As per the
Revised
Syllabus Effective
August 2005

Network Analysis

Second Edition

About the Authors

A. Sudhakar is presently working as a Faculty of Engineering and Computer Technology at Asian Institute of Medicine, Science and Technology, Malaysia. He is on leave from his previous post as Professor and Head of the Department of Electronics and Communication Engineering, RVR & JC College of Engineering, India. He obtained his B.Tech degree (1978–83) from V.R.Siddhartha Engineering college affiliated to Nagarjuna University and M.E from Jadavpur University, Calcutta during 1983-85. He completed his Ph.D from Andhra University, Visakhapatnam in the year 2002. He has over 18 years of teaching and research experience in the field of Electronics and Communication Engineering. He has published 18 papers in various National and International journals and has authored seven books, all published by Tata McGraw-Hill Publishing Company Limited. His fields of interest include Circuits and Signals & Systems.

Shyammohan S. Palli is the Professor and Head of the Department of Electrical and Electronics Engineering at Sir C.R. Reddy College of Engineering, Eluru affiliated to Andhra University. He obtained his B.Tech degree in Electrical Engineering from the College of Engineering, Kakinada, JNTU, during 1973–78. He completed his M.Tech specialising in Power Systems from the same college, during 1978–80. Prior to his current assignment he has worked as Lecturer at Siddaganga Institute of Technology, Tumkur in the year 1981, Lecturer, K.L. College of Engineering during 1981–87, Assistant Professor and Professor at RVR and JC College of Engineering, Guntur affiliated to Nagarjuna University during 1987–2001. His areas of interest include Electrical Machines, Networks and Power Electronics.

As per the
Revised
Syllabus Effective
August 2005

Network Analysis

Second Edition

A Sudhakar
Professor and Head
Department of Electronics and Communications Engineering
RVR & JC College of Engineering,
Guntur, Andhra Pradesh, INDIA

Shyammohan S Palli
Professor and Head
Department of Electrical and Electronics Engineering
Sir C.R. Reddy College of Engineering
Eluru, Andhra Pradesh, INDIA

Tata McGraw-Hill Publishing Company Limited
NEW DELHI

McGraw-Hill Offices
New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

Tata McGraw-Hill

Published by the Tata McGraw-Hill Publishing Company Limited,
7 West Patel Nagar, New Delhi 110 008.

Copyright © 2007, by Tata McGraw-Hill Publishing Company Limited.
No part of this publication may be reproduced or distributed in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise or stored in a database or retrieval system without the prior written permission of the publishers. The program listings (if any) may be entered, stored and executed in a computer system, but they may not be reproduced for publication.

This edition can be exported from India only by the publishers,
Tata McGraw-Hill Publishing Company Limited.

ISBN 0-07-063446-7

Head - Higher Education & School: *S. Raghothaman*
Executive Publisher-SEM & Tech Ed: *Vibha Mahajan*
Editorial Executive-SEM & Tech Ed: *Jacob Alexander*
Editorial Services Manager-SEM & Tech Ed: *Mini Narayanan*

Deputy General Manager - Higher Education & Sales: *Michael J. Cruz*
Asst. Product Manager-SEM & Tech Ed: *Biju Ganesan*

Asst. General Manager - Production: *B L Dogra*
Junior Executive - Production: *Anjali Razdan*

Information contained in this work has been obtained by Tata McGraw-Hill, from sources believed to be reliable. However, neither Tata McGraw-Hill nor its authors guarantee the accuracy or completeness of any information published herein, and neither Tata McGraw-Hill nor its authors shall be responsible for any errors, omissions, or damages arising out of use of this information. This work is published with the understanding that Tata McGraw-Hill and its authors are supplying information but are not attempting to render engineering or other professional services. If such services are required, the assistance of an appropriate professional should be sought.

Typeset at Script Makers, 19, A1-B, DDA Market, Paschim Vihar, New Delhi 110 063, and printed at Sai Printo Pack Pvt. Ltd, A-102/4 Phase II, Okhla Industrial Area, New Delhi - 110 020

Cover: SDR

DDLCRRBKRARLZ

The McGraw·Hill Companies

Dedicated
To
Our Parents
and
Students

Contents

15. TWO-PORT NETWORKS 15.1

16. S-DOMAIN ANALYSIS 16.1

17. FILTERS AND ALTENUATORS 17.1

Foreword

It gives me great pleasure to introduce *Network Analysis* by *Dr A. Sudhakar* and *Dr S. Shyammohan Palli*, publication of which heralds the completion of a book that caters completely and effectively to the students of JNTU.

The need for a good textbook for this subject can be easily understood. Numerous books are available to the students for the subject, but almost none of them have the right combination of simplicity, rigour, pedagogy and syllabus compatibility. These books usually do not address one or more of the specific problems faced by students of this subject in JNTU. There has always been a need for a good book relevant to the requirements of the students and dealing with all aspects of the course. I am sure that the present book will be able to fill this void.

The book has been organized and executed with lot of care and dedication. The authors have been outstanding teachers with vast experience and expertise in their chosen fields of interest. A conscious attempt has been made to simplify concepts to facilitate better understanding of the subject.

Dr Sudhakar and Dr Shyammohan deserve our praise and thanks for accomplishing this trying task. Tata McGraw-Hill, a prestigious publishing house, also deserves a pat on the back for doing an excellent job.

Dr K. Rajagopal
Vice-Chancellor
Jawaharlal Nehru Technological University
Hyderabad

Preface

This book is exclusively designed for use as a text for the course on *Network Analysis* offered to first year undergraduate engineering students of Jawaharlal Nehru Technological University (JNTU), Hyderabad. The primary goal of this text is to establish a firm understanding to the basic laws of electric circuits which develop a working knowledge of the methods of analysis used most frequently in further topics of electrical and electronics engineering. This book also provides a comprehensive insight into the principal techniques available for characterising circuits and networks theoretically.

Illustrative examples are interspersed throughout the book at their natural locations. With so many years of teaching, we have found that such illustrations permit a level of understanding otherwise unattainable. As an aid to both, the instructor and the student, objective questions and the tutorial problems provided at the end of each chapter progress from simple to complex. Answers to selected problems have been given to instill confidence in the reader. Due care is taken to see that the reader can easily start learning circuit analysis without prior knowledge of mathematics. As such, a student of first year B. Tech. will be able to follow the book without any difficulty.

All the elements with definitions, basic laws and different configurations of the resistive circuits have been introduced in the first chapter. Analysis of the D.C. resistive circuits have been discussed in Chapter 2. Graph theory has been written in an easy to understand manner. Network theorems on resistive circuits have been presented in Chapter 3. A.C. fundamentals have been introduced in Chapters 4 and 5 which include voltage-current relation of elements, complex impedance. Power and power factor concept is discussed in Chapter 6. Due emphasis has been laid on finding out the average and rms values of different waveforms in Chapter 4. The steady state analysis of A.C. circuits including network theorems have been discussed in Chapter 7. Problems, tutorials and objective questions on dependent sources have been included in Chapters 1 to 7. Resonance phenomena in series and parallel circuits and locus diagrams are presented in Chapter 8. A brief study of coupled and tuned circuits is introduced in Chapter 9. Magnetic circuits are also discussed in this chapter.

A brief discussion of differential equations is included in Chapter 10. The necessary mathematical background for transient analysis, the transient behavior of A.C and D.C circuits and their response has been discussed in Chapter 11. Laplace transforms and their applications are presented in Chapters 12 and13, respectively. Network functions and stability criteria have been discussed in Chapter 14. The parameters of two-port network and their inter-relations have been discussed in Chapter 15. A brief account of s-domain analysis is presented in Chapter 16. Various types of basic filters, attenuators and equalizers have been discussed in Chapter 17. The book also includes brief topics of Fourier series, Fourier Transforms and operator j . Six Model Question Papers and Solved May/June 2006 Question Papers (4 Sets) are provided at the end of the book.

Many people have helped us in producing this book. We extend our gratitude to them all for helping us in their own individual ways to shape the book into its final form. We would like to express our gratitude to the Management of R.V.R and J.C College of Engineering, particularly to the to the President Dr K. Basava Punnaiah, Secretary and Correspondent Dr M. Gopal Krishna. The management of Sir C. R. Reddy College of Engineering, particularly to the President M. Subashchandra Bose, Secretary G. Subbarao, Correspondent Dr V.V. Balakrishna Rao, Vice President Dr K.Sriramachandramurthy, Joint Secretary Dr K. Madhavarao, and Treasurer Raghunadha Rao for providing us a conducive work atmosphere. We are indebted to Dr P.S. Sankara Rao, Principal of R.V.R. and J.C. College of Engineering and Prof. A. Anand Kumar, Principal of Sir C.R Reddy College of Engineering for their support throughout the work. We are thankful to Mr. M. Ravindra Reddy, Prof. D. Dakshina Murthy, Sri. P.S. Somayajulu, D.S.R.K.V. Prasad, Ms. K. Swarna Sree, Sri. T. Sreerama Murthy, Sri. C.V. Gopalkrishna, Sri. Y.V. Narayana and many other colleagues for their invaluable suggestions. We also thank the students of the ECE Department, particularly N. Anand, S. Suresh Kumar of R.V.R. and J.C. College of Engineering and students of the EEE Department particularly P. Raghu Ram, N. Krisnakishore, V. Ramya, N. Nalini Gandhi of Sir C R Reddy College of Engineering who were involved directly or indirectly with the writing of this book. We are thankful to Mr D.S.R. Anjaneyulu and K. Srinivas for the error – free typing of the manuscript.

We wish to express our appreciation to the various members of Tata McGraw-Hill Publishing Company Limited who handled the book at different stages. We would like to express our sincere thanks to Vibha Mahajan, Jacob Alexander, Mini Narayanan, Michael J.Cruz, D. Naresh, S. Girish, and Anjili Razdan, for their valuable support.

Finally, we thank our family members Madhavi, Aparna and A.V. Yashwant, P. Siddartha and P. Yudhister whose invaluable support made the whole project possible.

A. SUDHAKAR
SHYAMMOHAN S. PALLI

Road Map to the Syllabus

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY
HYDERABAD

1 Year B.Tech. **2005-2006**

3 Periods/Week
6 Credits

NETWORK ANALYSIS

UNIT-I

Basic Voltage and Current relationships for R, L and C, 1st order Circuits, RL & RC, initial conditions for L and C, Kirchhoff's Laws, Ideal Voltage and Current sources. Network Topology: Definitions, Graph, Tree , Basic Cutset and Basic Tieset Matrices for planar networks, Formulation of network equations using loop and nodal methods of Analysis with dependent and independent Voltage and Current sources. Duality and Dual networks.

GO TO
Chapter 1: Circuit Elements and Kirchhoff's Laws
Chapter 2: Methods of Analysing Circuits
Chapter 3: Useful Theorems in Circuit Analysis

UNIT-II

Magnetic Circuits, Self and Mutual inductances, dot convention, impedance, reactance concept, Impedance transformation and coupled circuits, co-efficient of coupling, equivalent T for Magnetically coupled circuits, Ideal Transformer.

GO TO
Chapter 5: Complex Impedance
Chapter 9: Coupled Circuits
Chapter 16: s-Domain Analysis

UNIT-III

Steady state and transient analysis of RC, RL and RLC Circuits, Circuits with switches, step response, 2nd order series and parallel RLC Circuits, Root locus, damping factor, over damped, under damped, critically damped cases, quality factor and bandwidth for series and parallel resonance, resonance curves.

GO TO
Chapter 7: Steady State AC Analysis
Chapter 8: Frequency Domain Analysis
Chapter 11: Transients

UNIT-IV

Network Analysis using Laplace transform techniques, step, impulse and exponential excitation, response due to periodic excitation, RMS and average value of periodic waveforms.

GO TO
Chapter 4: Introduction to Alternating Currents and Voltages
Chapter 12: Introduction to Laplace Transform
Chapter 13: Application of the Laplace Transform in Circuit Analysis

UNIT-V

Network theorems, Tellegens, Superposition, Reciprocity, Thevinin's, Norton's, Max Power Transfer theorem. Milliman's Theorem (All without proof but with applications to network analysis) Complex Power, j Notation, phasor diagram, Sinusoidal steady state analysis, Duality in networks.

GO TO
Chapter 3: Useful Theorems in Circuit Analysis
Chapter 6: Power and Power Factor
Chapter 7: Steady State AC Analysis
Appendix C: The j Factor

UNIT-VI

Two-port network parameters, Z, Y, ABCD, h and g parameters, Characteristic impedance, Image transfer constant, image and iterative impedance, network function, driving point and transfer functions — using transformed (S) variables, Poles and Zeros.

GO TO **Chapter 15: Two-Port Networks**
Chapter 16: s-Domain Analysis

UNIT-VII

Standard T, π, L Sections, Characteristic impedance, image transfer constants, Design of Attenuators, impedance matching network, T and π Conversion.

GO TO **Chapter 17: Filters and Altenuators**

UNIT-VIII

LC Networks and Filters: Properties of LC Networks, Foster's Reactance theorem, design of constant K, LP, HP and BP Filters, Composite filter design.

GO TO **Chapter 17: Filters and Altenuators**

Chapter 1

Circuit Elements and Kirchhoff's Laws

1.1 VOLTAGE

According to the structure of an atom, we know that there are two types of charges: positive and negative. A force of attraction exists between these positive and negative charges. A certain amount of energy (work) is required to overcome the force and move the charges through a specific distance. All opposite charges possess a certain amount of potential energy because of the separation between them. The difference in potential energy of the charges is called the *potential difference*.

Potential difference in electrical terminology is known as voltage, and is denoted either by V or v. It is expressed in terms of energy (W) per unit charge (Q); i.e.

$$V = \frac{W}{Q} \quad \text{or} \quad v = \frac{dw}{dq}$$

dw is the small change in energy, and
dq is the small change in charge.
where energy (W) is expressed in joules (J), charge (Q) in coulombs (C), and voltage (V) in volts (V). One volt is the potential difference between two points when one joule of energy is used to pass one coulomb of charge from one point to the other.

Example 1.1 If 70 J of energy is available for every 30 C of charge, what is the voltage?

Solution $$V = \frac{W}{Q} = \frac{70}{30} = 2.33\text{ V}$$

1.2 CURRENT

There are free electrons available in all semiconductive and conductive materials. These free electrons move at random in all directions within the structure in the absence of external pressure or voltage. If a certain amount of voltage is applied across the material, all the free electrons move in one direction depending on the polarity of the applied voltage, as shown in Fig. 1.1.

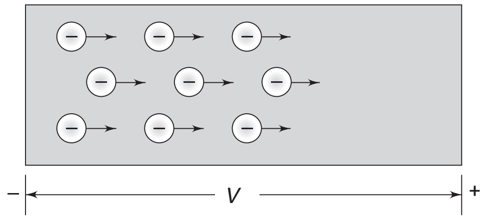


Fig. 1.1

This movement of electrons from one end of the material to the other end constitutes an electric current, denoted by either I or i. The conventional direction of current flow is opposite to the flow of – ve charges, i.e. the electrons.

Current is defined as the rate of flow of electrons in a conductive or semiconductive material. It is measured by the number of electrons that flow past a point in unit time. Expressed mathematically,

$$I = \frac{Q}{t}$$

where I is the current, Q is the charge of electrons, and t is the time, or

$$i = \frac{dq}{dt}$$

where dq is the small change in charge, and dt is the small change in time.

In practice, the unit *ampere* is used to measure current, denoted by A. One ampere is equal to one coulomb per second. One coulomb is the charge carried by 6.25×10^{18} electrons. For example, an ordinary 80 W domestic ceiling fan on 230 V supply takes a current of approximately 0.35 A. This means that electricity is passing through the fan at the rate of 0.35 coulomb every second, i.e. 2.187×10^{18} electrons are passing through the fan in every second; or simply, the current is 0.35 A.

Example 1.2 Five coulombs of charge flow past a given point in a wire in 2 s. How many amperes of current is flowing?

Solution
$$I = \frac{Q}{t} = \frac{5}{2} = 2.5 \text{ A}$$

1.3 POWER AND ENERGY

Energy is the capacity for doing work, i.e. energy is nothing but stored work. Energy may exist in many forms such as mechanical, chemical, electrical and so on. Power is the rate of change of energy, and is denoted by either P or p. If certain amount of energy is used over a certain length of time, then

$$\text{Power } (P) = \frac{\text{energy}}{\text{time}} = \frac{W}{t}, \text{ or}$$

$$p = \frac{dw}{dt}$$

where dw is the change in energy and dt is the change in time.

We can also write $$p = \frac{dw}{dt} = \frac{dw}{dq} \times \frac{dq}{dt}$$

$$= v \times i = vi \quad \text{W}$$

Energy is measured in joules (J), time in seconds (s), and power in watts (W).

By definition, one watt is the amount of power generated when on joule of energy is consumed in one second. Thus, the number of joules consumed in one second is always equal to the number of watts. Amounts of power less than one watt are usually expressed in fraction of watts in the field of electronics; viz. milliwatts (mW) and microwatts (μW). In the electrical field, kilowatts (kW) and megawatts (MW) are common units. Radio and television stations also use large amounts of power to transmit signals.

Example 1.3 What is the power in watts if energy equal to 50 J is used in 2.5 s?

Solution $$P = \frac{\text{energy}}{\text{time}} = \frac{50}{2.5} = 20 \text{ W}$$

1.4 THE CIRCUIT

An electric circuit consists of three parts: (1) energy source, such as battery or generator, (2) the load or sink, such as lamp or motor, and (3) connecting wires as shown in Fig. 1.2. This arrangement represents a simple circuit. A battery is connected to a lamp with two wires. The purpose of the circuit is to transfer energy from source (battery) to the load (lamp). And this is accomplished by the passage of electrons through wires around the circuit.

The current flows through the filament of the lamp, causing it to emit visible light. The current flows through the battery by chemical action. A closed circuit is defined as a circuit in which the current has a complete path to flow. When the current path is broken so that current cannot flow, the circuit is called an open circuit.

More specifically, interconnection of two or more simple circuit elements (viz. voltage sources, resistors, inductors and capacitors) is called an electric network.

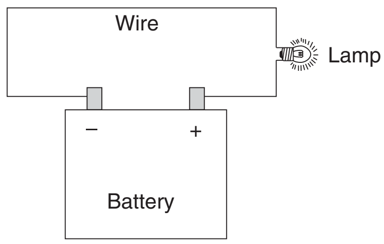


Fig. 1.2

If a network contains at least one closed path, it is called an electric circuit. By definition, a simple circuit element is the mathematical model of two terminal electrical devices, and it can be completely characterised by its voltage and current. Evidently then, a physical circuit must provide means for the transfer of energy.

Broadly, network elements may be classified into four groups, viz.

1. Active or passive
2. Unilateral or bilateral
3. Linear or nonlinear
4. Lumped or distributed

1.4.1 Active and Passive

Energy sources (voltage or current sources) are active elements, capable of delivering power to some external device. Passive elements are those which are capable only of receiving power. Some passive elements like inductors and capacitors are capable of storing a finite amount of energy, and return it later to an external element. More specifically, an active element is capable of delivering an average power greater than zero to some external device over an infinite time interval. For example, ideal sources are active elements. A passive element is defined as one that cannot supply average power that is greater than zero over an infinite time interval. Resistors, capacitors and inductors fall into this category.

1.4.2 Bilateral and Unilateral

In the bilateral element, the voltage-current relation is the same for current flowing in either direction. In contrast, a unilateral element has different relations between voltage and current for the two possible directions of current. Examples of bilateral elements are elements made of high conductivity materials in general. Vacuum diodes, silicon diodes, and metal rectifiers are examples of unilateral elements.

1.4.3 Linear and Nonlinear Elements

An element is said to be linear, if its voltage-current characteristic is at all times a straight line through the origin. For example, the current passing through a resistor is proportional to the voltage applied through it, and the relation is expressed as $V \propto I$ or $V = IR$. A linear element or network is one which satisfies the principle of superposition, i.e. the principle of homogeneity and additivity. An element which does not satisfy the above principle is called a nonlinear element.

1.4.4 Lumped and Distributed

Lumped elements are those elements which are very small in size and in which simultaneous actions takes place for any given cause at the same instant of time.

Typical lumped elements are capacitors, resistors, inductors and transformers. Generally the elements are considered as lumped when their size is very small compared to the wave length of the applied signal. Distributed elements, on the other hand, are those which are not electrically separable for analytical purposes. For example, a transmission line which has distributed resistance, inductance and capacitance along its length may extend for hundreds of miles.

1.5 RESISTANCE PARAMETER

When a current flows in a material, the free electrons move through the material and collide with other atoms. These collisions cause the electrons to lose some of their energy. This loss of energy per unit charge is the drop in potential across the material. The amount of energy lost by the electrons is related to the physical property of the material. These collisions restrict the movement of electrons. The property of a material to restrict the flow of electrons is called resistance, denoted by R. The symbol for the resistor is shown in Fig. 1.3.

Fig. 1.3

The unit of resistance is ohm (Ω). Ohm is defined as the resistance offered by the material when a current of one ampere flows between two terminals with one volt applied across it.

According to Ohm's law, the current is directly proportional to the voltage and inversely proportional to the total resistance of the circuit, i.e.

$$I = \frac{V}{R}$$

or

$$i = \frac{v}{R}$$

We can write the above equation in terms of charge as follows.

$$V = R\frac{dq}{dt}, \quad \text{or} \quad i = \frac{v}{R} = Gv$$

where G is the conductance of a conductor. The units of resistance and conductance are ohm (Ω) and mho ($\mho$) respectively.

When current flows through any resistive material, heat is generated by the collision of electrons with other atomic particles. The power absorbed by the resistor is converted to heat. The power absorbed by the resistor is given by

$$P = vi = (iR)i = i^2 R$$

where i is the current in the resistor in amps, and v is the voltage across the resistor in volts. Energy lost in a resistance in time t is given by

$$W = \int_0^t pdt = pt = i^2Rt = \frac{v^2}{R}t$$

where v is the volts
R is in ohms
t is in seconds and
W is in joules

Example 1.4 A 10 Ω resistor is connected across a 12 V battery. How much current flows through the resistor?

Solution

$$V = IR$$

$$I = \frac{V}{R} = \frac{12}{10} = 1.2 \text{ A}$$

1.6 INDUCTANCE PARAMETER

A wire of certain length, when twisted into a coil becomes a basic inductor. If current is made to pass through an inductor, an electromagnetic field is formed. A change in the magnitude of the current changes the electromagnetic field. Increase in current expands the fields, and decrease in current reduces it. Therefore, a change in current produces change in the electromagnetic field, which induces a voltage across the coil according to Faraday's law of electromagnetic induction.

The unit of inductance is *henry*, denoted by *H*. By definition, the inductance is one henry when current through the coil, changing at the rate of one ampere per second, induces one volt across the coil. The symbol for inductance is shown in Fig. 1.4.

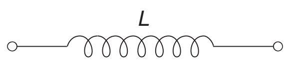


Fig. 1.4

The current-voltage relation is given by

$$v = L\frac{di}{dt}$$

where v is the voltage across inductor in volts, and i is the current through inductor in amps. We can rewrite the above equations as

$$di = \frac{1}{L} v\,dt$$

Integrating both sides, we get

$$\int_0^t di = \frac{1}{L}\int_0^t v\,dt$$

$$i(t) - i(0) = \frac{1}{L}\int_0^t v\,dt$$

$$i(t) = \frac{1}{L}\int_0^t v\,dt + i(0)$$

From the above equation we note that the current in an inductor is dependent upon the integral of the voltage across its terminals and the initial current in the coil, $i(0)$.

The power absorbed by inductor is

$$P = vi = Li\,\frac{di}{dt} \text{ watts}$$

The energy stored by the inductor is

$$W = \int_0^t p\,dt$$

$$= \int_0^t Li\,\frac{di}{dt}\,dt = \frac{Li^2}{2}$$

From the above discussion, we can conclude the following.

1. The induced voltage across an inductor is zero if the current through it is constant. That means an inductor acts as short circuit to dc.
2. A small change in current within zero time through an inductor gives an infinite voltage across the inductor, which is physically impossible. In a fixed inductor the current cannot change abruptly.
3. The inductor can store finite amount of energy, even if the voltage across the inductor is zero, and
4. A pure inductor never dissipates energy, only stores it. That is why it is also called a non-dissipative passive element. However, physical inductors dissipate power due to internal resistance.

Example 1.5 The current in a 2 H inductor varies at a rate of 2 A/s. Find the voltage across the inductor and the energy stored in the magnetic field after 2 s.

Solution

$$v = L\frac{di}{dt}$$

$$= 2 \times 4 = 8 \text{ V}$$

$$W = \frac{1}{2}Li^2$$

$$= \frac{1}{2} \times 2 \times (4)^2 = 16 \text{ J}$$

1.7 CAPACITANCE PARAMETER

Any two conducting surfaces separated by an insulating medium exhibit the property of a capacitor. The conducting surfaces are called *electrodes*, and the insulating medium is called *dielectric*. A capacitor stores energy in the form of an electric field that is established by the opposite charges on the two electrodes. The electric field is represented by lines of force between the positive and negative charges, and is concentrated within the dielectric. The amount of charge per unit voltage that is capacitor can store is its capacitance, denoted by *C*. The unit of capacitance is *Farad* denoted by *F*. By definition, one Farad is the amount of capacitance when one coulomb of charge is stored with one volt across the plates. The symbol for capacitance is shown in Fig. 1.5.

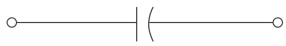


Fig. 1.5

A capacitor is said to have greater capacitance if it can store more charge per unit voltage and the capacitance is given by

$$C = \frac{Q}{V}, \quad \text{or} \quad C = \frac{q}{v}$$

(lower case letters stress instantaneous values)

We can write the above equation in terms of current as

$$i = C\frac{dv}{dt} \qquad \left(\because \; i = \frac{dq}{dt}\right)$$

where v is the voltage across capacitor, i is the current through it

$$dv = \frac{1}{C}\, idt$$

Integrating both sides, we have

$$\int_0^t dv = \frac{1}{C}\int_0^t idt$$

$$v(t) - v(0) = \frac{1}{C}\int_0^t idt$$

$$v(t) = \frac{1}{C}\int_0^t idt + v(0)$$

where $v(0)$ indicates the initial voltage across the capacitor.

From the above equation, the voltage in a capacitor is dependent upon the integral of the current through it, and the initial voltage across it.

The power absorbed by the capacitor is given by

$$p = vi = vC\frac{dv}{dt}$$

The energy stored by the capacitor is

$$W = \int_0^t pdt = \int_0^t vC\frac{dv}{dt}\, dt$$

$$W = \frac{1}{2}\, Cv^2$$

From the above discussion we can conclude the following

1. The current in a capacitor is zero if the voltage across it is constant; that means, the capacitor acts as an open circuit to dc.
2. A small change in voltage across a capacitance within zero time gives an infinite current through the capacitor, which is physically impossible. In a fixed capacitance the voltage cannot change abruptly.
3. The capacitor can store a finite amount of energy, even if the current through it is zero, and
4. A pure capacitor never dissipates energy, but only stores it; that is why it is called *non-dissipative passive element*. However, physical capacitors dissipate power due to internal resistance.

Example 1.6 A capacitor having a capacitance 2 μF is charged to a voltage of 1000 V. Calculate the stored energy in joules.

Solution $$W = \frac{1}{2} Cv^2 = \frac{1}{2} \times 2 \times 10^{-6} \times (1000)^2 = 1 \text{ J}.$$

1.8 ENERGY SOURCES

According to their terminal voltage-current characteristics, electrical energy sources are categorised into ideal voltage sources and ideal current sources. Further they can be divided into independent and dependent sources.

An ideal voltage source is a two-terminal element in which the voltage v_s is completely independent of the current i_s through its terminals. The representation of ideal constant voltage source is shown in Fig. 1.6(a).

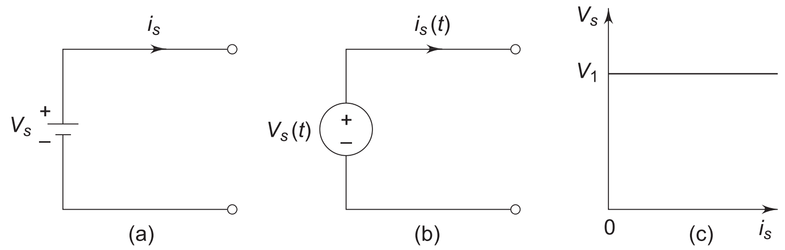


Fig. 1.6

If we observe the $v - i$ characteristics for an ideal voltage source as shown in Fig. 1.6(c) at any time, the value of the terminal voltage v_s is constant with respect to the value of current i_s. Whenever $v_s = 0$, the voltage source is the same as that of a short circuit. Voltage sources need not have constant magnitude; in many cases the specified voltage may be time-dependent like a sinusoidal waveform. This may be represented as shown in Fig. 1.6(b). In many practical voltage sources, the internal resistance is represented in series with the source as shown in Fig. 1.7(a). In this, the voltage across the terminals falls as the current through it increases, as shown in Fig. 1.7 (b).

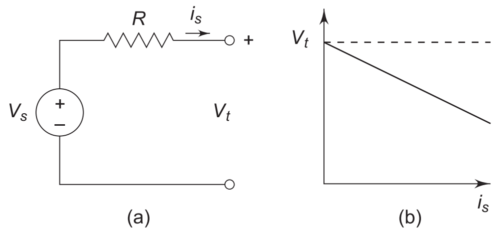


Fig. 1.7

The terminal voltage v_t depends on the source current as shown in Fig. 1.7(b), where $v_t = v_s - i_s R$.

An ideal constant current source is a two-terminal element in which the current i_s completely independent of the voltage v_s across its terminals. Like voltage sources we can have current sources of constant magnitude i_s or sources whose current varies with time $i_s(t)$. The representation of an ideal current source is shown in Fig. 1.8(a).

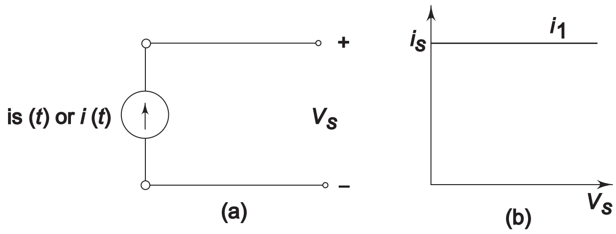


Fig. 1.8

If we observe the $v - i$ characteristics for an ideal current source as shown in Fig. 1.8(b), at any time the value of the current i_s is constant with respect to the voltage across it. In many practical current sources, the resistance is in parallel with a source as shown in Fig. 1.9(a). In this the magnitude of the current falls as the voltage across its terminals increases. Its terminal $v - i$ characteristics is shown in Fig. 1.9(b). The terminal current is given by $i_t = i_s - (v_s/R)$, where R is the internal resistance of the ideal current source.

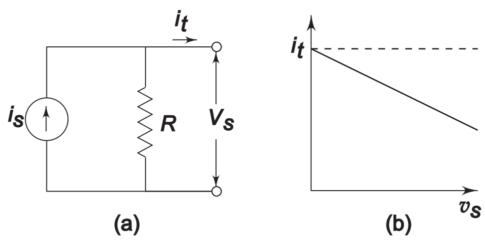


Fig. 1.9

The two types of ideal sources we have discussed are independent sources for which voltage and current are independent and are not affected by other parts of the circuit. In the case of dependent sources, the source voltage or current is not fixed, but is dependent on the voltage or current existing at some other location in the circuit.

Dependent or controlled sources are of the following types

(i) voltage controlled voltage source (VCVS)
(ii) current controlled voltage source (CCVS)
(iii) voltage controlled current source (VCCS)
(iv) current controlled current source (CCCS)

These are represented in a circuit diagram by the symbol shown in Fig. 1.10. These types of sources mainly occur in the analysis of equivalent circuits of transistors.

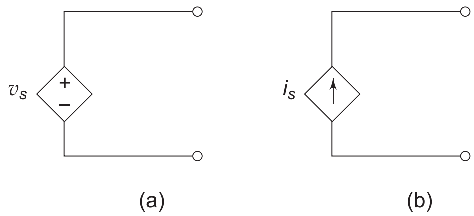


Fig. 1.10

1.9 KIRCHHOFF'S VOLTAGE LAW

Kirchhoff's voltage law states that the algebraic sum of all branch voltages around any closed path in a circuit is always zero at all instants of time. When the current passes through a resistor, there is a loss of energy and, therefore, a voltage drop. In any element, the current always flows from higher potential to lower potential. Consider the circuit in Fig. 1.11. It is customary to take the direction of current *I* as indicated in the figure, i.e. it leaves the positive terminal of the voltage source and enters into the negative terminal.

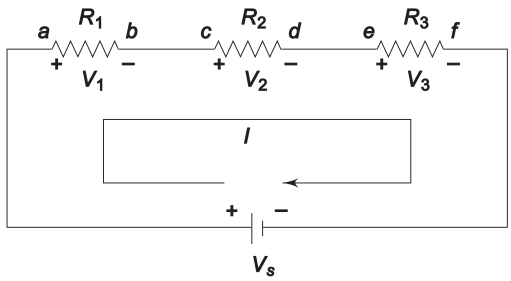


Fig. 1.11

As the current passes through the circuit, the sum of the voltage drop around the loop is equal to the total voltage in that loop. Here the polarities are attributed to the resistors to indicate that the voltages at points *a*, *c* and *e* are more than the voltages at *b*, *d* and *f*, respectively, as the current passes from *a* to *f*.

$\therefore \qquad V_s = V_1 + V_2 + V_3$

Consider the problem of finding out the current supplied by the source V in the circuit shown in Fig. 1.12.

Our first step is to assume the reference current direction and to indicate the polarities for different elements. (See Fig. 1.13).

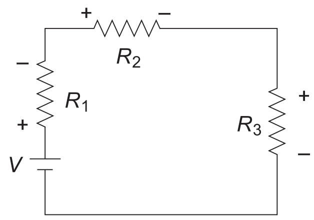


Fig. 1.12

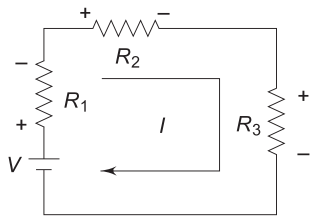


Fig. 1.13

By using Ohm's law, we find the voltage across each resistor as follows.

$$V_{R1} = IR_1,\ V_{R2} = IR_2,\ V_{R3} = IR_3$$

where V_{R1}, V_{R2} and V_{R3} are the voltages across R_1, R_2 and R_3, respectively. Finally, by applying Kirchhoff's law, we can form the equation

$$V = V_{R1} + V_{R2} + V_{R3}$$
$$V = IR_1 + IR_2 + IR_3$$

From the above equation the current delivered by the source is given by

$$I = \frac{V}{R_1 + R_2 + R_3}$$

Example 1.7 For the circuit shown in Fig. 1.14, determine the unknown voltage drop V_1.

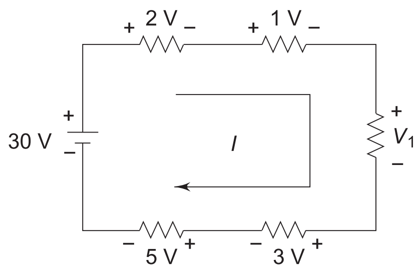


Fig. 1.14

Solution According to Kirchhoff's voltage law, the sum of the potential drops is equal to the sum of the potential rises;

Therefore, $\qquad 30 = 2 + 1 + V_1 + 3 + 5$

or $\qquad V_1 = 30 - 11 = 19$ V

Example 1.8 What is the current in the circuit shown in Fig. 1.15? Determine the voltage across each resistor.

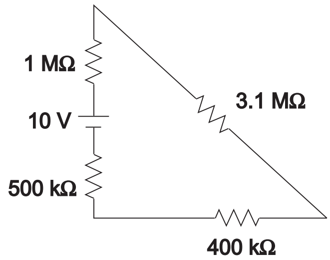


Fig. 1.15

Solution We assume current I in the clockwise direction and indicate polarities (Fig. 1.16). By using Ohm's law, we find the voltage drops across each resistor.

$$V_{IM} = I, \qquad V_{3.1M} = 3.1I$$
$$V_{500K} = 0.5I, \qquad V_{400K} = 0.4I$$

Now, by applying Kirchhoff's voltage law, we form the equation.

$$10 = I + 3.1\,I + 0.5\,I + 0.4\,I$$

or $$5\,I = 10$$

or $$I = 2\,\mu\text{A}$$

$\therefore$ Voltage across each resistor is as follows

$$V_{1M} = 1 \times 2 = 2.0\text{ V}$$
$$V_{3.1M} = 3.1 \times 2 = 6.2\text{ V}$$
$$V_{400K} = 0.4 \times 2 = 0.8\text{ V}$$
$$V_{500K} = 0.5 \times 2 = 1.0\text{ V}$$

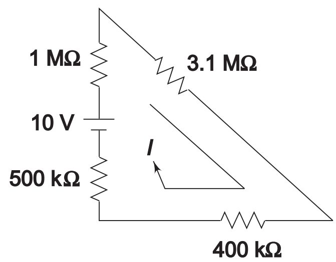


Fig. 1.16

Example 1.9 In the circuit given in Fig. 1.17, find (a) the current I, and (b) the voltage across 30 Ω.

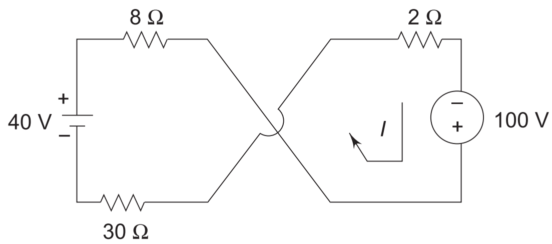


Fig. 1.17

Solution We redraw the circuit as shown in Fig. 1.18 and assume current direction and indicate the assumed polarities of resistors

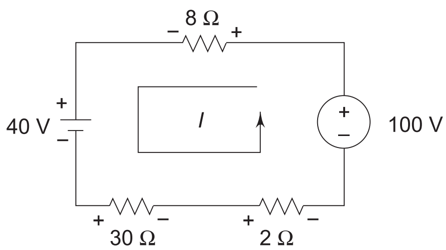


Fig. 1.18

By using Ohm's law, we determine the voltage across each resistor as

$$V_8 = 8I,\ V_{30} = 30I,\ V_2 = 2I$$

By applying Kirchhoff's law, we get

$$100 = 8I + 40 + 30I + 2I$$

$$40\ I = 60 \text{ or } I = \frac{60}{40} = 1.5 \text{ A}$$

$\therefore$ Voltage drop across 30 Ω = $V_{30} = 30 \times 1.5 = 45$ V

1.10 VOLTAGE DIVISION

The series circuit acts as a voltage divider. Since the same current flows through each resistor, the voltage drops are proportional to the values of resistors. Using this principle, different voltages can be obtained from a single source, called a voltage divider. For example, the voltage across a 40 Ω resistor is twice that of 20 Ω in a series circuit shown in Fig. 1.19.

In general, if the circuit consists of a number of series resistors, the total current is given by the total voltage divided by equivalent resistance. This is shown in Fig. 1.20.

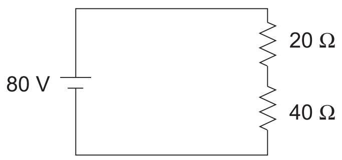


Fig. 1.19

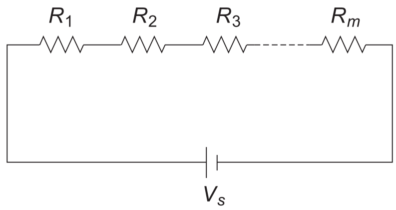


Fig. 1.20

The current in the circuit is given by $I = V_s/(R_1 + R_2 + \ldots + R_m)$. The voltage across any resistor is nothing but the current passing through it, multiplied by that particular resistor.

Therefore,
$$V_{R1} = IR_1$$
$$V_{R2} = IR_2$$
$$V_{R3} = IR_3$$
$$\vdots$$
$$V_{Rm} = IR_m$$

or
$$V_{Rm} = \frac{V_s\,(R_m)}{R_1 + R_2 + \ldots + R_m}$$

From the above equation, we can say that the voltage drop across any resistor, or a combination of resistors, in a series circuit is equal to the ratio of that resistance value to the total resistance, multiplied by the source voltage, i.e.

$$V_m = \frac{R_m}{R_T}\,V_s$$

where V_m is the voltage across mth resistor,
R_m is the resistance across which the voltage is to be determined and R_T is the total series resistance.

Example 1.10 What is the voltage across the 10 Ω resistor in Fig. 1.21.

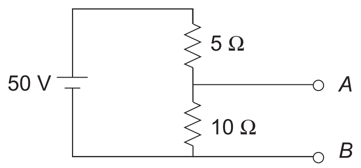


Fig. 1.21

Solution Voltage across 10 Ω = $V_{10} = 50 \times \dfrac{10}{10+5} = \dfrac{500}{15} = 33.3$ V

Example 1.11 Find the voltage between A and B in a voltage divider network shown in Fig. 1.22.

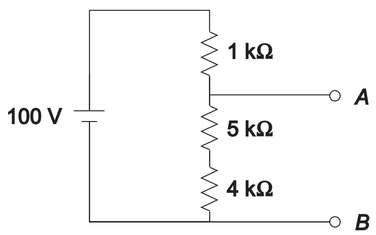


Fig. 1.22

Solution Voltage across 9 kΩ = $V_9 = V_{AB} = 100 \times \dfrac{9}{10} = 90$ V

1.11 POWER IN SERIES CIRCUIT

The total power supplied by the source in any series resistive circuit is equal to the sum of the powers in each resistor in series, i.e.

$$P_S = P_1 + P_2 + P_3 + \ldots + P_m$$

where m is the number of resistors in series, P_S is the total power supplied by source and P_m is the power in the last resistor in series. The total power in the series circuit is the total voltage applied to a circuit, multiplied by the total current. Expressed mathematically,

$$P_S = V_s I = I^2 R_T = \frac{V_s^2}{R_T}$$

where V_s is the total voltage applied, R_T is the total resistance, and I is the total current.

Example 1.12 Determine the total amount of power in the series circuit in Fig. 1.23.

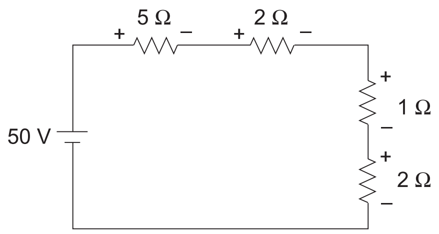


Fig. 1.23

Solution Total resistance = 5 + 2 + 1 + 2 = 10 Ω

We know $$P_S = \frac{V_s^2}{R_T} = \frac{(50)^2}{10} = 250 \text{ W}$$

Check We find the power absorbed by each resistor

$$\text{Current} = \frac{50}{10} = 5 \text{ A}$$

$$P_5 = (5)^2 \times 5 = 125 \text{ W}$$
$$P_2 = (5)^2 \times 2 = 50 \text{ W}$$
$$P_1 = (5)^2 \times 1 = 25 \text{ W}$$
$$P_2 = (5)^2 \times 2 = 50 \text{ W}$$

The sum of these powers gives the total power supplied by the source P_S = 250 W.

1.12 KIRCHHOFF'S CURRENT LAW

Kirchhoff's current law states that the sum of the currents entering into any node is equal to the sum of the currents leaving that node. The node may be an interconnection of two or more branches. In any parallel circuit, the node is a junction point of two or more branches. The total current entering into a node is equal to the current leaving that node. For example, consider the circuit shown in Fig. 1.24, which contains two nodes A and B. The total current I_T entering node A is divided into I_1, I_2 and I_3. These currents flow out of node A. According to Kirchhoff's current law, the current into node A is equal to the total current out of node A: that is, $I_T = I_1 + I_2 + I_3$. If we consider node B, all three currents I_1, I_2, I_3 are entering B, and the total current I_T is leaving node B, Kirchhoff's current law formula at this node is therefore the same as at node A.

$$I_1 + I_2 + I_3 = I_T$$

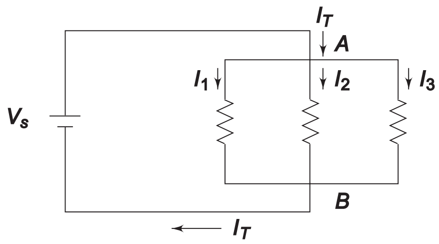


Fig. 1.24

In general, sum of the currents entering any point or node or junction equal to sum of the currents leaving from that point or node or junction as shown in Fig. 1.25.

$$I_1 + I_2 + I_4 + I_7 = I_3 + I_5 + I_6$$

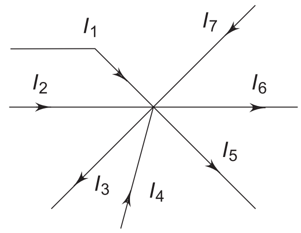


Fig. 1.25

If all of the terms on the right side are brought over to the left side, their signs change to negative and a zero is left on the right side, i.e.

$$I_1 + I_2 + I_4 + I_7 - I_3 - I_5 - I_6 = 0$$

This means that the algebraic sum of all the currents meeting at a junction is equal to zero.

Example 1.13 Determine the current in all resistors in the circuit shown in Fig. 1.26.

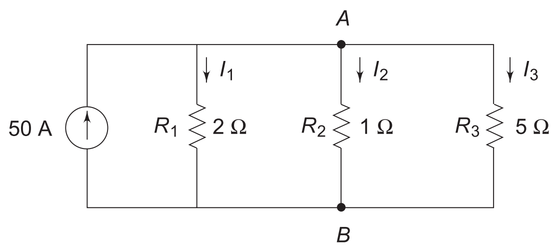


Fig. 1.26

Solution The above circuit contains a single node 'A' with reference node 'B'. Our first step is to assume the voltage V at node A. In a parallel circuit the same voltage is applied across each element. According to Ohm's law, the currents passing through each element are $I_1 = V/2$, $I_2 = V/1$, $I_3 = V/5$.

By applying Kirchhoff's current law, we have

$$I = I_1 + I_2 + I_3$$

$$I = \frac{V}{2} + \frac{V}{1} + \frac{V}{5}$$

$$50 = V\left[\frac{1}{2} + \frac{1}{1} + \frac{1}{5}\right] = V\,[0.5 + 1 + 0.2]$$

$$V = \frac{50}{1.7} = \frac{500}{17} = 29.41 \text{ V}$$

Once we know the voltage V at node A, we can find the current in any element by using Ohm's law.

The current in the 2 Ω resistor is $I_1 = 29.41/2 = 14.705$ A.

Similarly $$I_2 = \frac{V}{R_2} = \frac{V}{1} = 29.41 \text{ A}$$

$$I_3 = \frac{29.41}{5} = 5.882 \text{ A}$$

$\therefore$ $$I_1 = 14.7 \text{ A}, I_2 = 29.4 \text{ A, and } I_3 = 5.88 \text{ A}$$

Example 1.14 For the circuit shown in Fig. 1.27, find the voltage across the 10 Ω resistor and the current passing through it.

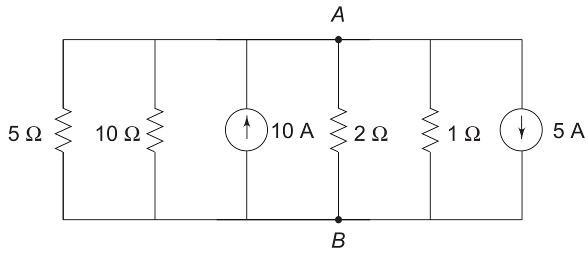


Fig. 1.27

Solution The circuit shown above is a parallel circuit, and consists of a single node A. By assuming voltage V at the node A w.r.t. B, we can find out the current in the 10 Ω branch. (See Fig. 1.28)

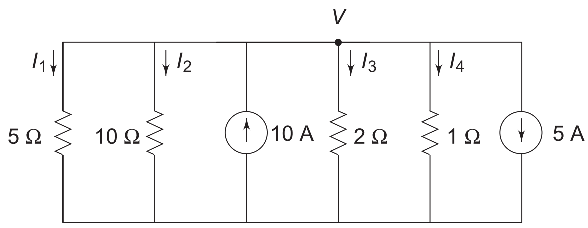


Fig. 1.28

According to Kirchhoff's current law,

$$I_1 + I_2 + I_3 + I_4 + 5 = 10$$

By using Ohm's law we have

$$I_1 = \frac{V}{5}; I_2 = \frac{V}{10}, I_3 = \frac{V}{2}, I_4 = \frac{V}{1}$$

$$\frac{V}{5} + \frac{V}{10} + \frac{V}{2} + V + 5 = 10$$

$$V\left[\frac{1}{5} + \frac{1}{10} + \frac{1}{2} + 1\right] = 5$$

$$V[0.2 + 0.1 + 0.5 + 1] = 5$$

$$V = \frac{5}{1.8} = 2.78 \text{ V}$$

$\therefore$ The voltage across the 10 Ω resistor is 2.78 V and the current passing through it is

$$I_2 = \frac{V}{10} = \frac{2.78}{10} = 0.278 \text{ A}$$

Example 1.15 Determine the current through resistance R_3 in the circuit shown in Fig. 1.29.

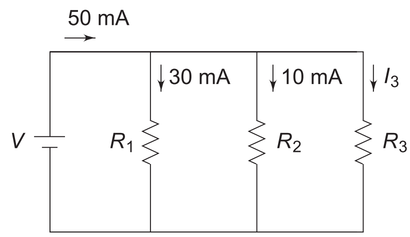


Fig. 1.29

Solution According to Kirchhoff's current law,

$$I_T = I_1 + I_2 + I_3$$

where I_T is the total current and I_1, I_2 and I_3 are the currents in resistances R_1, R_2 and R_3 respectively.

$\therefore$ $$50 = 30 + 10 + I_3$$

or $$I_3 = 10 \text{ mA}$$

1.13 PARALLEL RESISTANCE

When the circuit is connected in parallel, the total resistance of the circuit decreases as the number of resistors connected in parallel increases. If we consider *m* parallel branches in a circuit as shown in Fig. 1.30, the current equation is

$$I_T = I_1 + I_2 + \dots + I_m$$

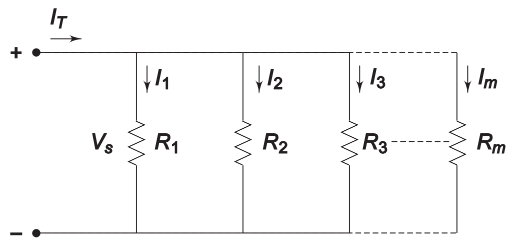


Fig. 1.30

The same voltage is applied across each resistor. By applying Ohm's law, the current in each branch is given by

$$I_1 = \frac{V_s}{R_1},\ I_2 = \frac{V_s}{R_2},\ \dots\ I_m = \frac{V_s}{R_m}$$

According to Kirchhoff's current law,

$$I_T = I_1 + I_2 + I_3 + \dots + I_m$$

$$\frac{V_s}{R_T} = \frac{V_s}{R_1} + \frac{V_s}{R_2} + \frac{V_s}{R_3} + \dots + \frac{V_s}{R_m}$$

From the above equation, we have

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_m}$$

Example 1.16 Determine the parallel resistance between points A and B of the circuit shown in Fig. 1.31.

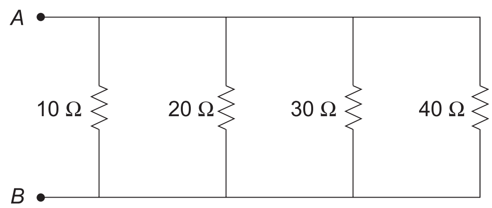


Fig. 1.31

Solution
$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$$

$$\frac{1}{R_T} = \frac{1}{10} + \frac{1}{20} + \frac{1}{30} + \frac{1}{40}$$

$$= 0.1 + 0.05 + 0.033 + 0.025 = 0.208$$

or
$$R_T = 4.8\ \Omega$$

1.14 CURRENT DIVISION

In a parallel circuit, the current divides in all branches. Thus, a parallel circuit acts as a current divider. The total current entering into the parallel branches is divided into the branches currents according to the resistance values. The branch having higher resistance allows lesser current, and the branch with lower resistance allows more current. Let us find the current division in the parallel circuit shown in Fig. 1.32.

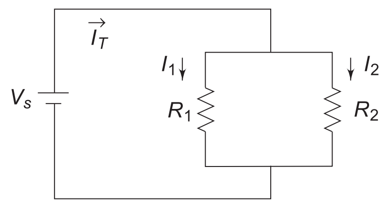


Fig. 1.32

The voltage applied across each resistor is V_s. The current passing through each resistor is given by

$$I_1 = \frac{V_s}{R_1},\ I_2 = \frac{V_s}{R_2}$$

If R_T is the total resistance, which is given by $R_1R_2/(R_1 + R_2)$,

Total current
$$I_T = \frac{V_s}{R_T} = \frac{V_s}{R_1 R_2}(R_1 + R_2)$$

or
$$I_T = \frac{I_1 R_1}{R_1 R_2}(R_1 + R_2) \text{ since } V_s = I_1 R_1$$

$$I_1 = I_T \cdot \frac{R_2}{R_1 + R_2}$$

Similarly,
$$I_2 = I_T \cdot \frac{R_1}{R_1 + R_2}$$

From the above equations, we can conclude that the current in any branch is equal to the ratio of the opposite branch resistance to the total resistance value, multiplied by the total current in the circuit. In general, if the circuit consists of *m* branches, the current in any branch can be determined by

$$I_i = \frac{R_T}{R_i + R_T} I_T$$

where I_i represents the current in the *i*th branch
R_i is the resistance in the *i*th branch
R_T is the total parallel resistance to the *i*th branch and
I_T is the total current entering the circuit.

Example 1.17 Determine the current through each resistor in the circuit shown in Fig. 1.33.

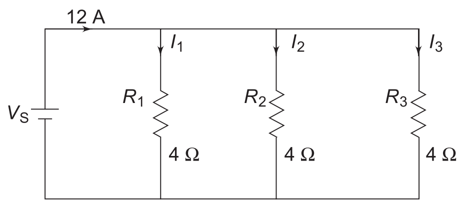


Fig. 1.33

Solution
$$I_1 = I_T \times \frac{R_T}{(R_1 + R_T)}$$

where
$$R_T = \frac{R_2 R_3}{R_2 + R_3} = 2\ \Omega$$

$\therefore$
$$R_1 = 4\ \Omega$$
$$I_T = 12\ \text{A}$$
$$I_1 = 12 \times \frac{2}{2+4} = 4\ \text{A}$$

Similarly,
$$I_2 = 12 \times \frac{2}{2+4} = 4\ \text{A}$$

and
$$I_3 = 12 \times \frac{2}{2+4} = 4\ \text{A}$$

Since all parallel branches have equal values of resistance, they share current equally.

1.15 POWER IN PARALLEL CIRCUIT

The total power supplied by the source in any parallel resistive circuit is equal to the sum of the powers in each resistor in parallel, i.e.

$$P_S = P_1 + P_2 + P_3 + \ldots + P_m$$

where m is the number of resistors in parallel, P_S is the total power and P_m is the power in the last resistor.

Additional Solved Problems

Problem 1.1 A resistor with a current of 3 A through it converts 500 J of electrical energy to heat energy in 12 s. What is the voltage across the resistor?

Solution

$$V = \frac{W}{Q}$$

$$Q = I \times t$$

$$= 3 \times 12 = 36 \text{ C}$$

$$V = \frac{500}{36} = 13.88 \text{ V}$$

Problem 1.2 A 5 Ω resistor has a voltage rating of 100 V. What is its power rating?

Solution

$$P = VI$$

$$I = V/R$$

$$P = \frac{V^2}{R} = \frac{(100)^2}{5} = 2000 \text{ W} = 2 \text{ kW}$$

Problem 1.3 Find the inductance of a coil through which flows a current of 0.2 A with an energy of 0.15 J.

Solution

$$W = \frac{1}{2} LI^2$$

$$L = \frac{2 \times W}{I^2} = \frac{2 \times 0.15}{(0.2)^2} = 7.5 \text{ H}$$

Problem 1.4 Find the inductance of a coil in which a current increases linearly from 0 to 0.2 A in 0.3 s, producing a voltage of 15 V.

Solution

$$v = L\frac{di}{dt}$$

Current in 1 s $= \dfrac{0.2}{0.3} = 0.66$ A

The current changes at a rate of 0.66 A/s,

$$\therefore \qquad L = \frac{v}{\left(\dfrac{di}{dt}\right)}$$

$$L = \frac{15\,\text{V}}{0.66\,\text{A/s}} = 22.73\text{ H}$$

Problem 1.5 When a dc voltage is applied to a capacitor, the voltage across its terminals is found to build up in accordance with $v_C = 50(1 - e^{-100t})$. After a lapse of 0.01 s, the current flow is equal to 2 mA.

(a) Find the value of capacitance in microfarads

(b) How much energy is stored in the electric field at this time?

Solution

(a) $$i = C\frac{dv_C}{dt}$$

where $v_C = 50(1 - e^{-100t})$

$$i = C\frac{d}{dt}\,50(1 - e^{-100t})$$

$$= C \times 50 \times 100e^{-100t}$$

At $t = 0.01$ s, $i = 2$ mA

$$C = \frac{2 \times 10^{-3}}{50 \times 100 \times e^{-100 \times 0.01}} = 1.089\ \mu\text{F}$$

(b) $$W = \frac{1}{2}\,Cv_C^2$$

At $t = 0.01$ s, $v_C = 50\ (1 - e^{-100 \times 0.01}) = 31.6$ V

$$W = \frac{1}{2} \times 1.089 \times 10^{-6} \times (31.6)^2$$

$$= 0.000543\text{ J}$$

Problem 1.6 Determine the total current in the circuit shown in Fig. 1.34.

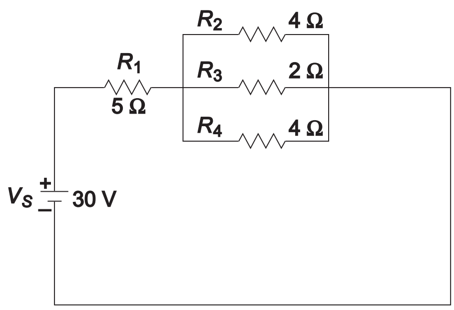


Fig. 1.34

Solution Resistances R_2, R_3 and R_4 are in parallel

$\therefore$ Equivalent resistance $R_5 = R_2 \| R_3 \| R_4$

$$= \frac{1}{1/R_2 + 1/R_3 + 1/R_4}$$

$\therefore$ $R_5 = 1\ \Omega$

R_1 and R_5 are in series,

$\therefore$ Equivalent resistance $R_T = R_1 + R_5 = 5 + 1 = 6\ \Omega$

And the total current $I_T = \dfrac{V_s}{R_T} = \dfrac{30}{6} = 5$ A

Problem 1.7 Find the current in the 10 Ω resistance, V_1, and source voltage V_s in the circuit shown in Fig. 1.35.

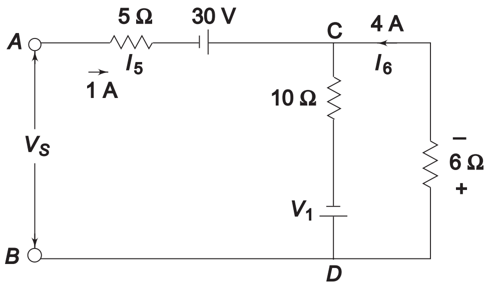


Fig. 1.35

Solution Assume voltage at node $C = V$

By applying Kirchhoff's current law, we get the current in the 10 Ω resistance

$$I_{10} = I_5 + I_6$$
$$= 4 + 1 = 5 \text{ A}$$

The voltage across the 6 Ω resistor is $V_6 = 24$ V

$\therefore$ Voltage at node C is $V_C = -24$ V.

The voltage across branch CD is the same as the voltage at node C.

Voltage across 10 Ω only = 10 × 5 = 50 V

So $V_C = V_{10} - V_1$

$-24 = 50 - V_1$

$\therefore$ $V_1 = 74$ V

Now, consider the loop CABD shown in Fig. 1.36

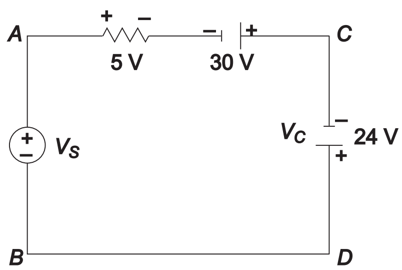


Fig. 1.36

If we apply Kirchhoff's voltage law we get

$$V_s = 5 - 30 - 24 = -49 \text{ V}$$

Problem 1.8 What is the voltage across A and B in the circuit shown in Fig. 1.37?

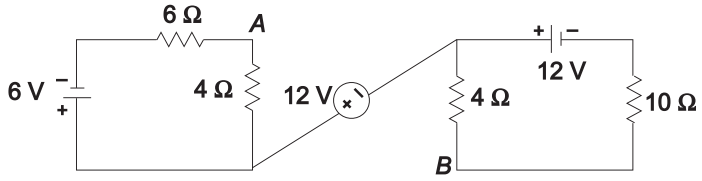


Fig. 1.37

Solution The above circuit can be redrawn as shown in Fig. 1.38.

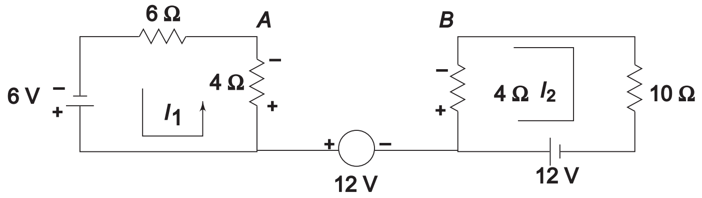


Fig. 1.38

Assume loop currents I_1 and I_2 as shown in Fig. 1.38.

$$I_1 = \frac{6}{10} = 0.6 \text{ A}$$

$$I_2 = \frac{12}{14} = 0.86 \text{ A}$$

V_A = Voltage drop across 4 Ω resistor = 0.6 × 4 = 2.4 V

V_B = Voltage drop across 4 Ω resistor = 0.86 × 4 = 3.44 V

The voltage between points A and B is the sum of voltages as shown in Fig. 1.39.

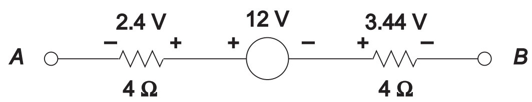


Fig. 1.39

$$\therefore \qquad V_{AB} = -2.4 + 12 + 3.44 = 13.04 \text{ V}$$

Problem 1.9 Determine the current delivered by the source in the circuit shown in Fig. 1.40.

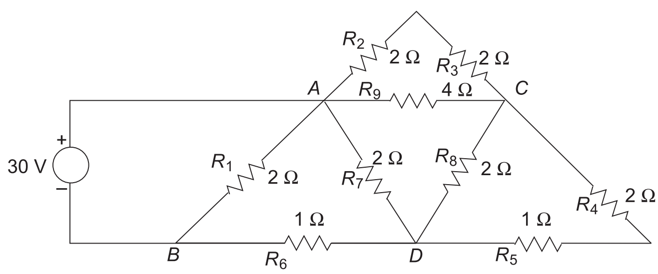


Fig. 1.40

Solution The circuit can be modified as shown in Fig. 1.41, where R_{10} is the series combination of R_2 and R_3.

$\therefore \quad R_{10} = R_2 + R_3 = 4\ \Omega$

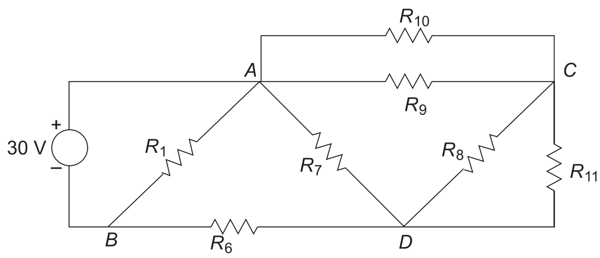


Fig. 1.41

R_{11} is the series combination of R_4 and R_5

$\therefore \quad R_{11} = R_4 + R_5 = 3\ \Omega$

Further simplification of the circuit leads to Fig 1.42 where R_{12} is the parallel combination of R_{10} and R_9.

$\therefore \quad R_{12} = (R_{10} \parallel R_9) = (4 \parallel 4) = 2\ \Omega$

Similarly, R_{13} is the parallel combination of R_{11} and R_8

$\therefore \quad R_{13} = (R_{11} \parallel R_8) = (3 \parallel 2) = 1.2\ \Omega$

In Fig. 1.42 as shown, R_{12} and R_{13} are in series, which is in parallel with R_7 forming R_{14}. This is shown in Fig. 1.43.

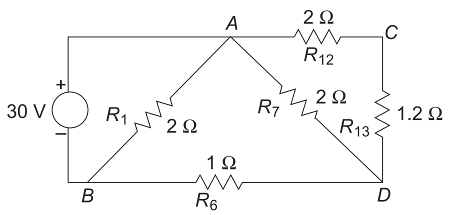


Fig. 1.42

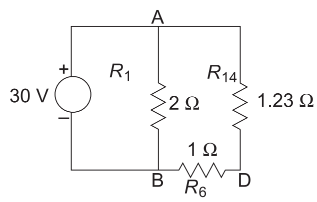


Fig. 1.43

$$\therefore \quad R_{14} = [(R_{12} + R_{13})//R_7]$$
$$= [(2 + 1.2)//2] = 1.23\ \Omega$$

Further, the resistances R_{14} and R_6 are in series, which is in parallel with R_1 and gives the total resistance

$$R_T = [(R_{14} + R_6)//R_1]$$
$$= [(1 + 1.23)//(2)] = 1.05\ \Omega$$

The current delivered by the source = 30/1.05 = 28.57 A

Problem 1.10 Determine the current in the 10 Ω resistance and find V_s in the circuit shown in Fig. 1.44.

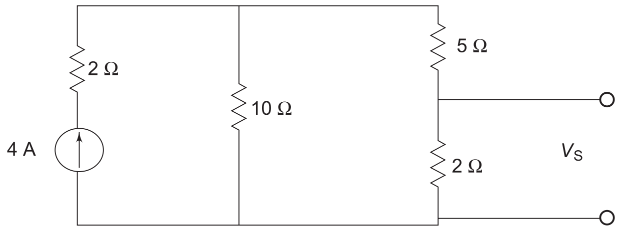


Fig. 1.44

Solution The current in 10 Ω resistance

$$I_{10} = \text{total current} \times (R_T)/(R_T + R_{10})$$

where R_T is the total parallel resistance.

$$I_{10} = 4 \times \frac{7}{17} = 1.65 \text{ A}$$

Similarly, the current in resistance R_5 is

$$I_5 = 4 \times \frac{10}{10 + 7} = 2.35 \text{ A}$$

or $\qquad 4 - 1.65 = 2.35$ A

The same current flows through the 2 Ω resistance.

∴ Voltage across 2 Ω resistance, $V_s = I_5 \times 2$

$$= 2.35 \times 2 = 4.7 \text{ V}$$

Problem 1.11 Determine the value of resistance R and current in each branch when the total current taken by the circuit shown in Fig. 1.45 is 6 A.

Solution The current in branch *ADB*

$I_{30} = 50/(25 + 5) = 1.66$ A

The current in branch *ACB* $I_{10+R} = 50/(10 + R)$.

According to Kirchhoff's current law

$$I_T = I_{30} + I_{(10+R)}$$

$$6\text{A} = 1.66 \text{ A} + I_{10+\text{R}}$$

∴ $$I_{10+R} = 6 - 1.66 = 4.34 \text{ A}$$

∴ $$\frac{50}{10 + R} = 4.34$$

$$10 + R = \frac{50}{4.34} = 11.52$$

$$R = 1.52 \, \Omega$$

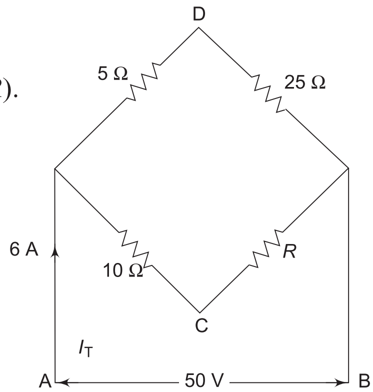


Fig. 1.45

Problem 1.12 Find the power delivered by the source in the circuit shown in Fig. 1.46.

Solution Between points $C(E)$ and D, resistances R_3 and R_4 are in parallel, which gives $R_8 = (R_3//R_4) = 2.5\ \Omega$

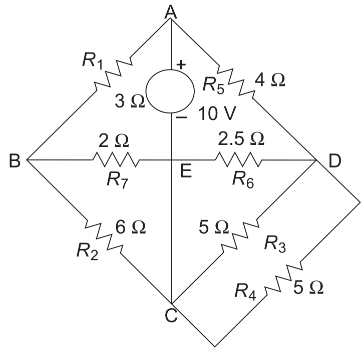


Fig. 1.46

Between points B and $C(E)$, resistances R_2 and R_7 are in parallel, which gives

$$R_9 = (R_2 \parallel R_7) = 1.5\ \Omega$$

Between points $C(E)$ and D, resistances R_6 and R_8 are in parallel and gives

$$R_{10} = (R_6 \parallel R_8) = 1.25\ \Omega$$

The series combination of R_1 and R_9 gives

$$R_{11} = R_1 + R_9 = 3 + 1.5 = 4.5\ \Omega$$

Similarly, the series combination of R_5 and R_{10} gives

$$R_{12} = R_5 + R_{10} = 5.25\ \Omega$$

The resistances R_{11} and R_{12} are in parallel, which gives

Total resistance = $(R_{11} \parallel R_{12}) = 2.42$ ohms

These reductions are shown in Figs. 1.47 (a), (b), (c) and (d).

Current delivered by the source = $\dfrac{10}{2.42} = 4.13$ A

Power delivered by the source = VI

$$= 10 \times 4.13 = 41.3\ \text{W}$$

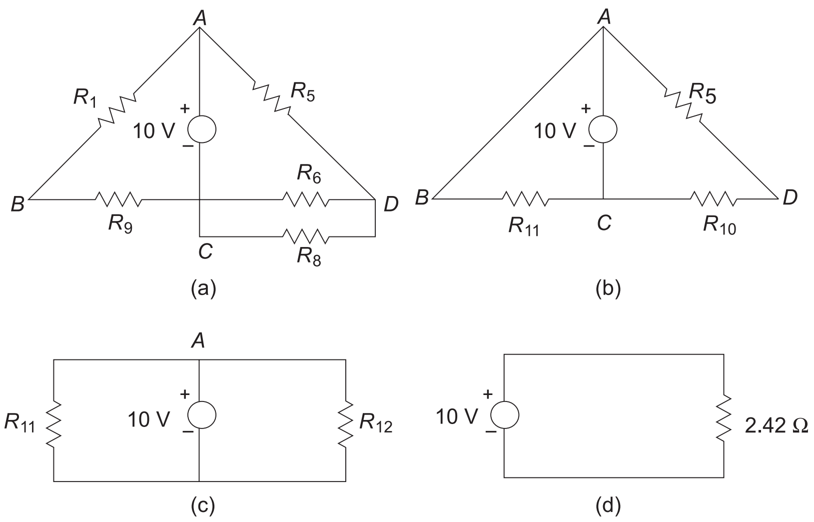


Fig. 1.47 (a, b, c and d)

Problem 1.13 Determine the voltage drop across the 10 Ω resistance in the circuit as shown in Fig. 1.48.

Solution The circuit is redrawn as shown in Fig. 1.49.

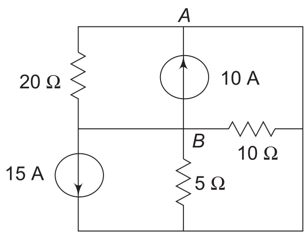


Fig. 1.48

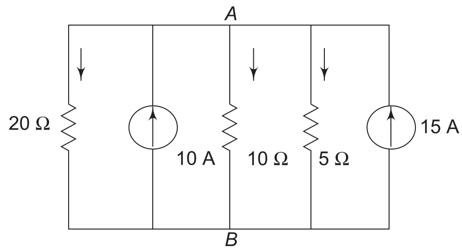


Fig. 1.49

This is a single node pair circuit. Assume voltage V_A at node A. By applying Kirchhoff's current law at node A, we have

$$\frac{V_A}{20} + \frac{V_A}{10} + \frac{V_A}{5} = 10 + 15$$

$$V_A\left[\frac{1}{20} + \frac{1}{10} + \frac{1}{5}\right] = 25 \text{ A}$$

$$V_A\ [0.05 + 0.1 + 0.2) = 25 \text{ A}$$

$$V_A = \frac{25}{0.35} = 71.42 \text{ V}$$

The voltage across 10 Ω is nothing but the voltage at node A.

$$\therefore \qquad V_{10} = V_A = 71.42 \text{ V}$$

Problem 1.14 In the circuit shown in Fig. 1.50 what are the values of R_1 and R_2, when the current flowing through R_1 is 1 A and R_2 is 5 A? What is the value of R_2 when the current flowing through R_1 is zero?

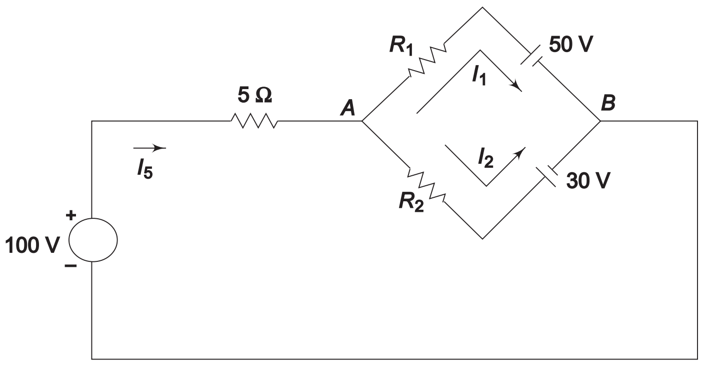


Fig. 1.50

Solution The current in the 5 Ω resistance

$$I_5 = I_1 + I_2 = 1 + 5 = 6 \text{ A}$$

Voltage across resistance 5 Ω is $V_5 = 5 \times 6 = 30$ V

The voltage at node A, $V_A = 100 - 30 = 70$ V

$$\therefore \quad I_2 = \frac{V_A - 30}{R_2} = \frac{70 - 30}{R_2}$$

$$R_2 = \frac{70 - 30}{I_2} = \frac{40}{5} = 8\ \Omega$$

Similarly, $$R_1 = \frac{70 - 50}{I_1} = \frac{20}{1} = 20\ \Omega$$

When $V_A = 50$ V, the current I_1 in resistance R_1 becomes zero.

$$\therefore \quad I_2 = \frac{50 - 30}{R_2}$$

where I_2 becomes the total current

$$\therefore \quad I_2 = \frac{100 - V_A}{5} = \frac{100 - 50}{5} = 10\ \text{A}$$

$$\therefore \quad R_2 = \frac{20}{I_2} = \frac{20}{10} = 2\ \Omega$$

Problem 1.15 Determine the output voltage V_{out} in the circuit shown in Fig. 1.51.

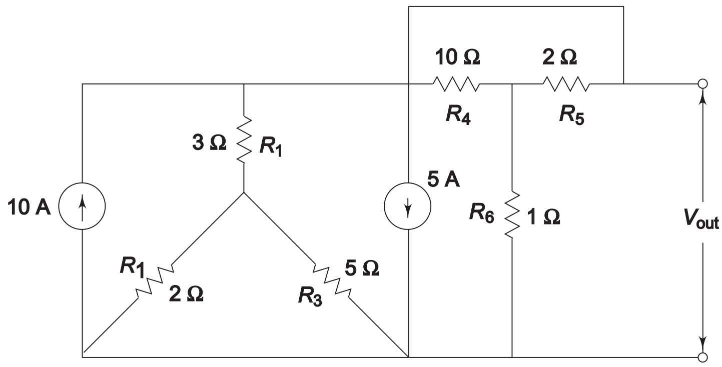


Fig. 1.51

Solution The circuit shown in Fig. 1.51 can be redrawn as shown in Fig. 1.52.

In Fig. 1.52, R_2 and R_3 are in parallel, R_4 and R_5 are in parallel. The complete circuit is a single node pair circuit. Assuming voltage V_A at node A and applying Kirchhoff's current law in the circuit, we have

$$10\text{A} - \frac{V_A}{4.43} - 5\text{A} - \frac{V_A}{2.67} = 0$$

$$\therefore \quad V_A\left[\frac{1}{4.43} + \frac{1}{2.67}\right] = 5\ \text{A}$$

$$V_A\,[0.225 + 0.375] = 5$$

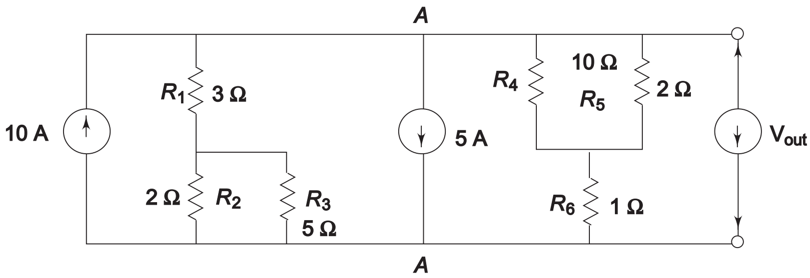


Fig. 1.52

$$\therefore \qquad V_A = \frac{5}{0.6} = 8.33 \text{ V}$$

$$V_{\text{out}} = V_A = 8.33 \text{ V}$$

Problem 1.16 Determine the voltage V_{AB} in the circuit shown in Fig. 1.53.

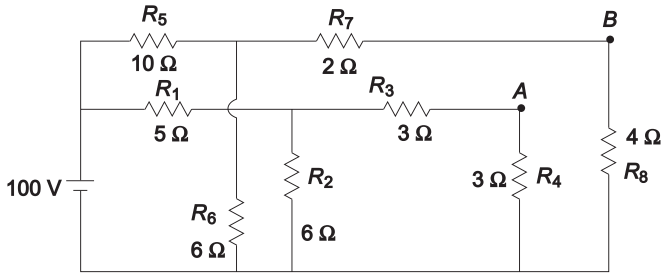


Fig. 1.53

Solution The circuit in Fig. 1.53 can be redrawn as shown in Fig. 1.54 (a).

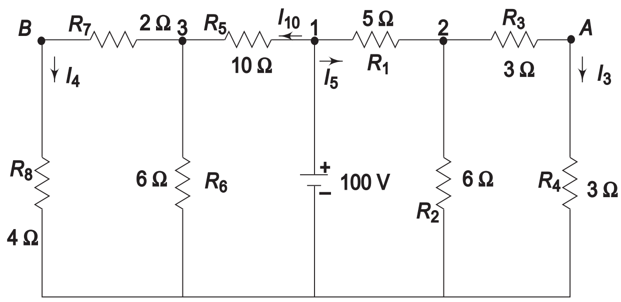


Fig. 1.54 (a)

At node 3, the series combination of R_7 and R_8 are in parallel with R_6, which gives $R_9 = [(R_7 + R_8)//R_6] = 3\ \Omega$.

At node 2, the series combination of R_3 and R_4 are in parallel with R_2, which gives $R_{10} = [(R_3 + R_4)//R_2] = 3\ \Omega$.

It is further reduced and is shown in Fig. 1.54 (b).

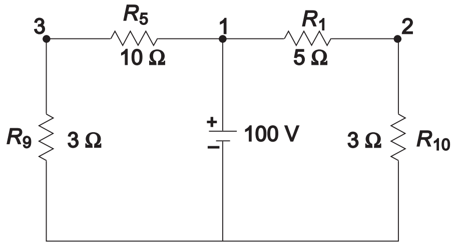


Fig. 1.54 (b)

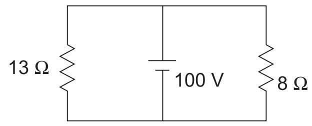


Fig. 1.54 (c)

Simplifying further we draw it as shown in Fig. 1.54 (c).

$$\text{Total current delivered by the source} = \frac{100}{R_T}$$

$$= \frac{100}{(13//8)} = 20.2 \text{ A}$$

Current in the 8 Ω resistor is $I_8 = 20.2 \times \dfrac{13}{13+8} = 12.5$ A

Current in the 13 Ω resistor is $I_{13} = 20.2 \times \dfrac{8}{13+8} = 7.69$ A

So $I_5 = 12.5$ A, and $I_{10} = 7.69$ A

Current in the 4 Ω resistance $I_4 = 3.845$ A

Current in the 3 Ω resistance $I_3 = 6.25$ A

$$V_{AB} = V_A - V_B$$

where $V_A = I_3 \times 3\ \Omega = 6.25 \times 3 = 18.75$ V

$V_B = I_4 \times 4\ \Omega = 3.845 \times 4 = 15.38$ V

$\therefore$ $V_{AB} = 18.75 - 15.38 = 3.37$ V

Problem 1.17 Determine the value of R in the circuit shown in Fig. 1.55, when the current is zero in the branch CD.

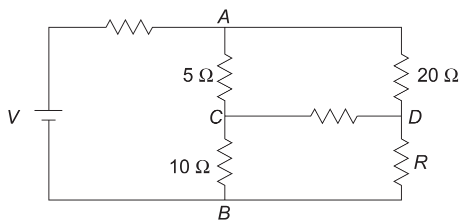


Fig. 1.55

Solution The current in the branch CD is zero, if the potential difference across CD is zero.

That means, voltage at point C = voltage at point D.

Since no current is flowing, the branch CD is open circuited. So the same voltage is applied across ACB and ADB

$$V_{10} = V_A \times \frac{10}{15}$$

$$V_R = V_A \times \frac{R}{20 + R}$$

$\therefore$ $$V_{10} = V_R$$

and $$V_A \times \frac{10}{15} = V_A \times \frac{R}{20 + R}$$

$\therefore$ $$R = 40\ \Omega$$

Problem 1.18 Find the power absorbed by each element in the circuit shown in Fig. 1.56.

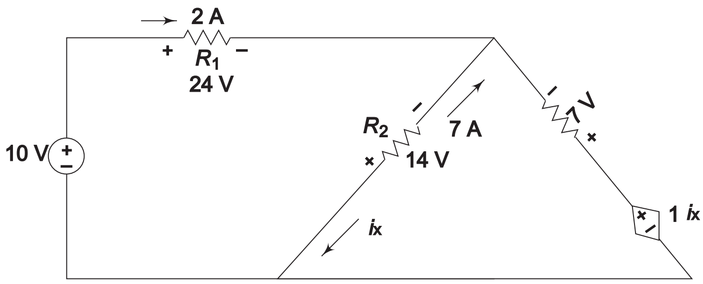


Fig. 1.56

Solution Power absorbed by any element = VI

where V is the voltage across the element and I is the current passing through that element

Here potential rises are taken as (–) sign.

Power absorbed by 10 V source = – 10 × 2 = – 20 W

Power absorbed by resistor R_1 = 24 × 2= 48 W

Power absorbed by resistor R_2 = 14 × 7 = 98 W

Power absorbed by resistor R_3 = – 7 × 9 = – 63 W

Power absorbed by dependent voltage source = (1 × – 7) × 9 = – 63 W

Problem 1.19 Show that the algebraic sum of the five absorbed power values in Fig. 1.57 is zero.

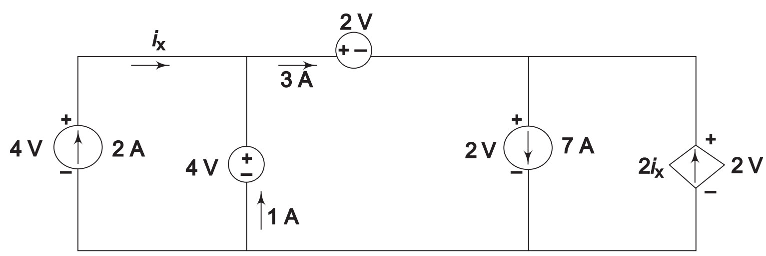


Fig. 1.57

Solution Power absorbed by 2 A current source = (– 4) × 2 = – 8 W
Power absorbed by 4 V voltage source = (– 4) × 10 = – 4 W
Power absorbed by 2 V voltage source = (2) × 3 = 6 W
Power absorbed by 7 A current source = (7) × 2 = 14 W
Power absorbed by $2i_x$ dependent current source = (– 2) × 2 × 2 = – 8 W
Hence, the algebraic sum of the five absorbed power values is zero.

Problem 1.20 For the circuit shown in Fig. 1.58, find the power absorbed by each of the elements.

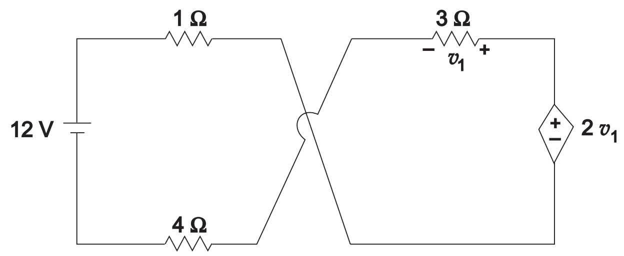


Fig. 1.58

Solution The above circuit can be redrawn as shown in Fig. 1.59.

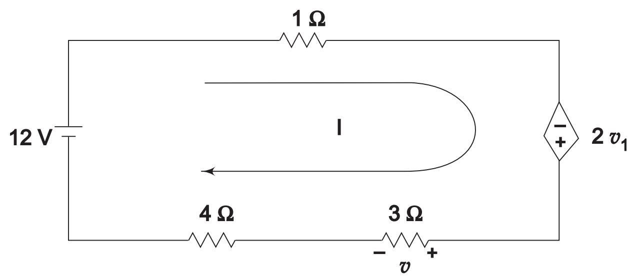


Fig. 1.59

Assume loop current I as shown in Fig. 1.59.
If we apply Kirchhoff's voltage law, we get

$$-12 + I - 2v_1 + v_1 + 4I = 0$$

The voltage across 3 Ω resistor is $v_1 = 3I$
Substituting v_1 in the loop equation, we get $I = 6$ A
Power absorbed by the 12 V source = (– 12) × 6 = – 72 W
Power absorbed by the 1 Ω resistor = 6 × 6 = 36 W
Power absorbed by $2v_1$ dependent voltage source

$$= (2v_1)I = 2 \times 3 \times 6 \times 6 = -216 \text{ W}$$

Power absorbed by 3 Ω resistor = $v_1 \times I = 18 \times 6 = 108$ W
Power absorbed by 4 Ω resistor = 4 × 6 × 6 = 144 W

Problem 1.21 For the circuit shown in Fig. 1.60, find the power absorbed by each element.

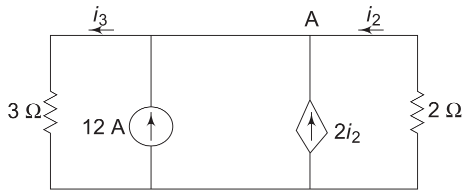


Fig. 1.60

Solution The circuit shown in Fig. 1.60 is a parallel circuit and consists of a single node *A*. By assuming voltage *V* at node A, we can find the current in each element.

According to Kirchhoff's current law

$$i_3 - 12 - 2i_2 - i_2 = 0$$

By using Ohm's law, we have

$$i_3 = \frac{V}{3},\ i_2 = \frac{-V}{2}$$

$$V\left[\frac{1}{3} + 1 + \frac{1}{2}\right] = 12$$

$$\therefore \qquad V = \frac{12}{1.83} = 6.56$$

$$i_3 = \frac{6.56}{3} = 2.187\text{A};\ i_2 = \frac{-6.56}{2} = -\,3.28\text{ A}$$

Power absorbed by the 3 Ω resistor = (+ 6.56) (2.187) = 14.35 W

Power absorbed by 12 A current source = (– 6.56) 12 = – 78.72 W

Power absorbed by $2i_2$ dependent current source

= (– 6.56) × 2 × (– 3.28) = 43.03 W

Power absorbed by 2 Ω resistor = (– 6.52) (– 3.28) = 21.51 W

Practice Problems

1.1 (i) Determine the current in each of the following cases
(a) 75 C in 1 s (b) 10 C in 0.5 s
(c) 5 C in 2 s
(ii) How long does it take 10 C to flow past a point if the current is 5 A?

1.2 A resistor of 30 Ω has a voltage rating of 500 V; what is its power rating?

1.3 A resistor with a current of 2 A through it converts 1000 J of electrical energy to heat energy in 15 s. What is the voltage across the resistor?

1.4 The filament of a light bulb in the circuit has a certain amount of resistance. If the bulb operates with 120 V and 0.8 A of current, what is the resistance of its filament?

1.5 Find the capacitance of a circuit in which an applied voltage of 20 V gives an energy store of 0.3 J.

1.6 A 6.8 kΩ resistor has burned out in a circuit. It has to be replaced with another resistor with the same ohmic value. If the resistor carries 10 mA, what should be its power rating?

1.7 If you wish to increase the amount of current in a resistor from 100 mA to 150 mA by changing the 20 V source, by how many volts should you change the source? To what new value should you set it?

1.8 A 12 V source is connected to a 10 Ω resistor.

(a) How much energy is used in two minutes?

(b) If the resistor is disconnected after one minute, does the power absorbed in resistor increase or decrease?

1.9 A capacitor is charged to 50 μC. The voltage across the capacitor is 150 V. It is then connected to another capacitor four times the capacitance of the first capacitor. Find the loss of energy.

1.10 The voltage across two parallel capacitors 5 μF and 3 μF changes uniformly from 30 to 75 V in 10 ms. Calculate the rate of change of voltage for (i) each capacitor, and (ii) the combination.

1.11 The following voltage drops are measured across each of three resistors in series: 5.5 V, 7.2 V and 12.3 V. What is the value of the source voltage to which these resistors are connected? If a fourth resistor is added to the circuit with a source voltage of 30 V. What should be the drop across the fourth resistor?

1.12 What is the voltage V_{AB} across the resistor shown in Fig. 1.61?

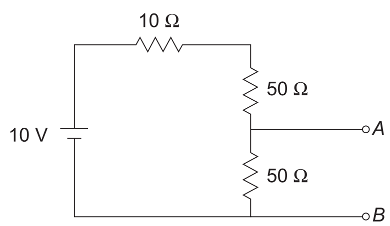


Fig. 1.61 (a)

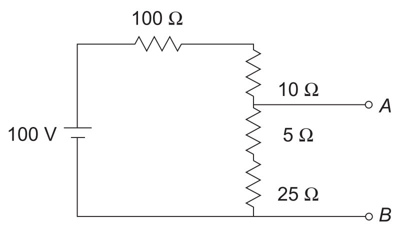


Fig. 1.61 (b)

1.13 The source voltage in the circuit shown in Fig. 1.62 is 100 V. How much voltage does each metre read?

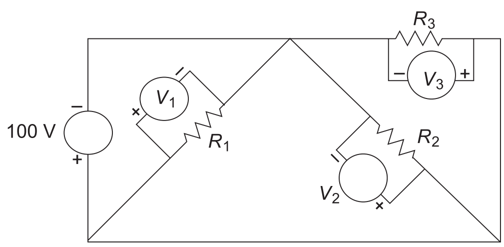


Fig. 1.62

1.14 Using the current divider formula, determine the current in each branch of the circuit shown in Fig. 1.63.

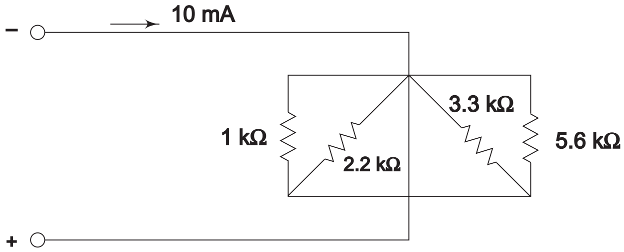


Fig. 1.63

1.15 Six light bulbs are connected in parallel across 110 V. Each bulb is rated at 75 W. How much current flows through each bulb, and what is the total current?

1.16 For the circuit shown in Fig. 1.64, find the total resistance between terminals *A* and *B*; the total current drawn from a 6 V source connected from *A* to *B*; and the current through 4.7 kΩ; voltage across 3 kΩ.

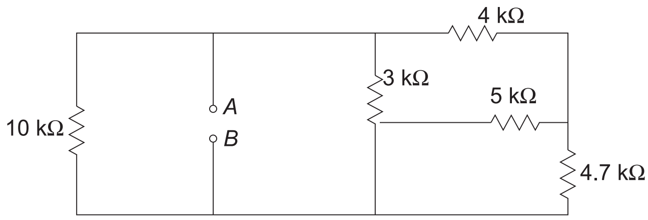


Fig. 1.64

1.17 For the circuit shown in Fig. 1.65, find the total resistance.

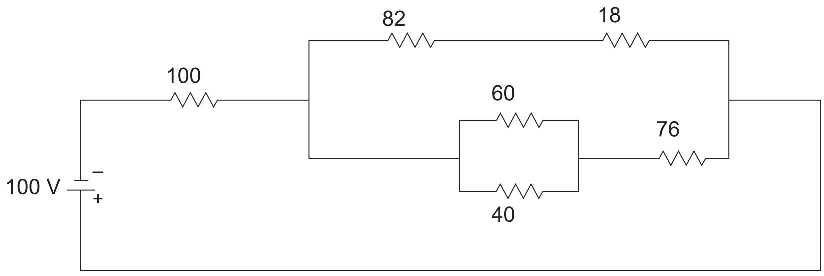


Fig. 1.65

1.18 The current in the 5 Ω resistance of the circuit shown in Fig. 1.66 is 5 A. Find the current in the 10 Ω resistor. Calculate the power consumed by the 5 Ω resistor.

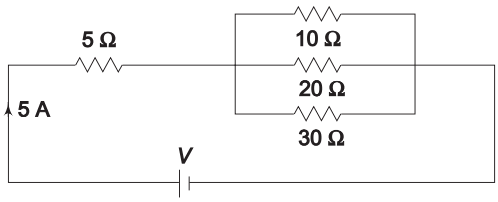


Fig. 1.66

1.19 A battery of unknown emf is connected across resistances as shown in Fig. 1.67. The voltage drop across the 8 Ω resistor is 20 V. What will be the current reading in the ammeter? What is the emf of the battery.

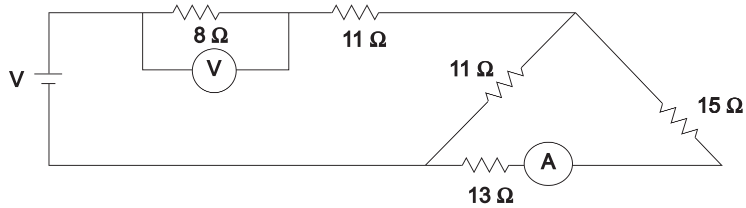


Fig. 1.67

1.20 An electric circuit has three terminals *A*, *B*, *C*. Between *A* and *B* is connected a 2 Ω resistor, between *B* and *C* are connected a 7 Ω resistor and 5 Ω resistor in parallel and between *A* and *C* is connected a 1 Ω resistor. A battery of 10 V is then connected between terminals *A* and *C*. Calculate (a) total current drawn from the battery (b) voltage across the 2 Ω resistor (c) current passing through the 5 Ω resistor.

1.21 Use Ohm's law and Kirchhoff's laws on the circuit given in Fig. 1.68, find V_{in}, V_s and power provided by the dependent source.

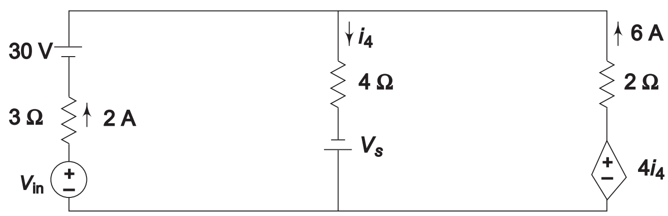


Fig. 1.68

1.22 Use Ohm's law and Kirchhoff's laws on the circuit given in Fig. 1.69, find all the voltages and currents.

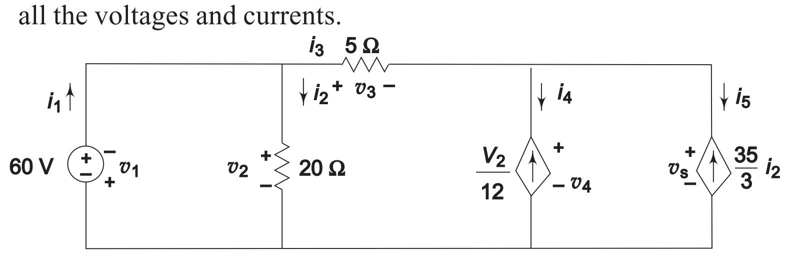


Fig. 1.69

1.23 Find the power absorbed by each element and show that the algebraic sum of powers is zero in the circuit shown in Fig. 1.70.

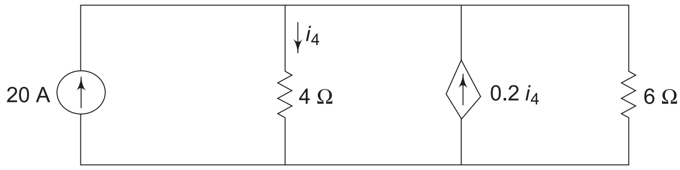


Fig. 1.70

1.24 Find the power absorbed by each element in the circuit shown in Fig. 1.71.

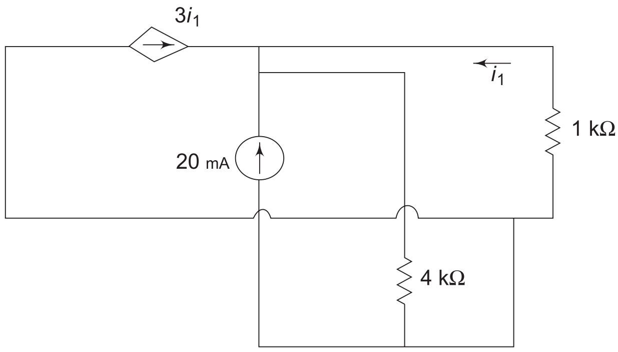


Fig. 1.71

Objective-type Questions

1.1 How many coulombs of charge do 50×10^{31} electrons possess?
(a) 80×10^{12} C (b) 50×10^{31} C
(c) 0.02×10^{-31} C (d) $1/80 \times 10^{12}$ C

1.2 Determine the voltage of 100 J/25 C.
(a) 100 V (b) 25 V
(c) 4 V (d) 0.25 V

1.3 What is the voltage of a battery that uses 800 J of energy to move 40 C of charge through a resistor?
(a) 800 V (b) 40 V
(c) 25 V (d) 20 V

1.4 Determine the current if a 10 coulomb charge passes a point in 0.5 seconds.
(a) 10 A (b) 20 A
(c) 0.5 A (d) 2 A

1.5 If a resistor has 5.5 V across it and 3 mA flowing through it, what is the power?
(a) 16.5 mW (b) 15 mW
(c) 1.83 mW (d) 16.5 W

1.6 Identify the passive element among the following
(a) Voltage source (b) Current source
(c) Inductor (d) Transistor

1.7 If a resistor is to carry 1 A of current and handle 100 W of power, how many ohms must it be? Assume that voltage can be adjusted to any required value.
(a) 50 Ω (b) 100 Ω
(c) 1 Ω (d) 10 Ω

1.8 A 100 Ω resistor is connected across the terminals of a 2.5 V battery. What is the power dissipation in the resistor?
(a) 25 W (b) 100 W
(c) 0.4 W (d) 6.25 W

1.9 Determine total inductance of a parallel combination of 100 mH, 50 mH and 10 mH.
(a) 7.69 mH (b) 160 mH
(c) 60 mH (d) 110 mH

1.10 How much energy is stored by a 100 mH inductance with a current of 1 A?
(a) 100 J (b) 1 J
(c) 0.05 J (d) 0.01 J

1.11 Five inductors are connected in series. The lowest value is 5 μH. If the value of each inductor is twice that of the preceding one, and if the inductors are connected in order ascending values. What is the total inductance?
(a) 155 μH (b) 155 H
(c) 155 mH (d) 25 μH

1.12 Determine the charge when $C = 0.001$ μF and $v = 1$ KV.
(a) 0.001 C (b) 1 μC
(c) 1 C (d) 0.001 C

1.13 If the voltage across a given capacitor is increased, does the amount of stored charge
(a) increase (b) decrease
(c) remain constant (d) is exactly doubled

1.14 A 1 μF, a 2.2 μF and a 0.05 μF capacitors are connected in series. The total capacitance is less than
(a) 0.07 (b) 3.25
(c) 0.05 (d) 3.2

1.15 How much energy is stored by a 0.05 μF capacitor with a voltage of 100 V?
(a) 0.025 J (b) 0.05 J
(c) 5 J (d) 100 J

1.16 Which one of the following is an ideal voltage source?
(a) voltage independent of current (b) current independent of voltage
(c) both (a) and (b) (d) none of the above

1.17 The following voltage drops are measured across each of three resistors in series: 5.2 V, 8.5 V and 12.3 V. What is the value of the source voltage to which these resistors are connected?
(a) 8.2 V (b) 12.3 V
(c) 5.2 V (d) 26 V

1.18 A certain series circuit has a 100 Ω, a 270 Ω, and a 330 Ω resistor in series. If the 270 Ω resistor is removed, the current

(a) increases (b) becomes zero
(c) decrease (d) remain constant

1.19 A series circuit consists of a 4.7 kΩ, 5.6 kΩ, 9 kΩ and 10 kΩ resistor. Which resistor has the most voltage across it?
(a) 4.7 kΩ (b) 5.6 kΩ
(c) 9 kΩ (d) 10 kΩ

1.20 The total power in a series circuit is 10 W. There are five equal value resistors in the circuit. How much power does each resistor dissipate?
(a) 10 W (b) 5 W
(c) 2 W (d) 1 W

1.21 When a 1.2 kΩ resistor, 100 Ω resistor, 1 kΩ resistor and 50 Ω resistor are in parallel, the total resistance is less than
(a) 100 Ω (b) 50 Ω
(c) 1 kΩ (d) 1.2 kΩ

1.22 If a 10 V battery is connected across the parallel resistors of 3 Ω, 5 Ω, 10 Ω and 20 Ω, how much voltage is there across 5 Ω resistor?
(a) 10 V (b) 3 V
(c) 5 V (d) 20 V

1.23 If one of the resistors in a parallel circuit is removed, what happens to the total resistance?
(a) decreases (b) increases
(c) remain constant (d) exactly doubles

1.24 The power dissipation in each of three parallel branches is 1 W. What is the total power dissipation of the circuit?
(a) 1 W (b) 4 W
(c) 3 W (d) zero

1.25 In a four branch parallel circuit, 10 mA of current flows in each branch. If one of the branch opens, the current in each of the other branches
(a) increases (b) decreases
(c) remains unaffected (d) doubles

1.26 Four equal value resistors are connected in parallel. Five volts are applied across the parallel circuit, and 2.5 mA are measured from the source. What is the value of each resistor?
(a) 4 Ω (b) 8 Ω
(c) 2.5 Ω (d) 5 Ω

1.27 Six light bulbs are connected in parallel across 110 V. Each bulb is related at 75 W. How much current flows through each bulb?
(a) 0.682 A (b) 0.7 A
(c) 75 A (d) 110 A

1.28 A 330 Ω resistor is in series with the parallel combination of four 1 kΩ resistors. A 100 V source is connected to the circuit. Which resistor has the most current through it.
(a) 330 Ω resistor
(b) parallel combination of three 1 kΩ resistors
(c) parallel combination of two 1 kΩ resistors
(d) 1 kΩ resistor

1.29 The current i_4 in the circuit shown in Fig. 1.72 is equal to

(a) 12 A (b) – 12 A

(c) 4 A (d) None of the above

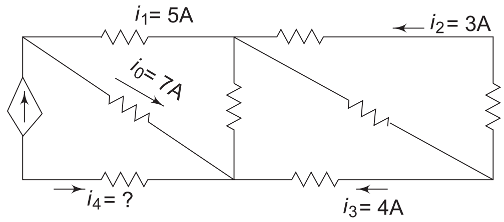


Fig. 1.72

1.30 The voltage V in Fig. 1.73 is equal to

(a) 3 V (b) – 3 V

(c) 5 V (d) None of the above

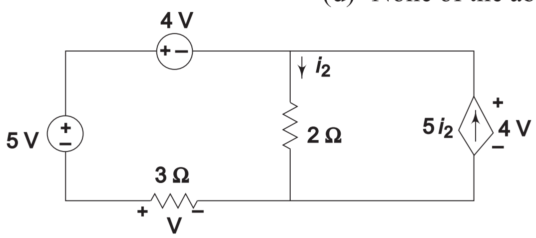


Fig. 1.73

1.31 The voltage V in Fig. 1.74 is always equal to

(a) 9 V (b) 5 V

(c) 1 V (d) None of the above

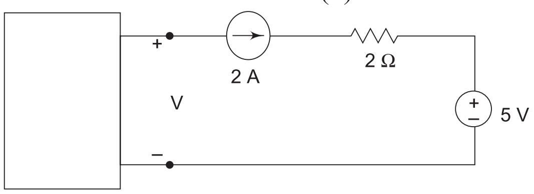


Fig. 1.74

1.32 The voltage V in Fig. 1.75 is

(a) 10 V (b) 15 V

(c) 5 V (d) None of the above.

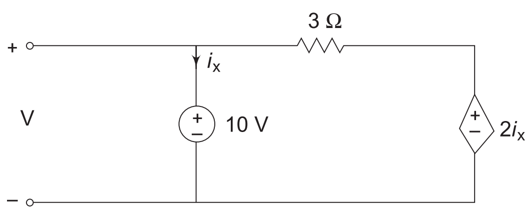


Fig. 1.75

Chapter 2

Methods of Analysing Circuits

2.1 INTRODUCTION

A division of mathematics called topology or graph theory deals with graphs of networks and provides information that helps in the formulation of network equations. In circuit analysis, all the elements in a network must satisfy Kirchhoff's laws, besides their own characteristics. Based on these laws, we can form a number of equations. These equations can be easily written by converting the network into a graph. Certain aspects of network behaviour are brought into better perspective if a graph of the network is drawn. If each element or a branch of a network is represented on a diagram by a line irrespective of the characteristics of the elements, we get a graph. Hence, network topology is network geometry. A network is an interconnection of elements in various branches at different nodes as shown in Fig. 2.1. The corresponding graph is shown in Fig. 2.2 (a).

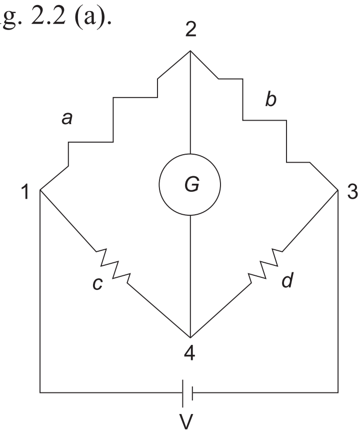


Fig. 2.1

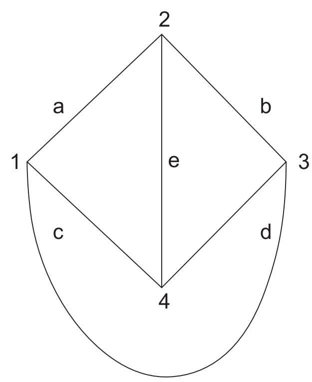


Fig. 2.2 (a)

The graphs shown in Figs 2.2 (b) and (c) are also graphs of the network in Fig. 2.1.

It is interesting to note that the graphs shown in Fig. 2.2 (a), (b) and (c) may appear to be different but they are topologically equivalent. A branch is represented by a line segment connecting a pair of nodes in the graph of a network. A node is a terminal of a branch, which is represented by a point. Nodes are the end points of branches. All these graphs have identical relationships between branches and nodes.

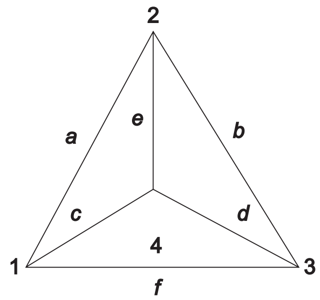


Fig. 2.2(b)

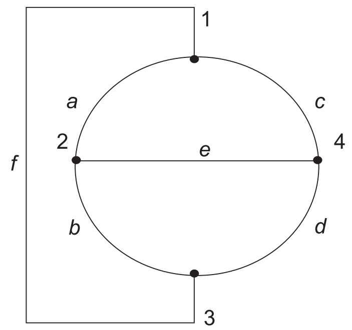


Fig. 2.2(c)

The three graphs in Fig. 2.2 have six branches and four nodes. These graphs are also called undirected. If every branch of a graph has a *direction* as shown in Fig. 2.3, then the graph is called a *directed graph*.

A node and a branch are incident if the node is a terminal of the branch. Nodes can be incident to one or more elements. The number of branches incident at a node of a graph indicates the degree of the node. For example, in Fig. 2.3 the degree of node 1 is three. Similarly, the degree of node 2 is three. If each element of the connected graph is assigned a direction as shown in Fig. 2.3 it is then said to be oriented. A graph is connected if and only if there is a path between every pair of nodes. A path is said to exist between any two nodes, for example 1 and 4 of the graph in Fig. 2.3, if it is possible to reach node 4 from node 1 by traversing along any of the branches of the graph. A graph can be drawn if there exists a path between any pair of nodes. A loop exists, if there is more than one path between two nodes.

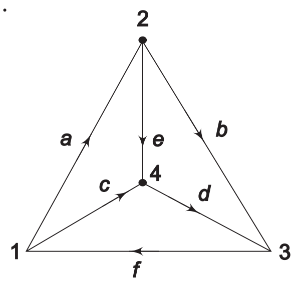


Fig. 2.3

Planar and Non-Planar Graphs

A graph is said to be planar if it can be drawn on a plane surface such that no two branches cross each other as shown in Fig. 2.2. On the other hand in a non-planar graph there will be branches which are not in the same plane as others, i.e. a non-planar graph cannot be drawn on a plane surface without a crossover. Figure 2.4 illustrates a non-planar graph.

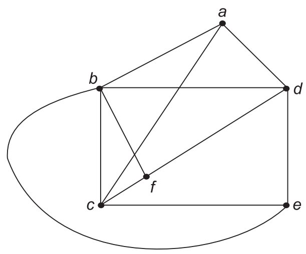


Fig. 2.4

2.2 TREE AND CO-TREE

A tree is a connected subgraph of a network which consists of all the nodes of the original graph but no closed paths. The graph of a network may have a number of trees. The number of nodes in a graph is equal to the number nodes in the tree. The number of branches in a tree is less than the number of branches in a graph. A graph is a tree if there is a unique path between any pair of nodes. Consider a graph with four branches and three nodes as shown in Fig. 2.5.

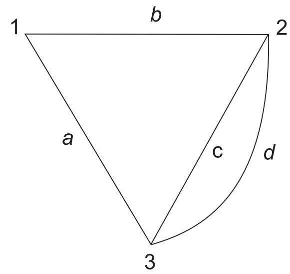


Fig. 2.5

Five open-ended graphs based on Fig. 2.5 are represented by Figs 2.6 (a) to (e). Since each of these open-ended graphs satisfies all the requirements of a tree, each graph in Fig. 2.6 is a tree corresponding to Fig. 2.5.

In Fig. 2.6, there is no closed path or loop; the number of nodes $n = 3$ is the same for the graph and its tree, where as the number of branches in the tree is only two. In general, if a tree contains n nodes, then it has $(n - 1)$ branches.

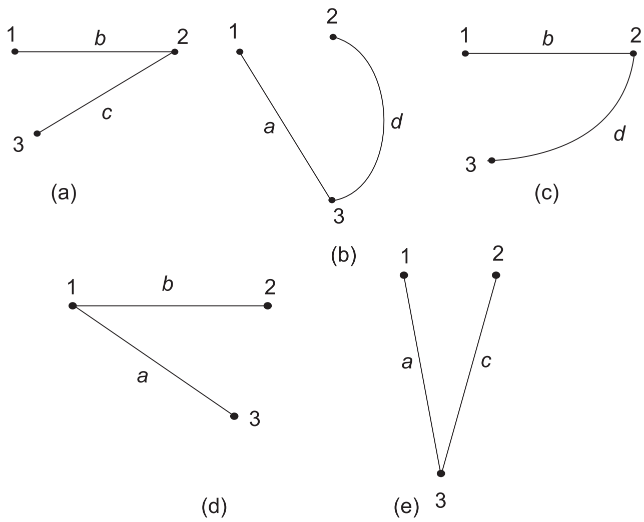


Fig. 2.6

In forming a tree for a given graph, certain branches are removed or opened. The branches thus opened are called links or *link branches*. The links for Fig. 2.6 (a) for example are *a* and *d* and for 2.6 (b) are *b* and *c*. The set of all links of a given tree is called the co-tree of the graph. Obviously, the branches *a*, *d* are a co-tree for Fig. 2.6 (a) and b, c are the co-tree. Similarly, for the tree in Fig. 2.6 (b), the branches *b*, *c* are the co-tree. Thus the link branches and the tree branches combine to form the graph of the entire network.

Example 2.1 For the given graph shown in Fig. 2.7 draw the number of possible trees.

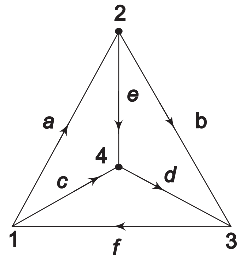


Fig. 2.7

Solution The number of possible trees for Fig. 2.7 are represented by Figs 2.8 (a) – (g).

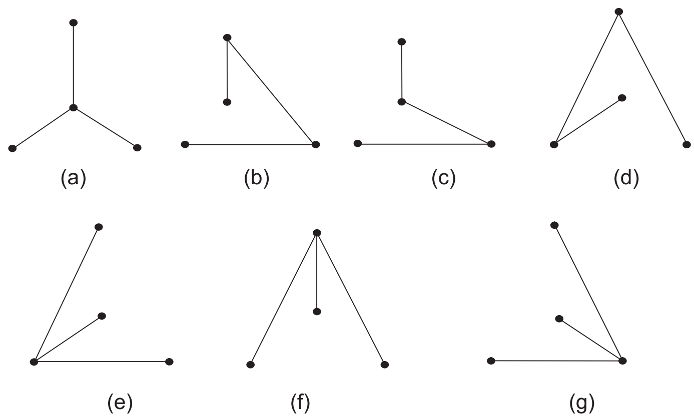


Fig. 2.8

2.3 TWIGS AND LINKS

The branches of a tree are called its 'twigs'. For a given graph, the complementary set of branches of the tree is called the co-tree of the graph. The branches of a co-tree are called links, i.e. those elements of the connected graph that are not included in the tree links and form a subgraph. For example, the set of branches (b, d, f) represented by dotted lines in Fig. 2.11 form a co-tree of the graph in Fig. 2.9 with respect to the tree in Fig. 2.10.

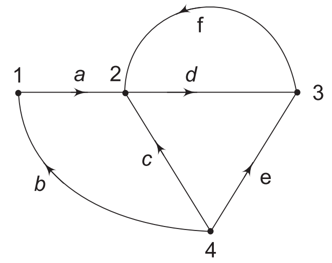


Fig. 2.9

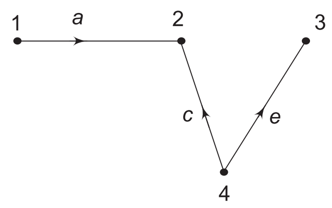


Fig. 2.10

The branches a, c and e are the twigs while the branches b, d and f are the links of this tree. It can be seen that for a network with b branches and n nodes, the number of twigs for a selected tree is $(n - 1)$ and the number of links I with respect to this tree is $(b - n + 1)$. The number of twigs $(n - 1)$ is known as the tree value of the graph. It is also called the *rank* of the tree. If a link is added to the tree, the resulting graph contains one closed path, called a loop. The addition of each subsequent link forms one or more additional loops. Loops which contain only one link are independent and are called basic loops.

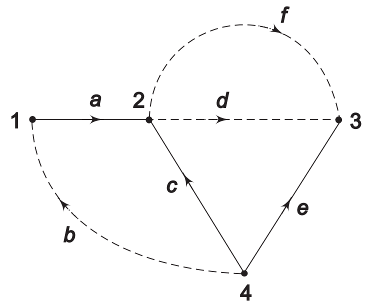


Fig. 2.11

2.4 INCIDENCE MATRIX (*A*)

The incidence of elements to nodes in a connected graph is shown by the element node incidence matrix (A). Arrows indicated in the branches of a graph result in an oriented or a directed graph. These arrows are the indication for the current flow or voltage rise in the network. It can be easily identified from an oriented graph regarding the incidence of branches to nodes. It is possible to have an analytical description of an oriented-graph in a matrix form. The dimensions of the matrix A is $n \times b$ where n is the number of nodes and b is number of branches. For a graph having n nodes and b branches, the complete incidence matrix A is a rectangular matrix of order $n \times b$.

In matrix A with n rows and b columns an entry a_{ij} in the i^{th} row and j^{th} column has the following values.

$$\left.\begin{aligned}
&a_{ij} = 1, \text{ if the } j^{\text{th}} \text{ branch is incident to and oriented away from the } i^{\text{th}} \text{ node}\\
&a_{ij} = -1, \text{ if the } j^{\text{th}} \text{ branch is incident to and oriented towards the } i^{\text{th}} \text{ node.}\\
&a_{ij} = 0, \text{ if the } j^{\text{th}} \text{ branch is not incident to the } i^{\text{th}} \text{ node.}
\end{aligned}\right\} \quad (2.1)$$

Figure 2.12 shows a directed graph.

Following the above convention its incidence matrix A is given by

$$
\begin{array}{cc}
 & \text{Nodes} \quad\quad\quad \text{Branches} \rightarrow \\
 & \downarrow \quad\quad a \quad b \quad c \quad d \quad e \quad f
\end{array}
$$

$$
A = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{bmatrix}
$$

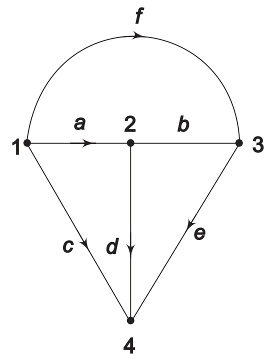


Fig. 2.12

The entries in the first row indicates that three branches a, c and f are incident to node 1 and they are oriented away from node 1 and therefore the entries a_{11}; a_{13} and a_{16} are + 1. Other entries in the 1st row are zero as they are not connected to node 1. Likewise, we can complete the incidence matrix for the remaining nodes 2, 3 and 4.

2.5 PROPERTIES OF INCIDENCE MATRIX A

Following properties are some of the simple conclusions from incidence matrix A.

1. Each column representing a branch contains two non-zero entries + 1 and – 1; the rest being zero. The unit entries in a column identify the nodes of the branch between which it is connected.
2. The unit entries in a row identify the branches incident at a node. Their number is called the degree of the node.
3. A degree of 1 for a row means that there is one branch incident at the node. This is commonly possible in a tree.
4. If the degree of a node is two, then it indicates that two branches are incident at the node and these are in series.
5. Columns of A with unit entries in two identical rows correspond to two branches with same end nodes and hence they are in parallel.
6. Given the incidence matrix A the corresponding graph can be easily constructed since A is a complete mathematical replica of the graph.
7. If one row of A is deleted the resulting $(n - 1) \times b$ matrix is called the reduced incidence matrix A_1. Given A_1, A is easily obtained by using the first property.

It is possible to find the exact number of trees that can be generated from a given graph if the reduced incidence matrix A_1 is known and the number of possible trees is given by Det $(A_1 A_1^T)$ where A_1^T is the transpose of the matrix A_1.

Example 2.2 Draw the graph corresponding to the given incidence matrix.

$$
A = \begin{bmatrix} -1 & 0 & 0 & 0 & +1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ -1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Solution There are five rows and eight columns which indicate that there are five nodes and eight branches. Let us number the columns from *a* to *h* and rows as 1 to 5.

$$\begin{array}{cc} & \begin{array}{cccccccc} A & b & c & d & e & f & g & h \end{array} \\ A = \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Mark the nodes corresponding to the rows 1, 2, 3, 4 and 5 as dots as shown in Fig. 2.13 (a). Examine each column of *A* and connect the nodes (unit entries) by a branch; label it after marking an arrow.

For example, examine the first column of *A*. There are two unit entries one in the first row and 2nd in the last row, hence connect branch *a* between node 1 and 5. The entry of A_{11} is – ve and that of A_{51} is + ve. Hence the orientation of the branch is away from node 5 and towards node 1 as per the convention. Proceeding in this manner we can complete the entire graph as shown in Fig. 2.13 (b).

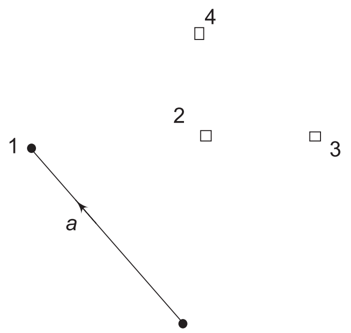


Fig. 2.13(a)

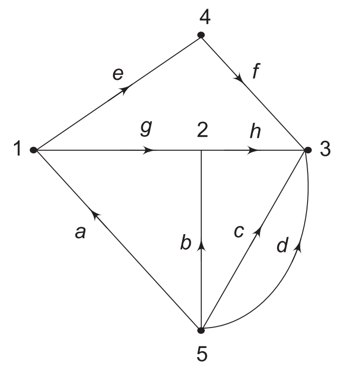


Fig. 2.13(b)

From the incidence matrix *A*, it can be verified that branches *c* and *d* are in parallel (property 5) and branches *e* and *f* are in series (property 4).

Example 2.3 Obtain the incidence matrix *A* from the following reduced incidence matrix A_1 and draw its graph.

$$[A_1] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Solution There are five rows and seven columns in the given reduced incidence matrix $[A_1]$. Therefore, the number of rows in the complete incidence matrix *A* will be 5 + 1 = 6. There will be six nodes and seven branches in the graph. The dimensions of matrix A is 6×7. The last row in A, i.e. 6th row for the matrix A can be obtained by using the first property of the incidence matrix. It is seen that the first column of $[A_1]$ has a single non-zero element – 1. Hence, the first element in the 6th row will be + 1 (– 1 + 1 = 0). Second column of A_1 has two non-zero elements + 1 and – 1, hence the 2nd element in the 6th row will be 0. Proceeding in this manner we can obtain the 6th row. The complete incidence matrix can therefore be written as

$$[A] = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We have seen that any one of the rows of a complete incidence matrix can be obtained from the remaining rows. Thus it is possible to delete any one row from A without loosing any information in A_1. Now the oriented graph can be constructed from the matrix A. The nodes may be placed arbitrarily. The number of nodes to be marked will be six. Taking node 6 as reference node the graph is drawn as shown in Fig. 2.14.

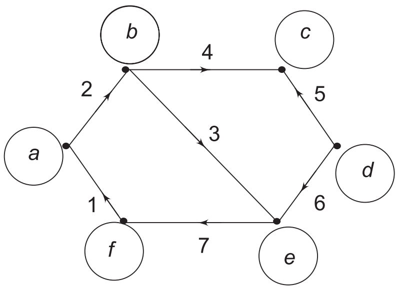


Fig. 2.14

2.6 INCIDENCE MATRIX AND KCL

Kirchhoff's current law (KCL) of a graph can be expressed in terms of the reduced incidence matrix as $A_1 I = 0$.

A_1, I is the matrix representation of KCL, where I represents branch current vectors I_1, I_2, $\cdots$ I_6.

Consider the graph shown in Fig. 2.15. It has four nodes a, b, c and d.

Let node d be taken as the reference node. The positive reference direction of the branch currents corresponds to the orientation of the graph branches. Let the branch currents be i_1, i_2, $\cdots$ i_6. Applying KCL at nodes a, b and c.

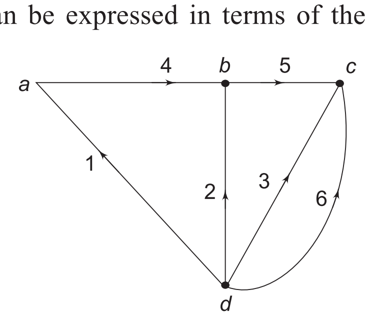


Fig. 2.15

$$- i_1 + i_4 = 0$$
$$- i_2 - i_4 + i_5 = 0$$
$$- i_3 = i_5 - i_6 = 0$$

These equations can be written in the matrix form as follows

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_1 \, I_b = 0 \qquad (2.2)$$

Here, I_b represents column matrix or a vector of branch currents.

$$I_b = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_b \end{bmatrix}$$

A_1 is the reduced incidence matrix of a graph with n nodes and b branches. And it is a $(n - 1) \times b$ matrix obtained from the complete incidence matrix of A deleting one of its rows. The node corresponding to the deleted row is called the reference node or datum node. It is to be noted that $A_1 \, I_b = 0$ gives a set of $n - 1$ linearly independent equations in branch currents $I_1, I_2, \cdots I_6$. Here $n = 4$. Hence, there are three linearly independent equations.

2.7 LINK CURRENTS: TIE-SET MATRIX

For a given tree of a graph, addition of each link between any two nodes forms a loop called the fundamental loop. In a loop there exists a closed path and a circulating current, which is called the link current. The current in any branch of a graph can be found by using link currents.

The fundamental loop formed by one link has a unique path in the tree joining the two nodes of the link. This loop is also called *f*-loop or a tie-set.

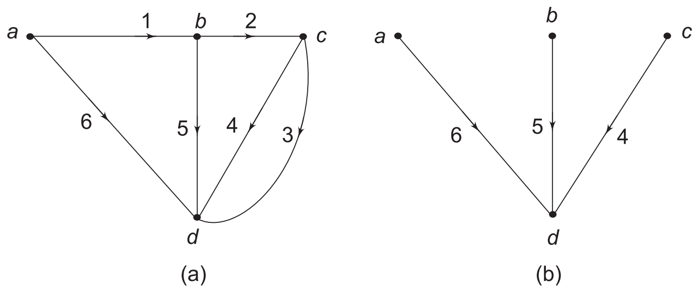


Fig. 2.16

Consider a connected graph shown in Fig. 2.16 (a). It has four nodes and six branches. One of its trees is arbitrarily chosen and is shown in Fig. 2.16 (b). The twigs of this tree are branches 4, 5 and 6. The links corresponding to this tree are branches 1, 2 and 3. Every link defines a fundamental loop of the network.

No. of nodes $n = 4$

No. of branches $b = 6$

No. of tree branches or twigs $= n - 1 = 3$

No. of link branches $I = b - (n - 1) = 3$

Let $i_1, i_2, \cdots i_6$ be the branch currents with directions as shown in Fig. 2.16 (a). Let us add a link in its proper place to the tree as shown in 2.16 (c). It is seen that a loop I_1 is formed by the branches 1, 5 and 6. There is a formation of link current, let this current be I_1. This current passes through the branches 1, 5 and 6. By convention a fundamental loop is given the same orientation as its defining link, i.e. the link current I_1 coincides with the branch current direction i_1 in *ab*. A tie set can also be defined as the set of branches that forms a closed loop in which the link current flows. By adding the other link branches 2 and 3, we can form two more fundamental loops or *f*-loops with link currents I_2 and I_3 respectively as shown in Figs 2.16 (d) and (e).

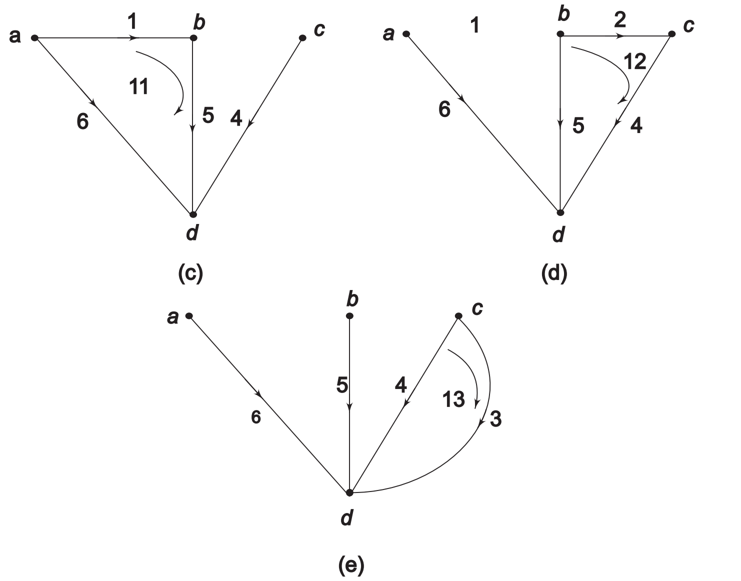


Fig. 2.16

2.7.1 Tie-Set Matrix

Kirchhoff's voltage law can be applied to the f-loops to get a set of linearly independent equations. Consider Fig. 2.17.

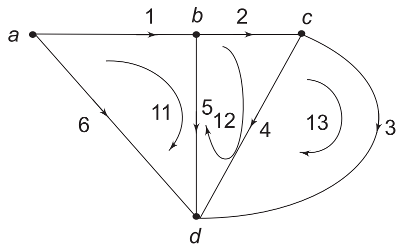


Fig. 2.17

There are three fundamental loops I_1, I_2 and I_3 corresponding to the link branches 1, 2 and 3 respectively. If V_1, V_2, $\cdots$ V_6 are the branch voltages the KVL equations for the three f-loops can be written as

$$\left.\begin{aligned} V_1 + V_5 - V_6 &= 0 \\ V_2 + V_4 - V_5 &= 0 \\ V_3 - V_4 &= 0 \end{aligned}\right\} \qquad (2.3)$$

In order to apply KVL to each fundamental loop, we take the reference direction of the loop which coincides with the reference direction of the link defining the loop.

The above equation can be written in matrix form as

loop branches$\rightarrow$ $3 \times 6 \times 6$

$$\begin{array}{c} \downarrow \\ I_1 \\ I_2 \\ I_3 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$B\ V_b = 0 \qquad (2.4)$$

where B is an $I \times b$ matrix called the tie-set matrix or fundamental loop matrix and V_b is a column vector of branch voltages.

The tie set matrix B is written in a compact form as $B\ [b_{ij}]$ (2.5)

The element b_{ij} of B is defined as

$b_{ij} = 1$ when branch b_j is in the f-loop I_i (loop current) and their reference directions coincide.

$b_{ij} = -1$ when branch b_j is in the f-loop I_i (loop current) and their reference directions are opposite.

$b_{ij} = 0$ when branch b_j is not in the f-loop I_i.

2.7.2 Tie-set Matrix and Branch Currents

It is possible to express branch currents as a linear combination of link current using matrix B.

If I_B and I_I represents the branch current matrix and loop current matrix respectively and B is the tie-set matrix, then

$$[I_b] = [B^T]\ [I_L] \qquad (2.6)$$

where $[B^T]$ is the transpose of the matrix $[B]$. Equation (6) is known as link current transformation equation.

Consider the tie-set matrix of Fig. 2.17

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The branch current vector $[I_b]$ is a column vector.

$$[I_b] = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}$$

The loop current vector $[I_L]$ is a column vector

$$[I_L] = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

Therefore the link current transformation equation is given by $[I_b] = [B^T]\,[I_L]$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The branch currents are

$$\begin{aligned} i_1 &= I_1 \\ i_2 &= I_2 \\ i_3 &= I_3 \\ i_4 &= I_2 - I_3 \\ i_5 &= I_1 - I_2 \\ i_6 &= -I_1 \end{aligned}$$

Example 2.4 For the electrical network shown in Fig. 2.18 (a) draw its topological graph and write its incidence matrix, tie-set matrix, link current transformation equation and branch currents.

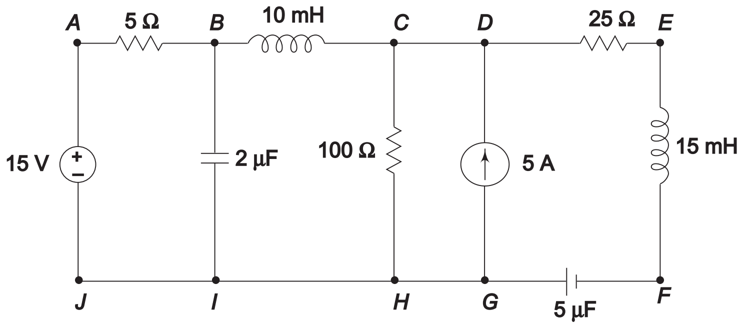


Fig. 2.18 (a)

Solution

Voltage source is short circuited, current source is open circuited, the points which are electrically at same potential are combined to form a single node. The graph is shown in Fig. 2.18 (b).

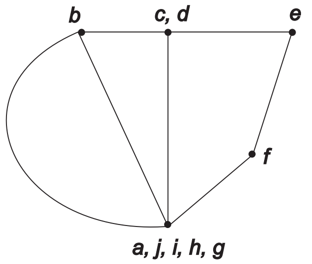


Fig. 2.18 (b)

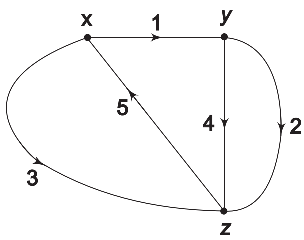


Fig. 2.18 (c)

Combining the simple nodes and arbitrarily selecting the branch current directions the oriented graph is shown in Fig. 2.18 (c). The simplified consists of three nodes. Let them be x, y and z and five branches 1, 2, 3, 4 and 5. The complete incidence matrix is given by

$$\begin{array}{cc} \text{Nodes} & \text{branches} \rightarrow \\ \downarrow & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

$$A = \begin{array}{c} x \\ y \\ z \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 1 \end{bmatrix}$$

Let us choose node z as the reference or datum node for writing the reduced incidence matrix A_1 or we can obtain A_1 by deleting the last row elements in A.

$$\begin{array}{cc} \text{nodes} & \text{branches} \\ \downarrow & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

$$A_1 = \begin{array}{c} x \\ y \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

For writing the tie-set matrix, consider the tree in the graph in Fig. 2.18 (c).

No. of nodes $n = 3$
No. of branches = 5
No. of tree branches or twigs
$= n - 1 = 2$
No. of link branches
$l = b - (n - 1)$
$= 5 - (3 - 1) = 3$

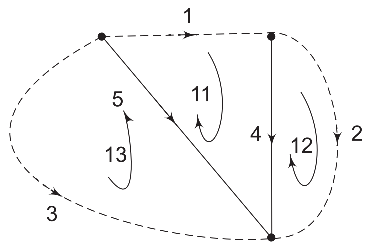


Fig. 2.18 (d)

The tree shown in Fig. 2.18 (d) consists of two branches 4 and 5 shown with solid lines and the link branches of the tree are 1, 2 and 3 shown with dashed lines. The tie-set matrix or fundamental loop matrix is given by

$$B = \begin{array}{c} \text{loop} \\ \downarrow \\ l_1 \\ l_2 \\ l_3 \end{array} \begin{array}{c} \text{branches} \rightarrow \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

To obtain the link current transformation equation and thereby branch currents the transpose of B should be calculated.

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The equation $[I_b] = [B^T]\,[I_L]$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The branch currents are given by

$$i_1 = I_1$$
$$i_2 = I_2$$
$$i_3 = I_3$$
$$i_4 = I_1 - I_2$$
$$i_5 = I_1 + I_3$$

2.8 CUT-SET AND TREE BRANCH VOLTAGES

A cut-set is a minimal set of branches of a connected graph such that the removal of these branches causes the graph to be cut into exactly two parts. The important property of a cut-set is that by restoring anyone of the branches of the cut-set the graph should become connected. A cut-set consists of one and only one branch of the network tree, together with any links which must be cut to divide the network into two parts.

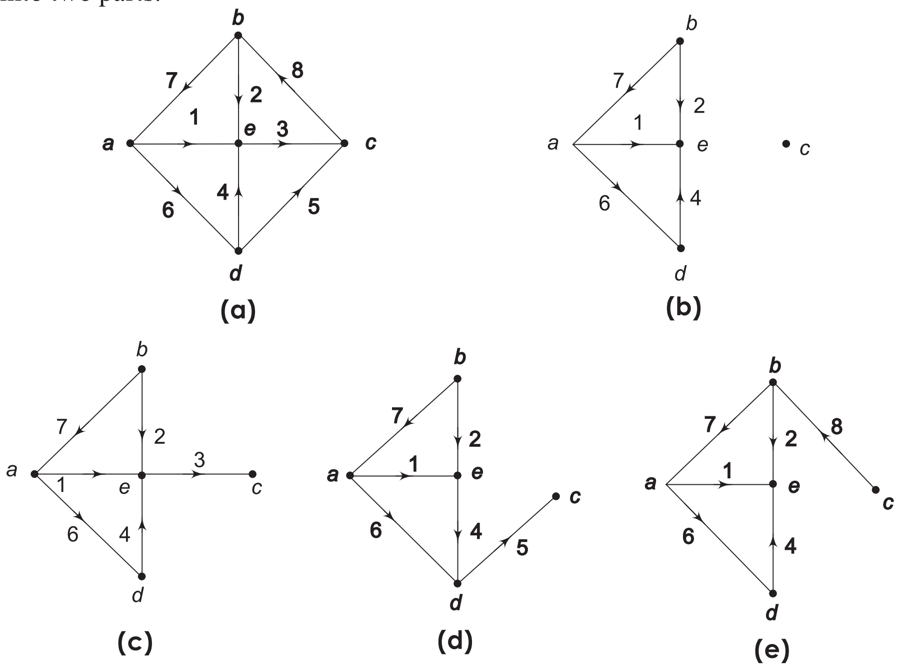


Fig. 2.19

Consider the graph shown in Fig. 2.19 (a).
if the branches 3, 5 and 8 are removed from the graph, we see that the connected graph of Fig. 2.19 (a) is separated into two distinct parts, each of which is connected as shown in Fig. 2.19 (b). One of the parts is just an isolated node. Now suppose the removed branch 3 is replaced, all others still removed. Figure 2.19 (c) shows the resultant graph. The graph is now connected. Likewise replacing the removed branches 5 and 8 of the set {3, 5, 8} one at a time, all other ones remaining removed, we obtain the resulting graphs as shown in Figs 2.19 (d) and (e). The set formed by the branches 3, 5 and 8 is called the cut-set of the connected graph of Fig. 2.19 (a).

2.8.1 Cut-Set Orientation

A cut-set is oriented by arbitrarily selecting the direction. A cut-set divides a graph into two parts. In the graph shown in Fig. 2.20, the cut-set is {2, 3}. It is represented by a dashed line passing through branches 2 and 3. This cut-set separates the graph into two parts shown as part-1 and part-2. We may take the orientation either from part-1 to part-2 or from part-2 to part-1.

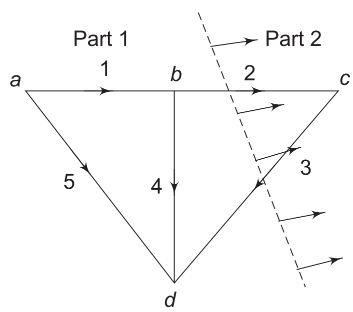


Fig. 2.20

The orientation of some branches of the cut-set may coincide with the orientation of the cut-set while some branches of the cut-set may not coincide. Suppose we choose the orientation of the cut-set {2, 3} from part-1 to part-2 as indicated in Fig. 2.20, then the orientation of branch 2 coincides with the cut-set, whereas the orientation of the branch 3 is opposite.

2.8.2 Cut-Set Matrix and KCL for Cut-Sets

KCL is also applicable to a cut-set of a network. For any lumped electrical network, the algebraic sum of all the cut-set branch currents is equal to zero. While writing the KCL equation for a cut-set, we assign positive sign for the current in a branch if its direction coincides with the orientation of the cut-set and a negative sign to the current in a branch whose direction is opposite to the orientation of the cut-set. Consider the graph shown in Fig. 2.21. It has five branches and four nodes. The branches have been numbered 1 through 5 and their orientations are also marked. The following six cut-sets are possible as shown in Fig. 2.22 (a)-(f).

