) = \mathbf{28.84\ kg/mol}$ **(Ans.) (e)**

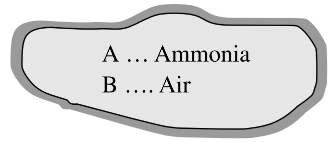
Comment Note that *mole fractions* are different from *mass fractions*.

EXAMPLE 14.2 *Estimate the mass diffusivity of ammonia in air (a) at 300 K and 1 atm and (b) at 350 K and 3 atm. Given:*

Species	Molecular weight	Molecular volume
Ammonia	17	25.8
Air	28.97	29.9

Solution

Schematic



Known Molecular weight and molecular volume of ammonia and air in the binary mixture.

Find Mass diffusivity, D_{AB} (a) at 1 atm, 300 K, and (b) at 3 atm, 350 K.

Assumption Ideal-gas behaviour.

Analysis (a) The binary diffusion coefficient (*or mass diffusivity*) is determined from

$$D_{AB}(\text{cm}^2/\text{s}) = 435.7 \frac{T^{3/2}}{P(\tilde{V}_A^{1/3} + \tilde{V}_B^{1/3})^2} \sqrt{\frac{1}{\tilde{M}_A} + \frac{1}{\tilde{M}_B}}$$

where T is absolute temperature in K, P is total system pressure in Pa and $\tilde{V}_A$ and $\tilde{V}_B$ are the molecular volumes of constituents A and B.

$$\therefore \qquad D_{AB} = \frac{435.7(300)^{3/2}}{(1.01325 \times 10^5)[25.8^{1/3} + 29.9^{1/3}]^2} \left[\frac{1}{17} + \frac{1}{28.97}\right]^{1/2}$$

$= \mathbf{0.186\ cm^2/s \quad or \quad 1.86 \times 10^{-5}\ m^2/s}$ **(Ans.) (a)**

(b) We note that $\dfrac{D_2}{D_1} = \left(\dfrac{T_2}{T_1}\right)^{3/2} \left(\dfrac{P_2}{P_1}\right)$

$$\therefore \qquad D_{AB}(3\ \text{atm},\ 350\ \text{K}) = D_{AB}(1\ \text{atm},\ 300\ \text{K}) \times \left(\frac{350\ \text{K}}{300\ \text{K}}\right)^{3/2} \times \frac{3\ \text{atm}}{1\ \text{atm}}$$

$= 0.186\ \text{cm}^2/\text{s} \times 1.26 \times 3 = \mathbf{0.703\ cm^2/s}$ **(Ans.) (b)**

(B) Stationary Media, Equimolar Counter-Diffusion, Evaporation in a Column

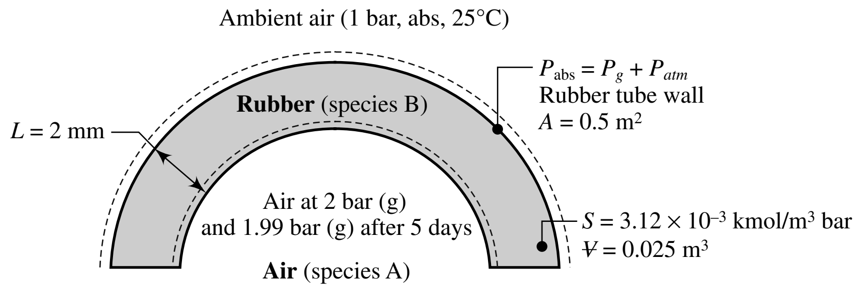
EXAMPLE 14.3 *The leakage of air from pneumatic tyres over a period of time is a common experience. The air pressure in a tyre reduces from 2 bar gauge to 1.99 bar gauge in five days. If the volume of air in the tube is 0.025 m^3, the surface area of the tube permitting diffusion is 0.5 m^2, and the wall thickness of the tube is 2 mm, estimate the mass diffusivity of air (Component A) in rubber (Component B). The ambient air temperature is 25°C. The solubility of air in rubber is 3.12 × 10^{-3} kmol/m^3 bar.*

Solution

Known Reduction in pressure in five days due to diffusion of air across the rubber-tube wall. Operating data are provided.

Find Mass diffusivity, D_{AB}(m^2/s).

Schematic



Assumptions (1) Steady-state, one-dimensional diffusion through a stationary medium. (2) The radius of curvature of tyre being large compared to its thickness, the tube wall can be approximated as a plane wall. (3) No chemical reactions.

Analysis Applying mass balance to a control volume: $\dot{M}_{A,\text{ stored}} = -\dot{M}_{A,\text{out}}$

or $$\frac{d}{dt}(\rho_A \forall) = -\dot{N}_A \tilde{M}_A \quad \text{or} \quad \frac{d}{dt}(C_A \tilde{M}_A \forall) = -\dot{N}_A \tilde{M}_A$$

or $$\frac{d}{dt}\left[\frac{P_A}{\overline{R}T}\forall\right] = -\dot{N}_A \quad \text{or} \quad \frac{dP_A}{dt} = -\frac{\dot{N}_A RT}{\forall} \quad \text{or} \quad -\frac{dP_A}{dt} = \frac{\dot{N}_A \overline{R}T}{\forall}$$

where $$-\frac{dP_A}{dt} = \frac{P_{A,\text{initial}} - P_{A,\text{final}}}{\Delta t} = \frac{(2.0 - 1.99)\,\text{bar}}{5 \times 24 \times 3600\,\text{s}} = 2.315 \times 10^{-8}\ \text{bar/s}$$

For diffusion of air through a stationary medium, $\dot{N}_A = AD_{AB}\dfrac{C_{A,1} - C_{A,2}}{L}$

where $$C_{A,1} = S \cdot P_{A_1} = \left(0.0312\frac{\text{kmol}}{\text{m}^3\,\text{bar}}\right)\left\{\left(\frac{2 + 1.99}{2}\right)\text{bar}(g) + 1\,\text{bar})\right\} = 0.009\ 34\ \text{kmol/m}^3$$

$$C_{A,2} = S \cdot P_{A_2} = (0.0312\ \text{kmol/m}^3\ \text{bar})\ (1\ \text{bar})$$

$$\frac{\dot{N}_A \overline{R}T}{\forall} = \frac{dP_A}{dt} = 2.315 \times 10^{-8}\ \text{bar/s}$$

∴ $$\dot{N}_A = \frac{(2.315 \times 10^{-8}\ \text{bar})(0.025\ \text{m}^3)}{(8.3143 \times 10^{-2}\ \text{bar m}^3/\text{kmol K})(298\ \text{K})} = 2.336 \times 10^{-11}\ \text{kmol/s}$$

Mass diffusivity,

$$D_{AB} = \frac{\dot{N}_A L}{A(C_{A,1} - C_{A,2})} = \frac{(2.336 \times 10^{-11}\ \text{kmol/s})(0.002\ \text{m})}{0.5\ \text{m}^2(0.00934 - 0.00312)\text{kmol/m}^3} = \mathbf{1.5 \times 10^{-11}\ m^2/s} \quad \textbf{(Ans.)}$$

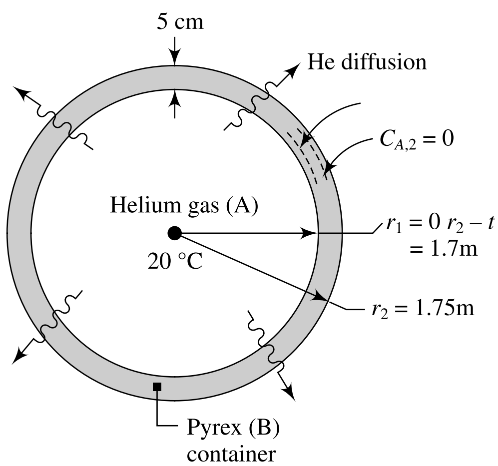
EXAMPLE 14.4 *Helium gas is stored at 20°C in a spherical container of 3.5 m outer diameter made of 5 cm thick Pyrex. The molar concentrations of helium in the Pyrex at the inner and outer surfaces are determined to be 0.000 75 and 0 kmol/m³, respectively. What is the mass flow rate of helium by diffusion through the Pyrex container? The binary diffusion coefficient of helium in Pyrex at the operation temperature is 4.5 × 10⁻¹⁵ m²/s.*

Solution

Known A spherical Pyrex container stores helium gas.

Find $\dot{m}_{\text{diff},A}$ or $\dot{m}_{He}(\text{kg}/s)$.

Schematic



Assumptions (1) Steady-state, one-dimensional (*radial*) diffusion through stationary container. (2) No chemical reaction in the Pyrex shell with no generation or depletion of helium.

Analysis The mass diffusion rate through a spherical shell is determined from

$$\dot{m}_{\text{diff},A} = \dot{m}_{He} = 4\pi r_1 r_2 \frac{(C_{A,1} - C_{A,2})\tilde{M}_{He}}{r_2 - r_1} D_{AB}$$

$$= \frac{4\pi(1.7\text{ m})(1.75\text{ m})}{0.05\text{ m}} \times 0.00075\frac{\text{kmol}}{\text{m}^3} \times 4.0\frac{\text{kg}}{\text{kmol}} \times 4.5 \times 10^{-15}\text{ m}^2/\text{s}$$

$= \mathbf{1.0 \times 10^{-14}}$ **kg/s** **(Ans.)**

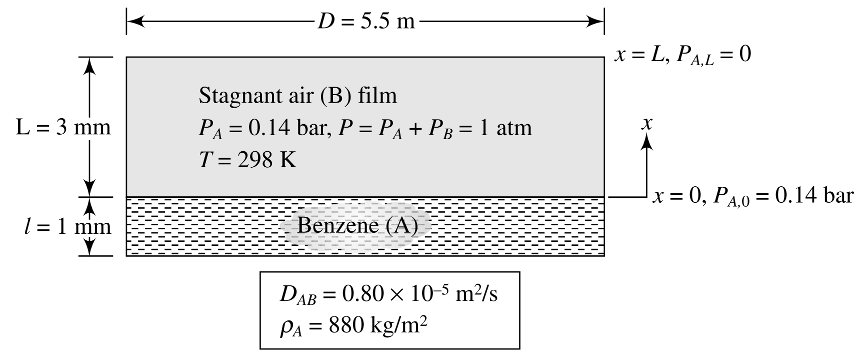
EXAMPLE 14.5 *An open tank of 5.5 m diameter, contains a 1 mm deep layer of benzene at 1 atm and 298 K, at its bottom. The vapour pressure of benzene in the tank is 0.14 bar and its diffusion takes place through a stagnant 3 mm thick air film. Determine the time taken for the entire benzene to evaporate, neglecting diffusive resistance of benzene beyond the air film. Take the binary diffusion coefficient,* D_{AB} *= 0.88 × 10⁻⁵ m²/s and the density of benzene, ρ = 880 kg/m³.*

Solution

Known A thin layer of benzene in a large tank exposed to stagnant air film.

Find Time taken for complete evaporation of benzene.

Schematic



Assumptions (1) Diffusion resistance of benzene beyond the air film is negligible. (2) Perfect gas behaviour (3) Steady-state, one-dimensional diffusion.

Analysis Total pressure, $P = P_A + P_B = 1$ atm $= 101.325$ kPa

Partial pressure of benzene vapour, $P_A = 0.14$ bar $= 14$ kPa

At $x = 0$ (surface of benzene layer): $P_{B,0} = \text{P} - P_{A,0} = 101.325 - 14.0 = 87.325$ kPa.

At $x = L$ (*stagnant air film thickness*): $P_{A,L} = 0$, $P_{B,L} = P = 101.325$ kPa

Molar diffusion rate, $\dfrac{\dot{N}_A}{A} = \dfrac{CD_{AB}}{L}\ln\dfrac{P_{B,L}}{P_{B,0}}$

where $$C = \frac{P}{\bar{R}T} = \frac{101.325\text{ kPa}}{8.3143\text{ kJ/kmol K} \times 298\text{ K}}\left|\frac{1\text{ kJ}}{1\text{ kPa m}^3}\right| = 0.0409\text{ kmol/m}^3$$

and $$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}(5.5\text{ m})^2 = 23.76\text{ m}^2$$

Hence, the mass diffusion rate, $\dot{m}_{\text{diff}} = \dot{N}_A\tilde{M}_A = \dfrac{CAD_{AB}\tilde{M}}{L}\ln\dfrac{P_{B,L}}{P_{B,0}}$

Molecular weight of benzene (C_6H_6) = (6 × 12) + (6 × 1) = 78 kg/kmol

$$\therefore \quad m_A = \frac{(0.0409\text{ kmol/m}^3)(23.76\text{ m}^2)(0.88\times10^{-5}\text{ m}^2/\text{s})(78\text{ kg/kmol})}{0.003\text{ m}}\ln\frac{101.325}{87.325}$$

$$= 0.0331\text{ kg/s}$$

Mass of benzene contained in the tank.

$$m_A = \text{Density} \times \text{Volume} = \rho A_c l = 880\text{ kg/m}^3 \times \frac{\pi}{4}(5.5\text{ m})^2 \times (1\times10^{-3}\text{ m}) = 20.91\text{ kg}$$

$\therefore$ Time required for the entire benzene to evaporate is

$$t = \frac{m_A}{\dot{m}_A} = \frac{20.9\text{ kg}}{0.0331\text{ kg/s}} = \mathbf{631.6\ s\ or\ 10.52\ min}$$ **(Ans.)**

(C) Transient Diffusion

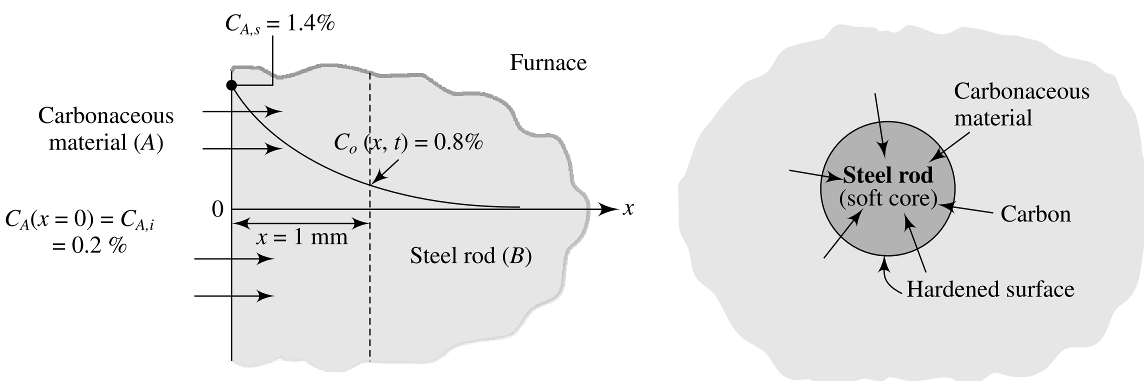
EXAMPLE 14.6 *A steel rod (0.2% carbon) is preheated to 900°C, and is packed in a carburizing mixture at 900°C. The concentration of carbon at the surface of the rod is maintained at 1.4% by the carburizing mixture. Calculate the time required for the percentage of carbon to be at least 0.8 at a depth of 1 mm. The value of the diffusion coefficient* $D_{AB} = 5.8 \times 10^{-10}$ *m*^{2}*/s.*

Solution

Known Case hardening of steel at the prescribed temperature.

Find Time required for the prescribed penetration depth.

Schematic



Assumptions (1) Diffusion coefficient is constant. (2) There are no homogeneous reactions. (3) Initially, the concentration of the species A is constant throughout the medium. (4) The steel rod is modelled as a semi-infinite medium.

Analysis For transient mass diffusion, $\dfrac{C_{A(x,t)} - C_{A,i}}{C_{A,s} - C_{A,i}} = \text{erfc}\left(\dfrac{x}{2\sqrt{D_{AB}t}}\right)$

where $C_{A,s} = 1.4\%$, $C_A(x, t) = 0.8\%$, $C_{A,i} = 0.2\%$

$x = 1$ mm, $D_{AB} = 5.8 \times 10^{-10}$ m^2/s

Hence, substituting the information above, we get

$$\frac{0.8 - 0.2}{1.4 - 0.2} = \text{erfc}\left[\frac{0.001}{2\sqrt{5.8 \times 10^{-10} \times t \times 3600}}\right]$$

where t is in h. or $\text{erfc}\left[\dfrac{0.346}{\sqrt{t}}\right] = 0.5$

From complementary error function tables (chapter 6), we note that for *erfc* (z) = 0.5, z = 0.477

Therefore, the time needed for the required penetration depth of 1 mm is

$$t = \left[\frac{0.346}{0.477}\right]^2 = \mathbf{0.526\ h} \qquad \textbf{(Ans.)}$$

(D) Convective Mass Transfer

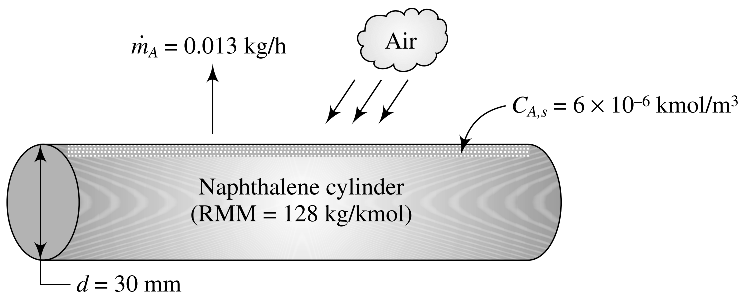
EXAMPLE 14.7 *A solid naphthalene cylinder of 30 mm diameter is exposed to an air stream with a mass sublimation rate of 0.013 kg/h per metre length and the saturated vapour concentration of naphthalene is 6×10^{-6} kmol/m^3, Find the convective mass-transfer coefficient. The relative molecular mass of naphthalene is 128 kg/kmol.*

Solution

Known Sublimation of naphthalene as air flows over it.

Find Mass-transfer coefficient, h_m (m/s).

Schematic



Assumptions (1) Steady-state conditions. (2) Naphthalene concentration in air is negligible.

Analysis Convection mass-transfer rate,

$$\dot{m}_A = h_m A_s(\rho_{A,s} - \rho_{A,\infty}) = h_m(\pi dl)(C_{A,s} - \overset{0}{\cancel{C_{A,\infty}}})\tilde{M}$$

or $$\dot{m}_A = h_m(\pi dl)(C_{A,s})(\tilde{M}_A)$$

Mass-transfer coefficient,

$$h_m = \frac{\dot{m}_A}{(\pi dl)(C_{A,s})(\tilde{M}_A)} = \frac{(0.013/3600)(\text{kg/s m})}{(\pi \times 30 \times 10^{-3}\text{ m} \times 1\text{ m})(6 \times 10^{-6}\text{ kmol/m}^3)(128\text{ kg/k mol})}$$

$$= \mathbf{0.05\ m/s}$$ **(Ans.)**

EXAMPLE 14.8 *An earthen pitcher (a 'matka') containing drinking water is wrapped with a moist cloth (to be maintained wet continually) and is kept in gentle breeze by a housewife on a typical day when the environment conditions are 1 atm, 31°C and 52% relative humidity. Estimate the temperature of the water when equilibrium conditions are attainted.*

Properties: Air (1 atm, 300 K): $C_p = 1.007$ kJ/kg °C, $\alpha = 22.5 \times 10^{-6}$ m^2/s

Saturated water vapour: (At an assumed $T_s = 23$°C): $P_g = 2.81$ kPa, $h_{fg} = 2447.0$ kJ/kg

$P_{g@31°C} = 4.496$ kPa. $D_{AB@300K} = 26 \times 10^{-6}$ m^2/s.

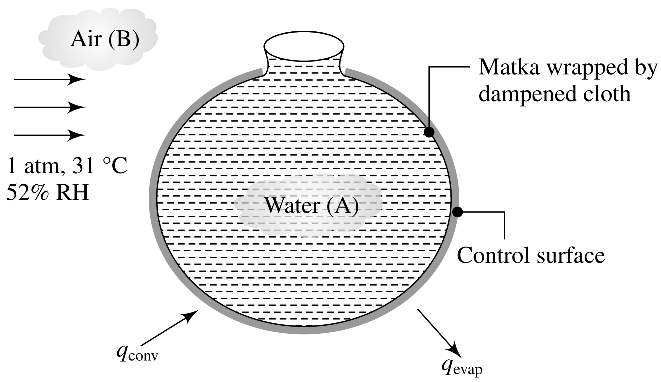
The molecular weights of air and water vapour are 28.966 and 18.016 kg/kmol.

Solution

Known Water in a *'matka'* being cooled in gentle breeze by evaporative cooling process under specified conditions.

Find Steady-state temperature of water in the vessel.

Schematic



Assumptions (1) Steady operating conditions. (2) Heat and mass transfer analogy is applicable. (3) Radiation effects are negligible. (4) Air and water vapour behave as ideal gases.

Analysis T_s = Temperature of saturated water vapour at control surface.
T_∞ = Temperature of water vapour far away from control surface.
Under equilibrium conditions:

$$\begin{pmatrix}\text{Heat gained by water by}\\ \text{convection heat transfer}\end{pmatrix} = \begin{pmatrix}\text{Heat lost by water by convection}\\ \text{mass transfer } (\textit{evaporation})\end{pmatrix}$$

$$\dot{Q}_{\text{conv, in}} = \dot{Q}_{\text{evap, out}} \quad \text{or} \quad h_A(T_\infty - T_s) = \dot{m}_v h_{fg} = h_m A(\rho_{v,s} - \rho_{v,\infty})h_{fg}$$

or
$$T_s = T_\infty - \frac{h_m}{h}[\rho_{v,s} - \rho_{v,\infty}]h_{fg} \tag{A}$$

Using Chilton–Colburn analogy, $h = (\rho C_p Le^{2/3})h_m$

Assuming water vapour as an ideal gas, $P = \rho RT$ or $P = \rho\left(\dfrac{\bar{R}}{M}\right)T$

$$\therefore \qquad \rho_{v,s} = \frac{P_{v,s}\tilde{M}_v}{\bar{R}T_s} \text{ and } \rho_{v,\infty} = \frac{P_{v,\infty}\tilde{M}_v}{\bar{R}T_\infty}$$

Substituting these values in Eq. (A):

$$T_s = T_\infty - \frac{1}{\rho C_p Le^{2/3}}\frac{\tilde{M}_v}{\bar{R}}\left[\frac{P_{v,s}}{T_s} - \frac{P_{v,\infty}}{T_\infty}\right]h_{fg}$$

Taking $\dfrac{1}{T_s} \approx \dfrac{1}{T_\infty} \approx \dfrac{1}{T}$ where $T = \dfrac{1}{2}(T_s + T_\infty)$ and $P = \rho\dfrac{\bar{R}}{\tilde{M}_a}T$ where P is the total pressure and $\tilde{M}_a$ is the molecular weight of air, we have

$$T_s = T_\infty - \frac{h_{fg}}{C_p Le^{2/3}} \frac{\tilde{M}_v}{\rho \bar{R} T}[P_{v,s} - P_{v,\infty}] = T_\infty - \frac{h_{fg}}{C_p Le^{2/3}} \frac{\tilde{M}_v}{P \tilde{M}_a}(P_{v,s} - P_{v,\infty})$$

With $\dfrac{\tilde{M}_v}{\tilde{M}_a} = \dfrac{18.016}{28.966} = 0.622,$ we have

$$\boxed{\boldsymbol{T_s = T_\infty - \frac{0.622\, h_{fg}}{C_p (Le)^{2/3}} \left[\frac{P_{g@T_s} - \phi P_{g@T_\infty}}{P}\right]}}$$

Lewis number, $Le = \dfrac{\alpha}{D_{AB}} = \dfrac{22.5 \times 10^{-6}\ \text{m}^2/\text{s}}{26 \times 10^{-6}\ \text{m}^2/\text{s}} = 0.8654$

Let us assume T_s = 23°C.

With $P_{g@T_s = 23°\text{C}} = 2.81\ \text{kPa},\ P_{g@T_\infty = 31°\text{C}} = 4.496\ \text{kPa}$

P = 101.3 kPa, h_{fg} = kJ/kg, C_p = 1.007 kJ/kg °C

ϕ = 0.52 and Le = 0.8654, we get

$$T_s = 31°\text{C} - \frac{0.622 \times 2447\ \text{kJ/kg}}{1.007\ \text{kJ/kg°C} \times (0.8654)^{2/3}} \left\{\frac{[2.81 - (0.52)(4.496)]\,\text{kPa}}{101.325\ \text{kPa}}\right\}$$

= **23.2°C** **(Ans.)**

As the assumed value of 23°C is quite close to the calculated value, the steady-state temperature of water in the *'matka'* will be

T_s = **23.2°C** **(Ans.)**

Comment A drop in temperature of water of about 8°C results *without* refrigeration due to evaporative cooling. The steady-state temperature reached is commonly called *wet-bulb temperature* and is the minimum temperature to which water can be cooled without mechanical refrigeration. In dry weather, when the relative humidity is very low, the cooling effect is more pronounced. For example, had there been 20% RH in 31°C air, the minimum water temperature reached (T_S) would be **16°C.**

EXAMPLE 14.9 *A solar pond 20 m × 20 m and 2-m deep is maintained isothermally at 30°C at a location where the atmospheric pressure is 1 atm. The ambient air conditions are 20°C and 65% relative humidity. Determine the rate of heat loss from the top surface of the pond by (a) radiation, (b) free convection, and (c) evaporation. Take the average temperature of the surrounding surfaces to be 15°C and the emissivity of liquid water is 0.95. The following properties may be used:*

Air: $\left(1\ atm,\ \text{T} = \dfrac{1}{2}(\text{T}_\infty + \text{T}_s) = \dfrac{20+30}{2} = 25°C\right)$

k = 0.02551 W/m K, *Pr = 0.7296* *α = 21.41 mm²/sν = 15.62 mm²/s*

Saturated water vapour (1 atm, 30°C)

h_{fg} *= 2430.5 kJ/kg* *ρ = 0.0304 m³/kg* P_g *= 4.246 kPa*

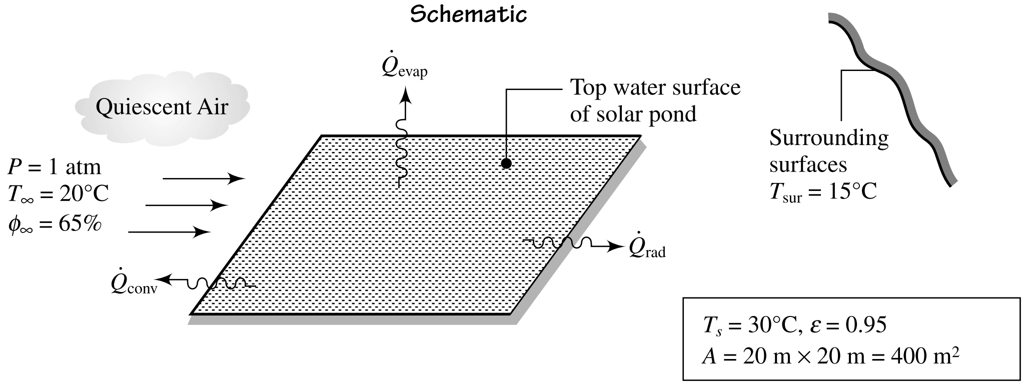
Air water vapour (25°C): $D_{AB} = 0.256 \times 10^{-4}\ m^2/s$

Solution

Known A large solar pond is exposed to still air and surrounding surfaces.

Find Heat loss by (a) radiation, (b) free convection, and (c) evaporation.

Schematic



Assumptions (1) Steady operating conditions. (2) Heat and mass transfer analogy is relevant because low mass flux conditions exist. (3) Water vapour and air are both ideal gases.

Analysis (a) The rate of heat loss by radiation from the water to the surrounding surfaces is determined from

$$\dot{Q}_{\text{rad}} = A\sigma\varepsilon[T_s^4 - T_{\text{sur}}^4] = (400 \text{ m}^2)(5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4)(0.95)[303.15^4 - 288.15^4]\text{K}^4$$

$= \mathbf{33430\ W}$ **(Ans.) (a)**

(b) The rate of heat loss by free (natural) convection is obtained from

$$\dot{Q}_{\text{conv}} = hA(T_s - T_\infty)$$

Let us first determine the Rayleigh number, Ra_L based on the characteristic length, L defined for a horizontal surface as

$$L \equiv \frac{A}{P} = \frac{20 \times 20 \text{ m}^2}{2[20 + 20]\text{m}} = 5 \text{ m}$$

Now, $Ra_L = Gr_L Pr$

where $Gr_L = \dfrac{g(\rho_\infty - \rho_s)L^3}{\rho v^2}$

Partial pressure of water vapour, $P_{v,\infty} = \phi_\infty P_g = 0.65 \times 4.246 \text{ kPa} = 2.76 \text{ kPa}$

At the pond's top surface: $P_{v,\text{S}} = P_{v,\text{sat}} = P_{g\ @\ 30°\text{C}} = 4.246 \text{ kPa}$

$$\rho_{v,s} = \frac{P_{v,s}}{R_v T_s} = \frac{4.246 \text{ kPa}}{(0.4615 \text{ kJ/kg K})(303.15 \text{ K})} = 0.0304 \text{ kg/m}^3$$

$$\rho_{a,s} = \frac{P_{a,s}}{R_a T_s} = \frac{(101.325 - 4.246)\text{kPa}}{(0.287 \text{ kJ/kg K})(303.15 \text{ K})} = 1.116 \text{ kg/m}^3$$

$\therefore$ $$\rho_s = \rho_{v,s} + \rho_{a,s} = 0.0304 + 1.116 = 1.146 \text{ kg/m}^3$$

and

Far away from the pond surface:

$$\rho_{v,\infty} = \frac{P_{v,\infty}}{R_v T_\infty} = \frac{2.76 \text{ kPa}}{(0.4615 \text{ kJ/kg K})(293.15 \text{ K})} = 0.0204 \text{ kg/m}^3$$

$$\rho_{a,\infty} = \frac{P_{a,\infty}}{R_a T_\infty} = \frac{(101.325 - 2.76)\text{kPa}}{(0.287 \text{ kJ/kg K})(293.15 \text{ K})} = 1.1715 \text{ kg/m}^3$$

$\therefore$ $$\rho_\infty = \rho_{v,\infty} + \rho_{a,\infty} = 0.0204 + 1.1715 = 1.192 \text{ kg/m}^3$$

$$\rho = \frac{1}{2}(\rho_s + \rho_\infty) = \frac{1}{2}(1.146 + 1.192) = 1.169 \text{ kg/m}^3$$

Substituting these values,

$$Gr_L = g(\rho_\infty - \rho_s)L^3/\rho v^2 = \frac{(9.81 \text{ m/s}^2)(1.192 - 1.146)\text{kg/m}^3 (5 \text{ m})^3}{(1.169 \text{ kg/m}^3)(15.62 \times 10^{-6} \text{ m}^2/\text{s})^2} = 1.98 \times 10^{11}$$

and $Ra_L = Gr_L Pr = (1.98 \times 10^{11})(0.7296) = 1.44 \times 10^{11}$

This is a case of hot horizontal surface facing up with $Ra_L > 10^7$, and the correlation to be used is

$$Nu_L = \frac{hL}{k} = 0.15(Ra_L)^{1/3}$$

and $$h_{\text{conv}} = \frac{0.15 \times 0.02551 \text{ W/m K}}{5 \text{ m}}(1.44 \times 10^{11})^{1/3} = 4.014 \text{ W/m}^2\text{K}$$

$\therefore$ $$\dot{Q}_{\text{conv}} = h_{\text{conv}} A(T_s - T_\infty)$$

$= (4.014 \text{ W/m}^2\text{K})(400 \text{ m}^2)(30 - 20)$°C or K **= 16 056 W** **(Ans.) (b)**

Using the analogy between heat and mass convection, $Sh = 0.15\ (\text{GrSc})^{1/3}$

where $$Sc = \frac{v}{D_{AB}} = \frac{15.62 \times 10^{-6} \text{ m}^2/\text{s}}{0.256 \times 10^{-4} \text{ m}^2/\text{s}} = 0.61$$

or $$\frac{h_m L}{D_{AB}} = 0.15(1.98 \times 10^{11} \times 0.61)^{1/3} = 741.5$$

$\therefore$ $h_m = (741.5)(0.256 \times 10^{-4} \text{ m}^2/\text{s})/5 \text{ m} = 0.0038 \text{ m/s}$

Now, $\dot{m}_v = h_m A(\rho_{v,s} - \rho_{v,\infty}) = (0.0038 \text{ m/s})(400 \text{ m}^2)(0.0304 - 0.0204) \text{ kg/m}^3 = 0.0151 \text{ kg/s}$

Hence, $\dot{Q}_{\text{evap}} = \dot{m}_v h_{fg} = (0.0151 \text{ kg/s})(2430.5 \times 10^3 \text{ J/Kg})$ **= 36 700 W** **(Ans.) (c)**

Total heat loss from the surface of the pond is

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{rad}} + \dot{Q}_{\text{conv}} + \dot{Q}_{\text{evap}} = 33430 + 16\ 056 + 36\ 700 = 86\ 186 \text{ W} = 86.2 \text{ kW}$$

EXAMPLE 14.10 *A dry-and wet-bulb thermometer records the following readings: Dry-bulb temperature (dbt) = 44°C. Wet-bulb temperature (wbt) = 28°C Determine the Schmidt number and Lewis number as also the specific humidities of the air far away from the wetted surface and near the wetted surface of the thermometer bulb. Hence, determine the relative humidity of air. Using the Carrier's equation given below, calculate the relative humidity for the sake of comparison.* $\text{P}_v = \text{P}_g^* - \dfrac{(\text{P} - \text{P}_g^*)(\text{T} - \text{T}^*)}{1810 - \text{T}^*(\text{K})}$

where T *and* T* *are dbt and wbt in kelvin.* P_g^* *is the saturated vapour pressure at* T**. Use the following data:* $D_{AB} = 2.7 \times 10^{-5}\ m^2/s$

Air (1 atm, 36 °C): $\rho = 1.142$ *kg/m3,* $C_p = 1.0063$ *kJ/kg °C,*
$\nu = 16.59 \times 10^{-6}\ m^2/s$, $\alpha = 23.54 \times 10^{-6}\ m^2/s$

Saturated water vapour: $P_g = P_{sat@44°C} = 9.111\ kPa$, $\rho_g = \dfrac{1}{16.02}\dfrac{\text{kg}}{\text{m}^3}$

$P_g^* = P_{sat@28°C} = 3.782\ kPa$, $\rho_g^* = \dfrac{1}{36.69}\dfrac{kg}{m^3}$

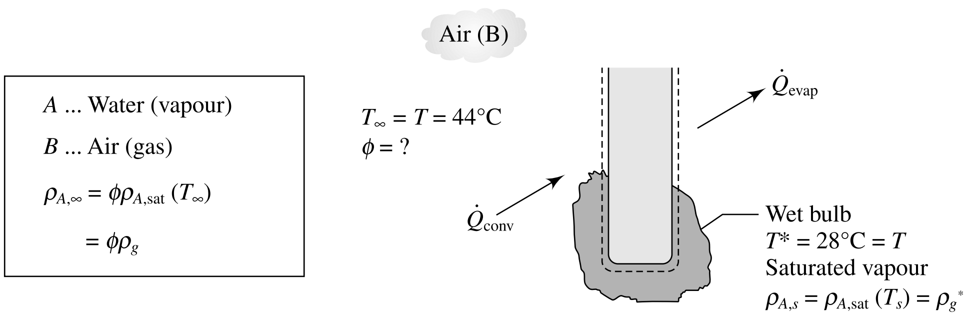
$h_{fg@28°C} = 2435.2\ kJ/kg$ ***[VTU, 2000]***

Solution

Known Dry-bulb and wet-bulb temperatures of atmospheric air.

Find Relative humidity, Sc, Le, ω, and ω^*.

Schematic



Assumptions (1) Steady operating conditions. (2) Negligible radiation and conduction along the thermometer. (3) Water vapour has ideal-gas behaviour. (4) Heat- and mass-transfer analogy is applicable.

Analysis Schmidt number, $Sc = \dfrac{\nu}{D_{AB}} = \dfrac{16.59 \times 10^{-6}\ \text{m}^2/\text{s}}{2.7 \times 10^{-5}\ \text{m}^2/\text{s}} = \mathbf{0.614}$ **(Ans.)**

Lewis number, $Le = \dfrac{\alpha}{D_{AB}} = \dfrac{23.54 \times 10^{-6}\ \text{m}^2/\text{s}}{2.7 \times 10^{-5}\ \text{m}^2/\text{s}} = \mathbf{0.872}$ **(Ans.)**

Incidentally, $\dfrac{Sc}{Le} = \dfrac{0.614}{0.872} = 0.704$

Applying energy balance on a thin layer of water at the wet-bulb surface,

$$\dot{Q}_{\text{sensible, transferred}} = \dot{Q}_{\text{latent, absorbed}}$$

or $$\dot{Q}_{\text{conv}} = \dot{m}_A h_{fg} \quad \text{or} \quad hA(T_\infty - T_s) = h_m A(\rho_{A,s} - \rho_{A,\infty})h_{fg}$$

or $$\frac{h}{h_m}\frac{(T_\infty - T_s)}{h_{fg}} = \rho_{A,\text{sat}}(T_s) - \phi\rho_{A,\text{sat}}(T_\infty) = \rho_g^* - \phi\rho_g$$

Relative humidity, $\phi = \left[\rho_g^* - \frac{(h/h_m)(T_\infty - T_s)}{h_{fg}}\right]\frac{1}{\rho_g}$

But using the analogy between heat and mass transfer,

$$\frac{h}{h_m} = \rho C_p (Le)^{2/3} = (1.142 \text{ kg/m}^3)(1006.3 \text{ J/kg°C})(0.872)^{2/3}$$

$$= 1048.9 \frac{\text{W/m}^2\,°\text{C}}{\text{m/s}} \text{ or } \frac{J}{\text{m}^3\,°\text{C}}$$

$$\therefore \qquad \phi = \frac{\rho_g^*}{\rho_g} - \frac{(h/h_m)(T_\infty - T_s)}{\rho_g h_{fg}}$$

$$= \frac{16.02 \text{ kg/m}^3}{36.69 \text{ kg/m}^3} - \frac{16.02 \text{ m}^3/\text{kg} \times 1048.9 \text{ J/m}^3\,°\text{C} \times (44-28)°\text{C}}{2435.2 \times 10^3 \text{ J/kg}}$$

$= 0.4366 - 0.1104 =$ **0.326 or 32.6%** **(Ans.)**

Specific humidities: $\omega = \omega_\infty = \frac{m_v(T_\infty)}{m_a} = \frac{P_v \forall_v}{R_v T_v} \times \frac{R_a T_a}{P_a \forall_a}$

As $\forall$ and T are same for air and water vapour,

$$\omega = \frac{P_v}{P_a}\frac{R_a}{R_v} = \frac{P_v}{P - P_v}\frac{0.287 \text{ kJ/kg K}}{0.4615 \text{ kJ/kg K}}$$

$$= \frac{0.622 P_{A,\infty}}{P - P_{A,\infty}} = \frac{0.622 \times \phi P_g}{P - \phi P_g} = \frac{0.622 \times 0.326 \times 9.111}{101.325 - (0.326 \times 9.111)}$$

$=$ **0.0576 kg H_2O/kg dry air** **(Ans.)**

and $$\omega_s = \omega^* = 0.622 \frac{P_g^*}{P - P_g^*} = \frac{0.622 \times 3.782}{101.325 - 3.782}$$

$=$ **0.0241 kg H_2O/kg dry air** **(Ans.)**

For comparison, using the Carrier equation,

$$P_v = 3.782 - \frac{(101.325 - 3.782)(44 - 28)}{1810 - (28 + 273.15)} = 2.7476 \text{ kPa}$$

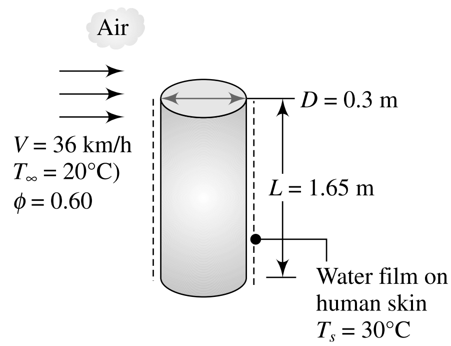
$$\therefore \qquad \phi = \frac{P_v}{P_g} = \frac{2.7476 \text{ kPa}}{9.111 \text{ kPa}} = 0.302 \quad \text{or} \quad 30.2\%$$

The two values of ϕ are quite comparable.