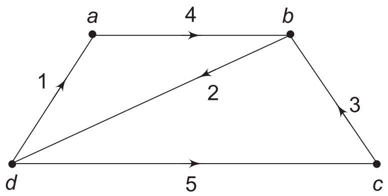


Fig. 2.21

Cut-set C_1 : {1, 4}; cut-set C_2 : {4, 2, 3}
Cut-set C_3 : {3, 5}; cut-set C_4 : {1, 2, 5}
Cut-set C_5 : {4, 2, 5} ; cut-set C_6 : {1, 2, 3}

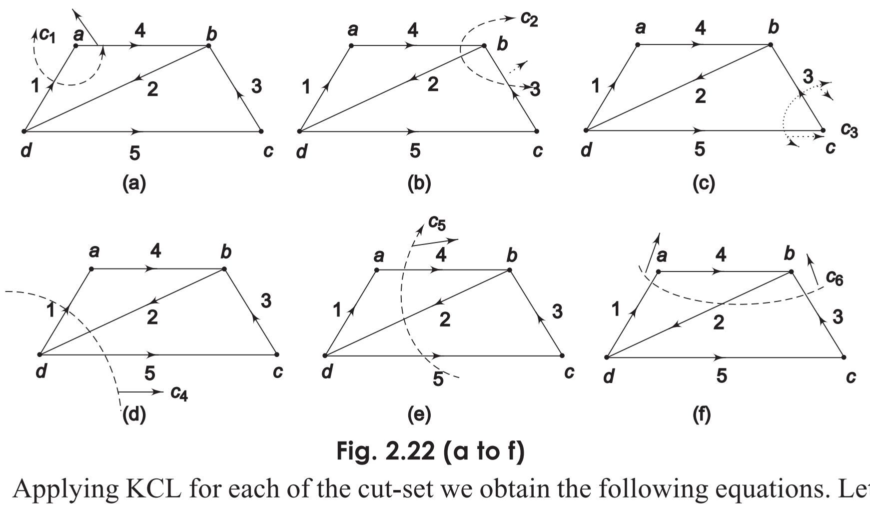


Fig. 2.22 (a to f)

Applying KCL for each of the cut-set we obtain the following equations. Let $i_1, i_2 \cdots i_6$ be the branch currents.

$$\left.\begin{array}{l} C_1 : i_1 - i_4 = 0 \\ C_2 : -i_2 + i_3 + i_4 = 0 \\ C_3 : -i_3 + i_5 = 0 \\ C_4 : i_1 - i_2 + i_5 = 0 \\ C_5 : -i_2 + i_4 + i_5 = 0 \\ C_6 : i_1 - i_2 + i_3 = 0 \end{array}\right\} \tag{2.7}$$

These equation can be put into matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$QI_b = 0 \tag{2.8}$$

where the matrix Q is called augmented cut-set matrix of the graph or all cut-set matrix of the graph. The matrix I_b is the branch-current vector.

The all cut-set matrix can be written as $Q = [q_{ij}]$.

where q_{ij} is the element in the i^{th} row and j^{th} column. The order of Q is number of cut-sets × number of branch as in the graph.

$$\left.\begin{array}{l} q_{ij} = 1, \text{ if branch } j \text{ in the cut-set } i \text{ and the orientation} \\ \text{coincides with each other} \\ q_{ij} = -1, \text{ if branch } j \text{ is in the cut-set } i \text{ and the orientation} \\ \text{is opposite.} \\ q_{ij} = 0, \text{ if branch } j \text{ is not present in cut-set } i. \end{array}\right\} \tag{2.9}$$

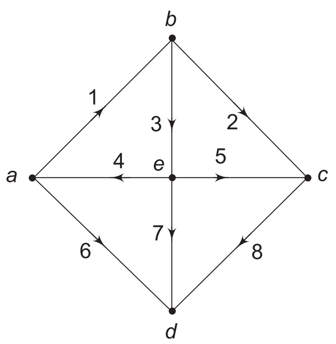


Fig. 2.23 (a)

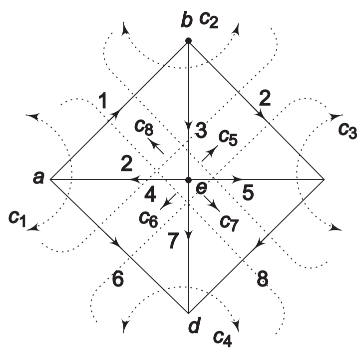


Fig. 2.23 (b)

Example 2.5 For the network-graph shown in Fig. 2.23 (a) with given orientation obtain the all cut-set (augmented cut-set) matrix.

Solution The graph has four nodes and eight branches. There are in all 12 possible cut-sets as shown with dashed lines in Figs 2.23 (b) and (c). The orientation of the cut-sets has been marked arbitrarily. The cut-sets are

C_1 : {1, 46}; C_2 {1, 2, 3}; C_3: {2, 5, 8}

C_4 : {6, 7, 8}; C_5 {1, 3, 5, 8}; C_6: {1, 4, 7, 8}

C_7 : {2, 5, 6, 7}; C_8 : {2, 3, 4, 6} C_9 : {1, 4, 7, 5, 2}

C_{10} : {2, 3, 4, 7, 8} ; C_{11} : {6, 4, 3, 5, 8}; C_{12} : {1, 3, 5, 7, 6}

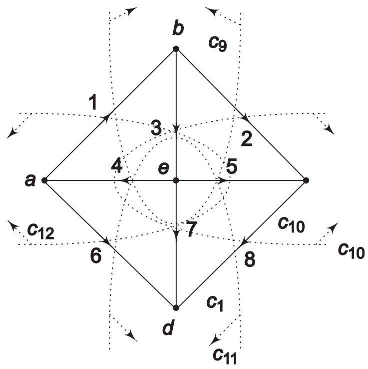


Fig. 2.23 (c)

Eight cut-sets C_1 to C_8 are shown if Fig. 2.23(b) and four cut-sets C_9 to C_{11} are shown in Fig. 2.23(c) for clarity.

As explained in section 2.8.2 with the help of equations 2.9, the all cut-set matrix Q is given by

$$
Q = \begin{array}{c} \text{Cut-sets} \\ \downarrow \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{array}
\begin{array}{c} \text{Branches} \rightarrow \\ \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 & -1 & -1 & -1 & 0
\end{bmatrix} \end{array}
$$

Matrix Q is a 12×8 matrix since there are 12 cut-sets and eight branches in the graph.

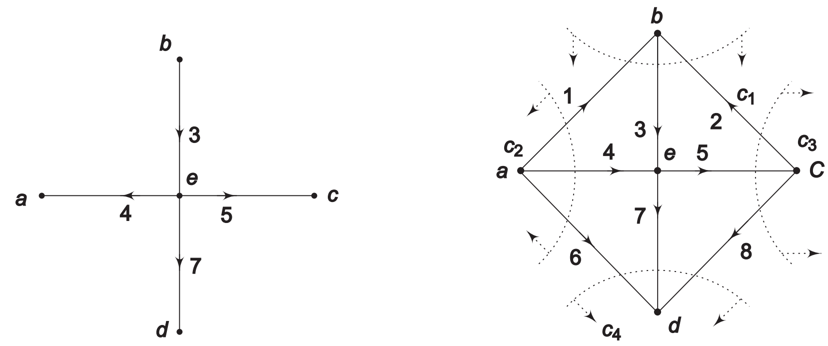
2.8.3 Fundamental Cut-Sets

Observe the set of equation 2.7 in Section 2.8.2 with respect to the graph in Fig. 2.22. Only first three equations are linearly independent, remaining equations can be obtained as a linear combination of the first three. The concept of fundamental cut-set (*f*-cut-set) can be used to obtain a set of linearly independent equations in branch current variables. The *f*-cut-sets are defined for a given tree of the graph. From a connected graph, first a tree is selected, and then a twig is selected. Removing this twig from the tree separates the tree into two parts. All the links which go from one part of the disconnected tree to the other, together with the twig of the selected tree will constitute a cut-set. This cut-set is called a fundamental cut-set or f-cut-set or the graph. Thus a fundamental cut-set of a graph with respect to a tree is a cut-set that is formed by one twig and a unique set of links. For each branch of the tree, i.e. for each twig, there will be a f-cut-set. So, for a connected graph having n nodes, there will be $(n-1)$ twigs in a tree, the number of f-cut-sets is also equal to $(n-1)$.

Fundamental cut-set matrix Q_f is one in which each row represents a cut-set with respect to a given tree of the graph. The rows of Q_1 correspond to the fundamental cut-sets and the columns correspond to the branches of the graph. The procedure for obtaining a fundamental cut-set matrix is illustrated in Example 2.6.

Example 2.6 Obtain the fundamental cut-set matrix Q_f for the network graph shown in Fig. 2.23 (a).

Solution A selected tree of the graph is shown in Fig. 2.24 (a).



(a) **(b)**

Fig. 2.24

The twigs of the tree are {3, 4, 5, 7}. The remaining branches 1, 2, 6 and 8 are the links, corresponding to the selected tree. Let us consider twig 3. The minimum number of links that must be added to twig 3 to form a cut-set C_1 is {1, 2}. This set is unique for C_1. Thus corresponding to twig 3. The f-cut-set C_1 is {1, 2, 3}. This is shown in Fig. 2.24 (b). As a convention the orientation of a cut-set is chosen to coincide with that of its defining twig. Similarly, corresponding to twig 4, the f-cut-set C_2 is {1, 4, 6} corresponding to twig 5, the f-cut-set C_3 is {2, 5, 8} and corresponding to twig 7, the f-cut-set is {6, 7, 8}. Thus the f-cut-set matrix is given by

$$\begin{array}{cc} \text{f-cut-sets} & \text{branches} \\ Q_f = \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} & \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \tag{2.11}$$

2.8.4 Tree Branch Voltages and *f*-Cut-Set Matrix

From the cut-set matrix the branch voltages can be expressed in terms of tree branch voltages. Since all tree branches are connected to all the nodes in the graph, it is possible to trace a path from one node to any other node by traversing through the tree-branches.

Let us consider Example 2.6, there are eight branches. Let the branch voltages be $V_1, V_2, \cdots V_8$. There are, four twigs, let the twig voltages be V_{t3}, V_{t4}, V_{t5} and V_{t7} for twigs 3, 4, 5 and 7 respectively.

We can express each branch voltage in terms of twig voltages as follows.

$$
\begin{aligned}
V_1 &= -V_3 - V_4 = -V_{t3} - V_{t4} \\
V_2 &= +V_3 + V_5 = +V_{t3} + V_{t5} \\
V_3 &= V_{t3} \\
V_4 &= V_{t4} \\
V_5 &= V_{t5} \\
V_6 &= V_7 - V_4 = V_{t7} - V_{t4} \\
V_7 &= V_{t7} \\
V_8 &= V_7 - V_5 = V_{t7} - V_{t5}
\end{aligned}
$$

The above equations can be written in matrix form as

$$
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} V_{t_3} \\ V_{t_4} \\ V_{t_5} \\ V_{t_7} \end{bmatrix} \tag{2.12}
$$

The first matrix on the right hand side of Eq. 2.12 is the transpose of the *f*-cut-set matrix Q_f given in Eq. 2.11 in Ex. 2.6. Hence, Eq. 2.12 can be written as

$$V_b = Q_f^T V_t. \tag{2.13}$$

Where V_b is the column matrix of branch-voltages V_t is the column matrix of twig voltages corresponding to the selected tree and Q_f^T in the transpose of *f*-cut-set matrix.

Equation 2.13 shows that each branch voltage can be expressed as a linear combination of the tree-branch voltages. For this purpose fundamental cut-set (*f*-cut-set) matrix can be used without writing loop equations.

2.9 MESH ANALYSIS

Mesh and nodal analysis are two basic important techniques used in finding solutions for a network. The suitability of either mesh or nodal analysis to a particular problem depends mainly on the number of voltage sources or current sources. If a network has a large number of voltage sources, it is useful to use mesh analysis; as this analysis requires that all the sources in a circuit be voltage sources. Therefore, if there are any current sources in a circuit they are to be converted into equivalent voltage sources, if, on the other hand, the network has more current sources, nodal analysis is more useful.

Mesh analysis is applicable only for planar networks. For non-planar circuits mesh analysis is not applicable. A circuit is said to be planar, if it can be drawn on a plane surface without crossovers. A non-planar circuit cannot be drawn on a plane surface without a crossover.

Figure 2.25 (a) is a planar circuit. Figure 2.25 (b) is a non-planar circuit and Fig. 2.25 (c) is a planar circuit which looks like a non-planar circuit. It has already been discussed that a loop is a closed path. A mesh is defined as a loop which does not contain any other loops within it. To apply mesh analysis, our first step is to check whether the circuit is planar or not and the second is to select mesh currents. Finally, writing Kirchhoff's voltage law equations in terms of unknowns and solving them leads to the final solution.

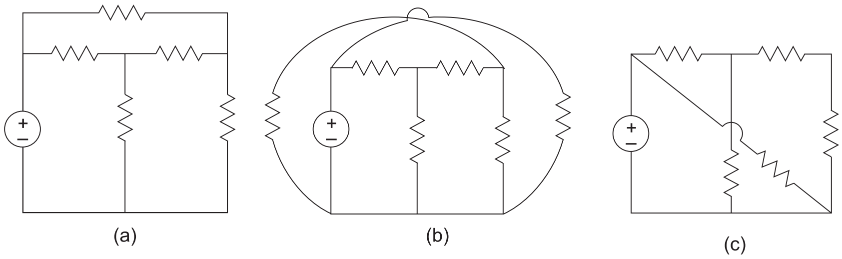


Fig. 2.25

Observation of the Fig. 2.26 indicates that there are two loops *abefa*, and *bcdeb* in the network. Let us assume loop currents I_1 and I_2 with directions as indicated in the figure. Considering the loop *abefa* alone, we observe that current I_1 is passing through R_1, and $(I_1 - I_2)$ is passing through R_2. By applying Kirchhoff's voltage law, we can write

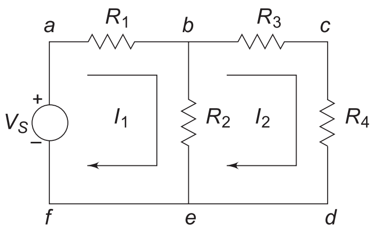


Fig. 2.26

$$V_s = I_1R_1 + R_2(I_1 - I_2)$$

Similarly, if we consider the second mesh *bcdeb*, the current I_2 is passing through R_3 and R_4, and $(I_2 - I_1)$ is passing through R_2. By applying Kirchhoff's voltage law around the second mesh, we have

$$R_2\,(I_2 - I_1) + R_3\,I_2 + R_4\,I_2 = 0$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(R_1 + R_2) - I_2\,R_2 = V_s$$

$$-\,I_1\,R_2 + (R_2 + R_3 + R_4)I_2 = 0$$

By solving the above equations, we can find the currents I_1 and I_2. If we observe Fig. 2.26, the circuit consists of five branches and four nodes, including the reference node. The number of mesh currents is equal to the number of mesh equations.

And the number of equations = branches – (nodes – 1). In Fig. 2.26, the required number of mesh currents would be 5 – (4 – 1) = 2.

In general, if we have B number of branches and N number of nodes including the reference node then the number of linearly independent mesh equations $M = B - (N - 1)$.

Example 2.7 Write the mesh current equations in the circuit shown in Fig. 2.27, and determine the currents.

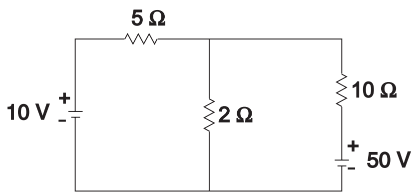


Fig. 2.27

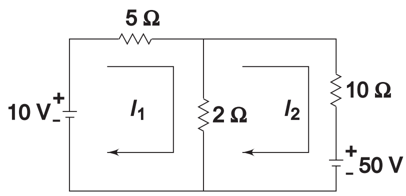


Fig. 2.28

Solution Assume two mesh currents in the direction as indicated in Fig. 2.28. The mesh current equations are

$$5I_1 + 2(I_1 - I_2) = 10$$

$$10I_2 + 2(I_2 - I_1) + 50 = 0$$

We can rearrange the above equations as

$$7I_1 - 2I_2 = 10$$

$$-2I_1 + 12I_2 = -50$$

By solving the above equations, we have

$$I_1 = 0.25\ A, \text{ and } I_2 = -4.125\ A$$

Here the current in the second mesh, I_2, is negative; that is the actual current I_2 flows opposite to the assumed direction of current in the circuit of Fig. 2.28.

Example 2.8 Determine the mesh current I_1 in the circuit shown in Fig. 2.29.

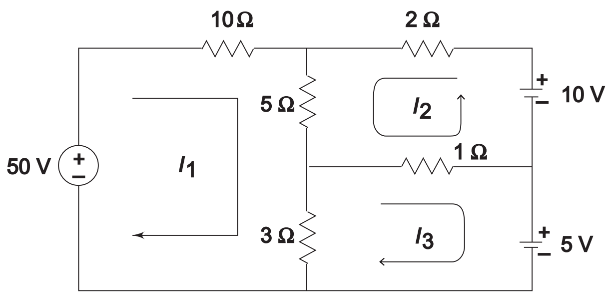


Fig. 2.29

Solution From the circuit, we can form the following three mesh equations

$$10I_1 + 5(I_1 + I_2) + 3(I_1 - I_3) = 50$$
$$2I_2 + 5(I_2 + I_1) + 1(I_2 + I_3) = 10$$
$$3(I_3 - I_1) + 1(I_3 + I_2) = -5$$

Rearranging the above equations we get

$$18I_1 + 5I_2 - 3I_3 = 50$$
$$5I_1 + 8I_2 + I_3 = 10$$
$$-3I_1 + I_2 + 4I_3 = -5$$

According to Cramer's rule

$$I_1 = \frac{\begin{vmatrix} 50 & 5 & -3 \\ 10 & 8 & 1 \\ -5 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{1175}{356}$$

or $\qquad I_1 = 3.3$ A

Similarly,

$$I_2 = \frac{\begin{vmatrix} 18 & 50 & -3 \\ 5 & 10 & 1 \\ -3 & -5 & 4 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{-355}{356}$$

or $\qquad I_2 = -0.997\ A$

$$I_3 = \frac{\begin{vmatrix} 18 & 5 & 50 \\ 5 & 8 & 10 \\ -3 & 1 & -5 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{525}{356}$$

or $\qquad I_3 = 1.47$ A

$\therefore \qquad I_1 = 3.3$ A, $\quad I_2 = -0.997$ A, $\quad I_3 = 1.47$ A

2.10 MESH EQUATIONS BY INSPECTION METHOD

The mesh equations for a general planar network can be written by inspection without going through the detailed steps. Consider a three mesh networks as shown in Fig. 2.30.

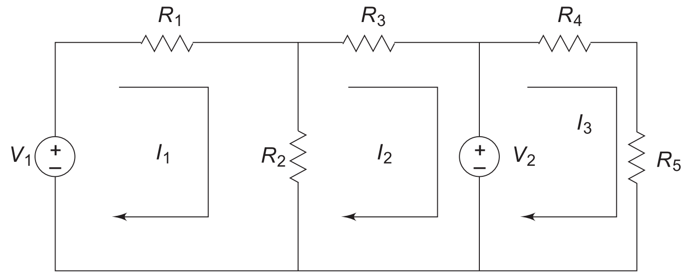


Fig. 2.30

The loop equations are

$$I_1R_1 + R_2(I_1 - I_2) = V_1 \quad (2.14)$$

$$R_2(I_2 - I_1) + I_2R_3 = - V_2 \quad (2.15)$$

$$R_4I_3 + R_5I_3 = V_2 \quad (2.16)$$

Reordering the above equations, we have

$$(R_1 + R_2)I_1 - R_2I_2 = V_1 \quad (2.17)$$

$$- R_2I_1 + (R_2 + R_3)I_2 = - V_2 \quad (2.18)$$

$$(R_4 + R_5)I_3 = V_2 \quad (2.19)$$

The general mesh equations for three mesh resistive network can be written as

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_a \quad (2.20)$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_b \quad (2.21)$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_c \quad (2.22)$$

By comparing the Eqs 2.17, 2.18 and 2.19 with Eqs 2.20, 2.21, and 2.22 respectively, the following observations can be taken into account.

1. The self resistance in each mesh.
2. The mutual resistances between all pairs of meshes and
3. The algebraic sum of the voltages in each mesh.

The self resistance of loop 1, $R_{11} = R_1 + R_2$, is the sum of the resistances through which I_1 passes.

The mutual resistance of loop 1, $R_{12} = - R_2$, is the sum of the resistances common to loop currents I_1 and I_2. If the directions of the currents passing through the common resistance are the same, the mutual resistance will have a positive sign; and if the directions of the currents passing through the common resistance are opposite then the mutual resistance will have a negative sign.

$V_a = V_1$ is the voltage which drives loop one. Here, the positive sign is used if the direction of the current is the same as the direction of the source. If the current direction is opposite to the direction of the source, then the negative sign is used.

Similarly, $R_{22} = (R_2 + R_3)$ and $R_{33} = R_4 + R_5$ are the self resistances of loops two and three, respectively. The mutual resistances $R_{13} = 0$, $R_{21} = -R_2$, $R_{23} = 0$, $R_{31} = 0$, $R_{32} = 0$ are the sums of the resistances common to the mesh currents indicated in their subscripts.

$V_b = -V_2$, $V_c = V_2$ are the sum of the voltages driving their respective loops.

Example 2.9 Write the mesh equations for the circuit shown in Fig. 2.31.

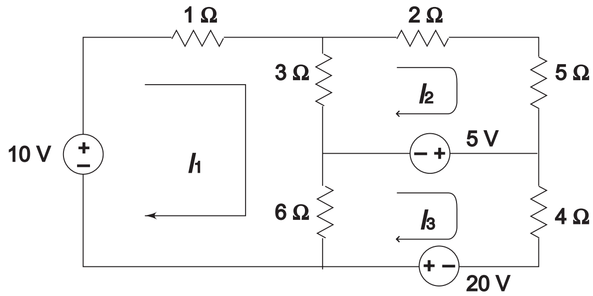


Fig. 2.31

Solution The general equations for three mesh network are

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_a \tag{2.23}$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_b \tag{2.24}$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_c \tag{2.25}$$

Consider Eq. 2.23

R_{11} = self resistance of loop 1 = (1 Ω + 3 Ω + 6 Ω) = 10 Ω

R_{12} = the mutual resistance common to loop 1 and loop 2 = – 3 Ω

Here, the negative sign indicates that the currents are in opposite direction

R_{13} = the mutual resistance common to loop 1 and 3 = – 6 Ω

V_a = + 10 V, the voltage driving the loop 1.

Here, the positive sign indicates the loop current I_1 is in the same direction as the source element.

Therefore, Eq. (2.23) can be written as

$$10I_1 - 3I_2 - 6I_3 = 10 \text{ V} \tag{2.26}$$

Consider Eq. (2.24)

R_{21} = mutual resistance common to loop 1 and loop 2 = – 3 Ω

R_{22} = self resistance of loop 2 = (3 Ω + 2 Ω + 5 Ω) = 10 Ω

R_{23} = 0, there is no common resistance between loop 2 and loop 3.

V_b = – 5 V, the voltage driving the loop 2.

Therefore, Eq. (2.24) can be written as

$$-3I_1 + 10I_2 = -5 \text{ V} \tag{2.27}$$

Consider Eq. (2.25)

R_{31} = mutual resistance common to loop 3 and loop 1 = – 6 Ω

R_{32} = mutual resistance common to loop 3 and loop 2 = 0

R_{33} = self resistance of loop 3 = (6 Ω + 4 Ω) = 10 Ω

V_c = the algebraic sum of the voltages driving loop 3
= (5 V + 20 V) = 25 V

Therefore, Eq. (2.25) can be written as

$$-6I_1 + 10I_3 = 25 \text{ V} \tag{2.28}$$

The three mesh equation are

$$10I_1 - 3I_2 - 6I_3 = 10 \text{ V}$$
$$-3I_1 + 10I_2 = -5 \text{ V}$$
$$-6I_1 + 10I_3 = 25 \text{ V}$$

2.11 SUPERMESH ANALYSIS

Suppose any of the branches in the network has a current source, then it is slightly difficult to apply mesh analysis straight forward because first we should assume an unknown voltage across the current source, writing mesh equations as before, and then relate the source current to the assigned mesh currents. This is generally a difficult approach. One way to overcome this difficulty is by applying the supermesh technique. Here we have to choose the kind of supermesh. A supermesh is constituted by two adjacent loops that have a common current source. As an example, consider the network shown in Fig. 2.32.

Here, the current source I is in the common boundary for the two meshes 1 and 2. This current source creates a supermesh, which is nothing but a combination of meshes 1 and 2.

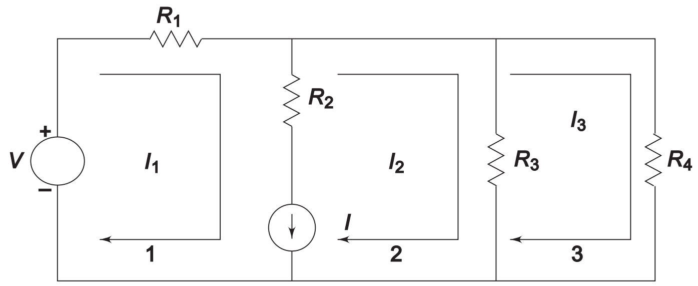


Fig. 2.32

$$R_1I_1 + R_3(I_2 - I_3) = \text{V}$$

or $$R_1I_1 + R_3I_2 - R_3I_3 = \text{V}$$

Considering mesh 3, we have

$$R_3(I_3 - I_2) + R_3I_3 = 0$$

Finally, the current I from current source is equal to the difference between two mesh currents, i.e.

$$I_1 - I_2 = I$$

We have, thus, formed three mesh equations which we can solve for the three unknown currents in the network.

Example 2.10 Determine the current in the 5 Ω resistor in the network given in Fig. 2.33.

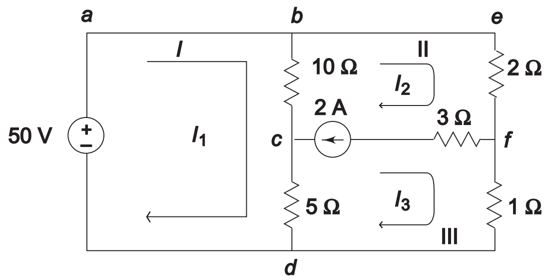


Fig. 2.33

Solution From the first mesh, i.e. *abcda*, we have

$$50 = 10(I_1 - I_2) + 5(I_1 - I_3)$$

or $$15I_1 - 10I_2 - 5I_3 = 50 \tag{2.29}$$

From the second and third meshes, we can form a supermesh

$$10(I_2 - I_1) + 2I_2 + I_3 + 5(I_3 - I_1) = 0$$

or $$-15I_1 + 12I_2 + 6I_3 = 0 \tag{2.30}$$

The current source is equal to the difference between II and III mesh currents, i.e.

$$I_2 - I_3 = 2\text{A} \tag{2.31}$$

Solving 2.29, 2.30 and 2.31, we have

$$I_1 = 19.99 \text{ A}, \quad I_2 = 17.33 \text{ A, and } I_3 = 15.33 \text{ A}$$

The current in the 5 Ω resistor = $I_1 - I_3$

$$= 19.99 - 15.33 = 4.66 \text{ A}$$

∴ The current in the 5 Ω resistor is 4.66 A.

Example 2.11 Write the mesh equations for the circuit shown in Fig. 2.34 and determine the currents, I_1, I_2 and I_3.

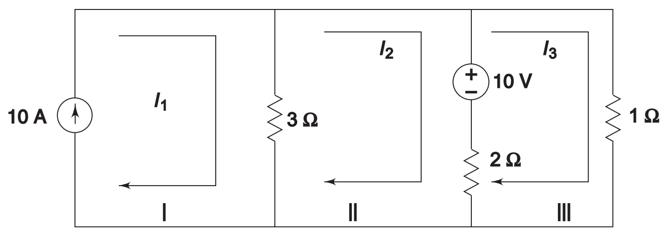


Fig. 2.34

Solution In Fig. 2.34, the current source lies on the perimeter of the circuit, and the first mesh is ignored. Kirchhoff's voltage law is applied only for second and third meshes.

From the second mesh, we have

$$3(I_2 - I_1) + 2(I_2 - I_3) + 10 = 0$$

or $$-3I_1 + 5I_2 - 2I_3 = -10 \qquad (2.32)$$

From the third mesh, we have

$$I_3 + 2(I_3 - I_2) = 10$$

or $$-2I_2 + 3I_2 = 10 \qquad (2.33)$$

From the first mesh,

$$I_1 = 10\text{ A} \qquad (2.34)$$

From the above three equations, we get

$$I_1 = 10\text{ A}, \quad I_2 = 7.27\text{ A}, \quad I_3 = 8.18\text{ A}$$

2.12 NODAL ANALYSIS

In the Chapter 1 we discussed simple circuits containing only two nodes, including the reference node. In general, in a N node circuit, one of the nodes is choosen as reference or datum node, then it is possible to write $N - 1$ nodal equations by assuming $N - 1$ node voltages. For example, a 10 node circuit requires nine unknown voltages and nine equations. Each node in a circuit can be assigned a number or a letter. The node voltage is the voltage of a given node with respect to one particular node, called the reference node, which we assume at zero potential. In the circuit shown in Fig. 2.35, node 3 is assumed as the reference node. The voltage at node 1 is the voltage at that node with respect to node 3. Similarly, the voltage at node 2 is the voltage at that node with respect to node 3. Applying Kirchhoff's current law at node 1; the current entering is equal to the current leaving. (See Fig. 2.36).

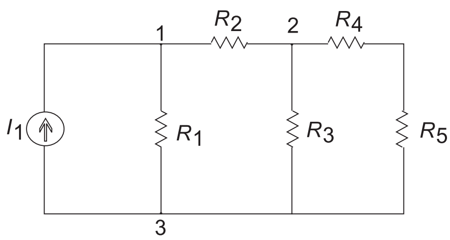


Fig. 2.35

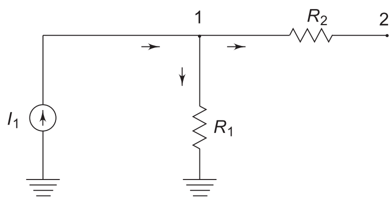


Fig. 2.36

$$I_1 = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2}$$

where V_1 and V_2 are the voltages at node 1 and 2, respectively. Similarly, at node 2, the current entering is equal to the current leaving as shown in Fig. 2.37.

$$\frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} + \frac{V_2}{R_4 + R_5} = 0$$

Rearranging the above equations, we have

$$V_1\left[\frac{1}{R_1} + \frac{1}{R_2}\right] - V_2\left[\frac{1}{R_2}\right] = I_1$$

$$-V_1\left[\frac{1}{R_2}\right] + V_2\left[\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4 + R_5}\right] = 0$$

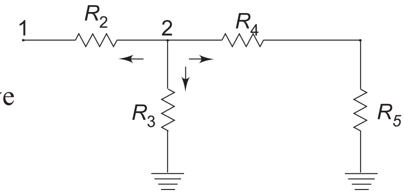


Fig. 2.37

From the above equations, we can find the voltages at each node.

Example 2.12 Write the node voltage equations and determine the currents in each branch for the network shown in Fig. 2.38.

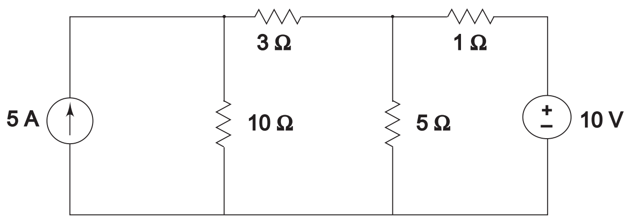


Fig. 2.38

Solution The first step is to assign voltages at each node as shown in Fig. 2.39.

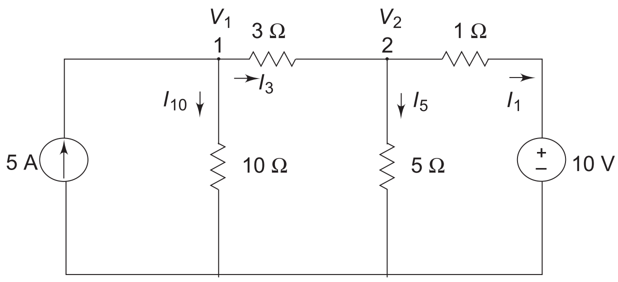


Fig. 2.39

Applying Kirchhoff's current law at node 1,

we have $$5 = \frac{V_1}{10} + \frac{V_1 - V_2}{3}$$

or $$V_1\left[\frac{1}{10} + \frac{1}{3}\right] - V_2\left[\frac{1}{3}\right] = 5 \qquad (2.35)$$

Applying Kirchhoff's current law at node 2, we have $$\frac{V_2 - V_1}{3} + \frac{V_2}{5} + \frac{V_2 - 10}{1} = 0$$

or $$-V_1\left[\frac{1}{3}\right] + V_2\left[\frac{1}{3} + \frac{1}{5} + 1\right] = 10 \qquad (2.36)$$

From Eqs 2.35 and 2.36, we can solve for V_1 and V_2 to get

$$V_1 = 19.85 \text{ V}, \; V_2 = 10.9 \text{ V}$$

$$I_{10} = \frac{V_1}{10} = 1.985 \text{ A}, \; I_3 = \frac{V_1 - V_2}{3} = \frac{19.85 - 10.9}{3} = 2.98 \text{ A}$$

$$I_5 = \frac{V_2}{5} = \frac{10.9}{5} = 2.18 \text{ A}, \; I_1 = \frac{V_2 - 10}{1} = 0.9 \text{ A}$$

Example 2.13 Determine the voltages at each node for the circuit shown in Fig. 2.40.

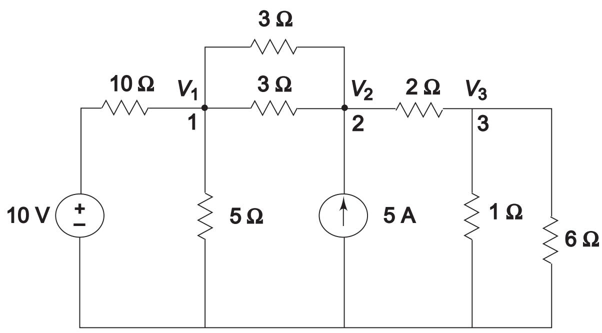


Fig. 2.40

Solution At node 1, assuming that all currents are leaving, we have

$$\frac{V_1 - 10}{10} + \frac{V_1 - V_2}{3} + \frac{V_1}{5} + \frac{V_1 - V_2}{3} = 0$$

or $$V_1\left[\frac{1}{10} + \frac{1}{3} + \frac{1}{5} + \frac{1}{3}\right] - V_2\left[\frac{1}{3} + \frac{1}{3}\right] = 1$$

$$0.96V_1 - 0.66V_2 = 1 \qquad (2.37)$$

At node 2, assuming that all currents are leaving except the current from current source, we have

$$\frac{V_2 - V_1}{3} + \frac{V_2 - V_1}{3} + \frac{V_2 - V_3}{2} = 5$$

$$-V_1\left[\frac{2}{3}\right] + V_2\left[\frac{1}{3} + \frac{1}{3} + \frac{1}{2}\right] - V_3\left[\frac{1}{2}\right] = 5$$

$$-0.66\, V_1 + 1.16\, V_2 - 0.5V_3 = 5 \qquad (2.38)$$

At node 3, assuming all currents are leaving, we have

$$\frac{V_3 - V_2}{2} + \frac{V_3}{1} + \frac{V_3}{6} = 0$$

$$-0.5\ V_2 + 1.66\ V_3 = 0 \tag{2.39}$$

Applying Cramer's rule, we get

$$V_1 = \frac{\begin{vmatrix} 1 & -0.66 & 0 \\ 5 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{7.154}{0.887} = 8.06\ \text{V}$$

Similarly,

$$V_2 = \frac{\begin{vmatrix} 0.96 & 1 & 0 \\ -0.66 & 5 & -0.5 \\ 0 & 0 & 1.66 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{9.06}{0.887} = 10.2\ \text{V}$$

$$V_3 = \frac{\begin{vmatrix} 0.96 & -0.66 & 1 \\ -0.66 & 1.16 & 5 \\ 0 & -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{2.73}{0.887} = 3.07\ \text{V}$$

2.13 NODAL EQUATIONS BY INSPECTION METHOD

The nodal equations for a general planar network can also be written by inspection, without going through the detailed steps. Consider a three node resistive network, including the reference node, as shown in Fig. 2.41.

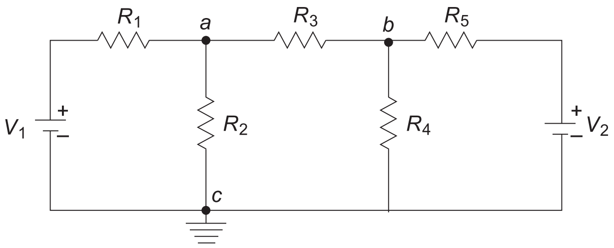


Fig. 2.41

In Fig. 2.41, the points a and b are the actual nodes and c is the reference node. Now consider the nodes a and b separately as shown in Fig. 2.42 (a) and (b).

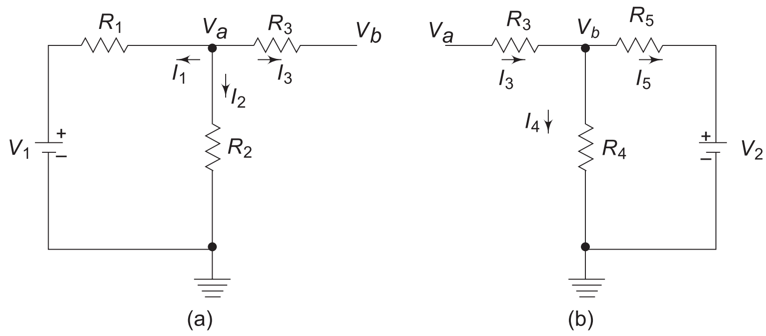


Fig. 2.42

In Fig. 2.42 (a), according to Kirchhoff's current law, we have

$$I_1 + I_2 + I_3 = 0$$

$$\therefore \qquad \frac{V_a - V_1}{R_1} + \frac{V_a}{R_2} + \frac{V_a - V_b}{R_3} = 0 \tag{2.40}$$

In Fig. 2.42 (b), if we apply Kirchhoff's current law, we get

$$I_4 + I_5 = I_3$$

$$\therefore \qquad \frac{V_b - V_a}{R_3} + \frac{V_b}{R_4} + \frac{V_b - V_2}{R_5} = 0 \tag{2.41}$$

Rearranging the above equations, we get

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) V_a - \left(\frac{1}{R_3}\right) V_b = \left(\frac{1}{R_1}\right) V_1 \tag{2.42}$$

$$\left(-\frac{1}{R_3}\right) V_a + \left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right) V_b = \frac{V_2}{R_5} \tag{2.43}$$

In general, the above equations can be written as

$$G_{aa}\, V_a + G_{ab}\, V_b = I_1 \tag{2.44}$$

$$G_{ba}\, V_a + G_{bb}\, V_b = I_2 \tag{2.45}$$

By comparing Eqs 2.42, 2.43 and Eqs 2.44, 2.45 we have the self conductance at node a, $G_{aa} = (1/R_1 + 1/R_2 + 1/R_3)$ is the sum of the conductances connected to node a. Similarly, $G_{bb} = (1/R_3 + 1/R_4 + 1/R_5)$, is the sum of the conductances connected to node b. $G_{ab} = (-\ 1/R_3)$, is the sum of the mutual conductances connected to node a and node b. Here all the mutual conductances have negative signs. Similarly, $G_{ba} = (-\ 1/R_3)$ is also a mutual conductance connected between nodes b and a. I_1 and I_2 are the sum of the source currents at node a and node b, respectively. The current which drives into the node has positive sign, while the current that drives away from the node has negative sign.

Example 2.14 For the circuit shown in Fig. 2.43, write the node equations by the inspection method.

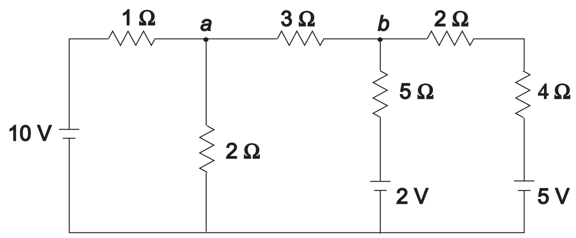


Fig. 2.43

Solution The general equations are

$$G_{aa}\, V_a + G_{ab}\, V_b = I_1 \tag{2.46}$$

$$G_{ba}\, V_a + G_{bb}\, V_b = I_2 \tag{2.47}$$

Consider Eq. 2.46

G_{aa} = (1 + 1/2 + 1/3) mho, the self conductance at node *a* is the sum of the conductances connected to node *a*.

G_{bb} = (1/6 + 1/5 + 1/3) mho the self conductance at node *b* is the sum of the conductances connected to node *b*.

G_{ab} = – (1/3) mho, the mutual conductance between nodes *a* and *b* is the sum of the conductances connected between nodes *a* and *b*.

Similarly, G_{ba} = – (1/3), the sum of the mutual conductances between nodes *b* and *a*.

$$I_1 = \frac{10}{1} = 10 \text{ A, the source current at node } a,$$

$$I_2 = \left(\frac{2}{5} + \frac{5}{6}\right) = 1.23 \text{ A, the source current at node } b.$$

Therefore, the nodal equations are

$$1.83\, V_a - 0.33\, V_b = 10 \tag{2.48}$$

$$-0.33\, V_a + 0.7\, V_b = 1.23 \tag{2.49}$$

2.14 SUPERNODE ANALYSIS

Suppose any of the branches in the network has a voltage source, then it is slightly difficult to apply nodal analysis. One way to overcome this difficulty is to apply the supernode technique. In this method, the two adjacent nodes that are connected by a voltage source are reduced to a single node and then the equations are formed by applying Kirchhoff's current law as usual. This is explained with the help of Fig. 2.44.

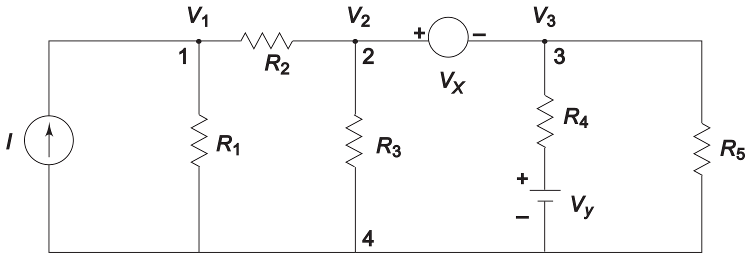


Fig. 2.44

It is clear from Fig. 2.44, that node 4 is the reference node. Applying Kirchhoff's current law at node 1, we get

$$I = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2}$$

Due to the presence of voltage source V_x in between nodes 2 and 3, it is slightly difficult to find out the current. The supernode technique can be conveniently applied in this case.

Accordingly, we can write the combined equation for nodes 2 and 3 as under.

$$\frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} + \frac{V_3 - V_y}{R_4} + \frac{V_3}{R_5} = 0$$

The other equation is

$$V_2 - V_3 = V_x$$

From the above three equations, we can find the three unknown voltages.

Example 2.15 Determine the current in the 5 Ω resistor for the circuit shown in Fig. 2.45.

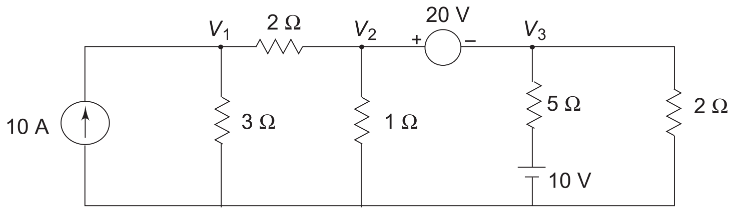


Fig. 2.45

Solution At node 1

$$10 = \frac{V_1}{3} + \frac{V_1 - V_2}{2}$$

or $$V_1\left[\frac{1}{3} + \frac{1}{2}\right] - \frac{V_2}{2} - 10 = 0$$

$$0.83\ V_1 - 0.5\ V_2 - 10 = 0 \tag{2.50}$$

At node 2 and 3, the supernode equation is

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} + \frac{V_3 - 10}{5} + \frac{V_3}{2} = 0$$

or
$$\frac{-V_1}{2} + V_2\left[\frac{1}{2} + 1\right] + V_3\left[\frac{1}{5} + \frac{1}{2}\right] = 2$$

$$-0.5\ V_1 + 1.5\ V_2 + 0.7\ V_3 - 2 = 0 \tag{2.51}$$

The voltage between nodes 2 and 3 is given by

$$V_2 - V_3 = 20 \tag{2.52}$$

The current in the 5 Ω resistor $I_5 = \dfrac{V_3 - 10}{5}$

Solving Eqs 2.50, 2.51 and 2.52, we obtain

$$V_3 = -8.42 \text{ V}$$

$\therefore$ Current $I_5 = \dfrac{-8.42 - 10}{5} = -3.68$ A (current towards node 3) i.e. the current flows towards node 3.

2.15 SOURCE TRANSFORMATION TECHNIQUE

In solving networks to find solutions one may have to deal with energy sources. It has already been discussed in Chapter 1 that basically, energy sources are either voltage sources or current sources. Sometimes it is necessary to convert a voltage source to a current source and vice-versa. Any practical voltage source consists of an ideal voltage source in series with an internal resistance. Similarly, a practical current source consists of an ideal current source in parallel with an internal resistance as shown in Fig. 2.46. R_v and R_i represent the internal resistances of the voltage source V_s, and current source I_s, respectively.

Any source, be it a current source or a voltage source, drives current through its load resistance, and the magnitude of the current depends on the value of the load resistance. Figure 2.47 represents a practical voltage source and a practical current source connected to the same load resistance R_L.

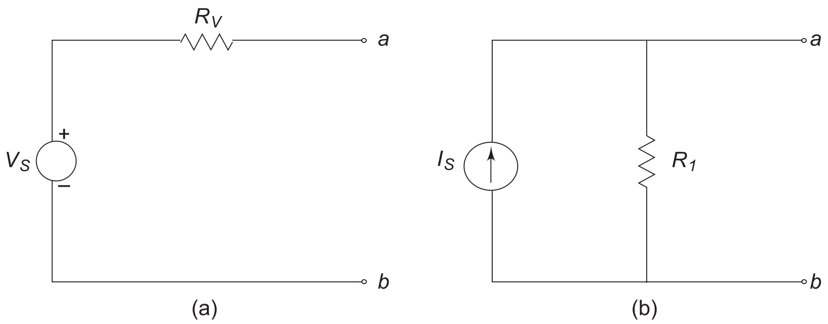


Fig. 2.46

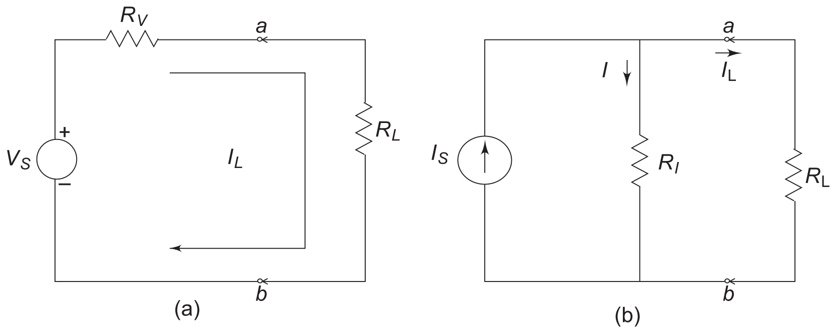


Fig. 2.47

From Fig. 2.47 (a), the load voltage can be calculated by using Kirchhoff's voltage law as

$$V_{ab} = V_s - I_L R_v$$

The open circuit voltage $V_{OC} = V_s$

The short circuit current $I_{SC} = \dfrac{V_s}{R_v}$

From Fig. 2.47 (b)

$$I_L = I_S - I = I_S - \frac{V_{ab}}{R_I}$$

The open circuit voltage $V_{ac} = I_S R_I$

The short circuit current $I_{SC} = I_S$

The above two sources are said to be equal, if they produce equal amounts of current and voltage when they are connected to identical load resistances. Therefore, by equating the open circuit voltages and short circuit currents of the above two sources we obtain

$$V_{ac} = I_s R_I = V_S$$

$$I_{SC} = I_S = \frac{V_s}{R_v}$$

It follows that $R_I = R_V = R_s \therefore V_s = I_S R_S$

where R_S is the internal resistance of the voltage or current source. Therefore, any practical voltage source, having an ideal voltage V_S and internal series resistance R_S can be replaced by a current source $I_S = V_S/R_S$ in parallel with an internal resistance R_S. The reverse transformation is also possible. Thus, a practical current source in parallel with an internal resistance R_S can be replaced by a voltage source $V_S = I_s R_s$ in series with an internal resistance R_S.

Example 2.16 Determine the equivalent voltage source for the current source shown in Fig. 2.48.

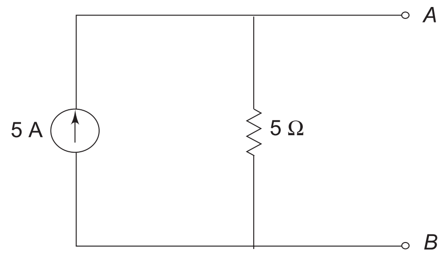


Fig. 2.48

Solution The voltage across terminals *A* and *B* is equal to 25 V. Since the internal resistance for the current source is 5 Ω, the internal resistance of the voltage source is also 5 Ω. The equivalent voltage source is shown in Fig. 2.49.

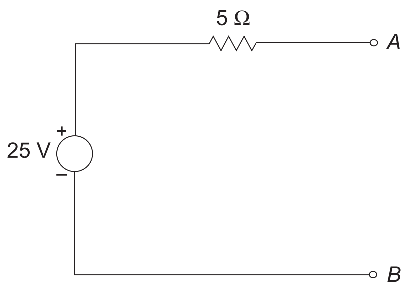


Fig. 2.49

Example 2.17 Determine the equivalent current source for the voltage source shown in Fig. 2.50.

Solution The short circuit current at terminals *A* and *B* is equal to

$$I = \frac{50}{30} = 1.66 \text{ A}$$

Since the internal resistance for the voltage source is 30 Ω, the internal resistance of the current source is also 30 Ω. The equivalent current source is shown in Fig. 2.51.

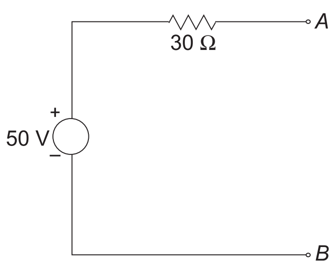


Fig. 2.50

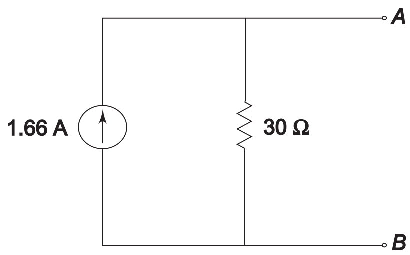


Fig. 2.51

Additional Solved Problems

Problem 2.1 Determine the power dissipation in the 4 Ω resistor of the circuit shown in Fig. 2.52 by using mesh analysis.

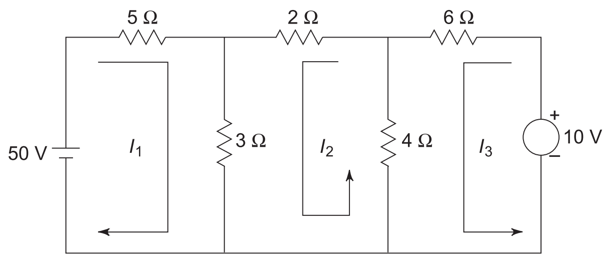


Fig. 2.52

Solution Power dissipated in the 4 Ω resistor is $P_4 = 4(I_2 - I_3)^2$
By using mesh analysis, we can find the currents I_2 and I_3.
From Fig. 2.52, we can form three equations.

From the given circuit in Fig. 2.52, we can obtain three mesh equations in terms of I_1, I_2 and I_3

$$8I_1 + 3I_2 = 50$$
$$3I_1 + 9I_2 - 4I_3 = 0$$
$$-4I_2 + 10I_3 = 10$$

By solving the above equations we can find I_1, I_2 and I_3.

$$I_2 = \frac{\begin{vmatrix} 8 & 50 & 0 \\ 3 & 0 & -4 \\ 0 & 10 & 10 \end{vmatrix}}{\begin{vmatrix} 8 & +3 & 0 \\ 3 & 9 & -4 \\ 0 & -4 & 10 \end{vmatrix}} = \frac{-1180}{502} = -2.35 \text{ A}$$

$$I_3 = \frac{\begin{vmatrix} 8 & 3 & 50 \\ 3 & 9 & 0 \\ 0 & -4 & 10 \end{vmatrix}}{\begin{vmatrix} 8 & 3 & 0 \\ 3 & 9 & -4 \\ 0 & -4 & 10 \end{vmatrix}} = \frac{30}{502} = 0.06 \text{ A}$$

The current in the 4 Ω resistor = $(I_2 - I_3)$

$$= (-2.35 - 0.06)\text{A} = -2.41 \text{ A}$$

Therefore, the power dissipated in the 4 Ω resistor, $P_4 = (2.41)^2 \times 4 = 23.23$ W.

Problem 2.2 Using mesh analysis, determine the voltage V_S which gives a voltage of 50 V across the 10 Ω resistor as shown in Fig. 2.53.

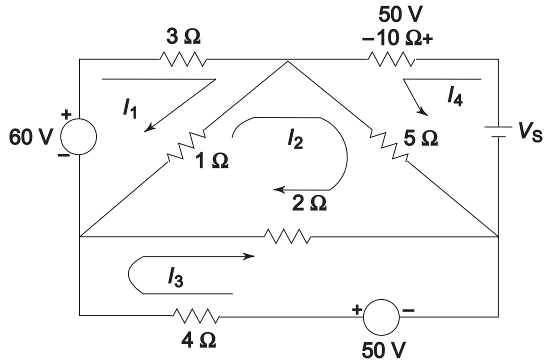


Fig. 2.53

Solution Since the voltage across the 10 Ω resistor is 50 V, the current passing through it is $I_4 = 50/10 = 5$ A.

From Fig. 2.53, we can form four equations in terms of the currents I_1, I_2, I_3 and I_4, as

$$4I_1 - I_2 = 60$$
$$-I_1 + 8I_2 - 2I_3 + 5I_4 = 0$$
$$-2I_2 + 6I_3 = 50$$
$$5I_2 + 15I_4 = V_S$$

Solving the above equations, using Cramer's rule, we get

$$I_4 = \frac{\begin{vmatrix} 4 & -1 & 0 & 60 \\ -1 & 8 & -2 & 0 \\ 0 & -2 & 6 & 50 \\ 0 & 5 & 0 & V_S \end{vmatrix}}{\begin{vmatrix} 4 & -1 & 0 & 0 \\ -1 & 8 & -2 & 5 \\ 0 & -2 & 6 & 0 \\ 0 & 5 & 0 & 15 \end{vmatrix}}$$

$$\Delta = 4\begin{vmatrix} 8 & -2 & 5 \\ -2 & 6 & 0 \\ 5 & 0 & 15 \end{vmatrix} + 1\begin{vmatrix} -1 & -2 & 5 \\ 0 & 6 & 0 \\ 0 & 0 & 15 \end{vmatrix}$$

$$= 4\{8(90) + 2(-30) + 5(-30)\} + 1\{-1(90)\}$$

$$\Delta = 1950.$$

$$\Delta_4 = 4\begin{vmatrix} 8 & -2 & 0 \\ -2 & 6 & 50 \\ 5 & 0 & V_S \end{vmatrix} + 1\begin{vmatrix} -1 & -2 & 0 \\ 0 & 6 & 50 \\ 0 & 0 & V_S \end{vmatrix} - 60\begin{vmatrix} -1 & 8 & -2 \\ 0 & -2 & 6 \\ 0 & 5 & 0 \end{vmatrix}$$

$$= 4\{8(6\,V_S) + 2(-\,2V_S - 250)\} + 1\{-\,1(6V_S)\} - 60\,\{-\,1\,(-\,30)\}$$

$$= 170\,V_S - 3800$$

$$I_4 = \frac{170V_S - 3800}{1950}$$

$$\therefore \qquad V_S = \frac{1950 \times I_4 + 3800}{170} = 79.7\text{ V}$$

Problem 2.3 Determine the voltage V which causes the current I_1 to be zero for the circuit shown in Fig. 2.54. Use Mesh analysis.

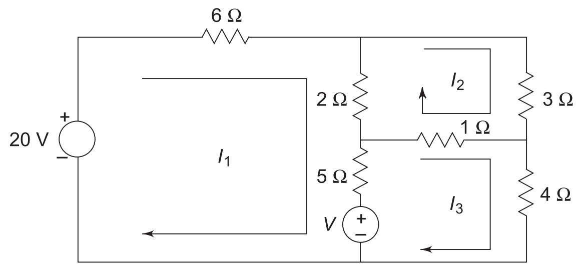


Fig. 2.54

Solution From Fig. 2.54 we can form three loop equations in terms of I_1, I_2, I_3 and V, as follows

$$13I_1 - 2I_2 - 5I_3 = 20 - V$$
$$-\,2I_1 + 6I_2 - I_3 = 0$$
$$-\,5I_1 - I_2 + 10I_3 = V$$

Using Cramer's rule, we get

$$I_1 = \frac{\begin{vmatrix} 20-V & -2 & -5 \\ 0 & 6 & -1 \\ V & -1 & +10 \end{vmatrix}}{\begin{vmatrix} 13 & -2 & -5 \\ -2 & +6 & -1 \\ -5 & -1 & +10 \end{vmatrix}}$$

$$\Delta_1 = (20 - V)\,(+\,60 - 1) + 2(V) - 5(-\,6V)$$
$$= 1180 - 27\,V$$

we have $\Delta = 557$

$$I_1 = \frac{\Delta_1}{557}$$

$$\therefore \qquad \Delta_1 = 0$$

$-27V + 1180 = 0$

$\therefore \qquad V = 43.7$ V

Problem 2.4 Determine the loop currents for the circuit shown in Fig. 2.55 by using mesh analysis.

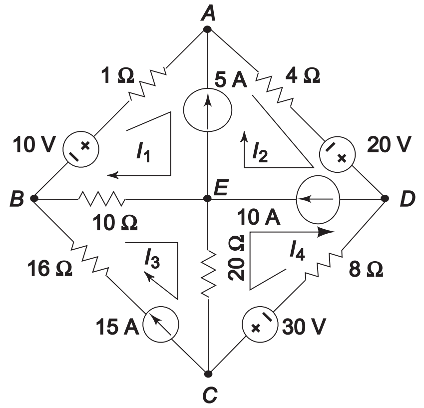


Fig. 2.55

Solution The branches AE, DE and BC consists of current sources. Here we have to apply supermesh analysis.

The combined supermesh equation is

$$10(I_1 - I_3) + I_1 - 10 + 4I_2 - 20$$
$$+ 8I_4 - 30 + 20\,(I_4 - I_3) = 0$$

or $\qquad 11I_1 + 4I_2 - 30I_3 + 28I_4 = 60$

In branch AE, $\qquad I_2 - I_1 = 5$ A

In branch BC, $\qquad I_3 = 15$ A

In branch DE, $\qquad I_2 - I_4 = 10$ A

Solving the above four equations, we can get the four currents I_1, I_2, I_3 and I_4 as

$$I_1 = 14.65 \text{ A}$$
$$I_2 = 19.65 \text{ A}, \quad I_3 = 15 \text{ A, and } I_4 = 9.65 \text{ A}$$

Problem 2.5 Determine the power delivered by the voltage source and the current in the 10 Ω resistor for the circuit shown in Fig. 2.56.

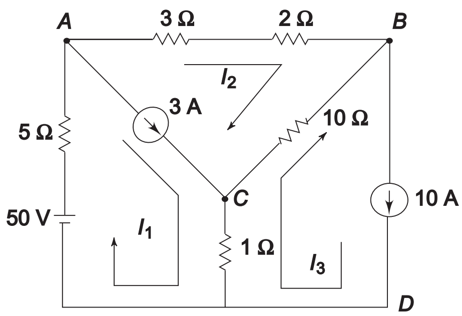


Fig. 2.56

Solution Since branches AC and BD consist of current sources, we have to use the supermesh technique.

The combined supermesh equation is

$$-50 + 5I_1 + 3I_2 + 2I_2 + 10(I_2 - I_3) + 1(I_1 - I_3) = 0$$

or
$$6I_1 + 15I_2 - 11I_3 = 50$$

or
$$I_1 - I_2 = 3 \text{ A or } I_3 = 10 \text{ A}$$

From the above equations we can solve for I_1, I_2 and I_3 follows

$$I_1 = 9.76 \text{ A}, \quad I_2 = 6.76 \text{ A}, \quad I_3 = 10 \text{ A}$$

Problem 2.6 Determine the voltage ratio V_{out}/V_{in} for the circuit shown in Fig. 2.57 by using nodal analysis.

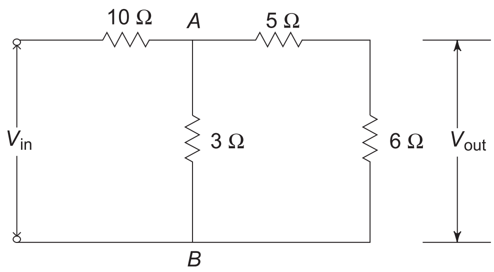


Fig. 2.57

Solution $I_{10} + I_3 + I_{11} = 0$

$$I_{10} = \frac{V_A - V_{in}}{10}$$

$$I_3 = \frac{V_A}{3}$$

$$I_{11} = \frac{V_A}{11}, \text{ or } \frac{V_{out}}{6}$$

$$\frac{V_A - V_{in}}{10} + \frac{V_A}{3} + \frac{V_A}{11} = 0$$

Also
$$\frac{V_A}{11} = \frac{V_{out}}{6}$$

$\therefore$
$$V_A = V_{out} \times 1.83$$

From the above equations $V_{out}/V_{in} = 1/9.53 = 0.105$

Problem 2.7 Find the voltages V in the circuit shown in Fig. 2.58 which makes the current in the 10 Ω resistor zero by using nodal analysis.

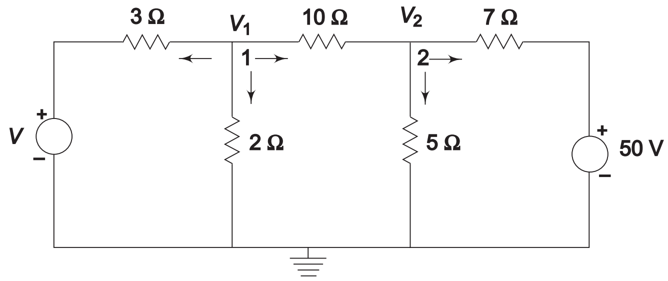


Fig. 2.58

Solution In the circuit shown, assume voltages V_1 and V_2 at nodes 1 and 2.

At node 1, the current equation in Fig. 2.59 (a) is

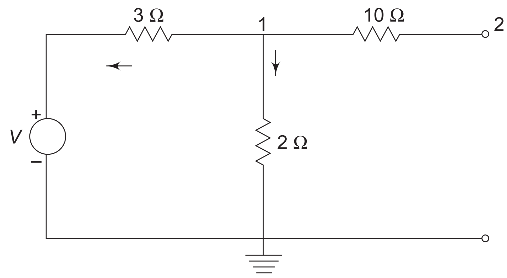


Fig. 2.59 (a)

$$\frac{V_1 - V}{3} + \frac{V_1}{2} + \frac{V_1 - V_2}{10} = 0$$

or $$0.93\ V_1 - 0.1\ V_2 = V/3$$

At node 2, the current equation in Fig. 2.59 (b) is

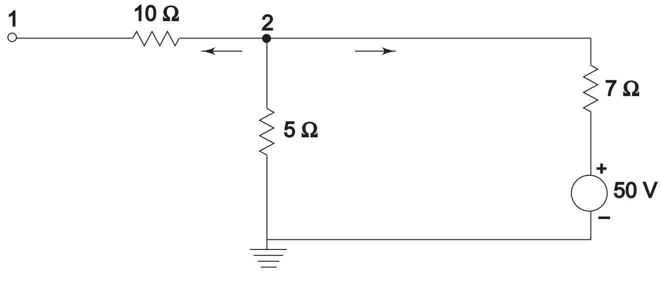


Fig. 2.59 (b)

$$\frac{V_2 - V_1}{10} + \frac{V_2}{5} + \frac{V_2 - 50}{7} = 0$$

or $$-\ 0.1\ V_1 + 0.443\ V_2 = 7.143$$

Since the current in 10 Ω resistor is zero, the voltage at node 1 is equal to the voltage at node 2.

$\therefore \qquad V_1 - V_2 = 0$

From the above three equations, we can solve for V

$$V_1 = 20.83 \text{ Volts and } V_2 = 20.83 \text{ volts}$$

$\therefore \qquad V = 51.87 \text{ V}$

Problem 2.8 Use nodal analysis to find the power dissipated in the 6 Ω resistor for the circuit shown in Fig. 2.60.

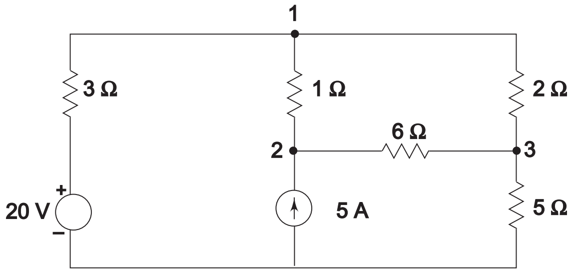


Fig. 2.60

Solution Assume voltage V_1, V_2 and V_3 at nodes 1, 2 and 3 as shown in Fig. 2.60.

By applying current law at node 1, we have

$$\frac{V_1 - 20}{3} + \frac{V_1 - V_2}{1} + \frac{V_1 - V_3}{2} = 0$$

or
$$1.83V_1 - V_2 - 0.5V_3 = 6.67 \qquad (2.53)$$

At node 2

$$\frac{V_2 - V_1}{1} + \frac{V_2 - V_3}{6} = 5 \text{ A}$$

or
$$-V_1 - 1.167V_2 - 0.167V_3 = 5 \qquad (2.54)$$

At node 3,

$$\frac{V_3 - V_1}{2} + \frac{V_3 - V_2}{6} + \frac{V_3}{5} = 0$$

or
$$-0.5\,V_1 - 0.167\,V_2 + 0.867\,V_3 = 0 \qquad (2.55)$$

Applying Cramer's rule to Eqs 2.53, 2.54 and 2.55, we have

$$V_2 = \frac{\Delta_2}{\Delta}$$

where
$$\Delta = \begin{vmatrix} 1.83 & -1 & -0.5 \\ -1 & -1.167 & -0.167 \\ -0.5 & -0.167 & 0.867 \end{vmatrix} = -2.64$$

$$\Delta_2 = \begin{vmatrix} 1.83 & 6.67 & -0.5 \\ -1 & 5 & -0.167 \\ -0.5 & 0 & 0.867 \end{vmatrix} = 13.02$$

$\therefore$ $$V_2 = \frac{13.02}{-2.64} = -4.93 \text{ V}$$

Similarly,

$$V_3 = \frac{\Delta_3}{\Delta}$$

$$\Delta_3 = \begin{vmatrix} 1.83 & -1 & 6.67 \\ -1 & -1.167 & 5 \\ -0.5 & -0.167 & 0 \end{vmatrix} = 1.25$$

$\therefore$ $$V_3 = \frac{1.25}{-2.64} = -0.47 \text{ V}$$

The current in the 6 Ω resistor is

$$I_6 = \frac{V_2 - V_3}{6}$$

$$= \frac{-4.93 + 0.47}{6} = -0.74 \text{ A}$$

The power absorbed or dissipated = $I_6^2 R_6$

$= (0.74)^2 \times 6$

$= 3.29$ W

Problem 2.9 Determine the power dissipated by 5 Ω resistor in the circuit shown in Fig. 2.61.

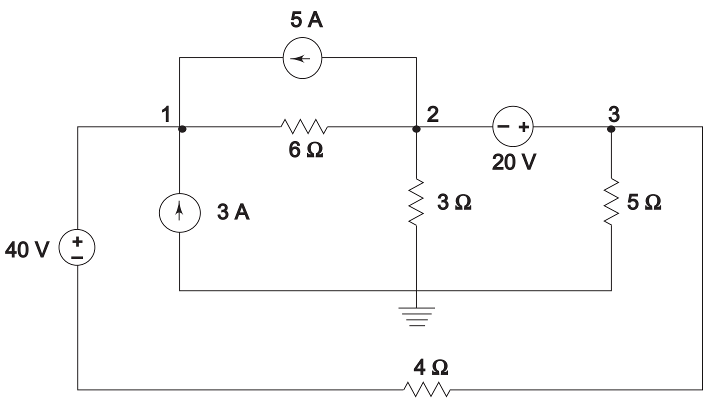


Fig. 2.61

Solution In Fig. 2.61, assume voltages V_1, V_2 and V_3 at nodes 1, 2 and 3. At node 1, the current law gives

$$\frac{V_1 - 40 - V_3}{4} + \frac{V_1 - V_2}{6} - 3 - 5 = 0$$

or $$0.42\ V_1 - 0.167\ V_2 - 0.25\ V_3 = 18$$

Applying the supernode technique between nodes 2 and 3, the combined equation at node 2 and 3 is

$$\frac{V_2 - V_1}{6} + 5 + \frac{V_2}{3} + \frac{V_3}{5} + \frac{V_3 + 40 - V_1}{4} = 0$$

or $$-0.42\ V_1 + 0.5\ V_2 + 0.45\ V_3 = -15$$

Also $$V_3 - V_2 = 20\ \text{V}$$

Solving the above three equations, we get

$$V_1 = 52.89\ \text{V},\ V_2 = -1.89\ \text{V and}$$
$$V_3 = 18.11\ \text{V}$$

$\therefore$ The current in the 5 Ω resistor $I_5 = \dfrac{V_3}{5}$

$$= \frac{18.11}{5} = 3.62\ \text{A}$$

The power absorbed by the 5 Ω resistor $P_5 = I_5^2 R_5$

$$= (3.62)^2 \times 5$$
$$= 65.52\ \text{W}$$

Problem 2.10 Find the power delivered by the 5 A current source in the circuit shown in Fig. 2.62 by using the nodal method.

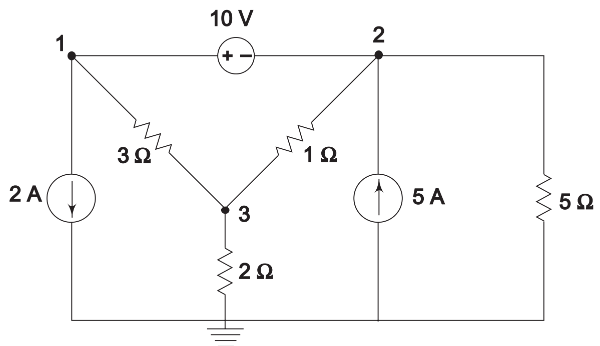


Fig. 2.62

Solution Assume the voltages V_1, V_2 and V_3 at nodes 1, 2, and 3, respectively. Here, the 10 V source is common between nodes 1 and 2. So applying the supernode technique, the combined equation at node 1 and 2 is

$$\frac{V_1 - V_3}{3} + 2 + \frac{V_2 - V_3}{1} - 5 + \frac{V_2}{5} = 0$$

or $\quad 0.34\ V_1 + 1.2\ V_2 - 1.34\ V_3 = 3$

At node 3, $$\frac{V_3 - V_1}{3} + \frac{V_3 - V_2}{1} + \frac{V_3}{2} = 0$$

or $\quad -0.34\ V_1 - V_2 + 1.83\ V_3 = 0$

Also $\quad V_1 - V_2 = 10$

Solving the above equations, we get

$$V_1 = 13.72\text{ V};\quad V_2 = 3.72\text{ V}$$

$$V_3 = 4.567\text{ V}$$

Hence the power delivered by the source (5 A) $= V_2 \times 5$

$$= 3.72 \times 5 = 18.6\text{ W}$$

Problem 2.11 Using source transformation, find the power delivered by the 50 V voltage source in the circuit shown in Fig. 2.63.

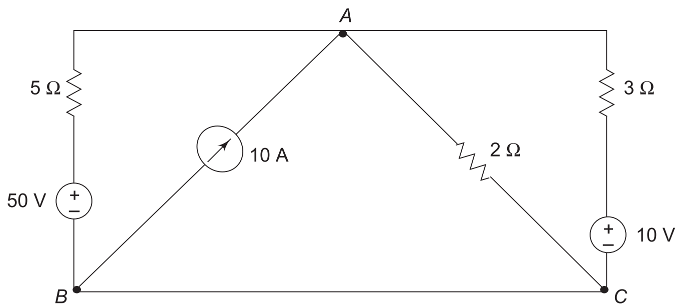


Fig. 2.63

Solution The current source in the circuit in Fig. 2.63 can be replaced by a voltage source as shown in Fig. 2.64.

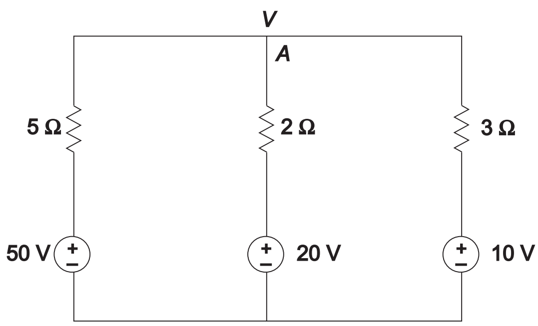


Fig. 2.64

$$\frac{V-50}{5}+\frac{V-20}{2}+\frac{V-10}{3}=0$$

$$V\,[0.2 + 0.5 + 0.33] = 23.33$$

or $$V=\frac{23.33}{1.03}=22.65\text{ V}$$

$\therefore$ The current delivered by the 50 V voltage source is $(50 - V)/5$

$$=\frac{50-22.65}{5}=5.47\text{ A}$$

Hence, the power delivered by the 50 V voltage source = $50 \times 5.47 = 273.5$ W

Problem 2.12 By using source transformation, source combination and resistance combination convert the circuit shown in Fig. 2.65 into a single voltage source and single resistance.

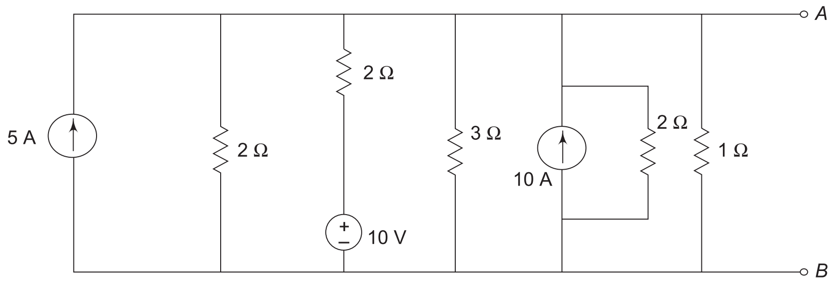


Fig. 2.65

Solution The voltage source in the circuit of Fig. 2.65 can be replaced by a current source as shown in Fig. 2.66 (a).

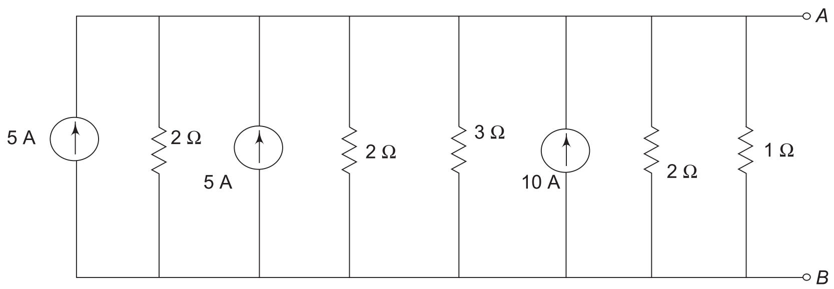


Fig. 2.66 (a)

Here the current sources can be combined into a single source. Similarly, all the resistances can be combined into a single resistance, as shown in Fig. 2.66 (b).

Figure 2.66 (b) can be replaced by single voltage source and a series resistance as shown in Fig. 2.66 (c).

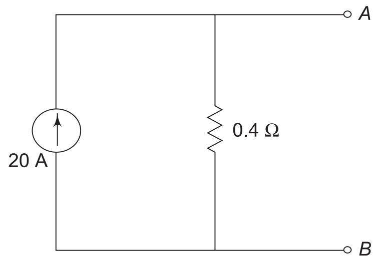


Fig. 2.66 (b)

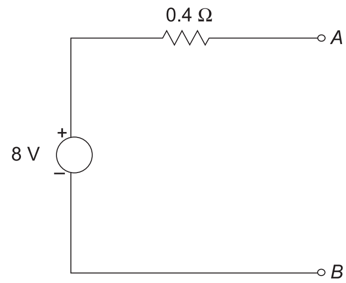


Fig. 2.66 (c)

Problem 2.13 For the circuit shown in Fig. 2.67 find the voltage across the 4 Ω resistor by using nodal analysis.

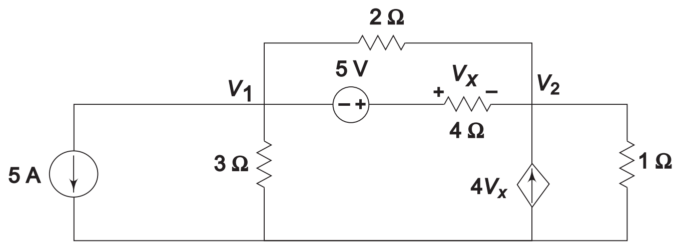


Fig. 2.67

Solution In the circuit shown, assume voltages V_1 and V_2 at nodes 1 and 2. At node 1, the current equation is

$$5 + \frac{V_1}{3} + \frac{V_1 + 5 - V_2}{4} + \frac{V_1 - V_2}{2} = 0$$

or $$1.08\, V_1 - 0.75\, V_2 = -\,6.25 \tag{2.56}$$

At node 2, the current equation is

$$\frac{V_2 - V_1 - 5}{4} + \frac{V_2 - V_1}{2} - 4V_x + \frac{V_2}{1} = 0$$

$$V_x = V_1 + 5 - V_2$$

or $$-4.75\ V_1 + 5.75\ V_2 = 21.25 \qquad (2.57)$$

Applying Cramer's rule to Eqs 2.56 and 2.57, we have

$$V_2 = \frac{\Delta_2}{\Delta}$$

where $$\Delta = \begin{vmatrix} 1.08 & -0.75 \\ -4.75 & 5.75 \end{vmatrix} = 2.65$$

$$\Delta_2 = \begin{vmatrix} 1.08 & -6.25 \\ -4.75 & 21.25 \end{vmatrix} = -6.74$$

$\therefore$ $$V_2 = \frac{\Delta_2}{\Delta} = \frac{-6.74}{2.65} = -2.54 \text{ V}$$

Similarly, $$V_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta_1 = \begin{vmatrix} -6.25 & -0.75 \\ 21.25 & 5.75 \end{vmatrix} = -20$$

$$V_1 = \frac{\Delta_1}{\Delta} = \frac{-20}{2.65} = -7.55 \text{ V}$$

The voltage across the 4 Ω resistor is

$$V_x = V_1 + 5 - V_2$$
$$= -.755 + 5 - (-2.54)$$
$$V_x = 0.01 \text{ volts}$$

Problem 2.14 For the circuit shown in Fig. 2.68, find the current passing through the 5 Ω resistor by using the nodal method.

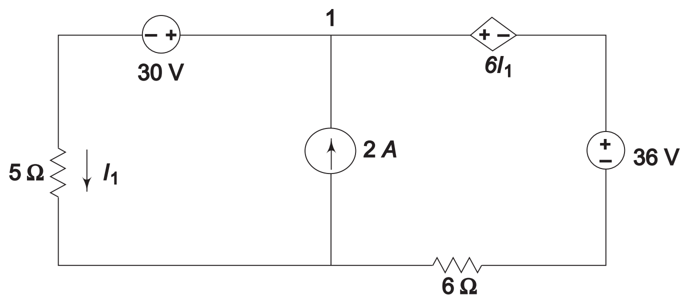


Fig. 2.68

Solution In the circuit shown, assume the voltage V at node 1.
At node 1, the current equation is

$$\frac{V-30}{5} - 2 + \frac{V-36-6I_1}{6} = 0$$

where $$I_1 = \frac{V-30}{5}$$

From the above equation

$$V = 48 \text{ V}$$

The current in 5 Ω resistor is

$$I_1 = \frac{V-30}{5} = 3.6 \text{ A}$$

Problem 2.15 In the circuit shown in Fig. 2.69, find the power delivered by 4 V source using mesh analysis and voltage across the 2 Ω resistor.

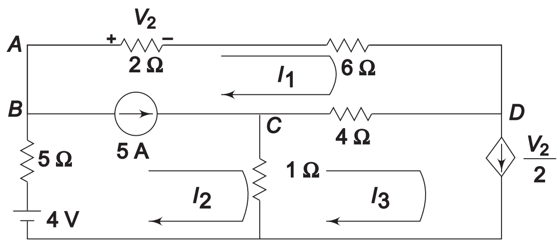


Fig. 2.69

Solution Since branches BC and DE consists of current sources, we use the supermesh technique.

The combined supermesh equation is

$$2I_1 + 6I_1 + 4(I_1 - I_3) - 4 + 5I_2 + I_2 - I_3 + 4(I_3 - I_1) + I_3 - I_2 = 0$$

or

$$8I_1 + 5I_2 = 4$$

In branch BC, $\quad I_2 - I_1 = 5$

In branch DE, $\quad I_3 = \dfrac{V_2}{2}$

Solving the above equations

$$I_1 = -1.62 \text{ A}; \quad I_2 = 3.38 \text{ A}$$

The voltage across the 2 Ω resistor $V_2 = 2I_1 = -3.24$ V
Power delivered by 4 V source $P_4 = 4I_2 = 4(3.38) = 13.52$ W

Problem 2.16 For the circuit shown in Fig. 2.70, find the current through the 10 Ω resistor by using mesh analysis.

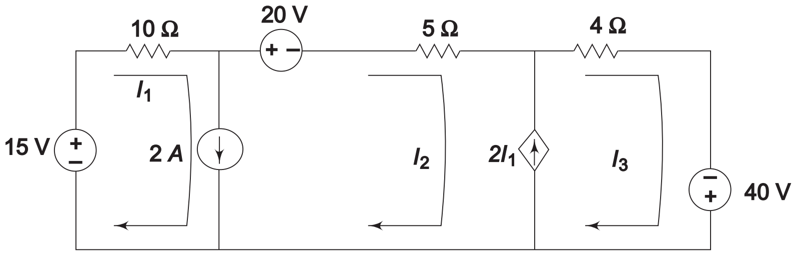


Fig. 2.70

Solution The parallel branches consist of current sources. Here we use supermesh analysis. The combined supermesh equation is.

or $$-15 + 10I_1 + 20 + 5I_2 + 4I_3 - 40 = 0$$

and $$10I_1 + 5I_2 + 4I_3 = 35$$

$$I_1 - I_2 = 2$$

$$I_3 - I_2 = 2I_1$$

Solving the above equations, we get

$$I_1 = 1.96 \text{ A}$$

The current in the 10 Ω resistor is $I_1 = 1.96$ A

Practice Problems

2.1 In the circuit shown in Fig. 2.71, use mesh analysis to find out the power delivered to the 4 Ω resistor. To what voltage should the 100 V battery be changed so that no power is delivered to the 4 Ω resistor?

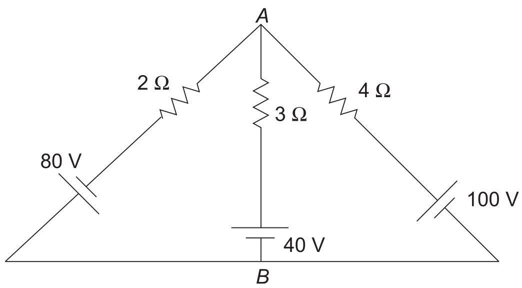


Fig. 2.71

2.2 Find the voltage between A and B of the circuit shown in Fig. 2.72 by mesh analysis.

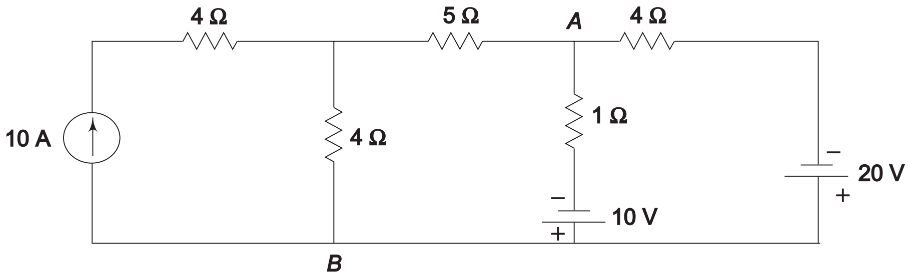


Fig. 2.72

2.3 In the circuit shown in Fig. 2.73, use nodal analysis to find out the voltage across 40 Ω and the power supplied by the 5 A source.

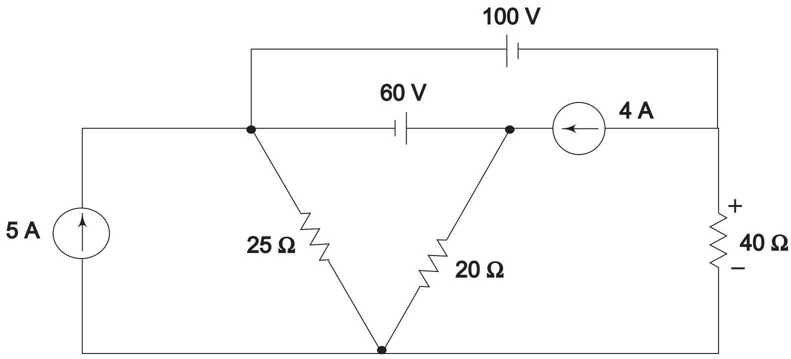


Fig. 2.73

2.4 In the network shown in Fig. 2.74, the resistance R is variable from zero to infinity. The current I through R can be expressed as $I = a + bV$, where V is the voltage across R as shown in the figure, and a and b are constants. Determine a and b.

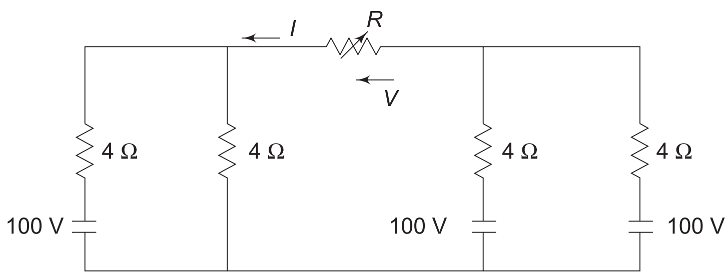


Fig. 2.74

2.5 Determine the currents in bridge circuit by using mesh analysis in Fig. 2.75.

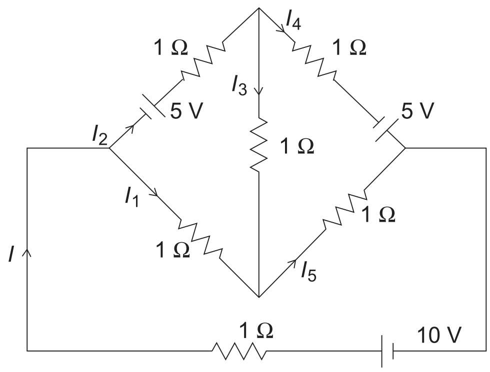


Fig. 2.75

2.6 Use nodal analysis in the circuit shown in Fig. 2.76 and determine what value of V will cause $V_{10} = 0$.

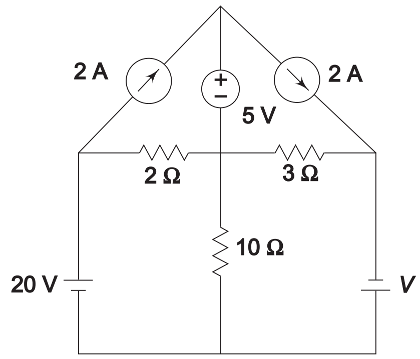


Fig. 2.76

2.7 For the circuit shown in Fig. 2.77, use mesh analysis to find the values of all mesh currents.

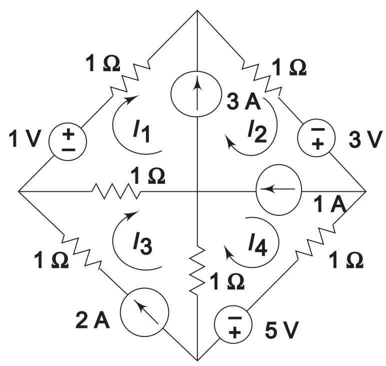


Fig. 2.77

2.8 For the circuit shown in Fig. 2.78, use node analysis to find the current delivered by the 24 V source.

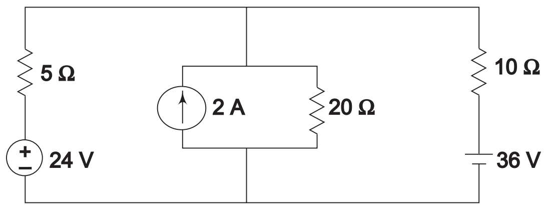


Fig. 2.78

2.9 Using mesh analysis, determine the voltage across the 10 kΩ resistor at terminals *A* and *B* of the circuit shown in Fig. 2.79.

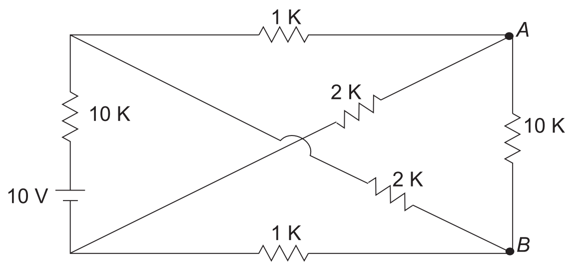


Fig. 2.79

2.10 Determine the current *I* in the circuit by using loop analysis in Fig. 2.80.

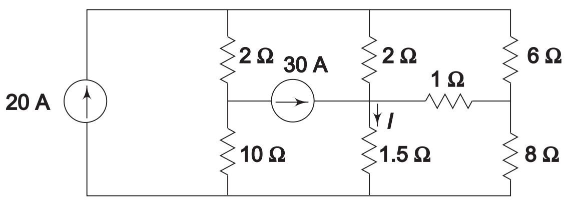


Fig. 2.80

2.11 Write nodal equations for the circuit shown in Fig. 2.81, and find the power supplied by the 10 V source.

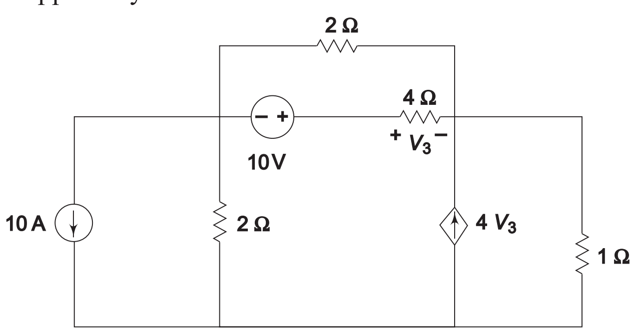


Fig. 2.81

2.12 Use nodal analysis to find V_2 in the circuit shown in Fig. 2.82.

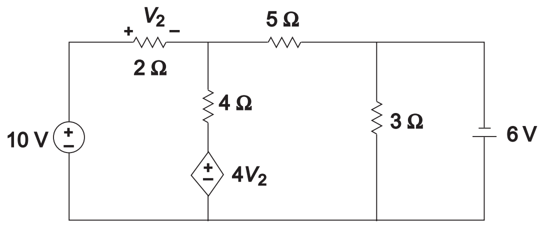


Fig. 2.82

2.13 Use mesh analysis to find V_x in the circuit shown in Fig. 2.83.

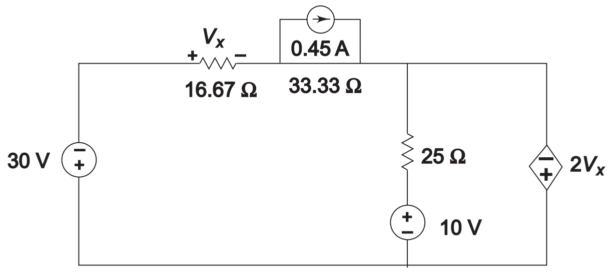


Fig. 2.83

2.14 For the circuit shown in Fig. 2.84, find the value of V_2 that will cause the voltage across 20 Ω to be zero by using mesh analysis.

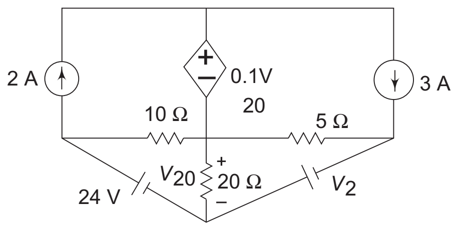


Fig. 2.84

Objective-type Questions

1. A tree has
   (a) a closed path (b) no closed paths
   (c) none
2. The number of branches in a tree is __________ the number of branches in a graph.
   (a) less than (b) more than
   (c) equal to

3. The tie-set schedule gives the relation between
   (a) branch currents and link currents
   (b) branch voltages and link currents
   (c) branch currents and link voltages
   (d) none of the above
4. The cut-set schedule gives the relation between
   (a) branch currents and link currents
   (b) branch voltages and tree branch voltages
   (c) branch voltages and link voltages
   (d) branch current and tree currents
5. Mesh analysis is based on
   (a) Kirchhoff's current law (b) Kirchhoff's voltage law
   (c) Both (d) None
6. If a network contains B branches, and N nodes, then the number of mesh current equations would be
   (a) $B-(N-1)$ (b) $N-(B-1)$
   (c) $B-N-1$ (d) $(B+N)-1$
7. A network has 10 nodes and 17 branches. The number of different node pair voltages would be
   (a) 7 (b) 9
   (c) 45 (d) 10
8. A practical voltage source consists of
   (a) an ideal voltage source in series with an internal resistance
   (b) an ideal voltage source in parallel with an internal resistance
   (c) both (a) and (b) are correct
   (d) none of the above
9. A practical current source consists of
   (a) an ideal current source in series with an impedance
   (b) an ideal current source in parallel with an impedance
   (c) both are correct
   (d) none of the above
10. A circuit consists of two resistances, R_1 and R_2, in parallel. The total current passing through the circuit is I_T. The current passing through R_1 is
    (a) $\dfrac{I_T\ R_1}{R_1+R_2}$ (b) $\dfrac{I_T\,(R_1+R_2)}{R_1}$
    (c) $\dfrac{I_T R_2}{R_1+R_2}$ (d) $\dfrac{I_T R_1+R_2}{R_2}$
11. A network has seven nodes and five independent loops. The number of branches in the network is
    (a) 13 (b) 12
    (c) 11 (d) 10
12. The nodal method of circuit analysis is based on
    (a) KVL and Ohm's law (b) KCL and Ohm's law
    (c) KCL and KVL (d) KCL, KVL and Ohm's law

13. The number of independent loops for a network with n nodes and b branches is
    (a) $n - 1$ (b) $b - n$
    (c) $b - n + 1$
    (d) independent of the number of nodes
14. The two electrical sub networks N_1 and N_2 are connected through three resistors as shown in Fig. 2.85. The voltage across the 5 Ω resistor and the 1 Ω resistor are given to be 10 V and 5 V respectively. The voltage across the 15 Ω resistor is
    (a) – 105 V (b) + 105 V
    (c) – 15 V (d) + 15 V

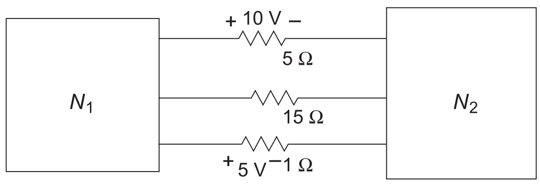


Fig. 2.85

15. Relative to a given fixed tree of a network
    (a) link currents form an independent set
    (b) branch currents form an independent set
    (c) link voltages form an independent set
    (d) branch voltages form an independent set

Chapter 3

Useful Theorems in Circuit Analysis

3.1 STAR-DELTA TRANSFORMATION

In the preceding chapter, a simple technique called the *source transformation technique* has been discussed. The star delta transformation is another technique useful in solving complex networks. Basically, any three circuit elements, i.e. resistive, inductive or capacitive, may be connected in two different ways. One way of connecting these elements is called the star connection, or the Y connection. The other way of connecting these elements is called the delta (Δ) connection. The circuit is said to be in star connection, if three elements are connected as shown in Fig. 3.1(a), when it appears like a star (Y). Similarly, the circuit is said to be in delta connection, if three elements are connected as shown in Fig. 3.1(b), when it appears like a delta (Δ).

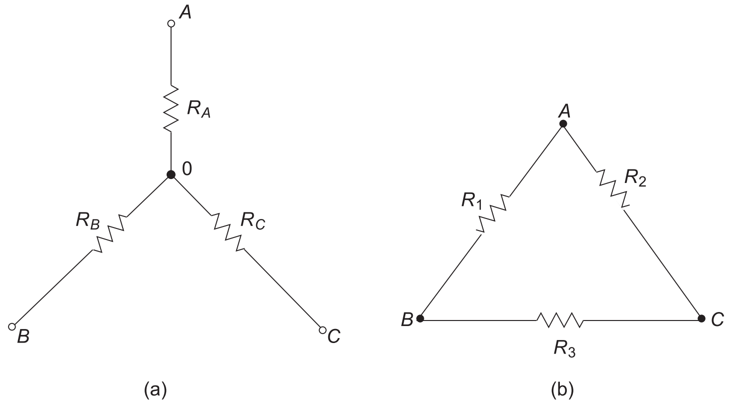


Fig. 3.1

The above two circuits are equal if their respective resistances from the terminals AB, BC and CA are equal. Consider the star connected circuit in Fig. 3.1(a); the resistance from the terminals AB, BC and CA respectively are

$$R_{AB}(Y) = R_A + R_B$$
$$R_{BC}(Y) = R_B + R_C$$
$$R_{CA}(Y) = R_C + R_A$$

Similarly, in the delta connected network in Fig. 3.1(b), the resistances seen from the terminals AB, BC and CA, respectively, are

$$R_{AB}(\Delta) = R_1 \parallel (R_2 + R_3) = \frac{R_1 (R_2 + R_3)}{R_1 + R_2 + R_3}$$

$$R_{BC}(\Delta) = R_3 \parallel (R_1 + R_2) = \frac{R_3 (R_1 + R_2)}{R_1 + R_2 + R_3}$$

$$R_{CA}(\Delta) = R_2 \parallel (R_1 + R_3) = \frac{R_2 (R_1 + R_3)}{R_1 + R_2 + R_3}$$

Now, if we equate the resistances of star and delta circuits, we get

$$R_A + R_B = \frac{R_1 (R_2 + R_3)}{R_1 + R_2 + R_3} \tag{3.1}$$

$$R_B + R_C = \frac{R_3 (R_1 + R_2)}{R_1 + R_2 + R_3} \tag{3.2}$$

$$R_C + R_{\mathrm{A}} = \frac{R_2 (R_1 + R_3)}{R_1 + R_2 + R_3} \tag{3.3}$$

Subtracting Eq. 3.2 from Eq. 3.1, and adding Eq. 3.3 to the resultant, we have

$$R_A = \frac{R_1 R_2}{R_1 + R_2 + R_3} \tag{3.4}$$

Similarly, $$R_B = \frac{R_1 R_3}{R_1 + R_2 + R_3} \tag{3.5}$$

and $$R_C = \frac{R_2 R_3}{R_1 + R_2 + R_3} \tag{3.6}$$

Thus, a delta connection of R_1, R_2 and R_3 may be replaced by a star connection of R_A, R_B and R_C as determined from Eqs 3.4, 3.5 and 3.6. Now if we multiply the Eqs 3.4 and 3.5, 3.5 and 3.6, 3.6 and 3.4, and add the three, we get the final equation as under:

$$R_A R_B + R_B R_C + R_C R_A = \frac{R_1^2 R_2 R_3 + R_3^2 R_1 R_2 + R_2^2 R_1 R_3}{(R_1 + R_2 + R_3)^2} \tag{3.7}$$

In Eq. 3.7 dividing the LHS by R_A, gives R_3; dividing it by R_B gives R_2, and doing the same with R_C, gives R_1.

Thus
$$R_1 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_C}$$

$$R_2 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_B}$$

and
$$R_3 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_A}$$

From the above results, we can say that a star connected circuit can be transformed into a delta connected circuit and vice-versa.

From Fig. 3.2 and the above results, we can conclude that any resistance of the delta circuit is equal to the sum of the products of all possible pairs of star resistances divided by the opposite resistance of the star circuit. Similarly, any resistance of the star circuit is equal to the product of two adjacent resistances in the delta connected circuit divided by the sum of all resistances in delta connected circuit.

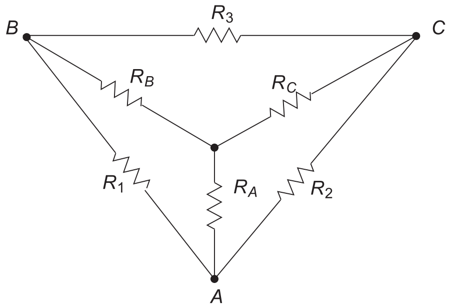


Fig. 3.2

Example 3.1 Obtain the star connected equivalent for the delta connected circuit shown in Fig. 3.3.

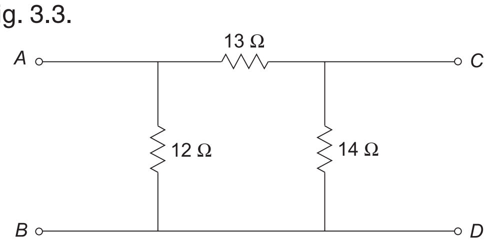
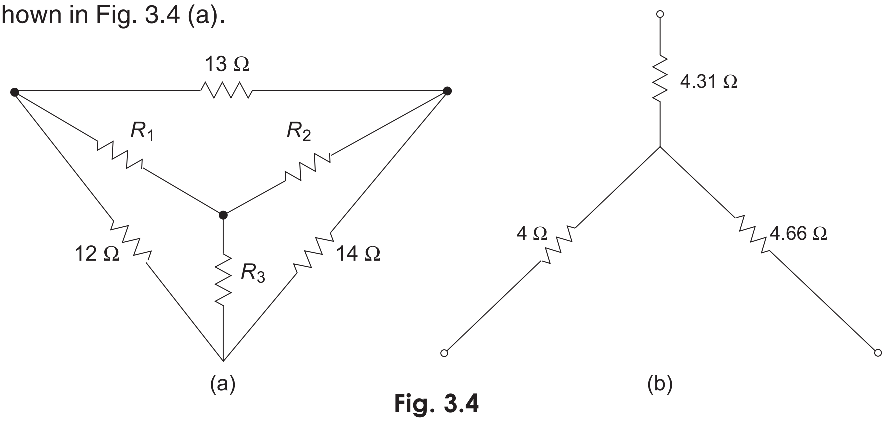


Fig. 3.3

Solution The above circuit can be replaced by a star connected circuit as shown in Fig. 3.4 (a).



(a) (b)

Fig. 3.4

Performing the Δ to Y transformation, we obtain

$$R_1 = \frac{13 \times 12}{14 + 13 + 12},\ R_2 = \frac{13 \times 14}{14 + 13 + 12}$$

and
$$R_3 = \frac{14 \times 12}{14 + 13 + 12}$$

$\therefore$
$$R_1 = 4\ \Omega,\ R_2 = 4.66\ \Omega,\ R_3 = 4.31\ \Omega$$

The star-connected equivalent is shown in Fig. 3.4 (b).

Example 3.2 Obtain the delta-connected equivalent for the star-connected circuit shown in Fig. 3.5.

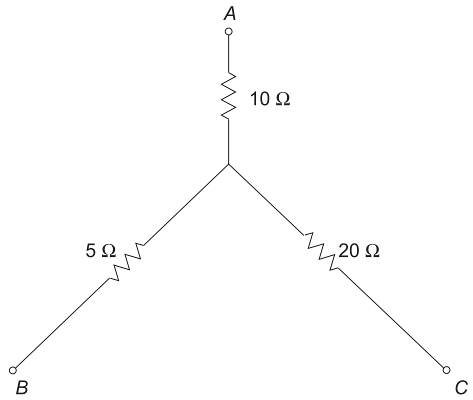


Fig. 3.5

Solution The above circuit can be replaced by a delta-connected circuit as shown in Fig. 3.6 (a).

Performing the Y to Δ transformation, we get from the Fig. 3.6 (a)

$$R_1 = \frac{20 \times 10 + 20 \times 5 + 10 \times 5}{20} = 17.5\ \Omega$$

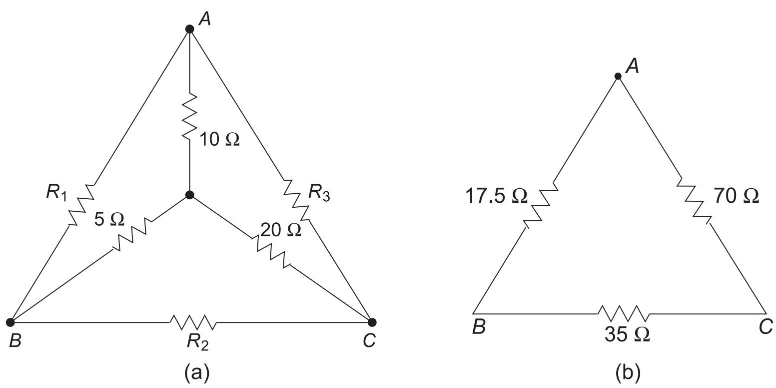


Fig. 3.6

$$R_2 = \frac{20 \times 10 + 20 \times 5 + 10 \times 5}{10} = 35\ \Omega$$

and
$$R_3 = \frac{20 \times 10 + 20 \times 5 + 10 \times 5}{5} = 70\ \Omega$$

The equivalent delta circuit is shown in Fig. 3.6 (b).

3.2 SUPERPOSITION THEOREM

The superposition theorem states that in any linear network containing two or more sources, the response in any element is equal to the algebraic sum of the responses caused by individual sources acting alone, while the other sources are non-operative; that is, while considering the effect of individual sources, other ideal voltage sources and ideal current sources in the network are replaced by short circuit and open circuit across their terminals. This theorem is valid only for linear systems. This theorem can be better understood with a numerical example.

Consider the circuit which contains two sources as shown in Fig. 3.7.

Now let us find the current passing through the 3 Ω resistor in the circuit. According to superposition theorem, the current I_2 due to the 20 V voltage source with 5 A source open circuited = 20/(5 + 3) = 2.5 A. (See Fig. 3.8)

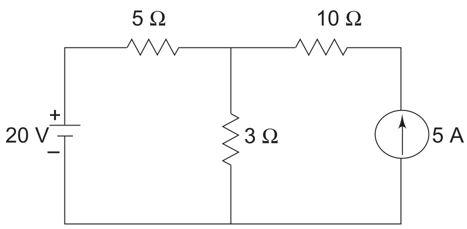


Fig. 3.7

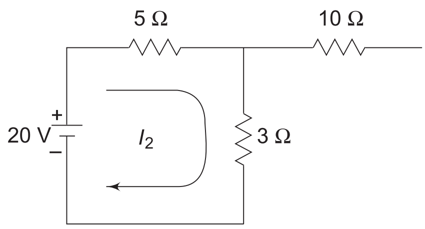


Fig. 3.8

The current I_5 due to 5 A source with 20 V source short circuited is

$$I_5 = 5 \times \frac{5}{(3 + 5)} = 3.125\text{ A}$$

The total current passing through the 3 Ω resistor is

$$(2.5 + 3.125) = 5.625\text{ A}$$

Let us verify the above result by applying nodal analysis.

The current passing in the 3 Ω resistor due to both sources should be 5.625 A.

Applying nodal analysis to Fig. 3.10, we have

$$\frac{V - 20}{5} + \frac{V}{3} = 5$$

$$V\left[\frac{1}{5} + \frac{1}{3}\right] = 5 + 4$$

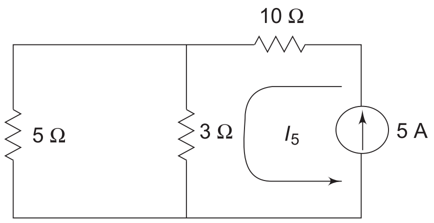


Fig. 3.9

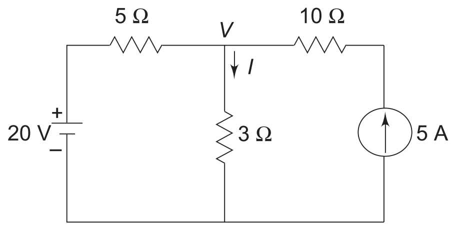


Fig. 3.10

$$V = 9 \times \frac{15}{8} = 16.875 \text{ V}$$

The current passing through the 3 Ω resistor is equal to $V/3$

i.e. $$I = \frac{16.875}{3} = 5.625 \text{ A}$$

So the superposition theorem is verified.

Let us now examine the power responses.

Power dissipated in the 3 Ω resistor due to voltage source acting alone

$$P_{20} = (I_{20})^2 R = (2.5)^2\ 3 = 18.75 \text{ W}$$

Power dissipated in the 3 Ω resistor due to current source acting alone

$$P_5 = (I_5)^2 R = (3.125)^2\ 3 = 29.29 \text{ W}$$

Power dissipated in the 3 Ω resistor when both the sources are acting simultaneously is given by

$$P = (5.625)^2 \times 3 = 94.92 \text{ W}$$

From the above results, the superposition of P_{20} and P_5 gives

$$P_{20} + P_5 = 48.04 \text{ W}$$

which is not equal to $P = 94.92$ W

We can, therefore, state that the superposition theorem is not valid for power responses. It is applicable only for computing voltage and current responses.

Example 3.3 Find the voltage across the 2 Ω resistor in Fig. 3.11 by using the super-position theorem.

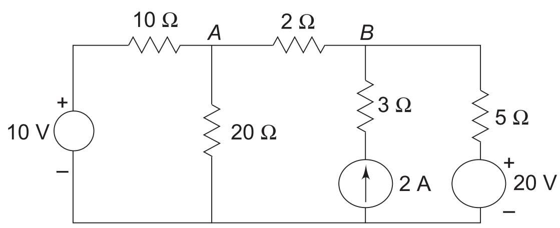


Fig. 3.11

Solution Let us find the voltage across the 2 Ω resistor due to individual sources. The algebraic sum of these voltages gives the total voltage across the 2 Ω resistor.

Our first step is to find the voltage across the 2 Ω resistor due to the 10 V source, while other sources are set equal to zero.

The circuit is redrawn as shown in Fig. 3.12 (a).

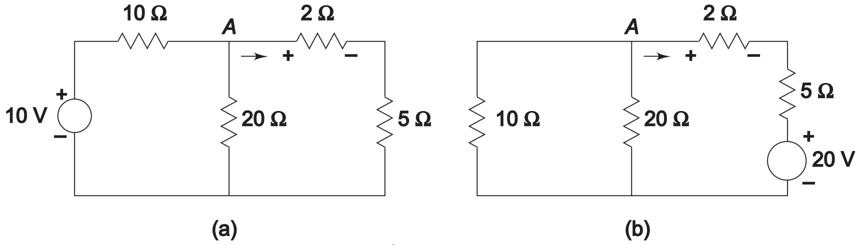


Fig. 3.12

Assuming a voltage V at node 'A' as shown in Fig. 3.12 (a), the current equation is

$$\frac{V-10}{10}+\frac{V}{20}+\frac{V}{7}=0$$

$$V\,[0.1+0.05+0.143]=1$$

or $$V=3.41\text{ V}$$

The voltage across the 2 Ω resistor due to the 10 V source is

$$V_2=\frac{V}{7}\times 2=0.97\text{ V}$$

Our second step is to find out the voltage across the 2 Ω resistor due to the 20 V source, while the other sources are set equal to zero. The circuit is redrawn as shown in Fig. 3.12 (b).

Assuming voltage V at node A as shown in Fig. 3.12 (b), the current equation is

$$\frac{V-20}{7}+\frac{V}{20}+\frac{V}{10}=0$$

$$V\,[0.143+0.05+0.1]=2.86$$

or $$V=\frac{2.86}{0.293}=9.76\text{ V}$$

The voltage across the 2 Ω resistor due to the 20 V source is

$$V_2=\left(\frac{V-20}{7}\right)\times 2=-\,2.92\text{ V}$$

The last step is to find the voltage across the 2 Ω resistor due to the 2 A current source, while the other sources are set equal to zero. The circuit is redrawn as shown in Fig. 3.12 (c).

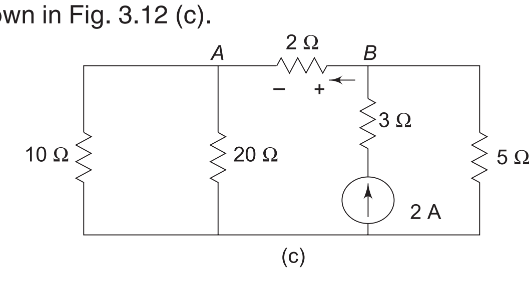


Fig. 3.12

The current in the 2 Ω resistor $= 2 \times \dfrac{5}{5 + 8.67}$

$$= \frac{10}{13.67} = 0.73 \text{ A}$$

The voltage across the 2 Ω resistor = 0.73 × 2 = 1.46 V

The algebraic sum of these voltages gives the total voltage across the 2 Ω resistor in the network

$$V = 0.97 - 2.92 - 1.46 = -3.41 \text{ V}$$

The negative sign of the voltage indicates that the voltage at 'A' is negative.

3.3 THEVENIN'S THEOREM

In many practical applications, it is always not necessary to analyse the complete circuit; it requires that the voltage, current, or power in only one resistance of a circuit be found. The use of this theorem provides a simple, equivalent circuit which can be substituted for the original network. Thevenin's theorem states that any two terminal linear network having a number of voltage current sources and resistances can be replaced by a simple equivalent circuit consisting of a single voltage source in series with a resistance, where the value of the voltage source is equal to the open circuit voltage across the two terminals of the network, and resistance is equal to the equivalent resistance measured between the terminals with all the energy sources are replaced by their internal resistances. According to Thevenin's theorem, an equivalent circuit can be found to replace the circuit in Fig. 3.13.

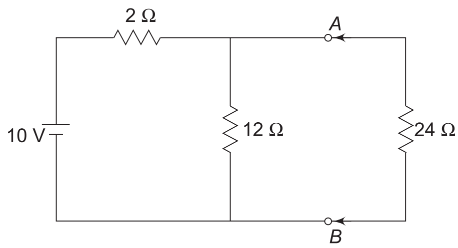


Fig. 3.13

In the circuit, if the load resistance 24 Ω is connected to Thevenin's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experienced in the original circuit. To verify this, let us find the current passing through the 24 Ω resistance due to the original circuit.

$$I_{24} = I_T \times \frac{12}{12 + 24}$$

where $$I_T = \frac{10}{2 + (12 \,||\, 24)} = \frac{10}{10} = 1 \text{ A}$$

∴ $$I_{24} = 1 \times \frac{12}{12 + 24} = 0.33 \text{ A}$$

The voltage across the 24 Ω resistor = 0.33 × 24 = 7.92 V. Now let us find Thevenin's equivalent circuit.

The Thevenin voltage is equal to the open circuit voltage across the terminals 'AB', i.e. the voltage across the 12 Ω resistor. When the load resistance is disconnected from the circuit, the Thevenin voltage

$$V_{Th} = 10 \times \frac{12}{14} = 8.57 \text{ V}$$

The resistance into the open circuit terminals is equal to the Thevenin resistance

$$R_{Th} = \frac{12 \times 2}{14} = 1.71 \ \Omega$$

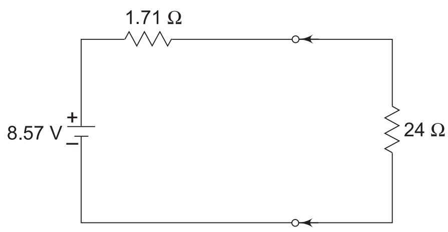


Fig. 3.14

Thevenin's equivalent circuit is shown in Fig. 3.14.

Now let us find the current passing through the 24 Ω resistance and voltage across it due to Thevenin's equivalent circuit.

$$I_{24} = \frac{8.57}{24 + 1.71} = 0.33 \text{ A}$$

The voltage across the 24 Ω resistance is equal to 7.92 V. Thus, it is proved that R_L (= 24 Ω) has the same values of current and voltage in both the original circuit and Thevenin's equivalent circuit.

Example 3.4 Determine the Thevenin's equivalent circuit across '*AB*' for the given circuit shown in Fig. 3.15.

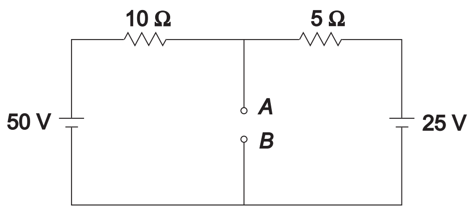


Fig. 3.15

Solution The complete circuit can be replaced by a voltage source in series with a resistance as shown in Fig. 3.16 (a)

where V_{Th} is the voltage across terminals *AB* and

R_{Th} is the resistance seen into the terminals *AB*.

To solve for V_{Th}, we have to find the voltage drops around the closed path as shown in Fig. 3.16 (b).

We have $50 - 25 = 10I + 5I$

or $15I = 25$

$\therefore$ $I = \dfrac{25}{15} = 1.67 \text{ A}$

Voltage across 10 Ω = 16.7 V

Voltage drop across 5 Ω = 8.35 V

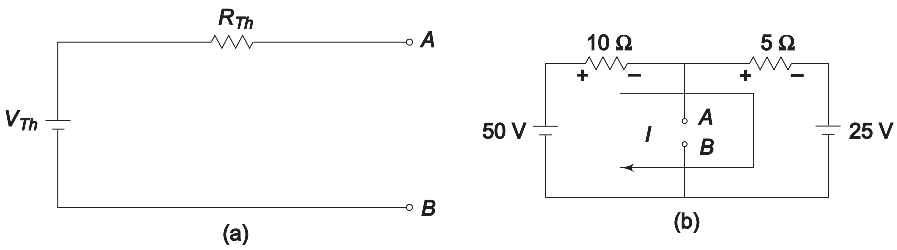


Fig. 3.16

or $$V_{\text{Th}} = V_{AB} = 50 - V_{10}$$
$$= 50 - 16.7 = 33.3 \text{ V}$$

To find R_{Th}, the two voltage sources are removed and replaced with short circuit. The resistance at terminals AB then is the parallel combination of the 10 Ω resistor and 5 Ω resistor; or

$$R_{\text{Th}} = \frac{10 \times 5}{15} = 3.33\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.16 (c).

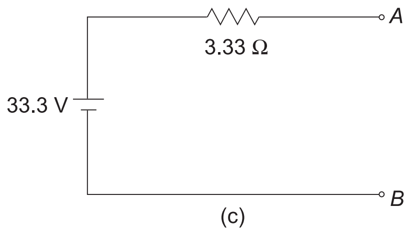


Fig. 3.16

3.4 NORTON'S THEOREM

Another method of analysing the circuit is given by *Norton's theorem*, which states that any two terminal linear network with current sources, voltage sources and resistances can be replaced by an equivalent circuit consisting of a current source in parallel with a resistance. The value of the current source is the short circuit current between the two terminals of the network and the resistance is the equivalent resistance measured between the terminals of the network with all the energy sources are replaced by their internal resistance.

According to Norton's theorem, an equivalent circuit can be found to replace the circuit in Fig. 3.17.

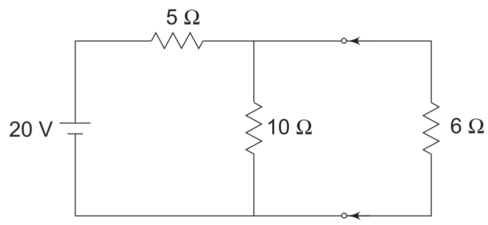


Fig. 3.17

In the circuit if the load resistance 6 Ω is connected to Norton's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experiences in the original circuit. To verify this, let us find the current passing through the 6 Ω resistor due to the original circuit.

$$I_6 = I_T \times \frac{10}{10+6}$$

where $$I_T = \frac{20}{5+(10\,||\,6)} = 2.285 \text{ A}$$

$\therefore$ $$I_6 = 2.285 \times \frac{10}{16} = 1.43 \text{ A}$$

i.e. the voltage across the 6 Ω resistor is 8.58 V. Now let us find Norton's equivalent circuit. The magnitude of the current in the Norton's equivalent circuit is equal to the current passing through short circuited terminals as shown in Fig. 3.18.

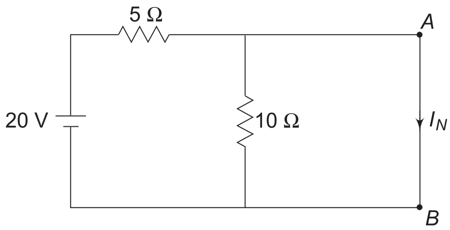


Fig. 3.18

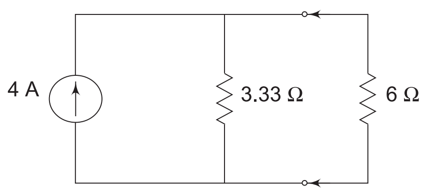


Fig. 3.19

Here $$I_N = \frac{20}{5} = 4 \text{ A}$$

Norton's resistance is equal to the parallel combination of both the 5 Ω and 10 Ω resistors

$$R_N = \frac{5 \times 10}{15} = 3.33\ \Omega$$

The Norton's equivalent source is shown in Fig. 3.19.

Now let us find the current passing through the 6 Ω resistor and the voltage across it due to Norton's equivalent circuit.

$$I_6 = 4 \times \frac{3.33}{6+3.33} = 1.43 \text{ A}$$

The voltage across the 6 Ω resistor = 1.43 × 6 = 8.58 V

Thus, it is proved that R_L (= 6 Ω) has the same values of current and voltage in both the original circuit and Norton's equivalent circuit.

Example 3.5 Determine Norton's equivalent circuit at terminals *AB* for the circuit shown in Fig. 3.20.

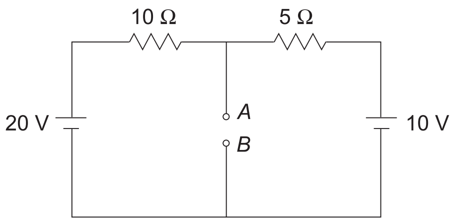


Fig. 3.20

Solution The complete circuit can be replaced by a current source in parallel with a single resistor as shown in Fig. 3.21 (a), where I_N is the current passing through the short circuited output terminals AB and R_N is the resistance as seen into the output terminals.

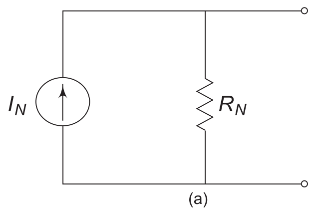


Fig. 3.21

To solve for I_N, we have to find the current passing through the terminals AB as shown in Fig. 3.21 (b).

From Fig. 3.21 (b), the current passing through the terminals AB is 4 A. The resistance at terminals AB is the parallel combination of the 10 Ω resistor and the 5 Ω resistor,

or $$R_N = \frac{10 \times 5}{10 + 5} = 3.33\ \Omega$$

Norton's equivalent circuit is shown in Fig. 3.21 (c).

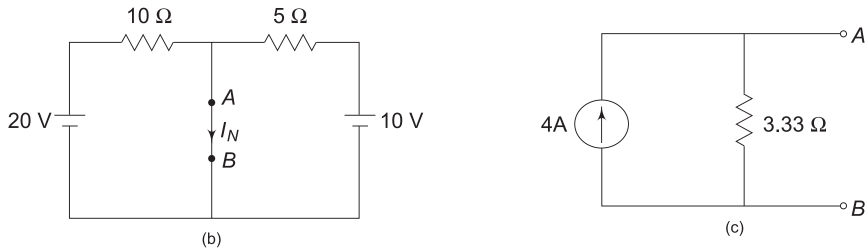


Fig. 3.21

3.5 RECIPROCITY THEOREM

In any linear bilateral network, if a single voltage source V_a in branch 'a' produces a current I_b in branch 'b', then if the voltage source V_a is removed and inserted in branch 'b' will produce a current I_b in branch 'a'. The ratio of response to excitation is same for the two conditions mentioned above. This is called the *reciprocity theorem*.

Consider the network shown in Fig. 3.22. AA' denotes input terminals and BB' denotes output terminals.

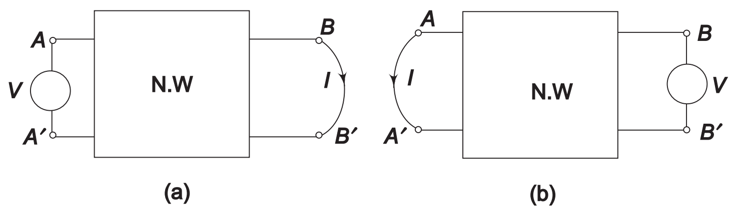


Fig. 3.22

The application of voltage V across AA' produces current I at BB'. Now if the positions of the source and responses are interchanged, by connecting the voltage source across BB', the resultant current I will be at terminals AA'. According to the reciprocity theorem, the ratio of response to excitation is the same in both cases.

Example 3.6 Verify the reciprocity theorem for the network shown in Fig. 3.23.

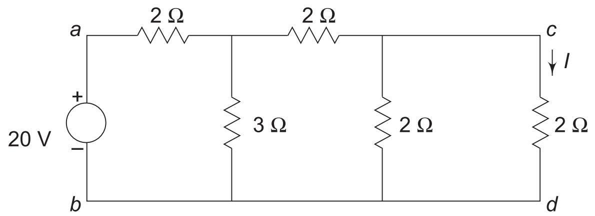
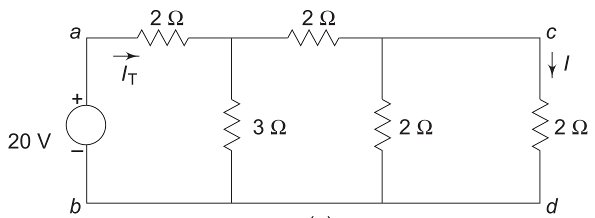


Fig. 3.23

Solution Total resistance in the circuit = 2 + [3 || (2 + 2 || 2)] = 3.5 Ω.
The current drawn by the circuit (See Fig. 3.24 (a))

$$I_T = \frac{20}{3.5} = 5.71\ \Omega$$

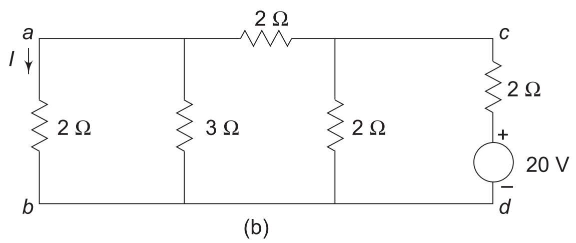


(a)

Fig. 3.24

The current in the 2 Ω branch *cd* is I = 1.43 A.

Applying the reciprocity theorem, by interchanging the source and response we get (See Fig. 3.24 (b)).



(b)

Fig. 3.24

Total resistance in the circuit = 3.23 Ω.

Total current drawn by the circuit = $\frac{20}{3.23}$ = 6.19 A

The current in the branch *ab* is I = 1.43 A

If we compare the results in both cases, the ratio of input to response is the same, i.e. (20/1.43) = 13.99.

3.6 COMPENSATION THEOREM

The *compensation theorem* states that any element in the linear, bilateral network, may be replaced by a voltage source of magnitude equal to the current passing through the element multiplied by the value of the element, provided the currents and voltages in other parts of the circuit remain unaltered. Consider the circuit shown in Fig. 3.25 (a). The element R can be replaced by voltage source V, which is equal to the current I passing through R multiplied by R as shown in Fig. 3.25 (b).

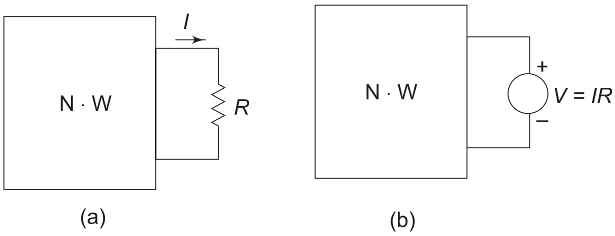


Fig. 3.25

This theorem is useful in finding the changes in current or voltage when the value of resistance is changed in the circuit. Consider the network containing a resistance R shown in Fig. 3.26 (a). A small change in resistance R, that is $(R + \Delta R)$, as shown in Fig. 3.26 (b) causes a change in current in all branches. This current increment in other branches is equal to the current produced by the voltage source of voltage $I.\ \Delta R$ which is placed in series with altered resistance as shown in Fig. 3.26 (c).

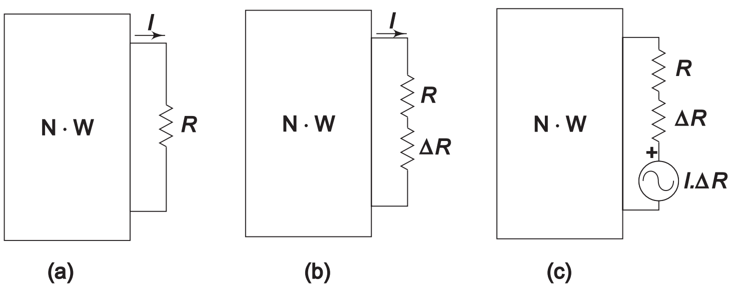


Fig. 3.26

Example 3.7 Determine the current flowing in the ammeter having 1 Ω internal resistance connected in series with a 3 Ω resistor as shown in Fig. 3.27.

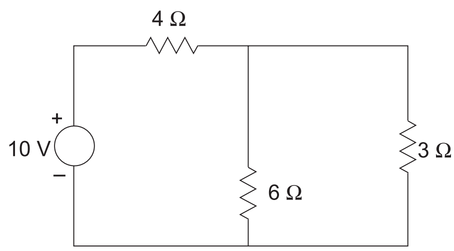


Fig. 3.27

Solution The current flowing through the 3 Ω branch is I_3 = 1.11 A. If we connect the ammeter having 1 Ω resistance to the 3 Ω branch, there is a change in resistance. The changes in currents in other branches then result as if a voltage source of voltage $I_3 \Delta R$ = 1.11 × 1 = 1.11 V is inserted in the 3 Ω branch as shown in Fig. 3.28.

Current due to this 1.11 V source is calculated as follows.

Current I'_3 = 0.17 A

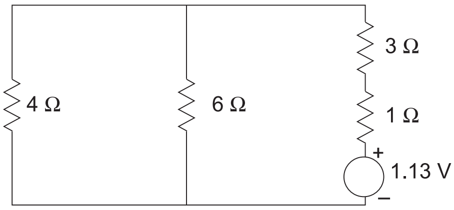


Fig. 3.28

This current is opposite to the current I_3 in the 3 Ω branch.

Hence the ammeter reading = (1.11 – 0.17) = 0.94 A.

3.7 MAXIMUM POWER TRANSFER THEOREM

Many circuits basically consist of sources, supplying voltage, current, or power to the load; for example, a radio speaker system, or a microphone supplying the input signals to voltage pre-amplifiers. Sometimes it is necessary to transfer maximum voltage, current or power from the source to the load. In the simple resistive circuit shown in Fig. 3.29, R_s is the source resistance. Our aim is to find the necessary conditions so that the power delivered by the source to the load is maximum.

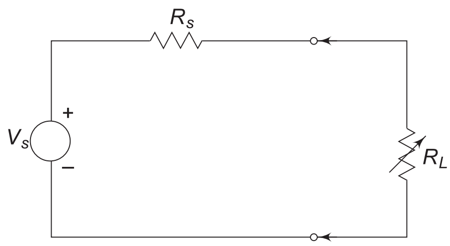


Fig. 3.29

It is a fact that more voltage is delivered to the load when the load resistance is high as compared to the resistance of the source. On the other hand, maximum current is transferred to the load when the load resistance is small compared to the source resistance.

For many applications, an important consideration is the maximum power transfer to the load; for example, maximum power transfer is desirable from the output amplifier to the speaker of an audio sound system. The maximum Power Transfer Theorem states that maximum power is delivered from a source to a load when the load resistance is equal to the source resistance. In Fig. 3.29, assume that the load resistance is variable.

Current in the circuit is $I = V_S/(R_S + R_L)$

Power delivered to the load R_L is $P = I^2R_L = V_S^2 R_L/(R_S + R_L)^2$

To determine the value of R_L for maximum power to be transferred to the load, we have to set the first derivative of the above equation with respect to R_L, i.e. when $\dfrac{dP}{dR_L}$ equals zero.

$$\frac{dP}{dR_L} = \frac{d}{dR_L}\left[\frac{V_S^2}{(R_S + R_L)^2} R_L\right]$$

$$= \frac{V_S^2 \{(R_S + R_L)^2 - (2R_L)(R_S + R_L)\}}{(R_S + R_L)^4}$$

$$\therefore \quad (R_S + R_L)^2 - 2R_L(R_S + R_L) = 0$$

$$R_S^2 + R_L^2 + 2R_S R_L - 2R_L^2 - 2R_S R_L = 0$$

$$\therefore \quad R_S = R_L$$

So, maximum power will be transferred to the load when load resistance is equal to the source resistance.

Example 3.8 In the circuit shown in Fig. 3.30 determine the value of load resistance when the load resistance draws maximum power. Also find the value of the maximum power.

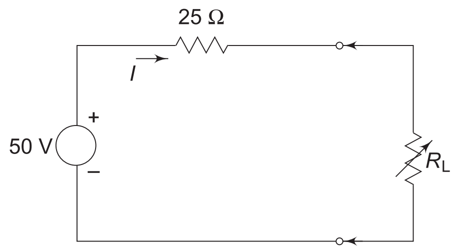


Fig. 3.30

Solution In Fig. 3.30, the source delivers the maximum power when load resistance is equal to the source resistance.

$R_L = 25\ \Omega$

The current $I = 50/(25 + R_L) = 50/50 = 1$ A

The maximum power delivered to the load $P = I^2 R_L$

$= 1 \times 25 = 25$ W

3.8 DUALS AND DUALITY

In an electrical circuit itself there are pairs of terms which can be interchanged to get new circuits. Such pair of dual terms are given below.

Current — Voltage
Open — Short
L — C
R — G
Series — Parallel
Voltage source — Current source
KCL — KVL

Consider a network containing R—L—C elements connected in series, and excited by a voltage source as shown in Fig. 3.31.

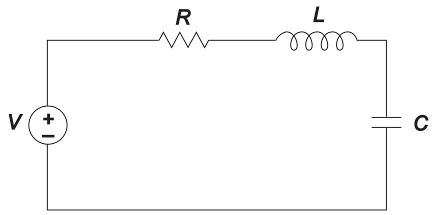


Fig. 3.31

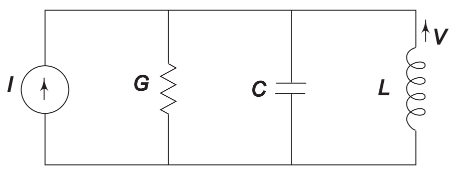


Fig. 3.32

The integrodifferential equation for the above network is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int idt = \text{V}$$

Similarly, consider a network containing R—L—C elements connected in parallel and driven by a current source as shown in Fig. 3.32.

The integrodifferential equation for the network in Fig. 3.32 is

$$i = Gv + C\frac{dv}{dt} + \frac{1}{L}\int vdt$$

If we observe both the equations, the solutions of these two equations are the same. These two networks are called *duals*.

To draw the dual of any network, the following steps are to be followed.

1. In each loop of a network place a node; and place an extra node, called the *reference node*, outside the network.
2. Draw the lines connecting adjacent nodes passing through each element, and also to the reference node, by placing the dual of each element in the line passing through original elements.

For example, consider the network shown in Fig. 3.33.

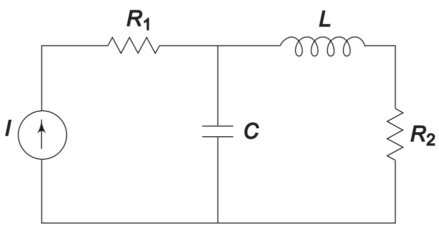


Fig. 3.33

Our first step is to place the nodes in each loop and a reference node outside the network.

Drawing the lines connecting the nodes passing through each element, and placing the dual of each element as shown in Fig. 3.34 (a) we get a new circuit as shown in Fig. 3.34 (b).

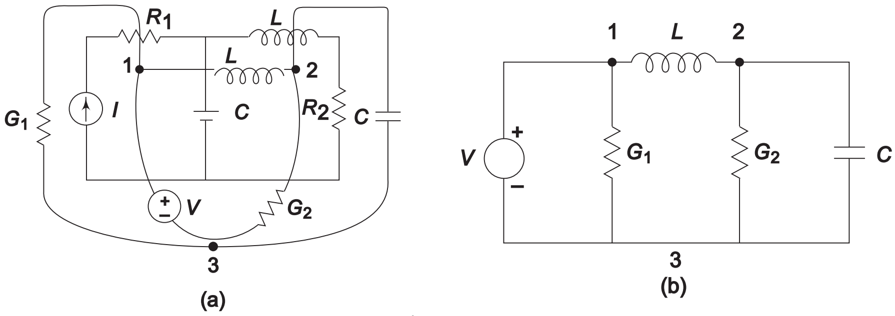


Fig. 3.34

Example 9.9 Draw the dual network for the given network shown in Fig. 3.35.

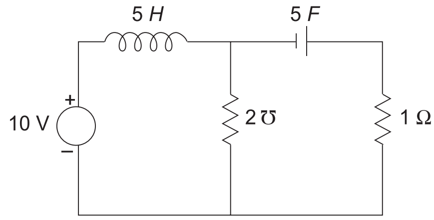


Fig. 3.35

Solution Place nodes in each loop and one reference node outside the circuit. Joining the nodes through each element, and placing the dual of each element in the line, we get the dual circuit as shown in Fig. 3.36 (a).

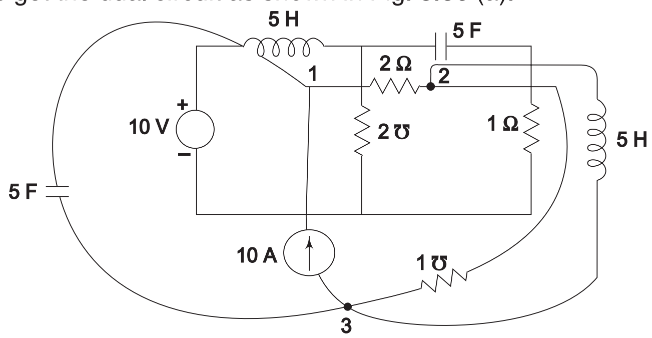


Fig. 3.36 (a)

The dual circuit is redrawn as shown in Fig. 3.36 (b)

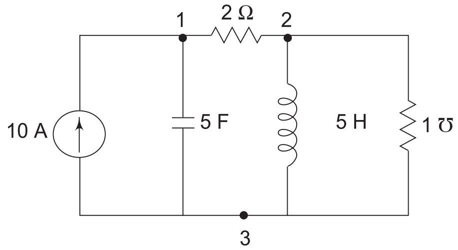


Fig. 3.36 (b)

3.9 TELLEGEN'S THEOREM

Tellegen's theorem is valid for any lumped network which may be linear or non-linear, passive or active, time-varying or time-invarient. This theorem states that in an arbitrary lumped network, the algebraic sum of the powers in all branches at any instant is zero. All branch currents and voltages in that network must satisfy Kirchhoff's laws. Otherwise, in a given network, the algebraic sum of the powers delivered by all sources is equal to the algebraic sum of the powers absorbed by all elements. This theorem is based on Kirchhoff's two laws, but not on the type of circuit elements.

Consider two networks N_1 and N_2, having the same graph with different types of elements between the corresponding nodes.

Then $$\sum_{K=1}^{b} v_{1K}\, i_{2K} = 0$$

and $$\sum_{K=1}^{b} v_{2K}\, i_{1K} = 0$$

To verify Tellegen's theorem, consider two circuits having same graphs as shown in Fig. 3.37.