Comment Latent heat of evaporation, h_{fg} is always evaluated at the surface temperature, i.e., wet-bulb temperature, because vaporization takes place at the wetted surface.

EXAMPLE 14.11 *A teenage boy in swimming trunks steps out of a swimming pool on a cold windy day. Approximating the teenager as a circular cylinder of 0.3 m diameter and 1.65 m long, the* *average skin surface temperature is 30°C. The wind speed is 36 km/h and the atmospheric air conditions are 20°C, 60 % relative humidity. Determine the rate of total heat loss from the drenched skin to air, neglecting radiation effects.*

Schematic



Solution

Known A boy with wet skin, idealized as a cylinder, loses heat under prescribed conditions.

Find Total heat loss by forced convection and convective mass transfer.

Assumptions (1) Water vapour is an ideal gas. (2) Unclothed body, i.e., direct exposure of skin to air. (3) Heat- and mass-transfer analogy is relevant.

Properties Air (1 atm, $T_\infty = 20°C$):

$k = 0.02514$ W/m K, $\nu = 15.16 \times 10^{-6} \text{m}^2/\text{s}$

$Pr = 0.7309$, $Pr(T_s = 30°C) = 0.7282$

***Saturated water vapour*:** ($T_\infty = 20°C$): $\rho_g = 0.0173 \text{ kg/m}^3$

($T_\infty = 30°C$): $\rho_g = 0.0304 \text{ kg/m}^3$, $h_{fg} = 2431$ kJ/kg

***Water vapour-air*:** $D_{AB} = 26 \times 10^{-6} \text{ m}^2/\text{s}$, $Sc = \dfrac{\nu}{D_{AB}} = 0.583$

Analysis: ***(a) Forced convection heat transfer:*** $\dot{Q}_{\text{conv}} = \bar{h}(\pi DL)(T_s - T_\infty)$

Reynolds number, $Re_D = \dfrac{VD}{\nu} = \left(\dfrac{36 \text{ m}}{3.6 \text{ s}}\right)(0.3 \text{ m})\left(\dfrac{10^6}{15.16 \text{ m}^2/\text{s}}\right) = 1.98 \times 10^{-5}$

Nusselt number, $\overline{Nu}_D = 0.26(Re_D)^{0.6}(Pr)^{0.37}\left(\dfrac{\text{Pr}}{Pr_s}\right)^{0.25}$

$$= 0.26(1.98 \times 10^5)^{0.6}(0.7309)^{0.37}\left(\frac{0.7309}{0.7282}\right)^{0.25} = 349$$

Therefore, $\bar{h} = \dfrac{k}{D} Nu = 349 \times \dfrac{0.02514 \text{ W/m K}}{0.3 \text{ m}} = 29.26 \text{ W/m}^2\text{K}$

$$\therefore \qquad \dot{Q}_{\text{conv}} = \left(29.26 \frac{W}{\text{m}^2\text{K}}\right)(\pi \times 0.3 \text{ m} \times 1.65 \text{ m})(30 - 20)°\text{C or K} = 455 \text{ W}$$

Convective mass transfer: $\dot{Q}_{\text{evap}} = \dot{m}_{\text{evap}} h_{fg}$

where $\dot{m}_{\text{evap}} = \bar{h}_m(\pi DL)[\rho_{v,s} - \rho_{v,\infty}]$

$\rho_{v,s} = \rho_{v,\text{sat}}$ at $(T_s = 30°C) = 0.0304 \text{ kg/m}^3$

$\rho_{v,\infty} = \phi\rho_{v,\text{sat}}$ at $(T_s = 20°C) = 0.6 \times 0.0173 \text{ kg/m}^3 = 0.01038 \text{ kg/m}^3$

To find $\bar{h}_m$, the analogy between heat and mass transfer can be used. Neglecting the *Pr* ratio term, the analogous form is

$$\overline{Sh}_D = 0.26\ (Re_D)^{0.6}(Sc)^{0.37} = 0.26\ (1.98 \times 10^5)^{0.6}(0.,583)^{0.37} = 320.8$$

Therefore, $\bar{h}_m = \overline{Sh}_D \dfrac{D_{AB}}{D} = 320.8 \times \dfrac{26 \times 10^{-6}\ \text{m}^2/\text{s}}{0.3\ \text{m}} = 0.0278\ \text{m/s}$

The rate of evaporation is

$$\dot{m}_{\text{evap}} = (0.0278\ \text{m/s})\ (\pi \times 0.3\ \text{m} \times 1.65\ \text{m})\ (0.0304 - 0.01038)\ \text{kg/m}^3$$
$$= 8.655 \times 10^{-4}\ \text{kg/s}$$

Hence, $\dot{Q}_{\text{evap}} = (8.655 \times 10^{-4}\ \text{kg/s})\ (2431 \times 10^3\ \text{J/kg}) = 2104\ \text{W}$

The rate of total heat loss is

$$\dot{Q} = \dot{Q}_{\text{conv}} + \dot{Q}_{\text{evap}} = 455\ \text{W} + 2104\ \text{W} = \mathbf{2559\ W \quad or \quad 2.56\ kW} \qquad \textbf{(Ans)}$$

Comment The total heat transfer comprises two contributions, viz., sensible (*heat transfer*) and latent (*mass transfer*).

Heat loss due to moisture transfer or evaporation far outweighs the heat loss due to forced convection. This is only to be expected when wet human skin is in direct contact with cold air.

EXAMPLE 14.12 *Benzene has been spilled on the floor and has spread to a length of 2.5 m. If a 1 mm deep film is formed, find the time taken for the benzene to evaporate completely. Air flows parallel to the surface at 1.2 m/s. Benzene and air are both at 25°C. The densities of benzene in the saturated vapour and liquid states are ρ_v = 0.417 and ρ_l = 900 kg/m^3*

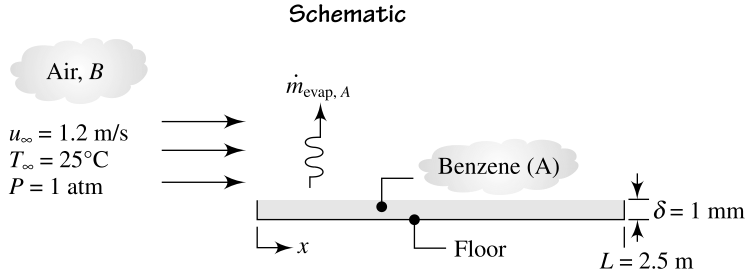
Properties of air at 1 atm, 25°C: DAB = 8.8 × 10^{-6} m^2/s.

Solution

Known Air flows parallel to and over a thin benzene layer.

Find Time required for complete evaporation.

Schematic



Assumptions (1) Steady operating conditions. (2) Negligible turbulence and smooth liquid surface. (3) Heat- and mass-transfer analogy is applicable. (4) Negligible benzene vapour concentration in free steam air. (5) Isothermal conditions prevail at 25°C.

Analysis Reynolds number,

$$Re_L = \frac{u_\infty L}{\nu} = \frac{1.2\ \text{m/s} \times 2.0\ \text{m}}{15.62 \times 10^{-6}\ \text{m}^2/\text{s}} = 1.5365 \times 10^5 \quad (< 5 \times 10^5) \qquad \Rightarrow \qquad \textit{Laminar flow.}$$

The appropriate correlation is, $Sh_L = 0.664(Re_L)^{1/2}(Sc)^{1/3} = \dfrac{h_m L}{D_{AB}}$

where $$Sc = \frac{\nu}{D_{AB}} = \frac{15.62\times10^{-6}\ \text{m}^2/\text{s}}{8.8\times10^{-6}\ \text{m}^2/\text{s}} = 1.775$$

Convection mass-transfer coefficient is,

$$h_m = 0.664 \times \frac{D_{AB}}{L} \times Re_L^{1/2} \times Sc^{1/3}$$

$$= 0.664 \times \frac{8.8\times10^{-6}\ \text{m}^2/\text{s}}{2.5\ \text{m}} \times (1.5365\times10^5)^{1/2} \times (1.775)^{1/3} = 1.11\times10^{-3}\ \text{m/s}$$

Mass, $m = \rho_l \forall = \rho_l A_s \delta$

Rate of mass diffusion of benzene in air, $\dot{m}_{\text{evap}} = -\dfrac{dm}{dt} = -\dfrac{d}{dt}(A_s\delta)\rho_l$

After time t, benzene will be completely evaporated and the layer thickness δ will become zero.

It follows that

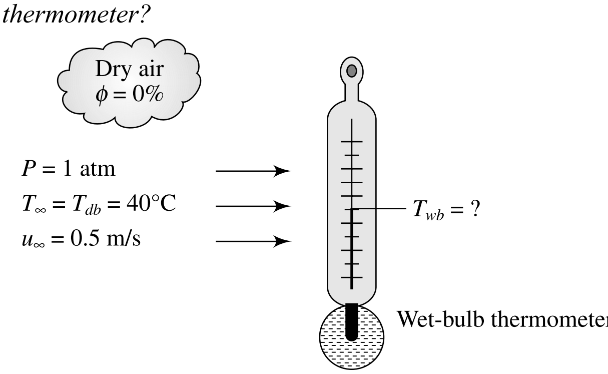
$$\dot{m}_{\text{evap}} = h_m A_s(\rho_{A,\text{sat}} - \underbrace{\rho_{A,\infty}}_{\substack{\text{negligible mass fraction}\\ \text{of vapour in air}}}) = -\frac{d\delta}{dt} A_s \rho_l$$

or $$h_m \rho_{A,\text{sat}} \int_0^t dt = -\rho_l \int_\delta^0 d\delta$$

Time required, $t = \dfrac{\rho_l \delta}{h_m \rho_{A,\text{sat}}} = \dfrac{900\ \text{kg/m}^3 \times 1\times10^{-3}\ \text{m}}{1.11\times10^{-3}\ \text{m/s}\times0.417\ \text{kg/m}^3}$ **= 1944 s = 32.4 min (Ans.)**

EXAMPLE 14.13 *Dry air at 1 atm and 40°C flows with a velocity of 0.5 m/s across a wet-bulb thermometer. What would be the reading on the thermometer?*

Solution



Known Dry air is blown over the wet bulb of a thermometer.

Find Wet-bulb temperature, T_{wb} (°C).

Assumptions (1) Air and water vapour are ideal gases. (2) Steady-state conditions.

Analysis Under steady operating conditions, we have

$$T_{wb} = T_{db} - \left(\frac{\tilde{M}_v}{\tilde{M}P}\right)\frac{(P_{v,s}\times h_{fg})_{@T_{wb}}}{(C_p Le^{2/3})_{@T_f}}$$

or $$T_{wb} = 40°\text{C} - \left(\frac{18.016}{28.97\times101.325}\right)\frac{(P_{\text{sat}}\, h_{fg})_{@T_{wb}}}{(C_p Le^{2/3})_{@T_f}}$$

As $\quad C_p = 1.007$ kJ/kg°C from 15°C to 70°C,

$$T_{wb} = 40 - \left(\frac{18}{28.97 \times 101.325 \times 1.007}\right) \times \frac{(P_{\text{sat}} h_{fg})_{@T_{wb}}}{(C_p Le^{2/3})_{@T_f}}$$

or
$$T_{wb} = 40 - \left\{0.0061 \times \frac{[P_{\text{sat}}(\text{kPa}) \times h_{fg}(\text{kJ/kg})]_{@T_{wb}}}{(\alpha/D_{AB})^{2/3}_{@T_f}}\right\}$$

Let
$$T_{wb} = 14\,°\text{C}: P_{\text{sat}} h_{fg} = 1.598 \times 2468.3 = 3944.34$$

$$T_f = \frac{1}{2}(T_{db} + T_{wb}) = (40 + 14)/2 = 27°\text{C or } 300.15\text{ K}$$

Mass diffusivity,

$$D_{AB} = \frac{1.87 \times 10^{-10}}{P(\text{atm})} T(K)^{2.072} = 1.87 \times 10^{-10} \times (300.15)^{2.072} = 2.54 \times 10^{-5}\text{ m}^2/\text{s}$$

Thermal diffusivity, $\alpha = 2.25 \times 10^{-5}$ m^2/s

$\therefore$
$$(\alpha/D_{AB})^{2/3} = \left(\frac{2.25}{2.54}\right)^{2/3} = 0.923$$

$\therefore$
$$T_{wb} = 40 - \frac{(3944.34 \times 0.0061)}{0.923} = 40 - 26 = 14\,°\text{C}$$

Since T_{wb} (calculated) = T_{wb} (assumed), the reading on the thermometer is

$\boldsymbol{T_{wb} = 14°\text{C}}$ **(Ans.)**

Points to Ponder

- The larger the molecular spacing, the higher is the rate of diffusion.
- In a binary ideal gas mixture of species A and B, $D_{AB} = D_{BA}$ and both increase with temperature.
- The pressure fraction of a species in an ideal-gas mixture is equivalent to the mole fraction of that species.
- Fick's law of diffusion is analogous to Newton's law of viscosity and Fourier's law of heat conduction.
- The diffusion coefficient in solid solutions is strongly dependent on temperature.
- Diffusion in solids is normally done at high temperatures to reduce the diffusion time.
- Penetration depth in transient mass diffusion is $\sqrt{\pi D_{AB} t}$.
- Mass convection between a surface and moving fluid involves both mass diffusion and bulk fluid motion.
- The density of a mixture is always equal to the sum of the densities of its constituents.
- The molar concentration of a mixture is always equal to the sum of the molar concentrations of its constituents.
- If the mole fractions of A and B are both 0.5, then the molar mass of the mixture is simply the arithmetic average of the molar masses of A and B.
- The mass and the mole fractions for a mixture of CO_2 and N_2O gases are identical.
- The free surface of a lake is exposed to the atmosphere. If the air at the lake surface is saturated, the mole fraction of water vapour in air at the lake surface will *not* be the same as the mole fraction of

water in the lake.

- For steady, one-dimensional mass diffusion through a wall, other things being equal, the higher the density of the wall, the higher the rate of mass transfer.
- The driving force for mass transfer is the concentration difference.
- According to Fick's law of diffusion, the mass flux is proportional to the concentration gradient.
- The SI unit of binary diffusion coefficient is m^2/s.
- Dependence of mass diffusivity, D_{AB} for dilute gases on pressure and temperature for dilute gases can be expressed as $D_{AB} \propto T^{3/2}/P$.
- The unit of solubility is kmol /m^3 bar.
- As Henry's constant increases with increasing temperature, the dissolved gasses in a liquid can be driven off by heating the liquid.
- When the penetration depth is small compared to the thickness of the solid, the solid can be treated as semi-infinite medium during transient mass diffusion.
- Permeability is a product of diffusion coefficient and solubility.
- Use of Schmidt number in mass transfer is analogous to Prandtl number in heat transfer.
- Diffusion coefficients are highest in gases and lowest in solids.
- Solubility × Diffusion coefficient = Permeability.
- When the Lewis number is unity, the thermal diffusivity and mass diffusivity are of the same order of magnitude.
- An impermeable surface in mass transfer corresponds to an adiabatic surface in heat transfer.
- Fick's law of diffusion is expressed on the mass and mole basis as $\dot{m}_{\text{diff},A} = -\rho A D_{AB}(dw_A/dx)$ and $\dot{N}_{\text{diff},A} = -CAD_{AB}(dy_A/dx)$, respectively. The diffusion coefficients D_{AB} in the two relations are same.
- If a gas mixture has constant total pressure P and temperature T throughout, the concentration of the mixture (molar density) C will always be constant but the mass concentration (density), ρ may not be constant unless the composition of the mixture remains unchanged.

GLOSSARY of Key Terms

• **Diffusion coefficient**	Constant of proportionality in the Fick's law and a transport property which is a measure of the ability of a gas to diffuse through another gas at constant and temperature pressure. It depends upon the temperature, pressure and composition of the binary mixture.
• **Relative humidity**	Ratio of the partial pressure of water vapour in the air stream to the partial pressure that would exist if the air stream was saturated with water vapour
• **Mass transfer**	The process of movement or traansport of a chemical species in a mixture from a region of higher concentration to a region of lower concentration.
• **Concentration**	In a multi-component mixture, the concentration (mass) in the mass of a species per unit volume of the mixture while the molar concentration is the number of moles per unit volume of the mixture.
• **Transient diffusion**	The process of diffusion of a species in a stationary medium in which the concentration at a given point varies with time.

• **Stefan flow**	Mass diffusion of a vapour through a stagnant (stationary) gas at constant pressure and temperaure.
• **Equimolar counter-diffusion**	A mass transfer process in a binary gas mixture in which two gases diffuse simultaneously in opposite directions
• **Fick's law of diffusion**	The mass flux (mass transfer rate by molecular diffusion per unit area) is directly proportional to the concentration gradient in a stationary medium in a specified direciton.
• **Reynolds analogy in mass transfer**	It expresses the mass transfer coefficient in terms of the friction factor.
• **Lewis number**	It is the ratio of thermal diffusivity to mass diffusivity. Also the ratio of Schmidt number to Prandtl number.
• **Schmidt number**	It is the ratio of kinematic viscosity to mass diffusivity analogous to Prandtl number in heat transfer.
• **Permeability**	It is the product of solutility of a gas and the mass diffusivity of the gas in a solid. It is a measure of the ability of a gas to penetrate a solid.
• **Sherwood number**	It is the ratio of the product of mass-transfer coefficient and characteristic length to the binary diffusion coefficient analogous to Nusselt number in heat transfer.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

14.1 The mass diffusivity defined by the Fick's law of diffusion in terms of mass concentration is not appropriate
(a) in a solid (b) in a dilute liquid solution
(c) in a dilute gaseous mixture (d) when the mixture density is not constant.

14.2 During the process of equimolar counter diffusion,
(a) the molar-average velocity of the gas mixture is zero
(b) the molar density and partial pressure of each gas vary linearly in the pipe connecting the reservoirs
(c) the mass-average velocity is not equal to zero
(d) all of the above

14.3 The mass transfer Biot number (Bi_m) is defined as $h_m L/\rho D_{AB}$. If R_{diff} is the resistance to species transfer by diffusion in the medium and R_{conv} is the resistance to species transfer by convection at the surface then Bi_m is
(a) $R_{\text{diff}}/R_{\text{conv}}$ (b) $R_{\text{conv}}/R_{\text{diff}}$ (c) $R_{\text{conv}}\ R_{\text{diff}}$ (d) $R_{\text{conv}} + R_{\text{diff}}$

14.4 Choose the incorrect statement:
During the isothermal evaporation of water vapour into still air, it is assumed that
(a) the total pressure remains constant
(b) air and water vapour behave like perfect gases
(c) the air movement creates a little turbulence
(d) the system is in steady state

14.5 In case of liquids, what is the binary diffusion coefficient proportional to?
(a) Pressure only (b) Temperature only (c) Volume only (d) All the above

14.6 The solubility of oxygen gas in rubber is 0.00312 kmol/m^3 bar. The mass density of oxygen at the interface is 0.35 kg/m^3. The partial pressure on the gas side at 25°C is
(a) 250 kPa (b) 100 kPa (c) 350 kPa (d) 400 kPa

14.7 A soda bottle contains a solution of water (H_2O) and carbon dioxide (CO_2). The concentration of the dissolved CO_2 gas in the liquid H_2O is 0.00076. The partial pressure of H_2O vapour in the gas volume at the top of the bottle is negligible. The Henry's constant at 300 K is 1710 bar. The pressure in the bottle is
(a) 2.25 bar (b) 25.6 bar (c) 1.87 bar (d) 1.3 bar

14.8 The case hardening of low carbon steel is done by the process of carburization at high temperature. The diffusion coefficient of carbon in steel at 1000°C is specified as 3×10^{-11} m^2/s The penetration depth of carbon in steel is required to be 1.3 mm. The estimated time required for this hardening process is
(a) 4.98 h (b) 11.2 h (c) 1.87 h (d) 36.8 h

14.9 Select the wrong statement below:
(a) Dissolution of sugar in a tea cup
(b) Diffusion of smoke into the atmosphere
(c) Boiling of water in a kettle
(d) Penetration of carbon in mild steel during case hardening

14.10 The dimensionless parameter that characterizes the fluid flow in which momentum and mass transfer processes occur simultaneously is
(a) Sherwood number (b) Schmidt number
(c) Lewis number (d) Stanton number

Answers

Multiple-Choice Questions

14.1 (d)	**14.2** (d)	**14.3** (a)	**14.4** (c)	**14.5** (b)	**14.6** (c)
14.7 (d)	**14.8** (a)	**14.9** (c)	**14.10** (b)		

REVIEW QUESTIONS

14.1 Define the process of *mass transfer*. How is it different from bulk fluid flow? What are the various mechanisms of mass transfer?

14.2 List some industrial and day-to-day applications of mass transfer.

14.3 What is the driving potential for (a) mass transfer, (b) heat transfer, (c) electric current flow, and (d) fluid flow?

14.4 Define various types of concentrations, velocities and fluxes used in mass transfer.

14.5 What is the effect of *temperature* and *pressure* on the *diffusion coefficient* of a binary gas mixture?

14.6 Define the following terms: *mass-average velocity*, *molar-average velocity*, *diffusion velocity*, *stationary medium*, and *moving medium*.

14.7 State Fick's law of diffusion for mass transfer. How does it closely resemble *Fourier's law of heat conduction*, *Ohm's law of electrical conduction*, and *Newton's law of viscosity*?

14.8 Show that kinematic viscosity, thermal diffusivity, and mass diffusivity have the same units.

14.9 Define the process of *equimolar counter diffusion* and derive an expression for the mass diffusion rate for a binary gas mixture.

14.10 What is Stefan flow? Develop an expression for Stefan's law (*isothermal evaporation of water*).

14.11 Derive the general mass-diffusion equation, on a mass basis, in a stationary medium in Cartesian coordinates.
14.12 What is *Henry's law* and *Henry's constant*?
14.13 Define *solubility* and *permeability*. How are they related to each other?
14.14 How is *mass-transfer Biot number* defined in the transient diffusion process?
14.15 Define the convective *mass-transfer coefficient*. What are its SI units?
14.16 What is the physical significane of the following dimensionless numbers? (a) *Schmidt number* (b) *Sherwood number* (c) *Lewis number*. Which dimensionless number does each corrospond to in heat transfer?
14.17 Discuss the analogy between heat and mass transfer, and explain how the mass-transfer coefficient can be evaluated from the relations for the heat-transfer coefficient.
14.18 The Chilton-Colburn analogy is expressed by the relation $f/2 = St\, Pr^{2/3} = St_{mass}\, Sc^{2/3}$ What are the names of the variables in it and under what conditions is the analogy valid?
14.19 What is the Lewis relation normally used in air-conditioning applications?
14.20 Discuss the *low mass flux approximation* in heat-mass convection analogy. Is the evaporation of water from a lake valid as a low-mass-flux model?

PRACTICE PROBLEMS

(A) Mixture Composition, Fick's Law, Diffusivity, Solubility, Henry's Law

14.1 Evaluate the binary diffusion coefficient D_{AB} for carbon dioxide (*Species A*) in air (*Species B*) at 1 atm and 27°C. Given:

Gas	Molecular Weight	Molecular Volume
Air	28.97	29.9
CO_2	44	34

Then, calculate D_{AB} for a pressure of 2 atm and 47°C. **[0.293 cm²/s]**

14.2 Estimate the binary diffusion coefficient (D_{AB}) of an air-carbon dioxide gas mixture at 1 atm and 298 K predicted by the following equation: $D_{AB} = \dfrac{1.8583 \times 10^{-3}}{P\sigma_{AB}^2\, \Omega_{D,AB}} T^{3/2}\left[\left(\dfrac{1}{\tilde{M}_A}\right)+\left(\dfrac{1}{\tilde{M}_A}\right)\right]^{1/2}$ where D_{AB} is in cm²/s, P in atm, and T in K, $\tilde{M}_A$ and $\tilde{M}_B$ are the molar masses of the two species A and B, σ_{AB} is the collision diameter in, Å and $\Omega_{D,AB}$ is the collision integral. The values of σ_{AB} and $\Omega_{D,AB}$ can be used from the following table.

Species	σ, Å	ε/k	Molar mass
Air	3.617	97	28.97
CO_2	3.996	190	44.01

kT/ε	2.0	2.1	2.2	2.3
$\Omega_{D,AB}$	1.075	1.057	1.041	1.026

Compare the result with the experimental value of $D_{AB} = 0.16 \times 10^{-4}$ m²/s. Determine D_{AB} if $P = 3$ atm and $T = 600$ K. **[0.152 cm²/s, 0.034 cm²/s]**

(B) Sationary Media, Equimolar Counter Diffusion, Evaporation in a Column, Moving Medium

14.3 A deep, narrow cylindrical vessel which is open at the top contains some toluene at the bottom. The air within the vessel is considered motionless, but there is sufficient air current at the top surface of the vessel that any toluene vapour arriving at the top surface is immediately removed to ensure zero toluene concentration at the top surface. The entire system is at 1 atm and 18.7°C. The saturated vapour pressure of toluene at the liquid surface in the vessel is 0.026 atm. Calculate the rate of evaporation of toluene into the air per unit area of the liquid surface if the distance between the liquid toluene surface and the top of the vessel is 1.5 m. The mass diffusivity of air-toluene vapour air at 25°C is 0.0844 cm^2/s. **[5.995×10^{-9} k mol/m^{2}s]**

14.4 Calculate the rate of burning of a coal particle of 2.5-mm diameter in an atmosphere of pure oxygen at 1200 K and 1 atm, assuming that a very large blanketing layer of CO_2 has formed around the particle. Assume that the combustion rate is such that all the oxygen reaching the surface is instantaneously consumed. Hence, the concentration of oxygen at the surface is effectively zero. Also, assume that the concentration of CO_2 far away is zero. The binary diffusion coefficient, $D_{AB} = 0.14 \times 10^{-4}$ m^2/s. **[2.47×10^{-7} kg/s]**

14.5 An open pan, 18 cm in diameter with 6 cm depth, contains water at 28°C and is exposed to dry atmospheric air. If the rate of diffusion of water vapour is 8.54×10^{-4} kg/h, estimate the diffusion coefficient of water in air. **[0.202 cm^2/s]**

(C) Transient Diffusion

14.6 Steel is carburized in a high-temperature process which depends on the transfer of carbon by diffusion. A steel component with an initial carbon content of 0.11 per cent by mass is case hardened in a furnace by exposing it to a carburising gas. The carbon-rich atmosphere in the furnace maintains the mass fraction at the exposed surface of the steel component at 0.011. The hardening process continues till the mass fraction of carbon is elevated to 0.50 per cent at a depth of 0.7 mm. The diffusion coefficient of carbon in steel is $D_{AB} = 2.67 \times 10^{-5}$ exp (–17400/T) (m^2/s) where T is in K. The time required for the component to be kept in the furnace is 9 h. Determine the temperature of the furnace. **[1179 K]**

(D) Convective Transfer

14.7 Consider a drop of water, of 0.3-mm diameter, travelling through air at a velocity of 5 m/s. If the air temperature is 20°C, determine the steady-state temperature of the drop. Neglect radiation and assume the relative humidity to be 50%. **[13.3°C]**

14.8 A swimming pool of dimensions 6 m by 12 m has its water surface temperature of 20°C. The ambient air conditions are 20°C and 30% relative humidity. The wind blows in the direction of the long (12 m) side of the pool with a speed of 7.2 km/h. Estimate the daily evaporative water loss from the pool. Use the following relation for evaluating $D_{\mathrm{H_2O(A)-Air(B)}}$:

$D_{AB} = (1.87 \times 10^{-10})\,(\mathrm{T}^{2.072}/\mathrm{P})\,((\mathrm{m}^2/\mathrm{s}))$ with T in K and P in atm.
Properties: *Dry air* (1 atm, 20°C): $\nu = 15.16 \times 10^{-6}$ m^2/s
Saturated water (1atm, 20°C): $P_g = 2.339$ kPa, $\rho_g = 0.0173$ kg/m^3 **[325 kg/day]**

14.9 Dry air at 1 atm and 40°C flows with a velocity of 0.5 m/s across a wet bulb thermometer. What would be the reading on the thermometer? **[14°C]**

14.10 In an experiment, dry air at 1 bar and 26°C is forced to flow over a body covered with naphthalene (RMM = 128.16 kg/kmol) with a free stream velocity of 2.0 m/s. The surface area of the body is 0.7 m^2. It is observed that sublimation of 100 g of naphthalene has taken 50 min. During the experiment, both the air and the body were maintained at 26°C, at which the mass diffusivity is $D_{AB} = 0.62 \times 10^{-5}$ m^2/s. (a) What is the mass convection coefficient? (b) Using the analogy between heat and mass transfer, calculate the average heat-transfer coefficient.

Properties of dry air at

$\rho = 1.165$ kg/m^3 $\quad C_p = 1.007$ kJ/kg °C $\quad \alpha = 2.33 \times 10^{-5}$ m^2/s

Saturated vapour pressure of naphthalene can be found from $P_{sat} = P \times 10^{E}$ where $E = 8.67 - (3766/T)$ with P in bar and T in K. **[211.5 W/m^2 °C]**

15 Multi-Dimensional Heat Conduction

15.1 ❑ INTRODUCTION

In Chapter 2, we analyzed steady-state one-dimensional heat conduction in plane walls cylinders, and spheres. There are many engineering applications, for which the assumption of one-dimensional heat conduction may be too much of a simplification or inappropriate. In that case, a multidimensional conduction problem must be solved. Such multidimensional heat conduction occurs in the block of an internal combustion engine, heat treatment of various metal parts, composite systems comprising materials with different thermal conductivities, large chimneys, and L-shaped bars.

The main objective of any heat-transfer analysis is the determination of the temperature distribution and the heat flow within and at the boundary of a given body. In solving two-dimensional problems, analytical, graphical, analogical, or numerical techniques are usually used. Analytical, analogical and graphical methods are normally used for simpler cases, while the numerical techniques are used to solve the more complex problems. The analytical solution requires basic knowledge of *Fourier's series, Bessel functions, Legendre polynomials, Laplace transform methods*, and *complex variable* theory. The knowledge of advanced mathematics is not necessary in the remaining three cases.

The general three-dimensional heat conduction equation is

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\overline{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{15.1}$$

For steady-state conditions, $\partial T/\partial t = 0$.

For no heat generation, $\overline{q}/k = 0$.

For two-dimensional heat conduction, $\partial^2 T/\partial z^2 = 0$.

The governing differential equation for two-dimensional steady-state conduction in a material with constant thermal conductivity and without heat generation becomes

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0} \tag{15.2}$$

An example of a two-dimensional problem is the temperature distribution in a horizontal section of the corner of a wall shown in Fig. 15.1. The points of equal temperature joined to form isothermal surfaces are shown and the lines of heat flow are indicated at right angles to these isothermals.

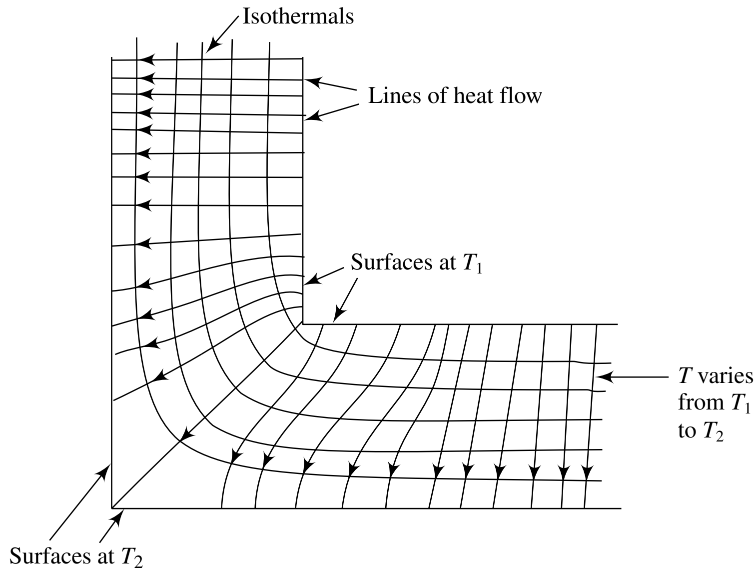


Fig. 15.1 *Heat flow in the corner of a wall: Two-dimensional heat conduction*

15.2 ❑ ANALYTICAL SOLUTION OF TWO-DIMENSIONAL HEAT CONDUCTION IN RECTANGULAR PLATES

The analytical method involves the exact solution of partial differential equations which can be used to compute the temperature at *any* point of interest in the medium.

The method involves the solution of the governing differential equation for conduction in the appropriate coordinate system subject to the initial and boundary conditions. However, rigorous and complex mathematical treatment, except in simple geometries, makes this method of limited use.

In this method, we start with the general differential equation for conduction in the required coordinate system and solve it in conjunction with given initial and boundary conditions to get the temperature field; then apply the Fourier's equation and get the heat flux at any desired point.

Let us consider a thin rectangular plate (Fig. 15.2) without heat generation and insulated at the top and bottom surfaces. For a thin plate, $\partial T/\partial z$ is negligible and temperature is a function of x and y only. Assume that three faces of this plate are maintained at a constant temperature $T = 0$, and the fourth face is held at a constant temperature T_s.

Let k be the thermal conductivity and let it be uniform, i.e., independent of both temperature and direction.

For steady equilibrium conditions, the temperature distribution must satisfy the following equation:

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0} \tag{15.3}$$

Equation (15.3) is a linear and homogeneous partial differential equation. The solution to this equation may be expressed as a product of two functions each of which are only functions of one independent variable as:

$$T(x, y) = X(x)Y(y) \tag{15.4}$$

where X is a function of x only and Y is a function of y only. Substituting this solution into Eq. (15.3), we get

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0$$

Dividing by XY, we get

$$\boxed{-\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{y}\frac{d^2Y}{dy^2}} \tag{15.5}$$

Since the variables in this equation are separated, each side is a constant. Taking this constant to be λ^2, we obtain the following two separate equations:

$$\boxed{\frac{d^2X}{dx^2}+\lambda^2X=0} \tag{15.6}$$

$$\boxed{\frac{d^2Y}{dy^2}-\lambda^2Y=0} \tag{15.7}$$

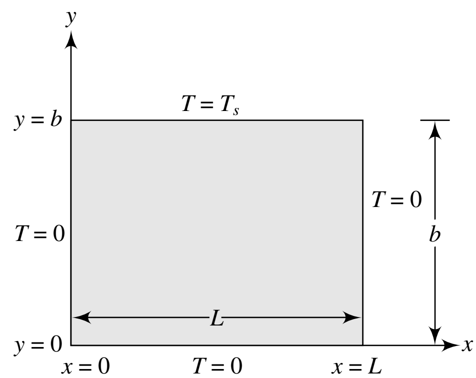


Fig. 15.2 *Two-dimensional conduction in a rectangular plate*

The general solution to Eq. (15.6) is

$$X=C_1\cos\lambda x+C_2\sin\lambda x$$

and the solution to Eq. (15.7) is

$$Y=C_3\,e^{-\lambda y}+C_4\,e^{\lambda y}$$

These solutions can be verified by substituting them into the corresponding differential equation. The general solution of Eq. (15.3) is

$$\boxed{T(x,y)=XY=(C_1\cos\lambda x+C_2\sin\lambda x)(C_3e^{-\lambda y}+C_4e^{\lambda y})} \tag{15.8}$$

where C_1, C_2, C_3, C_4, and λ are determined from the boundary conditions.
The boundary conditions are (BC_s) as follows:

1. $T = 0$ along $y = 0$.
2. $T = 0$ along $x = 0$.
3. $T = 0$ along $x = \mathrm{L}$.
4. $T = T_s$ along $y = b$

From BC (2), we get $C_1 = 0$ and from BC (1), we get

$$C_3 + C_4 = 0 \text{ or } C_3 = -C_4.$$

Therefore, $\quad T=C_2C_4\sin\lambda x(e^{\lambda y}-e^{-\lambda y})\quad$ or $\quad T=2C_2C_4\sin\lambda x\sinh\lambda y=C\sin\lambda x\sinh\lambda y$

where $C = 2C_2C_4$.
From BC (3), we get, $0 = C\sin\lambda L\sin\lambda y$
Therefore,

$$\sin\lambda L = 0 \qquad (\textit{for all values of } y)$$

$$\lambda=\frac{n\pi}{L} \qquad (n = 1,2,3,4,\ \ldots)$$

Therefore, $$T = C \sin \frac{n\pi x}{L} \sinh \lambda y \quad \text{or} \quad T = C \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \tag{15.9}$$

There exists a different solution for each integer n and each solution has a separate integration constant C_n. Summing these solutions, we get

$$\boxed{T(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}} \tag{15.10}$$

From the boundary condition (4), we get

$$T|_{y=b} = T_s = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi b}{L}$$

The problem is now reduced to one of a Fourier sine series.
Solving for C_n, we get

$$C_n = \frac{2}{n\pi} \frac{(1 - \cos n\pi)}{\sinh(n\pi b/L)} T_s \tag{15.11}$$

The solution, therefore, becomes

$$\boxed{T(x, y) = T_s \sum_{n=1}^{\infty} \frac{2(1 - \cos n\pi) \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}}{\sinh(n\pi b/L)}} \tag{15.12}$$

15.3 ❑ TWO-DIMENSIONAL HEAT CONDUCTION IN SEMI-INFINITE PLATE

Consider a semi-infinite plate as shown in Fig. 15.3. The length of the plate in the y-direction is extremely large. The procedure is exactly the same as given earlier.
The general differential equation is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

and its solution is of the form

$$\begin{aligned} T &= X(x)Y(y) \\ &= (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y}) \end{aligned}$$

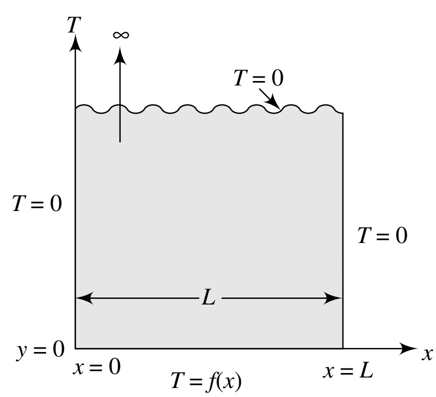


Fig. 15.3 *Two-dimensional conduction in a semi-infinite plate*

The boundary conditions needed to complete the problem formulation are

1. $T = 0$ at $x = 0$
2. $T = 0$ at $x = L$
3. $T = 0$ at $y = \infty$
4. $T = f(x)$ at $y = 0$

From the boundary condition (1), we get

$$0 = (C_1 + 0)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})$$

Hence, $C_1 = 0$.