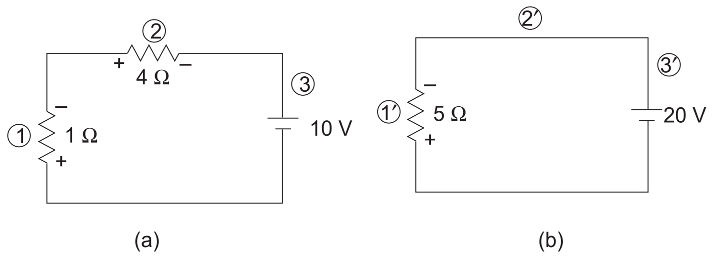


Fig. 3.37

In Fig. 3.37 (a)

$$i_1 = i_2 = 2\text{ A};\ i_3 = 2\text{ A}$$

and $$v_1 = -2\text{ V},\ v_2 = -8\text{ V},\ v_3 = 10\text{ V}$$

In Fig. 3.37 (b)

$$i_1^1 = i_2^1 = 4\text{ A};\ i_3^1 = 4\text{ A}$$

and $$v_1^1 = -20\text{ V};\ v_2^1 = 0\text{ V};\ v_3^1 = 20\text{ V}$$

Now $$\sum_{K=1}^{3} v_K\, i_K^1 = v_1\, i_1^1 + v_2\, i_2^1 + v_3\, i_3^1$$

$$= (-2)\,(4) + (-8)\,(4) + (10)\,(4) = 0$$

and $$\sum_{K=1}^{3} v_K^1\, i_K = v_1^1\, i_1 + v_2^1\, i_2 + v_3^1\, i_3$$

$$= (-20)\,(2) + (0)\,(2) + (20)\,(2) = 0$$

Similarly,

$$\sum_{K=1}^{3} v_K i_K = v_1 i_1 + v_2 i_2 + v_3 i_3$$

$$= (-2)(2) + (-8)(2) + (10)(2) = 0$$

and $$\sum_{K=1}^{3} v_K^1 i_K^1 = (-20)(4) + (0)(4) + (20)(4) = 0$$

This verifies Tellegen's theorem.

3.10 MILLMAN'S THEOREM

Millman's Theorem states that in any network, if the voltage sources $V_1, V_2, \cdots V_n$ in series with internal resistances $R_1, R_2, \cdots R_n$, respectively, are in parallel, then these sources may be replaced by a single voltage source V' in series with R' as shown in Fig. 3.38.

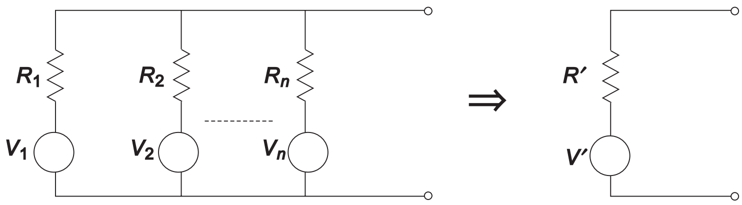


Fig. 3.38

where $$V = \frac{V_1G_1 + V_2G_2 + \cdots V_nG_n}{G_1 + G_2 + \cdots + G_n}$$

Here G_n is the conductance of the nth branch,

and $$R' = \frac{1}{G_1 + G_2 + \cdots + G_n}$$

A similar theorem can be stated for n current sources having internal conductances which can be replaced by a single current source I' in parallel with an equivalent conductance.

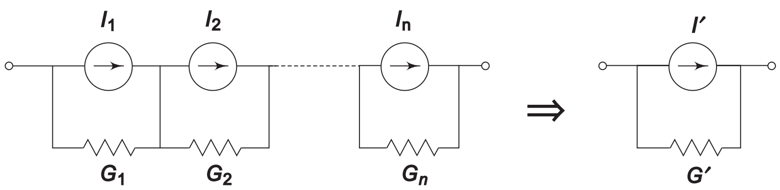


Fig. 3.39

where $$I' = \frac{I_1R_1 + I_2R_2 + \cdots I_nR_n}{R_1 + R_2 + \cdots + R_n}$$

and $$G' = \frac{1}{R_1 + R_2 + \cdots + R_n}$$

Example 3.10 Calculate the current I shown in Fig. 3.40 using Millman's Theorem.

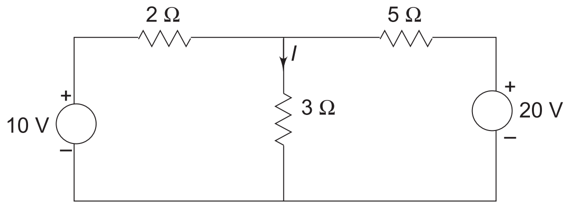


Fig. 3.40

Solution According to Millman's Theorem, the two voltage sources can be replaced by a single voltage source in series with resistance as shown in Fig. 3.41.

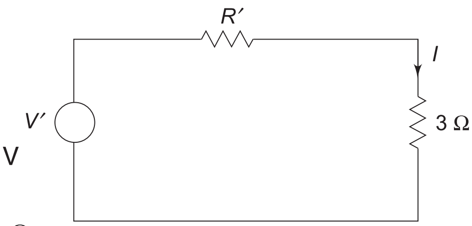


Fig. 3.41

We have $$V' = \frac{V_1G_1 + V_2G_2}{G_1 + G_2}$$

$$= \frac{[10(1/2) + 20(1/5)]}{1/2 + 1/5} = 12.86 \text{ V}$$

and $$R' = \frac{1}{G_1 + G_2} = \frac{1}{1/2 + 1/5} = 1.43\ \Omega$$

Therefore, the current passing through the 3 Ω resistor is

$$I = \frac{12.86}{3 + 1.43} = 2.9 \text{ A}$$

Additional Solved Problems

Problem 3.1 Determine the current drawn by the circuit shown in Fig. 3.42.

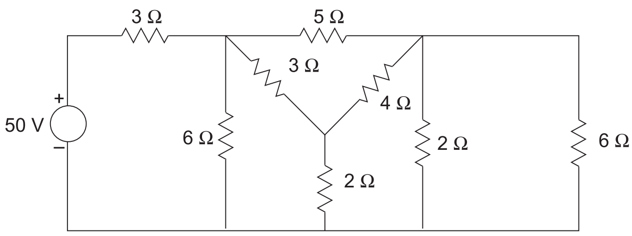
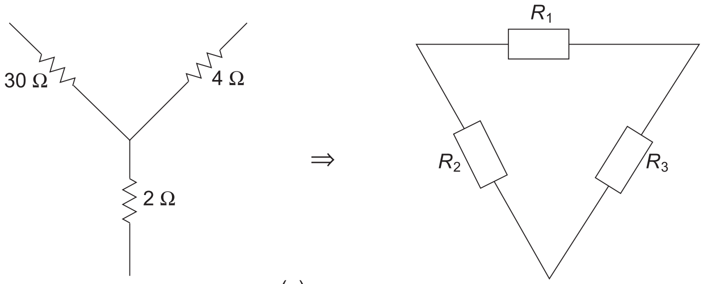


Fig. 3.42

Solution To simplify the network, the star circuit in Fig. 3.42 is converted into a delta circuit as shown under.



(a)

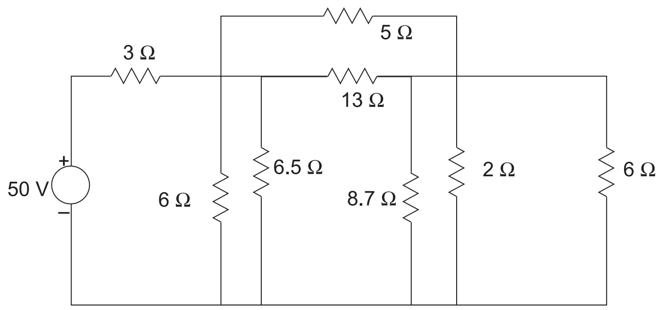
Fig. 3.43

$$R_1 = \frac{4 \times 3 + 4 \times 2 + 3 \times 2}{2} = 13\ \Omega$$

$$R_2 = \frac{4 \times 3 + 4 \times 2 + 3 \times 2}{4} = 6.5\ \Omega$$

$$R_3 = \frac{4 \times 3 + 4 \times 2 + 3 \times 2}{3} = 8.7\ \Omega$$

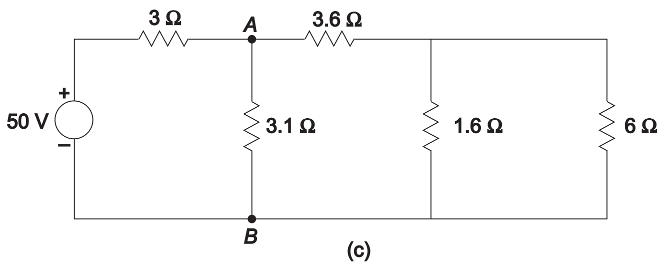
The original circuit is redrawn as shown in Fig. 3.43 (b).



(b)

Fig. 3.43

It is further simplified as shown in Fig. 3.43 (c). Here the resistors 5 Ω and 13 Ω are in parallel, 6 Ω and 6.5 Ω are in parallel, and 8.7 Ω and 2 Ω are in parallel.



(c)

Fig. 3.43

In the above circuit the resistors 6 Ω and 1.6 Ω are in parallel, the resultant of which is in series with 3.6 Ω resistor and is equal to $\left[3.6 + \dfrac{6 \times 1.6}{7.6}\right] = 4.9\ \Omega$ as shown in Fig. 3.43 (d).

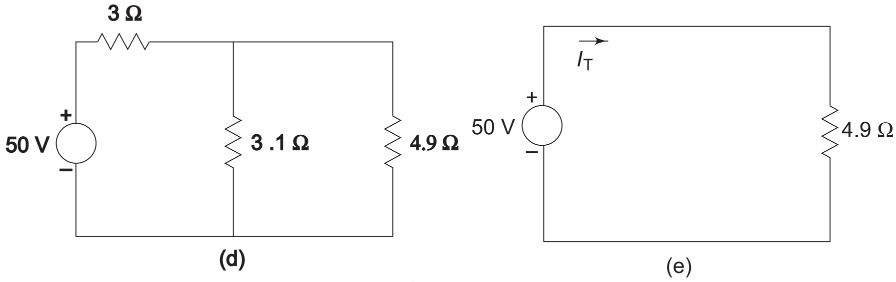


Fig. 3.43

In the above circuit 4.9 Ω and 3.1 Ω resistors are in parallel, the resultant of which is in series with 3 Ω resistor.

Therefore, the total resistance $R_T = 3 + \dfrac{3.1 \times 4.9}{8} = 4.9\ \Omega$

The current drawn by the circuit $I_T = 50/4.9 = 10.2$ A (See Fig. 3.43 (e)).

Problem 3.2 In Fig. 3.44 determine the equivalent resistance by using star-delta transformation.

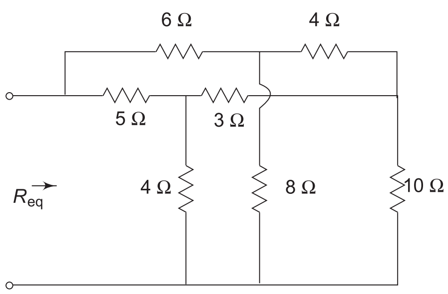


Fig. 3.44

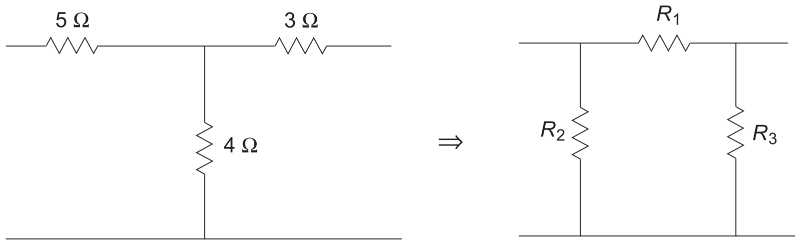
Solution In Fig. 3.44, we have two star circuits, one consisting of 5 Ω, 3 Ω and 4 Ω resistors, and the other consisting of 6 Ω, 4 Ω and 8 Ω resistors. We convert the star circuits into delta circuits, so that the two delta circuits are in parallel.

In Fig. 3.45 (a)

$$R_1 = \frac{5 \times 3 + 4 \times 3 + 5 \times 4}{4} = 11.75\ \Omega$$

$$R_2 = \frac{5 \times 3 + 4 \times 3 + 5 \times 4}{3} = 15.67\ \Omega$$

$$R_3 = \frac{5 \times 3 + 4 \times 3 + 5 \times 4}{5} = 9.4\ \Omega$$



(a)

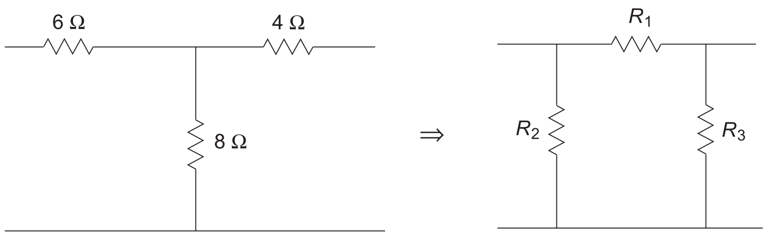
Fig. 3.45

Similarly, in Fig. 3.45 (b)

$$R_1 = \frac{6 \times 4 + 4 \times 8 + 8 \times 6}{8} = 13\ \Omega$$

$$R_2 = \frac{6 \times 4 + 4 \times 8 + 8 \times 6}{4} = 26\ \Omega$$

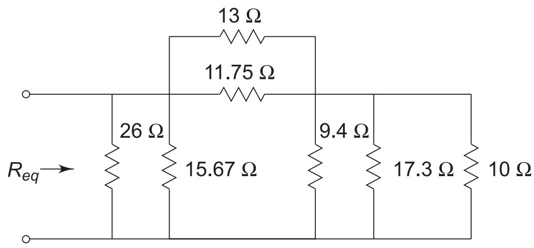
$$R_3 = \frac{6 \times 4 + 4 \times 8 + 8 \times 6}{6} = 17.3\ \Omega$$



(b)

Fig. 3.45

The simplified circuit is shown in Fig. 3.45 (c)



(c)

Fig. 3.45

In the above circuit, the three resistors 10 Ω, 9.4 Ω and 17.3 Ω are in parallel.

Equivalent resistance = (10 || 9.4 || 17.3) = 3.78 Ω

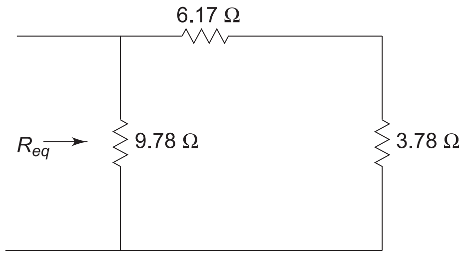
Resistors 13 Ω and 11.75 Ω are in parallel

Equivalent resistance = (13 || 11.75) = 6.17 Ω

Resistors 26 Ω and 15.67 Ω are in parallel

Equivalent resistance = (26 || 15.67) = 9.78 Ω

The simplified circuit is shown in Fig. 3.45 (d)



(d)

Fig. 3.45

From the above circuit, the equivalent resistance is given by

$$R_{\text{eq}} = (9.78) \parallel (6.17 + 3.78)$$
$$= (9.87) \parallel (9.95) = 4.93\ \Omega$$

Problem 3.3 For the resistive network shown in Fig. 3.46, find the current in each resistor, using the superposition principle.

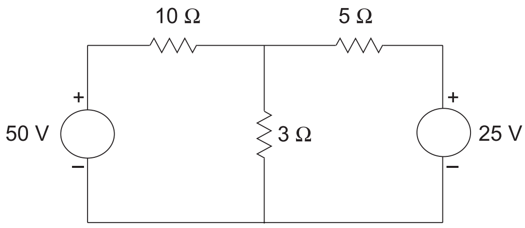


Fig. 3.46

Solution The current due to the 50 V source can be found in the circuit shown in Fig. 3.47 (a).

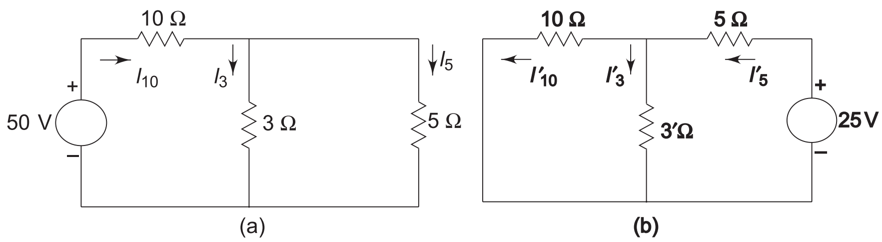


Fig. 3.47

$$\text{Total resistance } R_T = 10 + \frac{5 \times 3}{8} = 11.9\ \Omega$$

$$\text{Current in the 10 } \Omega \text{ resistor } I_{10} = \frac{50}{11.9} = 4.2\text{ A}$$

Current in the 3 Ω resistor $I_3 = 4.2 \times \dfrac{5}{8} = 2.63$ A

Current in the 5 Ω resistor $I_5 = 4.2 \times \dfrac{3}{8} = 1.58$ A

The current due to the 25 V source can be found from the circuit shown in Fig. 3.47 (b).

Total resistance $R_T = 5 + \dfrac{10 \times 3}{13} = 7.31\ \Omega$

Current in the 5 Ω resistor $I'_5 = \dfrac{25}{7.31} = 3.42$ A

Current in the 3 Ω resistor $I'_3 = 3.42 \times \dfrac{10}{13} = 2.63$ A

Current in the 10 Ω resistor $I'_{10} = 3.42 \times \dfrac{3}{13} = 0.79$ A

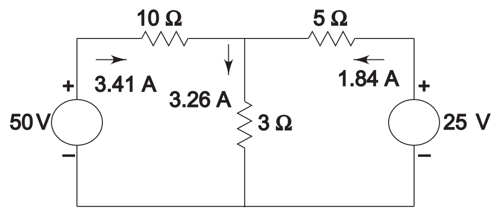
According to superposition principle

Current in the 10 Ω resistor $= I_{10} - I'_{10} = 4.2 - 0.79 = 3.41$ A

Current in the 3 Ω resistor $= I_3 + I'_3 = 2.63 + 2.63 = 5.26$ A

Current in the 5 Ω resistor $= I'_5 - I_5 = 3.42 - 1.58 = 1.84$ A

When both sources are operative, the directions of the currents are shown in Fig. 3.47 (c).



(c)

Fig. 3.47

Problem 3.4 Determine the voltage across the terminals AB in the circuit shown in Fig. 3.48.

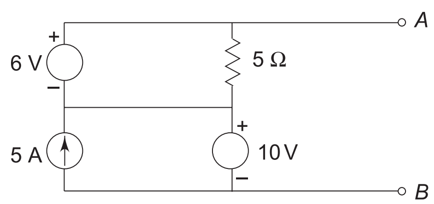


Fig. 3.48

Solution Voltage across AB is $V_{AB} = V_{10} + V_5$.

To find the voltage across the 5 Ω resistor, we have to use the superposition theorem.

Voltage across the 5 Ω resistor V_5 due to the 6 V source, when other sources are set equal to zero, is calculated using Fig. 3.49 (a).

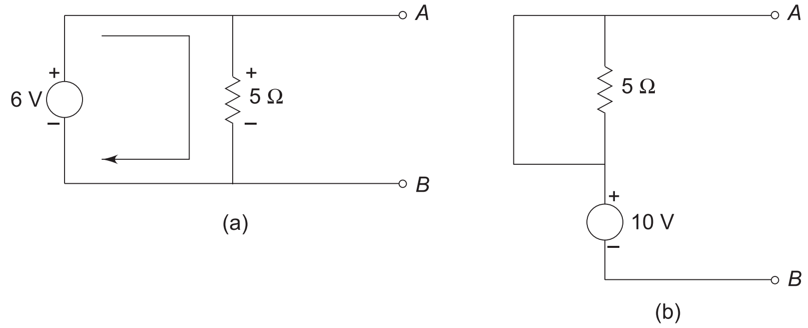


Fig. 3.49

$$V_5 = 6 \text{ V}$$

Voltage across the 5 Ω resistor V_5' due to the 10 V sources, when other sources are set equal to zero, is calculated using Fig. 3.49 (b).

$$V_5' = 0$$

Voltage across the 5 Ω resistor V_5'' due to the 5 A source only, is calculated using Fig. 3.49 (c).

$$V_5'' = 0$$

According to the superposition theorem,

Total voltage across the 5 Ω resistor

$$= 6 + 0 + 0 = 6 \text{ V}.$$

So the voltage across terminals *AB* is

$$V_{AB} = 10 + 6 = 16 \text{ V}$$

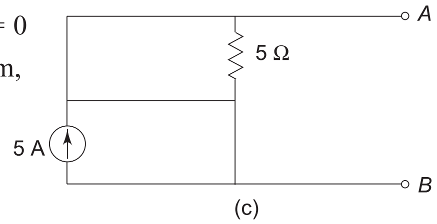


Fig. 3.49

Problem 3.5 Use Thevenin's theorem to find the current in 3 Ω resistor in Fig. 3.50.

Solution Current in the 3 Ω resistor can be found by using Thevenin's theorem.

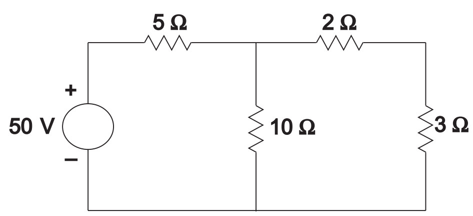


Fig. 3.50

In circuit shown in Fig. 3.51 (a) can be replaced by a single voltage source in series with a resistor as shown in Fig. 3.51 (b).

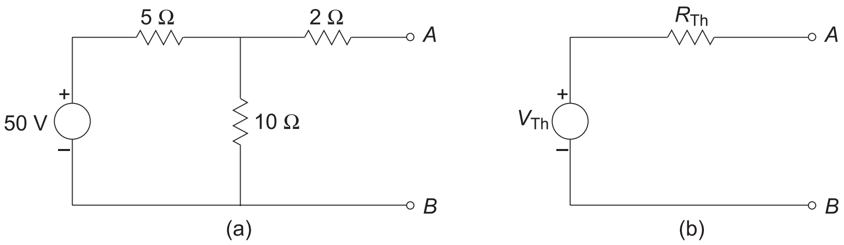


Fig. 3.51

$$V_{\text{Th}} = V_{AB} = \frac{50}{15} \times 10 = 33.3 \text{ V}$$

$R_{\text{Th}} = R_{AB}$, the resistance seen into the terminals AB

$$R_{AB} = 2 + \frac{5 \times 10}{15} = 5.33\ \Omega$$

The 3 Ω resistor is connected to the Thevenin equivalent circuit as shown in Fig. 3.51 (c).

Current passing through the 3 Ω resistor

$$I_3 = \frac{33.3}{5.33 + 3} = 4.00 \text{ A}$$

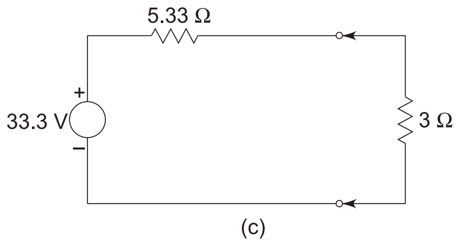


Fig. 3.51

Problem 3.6 Use Thevenin's theorem to find the current through the 5 Ω resistor in Fig. 3.52.

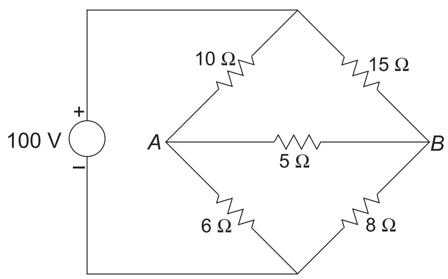


Fig. 3.52

Solution Thevenin's equivalent circuit can be formed by obtaining the voltage across terminals AB as shown in Fig. 3.53 (a).

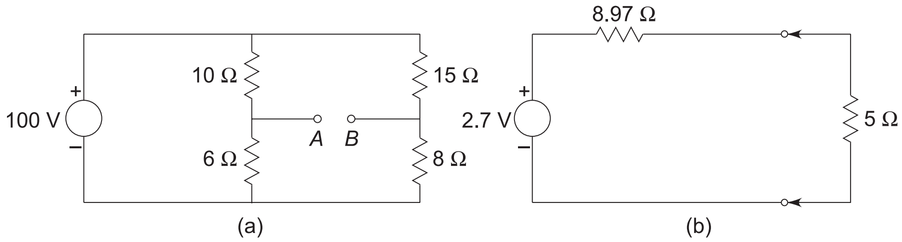


Fig. 3.53

Current in the 6 Ω resistor $I_6 = \dfrac{100}{16} = 6.25$ A

Voltage across the 6 Ω resistor $V_6 = 6 \times 6.25 = 37.5$ V

Current in the 8 Ω resistor $I_8 = \dfrac{100}{23} = 4.35$ A

Voltage across the 8 Ω resistor is $V_8 = 4.35 \times 8 = 34.8$ V

Voltage across the terminals AB is $V_{AB} = 37.5 - 34.8 = 2.7$ V

The resistance as seen into the terminals R_{AB}

$$= \frac{6 \times 10}{6 + 10} + \frac{8 \times 15}{8 + 15}$$

$$= 3.75 + 5.22 = 8.97\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.53 (b).

Current in the 5 Ω resistor $I_5 = \dfrac{2.7}{5 + 8.97} = 0.193$ A

Problem 3.7 Find Thevenin's equivalent circuit for the circuit shown in Fig. 3.54.

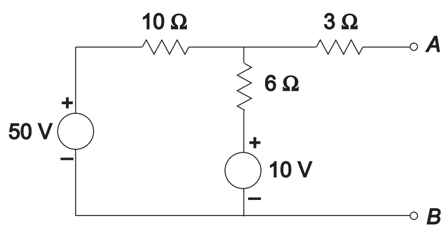


Fig. 3.54

Solution Thevenin's voltage is equal to the voltage across the terminals AB.

$\therefore$ $$V_{AB} = V_3 + V_6 + 10$$

Here the current passing through the 3 Ω resistor is zero.

Hence $V_3 = 0$

By applying Kirchhoff's law we have

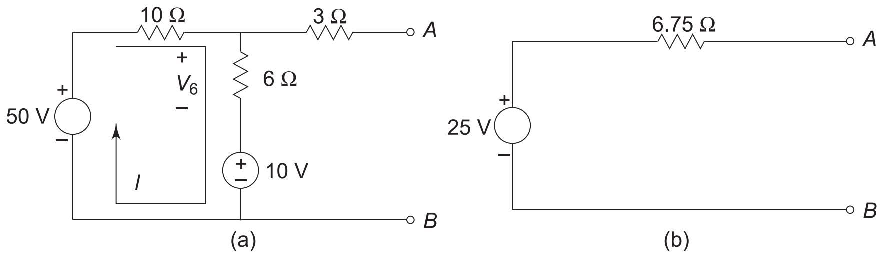


Fig. 3.55

$$50 - 10 = 10I + 6I$$

$$I = \frac{40}{16} = 2.5 \text{ A}$$

The voltage across 6 Ω is V_6 with polarity as shown in Fig. 3.55 (a), and is given by

$$V_6 = 6 \times 2.5 = 15 \text{ V}$$

The voltage across terminals *AB* is $V_{AB} = 0 + 15 + 10 = 25$ V.

The resistance as seen into the terminals *AB*

$$R_{AB} = 3 + \frac{10 \times 6}{10 + 6} = 6.75\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.55 (b).

Problem 3.8 Determine the Thevenin's equivalent circuit across terminals *AB* for the circuit in Fig. 3.56.

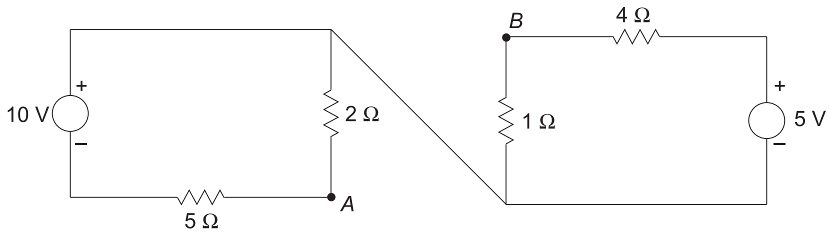


Fig. 3.56

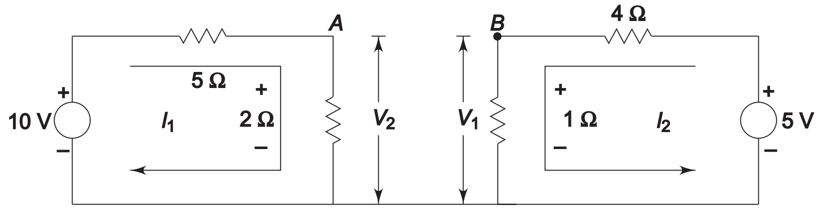
Solution The given circuit is redrawn as shown in Fig. 3.57 (a).

Voltage $V_{AB} = V_2 + V_1$

Applying Kirchhoff's voltage law to loop 1 and loop 2, we have the following

Voltage across the 2 Ω resistor $V_2 = 2 \times \frac{10}{7} = 2.85$ V

Voltage across the 1 Ω resistor $V_1 = 1 \times \frac{5}{5} = 1$ V

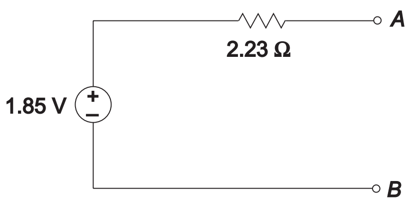


(a)

$$\therefore \quad V_{AB} = V_2 + V_1$$
$$= 2.85 - 1 = 1.85 \text{ V}$$

The resistance seen into the terminals AB

$$R_{AB} = (5 \parallel 2) + (4 \parallel 1)$$
$$= \frac{5 \times 2}{5 + 2} + \frac{4 \times 1}{4 + 1}$$
$$= 1.43 + 0.8 = 2.23 \ \Omega$$



(b)

Fig. 3.57

Thevenins's equivalent circuit is shown in Fig. 3.57 (b).

Problem 3.9 Determine Norton's equivalent circuit for the circuit shown in Fig. 3.58.

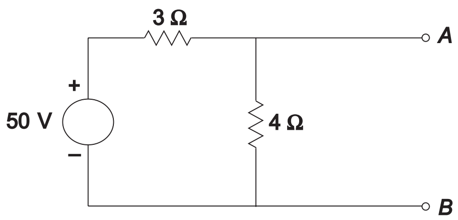


Fig. 3.58

Solution Norton's equivalent circuit is given by Fig. 3.59 (a).

where I_N = Short circuit current at terminals AB
R_N = Open circuit resistance at terminals AB

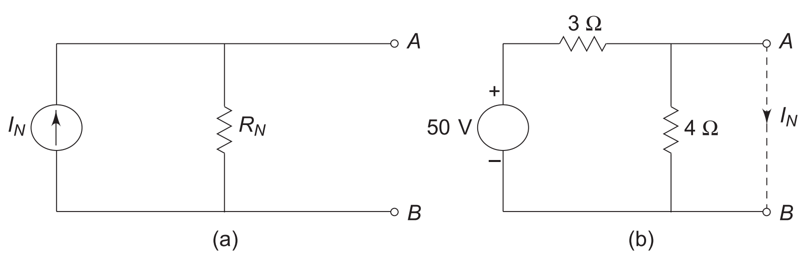


Fig. 3.59

The current I_N can be found as shown in Fig. 3.59 (b).

$$I_N = \frac{50}{3} = 16.7 \text{ A}$$

Norton's resistance can be found from Fig. 3.59 (c)

$$R_N = R_{AB} = \frac{3 \times 4}{3 + 4} = 1.71\ \Omega$$

Norton's equivalent circuit for the given circuit is shown in Fig. 3.59 (d).

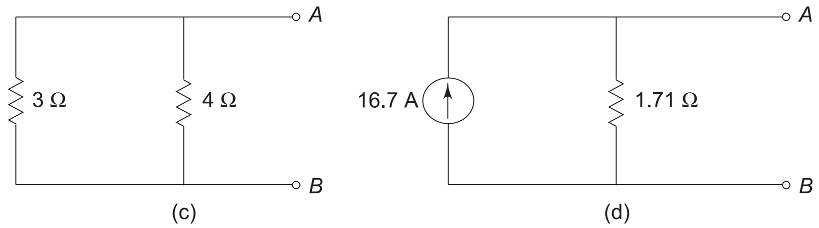


Fig. 3.59

Problem 3.10 Determine Norton's equivalent circuit for the given circuit shown in Fig. 3.60.

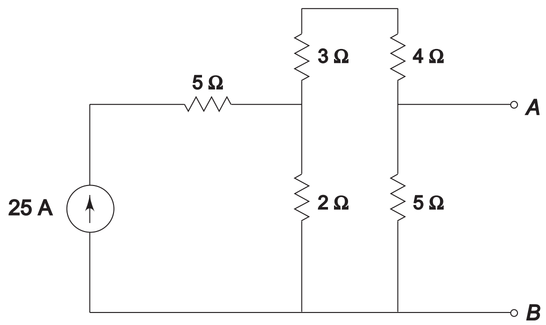


Fig. 3.60

Solution The short circuit current at terminals AB can be found from Fig. 3.61 (a) and Norton's resistance can be found from Fig. 3.61 (b).

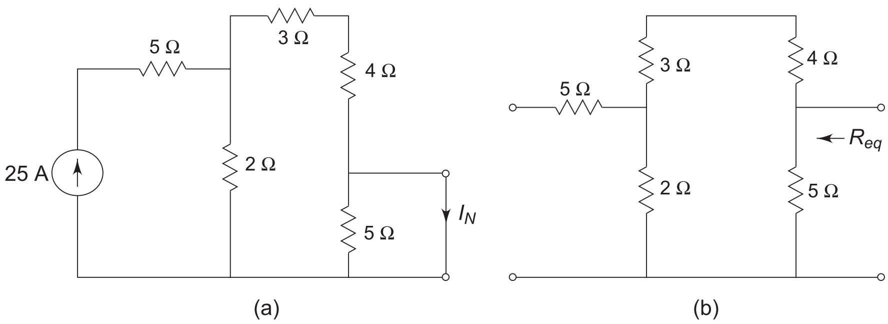


Fig. 3.61

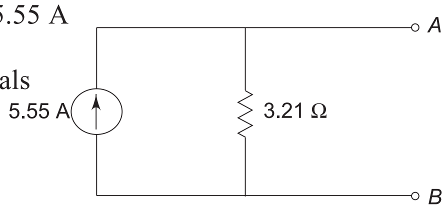
The current I_N is same as the current in the 3 Ω resistor or 4 Ω resistor.

$$I_N = I_3 = 25 \times \frac{2}{7+2} = 5.55 \text{ A}$$

The resistance as seen into the terminals AB is

$$R_{AB} = 5 \parallel (4+3+2)$$

$$= \frac{5 \times 9}{5+9} = 3.21\ \Omega$$



(c)

Fig. 3.61

Norton's equivalent circuit is shown in Fig. 3.61 (c).

Problem 3.11 Determine the current flowing through the 5 Ω resistor in the circuit shown in Fig. 3.62 by using Norton's theorem.

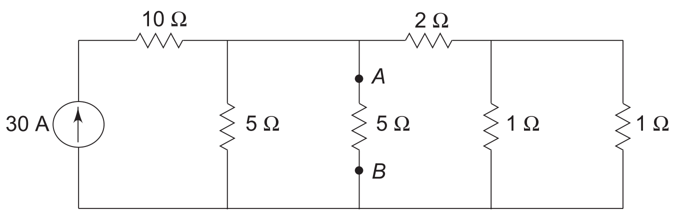


Fig. 3.62

Solution The short circuit current at terminals AB can be found from the circuit as shown in Fig. 3.63 (a). Norton's resistance can be found from Fig. 3.63 (b).

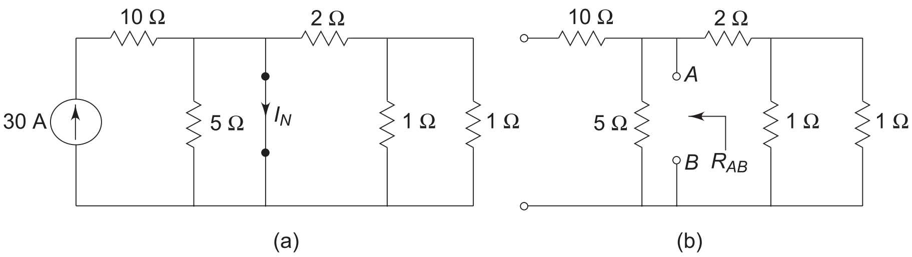


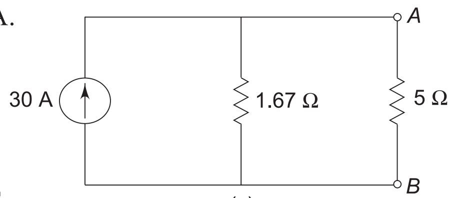
Fig. 3.63

In Fig. 3.63 (a), the current I_N = 30 A.

The resistance in Fig. 3.68 (b)

$$R_{AB} = 5 \parallel \left(2 + \frac{1 \times 1}{2}\right)$$

$$= 5 \parallel (2.5) = \frac{5 \times 2.5}{7.5} = 1.67\ \Omega$$



(c)

Fig. 3.63

Norton's equivalent circuit is shown in Fig. 3.63 (c)

$\therefore$ The current in the 5 Ω resistor

$$I_5 = 30 \times \frac{1.67}{6.67} = 7.51 \text{ A}$$

Problem 3.12 Replace the given network shown in Fig 3.64 by a single current source in parallel with a resistance.

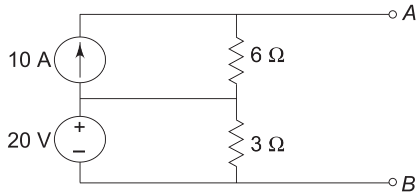


Fig. 3.64

Solution Here, using superposition technique and Norton's theorem, we can convert the given network.

We have to find a short circuit current at terminals *AB* in Fig. 3.65 (a) as shown

The current I'_N is due to the 10 A source. $I'_N = 10$ A

The current I''_N is due to the 20 V source (See Figs 3.65 (b) and (c))

$$I''_N = \frac{20}{6} = 3.33 \text{ A}$$

The current I_N is due to both the sources

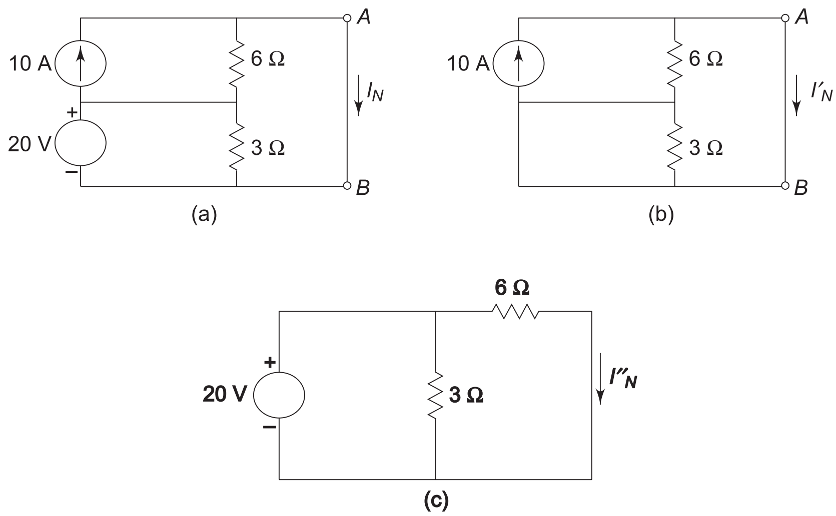


Fig. 3.65

$$I_N = I'_N + I''_N$$
$$= 10 + 3.33 = 13.33 \text{ A}$$

The resistance as seen from terminals AB

$$R_{AB} = 6\ \Omega \text{ (from the Fig. 3.65 (d))}$$

Hence, the required circuit is as shown in Fig. 3.65 (e).

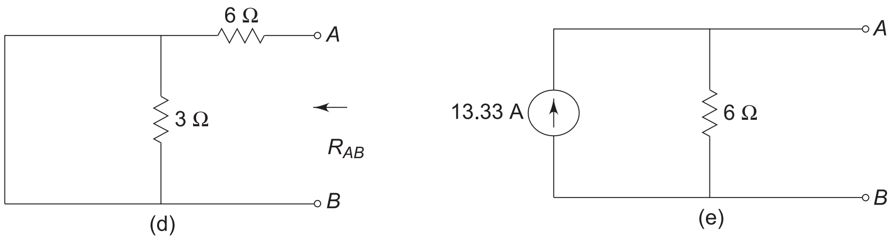


Fig. 3.65

Problem 3.13 Using the compensation theorem, determine the ammeter reading where it is connected to the 6 Ω resistor as shown in Fig. 3.66. The internal resistance of the ammeter is 2 Ω.

Solution The current flowing through the 5 Ω branch

$$I_5 = 20 \times \frac{3}{3 + 6.5} = 6.315 \text{ A}$$

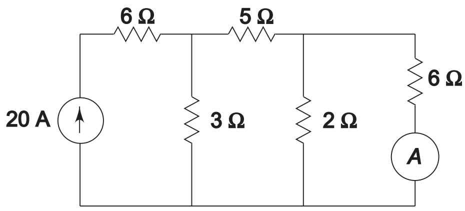


Fig. 3.66

So the current in the 6 Ω branch

$$I_6 = 6.315 \times \frac{2}{6 + 2} = 1.58 \text{ A}$$

If we connect the ammeter having 2 Ω internal resistance to the 6 Ω branch, there is a change in resistance. The changes in currents in other branches results if a voltage source of voltage $I_6\, \Delta R = 1.58 \times 2 = 3.16$ V is inserted in the 6 Ω branch as shown in Fig. 3.67.

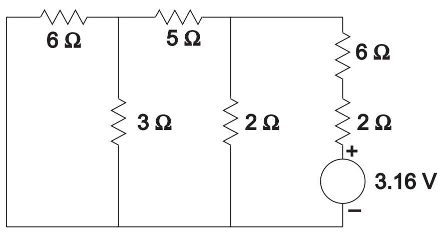


Fig. 3.67

The current due to this 3.16 V source is calculated.

The total impedance in the circuit

$$R_T = \{[(6 \parallel 3) + 5] \parallel [2]\} + \{6 + 2\}$$
$$= 9.56\ \Omega$$

The current due to 3.16 V source

$$I'_6 = \frac{3.16}{9.56} = 0.33\ \text{A}$$

This current is opposite to the current I_6 in the 6 Ω branch.

Hence, the ammeter reading = (1.58 – 0.33)

= 1.25 A

Problem 3.14 Verify the reciprocity theorem for the given circuit shown in Fig. 3.68.

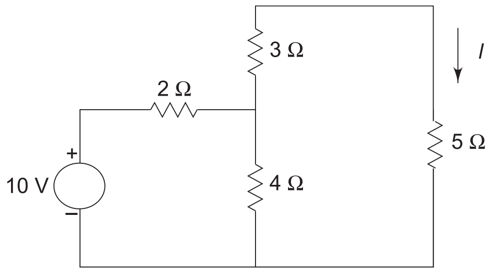


Fig. 3.68

Solution In Fig. 3.68, the current in the 5 Ω resistor is

$$I_5 = I_2 \times \frac{4}{8+4} = 2.14 \times \frac{4}{12} = 0.71\ \text{A}$$

where $$I_2 = \frac{10}{R_T}$$

and $$R_T = 4.67$$

$\therefore$ $$I_2 = \frac{10}{4.67} = 2.14\ \text{A}$$

We interchange the source and response as shown in Fig. 3.69.

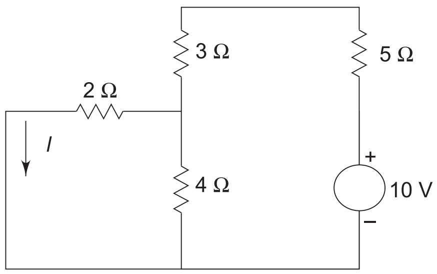


Fig. 3.69

In Fig. 3.69, the current in 2 Ω resistor is

$$I_2 = I_3 \times \frac{4}{4+2}$$

where $$I_3 = \frac{10}{R_T}$$

and $$R_T = 9.33\ \Omega$$

∴ $$I_3 = \frac{10}{9.33} = 1.07\ \text{A}$$

$$I_2 = 1.07 \times \frac{4}{6} = 0.71\ \text{A}$$

In both cases, the ratio of voltage to current is $\frac{10}{0.71} = 14.08$.

Hence the reciprocity theorem is verified.

Problem 3.15 Verify the reciprocity theorem in the circuit shown in Fig. 3.70.

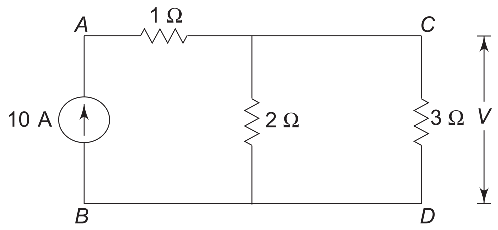


Fig. 3.70

Solution The voltage V across the 3 Ω resistor is

$$V = I_3 \times R$$

where $$I_3 = 10 \times \frac{2}{2+3} = 4\ \text{A}$$

∴ $$V = 4 \times 3 = 12\ \text{V}$$

We interchange the current source and response as shown in Fig. 3.71.

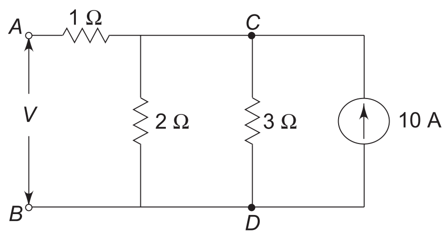


Fig. 3.71

To find the response, we have to find the voltage across the 2 Ω resistor

$$V = I_2 \times R$$

where $$I_2 = 10 \times \frac{3}{5} = 6 \text{ A}$$

$$\therefore \quad V = 6 \times 2 = 12 \text{ V}$$

In both cases, the ratio of current to voltage is the same, i.e. it is equal to 0.833. Hence the reciprocity theorem is verified.

Problem 3.16 Determine the maximum power delivered to the load in the circuit shown in Fig. 3.72.

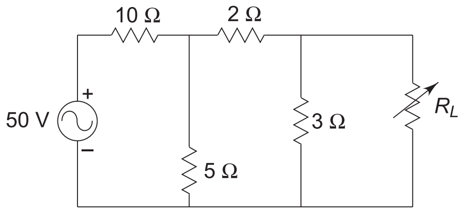


Fig. 3.72

Solution For the given circuit, let us find out the Thevenin's equivalent circuit across AB as shown in Fig. 3.73 (a).

The total resistance is

$$R_T = [\{(3 + 2) \parallel 5\} + 10]$$
$$= [2.5 + 10] = 12.5 \ \Omega$$

Total current drawn by the circuit is

$$I_T = \frac{50}{12.5} = 4 \text{ A}$$

The current in the 3 Ω resistor is

$$I_3 = I_T \times \frac{5}{5+5} = \frac{4 \times 5}{10} = 2 \text{ A}$$

Thevenin's voltage $V_{AB} = V_3 = 3 \times 2 = 6$ V

Thevenin's resistance $R_{\text{Th}} = R_{AB} = [((10 \parallel 5) + 2) \parallel 3] \ \Omega = 1.92 \ \Omega$

Thevenin's equivalent circuit is shown in Fig. 3.73 (b).

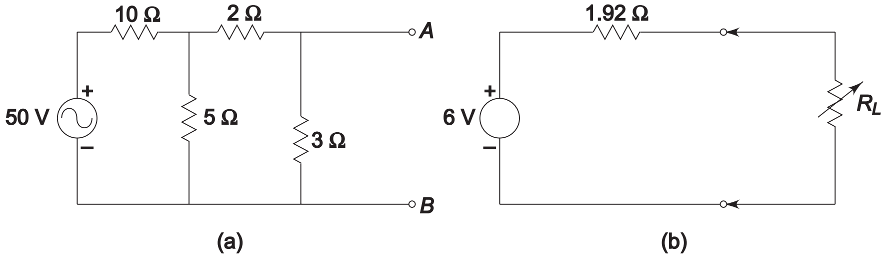


Fig. 3.73

From Fig. 3.73 (b), and maximum power transfer theorem

$$R_L = 1.92\ \Omega$$

$\therefore$ Current drawn by load resistance R_L

$$I_L = \frac{6}{1.92 + 1.92} = 1.56\text{ A}$$

Power delivered to the load $= I_L^2\, R_L$

$$= (1.56)^2 \times 1.92 = 4.67\text{ W}$$

Problem 3.17 Determine the load resistance to receive maximum power from the source; also find the maximum power delivered to the load in the circuit shown in Fig. 3.74.

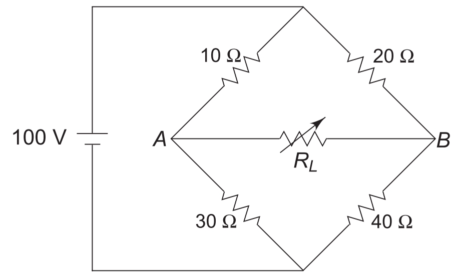
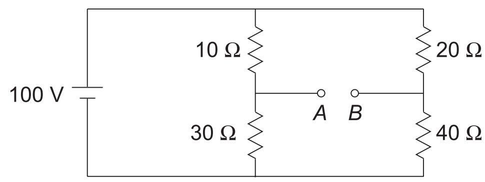


Fig. 3.74

Solution For the given circuit, we find out the Thevenin's equivalent circuit.

Thevenin's voltage across terminals A and B

$$V_{AB} = V_A - V_B$$



(a)

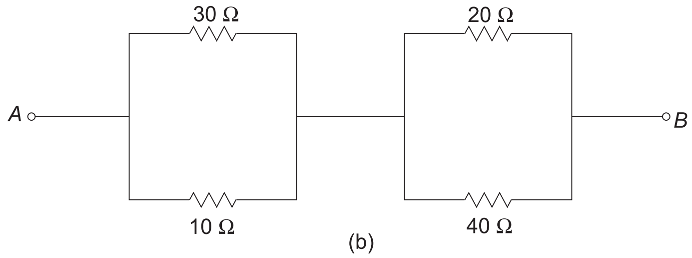
Fig. 3.75

Voltage at point A is $V_A = 100 \times \dfrac{30}{30 + 10} = 75\text{ V}$

Voltage at point B is $V_B = 100 \times \dfrac{40}{40 + 20} = 66.67\text{ V}$

$\therefore$ $$V_{AB} = 75 - 66.67 = 8.33\text{ V}$$

To find Thevenin's resistance the circuit in Fig. 3.75 (a) can be redrawn as shown in Fig. 3.75 (b).



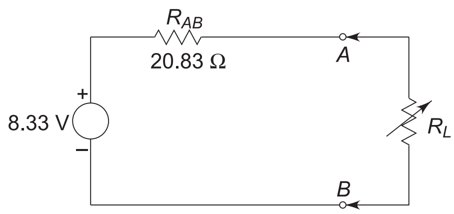
(b)

Fig. 3.75

From Fig. 3.75 (b), Thevenin's resistance

$$R_{AB} = [(30 \parallel 10) + (20 \parallel 40)]$$
$$= [7.5 + 13.33] = 20.83\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.75 (c).



(c)

Fig. 3.75

According to maximum power transfer theorem

$$R_L = 20.83\ \Omega$$

Current drawn by the load resistance

$$I_L = \frac{8.33}{20.83 + 20.83} = 0.2\ \text{A}$$

$\therefore$ Maximum power delivered to load $= I_L^2 R_L$

$$= (0.2)^2\ (20.83) = 0.833\ \text{W}$$

Problem 3.18 Draw the dual circuit for the given circuit shown in Fig. 3.76.

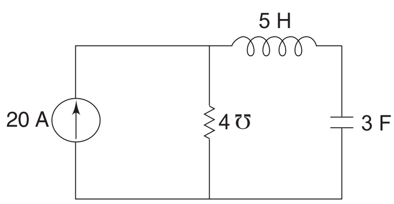
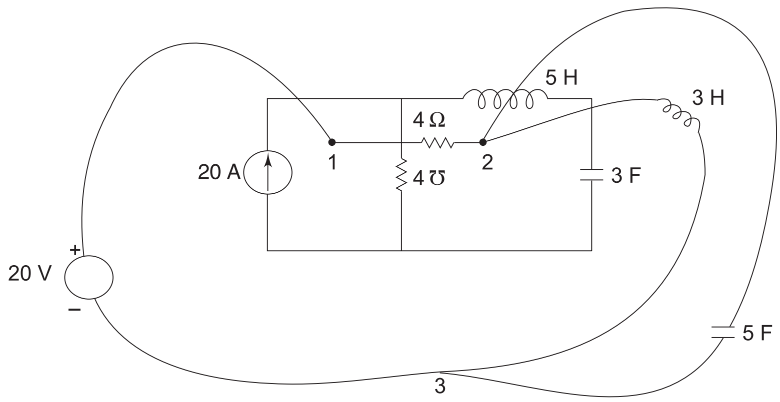


Fig. 3.76

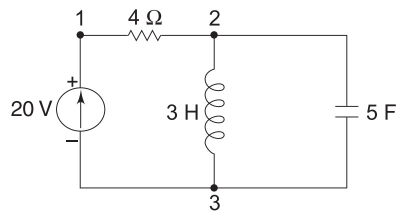
Solution Our first step is to place nodes in each loop, and a reference node outside the circuit.

Join the nodes with lines passing through each element and connect these lines with dual of each element as shown in Fig. 3.77 (a).

The dual circuit of the given circuit is shown in Fig. 3.77 (b).



(a)



(b)

Fig. 3.77

Problem 3.19 Draw the dual circuit of the Fig. 3.78 given below.

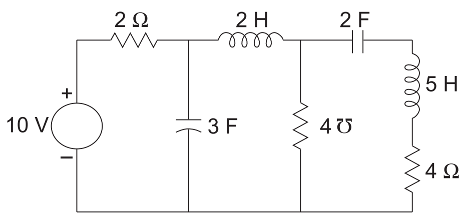
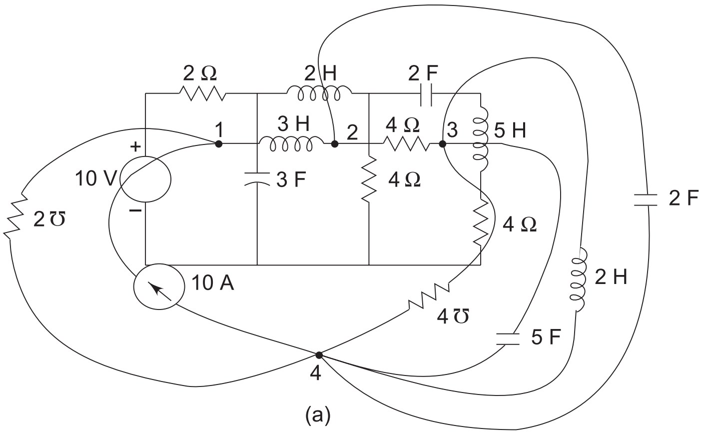


Fig. 3.78

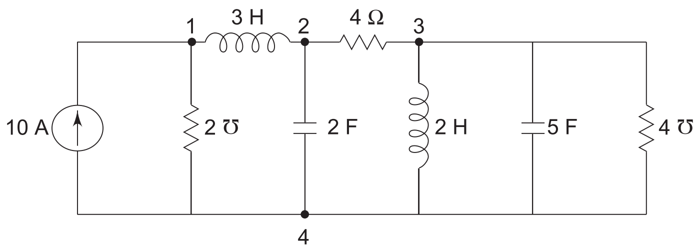
Solution Our first step is to mark nodes in each of the loop and a reference node outside the circuit.

Join the nodes with lines passing through each element and connect these lines with dual of each element as shown in Fig. 3.79 (a).

The dual circuit of given circuit is shown in Fig. 3.79 (b).



(a)



(b)

Fig. 3.79

Problem 3.20 For the circuit shown in Fig. 3.80, find the current i_4 using the superposition principle.

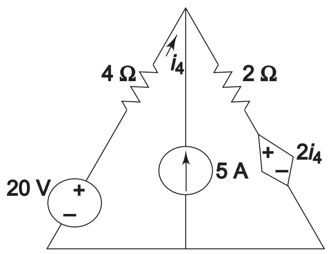


Fig. 3.80

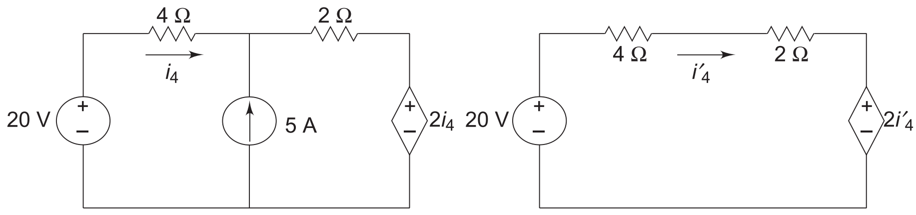
Solution The circuit can be redrawn as shown in Fig. 3.81 (a).

The current i'_4 due to the 20 V source can be found using the circuit shown in Fig. 3.81 (b).

Applying Kirchhoff's voltage law

$$-20 + 4i'_4 + 2i'_4 + 2i'_4 = 0$$

$$i'_4 = 2.5 \text{ A}$$



(a) (b)

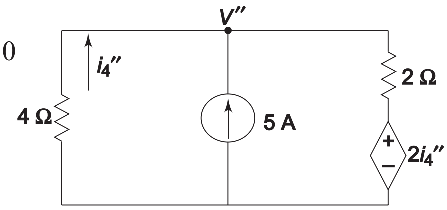
Fig. 3.81

The current i''_4 due to the 5 A source can be found using the circuit shown in Fig. 3.81 (c).

By assuming V'' at node shown in Fig. 3.81 (c) and applying Kirchhoff's current law

$$\frac{V''}{4} - 5 + \frac{V'' - 2i''_4}{2} = 0$$

$$I'_4 = \frac{-V''}{4}$$



(c)

Fig. 3.81

From the above equations

$$i''_4 = -1.25 \text{ A}$$

$\therefore$ Total current $i_4 = i'_4 + i''_4 = 1.25$ A

Problem 3.21 Determine the current through the 2 Ω resistor as shown in the Fig 3.82 by using the superposition theorem.

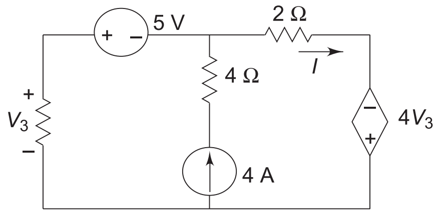
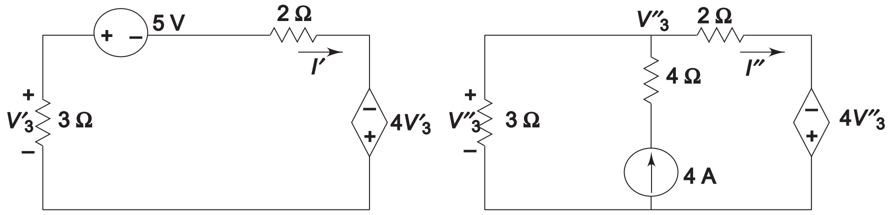


Fig. 3.82

Solution The current I' due to the 5 V source can be found using the circuit shown in Fig. 3.83 (a).



(a) (b)

Fig. 3.83

By applying Kirchhoff's voltage law, we have

$$3I' + 5 + 2I' - 4V_3' = 0$$

we know $V_3' = -3I'$

From the above equations

$$I' = -0.294 \text{ A}$$

The current I'' due to the 4 A source can be found using the circuit shown in Fig. 3.83 (b).

By assuming node voltage V''_3, we find

$$I' = \frac{V_3'' + 4V_3''}{2}$$

By applying Kirchhoff's current law at node we have

$$\frac{V_3''}{3} - 4 + \frac{V_3'' + 4V_3''}{2} = 0$$

$$V_3'' = 1.55 \text{ V}$$

$$\therefore \quad I'' = \frac{V_3'' + 4V_3''}{2} = 3.875 \text{ A}$$

Total current in the 2 Ω resistor $I = I' + I'' = -0.294 + 3.875$

$$\therefore \quad I = 3.581 \text{ A}$$

Problem 3.22 For the circuit shown in Fig. 3.84, obtain Thevenin's equivalent circuit.

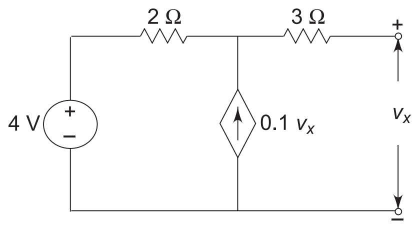


Fig. 3.84

Solution The circuit consists of a dependent source. In the presence of dependent source R_{Th} can be determined by finding v_{OC} and i_{SC}

$$\therefore \quad R_{Th} = \frac{v_{OC}}{i_{SC}}$$

Open circuit voltage can be found from the circuit shown in Fig. 3.85 (a)

Since the output terminals are open, current passes through the 2 Ω branch only.

$$v_x = 2 \times 0.1\, v_x + 4$$

$$v_x = \frac{4}{0.8} = 5 \text{ V}$$

Short circuit current can be calculated from the circuit shown in Fig. 3.85 (b).

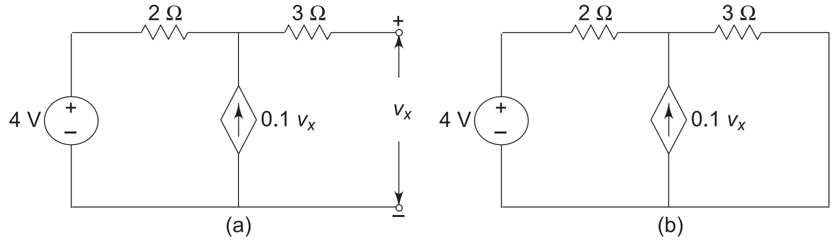


Fig. 3.85

Since $v_x = 0$, dependent current source is opened.

The current $i_{SC} = \dfrac{4}{2+3} = 0.8$ A

$\therefore \quad R_{\text{Th}} = \dfrac{v_{OC}}{i_{SC}} = \dfrac{5}{0.8} = 6.25\ \Omega$

The Thevenin's equivalent circuit is shown in Fig. 3.85 (c).

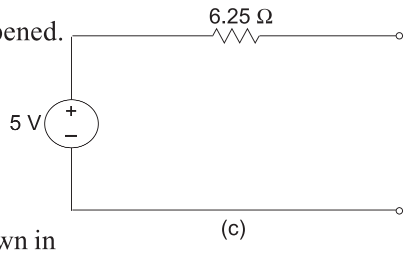


Fig. 3.85

Problem 3.24 For the circuit shown in Fig. 3.86, find the current i_2 in the 2 Ω resistor by using Thevenin's theorem.

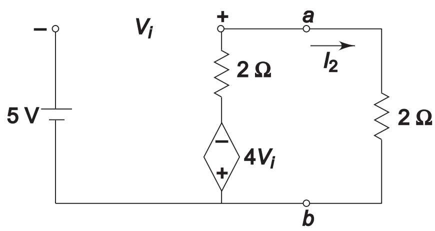


Fig. 3.86

Solution From the circuit, there is open voltage at terminals *ab* which is

$$V_{OC} = -4\ V_i$$

where $\quad V_i = -4V_i - 5$

$\therefore \quad V_i = -1$

Thevenin's voltage $V_{OC} = 4$ V

From the circuit, short circuit current is determined by shorting terminals *a* and *b*.

Applying Kirchhoff's voltage law, we have

$$4V_i + 2i_{SC} = 0$$

We know $\quad V_i = -5$

Substituting V_i in the above equation, we get

$$i_{SC} = 10 \text{ A}$$

$\therefore \quad R_{\text{Th}} = \dfrac{V_{OC}}{i_{SC}} = \dfrac{4}{10} = 0.4\ \Omega$

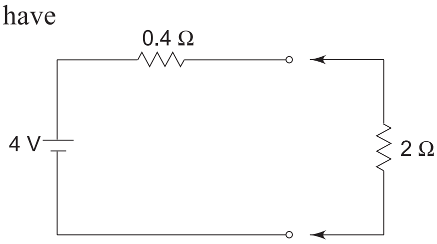


Fig. 3.87

The Thevenin's equivalent circuit is as shown in Fig. 3.87.

The current in the 2 Ω resistor $i_2 = \dfrac{4}{2.4} = 1.67$ A

Problem 3.25 For the circuit shown in Fig. 3.88, find Norton's equivalent circuit.

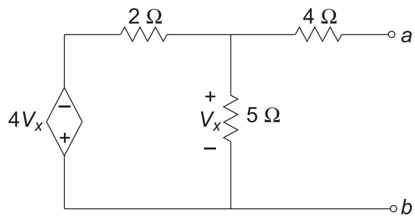


Fig. 3.88

Solution In the case of circuit having only dependent sources (without independent sources), both V_{OC} and i_{SC} are zero. We apply a 1 A source externally and determine the resultant voltage across it, and then find $R_{Th} = \dfrac{V}{1}$ or we can also apply the 1 V source externally and determine the current through it and then we find $R_{Th} = 1/i$.

By applying the 1 A source externally as shown in Fig. 3.89 (a).

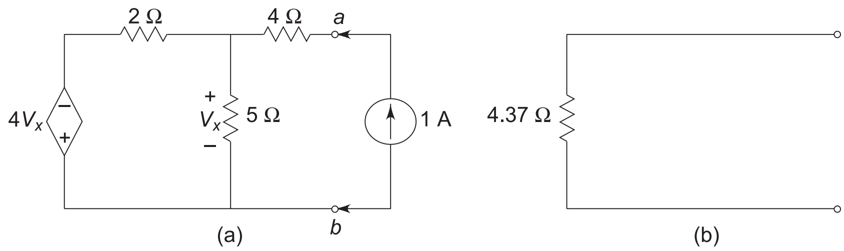


Fig. 3.89

and application of Kirchhoff's current law, we have

$$\frac{V_x}{5} + \frac{V_x + 4V_x}{2} = 1$$

$$V_x = 0.37 \text{ V}$$

The current in the 4 Ω branch is

$$\frac{V_x - V}{4} = -1$$

Substituting V_x in the above equation, we get

$$V = 4.37 \text{ V}$$

$$\therefore \qquad R_{Th} = \frac{V}{1} = 4.37\ \Omega$$

If we short circuit the terminals a and b we have

$$\frac{V_x - 4V_x}{2} = 0$$

$$V_x = 0$$

$$I_{SC} = \frac{V_x}{4} = 0$$

Therefore, Norton's equivalent circuit is as shown in Fig. 3.89 (b).

Practice Problems

3.1 For the bridge network shown in Fig. 3.90, determine the total resistance seen from terminals AB by using star-delta transformation.

3.2 Calculate the voltage across AB in the network shown in Fig. 3.91 and indicate the polarity of the voltage using star-delta transformation.

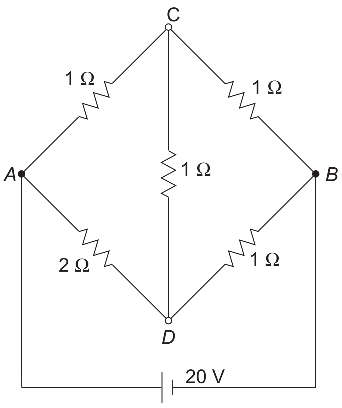


Fig. 3.90

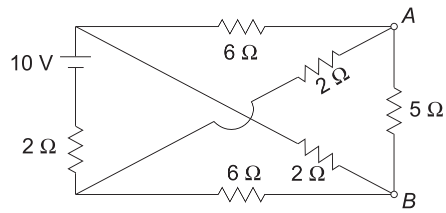


Fig. 3.91

3.3 Find the current I in the circuit shown in Fig. 3.92 by using the superposition theorem.

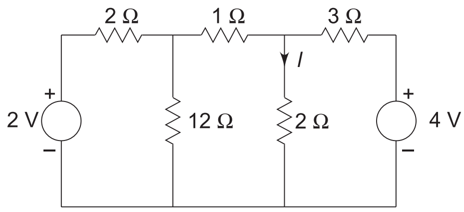


Fig. 3.92

3.4 Determine the current I in the circuit shown in Fig. 3.93 using the superposition theorem.

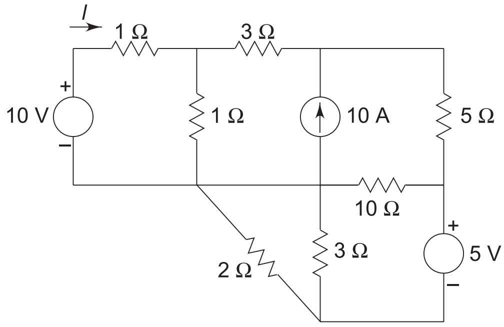


Fig. 3.93

3.5 Calculate the new current in the circuit shown in Fig. 3.94 when the resistor R_3 is increased by 30%.

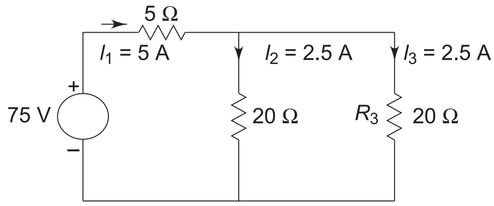


Fig. 3.94

3.6 Find the Thevenin's and Norton's equivalents for the circuit shown in Fig. 3.95 with respect to terminals *ab*.

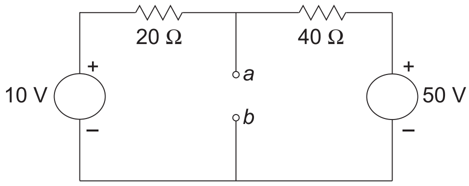


Fig. 3.95

3.7 Determine the Thevenin and Norton's equivalent circuits with respect to terminals *ab* for the circuit shown in Fig. 3.96.

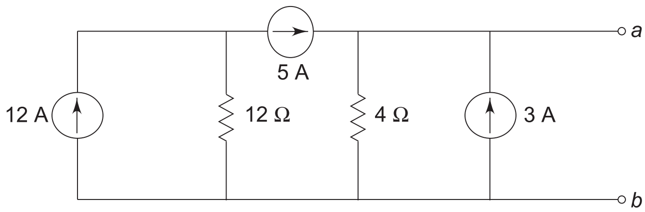


Fig. 3.96

3.8 By using source transformation or any other technique, replace the circuit shown in Fig. 3.97 between terminals *ab* with the voltage source in series with a single resistor.

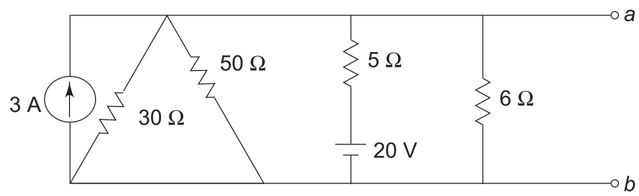


Fig. 3.97

3.9 For the circuit shown in Fig. 3.98, what will be the value of R_L to get the maximum power? What is the maximum power delivered to the load? What is the maximum voltage across the load? What is the maximum current in it?

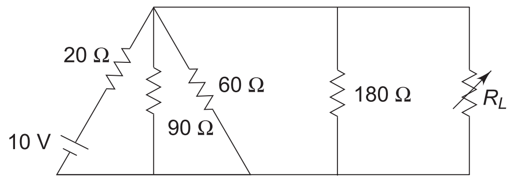


Fig. 3.98

3.10 For the circuit shown in Fig. 3.99 determine the value of R_L to get the maximum power. Also find the maximum power transferred to the load.

3.11 The circuit shown in Fig. 3.100 consists of dependent source. Use the superposition theorem to find the current I in the 3 Ω resistor.

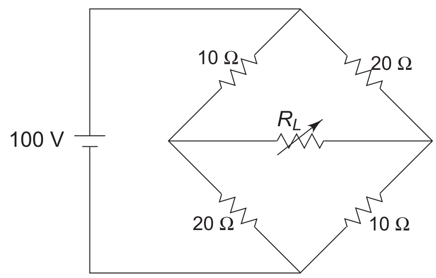


Fig. 3.99

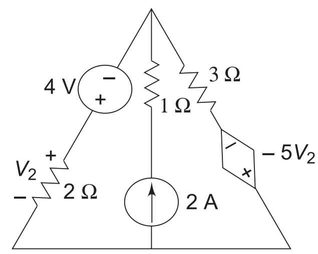


Fig. 3.100

3.12 Obtain the current passing through 2 Ω resistor in the circuit shown in Fig. 3.101 by using the superposition theorem.

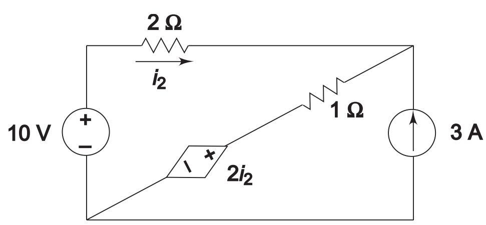


Fig. 3.101

3.13 Determine the current passing through 2 Ω resistor by using Thevenin's theorem in the circuit shown in Fig. 3.102.

3.14 Find Thevenin's equivalent circuit for the network shown in Fig. 3.103 and hence find the current passing through the 10 Ω resistor.

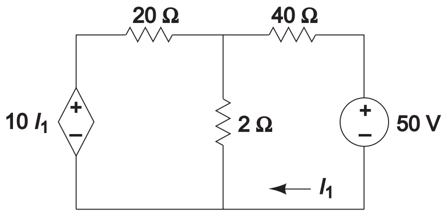


Fig. 3.102

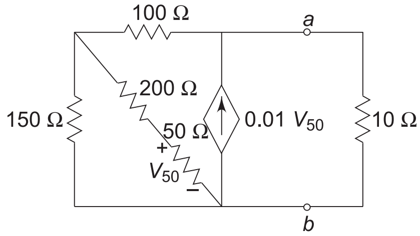


Fig. 3.103

3.15 Obtain Norton's equivalent circuit of the network shown in Fig. 3.104.

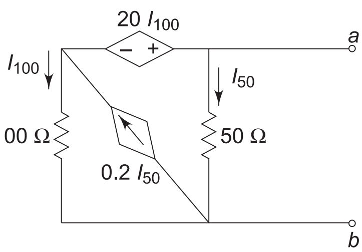


Fig. 3.104

Objective-type Questions

1. Three equal resistance of 3 Ω are connected in star. What is the resistance in one of the arms in an equivalent delta circuit?
   (a) 10 Ω (b) 3 Ω
   (c) 9 Ω (d) 27 Ω
2. Three equal resistances of 5 Ω are connected in delta. What is the resistance in one of the arms of the equivalent star circuit?
   (a) 5 Ω (b) 1.33 Ω
   (c) 15 Ω (d) 10 Ω
3. Superposition theorem is valid only for
   (a) linear circuits
   (b) non-linear circuits
   (c) both linear and non-linear
   (d) neither of the two
4. Superposition theorem is not valid for
   (a) voltage responses (b) current responses
   (c) power responses (d) all the three

5. Determine the current I in the circuit shown in Fig. 3.105. It is

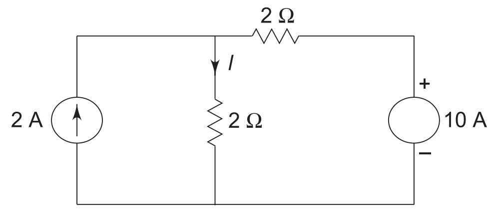


Fig. 3.105

(a) 2.5 A (b) 1 A
(c) 3.5 A (d) 4.5 A

6. Reduce the circuit shown in Fig. 3.106 to its Thevenin equivalent circuit as viewed from terminal A and B.
   (a) The circuit consists of 15 V battery in series with 100 kΩ
   (b) The circuit consists of 15 V battery in series with 22 kΩ
   (c) The circuit consists of 15 V battery in series with parallel combination of 100 kΩ and 22 kΩ
   (d) None of the above

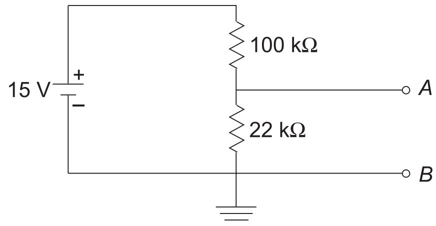


Fig. 3.106

7. Norton's equivalent circuit consists of
   (a) voltage source in parallel with impedance
   (b) voltage source in series with impedance
   (c) current source in series with impedance
   (d) current source in parallel with impedance
8. The reciprocity theorem is applicable to
   (a) linear networks only (b) bilateral networks only
   (c) linear/bilateral networks (d) neither of the two
9. Compensation theorem is applicable to
   (a) linear networks only (b) non-linear networks only
   (c) linear and non-linear networks (d) neither of the two
10. Maximum power is transferred when load impedance is
    (a) equal to source impedance
    (b) equal to half of the source impedance
    (c) equal to zero
    (d) none of the above
11. In the circuit shown in Fig. 3.107, what is the maximum power transferred to the load
    (a) 5 W (b) 2.5 W
    (c) 10 W (d) 25 W

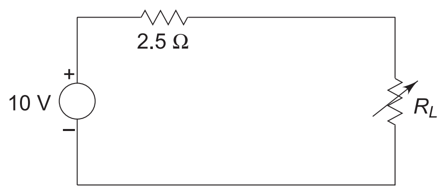


Fig. 3.107

12. Indicate the dual of series network consists of voltage source, capacitance, inductance in
    (a) parallel combination of resistance, capacitance and inductance
    (b) series combination of current source, capacitance and inductance.
    (c) parallel combination of current source, inductance and capacitance
    (d) none of the above
13. When the superposition theorem is applied to any circuit, the dependent voltage source in that circuit is always
    (a) opened (b) shorted
    (c) active (d) none of the above
14. Superposition theorem is not applicable to networks containing.
    (a) non-linear elements
    (b) dependent voltage sources
    (c) dependent current sources
    (d) transformers
15. Thevenins voltage in the circuit shown in Fig. 3.108 is
    (a) 3 V
    (b) 2.5 V
    (c) 2 V
    (d) 0.1 V

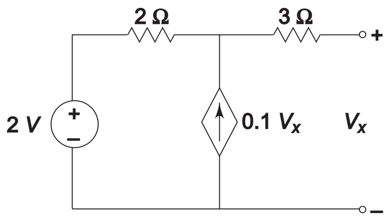


Fig. 3.108

16. Norton's current in the circuit shown in Fig. 3.109 is
    (a) $\frac{2i}{5}$
    (b) zero
    (c) infinite
    (d) None

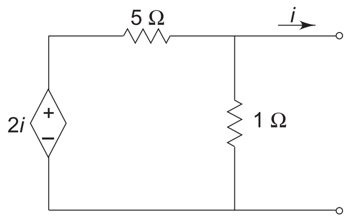


Fig. 3.109

17. A dc circuit shown in Fig. 3.110 has a voltage V, a current source *I* and several resistors. A particular resistor *R* dissipates a power of 4 W when *V* alone is active. The same resistor dissipates a power of 9 W when *I* alone is active. The power dissipated by *R* when both sources are active will be

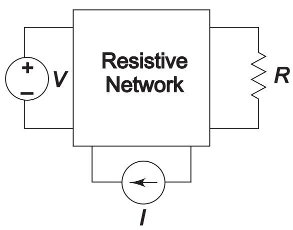


Fig. 3.110

    (a) 1 W (b) 5 W
    (c) 13 W (d) 25 W

Chapter 4

Introduction to Alternating Currents and Voltages

4.1 THE SINE WAVE

Many a time, alternating voltages and currents are represented by a sinusoidal wave, or simply a sinusoid. It is a very common type of alternating current (ac) and alternating voltage. The sinusoidal wave is generally referred to as a sine wave. Basically an alternating voltage (current) waveform is defined as the voltage (current) that fluctuates with time periodically, with change in polarity and direction. In general, the sine wave is more useful than other waveforms, like pulse, sawtooth, square, etc. There are a number of reasons for this. One of the reasons is that if we take any second order system, the response of this system is a sinusoid. Secondly, any periodic waveform can be written in terms of sinusoidal function according to Fourier theorem. Another reason is that its derivatives and integrals are also sinusoids. A sinusoidal function is easy to analyse. Lastly, the sinusoidal function is easy to generate, and it is more useful in the power industry. The shape of a sinusoidal waveform is shown in Fig. 4.1.

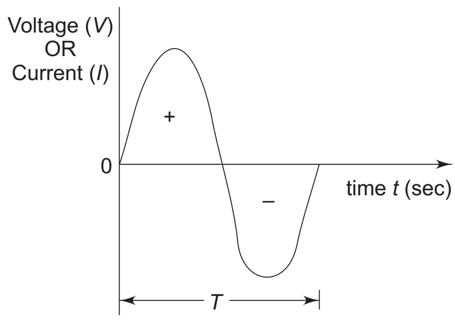


Fig. 4.1

The waveform may be either a current waveform, or a voltage waveform. As seen from the Fig. 4.1, the wave changes its magnitude and direction with time. If we start at time

$t = 0$, the wave goes to a maximum value and returns to zero, and then decreases to a negative maximum value before returning to zero. The sine wave changes with time in an orderly manner. During the positive portion of voltage, the current flows in one direction; and during the negative portion of voltage, the current flows in the opposite direction. The complete positive and negative portion of the wave is one cycle of the sine wave. Time is designated by t. The time taken for any wave to complete one full cycle is called the period (T). In general, any periodic wave constitutes a number of such cycles. For example, one cycle of a sine wave repeats a number of times as shown in Fig. 4.2. Mathematically it can be represented as $f(t) = f(t + T)$ for any t.

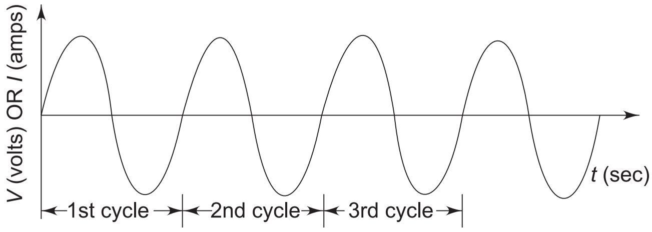


Fig. 4.2

The period can be measured in the following different ways (See Fig. 4.3).

1. From zero crossing of one cycle to zero crossing of the next cycle.
2. From positive peak of one cycle to positive peak of the next cycle, and
3. From negative peak of one cycle to negative peak of the next cycle.

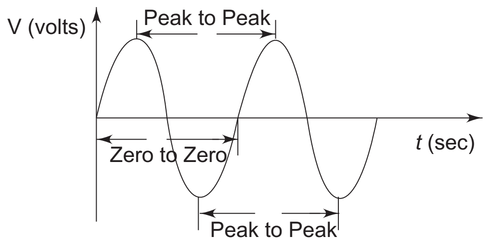


Fig. 4.3

The frequency of a wave is defined as the number of cycles that a wave completes in one second.

In Fig. 4.4 the sine wave completes three cycles in one second. Frequency is measured in hertz. One hertz is equivalent to one cycle per second, 60 hertz is 60 cycles per second and so on. In Fig. 4.4, the frequency denoted by f is 3 Hz,

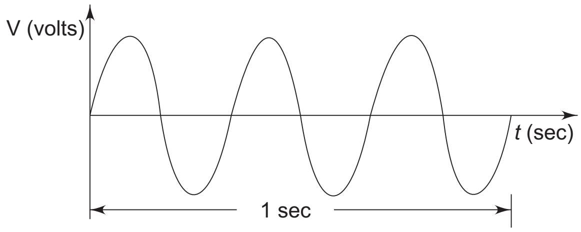


Fig. 4.4

that is three cycles per second. The relation between time period and frequency is given by

$$f = \frac{1}{T}$$

A sine wave with a longer period consists of fewer cycles than one with a shorter period.

Example 4.1 What is the period of sine wave shown in Fig. 4.5?

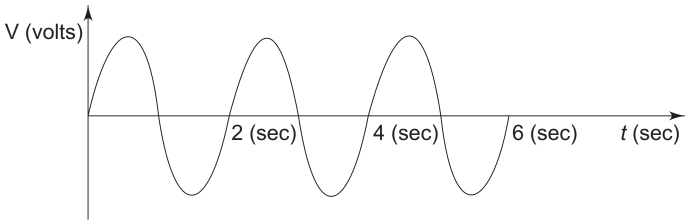


Fig. 4.5

Solution From Fig. 4.5, it can be seen the sine wave takes two seconds to complete one period in each cycle

$$T = 2\text{ s}$$

Example 4.2 The period of a sine wave is 20 milliseconds. What is the frequency.

Solution

$$f = \frac{1}{T}$$

$$= \frac{1}{20\text{ ms}} = 50\text{ Hz}$$

Example 4.3 The frequency of a sine wave is 30 Hz. What is its period.

Solution

$$T = \frac{1}{f}$$

$$= \frac{1}{30} = 0.03333\text{ s}$$

$$= 33.33\text{ ms}$$

4.2 ANGULAR RELATION OF A SINE WAVE

A sine wave can be measured along the X-axis on a time base which is frequency-dependent. A sine wave can also be expressed in terms of an angular measurement. This angular measurement is expressed in degrees or radians. A radian is defined as the angular distance measured along the circumference of a circle which is equal to the radius of the circle. One radian is equal to 57.3°. In a 360° revolution, there are 2π radians. The angular measurement of a sine wave is based on 360° or 2π radians for a complete cycle as shown in Figs. 4.6 (a) and (b).

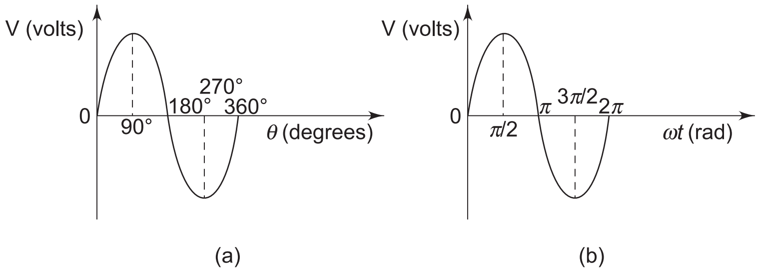


Fig. 4.6

A sine wave completes a half cycle in 180° or π radians; a quarter cycle in 90° or $\pi/2$ radians, and so on.

Phase of a Sine Wave

The phase of a sine wave is an angular measurement that specifies the position of the sine wave relative to a reference. The wave shown in Fig. 4.7 is taken as the reference wave.

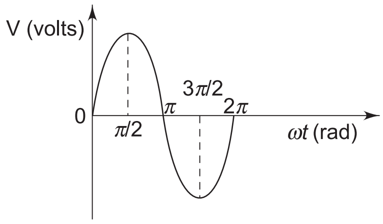


Fig. 4.7

When the sine wave is shifted left or right with reference to the wave shown in Fig. 4.7, there occurs a phase shift. Figure 4.8 shows the phase shifts of a sine wave.

In Fig. 4.8(a), the sine wave is shifted to the right by 90° ($\pi/2$ rad) shown by the dotted lines. There is a phase angle of 90° between A and B. Here the waveform B is lagging behind waveform A by 90°. In other words, the sine wave A is leading the waveform B by 90°. In Fig. 4.8(b) the sine wave A is lagging behind the waveform B by 90°. In both cases, the phase difference is 90°.

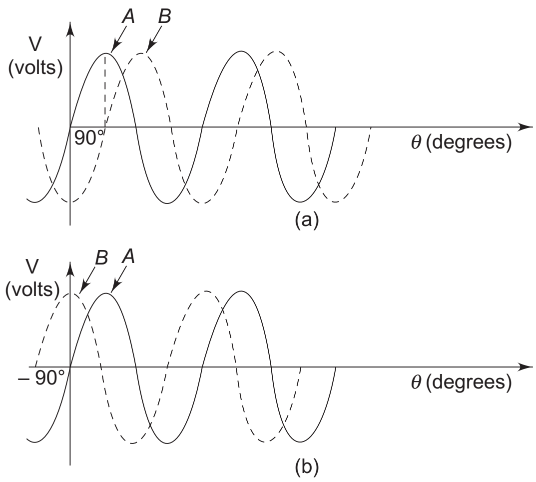


Fig. 4.8

Example 4.4 What are the phase angles between the two sine waves shown in Figs. 4.9(a) and (b)?

Solution In Fig. 4.9(a), sine wave *A* is in phase with the reference wave; sine wave *B* is out of phase, which lags behind the reference wave by 45°. So we say that sine wave *B* lags behind sine wave *A* by 45°.

In Fig. 4.9(b), sine wave *A* leads the reference wave by 90°; sine wave *B* lags behind the reference wave by 30°. So the phase difference between *A* and *B* is 120°, which means that sine wave *B* lags behind sine wave *A* by 120°. In other words, sine wave *A* leads sine wave *B* by 120°.

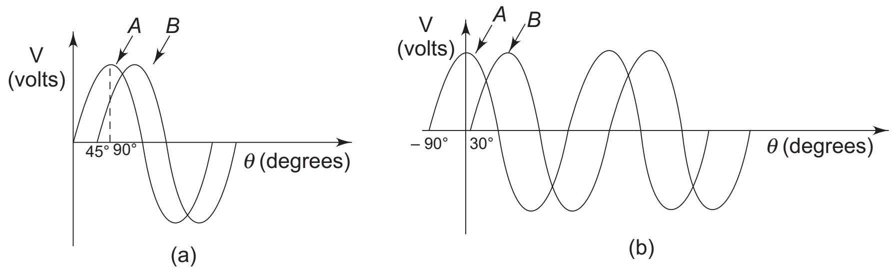


Fig. 4.9

4.3 THE SINE WAVE EQUATION

A sine wave is graphically represented as shown in Fig. 4.10(a). The amplitude of a sine wave is represented on vertical axis. The angular measurement (in degrees or radians) is represented on horizontal axis. Amplitude A is the maximum value of the voltage or current on the Y-axis.

In general, the sine wave is represented by the equation

$$v(t) = V_m \sin \omega t$$

The above equation states that any point on the sine wave represented by an instantaneous value $v(t)$ is equal to the maximum value times the sine of the angular frequency at that point. For example, if a certain sine wave voltage has peak value of 20 V, the instantaneous voltage at a point $\pi/4$ radians along the horizontal axis can be calculated as

$$v(t) = V_m \sin \omega t$$

$$= 20 \sin \left(\frac{\pi}{4}\right) = 20 \times 0.707 = 14.14 \text{ V}$$

When a sine wave is shifted to the left of the reference wave by a certain angle ϕ, as shown in Fig. 4.10 (b), the general expression can be written as

$$v(t) = V_m \sin (\omega t + \phi)$$

When a sine wave is shifted to the right of the reference wave by a certain angle ϕ, as shown in Fig. 4.10(c), the general expression is

$$v(t) = V_m \sin (\omega t - \phi)$$

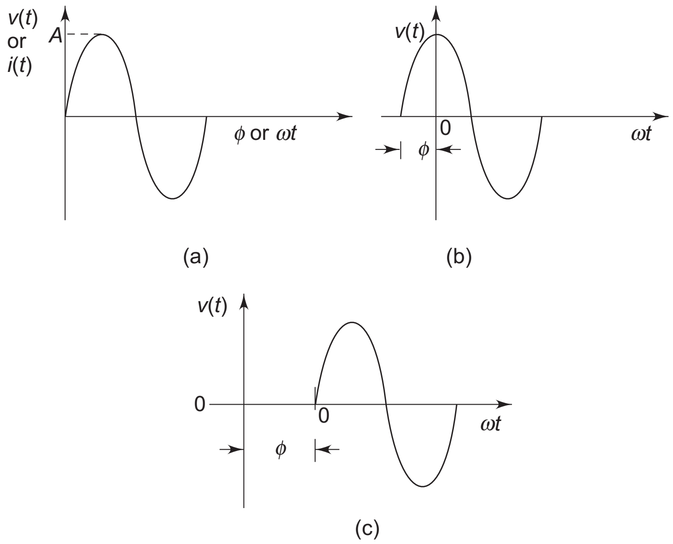


Fig. 4.10

Example 4.5 Determine the instantaneous value at the 90° point on the *X*-axis for each sine wave shown in Fig. 4.11.

Solution From Fig. 4.11, the equation for the sine wave *A*

$$v(t) = 10 \sin \omega t$$

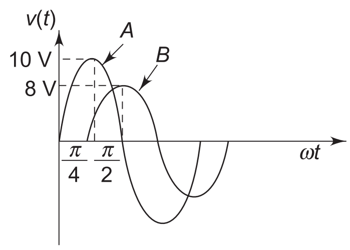


Fig. 4.11

The value at $\pi/2$ in this wave is

$$v(t) = 10 \sin \frac{\pi}{2} = 10 \text{ V}$$

The equation for the sine wave *B*

$$v(t) = 8 \sin (\omega t - \pi/4)$$

At $\omega t = \pi/2$

$$v(t) = 8 \sin \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 8 \sin 45° = 8\ (0.707) = 5.66 \text{ V}$$

4.4 VOLTAGE AND CURRENT VALUES OF A SINE WAVE

As the magnitude of the waveform is not constant, the waveform can be measured in different ways. These are instantaneous, peak, peak to peak, root mean square (rms) and average values.

4.4.1 Instantaneous Value

Consider the sine wave shown in Fig. 4.12. At any given time, it has some instantaneous value. This value is different at different points along the waveform.

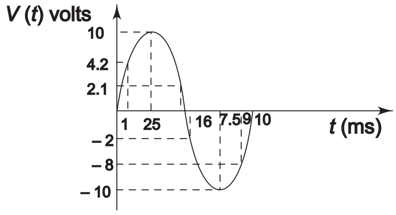


Fig. 4.12

In Fig. 4.12 during the positive cycle, the instantaneous values are positive and during the negative cycle, the instantaneous values are negative. In Fig. 4.12 shown at time 1 ms, the value is 4.2 V; the value is 10 V at 2.5 ms, – 2 V at 6 ms and – 10 V at 7.5 and so on.

4.4.2 Peak Value

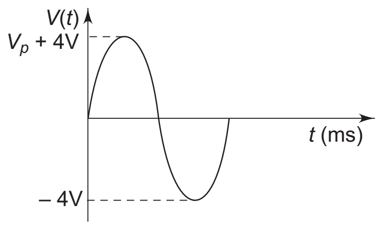


Fig. 4.13

The peak value of the sine wave is the maximum value of the wave during positive half cycle, or maximum value of wave during negative half cycle. Since the value of these two are equal in magnitude, a sine wave is characterised by a single peak value. The peak value of the sine wave is shown in Fig. 4.13; here the peak value of the sine wave is 4 V.

4.4.3 Peak to Peak Value

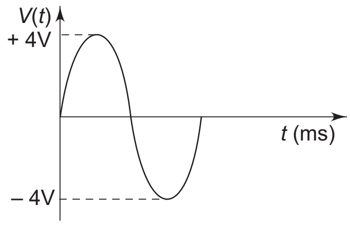


Fig. 4.14

The peak to peak value of a sine wave is the value from the positive to the negative peak as shown in Fig. 4.14. Here the peak to peak value is 8 V.

4.4.4 Average Value

In general, the average value of any function $v(t)$, with period T is given by

$$v_{\text{av}} = \frac{1}{T}\int_0^T v(t)\, dt$$

That means that the average value of a curve in the X-Y plane is the total area under the complete curve divided by the distance of the curve. The average value of a sine wave over one complete cycle is always zero. So the average value of a sine wave is defined over a half-cycle, and not a full cycle period.

The average value of the sine wave is the total area under the half-cycle curve divided by the distance of the curve.

The average value of the sine wave

$v(t) = V_P \sin \omega t$ is given by

$$v_{av} = \frac{1}{\pi}\int_0^{\pi} V_P \sin \omega t\, d(\omega t)$$

$$= \frac{1}{\pi}[- V_P \cos \omega t]_0^{\pi}$$

$$= \frac{2V_P}{\pi} = 0.637\ V_P$$

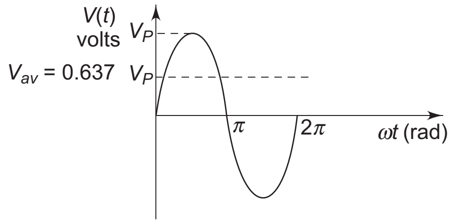


Fig. 4.15

The average value of a sine wave is shown by the dotted line in Fig. 4.15.

Example 4.6 Find the average value of a cosine wave $f(t) = \cos \omega t$ shown in Fig. 4.16.

Solution The average value of a cosine wave

$$v(t) = V_P \cos \omega t$$

$$V_{av} = \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} V_P \cos \omega t\ d(\omega t)$$

$$= \frac{1}{\pi} V_P(-\sin \omega t)_{\pi/2}^{3\pi/2}$$

$$= \frac{-V_P}{\pi}[-1-1] = \frac{2V_P}{\pi} = 0.637\ V_P$$

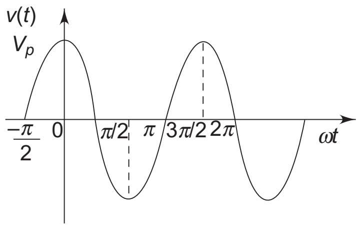


Fig. 4.16

4.4.5 Root Mean Square Value or Effective Value

The root mean square (rms) value of a sine wave is a measure of the heating effect of the wave. When a resistor is connected across a dc voltage source as shown in Fig. 4.17(a), a certain amount of heat is produced in the resistor in a given time. A similar resistor is connected across an ac voltage source for the same time as shown in Fig. 4.17(b). The value of the ac voltage is adjusted such that the same amount of heat is produced in the resistor as in the case of the dc source. This value is called the rms value.

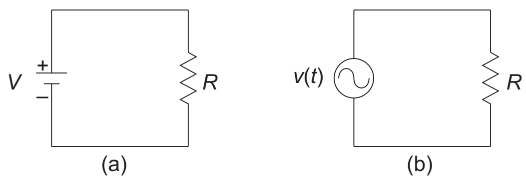


Fig. 4.17

That means the rms value of a sine wave is equal to the dc voltage that produces the same heating effect. In general, the rms value of any function with period T has an effective value given by

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T \overline{v(t)}^2\, dt}$$

Consider a function $v(t) = V_P \sin \omega t$

The rms value, $$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T (V_P \sin \omega t)^2\, d(\omega t)}$$

$$= \sqrt{\frac{1}{T}\int_0^{2\pi} V_P^2 \left[\frac{1-\cos 2\omega t}{2}\right] d(\omega t)}$$

$$= \frac{V_P}{\sqrt{2}} = 0.707\, V_P$$

If the function consists of a number of sinusoidal terms, that is

$$v(t) = V_0 + (V_{c1} \cos \omega t + V_{c2} \cos 2\,\omega t + \cdots) + (V_{s1} \sin \omega t + V_{s2} \sin 2\,\omega t + \cdots)$$

The rms, or effective value is given by

$$V_{\text{rms}} = \sqrt{V_0^2 + \frac{1}{2}(V_{c1}^2 + V_{c2}^2 + \cdots) + \frac{1}{2}(V_{s1}^2 + V_{s2}^2 + \cdots)}$$

Example 4.7 A wire is carrying a direct current of 20 A and a sinusoidal alternating current of peak value 20 A. Find the rms value of the resultant current in the wire.

Solution The rms value of the combined wave

$$= \sqrt{20^2 + \frac{20^2}{2}}$$

$$= \sqrt{400 + 200} = \sqrt{600} = 24.5 \text{ A}$$

4.4.6 Peak Factor

The peak factor of any waveform is defined as the ratio of the peak value of the wave to the rms value of the wave.

$$\text{Peak factor} = \frac{V_P}{V_{\text{rms}}}$$

$$\text{Peak factor of the sinusoidal waveform} = \frac{V_P}{V_P/\sqrt{2}} = \sqrt{2} = 1.414$$

4.4.7 Form Factor

Form factor of a waveform is defined as the ratio of rms value to the average value of the wave.

$$\text{Form factor} = \frac{V_{\text{rms}}}{V_{\text{av}}}$$

Form factor of a sinusoidal waveform can be found from the above relation.

For the sinusoidal wave, the form factor $= \dfrac{V_P/\sqrt{2}}{0.637\ V_P} = 1.11$

4.5 PHASE RELATION IN PURE RESISTOR

When a sinusoidal voltage of certain magnitude is applied to a resistor, a certain amount of sine wave current passes through it. We know the relation between $v(t)$ and $i(t)$ in the case of a resistor. The voltage/current relation in case of a resistor is linear,

i.e. $$v(t) = i(t)\,R$$

Consider the function

$$i(t) = I_m \sin \omega t = IM\,[I_m\, e^{j\omega t}] \text{ or } I_m \angle 0°$$

If we substitute this in the above equation, we have

$$v(t) = I_m R \sin \omega t = V_m \sin \omega t$$
$$= IM\,[V_m\, e^{j\omega t}] \text{ or } V_m \angle 0°$$

where $$V_m = I_m R$$

If we draw the waveform for both voltage and current as shown in Fig. 4.18, there is no phase difference between these two waveforms. The amplitudes of the waveform may differ according to the value of resistance.

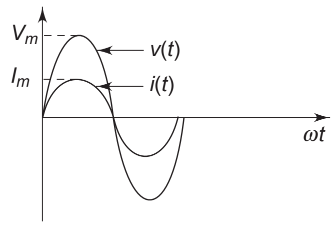


Fig. 4.18

As a result, in pure resistive circuits, the voltages and currents are said to be in phase. Here the term impedance is defined as the ratio of voltage to current function. With ac voltage applied to elements, the ratio of exponential voltage to the corresponding current (impedance) consists of magnitude and phase angles. Since the phase difference is zero in case of a resistor, the phase angle is zero. The impedance in case of resistor consists only of magnitude, i.e.

$$Z = \frac{V_m \angle 0°}{I_m \angle 0°} = R$$

4.6 PHASE RELATION IN A PURE INDUCTOR

Ad discussed earlier in Chapter 1, the voltage current relation in the case of an inductor is given by

$$v(t) = L\frac{di}{dt}$$

Consider the function $i(t) = I_m \sin \omega t = IM\,[I_m\, e^{j\omega t}]$ or $I_m \angle 0°$

$$v(t) = L\frac{d}{dt}(I_m \sin \omega t)$$
$$= L\omega I_m \cos \omega t = \omega L\, I_m \cos \omega t$$
$$v(t) = V_m \cos \omega t, \text{ or } V_m \sin(\omega t + 90°)$$
$$= IM\,[V_m\, e^{j(\omega t + 90°)}] \text{ or } V_m \angle 90°$$

where $V_m = \omega L\, I_m = X_L\, I_m$

and $e^{j90°} = j = 1 \angle 90°$

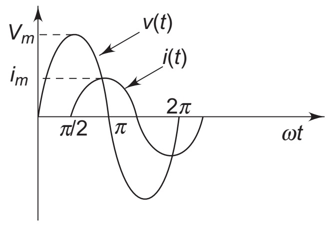


Fig. 4.19

If we draw the waveforms for both, voltage and current, as shown in Fig. 4.19, we can observe the phase difference between these two waveforms.

As a result, in a pure inductor the voltage and current are out of phase. The current lags behind the voltage by 90° in a pure inductor as shown in Fig. 4.20.

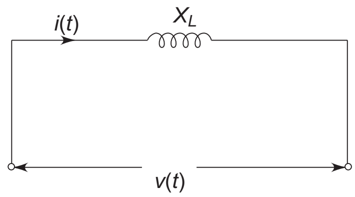


Fig. 4.20

The impedance which is the ratio of exponential voltage to the corresponding current, is given by

$$Z = \frac{V_m \sin(\omega t + 90°)}{I_m \sin \omega t}$$

where $V_m = \omega L I_m$

$$= \frac{I_m\, \omega L \sin(\omega t + 90°)}{I_m \sin \omega t} = \frac{\omega L I_m \angle 90°}{I_m \angle 0°}$$

$\therefore$ $Z = j\omega L = jX_L$

where $X_L = \omega L$ and is called the inductive reactance.

Hence, a pure inductor has an impedance whose value is ωL.

4.7 PHASE RELATION IN PURE CAPACITOR

As discussed in Chapter 1, the relation between voltage and current is given by

$$v(t) = \frac{1}{C}\int i(t)\, dt$$

Consider the function $i(t) = I_m \sin \omega t = IM\,[I_m\, e^{j\omega t}]$ or $I_m \angle 0°$

$$v(t) = \frac{1}{C}\int I_m \sin \omega t\, d(t)$$
$$= \frac{1}{\omega C} I_m [-\cos \omega t]$$

$$= \frac{I_m}{\omega C} \sin(\omega t - 90°)$$

$\therefore$ $$v(t) = V_m \sin(\omega t - 90°)$$
$$= IM\,[I_m\, e^{j(\omega t - 90°)}] \text{ or } V_m \angle -90°$$

where $$V_m = \frac{I_m}{\omega C}$$

$\therefore$ $$\frac{V_m \angle -90°}{I_m \angle 0°} = Z = \frac{-j}{\omega C}$$

Hence, the impedance is $Z = \dfrac{-j}{\omega C} = -jX_C$

where $X_C = \dfrac{1}{\omega C}$ and is called the capacitive reactance.

If we draw the waveform for both, voltage and current, as shown in Fig. 4.21, there is a phase difference between these two waveforms.

As a result, in a pure capacitor, the current leads the voltage by 90°. The impedance value of a pure capacitor

$$X_C = \frac{1}{\omega C}$$

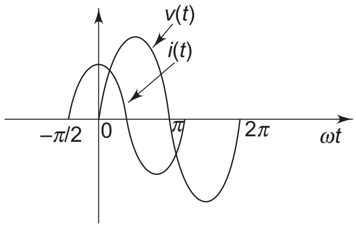


Fig. 4.21

Additional Solved Problems

Problem 4.1 Calculate the frequency for each of the following values of time period.

(a) 2 ms (b) 100 ms (c) 5 ms (d) 5 s

Solution The relation between frequency and period is given by

$$f = \frac{1}{T} \text{ Hz}$$

(a) Frequency $f = \dfrac{1}{20 \times 10^{-3}} = 50$ Hz

(b) Frequency $f = \dfrac{1}{100 \times 10^{-3}} = 10$ Hz

(c) Frequency $f = \dfrac{1}{5 \times 10^{-6}} = 200$ KHz

(d) Frequency $f = \dfrac{1}{5} = 0.2$ Hz

Problem 4.2 Calculate the period for each of the following values of frequency.

(a) 50 Hz (b) 100 KHz (c) 1 Hz (d) 2 MHz

Solution The relation between frequency and period is given by

$$f = \frac{1}{T} \text{ Hz}$$

(a) Time period $T = \frac{1}{f} = \frac{1}{50} = 0.02$ s

(b) Time period $T = \frac{1}{f} = \frac{1}{100 \times 10^3} = 10\ \mu$s

(c) Time period $T = \frac{1}{f} = \frac{1}{1} = 1$ s

(d) Time period $T = \frac{1}{f} = \frac{1}{2 \times 10^6} = 0.5\ \mu$s

Problem 4.3 A sine wave has a frequency of 50 KHz. How many cycles does it complete in 20 ms?

Solution The frequency of sine wave is 50 KHz.
That means in 1 second, a sine wave goes through 50×10^3 cycles.

In 20 ms the number of cycles $= 20 \times 10^{-3} \times 50 \times 10^3$

$= 1$ KHz

That means in 20 ms the sine wave goes through 10^3 cycles.

Problem 4.4 A sine wave has a peak value of 25 V. Determine the following values.

(a) rms (b) peak to peak (c) average

Solution (a) rms value of the sine wave

$$V_{\text{rms}} = 0.707\ V_P$$
$$= 0.707 \times 25 = 17.68 \text{ V}$$

(b) peak to peak value of the sine wave $V_{PP} = 2V_P$

$$V_{PP} = 2 \times 25 = 50 \text{ V}$$

(c) average value of the sine wave

$$V_{\text{av}} = 0.637\ V_P$$
$$= (0.637)25 = 15.93 \text{ V}$$

Problem 4.5 A sine wave has a peak value of 12 V. Determine the following values

(a) rms (b) average (c) crest factor (d) form factor

Solution (a) rms value of the given sine wave

$$= (0.707)12 = 8.48 \text{ V}$$

(b) average value of the sine wave = (0.637)12 = 7.64 V

(c) crest factor of the sine wave $= \dfrac{\text{Peak value}}{\text{rms value}}$

$$= \frac{12}{8.48} = 1.415$$

(d) Form factor $= \dfrac{\text{rms value}}{\text{average value}} = \dfrac{8.48}{7.64} = 1.11$

Problem 4.6 Sine wave 'A' has a positive going zero crossing at 45°. Sine wave 'B' has a positive going zero crossing at 60°. Determine the phase angle between the signals. Which of the signal lags behind the other?

Solution The two signals drawn are shown in Fig. 4.22.

From Fig. 4.22, the signal B lags behind signal A by 15°. In other words, signal A leads signal B by 15°.

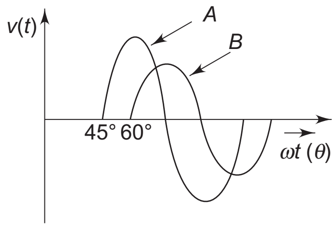


Fig. 4.22

Problem 4.7 One sine wave has a positive peak at 75°, and another has a positive peak at 100°. How much is each sine wave shifted in phase from the 0° reference? What is the phase angle between them?

Solution The two signals are drawn as shown in Fig. 4.23.

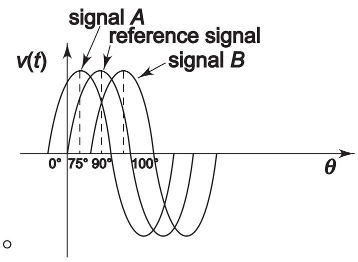


Fig. 4.23

The signal A leads the reference signal by 15°

The signal B lags behind the reference signal by 10°

The phase angle between these two signals is 25°

Problem 4.8 A sinusoidal voltage is applied to the resistive circuit shown in Fig. 4.24. Determine the following values.

(a) I_{rms} (b) I_{av} (c) I_P (d) I_{PP}

Solution The function given to the circuit shown is

$$v(t) = V_P \sin \omega t = 20 \sin \omega t$$

The current passing through the resistor

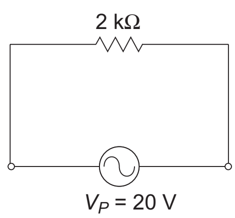


Fig. 4.24

$$i(t) = \frac{v(t)}{R}$$

$$i(t) = \frac{20}{2 \times 10^3} \sin \omega t$$

$$= 10 \times 10^{-3} \sin \omega t$$

$$I_P = 10 \times 10^{-3} \text{ A}$$

The peak value $I_P = 10$ mA
Peak to peak value $I_{PP} = 20$ mA

$$\text{rms value } I_{\text{rms}} = 0.707\ I_P$$
$$= 0.707 \times 10 \text{ mA} = 7.07 \text{ mA}$$
$$\text{Average value } I_{\text{av}} = (0.637)\ I_P$$
$$= 0.637 \times 10 \text{ mA} = 6.37 \text{ mA}$$

Problem 4.9 A sinusoidal voltage is applied to a capacitor as shown in Fig. 4.25. The frequency of the sine wave is 2 KHz. Determine the capacitive reactance.

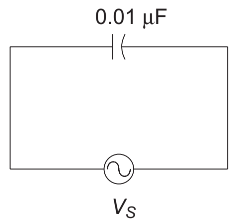


Fig. 4.25

Solution
$$X_C = \frac{1}{2\pi fC}$$
$$= \frac{1}{2\pi \times 2 \times 10^3 \times 0.01 \times 10^{-6}}$$
$$= 7.96 \text{ k}\Omega$$

Problem 4.10 Determine the rms current in the circuit shown in Fig. 4.26.

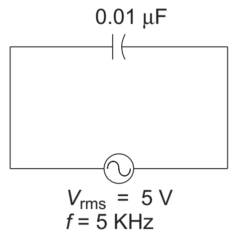


Fig. 4.26

Solution
$$X_C = \frac{1}{2\pi fC}$$
$$= \frac{1}{2\pi \times 5 \times 10^3 \times 0.01 \times 10^{-6}}$$
$$= 3.18 \text{ k}\Omega$$
$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_C} = \frac{5}{3.18 \text{ K}} = 1.57 \text{ mA}$$

Problem 4.11 A sinusoidal voltage is applied to the circuit shown in Fig. 4.27. The frequency is 3 KHz. Determine the inductive reactance.

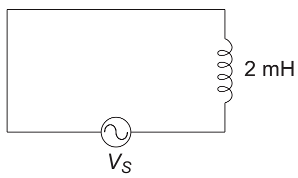


Fig. 4.27

Solution
$$X_L = 2\pi fL$$
$$= 2\pi \times 3 \times 10^3 \times 2 \times 10^{-3}$$
$$= 37.69\ \Omega$$

Problem 4.12 Determine the rms current in the circuit shown in Fig. 4.28.

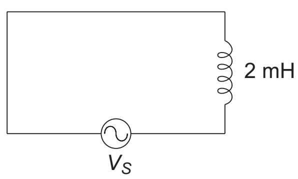


Fig. 4.28

Solution
$$X_L = 2\pi fL$$
$$= 2\pi \times 10 \times 10^3 \times 50 \times 10^{-3}$$
$$X_L = 3.141 \text{ k}\Omega$$
$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_L}$$
$$= \frac{10}{3.141 \times 10^3} = 3.18 \text{ mA}$$

Problem 4.13 Find the form factor of the half-wave rectified sine wave shown in Fig. 4.29.

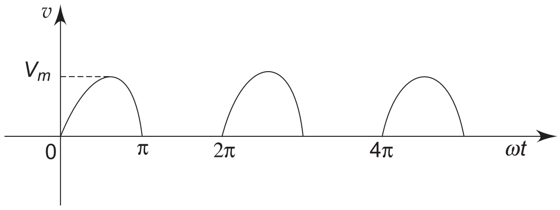


Fig. 4.29

Solution

$$v = V_m \sin \omega t, \quad \text{for } 0 < \omega t < \pi$$
$$= 0, \quad \text{for } \pi < \omega t < 2\pi$$

the period is 2π.

Average value

$$V_{av} = \frac{1}{2\pi}\left\{\int_0^{\pi} V_m \sin \omega t \, d(\omega t) + \int_{\pi}^{2\pi} 0 \, d(\omega t)\right\}$$
$$= 0.318\ V_m$$
$$V_{rms}^2 = \frac{1}{2\pi}\int_0^{\pi} (V_m \sin \omega t)^2 \, d(\omega t)$$
$$= \frac{1}{4} V_m^2$$
$$V_{rms} = \frac{1}{2} V_m$$

$$\text{Form factor} = \frac{V_{rms}}{V_{av}} = \frac{0.5\ V_m}{0.318\ V_m} = 1.572$$

Problem 4.14 Find the average and effective values of the saw tooth waveform shown in Fig. 4.30 below.

Solution From Fig. 4.30 shown, the period is T.

$$V_{av} = \frac{1}{T}\int_0^{T} \frac{V_m}{T} t \, dt$$
$$= \frac{1}{T}\frac{V_m}{T}\int_0^{T} t \, dt$$
$$= \frac{V_m}{T^2}\frac{t^2}{2} = \frac{V_m}{2}$$

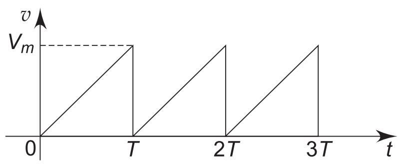


Fig. 4.30

Effective value

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^{T} v^2 \, dt}$$

$$= \sqrt{\frac{1}{T}\int_0^T \left[\frac{V_m}{T}t\right]^2 dt}$$

$$= \frac{V_m}{\sqrt{3}}$$

Problem 4.15 Find the average and rms value of the full wave rectified sine wave shown in Fig. 4.31.

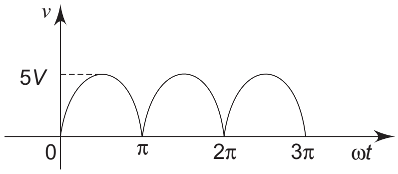


Fig. 4.31

Solution Average value $V_{av} = \frac{1}{\pi}\int_0^{\pi} 5 \sin \omega t \, d(\omega t)$

$= 3.185$

$$\text{Effective value or rms value} = \sqrt{\frac{1}{\pi}\int_0^{\pi} (5 \sin \omega t)^2 \, d(\omega t)}$$

$$= \sqrt{\frac{25}{2}} = 3.54$$

Problem 4.16 The full wave rectified sine wave shown in Fig. 4.32 has a delay angle of 60°. Calculate V_{av} and V_{rms}.

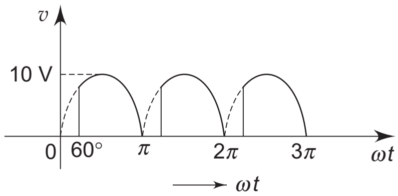


Fig. 4.32

Solution Average value $V_{av} = \frac{1}{\pi}\int_0^{\pi} 10 \sin (\omega t) \, d(\omega t)$

$$= \frac{1}{\pi}\int_{60°}^{\pi} 10 \sin \omega t \, d(\omega t)$$

$$V_{av} = \frac{10}{\pi}(-\cos \omega t)_{60}^{\pi} = 4.78$$

$$\text{Effective value } V_{rms} = \sqrt{\frac{1}{\pi}\int_0^{\pi} (10 \sin \omega t)^2 \, d(\omega t)}$$

$$= \sqrt{\frac{100}{\pi}\int_0^{\pi} \left(\frac{1 - \cos 2\omega t}{2}\right) d(\omega t)}$$

$= 6.33$

Problem 4.17 Find the form factor of the square wave as shown in Fig. 4.33.

Solution $v = 20$ for $0 < t < 0.01$

$= 0$ for $0.01 < t < 0.03$

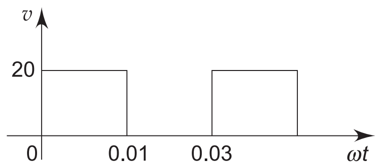


Fig. 4.33

The period is 0.03 sec.

$$\text{Average value } V_{av} = \frac{1}{0.03}\int_0^{0.01} 20\, dt$$

$$= \frac{20(0.01)}{0.03} = 6.66$$

$$\text{Effective value } V_{eff} = \sqrt{\frac{1}{0.03}\int_0^{0.01} (20)^2\, dt} = 66.6 = 0.816$$

$$\text{Form factor} = \frac{0.816}{6.66} = 0.123$$

Practice Problems

4.1 Calculate the frequency of the following values of period.

(a) 0.2 s (b) 50 ms

(c) 500 μs (d) 10 μs

4.2 Calculate the period for each of the values of frequency.

(a) 60 Hz (b) 500 Hz

(c) 1 KHz (d) 200 KHz

(e) 5 MHz

4.3 A certain sine wave has a positive going zero crossing at 0° and an rms value of 20 V. Calculate its instantaneous value at each of the following angles.

(a) 33° (b) 110°

(c) 145° (d) 325°

4.4 For a particular 0° reference sinusoidal current, the peak value is 200 mA; determine the instantaneous values at each of the following.

(a) 35° (b) 190°

(c) 200° (d) 360°

4.5 Sine wave A lags sine wave B by 30°. Both have peak values of 15 V. Sine wave A is the reference with a positive going crossing at 0°. Determine the instantaneous value of sine wave B at 30°, 90°, 45°, 180° and 300°.

4.6 Find the average values of the voltages across R_1 and R_2. In Fig. 4.34 values shown are rms.

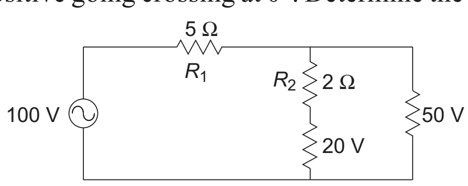


Fig. 4.34

4.7 A sinusoidal voltage is applied to the circuit shown in Fig. 4.35, determine rms current, average current, peak current, and peak to peak current.

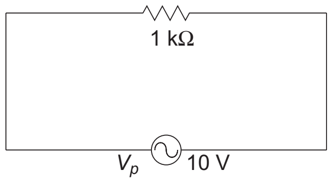


Fig. 4.35

4.8 A sinusoidal voltage of $v(t) = 50 \sin(500t)$ applied to a capacitive circuit. Determine the capacitive reactance, and the current in the circuit.

4.9 A sinusoidal voltage source in series with a dc source as shown in Fig. 4.36

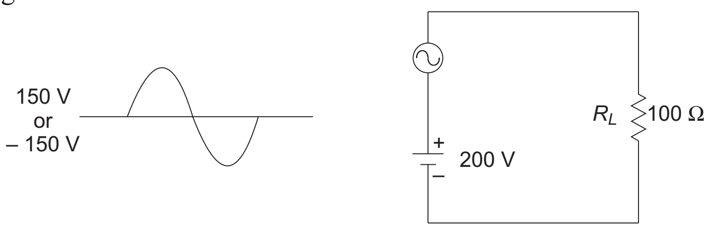


Fig. 4.36

Sketch the voltage across R_L. Determine the maximum current through R_L and the average voltage across R_L.

4.10 Find the effective value of the resultant current in a wire which carries a direct current of 10 A and a sinusoidal current with a peak value of 15 A.

4.11 An alternating current varying sinusoidally, with a frequency of 50 Hz, has an rms value of 20 A. Write down the equation for the instantaneous value and find this value at (a) 0.0025 s (b) 0.0125 s after passing through a positive maximum value. At what time, measured from a positive maximum value, will the instantaneous current be 14.14 A?

4.12 Determine the rms value of the voltage defined by

$$v = 5 + 5 \sin(314t + \pi/6).$$

4.13 Find the effective value of the function $v = 100 + 50 \sin \omega t$.

4.14 A full wave rectified sine wave is clipped at 0.707 of its maximum value as shown in Fig. 4.37. Find the average and effective values of the function.

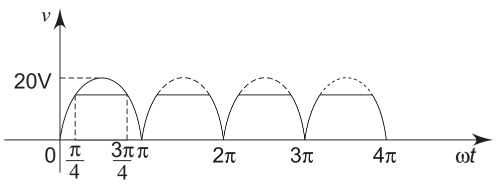


Fig. 4.37

4.15 Find the rms value of the function shown in Fig. 4.38 and described as follows

$$0 < t < 0.1 \qquad v = 40\,(1 - e^{-100t})$$

$$0.1 < t < 0.2 \qquad v = 40\, e^{-50(t - 0.1)}$$

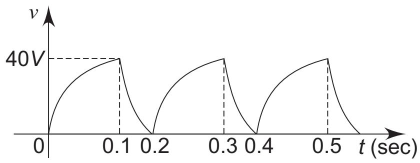


Fig. 4.38

4.16 Calculate average and effective values of the waveform shown in Fig. 4.39 and hence find from factor.

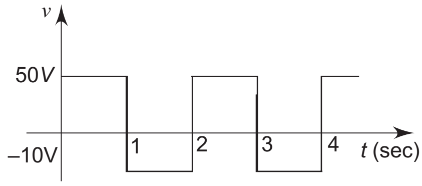


Fig. 4.39

4.17 A full wave rectified sine wave is clipped such that the effective value is 0.5 V_m as shown in Fig. 4.40. Determine the angle at which the wave form is clipped.

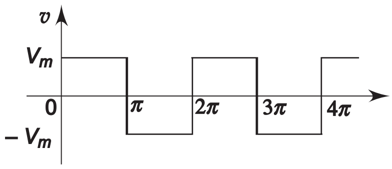


Fig. 4.40

Objective-type Questions

1. One sine wave has a period of 2 ms, another has a period of 5 ms, and other has a period of 10 ms. Which sine wave is changing at a faster rate?
   (a) sine wave with period 2 ms (b) sine wave with period of 5 ms
   (c) all are at the same rate (d) sine wave with period of 10 ms
2. How many cycles does a sine wave go through in 10 s when its frequency is 60 Hz?
   (a) 10 cycles (b) 60 cycles
   (c) 600 cycles (d) 6 cycles
3. If the peak value of a certain sine wave voltage is 10 V, what is the peak to peak value?
   (a) 20 V (b) 10 V
   (c) 5 V (d) 7.07 V

4. If the peak value of a certain sine wave voltage is 5 V, what is the rms value?
   (a) 0.707 V (b) 3.535 V
   (c) 5 V (d) 1.17 V
5. What is the average value of a sine wave over a full cycle?
   (a) V_m (b) $\dfrac{V_m}{\sqrt{2}}$
   (c) zero (d) $\sqrt{2}\,V_m$
6. A sinusoidal current has peak value of 12 A. What is its average value?
   (a) 7.64 A (b) 24 A
   (c) 8.48 A (d) 12 A
7. Sine wave *A* has a positive going zero crossing at 30°. Sine wave *B* has a positive going zero crossing at 45°. What is the phase angle between two signals?
   (a) 30° (b) 45°
   (c) 75° (d) 15°
8. A sine wave has a positive going zero crossing at 0° and an rms value of 20 V. What is its instantaneous value at 145°.
   (a) 7.32 V (b) 16.22 V
   (c) 26.57 V (d) 21.66 V
9. In a pure resistor, the voltage and current are
   (a) out of phase (b) in phase
   (c) 90° out of phase (d) 45° out of phase
10. The rms current through a 10 kΩ resistor is 5 mA. What is the rms voltage drop across the resistor?
    (a) 10 V (b) 5 V
    (c) 50 V (d) zero
11. In a pure capacitor, the voltage
    (a) is in phase with the current (b) is out of phase with the current
    (c) lags behind the current by 90° (d) leads the current by 90°
12. A sine wave voltage is applied across a capacitor; when the frequency of the voltage is increased, the current
    (a) increases (b) decreases
    (c) remains the same (d) is zero
13. The current in a pure inductor
    (a) lags behind the voltage by 90° (b) leads the voltage by 90°
    (c) is in phase with the voltage (d) lags behind the voltage by 45°
14. A sine wave voltage is applied across an inductor; when the frequency of voltage is increased, the current
    (a) increases (b) decreases
    (c) remains the same (d) is zero
15. The rms value of the voltage for a voltage function
    $v = 10 + 5\cos(628t + 30°)$ volts through a circuit is
    (a) 5 V (b) 10 V
    (c) 10.6 V (d) 15 V

16. For the same peak value, which is of the following wave will have the highest rms value
    (a) sine wave (b) square wave
    (c) triangular wave (d) half wave rectified sine wave
17. For 100 volts rms value triangular wave, the peak voltage will be
    (a) 100 V (b) 111 V
    (c) 141 V (d) 173 V
18. The form factor of dc voltage is
    (a) zero (b) infinite
    (c) unity (d) 0.5
19. For the half wave rectified sine wave shown in Fig. 4.41, the peak factor is

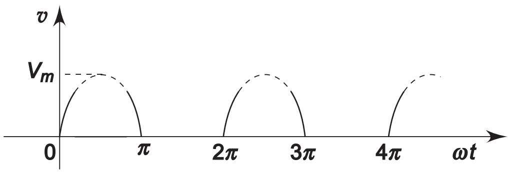


Fig. 4.41

    (a) 1.41 (b) 2.0
    (c) 2.82 (d) infinite
20. For the square wave shown in Fig. 4.42, the form factor is

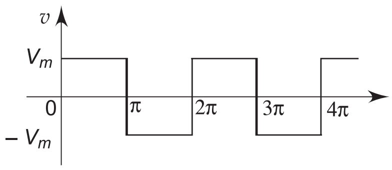


Fig. 4.42

    (a) 2.0 (b) 1.0
    (c) 0.5 (d) zero
21. The power consumed in a circuit element will be least when the phase difference between the current and voltage is
    (a) 0° (b) 30°
    (c) 90° (d) 180°
22. The voltage wave consists of two components: a 50 V dc component and a sinusoidal component with a maximum value of 50 volts. The average value of the resultant will be
    (a) zero (b) 86.6 V
    (c) 50 (d) none of the above

Chapter 5

Complex Impedance

5.1 IMPEDANCE DIAGRAM

So far our discussion has been confined to resistive circuits. Resistance restricts the flow of current by opposing free electron movement. Each element has some resistance; for example, an inductor has some resistance; a capacitance also has some resistance. In the resistive element, there is no phase difference between the voltage and the current. In the case of pure inductance, the current lags behind the voltage by 90 degrees, whereas in the case of pure capacitance, the current leads the voltage by 90 degrees. Almost all electric circuits offer impedance to the flow of current. Impedance is a complex quantity having real and imaginary parts; where the real part is the resistance and the imaginary part is the reactance of the circuit.

Consider the *RL* series circuit shown in Fig. 5.1. If we apply the real function $V_m \cos \omega t$ to the circuit, the response may be $I_m \cos \omega t$. Similarly, if we apply the imaginary function $jV_m \sin \omega t$ to the same circuit, the response is $jI_m \sin \omega t$. If we apply a complex function, which is a combination of real and imaginary functions, we will get a complex response.

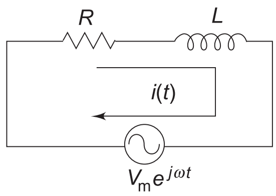


Fig. 5.1

This complex function is $V_m e^{j\omega t} = V_m (\cos \omega t + j \sin \omega t)$.
Applying Kirchhoff's law to the circuit shown in Fig. 5.1,

we get $$V_m e^{j\omega t} = Ri(t) + L\frac{di(t)}{dt}$$

The solution of this differential equation is

$$i(t) = I_m e^{j\omega t}$$

By substituting $i(t)$ in the above equation, we get

$$V_m\, e^{j\omega t} = R\, I_m\, e^{j\omega t} + L\,\frac{d}{dt}\,(I_m\, e^{j\omega t})$$

$$V_m\, e^{j\omega t} = R I_m\, e^{j\omega t} + L I_m\, j\omega\, e^{j\omega t}$$

$$V_m = (R + j\,\omega L)\, I_m$$

Impedance is defined as the ratio of the voltage to current function

$$Z = \frac{V_m\, e^{j\omega t}}{\dfrac{V_m}{R + j\omega L}\, e^{j\omega t}} = R + j\omega L$$

Complex impedance is the total opposition offered by the circuit elements to *ac* current, and can be displayed on the complex plane. The impedance is denoted by Z. Here the resistance R is the real part of the impedance, and the reactance X_L is the imaginary part of the impedance. The resistance R is located on the real axis. The inductive reactance X_L is located on the positive j axis. The resultant of R and X_L is called the complex impedance.

Figure 5.2 is called the impedance diagram for the *RL* circuit. From Fig. 5.2, the impedance $Z = \sqrt{R^2 + (\omega L)^2}$, and angle $\theta = \tan^{-1} \omega L/R$. Here, the impedance is the vector sum of the resistance and inductive reactance. The angle between impedance and resistance is the phase angle between the current and voltage applied to the circuit.

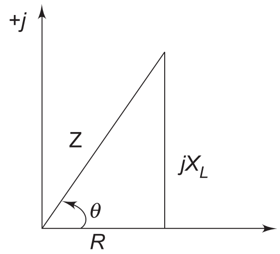


Fig. 5.2

Similarly, if we consider the RC series circuit, and apply the complex function $V_m\, e^{j\omega t}$ to the circuit in Fig. 5.3, we get a complex response as follows.

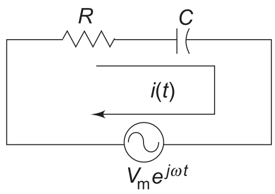


Fig. 5.3

Applying Kirchhoff's law to the above circuit, we get

$$V_m\, e^{j\omega t} = R\,i(t) + \frac{1}{C}\int i(t)\, dt$$

Solving this equation we get,

$$i(t) = I_m\, e^{j\omega t}$$

$$V_m\, e^{j\omega t} = R\, I_m\, e^{j\omega t} + \frac{1}{C} I_m \left(\frac{+1}{j\omega}\right) e^{j\omega t}$$

$$= \left[R I_m - \frac{j}{\omega C}\, I_m \right] e^{j\omega t}$$

$$V_m = \left(R - \frac{j}{\omega C} \right) I_m$$

The impedance

$$Z = \frac{V_m e^{j\omega t}}{V_m/[R - j/\omega C]\, e^{j\omega t}}$$

$$= [R - (j/\omega C)]$$

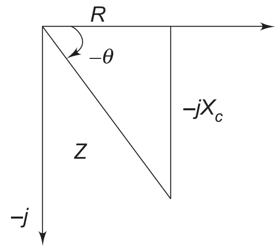


Fig. 5.4

Here impedance Z consists of resistance (R), which is the real part, and capacitive reactance ($X_C = 1/\omega C$), which is the imaginary part of the impedance. The resistance, R, is located on the real axis, and the capacitive reactance X_C is located on the negative j axis in the impedance diagram in Fig. 5.4.

Form Fig. 5.4, impedance $Z = \sqrt{R^2 + X_C^2}$ or $\sqrt{R^2 + (1/\omega C)^2}$ and angle $\theta = \tan^{-1}(1/\omega CR)$. Here, the impedance, Z, is the vector sum of resistance and capacitive reactance.The angle between resistance and impedance is the phase angle between the applied voltage and current in the circuit.

5.2 PHASOR DIAGRAM

A phasor diagram can be used to represent a sine wave in terms of its magnitude and angular position. Examples of phasor diagrams are shown in Fig. 5.5.

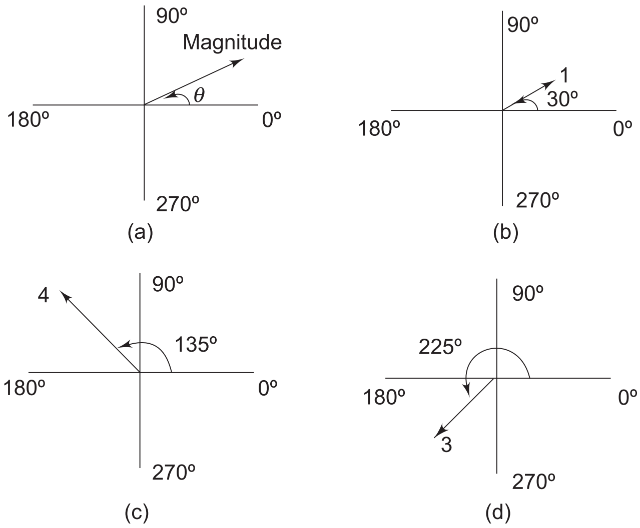


Fig. 5.5

In Fig. 5.5(a), the length of the arrow represents the magnitude of the sine wave; angle θ represents the angular position of the sine wave. In Fig. 5.5(b), the magnitude of the sine wave is one and the phase angle is 30°. In Fig. 5.5(c) and (d), the magnitudes are four and three, and phase angles are 135° and 225°, respectively. The position of a phasor at any instant can be expressed as a positive or negative angle. Positive angles are measured counterclockwise from 0°, whereas negative angles are measured clockwise from 0°. For a given positive angle θ, the corresponding negative angle is θ – 360°. This is shown in Fig. 5.6(a). In Fig. 5.6(b), the positive angle 135° of vector *A* can be represented by a negative angle – 225°, (135° – 360°).

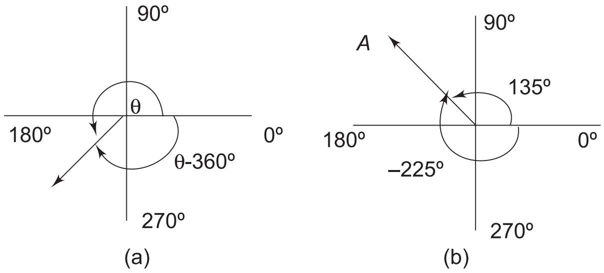


Fig. 5.6

A phasor diagram can be used to represent the relation between two or more sine waves of the same frequency. For example, the sine waves shown in Fig. 5.7(a) can be represented by the phasor diagram shown in Fig. 5.7(b).

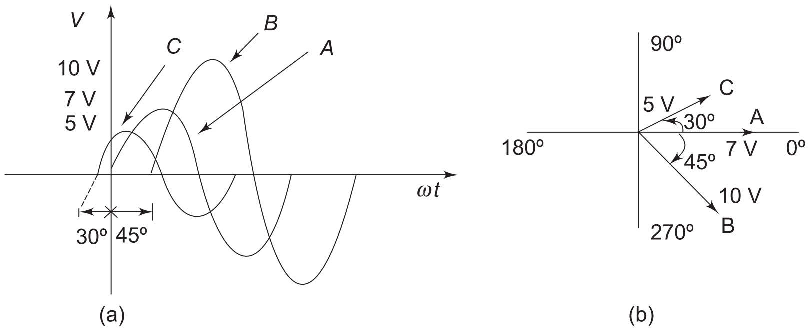


Fig. 5.7

In the above figure, sine wave *B* lags behind sine wave *A* by 45°; sine wave *C* leads sine wave *A* by 30°. The length of the phasors can be used to represent peak, rms, or average values.

Example 5.1 Draw the phasor diagram to represent the two sine waves shown in Fig. 5.8.

Solution The phasor diagram representing the sine waves is shown in Fig. 5.9. The length of the each phasor represents the peak value of the sine wave.

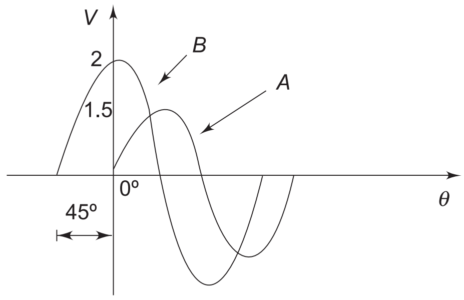


Fig. 5.8

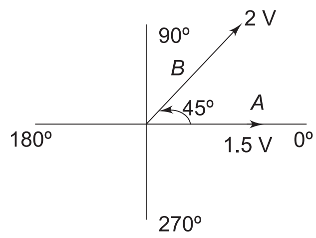


Fig. 5.9

5.3 SERIES CIRCUITS

The impedance diagram is a useful tool for analysing series ac circuits. Basically we can divide the series circuits as RL, RC and RLC circuits. In the analysis of series ac circuits, one must draw the impedance diagram. Although the impedance diagram usually is not drawn to scale, it does represent a clear picture of the phase relationships.

5.3.1 Series RL Circuit

If we apply a sinusoidal input to an RL circuit, the current in the circuit and all voltages across the elements are sinusoidal. In the analysis of the RL series circuit, we can find the impedance, current, phase angle and voltage drops. In Fig. 5.10 (a) the resistor voltage (V_R) and current (I) are in phase with each other, but lag behind the source voltage (V_S). The inductor voltage (V_L) leads the source voltage (V_S). The phase angle between current and voltage in a pure inductor is always 90°. The amplitudes of voltages and currents in the circuit are completely dependent on the values of elements (i.e. the resistance and inductive reactance). In the circuit shown, the phase angle is somewhere between zero and 90° because of the series combination of resistance with inductive reactance, which depends on the relative values of R and X_L.

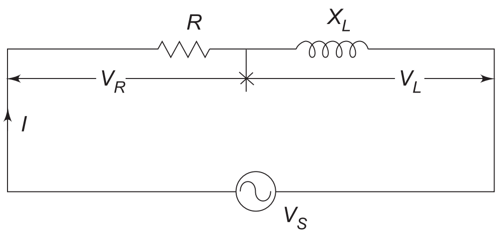


Fig. 5.10(a)

The phase relation between current and voltages in a series RL circuit is shown in Fig. 5.10(b).

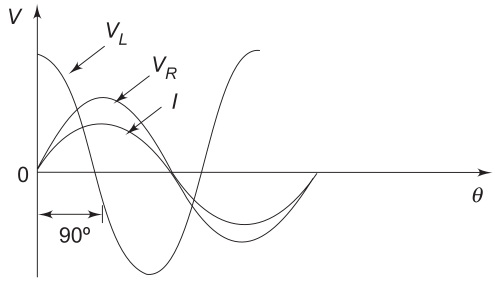


Fig. 5.10(b)

Here V_R and I are in phase. The amplitudes are arbitrarily chosen. From Kirchhoff's voltage law, the sum of the voltage drops must equal the applied voltage. Therefore, the source voltage V_S is the phasor sum of V_R and V_L.

$$\therefore \qquad V_S = \sqrt{V_R^2 + V_L^2}$$

The phase angle between resistor voltage and source voltage is

$$\theta = \tan^{-1} (V_L/V_R)$$

where θ is also the phase angle between the source voltage and the current. The phasor diagram for the series RL circuit that represents the waveforms in Fig. 5.10(c).