Thus, the general solution reduces to the following equation:

$$T = C_2 \sin \lambda x [C_3 e^{-\lambda y} + C_4 e^{\lambda y}] = \sin \lambda x [C e^{-\lambda y} + D e^{\lambda y}] \text{ where } C = C_2 C_3 \text{ and } D = C_2 C_4$$

From the boundary condition (3), we get

$$0 = \sin \lambda x \, (C e^{-\lambda \infty} + D e^{\lambda \infty}) = \sin \lambda x \, (0 + D e^{\lambda \infty}) \quad (15.13)$$

Thus, $D = 0$. Therefore,

$$T = C \sin \lambda x \, e^{-\lambda y}$$

Using the boundary condition (2), we have

$$0 = C \sin \lambda L \, e^{-\lambda y}$$

Therefore, $\sin \lambda L = 0 = \sin n\pi$ or $\lambda = \dfrac{n\pi}{L}$, where n is an integer

It follows that,

$$\boxed{T = \sum_{n=0}^{\infty} C_n \sin \frac{n\pi x}{L} \exp\left(\frac{-n\pi y}{L}\right)} \quad (15.14)$$

From the boundary condition (4), we have

$$f(x) = \sum_{n=0}^{\infty} C_n \sin \frac{n\pi x}{L}$$

which is a Fourier expansion of $f(x)$ in an infinite series of sine functions.

$$\therefore \qquad C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (15.15)$$

Thus, the final form of the solution is

$$\boxed{T = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\mathrm{L}} \exp\left(\frac{-n\pi y}{L}\right) \int_0^L f(x) \sin\left(\frac{n\pi x}{\mathrm{L}}\right) dx} \quad (15.16)$$

15.4 ❑ THE ANALOGICAL METHOD

Analogies have been found to exist between heat transfer and the flow of electrical current, and fluid flow which provide a useful tool for obtaining temperature distribution and the rate of heat transfer in a two-dimensional system. The two-dimensional Laplace equation applies to a temperature field. If the temperature T is replaced by the electric potential ϕ then the steady-state distribution of electric potential governing the voltage distribution in an electrical field is obtained. The differential equations associated with heat conduction and those governing the steady-state distribution of electric potential are analogous. These two equations are

$$\boxed{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0} \quad \boxed{\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0} \tag{15.17}$$

The analogy between electrical and temperature fields provides the basis for an experimental electrical analogy method. Such experimental arrangements are known as *analog field plotters*. A thin sheet of electrically conducting paper of high resistivity is used for the purpose. This sheet is cut to an exact geometric model of the two-dimensional heat conduction system. With regard to boundary conditions, a uniform surface temperature is modelled by maintaining a uniform voltage at the surface and insulated surfaces which are not connected to voltage sources.

An electric potential is then impressed on the model. Lines of constant voltage which correspond to isotherms are then found by using a millivoltmeter. The current flow lines which are perpendicular to the potential lines are drawn in free hand in such a way that the resulting network forms curvilinear squares. The current-flow lines can sometimes be determined by reversing the electrical boundary conditions.

Equipotential lines on the model surface that are plotted, correspond to isothermal lines in the temperature field.

Thus, the temperature distribution can be inferred.

In this method, a model of the given geometry is used with analogous boundary conditions.

15.5 ❑ GRAPHICAL METHOD

Graphical methods are used for two-dimensional problems with isothermal lines (constant temperature) lines and adiabatic (constant heat flux) boundaries. This is an approximate but versatile method applicable to many irregularly shaped two-dimensional geometries of practical interest and can give rapidly a reasonably good estimate of the temperature distribution. Here, temperature and heat flow lines are drawn by free hand, remembering that isothermal and heat-flow lines are orthogonal, thus forming curvilinear squares. Once such a *flux plot* is drawn, heat flow is easily calculated by applying Fourier's law to each *heat-flow lane*.

The basis of the graphical method is to draw, by *trial and error*, the heat-flow lines and isotherms in a curvilinear section in such a way that they are perpendicular at their points of intersection. This technique is called *flux plotting*. Consider an element of unit thickness of the material in a flux plot as shown in Fig. 15.4.

The heat-flow rate is

$$\dot{Q}=-k(\Delta x)(1)\frac{\Delta T}{\Delta y} \tag{15.18}$$

This heat flow will be the same through each flow line. If the sketch is drawn so that $\Delta x \approx \Delta y$, then the rate of heat transfer is proportional to ΔT across the element. Thus

$$\Delta T_{\text{overall}} = N\,\Delta T \tag{15.19}$$

where N is the number of temperature increments between the two boundaries at T_1 and T_2. Therefore, the total heat-transfer rate is

$$\dot{Q}=\frac{M}{N}k\,\Delta T_{\text{overall}}=\frac{M}{N}k\,(T_1-T_2) \tag{15.20}$$

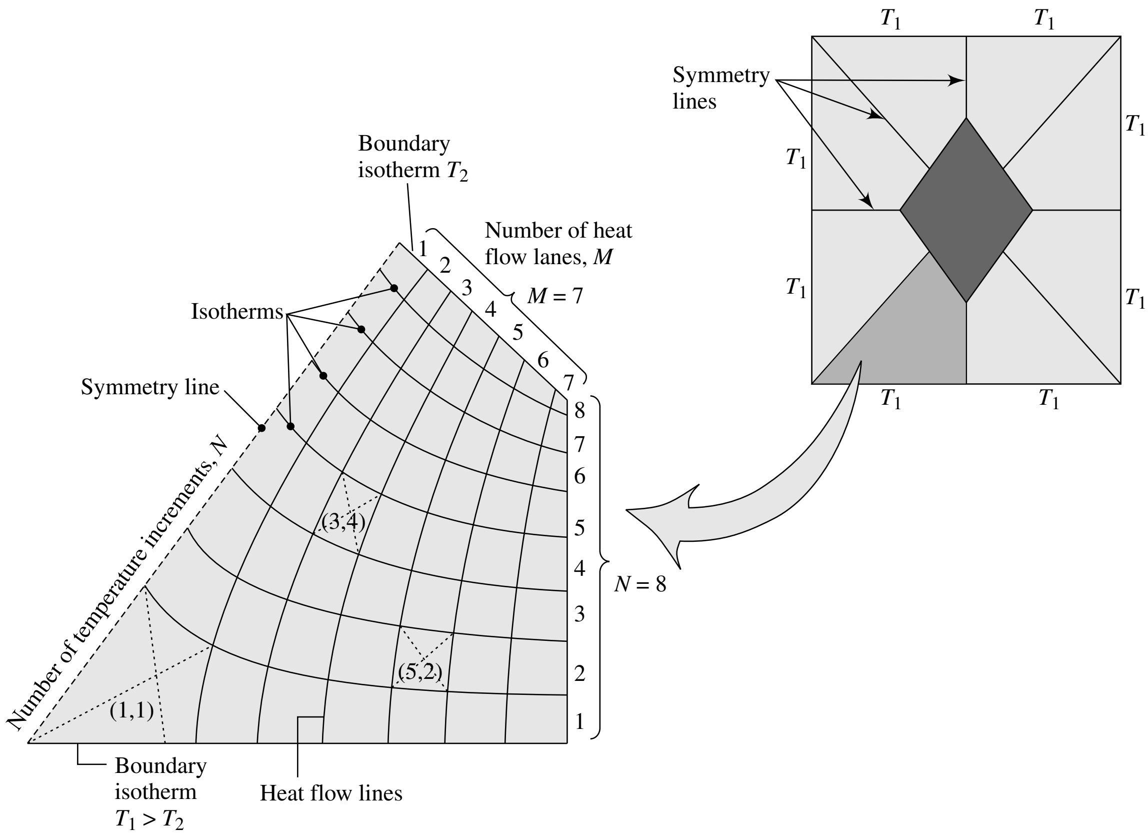


Fig. 15.4 *Arrangement of arbitrary isotherms and constant heat-flow lines*

where M is the number of heat flow lanes and $(T_1 - T_2)$, is the overall temperature difference between two isothermal boundaries. The ratio M/N, i.e., the number of flow lanes divided by the number of temperature increments, is called the *conduction shape factor*, S. Therefore, Eq. (15.20), reduces to

$$\boxed{\dot{Q} = S\,k\,(T_1 - T_2)} \quad \text{(W)} \tag{15.21}$$

15.6 ❑ CONDUCTION SHAPE FACTORS

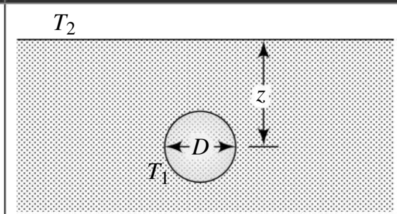
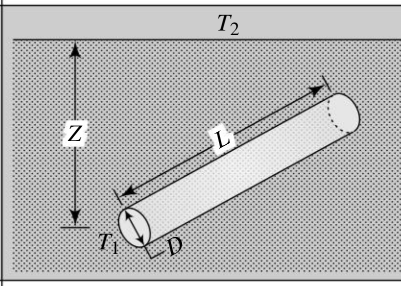
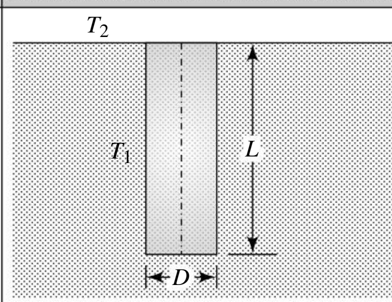
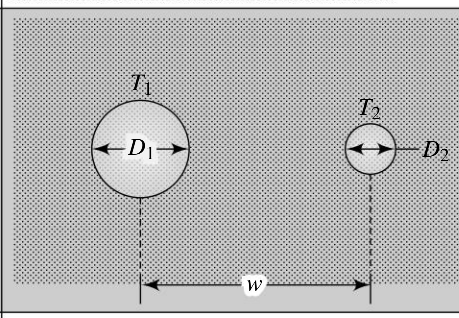
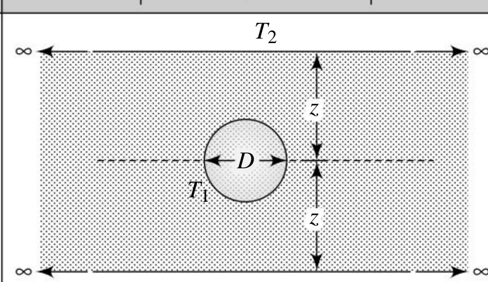
A simple, though approximate, approach to determining the heat-transfer rate by conduction in two dimensions is by using the conduction shape factors. In 2-D heat transfer, let the two surfaces of a solid medium are held at temperatures, T_1 and T_2 and k is the thermal conductivity of the medium. The heat-transfer rate with no heat generation within the medium is then given by

$$\boxed{\dot{Q} = k\,S\,(T_1 - T_2)} \tag{15.22}$$

where S is called *conduction shape factor* and has the dimension of length. It may be noted that for liquids and gases where convection is dominant, the above equation is not applicable.

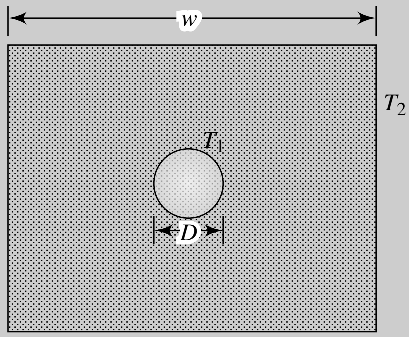
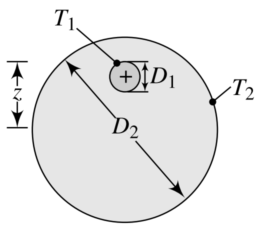
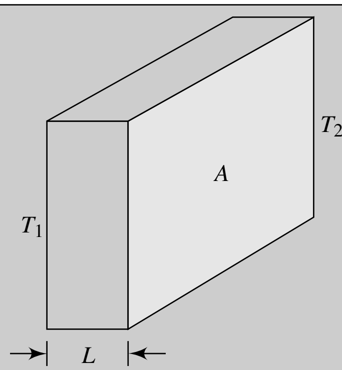
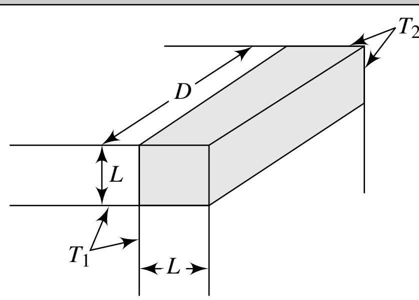
Table 15.1 gives conduction shape factors for some selected two-dimensional systems.

Table 15.1 *Conduction shape factors*

No.	System	Schematic	Restrictions	Shape Factor
1.	Isothermal sphere buried in a semi-infinite medium		$z > D/2$	$\dfrac{2\pi D}{1 - D/4z}$
2.	Horizontal isothermal cylinder of length L buried in a semi-infinite medium		$L >> D$ $L >> D$ $z >> 3D/2$	$\dfrac{2\pi L}{\cosh^{-1}(2z/D)}$ $\dfrac{2\pi L}{\ln(4z/D)}$
3.	Vertical cylinder in a semi-infinite medium		$L >> D$	$\dfrac{2\pi L}{\ln(4L/D)}$
4.	Conduction between two cylinders of length L in an infinite medium		$L >> D_1, D_2$ $L >> w$	$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}$
5.	Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		$z > D/2$ $L >> z$	$\dfrac{2\pi L}{\ln(8z/\pi D)}$

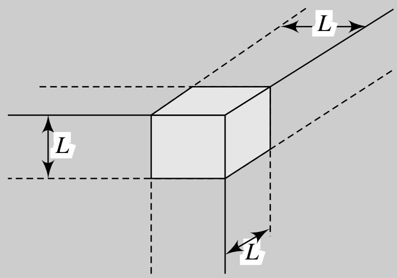
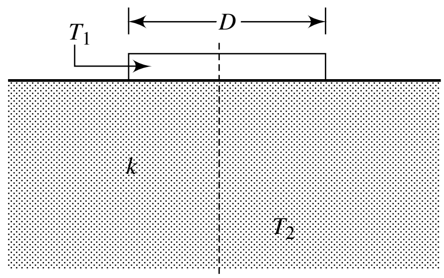
(*Contd.*)

(*Contd.*)

6.	Circular cylinder of length L centred in a square solid of equal length		$w > D$ $L >> w$	$\frac{2\pi L}{\ln(1.08w/D)}$
7.	Eccentric circular cylinder of length L in a cylinder of equal length		$L > D_2$	$\frac{2\pi L}{\cosh^{-1}\left(\frac{D_1^2 + D_2^2 - 4z^2}{2D_1D_2}\right)}$
8.	Plane wall		One-dimensional conduction	$\frac{A}{L}$
9.	Conduction through the edge of adjoining walls		$D > L/5$	0.54 D

(*Contd.*)

(*Contd.*)

10.	Conduction through corner of three walls with a temperature difference of ΔT_{1-2} across the walls		L << length and width of wall	0.15 L
11.	Disk of diameter D and T_1 on a semi-infinite medium of thermal conductivity k and T_2		None	2D

15.7 ❑ NUMERICAL METHOD

The main difference between the analytical solution method and the numerical solution method is that the former will give an equation from which the temperature may be obtained anywhere in the solid, whereas the latter will give values of temperatures at specified points only. With a widespread use of digital computers, the numerical analysis is energising as the method of choice for solving multidimensional heat transfer problems.

The numerical relaxation method was first introduced by Sir Richard Southwell and is used to solve a set of algebraic equations.

The step-wise procedure is given below:

- Subdivide the system into a number of small subvolumes and assign a reference number to each.
- Assume values of temperatures at the various nodes.
- Calculate the residuals at each node, using the assumed temperature.
- Relax the largest residual to zero by changing the corresponding nodal temperature by an appropriate amount.
- Change the residuals of the surrounding nodes to correspond with the temperature change in Step 4.
- Continue to relax residuals until all are as close to zero as desired.

15.7.1 • Finite Difference Form of Heat Equation

Consider a two-dimensional system such as a solid of constant thickness divided into equal increments in both x- and y-directions. The nodal points designated are shown in Fig. 15.5. The nodal network has m locations indicating the x-increment and the n locations indicating the y-increment. Let us focus our attention on a nodal point (m, n) and its four neighbouring points as shown in Fig. 15.6. Let $T_{m,n}$ be the temperature at the node m, n, $T_{m+1,n}$ at the node (m + 1), n, and so on. The *first derivative* of temperature with respect to x, i.e., temperature gradient at the node [m + (1/2)], n is

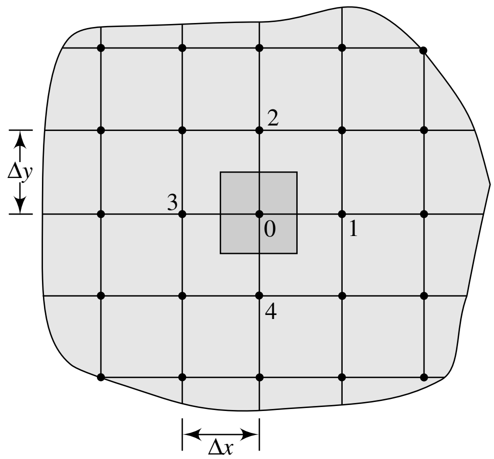


Fig. 15.5 *Network of curvilinear squares with nodal points*

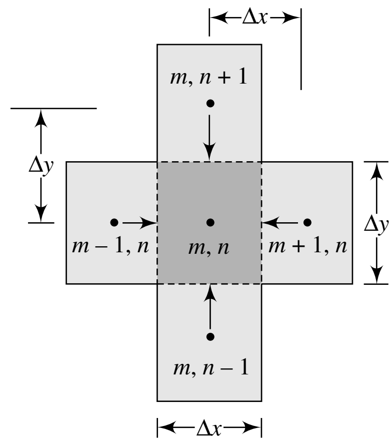


Fig. 15.6 *Conduction to an interior node from its neighbouring nodes*

$$T_{m,n} = \frac{1}{4}[T_{\text{left}} + T_{\text{right}} + T_{\text{top}} + T_{\text{bottom}}]$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+\frac{1}{2},n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

and at the node $[m - (1/2)]n$ is

$$\left.\frac{\partial T}{\partial x}\right|_{m-\frac{1}{2},n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Proceeding in a similar fashion, we can readily write

$$\left.\frac{\partial T}{\partial y}\right|_{m,\frac{n+1}{2}} \approx \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$\left.\frac{\partial T}{\partial y}\right|_{m,\frac{n-1}{2}} \approx \frac{T_{m,n} - T_{m,n-1}}{\Delta y}$$

The second derivatives at m, n nodal points may be approximated as

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{\left.\frac{\partial T}{\partial y}\right|_{m+1/2,n} - \left.\frac{\partial T}{\partial y}\right|_{m-1/2,n}}{\Delta x} = \frac{T_{m+1n} + T_{m-1n} - 2T_{m,n}}{(\Delta x)^2}$$

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{\left.\frac{\partial T}{\partial y}\right|_{m,\frac{n+1}{2}} - \left.\frac{\partial T}{\partial y}\right|_{m,\frac{n-1}{2}}}{\Delta y} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} \tag{15.23}$$

Substituting the values of $\partial^2T/\partial x^2$ and $\partial^2T/\partial y^2$ in the two-dimensional steady-state Laplace equation, we get

$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} = 0$$

For a square mesh, we have $\Delta x = \Delta y$.
Therefore, for an interior node in a two-dimensional solid, energy balance of Eq. (15.21) reduces to

$$\boxed{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0} \quad (15.24a)$$

or

$$\boxed{T_{\text{right}} + T_{\text{left}} + T_{\text{top}} + T_{\text{bottom}} = 4T_{m,n}} \quad (15.24b)$$

Equation (15.22) is the *finite difference* form of the two-dimensional heat conduction equation for the node (m, n) that is equidistant from its four adjacent nodes. This equation must be written for each node within the material and the equations are then solved for temperatures at the various nodes. Remember that the sum of the temperatures associated with the neighbouring nodes equal to four times the temperature of the node in question. In the method the right side of Eq. (15.24) is set equal to some residual R which we want to relax to zero.