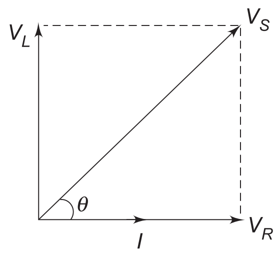


Fig. 5.10(c)

Example 5.2 To the circuit shown in Fig. 5.11, consisting a 1 kΩ resistor connected in series with a 50 mH coil, a 10 V rms, 10 kHz signal is applied. Find impedance Z, current I, phase angle θ, voltage across resistance V_R, and the voltage across inductance V_L.

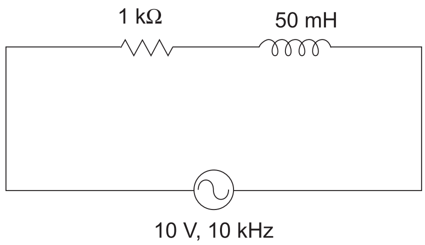


Fig. 5.11

Solution Inductive reactance $X_L = \omega L$

$$= 2\pi f L = (6.28)\ (10^4)\ (50 \times 10^{-3}) = 3140\ \Omega$$

In rectangular form,

Total impedance $Z = (1000 + j3140)\ \Omega$

$$= \sqrt{R^2 + X_L^2}$$

$$= \sqrt{(1000)^2 + (3140)^2} = 3295.4\ \Omega$$

Current $I = V_S/Z = 10/3295.4 = 3.03$ mA

Phase angle $\theta = \tan^{-1}(X_L/R) = \tan^{-1}(3140/1000) = 72.33°$

Therefore, in polar form total impedance $Z = 3295.4\ \angle 72.33°$

Voltage across resistance $V_R = IR$

$$= 3.03 \times 10^{-3} \times 1000 = 3.03\ \text{V}$$

Voltage across inductive reactance $V_L = IX_L$

$$= 3.03 \times 10^{-3} \times 3140 = 9.51\ \text{V}$$

Example 5.3 Determine the source voltage and the phase angle, if voltage across the resistance is 70 V and voltage across the inductive reactance is 20 V as shown in Fig. 5.12.

Solution In Fig. 5.12, the source voltage is given by

$$V_S = \sqrt{V_R^2 + V_L^2}$$

$$= \sqrt{(70)^2 + (20)^2} = 72.8\ \text{V}$$

The angle between current and source voltage is

$$\theta = \tan^{-1}(V_L/V_R) = \tan^{-1}(20/70) = 15.94°$$

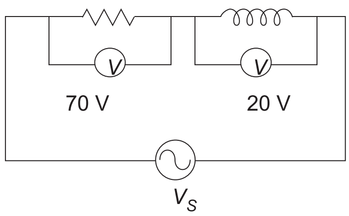


Fig. 5.12

5.3.2 Series RC Circuit

When a sinusoidal voltage is applied to an RC series circuit, the current in the circuit and voltages across each of the elements are sinusoidal. The series RC circuit is shown in Fig. 5.13 (a).

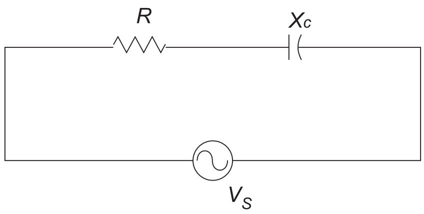


Fig. 5.13(a)

Here the resistor voltage and current are in phase with each other. The capacitor voltage lags behind the source voltage. The phase angle between the current and the capacitor voltage is always 90°. The amplitudes and the phase relations between the voltages and current depend on the ohmic values of the resistance and the capacitive reactance. The circuit is a series combination of both resistance and capacitance; and the phase angle between the applied voltage and the total current is somewhere between zero and 90°, depending on the relative values of the resistance and reactance. In a series RC circuit, the current is the same through the resistor and the capacitor. Thus, the resistor voltage is in phase with the current, and the capacitor voltage lags behind the current by 90° as shown in Fig. 5.13(b).

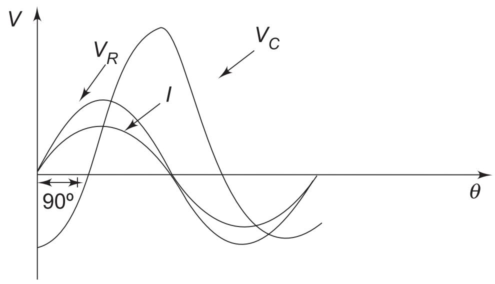


Fig. 5.13(b)

Here, I leads V_C by 90°. V_R and I are in phase. From Kirchhoff's voltage law, the sum of the voltage drops must be equal to the applied voltage. Therefore, the source voltage is given by

$$V_S = \sqrt{V_R^2 + V_C^2}$$

The phase angle between the resistor voltage and the source voltage is

$$\theta = \tan^{-1}(V_C/V_R)$$

Since the resistor voltage and the current are in phase, θ also represents the phase angle between the source voltage and current. The voltage phasor diagram for the series RC circuit, voltage and current phasor diagrams represented by the waveforms in Fig. 5.13(b) are shown in Fig. 5.13(c).

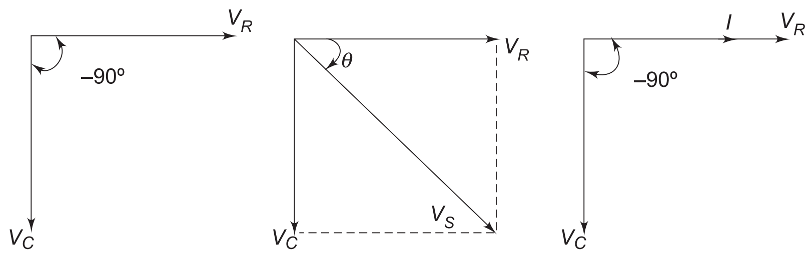


Fig. 5.13(c)

Example 5.4 A sine wave generator supplies a 500 Hz, 10 V rms signal to a 2 kΩ resistor in series with a 0.1 μF capacitor as shown in Fig. 5.14. Determine the total impedance Z, current I, phase angle θ, capacitive voltage V_C, and resistive voltage V_R.

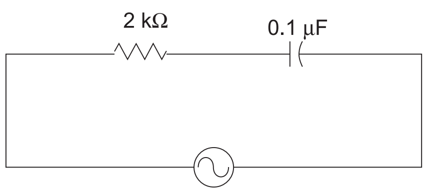


Fig. 5.14

Solution To find the impedance Z, we first solve for X_C

$$X_C = \frac{1}{2\pi fC} = \frac{1}{6.28 \times 500 \times 0.1 \times 10^{-6}}$$
$$= 3184.7\ \Omega$$

In rectangular form,

Total impedance $Z = (2000 - j3184.7)\ \Omega$

$$Z = \sqrt{(2000)^2 + (3184.7)^2}$$
$$= 3760.6\ \Omega$$

Phase angle $\theta = \tan^{-1}(-X_C/R) = \tan^{-1}(-3184.7/2000) = -57.87°$

Current $I = V_S/Z = 10/3760.6 = 2.66$ mA

Capacitive voltage $V_C = IX_C$

$$= 2.66 \times 10^{-3} \times 3184.7 = 8.47\text{ V}$$

Resistive voltage $V_R = IR$

$$= 2.66 \times 10^{-3} \times 2000 = 5.32\text{ V}$$

The arithmetic sum of V_C and V_R does not give the applied voltage of 10 volts. In fact, the total applied voltage is a complex quantity. In rectangular form,

Total applied voltage $V_S = 5.32 - j8.47$ V

In polar form

$$V_S = 10 \angle -57.87°\text{ V}$$

The applied voltage is complex, since it has a phase angle relative to the resistive current.

Example 5.5 Determine the source voltage and phase angle when the voltage across the resistor is 20 V and the capacitor is 30 V as shown in Fig. 5.15.

Solution Since V_R and V_C are 90° out of phase, they cannot be added directly. The source voltage is the phasor sum of V_R and V_C.

$$\therefore \quad V_S = \sqrt{V_R^2 + V_C^2} = \sqrt{(20)^2 + (30)^2} = 36\text{ V}$$

The angle between the current and source voltage is

$$\theta = \tan^{-1}(V_C/V_R) = \tan^{-1}(30/20) = 56.3°$$

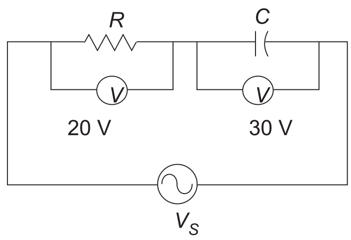


Fig. 5.15

5.3.3 Series R-L-C Circuit

A series RLC circuit is the series combination of resistance, inductance and capacitance. If we observe the impedance diagrams of series RL and series RC circuits as shown in Fig. 5.16(a) and (b), the inductive reactance, X_L, is displayed on the $+j$ axis and the capacitive reactance, X_C, is displayed on the $-j$ axis. These reactance are 180° apart and tend to cancel each other.

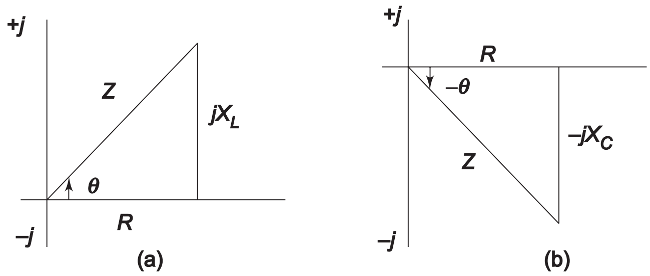


Fig. 5.16

The magnitude and type of reactance in a series RLC circuit is the difference of the two reactance. The impedance for an RLC series circuit is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$. Similarly, the phase angle for an RLC circuit is

$$\theta = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Example 5.6 In the circuit shown in Fig. 5.17, determine the total impedance, current I, phase angle θ, and the voltage across each element.
Solution To find impedance Z, we first solve for X_C and X_L

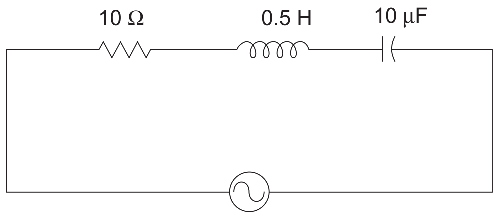


Fig. 5.17

$$X_C = \frac{1}{2\pi f C} = \frac{1}{6.28 \times 50 \times 10 \times 10^{-6}}$$

$$= 318.5\ \Omega$$

$$X_L = 2\pi f L = 6.28 \times 0.5 \times 50 = 157\ \Omega$$

Total impedance in rectangular form

$$Z = (10 + j157 - j318.5)\ \Omega$$

$$= 10 + j(157 - 318.5)\ \Omega = 10 - j161.5\ \Omega$$

Here, the capacitive reactance dominates the inductive reactance.

$$Z = \sqrt{(10)^2 + (161.5)^2}$$

$$= \sqrt{100 + 26082{\cdot}2} = 161.8\ \Omega$$

Current $$I = V_S/Z = \frac{50}{161.8} = 0.3\ \text{A}$$

Phase angle $\theta = \tan^{-1}[(X_L - X_C)/R] = \tan^{-1}(-161.5/10) = -86.45°$

Voltage across the resistor $V_R = IR = 0.3 \times 10 = 3$ V

Voltage across the capacitive reactance = $IX_C = 0.3 \times 318.5 = 95.55$ V

Voltage across the inductive reactance = $IX_L = 0.3 \times 157 = 47.1$ V

5.4 PARALLEL CIRCUITS

The complex number system simplifies the analysis of parallel ac circuits. In series circuits, the current is the same in all parts of the series circuit. In parallel ac circuits, the voltage is the same across each element.

5.4.1 Parallel RC Circuits

The voltages for an RC series circuit can be expressed using complex numbers, where the resistive voltage is the real part of the complex voltage and the capacitive voltage is the imaginary part. For parallel RC circuits, the voltage is the same across each component. Here the total current can be represented by a complex number. The real part of the complex current expression is the resistive current; the capacitive branch current is the imaginary part.

Example 5.7 A signal generator supplies a sine wave of 20 V, 5 KHz to the circuit shown in Fig. 5.18. Determine the total current I_T, the phase angle and total impedance in the circuit.

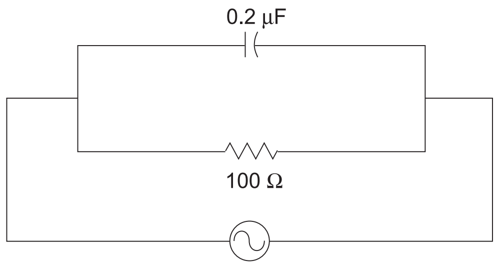


Fig. 5.18

Solution Capacitive reactance

$$X_C = \frac{1}{2\pi fC} = \frac{1}{6.28 \times 5 \times 10^3 \times 0.2 \times 10^{-6}} = 159.2\ \Omega$$

Since the voltage across each element is the same as the applied voltage, we can solve for the two branch currents.

∴ Current in the resistance branch

$$I_R = \frac{V_S}{R} = \frac{20}{100} = 0.2\ \text{A}$$

and current in the capacitive branch

$$I_C = \frac{V_S}{X_C} = \frac{20}{159.2} = 0.126 \text{ A}$$

The total current is the vector sum of the two branch currents.

$\therefore$ Total current I_T $= (I_R + jI_C)$ A = (0.2 + j0.13) A

In polar form $I_T = 0.24 \angle 33°$

So the phase angle θ between applied voltage and total current is 33°. It indicates that the total line current is 0.24 A and leads the voltage by 33°. Solving for impedance, we get

$$Z = \frac{V_S}{I_T} = \frac{20\angle 0°}{0.24\angle 33°} = 83.3 \angle -33° \ \Omega$$

5.4.2 Parallel RL Circuits

In a parallel RL circuit, the inductive current is imaginary and lies on the $-j$ axis. The current angle is negative when the impedance angle is positive. Here also the total current can be represented by a complex number. The real part of the complex current expression is the resistive current; and inductive branch current is the imaginary part.

Example 5.8 A 50 Ω resistor is connected in parallel with an inductive reactance of 30 Ω. A 20 V signal is applied to the circuit. Find the total impedance and line current in the circuit shown in Fig. 5.19.

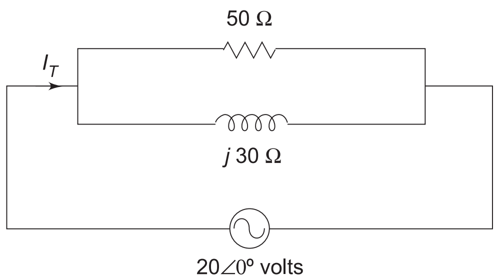


Fig. 5.19

Solution Since the voltage across each element is the same as the applied voltage,
current in the resistive branch,

$$I_R = \frac{V_S}{R} = \frac{20 \angle 0°}{50 \angle 0°} = 0.4 \text{ A}$$

current in the inductive branch

$$I_L = \frac{V_S}{X_L} = \frac{20 \angle 0°}{30 \angle 90°} = 0.66 \angle -90°$$

Total current is $I_T = 0.4 - j0.66$

In polar form, $I_T = 0.77 \angle -58.8°$

Here the current lags behind the voltage by 58.8°

Total impedance $Z = \dfrac{V_S}{I_T}$

$$= \frac{20 \angle 0°}{0.77 \angle -58.8°} = 25.97 \angle 58.8° \ \Omega$$

5.5 COMPOUND CIRCUITS

In many cases, ac circuits to be analysed consist of a combination of series and parallel impedances. Circuits of this type are known as series-parallel, or compound circuits. Compound circuits can be simplified in the same manner as a series-parallel dc circuit consisting of pure resistances.

Example 5.9 Determine the equivalent impedance of Fig. 5.20.

Solution In the circuit, Z_1 is in series with the parallel combination of Z_2 and Z_3.

where $Z_1 = (5 + j10)\ \Omega$,

$Z_2 = (2 - j4)\ \Omega$,

$Z_3 = (1 + j3)\ \Omega$

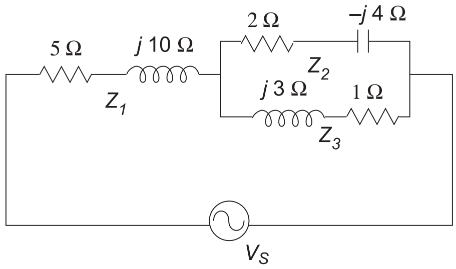


Fig. 5.20

The total impedance

$$Z_T = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}$$

$$= (5 + j10) + \frac{(2 - j4)(1 + j3)}{(2 - j4) + (1 + j3)}$$

$$= (5 + j10) + \frac{4.47 \angle -63.4° \times 3.16 \angle +71.5°}{3 - j1}$$

$$= (5 + j10) + \frac{14.12 \angle 81°}{3 - j1}$$

$$= (5 + j10 + \frac{14.12 \angle 8.1°}{3.16 \angle -18°}$$

$$= 5 + j10 + 4.46 \angle 26.1°$$

$$= 5 + j10 + 4 + j1.96$$

$$= 9 + j11.96$$

The equivalent circuit for the compound circuit shown in Fig. 5.20 is a series circuit containing 9 Ω of resistance and 11.96 Ω of inductive reactance. In polar form,

$$Z = 14.96 \angle 53.03°$$

The phase angle between current and applied voltage is

$$\theta = 53.03°$$

Example 5.10 In the circuit of Fig. 5.21, determine the values of the following (a) Z_T (b) I_T (c) θ.

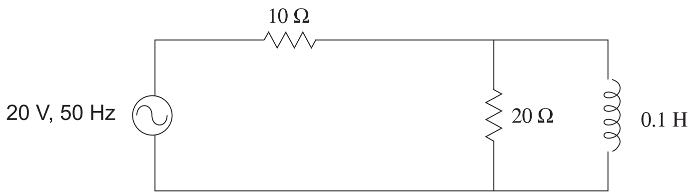


Fig. 5.21

Solution First, the inductive reactance is calculated.

$$X_L = 2\pi f L$$
$$= 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$

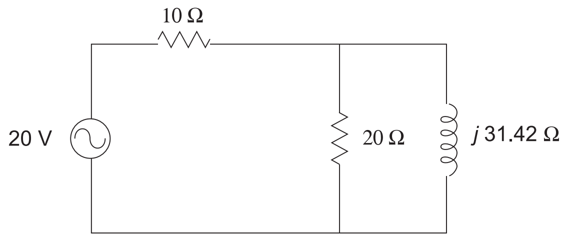


Fig. 5.22

In Fig. 5.22, the 10 Ω resistance is in series with the parallel combination of 20 Ω and $j31.42\ \Omega$

$$\therefore \qquad Z_T = 10 + \frac{(20)(j31.42)}{(20 + j31.42)}$$
$$= 10 + \frac{628.4 \angle 90°}{37.24 \angle 57.52°} = 10 + 16.87 \angle 32.48°$$
$$= 10 + 14.23 + j9.06 = 24.23 + j9.06$$

In polar form, $Z_T = 25.87 \angle 20.5°$
Here the current lags behind the applied voltage by 20.5°

Total current $$I_T = \frac{V_S}{Z_T}$$
$$= \frac{20}{25.87 \angle 20.5°} = 0.77 \angle -20.5°$$

The phase angle between voltage and current is

$$\theta = 20.5°$$

Additional Solved Problems

Problem 5.1 A signal generator supplies a 30 V, 100 Hz signal to a series circuit shown in Fig. 5.23. Determine the impedance, the line current and phase angle in the given circuit.

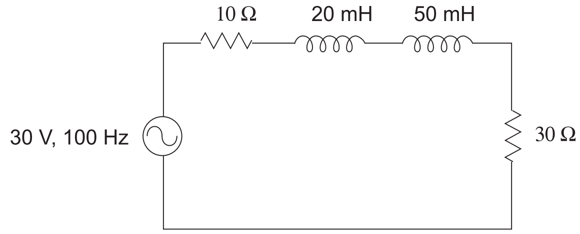


Fig. 5.23

Solution In Fig. 5.24, the resistances and inductive reactances can be combined.

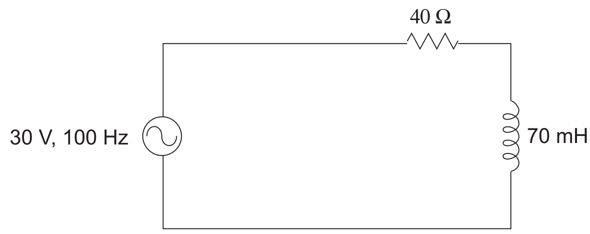


Fig. 5.24

First, we find the inductive reactance

$$X_L = 2\pi f L$$

$$= 2\pi \times 100 \times 70 \times 10^{-3} = 43.98\ \Omega$$

In rectangular form, the total impedance is

$$Z_T = (40 + j43.98)\ \Omega$$

Current $$I = \frac{V_S}{Z_T} = \frac{30\ \angle 0°}{40 + j43.98}$$

Here we are taking source voltage as the reference voltage

$\therefore$ $$I = \frac{30\ \angle 0°}{59.45\ \angle +47.7°} = 0.5\ \angle -47.7°\ \text{A}$$

The current lags behind the applied voltage by 47.7°
Hence, the phase angle between voltage and current

$$\theta = 47.7°$$

Problem 5.2 For the circuit shown in Fig. 5.25, find the effective voltages across resistance and inductance, and also determine the phase angle.

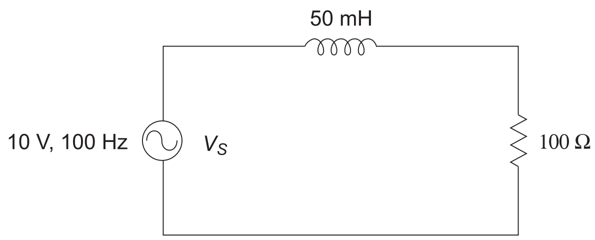


Fig. 5.25

Solution In rectangular form,

Total impedance $Z_T = R + jX_L$

where $X_L = 2\pi f L$

$$= 2\pi \times 100 \times 50 \times 10^{-3} = 31.42\ \Omega$$

$\therefore$ $Z_T = (100 + j31.42)\ \Omega$

$$\text{Current } I = \frac{V_S}{Z_T} = \frac{10\ \angle 0°}{(100 + j31.42)} = \frac{10\ \angle 0°}{104.8\ \angle 17.44°} = 0.095\ \angle{-17.44°}$$

Therefore, the phase angle between voltage and current

$$\theta = 17.44°$$

Voltage across resistance is $V_R = IR$

$$= 0.095 \times 100 = 9.5 \text{ V}$$

Voltage across inductive reactance is $V_L = IX_L$

$$= 0.095 \times 31.42 = 2.98 \text{ V}$$

Problem 5.3 For the circuit shown in Fig. 5.26, determine the value of impedance when a voltage of $(30 + j50)$ V is applied to the circuit and the current flowing is $(-5 + j15)$ A. Also determine the phase angle.

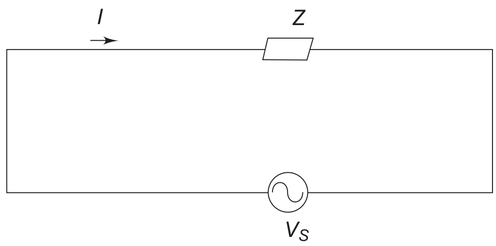


Fig. 5.26

Solution Impedance $Z = \dfrac{V_S}{I} = \dfrac{30 + j50}{-5 + j15}$

$$= \frac{58.31\ \angle 59°}{15.81\ \angle 108.43°} = 3.69\ \angle{-49.43°}$$

In rectangular form, the impedance $Z = 2.4 - j2.8$

Therefore, the circuit has a resistance of 2.4 Ω in series with capacitive reactance 2.8 Ω.

Phase angle between voltage and current is $\theta = 49.43°$. Here, the current leads the voltage by 49.43°.

Problem 5.4 A resistor of 100 Ω is connected in series with a 50 μF capacitor. Find the effective voltage applied to the circuit at a frequency of 50 Hz. The effective voltage across the resistor is 170 V. Also determine voltage across the capacitor and phase angle. (See Fig. 5.27)

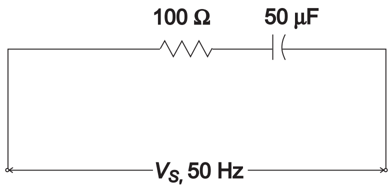


Fig. 5.27

Solution Capacitive reactance $X_C = \dfrac{1}{2\pi f C}$

$$= \frac{1}{2\pi \times 50 \times 50 \times 10^{-6}} = 63.66\ \Omega$$

Total impedance $Z_T = (100 - j63.66)\ \Omega$

Voltage across 100 Ω resistor is $V_R = 170$ V

Current in resistor, $I = \dfrac{170}{100} = 1.7$ A

Since the same current passes through capacitive reactance, the effective voltage across the capacitive reactance is

$$V_C = IX_C$$
$$= 1.7 \times 63.66 = 108.22\ \text{V}$$

The effective applied voltage to the circuit

$$V_S = \sqrt{V_R^2 + V_C^2}$$
$$= \sqrt{(170)^2 + (108.22)^2} = 201.5\ \text{V}$$

Total impedance in polar form

$$Z_T = 118.54\ \angle -32.48°$$

Therefore, the current leads the applied voltage by 32.48°.

Problem 5.5 For the circuit shown in Fig. 5.28, determine the total current, impedance Z and phase angle.

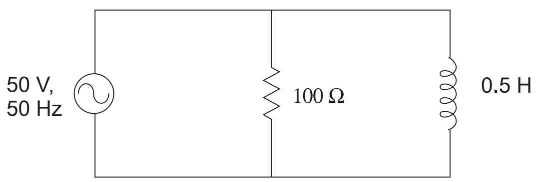


Fig. 5.28

Solution Here, the voltage across each element is the same as the applied voltage.

Current in resistive branch $I_R = \dfrac{V_S}{R} = \dfrac{50}{100} = 0.5$ A

Inductive reactance $X_L = 2\pi fL$

$= 2\pi \times 50 \times 0.5 = 157.06\ \Omega$

Current in inductive branch

$$I_L = \frac{V_S}{X_L} = \frac{50}{157.06} = 0.318 \text{ A}$$

Total current $I_T = \sqrt{I_R^2 + I_L^2}$

or $(0.5 - j0.318)\text{A} = 0.59 \angle{-32.5°}$

For parallel RL circuits, the inductive susceptance is

$$B_L = \frac{1}{X_L} = \frac{1}{157.06} = 0.0064 \text{ S}$$

Conductance $G = \dfrac{1}{100} = 0.01$ S

$\therefore$ Admittance $= \sqrt{G^2 + B_L^2} = \sqrt{(0.01)^2 + (0.0064)^2}$

$= 0.0118$ S

Converting to impedance, we get

$$Z = \frac{1}{Y} = \frac{1}{0.012} = 83.33\ \Omega$$

Phase angle $\theta = \tan^{-1}\left(\dfrac{R}{X_L}\right) = \tan^{-1}\left(\dfrac{100}{157.06}\right) = 32.48°$

Problem 5.6 Determine the impedance and phase angle in the circuit shown in Fig. 5.29.

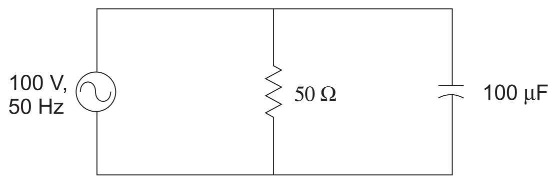


Fig. 5.29

Solution Capacitive reactance $X_C = \dfrac{1}{2\pi fC}$

$$= \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\ \Omega$$

Capacitive susceptance $B_C = \dfrac{1}{X_C}$

$$= \frac{1}{31.83} = 0.031 \text{ S}$$

Conductance $G = \dfrac{1}{R} = \dfrac{1}{50} = 0.02$ S

Total admittance $Y = \sqrt{G^2 + B_C^2}$

$$= \sqrt{(0.02)^2 + (0.031)^2}$$
$$= 0.037 \text{ S}$$

Total impedance $Z = \dfrac{1}{Y} = \dfrac{1}{0.037} = 27.02\ \Omega$

Phase angle $\theta = \tan^{-1}\left(\dfrac{R}{X_C}\right)$

$$= \tan^{-1}\left(\frac{50}{31.83}\right)$$
$$\theta = 57.52°$$

Problem 5.7 For the parallel circuit in Fig. 5.30, find the magnitude of current in each branch and the total current. What is the phase angle between the applied voltage and total current?

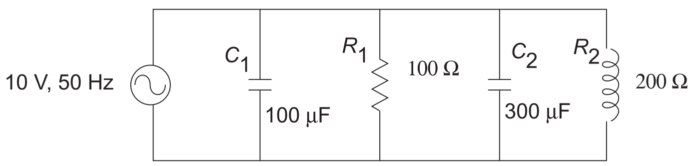


Fig. 5.30

Solution First let us find the capacitive reactances.

$$X_{C1} = \frac{1}{2\pi f C_1}$$
$$= \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\ \Omega$$
$$X_{C2} = \frac{1}{2\pi f C_2} = \frac{1}{2\pi \times 50 \times 300 \times 10^{-6}}$$
$$= 10.61\ \Omega$$

Here the voltage across each element is the same as the applied voltage.

Current in the 100 μF capacitor $I_C = \dfrac{V_S}{X_{C_1}}$

$$= \frac{10 \angle 0^\circ}{31.83 \angle -90^\circ} = 0.31 \angle 90^\circ \text{ A}$$

Current in the 300 μF capacitor $I_{C_2} = \dfrac{V_S}{X_{C_2}}$

$$= \frac{10 \angle 0^\circ}{10.61 \angle -90^\circ} = 0.94 \angle 90^\circ \text{ A}$$

Current in the 100 Ω resistor is $I_{R_1} = \dfrac{V_S}{R_1} = \dfrac{10}{100} = 0.1$ A

Current in the 200 Ω resistor is $I_{R_2} = \dfrac{V_S}{R_2} = \dfrac{10}{200} = 0.05$ A

Total current $I_T = I_{R_1} + I_{R_2} + j(I_{C_1} + I_{C_2})$

$$= 0.1 + 0.05 + j(0.31 + 0.94) = 1.26 \angle 83.2^\circ \text{ A}$$

The circuit shown in Fig. 5.30 can be simplified into a single parallel RC circuit as shown in Fig. 5.31.

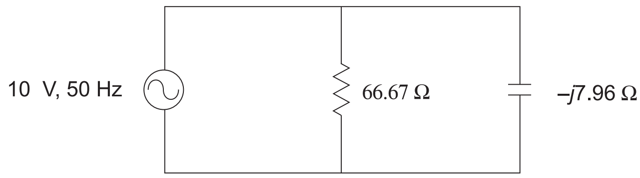


Fig. 5.31

In Fig. 5.30, the two resistances are in parallel and can be combined into a single resistance. Similarly, the two capacitive reactances are in parallel and can be combined into a single capacitive reactance.

$$R = \frac{R_1 R_2}{R_1 + R_2} = 66.67 \ \Omega$$

$$X_C = \frac{X_{C_1} X_{C_2}}{X_{C_1} + X_{C_2}} = 7.96 \ \Omega$$

Phase angle θ between voltage and current is

$$\theta = \tan^{-1}\left(\frac{R}{X_C}\right) = \tan^{-1}\left(\frac{66.67}{7.96}\right) = 83.19^\circ$$

Here the current leads the applied voltage by 83.19°.

Problem 5.8 For the circuit shown in Fig. 5.32, determine the total impedance, total current and phase angle.

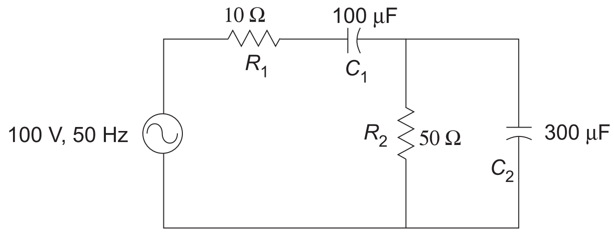


Fig. 5.32

Solution First, we calculate the magnitudes of the capacitive reactances.

$$X_{C_1} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\ \Omega$$

$$X_{C_2} = \frac{1}{2\pi \times 50 \times 300 \times 10^{-6}} = 10.61\ \Omega$$

We find the impedance of the parallel portion by finding the admittance.

$$G_2 = \frac{1}{R_2} = \frac{1}{50} = 0.02\ \text{S}$$

$$B_{C_2} = \frac{1}{X_{C_2}} = \frac{1}{10.61} = 0.094\ \text{S}$$

$$Y_2 = \sqrt{G_2^2 + B_{C_2}^2} = \sqrt{(0.02)^2 + (0.094)^2} = 0.096\ \text{S}$$

$$Z_2 = \frac{1}{Y_2} = \frac{1}{0.096} = 10.42\ \Omega$$

The phase angle associated with the parallel portion of the circuit

$$\theta_P = \tan^{-1}(R_2/X_{C_2}) = \tan^{-1}(50/10.61) = 78.02°$$

The series equivalent values for the parallel portion are

$$R_{eq} = Z_2 \cos\theta_P = 10.42 \cos(78.02°) = 2.16\ \Omega$$

$$X_{C(eq)} = Z_2 \sin\theta_P = 10.42 \sin(78.02°) = 10.19\ \Omega$$

The total resistance

$$R_T = R_1 + R_{eq}$$
$$= (10 + 2.16) = 12.16\ \Omega$$

$$X_{C_T} = X_{C_1} + X_{C(eq)}$$
$$= (31.83 + 10.19) = 42.02\ \Omega$$

Total impedance

$$Z_T = \sqrt{R_T^2 + X_{C_T}^2}$$
$$= \sqrt{(12.16)^2 + (42.02)^2} = 43.74\ \Omega$$

We can also find the total current by using Ohm's law

$$I_T = \frac{V_S}{Z_T} = \frac{100}{43.74} = 2.29 \text{ A}$$

The phase angle

$$\theta = \tan^{-1}\left(\frac{X_{C_T}}{R_T}\right)$$

$$= \tan^{-1}\left(\frac{42.02}{12.16}\right) = 73.86°$$

Problem 5.9 Determine the voltage across each element of the circuit shown in Fig. 5.33 and draw the voltage phasor diagram.

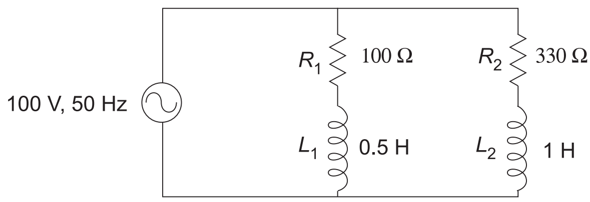


Fig. 5.33

Solution First we calculate X_{L_1} and X_{L_2}

$$X_{L_1} = 2\pi f L_1 = 2\pi \times 50 \times 0.5 = 157.08 \ \Omega$$
$$X_{L_2} = 2\pi f L_2 = 2\pi \times 50 \times 1.0 = 314.16 \ \Omega$$

Now we determine the impedance of each branch

$$Z_1 = \sqrt{R_1^2 + X_{L_1}^2} = \sqrt{(100)^2 + (157.08)^2} = 186.2 \ \Omega$$

$$Z_2 = \sqrt{R_2^2 + X_{L_2}^2} = \sqrt{(330)^2 + (314.16)^2} = 455.63 \ \Omega$$

The current in each branch

$$I_1 = \frac{V_S}{Z_1} = \frac{100}{186.2} = 0.537 \text{ A}$$

and
$$I_2 = \frac{V_S}{Z_2} = \frac{100}{455.63} = 0.219 \text{ A}$$

The voltage across each element

$$V_{R_1} = I_1 R_1 = 0.537 \times 100 = 53.7 \text{ V}$$
$$V_{L_1} = I_1 X_{L_1} = 0.537 \times 157.08 = 84.35 \text{ V}$$
$$V_{R_2} = I_2 R_2 = 0.219 \times 330 = 72.27 \text{ V}$$
$$V_{L_2} = I_2 X_{L_2} = 0.219 \times 314.16 = 68.8 \text{ V}$$

The angles associated with each parallel branch are now determined.

$$\theta_1 = \tan^{-1}\left(\frac{X_{L_1}}{R_1}\right) = \tan^{-1}\left(\frac{157.08}{100}\right) = 57.52°$$

$$\theta_2 = \tan^{-1}\left(\frac{X_{L_2}}{R_2}\right) = \tan^{-1}\left(\frac{314.16}{330}\right) = 43.59°$$

i.e. I_1 lags behind V_S by 57.52° and I_2 lags behind V_S by 43.59°

Here, V_{R_1} and I_1 are in phase and therefore, lag behind V_S by 57.52°
V_{R_2} and I_2 are in phase, and therefore lag behind V_S by 43.59°
V_{L_1} leads I_1 by 90°, so its angle is 90° – 57.52° = 32.48°
V_{L_2} leads I_2 by 90°, so its angle is 90° – 43.59° = 46.41°

The phase relations are shown in Fig. 5.34.

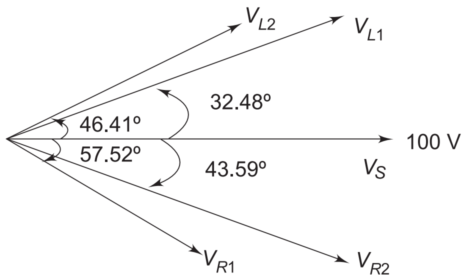


Fig. 5.34

Problem 5.10 In the series parallel circuit shown in Fig. 5.35, the effective value of voltage across the parallel parts of the circuits is 50 V. Determine the corresponding magnitude of V.

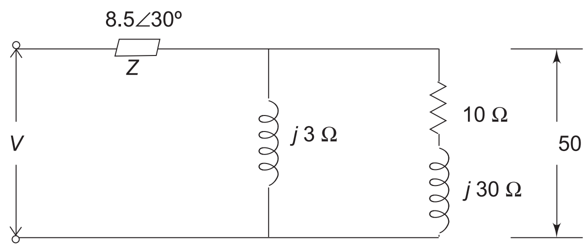


Fig. 5.35

Solution Here we can determine the current in each branch of the parallel part.

Current in the $j3\ \Omega$ branch, $I_1 = \dfrac{50}{3} = 16.67$ A

Current in $(10 + j30)\ \Omega$ branch, $I_2 = \dfrac{50}{31.62} = 1.58$ A

Total current $I_T = 16.67 + 1.58 = 18.25$ A

Total impedance $Z_T = 8.5\ \angle 30° + \dfrac{3\ \angle 90° \times (10 + j30)}{(10 + j30) + 3\ \angle 90°}$

$$= 8.5\ \angle 30° + \frac{3\ \angle 90° \times 31.62\ \angle 71.57°}{10 + j33}$$

$$= 7.36 + j4.25 + \frac{94.86\ \angle 161.57°}{34.48\ \angle 73.14°}$$

$$= 7.36 + j4.25 + 2.75\ \angle 88.43°$$

$$= 7.36 + j4.25 + 0.075 + j2.75$$

$$= (7.435 + j7)\ \Omega$$

$$= 10.21\ \angle 43.27°$$

In polar form, total impedance is $Z_T = 10.21\ \angle 43.27°$
The magnitude of applied voltage $V = I \times Z_T = 18.25 \times 10.21 = 186.33$ V.

Problem 5.11 For the series parallel circuit shown in Fig. 5.36, determine (a) the total impedance between the terminals a, b and state if it is inductive or capacitive (b) the voltage across in the parallel branch, and (c) the phase angle.

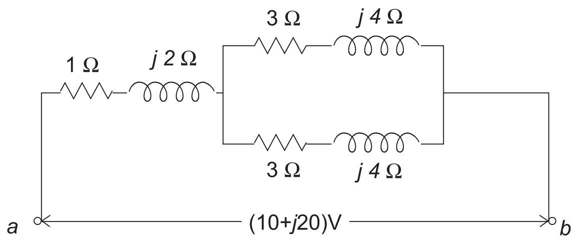


Fig. 5.36

Solution Here the parallel branch can be combined into a single branch

$$Z_P = (3 + j4)\ ||\ (3 + j4) = (1.5 + j2)\ \Omega$$

Total impedance $Z_T = 1 + j2 + 1.5 + j2 = (2.5 + j4)\ \Omega$

Hence the total impedance in the circuit is inductive
Total current in the circuit

$$I_T = \frac{V_S}{Z_T} = \frac{10 + j20}{2.5 + j4}$$

$$= \frac{22.36\ \angle 63.43°}{4.72\ \angle 57.99°}$$

$\therefore$ $I_T = 4.74\ \angle 5.44°$ A

i.e. the current lags behind the voltage by 57.99°
Phase angle $\theta = 57.99°$
Voltage across in the parallel branch

$$V_P = (1.5 + j2)\ 4.74\ \angle 5.44°$$

$$= 2.5 \times 4.74 \angle(5.44° + 53.13°)$$
$$= 11.85 \angle 58.57° \text{ V}$$

Problem 5.12 In the series parallel circuit shown in Fig. 5.37, the two parallel branches A and B are in series with C. The impedances are $Z_A = 10 + j8$, $Z_B = 9 - j6$, $Z_C = 3 + j2$ and the voltage across the circuit is $(100 + j0)$ V. Find the currents I_A, I_B and the phase angle between them.

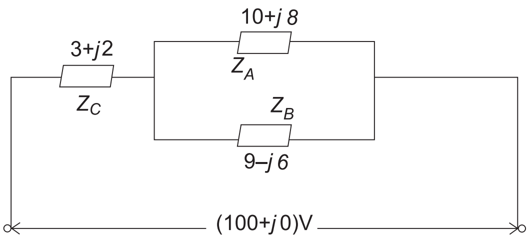


Fig. 5.37

Solution Total parallel branch impedance,

$$Z_P = \frac{Z_A Z_B}{Z_A + Z_B}$$
$$= \frac{(10 + j8)(9 - j6)}{19 + j2}$$
$$= \frac{12.8 \angle 38.66° \times 10.82 \angle -33.7}{19.1 \angle 6°} = 7.25 \angle -1.04°$$

In rectangular form,
Total parallel impedance $Z_P = 7.25 - j0.13$
This parallel impedance is in series with Z_C
Total impedance in the circuit

$$Z_T = Z_P + Z_C = 7.25 - j0.13 + 3 + j2 = (10.25 + j1.87)\ \Omega$$

Total current $\quad I_T = \dfrac{V_S}{Z_T}$

$$= \frac{(100 + j0)}{(10.25 + j1.87)} = \frac{100 \angle 0°}{10.42 \angle 10.34°}$$
$$= 9.6 \angle -10.34°$$

The current lags behind the applied voltage by 10.34°
Current in branch A is

$$I_A = I_T \frac{Z_B}{Z_A + Z_B}$$
$$= 9.6 \angle -10.34° \times \frac{(9 - j6)}{19 + j2}$$

$$= \frac{9.6 \angle -10.34° \times 10.82 \angle -33.7°}{19.1 \angle 6°}$$

$$= 5.44 \angle -50.04° \text{ A}$$

Current in branch B is I_B

$$I_B = I_T \times \frac{Z_A}{Z_A + Z_B}$$

$$= 9.6 \angle -10.34° \times \frac{10 + j8}{19 + j2}$$

$$= \frac{9.6 \angle -10.34° \times 12.8 \angle 38.66°}{19.1 \angle 6°}$$

$$= 6.43 \angle 22.32° \text{ A}$$

The angle between I_A and I_B,

$$\theta = (50.04° + 22.32°) = 72.36°$$

Problem 5.13 A series circuit of two pure elements has the following applied voltage and resulting current.

$$V = 15 \cos (200\, t - 30°) \text{ volts}$$

$$I = 8.5 \cos (200\, t + 15) \text{ volts}$$

Find the elements comprising the circuit.

Solution By inspection, the current leads the voltage by 30° + 15° = 45°. Hence the circuit must contain resistance and capacitance.

$$\tan 45 = \frac{1}{\omega CR}$$

$$1 = \frac{1}{\omega CR}, \quad \therefore \frac{1}{\omega C} = R$$

$$\frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \sqrt{R^2 + R^2} = \sqrt{2}\, R$$

$$\therefore \qquad R = \frac{15}{8.5 \times \sqrt{2}} = 1.248\ \Omega$$

$$\frac{1}{\omega C} = 1.248\ \Omega$$

and $$C = \frac{1}{200 \times 1.248} = 4 \times 10^{-3} \text{ F}$$

Problem 5.14 A resistor having a resistance of R = 10 Ω and an unknown capacitor are in series. The voltage across the resistor is $V_R = 50 \sin (1000\, t + 45°)$ volts. If the current leads the applied voltage by 60° what is the unknown capacitance C?

Solution Here, the current leads the applied voltage by 60°.

$$\tan 60° = \frac{1}{\omega CR}$$

Since $R = 10\ \Omega$

$$\omega = 1000 \text{ radians}$$

$$\tan 60° = \frac{1}{\omega CR}$$

$$C = \frac{1}{\tan 60 \times 1000 \times 10} = 57.7\ \mu\text{F}$$

Problem 5.15 A series circuit consists of two pure elements has the following current and voltage.

$$v = 100 \sin (2000\, t + 50°) \text{ V}$$
$$i = 20 \cos (2000\, t + 20°) \text{ A}$$

Find the elements in the circuit.

Solution We can write $i = 20 \sin (2000\, t + 20° + 90°)$

Since $\cos \theta = \sin (\theta + 90°)$

Current $i = 20 \sin (2000\, t + 110°)$ A

The current leads the voltage by 110° – 50° = 60°

and the circuit must consist of resistance and capacitance.

$$\tan \theta = \frac{1}{\omega CR}$$

$$\frac{1}{\omega C} = R \tan 60° = 1.73 \text{ R}$$

$$\frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \frac{100}{20}$$

$$R\sqrt{1 + (1.73)^2} = \frac{100}{20}$$

$$R\,(1.99) = 5$$

$$R = 2.5\ \Omega$$

and

$$C = \frac{1}{\omega\,(1.73 \text{ R})} = 115.6\ \mu\text{F}$$

Problem 5.16 A two branch parallel circuit with one branch of $R = 100\ \Omega$ and a single unknown element in the other branch has the following applied voltage and total current.

$$v = 2000 \cos (1000\, t + 45°) \text{ V}$$

$$I_T = 45 \sin (1000\,t + 135°) \text{ A}$$

Find the unknown element.

Solution Here, the voltage applied is same for both elements.

Current passing through resistor is $i_R = \dfrac{v}{R}$

$\therefore$ $$i_R = 20 \cos (1000\,t + 45°)$$

Total current $i_T = i_R + i_X$

Where I_X is the current in unknown element.

$$\begin{aligned} I_X &= i_T - i_R \\ &= 45 \sin (1000\,t + 135°) - 20 \cos (1000\,t + 45°) \\ &= 45 \sin (1000\,t + 135°) - 20 \sin (1000\,t + 135°) \end{aligned}$$

Current passing through the unknown element.

$$I_X = 25 \sin (1000\,t + 135°)$$

Since the current and voltage are in phase, the element is a resistor.

And the value of resistor

$$R = \frac{v}{i_X} = \frac{2000}{25} = 80\ \Omega$$

Problem 5.17 Find the total current to the parallel circuit with $L = 0.05$ H and $C = 0.667\ \mu$F with an applied voltage of $v = 200 \sin 5000t$ V.

Solution Current in the inductor $i_L = \dfrac{1}{L}\displaystyle\int v\,dt$

$\therefore$ $$\begin{aligned} i_L &= \frac{1}{0.05}\int 200 \sin 5000\,t \\ &= \frac{-200 \cos 5000t}{0.05 \times 5000} \\ i_L &= -\,0.8 \cos 5000\,t \end{aligned}$$

Current in the capacitor $i_C = C\dfrac{dv}{dt}$

$\therefore$ $$\begin{aligned} i_C &= 0.667 \times 10^{-6}\,\frac{d}{dt}(200 \sin 5000\,t) \\ i_C &= 0.667 \cos 500\,t \end{aligned}$$

Total current $i_T = i_L + i_C$

$$\begin{aligned} &= 0.667 \cos 5000\,t - 0.8 \cos 5000\,t \\ &= -\,0.133 \cos (5000\,t) \end{aligned}$$

Total current $i_T = 0.133 \sin (5000\,t - 90°)$ A

Practice Problems

5.1 For the circuit shown in Fig. 5.38, determine the impedance, phase angle and total current.

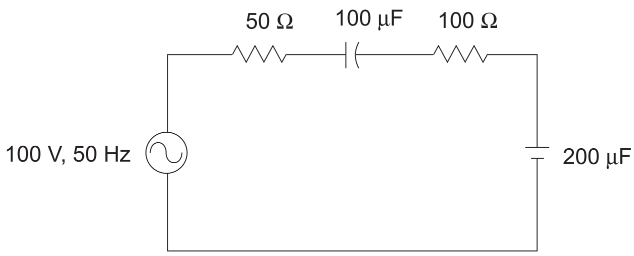


Fig. 5.38

5.2 Calculate the total current in the circuit in Fig. 5.39, and determine the voltage across resistor V_R, and across capacitor V_C.

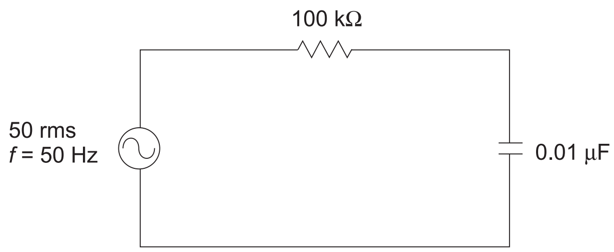


Fig. 5.39

5.3 Determine the impedance and phase angle in the circuit shown in Fig. 5.40.

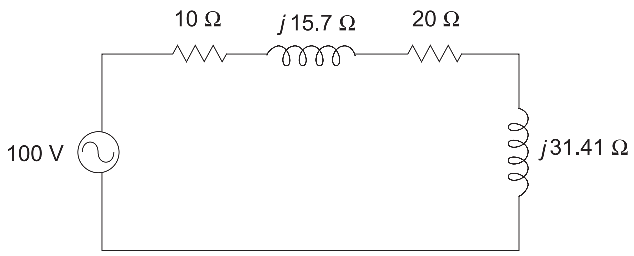


Fig. 5.40

5.4 Calculate the impedance at each of the following frequencies; also determine the current at each frequency in the circuit shown in Fig. 5.41.

(a) 100 Hz (b) 3 KHz

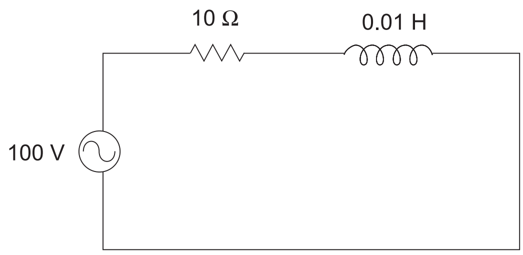


Fig. 5.41

5.5 A signal generator supplies a sine wave of 10 V, 10 KHz, to the circuit shown in Fig. 5.42. Calculate the total current in the circuit. Determine the phase angle θ for the circuit. If the total inductance in the circuit is doubled, does θ increase or decrease, and by how many degrees?

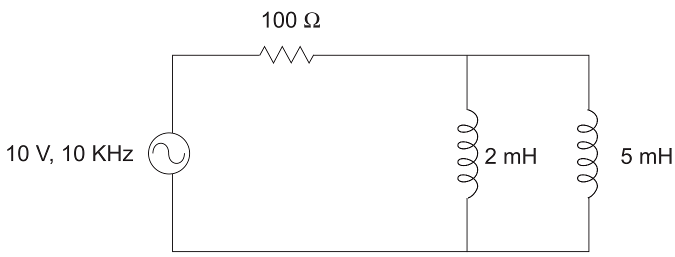


Fig. 5.42

5.6 For the circuit shown in Fig. 5.43, determine the voltage across each element. Is the circuit predominantly resistive or inductive? Find the current in each branch and the total current.

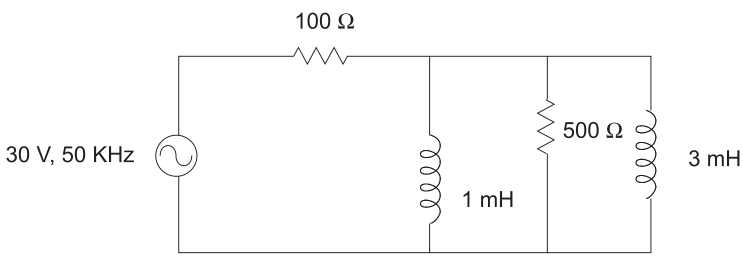


Fig. 5.43

5.7 Determine the total impedance Z_T, the total current I_T, phase angle θ, voltage across inductor L, and voltage across resistor R_3 in the circuit shown in Fig. 5.44.

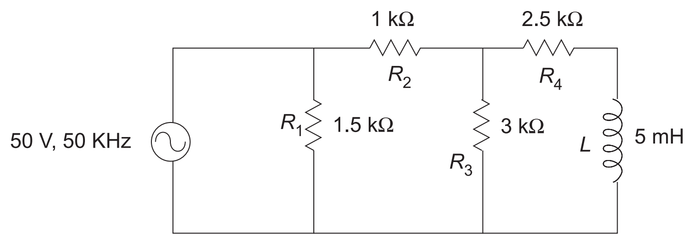


Fig. 5.44

5.8 For the circuit shown in Fig. 5.45, determine the value of frequency of supply voltage when a 100 V, 50 A current is supplied to the circuit.

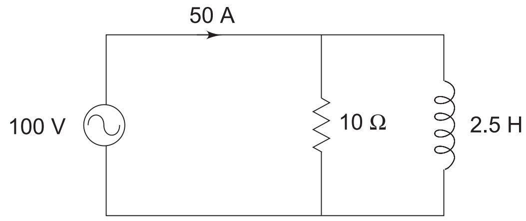


Fig. 5.45

5.9 A sine wave generator supplies a signal of 100 V, 50 Hz to the circuit shown in Fig. 5.46. Find the current in each branch, and total current. Determine the voltage across each element and draw the voltage phasor diagram.

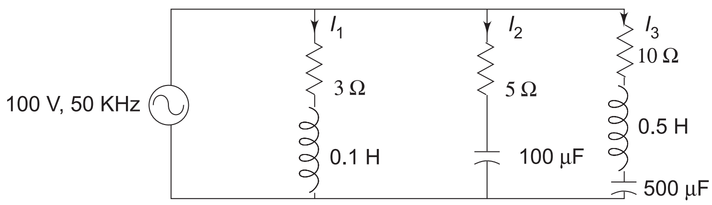


Fig. 5.46

5.10 Determine the voltage across each element in the circuit shown in Fig. 5.47. Convert the circuit into an equivalent series form. Draw the voltage phasor diagram.

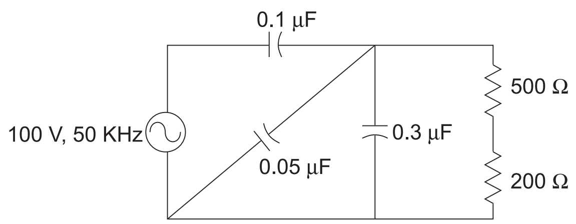


Fig. 5.47

5.11 For the circuit shown in Fig. 5.48, determine the total current I_T, phase angle θ and voltage across each element.

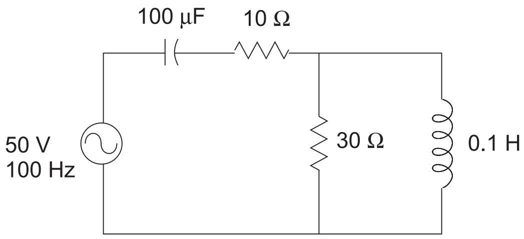


Fig. 5.48

5.12 For the circuit shown in Fig. 5.49, the applied voltage $v = V_m \cos \omega t$. Determine the current in each branch and obtain the total current in terms of the cosine function.

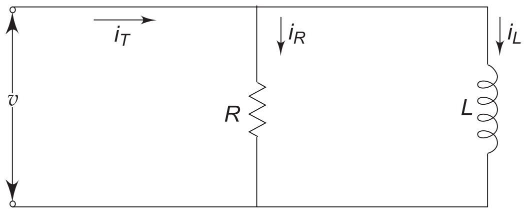


Fig. 5.49

5.13 For the circuit shown in Fig. 5.50, the voltage across the inductor is v_L = 15 sin 200 t. Find the total voltage and the angle by which the current lags the total voltage.

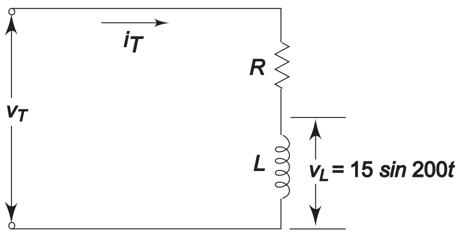


Fig. 5.50

5.14 In a parallel circuit having a resistance $R = 5\ \Omega$ and $L = 0.02$ H, the applied voltage is $v = 100 \sin (1000\,t + 50°)$ volts. Find the total current and the angle by which the current lags the applied voltage.

5.15 In the parallel circuit shown in Fig. 5.51, the current in the inductor is five times greater than the current in the capacitor. Find the element values.

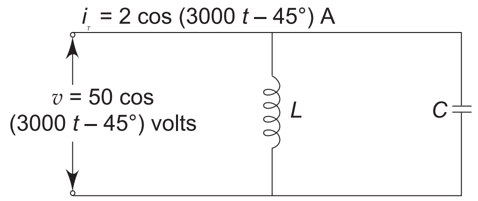


Fig. 5.51

5.16 In the parallel circuit shown in Fig. 5.52, the applied voltage is $v = 100 \sin 5000\, t$ V. Find the currents in each branch and also the total current in the circuit.

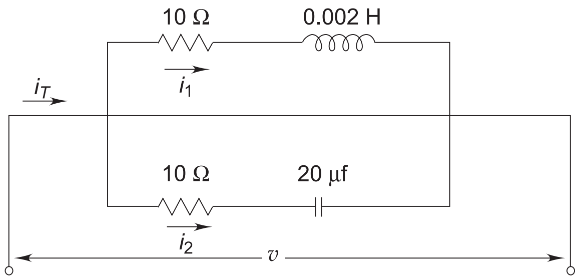


Fig. 5.52

5.17 For the circuit shown in Fig. 5.53, find the total current and the magnitude of the impedance.

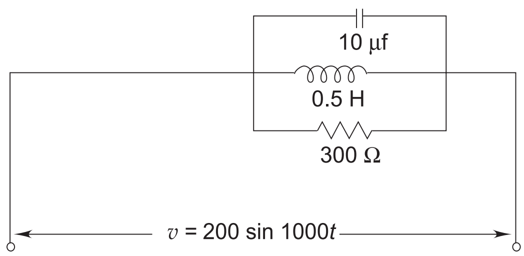


Fig. 5.53

Objective-type Questions

1. A 1 kHz sinusoidal voltage is applied to an RL circuit, what is the frequency of the resulting current?
   (a) 1 kHz (b) 0.1 kHz
   (c) 100 kHz (d) 2 kHz
2. A series RL circuit has a resistance of 33 kΩ, and an inductive reactance of 50 kΩ. What is its impedance and phase angle?

(a) 56.58 Ω, 59.9° (b) 59.9 kΩ, 56.58°
(c) 59.9 Ω, 56.58° (d) 5.99 Ω, 56.58°

3. In a certain *RL* circuit, $V_R = 2$ V and $V_L = 3$ V. What is the magnitude of the total voltage?
(a) 2 V (b) 3 V
(c) 5 V (d) 3.61 V

4. When the frequency of applied voltage in a series RL circuit is increased what happens to the inductive reactance?
(a) decreases (b) remains the same
(c) increases (d) becomes zero

5. In a certain parallel RL circuit, $R = 50$ Ω, and $X_L = 75$ Ω. What is the admittance?
(a) 0.024 S (b) 75 S
(c) 50 S (d) 1.5 S

6. What is the phase angle between the inductor current and the applied voltage in a parallel RL circuit?
(a) 0° (b) 45°
(c) 90° (d) 30°

7. When the resistance in an RC circuit is greater than the capacitive reactance, the phase angle between the applied voltage and the total current is closer to
(a) 90° (b) 0°
(c) 45° (d) 120°

8. A series RC circuit has a resistance of 33 kΩ, and a capacitive reactance of 50 kΩ. What is the value of the impedance.
(a) 50 kΩ (b) 33 kΩ
(c) 20 kΩ (d) 59.91 Ω

9. In a certain series RC circuit, $V_R = 4$ V and $V_C = 6$ V. What is the magnitude of the total voltage?
(a) 7.2 V (b) 4 V
(c) 6 V (d) 52 V

10. When the frequency of the applied voltage in a series RC circuit is increased what happens to the capacitive reactance?
(a) it increases (b) it decreases
(c) it is zero (d) remains the same

11. In a certain parallel RC circuit, $R = 50$ Ω and $X_C = 75$ Ω. What is Y?
(a) 0.01 S (b) 0.02 S
(c) 50 S (d) 75 S

12. The admittance of an RC circuit is 0.0035 S, and the applied voltage is 6 V. What is the total current?
(a) 6 mA (b) 20 mA
(c) 21 mA (d) 5 mA

13. What is the phase angle between the capacitor current and the applied voltage in a parallel RC circuit?
    (a) 90° (b) 0°
    (c) 45° (d) 180°
14. In a given series RLC circuit, X_C is 150 Ω, and X_L is 80 Ω, what is the total reactance? What is the type of reactance?
    (a) 70 Ω, inductive (b) 70 Ω, capacitive
    (c) 70 Ω, resistive (d) 150 Ω, capacitive
15. In a certain series RLC circuit V_R = 24 V, V_L = 15 V, and V_C = 45 V. What is the source voltage.
    (a) 38.42 V (b) 45 V
    (c) 15 V (d) 24 V
16. When R = 10 Ω, X_C = 18 Ω and X_L = 12 Ω, the current
    (a) leads the applied voltage
    (b) lags behind the applied voltage
    (c) is in phase with the voltage
    (d) is none of the above
17. A current $i = A \sin 500\, t$ A passes through the circuit shown in Fig. 5.54. The total voltage applied will be

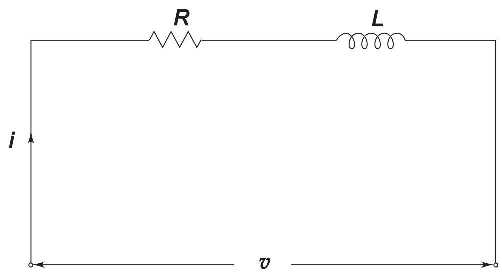


Fig. 5.54

    (a) $B \sin 500\, t$ (b) $B \sin (500\, t - \theta°)$
    (c) $B \sin (500\, t + \theta°)$ (d) $B \cos (200\, t + \theta°)$
18. A current of 100 mA through an inductive reactance of 100 Ω produces a voltage drop of
    (a) 1 V (b) 6.28 V
    (c) 10 V (d) 100 V
19. When a voltage $v = 100 \sin 5000\, t$ volts is applied to a series circuit of L = 0.05 H and unknown capacitance, the resulting current is $i = 2 \sin (5000\, t + 90°)$ amperes. The value of capacitance is
    (a) 66.7 pF (b) 6.67 pF
    (c) 0.667 μF (d) 6.67 μF
20. A series circuit consists of two elements has the following current and applied voltage.

$$i = 4 \cos (2000\, t + 11.32°) \text{ A}$$
$$v = 200 \sin (2000\, t + 50°) \text{ V}$$

The circuit elements are

(a) resistance and capacitance
(b) capacitance and inductance
(c) inductance and resistance
(d) both resistances

21. A pure capacitor of $C = 35\ \mu$F is in parallel with another signal circuit element. The applied voltage and resulting current are

$$v = 150 \sin 300\, t \text{ V}$$
$$i = 16.5 \sin (3000\, t + 72.4°) \text{ A}$$

The other element is

(a) capacitor of 30 μF
(b) inductor of 30 mH
(c) resistor of 30 Ω
(d) none of the above

Chapter 6

Power and Power Factor

6.1 INSTANTANEOUS POWER

In a purely resistive circuit, all the energy delivered by the source is dissipated in the form of heat by the resistance. In a purely reactive (inductive or capacitive) circuit, all the energy delivered by the source is stored by the inductor or capacitor in its magnetic or electric field during a portion of the voltage cycle, and then is returned to the source during another portion of the cycle, so that no net energy is transferred. When there is complex impedance in a circuit, part of the energy is alternately stored and returned by the reactive part, and part of it is dissipated by the resistance. The amount of energy dissipated is determined by the relative values of resistance and reactance.

Consider a circuit having complex impedance. Let $v(t) = V_m \cos \omega t$ be the voltage applied to the circuit and let $i(t) = I_m \cos(\omega t + \theta)$ be the corresponding current flowing through the circuit. Then the power at any instant of time is

$$\begin{aligned} P(t) &= v(t)\, i(t) \\ &= V_m \cos \omega t \, I_m \cos(\omega t + \theta) \end{aligned} \tag{6.1}$$

From Eq. 6.1, we get

$$P(t) = \frac{V_m I_m}{2} [\cos(2\omega t + \theta) + \cos \theta] \tag{6.2}$$

Equation 6.2 represents *instantaneous power*. It consists of two parts. One is a fixed part, and the other is time-varying which has a frequency twice that of the voltage or current waveforms. The voltage, current and power waveforms are shown in Figs 6.1 and 6.2.

Here, the negative portion (hatched) of the power cycle represents the power returned to the source. Figure 6.2 shows that the instantaneous power is negative

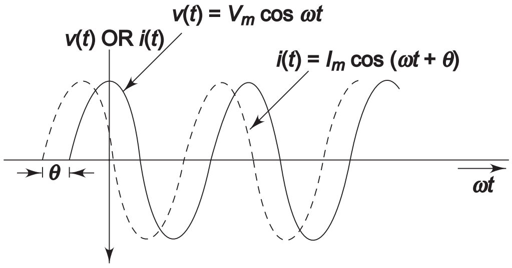


Fig. 6.1

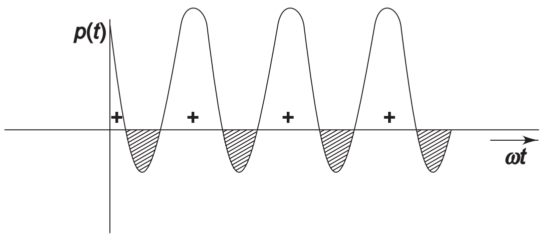


Fig. 6.2

whenever the voltage and current are of opposite sign. In Fig. 6.2, the positive portion of the power is greater than the negative portion of the power; hence the average power is always positive, which is almost equal to the constant part of the instantaneous power (Eq. 6.2). The positive portion of the power cycle varies with the phase angle between the voltage and current waveforms. If the circuit is pure resistive, the phase angle between voltage and current is zero; then there is no negative cycle in the $P(t)$ curve. Hence, all the power delivered by the source is completely dissipated in the resistance.

If θ becomes zero in Eq. 6.1, we get

$$\begin{aligned} P(t) &= v(t)\, i(t) \\ &= V_m I_m \cos^2 \omega t \\ &= \frac{V_m I_m}{2}(1 + \cos 2\omega t) \end{aligned} \tag{6.3}$$

The waveform for Eq. 6.3, is shown in Fig. 6.3, where the power wave has a frequency twice that of the voltage or current. Here the average value of power is $V_m I_m/2$.

When phase angle θ is increased, the negative portion of the power cycle increases and lesser power is dissipated.When θ becomes $\pi/2$, the positive and negative portions of the power cycle are equal. At this instant, the power dissipated in the circuit is zero, i.e. the power delivered to the load is returned to the source.

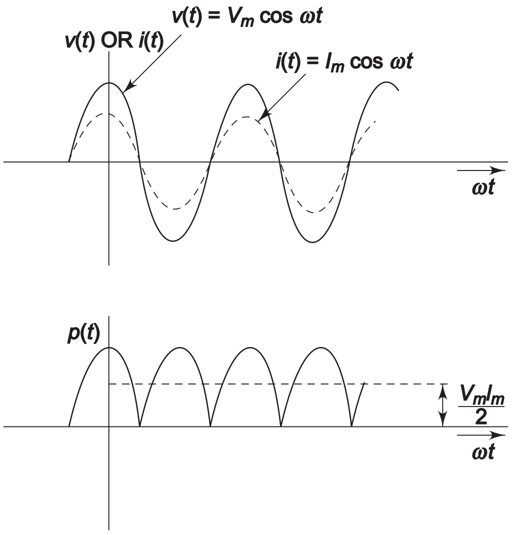


Fig. 6.3

6.2 AVERAGE POWER

To find the average value of any power function, we have to take a particular time interval from t_1 to t_2; by integrating the function from t_1 to t_2 and dividing the result by the time interval $t_2 - t_1$, we get the average power.

$$\text{Average power } P = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} P(t)\,dt \tag{6.4}$$

In general, the average value over one cycle is

$$P_{\text{av}} = \frac{1}{T}\int_0^T P(t)\,dt \tag{6.5}$$

By integrating the instantaneous power $P(t)$ in Eq. 6.5 over one cycle, we get average power

$$P_{\text{av}} = \frac{1}{T}\int_0^T \left\{\frac{V_m I_m}{2}[\cos(2\omega t + \theta) + \cos\theta]\,dt\right\}$$

$$= \frac{1}{T}\int_0^T \frac{V_m I_m}{2}[\cos(2\omega t + \theta)]\,dt + \frac{1}{T}\int_0^T \frac{V_m I_m}{2}\cos\theta\,dt \tag{6.6}$$

In Eq. 6.6, the first term becomes zero, and the second term remains. The average power is therefore

$$P_{\text{av}} = \frac{V_m I_m}{2}\cos\theta \text{ W} \tag{6.7}$$

We can write Eq. 6.7 as

$$P_{\text{av}} = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta \tag{6.8}$$

In Eq. 6.8, $V_m/\sqrt{2}$ and $I_m/\sqrt{2}$ are the effective values of both voltage and current.

$\therefore \qquad P_{\text{av}} = V_{\text{eff}}\, I_{\text{eff}} \cos\theta$

To get average power, we have to take the product of the effective values of both voltage and current multiplied by cosine of the phase angle between voltage and the current.

If we consider a purely resistive circuit, the phase angle between voltage and current is zero. Hence, the average power is

$$P_{\text{av}} = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R$$

If we consider a purely reactive circuit (i.e. purely capacitive or purely inductive), the phase angle between voltage and current is 90°. Hence, the average power is zero or $P_{\text{av}} = 0$.

If the circuit contains complex impedance, the average power is the power dissipated in the resistive part only.

Example 6.1 A voltage of $v(t) = 100 \sin \omega t$ is applied to a circuit. The current flowing through the circuit is $i(t) = 15 \sin(\omega t - 30°)$. Determine the average power delivered to the circuit.

Solution The phase angle between voltage and current is 30°.

Effective value of the voltage $V_{\text{eff}} = \dfrac{100}{\sqrt{2}}$

Effective value of the current $I_{\text{eff}} = \dfrac{15}{\sqrt{2}}$

Average power $\qquad P_{\text{av}} = V_{\text{eff}}\, I_{\text{eff}} \cos\theta$

$$= \frac{100}{\sqrt{2}} \times \frac{15}{\sqrt{2}} \cos 30°$$

$$= \frac{100 \times 15}{2} \times 0.866 = 649.5 \text{ W}$$

Example 6.2 Determine the average power delivered to the circuit consisting of an impedance $Z = 5 + j8$ when the current flowing through the circuit is $I = 5\angle 30°$.

Solution The average power is the power dissipated in the resistive part only.

or $\qquad P_{\text{av}} = \dfrac{I_m^2}{2} R$

Current $\qquad I_m = 5$ A

$\therefore \qquad P_{\text{av}} = \dfrac{5^2}{2} \times 5 = 62.5$ W

6.3 APPARENT POWER AND POWER FACTOR

The power factor is useful in determining useful power (true power) transferred to a load. The highest power factor is 1, which indicates that the current to a load is in phase with the voltage across it (i.e. in the case of resistive load). When the power factor is 0, the current to a load is 90° out of phase with the voltage (i.e. in case of reactive load).

Consider the following equation

$$P_{av} = \frac{V_m I_m}{2} \cos\theta \text{ W} \tag{6.9}$$

In terms of effective values

$$P_{av} = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos\theta$$

$$= V_{eff} I_{eff} \cos\theta \text{ W} \tag{6.10}$$

The average power is expressed in watts. It means the useful power transferred from the source to the load, which is also called true power. If we consider a dc source applied to the network, true power is given by the product of the voltage and the current. In case of sinusoidal voltage applied to the circuit, the product of voltage and current is not the true power or average power. This product is called *apparent power*. The apparent power is expressed in volt amperes, or simply VA.

$\therefore$ Apparent power $= V_{eff} I_{eff}$

In Eq. 6.10, the average power depends on the value of $\cos\theta$; this is called the *power factor* of the circuit.

$$\therefore \quad \text{Power factor (pf)} = \cos\theta = \frac{P_{av}}{V_{eff} I_{eff}}$$

Therefore, power factor is defined as the ratio of average power to the apparent power, whereas apparent power is the product of the effective values of the current and the voltage. Power factor is also defined as the factor with which the volt amperes are to be multiplied to get true power in the circuit.

In the case of sinusoidal sources, the power factor is the cosine of the phase angle between voltage and current

$$pf = \cos\theta$$

As the phase angle between voltage and total current increases, the power factor decreases. The smaller the power factor, the smaller the power dissipation. The power factor varies from 0 to 1. For purely resistive circuits, the phase angle between voltage and current is zero, and hence the power factor is unity. For purely reactive circuits, the phase angle between voltage and current is 90°, and hence the power factor is zero. In an RC circuit, the power factor is referred to as *leading* power factor because the current leads the voltage. In an RL circuit, the power factor is referred to as lagging power factor because the current lags behind the voltage.

Example 6.3 A sinusoidal voltage $v = 50 \sin \omega t$ is applied to a series RL circuit. The current in the circuit is given by $i = 25 \sin (\omega t - 53°)$. Determine (a) apparent power (b) power factor and (c) average power.

Solution (a) Apparent power $P = V_{\text{eff}}\, I_{\text{eff}}$

$$= \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$$

$$= \frac{50 \times 25}{2} = 625 \text{ VA}$$

(b) Power factor = $\cos \theta$

where θ is the angle between voltage and current

$$\theta = 53°$$

$\therefore$ Power factor = $\cos \theta = \cos 53° = 0.6$

(c) Average Power $P_{\text{av}} = V_{\text{eff}}\, I_{\text{eff}} \cos \theta$

$$= 625 \times 0.6 = 375 \text{ W}$$

6.4 REACTIVE POWER

We know that the average power dissipated is

$$P_{\text{av}} = V_{\text{eff}}\,[I_{\text{eff}} \cos \theta] \qquad (6.11)$$

From the impedance triangle shown in Fig. 6.4

$$\cos \theta = \frac{R}{|Z|} \qquad (6.12)$$

and $$V_{\text{eff}} = I_{\text{eff}}\, Z \qquad (6.13)$$

If we substitute Eqs (6.12) and (6.13) in Eq. (6.11), we get

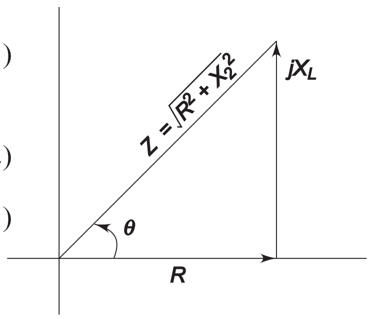


Fig. 6.4

$$P_{\text{av}} = I_{\text{eff}}\, Z \left[I_{\text{eff}} \frac{R}{Z} \right]$$

$$= I_{\text{eff}}^2 R \text{ watts} \qquad (6.14)$$

This gives the average power dissipated in a resistive circuit.

If we consider a circuit consisting of a pure inductor, the power in the inductor

$$P_r = i v_L \qquad (6.15)$$

$$= iL \frac{di}{dt}$$

Consider $$i = I_m \sin (\omega t + \theta)$$

Then $$P_r = I_m^2 \sin (\omega t + \theta)\, L\omega \cos (\omega t + \theta)$$

$$= \frac{I_m^2}{2} (\omega L) \sin 2 (\omega t + \theta)$$

$$\therefore \quad P_r = I_{\text{eff}}^2 (\omega L) \sin 2(\omega t + \theta) \tag{6.16}$$

From the above equation, we can say that the average power delivered to the circuit is zero. This is called *reactive* power. It is expressed in volt-amperes reactive (VAR).

$$P_r = I_{\text{eff}}^2 X_L \text{ VAR} \tag{6.17}$$

From Fig. 6.4, we have

$$X_L = Z \sin \theta \tag{6.18}$$

Substituting Eq. 6.18 in Eq. 6.17, we get

$$P_r = I_{\text{eff}}^2 Z \sin \theta$$
$$= (I_{\text{eff}} Z) I_{\text{eff}} \sin \theta$$
$$= V_{\text{eff}} I_{\text{eff}} \sin \theta \text{ VAR}$$

6.5 THE POWER TRIANGLE

A generalised impedance phase diagram is shown in Fig. 6.5. A phasor relation for power can also be represented by a similar diagram because of the fact that true power P_{av} and reactive power P_r differ from R and X by a factor I_{eff}^2, as shown in Fig. 6.5.

The resultant power phasor $I_{\text{eff}}^2 Z$, represents the apparent power P_a.
At any instant in time, P_a is the total power that appears to be transferred between the source and reactive circuit. Part of the apparent power is true power and part of it is reactive power.

$$\therefore \quad P_a = I_{\text{eff}}^2 Z$$

The power triangle is shown in Fig. 6.6.

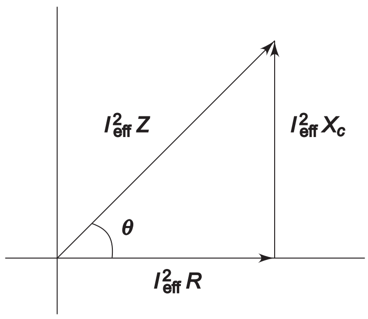


Fig. 6.5

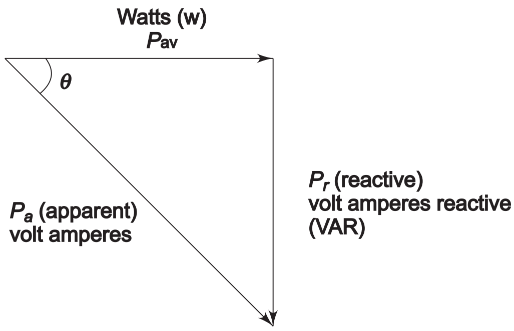


Fig. 6.6

From Fig. 6.6, we can write

$$P_{\text{true}} = P_a \cos \theta$$

or average power $P_{\text{av}} = P_a \cos \theta$
and reactive power $P_r = P_a \sin \theta$

Additional Solved Problems

Problem 6.1 In the circuit shown in Fig. 6.7, a voltage of $v(t) = 50 \sin(\omega t + 30°)$ is applied. Determine the true power, reactive power and power factor.

Solution The voltage applied to the circuit is

$$v(t) = 50 \sin(\omega t + 30°)$$

The current in the circuit is

$$I = \frac{V}{Z} = \frac{50\angle 30°}{10 + j30} = \frac{50\angle 30°}{31.6\angle 71.56°}$$
$$= 1.58\angle -41.56°\text{ A}$$

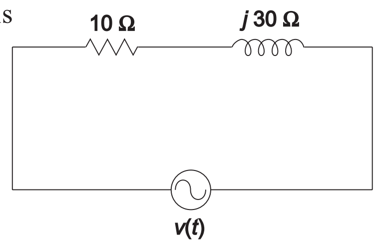


Fig. 6.7

The phasor diagram is shown in Fig. 6.8.
The phase angle between voltage and current $\theta = 71.56°$
Power factor = $\cos\theta = \cos 71.56° = 0.32$
True power or average power

$$P_{av} = V_{eff} I_{eff} \cos\theta$$
$$= \frac{50 \times 1.58}{\sqrt{2} \times \sqrt{2}} \cos 71.56°$$
$$= 12.49 \text{ W}$$

$$\text{Reactive power} = V_{eff} I_{eff} \sin\theta$$
$$= \frac{50 \times 1.58}{\sqrt{2} \times \sqrt{2}} \sin 71.56°$$
$$= 37.47 \text{ VAR}$$

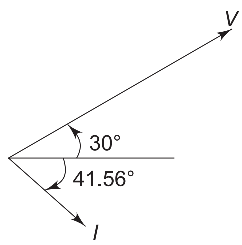


Fig. 6.8

Problem 6.2 Determine the circuit constants in the circuit shown in Fig. 6.9, if the applied voltage to the circuit $v(t) = 100 \sin(50t + 20°)$. The true power in the circuit is 200 W and the power factor is 0.707 lagging.

Solution Power factor = $\cos\theta = 0.707$

$\therefore$ The phase angle between voltage and current

$$\theta = \cos^{-1} 0.707 = 45°$$

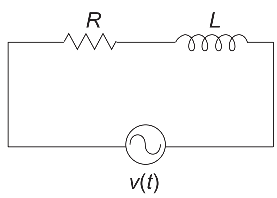


Fig. 6.9

Here the current lags behind the voltage by 45°.
Hence, the current equation is $i(t) = I_m \sin(50t - 25°)$
True power = $V_{eff} I_{eff} \cos\theta = 200$ W

$$I_{eff} = \frac{200}{V_{eff} \cos\theta}$$
$$= \frac{200}{(100/\sqrt{2}) \times 0.707} = 4 \text{ A}$$

$$I_m = 4 \times \sqrt{2} = 5.66 \text{ A}$$

$\therefore$ The current equation is $i(t) = 5.66 \sin (50t - 25°)$

The impedance of the circuit

$$Z = \frac{V}{I} = \frac{(100/\sqrt{2})\angle 20°}{(5.66/\sqrt{2})\angle -25°}$$

$\therefore$ $Z = 17.67 \angle 45° = 12.5 + j12.5$

Since $Z = R + jX_L = 12.5 + j12.5$

$\therefore$ $R = 12.5$ ohms, $X_L = 12.5$ ohms

$$X_L = \omega L = 12.5$$

$$L = \frac{12.5}{50} = 0.25 \text{ H}$$

Problem 6.3 A voltage $v(t) = 150 \sin 250t$ is applied to the circuit shown in Fig. 6.10. Find the power delivered to the circuit and the value of inductance in Henrys.

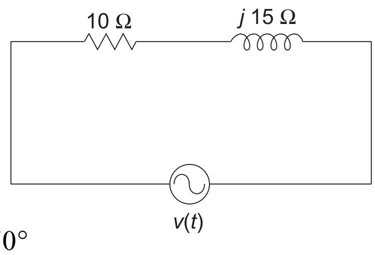


Fig. 6.10

Solution $Z = 10 + j15\ \Omega$

The impedance $Z = 18 \angle 56.3°$

The impedance of the circuit $Z = \dfrac{V}{I}$

$$18 \angle 56.3° = \frac{(150/\sqrt{2})\angle 0°}{I}$$

$\therefore$ Phasor current $I = \dfrac{150/\sqrt{2}}{18 \angle 56.3°} = 5.89 \angle -56.3°$

The current equation is $i(t) = 5.89\sqrt{2} \sin (250t - 56.3°)$

$= 8.33 \sin (250t - 56.3°)$

The phase angle between the current and the voltage

$$\theta = 56.3°$$

The power delivered to the circuit

$$P_{av} = VI \cos \theta$$

$$= \frac{150}{\sqrt{2}} \times \frac{8.33}{\sqrt{2}} \cos 56.3°$$

$$= 346.6 \text{ W}$$

The inductive impedance $X_L = 15\ \Omega$

$\therefore$ $\omega L = 15$

$\therefore$ $L = \dfrac{15}{250} = 0.06 \text{ H}$

Problem 6.4 Determine the power factor, true power, reactive power and apparent power in the circuit in Fig. 6.11.

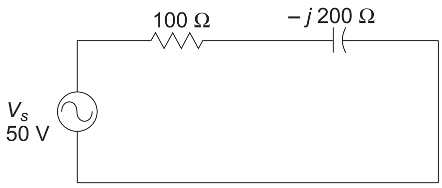


Fig. 6.11

Solution The impedance of the circuit

$$Z = \sqrt{R^2 + X_C^2}$$

$$= \sqrt{(100)^2 + (200)^2} = 223.6\ \Omega$$

The current $I = \dfrac{V_S}{Z} = \dfrac{50}{223.6} = 0.224$ A

The phase angle

$$\theta = \tan^{-1}\left(\frac{-X_C}{R}\right)$$

$$= \tan^{-1}\left(\frac{-200}{100}\right) = -\ 63.4°$$

∴ The power factor $pf = \cos\theta = \cos(63.4°) = 0.448$

The true power $P_{av} = VI\cos\theta$

$$= 50 \times 0.224 \times 0.448 = 5.01 \text{ W}$$

The reactive power $P_v = I^2 X_C$

$$= (0.224)^2 \times 200 = 10.03 \text{ VAR}$$

The apparent power

$$P_a = I^2 Z = (0.224)^2 \times 223.6 = 11.21 \text{ VA}$$

Problem 6.5 In a certain RC circuit, the true power is 300 W and the reactive power is 1000 W. What is the apparent power?

Solution The true power P_{true} or $P_{av} = VI\cos\theta$

$$= 300 \text{ W}$$

The reactive power $P_r = VI\sin\theta$

$$= 1000 \text{ W}$$

From the above results

$$\tan\theta = \frac{1000}{300} = 3.33$$

The phase angle between voltage and current, $\theta = \tan^{-1} 3.33 = 73.3°$

The apparent power $P_a = VI = \dfrac{300}{\cos 73.3°} = 1043.9$ VA

Problem 6.6 A sine wave of $v(t) = 200 \sin 50t$ is applied to a 10 Ω resistor in series with a coil. The reading of a voltmeter across the resistor is 120 V and across the coil, 75 V. Calculate the power and reactive volt-amperes in the coil and the power factor of the circuit.

Solution The rms value of the sine wave

$$V = \frac{200}{\sqrt{2}} = 141.4 \text{ V}$$

Voltage across the resistor, $V_R = 120$ V
Voltage across the coil, $V_L = 75$ V

$$\therefore \qquad IR = 120 \text{ V}$$

The current in resistor, $I = \dfrac{120}{10} = 12$ A

$$\text{Since} \qquad IX_L = 75 \text{ V}$$

$$\therefore \qquad X_L = \frac{75}{12} = 6.25\ \Omega$$

Power factor, $pf = \cos\theta = \dfrac{R}{Z}$

$$\text{where} \qquad Z = 10 + j6.25 = 11.8 \angle 32°$$

$$\therefore \qquad \cos\theta = \frac{R}{Z} = \frac{10}{11.8} = 0.85$$

True power $P_{\text{true}} = I^2 R = (12)^2 \times 10 = 1440$ W
Reactive power $P_r = I^2 X_L = (12)^2 \times 6.25 = 900$ VAR

Problem 6.7 For the circuit shown in Fig. 6.12, determine the true power, reactive power and apparent power in each branch. What is the power factor of the total circuit?

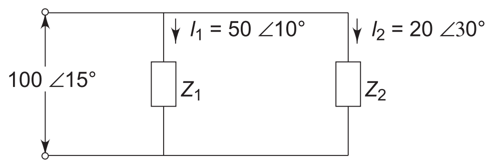


Fig. 6.12

Solution In the circuit shown in Fig. 6.12, we can calculate Z_1 and Z_2.

$$\text{Impedance} \qquad Z_1 = \frac{100 \angle 15°}{50 \angle 10°} = 2 \angle 5° = (1.99 + j0.174)\ \Omega$$

$$\text{Impedance} \qquad Z_2 = \frac{100 \angle 15°}{20 \angle 30°} = 5 \angle -15° = (4.83 - j1.29)\ \Omega$$

True power in branch Z_1 is $P_{t_1} = I_1^2 R = (50)^2 \times 1.99 = 4975$ W

Reactive power in branch Z_1, $P_{r_1} = I_1^2 X_L$

$$= (50)^2 \times 0.174 = 435 \text{ VAR}$$

Apparent power in branch Z_1, $P_{a_1} = I_1^2 Z$

$$= (50)^2 \times 2$$

$$= 2500 \times 2 = 5000 \text{ VA}$$

True power in branch Z_2, $P_{t_2} = I_2^2 R$

$$= (20)^2 \times 4.83 = 1932 \text{ W}$$

Reactive power in branch Z_2, $P_{r_2} = I_2^2 X_C$

$$= (20)^2 \times 1.29 = 516 \text{ VAR}$$

Apparent power in branch Z_2, $P_{a_2} = I_2^2 Z$

$$= (20)^2 \times 5 = 2000 \text{ VA}$$

Total impedance of the circuit, $Z = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$

$$= \frac{2\angle 5° \times 5 \times \angle -15°}{1.99 + j0.174 + 4.83 - j1.29}$$

$$= \frac{10\angle -10°}{6.82 - j1.116}$$

$$= \frac{10\angle -10°}{6.9\angle -9.29°} = 1.45\angle -0.71°$$

The phase angle between voltage and current, $\theta = 0.71°$

$\therefore$ Power factor $pf = \cos\theta$

$$= \cos 0.71° = 0.99 \text{ leading}$$

Problem 6.8 A voltage of $v(t) = 141.4 \sin \omega t$ is applied to the circuit shown in Fig. 6.13. The circuit dissipates 450 W at a lagging power factor, when the voltmeter and ammeter readings are 100 V and 6 A, respectively. Calculate the circuit constants.

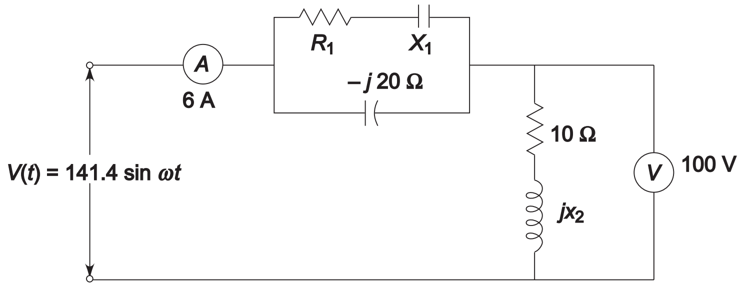


Fig. 6.13

Solution The magnitude of the current passing through $(10 + jX_2)\ \Omega$ is

$$I = 6 \text{ A}$$

The magnitude of the voltage across the $(10 + jX_2)$ ohms, $V = 100$ V. The magnitude of impedance $(10 + jX_2)$ is V/I.

Hence $$\sqrt{10^2 + X_2^2} = \frac{100}{6} = 16.67\ \Omega$$

$\therefore$ $$X_2 = \sqrt{(16.67)^2 - (10)^2} = 13.33\ \Omega$$

Total power dissipated in the circuit $= VI\cos\theta = 450$ W

$\therefore$ $$V = \frac{141.4}{\sqrt{2}} = 100\text{ V}$$

$$I = 6\text{ A}$$

$$100 \times 6 \times \cos\theta = 450$$

The power factor $pf = \cos\theta = \dfrac{450}{600} = 0.75$

$$\theta = 41.4°$$

The current lags behind the voltage by 41.4°

The current passing through the circuit, $I = 6\ \angle{-41.4°}$

The voltage across $(10 + j13.33)\ \Omega$, $V = 6\ \angle{-41.4°} \times 16.66\ \angle 53.1°$

$$= 100\ \angle 11.7°$$

The voltage across parallel branch, $V_1 = 100\ \angle 0° - 100\ \angle 11.7°$

$$= 100 - 97.9 - j20.27$$

$$= (2.1 - j20.27)\text{V} = 20.38\ \angle{-84.08°}$$

The current in $(-j20)$ branch, $I_2 = \dfrac{20.38\ \angle{-84.08°}}{20\ \angle{-90°}} = 1.02\ \angle{+5.92°}$

The current in $(R_1 - jX_1)$ branch, I_1

$$= 6\ \angle{-41.4°} - 1.02\ \angle 5.92° = 4.5 - j3.97 - 1.01 - j0.1$$

$$= 3.49 - j4.07 = 5.36\ \angle{-49.39°}$$

The impedance $$Z_1 = \frac{V_1}{I_1} = \frac{20.38\ \angle{-84.08°}}{5.36\ \angle{-49.39°}}$$

$$= 3.8\ \angle{-34.69°} = (3.12 - j2.16)\ \Omega$$

Since $$R_1 - jX_1 = (3.12 - j2.16)\ \Omega$$

$$R_1 = 3.12\ \Omega$$

$$X_1 = 2.16\ \Omega$$

Problem 6.9 Determine the value of the voltage source and power factor in the following network if it delivers a power of 100 W to the circuit shown in Fig. 6.14. Find also the reactive power drawn from the source.

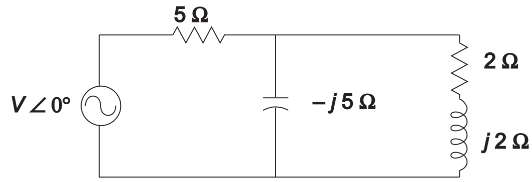


Fig. 6.14

Solution Total impedance in the circuit,

$$Z_{eq} = 5 + \frac{(2 + j2)(-j5)}{2 + j2 - j5}$$

$$= 5 + \frac{10 - j10}{2 - j3} = 5 + \frac{14.14\angle -45^\circ}{3.6\angle -56.3^\circ} = 5 + 3.93\ \angle 11.3^\circ$$

$$= 5 + 3.85 + j0.77 = 8.85 + j0.77 = 8.88\ \angle 4.97^\circ$$

Power delivered to the circuit, $P_T = I^2 R_T = 100$ W

$\therefore$ $$I^2 \times 8.85 = 100$$

Current in the circuit, $I = \sqrt{\dfrac{100}{8.85}} = 3.36$ A

Power factor $pf = \cos\theta = \dfrac{R}{Z}$

$$= \frac{8.85}{8.88} = 0.99$$

Since $$VI\cos\theta = 100 \text{ W}$$

$$V \times 3.36 \times 0.99 = 100$$

$\therefore$ $$V = \frac{100}{3.36 \times 0.99} = 30.06 \text{ V}$$

The value of the voltage source, $V = 30.06$ V

Reactive power $$P_r = VI\sin\theta$$

$$= 30.06 \times 3.36 \times \sin(4.97^\circ)$$

$$= 30.06 \times 3.36 \times 0.087 = 8.8 \text{ VAR}$$

Problem 6.10 For the circuit shown in Fig. 6.15, determine the circuit constants when a voltage of 100 V is applied to the circuit, and the total power absorbed is 600 W. The circuit constants are adjusted such that the currents in the parallel branches are equal and the voltage across the inductance is equal and in quadrature with the voltage across the parallel branch.

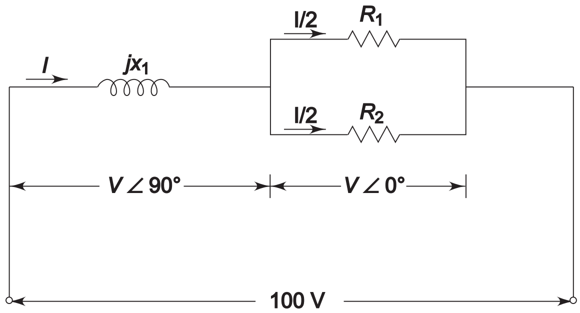


Fig. 6.15

Solution Since the voltages across the parallel branch and the inductance are in quadrature, the total voltage becomes $100\angle 45°$ as shown in Fig. 6.16.

Total voltage is $100\ \angle 45° = V + j0 + 0 + jV$

From the above result, $70.7 + j70.7 = V + jV$

$\therefore \qquad V = 70.7$

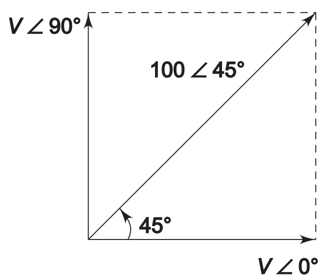


Fig. 6.16

If we take current as the reference, then current passing through the circuit is $I\angle 0°$. Total power absorbed by the circuit $= VI\cos\theta = 600$ W

or $\qquad 100 \times I \times \cos 45° = 600$ W

$\therefore \qquad I = 8.48$ A

Hence, the inductance, $X_1 = \dfrac{V\angle 90°}{I\angle 0°} = \dfrac{70.7\ \angle 90°}{8.48} = 8.33\ \angle 90°$

$\therefore \qquad X_1 = 8.33\ \Omega$

Current through the parallel branch, R_1 is $I/2 = 4.24$ A

Resistance, $R_1 = \dfrac{V\ \angle 0}{I/2\ \angle 0} = \dfrac{70.7}{4.24} = 16.67\ \Omega$

Current through parallel branch R_2 is $I/2 = 4.24$ A

Resistance is $R_2 = \dfrac{70.7}{4.24} = 16.67\ \Omega$

Problem 6.11 Determine the average power delivered by the $500\ \angle 0°$ voltage source in Fig. 6.17 and also dependent source.

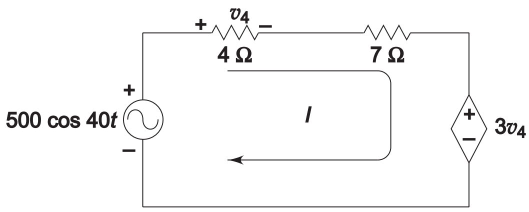


Fig. 6.17

Solution The current I can be determined by using Kirchhoff's voltage law.

$$I = \frac{500\ \angle 0° - 3v_4}{7+4}$$

where $v_4 = 4I$

$$I = \frac{500\ \angle 0°}{11} - \frac{12\,I}{11}$$

$$I = 21.73\ \angle 0°$$

Power delivered by the 500 $\angle 0°$ voltage source $= \dfrac{500 \times 21.73}{2} = 5.432$ kW

Power delivered by the dependent voltage source $= \dfrac{3v_4 \times I}{2} = \dfrac{3 \times 4I \times I}{2} = 2.833$ kW

Problem 6.12 Find the average power delivered by the dependent voltage source in the circuit shown in Fig. 6.18.

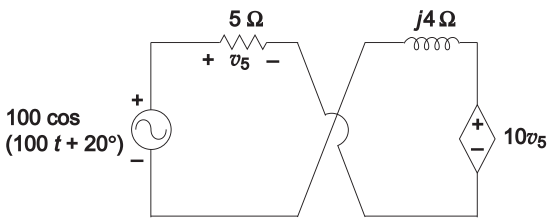


Fig. 6.18

Solution The circuit is redrawn as shown in Fig. 6.19.

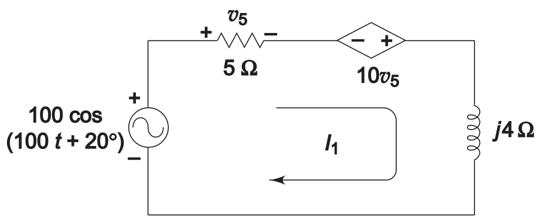


Fig. 6.19

Assume current I_1 flowing in the circuit.
The current I_1 can be determined by using Kirchhoff's voltage law.

$$I_1 = \frac{100\ \angle 20° + 10 \times 5\,I_1}{5 + j4}$$

$$I_1 - \frac{50\,I_1}{5+j4} = \frac{100\ \angle 20°}{5+j4}$$

$$I_1 = 2.213\ \angle - 154.9°$$

Average power delivered by the dependent source

$$= \frac{V_m\,I_m}{2}\cos\theta$$

$$= \frac{10 V_5 I_1}{2} \cos \theta$$

$$= \frac{50 \times (2.213)^2}{2} = 122.43 \text{ W}$$

Problem 6.13 For the circuit shown in Fig. 6.20, find the average power delivered by the voltage source.

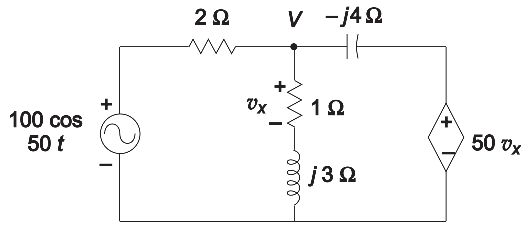


Fig. 6.20

Solution Applying Kirchhoff's current law at node

$$\frac{V - 100 \angle 0°}{2} + \frac{V}{1 + j3} + \frac{V - 50 V_x}{-j4} = 0$$

$$V_x = \frac{V}{1 + j3} \text{ volts}$$

Substituting in the above equation, we get

$$\frac{V - 100 \angle 0°}{2} + \frac{V}{1 + j3} + \frac{V}{-j4} - \frac{50 \text{V}}{(1 + j3)(-j4)} = 0$$

$$V = 14.705 \angle 157.5°$$

$$I = \frac{V - 100 \angle 0°}{2} = \frac{14.705 \angle 157.5° - 100 \angle 0°}{2}$$

$$= 56.865 \angle 177.18°$$

$$\text{Power delivered by the source} = \frac{100 \times 56.865 \cos 177.18°}{2}$$

$$= 2.834 \text{ kW}$$

Problem 6.14 For the circuit shown in Fig. 6.21, find the average power delivered by the dependent current source.

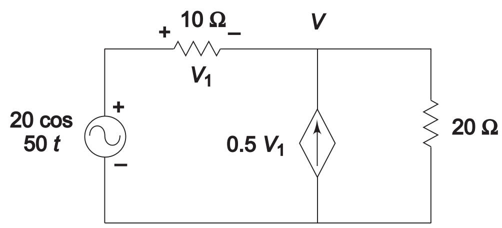


Fig. 6.21

Solution Applying Kirchhoff's current law at node

$$\frac{V - 20\,\angle 0^\circ}{10} - 0.5V_1 + \frac{V}{20} = 0$$

where $V_1 = 20\,\angle 0^\circ - V$

Substituting V_1 in the above equation, we get

$$V = 18.46\,\angle 0^\circ$$

$$V_1 = 1.54\,\angle 0^\circ$$

Average power delivered by the dependent source

$$\frac{V_m\,I_m\,\cos\theta}{2} = \frac{18.46 \times 0.5 \times 1.54}{2} = 7.107\text{ W}$$

Practice Problems

6.1 For the circuit shown in Fig. 6.22, a voltage of 250 sin ωt is applied. Determine the power factor of the circuit, if the voltmeter readings are V_1 = 100 V, V_2 = 125 V, V_3 = 150 V and the ammeter reading is 5 A.

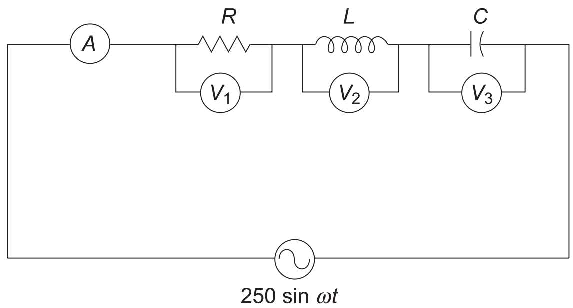


Fig. 6.22

6.2 For the circuit shown in Fig. 6.23, a voltage $v(t)$ is applied and the resulting current in the circuit $i(t)$ = 15 sin (ωt + 30°) amperes. Determine the active power, reactive power, power factor, and the apparent power.

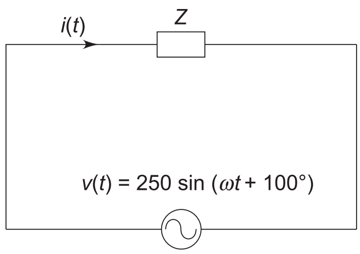


Fig. 6.23

6.3 A series RL circuit draws a current of $i(t) = 8 \sin (50t + 45°)$ from the source. Determine the circuit constants, if the power delivered by the source is 100 W and there is a lagging power factor of 0.707.

6.4 Two impedances, $Z_1 = 10 \angle -60°$ W and $Z_2 = 16 \angle 70°$ Ω are in series and pass an effective current of 5 A. Determine the active power, reactive power, apparent power and power factor.

6.5 For the circuit shown in Fig. 6.24, determine the value of the impedance if the source delivers a power of 200 W and there is a lagging power factor of 0.707. Also find the apparent power.

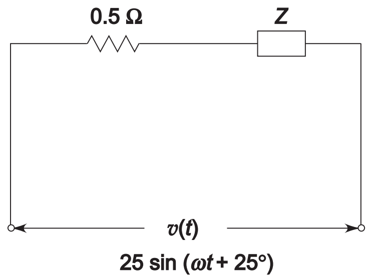


Fig. 6.24

6.6 A voltage of $v(t) = 100 \sin 500\, t$ is applied across a series R-L-C circuit where $R = 10$ Ω, $L = 0.05$ H and $C = 20$ μf. Determine the power supplied by the source, the reactive power supplied by the source, the reactive power of the capacitor, the reactive power of the inductor, and the power factor of the circuit.

6.7 For the circuit shown in Fig. 6.25 determine the power dissipated and the power factor of the circuit.

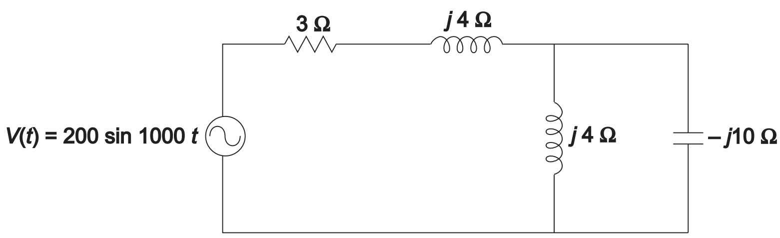


Fig. 6.25

6.8 For the circuit shown in Fig. 6.26, determine the power factor and the power dissipated in the circuit.

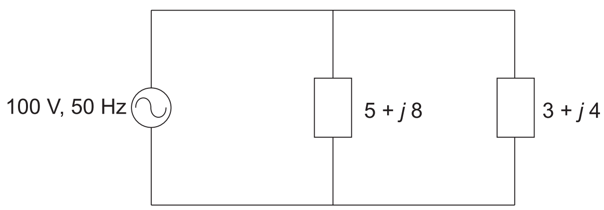


Fig. 6.26

6.9 For the circuit shown in Fig. 6.27, determine the power factor, active power, reactive power and apparent power.

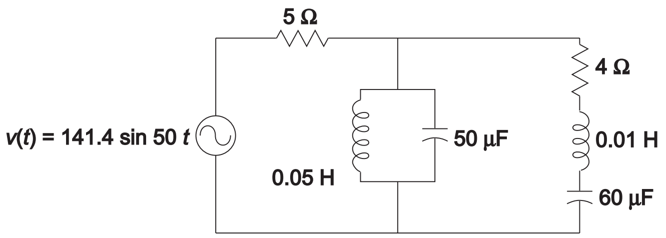


Fig. 6.27

6.10 In the parallel circuit shown in Fig. 6.28, the power in the 5 Ω resistor is 600 W and the total circuit takes 3000 VA at a leading power factor of 0.707. Find the value of impedance Z.

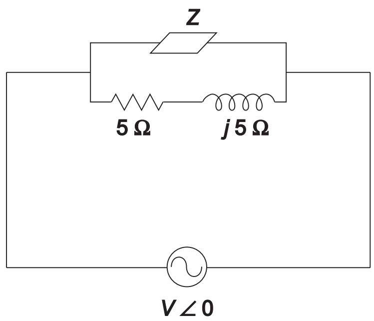


Fig. 6.28

6.11 For the parallel circuit shown in Fig. 6.29, the total power dissipated is 1000 W. Determine the apparent power, the reactive power, and the power factor.

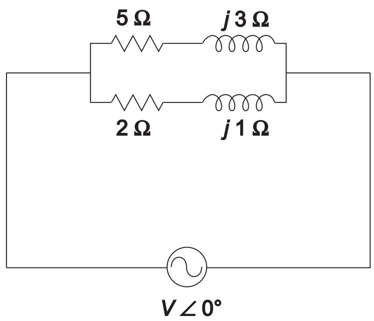


Fig. 6.29

6.12 A voltage source $v(t) = 150 \sin \omega t$ in series with 5 Ω resistance is supplying two loads in parallel, $Z_A = 60 \angle 30°$, and $Z_B = 50 \angle -25°$. Find the average power delivered to Z_A, the average power delivered to Z_B, the average power dissipated in the circuit, and the power factor of the circuit.

6.13 For the circuit shown in Fig. 6.30, determine the true power, reactive power and apparent power in each branch. What is the power factor of the total circuit?

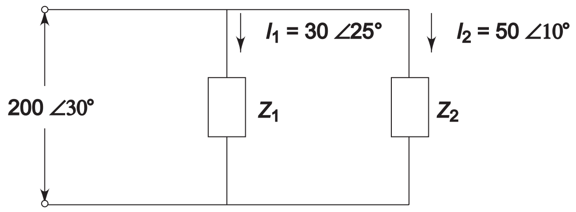


Fig. 6.30

6.14 Determine the value of the voltage source, and the power factor in the network shown in Fig. 6.31 if it delivers a power of 500 W to the circuit shown in Fig. 6.31. Also find the reactive power drawn from the source.

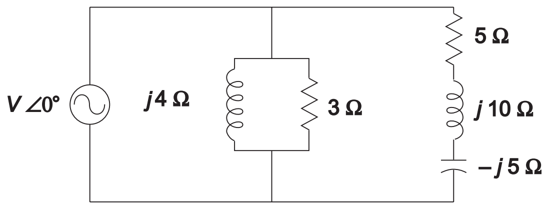


Fig. 6.31

6.15 Find the average power dissipated by the 500 Ω resistor shown in Fig. 6.32.

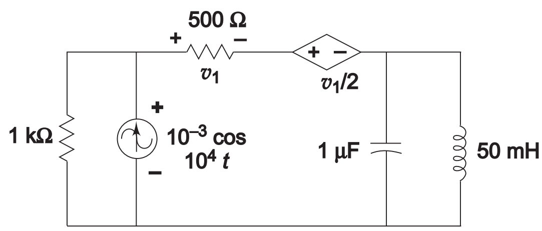


Fig. 6.32

6.16 Find the power dissipated by the voltage source shown in Fig. 6.33.

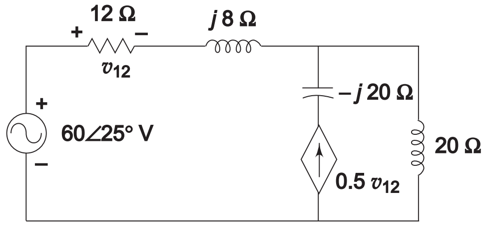


Fig. 6.33

6.17 Find the power delivered by current source shown in Fig. 6.34.

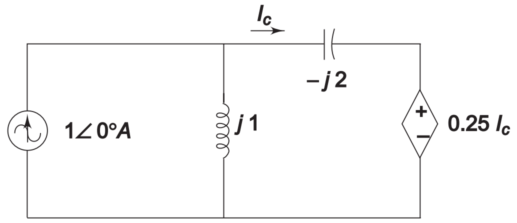


Fig. 6.34

6.18 For the circuit shown in Fig. 6.35, determine the power factor, active power, reactive power and apparent power.

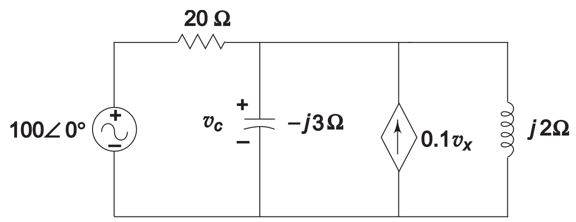


Fig. 6.35

Objective-type Questions

1. The phasor combination of resistive power and reactive power is called
   (a) true power (b) apparent power
   (c) reactive power (d) average power
2. Apparent power is expressed in
   (a) volt-amperes (b) watts
   (c) volt-amperes or watts (d) VAR
3. A power factor of '1' indicates
   (a) purely resistive circuit, (b) purely reactive circuit
   (c) combination of both, (a) and (b) (d) none of these
4. A power factor of 0 indicates
   (a) purely resistive element (b) purely reactive element
   (c) combination of both (a) and (b) (d) none of the above
5. For a certain load, the true power is 100 W and the reactive power is 100 VAR. What is the apparent power?
   (a) 200 VA (b) 100 VA
   (c) 141.4 VA (d) 120 VA
6. If a load is purely resistive and the true power is 5 W, what is the apparent power?

(a) 10 VA (b) 5 VA

(c) 25 VA (d) $\sqrt{50}$ VA

7. True power is defined as

(a) $VI \cos \theta$ (b) VI

(c) $VI \sin \theta$ (d) none of these

8. In a certain series RC circuit, the true power is 2 W, and the reactive power is 3.5 VAR. What is the apparent power?

(a) 3.5 VA (b) 2 VA

(c) 4.03 VA (d) 3 VA

9. If the phase angle θ is 45°, what is the power factor?

(a) cos 45° (b) sin 45°

(c) tan 45° (d) none of these

10. To which component in an RC circuit is the power dissipation due?

(a) capacitance (b) resistance

(c) both (d) none

11. A two element series circuit with an instantaneous current $I = 4.24 \sin (5000\ t + 45°)$ A has a power of 180 watts and a power factor of 0.8 lagging. The inductance of the circuit must have the value.

(a) 3 H (b) 0.3 H

(c) 3 mH (c) 0.3 mH

12. In the circuit shown in Fig. 6.36, if branch A takes 8 KVAR, the power of the circuit will be

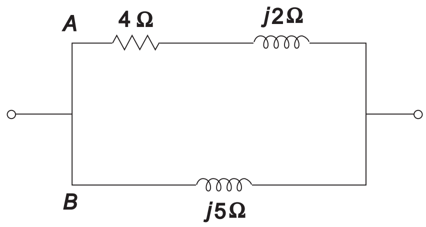


Fig. 6.36

(a) 2 kW (b) 4 kW

(c) 6 kW (d) 8 kW

13. In the circuit shown in Fig. 6.37, the voltage across 30 Ω resistor is 45 volts. The reading of the ammeter A will be

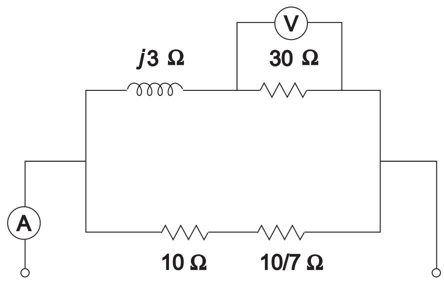


Fig. 6.37

(a) 10 A (b) 19.4 A
(c) 22.4 A (d) 28 A

14. In the circuit shown in Fig. 6.38, V_1 and V_2 are two identical sources of $10 \angle 90°$. The power supplied by V_1 is

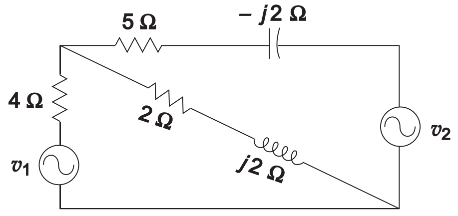


Fig. 6.38

(a) 6 W (b) 8.8 W
(c) 11 W (d) 16 W

Chapter 7

Steady State AC Analysis

7.1 MESH ANALYSIS

We have earlier discussed mesh analysis but have applied it only to resistive circuits. Some of the ac circuits presented in this chapter can also be solved by using mesh analysis. In Chapter 2, the two basic techniques for writing network equations for mesh analysis and node analysis were presented. These concepts can also be used for sinusoidal steady-state condition. In the sinusoidal steady-state analysis, we use voltage phasors, current phasors, impedances and admittances to write branch equations, KVL and KCL equations. For ac circuits, the method of writing loop equations is modified slightly. The voltages and currents in ac circuits change polarity at regular intervals. At a given time, the instantaneous voltages are driving in either the positive or negative direction. If the impedances are complex, the sum of their voltages is found by vector addition. We shall illustrate the method of writing network mesh equations with the following example.

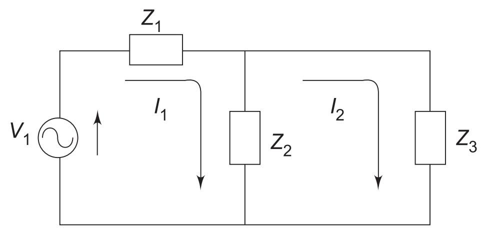


Fig. 7.1

Consider the circuit shown in Fig. 7.1, containing a voltage source and impedances.

The current in impedance Z_1 is I_1, and the current in Z_2, (assuming a positive direction downwards through the impedance) is $I_1 - I_2$. Similarly, the current in impedance Z_3 is I_2. By applying Kirchhoff's voltage law for each loop, we can get two equations. The voltage across any element is the product of the phasor current in the element and the complex impedance.

Equation for loop 1 is

$$I_1Z_1 + (I_1 - I_2)Z_2 = V_1 \tag{7.1}$$

Equation for loop 2, which contains no source is

$$Z_2(I_2 - I_1) + Z_3I_2 = 0 \tag{7.2}$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(Z_1 + Z_2) - I_2Z_2 = V_1 \tag{7.3}$$

$$- I_1Z_2 + I_2(Z_2 + Z_3) = 0 \tag{7.4}$$

By solving the above equations, we can find out currents I_1 and I_2. In general, if we have M meshes, B branches and N nodes including the reference node, we assume M branch currents and write M independent equations; then the number of mesh currents is given by $M = B - (N - 1)$.

Example 7.1 Write the mesh current equations in the circuit shown in Fig. 7.2, and determine the currents.

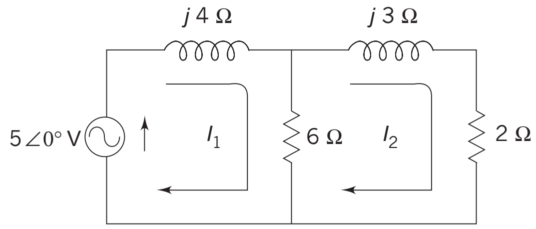


Fig. 7.2

Solution The equation for loop 1 is

$$I_1(j4) + 6(I_1 - I_2) = 5 \angle 0° \tag{7.5}$$

The equation for loop 2 is

$$6(I_2 - I_1) + (j_3)I_2 + (2)I_2 = 0 \tag{7.6}$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(6 + j4) - 6I_2 = 5 \angle 0° \tag{7.7}$$

$$- 6I_1 + (8 + j3)I_2 = 0 \tag{7.8}$$

Solving the above equations, we have

$$I_1 = \left[\frac{(8 + j3)}{6}\right] I_2$$

$$\left[\frac{(8 + j3)\,(6 + j4)}{6}\right] I_2 - 6I_2 = 5 \angle 0°$$

$$I_2 \left[\frac{(8 + j3)\,(6 + j4)}{6} - 6\right] = 5 \angle 0°$$

$$I_2\,[10.26 \angle 54.2° - 6 \angle 0°] = 5 \angle 0°$$

$$I_2\,[(6 + j8.32) - 6] = 5\,\angle 0°$$

$$I_2 = \frac{5\,\angle 0°}{8.32\,\angle 90°} = 0.6\,\angle{-90°}$$

$$I_1 = \frac{8.54\,\angle 20.5°}{6} \times 0.6\,\angle{-90°}$$

$$I_1 = 0.855\,\angle{-69.5°}$$

Current in loop 1, $I_1 = 0.855 = \angle{-}\,69.5°$

Current in loop 2, $I_2 = 0.6\,\angle{-}\,90°$

7.2 MESH EQUATIONS BY INSPECTION

In general, mesh equations can be written by observing any network. Consider the three mesh network shown in Fig. 7.3.

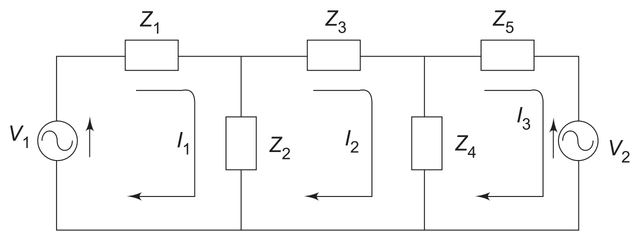


Fig. 7.3

The loop equations are

$$I_1 Z_1 + Z_2(I_1 - I_2) = V_1 \tag{7.9}$$

$$Z_2(I_2 - I_1) + Z_3 I_2 + Z_4(I_2 - I_3) = 0 \tag{7.10}$$

$$Z_4(I_3 - I_2) + Z_5 I_3 = -V_2 \tag{7.11}$$

By rearranging the above equations, we have

$$(Z_1 + Z_2)I_1 - Z_2 I_2 = V_1 \tag{7.12}$$

$$-Z_2 I_1 + (Z_2 + Z_3 + Z_4)I_2 - Z_4 I_3 = 0 \tag{7.13}$$

$$-Z_4 I_2 + (Z_4 + Z_5)I_3 = -V_2 \tag{7.14}$$

In general, the above equations can be written as

$$Z_{11} I_1 \pm Z_{12} I_2 + Z_{13} I_3 = V_a \tag{7.15}$$

$$\pm Z_{21} I_1 + Z_{22} I_2 \pm Z_{23} I_3 = V_b \tag{7.16}$$

$$\pm Z_{31} I_1 \pm Z_{32} I_2 + Z_{33} I_3 = V_c \tag{7.17}$$

If we compare the general equations with the circuit equations, we get the self impedance of loop 1

$$Z_{11} = Z_1 + Z_2$$

i.e. the sum of the impedances through which I_1 passes. Similarly, $Z_{22} = (Z_2 + Z_3 + Z_4)$, and $Z_{33} = (Z_4 + Z_5)$ are the self impedances of loops 2 and 3. This is equal to the sum of the impedances in their respective loops, through which I_2 and I_3 passes, respectively.

Z_{12} is the sum of the impedances common to loop currents I_1 and I_2. Similarly Z_{21} is the sum of the impedances common to loop currents I_2 and I_1. In the circuit shown in Fig. 7.3, $Z_{12} = -Z_2$, and $Z_{21} = -Z_2$. Here, the positive sign is used if both currents passing through the common impedance are in the same direction; and the negative sign is used if the currents are in opposite directions. Similarly, Z_{13}, Z_{23}, Z_{31}, Z_{32} are the sums of the impedances common to the mesh currents indicated in their subscripts. V_a, V_b and V_c are sums of the voltages driving their respective loops. Positive sign is used, if the direction of the loop current is the same as the direction of the source current. In Fig. 7.3, $V_b = 0$ because no source is driving loop 2. Since the source, V_2 drives against the loop current I_3, $V_c = -V_2$.

Example 7.2 For the circuit shown in Fig. 7.4, write the mesh equations using the inspection method.

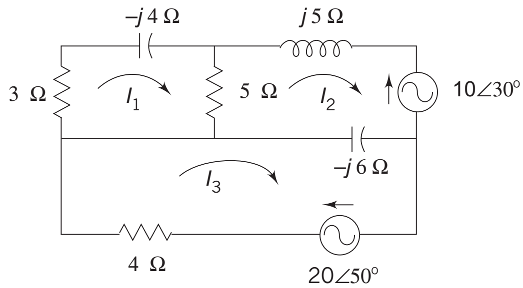


Fig. 7.4

Solution The general equations are

$$Z_{11} I_1 \pm Z_{12} I_2 \pm Z_{13} I_3 = V_a \tag{7.18}$$

$$\pm Z_{21} I_1 + Z_{22} I_2 \pm Z_{23} I_3 = V_b \tag{7.19}$$

$$\pm Z_{31} I_1 \pm Z_{32} I_2 + Z_{33} I_3 = V_c \tag{7.20}$$

Consider Eq. 7.18

Z_{11} = the self impedance of loop 1 = $(5 + 3 - j4)\ \Omega$

Z_{12} = the impedance common to both loop 1 and loop 2 = $-5\ \Omega$

The negative sign is used because the currents are in opposite directions.

$Z_{13} = 0$, because there is no common impedance between loop 1 and loop 3.

$V_a = 0$, because no source is driving loop 1.

∴ Equation 7.18 can be written as

$$(8 - j4)I_1 - 5I_2 = 0 \tag{7.21}$$

Now, consider Eq. 7.19

$Z_{21} = -5$, the impedance common to loop 1 and loop 2.

$Z_{22} = (5 + j5 - j6)$, the self impedance of loop 2.

$Z_{23} = -(-j6)$, the impedance common to loop 2 and loop 3.

$V_b = -10\angle 30°$, the source driving loop 2.

The negative sign indicates that the source is driving against the loop current, I_2.

Hence, Eq. 7.19 can be written as

$$-5I_1 + (5 - j1)I_2 + (j6)I_3 = -10 \angle 30° \tag{7.22}$$

Consider Eq. 7.20

$Z_{31} = 0$, there is no common impedance between loop 3 and loop 1

$Z_{32} = -(-j6)$, the impedance common to loop 2 and loop 3

$Z_{33} = (4 - j6)$, the self impedance of loop 3

$V_b = 20 \angle 50°$, the source driving loop 3

The positive sign is used because the source is driving in the same direction as the loop current 3. Hence, the equation can be written as

$$(j6)I_2 + (4 - j6)I_3 = 20 \angle 50° \tag{7.23}$$

The three mesh equations are

$$(8 - j4)I_1 - 5I_2 = 0$$
$$-5I_1 + (5 - j1)I_2 + (j6)I_3 = -10 \angle 30°$$
$$(j6)I_2 + (4 - j6)I_3 = 20 \angle 50°$$

7.3 NODAL ANALYSIS

The node voltage method can also be used with networks containing complex impedances and excited by sinusoidal voltage sources. In general, in an N node network, we can choose any node as the reference or datum node. In many circuits, this reference is most conveniently choosen as the common terminal or ground terminal. Then it is possible to write $(N-1)$ nodal equations using KCL. We shall illustrate nodal analysis with the following example.

Consider the circuit shown in Fig.7.5.

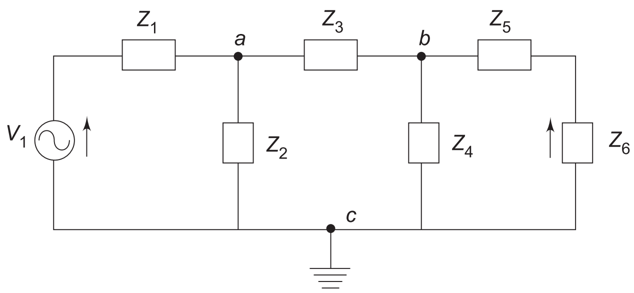


Fig. 7.5

Let us take a and b as nodes, and c as reference node. V_a is the voltage between nodes a and c. V_b is the voltage between nodes b and c. Applying Kirchhoff's current law at each node, the unknowns V_a and V_b are obtained.

In Fig. 7.6, node a is redrawn with all its branches, assuming that all currents are leaving the node a.

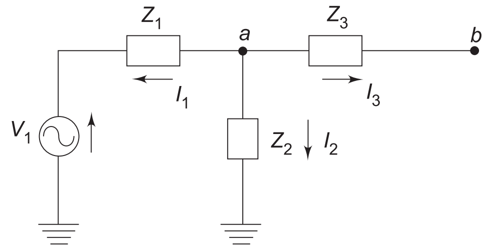


Fig. 7.6

In Fig. 7.6, the sum of the currents leaving node a is zero.

$\therefore$
$$I_1 + I_2 + I_3 = 0 \tag{7.24}$$

where
$$I_1 = \frac{V_a - V_1}{Z_1},\ I_2 = \frac{V_a}{Z_2},\ I_3 = \frac{V_a - V_b}{Z_3}$$

Substituting I_1, I_2 and I_3 in Eq. 1, we get

$$\frac{V_a - V_1}{Z_1} + \frac{V_a}{Z_2} + \frac{V_a - V_b}{Z_3} = 0 \tag{7.25}$$

Similarly, in Fig. 7.7, node b is redrawn with all its branches, assuming that all currents are leaving the node b.

In Fig. 7.7, the sum of the currents leaving the node b is zero.

$\therefore$
$$I_3 + I_4 + I_5 = 0 \tag{7.26}$$

where
$$I_3 = \frac{V_b - V_a}{Z_3},\ I_4 = \frac{V_b}{Z_4},\ I_5 = \frac{V_b}{Z_5 + Z_6}$$

Substituting I_3, I_4 and I_5 in Eq. 7.26

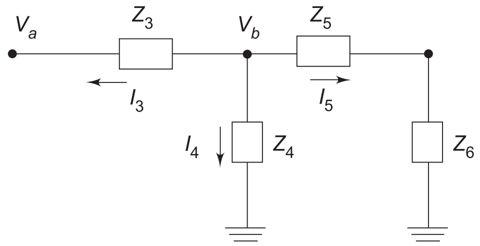


Fig. 7.7

we get
$$\frac{V_b - V_a}{Z_3} + \frac{V_b}{Z_4} + \frac{V_b}{Z_5 + Z_6} = 0 \tag{7.27}$$

Rearranging Eqs 7.25 and 7.27, we get

$$\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right) V_a - \left(\frac{1}{Z_3}\right) V_b = \left(\frac{1}{Z_1}\right) V_1 \tag{7.28}$$

$$-\left(\frac{1}{Z_3}\right)V_a + \left(\frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5 + Z_6}\right)V_b = 0 \qquad (7.29)$$

From Eqs 7.28 and 7.29, we can find the unknown voltages V_a and V_b.

Example 7.3 In the network shown in Fig. 7.8, determine V_a and V_b.

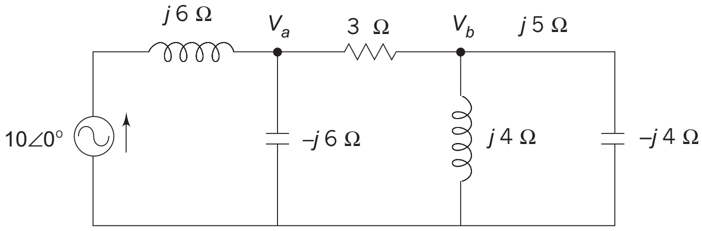


Fig. 7.8

Solution To obtain the voltage V_a at *a*, consider the branch currents leaving the node *a* as shown in Fig. 7.9 (a).

In Fig. 7.9(a), $I_1 = \dfrac{V_a - 10\angle 0°}{j6}$, $I_2 = \dfrac{V_a}{-j6}$, $I_3 = \dfrac{V_a - V_b}{3}$

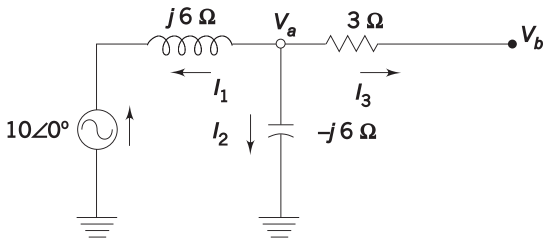


Fig. 7.9(a)

Since the sum of the currents leaving the node *a* is zero,

$$I_1 + I_2 + I_3 = 0$$

$$\frac{V_a - 10\angle 0°}{j6} + \frac{V_a}{-j6} + \frac{V_a - V_b}{3} = 0 \qquad (7.30)$$

$$\left(\frac{1}{j6} - \frac{1}{j6} + \frac{1}{3}\right)V_a - \frac{1}{3}V_b = \frac{10\angle 0°}{j6}$$

$$\therefore \quad \frac{1}{3}V_a - \frac{1}{3}V_b = \frac{10\angle 0°}{j6} \qquad (7.31)$$

To obtain the voltage V_b at *b*, consider the branch currents leaving node *b* as shown in Fig. 7.9 (b).

In Fig. 7.9(b), $I_3 = \dfrac{V_b - V_a}{3}$, $I_4 = \dfrac{V_b}{j4}$, $I_5 = \dfrac{V_b}{(j5 - j4)}$

Since the sum of the currents leaving node *b* is zero

$$I_3 + I_4 + I_5 = 0$$

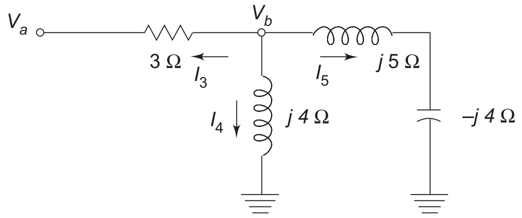


Fig. 7.9(b)

$$\frac{V_b - V_a}{3} + \frac{V_b}{j4} + \frac{V_b}{j1} = 0 \tag{7.32}$$

$$-\frac{1}{3} V_a + \left(\frac{1}{3} + \frac{1}{j4} + \frac{1}{j1}\right) V_b = 0 \tag{7.33}$$

From Eqs 7.31 and 7.33, we can solve for V_a and V_b.

$$0.33V_a - 0.33V_b = 1.67 \angle -90° \tag{7.34}$$

$$-0.33V_a + (0.33 - 0.25j - j)V_b = 0 \tag{7.35}$$

Adding Eqs 7.34 and 7.35 we get $(-1.25j)V_b = 1.67 \angle -90°$

$$-1.25 \angle 90° \; V_b = 1.67 \angle -90°$$

$$V_b = \frac{1.67 \angle -90°}{-1.25 \angle 90°}$$

$$= -1.34 \angle -180°$$

Substituting V_b in Eq. (7.34), we get

$$0.33V_a - (0.33)(-1.34 \angle -180°) = 1.67 \angle -90°$$

$$V_a = \frac{1.67 \angle -90°}{0.33} = -1.31 \text{ V}$$

$$V_a = 5.22 \angle -104.5° \text{ V}$$

Voltages V_a and V_b are $5.22 \angle -104.5°$ V and $-1.34 \angle -180°$ V respectively.

7.4 NODAL EQUATIONS BY INSPECTION

In general, nodal equations can also be written by observing the network. Consider a four node network including a reference node as shown in Fig. 7.10.

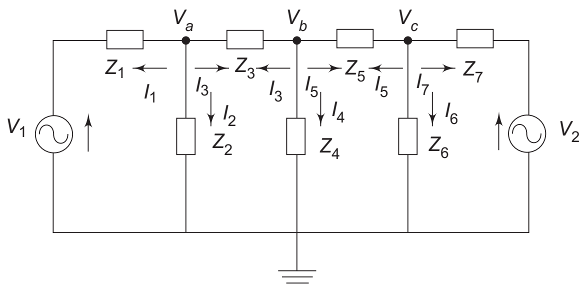


Fig. 7.10

Consider nodes a, b and c separately as shown in Figs 7.11(a), (b) and (c).

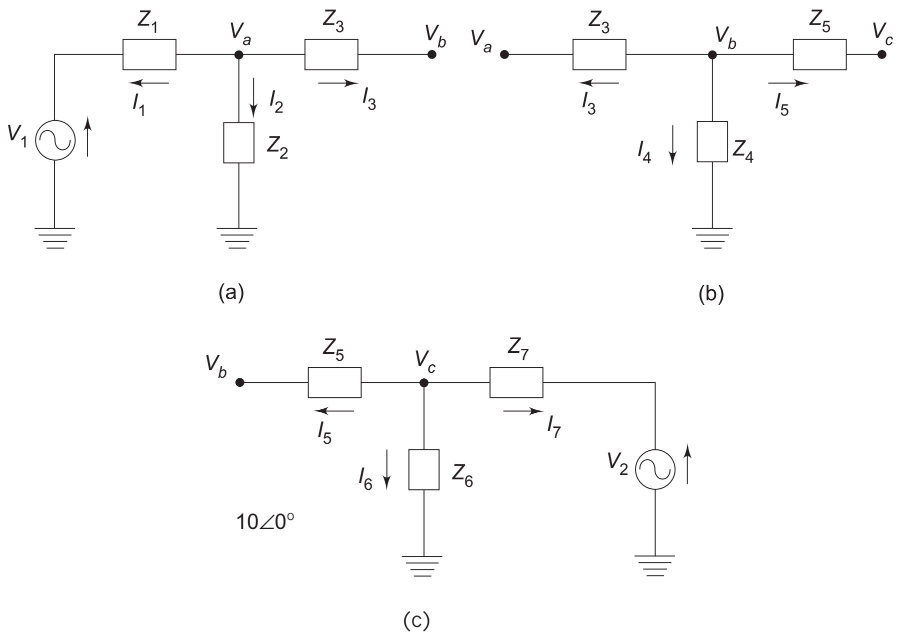


Fig. 7.11

Assuming that all the currents are leaving the nodes, the nodal equations at a, b and c are

$$I_1 + I_2 + I_3 = 0$$
$$I_3 + I_4 + I_5 = 0$$
$$I_5 + I_6 + I_7 = 0$$

$$\frac{V_a - V_1}{Z_1} + \frac{V_a}{Z_2} + \frac{V_a - V_b}{Z_3} = 0 \tag{7.36}$$

$$\frac{V_b - V_a}{Z_3} + \frac{V_b}{Z_4} + \frac{V_b - V_c}{Z_5} = 0 \tag{7.37}$$

$$\frac{V_c - V_b}{Z_5} + \frac{V_c}{Z_6} + \frac{V_c - V_2}{Z_7} = 0 \tag{7.38}$$

Rearranging the above equations, we get

$$\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right) V_a - \left(\frac{1}{Z_3}\right) V_a = \left(\frac{1}{Z_1}\right) V_1 \tag{7.39}$$

$$\left(\frac{-1}{Z_3}\right) V_a + \left(\frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5}\right) V_b - \left(\frac{1}{Z_5}\right) V_c = 0 \tag{7.40}$$

$$\left(\frac{-1}{Z_5}\right) V_b + \left(\frac{1}{Z_5} + \frac{1}{Z_6} + \frac{1}{Z_7}\right) V_c = \left(\frac{1}{Z_7}\right) V_2 \tag{7.41}$$

In general, the above equations can be written as

$$Y_{aa}V_{\mathrm{a}} + Y_{ab}V_b + Y_{ac}V_{\mathrm{c}} = I_1$$
$$Y_{ba}V_a + Y_{bb}V_b + Y_{bc}V_c = I_2$$
$$Y_{ca}V_a + Y_{cb}V_b + Y_{cc}V_c = I_3$$

If we compare the general equations with the circuit equations, the self admittance at node a is

$$Y_{aa} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

which is the sum of the admittances connected to node a.

Similarly, $Y_{bb} = \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5}$, and $Y_{cc} = \frac{1}{Z_5} + \frac{1}{Z_6} + \frac{1}{Z_7}$

are the self admittances at node b and node c, respectively. Y_{ab} is the mutual admittance between nodes a and b, i.e. it is the sum of all the admittances connecting nodes a and b. $Y_{ab} = -1/Z_3$ has a negative sign. All the mutual admittances have negative signs. Similarly, Y_{ac}, Y_{ba}, Y_{bc}, Y_{ca} and Y_{cb} are also mutual admittances. These are equal to the sums of the admittances connecting to nodes indicated in their subscripts. I_1 is the sum of all the source currents at node a. The current which drives into the node has a positive sign, while the current driving away from the node has a negative sign.

Example 7.4 For the circuit shown in Fig. 7.12, write the node equations by the inspection method.

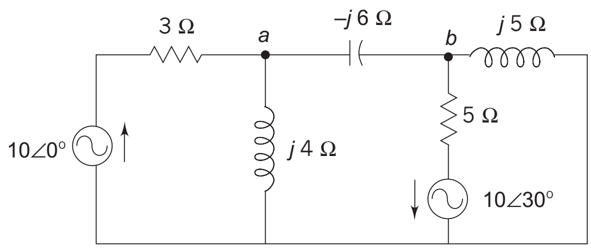


Fig. 7.12

Solution The general equations are

$$Y_{aa}\,V_a + Y_{ab}\,V_b = I_1 \tag{7.42}$$
$$Y_{ba}\,V_a + Y_{bb}\,V_b = I_2 \tag{7.43}$$

Consider Eq. 7.42

$$Y_{aa} = \frac{1}{3} + \frac{1}{j4} + \frac{1}{-j6}$$

The self admittance at node a is the sum of admittances connected to node a.

$$Y_{bb} = \frac{1}{-j6} + \frac{1}{5} + \frac{1}{j5}$$

The self admittance at node *b* is the sum of admittances connected to node *b*.

$$Y_{ab} = -\left(\frac{1}{-j6}\right)$$

The mutual admittance between nodes *a* and *b* is the sum of admittances connected between nodes *a* and *b*. Similarly, $Y_{ba} = -(-1/j6)$, the mutual admittance between nodes *b* and *a* is the sum of the admittances connected between nodes *b* and *a*.

$$I_1 = \frac{10\angle 0°}{3}$$

The source current at node *a*

$$I_2 = \frac{-10\angle 30°}{5}$$

the source current leaving at node *b*.

Therefore, the nodal equations are

$$\left(\frac{1}{3} + \frac{1}{j4} - \frac{1}{j6}\right)V_a - \left(\frac{-1}{j6}\right)V_b = \frac{10\angle 0°}{3} \tag{7.44}$$

$$-\left(\frac{-1}{j6}\right)V_a + \left(\frac{1}{5} + \frac{1}{j5} - \frac{1}{j6}\right)V_b = \frac{-10\angle 30°}{5} \tag{7.45}$$

7.5 SUPERPOSITION THEOREM

The superposition theorem also can be used to analyse ac circuits containing more than one source. The superposition theorem states that the response in any element in a circuit is the vector sum of the responses that can be expected to flow if each source acts independently of other sources. As each source is considered, all of the other sources are replaced by their internal impedances, which are mostly short circuits in the case of a voltage source, and open circuits in the case of a current source. This theorem is valid only for linear systems. In a network containing complex impedance, all quantities must be treated as complex numbers.

Consider a circuit which contains two sources as shown in Fig. 7.13.

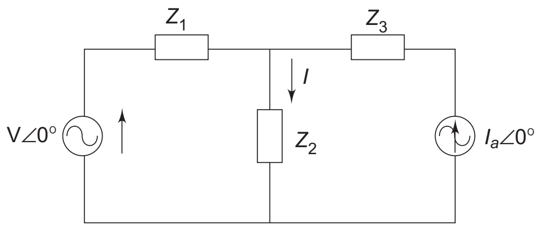


Fig. 7.13

Now let us find the current I passing through the impedance Z_2 in the circuit. According to the superposition theorem, the current due to voltage source V $\angle 0°$ V is I_1 with current source $I_a \angle 0°$ A open circuited.

$$I_1 = \frac{V \angle 0°}{Z_1 + Z_2}$$

The current due to $I_a \angle 0°$ A is I_2 with voltage source $V \angle 0°$ short circuited.

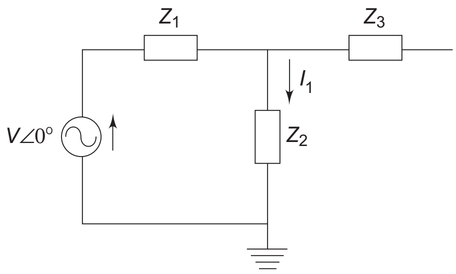


Fig. 7.14

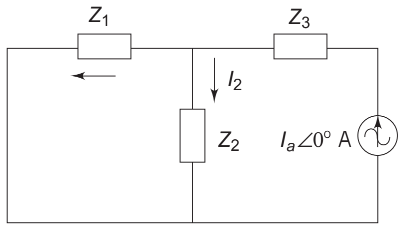


Fig. 7.15

$$\therefore \qquad I_2 = I_a \angle 0° \times \frac{Z_1}{Z_1 + Z_2}$$

The total current passing through the impedance Z_2 is

$$I = I_1 + I_2$$

Example 7.5 Determine the voltage across $(2 + j5)$ Ω impedance as shown in Fig. 7.16 by using the superposition theorem.

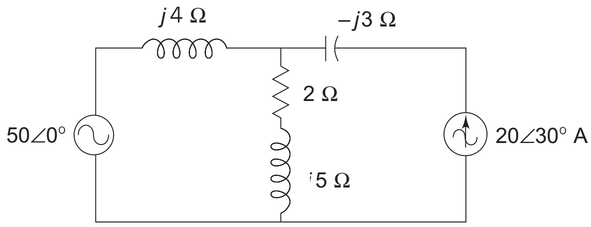


Fig. 7.16

Solution According to the superposition theorem, the current due to the 50 $\angle 0°$ V voltage source is I_1 as shown in Fig. 7.17 with current source 20 $\angle 30°$ A open circuited.

$$\text{Current} \qquad I_1 = \frac{50 \angle 0°}{2 + j4 + j5} = \frac{50 \angle 0°}{(2 + j9)}$$

$$= \frac{50 \angle 0°}{9.22 \angle 77.47°} = 5.42 \angle -77.47° \text{ A}$$

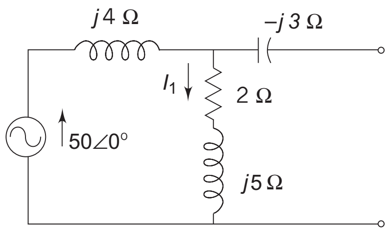


Fig. 7.17

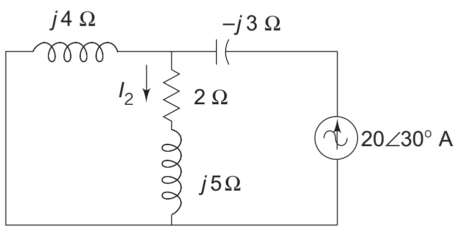


Fig. 7.18

Voltage across $(2 + j5)\ \Omega$ due to current I_1 is

$$V_1 = 5.42 \angle -77.47° (2 + j5)$$
$$= (5.38)(5.42) \angle -77.47° + 68.19°$$
$$= 29.16 \angle -9.28°$$

The current due to $20 \angle 30°$ A current source is I_2 as shown in Fig. 7.18, with voltage source $50 \angle 0°$ V short circuited.

Current $$I_2 = 20 \angle 30° \times \frac{(j4)\ \Omega}{(2 + j9)\ \Omega}$$

$$= \frac{20 \angle 30° \times 4 \angle 90°}{9.22 \angle 77.47°}$$

$\therefore$ $$I_2 = 8.68 \angle 120° - 77.47° = 8.68 \angle 42.53°$$

Voltage across $(2 + j5)\ \Omega$ due to current I_2 is

$$V_2 = 8.68 \angle 42.53° (2 + j5)$$
$$= (8.68)(5.38) \angle 42.53° + 68.19°$$
$$= 46.69 \angle 110.72°$$

Voltage across $(2 + j5)\ \Omega$ due to both sources is

$$V = V_1 + V_2$$
$$= 29.16 \angle -9.28° + 46.69 \angle 110.72°$$
$$= 28.78 - j4.7 - 16.52 + j43.67$$
$$= (12.26 + j38.97)\ \text{V}$$

Voltage across $(2 + j5)\ \Omega$ is $V = 40.85 \angle 72.53°$.

7.6 THEVENIN'S THEOREM

Thevenin's theorem gives us a method for simplifying a given circuit. The Thevenin equivalent form of any complex impedance circuit consists of an equivalent voltage source V_{Th}, and an equivalent impedance Z_{Th}, arranged as shown in Fig. 7.19. The values of equivalent voltage and impedance depend on the values in the original circuit.

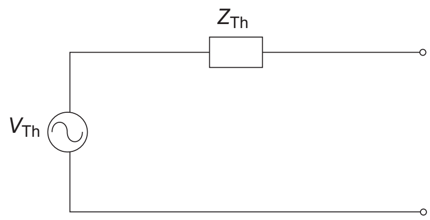


Fig. 7.19

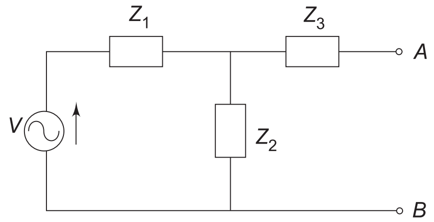


Fig. 7.20

Though the Thevenin equivalent circuit is not the same as its original circuit, the output voltage and output current are the same in both cases. Here, the Thevenin voltage is equal to the open circuit voltage across the output terminals, and impedance is equal to the impedance seen into the network across the output terminals.

Consider the circuit shown in Fig. 7.20.

Thevenin equivalent for the circuit shown in Fig. 7.20 between points A and B is found as follows.

The voltage across points A and B is the Thevenin equivalent voltage. In the circuit shown in Fig. 7.20, the voltage across A and B is the same as the voltage across Z_2 because there is no current through Z_3.

$$\therefore \qquad V_{Th} = V\left(\frac{Z_2}{Z_1 + Z_2}\right)$$

The impedance between points A and B with the source replaced by short circuit is the Thevenin equivalent impedance. In Fig. 7.20, the impedance from A to B is Z_3 in series with the parallel combination of Z_1 and Z_2.

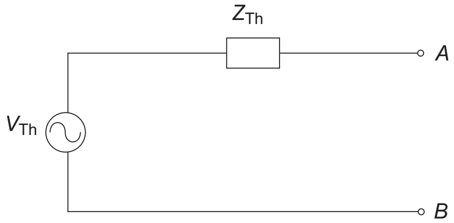


Fig. 7.21

$$\therefore \qquad Z_{Th} = Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2}$$

The Thevenin equivalent circuit is shown in Fig. 7.21.

Example 7.6 For the circuit shown in Fig. 7.22, determine Thevenin's equivalent between the output terminals.

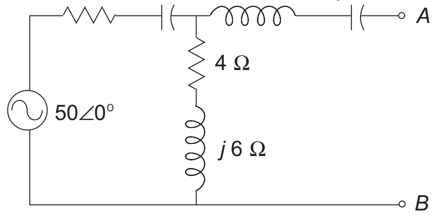


Fig. 7.22

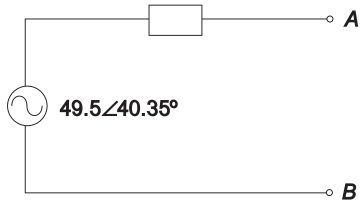


Fig. 7.23

Solution The Thevenin voltage, V_{Th}, is equal to the voltage across the $(4 + j6)\ \Omega$ impedance. The voltage across $(4 + j6)\ \Omega$ is

$$
\begin{aligned}
V &= 50\ \angle 0^\circ \times \frac{(4 + j6)}{(4 + j6) + (3 - j4)} \\
&= 50\ \angle 0^\circ \times \frac{4 + j6}{7 + j2} \\
&= 50\ \angle 0^\circ \times \frac{7.21 \angle 56.3^\circ}{7.28\ \angle 15.95^\circ} \\
&= 50\ \angle 0^\circ \times 0.99\ \angle 40.35^\circ \\
&= 49.5\ \angle 40.35^\circ\ \text{V}
\end{aligned}
$$

The impedance seen from terminals A and B is

$$
\begin{aligned}
Z_{Th} &= (j5 - j4) + \frac{(3 + j4)(4 + j6)}{(3 - j4)(4 + j6)} \\
&= j1 + \frac{5\ \angle 53.13^\circ \times 7.21 \angle 56.3^\circ}{7.28\ \angle 15.95^\circ} \\
&= j1 + 4.95\ \angle -12.78^\circ = j1 + 4.83 - j1.095 \\
&= 4.83 - j0.095
\end{aligned}
$$

$$\therefore \qquad Z_{Th} = 4.83\ \angle -1.13^\circ\ \Omega$$

The Thevenin equivalent circuit is shown in Fig. 7.23.

7.7 NORTON'S THEOREM

Another method of analysing a complex impedance circuit is given by Norton's theorem. The Norton equivalent form of any complex impedance circuit consists of an equivalent current source I_N and an equivalent impedance Z_N, arranged as shown in Fig. 7.24. The values of equivalent current and impedance depend on the values in the original circuit.

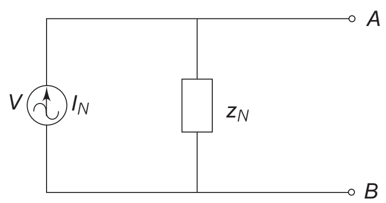


Fig. 7.24

Though Norton's equivalent circuit is not the same as its original circuit, the output voltage and current are the same in both cases; Norton's current is equal to the current passing through the short circuited output terminals and the value of impedance is equal to the impedance seen into the network across the output terminals.

Consider the circuit shown in Fig. 7.25.

Norton's equivalent for the circuit shown in Fig. 7.25 between points A and B is found as follows. The current passing through points A and B when it is short-circuited is the Norton's equivalent current, as shown in Fig. 7.26.

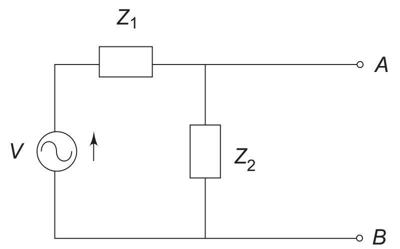


Fig. 7.25

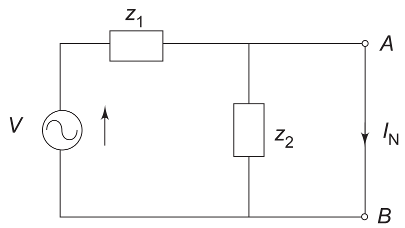


Fig. 7.26

Norton's current $I_N = V/Z_1$

The impedance between points A and B, with the source replaced by a short circuit, is Norton's equivalent impedance. In Fig. 7.25, the impedance from A to B, Z_2 is in parallel with Z_1.

$$\therefore \qquad Z_N = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Norton's equivalent circuit is shown in Fig. 7.27

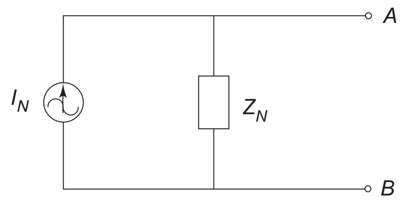


Fig. 7.27

Example 7.7 For the circuit shown in Fig. 7.28, determine Norton's equivalent circuit between the output terminals, AB.

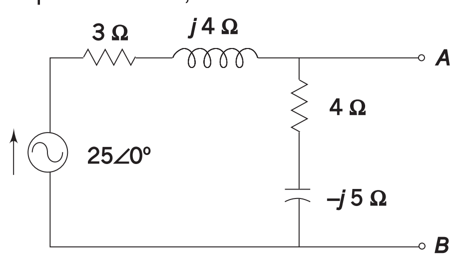


Fig. 7.28

Solution Norton's current I_N is equal to the current passing through the short circuited terminals AB as shown in Fig. 7.29.

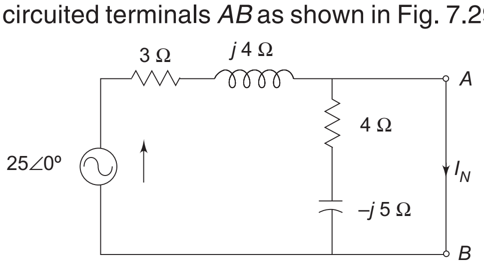


Fig. 7.29

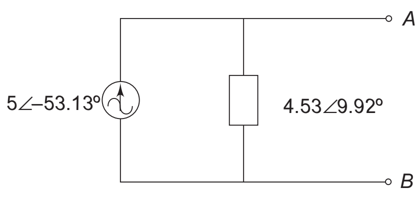


Fig. 7.30

The current through terminals AB is

$$I_N = \frac{25\angle 0^\circ}{3 + j4} = \frac{25\angle 0^\circ}{5\angle 53.13^\circ} = 5\angle -53.13^\circ$$

The impedance seen from terminals AB is

$$Z_N = \frac{(3 + j4)(4 - j5)}{(3 + j4) + (4 - j5)}$$

$$= \frac{5 \angle 53.13^\circ \times 6.4 \angle -51.34^\circ}{7.07 \angle -8.13^\circ} = 4.53 \angle 9.92^\circ$$

Norton's equivalent circuit is shown in Fig. 7.30.

7.8 MAXIMUM POWER TRANSFER THEOREM

In Chapter 3, the maximum power transfer theorem has been discussed for resistive loads. The maximum power transfer theorem states that the maximum power is delivered from a source to its load when the load resistance is equal to the source resistance. It is for this reason that the ability to obtain impedance matching between circuits is so important. For example, the audio output transformer must match the high impedance of the audio power amplifier output to the low input impedance of the speaker. Maximum power transfer is not always desirable, since the transfer occurs at a 50 per cent efficiency. In many situations, a maximum voltage transfer is desired which means that unmatched impedances are necessary. If maximum power transfer is required, the load resistance should equal the given source resistance. The maximum power transfer theorem can be applied to complex impedance circuits. If the source impedance is complex, then the maximum power transfer occurs when the load impedance is the complex conjugate of the source impedance.

Consider the circuit shown in Fig. 7.31, consisting of a source impedance delivering power to a complex load.

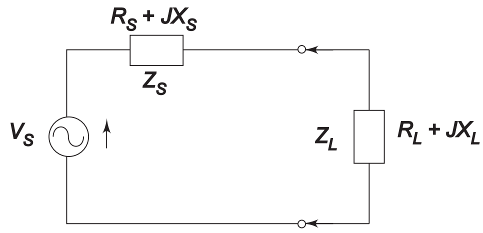


Fig. 7.31

Current passing through the circuit shown

$$I = \frac{V_s}{(R_s + jX_s) + (R_L + jX_L)}$$

Magnitude of current $I = |I| = \dfrac{V_s}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$

Power delivered to the circuit is

$$P = I^2 R_L = \frac{V_s^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

In the above equation, if R_L is fixed, the value of P is maximum when

$$X_s = -X_L$$

Then the power $$P = \frac{V_s^2 R_L}{(R_s + R_L)^2}$$

Let us assume that R_L is variable. In this case, the maximum power is transferred when the load resistance is equal to the source resistance (already discussed in Chapter 3). If $R_L = R_s$ and $X_L = -X_s$, then $Z_L = Z_s^*$. This means that the maximum power transfer occurs when the load impedance is equal to the complex conjugate of source impedance Z_s.

Example 7.8 For the circuit shown in Fig. 7.32, find the value of load impedance for which the source delivers maximum power. Calculate the value of the maximum power.

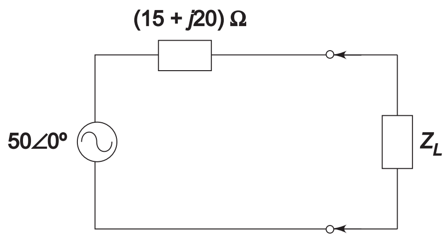


Fig. 7.32

Solution In the circuit shown in Fig. 7.32, the maximum power transfer occurs when the load impedance is complex conjugate of the source impedance

$$\therefore \quad Z_L = Z_s^* = 15 - j20$$

When $Z_L = 15 - j20$, the current passing through circuit is

$$I = \frac{V_s}{R_s + R_L} = \frac{50\angle 0^\circ}{15 + j20 + 15 - j20} = \frac{50\angle 0^\circ}{30} = 1.66\angle 0^\circ$$

The maximum power delivered to the load is

$$P = I^2 R_L = (1.66)^2 \times 15 = 41.33 \text{ W}$$

Additional Solved Problems

Problem 7.1 Using mesh analysis, determine the voltage V_s which gives a voltage of $30\angle 0^\circ$ V across the 30 Ω resistor shown in Fig. 7.33.

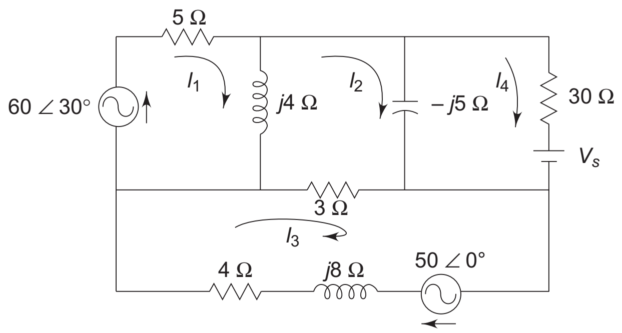


Fig. 7.33

Solution By the inspection method, we can have four equations from four loops.

$$(5 + j4)I_1 - (j4)I_2 = 60\ \angle 30° \tag{7.46}$$

$$(-j4)I_1 + (3 - j1)I_2 - 3I_3 + (j5)I_4 = 0 \tag{7.47}$$

$$-3I_2 + (7 + j8)I_3 = 50\ \angle 0° \tag{7.48}$$

$$(j5)I_2 + (30 - j5)I_4 = -V_s \tag{7.49}$$

Solving the above equations using Cramer's rule, we get

$$I_4 = \frac{\begin{vmatrix} (5+j4) & (-j4) & 0 & 60\ \angle 30° \\ (-j4) & (3-j1) & -3 & 0 \\ 0 & -3 & (7+j8) & 50\ \angle 0° \\ 0 & (j5) & 0 & -V_s \end{vmatrix}}{\begin{vmatrix} (5+j4) & (-j4) & 0 & 0 \\ (-j4) & (3-j1) & -3 & (J5) \\ 0 & -3 & (7+j8) & 0 \\ 0 & (j5) & 0 & (30-J5) \end{vmatrix}}$$

$$\Delta = (5+j4)\begin{vmatrix} (3-j1) & -3 & (j5) \\ -3 & (7+j8) & 0 \\ (j5) & 0 & (30-j5) \end{vmatrix}$$

$$+ (j4)\begin{vmatrix} (-j4) & -3 & (j5) \\ 0 & (7+j8) & 0 \\ 0 & 0 & (30-j5) \end{vmatrix}$$

$$= (5 + j4)\ \{(3 - j1)\ (7 + j8)\ (30 - j5) + 3\ [(-3)\ (30 - j5)]$$
$$+ j5\ [(-j5)\ (7 + j8)]\} + (j4)\ \{(-j4)\ (7 + j8)\ (30 - j5)\}$$

$$= (5 + j4)\ \{[3.16\ \angle{-18.4°} \times 10.6\ \angle 48.8° \times 30.4\ \angle{-9.46°}]$$
$$-9 \times 30.4\ \angle{-9.46°} + 25\ (10.6\ \angle 48.8°)\}$$
$$+ (j4)\ \{4\ \angle{-90°} \times 10.6\ \angle 48.8° \times 30.4\ \angle{-9.46°}\}$$

$$= (5 + j4)\ \{1018.27\ \angle 20.94° - 273.6\ \angle -9.46° + 265\ \angle 48.8°\}$$
$$+ j4\ \{1288.96\ \angle -50.66°\}$$
$$= (5 + j4)\ \{951 + j363.9 - 269.8 + j44.97 + 174.55 + j199.38\}$$
$$+ 4\ \angle 90°\ \{1288.96\ \angle -50.66°\}$$
$$= (5 + j4)\ \{855.75 + j608.25\} + 4\ \angle 90°\ \{1288.96\ \angle -50.66°\}$$
$$= 6.4\ \angle 38.6° \times 1049.9\ \angle 35.4° + 4\ \angle 90° \times 1288.96\ \angle -50.66°$$
$$= 6719.36\ \angle 74° + 5155.84\ \angle 39.34°$$
$$= 1852.1 + j6459 + 3987.5 + j3268.3$$
$$= 5839.6 + j9727.3$$
$$= 11345.5\ \angle 59°$$

$$\Delta_4 = (5 + j4)\begin{vmatrix} (3 - j1) & -3 & 0 \\ -3 & (7 + j8) & 50\ \angle 0° \\ (j5) & 0 & -V_s \end{vmatrix}$$

$$+ j4\begin{vmatrix} -j4 & -3 & 0 \\ 0 & (7 + j8) & 50\ \angle 0° \\ 0 & 0 & -V_s \end{vmatrix} - 60\ \angle 30° \begin{vmatrix} (-j4) & (3 - j1) & -3 \\ 0 & -3 & 7 + j8 \\ 0 & (j5) & 0 \end{vmatrix}$$

$$= (5 + j4)\ \{[(3 - j1)\ (7 + j8)\ (-V_s)] + 3[(3V_s) - (j5)\ 50\ \angle 0°]\}$$
$$+ (j4)\ \{(-j4)\ (7 + j8)\ (-V_s)\} - 60\ \angle 30°\ \{(-j4)\ (-j5)\ (7 + j8)\}$$
$$= 6.4\ \angle 38.6°\ \{[3.16\ \angle -18.4° \times 10.6\ \angle 48.8°\ (-V_s)]$$
$$+ [9V_s - (15j)\ 50\ \angle 0°]\}$$
$$+ 4\ \angle 90°\ \{4\ \angle -90° \times 10.6\ \angle 48.8°)\ (-V_s)\}$$
$$- 60\ \angle 30°\ \{4\ \angle -90° \times 5\ \angle -90° \times 10.6\ \angle 48.8°\}$$
$$= 6.4\ \angle 38.6°\ \{-33.49\ \angle 30.4°\ V_s\} + 6.4\ \angle 38.6° \times 9V_s$$
$$+ 4\ \angle 90°\ \{-42.4\ \angle -41.2°\ V_s\} - 60\ \angle 30°\ \{212\ \angle -131.2°\}$$
$$- 6.4\ \angle 38.6°\ \{+ 750\ \angle 90°\}$$
$$= V_s\ \{-214.33\ \angle 69° + 57.6\ \angle 38.6° - 169.6\ \angle 48.8°\}$$
$$- \{12720\ \angle -101.2° + 4800\ \angle 128.6°\}$$
$$= V_s\ \{-76.8 - j200 + 45 + j35.93 - 111.7 - j127.6\}$$
$$- \{-2470.6 - j12477.75 - 2994.6 + j3751.2\}$$
$$= V_s\ \{-143.5 - j291.67\} - \{-5465.2 - j8726.55\}$$

$$\therefore \qquad I_4 = \frac{(-143.5 - j291.67)\ V_s + (5465.2 + j8726.5)}{11345.5\ \angle 59°}$$

Since voltage across the 30 Ω resistor is 30 $\angle 0°$ V. Current passing through it is $I_4 = 1\ \angle 0°$ A

$$\therefore \qquad 1\ \angle 0° = \frac{(-143.5 - j291.67)\ V_s + (5465.2 + j8726.5)}{11345.5\ \angle 59°}$$

$$11345.5\ \angle 59° = 325\ \angle -116.19°\ V_s + 5465.2 + j8726.5$$

$$V_s = \frac{-5465.2 - j8726.5 + 5843.36 + j9724.99}{325\angle -116.19°}$$

$$= \frac{378.16 + j998.49}{325\ \angle -116.19°} = \frac{1067.7 + j69.26°}{325\ \angle -116.19°}$$

$$V_s = 3.29\ \angle 185.45°.$$

Problem 7.2 For the circuits shown in Fig. 7.34, determine the line currents I_R, I_Y and I_B using mesh analysis.

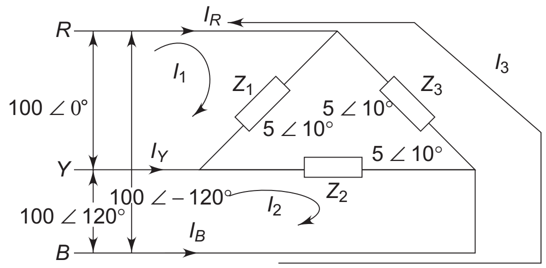


Fig. 7.34

Solution From Fig. 7.34, the three line currents are

$$I_R = I_1 - I_3$$
$$I_Y = I_2 - I_1$$
$$I_B = I_3 - I_2$$

Using the inspection method, the three loop equations are

$$5\ \angle 10°\ I_1 = 100\ \angle 0°$$
$$5\ \angle 10°\ I_2 = 100\ \angle 120°$$
$$5\ \angle 10°\ I_3 = 100\ \angle -120°$$

$$\therefore \qquad I_1 = \frac{100\ \angle 0°}{5\ \angle 10°} = 20\ \angle -10°$$

$$I_2 = \frac{100\ \angle 120°}{5\ \angle 10°} = 20\ \angle +110°$$

$$I_3 = \frac{100\ \angle -120°}{5\ \angle 10°} = 20\ \angle -130°$$

The line currents are

$$I_R = I_1 - I_3 = 20\ \angle -10° - 20\ \angle -130°$$
$$= 19.69 - j3.47 + 12.85 + j15.32$$
$$= 32.54 + j11.85 = 34.63\ \angle 20°$$
$$I_Y = I_2 - I_1 = 20\ \angle 110° - 20\ \angle -10°$$
$$= -6.84 + j18.79 - 19.69 + j3.47$$
$$= -26.53 + j22.26 = 34.63\ \angle 140°$$

$$\begin{aligned} I_B &= I_3 - I_2 = 20\ \angle -130° - 20\ \angle 110° \\ &= -12.85 - j15.32 + 6.84 - j18.79 \\ &= -6.01 - j34.11 = 34.63\ \angle -100° \end{aligned}$$

Problem 7.3 For the circuit shown in Fig. 7.35, determine the value of V_2 such that the current $(3 + j4)\ \Omega$ impedance is zero.

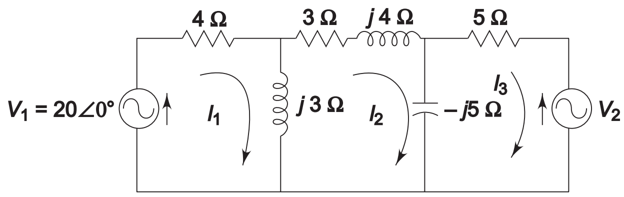


Fig. 7.35

Solution The three loop equations are

$$(4 + j3)\ I_1 - (j3)I_2 = 20\ \angle 0°$$

$$(-j3)I_1 + (3 + j2)I_2 + j5I_3 = 0$$

$$(j5)I_2 + (5 - j5)I_3 = -V_2$$

Since the current I_2 in $(3 + j4)\ \Omega$ is zero

$$I_2 = \frac{\Delta_2}{\Delta} = 0$$

$\therefore$ $$\Delta_2 = 0$$

where $$\Delta_2 = \begin{vmatrix} (4 + j3) & 20\ \angle 0° & 0 \\ (-j3) & 0 & j5 \\ 0 & -V_2 & (5 - j5) \end{vmatrix} = 0$$

$$(4 + j3)\ V_2\ (j5) - 20\ \angle 0°\ \{(-j3)\ (5 - j5)\} = 0$$

$$V_2 = \frac{20\ \angle 0°\ \{(-j3)\ (5 - j5)\}}{(j5)\ (4 + j3)}$$

$$= 20\ \angle 0°\ \frac{\{-15 - j15\}}{-15 + j20} = 20\ \angle 0° \times \frac{21.21\ \angle -135°}{25\ \angle 126.86°}$$

$$V_2 = 16.97\ \angle -261.85°\ \text{V}$$

Problem 7.4 For the circuit shown in Fig. 7.36, write the nodal equations using the inspection method and express them in matrix form.

Solution The number of nodes and reference node are selected as shown in Fig. 7.36, by assuming that all currents are leaving at each node.

At node a, $$\left(\frac{1}{4} + \frac{1}{-j1} + \frac{1}{1 + j1}\right) V_a - \left(\frac{1}{-j1}\right) V_b - \left(\frac{1}{1 + j1}\right) V_c = \frac{-50\ \angle 0°}{1 + j1}$$

At node b, $$-\left(\frac{1}{-j1}\right) V_a + \left(\frac{1}{3} + \frac{1}{-j1} + \frac{1}{j3}\right) V_b - \left(\frac{1}{j3}\right) V_c = \frac{20\ \angle 30°}{j3}$$

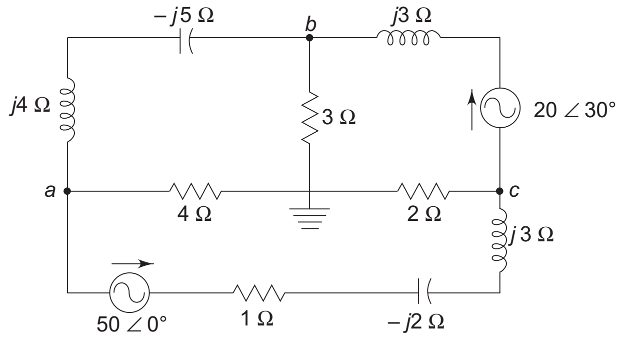


Fig. 7.36

At node c,

$$-\left(\frac{1}{1+j1}\right)V_a-\left(\frac{1}{j3}\right)V_b+\left(\frac{1}{2}+\frac{1}{j3}+\frac{1}{1+j1}\right)V_c=\frac{50\angle 0^\circ}{1+j1}-\frac{20\angle 30^\circ}{j3}$$

In matrix form, the nodal equations are

$$\begin{bmatrix} \frac{1}{4}+\frac{1}{(1+j1)}-\frac{1}{j1} & +\frac{1}{j1} & -\frac{1}{(1+j1)} \\ \frac{1}{j1} & \frac{1}{3}-\frac{1}{j1}+\frac{1}{j3} & -\frac{1}{j3} \\ -\frac{1}{(1+j1)} & -\frac{1}{(j3)} & \frac{1}{2}+\frac{1}{j3}+\frac{1}{1+j1} \end{bmatrix}\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$$

$$=\begin{bmatrix} \frac{-50\angle 0^\circ}{(1+j1)} \\ \frac{20\angle 30^\circ}{j3} \\ \left(\frac{50\angle 0^\circ}{1+j1}-\frac{20\angle 30^\circ}{j3}\right) \end{bmatrix}$$

Problem 7.5 For the circuit shown in Fig. 7.37, determine the voltage V_{AB}, if the load resistance R_L is infinite. Use node analysis.

Solution If the load resistance is infinite, no current passes through R_L. Hence R_L acts as an open circuit. If we consider A as a node and B as the reference node

$$\frac{V_A-20\angle 0^\circ}{3+2}+\frac{V_A-20\angle 90^\circ}{j4+3}=0$$

$$\frac{V_A}{5}+\frac{V_A}{(3+j4)}=\frac{20\angle 0^\circ}{5}+\frac{20\angle 90^\circ}{(3+j4)}$$

$$V_A\left[\frac{1}{5}+\frac{1}{3+j4}\right]=4\angle 0^\circ+\frac{20\angle 90^\circ}{5\angle 53.13^\circ}$$

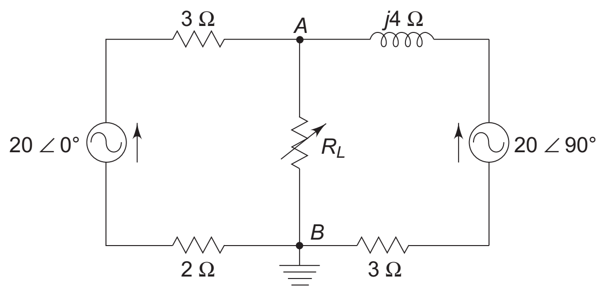


Fig. 7.37

$$= 4\ \angle 0° + 4\ \angle 36.87° = 4 + 3.19 + j2.4 = 7.19 + j2.4$$

$$V_A\ [0.2 + 0.12 - j0.16] = 7.19 + j2.4$$

$$V_A = \frac{7.19 + j2.4}{0.32 - j0.16} = \frac{7.58\ \angle 18.46°}{0.35\ \angle -26.56°}$$

Voltage across AB is $V_{AB} = V_A = 21.66\ \angle 45.02°$ V

Problem 7.6 For the circuit shown in Fig. 7.38, determine the power output of the source and the power in each resistor of the circuit.

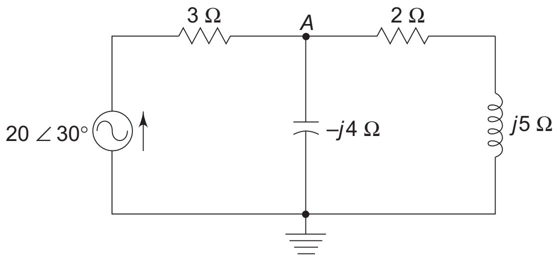


Fig. 7.38

Solution Assume that the voltage at node A is V_A

By applying nodal analysis, we have

$$\frac{V_A - 20\ \angle 30°}{3} + \frac{V_A}{-j4} + \frac{V_A}{2 + j5} = 0$$

$$V_A\left[\frac{1}{3} + \frac{1}{2 + j5} - \frac{1}{j4}\right] = \frac{20\ \angle 30°}{3}$$

$$V_A\ [0.33 + 0.068 + j0.078] = 6.67\ \angle 30°$$

$$\therefore \qquad V_A = \frac{6.67\ \angle 30°}{0.41\ \angle 11.09°} = 16.27\ \angle 18.91°$$

Current in the 2 Ω resistor

$$I_2 = \frac{V_A}{2 + j5} = \frac{16.27\ \angle 18.91°}{5.38\ \angle 68.19°}$$

$\therefore \qquad I_2 = 3.02 \angle -49.28°$

Power dissipated in the 2 Ω resistor

$$P_2 = I_2^2 R = (3.02)^2 \times 2 = 18.24 \text{ W}$$

Current in the 3 Ω resistor

$$I_3 = \frac{-20 \angle 30° + 16.27 \angle 18.91°}{3}$$

$$= -6.67 \angle 30° + 5.42 \angle 18.91°$$

$$= -5.78 - j3.34 + 5.13 + j1.76 = -0.65 - j1.58$$

$$I_3 = 1.71 \angle -112°$$

Power dissipated in the 3 Ω resistor

$$= (1.71)^2 \times 3 = 8.77 \text{ W}$$

Total power delivered by the source

$$= VI \cos \phi = 20 \times 1.71 \cos 142° = 26.95 \text{ W}$$

Problem 7.7 For the circuit shown in Fig. 7.39, determine the voltage V_{AB} using the superposition theorem.

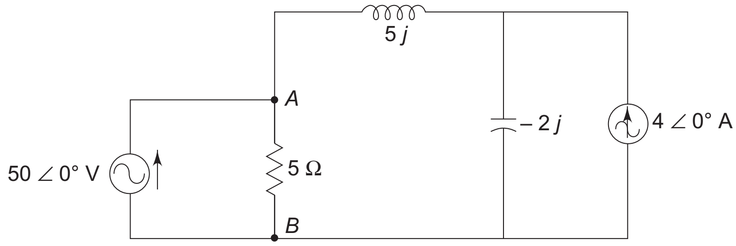


Fig. 7.39

Solution Let source 50 ∠0° V act on the circuit and set the source 4 ∠0° *A* equal to zero. If the current source is zero, it becomes open-circuited. Then the voltage across *AB* is $V_{AB} = 50 \angle 0°$.

Now set the voltage source 50 ∠0° V is zero, and is short circuited, or the voltage drop across *AB* is zero.

The total voltage is the sum of the two voltages.

$\therefore \qquad V_T = 50 \angle 0°$

Problem 7.8 For the circuit shown in Fig. 7.40, determine the current in $(2 + j3)$ Ω by using the superposition theorem.

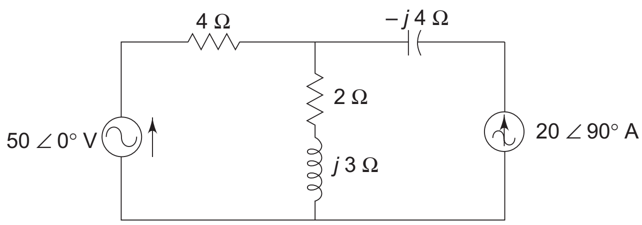


Fig. 7.40

Solution The current in $(2 + j3)\ \Omega$, when the voltage source $50\ \angle 0°$ acting alone is

$$I_1 = \frac{50\ \angle 0°}{(6 + j3)} = \frac{50\ \angle 0°}{6.7\ \angle 26.56°}$$

$\therefore$ $I_1 = 7.46\ \angle -26.56°$ A

Current in $(2 + j3)\ \Omega$, when the current source $20\ \angle 90°$ A acting alone is

$$I_2 = 20\ \angle 90° \times \frac{4}{(6 + j3)}$$

$$= \frac{80\ \angle 90°}{6.7\ \angle 26.56°} = 11.94\ \angle 63.44°\ \text{A}$$

Total current in $(2 + j3)\ \Omega$ due to both sources is

$$\begin{aligned} I &= I_1 + I_2 \\ &= 7.46\ \angle -26.56° + 11.94\ \angle 63.44° \\ &= 6.67 - j3.33 + 5.34 + j10.68 \\ &= 12.01 + j7.35 = 14.08\ \angle 31.46° \end{aligned}$$

Total current in $(2 + j3)\ \Omega$ is I = $14.08\ \angle 31.46°$.

Problem 7.9 For the circuit shown in Fig. 7.41, determine the load current by applying Thevenin's theorem.

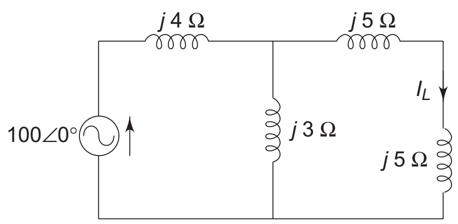


Fig. 7.41

Solution Let us find the Thevenin equivalent circuit for the circuit shown in Fig. 7.42(a).

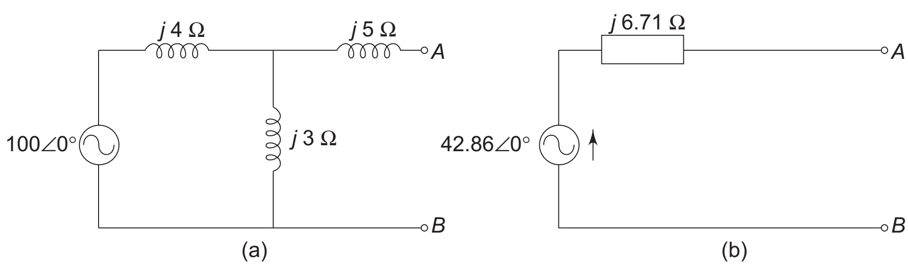


Fig. 7.42

Voltage across *AB* is the voltage across ($j3$) Ω

$$\therefore \qquad V_{AB} = 100\ \angle 0^\circ \times \frac{(j3)}{(j3)+(j4)}$$

$$= 100\ \angle 0^\circ\ \frac{(j3)}{j7} = 42.86\ \angle 0^\circ$$

Impedance seen from terminals *AB*

$$Z_{AB} = (j5) + \frac{(j4)(j3)}{j7}$$

$$= j5 + j1.71 = j6.71\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 7.42(b).

If we connect a load to Fig. 7.42(b), the current passing through ($j5$) Ω impedance is

$$I_L = \frac{42.86\ \angle 0^\circ}{(j6.71 + j5)} = \frac{42.86\ \angle 0^\circ}{11.71\ \angle 90^\circ} = 3.66\ \angle -90^\circ$$

Problem 7.10 For the circuit shown in Fig. 7.43, determine Thevenin's equivalent circuit.

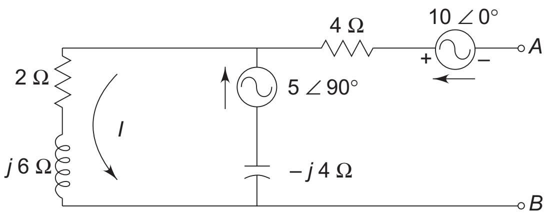


Fig. 7.43

Solution Voltage across ($-j4$) Ω is

$$V_{-j4} = \frac{5\ \angle 90^\circ}{(2 + j2)}\ (-j4)$$

$$= \frac{20\ \angle 0^\circ}{2.83\ \angle 45^\circ} = 7.07\ \angle -45^\circ$$

Voltage across *AB* is $V_{AB} = -V_{10} + V_5 - V_{-j4}$

$$= -10\ \angle 0^\circ + 5\ \angle 90^\circ - 7.07\ \angle -45^\circ$$

$$= j5 - 10 - 4.99 + j4.99$$

$$= -14.99 + j\,9.99$$

$$V_{AB} = 18\ \angle 146.31^\circ$$

The impedance seen from terminals *AB*, when all voltage sources are short circuited is

$$Z_{AB} = 4 + \frac{(2 + j6)(-j4)}{2 + j2}$$

$$= 4 + \frac{6.32\ \angle 71.56° \times 4\ \angle -90°}{2.83\ \angle 45°}$$

$$= 4 + 8.93\ \angle -63.44°$$

$$= 4 + 4 - j\ 7.98 = (8 - j7.98)\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 7.44.

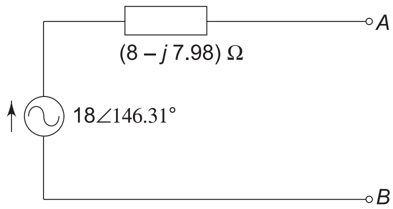


Fig. 7.44

Problem 7.11 For the circuit shown in Fig. 7.11, determine the load current I_L by using Norton's theorem.

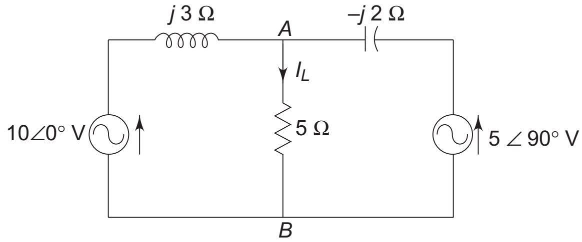


Fig. 7.45

Solution Norton's impedance seen from terminals AB is

$$Z_{AB} = \frac{(j3)(-j2)}{(j3)-(j2)} = \frac{6}{j1}$$

$\therefore$ $$Z_{AB} = 6\ \angle -90°$$

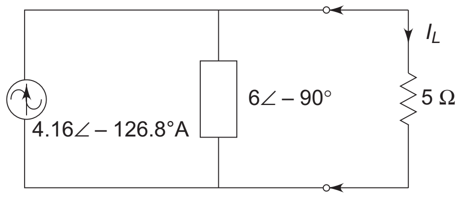
Current passing through AB, when it is shorted

$$I_N = \frac{10\ \angle 0°}{3\ \angle 90°} + \frac{5\ \angle 90°}{2\ \angle -90°}$$

$\therefore$ $$I_N = 3.33\ \angle -90° + 2.5\ \angle 180°$$

$$= -j3.33 - 2.5$$

$$I_N = 4.16\ \angle -126.8°$$



Norton's equivalent circuit is shown in Fig. 7.46

Fig. 7.46

Load current is $$I_L = I_N \times \frac{6\ \angle -90°}{5 + 6\ \angle -90°}$$

$$= 4.16\ \angle -126.8° \times \frac{6\ \angle -90°}{5 - j6}$$

$$= \frac{4.16 \times 6\ \angle -216.8°}{7.81\ \angle -50.19°}$$

$$= 3.19\ \angle -166.61°$$

Problem 7.12 For the circuit shown in Fig. 7.47, determine Norton's equivalent circuit.

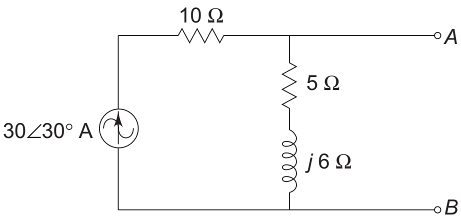


Fig. 7.47

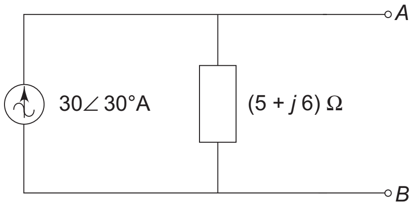


Fig. 7.48

Solution The impedance seen from the terminals when the source is reduced to zero

$$Z_{AB} = (5 + j6)\ \Omega$$

Current passing through the short circuited terminals, A and B, is

$$I_N = 30\ \angle 30^\circ\ \text{A}$$

Norton's equivalent circuit is shown in Fig. 7.48.

Problem 7.13 Convert the active network shown in Fig. 7.49 by a single voltage source in series with impedance.

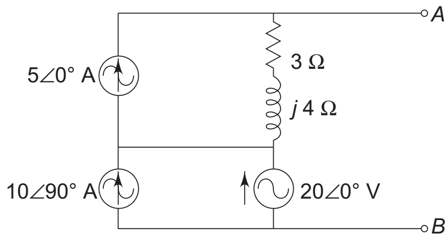


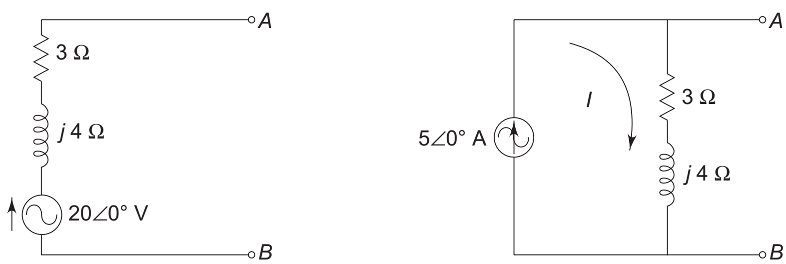
Fig. 7.49

Solution Using the superposition theorem, we can find Thevenin's equivalent circuit. The voltage across AB, with 20 $\angle 0^\circ$ V source acting alone, is V'_{AB}, and can be calculated from Fig. 7.50(a).

Since no current is passing through the $(3 + j4)\ \Omega$ impedance, the voltage

$$V'_{AB} = 20\ \angle 0^\circ$$

The voltage across AB, with 5 $\angle 0^\circ$ A source acting alone, is V'_{AB}, and can be calculated from Fig. 7.50(b).



(a) (b)

Fig. 7.50

$$V''_{AB} = 5\ \angle 0° \ (3 + j4) = 5\ \angle 0° \times 5\ \angle 53.13° = 25\ \angle 53.13° \text{ V}$$

The voltage across AB, with $10\ \angle 90°$ A source acting alone, is V'''_{AB}, and can be calculated from Fig. 7.50 (c).

$$V'''_{AB} = 0$$

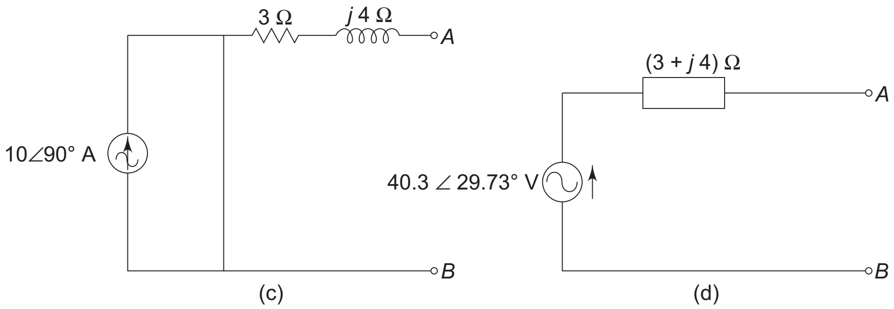


Fig. 7.50

According to the superposition theorem, the voltage across AB due to all sources is

$$V_{AB} = V'_{AB} + V''_{AB} + V'''_{AB}$$

$\therefore$ $$V_{AB} = 20\ \angle 0° + 25\ \angle 53.13° = 20 + 15 + j19.99$$
$$= (35 + j19.99) \text{ V} = 40.3\ \angle 29.73° \text{ V}$$

The impedance seen from terminals AB

$$Z_{\text{Th}} = Z_{AB} = (3 + j4)\ \Omega$$

$\therefore$ The required Thevenin circuit is shown in Fig. 7.50(d).

Problem 7.14 For the circuit shown in Fig. 7.51, find the value of Z that will receive maximum power; also determine this power.

Solution The equivalent impedance at terminals AB with the source set equal to zero is

$$Z_{AB} = \frac{5\,(j10)}{5 + j10} + \frac{7\,(-j20)}{(7 - j20)}$$
$$= \frac{50\ \angle 90°}{11.18\ \angle 63.43°} + \frac{140\ \angle -90°}{21.19\ \angle -70.7°}$$
$$= 4.47\ \angle 26.57° + 6.6\ \angle -19.3°$$
$$= 3.99 + j1.99 + 6.23 - j2.18$$
$$= 10.22 - j0.19$$

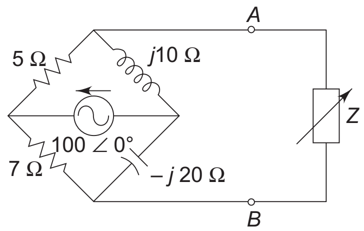


Fig. 7.51

The Thevenin equivalent circuit is shown in Fig. 7.52(a).

The circuit in Fig. 7.52(a) is redrawn as shown in Fig. 7.52(b).

Current $$I_1 = \frac{100\ \angle 0°}{5 + j10}$$

$$= \frac{100 \angle 0°}{11.18 \angle 63.43°} = 8.94 \angle -63.43° \text{ A}$$

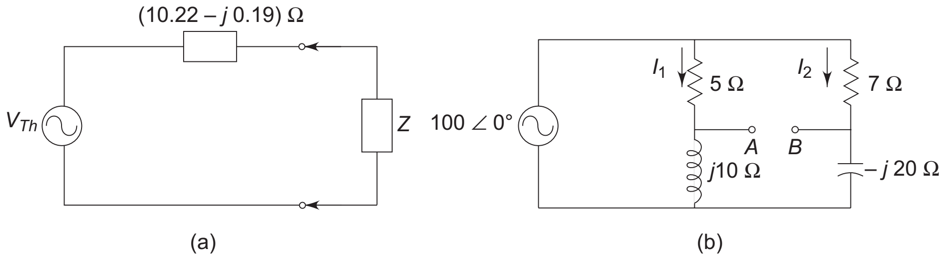


Fig. 7.52

Current $$I_2 = \frac{100 \angle 0°}{7 - j20} = \frac{100 \angle 0°}{21.19 \angle -70.7°} = 4.72 \angle 70.7°$$

Voltage at A, $V_A = 8.94 \angle -63.43° \times j10 = 89.4 \angle 26.57°$

Voltage at B, $V_B = 4.72 \angle 70.7° \times -j20 = 94.4 \angle -19.3°$

Voltage across terminals AB

$$\begin{aligned} V_{AB} &= V_A - V_B \\ &= 89.4 \angle 26.57° - 94.4 \angle -19.3° \\ &= 79.96 + j39.98 - 89.09 + j31.2 \\ &= -9.13 + j71.18 \\ V_{\text{Th}} &= V_{AB} = 71.76 \angle 97.3° \text{ V} \end{aligned}$$

To get maximum power, the load must be the complex conjugate of the source impedance.

$\therefore$ Load $Z = 10.22 + j0.19$

Current passing through the load Z

$$I = \frac{V_{\text{Th}}}{Z_{\text{Th}} + Z} = \frac{71.76 \angle 97.3°}{20.44} = 3.51 \angle 97.3°$$

Maximum power delivered to the load is

$$= (3.51)^2 \times 10.22 = 125.91 \text{ W}$$

Problem 7.15 For the circuit shown in Fig. 7.53, the resistance R_s is variable from 2 Ω to 50 Ω. What value of R_s results in maximum power transfer across the terminals AB?

Solution In the circuit shown the resistance R_L is fixed. Here, the maximum power transfer theorem does not apply. Maximum current flows in the circuit when R_s is minimum. For the maximum current

$$R_s = 2$$

But $Z_T = R_s - j5 + R_L = 2 - j5 + 20 = (22 - j5) = 22.56 \angle -12.8°$

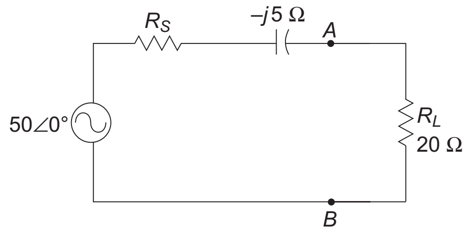


Fig. 7.53

$$\therefore \qquad I = \frac{V_s}{Z_T} = \frac{50\ \angle 0°}{22.56\ \angle -12.8°} = 2.22\ \angle 12.8°$$

Maximum power $P = I^2R = (2.22)^2 \times 20 = 98.6$ W

Problem 7.16 Determine the voltage V which results in a zero current through the $2 + j3\ \Omega$ impedance in the circuit shown in Fig. 7.54.

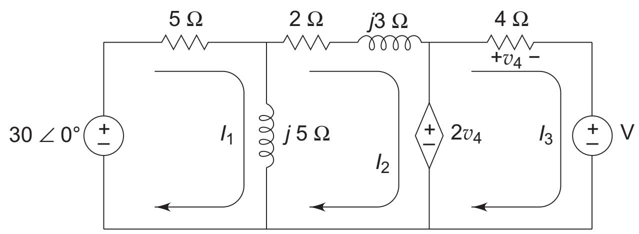


Fig. 7.54

Solution Choosing mesh currents as shown in Fig. 7.54, the three loop equations are

$$(5 + j5)\ I_1 - j5\ I_2 = 30\ \angle 0°$$
$$-j5\ I_1 + (2 + j8)\ I_2 = -2V_4$$
$$-2V_4 + V_4 + V = 0$$
$$V_4 = V$$

Since the current in $(2 + j3)\ \Omega$ is zero

$$I_2 = \frac{\Delta_2}{\Delta} = 0$$

Where
$$\Delta_2 = \begin{vmatrix} 5 + j5 & 30\ \angle 0° \\ -j5 & -2V \end{vmatrix} = 0$$

$$(5 + j5)\ (-2V) + (j5)\ 30\ \angle 0° = 0$$

$$V = \frac{30\ \angle 0°(j5)}{2\ (5 + j5)} = \frac{150\ \angle 90°}{14.14\ \angle 45°}$$

$$V = 10.608\ \angle 45° \text{ volts}$$

Problem 7.17 Find the value of voltage V which results in $V_0 = 5 \angle 0°$ V in the circuit shown in Fig. 7.56.

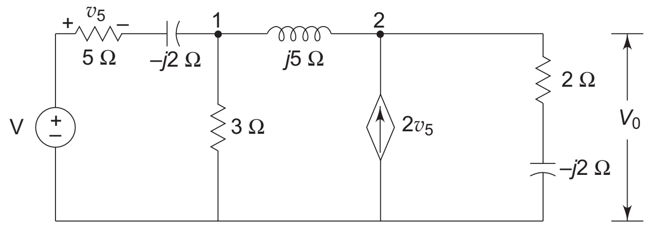


Fig. 7.56

Solution Assuming all currents are leaving the nodes, the nodal equations are

$$V_1\left[\frac{1}{5-j2}+\frac{1}{3}+\frac{1}{j5}\right]-V_2\left[\frac{1}{j5}\right]=\frac{V}{5-j2}$$

$$-V_1\left[\frac{1}{j5}\right]+V_2\left[\frac{1}{j5}+\frac{1}{2-j2}\right]=2V_5$$

where $$V_5=\left(\frac{V_1-V}{5-j2}\right)^5$$

The second equation becomes

$$V_1\left[\frac{-1}{j5}-\frac{10}{5-j2}\right]+V_2\left[\frac{1}{j5}+\frac{1}{2-j2}\right]=\frac{-10V}{5-j2}$$

$$V_0=V_2=\frac{\Delta_2}{\Delta}=5\angle 0°$$

$$\frac{\begin{vmatrix}\dfrac{1}{5-j2}+\dfrac{1}{3}+\dfrac{1}{j5} & \dfrac{V}{5-j2}\\[2ex] \dfrac{-1}{j5}-\dfrac{10}{5-j2} & \dfrac{-10\,V}{5-j2}\end{vmatrix}}{\begin{vmatrix}\dfrac{1}{5-j2}+\dfrac{1}{3}+\dfrac{1}{j5} & \dfrac{-1}{j5}\\[2ex] \dfrac{-1}{j5}-\dfrac{10}{5-j2} & \dfrac{1}{j5}+\dfrac{1}{2-j2}\end{vmatrix}}=5\angle 0°$$

The source voltage $V = 2.428 \angle{-88.74°}$ volts.

Problem 7.18 For the circuit shown in Fig. 7.57, find the current in the $j5\ \Omega$ inductance by using Thevenin's theorem.

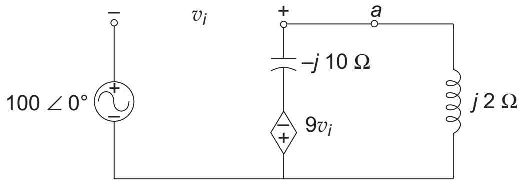


Fig. 7.57

Solution From the circuit shown in Fig. 7.57 the open circuit voltage at terminals a and b is

$$V_{oc} = -9\ V_i$$

where
$$V_i = -9V_i - 100\ \angle 0°$$
$$10V_i = -100\ \angle 0°$$
$$V_i = -10\ \angle 0°$$

Thevenin's voltage $V_{oc} = 90\ \angle 0°$

From the circuit, short circuit current is determined by shorting terminals a and b. Applying Kirchhoff's voltage law, we have

$$9V_i - j10\ i_{sc} = 0$$
$$i_{sc} = 90\ \angle 90°$$

$$\therefore \qquad Z_{\text{Th}} = \frac{V_{oc}}{I_{sc}} = \frac{90\ \angle 0°}{9\ \angle 90°}$$
$$= 1\ \angle -90°$$
$$Z_{\text{Th}} = -j1$$

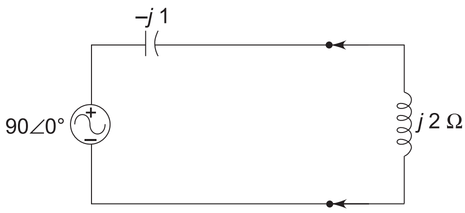


Fig. 7.58

The Thevenin's equivalent circuit is shown in Fig. 7.58

The current in the $j2\ \Omega$ inductor is $= \dfrac{90\ \angle 0°}{j1}$

$$= 90\ \angle -90°$$

Problem 7.19 For the circuit shown in Fig. 7.59, find the value of Z that will receive maximum power; also determine this power.

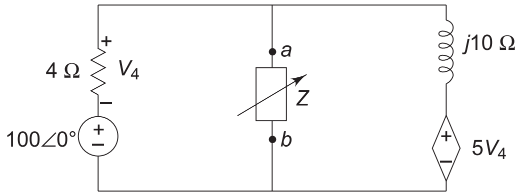


Fig. 7.59

Solution The equivalent impedance can be obtained by finding V_{oc} and i_{sc} at terminals *a b*. Assume that current *i* is passing in the circuit.

$$i = \frac{100\angle 0° - 5V_4}{4 + j10}$$

$$= \frac{100\angle 0°}{4 + j10} - \frac{5 \times 4i}{4 + j10}$$

$$i = 3.85 \angle -22.62°$$

$$V_{oc} = 100\angle 0° - 4 \times 3.85 \angle -22.62°$$

$$= 86 \angle 3.94°$$

$$i_{sc} = \frac{100\angle 0°}{4} = 25\angle 0°$$

Thevenin's equivalent impedance

$$Z_{\text{Th}} = \frac{V_{oc}}{i_{sc}} = 3.44\angle 3.94°$$

$$= 3.43 + j0.24$$

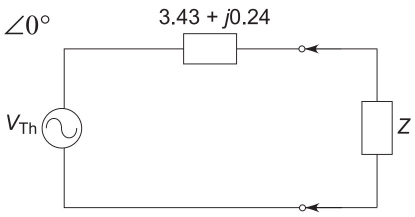


Fig. 7.60

The circuit is drawn as shown in Fig. 7.60.

To get maximum power, the load must be the complex conjugate of the source impedance.

$\therefore$ Load $Z = 3.43 - j0.24$

Current passing through load Z

$$I = \frac{V_{\text{Th}}}{Z_{\text{Th}} + Z} = \frac{8.6\angle 3.94°}{6.86} = 1.25\angle 3.94°$$

Maximum power delivered to the load is $(1.25)^2 \times 3.43 = 5.36$ W.

Practice Problems

7.1 For the circuit shown in Fig. 7.61, determine the value of current I_x in the impedance $Z = 4 + j5$ between nodes *a* and *b*.

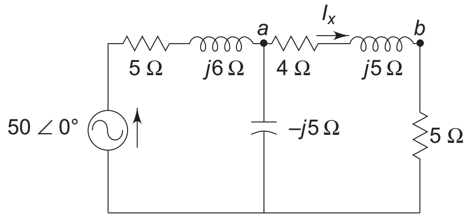


Fig. 7.61

7.2 Determine (i) the equivalent voltage generator and (ii) the equivalent current generator which may be used to represent the given network in Fig. 7.62 at the terminals *AB*.

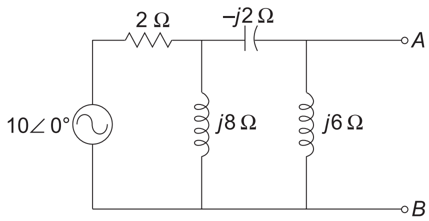


Fig. 7.62

7.3 For the circuit shown in Fig. 7.63, find the value of Z that will receive the maximum power. Also determine this power.

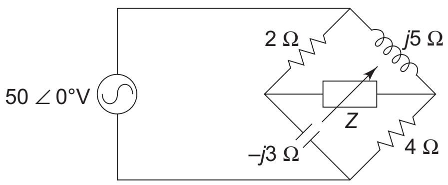


Fig. 7.63

7.4 Determine the voltage V_{ab} and V_{bc} in the network shown in Fig. 7.64 by loop analysis, where source voltage $e(t) = \sqrt{2} \times 100 \cos (314\, t + 45°)$.

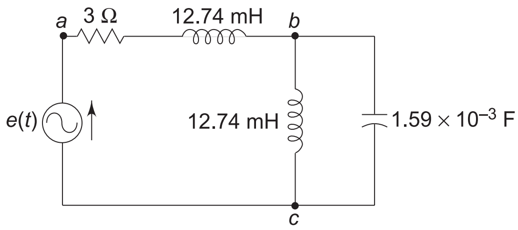


Fig. 7.64

7.5 Find the current in the 15 Ω resistor in the network shown in Fig. 7.65 by Thevenin's theorem.

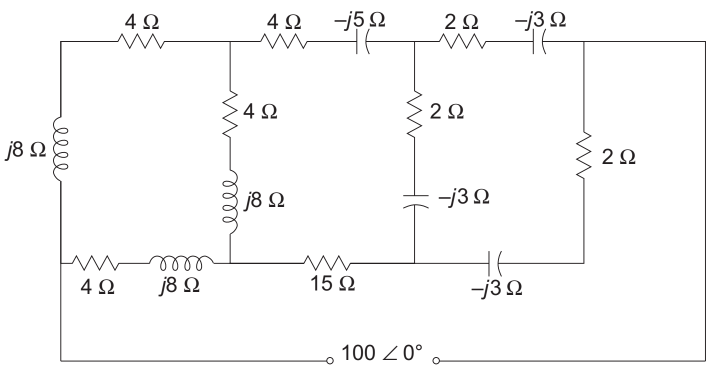


Fig. 7.65

7.6 Determine the power output of the voltage source by loop analysis for the network shown in Fig. 7.66. Also determine the power extended in the resistors.

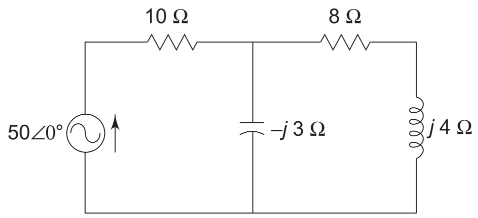


Fig. 7.66

7.7 In the circuit shown in Fig. 7.67, determine the power in the impedance $(2 + j5)\ \Omega$ connected between A and B using Norton's theorem.

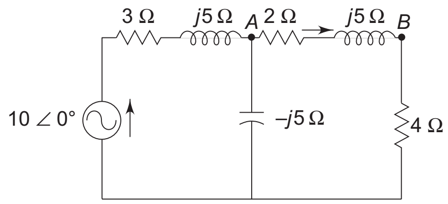


Fig. 7.67

7.8 Determine the value of source currents by loop analysis for the circuit shown in Fig. 7.68 and verify the results by using node analysis.

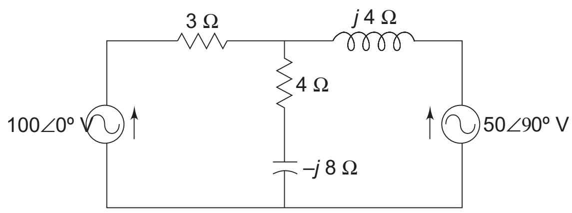


Fig. 7.68

7.9 Convert the active network shown in Fig. 7.69 by a single voltage source in series with an impedance, and also by a single current source in parallel with the impedance.

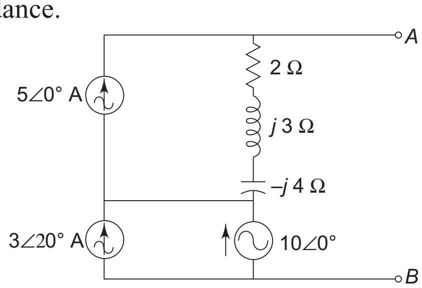


Fig. 7.69

7.10 Determine the power out of the source in the circuit shown in Fig. 7.70 by nodal analysis and verify the results by using loop analysis.

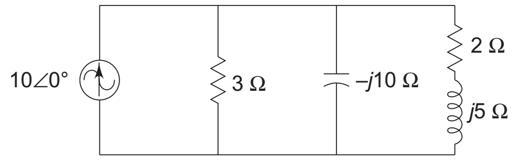


Fig. 7.70

7.11 For the circuit shown in Fig. 7.71, find the current in each resistor using the superposition theorem.

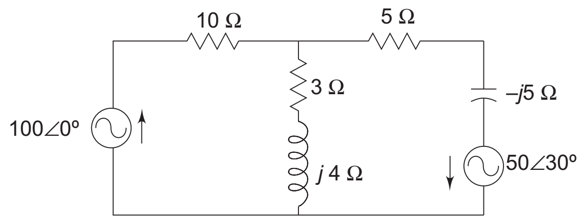


Fig. 7.71

7.12 Use Thevenin's theorem to find the current through the $(5 + j4)\ \Omega$ impedance in Fig. 7.72. Verify the results using Norton's theorem.

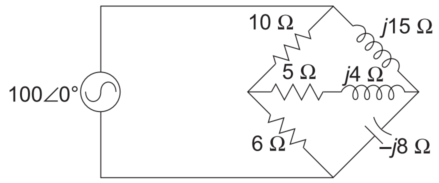


Fig. 7.72

7.13 Determine Thevenin's and Norton's equivalent circuits across terminals *AB*, in Fig. 7.73.

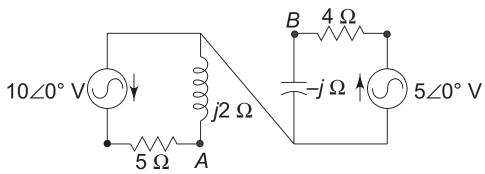


Fig. 7.73

7.14 Determine Norton's and Thevenin's equivalent circuits for the circuit shown in Fig. 7.74.

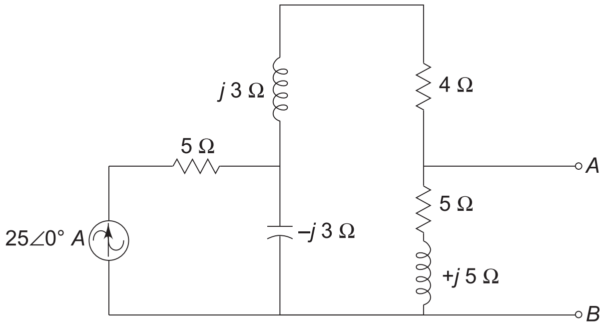


Fig. 7.74

7.15 Determine the maximum power delivered to the load in the circuit shown in Fig. 7.74.

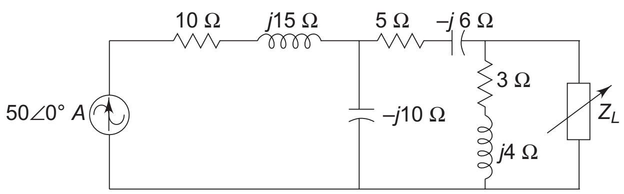


Fig. 7.75

7.16 For the circuit shown in Fig. 7.76, find the voltage across the dependent source branch by using mesh analysis.

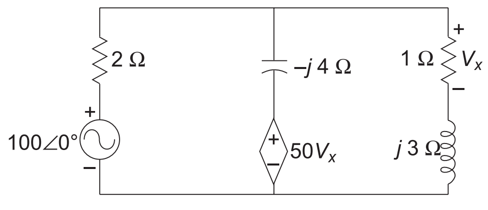


Fig. 7.76

7.17 Find Thevenin's equivalent for the network shown in Fig. 7.77.

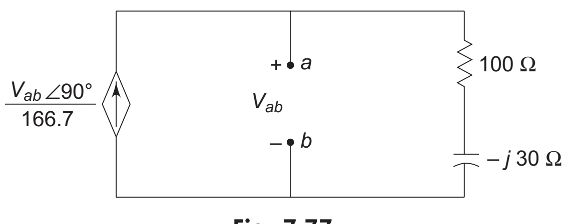


Fig. 7.77

7.18 For the circuit shown in Fig. 7.78, obtain the voltage across 500 Ω resistor.

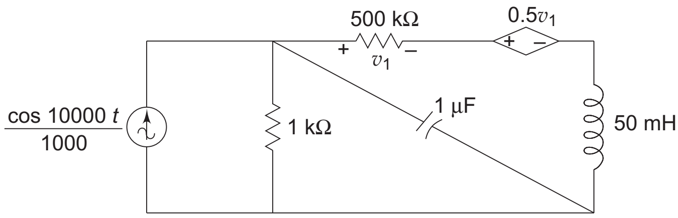


Fig. 7.78

7.19 For the circuit shown in Fig. 7.79, obtain the Thevenin's equivalent circuit at terminals *ab*.

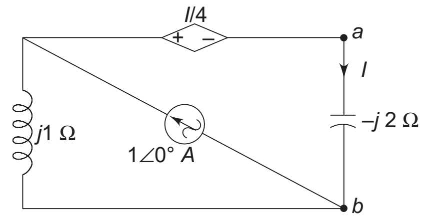


Fig. 7.79

Objective-type Questions

1. The superposition theorem is valid
   (a) only for ac circuits
   (b) only for dc circuits
   (c) For both, ac and dc circuits
   (d) neither of the two
2. When applying the superposition theorem to any circuit
   (a) the voltage source is shorted, the current source is opened
   (b) the voltage source is opened, the current source is shorted
   (c) both are opened
   (d) both are shorted
3. While applying Thevenin's theorem, the Thevenin's voltage is equal to
   (a) short circuit voltage at the terminals
   (b) open circuit voltage at the terminals
   (c) voltage of the source
   (d) total voltage available in the circuit
4. Thevenin impedance Z_{Th} is found
   (a) by short-circuiting the given two terminals
   (b) between any two open terminals
   (c) by removing voltage sources along with the internal resistances
   (d) between same open terminals as for V_{Th}

5. Thevenin impedance of the circuit at its terminals *A* and *B* in Fig. 7.80 is

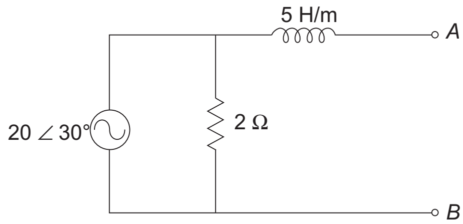


Fig. 7.80

(a) 5 H (b) 2 Ω
(c) 1.4 Ω (d) 7 H

6. Norton's equivalent form in any complex impedance circuit consists of
   (a) an equivalent current source in parallel with an equivalent resistance.
   (b) an equivalent voltage source in series with an equivalent conductance.
   (c) an equivalent current source in parallel with an equivalent impedance.
   (d) None of the above.
7. The maximum power transfer theorem can be applied
   (a) only to dc circuits (b) only to ac circuits
   (c) to both dc and ac circuits (d) neither of the two
8. In a complex impedance circuit, the maximum power transfer occurs when the load impedance is equal to
   (a) complex conjugate of source impedance
   (b) source impedance
   (c) source resistance
   (d) none of the above
9. Maximum power transfer occurs at a
   (a) 100% efficiency (b) 50% efficiency
   (c) 25% efficiency (d) 75% efficiency
10. In the circuit shown in Fig. 7.81, the power supplied by the 10 V source is

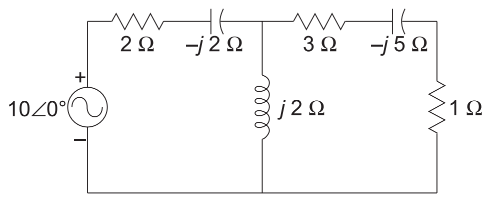


Fig. 7.81

(a) 6.6 W (b) 21.7 W
(c) 30 W (d) 36.7 W

11. The Thevenin equivalent impedance of the circuit in Fig. 7.82 is

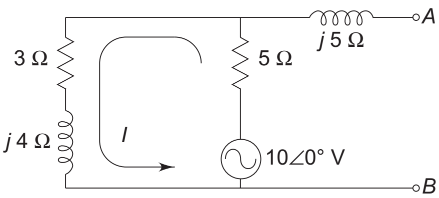


Fig. 7.82

(a) $(1 + j5)\ \Omega$ (b) $(2.5 + j25)\ \Omega$
(c) $(6.25 + j6.25)\ \Omega$ (d) $(2.5 + j6.25)\ \Omega$

12. A source has an emf of 10 V and an impedance of $500 + j100\ \Omega$. The amount of maximum power transferred to the load will be
(a) 0.5 mW (b) 0.05 mW
(c) 0.05 W (d) 0.5 W

13. For the circuit shown in Fig. 7.83, find the voltage across the dependent source.

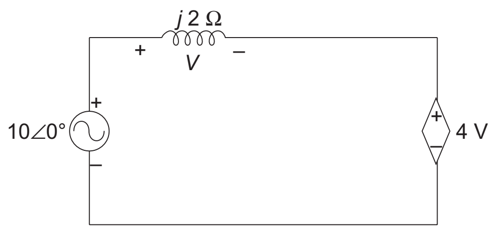


Fig. 7.83

(a) $8\ \angle 0°$ (b) $4\ \angle 0°$
(c) $4\ \angle 90°$ (d) $8\ \angle -90°$

Chapter 8

Frequency Domain Analysis

8.1 IMMITTANCE

Immittance is a general term used to represent both impedance and admittance. The volt-ampere relations of R, L, C elements and their combinations can be related by integro-differential equations, under sinusoidal steady state conditions. The ratio of response phasor to the excitation phasor associated with the sinusoidal voltage and current at a pair of terminals of a network is called immittance.

8.2 COMPLEX IMMITTANCE

Let the voltage applied to a linear lumped circuit be $v(t) = V_m \sin(\omega t + \phi_v)$ (8.1)

And the corresponding current under steady state condition be

$$i(t) = l_m \sin(\omega t + \phi i) \tag{8.2}$$

Phasors may be used to represent the sinusoidal voltages and currents. Immittance can be expressed as a ratio of two phasors which is a complex quantity. Hence, immittance is represented by a complex number. The complex number representing an immittance is invariant with respect to time.

The voltage phasor and current phasors of Eqs. 8.1 and 8.2 are given by

$$V = Ve^{j\phi v} = V\underline{\big|\phi v} = V(\cos \phi v + J \sin \phi v) \tag{8.3}$$

$$I = Ie^{j\phi i} = I\underline{\big|\phi L} = I(\cos \phi i + J \sin \phi i) \tag{8.4}$$

The excitation may be either voltage or current.

Consider the current excitation $i(t) = I \sin(\omega t + \phi)$ through R, L and C and the response phasor of voltage of these elements are shown in Fig. 8.1.

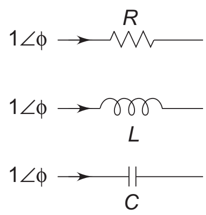


Fig. 8.1

$$V_R = RI\,(\cos\phi + J\sin\phi)$$
$$V_L = J\omega L_I\,(\cos\phi + J\sin\phi)$$
$$V_C = \frac{I}{J\omega c} = (\cos\phi + J\sin\phi)$$

The complex immittance of the lumped elements R, L and C can be defined as the ratio of response to excitation $\frac{V}{I}$.

$$R + J0 \text{ for resistance} \tag{8.5}$$
$$0 + J\omega L \text{ for inductance} \tag{8.6}$$
$$0 - \frac{J}{\omega c} \text{ for capacitance} \tag{8.7}$$

Equations 8.5, 8.6 and 8.7 represents the immittances of single elements, they are either purely real or purely imaginary. In general the immittance of combined elements of R, L, C, is a complex number. Hence, the ratio of voltage phasor to current phasor in a combined R, L, C network is defined as the impedance, Z. From Eq. 8.3 and 8.4

$$Z = \frac{Ve^{J\phi_v}}{Ie^{J\phi_i}} = \frac{V}{I}\, e^{J(\phi_v - \phi_i)}$$
$$= |\,Z\,|e^{J\phi} = Z\angle\phi = R + JX \tag{8.8}$$

where $|Z|$ is the magnitude of the complex impedance $\phi = (\phi_v - \phi_i)$ is the phase angle between voltage phasor and current phasor.

R is the effective resistance of the network and X is the effective reactance of the network. The reciprocal of complex impedance is the complex admittance.

$$Y = \frac{1}{Z} = \frac{Ie^{J\phi_i}}{Ve^{J\phi_v}} = \frac{I}{V}\, e^{J(\phi_i - \phi_v)}$$
$$= |Y|e^{J\theta} = Y\angle\theta = G + JB \tag{8.9}$$

where Y is the magnitude of the complex admittance, G is the effective conductance of the network and B is the effective susceptance of the network.

8.3 LOCI OF RLC NETWORKS

Locus diagrams can be drawn for reactance, susceptance, impedance and admittance when frequency of the sinusoidal excitation is varied. A phasor diagram may be drawn and is expanded to develop a curve known as a locus. Locus diagrams are useful in determining the behaviour or frequency response of an RLC circuit. Loci of these parameters furnish important information for use in circuit analysis. Such plots are particularly useful in the design of electric wave filters. The path traced by the terminus of the current phasor or voltage phasor when the frequency of the exciting source is changed is called the locus diagram. The term circle diagram identifies locus plots that are either circular or semicircular loci of terminus of a current phasor or voltage phasor. Circle diagrams are often employed as aids in analyzing the operating characteristics of circuits like equivalent circuit of transmission lines and some types of ac machines.

8.4 IMMITTANCE LOCI OF SINGLE ELEMENTS

From Eqs. 8.6 and 8.7, we can write the impedance (immittances) associated with inductance and capacitance respectively are

$$Z_L = 0 + J\omega L \tag{8.10}$$

$$Z_C = 0 + \frac{1}{J\omega c} \tag{8.11}$$

It can be observed that Z_L and Z_C are complex numbers and functions of the angular frequency (ω) of the actuating source.

Table 8.1

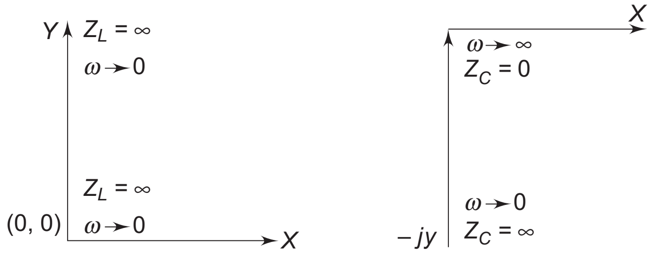
ω	Z_L	Z_C	Y_L	Y_C
0	0	$-\infty$	$-\infty$	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	∞	0	0	∞

The admittances (immittances) associated with the inductance and capacitance can also be represented by complex numbers and are also functions of angular frequency ω, given by

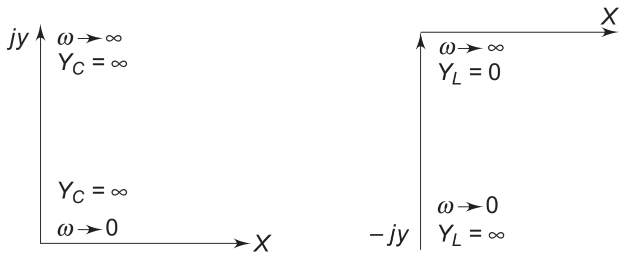
$$Y_L = 0 + \frac{1}{J\omega L} \tag{8.12}$$

$$Y_C = 0 + J\omega C \tag{8.13}$$

The impedances and admittances of single elements are purely imaginary even though they are complex. Any change in angular frequency will move the immittances along the y-axis in the x-y plane. The variations of the immittances, when the frequency is varied from zero to infinity is shown in Table 8.1. The locus plots of the impedances and admittances of single elements are shown in Fig. 8.2(a) through 8.2(d).



(a) Impedance Loci of L (b) Impedance of C



(c) Admittance Loci of C (d) Admittance Loci of L

Fig. 8.2

8.5 IMMITTANCE LOCI OF COMBINED ELEMENTS

8.5.1 Impedance Locus

When a series combination of RL or RC is connected across a source, the complex impedances (immittances) can be represented as follows

$$Z_{RL} = R + J\omega L \tag{8.14}$$

$$Z_{RC} = R + \frac{1}{J\omega c} \tag{8.15}$$

The complex impedances represented by Eq. 8.14 and 8.15 have both real and imaginary parts. The real parts of the equations are independent of the frequency whereas the imaginary part varies with the frequency, similar to the impedances of a single elements. Therefore, the locus Z_{RL} moves parallel to the y axis in the 1st quadrant of x-y plane with a constant real part as ω varies from zero to infinite. Similarly the locus of Z_{RC} moves in the fourth quadrant parallel to the $-y$ axis with a constant real part as ω varies from zero to infinite. These loci are shown in Figs 8.3 (a) and (b).

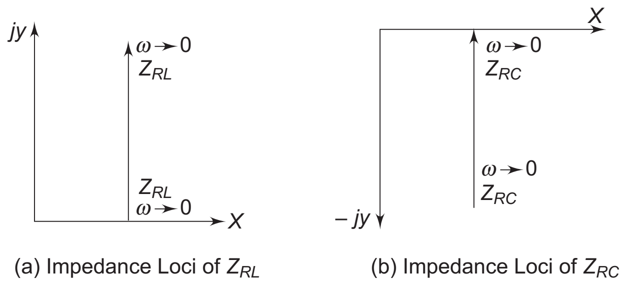


Fig. 8.3

8.5.2 Admittance Locus

The admittances of an RL and RC combination are given by

$$Y_{RL} = \frac{1}{Z_{RL}} = \frac{1}{R + J\omega L} \tag{8.16}$$

$$Y_{RC} = \frac{1}{Z_{RC}} = \frac{1}{R + \dfrac{1}{J\omega c}} \tag{8.17}$$

It can be shown that the admittance loci of series RL and RC circuits are circles. Rationalizing the Eq. 8.16,

$$Y_{RL} = \frac{R - J\omega L}{R^2 + (\omega L)^2} = \frac{R}{R^2 + (\omega L)^2} - J\frac{\omega L}{R^2 + (\omega L)^2} \tag{8.18}$$

Let the real component of Eq. 8.18 be x and imaginary component by y

$$\therefore \quad x = \frac{R}{R^2 + (\omega L)^2} \tag{8.19}$$

$$y = \frac{-\omega L}{R^2 + (\omega L)^2} \tag{8.20}$$

$$\frac{x}{y} = \frac{-R}{\omega L} \tag{8.21}$$

$$\omega = \frac{-R}{L} \times \frac{y}{x} \tag{8.22}$$

substituting the value of ω in x.

$$x = \frac{R}{R^2 + \dfrac{R^2}{L^2} \times \dfrac{y^2}{x^2} \times L^2} \tag{8.23}$$

$$xR^2 + \frac{y^2}{x} R^2 = R$$

$$R^2(x^2 + y^2) = xR$$

$$x^2 + y^2 - \frac{x}{R} = 0 \tag{8.24}$$

Add $\frac{1}{4R^2}$ on both sides of the above equation

$$x^2 + y^2 - \frac{x}{R} + \frac{1}{4R^2} = \frac{1}{4R^2} \tag{8.25}$$

Above equation can be written as

$$\left(x - \frac{1}{2R}\right)^2 + y^2 = \left(\frac{1}{2R}\right)^2 \tag{8.26}$$

Equation 8.26 describes a circle equation as $(x - a)^2 + (y - b)^2 = r^2$ where (a, b) represents centre point of circle and r is the radius.

Equation 8.26 can be used to draw the admittance loci by fixing centre at $\left(\frac{1}{2R}, 0\right)$ and radius $\left(\frac{1}{2R}\right)$ in the xy plane. When $\omega = 0$ the magnitude of the admittance assumes the maximum value of $\frac{1}{R}$ with a phase angle of zero with positive x-axis. As ω is increased from zero, the magnitude traces the path of a semi-circle in the fourth quadrant. When ω tends to infinity the magnitude tends to zero with a phase angle of – 90°, the locus diagram is shown in Fig. 8.4.

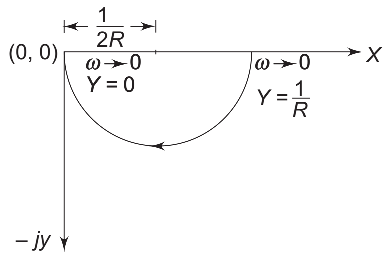


Fig. 8.4

Rationalizing the Eq. 8.17, we can write the admittance of a series RC circuit as

$$Y_{RC} = \frac{R}{R^2 + \left(\frac{1}{\omega c}\right)^2} + \frac{J}{\omega c\left[\left(R^2 + \left(\frac{1}{\omega c}\right)^2\right)\right]} \tag{8.27}$$

The above equation can be also represented as a circle equation by letting

$$x = \frac{R}{R^2 + \left(\frac{1}{\omega c}\right)^2} \tag{8.28}$$

and $$y = \frac{1/\omega c}{R^2 + \left(\frac{1}{\omega c}\right)^2} \tag{8.29}$$

$$\frac{x}{y} = \omega RC \quad \Rightarrow \quad \omega = \frac{x}{y}\frac{1}{RC} \tag{8.30}$$

substituting the value of win Eq. 8.28,

$$x^2 + y^2 - \frac{x}{R} = 0$$

Adding $\frac{1}{4R^2}$ on both sides, the above equation can be written as

$$\left(x - \frac{1}{2R}\right)^2 + y^2 = \left(\frac{1}{2R}\right)^2 \tag{8.31}$$

Above equation is identical to Eq. 8.26 which represents the equation of a circle. The admittance locus diagram of a series *RC* circuit is shown in Fig. 8.5. It is similar to the admittance locus of series *RL* except that it lies in the 1st quadrant of *x*-*y* plane.

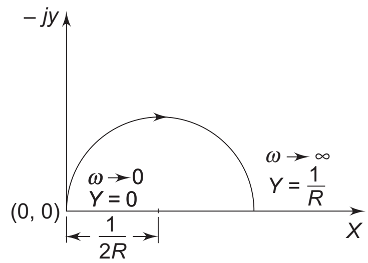


Fig. 8.5

8.6 LOCUS DIAGRAMS OF SERIES, PARALLEL CIRCUITS

Locus diagrams can also be drawn for series and parallel RLC circuits when one of its parameters is varied while the frequency and voltage kept constant.

8.6.1 Series Circuits

To discuss the basis of representing a series circuit by means of a circle diagram consider the circuit shown in Fig. 8.6(a). The analytical procedure is greatly simplified by assuming that inductance elements have no resistance and that capacitors have no leakage current.

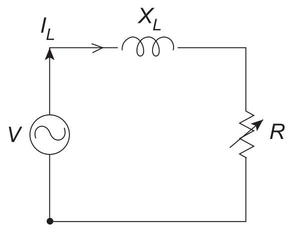


Fig. 8.6 (a)

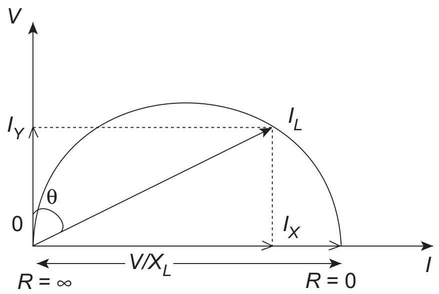


Fig. 8.6 (b)

The circuit under consideration has constant reactance but variable resistance. The applied voltage will be assumed with constant rms voltage V. The power factor angle is designated by θ. If $R = 0$, I_L is obviously equal to $\dfrac{V}{X_L}$ and has maximum value. Also I lags V by 90°. This is shown in Fig. 8.6(b). If R is increased from zero value, the magnitude of I becomes less than $\dfrac{V}{X_L}$ and θ becomes less than 90° and finally when the limit is reached, i.e. when R equals to infinity, I equals to zero and θ equals to zero. It is observed that the tip of the current vector represents a semicircle as indicated in Fig. 8.6(b).

In general

$$I_L = \frac{V}{Z}$$

$$\sin\theta = \frac{X}{Z}$$

or

$$Z = \frac{X_L}{\sin\theta}$$

$$I = \frac{V}{X_L}\sin\theta \tag{8.32}$$

For constant V and X, Eq. 8.32 is the polar equation of a circle with diameter $\dfrac{V}{X_L}$. Figure 8.6(b) shows the plot of Eq. 8.32 with respect to V as reference.

The active component of the current I_L in Fig. 8.6(b) is $OI_L \cos\theta$ which is proportional to the power consumed in the RL circuit. In a similar way we can draw the loci of current if the inductive reactance is replaced by a capacitive reactance as shown in Fig. 8.6(c). The current semicircle for the RC circuit with variable R will be on the left-hand side of the voltage vector OV with diameter

$\dfrac{V}{X_C}$ as shown in Fig. 8.6(d). The current vector OI_C leads V by $\theta°$. The active component of the current $I_C X$ in Fig. 8.6(d) is $OI_C \cos\theta$ which is proportional to the power consumed in the RC circuit.

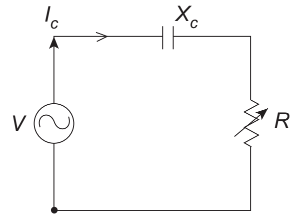


Fig. 8.6(c)

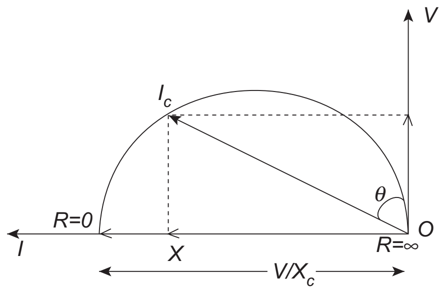


Fig. 8.6(d)

Circle Equations for an RL Circuit

(a) Fixed reactance and variable resistance The X-coordinate and Y-coordinate of I_L in Fig. 8.6(b) respectively are $I_X = I_L \sin\theta$; $I_y = I_L \cos\theta$.

Where $$I_L = \frac{V}{Z};\ \sin\theta = \frac{X_L}{Z};\ \cos\theta = \frac{R}{Z};\ Z = \sqrt{R^2 + X_L^2}$$

$\therefore$ $$I_X = \frac{V}{Z}\cdot\frac{X_L}{Z} = V\cdot\frac{X_L}{Z^2} \tag{8.33}$$

$$I_Y = \frac{V}{Z}\cdot\frac{R}{Z} = V\cdot\frac{R}{Z^2} \tag{8.34}$$

Squaring and adding Eqs. 8.33 and 8.34, we obtain

$$I_X^2 + I_Y^2 = \frac{V^2}{R^2 + X_L^2} \tag{8.35}$$

From Eq. 8.33

$$Z^2 = R^2 + X_L^2 = V\cdot\frac{X_L}{I_X}$$

$\therefore$ Equation 8.35 can be written as $I_X^2 + I_Y^2 = \dfrac{V}{X_L}\cdot I_X$

or $$I_X^2 + I_Y^2 - \frac{V}{X_L}\cdot I_X = 0$$

Adding $\left(\dfrac{V}{2X_L}\right)^2$ to both sides the above equation can be written as

$$\left(I_X - \frac{V}{2X_L}\right)^2 + I_Y^2 = \left(\frac{V}{2X_L}\right)^2 \tag{8.36}$$

Equation 8.36 represents a circle whose radius is $\dfrac{V}{2X_L}$ and the co-ordinates of the centre are $\dfrac{V}{2X_L}$, 0.

In a similar way we can prove that for a series RC circuit as in Fig. 8.6(c), with variable R, the locus of the tip of the current vector is a semi-circle and is given by

$$\left(I_X + \frac{V}{2X_C}\right)^2 + I_Y^2 = \frac{V^2}{4X_C^2} \tag{8.37}$$

The centre has co-ordinates of $-\dfrac{V}{2X_C}$, 0 and radius as $\dfrac{V}{2X_C}$.

(b) Fixed resistance, variable reactance Consider the series RL circuit with constant resistance R but variable reactance X_L as shown in Fig. 8.7(a).

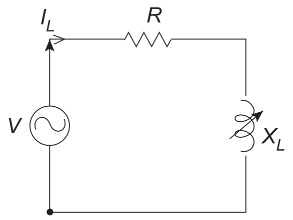


Fig. 8.7(a)

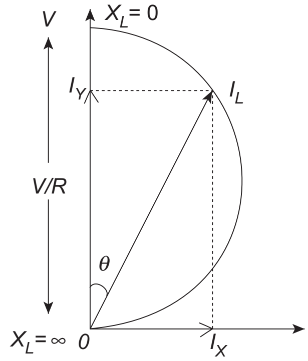


Fig. 8.7(b)

When $X_L = 0$; I_L assumes maximum value of $\dfrac{V}{R}$ and $\theta = 0$, the power factor of the circuit becomes unity; as the value X_L is increased from zero, I_L is reduced and finally when X_L is α, current becomes zero and θ will be lagging behind the voltage by 90° as shown in Fig. 8.7(b). The current vector describes a semicircle with diameter $\dfrac{V}{R}$ and lies in the right-hand side of voltage vector OV. The active component of the current $OI_L \cos\theta$ is proportional to the power consumed in the RL circuit.

Equation of Circle

Consider Eq. 8.35 $\qquad I_X^2 + I_Y^2 = \dfrac{V^2}{R^2 + X_L^2}$

From Eq. 8.34 $\qquad Z^2 = R^2 + X_L^2 = \dfrac{VR}{I_Y} \qquad (8.38)$

Substituting Eq. 8.38 in Eq. 8.35

$$I_X^2 + I_Y^2 = \frac{V}{R}\, I_Y \tag{8.39}$$

$$I_X^2 + I_Y^2 - \frac{V}{R}\, I_Y = 0$$

Adding $\left(\frac{V}{2R}\right)^2$ to both sides the above equation can be written as

$$I_X^2 + \left(I_Y - \frac{V}{2R}\right)^2 = \left(\frac{V}{2R}\right)^2 \tag{8.40}$$

Equation 8.39 represents a circle whose radius is $\frac{V}{2R}$ and the co-ordinates of the centre are 0; $\frac{V}{2R}$.

Let the inductive reactance in Fig. 8.7(a) be replaced by a capacitive reactance as shown in Fig. 8.8(a).

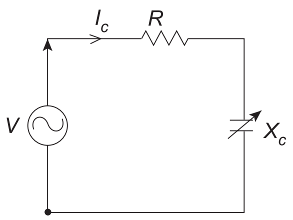


Fig. 2.8(a)

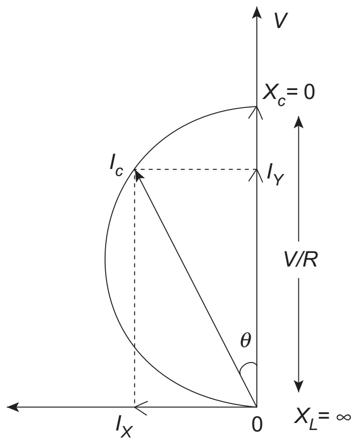


Fig. 2.8(b)

The current semicircle of a RC circuit with variable X_C will be on the left-hand side of the voltage vector OV with diameter $\frac{V}{R}$. The current vector OI_C leads V by $\theta°$. As before, it may be proved that the equation of the circle shown in Fig. 8.8(b) is

$$I_X^2 + \left(I_Y - \frac{V}{2R}\right)^2 = \left(\frac{V}{2R}\right)^2$$

Example 8.1 For the circuit shown in Fig. 8.9(a) plot the locus of the current, mark the range of I for maximum and minimum values of R, and the maximum power consumed in the circuit. Assume $X_L = 25\ \Omega$ and $R = 50\ \Omega$. The voltage is 200 V; 50 Hz.

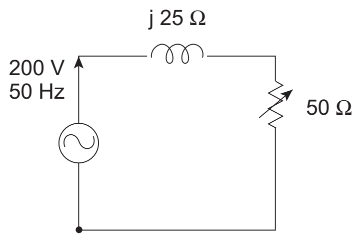


Fig. 8.9(a)

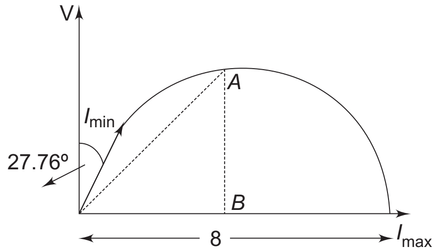


Fig. 8.9(b)

Solution

$$\text{Maximum value of current } I_{max} = \frac{200}{25} = 8 \text{ A}; \ \theta = 90°$$

$$\text{Minimum value of current } I_{min} = \frac{200}{\sqrt{(50)^2 + (25)^2}} = 3.777 \text{ A}; \ \theta = 27.76°$$

The locus of the current is shown in Fig. 8.9(b).

Power consumed in the circuit is proportional to $I \cos \theta$ for constant V. The maximum ordinate possible in the semicircle (AB in Fig. 8.9(b)) represents the maximum power consumed in the circuit. This is possible when $\theta = 45°$, under the condition power factor $\cos \theta = \cos 45° = \frac{1}{\sqrt{2}}$.

Hence, the maximum power consumed in the circuit = $V \times AB = V \times \frac{I_{max}}{L}$

$$I_{max} = \frac{V}{X_L} = 84 \text{ A}$$

$$P_{max} = \frac{V^2}{2X_L} = \frac{(200)^2}{2 \times 25} = 800 \text{ W}$$

Example 8.2 For the circuit shown in Fig. 8.9(a), if the reactance is variable, plot the range of I for maximum and minimum values of X_L and maximum power consumed in the circuit.

Solution

$$\text{Maximum value of current } I_{max} = \frac{200}{50} = 4 \text{ A}; \ q = 0$$

$$\text{Minimum value of current } I_{min} = \frac{200}{\sqrt{(50)^2 + (25)^2}}$$

$$= 3.777 \text{ A}; \ \theta = 27.76°$$

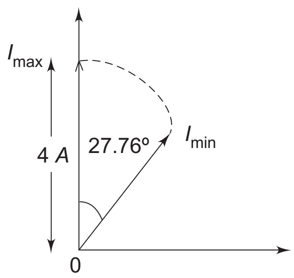


Fig. 8.10

The locus of current is shown in Fig. 8.23.

Maximum power will be when $I = 4$ A

Hence $P_{max} = 4 \times 200 = 800$ W

Example 8.3 For the circuit shown in Fig. 8.11(a) draw the locus of the current. Mark the range of I for maximum and minimum values. Assume $X_C = 50\ \Omega$; $R = 10\ \Omega$; $V = 400$ V.

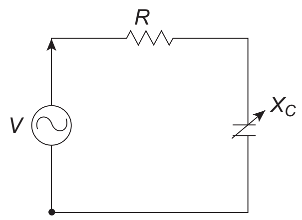


Fig. 8.11(a)

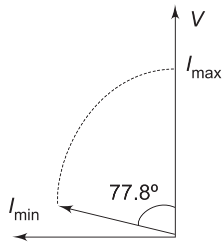


Fig. 8.11(b)

Solution

$$I_{max} = \frac{400}{10} = 40\text{ A; } \theta = 0°$$

$$I_{min} = \frac{400}{\sqrt{(50)^2 + (10)^2}} = 7.716\text{ A. } \theta = \tan^{-1} 5 = 77.8°$$

The locus of the current is shown in Fig. 8.11(b).

8.6.2 Parallel Circuits

(a) Variable XL Locus plots are drawn for parallel branches containing RLC elements in a similar way as for series circuits. Here we have more than one current locus. Consider the parallel circuit shown in Fig. 8.12(a). The quantities that may be varied are X_L, X_C, R_L and R_C for a given voltage and frequency.

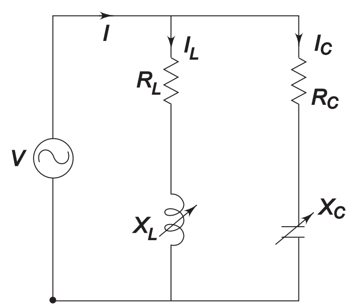


Fig. 8.12(a)

Let us consider the variation of X_L from zero to ∞. Let OV shown in Fig. 8.12(b), be the voltage vector, taken as reference. A current, I_C, will flow in the condenser branch whose parameters are held constant and leads V by an angle $\theta_C = \tan^{-1}\left(\frac{X_C}{R_C}\right)$, when $X_L = 0$, the current I_L, through the inductive branch is maximum and is given by $\frac{V}{R_L}$ and it is in phase with the applied voltage. When X_L is increased from zero, the current through the inductive branch I_L decreases and lags V by $\theta_L = \tan^{-1} \frac{X_L}{R_L}$ as shown in Fig. 8.12(b). For any value of I_L, the $I_L R_L$ drop and $I_L X_L$ drop must add at right angles to give the applied voltage. These drops are shown as OA and AV respectively. The locus of I_L is a semicircle, and the locus of I_LR_L drop is also a semicircle. When $X_L = 0$, i.e. I_L is maximum, I_L coincides with the diameter of its semicircle and I_LR_L drop also coincides with the diameter of its semi-circle as shown in the figure; both these semicircles are shown with dotted circles as OI_LB and OAV respectively.

Since the total current is $I_C + I_L$. For example, a particular value of I_C and I_L the total current is represented by OC on the total current semicircle. As X_L is varied, the locus of the resultant current is therefore, the circle $I_C CB$ as shown with thick line in the Fig. 8.12(b).

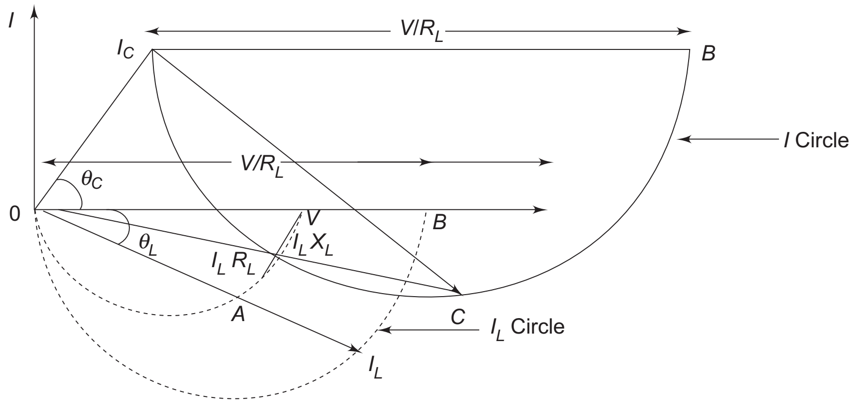


Fig. 8.12(b)

(b) Variable XC A similar procedure can be adopted as outlined above to draw the locus plots of I_1 and I when X_C is varying while R_L, R_C, X_L, V and f are held constant. The curves are shown in Fig. 8.12(c).

OV presents the voltage vector, OB is the maximum current through RC branch when $X_L = 0$; OI_L is the current through the R_L branch lagging OV by an angle $\theta_L = \tan^{-1} \dfrac{C_L}{R_L}$. As X_C is increased from zero, the current through the capacitive branch I_C decreases and leads V by $\theta_C = \tan^{-1} \dfrac{X_C}{R_C}$. For a particular I_C, the resultant current $I = I_L + I_C$ and is given by OC. The dotted semicircle OI_CB is the locus of the I_C, thick circle I_LCB is the locus of the resultant current.

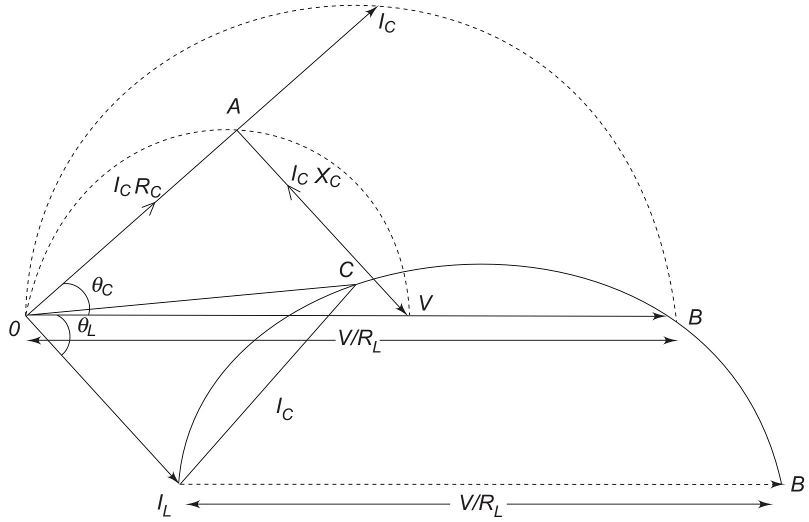


Fig. 8.12(c)

(c) Variable RL The locus of current for the variation of R_L in Fig. 8.13(a) is shown in Fig. 8.13(b). OV represents the reference voltage, $OI_L B$ represents the locus of I_L and $I_C CB$ represents the resultant current locus. Maximum $I_L = \dfrac{V}{X_L}$ is represented by OB.

(d) Variable RC The locus of currents for the variation of R_C in Fig. 8.14(a) is plotted in Fig. 8.14(b) where OV is the source voltage and semicircle OAB represents the locus of I_C. The resultant current locus is given by BCI_L.

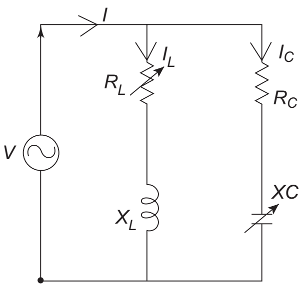


Fig. 8.13(a)

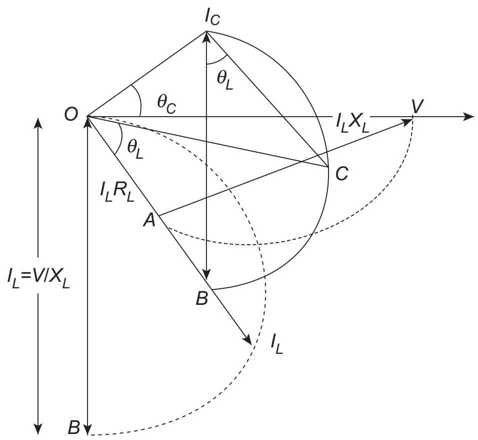


Fig. 8.13(b)

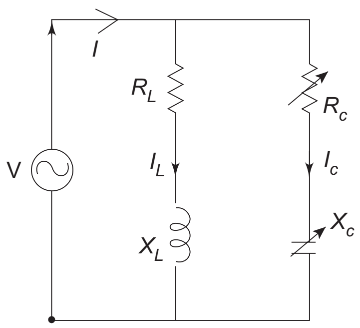


Fig. 8.14(a)

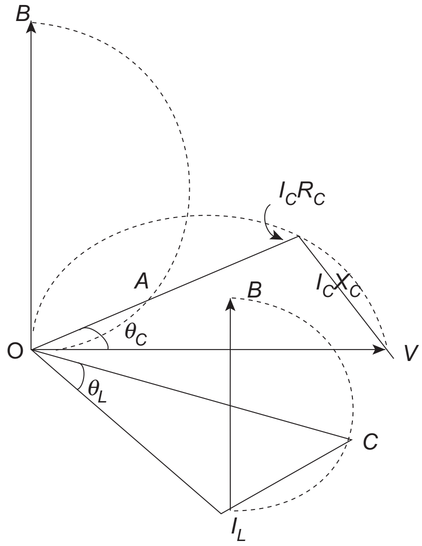


Fig. 8.14(b)

Example 8.4 For the parallel circuit shown in Fig. 8.15(a), draw the locus of I_1 and I. Mark the range of values for R_1 between 10 Ω and 100 Ω. Assume X_L = 25 Ω and R_2 = 25 Ω. The supply voltage is 200 V and frequency is 50 Hz, both held constant.

Solution

Let us take voltage as reference; on the positive X-axis. I_2 is given by $I_2 = \dfrac{200}{25}$ = 8A and is in phase with V.

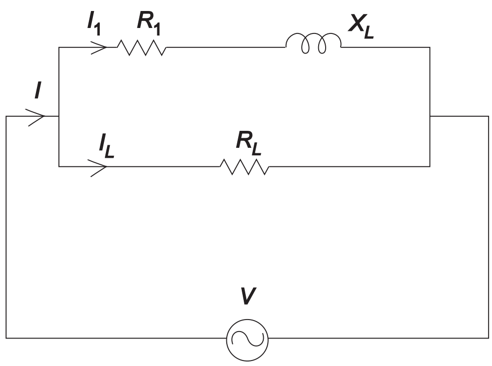


Fig. 8.15(a)

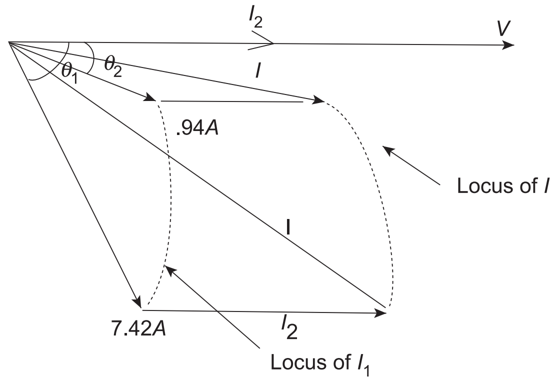


Fig. 8.15(b)

When $R_1 = 10\ \Omega$ $\quad I_1 = \dfrac{200}{\sqrt{(100+625)}} = 7.42$ A; $\theta_1 = \tan^{-1}\dfrac{25}{10} = 68.19°$

when $R_2 = 100\ \Omega$ $\quad I_1 = \dfrac{200}{\sqrt{(10000+625)}} = 1.94$ A; $\theta_2 = \tan^{-1}\dfrac{25}{100} = 14.0°$

The variation of I_1 and I are shown in Fig. 8.15(b).

Example 8.5 Draw the locus of I_2 and I for the parallel circuit shown in Fig. 8.16(a).

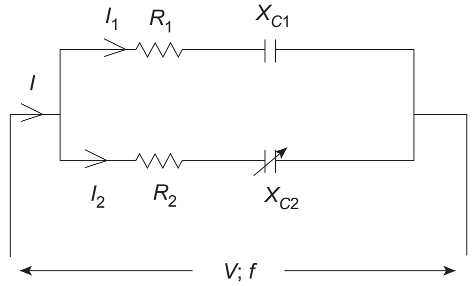


Fig. 8.16(a)

Solution

I_1 leads the voltage by a fixed angle θ_1 given by $\tan^{-1}\dfrac{X_C}{R_1}$

I_2 varies according to the value of X_{C_2}

I_2 is maximum when $X_{C_2} = 0$ and is in phase with V

I_2 is zero when $X_{C_2} = \infty$ as shown in Fig. 8.16(b).

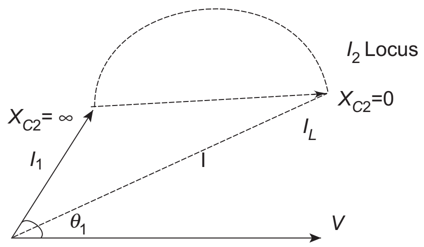


Fig. 8.16(b)

Example 8.6 For a parallel circuit shown in Fig. 8.17(a) plot the locus of currents.

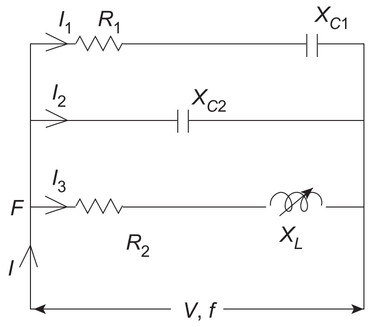


Fig. 8.17(a)

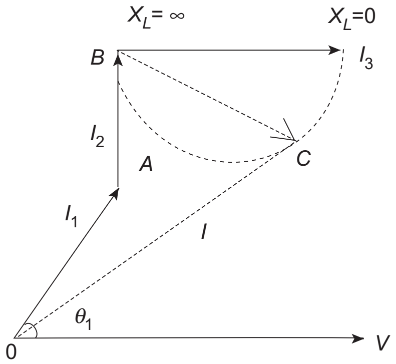


Fig. 8.17(b)

Current I_1 leads the voltage by a fixed angle θ_1 given by $\tan^{-1}\dfrac{X_C}{R_1}$, current I_2 leads the voltage by 90°. I_3 varies according to the value of X_L, when $X_L = 0$, I_3 is maximum and is given by $\dfrac{V}{R_L}$; is in phase with V; when $X_L = \infty$, I_3 is zero. Both these extremities are shown in Fig. 8.17(b). For a particular value of I_3 the total current I is given by $I_1 + I_2 + I_3 = OA + AB + BC$.

8.7 FREQUENCY RESPONSE OF RLC NETWORKS

The response of a circuit with sinusoidal excitation as a function of angular frequency ω is known as the *frequency response*. Frequency response means the steady state response of a system to sinusoidal input. Sinusoidal analysis provides us with the response of a network as a function of ω.

The frequency response of RLC circuits provides useful information and is of practical importance, as the RLC circuits possess filter properties.

The steady state response of a linear time-invariant RLC network to a sinusoidal input is sinusoidal with the same frequency as that of excitation.

As already mentioned in the preceding chapter, a network function $N(S)$, either driving point or transfer, may be expressed as a ratio of two polynomials.

Network functions describes the response of the circuits in the sinusoidal steady state. By letting $S = J\omega$, the network function can be expressed as a complex function. Its value is specified either in rectangular coordinates as

$$N(J\omega) = R(\omega) + JX(\omega)$$

or in polar form

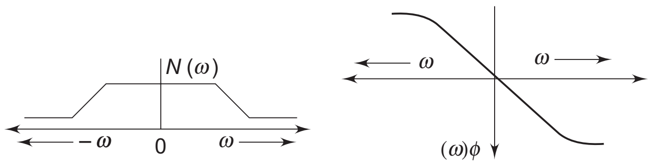
$$N(J\omega) = |N(J\omega)|\, e^{J\phi(\omega)} = N(J\omega)\ \angle\phi(\omega)$$

where $|N(J\omega)| = \sqrt{[R(\omega)]^2 + [x(\omega)]^2}$ is the magnitude response function

and $\phi(\omega) = \tan^{-1} \dfrac{x(\omega)}{R(\omega)}$ is the phase response function.

8.8 FREQUENCY RESPONSE PLOTS

Generally the frequency response is plotted as two curves (i) the magnitude plot (ii) phase angle plot. Both the curves are obtained by varying the angular frequency ω, from zero to infinity. A typical frequency response plots of a low-pass filter is shown in Fig. 8.18. The variation of ω from $-\infty$ to $+\infty$ is considered in this case.



(a) Amplitude response (b) Phase response

Fig. 8.18

Example 8.7 Draw the magnitude and phase plots of the voltage transfer function for the network shown in Fig. 8.19.

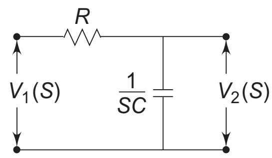


Fig. 8.19

Solution The voltage transfer function $G(S) = \dfrac{V_2(S)}{V_1(S)}$

$$\frac{V_2(S)}{V_1(S)} = \frac{1}{1 + RCS}$$

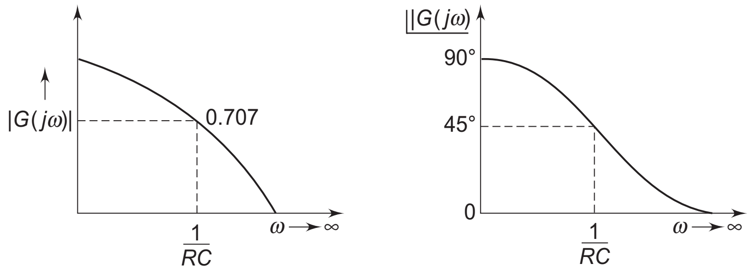
For sinusoidal steady state, put $S = J\omega$

$$G(J\omega) = \frac{1}{1 + J\omega RC}$$

$$= |G(J\omega)| \angle G\ (J\omega)$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}} \angle -\tan^{-1}(\omega RC)$$

The variation of magnitude and phase angle for varying ω from zero to $\propto$ is shown in Fig. 8.20. For small ω, the magnitude approaches unity while the phase becomes nearly zero. For large value of ω, the magnitude becomes 0, while the phase is – 90°. When $\omega = \frac{1}{RC}$, the magnitude is 0.707 and the phase is – 45°. This point is the half power point of the amplitude response.



(a) Magnitude plot (b) Phase plot

Fig. 8.20

Example 8.8 For the network shown in Fig. 8.21 draw the frequency response of $\frac{V_2}{V_1}$.

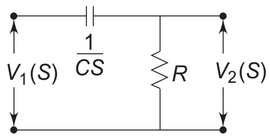


Fig. 8.21

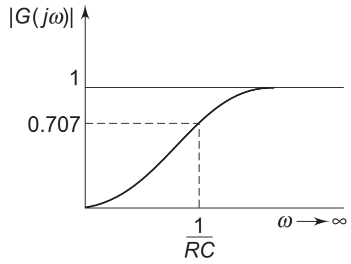
Solution $G(J\omega) = \frac{V_2\,(J\omega)}{V_1\,(J\omega)} = \frac{J\omega}{J\omega + \frac{1}{RC}}$

when $\omega = 0$; $|G(J\omega)| = 0$, $\angle G(J\omega) = 90°$

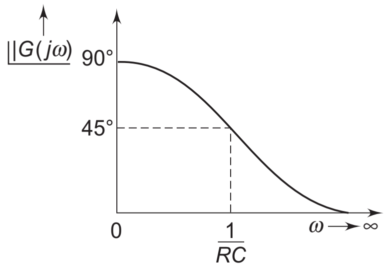
when $\omega = \frac{1}{RC}$; $|G(J\omega)| = 0.707$; $\angle G(J\omega) = 45°$

when $\omega \to \propto$; $|G(J\omega)| = 1$, $\angle G(J\omega) = 0°$

The plots are shown in Fig. 8.22.



(b) Magnitude plot



(b) Phase plot

Fig. 8.22

8.9 RESONANCE PHENOMENA

The most interesting part of the frequency response of RLC circuits is the resonance. If a sinusoidal voltage is applied to a circuit consisting of resistive and reactive elements under special circumstances, the impedance offered by the network is purely resistive, this phenomenon is called *resonance*. At resonance, the circuit voltage V and current I are in phase. The frequency at which resonance takes place is called the resonance frequency. The resonance phenomenon is due to the presence of energy storing elements L and C. The immittance loci are subject of interest under the resonance condition. The resonance may be classified into two groups.

(1) Series resonance (2) Parallel resonance

8.9.1 Series Resonance

In many electrical circuits, resonance is a very important phenomenon. The study of resonance is very useful, particularly in the area of communications. For example, the ability of a radio receiver to select a certain frequency, transmitted by a station and to eliminate frequencies from other stations is based on the principle of resonance. In a series RLC circuit, the current lags behind, or leads the applied voltage depending upon the values of X_L and X_C. X_L causes the total current to lag behind the applied voltage, while X_C causes the total current to lead the applied voltage. When $X_L > X_C$, the circuit is predominantly inductive, and when $X_C > X_L$, the circuit is predominantly capacitive. However, if one of the parameters of the series RLC circuit is varied in such a way that the current in the circuit is in phase with the applied voltage, then the circuit is said to be in resonance.

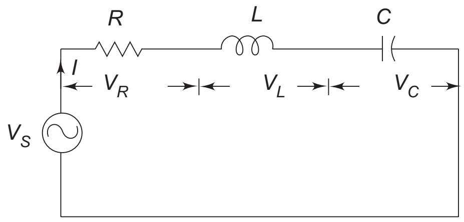


Fig. 8.23

Consider the series RLC circuit shown in Fig. 8.23.

The total impedance for the series RLC circuit is

$$Z = R + j(X_L - X_C) = R + j\left(\omega L - \frac{1}{\omega C}\right) \quad (8.41)$$

It is clear from the circuit that the current $I = V_S/Z$

The circuit is said to be in resonance if the current is in phase with the applied voltage. In a series RLC circuit, series resonance occurs when $X_L = X_C$. The frequency at which the resonance occurs is called the *resonant frequency*.

Since $X_L = X_C$, the impedance in a series RLC circuit is purely resistive. At the resonant frequency, f_r, the voltages across capacitance and inductance are equal in magnitude. Since they are 180° out of phase with each other, they cancel each other and, hence zero voltage appears across the LC combination.

At resonance

$$X_L = X_C \quad \text{i.e.} \quad \omega L = \frac{1}{\omega C}$$

Solving for resonant frequency, we get

$$2\pi f_r L = \frac{1}{2\pi f_r C}$$

$$f_r^2 = \frac{1}{4\pi^2 LC}$$

$$\therefore \quad f_r = \frac{1}{2\pi\sqrt{LC}} \quad (8.42)$$

In a series RLC circuit, resonance may be produced by varying the frequency, keeping L and C constant; otherwise, resonance may be produced by varying either L or C for a fixed frequency.

Example 8.9 For the circuit shown in Fig. 8.24, determine the value of capacitive reactance and impedance at resonance.

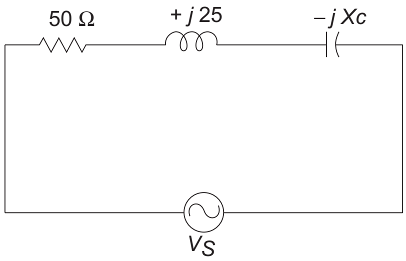


Fig. 8.24

Solution At resonance

$$X_L = X_C$$

Since $X_L = 25\ \Omega$

$$X_C = 25\ \Omega \quad \therefore \frac{1}{\omega C} = 25$$

The value of impedance at resonance is

$$Z = R$$

$$\therefore \quad Z = 50\ \Omega$$

Example 8.10 Determine the resonant frequency for the circuit shown in Fig. 8.25.

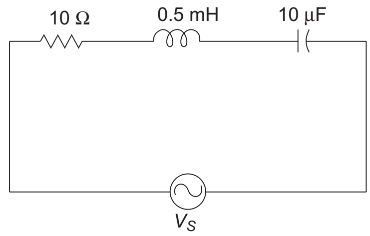


Fig. 8.25

Solution The resonant frequency is

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{10\times10^{-6}\times0.5\times10^{-3}}}$$

$$f_r = 2.25 \text{ kHz}$$

8.9.2 Impedance and Phase Angle of a Series Resonant Circuit

The impedance of a series RLC circuit is

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{8.43}$$

The variation of X_C and X_L with frequency is shown in Fig. 8.26.

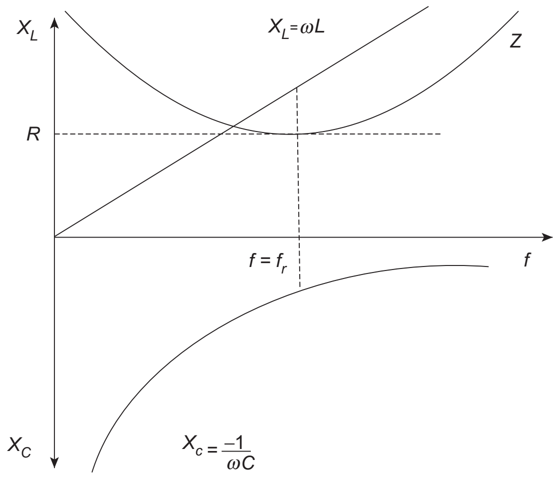


Fig. 8.26

At zero frequency, both X_C and Z are infinitely large, and X_L is zero because at zero frequency the capacitor acts as an open circuit and the inductor acts as a short circuit. As the frequency increases, X_C decreases and X_L increases. Since X_C is larger than X_L, at frequencies below the resonant frequency f_r, Z decreases along with X_C. At resonant frequency f_r, $X_C = X_L$, and $Z = R$. At frequencies above the resonant frequency f_r, X_L is larger than X_C, causing Z to increase. The phase angle as a function of frequency is shown in Fig. 8.27.

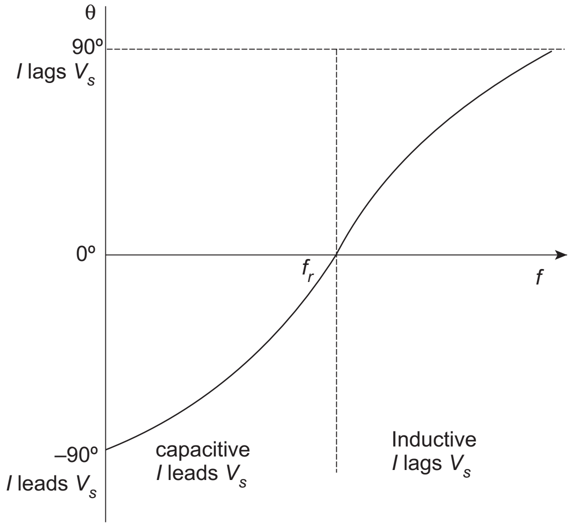


Fig. 8.27

At a frequency below the resonant frequency, current leads the source voltage because the capacitive reactance is greater than the inductive reactance. The phase angle decreases as the frequency approaches the resonant value, and is 0° at resonance. At frequencies above resonance, the current lags behind the source voltage, because the inductive reactance is greater than capacitive reactance. As the frequency goes higher, the phase angle approaches 90°.

Example 8.11 For the circuit shown in Fig. 8.28, determine the impedance at resonant frequency, 10 Hz above resonant frequency, and 10 Hz below resonant frequency.

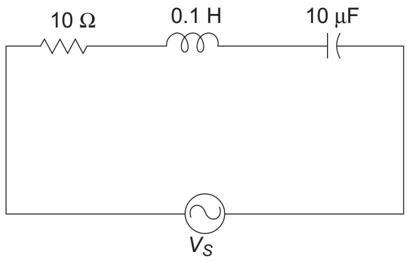


Fig. 8.28

Solution Resonant frequency $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{0.1 \times 10 \times 10^{-6}}} = 159.2 \text{ Hz}$$

At 10 Hz below $f_r = 159.2 - 10 = 149.2$ Hz

At 10 Hz above $f_r = 159.2 + 10 = 169.2$ Hz

Impedance at resonance is equal to R

$\therefore \quad Z = 10\ \Omega$

Capacitive reactance at 149.2 Hz is

$$X_{C_1} = \frac{1}{\omega_1 C} = \frac{1}{2\pi \times 149.2 \times 10^{-6} \times 10}$$

$\therefore \quad X_{C_1} = 106.6\ \Omega$

Capacitive reactance at 169.2 Hz is

$$X_{C_2} = \frac{1}{\omega_2 C} = \frac{1}{2\pi \times 169.2 \times 10 \times 10^{-6}}$$

$\therefore \quad X_{C_2} = 94.06\ \Omega$

Inductive reactance at 149.2 Hz is

$$X_{L_1} = \omega_2 L = 2\pi \times 149.2 \times 0.1 = 93.75\ \Omega$$

Inductive reactance at 169.2 Hz is

$$X_{L_2} = \omega_2 L = 2\pi \times 169.2 \times 0.1 = 106.31\ \Omega$$

Impedance at 149.2 Hz is

$$|Z| = \sqrt{R^2 + (X_{L_1} - X_{C_1})^2}$$

$$= \sqrt{(10)^2 + (93.75 - 106.6)^2} = 16.28\ \Omega$$

Here X_{C_1} is greater than X_{L_1}, so Z is capacitive.

Impedance at 169.2 Hz is

$$|Z| = \sqrt{R^2 + (X_{L_2} - X_{C_2})^2}$$

$$= \sqrt{(10)^2 + (106.31 - 94.06)^2} = 15.81\ \Omega$$

Here X_{L_1} is greater than X_{C_1}, so Z is inductive.

8.9.3 Voltages and Currents in a Series Resonant Circuit

The variation of impedance and current with frequency is shown in Fig. 8.29.

At resonant frequency, the capacitive reactance is equal to inductive reactance, and hence the impedance is minimum. Because of minimum impedance, maximum current flows through the circuit. The current variation with frequency is plotted.

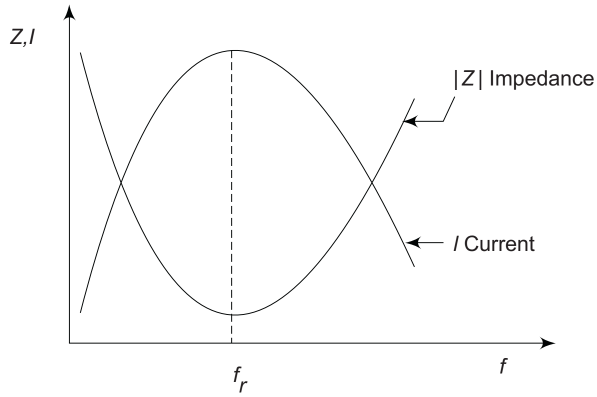


Fig. 8.29

The voltage drop across resistance, inductance and capacitance also varies with frequency. At $f = 0$, the capacitor acts as an open circuit and blocks current. The complete source voltage appears across the capacitor. As the frequency increases, X_C decreases and X_L increases, causing total reactance $X_C - X_L$ to decrease. As a result, the impedance decreases and the current increases. As the current increases, V_R also increases, and both V_C and V_L increase.

When the frequency reaches its resonant value f_r, the impedance is equal to R, and hence, the current reaches its maximum value, and V_R is at its maximum value.

As the frequency is increased above resonance, X_L continues to increase and X_C continues to decrease, causing the total reactance, $X_L - X_C$ to increase. As a result there is an increase in impedance and a decrease in current. As the current decreases, V_R also decreases, and both V_C and V_L decrease. As the frequency becomes very high, the current approaches zero, both V_R and V_C approach zero, and V_L approaches V_s.

The response of different voltages with frequency is shown in Fig. 8.30.

The drop across the resistance reaches its maximum when $f = f_r$. The maximum voltage across the capacitor occurs at $f = f_c$. Similarly, the maximum voltage across the inductor occurs at $f = f_L$.

The voltage drop across the inductor is

$$V_L = IX_L$$

where $$I = \frac{V}{Z}$$

$$\therefore \qquad V_L = \frac{\omega LV}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \tag{8.44}$$

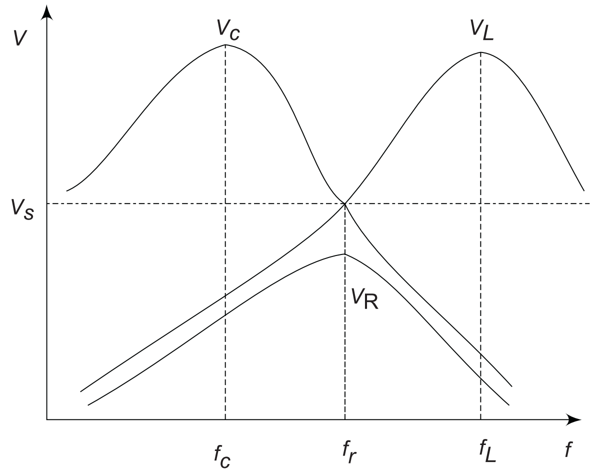


Fig. 8.30

To obtain the condition for maximum voltage across the inductor, we have to take the derivative of the above equation with respect to frequency, and make it equal to zero.

$$\therefore \qquad \frac{dV_L}{d\omega} = 0$$

If we solve for ω, we obtain the value of ω when V_L is maximum.

$$\frac{dV_L}{d\omega} = \frac{d}{d\omega}\left\{\omega LV\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-1/2}\right\}$$

$$\frac{LV\left(R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)^{-1/2} - \frac{\omega LV}{2}\left(R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)\left(2\omega L^2 - \frac{2}{\omega^3 C^2}\right)}{R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}} = 0$$

From this

$$R^2 - \frac{2L}{C} + 2/\omega^2 C^2 = 0$$

$$\therefore \qquad \omega L = \sqrt{\frac{2}{2LC - R^2C^2}} = \frac{1}{\sqrt{LC}}\sqrt{\frac{2}{2 - \frac{R^2C}{L}}}$$

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{1 - \frac{R^2C}{2L}}} \qquad (8.45)$$

Similarly, the voltage across the capacitor is

$$V_C = IX_C = \frac{I}{\omega C}$$

$$\therefore \qquad V_C = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \times \frac{1}{\omega C}$$

To get maximum value $\frac{dV_C}{d\omega} = 0$

If we solve for ω, we obtain the value of ω when V_C is maximum.

$$\frac{dV_C}{d\omega} = \omega C \frac{1}{2}\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-1/2} \left[2\left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 C}\right)\right] + \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, C = 0$$

From this

$$\omega_C^2 = \frac{1}{LC} - \frac{R^2}{2L}$$

$$\omega_C = \sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

$$\therefore \qquad f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L}} \tag{8.46}$$

The maximum voltage across the capacitor occurs below the resonant frequency; and the maximum voltage across the inductor occurs above the resonant frequency.

Example 8.12 A series circuit with $R = 10\ \Omega$, $L = 0.1$ H and $C = 50\ \mu$F has an applied voltage $V = 50\angle 0°$ with a variable frequency. Find the resonant frequency, the value of frequency at which maximum voltage occurs across the inductor and the value of frequency at which maximum voltage occurs across the capacitor.

Solution The frequency at which maximum voltage occurs across the inductor is

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{\left(1 - \frac{R^2 C}{2L}\right)}}$$

$$= \frac{1}{2\pi\sqrt{0.1 \times 50 \times 10^{-6}}}\sqrt{\frac{1}{1 - \left(\frac{(10)^2 \times 50 \times 10^{-6}}{2 \times 0.1}\right)}} = 72.08 \text{ Hz}$$

Similarly, $$f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1 \times 50 \times 10^{-6}} - \frac{(10)^2}{2 \times 0.1}}$$

$$= \frac{1}{2\pi}\sqrt{200000 - 500}$$

$$= 71.08 \text{ Hz}$$

Resonant frequency $$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1 \times 50 \times 10^{-6}}} = 71.18 \text{ Hz}$$

It is clear that the maximum voltage across the capacitor occurs below the resonant frequency and the maximum inductor voltage occurs above the resonant frequency.

8.9.4 Bandwidth of an RLC Circuit

The bandwidth of any system is the range of frequencies for which the current or output voltage is equal to 70.7% of its value at the resonant frequency, and it is denoted by *BW*. Figure 8.31 shows the response of a series RLC circuit.

Here the frequency f_1 is the frequency at which the current is 0.707 times the current at resonant value, and it is called the lower cut-off frequency. The frequency f_2 is the frequency at which the current is 0.707 times the current at resonant value (i.e. maximum value), and is called the *upper cut-off frequency*. The bandwidth, or *BW*, is defind as the frequency difference between f_2 and f_1.

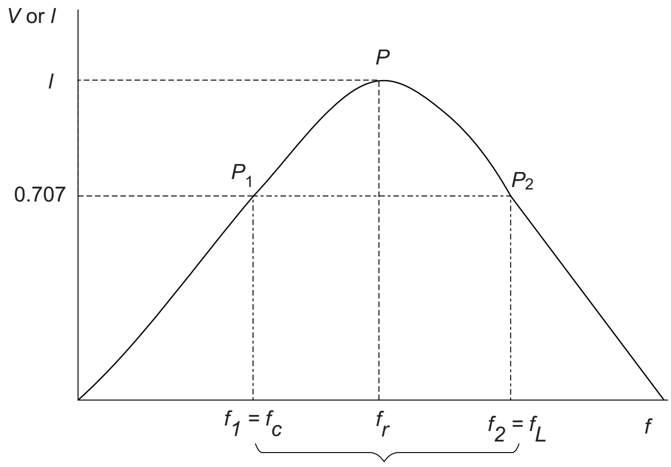


Fig. 8.31

$\therefore$ $$BW = f_2 - f_1$$

The unit of *BW* is hertz (Hz).