There are mainly four methods of solving a system of simultaneous, algebraic equations:

- Relaxation method
- Gauss–Siedel iteration method
- Matrix inversion method
- Gauss elimination method

15.8 ❑ RELAXATION METHOD

Numerical analysis using the relaxation method is quite suitable for the approximate determination of temperature fields in several practical situations. The residual errors at each point are successively reduced. Finally, acceptable accuracy is achieved.

If we carefully estimate the initial temperature then with moderate over-relaxation of points with large residuals, the number of iterations required can be considerably reduced.

The relaxation method is a numerical method for solving a set of algebraic equations and was first used by Sir Richard Southwell. The basic principles of the relaxation method are illustrated below by solving the following pair of algebraic equations:

$$12x - 13y = 232$$
$$6x - 25y = -155$$

To obtain a solution of the above equations, we rewrite the given equations as

$$R_1 = -12x + 13y + 232$$
$$R_2 = 6x - 25y + 155$$

where R_1 and R_2 are called *residuals*. Our aim in the *relaxation method* is to reduce the residuals systematically to zero or close to zero. The values of x and y which make the residuals zero are the desired values since they satisfy the given system of equations.

The *first* step in the solution is to choose the initial values of x and y. Let us choose $x = 0$ and $y = 0$. Then the residuals are $R_1 = 232$ and $R_2 = 155$.

The *second* step is to set up an *operation table* which will indicate the effect of a unit positive increment in the values of x and y on the residuals. The operation table for the present problem is shown in Table 15.2.

Table 15.2 *Operation table*

Increment	ΔR_1	ΔR_2
$\Delta x = 1$	–12	6
$\Delta y = 1$	13	–25

The *third* step is to reduce the currently largest residual to zero. Since the largest residual is $R_1 = 232$, $\Delta x = 19$ makes $R_1 = 4$ but R_2 will become 269. The next and subsequent steps concentrate on the currently largest residual to find the incremental changes in x and y to reduce the residuals to zero or as close as possible to zero. The subsequent calculations are summarized in tabular form in Table 15.3.

Table 15.3 *Relaxation Table I*

X	*Y*	R_1	R_2
0	0	232	155
$\Delta x = 19$		4	269
	$\Delta y = 11$	147	–6
$\Delta x = 13$		–9	72
	$\Delta y = 3$	30	–3
$\Delta x = 3$		–6	15
$x = 35$	$y = 14$	–6	–15

The last row of the table gives the values of x and y, which are correct to the nearest whole number. Then the calculations can be refined to obtain a solution to the first decimal place by using first decimal increments. Further refinements can be made in a similar manner depending on the required accuracy of the final solution. A summary of the first decimal and subsequent increment calculations are presented in tabular form in Table 15.4.

Table 15.4 *Relaxation Table II*

X	*Y*	R_1	R_2
35	14	–6	15
	$\Delta y = 0.6$	1.8	0
$\Delta x = 0.2$		–0.6	1.2
	$\Delta y = 0.05$	0.05	–0.05
$\Delta x = 35.2$	$y = 14.65$	0.05	–0.05

The residuals are almost zero. Hence, the final values of x and y are **35.2** and **14.65**, respectively.

15.8.1 • Shortcuts in Relaxation Method

The relaxation method (*explained above*) can also be applied to complex problems. However, by modifying the method, one can save considerable effort and time. These modifications are called shortcuts.

1. The first shortcut is to assign some initial values to the variables, instead of assigning zero values. The initial values can be guessed based on a close look at the physical problem.
2. The second shortcut, called *over-relaxation*, is used to change the residuals from a positive value to a negative value and vice versa, by choosing larger increments. However, one cannot quantify

the size of the increment to choose. The choice of the increment size may vary from person to person analyzing the problem.

3. The third shortcut, called *block operation*, is to change each of the variables by the same amount. The changes in the residuals due to the individual unit increments in the variables are added to obtain the effect of the *unit block operation.* This unit block operation can be used at any stage of calculation, but it is quite valuable in the initial step of the calculation. To apply this unit block operation in the initial step, we add the residuals and divide the sum by the effect of the unit block operation to obtain the size of the increment. If one uses the increment size thus obtained in a block operation, it will reduce the sum of the residuals to zero and the initial step considerably reduces the subsequent steps to obtain the final solution.

The method of applying the *unit block operation* is illustrated below by considering the earlier example. The operation table for the earlier example (*Table 15.2*) is given in *Table 15.5.*

Sum of residuals = 1 – 19 = –18 = effect of unit block operation.

Sum of residuals in the initial stage (with x = 0 and y = 0) = 232 + 155 = 387 (see *Table 15.3*)

Step size in the initial step $= \dfrac{387}{18} \approx 22$

Now, we can use a step size of 22 for both the variables x and y to apply the unit block operation. Table 15.6 illustrates the application of unit block operation.

The subsequent calculations can be carried out as in Table 15.3.

Table 15.5 *Initial-step increment size-calculation table*

Increment	ΔR_1	ΔR_2
$\Delta x = 1$	–12	6
$\Delta y = 1$	13	– 25
$\Delta x = 1$; $\Delta y = 1$	1	–19

Table 15.6 *Application of unit block operation*

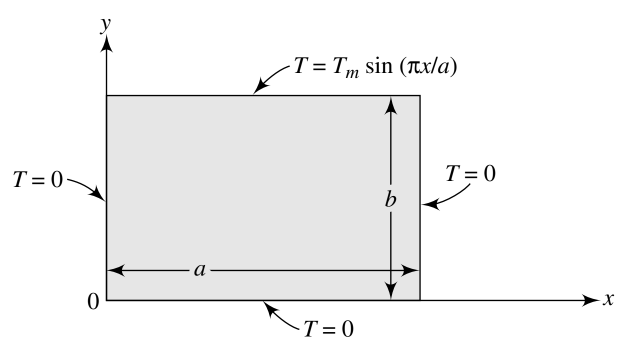
X	Y	R_1	R_2
0	0	232	155
$\Delta x = 22$	$\Delta y = 22$	254	–263

Illustrative Examples

(A) Two-Dimensional Steady-State Conduction

EXAMPLE 15.1 *In a two-dimensional domain shown in the figure, show that the temperature distribution is given by*

$$T = T_m \, sin\left(\frac{\pi x}{a}\right)\frac{sinh(\pi y/a)}{sinh(\pi b/a)}$$



Solution

Known A rectangular bar with three lateral sides maintained at T = 0 and the fourth side with sinusoidal temperature distribution.

Find Temperature distribution.

Assumptions (1) Steady-state, two-dimensional conduction.

Analysis Since there is no temperature gradient in the z-direction (*being very long*), the Laplace equation is.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{A}$$

subject to the boundary conditions

(i) $T(0, y) = 0 \qquad (0 < y < b)$

(ii) $T(a, y) = 0 \qquad (0 < y < b)$

(iii) $T(x, 0) = 0 \qquad (0 < x < a)$

(iv) $T(x, b) = T_m \sin\left(\frac{\pi x}{a}\right) \qquad (0 < x < a)$

Assume a solution of the form

$$T(x, y) = X(x)\, Y(y)$$

When substituted into the Laplace equation, this yields,

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} \tag{B}$$

The left side of (B), a function of x alone, can equal the right side, a function of y alone, only if both sides have a constant value, say λ^2 (> 0).

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad \frac{d^2Y}{dy^2} - \lambda^2 Y = 0$$

with general solutions.

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$
$$Y = C_3 \cosh \lambda y + C_4 \sinh \lambda y$$

so that $\quad T = (C_1 \cos \lambda x + C_2 \sin \lambda x)\,(C_3 \cosh \lambda y + C_4 \sinh \lambda y)$

Now applying the boundary conditions gives, (i) $C_1 = 0$, and (iii) $C_3 = 0$. Using these together with (ii) yields $0 = C_2C_4\,(\sin \lambda a)(\sinh \lambda y)$ which requires that

$$\sin \lambda a = 0 \quad \text{or} \quad \lambda = \frac{n\pi}{a} \qquad (\textit{n is a positive integer})$$

Since the original differential equation (A) is linear, the sum of any number of solutions constitutes a solution. Thus, T can be written as the sum of an infinite series:

$$T = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

where the constants have been combined.
Finally, the boundary condition (iv) gives

$$T_m \sin\frac{\pi x}{a} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \sin\frac{n\pi b}{a} \quad (0 < x < a)$$

which holds only if

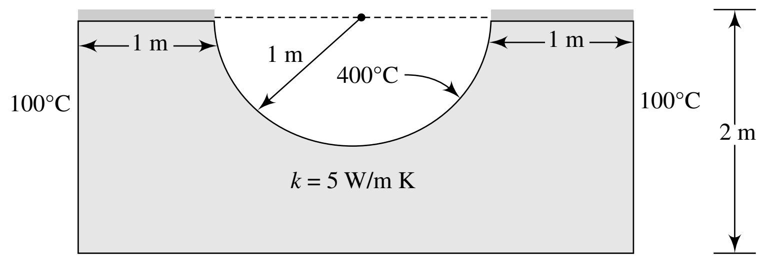
$$C_2 = C_3 = C_4 = \cdots = 0 \text{ and } C_1 = T_m / \sinh(\pi b/a)$$

Therefore,

$$\boxed{T = T_m \frac{\sin(\pi y/a)}{\sinh(\pi b/a)} \sin(\pi x/a)} \qquad \textbf{Hence, proved.}$$

(B) Graphical Method

EXAMPLE 15.2 *Determine the heat-flow rate per metre length from the inner to the outer surface of the object shown below using the graphical method (heat flux plot):*

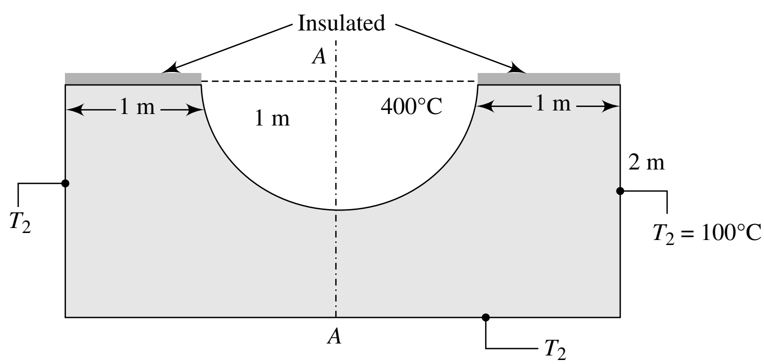


Solution

Known Surface thermal conditions and dimensions of an object with specified thermal conductivity.

Find Heat flow per unit length (flux plot method).

Schematic

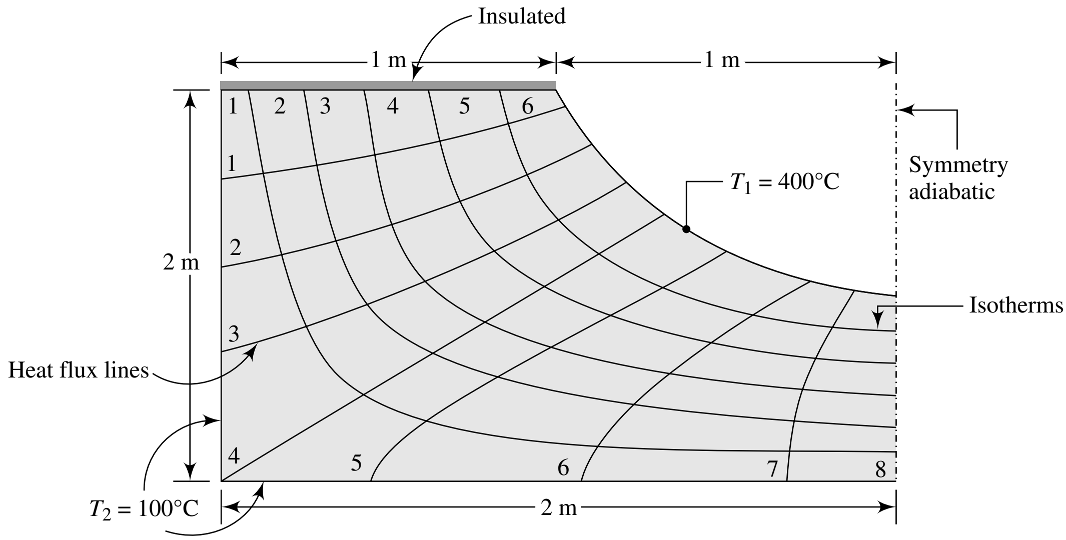


Assumptions (1) Two-dimensional, steady-state conduction. (2) Uniform thermal conductivity.

Analysis AA is the line of symmetry which is adiabatic. We draw the flux plot for the left half of the object. The shape factor, $S = \frac{M}{N} l$ and the heat flow per metre length is

$$\frac{\dot{Q}}{l} = k\left(\frac{S}{l}\right)(T_1 - T_2)$$

where l is the distance perpendicular to the plane of the figure.



We select $N = 6$. With 6 increments, we can approximate

$$\Delta T \equiv (T_1 - T_2)/N = (400 - 100)/6 = 50\,°\text{C}$$

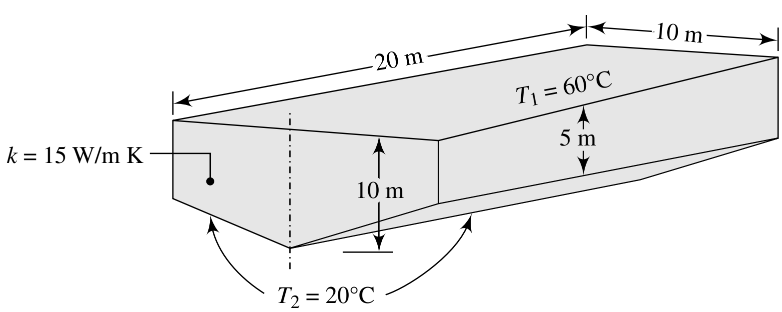
We draw the flux plot using the graphical method and count the number of heat flow channels, $M = 8$.
Hence, for the left half of the object,

Shape factor, $S = \dfrac{M}{N} l = \dfrac{8}{6} l = 1.33 l$

For the whole object, $S = 2 \times 1.33l = 2.666l$
Heat-transfer rate per unit length is

$$\frac{\dot{Q}}{l} = (5\text{ W/m K})\,(2.666)(400 - 100)\,°\text{C} \;\; = \mathbf{4000\ W} \quad \text{or} \quad \mathbf{4\ kW} \qquad \textbf{(Ans.)}$$

EXAMPLE 15.3 *Estimate the heat-transfer rate through the object ($\underline{k}$ = 15 W/m K) shown below using the flux plot method. Heat loss from the sides may be assumed negligible.*

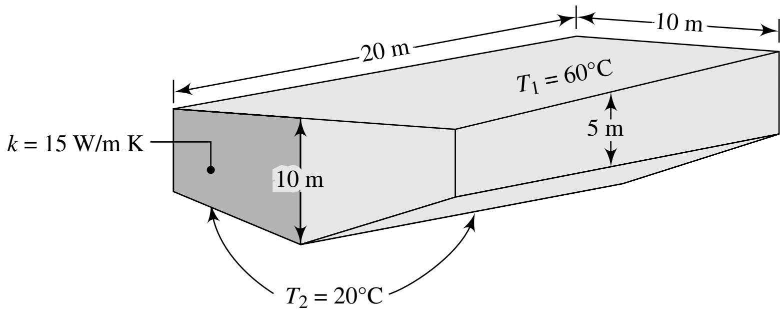


Solution

Known Shape and surface conditions of an object.

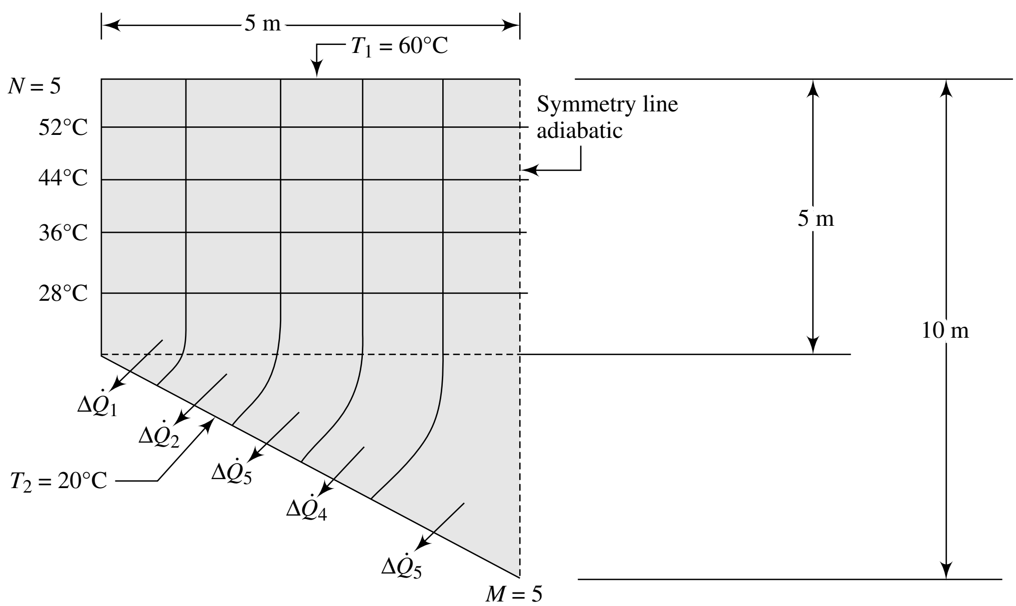
Find Heat flow rate.

Schematic



Assumptions (1) Two-dimensional, steady-state conduction. (2) Constant thermal conductivity.

Analysis We draw the flux plot for the cross-hatched symmetrical section and find the shape factor. For the entire object, the shape factor will be double this value.



We select 5 temperature increments i.e. N = 5 so that each

From the flux plot drawn, we find that the number of heat flow lanes, $M = 5$.

Hence, S for half-section $= \dfrac{M}{N} l = \dfrac{5}{5} \times 20 \text{ m} = 20 \text{ m}$

For the whole object, $S = 2 \times 20 \text{ m} = 40 \text{ m}$

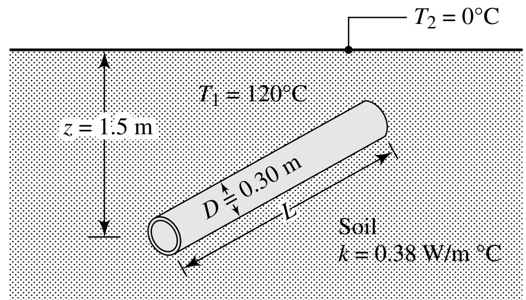
It follows that, $\dot{Q} = k\,S\,(T_1 - T_2) = (15 \text{ W/m K})\,(40 \text{ m})\big(60 - 20\big)°\text{C}$ = **24 kW** **(Ans.)**

(C) Conduction Shape Factors

EXAMPLE 15.4 *A 30 cm diameter steam pipe at a constant surface temperature of 120°C is buried horizontally with its centreline at a depth of 1.5 m below the earth's surface having a uniform temperature of 0°C. The thermal conductivity of the soil at that location is 0.38 W/m °C. Using a flux plot, determine the heat loss per linear metre of pipe. Check your result against that obtained by using the appropriate shape factor.*

Solution

Schematic



Known A horizontal isothermal pipe laid beneath the earth carries steam at a specified temperature.

Find Flux plot. Conduction shape factor, S.