If the current at P_1 is $0.707I_{max}$, the impedance of the circuit at this point is $\sqrt{2}\,R$, and hence

$$\frac{1}{\omega_1 C} - \omega_1 L = R \tag{8.47}$$

Similarly,
$$\omega_2 L - \frac{1}{\omega_2 C} = R \tag{8.48}$$

If we equate both the above equations, we get

$$\frac{1}{\omega_1 C} - \omega_1 L = \omega_2 L - \frac{1}{\omega_2 C}$$

$$L(\omega_1 + \omega_2) = \frac{1}{C}\left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}\right)$$

From Eq. 8.3, we get

$$\omega_1 \omega_2 = \frac{1}{LC}$$

we have
$$\omega_r^2 = \frac{1}{LC}$$

$\therefore$
$$\omega_r^2 = \omega_1 \omega_2 \tag{8.49}$$

If we add Eqs 8.47 and 8.48, we get

$$\frac{1}{\omega_1 C} - \omega_1 L + \omega_2 L - \frac{1}{\omega_2 C} = 2R$$

$$(\omega_2 - \omega_1)L + \frac{1}{C}\left(\frac{\omega_2 + \omega_1}{\omega_1 \omega_2}\right) = 2R$$

Since
$$C = \frac{1}{\omega_r^2 L}$$

and
$$\omega_1 \omega_2 = \omega_r^2$$

$$(\omega_2 - \omega_1)L + \frac{\omega_r^2 L(\omega_2 - \omega_1)}{\omega_r^2} = 2R \tag{8.50}$$

From Eq. 8.50, we have

$$\omega_2 - \omega_1 = \frac{R}{L} \tag{8.51}$$

$\therefore$
$$f_2 - f_1 = \frac{R}{2\pi L} \tag{8.52}$$

or
$$BW = \frac{R}{2\pi L} \tag{8.53}$$

From Fig. 8.31, we have

$$f_2 - f_1 = \frac{R}{2\pi L}$$

$\therefore$
$$f_r - f_1 = \frac{R}{4\pi L}$$

$$f_2 - f_r = \frac{R}{4\pi L}$$

The lower frequency limit $$f_1 = f_r - \frac{R}{4\pi L} \quad (8.54)$$

The upper frequency limit $$f_2 = f_r + \frac{R}{4\pi L} \quad (8.55)$$

If we divide the equation on both sides by f_r, we get

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} \quad (8.56)$$

Here an important property of a coil is defined. It is the ratio of the reactance of the coil to its resistance. This ratio is defined as the Q of the coil. Q is known as a figure of merit, it is also called quality factor and is an indication of the quality of a coil.

$$Q = \frac{X_L}{R} = \frac{2\pi f_r L}{R} \quad (8.57)$$

If we substitute Eq. (8.56) in Eq. (8.57), we get

$$\frac{f_2 - f_1}{f_r} = \frac{1}{Q} \quad (8.58)$$

The upper and lower cut-off frequencies are sometimes called the *half-power frequencies*. At these frequencies the power from the source is half of the power delivered at the resonant frequency.

At resonant frequency, the power is

$$P_{\text{max}} = I_{\text{max}}^2 R$$

At frequency f_1, the power is $P_1 = \left(\frac{I_{\text{max}}}{\sqrt{2}}\right)^2 R = \frac{I_{\text{max}}^2 R}{2}$

Similarly, at frequency f_2, the power is

$$P_2 = \left(\frac{I_{\text{max}}}{\sqrt{2}}\right)^2 R$$

$$= \frac{I_{\text{max}}^2 R}{2}$$

The response curve in Fig. 8.31 is also called the *selectivity curve* of the circuit. Selectivity indicates how well a resonant circuit responds to a certain frequency and eliminates all other frequencies. The narrower the bandwidth, the greater the selectivity.

Example 8.13 Determine the quality factor of a coil for the series circuit consisting of $R = 10\ \Omega$, $L = 0.1$ H and $C = 10\ \mu$F.

Solution Quality factor $Q = \frac{f_r}{BW}$

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1 \times 10 \times 10^{-6}}} = 159.2 \text{ Hz}$$

At lower half power frequency, $X_C > X_L$

$$\frac{1}{2\pi f_1 C} - 2pf_1 L = R$$

From which $$f_1 = \frac{-R + \sqrt{R^2 + 4L/C}}{4\pi L}$$

At upper half power frequency $X_L > X_C$

$$2\pi f_2 L - \frac{1}{2\pi f_2 C} = R$$

From which $$f_2 = \frac{+R + \sqrt{R^2 + 4L/C}}{4\pi L}$$

Bandwidth $$BW = f_2 - f_1 = \frac{R}{2\pi L}$$

Hence $$Q_0 = \frac{f_r}{BW} = \frac{2\pi f_r L}{R} = \frac{2 \times \pi \times 159.2 \times 0.1}{10}$$

$$Q_0 = \frac{f_r}{BW} = 10$$

8.9.5 The Quality Factor (Q) and its Effect on Bandwidth

The quality factor, Q, is the ratio of the reactive power in the inductor or capacitor to the true power in the resistance in series with the coil or capacitor.

The quality factor

$$Q = 2\pi \times \frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}}$$

In an inductor, the max energy stored is given by $\frac{LI^2}{2}$

Energy dissipated per cycle $= \left(\frac{I}{\sqrt{2}}\right)^2 R \times T = \frac{I^2 RT}{2}$

$\therefore$ Quality factor of the coil $Q = 2\pi \times \dfrac{\frac{1}{2} LI^2}{\frac{I^2 R}{2} \times \frac{1}{f}}$

$$= \frac{2\pi f L}{R} = \frac{\omega L}{R}$$

Similarly, in a capacitor, the max energy stored is given by $\frac{CV^2}{2}$

The energy dissipated per cycle = $(I/\sqrt{2})^2 R \times T$

The quality factor of the capacitance circuit

$$Q = \frac{2\pi \frac{1}{2} C \left(\frac{I}{\omega C}\right)^2}{\frac{I^2}{2} R \times \frac{1}{f}} = \frac{1}{\omega CR}$$

In series circuits, the quality factor $Q = \dfrac{\omega L}{R} = \dfrac{1}{\omega CR}$ (8.59)

We have already discussed the relation between bandwidth and quality factor, which is $Q = \dfrac{f_r}{BW}$.

A higher value of circuit Q results in a smaller bandwidth. A lower value of Q causes a larger bandwidth.

Example 8.14 For the circuit shown in Fig. 8.32, determine the value of Q at resonance and bandwidth of the circuit.

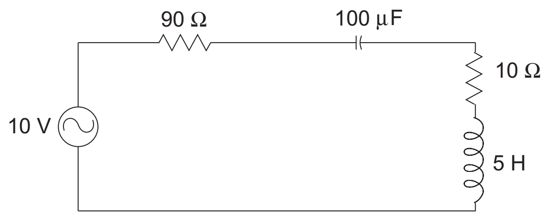


Fig. 8.32

Solution The resonant frequency,

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{5 \times 100 \times 10^{-6}}}$$

$$= 7.12 \text{ Hz}$$

Quality factor $Q = X_L/R = 2\pi f_r L/R$

$$= \frac{2\pi \times 7.12 \times 5}{100} = 8.24$$

Bandwidth of the circuit is $BW = \dfrac{f_r}{Q} = \dfrac{7.12}{2.24} = 3.178$ Hz

8.9.6 Magnification in Resonance

If we assume that the voltage applied to the series RLC circuit is V, and the current at resonance is I, then the voltage across L is $V_L = IX_L = (V/R)\ \omega_r L$

Similarly, the voltage across C

$$V_C = IX_C = \frac{V}{R\omega_r C}$$

Since $\quad Q = 1/\omega_r CR = \omega_r L/R$

where ω_r is the frequency at resonance.

Therefore $\quad V_L = VQ$

$$V_C = VQ$$

The ratio of voltage across either L or C to the voltage applied at resonance can be defined as magnification.

$\therefore$ Magnification $= Q = V_L/V$ or V_C/V (8.60)

8.10 PARALLEL RESONANCE

Basically, parallel resonance occurs when $X_C = X_L$. The frequency at which resonance occurs is called the *resonant* frequency. When $X_C = X_L$, the two branch currents are equal in magnitude and 180° out of phase with each other. Therefore, the two currents cancel each other out, and the total current is zero. Consider the circuit shown in Fig. 8.33. The condition for resonance occurs when $X_L = X_C$.

In Fig. 8.33, the total admittance

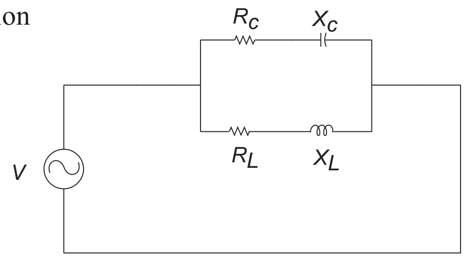


Fig. 8.33

$$Y = \frac{1}{R_L + j\omega L} + \frac{1}{R_C - (j/\omega C)}$$

$$= \frac{R_L - j\omega L}{R_L^2 + \omega^2 L^2} + \frac{R_C + (j/\omega C)}{R_C^2 + \dfrac{1}{\omega^2 C^2}}$$

$$= \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_C}{R_C^2 + \dfrac{1}{\omega^2 C^2}} + j\left\{\left[\frac{1/\omega C}{R_C^2 + \dfrac{1}{\omega^2 C^2}}\right] - \left[\frac{\omega L}{R_L^2 + \omega^2 L^2}\right]\right\} \quad (8.61)$$

At resonance the susceptance part becomes zero

$$\therefore \quad \frac{\omega_r L}{R_L^2 + \omega_r^2 L^2} = \frac{\dfrac{1}{\omega_r C}}{R_C^2 + \dfrac{1}{\omega_r^2 C^2}} \quad (8.62)$$

$$\omega_r L\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{\omega_r C}\,[R_L^2 + \omega_r^2 L^2]$$

$$\omega_r^2\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{LC}\,[R_L^2 + \omega_r^2 L^2]$$

$$\omega_r^2 R_C^2 - \frac{\omega_r^2 L}{C} = \frac{1}{LC} R_L^2 - \frac{1}{C^2}$$

$$\omega_r^2\left[R_C^2-\frac{L}{C}\right]=\frac{1}{LC}\left[R_L^2-\frac{L}{C}\right]$$

$$\omega_r=\frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2-(L/C)}{R_C^2-(L/C)}} \tag{8.63}$$

The condition for resonant frequency is given by Eq. 8.63. As a special case, if $R_L = R_C$, then Eq. 8.63 becomes

$$\omega_r=\frac{1}{\sqrt{LC}}$$

Therefore $$f_r=\frac{1}{2\pi\sqrt{LC}}$$

Example 8.15 Find the resonant frequency in the ideal parallel *LC* circuit shown in Fig. 8.34.

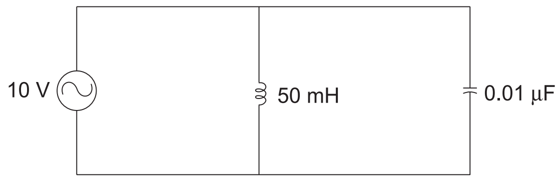


Fig. 8.34

Solution $f_r=\dfrac{1}{2\pi\sqrt{LC}}=\dfrac{1}{2\pi\sqrt{50\times10^{-3}\times0.01\times10^{-6}}}$ = 7117.6 Hz

8.10.1 Resonant Frequency for a Tank Circuit

The parallel resonant circuit is generally called a tank circuit because of the fact that the circuit stores energy in the magnetic field of the coil and in the electric field of the capacitor. The stored energy is transferred back and forth between the capacitor and coil and vice-versa. The tank circuit is shown in Fig. 8.35. The circuit is said to be in resonant condition when the susceptance part of admittance is zero.

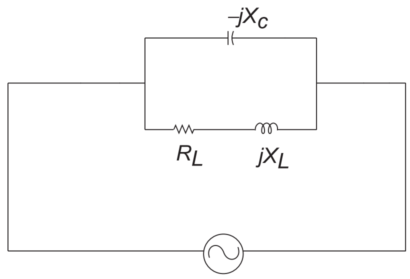


Fig. 8.35

The total admittance is $Y=\dfrac{1}{R_L+jX_L}+\dfrac{1}{-jX_C}$ (8.64)

Simplifying Eq. 8.64, we have

$$Y = \frac{R_L - jX_L}{R_L^2 + X_L^2} + \frac{j}{X_C}$$

$$= \frac{R_L}{R_L^2 + X_L^2} + j\left[\frac{1}{X_C} - \frac{X_L}{R_L^2 + X_L^2}\right]$$

To satisfy the condition for resonance, the susceptance part is zero.

$$\therefore \quad \frac{1}{X_C} = \frac{X_L}{R_L^2 + X_L^2} \tag{8.65}$$

$$\omega C = \frac{\omega L}{R_L^2 + \omega^2 L^2} \tag{8.66}$$

From Eq. 8.66, we get

$$R_L^2 + \omega^2 L^2 = \frac{L}{C}$$

$$\omega^2 L^2 = \frac{L}{C} - R_L^2$$

$$\omega^2 = \frac{1}{LC} - \frac{R_L^2}{L^2}$$

$$\therefore \quad \omega = \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}} \tag{8.67}$$

The resonant frequency for the tank circuit is

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}} \tag{8.68}$$

Example 8.16 For the tank circuit shown in Fig. 8.36, find the resonant frequency.

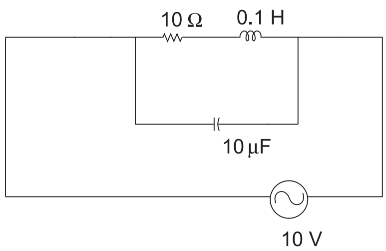


Fig. 8.36

Solution The resonant frequency

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1 \times 10 \times 10^{-6}} - \frac{(10)^2}{(0.1)^2}}$$

$$= \frac{1}{2\pi}\sqrt{(10)^6 - (10)^2} = \frac{1}{2\pi}\ (994.98) = 158.35 \text{ Hz}$$

8.10.2 Variation of Impedance with Frequency

The impedance of a parallel resonant circuit is maximum at the resonant frequency and decreases at lower and higher frequencies as shown in Fig. 8.37.

At very low frequencies, X_L is very small and X_C is very large, so the total impedance is essentially inductive. As the frequency increases, the impedance also increases, and the inductive reactance dominates until the resonant frequency is reached. At this point $X_L = X_C$ and the impedance is at its maximum. As the frequency goes above resonance, inductive reactance dominates and the impedance decreases.

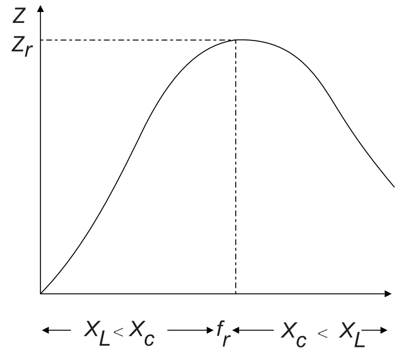


Fig. 8.37

8.10.3 Q Factor of Parallel Resonance

Consider the parallel RLC circuit shown in Fig. 8.38.

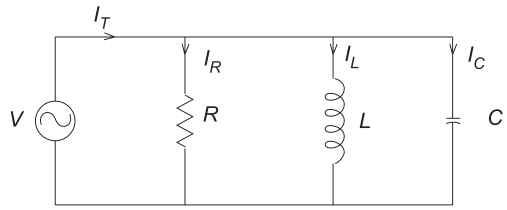


Fig. 8.38

In the circuit shown, the condition for resonance occurs when the susceptance part is zero.

Admittance $$Y = G + jB \tag{8.69}$$

$$= \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

$$= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \tag{8.70}$$

The frequency at which resonance occurs is

$$\omega_r C - \frac{1}{\omega_r L} = 0 \tag{8.71}$$

$$\omega_r = \frac{1}{\sqrt{LC}} \tag{8.72}$$

The voltage and current variation with frequency is shown in Fig. 8.39. At resonant frequency, the current is minimum.

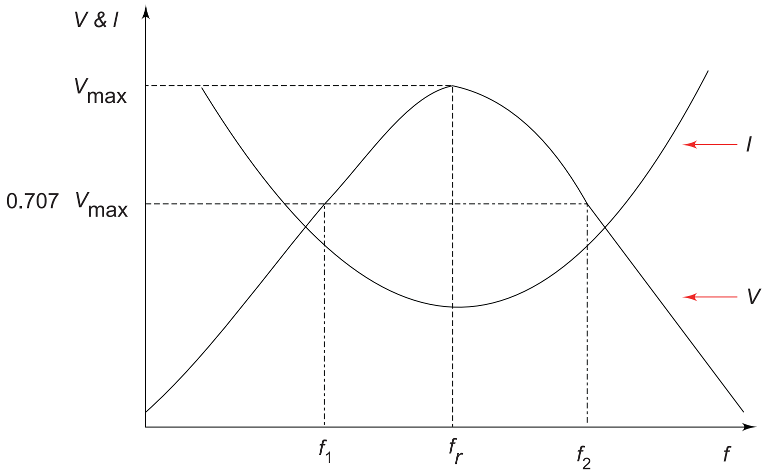


Fig. 8.39

The bandwidth, $BW = f_2 - f_1$

For parallel circuit, to obtain the lower half power frequency,

$$\omega_1 C - \frac{1}{\omega_1 L} = -\frac{1}{R} \tag{8.73}$$

From Eq. 8.73, we have

$$\omega_1^2 + \frac{\omega_1}{RC} - \frac{1}{LC} = 0 \tag{8.74}$$

If we simplify Eq. 8.74, we get

$$\omega_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{8.75}$$

Similarly, to obtain the upper half power frequency

$$\omega_2 C - \frac{1}{\omega_2 L} = \frac{1}{R} \tag{8.76}$$

From Eq. 8.76, we have

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{8.77}$$

Bandwidth $$BW = \omega_2 - \omega_1 = \frac{1}{RC}$$

The quality factor is defined as $Q_r = \dfrac{\omega_r}{\omega_2 - \omega_1}$

$$Q_r = \frac{\omega_r}{1/RC} = \omega_r RC$$

In other words,

$$Q = 2\pi \times \frac{\text{maximum energy stored}}{\text{Energy dissipated/cycle}}$$

In the case of an inductor,

The maximum energy stored $= \dfrac{1}{2}\, LI^2$

Energy dissipated per cycle $= \left(\dfrac{I}{\sqrt{2}}\right)^2 \times R \times T$

The quality factor $\quad Q = 2\pi \times \dfrac{1/2\,(LI^2)}{\dfrac{I^2}{2} R \times \dfrac{1}{f}}$

$$\therefore \qquad Q = 2\pi \times \frac{\dfrac{1}{2} L \left(\dfrac{V}{\omega L}\right)^2 R}{\dfrac{V^2}{2} \times \dfrac{1}{f}}$$

$$= \frac{2\pi f L R}{\omega^2 L^2} = \frac{R}{\omega L}$$

For a capacitor, maximum energy stored = $1/2\,(CV^2)$

Energy dissipated per cycle $= P \times T = \dfrac{V^2}{2 \times R} \times \dfrac{1}{f}$

The quality factor $\qquad Q = 2\pi \times \dfrac{1/2\,(CV^2)}{\dfrac{V^2}{2R} \times \dfrac{1}{f}}$

$$= 2\pi f CR = \omega CR$$

8.10.4 Magnification

Current magnification occurs in a parallel resonant circuit. The voltage applied to the parallel circuit, $V = IR$

Since $\qquad I_L = \dfrac{V}{\omega_r\, L} = \dfrac{IR}{\omega_r\, L} = IQ_r$

For the capacitor, $I_C = \dfrac{V}{1/\omega_r\, C} = IR\omega_r C = IQ_r$

Therefore, the quality factor $Q_r = I_L/I$ or I_C/I

8.10.5 Reactance Curves in Parallel Resonance

The effect of variation of frequency on the reactance of the parallel circuit is shown in Fig. 8.40.

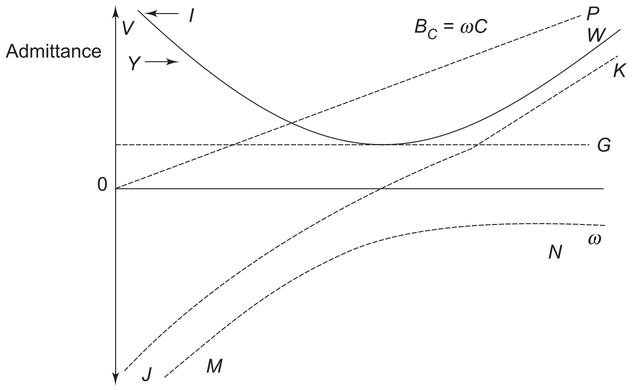


Fig. 8.40

The effect of inductive susceptance,

$$B_L = \frac{-1}{2\pi f L}$$

Inductive susceptance is inversely proportional to the frequency or ω. Hence it is represented by a rectangular hyperbola, *MN*. It is drawn in fourth quadrant, since B_L is negative. Capacitive susceptance, $B_C = 2\pi fC$. It is directly proportional to the frequency f or ω. Hence it is represented by *OP*, passing through the origin. Net susceptance $B = B_C - B_L$. It is represented by the curve JK, which is a hyperbola. At point ω_r, the total susceptance is zero, and resonance takes place. The variation of the admittance *Y* and the current *I* is represented by curve *VW*. The current will be minimum at resonant frequency.

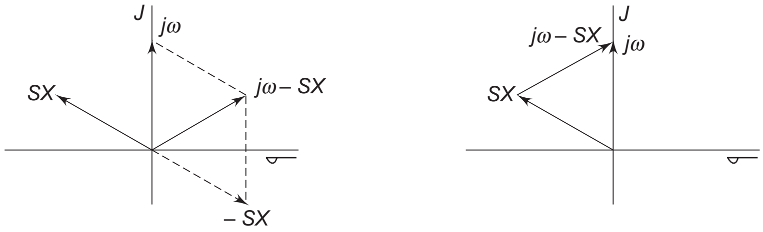
8.11 FREQUENCY RESPONSE FROM POLES AND ZEROS

The amplitude and phase response with respect to angular frequency can be obtained from the pole-zero diagram of a system function. Any network function $G(S)$ can be expressed in the form of a quotient of polynomials in S, each numerator and denominator polynomial can be expressed as a product of factors of the form $(S - Sx)$, where Sx may be either a pole or a zero. In sinusoidal steady state $S = J\omega$, then the factor becomes $(J\omega - Sx)$.

In complex S-plane, both $J\omega$ and Sx are phasor. The phasor difference $(J\omega - Sx)$ also represents another phasor. Let us consider a complex zero Sx and an imaginary phasor $J\omega$ as shown in Fig. 8.41.

It is seen from Fig. 8.41, the phasor difference $(J\omega - Sx)$ is a phasor directed from Sx to $J\omega$. As we increase the value of ω, the phasor $(J\omega - Sx)$ also changes its length and the angle it subtends with positive x axis in counter clockwise direction. Generally, any network function $G(S)$ contains such factors, some in the numerator and some in the denominator as given by the system function.

$$G(J\omega) = \frac{K(J\omega - Zo)(J\omega - Z_1)\ldots(J\omega - Zn)}{(J\omega - Po)(J\omega - P_1)\ldots(J\omega - Pm)}$$



(a) (b)

Fig. 8.41

Each one of the factors $(J\omega - Zi)$, $i = 0, 1, 2 \ldots n$; $(J\omega - Pj)$, $j = 0, 1, 2, \ldots m$ corresponds to a vector from the zero Zi or pole Pi directed to a point $J\omega$ on the imaginary axis. The variation of the function $G(J\omega)$ with ω may be obtained by studying the variation of the individual factors. $(J\omega - Zi)$ and $(J\omega - Pj)$, each changing in a pattern determined by the position of Sx with respect to the imaginary axis.

In general, the amplitude response $N(\omega)$ can be expressed in terms of the following equation.

$$N(\omega) = \frac{\prod_{i=0}^{n} \text{vector magnitudes from zeros to the point on the } J\omega \text{ axis}}{\prod_{J=0}^{m} \text{vector magnitudes from the poles to the point on the } J\omega \text{ axis}}$$

Similarly the phase response $\phi(\omega)$ can be expressed as

$$\phi(\omega) = \sum_{i=0}^{n} \text{ angles of the vectors from zero to } J\omega \text{ axis}$$

$$- \sum_{J=0}^{m} \text{ angles of the vectors from the poles to the } J\omega \text{ axis.}$$

Example 8.17 Draw the frequency response for the admittance of the *RC* network shown in Fig. 8.42 using pole zero concept.

Solution The admittance of the network is given by

$$Y_{RC} = \frac{CS}{(1 + RCS)} = \frac{S}{R\left(S + \frac{1}{RC}\right)}$$

$$= K\frac{S}{\left(S + \frac{1}{RC}\right)}; \text{ where } K = \frac{1}{R}$$

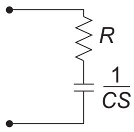


Fig. 8.42

The admittance function has a zero at origin and a pole at $-\frac{1}{R}$ as shown in Fig. 8.43(a).

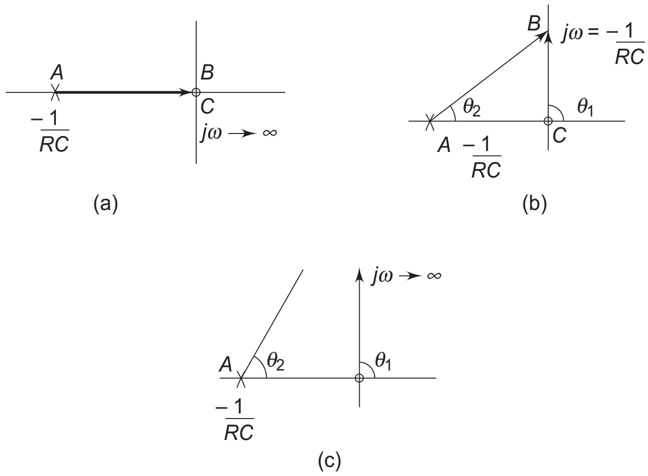


Fig. 8.43

Consider the variation of frequency from $J\omega = 0$ to $J\omega = \dfrac{1}{RC}$ to $J\omega = \propto$.

For every value of ω, the phasors are to be constructed from poles and zeroes of the function. The magnitude of the admittance function can be represented by

$$N(\omega) = K\,\frac{|a|}{|b|}$$

Where $|a|$ is the length of the phasor drawn from zero at $S = 0$ to $J\omega$ under consideration. $|b|$ is the length of the phasor drawn from pole at $S = -\dfrac{1}{RC}$ to $J\omega$ under consideration. The phase angle $\phi(\omega)$ of the admittance function can represented by

$$\phi(\omega) = \frac{\angle\theta_1}{\angle\theta_2}$$

$\angle\theta_1$ is the angle between zero and positive x-axis.

$\angle\theta_2$ is the angle between pole and positive x-axis.

From Fig. 8.43(a) when $J\omega = 0$

$\lvert a\rvert = 0;$	$\angle\theta_1 = 90°$	$N(\omega) = 0$
$\lvert b\rvert = \dfrac{1}{RC}$	$\angle\theta_2 = 0°$	$\phi(\omega) = 90°$

From Fig. 8.43(b) when $J\omega = \dfrac{1}{RC}$

$\lvert a\rvert = \dfrac{1}{RC}$	$\angle\theta_1 = 90°$	$N(\omega) = \dfrac{1}{\sqrt{2R}}$
$\lvert b\rvert = \dfrac{\sqrt{2}}{RC}$	$\angle\theta_2 = 45°$	$\phi(\omega) = 45°$

From Fig. 8.43(c) when $\omega \to \propto$

$$|a| = \to \propto \qquad \theta_1 = 90° \qquad N(\omega) = \frac{1}{R}$$

$$|b| = \to \propto \qquad \theta_2 = 90° \qquad \phi(\omega) = 0$$

It can be observed that the magnitude changes from 0 to $\frac{1}{R}$ as ω changes from zero to infinity. The complex locus of this variation is shown in Fig. 8.44.

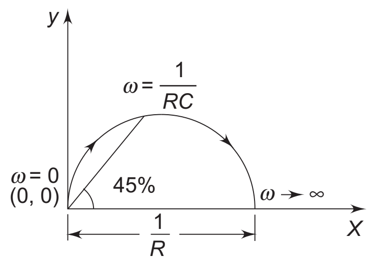


Fig. 8.44

This is similar to the one shown in Fig. 8.5.

Example 8.18 Find the amplitude and phase response from the poles and zeros of the network function given by

$$N(S) = \frac{2S}{S^2 + 4S + 5}$$

Solution $N(S) = \dfrac{2S}{(S + 2 + J1)(S + 2 - J1)}$

The poles are at $S = -2 - J1$ and $S = -2 + J1$ and zero at origin as shown in Fig. 8.45.

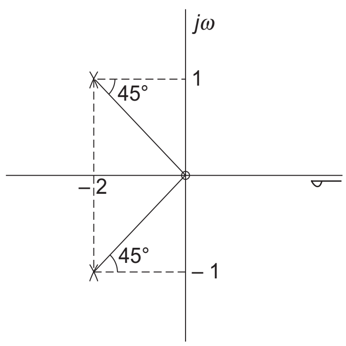


Fig. 8.45 (a)

Let us find the amplitude and phase at $\omega = 0$. Draw the vectors from poles and zeros of $N(S)$ to the point $\omega = 0$ as shown in Fig. 8.45(a). Let the length of vector from zero to $J\omega$ be a, from pole $(-2 + J1)$ to $J\omega$ be b and from $(-1 - J1)$ to $J\omega$ be c and the corresponding angles be θ_1, θ_2 and θ_3 respectively.

From the pole zero diagram of Fig. 8.45(a), we can find

$$N(J0) = 2\left(\frac{a}{b \times c}\right) = 2\left(\frac{0}{\sqrt{5} \times \sqrt{5}}\right) = 0$$

$$\phi(J0) = \theta_1 - \theta_2 - \theta_3 = 90° - 45° + 45° = 90°$$

Similarly, we can find the amplitude and phase from pole zero diagram for different values of ω as explained below.

For $\omega = 1$, the vectors are shown in Fig. 8.45(b).

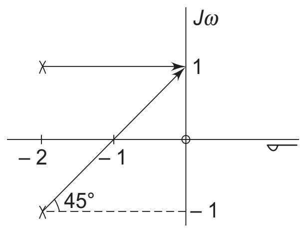


Fig. 8.45 (b)

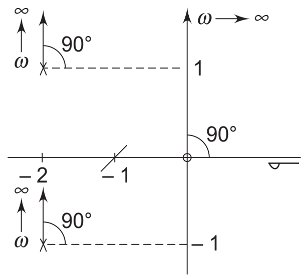


Fig. 8.45 (c)

$$N(J1) = 2\left(\frac{1}{2 \times 2\sqrt{2}}\right) = 0.3535$$

$$\phi(J1) = 90° - 0° - 45° = 45°$$

For $\omega = \propto$, the vectors are shown in Fig. 8.45(c). At very high frequency as ω tends to infinity all vectors are approximately equal to infinity, the magnitude tends to zero.

$$N(\omega \propto) \approx \frac{2\omega_\propto}{\omega_\propto^2} = \frac{2}{\omega_\propto}$$

whereas the phase would be

$$\phi(\omega_\propto) = 90° - 90° - 90° = -90°$$

With the information available for three points, we can roughly estimate the frequency response of the network function. The sketch is shown in Fig. 8.46. Extending the analysis further, the values are given in Table 8.2.

Table 8.2

ω	$N(\omega)$	$\theta(\omega)$
0	0	90°
1	0.3535	45°
2	0.49	7.6°
4	0.37	– 23°
∝	0	– 90°

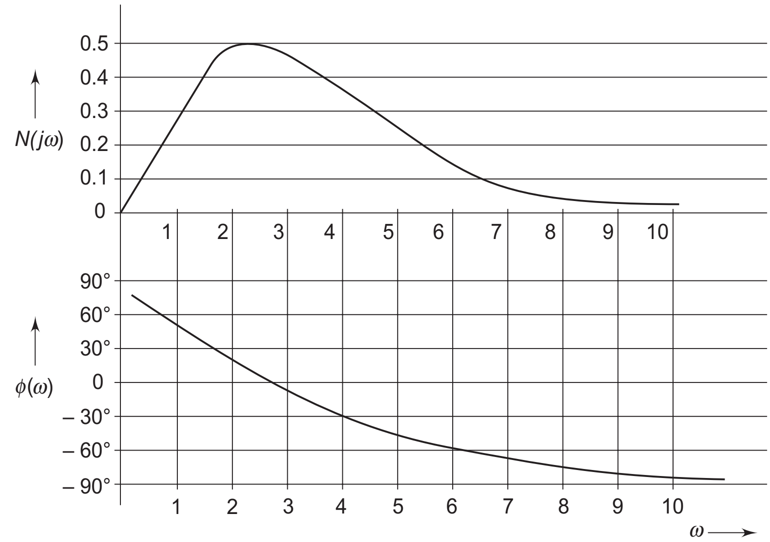


Fig. 8.46

8.12 BODE PLOTS

The Bode plot or corner plot of a function $GH(J\omega)$ is a graphical representation of the magnitude and phase angle of $GH(J\omega)$ with respect to frequency ω. This techniques is used in the analysis of feedback control systems. The Bode plot is also known as the **asymptotic plot** of $GH(J\omega)$. Because these plots can be constructed by using straight line approximations that are asymptotic to the actual plots. Bode plots consist of two graphs. Logarithmic scales are used for the frequency ω and for the magnitude of $GH(J\omega)$ to simplify their construction and manipulation. The use of logarithmic scale facilitates the representation of the magnitude and phase of the function over wide range of frequencies. Multiplication of magnitudes can be converted to addition. Both the low frequency and high frequency characteristics of a transfer function can be easily studied. Because of logarithmic scale, it is not possible to represented the frequency scale right down to zero.

The magnitude $|GH(J\omega)|$ of a transfer function of any value of frequency ω is plotted on a logarithmic scale in decible (db) units. Hence the magnitude of $GH(J\omega)$ in db is obtained by multiplying the logarithm to the base 10 of $|GH(J\omega)|$ by 20 where db = 20 $\log_{10} |GH(J\omega)|$. Since, open loop or closed loop transfer functions of a control system are usually expressed in terms of Products of factors like $S^{\pm n}$, $(1 + ST)^{\pm n}$. It is convenient to draw the plots as the logarithm of a product of factors is equal to the sum of the db magnitudes of the individual factors.

8.12.1 Factors of $GH(J\omega)$

In general $GH(J\omega)$ can contain the following factors.

1. Constant factor K.

2. Integral and derivative factors of order m: $(J\omega)^{\mp m}$
3. First order factors of order n: $(1 + J\omega T)^{\mp n}$
4. Quadratic factors $\left[1 + 2\xi \dfrac{\omega}{\omega_n} + \left(J \dfrac{\omega}{\omega_n}\right)^n\right]^{\mp 1}$

Let us sketch the Bode plot of different factors.

8.12.2 Constant Factor (K)

Real constant K has a magnitude |K|, a phase angle of 0°. If K is positive, and – 180° if K is negative. Therefore, the Bode plots for K are simply horizontal straight lines as shown in Fig. 8.47.

The effect of varying the gain is to raise or lower the log magnitude but has no effect on phase angle.

$$K_{dB} = 20 \log_{10} K = \text{constant}$$
$$\angle K = 0 \text{ If } K > 0$$
$$= -180° \text{ If } K < 0$$

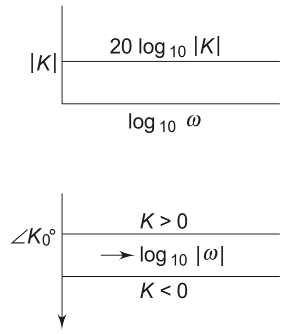


Fig. 8.47

8.12.3 Integral and Derivative Factors $(J\omega)^{\mp m}$

Poles and zeroes at the origin: $(J\omega)^{\mp 1}$
The magnitude of $(J\omega)^{-1}$ in dB is given by

$$20 \log \left|\frac{1}{J\omega}\right| = -20 \log \omega \text{ dB}$$

Phase angle of $\dfrac{1}{J\omega}$ = – 90° constant at all values of ω

For $\omega = 1$, log magnitude is 0 dB
For $\omega = 10$ log magnitude is – 20 dB
For $\omega = 100$ log magnitude is – 40 dB
For $\omega = 0.1$ log magnitude is 20 dB

Thus, the log magnitude plot for the function $\frac{1}{J\omega}$ is a straight line with a slope of – 20 dB per decade of frequency. The Bode plot for the function is shown in Fig. 8.48.

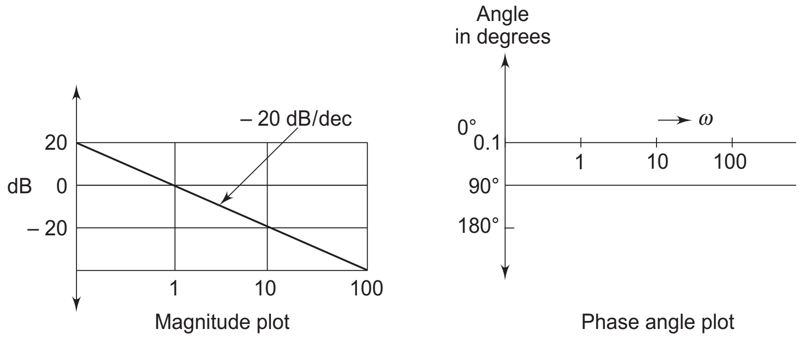


Fig. 8.48

Similarly, the magnitude of $(J\omega) = 20 \log \omega$ dB and phase angle of $(J\omega) = 90°$ constant at all values of ω.

for $\omega = 0.1$ log magnitude is – 20 dB

for $\omega = 1$ log magnitude is 0 dB

for $\omega = 10$ log magnitude is 20 dB

for $\omega = 100$ log magnitude is 40 dB

Thus, the log magnitude plot for the function $(J\omega)$ is a straight line with a slope of 20 dB per decade of frequency. The Bode plot for the function is shown in Fig. 8.49.

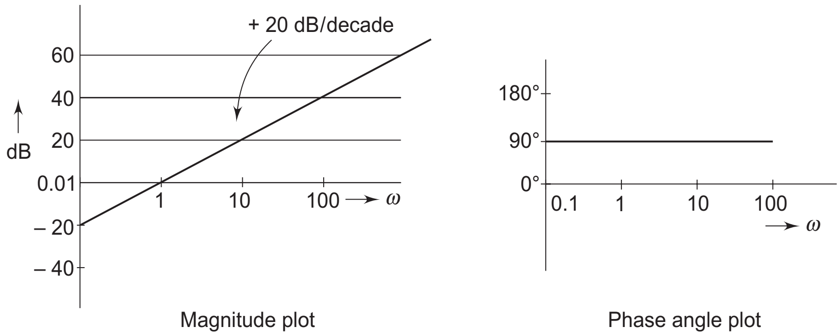


Fig. 8.49

If the transfer function contain the $\left(\frac{1}{J\omega}\right)^m$ or $(J\omega)^m$, that is the multiple poles and zeroes at origin. The log magnitude becomes as follows:

$$20 \log \left|\frac{1}{(J\omega)^m}\right| = -m \times 20 \log |J\omega| = -20\, m \log \omega \text{ dB}$$

$$20 \log (J\omega) = m \times 20 \log |J\omega| = +20\, m \log \omega \text{ dB}$$

The Bode plots for various values of m is shown in Figs. 8.50 and 8.51 respectively for the functions $\left(\frac{1}{J\omega}\right)^m$ and $(J\omega)^m$.

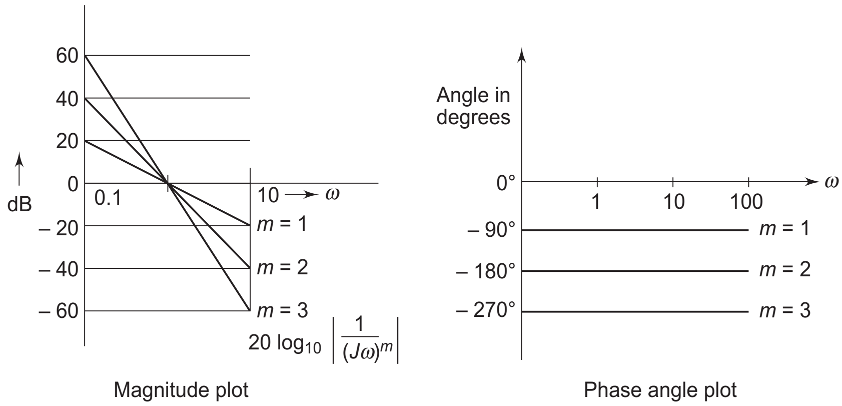


Fig. 8.50

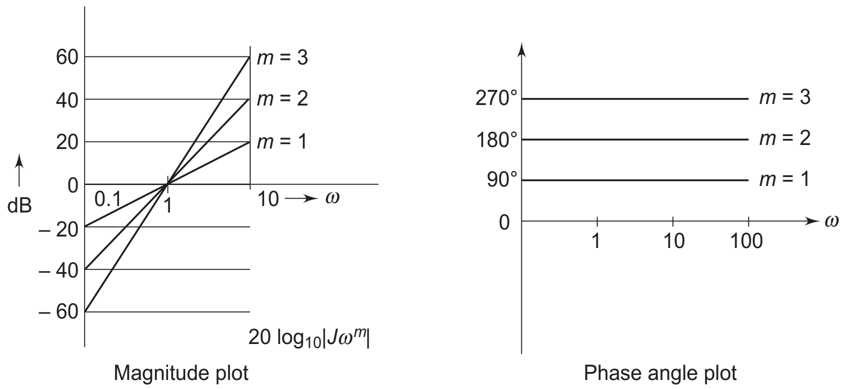


Fig. 8.51

8.12.4 First Order Factors $(1 + J\omega T)^{\mp 1}$

Simple Pole Consider the function $\frac{1}{(1+J\omega T)}$; the log magnitude in dB is given by

$$20 \log_{10} \left|\frac{1}{1+J\omega T}\right| = -20 \log_{10} \sqrt{1+\omega^2 T^2} \text{ dB}$$

In order to obtain asymptotic approximations of the above function, we have to consider both very large and very small values of ω.

For very low frequencies $\omega << \frac{1}{T}$ we can approximate

$$-20 \log_{10} \sqrt{1+\omega^2 T^2} \text{ as } -20 \log_{10} 1 = 0 \text{ dB}$$

For high frequencies $\omega >> \frac{1}{T}$ we can approximate

$$-20 \log_{10} \sqrt{1+\omega^2 T^2} \text{ as } -20 \log \omega T \text{ dB}$$

At $\omega = \frac{1}{T}$ known as corner frequency or break point frequency; $-20 \log 1$ dB = 0

At $\omega = \frac{10}{T}$; $-20 \log 10$ dB = -20 dB

At $\omega = \frac{100}{T}$; $-20 \log 100$ dB = -40 dB

Thus, an approximate log magnitude plot consists essentially of 2 straight lines meeting at corner frequency $\omega = \frac{1}{T}$; the value of $-20 \log \omega T$ dB decreases by 20 dB for every decade of ω for $\omega >> \frac{1}{T}$. Hence, the log magnitude curve is zero up to the corner frequency $\omega = \frac{1}{T}$ and is a straight line with a slope of -20 dB/decade from $\omega = \frac{1}{T}$. The phase angle ϕ of

$$\frac{1}{(1+J\omega T)} = -\tan^{-1} \omega T$$

For $\omega T >> 1$; $\phi = -\tan^{-1} \propto = -90°$

For $\omega = 0$ the phase angle $\phi = 0°$

At corner frequency $\omega = \frac{1}{T}$; $\phi = -45°$

Thus, the phase angle ϕ plot varies from 0 to 90°, its value is 45° at corner frequency. The exact plot has the value of log magnitude of ≈ -3 dB at $\omega = \frac{1}{T}$. Thus, the error between the straight-line approximation and the actual magnitude curve is -3 dB. The Bode plot of $\frac{1}{(1+J\omega T)}$ is shown in Fig. 8.52.

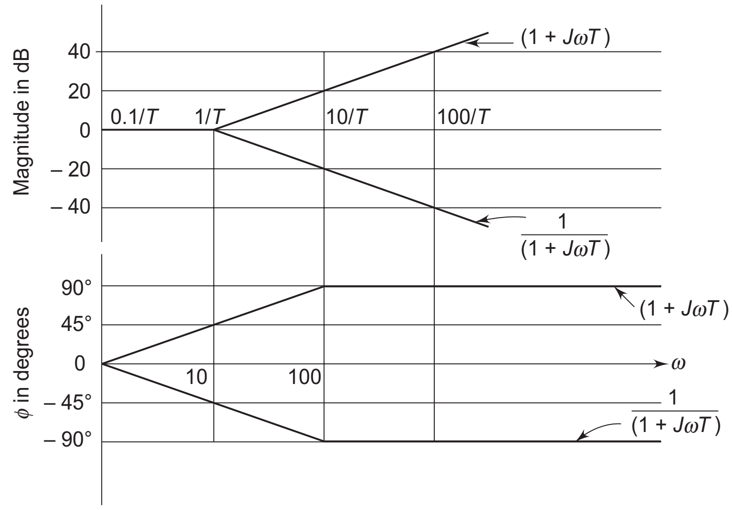


Fig. 8.52

8.12.5 Simple Zero

Consider the function $(1 + J\omega T)$. The magnitude in dB is $20 \log_{10} \sqrt{1+\omega^2 T^2}$ = $10 \log (1 + \omega^2 T^2)$ dB. The phase angle $\phi = \tan^{-1} (\omega T)$. In a similar manner to simple pole, for very low frequencies, log magnitude is approximated as zero, and phase angle $\phi = 0$. Also for high frequencies, the log magnitude is approximated as $20 \log \omega T$ and phase angle $\phi = 90°$. At corner frequency $\omega = \frac{1}{T}$; the phase angle $\phi = 45°$. The Bode plot of $(1 + J\omega T)$ is shown in Fig. 8.52. The corner frequency is the intersect of the high frequency approximate plot and the low frequency approximate plot which is the 0 dB axis. The actual plot of $(1 + J\omega T)$ is a smooth curve, and deviates only slightly from the straight line. A straight line approximation is a reasonable one. The error between the actual magnitude curve and the straight-line asymptotes is symmetrical with respect to the corner frequency $\omega = \frac{1}{T}$ and the error is 3 dB. At the corner frequency and 1 dB at 1 octave above $\left(\omega = \frac{2}{T}\right)$ and 1 octave below $\left(\omega = \frac{1}{2T}\right)$, the corner frequency. The actual magnitude curve can be obtained by adding the errors to the asymptotic plot at the corner frequency and one octave above and below the corner frequency.

Example 8.19 Draw the asymptotic Bode plot for the transfer function

$$GH(S) = \frac{10(1+0.5S)}{S(1+0.1S)(1+0.2S)}$$

Solution $G(J\omega) = \dfrac{10(1+0.5J\omega)}{J\omega(1+0.1J\omega)(1+0.2J\omega)}$

Identify the corner frequencies:

Positive corner frequency (p.c.f.) = $\dfrac{1}{0.5} = 2$

Negative corner frequencies (n.c.f.) = $\dfrac{1}{0.1} = 10;\ \dfrac{1}{0.2} = 5$

Magnitude of constant K = 10

Log magnitude of K = 20 $\log_{10}$ 10 = 20 dB

There are five factors in the function (1) constant factor (2) simple pole (3). First order factor in numerator with p.c.f. of 2(4) first order factor in denominator with n.c.f. of 5 and (5) first order factor in denominator with n.c.f. of 10.

The Bode plots of all the factor are shown in dashed lines and total plot is shown in solid line in Fig. 8.53.

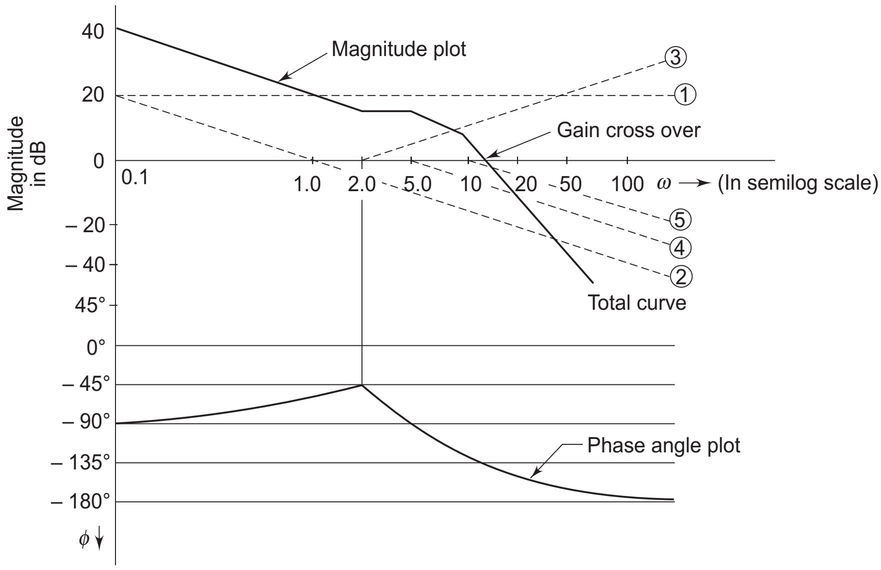


Fig. 8.53

From the figure it can be observed that, the curve starts at – 90° at low frequency and has the values of – 45° at $\omega = 2$; – 95° at $\omega = 5$; – 135° at $\omega = 10$ and reaches – 180° at higher values of ω; as ω tending to infinity.

8.12.6 Gain Cross Over and Phase Cross Over

Gain crossover The point at which the magnitude of $GH(J\omega)$ is unity, i.e. log magnitude of $GH(J\omega) = 0$ db, is called the gain cross over frequency. From the Bode magnitude plot, it is the point at which the curve crosses the 0 dB axis as shown in Fig. 8.54.

Phase crossover It is the frequency at which the Bode phase angle curve crosses the – 180° axis as shown in Fig. 8.54.

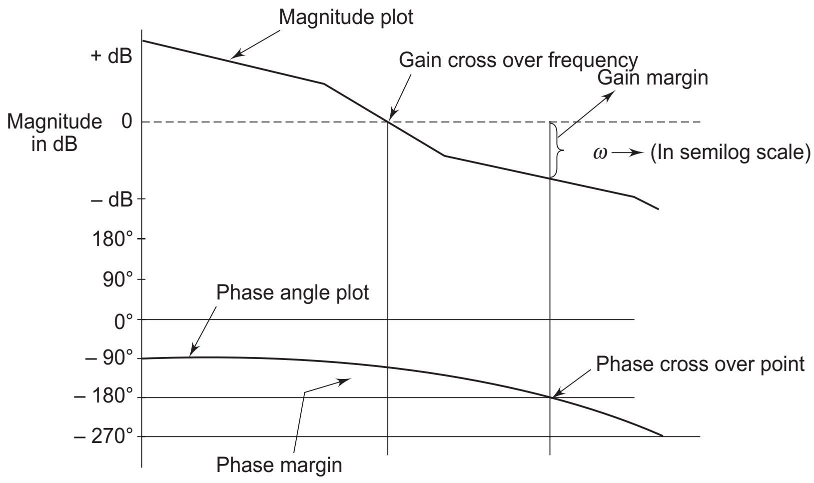


Fig. 8.54

8.12.7 Gain Margin and Phase Margin

Gain margin It is the amount of gain in dBs, that can be allowed to increase in the loop before the closed loop system reaches instability. It can be obtained from the Bode plot. The gain measured at the phase cross over frequency (ω_c) in dBs as shown in Fig. 8.54.

$$\text{G.M.} = -|GH(J\omega c)| \text{ dB}$$

Phase margin It is the angle in degrees between phase curve and –180° axis at the gain cross over frequency as shown in Fig. 8.54.

Phase margin is the angle by which the phase curve must be shifted so that it will pass through the –180° axis at the gain cross over frequency. If the phase angle curve never crosses the –180° axis, the system is always stable. The problem in Example 8.17 is a stable system.

Solved Problems

Problem 8.1 For the circuit shown in Fig. 8.55, determine the frequency at which the circuit resonates. Also find the voltage across the inductor at resonance and the Q factor of the circuit.

Solution The frequency of resonance occurs when $X_L = X_C$

$$\omega L = \frac{1}{\omega C}$$

$$\therefore \qquad \omega = \frac{1}{\sqrt{LC}} \text{ radians/sec}$$

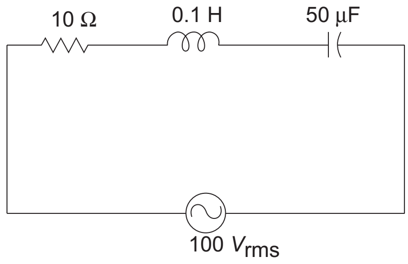


Fig. 8.55

$$= \frac{1}{\sqrt{0.1 \times 50 \times 10^{-6}}} = 447.2 \text{ rad/sec}$$

$$f_r = \frac{1}{2\pi}(447.2) = 71.17 \text{ Hz}$$

The current passing through the circuit at resonance,

$$I = \frac{V}{R} = \frac{100}{10} = 10\text{A}$$

The voltage drop across the inductor

$$V_L = IX_L = I\omega L$$
$$= 10 \times 447.2 \times 0.1 = 447.2 \text{ V}$$

The quality factor $Q = \dfrac{\omega L}{R} = \dfrac{447.2 \times 0.1}{10} = 4.47$

Problem 8.2 A series RLC circuit has a quality factor of 5 at 50 rad/sec. The current flowing through the circuit at resonance is 10 A and the supply voltage is 100 V. The total impedance of the circuit is 20 Ω. Find the circuit constants.

Solution The quality factor $Q = 5$

At resonance the impedance becomes resistance.

The current at resonance is $I = \dfrac{V}{R}$

$$10 = \frac{100}{R}$$

$\therefore$ $R = 10\ \Omega$

$$Q = \frac{\omega L}{R}$$

Since $Q = 5, R = 10$

$$\omega L = 50$$

$\therefore$ $L = \dfrac{50}{\omega} = 1 \text{ H}$

Total impedance is $Z = \sqrt{R^2 + (X_L - X_C)^2}$

$$(20) = \sqrt{(10)^2 + (X_L - X_C)^2}$$

$\therefore$ $X_L - X_C = \pm\sqrt{(20)^2 - (10)^2} = \pm 17.3\ \Omega$

So $X_C - X_L = 17.3\ \Omega$

$\therefore$ $X_C = 17.3 + \omega L = 17.3 + 50 = 67.3\ \Omega$

$\therefore$ $\dfrac{1}{2\pi f C} = 67.3$

$\therefore$ $C = \dfrac{1}{50 \times 67.3} = 2.97 \times 10^{-4} \text{ F}$

Problem 8.3 A voltage $v(t) = 10 \sin \omega t$ is applied to a series RLC circuit. At the resonant frequency of the circuit, the maximum voltage across the capacitor is found to be 500 V. Moreover, the bandwidth is known to be 400 rad/sec and the impedance at resonance is 100 Ω. Find the resonant frequency. Also find the values of L and C of the circuit.

Solution The applied voltage to the circuit is

$$V_{\max} = 10 \text{ V}$$

$$V_{\text{rms}} = \frac{10}{\sqrt{2}} = 7.07 \text{ V}$$

The voltage across capacitor $V_C = 500$ V

The magnification factor $Q = \dfrac{V_C}{V} = \dfrac{500}{7.07} = 70.7$

The bandwidth $BW = 400$ rad/sec

$$\omega_2 - \omega_1 = 400 \text{ rad/sec}$$

The impedance at resonance $Z = R = 100\ \Omega$

Since $$Q = \frac{\omega_r}{\omega_2 - \omega_1}$$

$$\omega_r = Q(\omega_2 - \omega_1) = 28280 \text{ rad/sec}$$

$$f_r = \frac{28280}{2\pi} = 4499 \text{ Hz}$$

The bandwidth $\omega_2 - \omega_1 = \dfrac{R}{L}$

$\therefore$ $$L = \frac{R}{\omega_2 - \omega_1} = \frac{100}{400} = 0.25 \text{ H}$$

Since $$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$C = \frac{1}{(2\pi f_r)^2 \times L} = \frac{1}{2\pi \times (4499)^2 \times 0.25} = 5\ \mu\text{F}$$

Problem 8.4 Find the value of L at which the circuit resonates at a frequency of 1000 rad/sec in the circuit shown in Fig. 8.56.

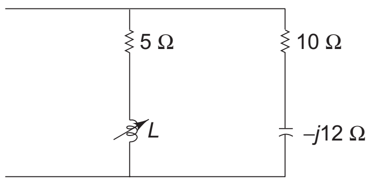


Fig. 8.56

Solution $$Y = \frac{1}{10 - j12} + \frac{1}{5 + jX_L}$$

$$Y = \frac{10 + j12}{10^2 + 12^2} + \frac{5 - jX_L}{25 + X_L^2}$$

$$= \frac{10}{10^2 + 12^2} + \frac{5}{25 + X_L^2} + j\left[\frac{12}{10^2 + 12^2} - \frac{X_L}{25 + X_L^2}\right]$$

At resonance the susceptance becomes zero.

Then
$$\frac{X_L}{25+X_L^2}=\frac{12}{10^2+12^2}$$
$$12X_L^2 - 244\,X_L + 300 = 0$$

From the above equation

$$X_L^2 - 20.3\,X_L + 25 = 0$$

$$X_L = \frac{+\,20.3 \pm \sqrt{(20.3)^2 - 4\times 25}}{2}$$

$$= \frac{20.3+\sqrt{412-100}}{2} \text{ or } \frac{20.3-\sqrt{412-100}}{2}$$

$$= 18.98\ \Omega \text{ or } 1.32\ \Omega$$

$$\therefore \qquad X_L = \omega L = 18.98 \text{ or } 1.32\ \Omega$$

$$L = \frac{18.98}{1000} \text{ or } \frac{1.32}{1000}$$

$$L = 18.98 \text{ mH or } 1.32 \text{ mH}$$

Problem 8.5 Two impedances $Z_1 = 20 + j10$ and $Z_2 = 10 - j30$ are connected in parallel and this combination is connected in series with $Z_3 = 30 + jX$. Find the value of X which will produce resonance.

Solution Total impedance is

$$Z = Z_3 + (Z_1 \,\|\, Z_2)$$

$$= (30 + jX) + \left\{\frac{(20+j10)(10-j30)}{20+j10+10-j30}\right\}$$

$$= (30 + jX) + \frac{200 - j600 + j100 + 300}{30 - j20}$$

$$= 30 + jX + \left(\frac{500 - j500}{30 - j20}\right)$$

$$= 30 + jX + \left[\frac{500(1-j)(30+20j)}{(30)^2+(20)^2}\right]$$

$$= (30 + jX) + \left[\frac{500(30+20j-30j+20)}{900+400}\right]$$

$$= 30 + jX + \frac{5}{13}(50 - j10)$$

$$= \left(30 + \frac{5}{13}\times 50\right) + j\left(X - \frac{5}{13}\times 10\right)$$

At resonance, the imaginary part is zero

$$\therefore \quad X - \frac{50}{13} = 0$$

$$X = \frac{50}{13} = 3.85\ \Omega$$

Problem 8.6 A 50 Ω resistor is connected in series with an inductor having internal resistance, a capacitor and 100 V variable frequency supply as shown in Fig. 8.57. At a frequency of 200 Hz, a maximum current of 0.7 A flows through the circuit and voltage across the capacitor is 200 V. Determine the circuit constants.

Solution At resonance, current in the circuit is maximum

$$I = 0.7\ \text{A}$$

Voltage across capacitor is $V_C = IX_C$

Since $\quad V_C = 200, I = 0.7$

$$X_C = \frac{1}{\omega C}$$

$$\omega C = \frac{0.7}{200}$$

$$\therefore \quad C = \frac{0.7}{200 \times 2\pi \times 200} = 2.785\ \mu\text{F}$$

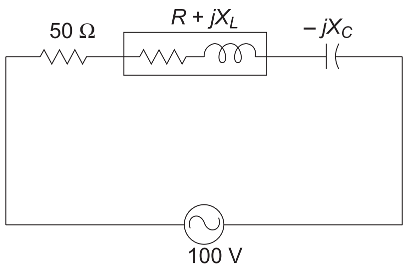


Fig. 8.57

At resonance

$$X_L - X_C = 0$$

$$\therefore \quad X_L = X_C$$

Since $\quad X_C = \dfrac{1}{\omega C} = \dfrac{200}{0.7} = 285.7\ \Omega$

$$X_L = \omega L = 285.7\ \Omega$$

$$\therefore \quad L = \frac{285.7}{2\pi \times 200} = 0.23\ \text{H}$$

At resonance, the total impedance

$$Z = R + 50$$

$$\therefore \quad R + 50 = \frac{V}{I} = \frac{100}{0.7}$$

$$R + 50 = 142.86\ \Omega$$

$$\therefore \quad R = 92.86\ \Omega$$

Problem 8.7 In the circuit shown in Fig. 8.58, a maximum current of 0.1 A flows through the circuit when the capacitor is at 5 μF with a fixed frequency and a voltage of 5 V. Determine the frequency at which the circuit resonates, the bandwidth, the quality factor Q and the value of resistance at resonant frequency.

Solution At resonance, the current is maximum in the circuits

$$I = \frac{V}{R}$$

$$\therefore \qquad R = \frac{V}{I} = \frac{5}{0.1} = 50\ \Omega$$

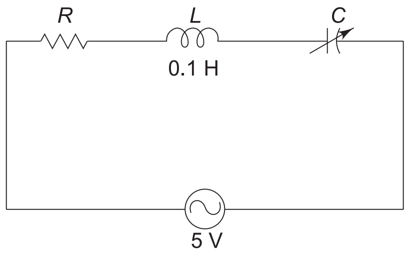


Fig. 8.58

The resonant frequency is

$$\omega_r = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{0.1 \times 5 \times 10^{-6}}} = 1414.2 \text{ rad/sec}$$

$$f_r = \frac{1414.2}{2\pi} = 225 \text{ Hz}$$

The quality factor is $Q = \dfrac{\omega L}{R} = \dfrac{1414.2 \times 0.1}{50} = 28$

Since $\qquad \dfrac{f_r}{BW} = Q$

The bandwidth $BW = \dfrac{f_r}{Q} = \dfrac{225}{2.8} = 80.36$ Hz

Problem 8.8 In the circuit shown in Fig. 8.59, determine the circuit constants when the circuit draws a maximum current at 10 μF with a 10 V, 100 Hz supply. When the capacitance is changed to 12 μF, the current that flows through the circuit becomes 0.707 times its maximum value. Determine Q of the coil at 900 rad/sec. Also find the maximum current that flows through the circuit.

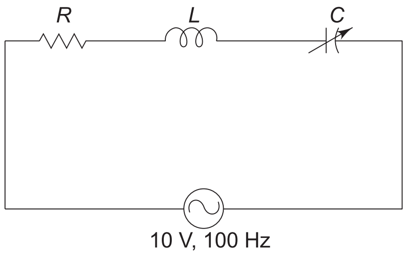


Fig. 8.59

Solution At resonant frequency, the circuit draws maximum current. So, the resonant frequency $f_r = 100$ Hz

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$L = \frac{1}{C \times (2\pi f_r)^2}$$

$$= \frac{1}{10 \times 10^{-6} (2\pi \times 100)^2} = 0.25 \text{ H}$$

We have
$$\omega L - \frac{1}{\omega C} = R$$

$$900 \times 0.25 - \frac{1}{900 \times 12 \times 10^{-6}} = R$$

$\therefore$
$$R = 132.4\ \Omega$$

The quality factor $Q = \dfrac{\omega L}{R} = \dfrac{900 \times 0.25}{132.4} = 1.69$

The maximum current in the circuit is $I = \dfrac{10}{132.4} = 0.075$ A

Problem 8.9 In the circuit shown in Fig. 8.60, the current is at its maximum value with capacitor value $C = 20\ \mu$F and 0.707 times its maximum value with $C = 30\ \mu$F. Find the value of Q at $\omega = 500$ rad/sec, and circuit constants.

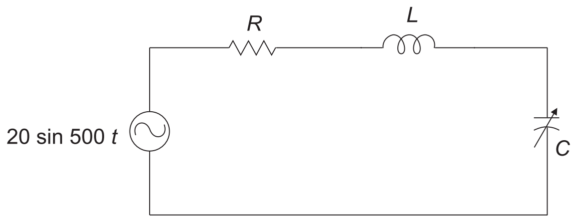


Fig. 8.60

Solution The voltage applied to the circuit is $V = 20$ V. At resonance, the current in the circuit is maximum. The resonant frequency $\omega_r = 500$ rad/sec.

Since
$$\omega_r = \frac{1}{\sqrt{LC}}$$

$\therefore$
$$L = \frac{1}{\omega_r^2 C} = \frac{1}{(500)^2 \times 20 \times 10^{-6}} = 0.2 \text{ H}$$

Since we have

$$\omega L - \frac{1}{\omega C} = R$$

$$500 \times 0.2 - \frac{1}{500 \times 30 \times 10^{-6}} = R$$

$\therefore$
$$R = 100 - 66.6 = 33.4$$

The quality factor is $Q = \dfrac{\omega L}{R} = \dfrac{500 \times 0.2}{33.4} = 2.99$

Problem 8.10 In the circuit shown in Fig. 8.61, an inductance of 0.1 H having a Q of 5 is in parallel with a capacitor. Determine the value of capacitance and coil resistance at resonant frequency of 500 rad/sec.

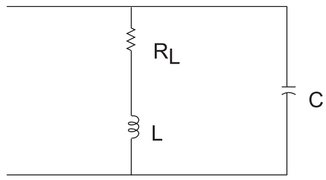


Fig. 8.61

Solution The quality factor $Q = \dfrac{\omega_r L}{R}$

Since $L = 0.1$ H, $Q = 5$ and

$$\omega_r = 500 \text{ rad/sec}$$

$$Q = \frac{500 \times 0.1}{R}$$

$$\therefore \quad R = \frac{500 \times 0.1}{5} = 10\ \Omega$$

Since

$$\omega_r^2 = \frac{1}{LC}$$

$$(500)^2 = \frac{1}{0.1 \times C}$$

$\therefore$ The capacitance value $C = \dfrac{1}{0.1 \times (500)^2} = 40\ \mu\text{F}$

Problem 8.11 A series RLC circuit consists of a 50 Ω resistance, 0.2 H inductance and 10 μF capacitor with an applied voltage of 20 V. Determine the resonant frequency. Find the Q factor of the circuit. Compute the lower and upper frequency limits and also find the bandwidth of the circuit.

Solution Resonant frequency

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 10 \times 10^{-6}}} = 112.5 \text{ Hz}$$

Quality factor $Q = \dfrac{\omega L}{R} = \dfrac{2\pi \times 112.5 \times 0.2}{50} = 2.83$

Lower frequency limit

$$f_1 = f_r - \frac{R}{4\pi L} = 112.5 - \frac{50}{4 \times \pi \times 0.2} = 92.6 \text{ Hz}$$

Upper frequency limit

$$f_2 = f_r + \frac{R}{4\pi L} = 112.5 + \frac{50}{4\pi \times 0.2} = 112.5 + 19.89 = 132.39 \text{ Hz}$$

Bandwidth of the circuit

$$BW = f_2 - f_1 = 132.39 - 92.6 = 39.79 \text{ Hz}$$

Practice Problems

8.1 For the circuit shown in Fig. 8.62, determine the frequency at which the circuit resonates. Also find the voltage across the capacitor at resonance, and the Q factor of the circuit.

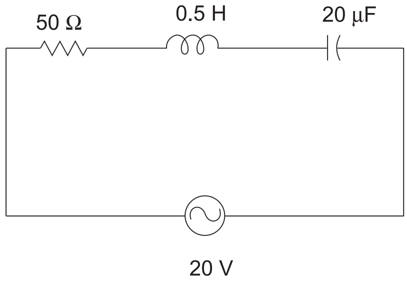


Fig. 8.62

8.2 A series RLC circuit has a quality factor of 10 at 200 rad/sec. The current flowing through the circuit at resonance is 0.5 A and the supply voltage is 10 V. The total impedance of the circuit is 40 Ω. Find the circuit constants.

8.3 The impedance $Z_1 = (5 + j3)\ \Omega$ and $Z_2 = (10 - j30)\ \Omega$ are connected in parallel as shown in Fig. 8.63. Find the value of X_3 which will produce resonance at the terminals a and b.

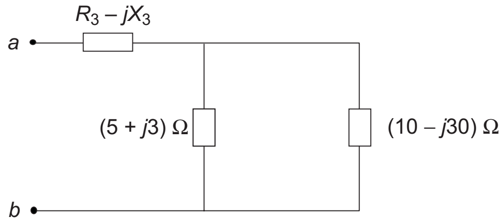


Fig. 8.63

8.4 A RLC series circuit is to be chosen to produce a magnification of 10 at 100 rad/sec. The source can supply a maximum current of 10 A and the supply voltage is 100 V. The power frequency impedance of the circuit should not be more than 14.14 Ω. Find the values of R, L and C.

8.5 A voltage $v(t) = 50 \sin \omega t$ is applied to a series RLC circuit. At the resonant frequency of the circuit, the maximum voltage across the capacitor is found to be 400 V. The bandwidth is known to be 500 rad/sec and the impedance at resonance is 100 Ω. Find the resonant frequency, and compute the upper and lower limits of the bandwidth. Determine the values of L and C of the circuit.

8.6 A current source is applied to the parallel arrangement of R, L and C where $R = 12\ \Omega$, $L = 2$ H and $C = 3\ \mu$F. Compute the resonant frequency in rad/sec. Find the quality factor. Calculate the value of bandwidth.Compute the lower and upper frequency of the bandwidth. Compute the voltage appearing across the parallel elements when the input signal is $i(t) = 10 \sin 1800\, t$.

8.7 For the circuit shown in Fig. 8.64, determine the value of R_C for which the given circuit resonates.

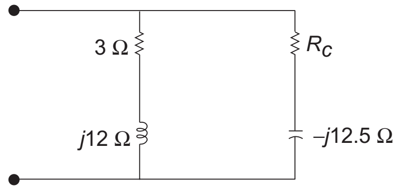


Fig. 8.64

8.8 For the circuit shown in Fig. 8.65, the applied voltage $v(t) = 15 \sin 1800t$. Determine the resonant frequency. Calculate the quality factor and bandwidth. Compute the lower and upper limits of the bandwidth.

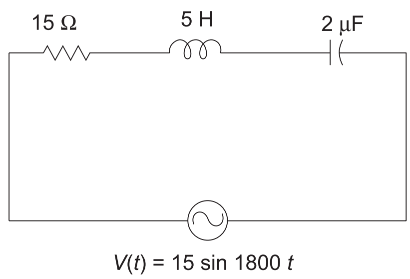


Fig. 8.65

8.9 In the circuit shown in Fig. 8.66, the current is at its maximum value with inductor value $L = 0.5$ H, and 0.707 times of its maximum value with $L = 0.2$ H. Find the value of Q at $\omega = 200$ rad/sec and circuit constants.

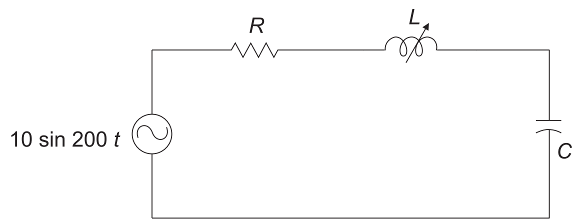


Fig. 8.66

8.10 The voltage applied to the series RLC circuit is 5 V. The Q of the coil is 25 and the value of the capacitor is 200 PF. The resonant frequency of the circuit is 200 kHz. Find the value of inductance, the circuit current and the voltage across the capacitor.

Objective-type Questions

1. What is the total reactance of a series RLC circuit at resonance?
   (a) equal to X_L (b) equal to X_C
   (c) equal to R (d) zero
2. What is the phase angle of a series RLC circuit at resonance?
   (a) zero (b) 90°
   (c) 45° (d) 30°
3. In a series circuit of $L = 15$ mH and $C = 0.015\ \mu$F and $R = 80\ \Omega$, what is the impedance at the resonant frequency?
   (a) (15 mH) ω (b) (0.015 F) ω
   (c) 80 Ω (d) $1/(\omega \times (0.015))$
4. In a series RLC circuit operating below the resonant frequency, the current
   (a) I leads V_S (b) I lags behind V_S
   (c) I is in phase with V_S
5. In a series RLC circuit, if C is increased, what happens to the resonant frequency?
   (a) It increases (b) It decreases
   (c) It remains the same (d) It is zero
6. In a certain series resonant circuit, $V_C = 150$ V, $V_L = 150$ V and $V_R = 50$ V. What is the value of the source voltage?
   (a) zero (b) 50 V
   (c) 150 V (d) 200 V
7. A certain series resonant circuit has a bandwidth of 1000 Hz. If the existing coil is replaced by a coil with a lower Q, what happens to the bandwidth?
   (a) It increases (b) It decreases
   (c) It is zero (d) It remains the same
8. In a parallel resonance circuit, why does the current lag behind the source voltage at frequencies below resonance?
   (a) because the circuit is predominantly resistive
   (b) because the circuit is predominantly inductive
   (c) because the circuit is predominantly capacitive
   (d) none of the above
9. In order to tune a parallel resonant circuit to a lower frequency, the capacitance must
   (a) be increased (b) be decreased
   (c) be zero (d) remain the same
10. What is the impedance of an ideal parallel resonant circuit without resistance in either branch?
    (a) zero (b) inductive
    (c) capacitive (d) infinite
11. If the lower cut-off frequency is 2400 Hz and the upper cut-off frequency

is 2800 Hz, what is the bandwidth?

(a) 400 Hz (b) 2800 Hz
(c) 2400 Hz (d) 5200 Hz

12. What values of L and C should be used in a tank circuit to obtain a resonant frequency of 8 kHz? The bandwidth must be 800 Hz. The winding resistance of the coil is 10 Ω.

(a) 2 mH, 1 μF (b) 10 H, 0.2 μF
(c) 1.99 mH, 0.2 μF (d) 1.99 mH, 10 μF

Chapter 9

Coupled Circuits

9.1 INTRODUCTION

Two circuits are said to be 'coupled' when energy transfer takes place from one circuit to the other when one of the circuits is energised. There are many types of couplings like conductive coupling as shown by the potential divider in Fig. 9.1(a), inductive or magnetic coupling as shown by a two winding transformer in Fig. 9.1(b) or conductive and inductive coupling as shown by an auto transformer in Fig. 9.1(c). A majority of the electrical circuits in practice are conductively or electromagnetically coupled. Certain coupled elements are frequently used in network analysis and synthesis. Transformer, transistor and electronic pots, etc. are some among these circuits. Each of these elements may be represented as a two port network as shown in Fig. 9.1(d).

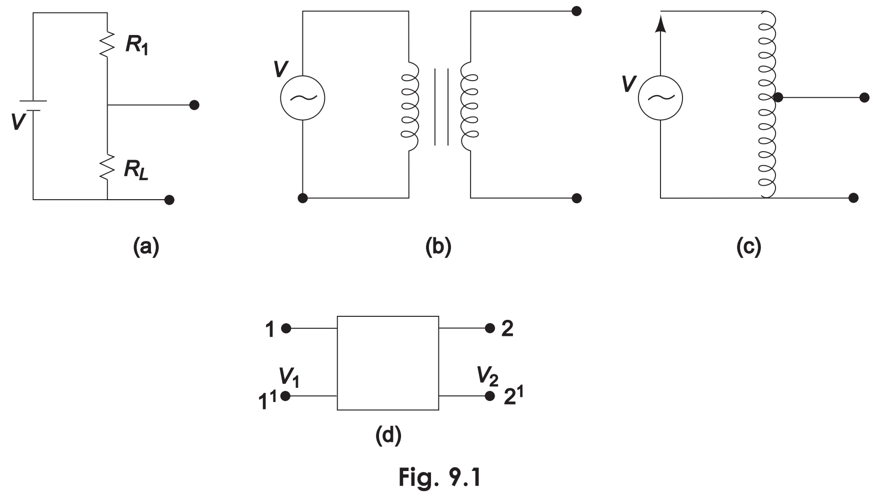


Fig. 9.1

9.2 CONDUCTIVITY COUPLED CIRCUIT AND MUTUAL IMPEDANCE

A conductively coupled circuit which does not involve magnetic coupling is shown in Fig. 9.2(a).

In the circuit shown the impedance Z_{12} or Z_{21} common to loop 1 and loop 2 is called *mutual impedance*. It may consists of a pure resistance, a pure inductance, a pure capacitance or a combination of any of these elements. Mesh analysis, nodal analysis or Kirchhoff's laws can be used to solve these type of circuits as described in Chapter 7.

The general definition of mutual impedance is explained with the help of Fig. 9.2 (b).

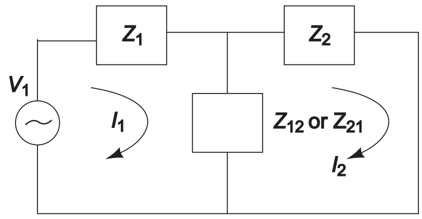


Fig. 9.2(a)

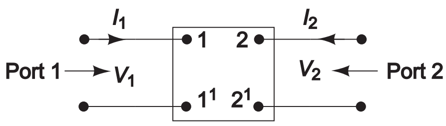


Fig. 9.2(b)

The network in the box may be of any configuration of circuit elements with two ports having two pairs of terminals 1-1' and 2-2' available for measurement. The mutual impedance between port 1 and 2 can be measured at 1-1' or 2-2'. If it is measured at 2-2'. It can be defined as the voltage developed (V_2) at 2–2' per unit current (I_1) at port 1-1'. If the box contains linear bilateral elements, then the mutual impedance measured at 2-2' is same as the impedance measured at 1-1' and is defined as the voltage developed (V_1) at 1-1' per unit current (I_2) at port 2-2′.

Example 9.1 Find the mutual impedance for the circuit shown in Fig. 9.3.

Solution Mutual impedance is given by

$$\frac{V_2}{I_1} \text{ or } \frac{V_1}{I_2}$$

$$V_2 = \frac{3}{2} I_1 \text{ or } \frac{V_2}{I_1} = 1.5\ \Omega$$

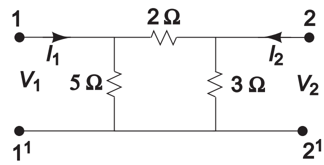


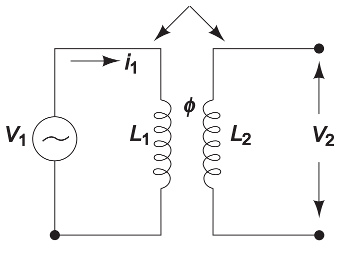
Fig. 9.3

or
$$V_1 = 5 \times I_2 \times \frac{3}{10} \text{ or } \frac{V_2}{I_2} = 1.5\ \Omega$$

9.3 MUTUAL INDUCTANCE

The property of inductance of a coil was introduced in Section 1.6. A voltage is induced in a coil when there is a time rate of change of current through it. The inductance parameter L, is defined in terms of the voltage across it and the time rate of change of current through it $v(t) = L\frac{di(t)}{dt}$, where $v(t)$ is the voltage across the coil, $I(t)$ is the current through the coil and L is the inductance of the coil. Strictly speaking, this definition is of self-inductance and this is considered as a circuit element with a pair of terminals. Whereas a circuit element "mutual inductor" does not exist. Mutual inductance is a property associated with two or more coils or inductors which are in close proximity and the presence of common magnetic flux which links the coils. A transformer is such a device whose operation is based on mutual inductance.

Let us consider two coils, L_1 and L_2 as shown in Fig. 9.4(a), which are sufficiently close together, so that the flux produced by i_1 in coil L_1 also link coil L_2. We assume that the coils do not move with respect to one another, and the medium in which the flux is established has a constant permeability. The two coils may be also arranged on a common magnetic core, as shown in Fig. 9.4(b). The two coils are said to be magnetically coupled, but act as a separate circuits. It is possible to relate the voltage induced in one coil to the time rate of change of current in the other coil. When a voltage v_1 is applied across L_1, a current i_1 will start flowing in this coil, and produce a flux ϕ. This flux also links coil L_2. If i_1 were to change with respect to time, the flux 'ϕ' would also change with respect to time. The time-varying flux surrounding the second coil, L_2 induces an emf, or voltage, across the terminals of L_2; this voltage is proportional to the time rate of change of current flowing through the first coil L_1. The two coils, or circuits, are said to be inductively coupled, because of this property they are called coupled elements or coupled circuits and the induced voltage, or emf is called the voltage/emf of mutual induction and is given by $v_2(t) = M_1\frac{di_1(t)}{dt}$ volts, where v_2 is the voltage induced in coil L_2 and M_1 is the coefficient of proportionality, and is called the coefficient of mutual inductance, or simple mutual inductance.



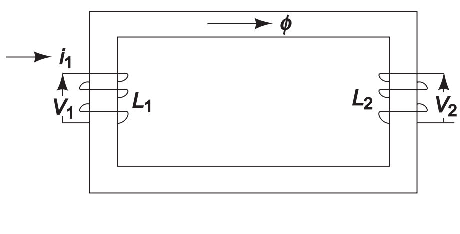


Fig. 9.4

If current i_2 is made to pass through coil L_2 as shown in Fig. 9.4(c) with coil L_1 open, a change of i_2 would cause a voltage v_1 in coil L_1, given by $v_1(t) = M_2 \dfrac{di_2(t)}{dt}$.

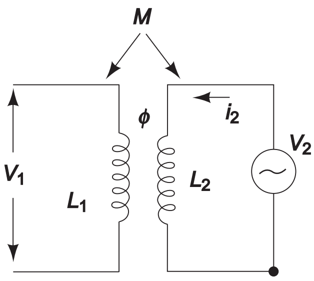


Fig. 9.4(c)

In the above equation, another coefficient of proportionality M_2 is involved. Though it appears that two mutual inductances are involved in determining the mutually induced voltages in the two coils, it can be shown from energy considerations that the two coefficients are equal and, therefore, need not be represented by two different letters. Thus $M_1 = M_2 = M$.

$$\therefore \qquad v_2(t) = M\frac{di_1(t)}{dt} \text{ Volts}$$

$$v_1(t) = M\frac{di_2(t)}{dt} \text{ Volts}$$

In general, in a pair of linear time invariant coupled coils or inductors, a non-zero current in each of the two coils produces a mutual voltage in each coil due to the flow of current in the other coil. This mutual voltage is present independently of, and in addition to, the voltage due to self induction. Mutual inductance is also measured in Henrys and is positive, but the mutually induced voltage, $M\dfrac{di}{dt}$ may be either positive or negative, depending on the physical construction of the coil and reference directions. To determine the polarity of the mutually induced voltage (i.e. the sign to be used for the mutual inductance), the dot convention is used.

9.4 DOT CONVENTION

Dot convention is used to establish the choice of correct sign for the mutually induced voltages in coupled circuits.

Circular dot marks and/or special symbols are placed at one end of each of two coils which are mutually coupled to simplify the diagrammatic representation of the windings around its core.

Let us consider Fig. 9.5, which shows a pair of linear, time invariant, coupled inductors with self inductances L_1 and L_2 and a mutual inductance M. If these inductions form a portion of a network, currents i_1 and i_2 are shown, each arbitrarily assumed entering at the dotted terminals, and voltages v_1 and v_2 are developed across the inductors. The voltage across L_1 is, thus composed of two parts and is given by

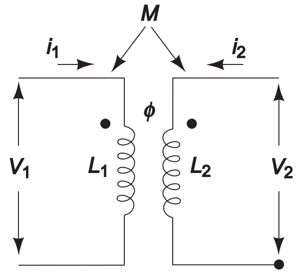


Fig. 9.5

$$v_1(t) = L_1 \frac{di_1(t)}{dt} \pm M \frac{di_2(t)}{dt}$$

The first term on the RHS of the above equation is the self induced voltage due to i_1, and the second term represents the mutually induced voltage due to i_2.

Similarly, $$v_2(t) = L_2 \frac{di_2(t)}{dt} \pm M \frac{di_1(t)}{dt}$$

Although the self-induced voltages are designated with positive sign, mutually induced voltages can be either positive or negative depending on the direction of the winding of the coil and can be decided by the presence of the *dots* placed at one end of each of the two coils. The convention is as follows.

If two terminals belonging to different coils in a coupled circuit are marked identically with dots then for the same direction of current relative to like terminals, the magnetic flux of self and mutual induction in each coil add together. The physical basis of the dot convention can be verified by examining Fig. 9.6. Two coils *ab* and *cd* are shown wound on a common iron core.

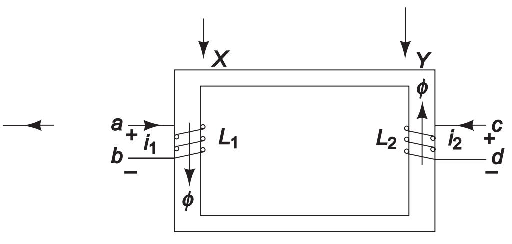


Fig. 9.6

It is evident from Fig.9.6 that the direction of the winding of the coil *ab* is clock-wise around the core as viewed at *X*, and that of *cd* is anti-clockwise as viewed at *Y*. Let the direction of current i_1 in the first coil be from *a* to *b*, and increasing with time. The flux produced by i_1 in the core has a direction which may be found by right hand rule, and which is downwards in the left limb of the core. The flux also increases with time in the direction shown at *X*. Now suppose that the current i_2 in the second coil is from *c* to *d*, and increasing with time. The application of the right hand rule indicates that the flux produced by i_2 in the core has an upward direction in the right limb of the core. The flux also increases with time in the direction shown at Y. The assumed currents i_1 and i_2 produce flux in the core that are additive. The terminals a and c of the two coils attain similar polarities simultaneously. The two simultaneously positive terminals are identified by two dots by the side of the terminals as shown in Fig. 9.7.

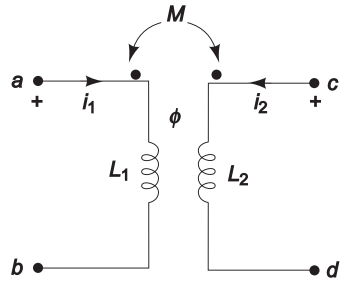


Fig. 9.7

The other possible location of the dots is the other ends of the coil to get additive fluxes in the core, i.e. at b and d. It can be concluded that the mutually induced voltage is positive when currents i_1 and i_2 both enter (or leave) the windings by the dotted terminals. If the current in one winding enters at the dot-marked terminals and the current in the other winding leaves at the dot-marked terminal, the voltages due to self and mutual induction in any coil have opposite signs.

Example 9.2 Using dot convention, write voltage equations for the coils shown in Fig. 9.8.

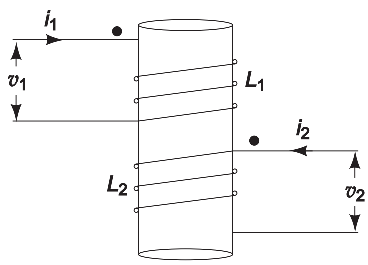


Fig. 9.8

Solution Since the currents are entering at the dot marked terminals the mutually induced voltages or the sign of the mutual inductance is positive; using the sign convention for the self inductance, the equations for the voltages are

$$v_1 = L_1 \frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} + M\frac{di_1}{dt}$$

Example 9.3 Write the equation for voltage v_0 for the circuits shown in Fig. 9.9.

Solution v_0 is assumed positive with respect to terminal C and the equation is given by

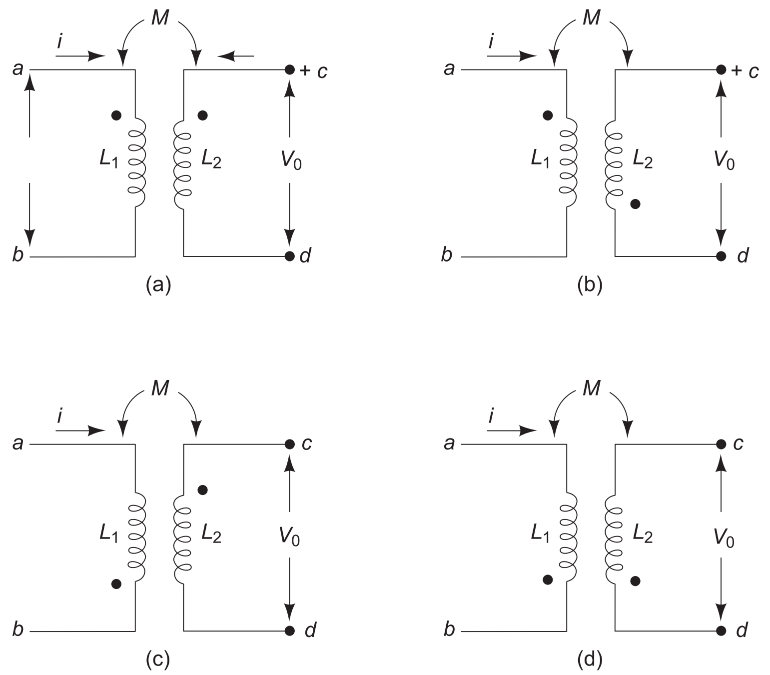


Fig. 9.9

(a) $v_0 = M\dfrac{di}{dt}$ (b) $v_0 = -M\dfrac{di}{dt}$

(c) $v_0 = -M\dfrac{di}{dt}$ (d) $v_0 = M\dfrac{di}{dt}$

9.5 COEFFICIENT OF COUPLING

The amount of coupling between the inductively coupled coils is expressed in terms of the coefficient of coupling, which is defined as $K = M/\sqrt{L_1 L_2}$

where M = mutual inductance between the coils
L_1 = self inductance of the first coil, and
L_2 = self inductance of the second coil

Coefficient of coupling is always less than unity, and has a maximum value of 1 (or 100%). This case, for which $K = 1$, is called perfect coupling, when the entire flux of one coil links the other. The greater the coefficient of coupling between the two coils, the greater the mutual inductance between them, and vice-versa. It can be expressed as the fraction of the magnetic flux produced by the current in one coil that links the other coil.

For a pair of mutually coupled circuits shown in Fig. 9.10, let us assume initially that i_1, i_2 are zero at $t = 0$.

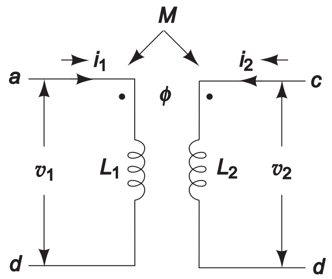


Fig. 9.10

then $$v_1(t) = L_1 \frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$$

and $$v_2(t) = L_2 \frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$$

Initial energy in the coupled circuit at $t = 0$ is also zero. The net energy input to the system shown in Fig. 9.10 at time t is given by

$$W(t) = \int_0^t [v_1(t)\, i_1(t) + v_2(t)\, i_2(t)]\, dt$$

Substituting the values of $v_1(t)$ and $v_2(t)$ in the above equation yields

$$W(t) = \int_0^t \left[L_1\, i_1(t)\frac{di_1(t)}{dt} + L_2\, i_2(t)\frac{di_2(t)}{dt} + M(i_1(t))\frac{di_2(t)}{dt} + i_2(t)\frac{di_1(t)}{dt}\right] dt$$

From which we get

$$W(t) = \frac{1}{2} L_1[i_1(t)]^2 + \frac{1}{2} L_2[i_2(t)]^2 + M[i_1(t)i_2(t)]$$

If one current enters a dot-marked terminal while the other leaves a dot marked terminal, the above equation becomes

$$W(t) = \frac{1}{2} L_1[i_1(t)]^2 + \frac{1}{2} L_2[i_2(t)]^2 - M[i_1(t)i_2(t)]$$

According to the definition of passivity, the net electrical energy input to the system is non-negative. $W(t)$ represents the energy stored within a passive network, it cannot be negative.

$\therefore$ $W(t) \geq 0$ for all values of i_1, i_2; L_1, L_2 or M

The statement can be proved in the following way. If i_1 and i_2 are both positive or negative, $W(t)$ is positive. The other condition where the energy equation could be negative is

$$W(t) = \frac{1}{2}\ L_1[i_1(t)]^2 + \frac{1}{2}\ L_2[i_2(t)]^2 - M[i_1(t)\ i_2(t)]$$

The above equation can be rearranged as

$$W(t) = \frac{1}{2}\left(\sqrt{L_1}\,i_1 - \frac{M}{\sqrt{L_1}}\,i_2\right)^2 + \frac{1}{2}\left(L_2 - \frac{M^2}{L_1}\right)i_2^2$$

The first term in the parenthesis of the right side of the above equation is positive for all values of i_1 and i_2, and, thus, the last term cannot be negative; hence

$$L_2 - \frac{M^2}{L_1} \geq 0$$

$$\frac{L_1 L_2 - M^2}{L_1} \geq 0$$

$$L_1 L_2 - M^2 \geq 0$$

$$\sqrt{L_1 L_2} \geq M$$

$$M \leq \sqrt{L_1 L_2}$$

Obviously the maximum value of the mutual inductance is $\sqrt{L_1 L_2}$. Thus, we define the coefficient of coupling for the coupled circuit as

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

The coefficient, K, is a non negative number and is independent of the reference directions of the currents in the coils. If the two coils are a great distance apart in space, the mutual inductance is very small, and K is also very small. For iron-core coupled circuits, the value of K may be as high as 0.99, for air-core coupled circuits, K varies between 0.4 to 0.8.

Example 9.4 Two inductively coupled coils have self inductances L_1 = 50 mH and L_2 = 200 mH. If the coefficient of coupling is 0.5 (i), find the value of mutual inductance between the coils, and (ii) what is the maximum possible mutual inductance?

Solution (i) $$M = K\sqrt{L_1 L_2}$$

$$= 0.5\ \sqrt{50 \times 10^{-3} \times 200 \times 10^{-3}} = 50 \times 10^{-3}\ \text{H}$$

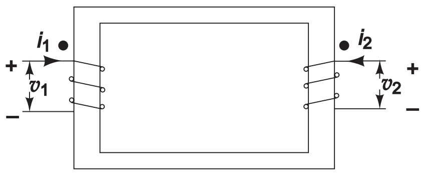
(ii) Maximum value of the inductance when $K = 1$,

$$M = \sqrt{L_1 L_2} = 100\ \text{mH}$$

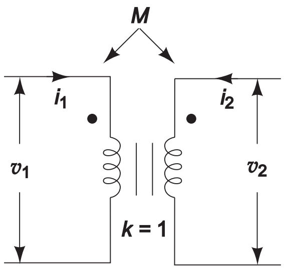
9.6 IDEAL TRANSFORMER

Transfer of energy from one circuit to another circuit through mutual induction is widely utilised in power systems. This purpose is served by transformers. Most often, they transform energy at one voltage (or current) into energy at some other voltage (or current).

A transformer is a static piece of apparatus, having two or more windings or coils arranged on a common magnetic core. The transformer winding to which the supply source is connected is called the *primary*, while the winding connected to load is called the *secondary*. Accordingly, the voltage across the primary is called the primary voltage, and that across the secondary, the secondary voltage. Correspondingly i_1 and i_2 are the currents in the primary and secondary windings. One such transformer is shown in Fig. 9.11(a). In circuit diagrams, ideal transformers are represented by Fig. 9.11(b). The vertical lines between the coils represent the iron core; the currents are assumed such that the mutual inductance is positive. An ideal transformer is characterised by assuming (i) zero power dissipation in the primary and secondary windings, i.e. resistances in the coils are assumed to be zero, (ii) the self inductances of the primary and secondary are extremely large in comparison with the load impedance, and (iii) the coefficient of coupling is equal to unity, i.e. the coils are tightly coupled without having any leakage flux. If the flux produced by the current flowing in a coil links all the turns, the self inductance of either the primary or secondary coil is proportional to the square of the number of turns of the coil. This can be verified from the following results.



(a)



(b) $N_1 : N_2$

Fig. 9.11

The magnitude of the self induced emf is given by

$$v = L\frac{di}{dt}$$

If the flux linkages of the coil with N turns and current are known, then the self induced emf can be expressed as

$$v = N\frac{d\phi}{dt}$$

$$L\frac{di}{dt} = N\frac{d\phi}{dt}$$

$$L = N\frac{d\phi}{di}$$

But $$\phi = \frac{Ni}{\text{reluctance}}$$

$\therefore$ $$L = N\frac{d}{di}\left(\frac{Ni}{\text{reluctance}}\right)$$

$$L = \frac{N^2}{\text{reluctance}}$$

$$L \alpha N^2$$

From the above relation it follows that

$$\frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = a^2$$

where $a = N_2/N_1$ is called the *turns ratio* of the transformer. The turns ratio, a, can also be expressed in terms of primary and secondary voltages. If the magnetic permeability of the core is infinitely large then the flux would be confined to the core. If ϕ is the flux through a single turn coil on the core and N_1, N_2 are the number of turns of the primary and secondary, respectively, then the total flux through windings 1 and 2, respectively, are

$$\phi_1 = N_1\,\phi;\ \phi_2 = N_2\,\phi$$

Also we have $$v_1 = \frac{d\phi_1}{dt},\ \text{and } v_2 = \frac{d\phi_2}{dt}$$

so that $$\frac{v_2}{v_1} = \frac{N_2\dfrac{d\phi}{dt}}{N_1\dfrac{d\phi}{dt}} = \frac{N_2}{N_1}$$

Figure 9.12 shows an ideal transformer to which the secondary is connected to a load impedance Z_L. The turns ratio $\dfrac{N_2}{N_1} = a$.

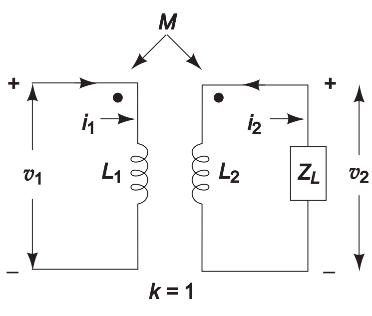


Fig. 9.12

The ideal transformer is a very useful model for circuit calculations, because with few additional elements like R, L and C, the actual behaviour of the physical transformer can be accurately represented. Let us analyse this transformer with sinusoidal excitations. When the excitations are sinusoidal voltages or currents, the steady state response will also be sinusoidal. We can use phasors for representing these voltages and currents. The input impedance of the transformer can be determined by writing mesh equations for the circuit shown in Fig. 9.12.

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 \tag{9.1}$$

$$0 = -j\omega M I_1 + (Z_L + j\omega L_2) I_2 \tag{9.2}$$

where V_1, V_2 are the voltage phasors, and I_1, I_2 are the current phasors in the two windings. $j\omega L_1$ and $j\omega L_2$ are the impedances of the self inductances and $j\omega M$ is the impedance of the mutual inductance, ω is the angular frequency.

from Eq. 9.2 $\qquad I_2 = \dfrac{j\omega M I_1}{(Z_L + j\omega L_2)}$

Substituting in Eq. 9.1, we have

$$V_1 = I_1 j\omega L_1 + \frac{I_1 \omega^2 M^2}{Z_L + j\omega L_2}$$

The input impedance $Z_{\text{in}} = \dfrac{V_1}{I_1}$

$$\therefore \qquad Z_{\text{in}} = j\omega L_1 + \frac{\omega^2 M^2}{(Z_L + j\omega L_2)}$$

When the coefficient of coupling is assumed to be equal to unity,

$$M = \sqrt{L_1 L_2}$$

$$\therefore \qquad Z_{\text{in}} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{(Z_L + j\omega L_2)}$$

We have already established that $\dfrac{L_2}{L_1} = a^2$

where a is the turns ratio N_2/N_1

$$\therefore \qquad Z_{\text{in}} = j\omega L_1 + \frac{\omega^2 L_1^2 a^2}{(Z_L + j\omega L_2)}$$

Further simplication leads to

$$Z_{\text{in}} = \frac{(Z_L + j\omega L_2)\, j\omega L_1 + \omega^2 L_1^2 a^2}{(Z_L + j\omega L_2)}$$

$$Z_{\text{in}} = \frac{j\omega L_1 Z_L}{(Z_L + j\omega L_2)}$$

As L_2 is assumed to be infinitely large compared to Z_L

$$Z_{\text{in}} = \frac{j\omega L_1 Z_L}{j\omega a^2 L_1} = \frac{Z_L}{a^2} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

The above result has an interesting interpretation, that is the ideal transformers change the impedance of a load, and can be used to match circuits with different impedances in order to achieve maximum power transfer. For example, the input impedance of a loudspeaker is usually very small, say 3 to 12 Ω, for connecting directly to an amplifier. The transformer with proper turns ratio can be placed between the output of the amplifier and the input of the loudspeaker to match the impedances as shown in Fig. 9.13.

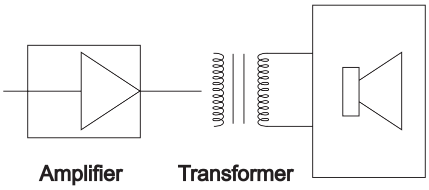


Fig. 9.13

Example 9.5 An ideal transformer has $N_1 = 10$ turns, and $N_2 = 100$ turns. What is the value of the impedance referred to as the primary, if a 1000 Ω resistor is placed across the secondary?

Solution The turns ratio $a = \dfrac{100}{10} = 10$

$$Z_{\text{in}} = \frac{Z_L}{a^2} = \frac{1000}{100} = 10\ \Omega$$

The primary and secondary currents can also be expressed in terms of turns ratio. From Eq. 9.2, we have

$$I_1\, j\omega M = I_2 (Z_L + j\omega L_2)$$

$$\frac{I_1}{I_2} = \frac{Z_L + j\omega L_2}{j\,\omega M}$$

When L_2 is very large compared to Z_L,

$$\frac{I_1}{I_2} = \frac{j\omega L_2}{j\,\omega M} = \frac{L_2}{M}$$

Substituting the value of $M = \sqrt{L_1 L_2}$ in the above equation $\dfrac{I_1}{I_2} = \dfrac{L_2}{M}$

$$\frac{I_1}{I_2} = \frac{L_2}{\sqrt{L_1 L_2}} = \sqrt{\frac{L_2}{L_1}}$$

$$\frac{I_1}{I_2} = \sqrt{\frac{L_2}{L_1}} = a = \frac{N_2}{N_1}$$

Example 9.6 An amplifier with an output impedance of 1936 Ω is to feed a loudspeaker with an impedance of 4 Ω.

(a) Calculate the desired turns ratio for an ideal transformer to connect the two systems.
(b) An rms current of 20 mA at 500 Hz is flowing in the primary. Calculate the rms value of current in the secondary at 500 Hz.
(c) What is the power delivered to the load?

Solution (a) To have maximum power transfer the output impedance of the

$$\text{amplifier} = \frac{\text{Load impedance}}{a^2}$$

$$\therefore \qquad 1936 = \frac{4}{a^2}$$

$$\therefore \qquad a = \sqrt{\frac{4}{1936}} = \frac{1}{22}$$

or
$$\frac{N_2}{N_1} = \frac{1}{22}$$

(b) $I_1 = 20$ mA

We have $\dfrac{I_1}{I_2} = a$

RMS value of the current in the secondary winding

$$= \frac{I_1}{a} = \frac{20 \times 10^{-3}}{1/22} = 0.44 \text{ A}$$

(c) The power delivered to the load (speaker)

$$= (0.44)^2 \times 4 = 0.774 \text{ W}$$

The impedance changing properties of an ideal transformer may be utilised to simplify circuits. Using this property, we can transfer all the parameters of the primary side of the transformer to the secondary side, and *vice-versa*. Consider the coupled circuit shown in Fig. 9.14(a).

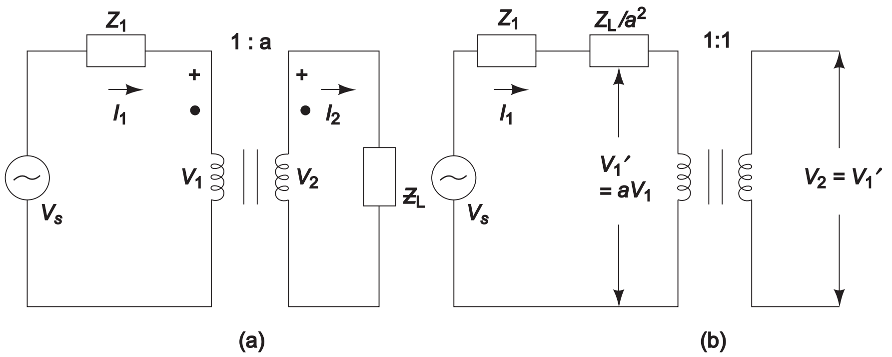


Fig. 9.14

To transfer the secondary side load and voltage to the primary side, the secondary voltage is to be divided by the ratio, a, and the load impedance is to be divided by a^2. The simplified equivalent circuits is shown in Fig. 9.14(b).

Example 9.7 For the circuit shown in Fig. 9.15 with turns ratio, $a = 5$, draw the equivalent circuit referring (a) to primary and (b) secondary. Take source resistance as 10 Ω.

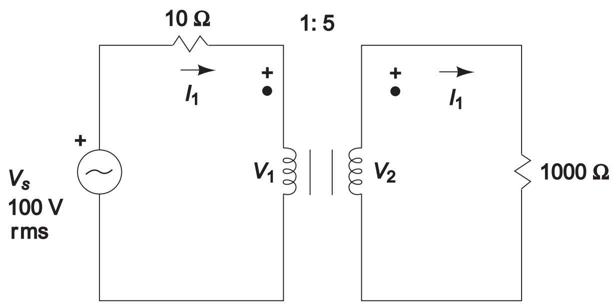


Fig. 9.15

Solution (a) Equivalent circuit referred to primary is as shown in Fig. 9.16(a).

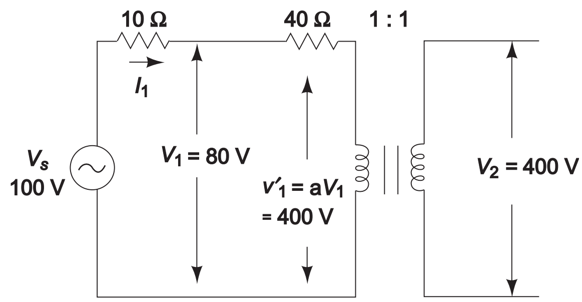


Fig. 9.16(a)

(b) Equivalent circuit referred to secondary is as shown in Fig. 9.16(b)

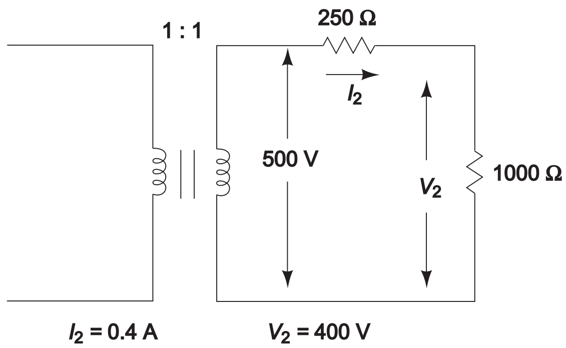


Fig. 9.16(b)

9.7 ANALYSIS OF MULTI-WINDING COUPLED CIRCUITS

Inductively coupled multi-mesh circuits can be analysed using Kirchhoff's laws and by loop current methods. Consider Fig. 9.17, where three coils are inductively coupled. For such a system of inductors we can define a inductance matrix L as

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

where L_{11}, L_{22} and L_{33} are self inductances of the coupled circuits, and $L_{12} = L_{21}$; $L_{23} = L_{32}$ and $L_{13} = L_{31}$ are mutual inductances. More precisely, L_{12} is the mutual inductance between coils 1 and 2, L_{13} is the mutual inductance between coils 1 and 3, and L_{23} is the mutual inductance between coils 2 and 3. The inductance matrix has its order equal to the number of inductors and is symmetric. In terms of voltages across the coils, we have a voltage vector related to i by

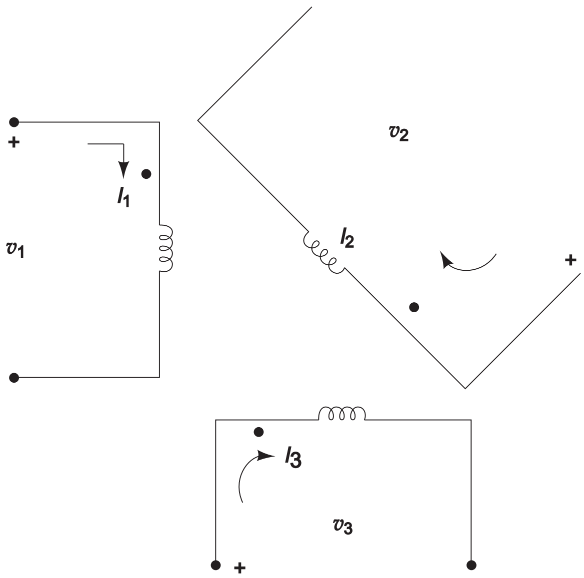


Fig. 9.17

$$[v] = [L]\left[\frac{di}{dt}\right]$$

where v and i are the vectors of the branch voltages and currents, respectively. Thus, the branch volt-ampere relationships of the three inductors are given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix}$$

Using KVL and KCL, the effective inductances can be calculated. The polarity for the inductances can be determined by using passivity criteria, whereas the signs of the mutual inductances can be determined by using the dot convention.

Example 9.8 For the circuit shown in Fig. 9.18, write the inductance matrix.

Solution Let L_1, L_2 and L_3 be the self inductances, and $L_{12} = L_{21}$, $L_{23} = L_{32}$ and $L_{13} = L_{31}$ be the mutual inductances between coils, 1, 2, 2, 3 and 1, 3, respectively.

$L_{12} = L_{21}$ is positive, as both the currents are entering at dot marked terminals, whereas $L_{13} = L_{31}$ and $L_{23} = L_{32}$ are negative.

$\therefore$ The inductance matrix is $L = \begin{bmatrix} L_1 & L_{12} & -L_{13} \\ L_{21} & L_2 & -L_{23} \\ -L_{31} & -L_{32} & L_3 \end{bmatrix}$

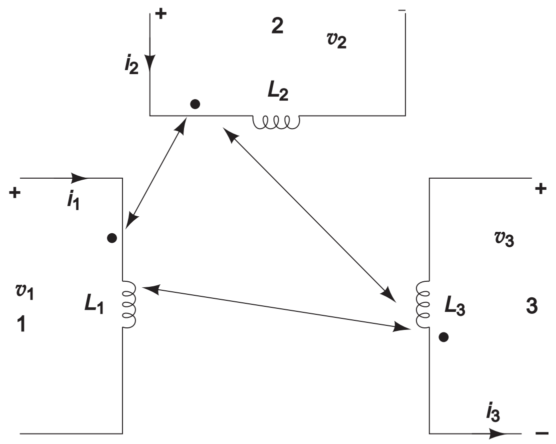


Fig. 9.18

9.8 SERIES CONNECTION OF COUPLED INDUCTORS

Let there be two inductors connected in series, with self inductances L_1 and L_2 and mutual inductance of M. Two kinds of series connections are possible; series aiding as in Fig. 9.19(a), and series opposition as in Fig. 9.19(b).

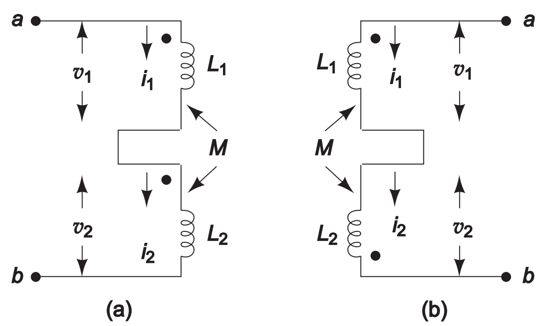


Fig. 9.19

In the case of series aiding connection, the currents in both inductors at any instant of time are in the same direction relative to like terminals as shown in

Fig. 9.19(a). For this reason, the magnetic fluxes of self induction and of mutual induction linking with each element add together.

In the case of series opposition connection, the currents in the two inductors at any instant of time are in opposite direction relative to like terminals as shown in Fig. 9.19(b). The inductance of an element is given by $L = \phi/i$, where ϕ is the flux produced by the inductor.

$\therefore$ $$\phi = Li$$

For the series aiding circuit, if ϕ_1 and ϕ_2 are the flux produced by the coils 1 and 2, respectively, then the total flux

$$\phi = \phi_1 + \phi_2$$

where $$\phi_1 = L_1 i_1 + M i_2$$

$$\phi_2 = L_2 i_2 + M i_1$$

$\therefore$ $$\phi = Li = L_1 i_1 + M i_2 + L_2 i_2 + M i_1$$

Since $$i_1 = i_2 = i$$

$$L = L_1 + L_2 + 2M$$

Similarly, for the series opposition

$$\phi = \phi_1 + \phi_2$$

where $$\phi_1 = L_1 i_1 - M i_2$$

$$\phi_2 = L_2 i_2 - M i_1$$

$$\phi = Li = L_1 i_1 - M i_2 + L_2 i_2 - M i_1$$

Since $$i_1 = i_2 = i$$

$$L = L_1 + L_2 - 2M$$

In general, the inductance of two inductively coupled elements in series is given by $L = L_1 + L_2 \pm 2M$.

Positive sign is applied to the series aiding connection, and negative sign to the series opposition connection.

Example 9.9 Two coils connected in series have an equivalent inductance of 0.4 H when connected in aiding, and an equivalent inductance 0.2 H when the connection is opposing. Calculate the mutual inductance of the coils.

Solution When the coils are arranged in aiding connection, the inductance of the combination is $L_1 + L_2 + 2M = 0.4$; and for opposing connection, it is $L_1 + L_2 - 2M = 0.2$. Solving the two equations, we get

$$4M = 0.2 \text{ H}$$

$$M = 0.05 \text{ H}$$

9.9 PARALLEL CONNECTION OF COUPLED COILS

Consider two inductors with self inductances L_1 and L_2 connected parallel which are mutually coupled with mutual inductance M as shown in Fig. 9.20.

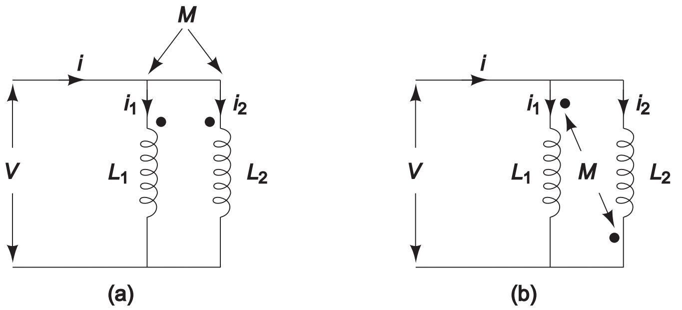


Fig. 9.20

Let us consider Fig. 9.20(a) where the self induced emf in each coil assists the mutually induced emf as shown by the dot convention.

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} \tag{9.3}$$

The voltage across the parallel branch is given by

$$v = L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} \text{ or } L_2 \frac{di_2}{dt} + M\frac{di_1}{dt}$$

also
$$L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = L_2\frac{di_2}{dt} + M\frac{di_1}{dt}$$

$$\frac{di_1}{dt}(L_1 - M) = \frac{di_2}{dt}(L_2 - M)$$

$\therefore$
$$\frac{di_1}{dt} = \frac{di_2}{dt}\frac{(L_2 - M)}{(L_1 - M)} \tag{9.4}$$

Substituting Eq. 9.4 in Eq. 9.3, we get

$$\frac{di}{dt} = \frac{di_2}{dt}\frac{(L_2 - M)}{(L_1 - M)} + \frac{di_2}{dt} = \frac{di_2}{dt}\left[\frac{(L_2 - M)}{L_1 - M} + 1\right] \tag{9.5}$$

If L_{eq} is the equivalent inductance of the parallel circuit in Fig. 9.20 (a) then v is given by

$$v = L_{eq}\frac{di}{dt}$$

$$L_{eq}\frac{di}{dt} = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L_{eq}}\left[L_1\frac{di_1}{dt} + M\frac{di_2}{dt}\right]$$

Substituting Eq. 9.4 in the above equation we get

$$\frac{di}{dt} = \frac{1}{L_{\text{eq}}}\left[L_1 \frac{di_2\,(L_2 - M)}{dt(L_2 - M)} + M\frac{di_2}{dt}\right]$$

$$= \frac{1}{L_{\text{eq}}}\left[L_1 \frac{(L_2 - M)}{(L_1 - M)} + M\right]\frac{di_2}{dt} \qquad (9.6)$$

Equating Eq. 9.6 and Eq. 9.5, we get

$$\frac{L_2 - M}{L_2 - M} + 1 = \frac{1}{L_{\text{eq}}}\left[L_1\left(\frac{L_2 - M}{L_1 - M}\right) + M\right]$$

Rearranging and simplifying the above equation results in

$$L_{\text{eq}} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

If the voltage induced due to mutual inductance oppose the self induced emf in each coil as shown by the dot convention in Fig. 9.20(b), the equivalent inductance is given by

$$L_{\text{eq}} = \frac{L_1L_2 - M^2}{L_1 + L_2 + 2M}$$

9.10 TUNED CIRCUITS

Tuned circuits are, in general, single tuned and double tuned. Double tuned circuits are used in radio receivers to produce uniform response to modulated signals over a specified bandwidth; double tuned circuits are very useful in communication system.

9.10.1 Single Tuned Circuit

Consider the circuit in Fig. 9.21. A tank circuit (i.e. a parallel resonant circuit) on the secondary side is inductively coupled to coil (1) which is excited by a source, v_i. Let R_s be the source resistance and R_1, R_2 be the resistances of coils, 1 and 2, respectively. Also let L_1, L_2 be the self inductances of the coils, 1 and 2, respectively.

Let $\qquad R_s + R_1 + j\omega L_1 = R_s$

with the assumption that $R_s \gg R_1 \gg j\omega L_1$
The mesh equations for the circuit shown in Fig. 9.21 are

$$i_1R_s - j\omega Mi_2 = v_i$$

$$-j\omega Mi_1 + \left(R_2 + j\omega L_2 - \frac{j}{\omega C}\right)i_2 = 0$$

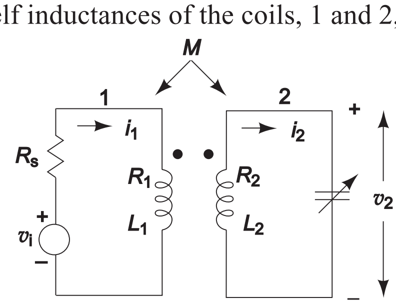


Fig. 9.21

$$i_2 = \begin{vmatrix} R_s & v_i \\ -j\omega M & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} R_s & (-j\omega M) \\ (-j\omega M) & \left(R_2 + j\omega L_2 - \dfrac{j}{\omega C}\right) \end{vmatrix}$$

or $$i_2 = \frac{j v_i \omega M}{R_s\left(R_2 + j\omega L_2 - \dfrac{j}{\omega C}\right) + \omega^2 M^2}$$

The output voltage $v_o = i_2 \cdot \dfrac{1}{j\omega C}$

$$v_o = \frac{j v_i \omega M}{j\omega C\left\{R_s\left[R_2 + \left(j\omega L_2 - \dfrac{1}{\omega C}\right)\right] + \omega^2\ M^2\right\}}$$

The voltage transfer function, or voltage amplification, is given by

$$\frac{v_o}{v_i} = A = \frac{M}{C\left\{R_s\left[R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C}\right)\right] + \omega^2\ M^2\right\}}$$

When the secondary side is tuned, i.e. when the value of the frequency ω is such that $\omega L_2 = 1/\omega C$, or at resonance frequency ω_r, the amplification is given by

$$A = \frac{v_o}{v_i} = \frac{M}{C\left[R_s R_2 + \omega_r^2\ M^2\right]}$$

the current i_2 at resonance $i_2 = \dfrac{j v_i\, \omega_r\, M}{R_s R_2 + \omega_r^2\ M^2}$

Thus, it can be observed that the output voltage, current and amplification depends on the mutual inductance M at resonance frequency, when $M = K\sqrt{L_1 L_2}$. The maximum output voltage or the maximum amplification depends on M. To get the condition for maximum output voltage, make $dv_o/dM = 0$.

$$\frac{dv_o}{dM} = \frac{d}{dM}\left[\frac{v_i M}{C\left[R_s R_2 + \omega_r^2 M^2\right]}\right]$$
$$= 1 - 2M^2\ \omega_r^2\,[R_s R_2 + \omega_r^2 M^2]^{-1} = 0$$

From which, $$R_s R_2 = \omega_r^2 M^2$$

or $$M = \sqrt{\frac{R_s R_2}{\omega_r}}$$

From the above value of M, we can calculate the maximum output voltage. Thus

$$v_{oM} = \frac{v_i}{2\omega_r C\sqrt{R_s R_2}},$$

or the maximum amplification is given by

$$A_m = \frac{1}{2\omega_r C\sqrt{R_s R_2}} \text{ and } i_2 = \frac{j v_i}{2\sqrt{R_s R_2}}$$

The variation of the amplification factor or output voltage with the coefficient of coupling is shown in Fig. 9.22.

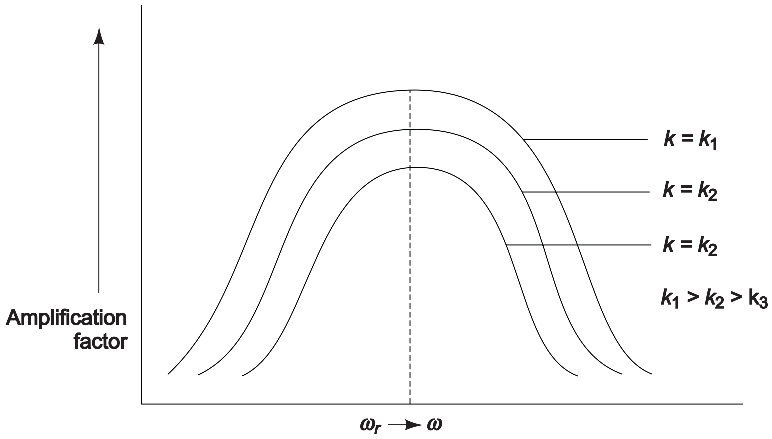


Fig. 9.22

Example 9.10 Consider the single tuned circuit shown in Fig. 9.23 and determine (i) the resonant frequency (ii) the output voltage at resonance (iii) and the maximum output voltage. Assume $R_s \gg \omega_r L_1$, and $K = 0.9$.

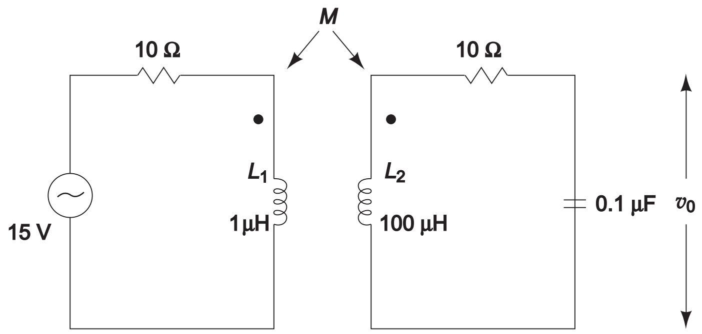


Fig. 9.23

Solution

$$M = K\sqrt{L_1 L_2}$$

$$= 0{\cdot}9\sqrt{1\times 10^{-6}\times 100\times 10^{-6}}$$

$$= 9\ \mu\text{H}$$

(i) Resonance frequency

$$\omega_r = \frac{1}{\sqrt{L_2 C}} = \frac{1}{\sqrt{100\times 10^{-6}\times 0.1\times 10^{-6}}}$$

$$= \frac{10^6}{\sqrt{10}} \text{ rad/sec.}$$

or $$f_r = 50.292 \text{ kHz}$$

The value of $$\omega_r L_1 = \frac{10^6}{\sqrt{10}} 1 \times 10^{-6} = 0.316$$

Thus the assumption that $\omega_r L_1 \ll R_1$ is justified.

(ii) Output voltage

$$v_o = \frac{M v_i}{C\left[R_s R_2 + \omega_r^2 M\right]}$$

$$= \frac{9 \times 10^{-6} \times 15}{0.1 \times 10^{-6}\left[10 \times 10 + \left(\frac{10^6}{\sqrt{10}}\right)^2 \times 9 \times 10^{-6}\right]}$$

$$= 1.5 \text{ mV}$$

(iii) Maximum value of output voltage

$$v_{oM} = \frac{v_i}{2\omega_r C\sqrt{R_s R_2}}$$

$$= \frac{15}{2 \times \frac{10^6}{\sqrt{10}} \times 0.1 \times 10^{-6}\sqrt{100}}$$

$$v_{oM} = 23.7 \text{ V}$$

9.10.2 DOUBLE TUNED COUPLED CIRCUITS

Figure 9.24 shows a double tuned transformer circuit involving two series resonant circuits.

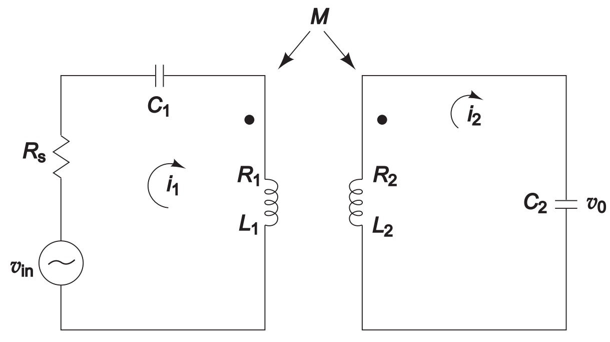


Fig. 9.24

For the circuit shown in the figure, a special case where the primary and secondary resonate at the same frequency ω_r, is considered here, i.e $$\omega_r^2 = \frac{1}{L_1C_1} = \frac{1}{L_2C_2}$$

The two mesh equations for the circuit are

$$v_{\text{in}} = i_1\left(R_s + R_1 + j\omega L_1 - \frac{j}{\omega C_1}\right) - i_2 j\omega M$$

$$0 = -j\omega M i_1 + i_2\left(R_2 + j\omega L_2 - \frac{j}{\omega C_2}\right)$$

From which

$$i_2 = \frac{v_{\text{in}} j\omega M}{\left[(R_s + R_1) + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2}$$

also $\omega_r = \dfrac{1}{\sqrt{L_1C_1}} = \dfrac{1}{\sqrt{L_2\,C_2}}$ at resonance

$$v_o = \frac{v_{\text{in}} M}{C_2\left[(R_s + R_1)R_2 + \omega_r^2 M^2\right]}$$

or $$v_o = A v_{\text{in}}$$

where A is the amplification factor given by

$$A = \frac{M}{C_2\left[(R_1 + R_s)R_2 + \omega_r^2 M^2\right]}$$

The maximum amplification or the maximum output voltage can be obtained by taking the first derivative of v_o with respect to M, and equating it to zero.

$$\therefore \quad \frac{dv_o}{dM} = 0, \text{ or } \frac{dA}{dM} = 0$$

$$\frac{dA}{dM} = (R_1 + R_s)R_2 + \omega_r^2 M^2 - 2M^2\omega_r^2 = 0$$

$$\omega_r^2 M^2 = R_2(R_1 + R_s)$$

$$M_c = \frac{\sqrt{R_2(R_1 + R_s)}}{\omega_r}$$

where M_c is the critical value of mutual inductance. Substituting the value of M_c in the equation of v_o, we obtain the maximum output voltage as

$$|v_o| = \frac{v_{\text{in}}}{2\omega_r^2 C_2 M_c} = \frac{v_{\text{in}}}{2\omega_r C_2\sqrt{R_2(R_1 + R_s)}}$$

and $$|i_2| = \frac{v_{\text{in}}}{2\omega_r M_c} = \frac{v_{\text{in}}}{2\sqrt{R_2(R_1 + R_s)}}$$

By definition, $M = K\sqrt{L_1 L_2}$, the coefficient of coupling, K at $M = M_c$ is called the critical coefficient of coupling, and is given by $K_c = M_c/\sqrt{L_2 L_1}$.

The critical coupling causes the secondary current to have the maximum possible value. At resonance, the maximum value of amplification is obtained by changing M, or by changing the coupling coefficient for a given value of L_1 and L_2. The variation of output voltage with frequency for different coupling coefficients is shown in Fig. 9.25.

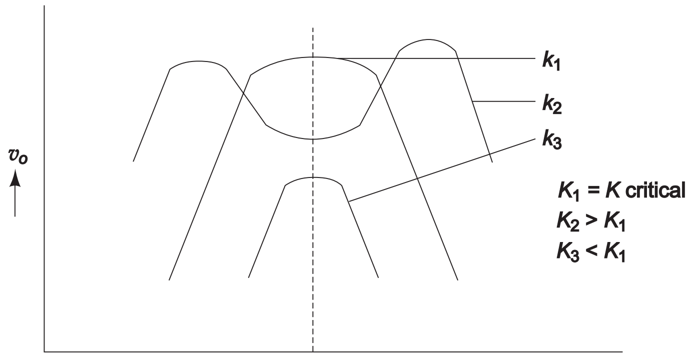


Fig. 9.25

9.11 ANALYSIS OF MAGNETIC CIRCUITS

The presence of charges in space or in a medium creates an electric field, similarly the flow of current in a conductor sets up a magnetic field. Electric field is represented by electric flux lines, magnetic flux lines are used to describe the magnetic field. The path of the magnetic flux lines is called the magnetic circuit. Just as a flow of current in the electric circuit requires the presence of an electromotive force, so the production of magnetic flux requires the presence of magneto-motive force (mmf). We now discuss some properties related to magnetic flux.

(i) Flux density (B)

The magnetic flux lines start and end in such a way that they form closed loops. Weber (Wb) is the unit of magnetic flux (ϕ). Flux density (B) is the flux per unit area. Tesla (T) or Wb/m^2 is the unit of flux density.

$$B = \frac{\Phi}{A} \text{ Wb/m}^2 \text{ or Tesla}$$

where B is a quantity called magnetic flux density in teslas, ϕ is the total flux in webers and A is the area perpendicular to the lines in m^2.

(ii) Magneto-motive force MMF ($\Im$)

A measure of the ability of a coil to produce a flux is called the *magneto-motive force*. It may be considered as a magnetic pressure, just as emf is considered as an electric pressure. A coil with N turns which is carrying a current of I amperes constitutes a magnetic circuit and produces an mmf of NI ampere turns. The source of flux (ϕ) in the magnetic circuit is the mmf. The flux produced in the circuit depends on mmf and the length of the circuit.

(iii) Magnetic field strength (H)

The magnetic field strength of a circuit is given by the mmf per unit length.

$$H = \frac{\Im}{l} = \frac{NI}{l} \text{ AT/m}$$

The magnetic flux density (B) and its intensity (field strength) in a medium can be related by the following equation

$$B = \mu H$$

where $\mu = \mu_0 \mu_r$ is the permeability of the medium in Henrys/metre (H/m),

μ_0 = absolute permeability of free space and is equal to $4\pi \times 10^{-7}$ H/m

and μ_r = relative permeability of the medium.

Relative permeability is a non-dimensional numeric which indicates the degree to which the medium is a better conductor of magnetic flux as compared to free space. The value of $\mu_r = 1$ for air and non-magnetic materials. It varies from 1,000 to 10,000 for some types of ferro-magnetic materials.

(iv) Reluctance ($\Re$)

It is the property of the medium which opposes the passage of magnetic flux. The magnetic reluctance is analogous to resistance in the electric circuit. Its unit is AT/Wb. Air has a much higher reluctance than does iron or steel. For this reason, magnetic circuits used in electrical machines are designed with very small air gaps.

According to definition, reluctance = $\dfrac{\text{mmf}}{\text{flux}}$

The reciprocal of reluctance is known as permeance $\dfrac{1}{\Re} = \dfrac{\phi}{\Im}$

Thus reluctance is a measure of the opposition offered by a magnetic circuit to the setting up of the flux. The reluctance of the magnetic circuit is given by $\Re = \dfrac{1}{\mu}\dfrac{l}{a}$.

where I is the length, a is the cross-sectional area of the magnetic circuit and μ is the permeability of the medium.

From the above equations

$$\frac{1}{\mu} \cdot \frac{l}{a} = \frac{\Im}{\phi}$$

or

$$\frac{\Im}{1} = \frac{1}{\mu} \cdot \frac{\phi}{a}$$

$$\frac{NI}{l} = \frac{1}{\mu} \cdot B$$

$$H = \frac{1}{\mu} \cdot B$$

or $$B = \mu \mathrm{H}$$

9.12 SERIES MAGNETIC CIRCUIT

A series magnetic circuit is analogous to a series electric circuit. Kirchhoff's laws are applicable to magnetic circuits also. Consider a ring specimen having a magnetic path of l meters, area of cross-section $(A)m^2$ with a mean radius of R meters having a coil of N turns carrying I amperes wound uniformly as shown in Fig. 9.26. MMF is responsible for the establishment of flux in the magnetic medium. This mmf acts along the magnetic lines of force. The flux produced by the circuit is given by

$$\phi = \frac{\text{MMF}}{\text{Reluctance}}$$

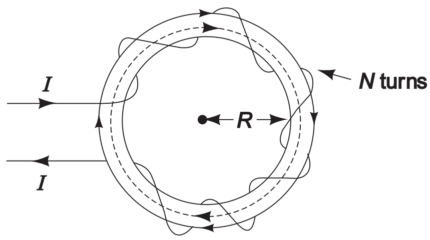


Fig. 9.26

The magnetic field intensity of the ring is given by $H = \frac{\text{mmf}}{l} = \frac{NI}{l} = \text{AT/m}$

Where l is the mean length of the magnetic path and is given by $2\pi R$.

Flux density $B = \mu_o \mu_r H = \mu_o \mu_r \frac{NI}{l}$ Wb/m^2

$$\text{Flux } \phi = \mu \text{HA Webers}$$

$$= \mu_0 \mu_r \frac{NI}{l} \times A \text{ Wb}$$

$$\phi = \frac{NI}{l/\mu_0 \mu_r A} \text{ Wb}$$

NI is the mmf of the magnetic circuit, which is analogous to emf in electric circuit. $l/\mu_0 \mu_r A$ is the reluctance of the magnetic circuit which is analogous to resistance in electric circuit.

9.13 COMPARISON OF ELECTRIC AND MAGNETIC CIRCUITS

A series electric and magnetic circuits are shown in Figs. 9.27(a) and (b) respectively.

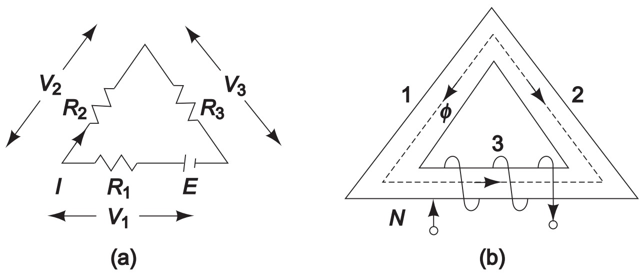


Fig. 9.27

Figure 9.27(a) represents an electric circuit with three resistances connected in series, the dc source E drives the current I through all the three resistances whose voltage drops are V_1, V_2 and V_3. Hence, $E = V_1 + V_2 + V_3$, also $E = I(R_1 + R_2 + R_3)$. We also know that $R = \dfrac{\rho l}{a}$, where ρ is the specific resistance of the material, l is the length of the wire of the resistive material and a is the area of cross-section of the wire.

The drop across each resistor $V = RI = \rho l \dfrac{I}{a}$

or $$\frac{V}{l} = \rho \frac{I}{a}$$

Voltage drop per unit length = specific resistance × current density.

Let us consider the magnetic circuit in Fig. 9.27(b). The MMF of the circuit is given by $\Im = NI$, drives the flux ϕ around the three parts of the circuit which are in series. Each part has a reluctance $\Re = \dfrac{1}{\mu} \cdot \dfrac{l}{a}$, where l is the length and a is the area of cross-section of each arm. The mmf of the magnetic circuit is given by $\Im = \Im_1 + \Im_2 + \Im_3$. $\Im = \phi\,(\Re_1 + \Re_2 + \Re_3)$ where $\Re_1$ $\Re_2$ and $\Re_3$ are the reluctances of the portion 1, 2 and 3 respectively.

Also $$\Im = \frac{1}{\mu} \cdot \frac{l}{a} \cdot \phi$$

$$\frac{\Im}{l} = \frac{1}{\mu} \cdot \frac{\phi}{a}$$

$$H = \frac{1}{\mu} \cdot B.$$

$\frac{1}{\mu}$ can be termed as *reluctance* of a cubic metre of magnetic material from which, the above equation gives the mmf per unit length (intensity) which is analogous to the voltage per unit length. Parallels between electric-circuit and magnetic-circuit quantities are shown in Table 9.1.

Thus, it is seen that the magnetic reluctance is analogous to resistance, mmf is analogous to emf and flux is analogous to current. These analogies are useful in magnetic circuit calculations. Though we can draw many parallels between the two circuits, the following differences do exists.

The electric current is a true flow but there is no flow in a magnetic flux. For a given temperature, ρ is independent of the strength of the current, but μ is not independent of the flux.

In an electric circuit energy is expended so long as the current flows, but in a magnetic circuit energy is expended only in creating the flux, and not in maintaining it. Parallels between the quantities are shown in Table 9.1.

Table 9.1 Analogy between magnetic and electric circuit

Electric circuit	*Magnetic circuit*
Exciting force = emf in volts	mmf in AT
Response = current in amps	flux in webers
Voltage drop = VI volts	mmf drop = $\Re\phi$ AT
Electric field density = $\frac{V}{l}$ volt/m	Magnetic field Intensity = $\frac{\Im}{1}$ AT/m
Current(I) = $\frac{E}{R}$ A	Flux (ϕ) = $\frac{\Im}{R}$ Web
Current density(J) = $\frac{I}{a}$ Amp/m^2	Flux density (B) = $\frac{\phi}{A}$ Web/m^2
Resistance (R) = $\frac{\rho_l}{a}$ ohm	Reluctance ($\Re$) = $\frac{1}{\mu}\cdot\frac{l}{a}$ AT/Web
Conductance (G) = $\frac{1}{R}$ Mho	Permeance = $\frac{1}{\Re} = \frac{\mu a}{\mu}\cdot\frac{l}{a}$ Web/AT

9.14 MAGNETIC LEAKAGE AND FRINGING

Figure 9.28 shows a magnetised iron ring with a narrow air gap, and the flux which crosses the gap can be regarded as useful flux. Some of the total flux produced by the ring does not cross the air gap, but instead takes a shorter route as shown in Fig. 9.28 and is known as *leakage flux*. The flux while crossing the air gap bulges outwards due to variation in reluctance. This is known as *fringing*. This is because the lines of force repel each other when passing through the air as a result the flux density in the air gap decreases. For the purpose of calculation it

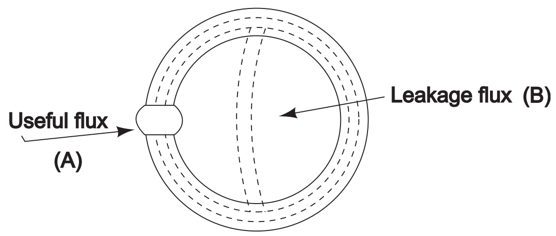


Fig. 9.28

is assumed that the iron carries the whole of the total flux throughout its length. The ratio of total flux to useful flux is called the *leakage coefficient* or leakage factor.

Leakage factor = Total flux/useful flux.

Example 9.11 A coil of 100 turns is wound uniformly over a insulator ring with a mean circumference of 2 m and a uniform sectional area of 0.025 cm^2. If the coil is carrying a current of 2 A. Calculate (a) the mmf of the circuit, (b) magnetic field intensity (c) flux density (d) the total flux.

Solution

(a) mmf = NI = 100 × 2 = 2000 AT

(b) $H = \dfrac{\text{mmf}}{l} = \dfrac{2000}{2} = 1000$ AT/m

(c) $B = \mu_0 H = 4\pi \times 10^{-7} \times 1000 = 1.2565$ mWb/m^2.

(d) $\phi = B \times A = 1.2565 \times 10^{-3} \times 0.025 \times 10^{-4} = 0.00314 \times 10^{-6}$ Wb

Example 9.12 Calculate the mmf required to produce a flux of 5 mWb across an air gap of 2.5 mm of length having an effective area of 100 cm^2 of a cast steel ring of mean iron path of 0.5 m and cross-sectional area of 150 cm^2 as shown in Fig. 9.29. The relative permeability of the cast steel is 800. Neglect leakage flux.

Solution

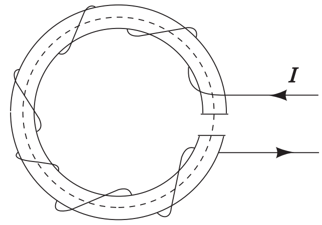


Fig. 9.29

Area of the gap = 100×10^{-4} m^2

Flux density of the gap = $\dfrac{5 \times 10^{-3} \times 10^4}{100} = 0.5$ T

H of the gap = $\dfrac{B}{\mu_0} = \dfrac{0.5}{4\pi \times 10^{-7}}$

$= 0.39 \times 10^6$ A/m

Length of the gap = 2.5×10^{-3} m

mmf required for the gap = $0.39 \times 10^6 \times 2.5 \times 10^{-3} = 975$ AT

Flux density in the cast steel ring is = $\dfrac{\phi}{\text{Area}}$

$$= \frac{5 \times 10^{-3} \times 10^4}{100}$$

$$= 0.333 \text{ T}$$

$$\therefore \qquad H = \frac{B}{\mu_0 \mu_r} = \frac{0.333}{4\pi\, 10^{-7} \times 800} = 332 \text{ A T/m}$$

Length of the cast steel path = 0.5 m

The required mmf for the cast steel to produce the necessary flux = 0.5 × 332 = 166 AT

Therefore total mmf = 975 + 166 = 1141 AT

9.15 COMPOSITE SERIES CIRCUIT

Consider a toroid composed of three different magnetic materials of different permeabilities, areas and lengths excited by a coil of *N* turns.

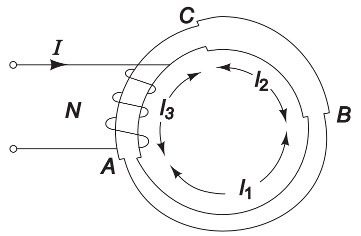


Fig. 9.30

With a current of *I* amperes as shown in Fig. 9.30. The lengths of sections *AB*, *BC* and *CA* are I_1, I_2 and I_3 respectively. Each section will have its own reluctance and permeability. Since all of them are joined in series, the total reluctance of the combined magnetic circuit is given by

$$\Re_{\text{Total}} = \frac{1}{\mu A}$$

$$= \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} + \frac{l_3}{\mu_3 A_3}$$

The flux produced in the circuit is given by $\phi = \dfrac{\text{mmf}}{\text{Total reluctance}}$ Wb

$$\therefore \qquad \phi = \left[\frac{NI}{\dfrac{l_1}{\mu_1 A_1} + \dfrac{l_2}{\mu_2 A_2} + \dfrac{l_3}{\mu_3 A_3}} \right] \text{Wb}$$

9.16 PARALLEL MAGNETIC CIRCUIT

We have seen that a series magnetic circuit carries the same flux and the total mmf required to produce a given quantity of flux is the sum of the mmf's for the separate parts. In a parallel magnetic circuit, different parts of the circuit are in parallel. For such circuits the Kirchhoff's laws, in their analogous magnetic form can be applied for the analysis. Consider an iron core having three limbs A, B and C as shown in Fig. 9.31(a). A Coil with N turns is arranged around limb A which carries a current I amperes. The flux produced by the coil in limb A. ϕ_A is divided between limbs B and C and each equal to $\phi_A/2$. The reluctance offered by the two parallel paths is equal to the half the reluctance of each path (Assuming equal lengths and cross sectional areas). Similar to Kirchhoff's current law in an electric circuit, the total magnetic flux directed towards a junction in a magnetic circuit is equal to the sum of the magnetic fluxes directed away from that junction. Accordingly $\phi_A = \phi_B + \phi_C$ or $\phi_A - \phi_B - \phi_C = 0$. The electrical equivalent of the above circuit is shown in Fig. 9.31(b). Similar to Kirchhoff's second law, in a closed magnetic circuit, the resultant mmf is equal to the algebraic sum of the products of field strength and the length of each part in the closed path. Thus applying the law to the first loop in Fig. 9.31(a), we get

$$NI = H_A l_A + H_B l_B$$

or

$$NI = \phi_A \Re_A + \phi_B \Re_B$$

The mmf across the two parallel paths is identical.
Therefore NI is also equal to

$$NI = \phi_A \Re_A + \phi_C \Re_C$$

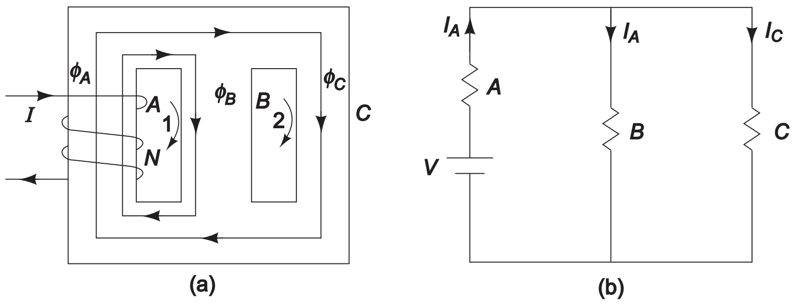


Fig. 9.31

Additional Solved Problems

Problem 9.1 In the circuit shown in Fig. 9.32, write the equation for the voltages across the coils ab and cd; also mention the polarities of the terminals.

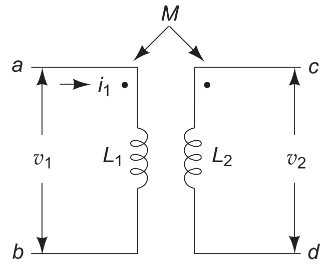


Fig. 9.32

Solution Current i_1 is only flowing in coil *ab*, whereas coil *cd* is open. Therefore, there is no current in coil *cd*. The emf due to self induction is zero on coil *cd*.

$$\therefore \qquad v_2(t) = M\frac{di_1(t)}{dt} \text{ with } C \text{ being positive}$$

Similarly the emf due to mutual induction in coil *ab* is zero.

$$\therefore \qquad v_1(t) = L\frac{di_1(t)}{dt}$$

Problem 9.2 In the circuit shown in Fig. 9.33, write the equation for the voltages v_1 and v_2. L_1 and L_2 are the coefficients of self inductances of coils 1 and 2, respectively, and M is the mutual inductance.

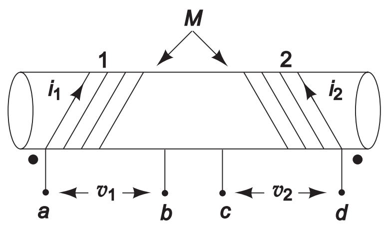


Fig. 9.33

Solution In the figure, *a* and *d* are like terminals.
Currents i_1 and i_2 are entering at dot marked terminals.

$$v_1 = L_1\frac{di_1(t)}{dt} + \frac{M\,di_2(t)}{dt}$$

$$v_2 = L_2\frac{di_2(t)}{dt} + \frac{M\,di_1(t)}{dt}$$

Problem 9.3 In Fig. 9.34, $L_1 = 4$ H; $L_2 = 9$ H, K = 0.5, $i_1 = 5\cos(50t - 30°)$ A, $i_2 = 2\cos(50t - 30°)$ A. Find the values of (a) v_1; (b) v_2, and (c) the total energy stored in the system at $t = 0$.

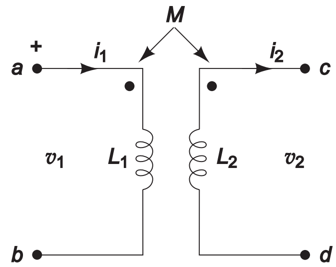


Fig. 9.34

Solution Since the current in coil *ab* is entering at the dot marked terminal, whereas in coil *cd* the current is leaving, we can write the equations as

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$v_2 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

$$M = K\sqrt{L_1 L_2} = 0.5\sqrt{36} = 3$$

(a) $v_1 = 4 \dfrac{d}{dt}$ [5 cos (50t – 30°) – 3$\dfrac{d}{dt}$ [2 cos (50t – 30°)]

v_1 = 20 [– sin (50t – 30°) × 50]– 6 [– sin (50t – 30°) 50]

at t = 0

v_1 = 500 – 150 = 350 V

(b) $v_2 = -3 \dfrac{d}{dt}$ [5 cos (50t – 30°)] + 9$\dfrac{d}{dt}$ [2 cos (50t – 30°)]

= – 15 [– sin (50t – 30°) × 50] + 18 [– sin (50t – 30°) 50]

at t = 0

v_2 = – 375 + 450 = 75 V

(c) The total energy stored in the system

$$W(t) = \frac{1}{2} L_1[i_1(t)]^2 + \frac{1}{2} L_2[i_2(t)]^2 - M[i_1(t)i_2(t)]$$

$$= \frac{1}{2} \times 4[5\cos(50t - 30°)]^2 + \frac{1}{2} \times 9[2\cos(50t - 30°)]^2$$

$$- 3\,[5\cos(50t - 30°) \times 2\cos(50t - 30°)]$$

at t = 0 $W(t)$ = 28.5 j

Problem 9.4 For the circuit shown in Fig. 9.35, write the mesh equations.

Solution There exists mutual coupling between coil 1 and 3, and 2 and 3. Assuming branch currents i_1, i_2 and i_3 in coils 1, 2 and 3, respectively, the equation for mesh 1 is

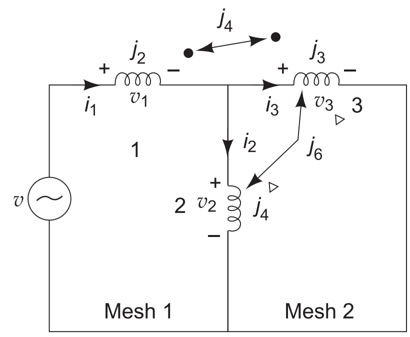


Fig. 9.35

$$v = v_1 + v_2$$

$$v = i_1 j_2 - i_3 j_4 + i_2 j_4 - i_3 j_6 \tag{9.7}$$

$j_4 i_3$ is the mutual inductance drop between coils (1) and (3), and is considered negative according to dot convention and $i_3 j_6$ is the mutual inductance drop between coils 2 and 3.

For the 2nd mesh $\quad 0 = - v_2 + v_3$

$$= - (j_4 i_2 - j_6 i_3) + j_3 i_3 - j_6 i_2 - j_4 i_1 \tag{9.8}$$

$$= - j_4 i_1 - j_{10} i_2 + j_9 i_3 \tag{9.9}$$

$$i_1 = i_3 + i_2$$

Problem 9.5 Calculate the effective inductance of the circuit shown in Fig. 9.36 across terminals a and b.

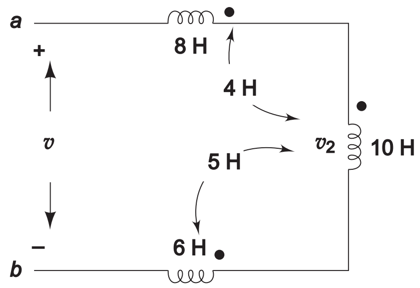


Fig. 9.36

Solution Let the current in the circuit be i

$$v = 8\frac{di}{dt} - 4\frac{di}{dt} + 10\frac{di}{dt} - 4\frac{di}{dt} + 5\frac{di}{dt} + 6\frac{di}{dt} + 5\frac{di}{dt}$$

or
$$\frac{di}{dt}[34 - 8] = 26\frac{di}{dt} = v$$

Let L be the effective inductance of the circuit across ab. Then the voltage across $ab = v = L\dfrac{di}{dt} = 26\dfrac{di}{dt}$.

Hence, the equivalent inductance of the circuit is given by 26 H.

Problem 9.6 For the circuit shown in Fig. 9.37, find the ratio of output voltage to the source voltage.

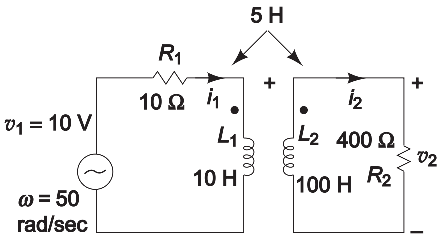


Fig. 9.37

Solution Let us consider i_1 and i_2 as mesh currents in the primary and secondary windings.

As the current i_1 is entering at the dot marked terminal, and current i_2 is leaving the dot marked terminal, the sign of the mutual inductance is to be negative. Using Kirchhoff's voltage law, the voltage equation for the first mesh is

$$i_1(R_1 + j\omega L_1) - i_2 j\omega M = v_1$$

$$i_1(10 + j500) - i_2 j250 = 10 \tag{9.10}$$

Similarly, for the 2nd mesh

$$i_2(R_2 + j\omega L_2) - i_1 j\omega M = 0$$

$$i_2(400 + j5000) - i_1 j250 = 0 \tag{9.11}$$

$$i_2 = \frac{\begin{vmatrix} (10 + j500) & 10 \\ -j250 & 0 \end{vmatrix}}{\begin{vmatrix} (10 + j500) & -j250 \\ -j250 & (400 + j5000 \end{vmatrix}}$$

$$i_2 = 0.00102 \angle -84.13°$$

$$v_2 = i_2 \times R_2$$

$$= 0.00102 \angle -84.13° \times 400$$

$$= 0.408 \angle -84.13°$$

$$\frac{v_2}{v_1} = \frac{0.408}{10} \angle -84.13°$$

$$\frac{v_2}{v_1} = 40.8 \times 10^{-3} \angle -84.13°$$

Problem 9.7 Calculate the effective inductance of the circuit shown in Fig. 9.38 across *AB*.

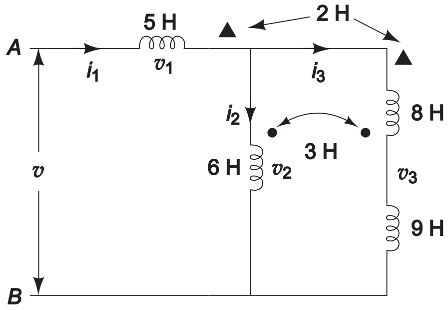


Fig. 9.38

Solution The inductance matrix is

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 17 \end{bmatrix}$$

From KVL $\qquad v = v_1 + v_2 \qquad (9.12)$

and $\qquad v_2 = v_3 \qquad (9.13)$

From KCL $\qquad i_1 = i_2 + i_3 \qquad (9.14)$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 17 \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix}$$

$$v_1 = 5\frac{di_1}{dt} - 2\frac{di_3}{dt} \qquad (9.15)$$

and

$$v_2 = 6\frac{di_2}{dt} - 3\frac{di_3}{dt} \qquad (9.16)$$

$$v_3 = -2\frac{di_1}{dt} - 3\frac{di_2}{dt} + 17\frac{di_3}{dt} \qquad (9.17)$$

From Eq. 9.12, we have

$$v = v_1 + v_2$$

$$= 5\frac{di_1}{dt} - 2\frac{di_3}{dt} + 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$$

$$v = 5\frac{di_1}{dt} + 6\frac{di_2}{dt} - 5\frac{di_3}{dt} \qquad (9.18)$$

From Eq. 9.14,

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \tag{9.19}$$

Substituting Eq. 9.19 in Eq. 9.17, we have

$$v_3 = -2\left[\frac{di_2}{dt} + \frac{di_3}{dt}\right] - 3\left[\frac{di_2}{dt}\right] + 17\left[\frac{di_3}{dt}\right]$$

or
$$-5\frac{di_2}{dt} + 15\frac{di_3}{dt} = v_3 \tag{9.20}$$

Multiplying Eq. 9.16 by 5, we get

$$30\frac{di_2}{dt} - 15\frac{di_3}{dt} = 5v_2 \tag{9.21}$$

Adding Eqs. (9.20) and (9.21), we get

$$25\frac{di_2}{dt} = v_3 + 5v_2$$

$$25\frac{di_2}{dt} = 6v_2$$

$$= 6v_3, \text{ since } v_2 = v_3$$

or
$$v_2 = \frac{25}{6}\frac{di_2}{dt}$$

From Eq. 9.16

$$\frac{25}{6}\frac{di_2}{dt} = 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$$

from which
$$\frac{di_2}{dt} = \frac{18}{11}\frac{di_3}{dt}$$

From Eq. 9.19

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{11}{18}\frac{di_2}{dt} = \frac{29}{18}\frac{di_2}{dt}$$

Substituting the values of $\frac{di_2}{dt}$ and $\frac{di_3}{dt}$ in Eq. 9.18 yields

$$v = 5\frac{di_1}{dt} + 6\frac{18}{29}\frac{di_1}{dt} - 5\frac{11}{18}\frac{di_2}{dt}$$

$$= 5\frac{di_1}{dt} + \frac{108}{29}\frac{di_1}{dt} - \frac{55}{18}\frac{18}{29}\frac{di_1}{dt}$$

$$v = \frac{198}{29}\frac{di_1}{dt} = 6.827\,\frac{di_1}{dt}$$

$\therefore$ equivalent inductance across AB = 6.827 H

Problem 9.8 Write the mesh equations for the network shown in Fig. 9.39.

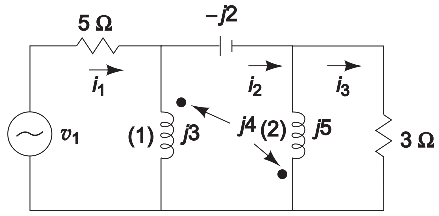


Fig. 9.39

Solution The circuit contains three meshes. Let us assume three loop currents i_1, i_2 and i_3.

For the first mesh

$$5i_1 + j3(i_1 - i_2) + j4(i_3 - i_2) = v_1 \tag{9.22}$$

The drop due to self inductance is $j3(i_1 - i_2)$ is written by considering the current $(i_1 - i_2)$ entering at dot marked terminal in the first coil, $j4(i_3 - i_2)$ is the mutually induced voltage in coil 1 due to current $(i_3 - i_2)$ entering at dot marked terminal of coil 2.

Similarly, for the 2nd mesh,

$$j3(i_2 - i_1) + j5(i_2 - i_3) - j2\,i_2 + j_4(i_2 - i_3) + j4(i_2 - i_1) = 0 \tag{9.23}$$

$j4(i_2 - i_1)$ is the mutually induced voltage in coil 2 due to the current in coil 1, and $j4(i_2 - i_3)$ is the mutually induced voltage in coil 1 due to the current in coil 2.

For the third mesh,

$$3i_3 + j5(i_3 - i_2) + j4(i_1 - i_2) = 0 \tag{9.24}$$

Further simplification of Eqs. 9.22, 9.23 and 10.24 leads to

$$(5 + j3)i_1 - j7i_2 + j4i_3 = v_1 \tag{9.25}$$

$$-j7i_1 + j14i_2 - j9i_3 = 0 \tag{9.26}$$

$$j4i_1 - j9i_2 + (3 + j5)i_3 = 0 \tag{9.27}$$

Problem 9.9 The inductance matrix for the circuit of three series connected coupled coils is given in Fig. 9.40. Find the inductances, and indicate the dots for the coils.

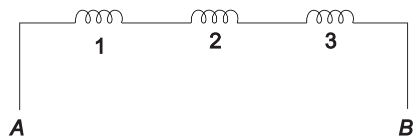


Fig. 9.40

$$L = \begin{bmatrix} 4 & -4 & 1 \\ -4 & 2 & -3 \\ 1 & -3 & 6 \end{bmatrix}$$

All elements are in henrys

Solution The diagonal elements (4, 2, 6) in the matrix represent the self inductances of the three coils 1, 2 and 3, respectively. The second element in the 1st row (– 4) is the mutual inductance between coil 1 and 2, the negative sign indicates that the current in the first coil enters the dotted terminal, and the current in the second coil enters at the undotted terminal. Similarly, the remaining elements are fixed. The values of inductances and the dot convention is shown in Fig. 9.41.

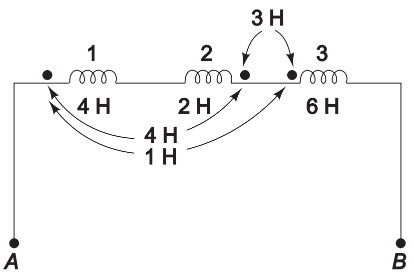


Fig. 9.41

Problem 9.10 Find the voltage across the 10 Ω resistor for the network shown in Fig. 9.42.

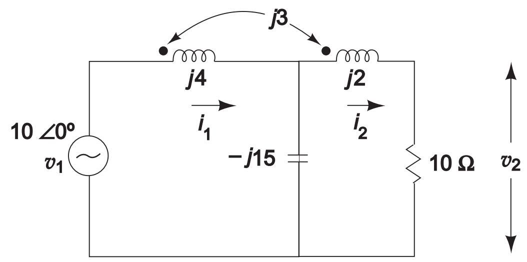


Fig. 9.42

Solution From Fig. 9.42, it is clear that

$$v_2 = i_2\, 10 \tag{9.28}$$

Mesh equation for the first mesh is

$$j4i_1 - j15\,(i_1 - i_2) + j3i_2 = 10\,\angle 0^\circ$$

$$-j11i_1 + j18i_2 = 10\,\angle 0^\circ \tag{9.29}$$

Mesh equation for the 2nd mesh is

$$j2i_2 + 10i_2 - j15(i_2 - i_1) + j3i_1 = 0$$

$$j18i_1 - j13i_2 + 10i_2 = 0$$
$$j18i_1 + i_2(10 - j13) = 0 \tag{9.30}$$

Solving for i_2 from Eqs. 9.29 and 9.30, we get

$$i_2 = \begin{bmatrix} -j11 & 10\angle 0° \\ j18 & 0 \end{bmatrix} \Big/ \begin{bmatrix} -j11 & j18 \\ j18 & 10-j3 \end{bmatrix}$$
$$= \frac{-180\ \angle 90°}{291 - j110}$$
$$= \frac{-180\ \angle 90°}{311\ \angle 20.70°} = -\ 0.578\ \angle 110.7°$$

$\therefore$ $$v_2 = i_2\ 10 = -\ 5.78\ \angle 110.7°$$
$$|v_2| = 5.78$$

Problem 9.11 The resonant frequency of the tuned circuit shown in Fig. 9.43 is 1000 rad/sec. Calculate the self inductances of the two coils and the optimum value of the mutual inductance.

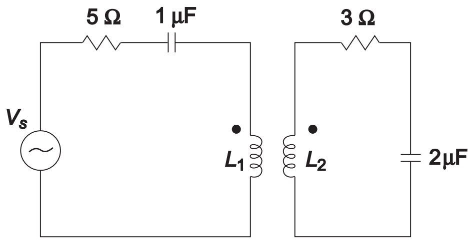


Fig. 9.43

Solution From Section 9.7, we know that

$$\omega_r^2 = \frac{1}{L_1 C_1} = \frac{1}{L_2 C_2}$$
$$L_1 = \frac{1}{\omega_r^2 C_1} = \frac{1}{(1000)^2\, 1 \times 10^{-6}} = 1\text{ H}$$
$$L_2 = \frac{1}{\omega_r^2 C_2} = \frac{1}{(1000)^2 \times 2 \times 10^{-6}} = 0.5\text{ H}$$

Optimum value of the mutual inductance is given by

$$M_{\text{optimum}} = \frac{\sqrt{R_1 R_2}}{\omega_r}$$

where R_1 and R_2 are the resistances of the primary and secondary coils

$$M = \frac{\sqrt{15}}{1000} = 3.87\text{ mH}$$

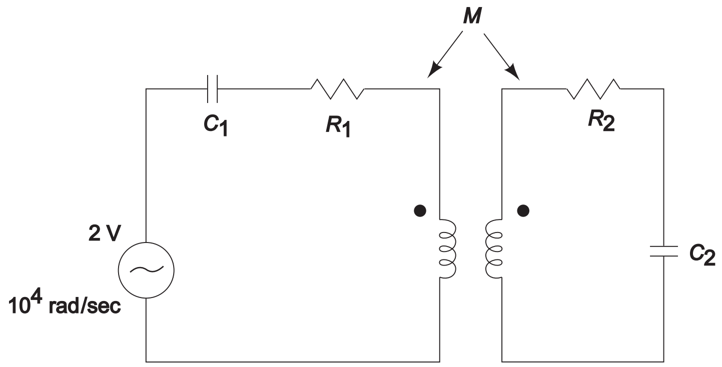


Fig. 9.44

Problem 9.12 The tuned frequency of a double tuned circuit shown in Fig. 9.44 is 10^4 rad/sec. If the source voltage is 2 V and has a resistance of 0.1 Ω, calculate the maximum output voltage at resonance if $R_1 = 0.01$ Ω, $L_1 = 2\ \mu$H; $R_2 = 0.1$ Ω, and $L_2 = 25\ \mu$H.

Solution

The maximum output voltage $v_0 = \dfrac{v_i}{2\omega_r^2 C_2 M_c}$

where M_c is the critical value of the mutual inductance given by

$$M_c = \frac{\sqrt{R_2 (R_1 + R_s)}}{\omega_r}$$

$$M_c = \frac{\sqrt{0.1\,(0.01 + 0.1)}}{10^4} = 10.48\ \mu\text{H}$$

At resonance $$\omega_r^2 = \frac{1}{L_2 C_2}$$

$$C_2 = \frac{1}{\omega_r^2 L_2} = \frac{1}{(10^4)^2 \times 25 \times 10^{-6}} = 0.4 \times 10^{-3}\ \text{F}$$

$$v_0 = \frac{2}{2(10^4)^2 \times 0.4 \times 10^{-3} \times 10.48 \times 10^{-6}}$$

$$= 2.385\ \text{V}$$

Problem 9.13 An iron ring 10 cm dia and 15 cm^2 in cross-section is wound with 250 turns of wire for a flux density of 1.5 Web/m^2 and permeability 500. Find the exciting current, the inductance and stored energy. Find corresponding quantities when there is a 2 mm air gap.

Solution

(a) *Without air gap*

Length of the flux path = $\pi D = \pi \times 10 = 31.41$ cm

$= 0.3141$ m

Area of flux path = 15 cm^2 = 15×10^{-4} m^2

$$\text{mmf} = \text{A.T}$$

$$A = \frac{\text{mmf}}{T}$$

$$H = \frac{B}{\mu_0 \mu_r} = \frac{1.5}{4\pi \times 10^{-7} \times 500} = 2387$$

mmf = $H \times l = 2387 \times 0.3141 = 750$ AT

$$\text{Exciting current} = \frac{\text{mmf}}{T} = \frac{750}{250} = 3\text{ A}$$

$$\text{Reluctance} = \frac{l}{\mu_0 \mu_r A} = \frac{0.3141}{4\pi 10^{-7} \times 500 \times 15 \times 10^{-4}}$$

$= 333270$

$$\text{Self Inductance} = \frac{N^2}{\text{Reluctance}} = \frac{(250)^2}{333270} = 0.1875\text{ H}$$

$$\text{Energy} = \frac{1}{2} LI^2 = \frac{1}{2} \times 0.1875 \times (3)^2$$

$= 0.843$ Joules

(b) *With air gap*

$$\text{Reluctance of the gap} = \frac{l}{\mu_0 A} = \frac{2 \times 10^{-3}}{4\pi \times 10^{-7} \times 15 \times 10^{-4}}$$

$= 1.06 \times 10^6$ A/Wb

Total reluctance = $(0.333 + 1.06)\,10^6 = 1.393 \times 10^6$ A/Wb

mmf = $\phi \times$ reluctance

$= 1.5 \times 15 \times 10^{-4} \times 1.393 \times 10^6$

$= 3134$ AT

$$\text{Exciting current} = \frac{3134}{250} = 12.536\text{ A}$$

$$L = \frac{N^2}{\Re} = \frac{(250)^2}{1.393 \times 10^6} = 44.8\text{ mH}$$

$$\text{Energy} = \frac{1}{2} LI^2$$

$$= \frac{1}{2} \times 44.8 \times 10^{-3} \times (12.536)^2$$

$$= 3.52 \text{ Joules}$$

Problem 9.14 A 700 turn coil is wound on the central limb of the cast steel frame as shown in Fig. 9.45. A total flux of 1.8 m Wb is required in the gap. What is the current required? Assume that the gap density is uniform and that all lines pass straight across the gap. All dimensions are in centimeters. Assume μ_r as 600.

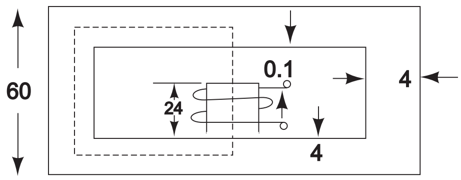


Fig. 9.45

Solution Each of the side limbs carry half the total flux as their reluctances are equal.

Total mmf required is equal to the sum of the mmf required for gap, central limb and side limb.

Reluctance of gap and central limb are in series and they carry the same flux.

Air gap

$$\phi_g = 1.8 \times 10^{-3} \text{ Wb}$$

$$A_g = 4 \times 4 \times 10^{-4} \text{ m}^2$$

$$B_g = \frac{1.8 \times 10^{-3}}{16 \times 10^{-4}} = 1.125 \text{ Wb/m}^2$$

$$H_g = \frac{B_g}{\mu_0} = \frac{1.125}{4\pi \times 10^{-7}} = 8.95 \times 10^5 \text{ AT/m}$$

Required mmf for the gap $= H_g l_g$

$$= 8.95 \times 10^5 \times 0.001 = 895 \text{ AT}$$

Central Limb

$$\phi_c = 1.8 \times 10^{-3} \text{ Wb}$$

$$A_c = 4 \times 4 \times 10^{-4} \text{ m}^2$$

$$B_c = 1.125 \text{ Wb/m}^2$$

$$H_c = \frac{B_c}{\mu_0 \mu_r} = \frac{1.125}{4\pi \times 10^{-7} \times 600} = 1492 \text{ AT/m}$$

Required mmf for central limb $= H_c l_c$

$$= 1492 \times 0.24 = 358 \text{ AT}$$

Side Limb:

$$\phi_s = \frac{1}{2} \times \text{flux in central limb}$$

$$= \frac{1}{2} \times 1.8 \times 10^{-3} = 0.9 \times 10^{-3} \text{ Wb}$$

$$B_s = \frac{0.9 \times 10^{-3}}{16 \times 10^{-4}} = 0.5625 \text{ Wb/m}^2$$

$$H_s = \frac{B_s}{\mu_0 \mu_r} = \frac{0.5625}{4\pi \times 10^{-7} \times 600} = 746 \text{ AT/m}$$

Required mmf for side limb $= H_s l_s$

$$= 746 \times 0.6 = 447.6 \cong 448$$

Total mmf = 895 + 358 + 448 = 1701 AT

Required current $= \dfrac{1701}{700} = 2.43$ A

Practice Problems

9.1 Using the dot convention, write the voltage equations for the coils shown in Fig. 9.46.

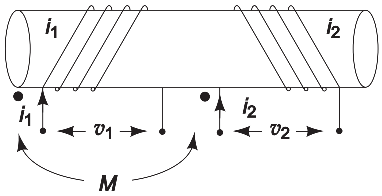


Fig. 9.46

9.2 Two inductively coupled coils have self inductances $L_1 = 40$ mH and $L_2 = 150$ mH. If the coefficient of coupling is 0.7, (i) find the value of mutual inductance between the coils, and (ii) the maximum possible mutual inductance.

9.3 For the circuit shown in Fig. 9.47 write the inductance matrix.

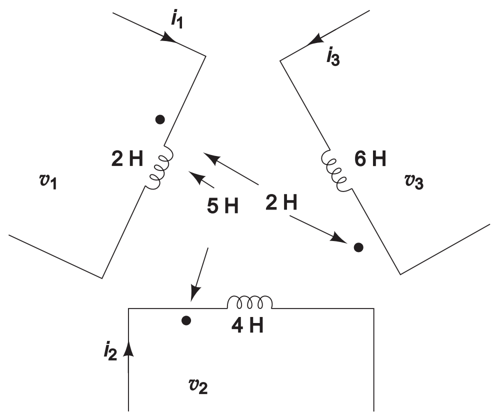


Fig. 9.47

9.4 Two coils connected in series have an equivalent inductance of 0.8 H when connected in aiding, and an equivalent inductance of 0.5 H when the connection is opposing. Calculate the mutual inductance of the coils.

9.5 In Fig. 9.48, $L_1 = 2$ H; $L_2 = 6$ H; $K = 0.5$; $i_1 = 4 \sin (40t - 30°)$ A; $i_2 = 2 \sin (40t - 30°)$ A. Find the values of (i) v_1 and (ii) v_2.

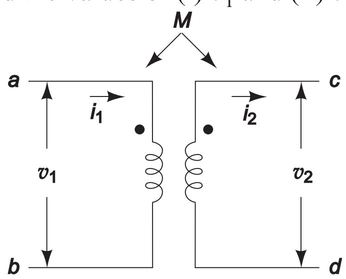


Fig. 9.48

9.6 For the circuit shown in Fig. 9.49, write the mesh equations.

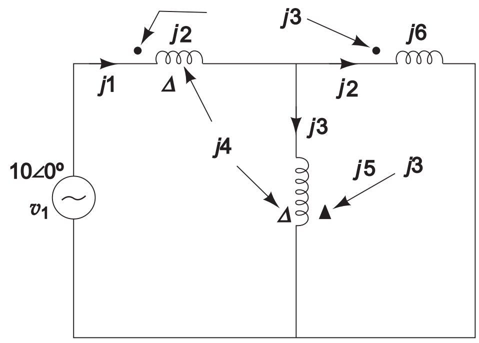


Fig. 9.49

9.7 Calculate the effective inductance of the circuit shown in Fig. 9.50 across *XY*.

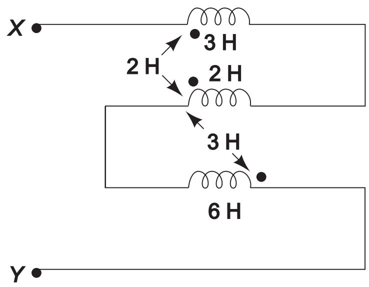


Fig. 9.50

9.8 For the circuit shown in Fig. 9.51, find the ratio of output voltage to the input voltage.

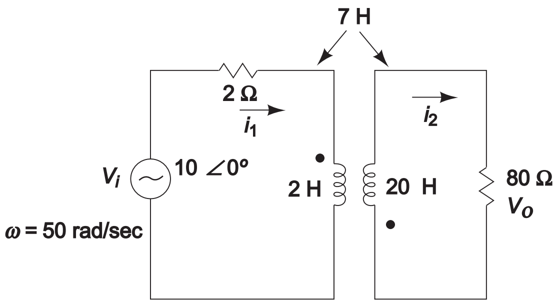


Fig. 9.51

9.9 Calculate the effective inductance of the circuit shown in Fig. 9.52.

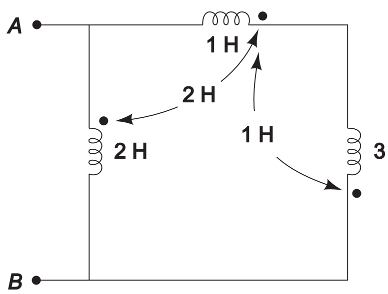


Fig. 9.52

9.10 Write the mesh equations for the network shown in Fig. 9.53.

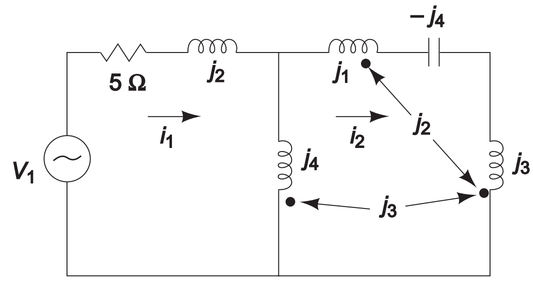


Fig. 9.53

9.11 Find the source voltage if the voltage across the 100 ohms is 50 V for the network in the Fig. 9.54.

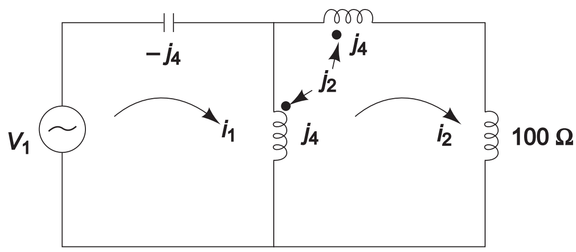


Fig. 9.54

9.12 The inductance matrix for the circuit of a three series connected coupled coils is given below. Find the inductances and indicate the dots for the coils.

$$L = \begin{bmatrix} 8 & -2 & 1 \\ -2 & 4 & -6 \\ 1 & -6 & 6 \end{bmatrix}$$

Objective-type Questions

1. Mutual inductance is a property associated with
   (a) only one coil
   (b) two or more coils
   (c) two or more coils with magnetic coupling
2. Dot convention in coupled circuits is used
   (a) to measure the mutual inductance
   (b) to determine the polarity of the mutually induced voltage in coils
   (c) to determine the polarity of the self induced voltage in coils

3. Mutually induced voltage is present independently of, and in addition to, the voltage due to self induction.
   (a) true (b) false
4. Two terminals belonging to different coils are marked identically with dots, if for the different direction of current relative to like terminals the magnetic flux of self and mutual induction in each circuit add together.
   (a) true (b) false
5. The maximum value of the coefficient of coupling is
   (a) 100% (b) more than 100%
   (c) 90%
6. The case for which the coefficient of coupling $K = 1$ is called perfect coupling
   (a) true (b) false
7. The maximum possible mutual inductance of two inductively coupled coils with self inductances $L_1 = 25$ mH and $L_2 = 100$ mH is given by
   (a) 125 mH (b) 75 mH
   (c) 50 mH
8. The value of the coefficient of coupling is more for aircored coupled circuits compared to the iron core coupled circuits.
   (a) true (b) false
9. Two inductors are connected as shown in Fig. 9.55. What is the value of the effective inductance of the combination.

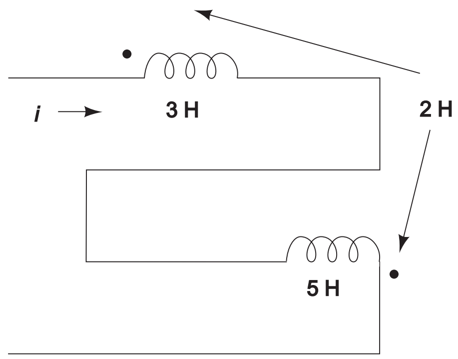


Fig. 9.55

   (a) 8 H (b) 10 H
   (c) 4 H
10. Two coils connected in series have an equivalent inductance of 3 H when connected in aiding. If the self inductance of the first coil is 1 H, what is the self inductance of the second coil (Assume $M = 0.5$ H)
   (a) 1 H (b) 2 H
   (c) 3 H

11. For Fig. 9.56 shown below, the inductance matrix is given by

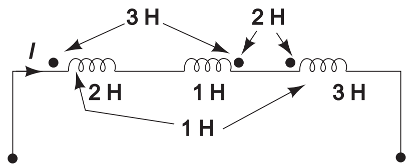


Fig. 9.56

(a) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

Chapter 10

Differential Equations

10.1 BASIC CONCEPTS

Differential equations which denote rates of change, occur in various branches of science and engineering. We make use of differential equations, for example, to determine the motion of a rocket or a satellite, to determine the charge or current in an electric circuit, or to determine the vibrations of a wire or membrane. The mathematical formulation of the above problems gives rise to differential equations.

A differential equation is one which involves derivatives of one or more dependent variables with respect to one or more independent variables. Differential equations are classified according to the variables and derivatives involved in them. Ordinary differential equations are those which involve ordinary derivatives of one or more dependent variables with respect to a single independent variable. For example,

$$dy = \sin x \, dx \tag{10.1}$$

$$\frac{d^3x}{dt^4} + 3\frac{d^2x}{dt^2} + 5x = \cos t \tag{10.2}$$

In Eq. 10.1, x is an independent variable, and y is a dependent variable. In Eq. 10.2, variable t is an independent variable and x is a dependent variable. Partial differential equations are those which involve partial derivatives of one or more dependent variables with respect to more than one independent variables. For example,

$$\frac{\partial v}{\partial u} + \frac{\partial v}{\partial t} = v \tag{10.3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{10.4}$$

In Eq. 10.3, variables u and t are independent, and v is a dependent variable.

In Eq. 10.4, variables x, y, and z are independent, whereas v is a dependent variable.

The order of differential equation is the order of the highest derivative in it. Equation 10.1 is a first order differential equation, since the highest derivative involved is the first order. Similarly, Eq. 10.2 is of the 3rd order. Equations 10.3 and 10.4 are of the first and second order, respectively.

The degree of a differential equation is the degree of the derivative of the highest order; for example,

$$\frac{d^2x}{dt^2} = \left[1 + \left(\frac{dx}{dt}\right)^2\right]^{1/2} \tag{10.5}$$

Equation 10.5 is of the second degree, since when the radical is removed, it becomes

$$\left(\frac{d^2x}{dt^2}\right)^2 = \left[1 + \left(\frac{dx}{dt}\right)^2\right] \tag{10.6}$$

Differential equations are further classified as *linear* and *non-linear*.

A linear ordinary differential equation of the order n, in the dependent variable x and the independent variable t, is given in the form

$$a_0(t)\frac{d^nx}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = c(t) \tag{10.7}$$

where a_0 is not identically zero. The order of the equation is n. The term $c(t)$ is the *forcing function* and is independent of $x(t)$. When $c(t)$ is zero, the equation is said to homogeneous; otherwise, it is non-homogeneous. A differential equation is said to be linear, when the dependent variable x and its derivatives occur in the first degree only, and no products of x and its derivatives are present in the equation.

For example,

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 5x = 0 \tag{10.8}$$

In Eq. 10.8, the dependent variable, x, and its derivatives are of the first degree only. A non-linear ordinary differential equation is defined as an equation which is not linear.

For example,

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 5x^2 = 0 \tag{10.9}$$

$$\frac{d^2x}{dt^2} + 5\left(\frac{dx}{dt}\right)^2 + 7x = 0 \tag{10.10}$$

In Eq. 10.7, if $a_0(t)$, $a_1(t)$ … $a_n(t)$ are constants, the equation is said to be linear with constant coefficients; otherwise, the equation is said to be linear with variable coefficients.

10.2 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Consider an *n*th order homogeneous linear differential equation with constant coefficients,

$$a_0 \frac{d^n x}{dt} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1} \frac{dx}{dt} + a_n x = 0 \tag{10.11}$$

where a_0, a_1 … a_n are real constants.

Now we shall find the solution of Eq. 10.11 of the form $x = e^{mt}$. By assuming that $x = e^{mt}$ is a solution for certain m, we have

$$\frac{dx}{dt} = me^{mt}$$

$$\frac{d^2x}{dt^2} = m^2e^{mt}$$

$$\vdots$$

$$\frac{d^nx}{dt^n} = m^ne^{mt}$$

Substituting in Eq. 10.11, we get

$$a_0m^ne^{mt} + a_1m^{n-1}e^{mt} + \ldots + a_ne^{mt} = 0$$

or $$e^{mt}(a_0m^n + a_1m^{n-1} + \ldots + a_n) = 0$$

where $$a_0m^n + a_1m^{n-1} + \ldots + a_n = 0 \tag{10.12}$$

This is called the *auxiliary*, or the characteristic equation of the given differential equation. Three cases might occur in the auxiliary equation which are, subject to the roots of Eq. 10.12 being real and distinct, real and repeated, or complex.

Case 1 Distinct real roots

If the roots of the Eq. 10.12, m_1, m_2 … m_n are real and distinct, the general solution of Eq. 10.11 is

$$x = k_1e^{m_1t} + k_2e^{m_2t} + \ldots + k_ne^{m_nt}$$

where k_1, k_2 … k_n are arbitrary constants.

k_1, k_2 … k_n values can be determined by using initial conditions.

Example 10.1 Find the solution for the following equation

$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$$

given the initial conditions

$$x''(0) = 0,\ x'(0) = 2,\ x(0) = 1$$

Solution The characteristic equation is

$$m^3 + 2m^2 - m - 2 = 0$$

By taking factors, we have

$$(m + 1)\,(m^2 + m - 2) = 0$$

or $$(m + 1)\,(m - 1)\,(m + 2) = 0$$

Thus, the roots are distinct, real numbers

$$m_1 = 1,\ m_2 = -1,\ m_3 = -2$$

The general solution is

$$x = k_1\, e^{-t} + k_2 e^{t} + k_3 e^{-2t}$$

At $t = 0$, $$k_1 + k_2 + k_3 = 1 \tag{10.13}$$

$$x' = -k_1\, e^{-t} + k_2 e^{t} - 2k_3\, e^{-2t}$$

At $t = 0$, $$-k_1 + k_2 - 2k_3 = 2 \tag{10.14}$$

$$x'' = k_1 e^{-t} + k_2 e^{t} + 4k_3\, e^{-2t}$$

At $t = 0$, $$k_1 + k_2 + 4k_3 = 0 \tag{10.15}$$

Solving Eqs. 10.13, 10.14 and 10.15 we get

$$k_3 = -\frac{1}{3}$$

$$k_2 = \frac{4}{3} \text{ and } k_1 = 0$$

The solution for the differential equation is therefore

$$x = \frac{4}{3}e^{t} - \frac{1}{3}e^{-2t}$$

Case 2 Roots are real and repeated

If the roots of Eq. 10.12 are the double real root m, and $(n - 2)$ distinct real roots.

$$m_1, m_2 \ldots m_{n-2}$$

then the linearly independent solutions of Eq. 10.11 are

$$e^{mt},\ te^{mt},\ e^{m_1 t},\ e^{m_2 t} \ldots e^{m_{n-2} t}$$

And the general solution may be written as

$$x = k_1 e^{mt} + k_2 te^{mt} + k_3 e^{m_1 t} + k_4 e^{m_2 t} + \ldots + k_n e^{m_{n-2} t}$$

Similarly, if Eq. 10.12 has a triple real root m, the general solution is $(c_1 + c_2 t + c_3 t^2)\, e^{mt}$.

Example 10.2 Find the general solution for the differential equation

$$\frac{d^3x}{dt^3} + 11\frac{d^2x}{dt^2} + 35\frac{dx}{dt} + 25x = 0$$

Solution The auxiliary equation is

$$m^3 + 11m^2 + 35m + 25 = 0$$

By taking factors, we have

$$(m + 1)\,(m + 5)^2 = 0$$

or $$(m + 1)\,(m + 5)\,(m + 5) = 0$$

$\therefore$ The general solution is

$$x(t) = (k_1 + k_2 t)\, e^{-5t} + k_3 e^{-t}$$

Case 3 Roots are Complex Conjugate

Consider the auxiliary equation which has the complex number $a + jb$ as a non-repeated root. The corresponding part of the general solution is $p_1 e^{(a + jb)t} + p_2 e^{(a - jb)t}$, where p_1 and p_2 are arbitrary constants.

$$\begin{aligned} p_1 e^{(a + jb)t} + p_2 e^{(a - jb)t} &= p_1 e^{at}\, e^{jbt} + p_2 e^{at}\, e^{-jbt} \\ &= e^{at}\, [p_1 e^{jbt} + p_2 e^{-jbt}] \\ &= e^{at}\, [p_1(\cos bt + j \sin bt) + p_2\, (\cos bt - j \sin bt)] \\ &= e^{at}\, [(p_1 + p_2) \cos bt + j(p_1 - p_2) \sin bt] \\ &= e^{at}\, [k_1 \sin bt + k_2 \cos bt] \end{aligned}$$

where $$k_1 = j(p_1 - p_2) \text{ and } k_2 = (p_1 + p_2)$$

are the new arbitrary constants.

If however, $(a + jb)$ and $(a - jb)$ are each n roots of the auxiliary equation, the corresponding general solution may be written as

$$\begin{aligned} x = e^{at}\, [&(k_1 + k_2 t + k_3 t^2 + \ldots + k_n t^{n-1}) \sin bt \\ &+ (k_{n+1} + k_{n+2} t + k_{n+3} t^2 + \ldots + k_{2n} t^{n-1}) \cos bt] \end{aligned}$$

Example 10.3 Find the general solution of

$$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 20x = 0$$

Solution The auxiliary equation is

$$m^2 - 5m + 20 = 0$$

The roots are $$m = \frac{5 \pm \sqrt{25 - 80}}{2}$$

$$m = 2.5 \pm j3.7$$

Here the roots are conjugate complex numbers $a + jb$

where $a = 2.5$, $b = 3.7$

The general solution may be written as

$$x = e^{2.5t}\, (c_1 \sin 3.7t + c_2 \cos 3.7t)$$

10.3 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

Now let us consider the following non-homogeneous differential equation,

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}\frac{dx}{dt} + a_n x = f(t)$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants, and $f(t)$ is a function of time.

The general solution may be written

$$x = x_c + x_p$$

where x_c is the complementary function, and x_p is the particular integral. Since x_c is the general solution of the corresponding homogeneous equation with $f(t)$ replaced by zero, we have to find out the particular integral x_p. The particular integral can be calculated by the method of undetermined coefficients. This method is useful to equations

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_n x = c(t)$$

when $c(t)$ is such that the form of a particular solution x_p of the above equation may be guessed. For example, $c(t)$ may be a single power of t, a polynomial, an exponential, a sinusoidal function, or a sum of such functions. The method consists in assuming for x_p an expression similar to that of $c(t)$, containing unknown coefficients which are to be determined by inserting x_p and its derivatives in the original equation.

Example 10.4 Find the particular integral for the differential equation

$$\frac{d^2 x}{dt^2} - 10\frac{dx}{dt} + 5x = 10e^{-3t}$$

Solution By assuming $x_p = ke^{-3t}$, and substituting x_p'', x_p', and x_p into the differential equation

$$9ke^{-3t} + 30ke^{-3t} + 5ke^{-3t} = 10e^{-3t}$$

$\therefore$ $$9k + 30k + 5k = 10$$

or $$k = \frac{10}{44} = 0.23$$

Therefore, the particular integral is $x_p = 0.23e^{-3t}$

Example 10.5 Find the particular integral for the differential equation

$$\frac{d^2 x}{dt^2} + 2x = 5t^2$$

Solution If the driving function is the power of t, then we have to assume the particular solution as

$$x_p = k_1 t^2 + k_2 t + k_3$$

Then $$x_p'' = 2k_1$$

Substituting x_p'' and x_p in the given differential equation, we have

$$2k_1 + 2k_1 t^2 + 2k_2 t + 2k_3 = 5t^2$$

Comparing the coefficients

$$2k_1 = 5 \text{ and } 2k_2 = 0,\ 2k_1 + 2k_3 = 0$$

$$k_1 = 2.5 \text{ and } k_3 = -2.5,\ k_2 = 0$$

∴ The particular integral

$$x_p = 2.5t^2 - 2.5$$

Example 10.6 Find the particular integral for the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x = 20 \sin t$$

Solution If the driving function is sine or cosine function, the particular solution is to be assumed as

$$x_p = k_1 \cos t + k_2 \sin t$$

Then $\quad x'_p = - k_1 \sin t + k_2 \cos t$

$$x''_p = - k_1 \cos t - k_2 \sin t$$

$$- k_1 \cos t - k_2 \sin t - k_1 \sin t + k_2 \cos t + 3k_1 \cos t + 3k_2 \sin t = 20 \sin t$$

Comparing the cosine terms and sine terms in the above equation, we have

$$2k_1 + k_2 = 0$$

$$- k_1 + 2k_2 = 20$$

From which $\quad k_1 = - 4,\ k_2 = 8$

Substituting the values of k_1 and k_2 in particular integral

∴ $\quad x_p = - 4 \cos t + 8 \sin t$

This method of undetermined coefficients may be applied to forcing functions of the following.

1. $c(t) = A$
2. $c(t) = A(k_1t^n + k_2t^{n-1} + k_3t^{n-3} + \ldots + k_n)$
3. $c(t) = e^{mt}$; m is real or complex
4. Any function formed by multiplying terms of type 1, 2, or 3.

10.4 APPLICATIONS TO ELECTRICAL CIRCUITS

In this section, we consider the application of differential equations to circuits containing a source, resistors, inductors and capacitors. Before discussing the formation of differential equation for the circuits, let us discuss the v-i relationships for basic network elements.

Resistor The resistor shown in Fig. 10.1(a) has the following relation between voltage and current.

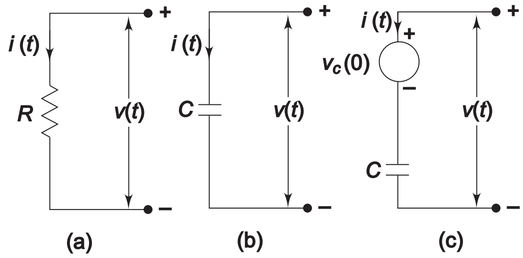


Fig. 10.1

$$v(t) = Ri(t)$$

where R is given in ohms.

Capacitor For the capacitor shown in Fig. 10.1(b), the v–i relationships are

$$i(t) = C\frac{dv(t)}{dt}$$

or

$$v(t) = \frac{1}{C}\int_0^t i(t)\, dt + v_C(0)$$

where $v_C(0)$ is the initial voltage across the capacitor. The capacitor can be represented as shown in Fig. 10.1(c).

Inductor For the inductor shown in Fig. 10.2(a), the v–i relationships are

$$i(t) = \frac{1}{L}\int_0^t v(t)dt + i_L(0)$$

or

$$v(t) = L\frac{di}{dt}$$

where $i_L(0)$ is the initial current passing through the circuit. The inductor can be represented as shown in Fig. 10.2(b).

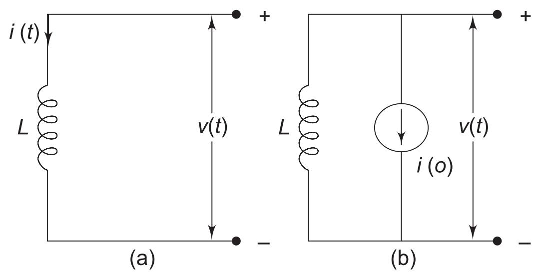


Fig. 10.2

We now consider the circuit shown in Fig. 10.3

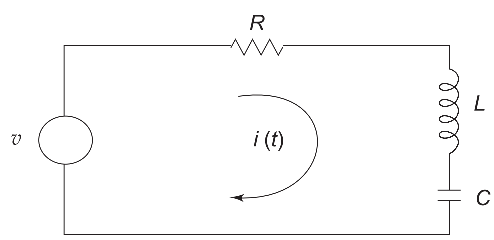


Fig. 10.3

By applying Kirchhoff's law to the circuit in Fig. 10.3, we have

$$v = Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\, dt + v_C(0)$$

If the capacitor has no initial charge, the above equation becomes

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = v$$

Differentiating the above equation, we get

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv}{dt}$$

This is a second order linear differential equation in the single dependent variable, i.

Example 10.7 The circuit shown in Fig. 10.4 consists of series R, L elements which are 5 Ω and 0.1 H, respectively. If the initial current is zero, find the current at time $t > 0$.

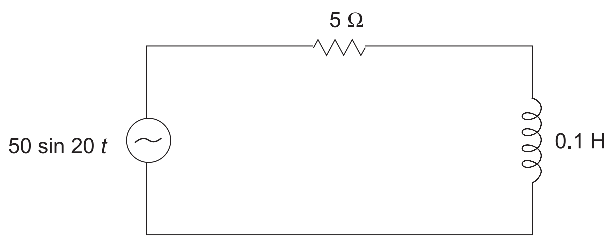


Fig. 10.4

Solution By applying Kirchhoff's laws, we have

$$\frac{1}{10}\frac{di}{dt} + 5i = 50 \sin 20t$$

or $$\frac{di}{dt} + 50i = 500 \sin 20t$$

$$(D + 50)i = 500 \sin 20t$$

$$i = i_c + i_p$$

$$i_c = ce^{-50t}$$

and $$i_p = A \cos 20t + B \sin 20t$$

$$i'_p = -20\,A \sin 20t + 20B \cos 20t$$

Substituting in the differential equation, we get

$$-20A \sin 20t + 20B \cos 20t + 50A \cos 20t + 50B \sin 20t = 500 \sin 20t$$

Comparing the coefficients, we have

$$-20A + 50B = 500$$

$$50A + 20B = 0$$

From which, $A = -3.45$ and $B = 8.62$

$\therefore$ $$i_p = -3.45 \cos 20t + 8.62 \sin 20t$$

The complete solution is

$$i = i_c + i_p$$
$$= ce^{-50t} - 3.45 \cos 20t + 8.62 \sin 20t$$

Applying the condition $i = 0$ when $t = 0$, we find

$$c = 3.45$$

Thus the solution is

$$i = 3.45\, e^{-50t} - 3.45 \cos 20t + 8.62 \sin 20t$$

Example 10.8 The circuit shown in Fig. 10.5 has series R, L, C elements which are 2Ω, $\frac{1}{10}$ H and $\frac{1}{260}$ F respectively. If the initial current and initial charge on the capacitor are both zero, find the charge on the capacitor at any time $t > 0$.

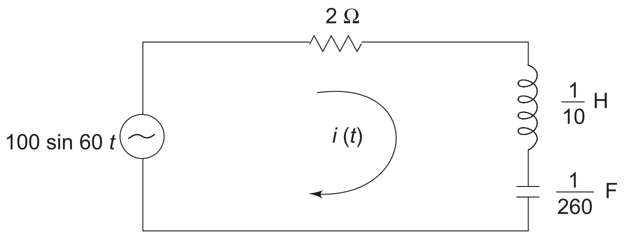


Fig. 10.5

Solution By applying Kirchhoff's laws, we have

$$\frac{1}{10}\frac{di}{dt} + 2i + 260\int idt = 100 \sin 60t$$

Since $i = \frac{dq}{dt}$, this reduces to

$$\frac{1}{10}\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 260q = 100 \sin 60t$$

or
$$\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 2600q = 1000 \sin 60t$$

Since the charge on the capacitor is zero,

$$q(0) = 0$$

Since the initial current is zero and $i = \frac{dq}{dt}$

$$q'(0) = 0$$

The complete solution is $q = q_c + q_p$

The roots of characteristic equation are $-10 \pm j50$

$\therefore$ The complementary function becomes

$$q_c = e^{-10t}(c_1 \sin 50t + c_2 \cos 50t)$$

By assuming a particular integral, we have

$$q_p = A \sin 60t + B \cos 60t$$

Differentiating and substituting in the differential equation, we get

$$A = \frac{-25}{61} \text{ and } B = \frac{-30}{61}$$

The general solution is

$$q = e^{-10t}(c_1 \sin 50t + c_2 \cos 50t) - \frac{25}{61} \sin 60t \frac{-30}{61} \cos 60t$$

Differentiating once, and substituting initial conditions, we get

$$c_1 = \frac{36}{61} \text{ and } c_2 = \frac{30}{61}$$

∴ The complete solution is

$$q = e^{-10t}\left(\frac{36}{61} \sin 50t + \frac{30}{61} \cos 50t\right) - \frac{25}{61} \sin 60t - \frac{30}{61} \cos 60t$$

Additional Solved Problems

Problem 10.1 Determine the general solution for the differential equation.

$$\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} - \frac{dx}{dt} + 3x = 0$$

given the initial conditions

$$x''(0) = 3;\ x'(0) = 1,\ x(0) = 0$$

Solution The auxiliary equation is

$$m^3 - 3m^2 - m + 3 = 0$$

By taking factors, we have

$$(m + 1)(m^2 - 4m + 3) = 0$$

or

$$(m + 1)(m - 1)(m - 3) = 0$$

Thus, the roots are distinct and real numbers

$$m_1 = -1,\ m_2 = 1,\ m_3 = 3$$

The general solution is

$$x = k_1e^{-t} + k_2e^{t} + k_3e^{3t}$$

At $t = 0$,

$$k_1 + k_2 + k_3 = 0 \tag{10.16}$$

$$x' = -k_1e^{-t} + k_2e^{t} + 3k_3e^{3t}$$

At $t = 0$,

$$-k_1 + k_2 + 3k_3 = 1 \tag{10.17}$$

$$x'' = k_1e^{-t} + k_2e^{t} + 9k_3e^{3t}$$

At $t = 0$

$$k_1 + k_2 + 9k_3 = 3 \tag{10.18}$$

Solving Eqs 10.16, 10.17 and 10.18, we get

$$k_1 = \frac{-1}{8},\ k_2 = \frac{-1}{4},\ k_3 = \frac{3}{8}$$

Thus, the solution for the differential equation is

$$x = \frac{-1}{8}e^{-t} - \frac{1}{4}e^{t} + \frac{3}{8}e^{3t}$$

Problem 10.2 Find the general solution for the differential equation

$$\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + 32x = 0$$

Solution The auxiliary equation is

$$m^3 - 6m^2 + 32 = 0$$

By taking factors, we have

$$(m + 2)\,(m - 4)^2 = 0$$

or $$(m + 2)\,(m - 4)\,(m - 4) = 0$$

Thus, the roots are real and repeated

$$m_1 = -2,\ m_2 = 4,\ m_3 = 4$$

The general solution is

$$x(t) = (k_1 + k_2t)e^{+4t} + k_3e^{-2t}$$

Problem 10.3 Find the general solution for the differential equation

$$\frac{d^4x}{dt^4} - 4\frac{d^3x}{dt^3} + 14\frac{d^2x}{dt^2} - 20\frac{dx}{dt} + 25x = 0$$

Solution The auxiliary equation is

$$m^4 - 4m^3 + 14m^2 - 20m + 25 = 0$$

The roots of the characteristic equation are

$$(1 + j2),\ (1 - j2),\ (1 + j2)\ (1 - j2)$$

Since each pair of conjugate complex roots is double, the general solution is

$$x(t) = e^t\,[(k_1 + k_2t)\sin 2t + (k_3 + k_4t)\cos 2t]$$

or $$x(t) = k_1e^t \sin 2t + k_2t\,e^t \sin 2t + k_3e^t \cos 2t + k_4t\,e^t \cos 2t$$

Problem 10.4 Determine the general solution for the differential equation

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 25x = 0$$

$$x(0) = 0,\ x'(0) = -1$$

Solution The auxiliary equation is

$$m^2 - 6m + 25 = 0$$

The roots are $$m = \frac{6 \pm \sqrt{36 - 100}}{2} = 3 \pm j4$$

Here the roots are the conjugate complex numbers.
The general solution of the differential equations

$$x(t) = e^{3t}(k_1 \sin 4t + k_2 \cos 4t) \tag{10.19}$$

Differentiating once, we get

$$x'(t) = e^{3t}\,[(3k_1 - 4k_2) \sin 4t + (4k_1 + 3k_2) \cos 4t] \tag{10.20}$$

At $\quad t = 0,\ x(0) = 0$

Substituting in Eq. 10.19, we get

$$(k_1 \sin 0 + k_2 \cos 0)\, e^0 = 0$$

$$k_2 = 0 \tag{10.21}$$

Similarly, at $t = 0\ x'(0) = -1$
Substituting in Eq. 10.20, we get

$$e^0\,[4k_1 + 3k_2] = -1 \tag{10.22}$$

Solving Eqs 10.21 and 10.22, we get

$$4k_1 = -1 \text{ i.e., } k_1 = -\frac{1}{4}$$

The solution for the differential equation is

$$x(t) = -\frac{1}{4}\, e^{3t} \sin 4t$$

Problem 10.5 Find the general solution for the differential equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 20e^t - 50 \cos t$$

Solution The corresponding homogeneous equation is

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0$$

The complementary function is

$$x_c = k_1 e^{-t} + k_2 e^{3t}$$

If the driving function is $20e^t - 50 \cos t$, then we assume

$$x_p = Ae^t + B \sin t + C \cos t$$

as a particular solution, or

$$x'_p = Ae^t + B \cos t - C \sin t$$

$$x''_p = Ae^t - B \sin t - C \cos t$$

Substituting the above in differential equation, we get

$$(Ae^t - B \sin t - C \cos t) - 2(Ae^t + B \cos t - C \sin t)$$
$$- 3(Ae^t + B \sin t + C \cos t) = 20\, e^t - 50 \cos t$$

Comparing exponential, sine and cosine terms on both sides

$$A - 2A - 3A = 20 \tag{10.23}$$

$$-B + 2C - 3B = 0 \tag{10.24}$$

$$-C - 2B - 3C = -50 \tag{10.25}$$

From the above equations, we get

$$A = -5, B = 5, C = 10$$

Thus, the particular solution is

$$x_p = -5e^t + 5\sin t + 10\cos t$$

Therefore, the complete solution is

$$\begin{aligned} x &= x_c + x_p \\ &= k_1e^{-t} + k_2e^{3t} - 5e^t + 5\sin t + 10\cos t \end{aligned}$$

Problem 10.6 Find the general solution of the differential equation,

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 2t^2 + e^t + 2te^t + 4e^{3t}$$

Solution The corresponding homogeneous equation is

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 0$$

The auxiliary equation is

$$(m^2 - m - 6) = 0$$

The roots of the equation are

$$(m - 3)(m + 2) = 0$$

Thus, the complementary function is

$$x_c = k_1e^{3t} + k_2e^{-2t}$$

To find the particular solution, we assume

$$x_p = p_1t^2 + p_2t + p_3 + p_4e^{3t} + p_5t^2e^t + p_6te^t$$

From this, we have

$$\begin{aligned} x'_p &= 2tp_1 + p_2 + 3p_4e^{3t} + 2p_5te^t + p_5t^2e^t + p_6te^t + p_6e^t \\ x''_p &= 2p_1 + 9p_4e^{3t} + 2p_5e^t + 2p_5te^t + p_5t^2e^t + 2p_5te^t \\ &\quad + p_6te^t + p_6e^t + p_6e^t \end{aligned}$$

Substituting x_p, x'_p and x''_p into differential equation and equating coefficients of like terms, we get

$$p_1 = 1, p_2 = 3, p_3 = 3.5, p_4 = 2, p_5 = -1, p_6 = -3$$

Thus, the particular integral is

$$x_p = t^2 + 3t + 3.5 + 2e^{3t} - t^2e^t - 3te^t$$

Therefore, the general solution is

$$x = x_c + x_p = k_1e^{3t} + k_2e^{-2t} + t^2 + 3t + 3.5 + 2e^{3t} - t^2e^t - 3te^t$$

Problem 10.7 Find the general solution of differential equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 35x = t\sin t$$

where $x(0) = 5$; $x'(0) = 3$

Solution The corresponding homogeneous equation is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 35x = 0$$

The auxiliary equation is

$$(m^2 + 2m - 35) = 0$$

The roots of the equation are

$$(m + 7)(m - 5) = 0$$

Thus the complementary function is

$$x_c = k_1 e^{-7t} + k_2 e^{+5t}$$

$$x_c' = -7k_1 e^{-7t} + 5k_2 e^{+5t}$$

At $t = 0$, $x_c(0) = 5$

$$k_1 + k_2 = 5$$

At $t = 0$, $x_c'(0) = 3$

$$-7k_1 + 5k_2 = 3$$

Solving the above equations, we get

$$k_1 = 1.83,\ k_2 = 3.17$$

Therefore, the complementary function is

$$x_c = 1.83\ e^{-7t} + 3.17\ e^{-5t}$$

To find the particular solution, we assume

$$x_p = p_1 t \sin t + p_2 t \cos t + p_3 \sin t + p_4 \cos t$$

Then $x_p' = p_1 \sin t + p_1 t \cos t + p_2 \cos t - p_2 t \sin t + p_3 \cos t - p_4 \sin t$

$$x_p'' = p_1 \cos t - p_1 t \sin t + p_1 \cos t - p_2 \sin t - p_2 \sin t - p_2 t \cos t - p_3 \sin t - p_4 \cos t$$

Substituting x_p, x_p' and x_p'' into differential equation and equating coefficients of like terms, we get

$$p_1 = 1.01;\ p_2 = 0.056;\ p_3 = 0.05;\ p_4 = 0.062$$

Thus the particular integral is

$$x_p = 1.01t \sin t + 0.056t \cos t + 0.05 \sin t + 0.062 \cos t$$

Therefore, the complete solution is

$$x = x_c + x_p = 1.83e^{-7t} + 3.17e^{-5t} + 1.01t \sin t + 0.056t \cos t + 0.05 \sin t + 0.062 \cos t$$

Problem 10.8 For the series RL circuit shown in Fig. 10.6, find the current at time $t > 0$. The switch is closed at $t = 0$. Assume the initial current in the circuit is zero.

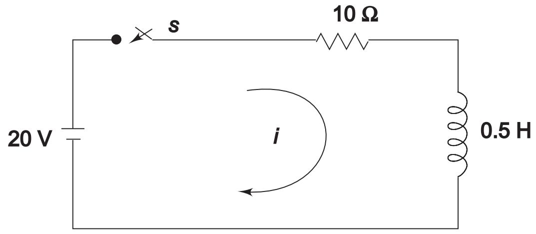


Fig. 10.6

Solution By applying Kirchhoff's law to the circuit, we have

$$0.5\frac{di}{dt} + 10i = 20$$

or $$\frac{di}{dt} + 20i = 40$$

The auxiliary equation is

$$(m + 20) = 0$$

Therefore, the complementary function

$$i_c = k_1 e^{-20t}$$

The particular integral is

$$i_p = 20e^{-20t} \int e^{20t}\, dt = \frac{20}{20} = 1$$

Therefore, the complete solution is

$$i = i_c + i_p = k_1 e^{-20t} + 1$$

At $t = 0$, $i(0) = 0$

$\therefore$ $$k_1 = -1$$

The complete solution is

$$i = (1 - e^{-20t})\ \text{A}$$

Problem 10.9 For the circuit shown in Fig. 10.7, find the current at $t > 0$. The switch is closed at $t = 0$. Assume no initial charge on the capacitor.

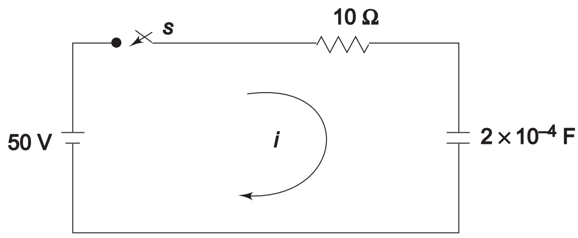


Fig. 10.7

Solution By applying Kirchhoff's law to the circuit, we have

$$10i + \frac{1}{2 \times 10^{-4}} \, idt = 50$$

Differentiating the above equation, we get

$$10\frac{di}{dt} + \frac{i}{2 \times 10^{-4}} = 0$$

$$\frac{di}{dt} + 0.5 \times 10^3 i = 0$$

The auxiliary equation is

$$(m + 500) = 0$$

Since, the equation is a linear homogeneous one, there is no particular integral. Therefore, the complementary function is

$$i = k_1 e^{-500t}$$

At $t = 0$, the current passing through the circuit is $i = \dfrac{V}{R} = \dfrac{50}{10} = 5\text{A}$

$\therefore$ $i(0) = 5$

At $t = 0$, $k_1 = 5$

The current equation becomes

$$i = 5e^{-500t}$$

Problem 10.10 For the circuit shown in Fig. 10.8, determine the current at any time $t > 0$. The switch is closed at $t = 0$. Assume that initial current and initial charge on the capacitor are zero.

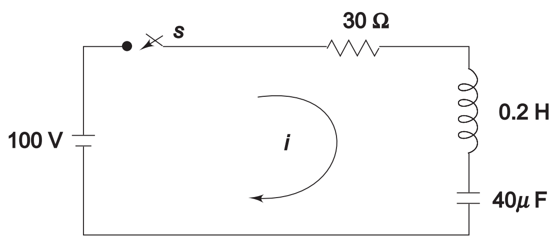


Fig. 10.8

Solution By applying Kirchhoff's law, we have

$$30i + 0.2\frac{di}{dt} + \frac{1}{40 \times 10^{-6}} \int idt = 100$$

Differentiating the above equation, we have

$$\frac{d^2 i}{dt^2} + 150\frac{di}{dt} + \frac{1}{8 \times 10^{-6}} i = 0$$

The roots of the auxiliary equation are

$$m_1 = -75 + j345.5$$
$$m_2 = -75 - j345.5$$

Hence, the current is

$$i = e^{-75t}(c_1 \cos 345.5t + c_2 \sin 345.5t)\text{A}$$

At $t = 0$, $\quad i(0) = 0$

$\therefore \quad c_1 = 0$

$$i' = c_2\{e^{-75t}(345.5) \cos 345.5t + e^{-75t} (-75) \sin 345.5t\}$$

At $t = 0$, the complete voltage appears across inductor

$$0.2\frac{di}{dt} = 100$$

$$\therefore \quad \frac{di}{dt} = 500$$

$\therefore$ At $t = 0$, $i'(0) = 500$

$$500 = c_2 (345.5)$$
$$c_2 = 1.45$$

Thus the required current is

$$i = 1.45e^{-75t} \sin 345.5t \text{ A}$$

Practice Problems

10.1 Find the general solution of each of the following differential equations.

(a) $4\dfrac{d^2x}{dt^2} - 12\dfrac{dx}{dt} + 5x = 0$

(b) $4\dfrac{d^2x}{dt^2} + x = 0$

10.2 Find the general solution of each of the following differential equations.

(a) $\dfrac{d^5x}{dt^5} - 2\dfrac{d^4x}{dt^4} + \dfrac{d^3x}{dt^3} = 0$

(b) $\dfrac{d^4x}{dt^4} + 6\dfrac{d^3x}{dt^3} + 15\dfrac{d^2x}{dt^2} + 20\dfrac{dx}{dt} + 12x = 0$

(c) $\dfrac{d^4x}{dt^4} = 0$

10.3 Find the general solution of each of the following differential equations.

(a) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 8x = 0$

where $x(0) = 2x'(0) = 4$

(b) $9\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + x = 0$

$x(0) = 4;\ x'(0) = -1$

(c) $4\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 37x = 0$

where $x(0) = 3,\ x'(0) = -2$

(d) $\dfrac{d^3x}{dt^3} - 5\dfrac{d^2x}{dt^2} + 9\dfrac{dx}{dt} - 5x = 0$

$x(0) = 0,\ x'(0) = 1,\ x'(0) = 3$

10.4 Solve the following differential equations

(a) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x = \sin 2t$

(b) $\dfrac{d^3x}{dt^3} + 3\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + x = e^{-t}$

(c) $\dfrac{d^3x}{dt^3} - 3\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} - 2x = e^{t} + \cos t$

(d) $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = 3t^2 e^{2t} \sin 2t$

(e) $\dfrac{d^4x}{dt^4} + 2\dfrac{d^2x}{dt^2} + x = t^2 \cos^2 t$

10.5 Solve the following differential equations

(a) $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 3x = 4te^{-3t}$

$x(0) = 6x'(0) = 3$

(b) $\dfrac{d^2x}{dt^2} + 4x = 3te^{t} + 2e^{t} - \sin t$

$x(0) = 1,\ x'(0) = 0,\ x'(0) = 2$

(c) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 9x = 8t^2 + 3 - 6e^{2t}$

$x''(0) = 3,\ x'(0) = 0,\ x(0) = 3$

(d) $\dfrac{d^3x}{dt^3} - 6\dfrac{d^2x}{dt^2} + 9\dfrac{dx}{dt} - 4x = 2te^{2t} + 6e^{t}$

$x(0) = 1,\ x'(0) = 0$

10.6 For the circuit shown in Fig. 10.9, determine the current at any time $t > 0$. The switch is closed at $t = 0$. Assume no initial charge on the capacitor.

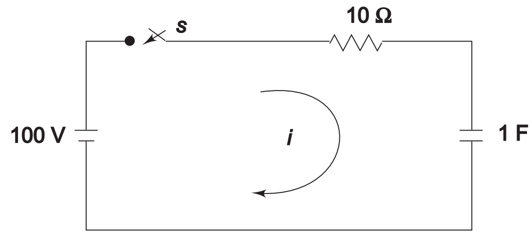


Fig. 10.9

10.7 For the circuit shown in Fig. 10.10, determine the current at any time $t > 0$. The switch is closed at $t = 0$. Assume no initial current in the circuit.

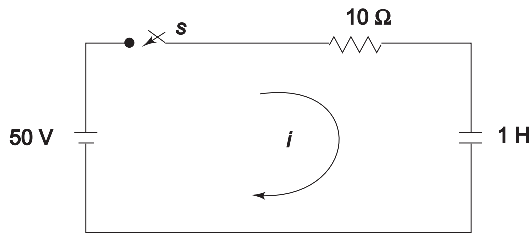


Fig. 10.10

10.8 For the circuit shown in Fig. 10.11, determine the current at any time $t > 0$. The switch is closed at $t = 0$. Assume no initial charge on the capacitor.

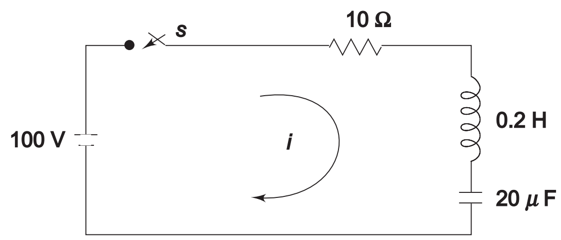


Fig. 10.11

10.9 For the circuit shown in Fig. 10.12, determine the current at any time $t > 0$. The switch is closed at $t = 0$.

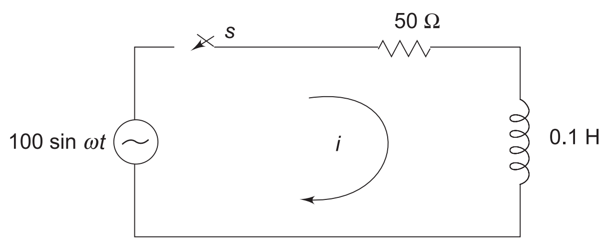


Fig. 10.12

10.10 For the circuit shown in Fig. 10.13, determine the current at any time $t > 0$. The switch is closed at $t = 0$. Assume no initial charge on the capacitor.

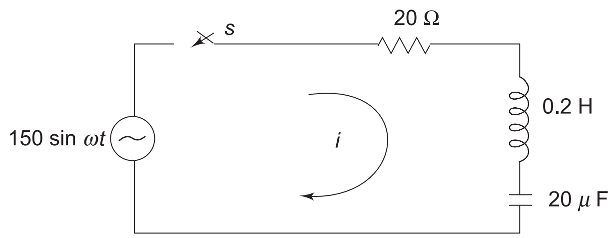


Fig. 10.13

Objective-type Questions

1. The degree of the differential equation $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = 5\,\dfrac{d^2y}{dx^2}$ is
   (a) two (b) three
   (c) one (d) four

2. The particular integral of the equation $\dfrac{d^4y}{dx^4} + 4y = x^4$ will be
   (a) $\dfrac{1}{4}(x^4 - 6)$ (b) $\dfrac{1}{4}(x^6 - 6)$
   (c) $\dfrac{1}{4}(x^6 + 6)$ (d) $\dfrac{1}{4}(x + 6)$

3. The differential equation $\dfrac{d^2y}{dx^2} + \dfrac{a}{x}\dfrac{dy}{dx} + k^2y = 0$, where a is any constant, can be expressed as $y(x) = x^n[c_1J_n(kx) + c_2J_{-n}(kx)]$ where n is
   (a) an odd integer (b) an integer
   (c) an even integer (d) a fraction

4. The Bessal's differential equation $xy'' + xy' + xy = 0$ is a
   (a) linear non-homogeneous equation
   (b) non-linear equation
   (c) non-linear homogeneous equation with constant coefficients
   (d) linear homogeneous with variable coefficients

5. The general solution of $(D^2 + 4)y = 0$ is
   (a) $y = A\cos(2x + B)$ (b) $y = Ae^{2x} + Be^{-2x}$
   (c) $y = A\cos 2x + B\sin 2x$ (d) $y = e^{2x}(A - Bx)$

6. The complementary function of $(D^2 + 9)y = 1$ is

   (a) $\frac{1}{9}$ (b) $\frac{-1}{9}$

   (c) $c_1e^{3x} + c_2e^{-3x}$ (d) $c_1 \sin 3x + c_2 \cos 3x$

7. The solution of the differential equation

$$y''(t) - 2y'(t) + y(t) = 1 \text{ is}$$

   (a) $y(t) = c_1e^t + c_2e^{-t} + 1$ (b) $y(t) = (c_1 + c_2t)e^t + 1$

   (c) $y(t) = (c_1 + c_2t)e^{-t} + 1$ (d) $y(t) = (c_1 + c_2)te^t - 1$

8. The differential equation of an electric current containing resistance R and a capacitor C in series with the voltage source V is

   (a) $\frac{dV}{dt} = Ri + \int \frac{1}{C} idt$ (b) $\frac{dV}{dt} = R\frac{di}{dt} + \int \frac{1}{C} idt$

   (c) $\frac{dV}{dt} = R\frac{di}{dt} + \frac{i}{C}$ (d) $V = R\frac{di}{dt} + \frac{i}{C}$

9. The particular integral of differential equation

$$3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x \text{ is}$$

   (a) x (b) $\frac{x}{2}$

   (c) $\frac{x}{3}$ (d) x^4

10. The differential equation of an electric current containing resistance R and an inductor L in series with a constant voltage source V is

   (a) $V = R\int idt + Li$ (b) $V = Ri + L\int \left(\frac{di}{dt}\right) dt$

   (c) $V = Ri + L\int idt$ (d) $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} = 0$

Chapter 11

Transients

11.1 STEADY STATE AND TRANSIENT RESPONSE

A circuit having constant sources is said to be in steady state if the currents and voltages do not change with time. Thus, circuits with currents and voltages having constant amplitude and constant frequency sinusoidal functions are also considered to be in a steady state. That means that the amplitude or frequency of a sinusoid never changes in a steady state circuit.

In a network containing energy storage elements, with change in excitation, the currents and voltages change from one state to other state. The behaviour of the voltage or current when it is changed from one state to another is called the *transient state*. The time taken for the circuit to change from one steady state to another steady state is called the *transient time*. The application of KVL and KCL to circuits containing energy storage elements results in differential, rather than algebraic, equations. When we consider a circuit containing storage elements which are independent of the sources, the response depends upon the nature of the circuit and is called the *natural response*. Storage elements deliver their energy to the resistances. Hence the response changes with time, gets saturated after some time, and is referred to as the *transient response*. When we consider sources acting on a circuit, the response depends on the nature of the source or sources. This response is called *forced response*. In other words, the complete response of a circuit consists of two parts: the forced response and the transient response. When we consider a differential equation, the complete solution consists of two parts: the complementary function and the particular solution. The complementary function dies out after short interval, and is referred to as the transient response or source free response. The particular solution is the steady state response, or the forced response. The first step in finding the complete solution of a circuit is to form a differential equation for the circuit. By obtaining the differential equation, several methods can be used to find out the complete solution.

11.2 DC RESPONSE OF AN R-L CIRCUIT

Consider a circuit consisting of a resistance and inductance as shown in Fig. 11.1. The inductor in the circuit is initially uncharged and is in series with the resistor. When the switch S is closed, we can find the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

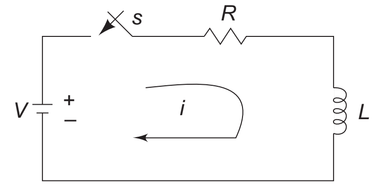


Fig. 11.1

$$V = Ri + L\frac{di}{dt} \tag{11.1}$$

or

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \tag{11.2}$$

In the above equation, the current i is the solution to be found and V is the applied constant voltage. The voltage V is applied to the circuit only when the switch S is closed. The above equation is a linear differential equation of first order. Comparing it with a non-homogeneous differential equation

$$\frac{dx}{dt} + Px = K \tag{11.3}$$

whose solution is

$$x = e^{-pt}\int Ke^{+Pt}\,dt + ce^{-Pt} \tag{11.4}$$

where c is an arbitrary constant. In a similar way, we can write the current equation as

$$i = ce^{-(R/L)t} + e^{-(R/L)t}\int \frac{V}{L}e^{(R/L)t}\,dt$$

$$\therefore \qquad i = ce^{-(R/L)t} + \frac{V}{R} \tag{11.5}$$

To determine the value of c in Eq. 11.5, we use the initial conditions. In the circuit shown in Fig. 11.1, the switch S is closed at $t = 0$. At $t = 0^-$, i.e. just before closing the switch S, the current in the inductor is zero. Since the inductor does not allow sudden changes in currents, at $t = 0^+$ just after the switch is closed, the current remains zero.

Thus at $\qquad t = 0,\ i = 0$

Substituting the above condition in Eq. 11.5, we have

$$0 = c + \frac{V}{R}$$

Hence $$c = -\frac{V}{R}$$

Substituting the value of c in Eq. 5, we get

$$i = \frac{V}{R} - \frac{V}{R}\exp\left(-\frac{R}{L}t\right)$$

$$i = \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right) \tag{11.6}$$

Equation 11.6 consists of two parts, the steady state part V/R, and the transient part $(V/R)e^{-(R/L)t}$. When switch S is closed, the response reaches a steady state value after a time interval as shown in Fig. 11.2.

Here the transition period is defined as the time taken for the current to reach its final or steady state value from its initial value. In the transient part of the solution, the quantity L/R is important in describing the curve since L/R is the time required for the current to reach from its initial value of zero to the final value V/R. The time constant of a function $\frac{V}{R}e^{-\left(\frac{R}{L}\right)t}$ is the time at which the exponent of e is unity, where e is the base of the natural logarithms. The term L/R is called the *time constant* and is denoted by τ

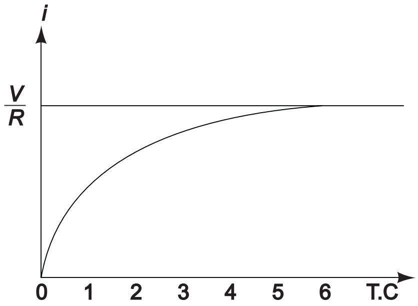


Fig. 11.2

$$\therefore \qquad \tau = \frac{L}{R} \text{ sec}$$

$\therefore$ The transient part of the solution is

$$i = -\frac{V}{R}\exp\left(-\frac{R}{L}t\right) = -\frac{V}{R}e^{-t/\tau}$$

At one TC, i.e. at one time constant, the transient term reaches 36.8 percent of its initial value.

$$i(\tau) = -\frac{V}{R}e^{-t/\tau} = -\frac{V}{R}e^{-1} = -0.368\frac{V}{R}$$

Similarly,

$$i(2\tau) = -\frac{V}{R}e^{-2} = -0.135\frac{V}{R}$$

$$i(3\tau) = -\frac{V}{R}e^{-3} = -0.0498\ \frac{V}{R}$$

$$i(5\tau) = -\frac{V}{R}e^{-5} = -0.0067\ \frac{V}{R}$$

After 5 TC, the transient part reaches more than 99 percent of its final value.

In Fig. 11.1, we can find out the voltages and powers across each element by using the current.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R}\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

$$\therefore \qquad v_R = V\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

Similarly, the voltage across the inductance is

$$v_L = L\frac{di}{dt}$$

$$= L\frac{V}{R} \times \frac{R}{L}\exp\left(-\frac{R}{L}t\right) = V\exp\left(-\frac{R}{L}t\right)$$

The response are shown in Fig. 11.3

Power in the resistor is

$$p_R = v_R i = V\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\frac{V}{R}$$

$$= \frac{V^2}{R}\left(1 - 2\exp\left(-\frac{R}{L}t\right) + \exp\left(-\frac{2R}{L}t\right)\right)$$

Power in the inductor is

$$p_L = v_L i = V\exp\left(-\frac{R}{L}t\right) \times \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right)$$

$$= \frac{V^2}{R}\left(\exp\left(-\frac{R}{L}t\right) - \exp\left(-\frac{2R}{L}t\right)\right)$$

The responses are shown in Fig. 11.4.

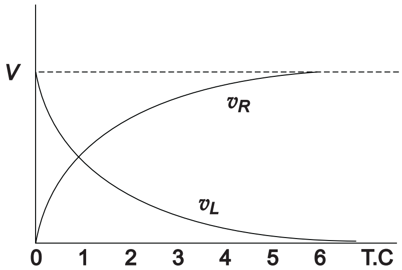


Fig. 11.3

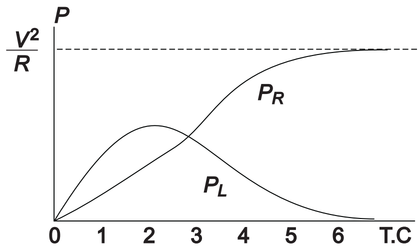


Fig. 11.4

Example 11.1 A series RL circuit with $R = 30\ \Omega$ and $L = 15$ H has a constant voltage $V = 60$ V applied at $t = 0$ as shown in Fig. 11.5. Determine the current i, the voltage across resistor and the voltage across the inductor.

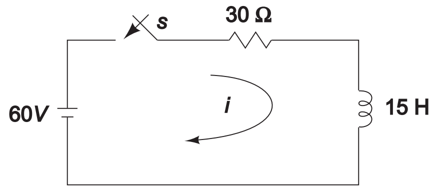


Fig. 11.5

Solution By applying Kirchhoff's voltage law, we get

$$15\frac{di}{dt} + 30i = 60$$

$$\therefore \qquad \frac{di}{dt} + 2i = 4$$

The general solution for a linear differential equation is

$$i = ce^{-Pt} + e^{-Pt}\int Ke^{Pt}\,dt$$

where $P = 2$, $K = 4$

$$\therefore \qquad i = ce^{-2t} + e^{-2t}\int 4e^{2t}\,dt$$

$$\therefore \qquad i = ce^{-2t} + 2$$

At $t = 0$, the switch S is closed.

Since the inductor never allows sudden changes in currents. At $t = 0^+$ the current in the circuit is zero.

Therefore at $t = 0^+$, $i = 0$

$$\therefore \qquad 0 = c + 2$$

$$\therefore \qquad c = -2$$

Substituting the value of c in the current equation, we have

$$i = 2(1 - e^{-2t})\text{ A}$$

Voltage across resistor $v_R = iR$

$$= 2(1 - e^{-2t}) \times 30 = 60\,(1 - e^{-2t})\text{ V}$$

Voltage across inductor $v_L = L\dfrac{di}{dt}$

$$= 15 \times \frac{d}{dt}\,2(1 - e^{-2t}) = 30 \times 2e^{-2t} = 60e^{-2t}\text{ V}$$

11.3 DC RESPONSE OF AN R-C CIRCUIT

Consider a circuit consisting of resistance and capacitance as shown in Fig. 11.6. The capacitor in the circuit is initially uncharged, and is in series with a resistor. When the switch S is closed at $t = 0$, we can determine the complete solution for the current. Application of the Kirchhoff's voltage law to the circuit results in the following differential equation.

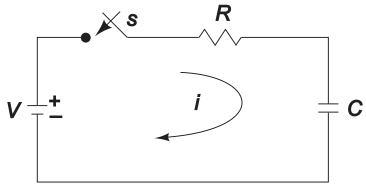


Fig. 11.6

$$V = Ri + \frac{1}{C}\int i\,dt \tag{11.7}$$

By differentiating the above equation, we get

$$0 = R\frac{di}{dt} + \frac{i}{C} \tag{11.8}$$

or
$$\frac{di}{dt} + \frac{1}{RC}i = 0 \tag{11.9}$$

Equation 11.9 is a linear differential equation with only the complementary function. The particular solution for the above equation is zero. The solution for this type of differential equation is

$$i = ce^{-t/RC} \tag{11.10}$$

Here, to find the value of c, we use the initial conditions.

In the circuit shown in Fig. 11.6, switch S is closed at $t = 0$. Since the capacitor never allows sudden changes in voltage, it will act as a short circuit at $t = 0^+$. So, the current in the circuit at $t = 0^+$ is V/R

$\therefore$ At $t = 0$, the current $i = \dfrac{V}{R}$

Substituting this current in Eq. 11.10, we get

$$\frac{V}{R} = c$$

$\therefore$ The current equation becomes

$$i = \frac{V}{R}e^{-t/RC} \tag{11.11}$$

When switch S is closed, the response decays with time as shown in Fig. 11.7.

In the solution, the quantity RC is the time constant, and is denoted by τ,

where $\tau = RC$ sec

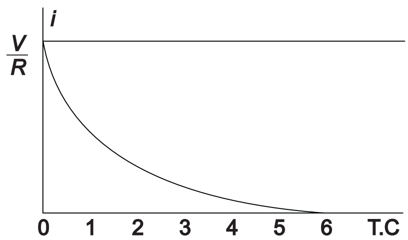


Fig. 11.7

After 5 TC, the curve reaches 99 per cent of its final value. In Fig. 11.6, we can find out the voltage across each element by using the current equation.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R}\, e^{-(1/RC)t}\,;\; v_R = Ve^{-\frac{t}{RC}}$$

Similarly, voltage across the capacitor is

$$v_C = \frac{1}{C}\int i\,dt$$

$$= \frac{1}{C}\int \frac{V}{R}\, e^{-t/RC}\,dt$$

$$= -\left(\frac{V}{RC} \times RC\, e^{-t/RC}\right) + c = -\,Ve^{-t/RC} + c$$

At $t = 0$, voltage across capacitor is zero

$\therefore \quad c = V$

$\therefore \quad v_C = V(1 - e^{-t/RC})$

The responses are shown in Fig. 11.8.

Power in the resistor

$$p_R = v_R i = Ve^{-t/RC} \times \frac{V}{R}\, e^{-t/RC} = \frac{V^2}{R}\, e^{-2t/RC}$$

Power in the capacitor

$$p_C = v_C i = V(1 - e^{-t/RC})\, \frac{V}{R}\, e^{-t/RC}$$

$$= \frac{V^2}{R}\,(e^{-t/RC} - e^{-2t/RC})$$

The responses are shown in Fig. 11.9.

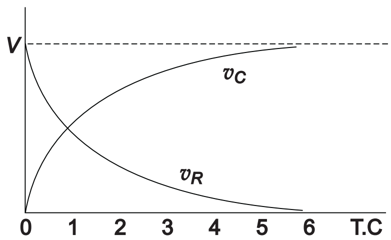


Fig. 11.8

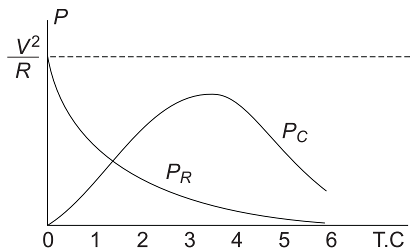


Fig. 11.9

Example 11.2 A series RC circuit consists of resistor of 10 Ω and capacitor of 0.1 F as shown in Fig. 11.10. A constant voltage of 20 V is applied to the circuit at $t = 0$. Obtain the current equation. Determine the voltages across the resistor and the capacitor.

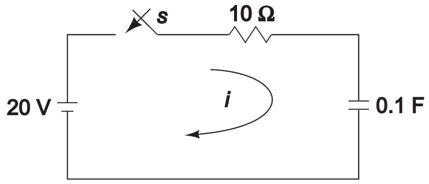


Fig. 11.10

Solution BY applying Kirchhoff's law, we get

$$10i + \frac{1}{0.1}\int i\,dt = 20$$

Differentiating with respect to t we get

$$10\frac{di}{dt} + \frac{i}{0.1} = 0$$

$$\therefore \qquad \frac{di}{dt} + i = 0$$

The solution for the above equation is $i = ce^{-t}$

At $t = 0$, switch S is closed. Since the capacitor does not allow sudden changes in voltage, the current in the circuit is $i = V/R = 20/10 = 2$ A.

At $t = 0$, $i = 2$ A.

$\therefore$ The current equation $i = 2e^{-t}$

Voltage across the resistor is $v_R = i \times R = 2e^{-t} \times 10 = 20e^{-t}$ V

Voltage across the capacitor is $v_C = V\left(1 - e^{-\frac{t}{RC}}\right)$

$$= 20\,(1 - e^{-t})\text{ V}$$

11.4 DC RESPONSE OF AN R-L-C CIRCUIT

Consider a circuit consisting of resistance, inductance and capacitance as shown in Fig. 11.11. The capacitor and inductor are initially uncharged, and are in series with a resistor. When switch S is closed at $t = 0$, we can determine the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

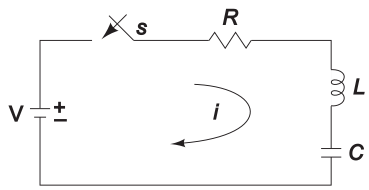


Fig. 11.11

$$V = Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt \qquad (11.12)$$

By differentiating the above equation, we have

$$0 = R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C}i \qquad (11.13)$$

or $$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0 \tag{11.14}$$

The above equation is a second order linear differential equation, with only complementary function. The particular solution for the above equation is zero. Characteristic equation for the above differential equation is

$$\left(D^2 + \frac{R}{L} D + \frac{1}{LC}\right) = 0 \tag{11.15}$$

The roots of Eq. 11.15 are

$$D_1, D_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $$K_1 = -\frac{R}{2L} \text{ and } K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$D_1 = K_1 + K_2 \text{ and } D_2 = K_1 - K_2$$

Here K_2 may be positive, negative or zero.

K_2 is positive, when $\left(\frac{R}{2L}\right)^2 > 1/LC$

The roots are real and unequal, and give the over damped response as shown in Fig. 11.12. Then Eq. 11.14 becomes

$$[D - (K_1 + K_2)]\,[D - (K_1 - K_2)]\, i = 0$$

The solution for the above equation is

$$i = c_1 e^{(K_1 + K_2)t} + c_2\, e^{(K_1 - K_2)t}$$

The current curve for the overdamped case is shown in Fig. 11.12.

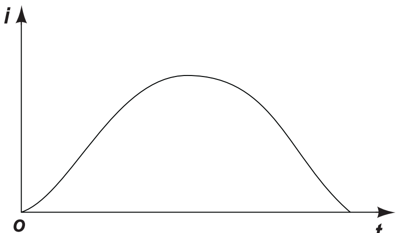


Fig. 11.12

K_2 is negative, when $(R/2L)^2 < 1/LC$

The roots are complex conjugate, and give the underdamped response as shown in Fig. 11.13. Then Eq. 11.14 becomes

$$[D - (K_1 + jK_2)]\,[D - (K_1 - jK_2)]i = 0$$

The solution for the above equation is

$$i = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

The current curve for the underdamped case is shown in Fig. 11.13.

K_2 is zero, when $(R/2L)^2 = 1/LC$

The roots are equal, and give the critically damped response as shown in Fig. 11.14. Then Eq. 11.14 becomes

$$(D - K_1)\,(D - K_1)i = 0$$

The solution for the above equation is

$$i = e^{K_1 t}\,(c_1 + c_2 t)$$

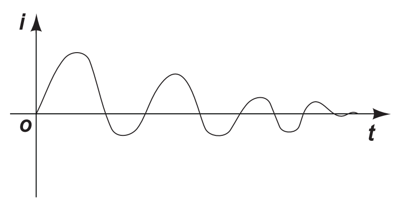


Fig. 11.13

The current curve for the critically damped case is shown in Fig. 11.14.

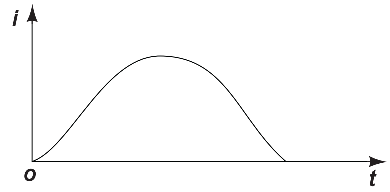


Fig. 11.14

Example 11.3 The circuit shown in Fig. 11.15 consists of resistance, inductance and capacitance in series with a 100 V constant source when the switch is closed at $t = 0$. Find the current transient.

Solution At $t = 0$, switch S is closed when the 100 V source is applied to the circuit and results in the following differential equation.

$$100 = 20i + 0.05\frac{di}{dt} + \frac{1}{20 \times 10^{-6}}\int i dt \tag{11.16}$$

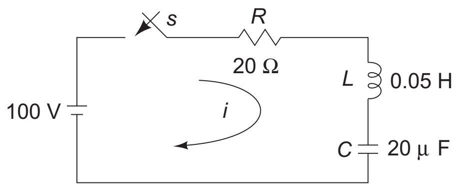


Fig. 11.15

Differentiating the Eq. 11.16, we get

$$0.05\frac{d^2i}{dt^2} + 20\frac{di}{dt} + \frac{1}{20 \times 10^{-6}}i = 0$$

$$\frac{d^2i}{dt^2} + 400\frac{di}{dt} + 10^6 i = 0$$

$$\therefore \quad (D^2 + 400D + 10^6)i = 0$$

$$D_1, D_2 = -\frac{400}{2} \pm \sqrt{\left(\frac{400}{2}\right)^2 - 10^6}$$

$$= -200 \pm \sqrt{(200)^2 - 10^6}$$

$$D_1 = -200 + j979.8$$

$$D_2 = -200 - j979.8$$

Therefore the current

$$i = e^{+k_1 t}[c_1 \cos K_2 t + c_2 \sin K_2 t)]$$

$$i = e^{-200t}[c_1 \cos 979.8t + c_2 \sin 979.8t)] \text{ A}$$

At $t = 0$, the current flowing through the circuit is zero

$$i = 0 = (1)[c_1 \cos 0 + c_2 \sin 0]$$

$\therefore \quad c_1 = 0$

$\therefore \quad i = e^{-200t}\, c_2 \sin 979.8t$ A

Differentiating, we have

$$\frac{di}{dt} = c_2\,[e^{-200t}\, 979.8 \cos 979.8t + e^{-200t}\,(-200) \sin 979.8t)]$$

At $t = 0$, the voltage across inductor is 100 V

$$\therefore \quad L\frac{di}{dt} = 100$$

or

$$\frac{di}{dt} = 2000$$

At $t = 0$

$$\frac{di}{dt} = 2000 = c_2\, 979.8 \cos 0$$

$$\therefore \quad c_2 = \frac{2000}{979.8} = 2.04$$

The current equation is

$$i = e^{-200t}\,(2.04 \sin 979.8t) \text{ A}$$

11.5 SINUSOIDAL RESPONSE OF R-L CIRCUIT

Consider a circuit consisting of resistance and inductance as shown in Fig. 11.16. The switch, S, is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V\cos(\omega t + \theta)$ is applied to the series R-L circuit, where V is the amplitude of the wave and θ is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

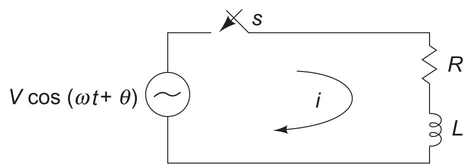


Fig. 11.16

$$V\cos(\omega t + \theta) = Ri + L\frac{di}{dt} \tag{11.17}$$

$$\therefore \quad \frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t + \theta)$$

The corresponding characteristic equation is

$$\left(D + \frac{R}{L}\right)i = \frac{V}{L}\cos(\omega t + \theta) \tag{11.18}$$

For the above equation, the solution consists of two parts, viz. complementary function and particular integral.

The complementary function of the solution i is

$$i_c = ce^{-t(R/L)} \tag{11.19}$$

The particular solution can be obtained by using undetermined co-efficients.

By assuming $$i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta) \tag{11.20}$$

$$i_p' = -A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) \tag{11.21}$$

Substituting Eqs 11.20 and 11.21 in Eq. 11.18, we have

$$\{-A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) + \frac{R}{L}\{A\cos(\omega t+\theta)$$

$$+ B\sin(\omega t+\theta)\} = \frac{V}{L}\cos(\omega t+\theta)$$

or $$\left(-A\omega + \frac{BR}{L}\right)\sin(\omega t+\theta) + \left(B\omega + \frac{AR}{L}\right)\cos(\omega t+\theta) = \frac{V}{L}\cos(\omega t+\theta)$$

Comparing cosine terms and sine terms, we get

$$-A\omega + \frac{BR}{L} = 0$$

$$B\omega + \frac{AR}{L} = \frac{V}{L}$$

From the above equations, we have

$$A = V\frac{R}{R^2+(\omega L)^2}$$

$$B = V\frac{\omega L}{R^2+(\omega L)^2}$$

Substituting the values of A and B in Eq. 11.20, we get

$$i_p = V\frac{R}{R^2+(\omega L)^2}\cos(\omega t+\theta) + V\frac{\omega L}{R^2+(\omega L)^2}\sin(\omega t+\theta) \tag{11.22}$$

Putting $$M\cos\phi = \frac{VR}{R^2+(\omega L)^2}$$

and $$M\sin\phi = V\frac{\omega L}{R^2+(\omega L)^2},$$

to find M and ϕ, we divide one equation by the other

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\omega L}{R}$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi + M^2\sin^2\phi = \frac{V^2}{R^2+(\omega L)^2}$$

or $$M = \frac{V}{\sqrt{R^2+(\omega L)^2}}$$

$\therefore$ The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2+(\omega L)^2}}\cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right) \tag{11.23}$$

The complete solution for the current $i = i_c + i_p$

$$i = ce^{-t(R/L)} + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Since the inductor does not allow sudden changes in currents, at $t = 0$, $i = 0$

$$\therefore \qquad c = -\frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)$$

The complete solution for the current is

$$i = e^{-(R/L)t}\left[\frac{-V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)\right]$$
$$+ \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Example 11.4 In the circuit shown in Fig. 11.17, determine the complete solution for the current, when switch S is closed at $t = 0$. Applied voltage is $v(t) = 100 \cos(10^3 t + \pi/2)$. Resistance $R = 20\ \Omega$ and inductance $L = 0.1$ H.

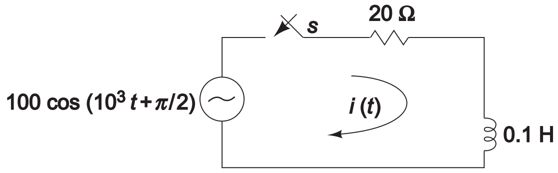


Fig. 11.17

Solution By applying Kirchhoff's voltage law to the circuit, we have

$$20i + 0.1\frac{di}{dt} = 100\cos(10^3 t + \pi/2)$$

$$\frac{di}{dt} + 200i = 1000\cos(1000t + \pi/2)$$

$$(D = 200)i = 1000\cos(1000t + \pi/2)$$

The complementary function $i_c = ce^{-200t}$

By assuming particular integral as

$$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta)$$

we get

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

where $\omega = 1000$ rad/sec $V = 100$ V

$$\theta = \pi/2$$

$$L = 0.1\text{ H},\ R = 20\ \Omega$$

Substituting the values in the above equation, we get

$$i_p = \frac{100}{\sqrt{(20)^2 + (1000 \times 0.1)^2}} \cos\left(1000t + \frac{\pi}{2} - \tan^{-1}\frac{100}{20}\right)$$

$$= \frac{100}{101.9} \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

$$= 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

The complete solution is

$$i = ce^{-200t} + 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

At $t = 0$, the current flowing through the circuit is zero, i.e. $i = 0$

$$\therefore \qquad c = -0.98 \cos\left(\frac{\pi}{2} - 78.6°\right)$$

$\therefore$ The complete solution is

$$i = \left[-0.98 \cos\left(\frac{\pi}{2} - 78.6°\right)\right] e^{-200t} + 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

11.6 SINUSOIDAL RESPONSE OF R-C CIRCUIT

Consider a circuit consisting of resistance and capacitance in series as shown in Fig. 11.18. The switch, S, is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V \cos(\omega t + \theta)$ is applied to the R-C circuit, where V is the amplitude of the wave and θ is the phase angle. Applying Kirchhoff's voltage law to the circuit results in the following differential equation.

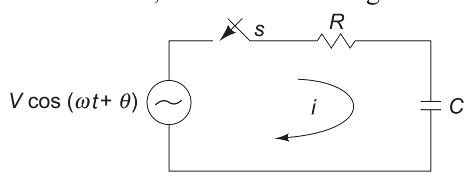


Fig. 11.18

$$V \cos(\omega t + \theta) = Ri + \frac{1}{C}\int i\,dt \tag{11.24}$$

$$R\frac{di}{dt} + \frac{i}{C} = -V\omega \sin(\omega t + \theta)$$

$$\left(D + \frac{1}{RC}\right)i = -\frac{V\omega}{R}\sin(\omega t + \theta) \tag{11.25}$$

The complementary function $i_C = ce^{-t/RC}$ (11.26)

The particular solution can be obtained by using undetermined coefficients.

$$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta) \tag{11.27}$$

$$i'_P = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta) \tag{11.28}$$

Substituting Eqs 11.27 and 11.28 in Eq. 11.25, we get

$$\{-A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)\}$$

$$+ \frac{1}{RC}\{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\}$$

$$= -\frac{V\omega}{R}\sin(\omega t + \theta)$$

Comparing both sides, $$-A\omega + \frac{B}{RC} = -\frac{V\omega}{R}$$

$$B\omega + \frac{A}{RC} = 0$$

From which,

$$A = \frac{VR}{R^2 + \left(\frac{1}{\omega c}\right)^2}$$

and $$B = \frac{-V}{\omega C\left[R^2 + \left(\frac{1}{\omega c}\right)^2\right]}$$

Substituting the values of A and B in Eq. 11.27, we have

$$i_p = \frac{VR}{R^2 + \left(\frac{1}{\omega c}\right)^2}\cos(\omega t + \theta) + \frac{-V}{\omega C\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}\sin(\omega t + \theta)$$

Putting $$M\cos\phi = \frac{VR}{\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$

and $$M\sin\phi = \frac{V}{\omega C\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$

To find M and ϕ, we divide one equation by the other,

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{1}{\omega CR}$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi + M^2\sin^2\phi = \frac{V^2}{\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$

$\therefore$ $$M = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \tag{11.29}$$

The complete solution for the current $i = i_c + i_p$

$$\therefore \qquad i = ce^{-(t/RC)} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \qquad (11.30)$$

Since the capacitor does not allow sudden changes in voltages at $t = 0$, $i = \frac{V}{R}$ $\cos\theta$

$$\therefore \qquad \frac{V}{R}\cos\theta = c + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$c = \frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

The complete solution for the current is

$$i = e^{-(t/RC)}\left[\frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)\right]$$

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \qquad (11.31)$$

Example 11.5 In the circuit shown in Fig. 11.19, determine the complete solution for the current when switch S is closed at $t = 0$. Applied voltage is $v(t) = 50\cos\left(10^2 t + \frac{\pi}{4}\right)$. Resistance $R = 10\ \Omega$ and capacitance $C = 1\ \mu F$.

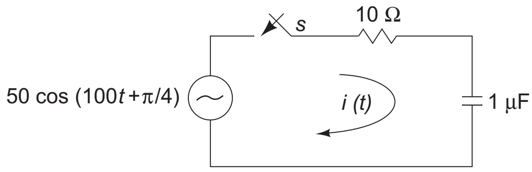


Fig. 11.19

Solution By applying Kirchhoff's voltage law to the circuit, we have

$$10i + \frac{1}{1\times 10^{-6}}\int i\,dt = 50\cos\left(100t + \frac{\pi}{4}\right)$$

$$10\frac{di}{dt}+\frac{i}{1\times 10^{-6}} = -5(10)^3 \sin\left(100t+\frac{\pi}{4}\right)$$

$$\frac{di}{dt}+\frac{i}{10^{-5}} = -500 \sin\left(100t+\frac{\pi}{4}\right)$$

$$\left(D+\frac{1}{10^{-5}}\right)i = -500 \sin\left(100t+\frac{\pi}{4}\right)$$

The complementary function is $i_C = ce^{-t/10^{-5}}$. By assuming particular integral as $i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta)$,

we get
$$i_p = \frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}\cos\left(\omega t+\theta+\tan^{-1}\frac{1}{\omega CR}\right)$$

where $\omega = 100$ rad/sec $\quad \theta = \pi/4$

$C = 1\mu$F $\quad R = 10\ \Omega$

Substituting the values in the above equation, we have

$$i_p = \frac{50}{\sqrt{(10)^2+\left(\frac{1}{100\times 10^{-6}}\right)^2}}\cos\left(\omega t+\frac{\pi}{4}+\tan^{-1}\frac{1}{100\times 10^{-6}\times 10}\right)$$

$$i_p = 4.99\times 10^{-3}\cos\left(100t+\frac{\pi}{4}+89.94°\right)$$

At $t = 0$, the current flowing through the circuit is

$$\frac{V}{R}\cos\theta = \frac{50}{10}\cos\pi/4 = 3.53\text{ A}$$

$$i = \frac{V}{R}\cos\theta = 3.53\text{ A}$$

$$\therefore \quad i = ce^{-t/10^{-5}} + 4.99\times 10^{-3}\cos\left(100t+\frac{\pi}{4}+89.94°\right)$$

At $\quad t = 0$

$$c = 3.53 - 4.99\times 10^{-3}\cos\left(\frac{\pi}{4}+89.94°\right)$$

Hence the complete solution is

$$i = \left[3.53 - 4.99\times 10^{-3}\cos\left(\frac{\pi}{4}+89.94°\right)\right]e^{-(t/10^{-5})}$$
$$+\ 4.99\times 10^{-3}\cos\left(100t+\frac{\pi}{4}+89.94°\right)$$

11.7 SINUSOIDAL RESPONSE OF R-L-C CIRCUIT

Consider a circuit consisting of resistance, inductance and capacitance in series as shown in Fig. 11.20. Switch S is closed at $t = 0$. At $t = 0$, a sinusoidal voltage

$V\cos(\omega t + \theta)$ is applied to the RLC series circuit, where V is the amplitude of the wave and θ is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

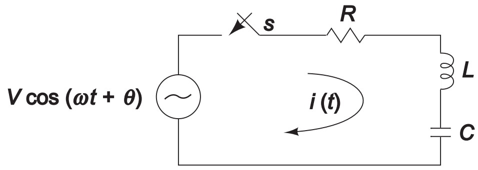


Fig. 11.20

$$V\cos(\omega t + \theta) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt \tag{11.32}$$

Differentiating the above equation, we get

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + i/C = -V\omega\sin(\omega t + \theta)$$

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = -\frac{V\omega}{L}\sin(\omega t + \theta) \tag{11.33}$$

The particular solution can be obtained by using undetermined coefficients. By assuming

$$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta) \tag{11.34}$$

$$i'_p = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta) \tag{11.35}$$

$$i''_p = -A\omega^2\cos(\omega t + \theta) - B\omega^2\sin(\omega t + \theta) \tag{11.36}$$

Substituting ip, i'_p and i''_p in Eq. 11.33, we have

$$\{-A\omega^2\cos(\omega t + \theta) - B\omega^2\sin(\omega t + \theta)\}$$

$$+\frac{R}{L}\{-A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)\}$$

$$+\frac{1}{LC}\{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\} = -\frac{V\omega}{L}\sin(\omega t + \theta) \tag{11.37}$$

Comparing both sides, we have

Sine coefficients.

$$-B\omega^2 - A\frac{\omega R}{L} + \frac{B}{LC} = -\frac{V\omega}{L}$$

$$A\left(\frac{\omega R}{L}\right) + B\left(\omega^2 - \frac{1}{LC}\right) = \frac{V\omega}{L} \tag{11.38}$$

Cosine coefficients

$$-A\omega^2 + B\frac{\omega R}{L} + \frac{A}{LC} = 0$$

$$A\left(\omega^2 - \frac{1}{LC}\right) - B\left(\frac{\omega R}{L}\right) = 0 \tag{11.39}$$

Solving Eqs 11.38 and 11.39, we get

$$A = \frac{V \times \dfrac{\omega^2 R}{L^2}}{\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

$$B = \frac{\left(\omega^2 - \dfrac{1}{LC}\right) V\omega}{L\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

Substituting the values of A and B in Eq. 11.34, we get

$$i_p = \frac{V \dfrac{\omega^2 R}{L^2}}{\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]} \cos(\omega t + \theta)$$

$$+ \frac{\left(\omega^2 - \dfrac{1}{LC}\right) V\omega}{L\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]} \sin(\omega t + \theta) \qquad (11.40)$$

Putting $$M \cos \phi = \frac{V \dfrac{\omega^2 R}{L^2}}{\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2}$$

and $$M \sin \phi = \frac{V\left(\omega^2 - \dfrac{1}{LC}\right)\omega}{L\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

To find M and ϕ we divide one equation by the other

or $$\frac{M \sin \phi}{M \cos \phi} = \tan \phi = \frac{\left(\omega L - \dfrac{1}{\omega C}\right)}{R}$$

$$\phi = \tan^{-1}\left[\left(\omega L - \frac{1}{\omega C}\right) \Big/ R\right]$$

Squaring both equations and adding, we get

$$M^2 \cos^2 \phi + M^2 \sin^2 \phi = \frac{V^2}{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}$$

$\therefore \qquad M = \dfrac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}}$

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\frac{\left(\dfrac{1}{\omega C} - \omega L\right)}{R}\right] \qquad (11.41)$$

The complementary function is similar to that of DC series RLC circuit. To find out the complementary function, we have the characteristic equation

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0 \qquad (11.42)$$

The roots of Eq. 11.42, are

$$D_1, D_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $\qquad K_1 = -\dfrac{R}{2L}$ and $K_2 = \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}}$

$\therefore \qquad D_1 = K_1 + K_2$ and $D_2 = K_1 - K_2$

K_2 becomes positive, when $(R/2L)^2 > 1/LC$

The roots are real and unequal, which gives an overdamped response. Then Eq. 11.42 becomes

$$[D - (K_1 + K_2)]\,[D - (K_1 - K_2)]i = 0$$

The complementary function for the above equation is

$$i_c = c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$$

Therefore, the complete solution is

$$i = i_c + i_p$$
$$= c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$$
$$+ \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

K_2 becomes negative, when $\left(\dfrac{R}{2L}\right)^2 < \dfrac{1}{LC}$

Then the roots are complex conjugate, which gives an underdamped response. Equation 11.42 becomes

$$[D-(K_1+jK_2)]\,[D-(K_1-jK_2)]i=0$$

The solution for the above equation is

$$i_c = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

Therefore, the complete solution is

$$i = i_c + i_p$$

$$\therefore \quad i = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

$$+ \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

K_2 becomes zero, when $\left(\dfrac{R}{2L}\right)^2 = 1/LC$

Then the roots are equal which gives critically damped response. Then, Eq. 11.42 becomes $(D-K_1)\,(D-K_1)i=0$.

The complementary function for the above equation is

$$i_c = e^{K_1 t}\,(c_1 + c_2 t)$$

Therefore, the complete solution is $i = i_c + i_p$

$$\therefore \quad i = e^{K_1 t}\,[c_1 + c_2 t]$$

$$+ \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

Example 11.6 In the circuit shown in Fig. 11.21, determine the complete solution for the current, when the switch is closed at $t = 0$. Applied voltage is $v(t) = 400 \cos\left(500t + \dfrac{\pi}{4}\right)$. Resistance $R = 15\ \Omega$, inductance $L = 0.2$ H and capacitance $C = 3\mu$F.

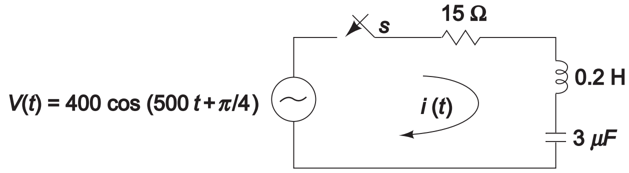


Fig. 11.21

Solution By applying Kirchhoff's voltage law to the circuit,

$$15i(t) + 0.2\frac{di(t)}{dt} + \frac{1}{3\times 10^{-6}}\int i(t)\,dt = 400\cos\left(500t + \frac{\pi}{4}\right)$$

Differentiating the above equation once, we get

$$15\frac{di}{dt}+0.2\frac{d^2i}{dt}+\frac{i}{3\times10^{-6}}=-2\times10^5\sin\left(500t+\frac{\pi}{4}\right)$$

$$(D^2+75D+16.7\times10^5)i=\frac{-2\times10^5}{0.2}\sin\left(500t+\frac{\pi}{4}\right)$$

The roots of the characteristic equation are

$$D_1=-37.5+j1290 \text{ and } D_2=-37.5-j1290$$

The complementary current

$$i_c=e^{-37.5t}(c_1\cos 1290t+c_2\sin 1290t)$$

Particular solution is

$$i_p=\frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}\cos\left[\omega t+\theta+\tan^{-1}\left(\frac{1}{\omega CR}-\frac{\omega L}{R}\right)\right]$$

$$\therefore \quad i_p=0.71\cos\left(500t+\frac{\pi}{4}+88.5°\right)$$

The complete solution is

$$i=e^{-37.5t}(c_1\cos 1290t+c_2\sin 1290t)+0.71\cos(500t+45°+88.5°)$$

At $t=0$, $i_0=0$

$$\therefore \quad c_1=-0.71\cos(133.5°)=+0.49$$

Differentiating the current equation, we have

$$\frac{di}{dt}=e^{-37.5t}(-1290c_1\sin 1290t+c_2 1290\cos 1290t)$$
$$-37.5e^{-37.5t}(c_1\cos 1290t+c_2\sin 1290t)$$
$$-0.71\times500\sin(500t+45°+88.5°)$$

At $t=0$, $\frac{di}{dt}=1414$

$$\therefore \quad 1414=1290c_2-37.5\times0.49-0.71\times500\sin(133.5°)$$
$$1414=1290c_2-18.38-257.5$$
$$\therefore \quad c_2=1.31$$

The complete solution is

$$i=e^{-37.5t}(0.49\cos 1290t+1.31\sin 1290t)+0.71\cos(500t+133.5°)$$

Solved Problems

Problem 11.1 For the circuit shown in Fig. 11.22, find the current equation when the switch is changed from position 1 to position 2 at $t=0$.

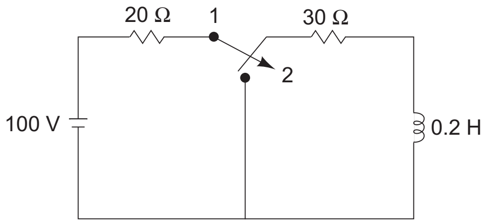


Fig. 11.22

Solution When the switch is at position 2, the current equation can be written by using Kirchhoff's voltage law as

$$30i(t) + 0.2\frac{di(t)}{dt} = 0$$

$$\left(D + \frac{30}{0.2}\right)i = 0$$

$$(D + 150)i = 0$$

$$\therefore \qquad i = c_1 e^{-150t}$$

At $t = 0$, the switch is changed to position 2, i.e. $i(0) = c_1$.

At $t = 0$, the initial current passing through the circuit is the same as the current passing through the circuit when the switch is at position 1. At $t = 0^-$, the switch is at position 1, and the current passing through the circuit $i = 100/50 = 2$ A.

At $t = 0^+$, the switch is at position 2. Since the inductor does not allow sudden changes in current, the same current passes through the circuit. Hence the initial current passing through the circuit, when the switch is at position 2 is $i(0^+) = 2$A.

$$\therefore \qquad c_1 = 2\text{ A}$$

Therefore the current $i = 2e^{-150t}$

Problem 11.2 For the circuit shown in Fig. 11.23, find the current equation when the switch is opened at $t = 0$.

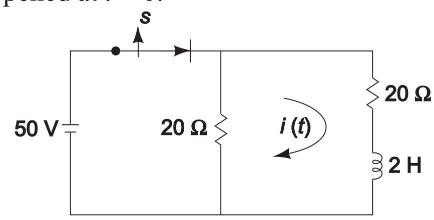


Fig. 11.23

Solution At $t = 0$, switch S is opened. By using Kirchhoff's voltage law, the current equation can be written as

$$20i + 20i + 2\frac{di}{dt} = 0$$

$$40i + 2\frac{di}{dt} = 0$$

$\therefore \qquad D + 20i = 0$

The solution for the above equation is

$$i = c_1\, e^{-20t}$$

When the switch has been closed for a time, since the inductor acts as short circuit for dc voltages, the current passing through the inductor is 2.5 A.

That means, just before the switch is opened, the current passing through the inductor is 2.5 A. Since the current in the inductor cannot change instantaneously, $i(0^+)$ is also equal to 2.5 A.

At $t = 0 \qquad c_1 = i(0^+) = 2.5$

Therefore, the final solution is $i(t) = 2.5e^{-20t}$

Problem 11.3 For the circuit shown in Fig. 11.24, find the current equation when the switch is opened at $t = 0$.

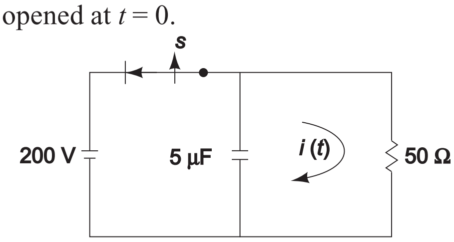


Fig. 11.24

Solution By using Kirchhoff's voltage law, the current equation is given by

$$\frac{1}{5 \times 10^{-6}} \int idt + 50i = 0$$

Differentiating the above equation once, we get

$$50\,\frac{di}{dt} + \frac{1}{5 \times 10^{-6}}\, i = 0$$

$$\therefore \qquad \left(D + \frac{1}{250 \times 10^{-6}}\right) i = 0$$

$$\therefore \qquad i = c_1 \exp\left(\frac{-1}{250 \times 10^{-6}}\, t\right) \qquad (11.43)$$

At $t = 0^-$, just before the switch S is opened, the voltage across the capacitor is 200 V. Since the voltage across the capacitor cannot change instantly, it remains equal to 200 V at $t = 0^+$. At that instant, the current through the resistor is

$$i(0^+) = \frac{200}{50} = 4\text{A}$$

In Eq. 11.43, the current is $i(0^+)$ at $t = 0$

$$\therefore \qquad c_1 = 4 \text{ A}$$

Therefore, the current equation is

$$i = 4 \exp\left(\frac{-1}{250 \times 10^{-6}}\, t\right) \text{A}$$

Problem 11.4 For the circuit shown in Fig. 11.25, find the current equation when the switch S is opened at $t = 0$.

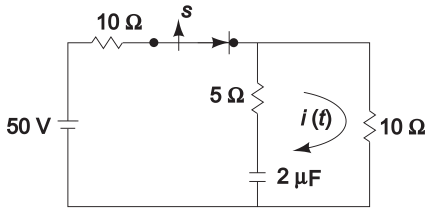


Fig. 11.25

Solution By using Kirchhoff's voltage law, the current equation is given by

$$\frac{1}{2 \times 10^{-6}} \int idt + 5i + 10i = 0$$

Differentiating the above equation, we have

$$15\, \frac{di}{dt} + \frac{i}{2 \times 10^{-6}} = 0$$

$$\left(D + \frac{1}{30 \times 10^{-6}}\right) i = 0$$

$$\therefore \qquad i = c_1 \exp\left(\frac{-1}{30 \times 10^{-6}}\right) t$$

At $t = 0^{-}$, just before switch S is opened, the current through 10 ohms resistor is 2.5 A. The same current passes through 10 Ω at $t = 0^{+}$

$\therefore \qquad i(0^{+}) = 2.5$ A

At $t = 0 \qquad i(0^{+}) = 2.5$ A

$\therefore \qquad c_1 = 2.5$

The complete solution is $i = 2.5 \exp\left(\frac{-1}{30 \times 10^{-6}}\, t\right)$

Problem 11.5 For the circuit shown in Fig. 11.26, find the complete expression for the current when the switch is closed at $t = 0$.

Solution By using Kirchhoff's law, the differential equation when the switch is closed at $t = 0$ is given by

$$20i + 0.1\, \frac{di}{dt} = 100$$

$$(D + 200)i = 1000$$

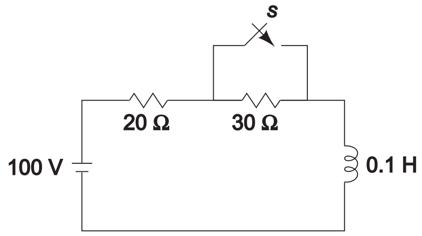


Fig. 11.26

$$i = c_1 e^{-200t} + e^{-200t} \int 1000 e^{200t}\, dt$$

$\therefore$ $$i = c_1\, e^{-200t} + 5$$

At $t = 0^-$, the current passing through the circuit is $i(0^-) = \dfrac{100}{50} = 2$ A. Since, the inductor does not allow sudden changes in currents, at $t = 0^+$, the same current passes through circuit.

$\therefore$ $\quad i(0^+) = 2$ A

At $t = 0$ $\quad i(0^+) = 2$

$\therefore$ $\quad c_1 = -3$

The complete solution is $i = -3e^{-200t} + 5$ A

Problem 11.6 The circuit shown in Fig. 11.27, consists of series RL elements with $R = 150\ \Omega$ and $L = 0.5$ H. The switch is closed when $\phi = 30°$. Determine the resultant current when voltage $V = 50 \cos(100t + \phi)$ is applied to the circuit at $\phi = 30°$.

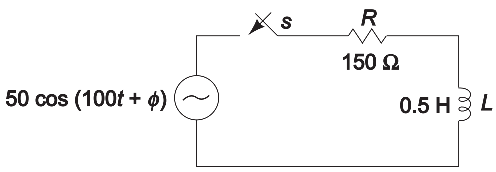


Fig. 11.27

Solution By using Kirchhoff's laws, the differential equation, when the switch is closed at $\phi = 30°$ is

$$150i + 0.5\frac{di}{dt} = 50 \cos(100t + \phi)$$

$$0.5Di + 150i = 50 \cos(100t + 30°)$$

$$(D + 300)i = 100 \cos(100t + 30°)$$

The complementary current $i_c = ce^{-300t}$

To determine the particular current, first we assume a particular current

$$i_p = A \cos(100t + 30°) + B \sin(100t + 30°)$$

Then $$i'_p = -100\,A \sin(100t + 30°) + 100\,B \cos(100t + 30°)$$

Substituting i_p and i'_p in the differential equation and equating the coefficients, we get

$$-100\,A \sin(100t + 30°) + 100B \cos(100t + 30°) + 300\,A \cos$$
$$(100t + 30°) + 300B \sin(100t + 30°) = 100 \cos(100t + 30°)$$
$$-100\,A + 300\,B = 0$$
$$300\,A + 100\,B = 100$$

From the above equation, we get

$$A = 0.3 \text{ and } B = 0.1$$

The particular current is

$$i_p = 0.3 \cos(100t + 30°) + 0.1 \sin(100t + 30°)$$

$\therefore$ $$i_p = 0.316 \cos(100t + 11.57°)\text{ A}$$

The complete equation for the current is $i = i_p + i_c$

$\therefore$ $$i = ce^{-300t} + 0.316 \cos(100t + 11.57°)$$

At $t = 0$, the current $i_0 = 0$

$\therefore$ $$c = -0.316 \cos(11.57°) = -0.309$$

Therefore, the complete solution for the current is

$$i = -0.309e^{-300t} + 0.316 \cos(100t + 11.57°)\text{ A}$$

Problem 11.7 The circuit shown in Fig. 11.28, consists of series RC elements with $R = 15\ \Omega$ and $C = 100\ \mu$F. A sinusoidal voltage $v = 100 \sin(500t + \phi)$ volts is applied to the circuit at time corresponding to $\phi = 45°$. Obtain the current transient.

Solution By using Kirchhoff's laws, the differential equation is

$$15i + \frac{1}{100 \times 10^{-6}} \int idt = 100 \sin(500t + \phi)$$

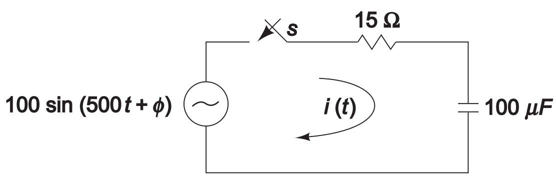


Fig. 11.28

Differentiating once, we have

$$15\frac{di}{dt} + \frac{1}{100 \times 10^{-6}} i = (100)(500) \cos(500t + \phi)$$

$$\left(D + \frac{1}{1500 \times 10^{-6}}\right) i = 3333.3 \cos(500t + \phi)$$

$$(D + 666.67)i = 3333.3 \cos(500t + \phi)$$

The complementary function $i_c = ce^{-666.67t}$

To determine the particular current, first we assume a particular current

$$i_p = A\cos(500t + 45°) + B\sin(500t + 45°)$$

$$i'_p = -500\,A\sin(500t + 45°) + 500\,B\cos(500t + 45°)$$

Substituting i_p and i'_p in the differential equation, we get

$$-500\,A\sin(500t + 45°) + 500\,B\cos(500t + 45°)$$
$$+\,666.67A\cos(500t + 45°) + 666.67B\sin(500t + 45°)$$
$$= 3333.3\cos(500t + \phi)$$

By equating coefficients, we get

$$500\,B + 666.67\,A = 3333.3$$

$$666.67B - 500\,A = 0$$

From which, the coefficients

$$A = 3.2;\ B = 2.4$$

Therefore, the particular current is

$$i_p = 3.2\cos(500t + 45°) + 2.4\sin(500t + 45°)$$

$$i_p = 4\sin(500t + 98.13°)$$

The complete equation for the current is

$$i = i_c + i_p$$

$$i = ce^{-666.67t} + 4\sin(500t + 98.13°)$$

At $t = 0$, the differential equation becomes

$$15i = 100\sin 45°$$

$$i = \frac{100}{15}\sin 45° = 4.71\text{ A}$$

$\therefore$ At $t = 0$

$$4.71 = c + 4\sin(98.13°)$$

$$\therefore \quad c = 0.75$$

The complete current is

$$i = 0.75\,e^{-666.67t} + 4\sin(500t + 98.13°)$$

Problem 11.8 The circuit shown in Fig. 11.29 consisting of series RLC elements with $R = 10\ \Omega$, $L = 0.5$ H and $C = 200\ \mu$F has a sinusoidal voltage $v = 150\sin(200t + \phi)$. If the switch is closed when $\phi = 30°$, determine the current equation.

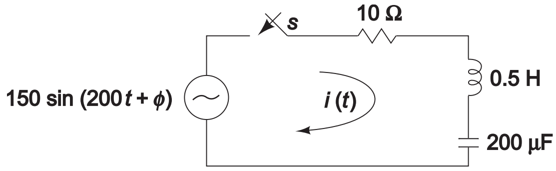


Fig. 11.29

Solution By using Kirchhoff's laws, the differential equation is

$$10i + 0.5\frac{di}{dt} + \frac{1}{200\times 10^{-6}}\int i dt = 150 \sin (200t + \phi)$$

Differentiating once, we have

$$(D^2 + 20D + 10^4)i = 60000 \cos (200t + \phi)$$

The roots of the characteristics equation are

$$D_1 = -10 + j99.49 \text{ and } D_2 = -10 - j99.49$$

The complementary function is

$$i_c = e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49)$$

We can find the particular current by using the undetermined coefficient method.

Let us assume

$$i_p = A \cos (200t + 30°) + B \sin (200t + 30°)$$
$$i'_p = -200 A \sin (200t + 30°) + 200 B \cos (200t + 30°)$$
$$i''_p = -(200)^2 A \cos (200t + 30°) - (200)^2 B \sin (220t + 30°)$$

Substituting these values in the equation, and equating the coefficients, we get

$$A = 0.1 \quad B = 0.067$$

Therefore, the particular current is

$$i_p = 1.98 \cos (200t - 52.4°) \text{ A}$$

The complete current is

$$i = e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49t) + 1.98 \cos (200t - 52.4°) \text{ A}$$

From the differential equation at $t = 0$, $i_0 = 0$ and $\frac{di}{dt} = 300$

$\therefore$ At $t = 0$

$$c_1 = -1.98 \cos (-52.4°) = -1.21$$

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-10t}(-99.49c_1 \sin 99.49t + 99.49c_2 \cos 99.49t)$$
$$-200(1.98) \sin (200t - 52.4°) - 10e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49t)$$

At $t = 0$, $\frac{di}{dt} = 300$ and $c_1 = -1.21$

$$300 = 99.49\, c_2 - 396 \sin (-52.4°) - 10(-1.21)$$
$$300 = 99.49\, c_2 + 313.7 + 11.1$$
$$c_2 = -25.8$$

Therefore, the complete current equation is

$$i = e^{-10t}(0.07 \cos 99.49t - 25.8 \sin 99.49t) + 1.98 \cos (200t - 52.4°) \text{ A}$$

Problem 11.9 For the circuit shown in Fig. 11.30, determine the transient current when the switch is moved from position 1 to position 2 at $t = 0$. The circuit is in steady state with the switch in position 1. The voltage applied to the circuit is $v = 150 \cos(200t + 30°)$ V.

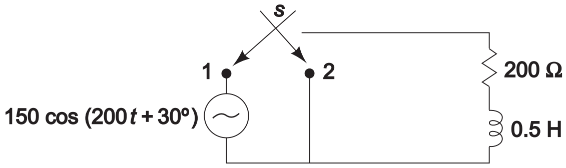


Fig. 11.30

Solution When the switch is at position 2, by applying Kirchhoff's law, the differential equation is

$$200i + 0.5\frac{di}{dt} = 0$$

$$(D + 400)i = 0$$

∴ The transient current is

$$i = ce^{-400t}$$

At $t = 0$, the switch is moved from position 1 to position 2. Hence the current passing through the circuit is the same as the steady state current passing through the circuit when the switch is in position 1.

When the switch is in position 1, the current passing through the circuit is

$$i = \frac{v}{z} = \frac{150\angle 30°}{R + j\omega L}$$

$$= \frac{150\angle 30°}{200 + j(200)(0.5)} = \frac{150\angle 30°}{223.6\angle 26.56°} = 0.67\angle 3.44°$$

Therefore, the steady state current passing through the circuit when the switch is in position 1 is

$$i = 0.67\cos(200t + 3.44°)$$

Now substituting this equation in transient current equation, we get

$$0.67\cos(200t + 3.44°) = ce^{-400t}$$

At $t = 0$; $c = 0.67\cos(3.44°) = 0.66$

Therefore, the current equation is $i = 0.66e^{-400t}$

Problem 11.10 In the circuit shown in Fig. 11.31, determine the current equations for i_1 and i_2 when the switch is closed at $t = 0$.

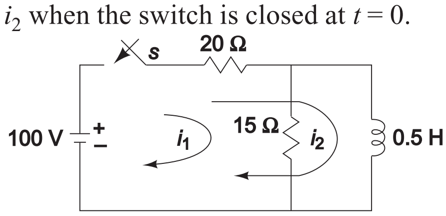


Fig. 11.31

Solution By applying Kirchhoff's laws, we get two equations

$$35i_1 + 20i_2 = 100 \tag{11.44}$$

$$20i_1 + 20i_2 + 0.5\,\frac{di_2}{dt} = 100 \tag{11.45}$$

From Eq. 11.44, we have

$$35i_1 = 100 - 20i_2$$

$$i_1 = \frac{100}{35} - \frac{20}{35}i_2$$

Substituting i_1 in Eq. 11.45, we get

$$20\left(\frac{100}{35} - \frac{20}{35}i_2\right) + 20i_2 + 0.5\,\frac{di_2}{dt} = 100 \tag{11.46}$$

$$57.14 - 11.43i_2 + 20i_2 + 0.5\,\frac{di_2}{dt} = 100$$

$$(D + 17.14)i_2 = 85.72$$

From the above equation,

$$i_2 = ce^{-17.14t} + 5$$

Loop current i_2 passes through inductor and must be zero at $t = 0$

At $t = 0$, $i_2 = 0$

$\therefore \quad c = -5$

$\therefore \quad i_2 = 5(1 - e^{-17.14t})$ A

and the current $\quad i_1 = 2.86 - \{0.57 \times 5(1 - e^{-17.14t})\}$

$\quad = (0.01 + 2.85\,e^{-17.14t})$ A

Problem 11.11 For the circuit shown in Fig. 11.32, find the current equation when the switch is changed from position 1 to position 2 at $t = 0$.

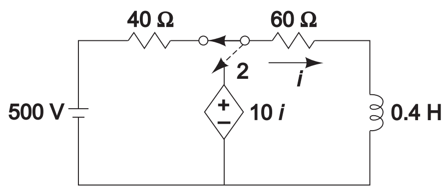


Fig. 11.32

Solution By using Kirchhoff's voltage law, the current equation is given by

$$60i + 0.4\,\frac{di}{dt} = 10i$$

At $t = 0^-$, the switch is at position 1, the current passing through the circuit is

$$i(0^-) = \frac{500}{100} = 5 \text{ A}$$

$$0.4\frac{di}{dt} + 50i = 0$$

$$\left(D + \frac{50}{0.4}\right)i = 0$$

$$i = ce^{-125t}$$

At $t = 0$, the initial current passing through the circuit is same as the current passing through the circuit when the switch is at position 1.

At $t = 0$, $i(0) = i(0^-) = 5$ A

At $t = 0$, $c = 5$ A

$\therefore$ The current $I = 5e^{-125t}$

Problem 11.12 For the circuit shown in Fig. 11.33, find the current equation when the switch S is opened at $t = 0$.

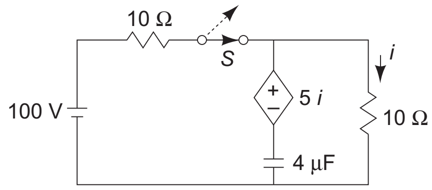


Fig. 11.33

Solution When the switch is closed for a long time,

$$\text{At } t = 0^-, \text{ the current } i(0^-) = \frac{100}{20} = 5 \text{ A}$$

When the switch is opened at $t = 0$, the current equation by using Kirchhoff's voltage law is given by

$$\frac{1}{4 \times 10^{-6}}\int i\,dt + 10i = 5i$$

$$\frac{1}{4 \times 10^{-6}}\int i\,dt + 5i = 0$$

Differentiating the above equation

$$5\frac{di}{dt} + \frac{1}{4 \times 10^{-6}}i = 0$$

$$\left(D + \frac{1}{20 \times 10^{-6}}\right)i = 0$$

$$\therefore \qquad i = ce^{\frac{-1}{20 \times 10^{-6}}t}$$

At $t = 0^-$, just before switch S is opened, the current passing through the 10 Ω resistor is 5 A. The same current passes through 10 Ω at $t = 0$.