Assumptions (1) Steady-state conditions. (2) Two-dimensional conduction. (3) Temperature drop across the pipe wall is negligible. (4) Constant soil thermal conductivity. (5) The earth is a semi-infinite medium.

Analysis The pipe carrying steam loses heat to the soil surrounding it.

Heat-transfer rate, $\dot{Q} = S\,k\,(T_1 - T_2)$

where, using the analytical expression the conduction shape factor is

$$S = \frac{2\pi L}{\cosh^{-1}(2z/D)} \quad (L >> D)$$

$$= \frac{2\pi(1\text{ m})}{\cosh^{-1}(2 \times 1.5\text{ m}/0.3\text{ m})} = 2.1\text{ m}$$

$\therefore$ $\dot{Q}$ = (2.1 m) (0.38 W/m °C) (120 – 0) °C = **95.8 W** **(Ans.)**

For the flux plot:

$$S = \frac{\text{Total heat flow lanes} \times \text{Depth of segment}}{\text{Number of temperature increments}} = \frac{M}{N}l$$

Only one half of the heat flow field is shown because of the symmetry of the configuration. We have arbitrarily selected 8 temperature increments of 15°C each, given the overall temperature difference of (120 – 0)°C or 120°C. There are 18 heat flow lanes. Therefore,

$$S = \frac{M}{N} = \frac{18}{8} = 2.25 \qquad \text{(per unit length)}$$

and $\dot{Q} = S\,k\,(T_1 - T_2)$ = (2.25 m) (0.38 W/m °C) (120 – 0)°C = **102.6 W**

Comments The answers obtained by flux plotting compare favourably with those obtained by using the analytical expression.

EXAMPLE 15.5 *A 25 m long and 12 cm diameter hot-water pipe is buried in the soil 60 cm below the ground surface. The outer surface temperature of the pipe is 70°C. Taking the surface temperature of the earth to be 10°C and the thermal conductivity of the soil at that location to be 0.55 W/m K, determine the rate of heat loss from the pipe.*

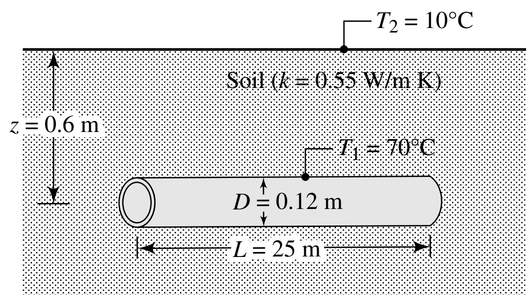
Solution

Known The hot water pipe is buried in the soil.

Find Rate of heat loss from the pipe.

Assumptions (1) Steady operating conditions prevail. (2) Constant thermal conductivity of the soil. (3) Two-dimensional heat conduction (no change in the axial direction).

Schematic



Analysis The conduction shape factor for this configuration is, $S = \dfrac{2\pi L}{\ln\left(4\,z/D\right)}$

Since $z > 1.5\,D$ where z is the distance of the pipe from the ground surface and D is the diameter of the pipe. Substituting the relevant values,

$$S = \frac{2\pi(25\text{ m})}{\ln(4\times 0.60\text{ m}/0.12\text{ m})} = 52.43\text{ m}$$

Steady-state heat-transfer rate from the pipe is

$$\dot{Q} = k\,S\left(T_1 - T_2\right) = (0.55\text{ W/m K})\,(52.43\text{ m})\,(70 - 10)\text{ K} = \mathbf{1730\ W} \quad \textbf{(Ans.)}$$

Comment The heat is transferred to the ground surface by conduction through the soil and then dissipated to the atmosphere by convection and radiation.

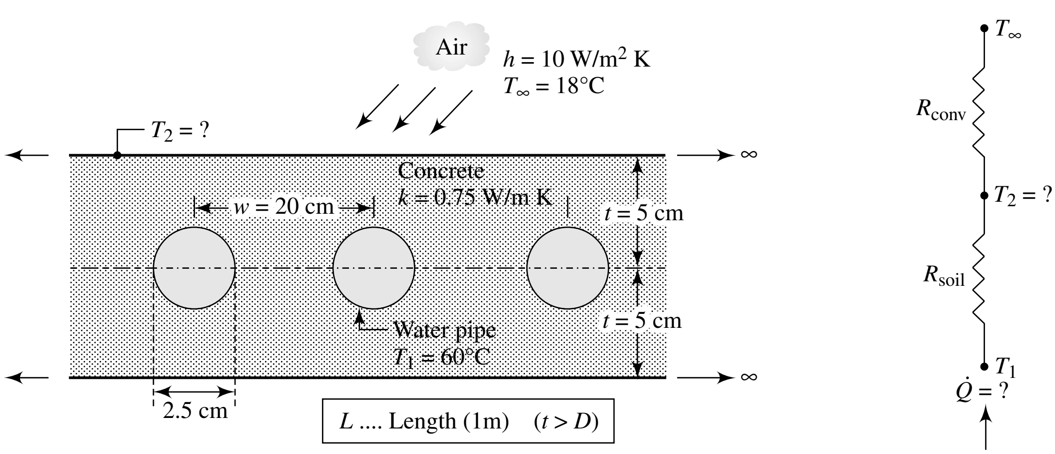
EXAMPLE 15.6 *Hot water pipes of 2.5 cm diameter, are located at the midplane of a 10 cm thick concrete floor (k = 0.75 W/m °C) and are spaced 20 cm apart centre to centre. Water flowing through the pipes is 60°C. The ambient air temperature is 18°C and the convection heat transfer coefficient is 10 W/m² K. Determine (a) the heat-transfer rate per m length of the pipe, and (b) the surface temperature of the concrete.*

Solution

Known Hot-water pipes are placed parallel to each other at the centreline of a concrete floor exposed to convective environment.

Find (a) Heat-transfer rate, (b) Concrete surface temperature per unit length.

Schematic



Assumptions (1) Steady operating conditions exist. (2) Constant soil properties and uniform air side heat-transfer coefficient. (3) Water-side convection coefficient is very high. (4) Two-dimensional conduction in soil.

Analysis For the prescribed geometrical configuration, the conduction shape factor is

$$S = \frac{2\pi L}{\ln\left[\left(\frac{2w}{\pi D}\right)\sinh\left(\frac{\pi t}{w}\right)\right]} = \frac{2\pi \times 1\text{ m}}{\ln\left\{\left(\frac{2\times 20\text{ cm}}{\pi\times 2.5\text{ cm}}\right)\sinh\left(\frac{\pi\times 5\text{ cm}}{20\text{ cm}}\right)\right\}} = 2.0\text{ m} \quad (t > D)$$

Heat-loss rate from water to air is

$$\dot{Q} = \frac{\Delta T_{\text{overall}}}{R_{\text{total}}} = \frac{T_1 - T_\infty}{R_{\text{soil}} + R_{\text{conv}}}$$

Thermal resistances

Conduction resistance through soil

$$= \frac{1}{Sk} = \frac{1}{(2\text{ m})(0.75\text{ W/m K})} = 0.666\text{ K/W}$$

Convection resistance (*air side*)

$$= \frac{1}{hA} = \frac{1}{h(DL)} = \frac{1}{(10\text{ W/m}^2\text{ K})\,(0.025\text{ m}\times 1\text{ m})} = 4.0\text{ K/W}$$

∴ $R_{\text{total}} = 0.666 + 4.0 = 4.666$ K/W

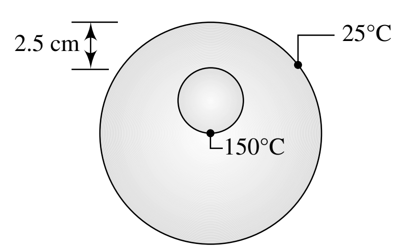
∴ $\dot{Q} = \frac{(60-18)\,°\text{C} \quad \text{or} \quad K}{4.666\text{ K/W}} = \mathbf{9.0\ W}$ **(Ans.) (a)**

Also $\dot{Q} = \frac{T_2 - T_\infty}{R_{\text{conv}}}$

∴ Concrete floor temperature,

$T_2 = T_\infty + \dot{Q}\cdot R_{\text{conv}} = 18°\text{C} + (9\text{ W})\,(4\text{ K/W}) = \mathbf{54°C}$ **(Ans.) (b)**

EXAMPLE 15.7 *Determine the rate of heat transfer per m length from a 5 cm OD pipe at 150°C placed eccentrically within a large cylinder of rockwool as shown in the figure. The outside diameter of the larger cylinder of rockwool is 15 cm and the surface temperature is 25°C. Take k for rockwool = 0.065 W/m °C.*



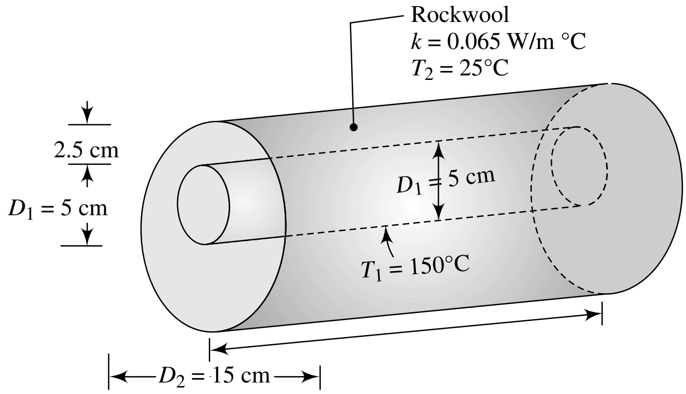
Solution

Known Eccentric insulation to prevent heat loss from a pipe with a specified geometry.

Find Heat-transfer rate per m length.

Assumptions (1) Pipe and larger cylinder are thin-walled (wall resistance is negligible). (2) Constant properties. (3) Steady-state conditions prevail. (4) Two-dimensional conduction.

Schematic



Analysis Conduction shape factor,

$$S = \frac{2\pi L}{\cosh^{-1}\left\{\dfrac{D_1^2 + D_2^2 - 4z^2}{2D_1 D_2}\right\}}$$

Substituting numerical values,

$$S = \frac{2\pi \times 1\ \text{m}}{\cosh^{-1}\left\{\dfrac{5^2 + 15^2 - 4(2.5)^2}{2 \times 5 \times 15}\right\}} = \frac{2\pi}{\cosh^{-1} 1.5} = 6.53\ \text{m}$$

Heat loss per metre length of the pipe is

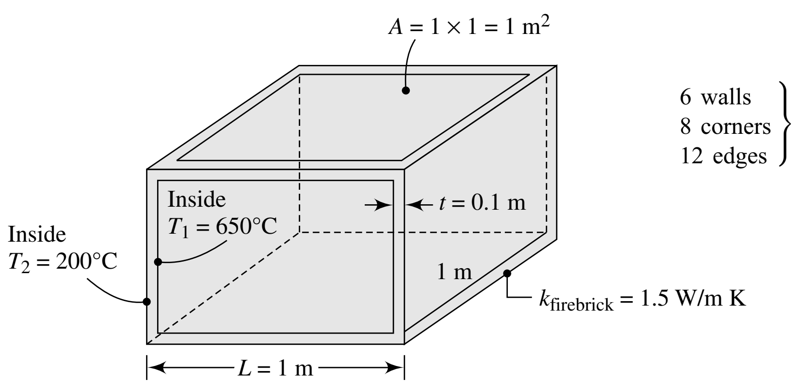
$\dot{Q} = Sk(T_1 - T_2)$ = (6.53 m) (0.065 W/m °C) (150 – 25)°C = **53.0 W** **(Ans.)**

EXAMPLE 15.8 *A small cubical furnace, 1 m × 1 m × 1 m on the inside, is made of 10 cm thick fireclay brick (k = 1.5 W/m K). Calculate the rate of heat loss if the inside is maintained at 650°C and the outside at 200°C.*

Solution

Known A cubical furnace made of fireclay brick has specified inside and outside temperatures.

Find Heat transfer rate.



Assumptions (1) Steady-state conditions. (2) Constant properties. (3) Isothermal surfaces.

Analysis Conduction shape factor for the specified geometry:

Per wall: $S = \dfrac{A}{t} = \dfrac{1 \times 1\,\text{m}^2}{0.1\,\text{m}} = 10$ m

Per edge: $S = 0.54\,L = 0.54$ (1 m) = 0.54 m

Per corner: $S = 0.15\,t = 0.15$ (0.1 m) = 0.015 m

For the *six* walls, *twelve* edges and *eight* corners,

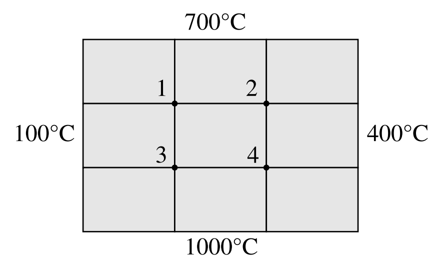
$$S = (6 \times 10) + (12 \times 0.54) + (8 \times 0.15) = 66.6 \text{ m}$$

and the rate of heat loss is

$$\dot{Q} = S\,k\,(T_1 - T_2) = (66.6 \text{ m})(1.5 \text{ W/m K})(650 - 200)°\text{C or K}$$
$$= \mathbf{45 \times W \text{ or } 45 \text{ kW}}$$ **(Ans.)**

(D) Numerical Methods: Relaxation Technique

EXAMPLE 15.9 *Calculate, using the relaxation method, the steady-state temperature at the four interior nodes of the square plate shown:*

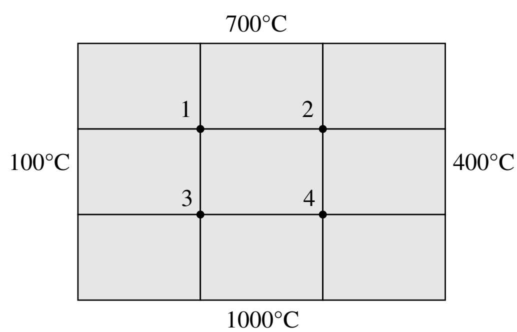


Solution

Known A square plate subjected to uniform surface temperature conditions.

Find Temperatures at the specified four nodes.

Schematic



Assumptions (1) Two-dimensional conduction. (2) Steady-state conditions.

Analysis The residuals for the four nodes 1, 2, 3 and 4 are:

***Node 1*:** $R_1 = 700 + 100 + T_2 + T_3 - 4\,T_1$

$R_1 = 800 + T_2 + T_3 - 4T_1$

***Node 2*:** $R_2 = 700 + 400 + T_1 + T_4 - 4\,T_2$

$R_2 = 1100 + T_1 + T_4 - 4T_2$

***Node 3*:** $R_3 = 1000 + 100 + T_1 + T_4 - 4\,T_3$

$R_3 = 1100 + T_1 + T_4 - 4\,T_3$

Node 4: $R_4 = 1000 + 400 + T_2 + T_3 - 4\,T_4$

$$R_4 = 1400 + T_2 + T_3 - 4\,T_4$$

Initial guess for node temperatures T, T_2, T_3, and T_4:

T_1 = 400°C T_3 = 500°C

T_2 = 500°C T_4 = 600°C

The residuals are then calculated to be

$$R_1 = 800 + 500 + 500 - (4 \times 400) = 200$$

$$R_2 = 1100 + 400 + 600 - (4 \times 500) = 100$$

$$R_3 = 1100 + 400 + 600 - (4 \times 500) = 100$$

$$R_4 = 1400 + 500 + 500 - (4 \times 600) = 0$$

We now calculate the initial step size and prepare the table for unit block operation below:

	T_1	R_1	T_2	R_2	T_3	R_3	T_4	R_4
	400	200	500	100	500	100	600	0
$\Delta T_1 = 1$		– 4		1		1		
$\Delta T_2 = 1$		1		– 4				1
$\Delta T_3 = 1$		1				– 4		1
$\Delta T_4 = 1$				1		1		– 4
Total with $\Delta T_1 = \Delta T_2 = \Delta T_3 = \Delta T_4 = 1$		–2		–2		–2		–2

Sum of residuals as a result of unit block operation

= (–2) + (–2) + (–2) + (–2) = – 8

Sum of residuals with initial guesses of the four node temperatures is

200 + 100 + 100 – 0 = 400

Step size = 400/(–8) = **–50**

Relaxation table is now prepared.

	T_1	R_1	T_2	R_2	T_3	R_3	T_4	R_4
Initial values	400	200	500	100	500	100	600	0
Block (50)	+50		+50		+50		+50	
		100		0		0		–100
	+25	0		+25		+25		–100
		$\boxed{0}$		$\boxed{0}$		$\boxed{0}$	–25	$\boxed{0}$
	475		**550**		**550**		**625**	

With an increment of 50 for T_1, T_2, T_3, and T_4, the new residuals are

$\boldsymbol{R_1}$ **:** $200 + \left[T_2 + T_3 - 4\,T_1\right] = 200 + \left[50 + 50 - 4 \times 50\right] = 200 - 100 = \mathbf{100}$

Similarly,

***R_2* :** $100 + (T_1 + T_4 - 2T_2) = 100 + (-100) = \mathbf{0}$

***R_3* :** $100 + (-100) = \mathbf{0}$

***R_4* :** $0 - 100 = \mathbf{100}$

Our objective should be to reduce the maximum value of the residual to zero.

For $T_1 = 25,\ R_1 = -4T_1 + 100 = 0$

$R_2 = 25 + 0 = 25$

$R_3 = 25 + 0 = 25$

$R_4 = -100$

To reduce R_4 to 0, we take $T_4 = -25$, so that $R_4 = -100 + [-4\ (-25)] = 0$
R_3 becomes $25 + (-25) = 0$ and R_2 also becomes $25 + (-25) = 0$. R_1 remains unaffected and is 0. Since R_1, R_2, R_3 and R_4 have zero values now, we add up the values of node temperatures to get the final answer.
Finally,

$$\mathbf{T_1 = 475^\circ C, \quad T_2 = 550^\circ C, \quad T_3 = 550^\circ C} \quad \text{and} \quad \mathbf{T_4 = 625^\circ C} \qquad \textbf{(Ans.)}$$

Check:

$$R_1 = 800 + 550 + 550 - (4 \times 475) = 0$$

$$R_2 = 1100 + 475 + 625 - (4 \times 550) = 0$$

$$R_3 = 1100 + 475 + 625 - (4 \times 550) = 0$$

$$R_4 = 1400 + 550 + 550 - (4 \times 625) = 0$$

Points to Ponder

- The analytical method used to deal with two-dimensional conduction problems yields solutions that are exact at any point in the conducting material.
- The numerical method of solution gives temperatures only at specified points in the solid involving two dimensional steady state conduction.
- If the number of heat-flow lanes and the number of temperature increments are 15 and 5 each, then the conduction shape factor is equal to 3.
- When two different physical phenomena can be described by the same equation, the two phenomena are said to be analogous.
- In the numerical method of analysis, the differential equation and the boundary conditions are expressed as a set of algebraic equations.
- Multi-dimensional steady-state conduction heat-transfer problems can be solved *analytically*, *graphically*, *analogically* or *numerically*.
- Separation of variables technique can be applied to solve linear partial differential equations provided the differential equation can be expressed in a product form satisfying the prescribed boundary conditions.

- Graphical methods give an approximate solution to steady-state, two-dimensional conduction heat-transfer problems.
- Isotherms are constant temperature lines and are perpendicular to insulated surfaces in the graphical solution.
- Heat-flow lines are always perpendicular to isotherms and bisect the corners of isothermal boundaries.

GLOSSARY of Key Terms

• **Conduction shape factor**	It is a characteristic of the specific geometry defined as the heat-transfer rate per unit thermal conductivity and per unit temperature difference and has the dimension of length
• **Flux plotting**	It is an approximate graphical method to solve two-dimensional conduction problems with isothermal and adiabatic boundaries.
• **Isotherms**	Lines of constant temperature.
• **Adiabats**	Lines perpendicular to heat-flow lines without heat flow. Also called *lines of symmetry*.
• **Curvilinear square**	Squares in which isotherms and heat-flow lines are orthogonal to each other.
• **Numerical analysis**	It involves the development of an approximate solution to the heat conduction equation for complex geometries in multiple dimension.

OBJECTIVE-TYPE QUESTIONS

• Multiple-Choice Questions

15.1 Select the correct answer:
Isotherms are constant temperature lines and they are
(a) parallel to insulated surfaces
(b) parallel to heat-flow lines
(c) perpendicular to both insulated surfaces and heat-flow lines
(d) bisecting the corners of isothermal boundaries

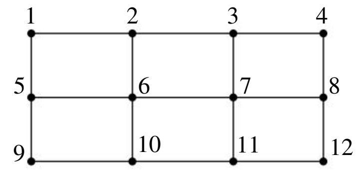
15.2 Select the incorrect answer:
(a) Diagonals of curvilinear squares in a flux plot bisect each other at 90° but do not bisect the corners
(b) Flow lines are perpendicular to isothermal boundaries
(c) Isotherms are perpendicular to insulated boundaries
(d) Isotherms and flow lines form a network of curvilinear squares.

15.3 If the number of isotherms is M and the number of heat flow lanes is N, the conduction shape factor S is defined as
(a) M/N (b) N/M (c) $(M - 1)/N$ (d) $(N - 1)M$

15.4 Conduction shape factor has the SI unit of
(a) m^{-1} (b) m^2 (c) m (d) dimensionless

15.5 Using the graphical method to solve a two-dimensional heat conduction problem, if N is the number of heat-flow lanes and M is the number of squares in each flow lane for the entire configuration, the conduction shape factor S is
(a) M/N (b) N/M (c) MN (d) MN^2

15.6 The correct steady-state finite difference heat conduction equation of the node 7 of the rectangular solid is shown in the following figure:



(a) $T_7 = \frac{1}{2}(T_2 + T_3 + T_{10} + T_{11})$

(b) $T_7 = \frac{1}{2}(T_6 + T_8 + T_3 + T_{11})$

(c) $T_7 = \frac{1}{4}(T_6 + T_3 + T_8 + T_{11})$

(d) $T_7 = \frac{1}{4}(T_2 + T_4 + T_{10} + T_{12})$

Answers

Multiple-Choice Questions

15.1 (c) **15.2** (a) **15.3** (c) **15.4** (c) **15.5** (a) **15.6** (c)

REVIEW QUESTIONS

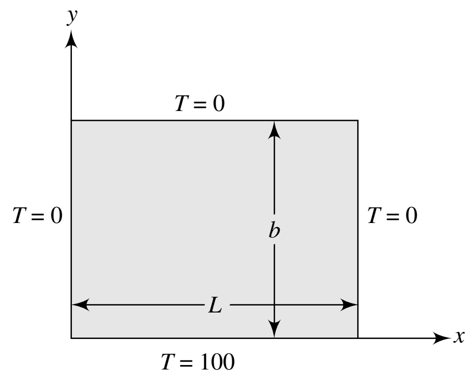
15.1 Define the conduction shape factor. How is it related to the thermal resistance?
15.2 What is the significance of conduction shape factors in engineering?
15.3 What are the methods of analyzing heat conduction in two- and three-dimensional systems?
15.4 What are the difficulties associated with the analytical method of analysis of two- and three-dimensional heat conduction problems?
15.5 What is the advantage associated with the graphical and numerical methods of analysis of two- and three-dimensional heat conduction problems?
15.6 What advantage is obtained by choosing $\Delta x = \Delta y$ in the graphical method of analyzing the heat conduction equation?
15.7 When do we say that two different physical phenomena are analogous?
15.8 What is the basic procedure in setting up a numerical solution to a two-dimensional conduction problem?
15.9 Having developed the finite difference equations for a conduction problem, what methods can be used to obtain a solution?
15.10 What purpose does an *operation table* serve in the relaxation technique?
15.11 Name a few shortcuts which can be introduced in the relaxation technique to save considerable effort and time.
15.12 What is meant by *block operation* in the relaxation technique?

PRACTICE PROBLEMS

(A) Two-Dimensional Steady-State conduction

15.1 A rectangular plate of 1 m width in the x direction, infinite in the y direction, has a temperature distribution given by $T(x,0) = 100 \sin \pi x$ imposed on the $y = 0$ edge. Determine the temperature distribution $T(x, y)$. **[$T(x, y) = 100e^{-\pi y} \sin \pi x$]**

15.2 Compute the temperature distribution in an infinitely long two-dimensional rectangular bar, shown in the figure.

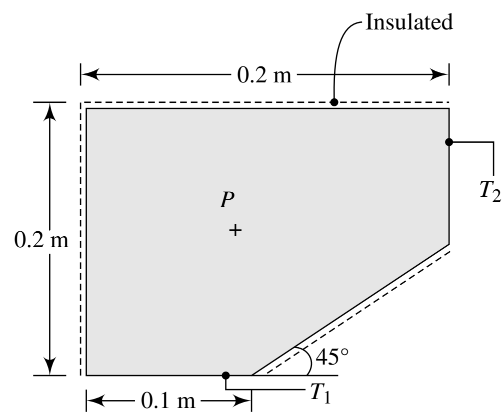


$$\left[\boldsymbol{T = \frac{400}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin(2n+1) \frac{\pi x}{L} \sinh \frac{(b-y)(2n+1)\pi}{L} \text{cosech} \frac{(2n+1)\pi b}{L}} \right]$$

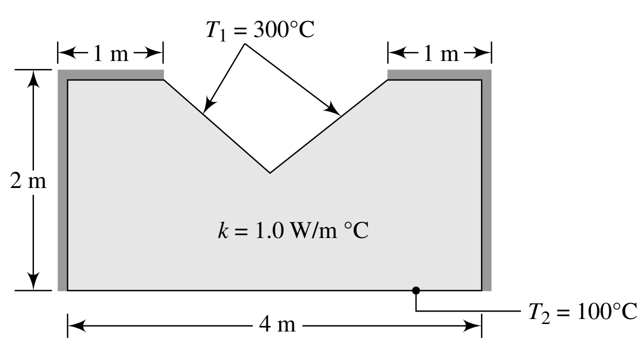
(B) Graphical Method

15.3 A structural member fabricated from a material with a thermal conductivity of 60 W/m°C has the cross section shown in the following figure. The temperature of the end faces are T_1 = 100°C and T_2 = 0°C, while the remaining sides are insulated. (a) Estimate the temperature at the position *P* (b) Using the graphical flux plot method, estimate the conduction shape factor and the heat transfer rate through the strut per metre length.

[(a) 50°C (b) 0.53 *L*, 3180 W/m]



15.4 Determine the shape factor and the heat flow per unit depth of the V-grooved channel shown below, using a flux plot: **[2.34 *l* , 468 W/m]**

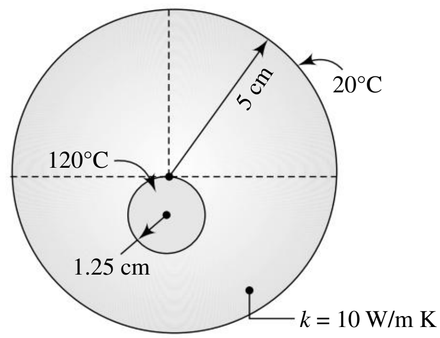


(C) Conduction Shape Factors

15.5 Saturated steam at 7 bar (T_{sat} = 165°C, h_{fg} = 2066.3 kJ/kg) is piped vertically into the earth at a mass flow rate of 250 kg/h as a part of an oil reclamation project. The thermal conductivity of the earth is 1.4 W/m K and the outside diameter of the pipe is 10 cm. The ground surface temperature is 15°C. How deep will the steam travel before it condenses completely? **[1170 m]**

15.6 A 25-mm-OD heating rod is eccentrically embedded in a 100-mm-OD cylinder as shown in figure. For the conditions shown, (a) determine the heat flow from the heating rod per unit length. (b) If the same heat loss takes place under insulation without any eccentricity, what will be the outer diameter of insulation for the same values of temperature.

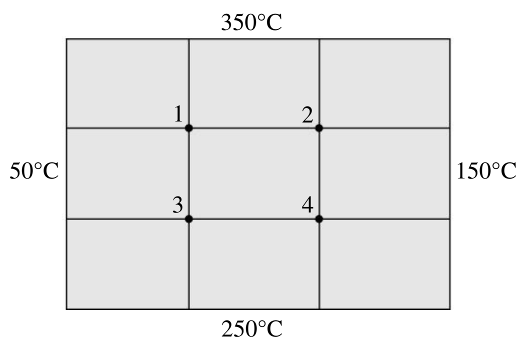
[(a) 4.77 kW (b) 9.33 cm]



15.7 A radioactive sample is to be stored in a protective box with 10 cm thick walls having inside dimensions of 4 cm × 4 cm × 12 cm. The radiation emitted by the sample is completely absorbed at the inner surface of the box, which is made of concrete (k = 1.37 W/m°C. If the outside temperature of the box is 25°C, but the inside temperature is limited to a maximum of 50°C, estimate the maximum allowable rate of radiation from the sample. **[26.6 W]**

(D) Relaxation Method

15.8 By using the relaxation method, determine the temperatures at the points 1, 2, 3, and 4 under steady operating conditions.



[T_1 = 200°C, T_2 = 225°C, T_3 = 175°C, T_4 = 200°C]

Appendix

Table A-1 *Thermophysical Properties of Dry Air at Atmospheric Pressure*

T, °C	μ, kg/m^3	C_p, kJ/kg K	k (W/m K)	$\mu \times 10^6$ N s/m^2	$\nu \times 10^6$ m^2/s	$\alpha \times 10^6$ m^2/s	Pr
–50	1.584	1.013	0.0204	12.7	9.23	14.6	0.728
–40	1.515	1.013	0.0212	13.8	10.04	15.2	0.728
–30	1.453	1.013	0.0220	14.9	10.8	15.7	0.723
–20	1.395	1.009	0.0228	16.2	12.79	16.2	0.716
–10	1.342	1.009	0.0236	17.4	12.43	16.7	0.712
0	1.293	1.005	0.0244	18.8	13.28	17.2	0.707
10	1.247	1.005	0.0251	20.0	14.16	17.6	0.705
20	1.205	1.005	0.0259	21.4	15.06	18.1	0.703
30	1.165	1.005	0.0267	22.9	16.00	18.6	0.701
40	1.128	1.005	0.0276	24.3	16.96	19.1	0.699
50	1.093	1.005	0.0283	25.7	17.95	19.6	0.698
60	1.060	1.005	0.0290	27.2	18.97	20.1	0.696
70	1.029	1.009	0.0296	28.6	20.02	20.6	0.694
80	1.000	1.009	0.0305	30.2	21.09	21.1	0.692
90	0.972	1.009	0.0313	31.9	22.10	21.5	0.690
100	0.946	1.009	0.0321	33.6	23.13	21.9	0.688
120	0.898	1.009	0.0334	36.8	25.45	22.8	0.686
140	0.854	1.013	0.0349	40.3	27.8	23.7	0.684
160	0.815	1.017	0.0364	43.9	30.09	24.5	0.682
180	0.779	1.022	0.0378	47.5	32.49	25.3	0.681
200	0.746	1.026	0.0393	51.4	34.85	26.0	0.680
250	0.674	1.038	0.0427	61.0	40.61	27.4	0.677
300	0.615	1.047	0.0460	71.6	48.33	29.7	0.674
350	0.566	1.059	0.0491	81.9	55.46	31.4	0.676
400	0.524	1.068	0.0521	93.1	63.09	33.0	0.678
500	0.456	1.093	0.0574	115.3	79.38	36.2	0.687
600	0.404	1.114	0.0622	138.3	96.89	39.1	0.699
700	0.362	1.135	0.0671	163.4	115.4	41.8	0.706
800	0.329	1.156	0.0718	188.8	134.8	44.3	0.713
900	0.301	1.172	0.0763	216.2	155.1	46.7	0.717
1000	0.277	1.185	0.0807	245.9	177.1	49.0	0.719
1100	0.257	1.197	0.0850	276.2	199.3	51.2	0.722
1200	0.239	1.210	0.0915	316.5	233.7	53.5	0.724

Table A-2 *Physical Properties of Saturated Water*

T, °C	ρ, kg/m³	C_p, kJ/kg K	k(W/m K)	α × 10⁶ m²/s	μ × 10⁶, N s/m²	ν × 10⁶, m²/s	β × 10⁴ 1/K	Pr
0	999.9	4.212	0.551	0.131	1788	1.789	–0.63	13.67
10	999.7	4.191	0.574	0.137	1306	1.306	+0.70	9.52
20	998.2	4.183	0.599	0.143	1004	1.006	1.82	7.02
30	995.7	4.174	0.618	0.149	801.5	0.805	3.21	5.42
40	992.2	4.174	0.635	0.153	653.3	0.659	3.87	4.31
50	988.1	4.174	0.648	0.157	549.4	0.556	4.49	3.54
60	983.1	4.179	0.659	0.160	469.9	0.478	5.11	2.98
70	977.8	4.187	0.668	0.163	406.1	0.415	5.70	2.55
80	971.8	4.195	0.674	0.166	355.1	0.365	6.32	2.21
90	965.3	4.208	0.680	0.168	314.9	0.326	6.95	1.95
100	958.4	4.220	0.683	0.169	282.5	0.295	7.52	1.75
110	951.0	4.233	0.685	0.170	259.0	0.272	8.08	1.60
120	943.1	4.250	0.686	0.171	237.4	0.252	8.64	1.47
130	934.8	4.266	0.686	0.172	217.8	0.233	9.19	1.36
140	926.1	4.287	0.685	0.172	201.1	0.217	9.72	1.26
150	917.0	4.313	0.684	0.173	186.4	0.203	10.3	1.17
160	907.0	4.346	0.683	0.173	173.6	0.191	10.7	1.10
170	897.3	4.380	0.679	0.173	162.8	0.181	11.3	1.05
180	886.9	4.417	0.674	0.172	153.0	0.173	11.9	1.00
190	876.0	4.459	0.670	0.171	144.2	0.165	12.6	0.96
200	863.0	4.505	0.663	0.170	136.4	0.158	13.3	0.93
210	852.8	4.555	0.655	0.169	130.5	0.153	14.1	0.91
220	840.3	4.614	0.645	0.166	124.6	0.148	14.8	0.89
230	827.3	4.681	0.637	0.164	119.7	0.145	15.9	0.88
240	813.6	4.756	0.628	0.162	114.8	0.141	16.8	0.87
250	799.0	4.844	0.618	0.159	109.9	0.137	18.1	0.86
260	784.0	4.949	0.605	0.156	105.9	0.135	19.7	0.87
270	767.9	5.070	0.590	0.151	102.0	0.133	21.6	0.88
280	750.7	5.230	0.574	0.146	98.1	0.131	23.7	0.90
290	732.3	5.485	0.558	0.139	94.2	0.129	26.2	0.93
300	712.5	5.736	0.540	0.132	91.2	0.128	29.2	0.97
310	691.1	6.071	0.523	0.125	88.3	0.128	32.9	1.03
320	667.1	6.574	0.506	0.115	85.3	0.128	38.2	1.11
330	640.2	7.244	0.484	0.104	81.4	0.127	43.3	1.22
340	610.1	8.165	0.457	0.917	77.5	0.127	53.4	1.39
350	574.4	9.504	0.430	0.788	72.6	0.126	66.8	1.60
360	528.0	13.984	0.395	0.536	66.7	0.126	109	2.35
370	450.5	40.321	0.337	0.186	56.9	0.126	164	6.79

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Nomenclature

Many symbols have more than one meaning. The context will indicate the specific meaning.

a	Speed of sound, m/s
Bi	Biot number
C_p	Specific heat, kJ/kg °C or kJ/kg K or J/kg K
C	Thermal capacity, W/°C
c_o	Speed of light, m/s
C_f	Friction coefficient
d, D	Diameter, m
D_h	Hydraulic diameter, m
E, E_λ	Emissive power, W/m^2 (*total*) or W/m^2 µm (*spectral*)
E_{st}	Total energy stored, J
erf	Error function
erfc	Complementary error function
F_{ij}	Shape factor between surface *i* and surface *j*
f	Frequency, Hz
f	Friction factor
f_l	Fraction of blackbody radiation between wavelengths 0 and l
Fo	Fourier number
g	Acceleration due to gravity, m/s^2
$\overline{q}$	Heat generation rate, W/m^3
G	Irradiation, W/m^2
Gr	Grashof number
Gz	Graetz number
h	Heat transfer coefficient, W/m^2 °C or W/m^2 K
h	Planck's constant, J s
h_r	Radiation heat-transfer coefficient, W/m^2 °C or W/m^2 K
h_{sf}	Latent heat of melting/solidification, kJ/kg
I_0, I_1	Modified Bessel function of order zero and order one, respectively
I	Radiation intensity, W/m^2 sr (*total*) or W/m^2 sr mm (*monochromatic*)
J	Radiosity, W/m^2
k	Thermal conductivity, W/m °C or W/m K
L	Length, m
L_c	Characteristic length, m,
LMTD	Logarithmic mean temperature difference, °C or K
L_m, L_e	Mean (*effectives*) beam length, m
M	Fin parameter for a uniform area fin, m^{-1}
M	Mach number
M	Mass, kg
NTU	Number of transfer units
Nu	Nusselt number
P	Perimeter, m
P	Pressure, atm
Pe	Peclet number
Pr	Prandtl number
Q	Heat flux, W/m^2

$\dot{Q}$	Heat transfer rate, W
R_e	Electrical resistance, W
R	Radial coordinate, m
R	Radius, m
R	Ratio of heat capacities in heat exchangers
R	Thermal resistance, °C/W or m^2 °C/W
Ra	Rayleigh number
Re	Reynolds number
R_f	Fouling resistance, m^2 °C/W
S	Conduction shape factor
G_s	Solar constant, W/m^2
A_s	Surface area, m^2
S_D	Diagonal pitch, m
S_L	Longitudinal pitch, m
St	Stanton number
S_T	Transverse pitch, m
T	Temperature, °C or K
t	Time, s
τ	Transmissivity or transmittance
U_∞, u_∞	Free stream velocity, m/s
U	Overall heat transfer coefficient, W/m^2 °C or W/m^2 K
U	x-component of velocity, m/s
V	y-component of velocity, m/s
$\forall$	Volume, m^3
V	Arrange velocity
x	x-coordinate, m
y	y-coordinate, m
z	z-coordinate, m

GREEK SYMBOLS

α	Absorptivity
α	Thermal diffusivity, m^2/s
β	Isobaric volumetric expansion coefficient, K^{-1}
δ	Boundary layer thickness, m
δ	Depth of penetration, m
δ_t	Thermal boundary layer thickness, m
ϕ	Phase angle, rad
ε	Eddy viscosity
ε	Effectiveness of a fin
ε	Emissivity
ε	Heat exchanger effectiveness
α_H	Eddy diffusivity of heat
η	Similarity variable
η	Fin efficiency
κ	Absorption coefficient, m^{-1}
μ	Cosine of angle with respect to normal
ν	Kinematic viscosity, m^2/s
ν	Photon frequency, Hz
θ	Angle, degree or radian
θ	Non-dimensional temperature
ρ	Density, kg/m^3
ρ	Radius ratio
ρ	Reflectivity
ρ	Resistivity, Ω-m
σ	Stefan-Boltzmann constant, W/m^2 K^4
τ	Shear stress, Pa
τ	Time constant, s
τ	Transmissivity
ω	Circular frequency, rad/s
ω	Solid angle, sr

SUBSCRIPTS

f	Fluid
∞	Pertaining to the ambient
rad	Pertaining to radiation
r	Radial
sur	Pertaining to surroundings
1, 2, etc.	Pertaining to a specific position

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