$\therefore$ At $\quad t = 0,\ i(0) = 5\ A$

At $\quad t = 0,\ c_1 = 5\ A$

The current equation is $i = 5e^{\frac{-t}{20 \times 10^{-6}}}$

Problem 11.13 For the circuit shown in Fig. 11.34, find the current in the 20 Ω when the switch is opened at $t = 0$.

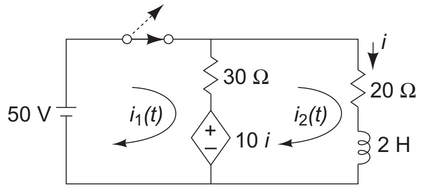


Fig. 11.34

Solution When the switch is closed, the loop current i_1 and i_2 are flowing in the circuit.

The loop equations are $30(i_1 - i_2) + 10i_2 = 50$

$$30(i_2 - i_1) + 20i_2 = 10i_2$$

From the above equations, the current in the 20 Ω resistor $i_2 = 2.5$ A.

The same initial current is flowing when the switch is opened at $t = 0$.

When the switch is opened the current equations

$$30i + 20i + 2\frac{di}{dt} = 10i$$

$$40i + \frac{2di}{dt} = 0$$

$$(D + 20)i = 0$$

$$i = ce^{-20t}$$

At $t = 0$, the current $i(0) = 2.5$ A

$\therefore$ At $t = 0$, $c = 2.5$

The current in the 20 Ω resistor is $i = 2.5\ e^{-20t}$.

Problem 11.14 For the circuit shown in Fig. 11.35, find the current equation when the switch is opened at $t = 0$.

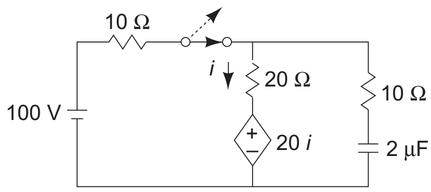


Fig. 11.35

Solution When the switch is closed, the current in the 20 Ω resistor i can be obtained using Kirchhoff's voltage law.

$$10i + 20i + 20i = 100$$
$$50i = 100, \therefore\ i = 2\text{ A}$$

The same initial current passes through the 20 Ω resistor when the switch is opened at $t = 0$.

The current equation is

$$20i + 10i + \frac{1}{2 \times 10^{-6}} \int idt = 20i$$

$$10i + \frac{1}{2 \times 10^{-6}} \int idt = 0$$

Differentiating the above equation, we get

$$10\frac{di}{dt} + \frac{1}{2 \times 10^{-6}} i = 0$$

$$\left(D + \frac{1}{20 \times 10^{-6}}\right) i = 0$$

The solution for the above equation is

$$i = ce^{\frac{-1}{20 \times 10^{-6}} t}$$

At $t = 0$, $i(0) = i(0^-) = 2$ A

∴ At $t = 0$, $c = 2$ A

The current equation is

$$i = 2e^{\frac{-1}{20 \times 10^{-6}} t}$$

Practice Problems

11.1 (a) What do you understand by transient and steady state parts of response? How can they be identified in a general solution?

(b) Obtain an expression for the current $i(t)$ from the differential equation

$$\frac{d^2 i(t)}{dt^2} + 10\frac{di(t)}{dt} + 25i(t) = 0$$

with initial conditions

$$i(0^+) = 2 \quad \frac{di(0^+)}{dt} = 0$$

11.2 A series circuit shown in Fig. 11.36, comprising resistance 10 Ω and inductance 0.5 H, is connected to a 100 V source at $t = 0$. Determine the complete expression for the current $i(t)$.

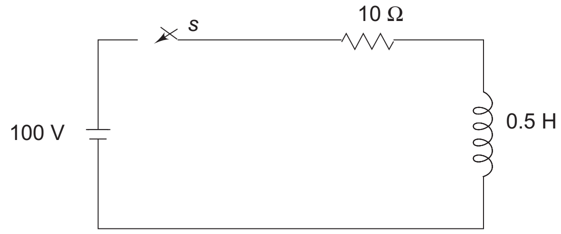


Fig. 11.36

11.3 In the network shown in Fig. 11.37, the capacitor c_1 is charged to a voltage of 100 V and the switch S is closed at $t = 0$. Determine the current expression i_1 and i_2.

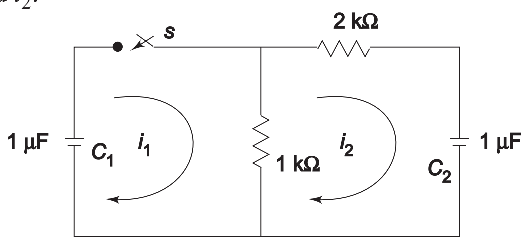


Fig. 11.37

11.4 A series RLC circuit shown in Fig. 11.38, comprising $R = 10\ \Omega$, $L = 0.5$ H and $C = 1\ \mu$F, is excited by a constant voltage source of 100 V. Obtain the expression for the current. Assume that the circuit is relaxed initially.

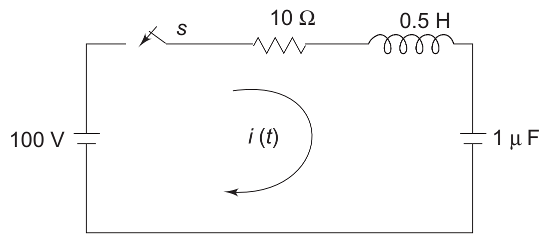


Fig. 11.38

11.5 In the circuit shown in Fig. 11.39, the initial current in the inductance is 2 A and its direction is as shown in the figure. The initial charge on the capacitor is 200 C with polarity as shown when the switch is closed. Determine the current expression in the inductance.

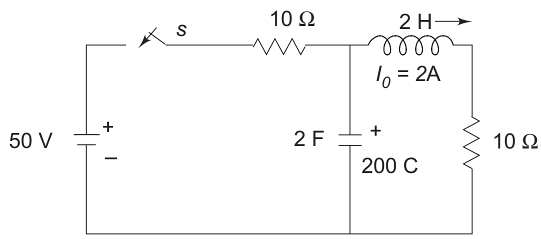


Fig. 11.39

11.6 In the circuit shown in Fig. 11.40, the switch is closed at $t = 0$ with zero capacitor voltage and zero inductor current. Determine V_1 and V_2 at $t = 0^+$.

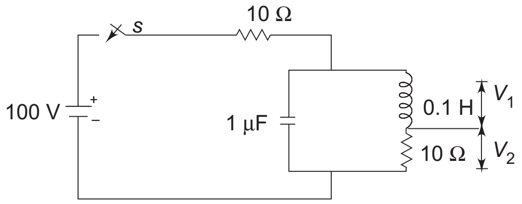


Fig. 11.40

11.7 In the network shown in Fig. 11.41, the switch is moved from position 1 to position 2 at $t = 0$. The switch is in position 1 for a long time. Determine the current expression $i(t)$.

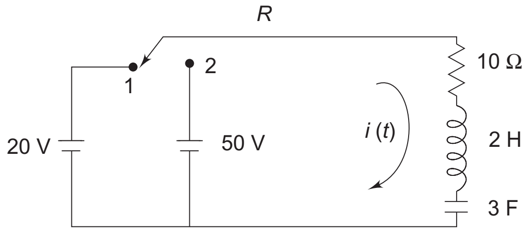


Fig. 11.41

11.8 In the network shown in Fig. 11.42, determine the current expression for $i_1(t)$ and $i_2(t)$ when the switch is closed at $t = 0$. The network has no initial energy.

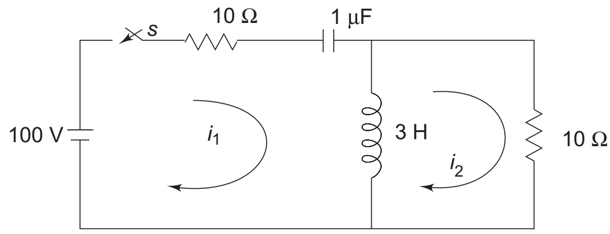


Fig. 11.42

11.9 In the network shown in Fig. 11.43, the switch is closed at $t = 0$ and there is no initial charge on either of the capacitances. Find the resulting current $i(t)$.

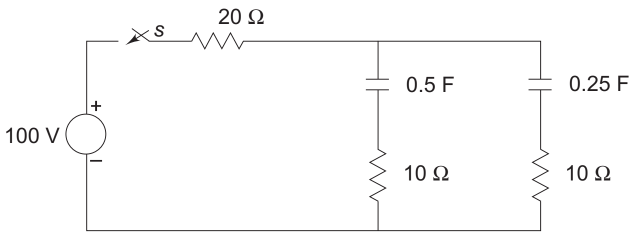


Fig. 11.43

11.10 In the RC circuit shown in Fig. 11.44, the capacitor has an initial charge $q_0 = 25 \times 10^{-6}$ C with polarity as shown. A sinusoidal voltage $v = 100 \sin (200t + \phi)$ is applied to the circuit at a time corresponding to $\phi = 30°$. Determine the expression for the current $i(t)$.

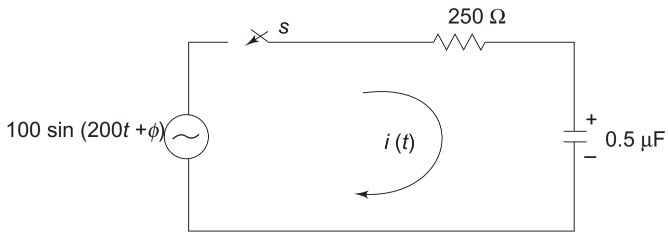


Fig. 11.44

11.11 In the network shown in Fig. 11.45, the switch is moved from position 1 to position 2 at $t = 0$. The switch is in position 1 for a long time. Initial charge on the capacitor is 7×10^{-4} coulombs. Determine the current expression $i(t)$.

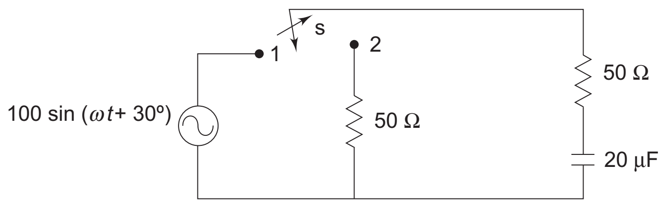


Fig. 11.45

11.12 In the network shown in Fig. 11.46, the switch is moved from position 1 to position 2 at $t = 0$. Determine the current expression.

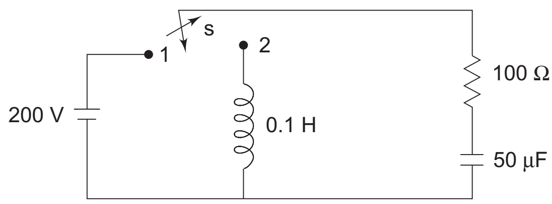


Fig. 11.46

11.13 In the network shown in Fig. 11.47, find $i_2(t)$ for $t > 0$, if $i_1(0) = 5$ A.

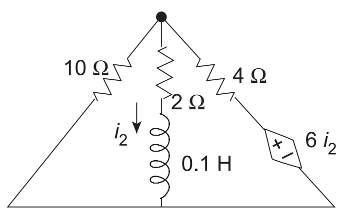


Fig. 11.47

11.14 For the circuit shown in Fig. 11.48, find v_5, if the switch is opened for $t > 0$.

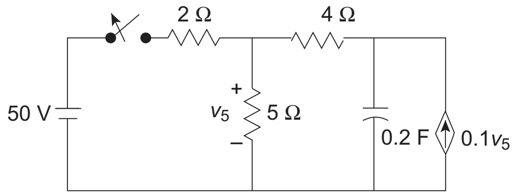


Fig. 11.48

11.15 Calculate the voltage $v_1(t)$ across the inductance for $t > 0$ in the circuit shown in Fig. 11.49.

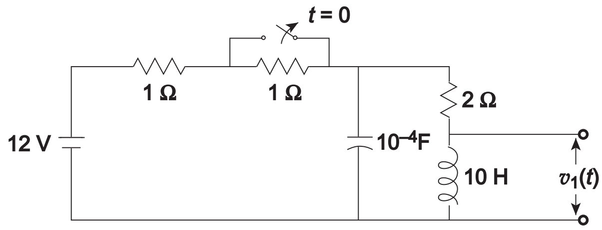


Fig. 11.49

11.16 The network shown in Fig. 11.50 is initially under steady state condition with the switch in position 1. The switch is moved from position 1 to position 2 at $t \neq 0$. Calculate the current $i(t)$ through R_1 after switching.

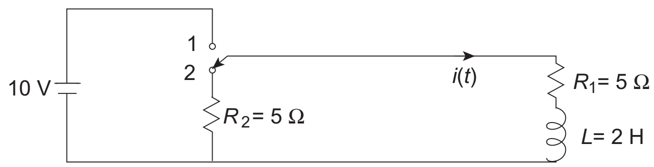


Fig. 11.50

Objective-type Questions

1. Transient behaviour occurs in any circuit when
   (a) there are sudden changes of applied voltage.
   (b) the voltage source is shorted.
   (c) the circuit is connected or disconnected from the supply.
   (d) all of the above happen.
2. The transient response occurs
   (a) only in resistive circuits (b) only in inductive circuits
   (c) only in capacitive circuits (d) both in (b) and (c).
3. Inductor does not allow sudden changes
   (a) in currents (b) in voltages
   (c) in both (a) and (b) (d) in none of the above

4. When a series RL circuit is connected to a voltage V at $t = 0$, the current passing through the inductor L at $t = 0^+$ is
   (a) $\frac{V}{R}$ (b) infinite
   (c) zero (d) $\frac{V}{L}$
5. The time constant of a series RL circuit is
   (a) LR (b) $\frac{L}{R}$
   (c) $\frac{R}{L}$ (d) $e^{-R/L}$
6. A capacitor does not allow sudden changes
   (a) in currents (b) in voltages
   (c) in both currents and voltages (d) in neither of the two
7. When a series RC circuit is connected to a constant voltage at $t = 0$, the current passing through the circuit at $t = 0^+$ is
   (a) infinite (b) zero
   (c) $\frac{V}{R}$ (d) $\frac{V}{\omega C}$
8. The time constant of a series RC circuit is
   (a) $\frac{1}{RC}$ (b) $\frac{R}{C}$
   (c) RC (d) e^{-RC}
9. The transient current in a loss-free LC circuit when excited from an ac source is an ________ sine wave
   (a) undamped (b) overdamped
   (c) under damped (d) critically damped.
10. Transient current in an RLC circuit is oscillatory when
   (a) $R = 2\sqrt{L/C}$ (b) $R = 0$
   (c) $R > 2\sqrt{L/C}$ (d) $R < 2\sqrt{L/C}$
11. The initial current in the circuit shown in Fig. 11.51 when the switch is opened for $t > 0$ is

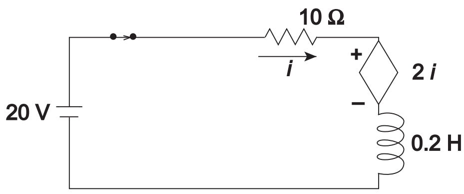


Fig. 11.51

   (a) 1.67 A (b) 3 A
   (c) 0 A (d) 2 A

12. The initial current in the circuit shown in Fig. 11.52 below when the switch is opened for $t > 0$ is

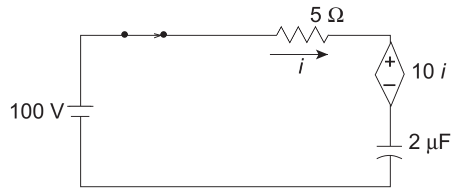


Fig. 11.52

(a) 1.5 A (b) 0 A
(c) 2 A (d) 10 A

13. For the circuit shown in Fig. 11.53 the current in the 10 Ω resistor when the switch is changed from 1 to 2 is

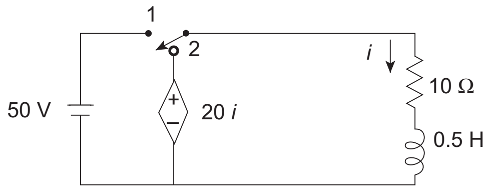


Fig. 11.53

(a) $5\,e^{+20t}$ (b) $5\,e^{-20t}$
(c) $20\,e^{+5t}$ (d) $20e^{-5t}$

14. For the circuit shown in Fig. 11.54, the current in the 5 Ω resistor when the switch is changed from 1 to 2 is

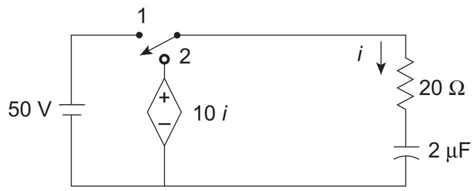


Fig. 11.54

(a) $2.5e^{\frac{1}{2\times 10^{-6}}}$ (b) 0
(c) $2.5\,e^{-10t}$ (d) $5e^{-5t}$

Chapter 12

Introduction to the Laplace Transform

12.1 DEFINITION OF THE LAPLACE TRANSFORM

The Laplace transform is a powerful analytical technique that is widely used to study the behaviour of linear, lumped parameter circuits. Laplace transforms are useful in engineering, particularly when the driving function has discontinuities and appears for a short period only.

In circuit analysis, the input and output functions do not exist forever in time. For causal functions, the function can be defined as $f(t)\,u(t)$. The integral for the Laplace transform is taken with the lower limit at $t = 0$ in order to include the effect of any discontinuity at $t = 0$.

Consider a function $f(t)$ which is to be continuous and defined for values of $t \geq 0$. The Laplace transform is then

$$\mathscr{L}[f(t)] = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, u(t)\, dt = \int_{0}^{\infty} f(t) e^{-st}\, dt \tag{12.1}$$

$f(t)$ is a continuous function for $t \geq 0$ multiplied by e^{-st} which is integrated with respect to t between the limits 0 and ∞. The resultant function of the variables is called Laplace transform of $f(t)$. Laplace transform is a function of independent variable s corresponding to the complex variable in the exponent of e^{-st}. The complex variable S is, in general, of the form $S = \sigma + j\omega$ and σ and ω being the real and imaginary parts respectively. For a function to have a Laplace transform, it must satisfy the condition $\int_{0}^{\infty} f(t)\, e^{-st}\, dt < \infty$. Laplace transform changes the time domain function $f(t)$ to the frequency domain function $F(s)$. Similarly, inverse

Laplace transformation converts frequency domain function $F(s)$ to the time domain function $f(t)$ as follows.

$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{-j}^{+j} F(s)\, e^{st}\, ds \tag{12.2}$$

Here, the inverse transform involves a complex integration. $f(t)$ can be represented as a weighted integral of complex exponentials. We will denote the transform relationship between $f(t)$ and $F(s)$ are

$$f(t) \xleftrightarrow{\mathscr{L}} F(s).$$

In Eq. 8.1, if the lower limit is 0 then the transform is referred to as one-sided, or unilateral, Laplace transform. In the two-sided, or bilateral, Laplace transform, the lower limit is $-\infty$.

In the following discussion, we divide the Laplace transforms into two types: functional transforms and operational transforms. A functional transform is the Laplace transform of a specific function, such as $\sin \omega t$, t, e^{-at}, and so on. An operational transform defines a general mathematical property of the Laplace transform, such as binding the transform of the derivative of $f(t)$. Before considering functional and operational transforms, we used to introduce the step and impulse functions.

12.2 THE STEP FUNCTION

In switching operations abrupt changes may occur in current and voltages. On some functions discontinuity may appear at the origin. We accommodate these discontinuities mathematically by introducing the step and impulse functions.

Figure 12.1 shows the step function. It is zero for $t < 0$. It is denoted by $k\,u(t)$.

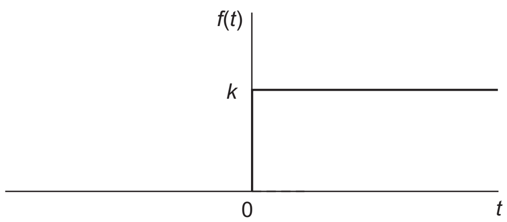


Fig. 12.1

Mathematically it is defined as

$$k\,u(t) = 0,\ t < 0$$

$$k\,u(t) = k,\ t > 0 \tag{12.3}$$

If k is 1, the function defined by Eq. (8.3) is the unit step. The step function is not defined at $t = 0$. In situations where we need to define the transition between 0^- and 0^+, we assume that it is linear and that

$$k\,u(0) = 0.5\ \mathrm{K} \tag{12.4}$$

Figure 12.2 shows the linear transition from 0^- to 0^+.

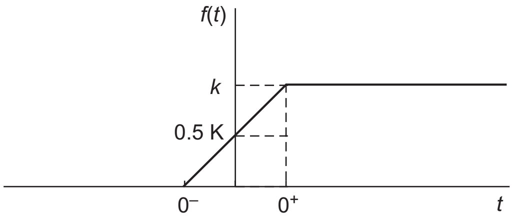


Fig. 12.2

A discontinuity may occur at sometime other than $t = 0$, for example, in sequential switching. The step function occuring at $t = a$ when $a > 0$ is shown in Fig. 12.3. A step occurs at $t = a$ is expressed as $k\,u(t - a)$. Thus

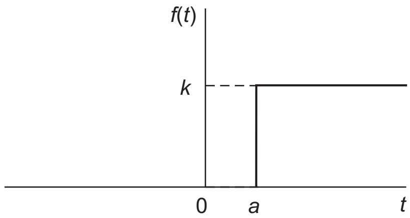


Fig. 12.3

$$k\,u(t - a) = 0,\ t < a$$
$$k\,u(t - a) = k,\ t > a \tag{12.5}$$

If $a > 0$, the step occurs to the right of the origin, and if $a < 0$, the step occurs to the left of the origin. Step function is 0 when the argument $t - a$ is negative, and it is k when the argument is positive.

A step function equal to k for $t < a$ is written as $k\,u(a - t)$. Thus

$$k\,u(a - t) = k,\ t < a$$
$$k\,u(a - t) = 0,\ t > 0 \tag{12.6}$$

The discontinuity is to the left of the origin when $a < 0$. A step function $k\,u(a - t)$ for $a > 0$ is shown in Fig. 12.4.

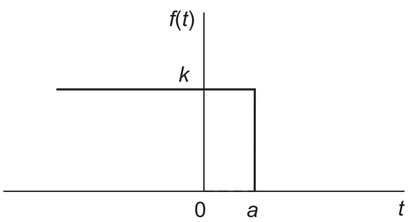


Fig. 12.4

Step function is useful to define a finite-width pulse, by adding two step functions. For example, the function $k[u(t - 1) - u(t - 3)]$ has the value k for $1 < t < 3$ and the value 0 everywhere else, so it is a finite-width pulse of height k initiated at $t = 1$ and terminated at $t = 3$. Here, $u(t - 1)$ is a function "turning on" the constant value k at $t = 1$, and the step function $- u(t = 3)$ as "turning off" the constant value k at $t = 3$. We use step functions to turn on and turn off linear functions.

Example 12.1 Use step functions to write an expression for the function shown in Fig. 12.5.

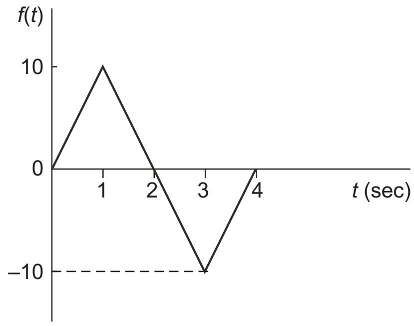


Fig. 12.5

Solution The function shown in Fig. 12.5 is made up of linear segments with break points at 0, 1, 3 and 4 seconds. The above Fig. 8.5 consists of three linear segments as shown in Fig. 12.6.

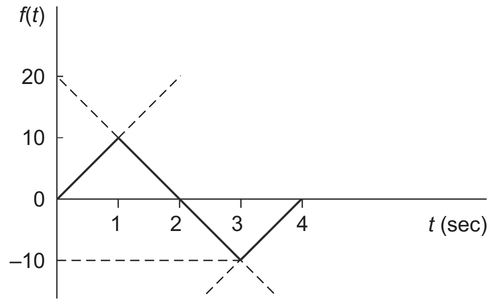


Fig. 12.6

(i) $f_1(t) = 10t \quad \text{for} \quad 0 < t < 1$
(ii) $f_2(t) = -10t + 20 \quad \text{for} \quad 1 < t < 3$
(iii) $f_3(t) = 20t - 40 \quad \text{for} \quad 3 < t < 4$ (12.7)

We use the step function to initiate and terminate these linear segments at the proper times.

(i) $f_1(t) = 10t[u(t) - u(t-1)]$, this function turn on at $t = 0$, turn off at $t = 1$.
(ii) $f_2(t) = (-10t + 20)\,[u(t-1) - u(t-3)]$, this function turn on at $t = 1$, turn off at $t = 3$.
(iii) $f_3(t) = (20t - 40)\,[u(t-3) - u(t-4)]$, this function turn on at $t = 3$, turn off at $t = 4$.

The expression for $f(t)$ is

$$f(t) = 10t[u(t) - u(t-1)] + (-10t + 20)\,[u(t-1) - u(t-3)] + (20t - 40)\,[u(t-3) - u(t-4)] \qquad (12.8)$$

Example 12.2 Use step function to write the expression for the following function.

Solution The function shown in Fig. 12.7 is a combination of linear segments at break points 0, 2, 6, 8. To construct this function, we must add and subtract linear functions of the proper slope. We use the step function to start and terminate these linear segments at the proper times.

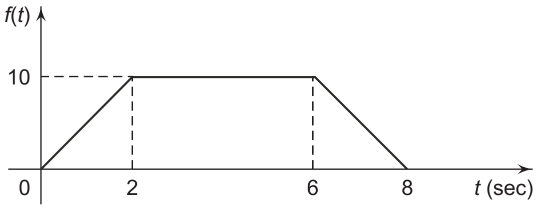


Fig. 12.7

Figure 12.7 consists of the three linear segments with the following equations.

$$\begin{aligned} f_1(t) &= 5t && \text{for} \quad 0 < t < 2 \\ f_2(t) &= 10 && \text{for} \quad 2 < t < 6 \\ f_3(t) &= -5t + 40 && \text{for} \quad 6 < t < 8 \end{aligned} \tag{12.9}$$

using step function, the above equations can be written as

$$\begin{aligned} f_1(t) &= 5t\,[u(t) - u(t-2)] \\ f_2(t) &= 10\,[u(t-2) - u(t-6)] \\ f_3(t) &= (-5t + 40)\,[u(t-6) - u(t-8)] \end{aligned} \tag{12.10}$$

The expression for $f(t)$ is

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) + f_3(t) \\ f(t) &= 5t\,[u(t) - u(t-2)] + 10\,[u(t-2) - u(t-6)] \\ &\quad + (-5t + 40)\,[u(t-6) - u(t-8)] \end{aligned} \tag{12.11}$$

Example 12.3 Use step function to write the expression for the following waveform.

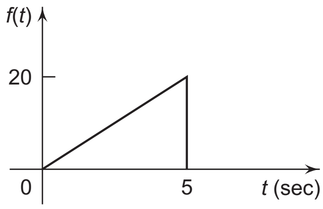


Fig. 12.8

Solution The waveform shown in Fig. 12.8 starts at $t = 0$ and ends at $t = 5$ sec. The equation for the above waveform is $f(t) = 4t$. Interms of unit function the waveform can be expressed as

$$f(t) = 4t\,[u(t) - u(t-5)] \tag{12.12}$$

Example 12.4 Use step function to write the expression for the following sinusoidal waveform.

Solution The sine wave shown in Fig. 12.9 originates at $t = 0$ and ends at $t = 2$ sec. The wave equation

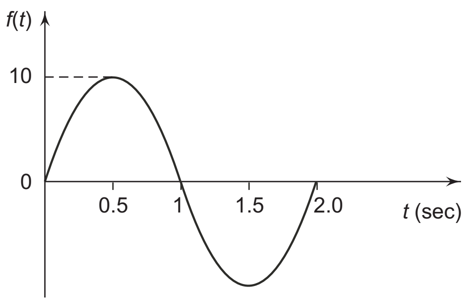


Fig. 12.9

$$f(t) = 10 \sin \omega t \quad \text{for} \quad 0 < t < 2$$

Interms of unit step functions, the equation

$$f(t) = 10 \sin \omega t\,[u(t) - u(t-2)] \tag{12.13}$$

Example 12.5 Use step function to write the expression for the function shown in Fig. 12.10.

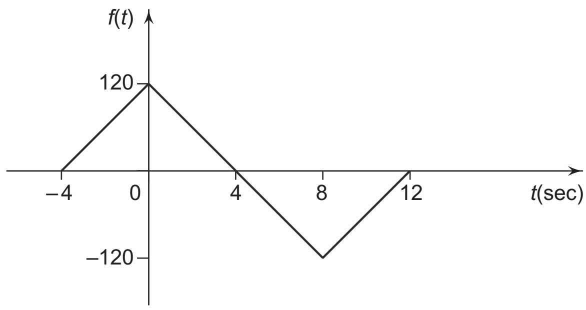


Fig. 12.10

Solution The waveform in Fig. 12.10 consists of three linear segments. The function $f(t)$ is defined as follows.

$$f_1(t) = 80t + 120 \quad \text{for} \quad -4 < t < 0$$
$$f_2(t) = -30t + 120 \quad \text{for} \quad 0 < t < 8$$
$$f_3(t) = 30t - 360 \quad \text{for} \quad 8 < t < 12 \tag{12.14}$$

Interms of unit step function

$$f_1(t) = (80t + 120)\,[u(t + u) - u(t)]$$
$$f_2(t) = (-30t + 120)\,[u(t) - u(t-8)]$$
$$f_3(t) = (30t - 360)\,[u(t-8) - u(t-12)] \tag{12.15}$$

The expression for $f(t)$ is

$$f(t) = f_1(t) + f_2(t) + f_3(t)$$
$$f(t) = (80\text{t} + 120)\,[u(t+4) - \text{u}(t)] + (-30t + 120)\,[u(t) - u(t-8)]$$
$$+ (30t - 360)\,[u(t-8) - u(t-12)] \tag{12.16}$$

12.3 THE IMPULSE FUNCTION

An impulse is a signal of infinite amplitude and zero duration. In general, an impulse signal doesn't exist in nature, but some circuit signals come very close to approximating this definition. Due to switching operations impulsive voltages and currents occur in circuit analysis. The impulse function enables us to define the derivative at a discontinuity, and thus to define the Laplace transform of that derivative.

To define derivative of a function at a discontinuity, consider that the function varies linearly across the discontinuity as shown in Fig. 12.11.

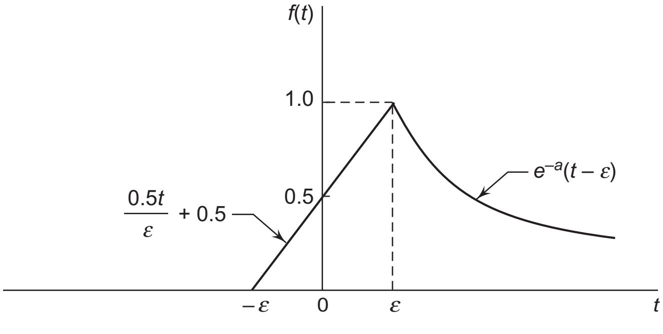


Fig. 12.11

In the Fig. 12.11 shown as $\varepsilon \to 0$, an abrupt discontinuity occurs at the origin. When we differentiate the function, the derivative between $-\varepsilon$ and $+\varepsilon$ is constant at a value of $\dfrac{1}{2\varepsilon}$. For $t > \varepsilon$, the derivative is $-ae^{-a(t-\varepsilon)}$. The derivative of the function shown in Fig. 12.11 is shown in Fig. 12.12.

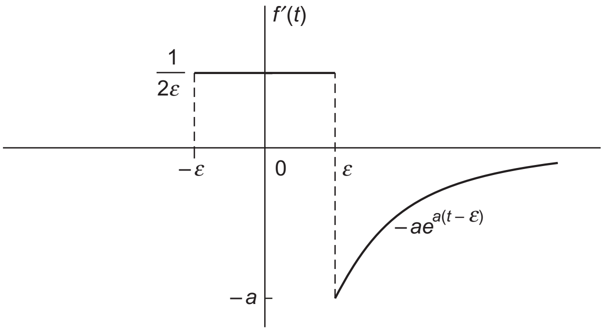


Fig. 12.12

As ε approaches zero, the value of $f'(t)$ between $\pm\,\varepsilon$ approaches infinity. At the same time, the duration of this large value is approaching zero. Furthermore, the area under $f'(t)$ between $\pm\,\varepsilon$ remains constant as $\varepsilon \to 0$. In this example, the area is unity. As ε approaches zero, we say that the function between $\pm\,\varepsilon$ approaches a unit impulse function; denoted $\delta(t)$. Thus the derivative of $f(t)$ at the origin approaches a unit impulse function as ε approaches zero, or

$$f'(0) \to \delta(t) \text{ as } \varepsilon \to 0$$

If the area under the impulse function curve is other than unity, the impulse function is denoted by $K\,\delta(t)$, where K is the area. K is often referred to as the strength of the impulse function.

Mathematically, the impulse function is defined

$$\int_{-\infty}^{\infty} K\,\delta(t)\,dt = k \tag{12.17}$$

$$\delta(t) = 0,\ t \neq 0 \tag{12.18}$$

Equation (12.17) states that the area under the impulse function is constant. This area represents the strength of the impulse. Equation (8.18) states that the impulse is zero everywhere except at $t = 0$. An impulse that occurs at $t = a$ is denoted $K\,\delta(t-a)$. The graphical symbol is shown in Fig. 12.13. The impulse $K\,\delta(t-a)$ is also shown in Fig. 12.13.

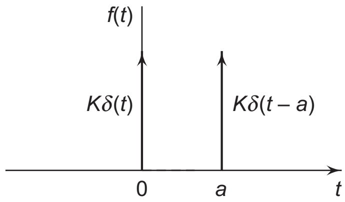


Fig. 12.13

An important property of the impulse function is the shifting property, which is expressed as

$$\int_{-\infty}^{\infty} f(t)\ \delta(t-a)\ dt = f(a) \tag{12.19}$$

Equation 12.19 shows that the impulse function shifts out everything except the value of $f(t)$ at $t = a$. The value of $\delta(t-a)$ is zero everywhere except at $t = a$, and hence the integral can be written

$$I = \int_{-\infty}^{\infty} f(t)\ \delta(t-a)\ dt = \int_{a-\epsilon}^{a+\epsilon} f(t)\ \delta(t-a)\ dt \tag{12.20}$$

But because $f(t)$ is continuous at a, it takes on the value $f(a)$ as $t \to a$, so

$$I = \int_{a-\epsilon}^{a+\epsilon} f(a)\ \delta(t-a)\ dt = f(a) \int_{a-\epsilon}^{a+\epsilon} \delta(t-a)\ dt = f(a) \tag{12.21}$$

We use the shifting property of the impulse function to find its Laplace transform.

$$\mathscr{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)\ e^{-st}\ dt = \int_{0^-}^{\infty} \delta(t)\ dt = 1 \tag{12.22}$$

which is an important Laplace transform pair that we make good use of the circuit analysis.

We can also define the derivatives of the impulse function and the Laplace transform of these derivatives.

The function illustrated in Fig. 12.14(a) generates an impulse function as $\varepsilon \to 0$. Figure 8.14(b) shows the derivative of the impulse generating function, which is defined as the derivative of the impulse $[\delta'(t)]$ as $\in\ \to 0$. The derivative of the impulse function sometimes is referred to as a moment function, or unit doublet.

To find the Laplace transform of $\delta'(t)$, we simply apply the defining integral to the function shown in Fig. 12.14 (b) and, after integrating, let $\varepsilon \to 0$. Then,

$$\mathscr{L}\{\delta'(t)\} = \lim_{\varepsilon \to 0}\left[\int_{-\varepsilon}^{0^-} \frac{1}{\varepsilon^2} e^{-st} dt + \int_{0^+}^{\varepsilon}\left(\frac{-1}{\varepsilon^2}\right) e^{-st} dt\right]$$

$$= \lim_{\varepsilon \to 0} \frac{e^{s\varepsilon} + e^{-s\varepsilon} - 2}{s\varepsilon^2}$$

$$= \lim_{\varepsilon \to 0} \frac{se^{s\varepsilon} - se^{-s\varepsilon}}{2\varepsilon s}$$

$$= \lim_{\varepsilon \to 0} \frac{s^2 e^{s\varepsilon} + s^2 e^{-s\varepsilon}}{2s}$$

$$= s \tag{12.23}$$

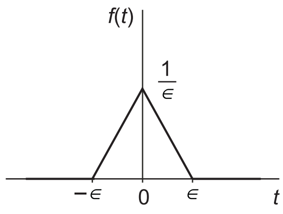


Fig. 12.14(a)

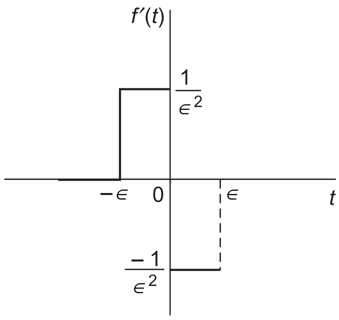


Fig. 12.14(b)

For the nth derivative of the impulse function, we find that its Laplace transform simply is s^n; that is,

$$\mathscr{L}\{\delta^n(t)\} = s^n \tag{12.24}$$

An impulse function can be thought of as a derivative of a step function, that is

$$\delta(t) = \frac{du(t)}{dt} \tag{12.25}$$

Figure 12.15(a) approaches a unit step function as $\varepsilon \to 0$. The function shown in Fig. 8.15(b), the derivative of the function in 12.15 (b), approaches a unit impulse as $\varepsilon \to 0$.

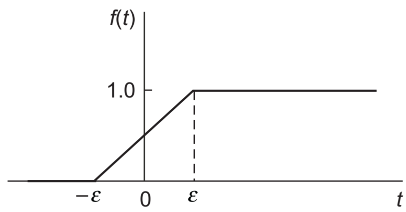


Fig. 12.15(a)

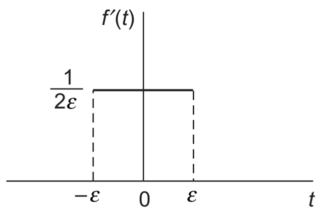


Fig. 12.15(b)

The impulse function is an extremely useful concept in circuit analysis where discontinuities occur at the origin.

Example 12.6 (a) Find the area under the function shown in Fig. 12.16. (b) What is the duration of the function when $\varepsilon = 0$? (c) What is the magnitude of $f(0)$ when $\varepsilon = 0$?

Solution (a) Area under the function is

$$A = \int_{-\varepsilon}^{\varepsilon} f(t)\, dt \qquad (12.26)$$

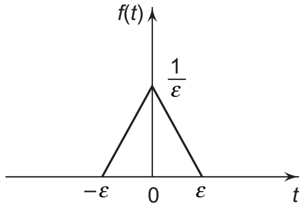


Fig. 12.16

$$= \int_{-\varepsilon}^{0} f(t)\, dt + \int_{0}^{\varepsilon} f(t)\, dt$$

$$= \int_{-\varepsilon}^{0} \left(\frac{t}{\varepsilon^2} + \frac{1}{\varepsilon}\right) dt + \int_{0}^{\varepsilon} \left(\frac{-1}{\varepsilon^2} t + \frac{1}{\varepsilon}\right) dt$$

$$= \left[\frac{t^2}{2\varepsilon^2} + \frac{t}{\varepsilon}\right]_{-\varepsilon}^{0} + \left[\frac{-1}{\varepsilon}\frac{t^2}{2} + \frac{t}{\varepsilon}\right]_{0}^{\varepsilon} = 1 \qquad (12.27)$$

(b) As $\varepsilon \to 0$, the above function shown in Fig. 8.16 becomes an impulse function. The duration of the function becomes zero.

(c) For an impulse function, the magnitude becomes infinite. Therefore as $\varepsilon \to 0$, the magnitude of the above function becomes infinite.

12.4 FUNCTIONAL TRANSFORMS

A functional transform is simply the Laplace transform of a specified function of t. Because we are limiting our introduction to the unilateral, or one-sided, Laplace transform, we define all functions to be zero for $t < 0^-$.

(i) The unit step function $f(t) = u(t)$ (12.28)

where $u(t) = 1$ for $t > 0$

$= 0$ for $t < 0$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty 1 e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

$$\mathcal{L}[u(t)] = \frac{1}{s} \qquad (12.29)$$

(ii) Exponential function $f(t) = e^{-at}$ (12.30)

$$\mathcal{L}(e^{-at}) = \int_0^\infty e^{-at} \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s+a)t} = \frac{-1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty = \frac{1}{s+a}$$

$$\mathscr{L}[e^{-at}] = \frac{1}{s+a} \tag{12.31}$$

(iii) The cosine function: $\cos \omega t$ (12.32)

$$\mathscr{L}(\cos \omega t) = \int_0^\infty \cos \omega t \, e^{-st} \, dt$$

$$= \int_0^\infty e^{-st} \left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-j\omega)t} dt + \int_0^\infty e^{-(s+j\omega)t} dt\right]$$

$$= \frac{1}{2}\left[-\frac{e^{-(s-j\omega)t}}{s-j\omega}\right]_0^\infty + \frac{1}{2}\left[-\frac{e^{-(s+j\omega)t}}{s+j\omega}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

$$\therefore \quad \mathscr{L}(\cos \omega t) = \frac{s}{s^2+\omega^2} \tag{12.33}$$

(iv) The sine function: $\sin \omega t$ (12.34)

$$\mathscr{L}(\sin \omega t) = \int_0^\infty \sin \omega t \, e^{-st} \, dt$$

$$= \int_0^\infty e^{-st} \frac{[e^{j\omega t} - e^{-j\omega t}]}{2j} \, dt$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t} dt - \int_0^\infty e^{-(s+j\omega)t} dt\right]$$

$$= \frac{1}{2j}\left\{\left[-\frac{e^{(s-j\omega)t}}{s-j\omega}\right]_0^\infty + \left[\frac{e^{-(s+j\omega)t}}{s+j\omega}\right]_0^\infty\right\}$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

$$\therefore \quad \mathscr{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2} \tag{12.35}$$

(v) The function t^n, where n is a positive integer

$$\mathscr{L}(t^n) = \int_0^\infty t^n \cdot e^{-st} \, dt \tag{12.36}$$

$$= \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \, n \, t^{n-1} \, dt$$

$$= \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt$$

$$= \frac{n}{s} \mathscr{L}(t^{n-1}) \tag{12.37}$$

Similarly $\quad \mathscr{L}(t^{n-1}) = \dfrac{n-1}{s} \mathscr{L}(t^{n-2})$

By taking Laplace transformations of t^{n-2}, t^{n-3} ... and substituting in the above equation, we get

$$\mathscr{L}(t^n) = \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s} \ldots\ldots \frac{2}{s}\frac{1}{s} \mathscr{L}(t^{n-m})$$

$$= \frac{\angle n}{s^n} \mathscr{L}(t^0) = \frac{\angle n}{s^n} \times \frac{1}{s} = \frac{\angle n}{s^{n+1}}$$

$$\therefore \qquad \mathscr{L}(t) = \frac{1}{s^2} \tag{12.38}$$

(vi) The hyperbolic sine and cosine function

$$\mathscr{L}(\cosh at) = \int_0^\infty \cosh at\, e^{-st}\, dt \tag{12.39}$$

$$= \int_0^\infty \left[\frac{e^{at} + e^{-at}}{2}\right] e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\, dt + \frac{1}{2}\int_0^\infty e^{-(s+a)t}\, dt$$

$$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2 - a^2} \tag{12.40}$$

Similarly,

$$\mathscr{L}(\sinh at) = \int_0^\infty \sinh (at)\, e^{-st}\, dt \tag{12.41}$$

$$= \int_0^\infty \left[\frac{e^{at} - e^{-at}}{2}\right] e^{-st}\, dt$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2 - a^2} \tag{12.42}$$

List of Laplace Transform Pairs See Table 12.1

Table 12.1

Type	*f(t)*	*F(s)*
Impulse	$\delta(t)$	1
Step	$U(t)$	$\frac{1}{s}$

Table 12.1 *(Contd.)*

Type	*f(t)*	*F(s)*
ramp	t	$\frac{1}{s^2}$
exponential	e^{-at}	$\frac{1}{s+a}$
sine	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
cosine	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
Hyperbolic sine	$\sinh at$	$\frac{a}{s^2-a^2}$
Hyperbolic cosine	$\cosh at$	$\frac{s}{s^2-a^2}$
damped ramp	te^{-at}	$\frac{1}{(s+a)^2}$
damped sine	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
damped cosine	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

12.5 OPERATIONAL TRANSFORMS

Operational transforms indicate how mathematical operations performed on either $f(t)$ or $F(s)$ are converted into the opposite domain. The operations of primary interest are

(1) multiplication by a constant
(2) addition (subtraction)
(3) differentiation
(4) integration
(5) translation in the time domain
(6) translation in the frequency domain and
(7) scale charging.

Multiplication by Constant

From the defining integral, if

$$\mathscr{L}[f(t)] = F(s),$$

then $$\mathscr{L}\{K f(t)\} = K F(s) \tag{12.43}$$

Consider a function $f(t)$ multiplied by a constant K.

The Laplace transform of this function is given by

$$\mathscr{L}\{K f(t)\} = \int_0^\infty K f(t)\, e^{-st}\, dt \tag{12.44}$$

$$= K \int_0^\infty f(t)\, e^{-st}\, dt = KF(s) \tag{12.45}$$

This property is called linearity property.

Addition (Subtraction)

Addition (subtraction) in the time domain translates into addition (subtraction) in the frequency domain.

Thus if

$$f_1(t) \xleftrightarrow{\mathscr{L}} F_1(s) \text{ and}$$

$$f_2(t) \xleftrightarrow{\mathscr{L}} F_2(s), \text{ then}$$

$$\mathscr{L}\,[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s) \tag{12.46}$$

Consider two functions $f_1(t)$ and $f_2(t)$. The Laplace transform of the sum or difference of these two functions is given by

$$\mathscr{L}\,\{f_1(t) \pm f_2(t)\} = \int_0^\infty \{f_1(t) \pm f_2(t)\}\, e^{-st}\, dt$$

$$= \int_0^\infty f_1(t)\, e^{-st}\, dt \pm \int_0^\infty f_2(t)\, e^{-st}\, dt$$

$$= F_1(s) \pm F_2(s)$$

$$\therefore \qquad \mathscr{L}\,\{f_1(t) \pm f_2(t)\} = F_1(s) \pm F_2(s) \tag{12.47}$$

The Laplace transform of the sum of the two or more functions is equal to the sum of transforms of the individual function. This is called superposition property.

If we can use the linearity and superposition properties jointly, we have

$$\mathscr{L}\,[K_1 f_1(t) + K_2 f_2(t)] = K_1\, \mathscr{L}\,[f_1(t)] + K_2\, \mathscr{L}\,[F_2(t)]$$

$$= K_1\, F_1(s) + K_2\, F_2(s) \tag{12.48}$$

Example 12.7 Find the Laplace transform of the function

$$f(t) = 4t^3 + t^2 - 6t + 7 \tag{12.49}$$

Solution

$$\mathscr{L}\,(4t^3 + t^2 - 6t + 7) = 4\mathscr{L}\,(t^3) + \mathscr{L}\,(t^2) - 6\mathscr{L}\,(t) + 7\mathscr{L}\,(1)$$

$$= 4 \times \frac{\angle 3}{s^4} + \frac{\angle 2}{s^3} - 6\frac{\angle 1}{s^2} + 7\frac{1}{s}$$

$$= \frac{24}{s^4} + \frac{2}{s^3} - \frac{6}{s^2} + \frac{7}{s} \tag{12.50}$$

Example 12.8 Find the Laplace transform of the function

$$f(t) = \cos^2 t. \tag{12.51}$$

Solution $\mathscr{L}(\cos^2 t) = \mathscr{L}\left(\frac{1+\cos 2t}{2}\right)$

$$= \mathscr{L}\left(\frac{1}{2}\right) + \mathscr{L}\left(\frac{\cos 2t}{2}\right) = \frac{1}{2}\,[\mathscr{L}(1) + \mathscr{L}(\cos 2t)]$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+4)} = \frac{2s^2+4}{2s(s^2+4)} \tag{12.52}$$

Example 12.9 Find the Laplace transform of the function

$$f(t) = 3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t \tag{12.53}$$

Solution $\mathscr{L}(3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t)$

$$= 3\mathscr{L}(t^4) - 2\mathscr{L}(t^3) + 4\mathscr{L}(e^{-3t}) - 2\mathscr{L}(\sin 5t) + 3\mathscr{L}(\cos 2t)$$

$$= 3\frac{\angle 4}{s^5} - 2\frac{\angle 3}{s^4} + 4\frac{1}{s+3} - 2\times\frac{5}{s^2+25} + 3\times\frac{s}{s^2+4}$$

$$= \frac{72}{s^5}\,\frac{-12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4} \tag{12.54}$$

Differentiation: If a function $f(t)$ is piecewise continuous, then the Laplace transform of its derivative $\frac{d}{dt}\,[f(t)]$ is given by

$$\mathscr{L}\left[\frac{df(t)}{dt}\right] = SF(s) - f(0) \tag{12.55}$$

By definition

$$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = \int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,d\{f(t)\} \tag{12.56}$$

Integrating by parts, we get

$$= [e^{-st}f(t)]_0^\infty + \int_0^\infty se^{-st}f(t)\,dt$$

$$= -f(0) + SF(s) \tag{12.57}$$

Hence we have

$$\mathscr{L}\,[f'(t)] = SF(s) - f(0) \tag{12.58}$$

This is applicable to higher order derivatives also. The Laplace transform of second derivative of $f(t)$ is

$$\mathscr{L}\,[f''(t)] = \mathscr{L}\left[\frac{d}{dt}(f'(t))\right]$$

$$= S\mathscr{L}[f'(t)] - f'(0) = S\{SF(s) - f(0)\} - f'(0)$$
$$= S^2 F(s) - Sf(0) - f'(0) \tag{12.59}$$

where $f'(0)$ is initial value of first derivative of $f(t)$. We find the Laplace transform of the nth derivative by successively applying the proceeding process, which leads to the general result.

$$\mathscr{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = S^n F(s) - S^{n-1} f(0^-) - S^{n-2}\frac{df(0)}{dt}$$
$$- S^{n-3}\frac{d^2 f(0^-)}{dt^2} - \dots - \frac{d^{n-1}}{dt^{n-1}} f(0^-) \tag{12.60}$$

Example 12.10 Using the formula for Laplace transform of derivatives, obtain the Laplace transform of (a) sin $3t$ (b) t^3.

Solution (a) Let $f(t) = \sin 3t$

$$f'(t) = 3\cos 3t$$
$$f''(t) = -9\sin 3t$$
$$\mathscr{L}[f''(t)] = s^2[\mathscr{L} f(t)] - sf(0) - f'(0) \tag{12.61}$$
$$f(0) = 0,\ f'(0) = 3$$
$$\mathscr{L}[f''(t)] = \mathscr{L}[-9\sin 3t]$$

Substituting in Eq. 12.61, we get

$$\mathscr{L}[-9\sin 3t] = s^2\,\mathscr{L}[f(t)] - 3$$
$$\mathscr{L}[-9\sin 3t] - s^2[\mathscr{L}(\sin 3t)] = -3$$
$$\mathscr{L}[(s^2+9)\sin 3t] = 3 \ \therefore\ \mathscr{L}(\sin 3t) = \frac{3}{s^2+9} \tag{12.62}$$

(b) Let $f(t) = t^3$ (12.63)

Differentiating successively, we get

$$f'(t) = 3t^2,\ f''(t) = 6t,\ f'''(t) = 6$$

By using differentiation theorem, we get

$$\mathscr{L}[f'''(t)] = s^3\,\mathscr{L}[f(t)] - s^2 f(0) - sf'(0) - f''(0) \tag{12.64}$$

Substituting all initial conditions, we get

$$\mathscr{L}[f'''(t)] = s^3\,\mathscr{L}[f(t)]$$
$$\mathscr{L}[6] = s^3\,\mathscr{L}[f(t)]$$
$$\frac{6}{s} = s^3\,\mathscr{L}[f(t)]$$
$$F(s) = \mathscr{L}[f(t)] = \frac{6}{s^4} \tag{12.65}$$

Integration

If a function $f(t)$ is continuous, then the Laplace transform of its integral $\int f(t)\,dt$ is given by

$$\mathscr{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}\ F(s) \tag{12.66}$$

By definition

$$\mathscr{L}\left[\int_0^t f(t)dt\right] = \int_0^\infty \left[\int_0^t f(t)dt\right] e^{-st}\ dt \tag{12.67}$$

Integrating by parts, we get

$$= \left[\frac{e^{-st}}{-s}\int_0^t f(t)dt\right]_0^\infty + \frac{1}{s}\int_0^\infty\ e^{-st} f(t)\ dt \tag{12.68}$$

Since, the first term is zero, we have

$$\mathscr{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}\ \mathscr{L}\,[f(t)] = \frac{F(s)}{s} \tag{12.69}$$

Example 12.11 Find the Laplace transform of ramp function $r(t) = t$.

Solution We know that $\int_0^t u(t) = r(t) = t$ (12.70)

Integration of unit step function gives the ramp function.

$$\mathscr{L}\,[r(t)] = \mathscr{L}\left[\int_0^t u(t)dt\right]$$

Using the integration theorem, we get

$$\mathscr{L}\left[\int_0^t u(t)dt\right] = \frac{1}{s}\ \mathscr{L}\,[u(t)] = \frac{1}{s^2}$$

Since $\mathscr{L}\,[u(t)] = \dfrac{1}{s}$ (12.71)

Differentiation of Transforms It the Laplace transform of the function $f(t)$ exists, then the derivative of the corresponding transform with respect to s in the frequency domain is equal to its multiplication by t in the time domain.

i.e. $\mathscr{L}\,[t\,f(t)] = \dfrac{-d}{ds}\ F(s)$ (12.72)

By definition

$$\frac{d}{ds}\ F(s) = \frac{d}{ds}\int_0^\infty f(t)\ e^{-st}\ dt \tag{12.73}$$

Since s and t are independent variables, and the limits 0, ∞ are constants not depending on s, we can differentiate partially with respect to s within the integration and then integrate the function obtained with respect to t.

$$\frac{d}{ds} F(s) = \frac{d}{ds}\int_0^\infty [f(t)\, e^{-st}]\, dt$$

$$= \int_0^\infty f(t)\, [-\, te^{-st}]\, dt$$

$$= -\int_0^\infty \{tf(t)\}\, e^{-st}\, dt = -\,\mathcal{L}\,[tf(t)]$$

Hence $$\mathcal{L}\,[t\, f(t)] = \frac{-d}{ds} F(s) \qquad (12.74)$$

Example 12.12 Find the Laplace transform of function

$$f(t) = t \sin 2t \qquad (12.75)$$

Solution Let $f_1(t) = \sin 2t$

$$\mathcal{L}\,[f_1(t)] = \mathcal{L}\,[\sin 2t] = F_1(s)$$

where $$F_1(s) = \frac{2}{s^2+4}$$

$$\mathcal{L}\,[t\, f_1(t)] = \mathcal{L}\,[t \sin 2t] = \frac{-d}{ds}\left[\frac{2}{s^2+4}\right] = +\frac{4s}{(s^2+4)^2} \qquad (12.76)$$

Integration of Transforms

If the Laplace transform of the function $f(t)$ exists, then the integral of corresponding transform with respect to s in the complex frequency domain is equal to its division by t in the time domain.

i.e. $$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds \qquad (12.77)$$

i.e. $$f(t) \leftrightarrow F(s)$$

$$F(s) = \mathcal{L}\,[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt \qquad (12.78)$$

Integrating both sides from s to ∞

$$\int_s^\infty F(s)\, ds = \int_s^\infty \left[\int_0^\infty f(t) e^{-st} dt\right] ds \qquad (12.79)$$

By changing the order of integration, we get

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st} ds\right] dt \tag{12.80}$$

$$= \int_0^{\infty} f(t)\left(\frac{e^{-st}}{t}\right) dt$$

$$= \int_0^{\infty} \left[\frac{f(t)}{t}\right] e^{-st}\, dt = \mathscr{L}\left[\frac{f(t)}{t}\right] \tag{12.81}$$

$$\int_0^{\infty} F(s)\, ds = \mathscr{L}\left[\frac{f(t)}{t}\right] \tag{12.82}$$

Example 12.13 Find the Laplace transform of the function

$$f(t) = \frac{2-2e^{-2t}}{t}$$

Solution Let $f_1(t) = 2 - 2e^{-2t}$ then

$$\mathscr{L}[f_1(t)] = \mathscr{L}[2 - 2e^{-2t}] \tag{12.83}$$

$$= \mathscr{L}(2) - \mathscr{L}(2e^{-2t}) = \frac{2}{s} - \frac{2}{s+2}$$

$$= \frac{2s+4-2s}{s(s+2)} = \frac{4}{s(s+2)}$$

Hence $$\mathscr{L}\left[\frac{2-2e^{-2t}}{t}\right] = \int_0^{\infty} F_1(s)\, ds$$

$$= \int_s^{\infty} \frac{4}{s(s+2)} ds \tag{12.84}$$

By taking the partial fraction expansion (discussed in later section), we get

$$\frac{4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{2}{s} - \frac{2}{s+2}$$

$\therefore$ $$\mathscr{L}\left[\frac{2-e^{-2t}}{t}\right] = \int_s^{\infty} \mathscr{L}[2-2e^{-2t}]ds$$

$$= \int_s^{\infty} \frac{2}{s} ds - \int_s^{\infty} \frac{2}{s+2} ds$$

$$= [2\log s - 2\log(s+2)]_S^{\infty}$$

$$= \left[2\log\frac{1}{1+\frac{2}{s}}\right]_s^{\infty} = -2\log\left(\frac{s}{s+2}\right)$$

$$\mathcal{L}\left[\frac{2-2e^{-2t}}{t}\right] = 2\log\left(\frac{s+2}{s}\right) \tag{12.85}$$

Translation in the Time Domain

If the function $f(t)$ has the transform $F(s)$, then the Laplace transform of $f(t-a)\,(t-a)$ is $e^{-as}\,F(s)$. By definition

$$\mathcal{L}\,[f(t-a)\,u(t-a)] = \int_0^{\infty} [f(t-a)\,u(t-a)]\,e^{-st}\,dt \tag{12.86}$$

Since $\quad f(t-a)\,u(t-a) = 0$ for $t < a$

$$= f(t-a) \text{ for } t > a$$

$$\therefore \quad \mathcal{L}\,[f(t-a)\,u(t-a)] = \int_{\omega}^{\infty} f(t-a)\,e^{-st}\,dt \tag{12.87}$$

Put $t - a = \tau$ then $\tau + a = t$

$$dt = d\tau$$

Therefore, the above becomes

$$\mathcal{L}\,[f(t-a)\,u(t-a)] = \int_0^{\infty} f(t)\,e^{-s(\tau+a)}\,d\tau \tag{12.88}$$

$$= e^{-as}\int_0^{\infty} f(\tau)\,e^{-st}\,d\tau = e^{-as}\,F(s)$$

$$\therefore \quad \mathcal{L}\,[f(t-a)\,u(t-a)] = e^{-as}\,F(s) \tag{12.89}$$

Translation in the time domain corresponds to multiplication by an exponential in the frequency domain.

Example 12.14 If $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$, determine the Laplace transform of $[u(t) - u(t-a)]$.

Solution The function $f(t) = u(t) - u(t-a)$ is shown in Fig. 12.17.

$$\mathcal{L}\,[f(t)] = \mathcal{L}\,[u(t) - u(t-a)] \tag{12.90}$$

$$= \mathcal{L}\,[u(t)] - \mathcal{L}\,[u(t-a)]$$

$$= \frac{1}{s} - e^{-as}\,\frac{1}{s} = \frac{1}{s}\,(1 - e^{-as})$$

$$\mathcal{L}\,[f(t)] = \frac{1}{s}\,(1 - e^{-as}) \tag{12.91}$$

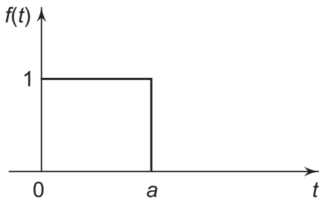


Fig. 12.17

Translation in the Frequency Domain

If the function $f(t)$ has the transform $F(s)$, then Laplace transform of $e^{-at} f(t)$ is $F(s + a)$.

By definition,
$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \tag{12.92}$$

and therefore,
$$F(s + a) = \int_0^\infty f(t)\, e^{-(s+a)t}\, dt \tag{12.93}$$

$$= \int_0^\infty e^{-at} f(t)\, e^{-st}\, dt = \mathscr{L}\,[e^{-at} f(t)] \tag{12.94}$$

$$\therefore \quad F(s + a) = \mathscr{L}\,[e^{-at} f(t)] \tag{12.95}$$

Similarly, we have

$$\mathscr{L}\,[e^{at} f(t)] = F(s - a) \tag{12.96}$$

Translation in the frequency domain corresponds to multiplication by an exponential in the time domain.

Example 12.15 Find the Laplace transform of $e^{at} \sin bt$.

Solution Let
$$f(t) = \sin bt \tag{12.97}$$

$$\mathscr{L}\,[f(t)] = \mathscr{L}\,[\sin bt] = \frac{b}{s^2 + b^2}$$

Since
$$\mathscr{L}\,[e^{at} f(t)] = F(s - a)$$

$$\mathscr{L}\,[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2} \tag{12.98}$$

Example 12.16 Find the Laplace transform of $(t + 2)^2\, e^t$

Solution Let
$$f(t) = (t + 2)^2 = t^2 + 2t + 4 \tag{12.99}$$

$$\mathscr{L}\,[f(t)] = \mathscr{L}\,[t^2 + 2t + 4] = \frac{2}{s^3} + \frac{2}{s^2} + \frac{4}{s}$$

Since
$$\mathscr{L}\,[e^{at} f(t)] = F(s - a)$$

$$\mathscr{L}\,[e^{t} f(t)] = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{4}{s-1} \tag{12.100}$$

Scale Changing

The scale change property gives the relationship between $f(t)$ and $F(s)$ when the time variable is multiplied by a positive constant.

$$\mathscr{L}\,\{f(at)\} = \frac{1}{a} F\left(\frac{s}{s}\right),\ a > 0 \tag{12.101}$$

By definition

$$\mathscr{L}[f(at)] = \int_0^\infty f(at)\, e^{-st}\, dt \tag{12.102}$$

Put $at = \tau$

$$dt = \frac{1}{a}\, d\tau$$

$$\mathscr{L}[f(at)] = \int_0^\infty f(\tau) e^{-\frac{s}{a}\tau} \cdot \frac{1}{a} d\tau$$

$$= \frac{1}{a}\int_0^\infty f(\tau) e^{-\frac{s}{a}\tau} d\tau$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right) \tag{12.103}$$

List of Operational Transforms

See Table 12.2

Table 12.1

Operation	*f(t)*	*F(s)*
Multiplication by a constant	$K f(t)$	$K F(s)$
Addition/Subtraction	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
First derivative (time)	$\frac{df(t)}{dt}$	$SF(s) - f(0)$
Second derivative (time)	$\frac{d^2 f(t)}{dt^2}$	$S^2F(s) - Sf(0) - \frac{df(0)}{dt}$
nth derivative (time)	$\frac{d^n f(t)}{dt^n}$	$S^n F(s) - S^{n-1} f(0) - S^{n-2} f'(0)$ $- S^{n-3} f''(0) \ldots f_{(0)}^{n-1}$
Operation	$f(t)$	$F(s)$
Time integral	$\int_0^t f(t)\, dt$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)\, u(t-a),\ a > 0$	$e^{-as} F(s)$
Translation in frequency	$e^{-at} f(t)$	$F(s+a)$
Scale changing	$f(at),\ a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
First derivative (s)	$t\, f(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
S integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\, du$

12.6 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

Periodic functions appear in many practical problems. Let function $f(t)$ be a periodic function which satisfies the condition $f(t) = f(t + T)$ for all $t > 0$ where T is period of the function.

$$\mathscr{L}[f(t)] = \int_0^T f(t)\, e^{-st}\, dt + \int_T^{2T} f(t)\, e^{-st}\, dt + \ldots$$

$$+ \int_{nT}^{(n+1)T} f(t)\, e^{-st}\, dt + \ldots \qquad (12.104)$$

$$= \int_0^T f(t)\, e^{-st}\, dt + \int_0^T f(t)\, e^{-st}\, e^{-sT}\, dt + \ldots$$

$$+ \int_0^T f(t)\, e^{-st}\, e^{-nsT}\, dt + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots + e^{-nsT} + \ldots) \int_0^T f(t)\, e^{-st}\, dt \qquad (12.105)$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt \qquad (12.106)$$

Example 12.17 Find the transform of the waveform shown in Fig. 12.18.

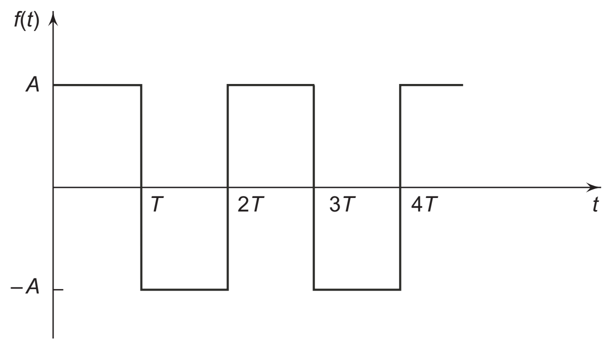


Fig. 12.18

Solution Here the period is $2T$

$$\mathscr{L}[f(t)] = \frac{1}{1 - e^{-2sT}} \int_0^{2T} f(t)\, e^{-st}\, dt \qquad (12.107)$$

$$= \frac{1}{1 - e^{-2sT}} \left[\int_0^T A e^{-st} dt + \int_T^{2T} (-A) e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2sT}}\left[-\frac{A}{s}e^{-st}\Big)_0^T + \frac{A}{s}e^{-st}\Big)_T^{2T} \right]$$

$$= \frac{1}{1-e^{-2sT}}\left[-\frac{A}{s}(e^{-sT}-1) + \frac{A}{s}(e^{-2sT}-e^{-sT}) \right]$$

$$= \frac{1}{1-e^{-2sT}}\left[\frac{A}{s}(1-e^{-sT})^2 \right] = \frac{A}{s}\left(\frac{1-e^{-sT}}{1+e^{-sT}} \right)$$

$$\therefore \qquad \mathscr{L}[f(t)] = \frac{A}{s}\left(\frac{1-e^{-sT}}{1+e^{-sT}} \right) \qquad (12.108)$$

12.7 INVERSE TRANSFORMS

So far, we have discussed Laplace transform of a functions $f(t)$. If the function is a rational function of s, which can be expressed in the form of a ratio of two polynomial in s such that no non-integral powers of s appear in the polynomials. Infact, for linear, lumped parameter circuits whose component values are constant, the s-domain expressions for the unknown voltages and currents are always rational functions of s. If we can inverse transform rational functions of s, we can solve for the time domain expressions for the voltages and currents.

In general, we need to find the inverse transform of a function that has the form.

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0} \qquad (12.109)$$

The coefficients a and b are real constants, and the exponents m and n are positive integers. The ratio $\frac{N(s)}{D(s)}$ is called a proper rational function if $m > n$, and an improper rational function if $m \le n$. Only a proper rational function can be expanded as a sum of partial fractions.

Partial Fraction Expansion: Proper Rational Functions

A proper rational function is expanded into a sum of partial fractions by writing a term or a series of terms for each root of $D(s)$. Thus $D(s)$ must be in factored form before we can make a partial fraction expansion. The roots of $D(s)$ are either (1) real and distinct (2) complex and distinct (3) real and repeated or (H) complex and repeated.

(i) When the roots are real and distinct

In this case $\qquad F(s) = \frac{N(s)}{D(s)} \qquad (12.110)$

where $\qquad D(s) = (s-a)(s-b)(s-c) \qquad (12.111)$

Expanding $F(s)$ into partial fractions, we get

$$F(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} \tag{12.112}$$

To obtain the constant A, multiplying Eq. (12.112) with $(s - a)$ and putting $s = a$, we get

$$F(s)\,(s-a)\big|_{s=a} = A$$

Similarly, we can get the other constants.

$$B = (s-b)\,F(s)\big|_{s=b}$$

$$C = (s-c)\,F(s)\big|_{s=c}$$

Example 12.18 Determine the partial fraction expansion for

$$F(s) = \frac{s^2+s+1}{s(s+5)(s+3)}$$

Solution

$$F(s) = \frac{s^2+s+1}{s(s+5)(s+3)} \tag{12.113}$$

$$\frac{s^2+s+1}{s(s+5)(s+3)} = \frac{A}{s} + \frac{B}{s+5} + \frac{C}{s+3} \tag{12.114}$$

$$A = sF(s)|_{s=0} = \left.\frac{s^2+s+1}{(s+5)(s+3)}\right|_{s=0} = \frac{1}{15}$$

$$B = (s+5)\,F(s)|_{s=-5} = \left.\frac{s^2+s+1}{s(s+3)}\right|_{s=-5} = 2.1$$

$$C = (s+3)\,F(s)|_{s=-3} = \left.\frac{s^2+s+1}{s(s+5)}\right|_{s=-3} = -1.17$$

$$\frac{s^2+s+1}{s(s+5)(s+3)} = \frac{1}{15s} + \frac{2.1}{s+5} - \frac{1.17}{s+3} \tag{12.115}$$

(ii) When roots are real and repeated

In this case $\quad F(s) = \dfrac{N(s)}{D(s)}$

where $\quad D(s) = (s-a)^n\,D_1(s)$

The partial fraction expansion of $F(s)$ is

$$F(s) = \frac{A_0}{(s-a)^n} + \frac{A_1}{(s-a)^{n-1}} + \ldots + \frac{A_{n-1}}{s-a} + \frac{N_1(s)}{D_1(s)} \tag{8.115}$$

where $\dfrac{N_1(s)}{D_1(s)}$ represents the remainder terms of expansion.

To obtain the constant A_0, A_1, ... A_{n-1}, let us multiply both sides of Eq. (12.115) by $(s-a)^n$.

Thus

$$(s-a)^n F(s) = F_1(s) = A_0 + A_1 (s-a) + A_2 (s-a)^2 + \ldots$$
$$+ A_{n-1} (s-a)^{n-1} + R(s) (s-a)^n \quad (12.117)$$

where $R(s)$ indicates the remainder terms

Putting $s = a$, we get

$$A_0 = (s-a)^n F(s)|_{s=a}$$

Differentiating Eq. (12.116) with respect to s, and putting $s = a$, we get

$$A_1 = \frac{d}{ds} F_1(s)|_{s=a}$$

Similarly $$A_2 = \frac{1}{2!}\frac{d^2}{ds^2} F_1(s)|_{s=a}$$

In general, $$A_n = \frac{1}{n!}\frac{d^n F_1(s)}{ds^n}\bigg|_{s=a} \quad (12.118)$$

Example 12.19 Determine the partial fraction expansion for

$$F(s) = \frac{s-5}{s(s+2)^2}$$

Solution $$F(s) = \frac{s-5}{s(s+2)^2} = \frac{A}{s} + \frac{B}{(s+2)^2} + \frac{B_1}{s+2}$$

$$A = F(s)\ S|_{s=0} = \frac{s-5}{(s+2)^2}\bigg|_{s=0} = -\frac{5}{4} = -1.25$$

$$N_1(s) = (s+2)^2 F(s) = \frac{s-5}{s}$$

$$B_0 = F(s)\ (s+2)^2|_{s=2} = \frac{s-5}{s}\bigg|_{s=-2} = 3.5$$

$$B_1 = \frac{d}{ds} F_1(s)|_{s=-2}$$

$$= \frac{d}{ds}\left(1-\frac{5}{3}\right)\bigg|_{s=-2} = \frac{5}{s^2}\bigg|_{s=-2} = \frac{5}{4} = 1.25$$

(iii) When roots are distinct complex roots of $D(s)$

Consider a function $$F(s) = \frac{N(s)}{D(s)(s-\alpha+j\beta)(s-\alpha-j\beta)} \quad (8.118)$$

The partial fraction expansion of $F(s)$ is

$$F(s) = \frac{A}{s-\alpha-j\beta} + \frac{B}{s-\alpha+j\beta} + \frac{N_1(s)}{D_1(s)} \quad (12.120)$$

where $\dfrac{N_1(s)}{D_1(s)}$ is the remainder term.

Multiplying Eq. (12.118) by $(s-\alpha-j\beta)$ and putting $S = \alpha + j\beta$, we get

$$A = \frac{N(\alpha + j\beta)}{D_1(\alpha + j\beta)(+2j\beta)} \tag{12.121}$$

Similarly

$$B = \frac{N(\alpha - j\beta)}{(-2j\beta)D_1(\alpha - j\beta)} \tag{12.122}$$

In general, $B = A^*$ where A^* is complex conjugate of A.

If we denote the inverse transform of the complex conjugate terms as $f(t)$

$$f(t) = \mathscr{L}^{-1}\left[\frac{A}{s-\alpha-j\beta} + \frac{B}{s-\alpha+j\beta}\right]$$

$$= \mathscr{L}^{-1}\left[\frac{A}{s-\alpha-j\beta} + \frac{A^*}{s-\alpha+j\beta}\right] \tag{12.123}$$

where A and A^* are conjugate terms.

If we denote $A = C + jD$, then

$$B = C - jD = A^*$$

$$\therefore \quad f(t) = e^{\alpha t}\,(Ae^{j\beta t} + A^*\, e^{-j\beta t}) \tag{12.124}$$

Example 12.20 Find the inverse transform of the function

$$F(s) = \frac{s+5}{s(s^2+2s+5)}$$

Solution

$$F(s) = \frac{s+5}{s(s^2+2s+5)} \tag{12.125}$$

By taking partial fractions, we have

$$F(s) = \frac{s+5}{s(s^2+2s+5)} = \frac{A}{s} + \frac{B}{s+1-j2} + \frac{B^*}{s+1+j2} \tag{12.126}$$

$$A = F(s)\, s|_{s=0} = \frac{s+5}{s^2+2s+5} = 1$$

$$B = F(s)\,(s+1-j2)|_{s=-1+j2} = \frac{s+5}{s(s+1+j2)}\Big|_{s=-1+j2}$$

$$= \frac{4+j2}{(-1+j2)j4} = \frac{2+j}{2j(-1+j2)} = \frac{2+j}{-2j-4} = \frac{-1}{2}$$

$$B^* = F(s)\,(s+1+j2)|_{s=-1-j2} = \frac{s+5}{s(s+1-j2)}\Big|_{s=-1-j2}$$

$$= \frac{-1 - j2 + 5}{(-1 - j2)(-1 - j2 + 1 - j2)}$$

$$= \frac{4 - j2}{-(1 + j2)(j4)} = \frac{4 - j2}{4j - 8} = \frac{2(2 - j)}{-4(2 - j)} = \frac{-1}{2}$$

$$\therefore \qquad F(s) = \frac{1}{s} - \frac{1}{2(s + 1 - j2)} - \frac{1}{2(s + 1 + j2)} \qquad (12.127)$$

The inverse transform of $F(s)$ is $f(t)$

$$f(t) = \mathscr{L}^{-1}[F(s)] = \mathscr{L}^{-1}\left[\frac{1}{s} - \frac{1}{2(s + 1 - j2)} - \frac{1}{2(s + 1 + j2)}\right]$$

$$= \mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{(s + 1 - j2)}\right] - \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s + 1 + j2}\right]$$

$$= 1 - \frac{1}{2}e^{(-1 + j2)t} - \frac{1}{2}e^{(-1 - j2)t} \qquad (12.128)$$

(iv) when roots are repeated and complex of $D(S)$

The complex roots always appear in conjugate pairs and that the coefficients associated with a conjugate pair are also conjugate, so that only half the Ks need to be evolved.

Consider the function $F(s) = \dfrac{768}{(s^2 + 6s + 2s)^2}$ (12.129)

By factoring the denominator polynomial, we have

$$F(s) = \frac{768}{(s + 3 - j4)^2 (s + 3 + j4)^2}$$

$$= \frac{K_1}{(s + 3 - j4)^2} + \frac{K_2}{s + 3 - j4}$$

$$+ \frac{K_1^*}{(s + 3 + j4)^2} + \frac{K_2^*}{s + 3 + j4} \qquad (12.130)$$

Now we need to evaluate only K_1 and K_2, because K_1^* and K_2^* are conjugate values.

The value of K_1, is

$$K_1 = \frac{768}{(s + 3 + j4)^2}\bigg|_{s = -3 + j4} = \frac{768}{(j8)^2} = -12 \qquad (12.131)$$

The value of K_2 is

$$K_2 = \frac{d}{ds}\left[\frac{768}{(s + 3 + j4)^2}\right]_{s = -3 + j4}$$

$$= -\frac{2(768)}{(s + 3 + j4)^3}\bigg|_{s = -3 + j4} = -\frac{2(768)}{(j8)^3}$$

$$= -j^3 = 3 \angle -90^\circ \qquad (12.132)$$

From Eq. (12.131) and (12.132)

$$K_1^* = -12,\ K_2^* = j3 = 3\ \angle 90^\circ \tag{12.133}$$

are now group *f* the partial fraction expansion by conjugate terms to obtain

$$F(s) = \left[\frac{-12}{(s+3-j4)^2} + \frac{-12}{(s+3+j4)^2}\right] + \left(\frac{3\angle -90^\circ}{s+3-j4} + \frac{3\angle 90^\circ}{s+3+j4}\right) \tag{12.134}$$

Inverse transform of the above function is

$$f(t) = [-24t\, e^{-3t}\cos 4t + 6e^{-3t}\cos(4t - 90^\circ)]\, u(t) \tag{12.135}$$

Table 12.3 *Useful transform pairs*

Nature of roots	*F(s)*	*f(t)*
Distinct real	$\frac{k}{s+a}$	$Ke^{-at}\, u(t)$
Repeated real	$\frac{k}{(s+a)^2}$	$Kt\, e^{-at}\, u(t)$
Distinct complex	$\frac{k}{s+\alpha-j\beta} + \frac{k^*}{s+\alpha+j\beta}$	$2\lvert k\rvert e^{-\alpha t}\cos(\beta t+\theta)\, u(t)$
Repeated complex	$\frac{k}{(s+\alpha-j\beta)^2} + \frac{k^*}{(s+\alpha+j\beta)^2}$	$2t\lvert k\rvert e^{-\alpha t}\cos(\beta t+\theta)\, u(t)$

Partial Fraction Expansion: Improper Rational Function

An improper rational function can always be expanded into a polynomial plus a proper rational function. The polynomial is then inverse-transformed into impulse functions and derivatives of impulse functions.

Consider a function

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200\, s + 300}{s^2 + 9s + 20} \tag{12.136}$$

Dividing the denominator into the numerator until the remainder is proper rational function gives

$$F(s) = s^2 + 4s + 10 + \frac{30s+100}{s^2+9s+20} \tag{12.137}$$

Now we expand the proper rational function into a sum of partial fractions

$$\frac{30s+100}{s^2+9s+20} = \frac{30s+100}{(s+4)(s+5)} = \frac{-20}{s+4} + \frac{50}{s+5} \tag{12.138}$$

Substituting Eq. (12.137) into Eq. (12.136) yields

$$F(s) = s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5} \tag{12.139}$$

By taking inverse transform, we get

$$f(t) = \frac{d^2\delta(t)}{dt^2} + 4\,\frac{d\delta(t)}{dt} + 10\ \delta(t) - (20e^{-4t} - 50e^{-5t})\,u(t) \qquad (12.140)$$

12.8 INITIAL AND FINAL VALUE THEOREMS

The initial and final value theorems are useful because they enable us to determine from $F(s)$ the behaviour of $f(t)$ at 0 and ∞. Hence we can check the initial and final values of $f(t)$ to see if they conform with known circuit behaviour, before actually finding the inverse transform of $F(s)$.

The initial-value theorem states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} SF(s) \qquad (12.141)$$

and the final-value theorem states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} SF(s) \qquad (12.142)$$

The initial-value theorem is based on the assumption that $f(t)$ contains no impulse functions.

To prove, initial value theorem, we start with the operational transform of the first derivative.

$$\mathscr{L}\left[\frac{df}{dt}\right] = SF(s) - f(0)$$

$$= \int_0^\infty \frac{df}{dt}\, e^{-st}\, dt \qquad (12.143)$$

Now we take the limit as $s \to \infty$

$$\lim_{s\to\infty} [SF(s) - f(0)] = \lim_{s\to\infty} \int_0^\infty \frac{df}{dt}\, e^{-st} dt \qquad (12.144)$$

The right hand side of the above equation becomes zero as $s \to \infty$

$$\therefore \quad \lim_{s\to\infty} [SF(s) - f(0)] = 0$$

$$\lim_{s\to\infty} SF(s) = f(0) = \lim_{t\to 0} f(t) \qquad (12.145)$$

The proof of the final value theorem also starts with Eq. (12.142). Here we take the limit as $s \to 0$.

$$\lim_{s\to 0} [SF(s) - f(0)] = \lim_{s\to 0}\left(\int_0^\infty \frac{df}{dt}\, e^{-st} dt\right) \qquad (12.146)$$

$$\lim_{s\to 0} [SF(s) - f(0)] = [f(t)]_0^\infty \qquad (12.147)$$

$$\lim_{s\to 0} SF(s) - f(0) = \lim_{t\to\infty} f(t) - f(0)$$

Since $f(0)$ is not a function of s, it gets cancelled from both sides.

$$\therefore \qquad \lim_{t\to\infty} f(t) = \lim_{s\to\infty} SF(s) \tag{12.148}$$

The final-value theorem is useful only if $f(\infty)$ exists.

The Application of Initial and Final Value Theorems

Consider the transform pair given by

$$\frac{100\,(s+3)}{(s+6)\,(s^2+6s+25)} \leftrightarrow [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13°)]\,u(t) \tag{12.149}$$

The initial value theorem gives

$$\lim_{s\to\infty} SF(s) = \lim_{s\to\infty} \frac{100\,s^2\left(1+\dfrac{3}{s}\right)}{s^3\left[1+\dfrac{6}{s}\right]\left[1+\dfrac{6}{s}+\dfrac{25}{s^2}\right]} = 0 \tag{12.150}$$

$$\lim_{t\to 0} f(t) = [-12 + 20\cos(-53.13°)]\,(1) = -12 + 12 = 0 \tag{12.151}$$

The final value theorem gives

$$\lim_{s\to 0} SF(s) = \lim_{s\to 0} \frac{100\,s(s+3)}{(s+6)\,(s^2+6s+25)} = 0 \tag{12.152}$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [-\,12e^{-6t} + 20e^{-3t}\cos(4t - 53.13°)]\,u(t) = 0 \tag{12.153}$$

Example 12.21 Verify the initial value theorem for the following functions

(i) $5e^{-4t}$ (ii) $2 - e^{5t}$

Solution (i) Let $f(t) = 5e^{-4t}$ (12.154)

then

$$F(s) = \frac{5}{s+4}$$

$$SF(s) = \frac{5s}{s+4}$$

$$\lim_{s\to\infty} SF(s) = \lim_{s\to\infty} \frac{5}{1+4/s} = 5$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} 5e^{-4t} = 5 \tag{12.155}$$

Hence the initial value theorem is proved.

(ii) Let $f(t) = 2 - e^{5t}$ (12.156)

then

$$F(s) = \mathscr{L}(2 - e^{5t}) = \mathscr{L}(2) - \mathscr{L}(e^{5t})$$

$$= \frac{2}{s} - \frac{1}{s-5} = \frac{s-10}{s\,(s-5)}$$

$$SF(s) = \frac{s-10}{s-5}$$

$$\lim_{s\to\infty} SF(s) = \frac{\left(1-\frac{10}{s}\right)}{\left(1-\frac{5}{s}\right)} = 1$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} (2 - e^{5t}) = 1 \tag{12.157}$$

Hence the initial value theorem is proved.

Example 12.22 Verify the final value theorem for the following functions

(i) $2 + e^{-3t} \cos 2t$ (ii) $6(1 - e^{-t})$

Solution (i) Let $f(t) = 2 + e^{-3t} \cos 2t$ (12.158)

then $$F(s) = \frac{2}{s} + \frac{s+3}{(s+3)^2+4}$$

$$SF(s) = 2 + \frac{s^2}{(s+3)^2+4} + \frac{3s}{(s+3)^2+4}$$

$$\lim_{s\to 0} SF(s) = \lim_{s\to 0}\left[2 + \frac{s(s+3)}{(s+3)^2+4}\right] = 2$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [2 + e^{-3t} \cos 2t] = 2 \tag{12.159}$$

Hence the final value theorem is proved.

(ii) Let $f(t) = 6(1 - e^{-t})$ (12.160)

then $$F(s) = \frac{6}{s} - \frac{6}{s+1} = \frac{6}{s(s+1)}$$

$$SF(s) = \frac{6}{s+1}$$

$$\lim_{s\to 0} SF(s) = 6$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 6(1 - e^{-t}) = 6 \tag{12.161}$$

Hence the final value theorem is proved.

Solved Problems

Problem 12.1 For the waveform shown in Fig. 12.19, write the expression and find the Laplace transform.

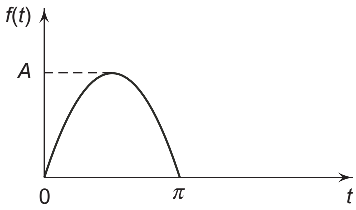


Fig. 12.19

Solution The waveform shown in Fig. 12.19 starts at $t = 0$ and ends at $t = \pi$. The equation for the above waveform is $f(t) = A \sin t$. In terms of unit function the waveform can be expressed as

$$f(t) = A \sin t\, [u(t) - u(t - \pi)] \tag{12.162}$$

By definition, we have

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty A \sin t\, [u(t) - u(t - \pi)]\, e^{-st}\, dt \tag{12.163}$$

$$= \int_0^\infty A \sin t\, u(t)\, e^{-st}\, dt - \int_0^\infty A \sin u(t - \pi)^{-st}\, edt$$

Since $f(t) = 0$ for $t > \pi$, the second term becomes zero

$$\therefore \quad \mathscr{L}[f(t)] = \int_0^\pi A \sin t\, e^{-st}\, dt \tag{12.164}$$

$$= A \frac{e^{-st}}{(s^2 + 1)} [- s \sin t - \cos t]_0^\pi$$

$$= 2A \frac{e^{-s\pi} - 1}{(s^2 + 1)} \tag{12.165}$$

Problem 12.2 For the waveform shown in Fig. 12.20 write the expression using step functions and obtain the Laplace transform.

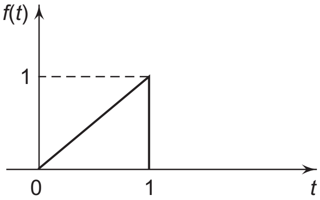


Fig. 12.20

Solution The waveform shown in Fig. 12.20 starts at $t = 0$ and ends $t = 1$ sec. The equation for the above waveform is $f(t) = t$. In terms of unit functions, the waveform can be expressed as

$$f(t) = t[u(t) - u(t-1)] \tag{12.166}$$

By definition

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty t[u(t) - u(t-1)]\, e^{-st}\, dt \tag{12.167}$$

Since $f(t) = 0$ for $t > 1$, the second term becomes zero

$$\mathcal{L}[f(t)] = \int_0^\infty t\, u(t)\, e^{-st}\, dt$$

$$= \int_0^1 t\, e^{-st}\, dt$$

$$= t\int_0^1 e^{-st}\, dt - \int_0^1 \frac{e^{-st}}{-s}\, dt \tag{12.168}$$

$$= t\left.\frac{e^{-st}}{-s}\right)_0^1 - \left.\frac{e^{-st}}{s^2}\right)_0^1$$

$$= \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2} - e^{-s}\left[\frac{1}{s} + \frac{1}{s^2}\right] \tag{12.169}$$

Problem 12.3 Determine the Laplace transform of $\frac{d^2y}{dt^2}$ if $y = t^2$.

Solution From the differentiation property

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2y - sy(0) - \frac{dy(0)}{dt} \tag{12.170}$$

$$\mathcal{L}(y) = \mathcal{L}(t^2) = \frac{2}{s^3}$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^2) = 2t$$

$$\frac{dy(0)}{dt} = 0$$

From the function $y = t^2$

$$y(0) = 0$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2\left\{\frac{2}{s^3}\right\} - s(0) - 0$$

$$= \frac{2}{s} \tag{12.171}$$

Problem 12.4 Determine the Laplace transform of $y = e^{-6t} + t$

Solution

$$\mathscr{L}[y] = \mathscr{L}[e^{-6t} + t]$$

$$= \mathscr{L}[e^{-6t}] + \mathscr{L}[t]$$

$$= \frac{1}{s+6} + \frac{1}{s^2}$$

$$= \frac{s^2 + s + 6}{s^2 (s+6)} = \frac{s^2 + s + 6}{s^3 + 6s^2} \tag{12.172}$$

Problem 12.5 Determine the inverse Laplace transform of

$$\left\{\frac{6}{s+1} + \frac{14}{s}\right\}$$

Solution

$$\mathscr{L}^{-1}\left\{\frac{6}{s+1} + \frac{14}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{6}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{14}{s}\right\} \tag{12.173}$$

$$= 6\,\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{14}{s}\right\}$$

$$= 6e^{-t} + 14 \tag{12.174}$$

Problem 12.6 Determine the inverse Laplace transform of

$$\left\{\frac{4}{s^2 + 64}\right\}$$

Solution

$$\frac{4}{s^2 + 64} = \frac{4}{s^2 + 8^2} \tag{12.175}$$

$$= \frac{(4)}{8}\left\{\frac{8}{s^2 + 8^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{8}{s^2 + 8^2}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{4}{s^2 + 64}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{2}\right)\frac{8}{s^2 + 8^2}\right\} \tag{12.176}$$

$$= \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{8}{s^2+8^2}\right\}$$

$$= \frac{1}{2}\sin 8t \qquad (12.177)$$

Problem 12.7 Determine the inverse Laplace transform of the function

$$F(s) = \frac{s-3}{s^s+4s+13}$$

Solution

$$F(s) = \frac{s-3}{s^2+4s+13} = \frac{s-3}{(s+2)^2+9} = \frac{(s+2)-5}{(s+2)^2+9} \qquad (12.178)$$

We can write the above equation as

$$\frac{s+2}{(s+2)^2+9} - \frac{5}{(s+2)^2+9}$$

By taking the inverse transform, we get

$$\mathscr{L}^{-1}F(s) = \mathscr{L}^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - \mathscr{L}^{-1}\left[\frac{5}{(s+2)^2+9}\right]$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$$

$$= \frac{e^{-2t}}{3}[3\cos 3t - 5\sin 3t] \qquad (12.179)$$

Problem 12.8 Find the inverse transform of the following

(a) $\log\left(\frac{s+5}{s+6}\right)$ (b) $\frac{1}{(s^2+5^2)^2}$

Solution (a) Let $F(s) = \log\left(\frac{s+5}{s+6}\right)$ (12.180)

Then $$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\log\left(\frac{s+5}{s+6}\right)\right] = \frac{1}{s+5} - \frac{1}{s+6}$$

We know that

$$\mathscr{L}^{-1}\left[\frac{d}{ds}F(s)\right] = -t\,f(t) \qquad (12.181)$$

$\therefore$ $$\mathscr{L}^{-1}\left[\frac{d}{ds}F(s)\right] = \mathscr{L}^{-1}\left[\frac{1}{s+5} - \frac{1}{s+6}\right] = e^{-5t} - e^{-6t}$$

Hence $$-tf(t) = e^{-5t} - e^{-6t}$$

$$f(t) = \frac{e^{-6t} - e^{-5t}}{t} \qquad (12.182)$$

(b)
$$\text{Let } F(s) = \frac{1}{(s^2+5^2)^2} \tag{12.183}$$

$$\frac{1}{(s^2+5^2)^2} = \frac{1}{s}\cdot\frac{s}{(s^2+5^2)^2}$$

Therefore
$$\mathscr{L}^{-1}\left[\frac{1}{(s^2+5^2)^2}\right] = \mathscr{L}^{-1}\left[\frac{1}{s}\frac{s}{(s^2+5^2)^2}\right] \tag{12.184}$$

According to integration theorem

$$\mathscr{L}^{-1}\left[\frac{1}{s}\frac{s}{(s^2+5^2)^2}\right] = \int_0^t \left[\mathscr{L}^{-1}\frac{s}{(s^2+5^2)}\right] dt \tag{12.185}$$

if
$$\mathscr{L}^{-1}\left[\frac{1}{s}\frac{s}{(s^2+5^2)^2}\right] = \int_0^t \left[\mathscr{L}^{-1}\frac{s}{(s^2+5^2)^2}\right] dt$$

if
$$\mathscr{L}[f(t)] = F(s), \text{ then } \mathscr{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$$

Here
$$\int_s^\infty \frac{s}{(s^2+5^2)^2}\, ds = \frac{-1}{2}\left[\frac{1}{s^2+5^2}\right]_s^\infty = \frac{1}{2}\frac{1}{s^2+5^2}$$

Therefore
$$\frac{f(t)}{t} = \mathscr{L}^{-1}\left(\frac{1}{2}\cdot\frac{1}{s^2+5^2}\right) = \frac{1}{10}\sin 5t$$

$\therefore$
$$f(t) = \frac{t}{10}\sin 5t$$

or
$$\mathscr{L}^{-1}\left[\frac{1}{s}\frac{s}{(s^2+5^2)^2}\right] = \int_0^t \frac{t\sin 5t}{10}\, dt \tag{12.186}$$

$$= \frac{1}{10}\left[t\left(\frac{-\cos 5t}{5}\right) + \frac{\sin 5t}{25}\right]_0^f$$

$$= \frac{1}{250}[\sin 5t - 5t\cos 5t] \tag{12.187}$$

Problem 12.9 Find the Laplace transform of the full wave rectified output as shown in Fig. 12.21.

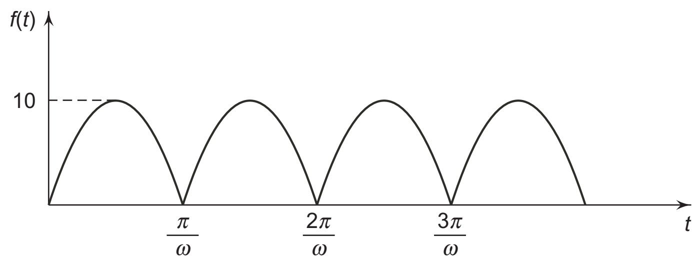


Fig. 12.21

Solution We have

$$f(t) = 10 \sin \omega t \text{ for } 0 < t < \frac{\pi}{\omega} \tag{12.188}$$

Hence

$$\mathscr{L}[f(t)] = \int_0^{\pi/\omega} \frac{\left(e^{-st} f(t)\right)dt}{1-e^{\frac{-s\pi}{w}}} \tag{12.189}$$

$$= \int_0^{\pi/w} \frac{\left(e^{-st}\, 10 \sin \omega t\right)dt}{1-e^{\frac{-s\pi}{\omega}}}$$

$$= \frac{10}{1-e^{\frac{-s\pi}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s \sin \omega t - \omega \cos \omega t)\right]_0^{\pi/\omega}$$

$$= \frac{10}{\left(1-e^{\frac{-s\pi}{\omega}}\right)(s^2+\omega^2)}\left[\omega\, e^{\frac{-s\pi}{\omega}} + \omega\right]$$

$$= \frac{10\omega}{s^2+\omega^2} \frac{\left(1+e^{\frac{-s\pi}{\omega}}\right)}{\left(1-e^{\frac{-s\pi}{\omega}}\right)}$$

$$= \frac{10\omega}{s^2+\omega^2} \frac{e^{\frac{s\pi}{2\omega}} + e^{\frac{-s\pi}{2\omega}}}{e^{\frac{s\pi}{2\omega}} - e^{\frac{-s\pi}{2\omega}}}$$

$$= \frac{10\omega}{s^2+\omega^2} \cosh\left(\frac{s\pi}{2\omega}\right) \tag{12.190}$$

Problem 12.10 Find the Laplace transform of the square wave shown in Fig. 12.22.

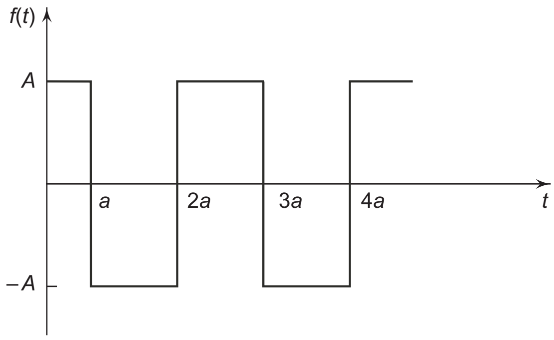


Fig. 12.22

Solution We have

$$f(t) = A, \qquad 0 < t < a$$
$$= -A, \qquad a < t < 2a \tag{12.191}$$

$$\mathscr{L}[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a Ae^{-st}\,dt + \int_a^{2a}(-A)e^{-st}\,dt\right]$$
$$= \frac{A}{s}\frac{\left(1-2e^{-as}+e^{-2as}\right)}{1-e^{-2as}}$$
$$= \frac{A}{s}\frac{\left(1-e^{-as}\right)^2}{\left(1+e^{-as}\right)\left(1-e^{-as}\right)} = \frac{A}{s}\tanh\left(\frac{as}{2}\right) \tag{12.192}$$

Problem 12.11 Determine the form of the partial fraction expansion for the proper fraction.

$$\frac{s-1}{(s+9)^2\,(s+4)\,(s^2+3s+2)\,(s+7)^2}$$

Solution

$$\frac{s-1}{(s+9)^2\,(s+4)\,(s^2+3s+2)\,(s+7)^2} = \frac{K_1}{(s+9)^2} + \frac{K_2}{s+9}$$
$$+\frac{K_3}{s+4} + \frac{K_4 s + K_5}{s^2+3s+2} + \frac{K_6}{(s+7)^2} + \frac{K_7}{s+7} \tag{12.193}$$

Alternatively $(s^2 + 3s + 2)$ can be factored and written as $(s + 2)\,(s + 1)$, and the resulting partial fraction expansion can be written as

$$\frac{s-1}{(s+9)^2\,(s+4)\,(s+2)\,(s+1)\,(s+7)^2} = \frac{K_1}{(s+9)^2} + \frac{K_2}{(s+9)} + \frac{K_3}{(s+4)} + \frac{K_4}{s+2}$$
$$\frac{K_5}{s+1} + \frac{K_6}{(s+7)^2} + \frac{K_7}{s+7} \tag{12.194}$$

Problem 12.12 Determine the form for a partial fraction expansion for the improper fraction.

$$\frac{6s^3+100s^2+85s+52}{s^3+7s^2+14s+8}$$

Solution Because the expression is not a proper fraction, it cannot be expanded into partial fractions. However, if the denominator is divided into the numerator, part of the expression becomes a proper fraction, and that part can be expanded.

$$\frac{6s^3+100s^2+85s+52}{s^2+7s^2+14s+8} = 6 + \frac{58s^2+s+4}{s^3+7s^2+14s+8}$$

$$= 6 + \frac{58s^2 + s + 4}{(s+1)(s+2)(s+4)} \tag{12.195}$$

$$\therefore \quad \frac{6s^3 + 100s^2 + 85s + 52}{s^3 + 7s^2 + 14s + 8} = 6 + \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+4} \tag{12.196}$$

Problem 12.13 Determine the inverse Laplace transform of $F(s)$, where

$$F(s) = \frac{s}{(s+1)^2 (s+4)}$$

Solution From the rules given for expanding proper fractions

$$\frac{s}{(s+1)^2 (s+4)} = \frac{K_1}{(s+1)^2} + \frac{K_2}{s+1} + \frac{K_3}{s+4} \tag{12.197}$$

$$K_1 = F(s)\,(s+1)^2|_{s=1}$$

$$= \frac{s}{s+4}\Big|_{s=-1} = \frac{-1}{3}$$

$$K_3 = F(s)\,(s+4)|_{s=-4}$$

$$= \frac{s}{(s+1)^2} = \frac{-4}{9}$$

We can determine K_2 by putting the right side of the equation on a common denominator and equating numerators.

$$\frac{s}{(s+1)^2 (s+4)} = \frac{K_1 (s+4) + K_2 (s+1)(s+4) + K_3 (s+1)^2}{(s+1)^2 (s+4)} \tag{12.198}$$

Since $\qquad K_1 = \frac{-1}{3}$ and $K_3 = \frac{-4}{9}$

equating numerators results in

$$s = \left(\frac{-11}{9} + 5K_2\right)s + \left(K_2 - \frac{4}{9}\right)s^2 + \left(\frac{-16}{9} + 4\,K_2\right)$$

equating coefficients results in

$$\frac{-11}{9} + 5K_2 = 1$$

$$K_2 = \frac{4}{9}$$

$$F(s) = \frac{s}{(s+1)^2 (s+4)} = \frac{\left(\frac{-1}{3}\right)}{(s+1)^2} + \frac{(4/9)}{s+1} + \frac{(-4/9)}{s+4}$$

$$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{\frac{-1}{3}}{(s+1)^2} + \frac{(4/9)}{s+1} + \frac{(-4/9)}{s+4}\right\}$$

From the rules

$$f(t) = \frac{-1}{3}\, t\, e^{-t} + \frac{4}{9}\, e^{-t} - \frac{4}{9}\, e^{-4t} \tag{12.199}$$

Problem 12.14 Expand the following proper fraction into partial fraction.

$$F(s) = \frac{1}{(s^2 + 3s + 2)\,(s + 4)}$$

Solution By expanding proper fractions

$$\frac{1}{(s^2 + 3s + 2)\,(s + 4)} = \frac{K_1 s + K_2}{(s^2 + 3s + 4)} + \frac{K_3}{s + 4} \tag{12.200}$$

$$K_3 = F(s)\,(s + 4)\big|_{s = -4}$$

$$= \frac{1}{(s^2 + 3s + 2)}\Big|_{s = -4}$$

$$K_3 = \frac{1}{6}$$

Putting the right hand side of the expanded equation over a common denominator to determine K_1 and K_2 results in

$$\frac{1}{(s^2 + 3s + 2)\,(s + 4)} = \frac{K_1 s + K_2}{(s^2 + 3s + 4)} + \frac{1/6}{s + 4}$$

$$= \frac{K_1 s^2 + 4s\,K_1 + K_2 s + 4K_2 + \left(\frac{1}{6}\right)s^2 + \left(\frac{1}{2}\right)s + 1/3}{(s^2 + 3s + 2)\,(s + 4)}$$

$$= \frac{\left(K_1 + \frac{1}{6}\right)s^2 + \left(4K_1 + 4K_2 + \frac{1}{2}\right)s + \left(4K_2 + 1/3\right)}{(s^2 + 3s + 2)\,(s + 4)} \tag{12.201}$$

Equating numerators, we get

$$K_1 + \frac{1}{6} = 0$$

$$4K_1 + K_2 + \frac{1}{2} = 0$$

$$4K_2 + \frac{1}{3} = 1$$

From the above equations, we get

$$K_1 = \frac{-1}{6},\; K_2 = \frac{1}{6}$$

The required partial fraction expansion is

$$F(s) = \frac{\left(\frac{-1}{6}\right)s + \left(\frac{1}{6}\right)}{s^2 + 3s + 2} + \frac{(1/6)}{s+4} \tag{12.202}$$

Problem 12.15 Determine the inverse Laplace transform of the following function

$$F(s) = \frac{96\,(s^2 + 17s + 60)}{s^3 + 14s^2 + 48\,s}$$

Solution The function $F(s)$ can be factorised as given

$$F(s) = \frac{96\,(s+5)\,(s+12)}{s\,(s+8)\,(s+6)} \tag{12.203}$$

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

To find K_1, we multiply both sides by s and then put $s = 0$

$$K_1 = F(s)\,s|_{s=0} = \frac{96\,(s+5)\,(s+12)}{(s+8)\,(s+6)}\Big|_{s=0} = 120$$

To find the value of K_2, we multiply both sides by $s + 8$ and then evaluate both sides at $s = -8$

$$K_2 = F(s)\,(s+8)|_{s=-8} = \frac{96\,(s+5)\,(s+12)}{s(s+6)}\Big|_{s=-8} = -72$$

Then K_3 is

$$\frac{96\,(s+5)\,(s+12)}{s(s+8)}\Big|_{s=-6} = K_3 = 48$$

Therefore

$$\frac{96\,(s+5)\,(s+12)}{s(s+8)\,(s+6)} = \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8} \tag{12.204}$$

By taking inverse transform of the above function, we get

$$\mathscr{L}^{-1}\left\{\frac{96\,(s+5)\,(s+12)}{s(s+8)\,(s+6)}\right\} = \mathscr{L}^{-1}\left\{\frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8}\right\}$$

$$= 120 + 48e^{-6t} - 72e^{-8t} \tag{12.205}$$

Problem 12.16 Determine the inverse Laplace transform of the following function

$$F(s) = \frac{100\,(s+3)}{(s+6)\,(s^2 + 6s + 25)}$$

Solution By factoring denominator, we have

$$\frac{100\,(s+3)}{(s+6)\,(s^2 + 6s + 25)} = \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4} \tag{12.206}$$

To find K_1, K_2 and K_3, we use the same process as before:

$$K_1 = \frac{100\,(s+3)}{s^2+6s+25}\Big|_{s=-6} = \frac{100\,(-3)}{25} = -12$$

$$K_2 = \frac{100\,(s+3)}{(s+6)\,(s^2+3+j4)}\Big|_{s=-3+j4} = \frac{100\,(j4)}{(3+j4)(j8)}$$

$$K_3 = \frac{100\,(s+3)}{(s+6)\,(s^2+3-j4)}\Big|_{s=-3-j4} = \frac{100\,(-j4)}{(3-j4)(-j8)}$$

Then

$$\frac{100\,(s+3)}{(s+6)\,(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle-53.13^\circ}{s+3-j4} + \frac{10\angle53.13^\circ}{s+3+j4} \tag{12.207}$$

By taking inverse Laplace transform, we get

$$\mathcal{L}^{-1}\left\{\frac{100\,(s+3)}{(s+6)\,(s^2+6s+25)}\right\} = -12e^{-6t} + 10e^{-j53.13^\circ}\cdot e^{-(3-j4)t} + 10e^{j53.13^\circ}\cdot e^{-(3+j4)t}$$

By simplifying, we get

$$\mathcal{L}^{-1}\left\{\frac{100\,(s+3)}{(s+6)\,(s^2+6s+25)}\right\} = \{-12e^{-6t} + 20e^{-3t}\cos(4t-53.13^\circ)\} \tag{12.208}$$

Problem 12.17 Obtain inverse Laplace transform of the following function

$$F(s) = \frac{100\,(s+25)}{s\,(s+5)^3}$$

Solution By factorising the denominator, we have

$$\frac{100\,(s+25)}{s\,(s+5)^3} = \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5} \tag{12.209}$$

we find K_1, as

$$K_1 = \frac{100\,(s+25)}{(s+5)^3}\Big|_{s=0} = 20$$

To find K_2, we multiply both sides by $(s+5)^3$ and then evaluate both sides at -5.

$$\frac{100\,(s+25)}{s}\Big|_{s=-5} = \frac{K_1(s+5)^3}{s}\Big|_{s=-5} + K_2 + K_3\,(s+5)\big|_{s=-5} + K_4(s+5)^2\big|_{s=-5}$$

$$\therefore \qquad K_2 = -400$$

To find K_3, we first multiply both sides by $(s+5)^3$. Next we differentiate both sides once with respect to s and then evaluate at $s = -5$.

$$\frac{d}{ds}\left[\frac{100\,(s+25)}{s}\right]_{s=-5} = \frac{d}{ds}\left[\frac{K_1\,(s+5)^3}{s}\right]_{s=-5}$$

$$+\frac{d}{ds}[K_2]_{s=-5}$$

$$+\frac{d}{ds}[K_3(s+5)]_{s=-5}$$

$$+\frac{d}{ds}[K_4(s+5)^2]_{s=-5}$$

$$100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5} = K_3 = -100$$

To find K_4, we first multiply both sides by $(s + 5)^3$. Next we differentiate both sides twice with respect to s and then evaluate both sides at $s = -5$. After simplifying the first derivative, the second derivative becomes

$$100\,\frac{d}{ds}\left[\frac{-25}{s^2}\right]_{s=-5} = K_1\,\frac{d}{ds}\left[\frac{(s+5)^2\,(2s-5)}{s^2}\right]_{s=-5} + 0 + \frac{d}{ds}[K_3]_{s=-5}$$

$$+\frac{d}{ds}[2K_4(s+5)]_{s=-5}$$

or

$$-40 = 2K_4$$

$$K_4 = -20$$

Then

$$\frac{100\,(s+25)}{s\,(s+5)^3} = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5} \tag{12.210}$$

By taking inverse transform we get

$$\mathscr{L}^{-1}\left\{\frac{100\,(s+25)}{s\,(s+5)^3}\right\} = 20 - 200t^2\,e^{-5t} - 100te^{-5t} - 20e^{-5t} \tag{12.211}$$

Problem 12.18 Verify the initial and final value theorems for the function $f(t) = e^{-t}(\sin 3t + \cos 5t)$

Solution

$$f(t) = e^{-t}(\sin 3t + \cos 5t) \tag{12.212}$$

$$F(s) = \mathscr{L}[f(t)] = \mathscr{L}[e^{-t}(\sin 3t + \cos 5t)] \tag{12.213}$$

Since $$\mathscr{L}(e^{-t}\sin 3t) = \frac{3}{(s+1)^2+3^2}$$

and $$\mathscr{L}(e^{-t}\cos 5t) = \frac{s+1}{(s+1)^2+5^2}$$

$\therefore$ $$F(s) = \mathscr{L}[f(t)] = \frac{3}{(s+1)^2+3^2} + \frac{s+1}{(s+1)^2+5^2} \tag{12.214}$$

According to initial value theorem

$$\underset{t\to 0}{\mathscr{L}t}\ f(t) = \underset{s\to\infty}{\mathscr{L}t}\ SF(s)$$

$$F(s) = \frac{3}{s^2+2s+10} + \frac{s+1}{s^2+2s+26}$$

$$SF(s) = \frac{3s}{s^2\left(1+\frac{2}{s}+\frac{10}{s^2}\right)} + \frac{s^2+s}{s^2\left(1+\frac{2}{s}+\frac{26}{s^2}\right)}$$

$$= \frac{3}{s\left(1+\frac{2}{s}+\frac{10}{s^2}\right)} + \frac{1}{1+\frac{2}{8}+\frac{26}{s^2}} + \frac{1}{s\left(1+\frac{2}{s}+\frac{26}{s^2}\right)} \qquad (12.215)$$

$$\lim_{s\to\infty} SF(s) = 1$$

$$f(t) = e^{-t}(\sin 3t + \cos 5t)$$

$$\lim_{t\to 0} f(t) = 1$$

Hence the initial value theorem is proved.
According to the final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} SF(s)$$

$$\lim_{s\to 0} SF(s) = 0$$

$$\lim_{t\to\infty} f(t) = 0$$

Hence the final value theorem is proved.

Problem 12.19 Find the value of $i(0+)$ using the initial value theorem for the function given

$$I(s) = \frac{2s+3}{(s+1)(s+3)}$$

Verify the result by solving it for $i(t)$.

Solution The initial value theorem is given by

$$\lim_{t\to 0} i(t) = \lim_{s\to\infty} SI(s)$$

$$= \lim_{s\to\infty} \frac{s(2s+3)}{(s+1)(s+3)} \qquad (12.216)$$

Taking S common and putting $S = \infty$, we get

$$\lim_{s\to\infty} \frac{s^2\left(2+\frac{3}{s}\right)}{s^2\left(1+\frac{1}{s}\right)\left(1+\frac{3}{s}\right)} = 2 \qquad (12.217)$$

To verify the result, we solve for $i(t)$ and put $t \to \infty$.
Taking partial fractions

$$I(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

where $$A = (s+1)\left.\frac{2s+3}{(s+1)(s+3)}\right|_{s=-1} = \frac{1}{2}$$

$$B = (s+3)\left.\frac{2s+3}{(s+1)(s+3)}\right|_{s=-3} = \frac{3}{2}$$

$\therefore$ $$I(s) = \frac{1}{2(s+1)} + \frac{3}{2(s+3)} \qquad (12.218)$$

Taking inverse transform, we get

$$i(t) = \frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \qquad (12.219)$$

Practice Problems

12.1 Use step functions to write the expression for the function shown in Fig. 12.23.

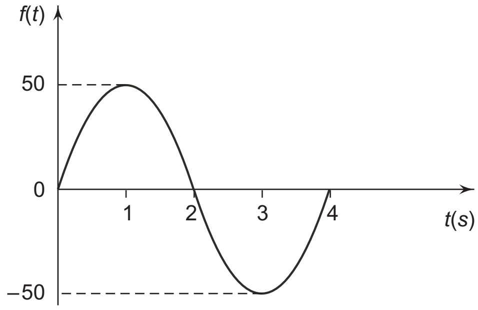


Fig. 12.23

12.2 Step functions can be used to define a window function. Thus $u(t-1) - u(t-u)$ defines a window 1 unit high and 3 units wide located on the time axis between 1 and 4.
A function $f(t)$ is defined as follows

$$\begin{aligned} f(t) &= 0,\ t \le 0 \\ &= 30t,\ 0 \le t \le 2s \\ &= 60,\ 2\text{s} \le t \le 4s \\ &= 60,\ \cos\left(\frac{\pi}{4}t - \pi\right),\ 4s \le t \le 8s \\ &= 30t - 300, \quad 8s \le t \le 10s \\ &= 0 \quad 10s \le t \le \infty \end{aligned}$$

Sketch $f(t)$ over the internal $-2s \le t \le 12s$.

12.3 Find $f(t)$ if $f(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(\omega)\, e^{jtw}\, d\omega$

$$F(\omega) = \frac{4 + j\omega}{9 + j\omega}\ \pi\, d(\omega)$$

12.4 Make a sketch of $f(t)$ for $-25s \le t \le 25s$ when $f(t)$ is given by the following expression:

$$f(t) = -(20t + 400)\, u(t + 20) + (40t + 400)\, u(t + 10)$$
$$+ (400 - 40t)\, u(t - 10) + (20t - 400)\, u(t - 20)$$

12.5 Evaluate the following integrals

(a) $I = \displaystyle\int_{-1}^{3} (t^3 + 2)\, [\delta(t) + 8\delta(t - 1)]\, dt$

(b) $I = \displaystyle\int_{-1}^{2} t^2\, [\delta(t) + \delta(t + 1.5) + \delta(t - 3)]\, dt$

12.6 Explain why the following function generates an impulse function as $\varepsilon \to 0$

$$f(t) = \frac{\varepsilon/\pi}{\varepsilon^2 + t^2}, \qquad -\infty \le t \le \infty$$

12.7 Find the Laplace transforms of the following functions

(a) $t^3 + at^2 + bt + 3$ (b) $\sin^2 5t$
(c) $e^{5t + 6}$ (d) $\cosh^2 3t$

12.8 Find the Laplace transform of each of the following functions

(a) te^{-at} (b) $\sin \omega t$
(c) $\sin(\omega t + \theta)$ (d) cosht
(e) $\cosh(t + \theta)$

12.9 Use the appropriate operational transform to find the Laplace transform of each function

(a) $t^2\, e^{-at}$ (b) $\dfrac{d}{dt}(e^{-at} \sinh \beta t)$

(c) $t \cos \omega t$

12.10 Find the inverse transforms of the following functions

(a) $\dfrac{1}{s^2 + 9}$ (b) $\dfrac{2\pi}{s + \pi}$

(c) $\dfrac{8}{(s + 3)(s + 5)}$ (d) $\dfrac{5}{s^2 + 9}$

(e) $\dfrac{K_1}{s} + \dfrac{K_2}{s^2} + \dfrac{K_3}{s^3}$

12.11 Find the inverse transforms of the following functions

(a) $\dfrac{5s+4}{(s-1)(s^2+2s+5)}$ (b) $\dfrac{4s+2}{s^2+2s+5}$

(c) $\dfrac{s}{s^2-2s+5}$ (d) $\dfrac{s(s+1)}{s^2+4s+5}$

12.12 Find the Laplace transform for (a) and (b)

(a) $f(t) = \dfrac{d}{dt}(e^{-at}\sin \omega t)$

(b) $f(t) = \displaystyle\int_{0-}^{t} e^{-ax}\cos \omega x\, dx$

(c) Verify the results obtained in (a) and (b) by first carrying out the mathematical operation and the finding the Laplace transform.

12.13 (a) Show that $\mathscr{L}[f(at)] = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

(b) Show that if $F(s) = \mathscr{L}[f(t)]$ and $\left[\dfrac{f(t)}{t}\right]$ is Laplace-transformable, then

$$\int_s^{\infty} F(u)\, du = \mathscr{L}\left\{\frac{f(t)}{t}\right\}$$

Hint: Use the determing integral to write

$$\int_s^{\infty} F(u)\, du = \int_s^{\infty}\left(\int_0^{\infty} f(t)\, e^{-ut}\, dt\right)du$$

and then reverse the order of integration.

12.14 Find the transforms of the following functions

(a) $t\, e^{-2t}\sin 2t + \dfrac{\cos 2t}{t}$ (b) $\log\left[\dfrac{s^2-1}{s(s+1)}\right]$

(c) $(1 + 2t\, e^{-5t})3$ (d) $\dfrac{s+4}{(s^2+5s+12)^2}$

12.15 Find $f(t)$ if $F(s) = \dfrac{10(s^2+119)}{(s+5)(s^2+10s+169)}$

12.16 Find $f(t)$ for each of the following functions

(a) $F(s) = \dfrac{18s^2+66s+54}{(s+1)(s+2)(s+3)}$ (b) $F(s) = \dfrac{11s^2+172s+700}{(s+2)(s^2+12s+100)}$

(c) $F(s) = \dfrac{56s^2+112s+5000}{s(s^2+14s+625)}$

12.17 Find the Laplace transform of a sawtooth waveform $f(t)$ which is periodic, with period equal to unity, and is given by $f(t)$ = at for $0 < t < 1$.

12.18 Find the Laplace transform of the periodic waveform shown in Fig. 12.24.

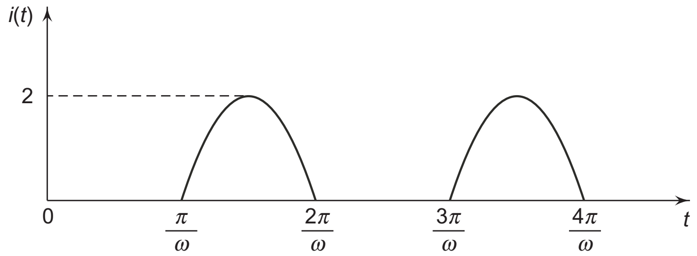


Fig. 12.24

12.19 Find $f(t)$ of the following functions

(a) $F(s) = \dfrac{40}{(s^2 + 4s + 5)^2}$ (b) $F(s) = \dfrac{5s^2 + 29s + 32}{(s+2)(s+4)}$

(c) $F(s) = \dfrac{2s^3 + 8s^2 + 2s - 4}{s^2 + 5s + 4}$

12.20 Apply the initial and final value theorems to each transform pair in Problem 12.16.

12.21 Use the initial and final value theorems to find the initial and final values of $f(t)$ for the following functions.

(a) $F(s) = \dfrac{7s^2 + 63s + 134}{(s+3)(s+4)(s+5)}$ (b) $F(s) = \dfrac{4s^2 + 7s + 1}{s(s+1)^2}$

(c) $F(s) = \dfrac{40}{(s^2 + 4s + 5)^2}$

12.22 For the given function $f(t) = 3u(t) + 2e^{-t}$, find its final value $f(\infty)$ using final value theorem.

Objective-type Questions

1. Laplace transform analysis gives
   (a) time domain response only
   (b) frequency domain response only
   (c) both (a) and (b)
   (d) none
2. The Laplace transform of a unit step function is
   (a) $\dfrac{1}{s}$ (b) 1

(c) $\dfrac{1}{s^2}$ (d) $\dfrac{1}{s+a}$

3. The Laplace transform of the first derivative of a function $f(t)$ is
   (a) $F(s)/s$ (b) $SF(s) - f(0)$
   (c) $F(s) - f(0)$ (d) $f(0)$
4. The Laplace transform of the integral of function $f(t)$ is
   (a) $\dfrac{1}{s}F(s)$ (b) $SF(s) - f(0)$
   (c) $F(s) - f(0)$ (d) $f'(0)$
5. The Laplace transform of $e^{5t} f(t)$ is
   (a) $F(s)$ (b) $F(s-1)$
   (c) $F\left(\dfrac{s}{5}\right)$ (d) $F(s-5)$
6. The inverse Laplace transform of $\dfrac{1}{s}(1 - e^{-as})$ is
   (a) $u(t) - u(t-a)$ (b) $u(t)$
   (c) $u(t-a)$ (d) zero
7. The inverse transform of $\dfrac{6}{s^4}$ is
   (a) 3 (b) t^2
   (c) t^3 (d) $3t$
8. The inverse transform of $2\log\left(\dfrac{s+2}{s}\right)$ is
   (a) $\dfrac{2-e^{-2t}}{t}$ (b) $\dfrac{e^{-2t}}{t}$
   (c) $\dfrac{2}{t}$ (d) $\dfrac{2+e^{-2t}}{t}$
9. The Laplace transform of a square wave with amplitude of peak value A and period T is
   (a) $\dfrac{1+e^{-sT}}{1-e^{-sT}}$ (b) $\dfrac{A}{s}\left(\dfrac{1-e^{-sT}}{1+e^{-sT}}\right)$
   (c) $\dfrac{A}{s}\left(\dfrac{1+e^{sT}}{1-e^{sT}}\right)$ (d) $\dfrac{A}{s}\left(\dfrac{1-e^{sT}}{1+e^{sT}}\right)$
10. The inverse Laplace transform of the function $\dfrac{s+5}{(s+1)(s+3)}$ is
    (a) $2e^{t} - e^{-3t}$ (b) $2e^{-t} + e^{-3t}$
    (c) $e^{-t} - 2e^{-3t}$ (d) $e^{-t} + 2e^{-3t}$

11. The Laplace transform of a unit ramp function at $t = a$ is

    (a) $\frac{1}{(s+a)^2}$ (b) $\frac{e^{-as}}{(s+a)^2}$

    (c) $\frac{e^{-as}}{s^2}$ (d) $\frac{a}{s^2}$

12. The initial value of $\frac{2s+1}{s^4+8s^3+16s^2+s}$ is

    (a) 2 (b) infinite

    (c) zero (d) 1

13. The initial value of $20 - 10t - e^{25t}$ 1 is

    (a) 20 (b) 19

    (c) 10 (d) 25

14. $\mathscr{L}[f(t)] = \frac{2(s+1)}{s^2+2s+5}$, then $f(0+)$ and $f(\infty)$ are given by

    (a) 0, 2 respectively (b) 2, 0 respectively

    (c) 0, 1 respectively (d) $\frac{2}{5}$, 0 respectively

15. The final value theorem is used to find the

    (a) steady state value of the system output
    (b) initial value of the system output
    (c) transient behaviour of the system output
    (d) none of these

Chapter 13

Application of the Laplace Transform in Circuit Analysis

The Laplace transform is an attractive tool in circuit analysis. It transforms a set of linear constant-coefficient differential equations into a set of linear polynomial equations. It automatically introduces into the polynomial equations the initial values of the current and voltage variables. In the circuit analysis, we can develop the *s*-domain circuit models for various elements and *s*-domain equations can be written directly.

13.1 CIRCUIT ELEMENTS IN THE S-DOMAIN

For any element, we write the time-domain equation that relates the terminal voltage to the terminal current. Then, we take the Laplace transform of the time-domain equation. This gives an algebraic relation between *s*-domain current and voltage. The dimensions of a transformed voltage is volt-seconds, and the dimension of a transformed current is ampere-seconds. A voltage to current ratio in the *s*-domain carries the dimension of volts per ampere. An impedance in the *s*-domain is measured in ohms, and admittance is measured in Siemens.

A Resistor in the s-Domain

Consider the resistive element shown in Fig. 13.1 From ohm's law,

$$v = Ri \tag{13.1}$$

The Laplace transform of Eq. (13.1) is

$$V = RI \tag{13.2}$$

where $\qquad V = \mathscr{L}[v] \quad \text{and} \quad I = \mathscr{L}[i]$

Equation (13.2) states that the *s*-domain equivalent circuit of a resistor is simply resistance of *R* ohms that carries a current of *I* ampere seconds and has a terminal voltage of *V* volt-seconds.

Figures 13.1 (a) and (b) show the time and frequency domain circuits of the resistor respectively.

(a) (b)

Fig. 13.1

The resistance element does not change while going from the time domain to the frequency domain.

An Inductor in the s-Domain

Consider an inductor shown in Fig. 13.2 with an initial current of I_0 amperes.

The time domain relation between voltage and current is

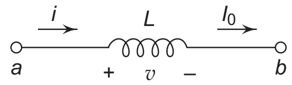


Fig. 13.2

$$v = \mathscr{L}\frac{di}{dt} \tag{13.3}$$

The Laplace transform of Eq. (13.3) gives

$$V = L\,[SI - i(0)] = SLI - LI_0 \tag{13.4}$$

The above equation satisfies two circuits. The first consists of an impedance of *SL* ohms in series with an independent voltage source of LI_0 volt-seconds as shown in Fig. 13.3(a).

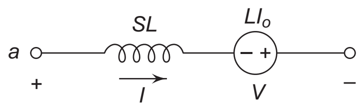


Fig. 13.3(a)

The second *s*-domain equivalent circuit that satisfies Eq. (13.4) consists of an impedance of *SL* ohms in parallel with an independent current source of I_0/S ampere-seconds, as shown in Fig. 13.3(b).

By solving Eq. (13.4) for the current *I*, we can construct the circuit shown in Fig. (13.3(b))

$$I = \frac{V + LI_0}{SL} = \frac{V}{SL} + \frac{I_0}{S} \tag{13.5}$$

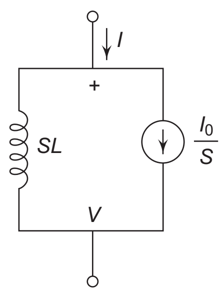


Fig. 13.3(b)

If the initial energy stored in the inductor is zero, i.e. if $I_0 = 0$, the s-domain equivalent circuit of the inductor reduces to an inductor with an impedance of SL ohms as shown in Fig. 13.4.

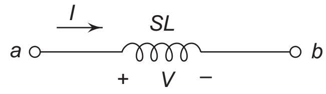


Fig. 13.4

A Capacitor in the s-domain

Consider an initially charged capacitor shown in Fig.13.5. The initial voltage on the capacitor is V_0 volts.

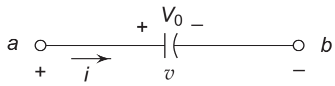


Fig. 13.5

The voltage current relation in the time domain is

$$i = c\frac{dv}{dt} \tag{13.6}$$

By taking Laplace transforms both sides, we get

$$I = C\,[SV - v(0)]$$

$$I = SCV - CV_0 \tag{13.7}$$

The Eq. (13.7) represents two circuits. First, the parallel equivalent circuit for capacitor initially charged to V_0 volts is shown in Fig. 13.6.

Secondly, the series equivalent circuit can be derived for the charged capacitor by solving Eq. (13.7) for V.

$$V = \left(\frac{1}{SC}\right)I + \frac{V_0}{S} \tag{13.8}$$

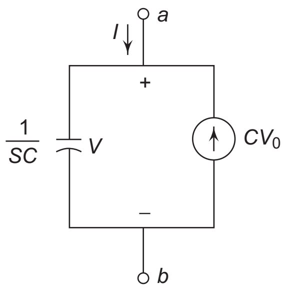


Fig. 13.6

Figure (13.7) shows the circuit that satisfies Eq. (13.8)

The s-domain circuit for a capacitor when the initial voltage is zero is shown in Fig. 13.8.

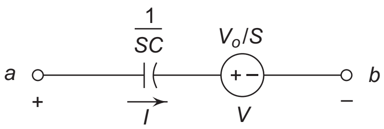


Fig. 13.7

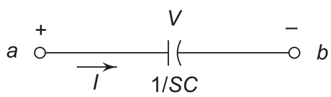


Fig. 13.8(a)

13.2 APPLICATIONS

In this section we illustrate how to use the Laplace transform to determine the transient behaviour of several linear lumped parameter circuits. In analysis familiar circuits, the Laplace transform approach yields the same results like the time domain analysis. In all the examples, the ease of manipulating algebraic equations instead of differential equations should be apparent.

The Natural Response of an *RC* Circuit

In this section, we find the natural response of an *RC* circuit through Laplace transform techniques. Consider the capacitor discharge circuit shown in Fig. 13.8b. Assume the capacitor is initially charged to V_0 volts. The series equivalent *s*-domain circuit is shown in 13.9.

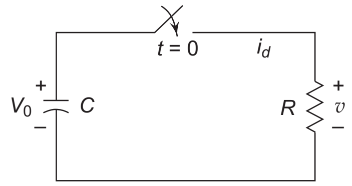


Fig. 13.8(b)

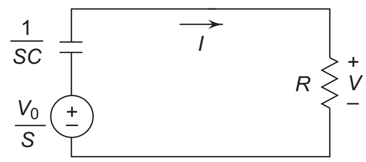


Fig. 13.9

From the circuit shown in Fig. 13.9, applying Kirchhoff's voltage law around the loop, we have

$$\frac{V_0}{S} = \frac{1}{SC} I + RI \tag{13.9}$$

Solving for the above equation yields

$$I = \frac{CV_0}{RCS+1} = \frac{V_0/R}{S+\left(\dfrac{1}{RC}\right)} \tag{13.10}$$

By taking the inverse transform of the Eq. (13.10), we get

$$i = \frac{V_0}{R} e^{\frac{-t}{RC}} \tag{13.11}$$

we can determine *v* by simply applying ohm's law from the circuit

$$v = Ri = V_0 e^{\frac{-t}{RC}} \tag{13.12}$$

Now we can use the parallel equivalent circuit of Fig. 13.8b. Figure 13.10 shows the new *s*-domain equivalent circuit.

By taking mode voltage equation, we get

$$\frac{V}{R} + SCV = CV_0 \tag{13.13}$$

Solving Eq. (9.13) for *V* gives

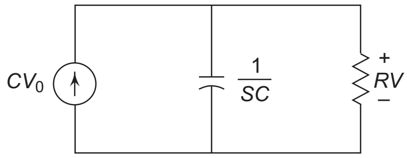


Fig. 13.10

$$V = \frac{V_0}{S+\dfrac{1}{RC}} \tag{13.14}$$

By taking inverse transform, we get

$$v = V_0\, e^{\frac{-t}{RC}} = V_0\, e^{\frac{-t}{\tau}} \tag{13.15}$$

where τ is the time constant $\tau = RC$

The Step Response of a Parallel Circuit

Consider the parallel *RLC* circuit shown in Fig. 13.11. We can find the expression for i_L after the constant current source is switched across the parallel elements. The initial energy stored in the circuit is zero.

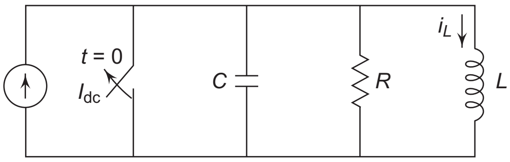


Fig. 13.11

The *s*-domain equivalent circuit is shown in Fig. 13.12. Here, an independent source can be transformed easily from the time domain to the frequency domain. Opening the switch results in a step change in the current applied to the circuit.

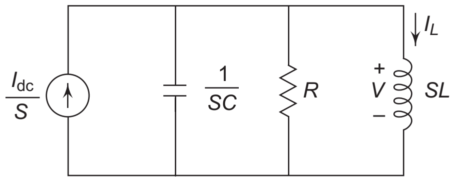


Fig. 13.12

By applying Kirchhoff's current law, we get

$$SCR + \frac{V}{R} + \frac{V}{SL} = \frac{I_{dc}}{S}. \tag{13.16}$$

Solving Eq. (13.16) for V gives

$$V = \frac{I_{dc}/c}{S^2 + \left(\frac{1}{RC}\right)S + \frac{1}{LC}} \tag{13.17}$$

We know the current in inductor I_L

$$I_L = \frac{V}{SL} \tag{13.18}$$

Substituting Eq. (13.17) into Eq. (13.18) gives

$$I_L = \frac{I_{dc}/LC}{S\left[S^2 + \left(\frac{1}{RC}\right)S + \frac{1}{LC}\right]} \tag{13.19}$$

By taking the inverse transform, we can obtain i_L.

The Transient Response of a Parallel *RLC* Circuit

The transient behaviour of a circuit arises from replacing the *dc* current source in the circuit shown in Fig. 13.11 with a sinusoidal current source. The new current source is

$$i_g = I_m \cos \omega t \tag{13.20}$$

The s-domain expression for the source current is

$$I_g = \frac{SI_M}{S^2 + \omega^2} \tag{13.21}$$

The voltage across the parallel elements is

$$V = \frac{(I_g/C)S}{S^2 + \left(\frac{1}{RC}\right)S + \left(\frac{1}{LC}\right)} \tag{13.22}$$

Substituting Eq. (13.21) into Eq. (13.22) results in

$$V = \frac{(I_m/C)S^2}{\left(S^2 + \omega^2\right)\left[S^2 + \left(\frac{1}{RC}\right)S + \left(\frac{1}{LC}\right)\right]} \tag{13.23}$$

from which

$$I_L = \frac{V}{SL} = \frac{(I_m/LC)S}{\left(S^2 + \omega^2\right)\left[S^2 + \left(\frac{1}{RC}\right)S + \left(\frac{1}{LC}\right)\right]} \tag{13.24}$$

The Use of Thevenin's Equivalent

In this section we show how to use Thevenin's equivalent in the s-domain. Consider a circuit shown in Fig. 13.13. We find the capacitor current that results from closing the switch. The energy stored in the circuit prior to closing is zero.

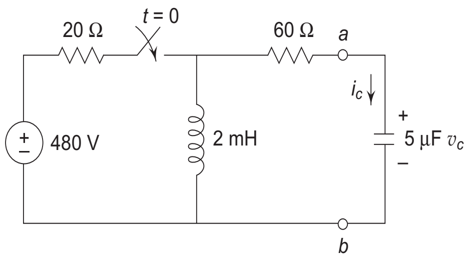


Fig. 13.13

To find i_c, we first construct the s-domain equivalent circuit and the find the Thevenin equivalent of this circuit with respect to the terminals of the capacitor. Fig. 13.14 shows the s-domain circuit.

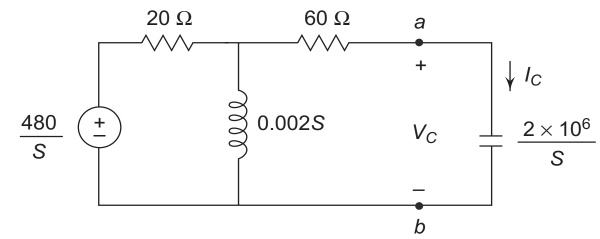


Fig. 13.14

The open circuit voltage across terminals a, b is

$$V_{\text{Th}} = \frac{\left(\frac{480}{S}\right)(0.0025)}{20 + 0.002S} = \frac{480}{S + 10^4} \tag{13.25}$$

The Thevenin impedance seen from terminals a and b equals the 60 Ω resistor in series with the parallel combination of the 20 Ω resistor and the 2 mH inductor. Thus

$$Z_{\text{Th}} = 60 + \frac{0.0025(20)}{20 + 0.002S} = \frac{80(S + 7500)}{S + 10^4} \tag{13.26}$$

A simplified version of the Thevenin equivalent circuit is shown in Fig. 13.15.

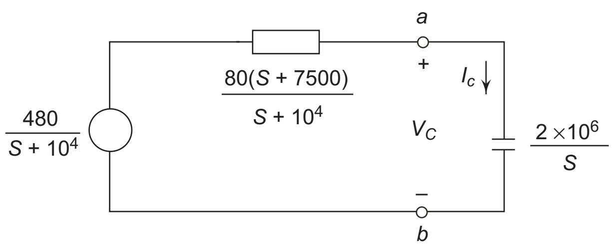


Fig. 13.15

Thus the capacitor current I_C equals the Thevenin voltage divided by the total series impedance.

Thus,

$$I_C = \frac{480/\left(S + 10^4\right)}{\left[80(S + 7500)/\left(S + 10^4\right) + \left[\left(2 \times 10^5\right)/S\right]\right]} \tag{13.27}$$

We simplify Eq. (13.27) to

$$I_C = \frac{6S}{S^2 + 10{,}000S + 25 \times 10^6}$$

$$= \frac{6S}{(S + 5000)^2} \tag{13.28}$$

By taking partial fraction expansion, we get

$$I_C = \frac{-3000}{(S+5000)^2} + \frac{6}{(S+5000)} \tag{13.29}$$

By taking inverse transform, we get

$$i_c = (-\,30{,}000t\, e^{-5000t} + 6\, e^{-5000t})\ A \tag{13.30}$$

Now the voltage across capacitors is

$$V_C = \frac{I_C}{SC} = \frac{2\times10^5}{S}\ \frac{6S}{(S+5000)^2} = \frac{12\times10^5}{(S+5000)^2} \tag{13.31}$$

By taking inverse transform, we get

$$v_c = 12\times10^5\ t\ e^{-5000t} \tag{13.32}$$

A Circuit with Mutual Inductance

In this section we illustrate example how to use the Laplace transform to analyze the transient response of a circuit that contains mutual inductance as shown in Fig. 13.16.

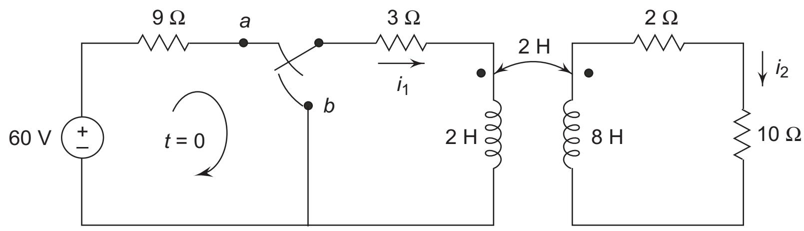


Fig. 13.16

To make-before break switch has been in position 'a' for a long time. At $t = 0$, the switch moves instantaneously to position b. The problem is to derive the time-domain expression for i_2.

We begin by redrawing the circuit in Fig. 13.16, with the switch in position b and the magnetically coupled coils replaced with a T equivalent circuit as shown in Fig. 13.17.

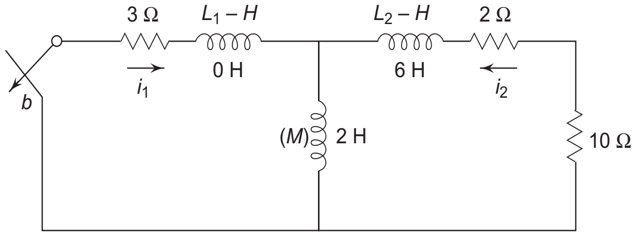


Fig. 13.17

The s-domain equivalent circuit for the circuit of Fig. 13.17 is shown in Fig. 13.18.

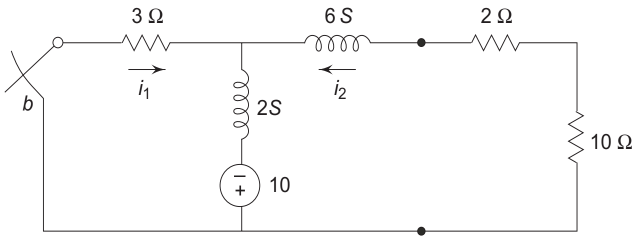


Fig. 13.18

The initial currents are

$$i_1(0) = \frac{60}{12} = 5\text{A} \tag{13.33}$$

$$i_2(0) = 0 \tag{13.34}$$

The initial value of the current in the 2H inductor is

$$i_1(0) + i_2(0) = 5 \text{ A} \tag{13.35}$$

The S domain mesh equations in Fig. 13.18 are

$$(3 + 2S)\, I_1 + 2SI_2 = 0 \tag{13.36}$$

$$2SI_1 + (12 + 8S)I_2 = 10 \tag{13.37}$$

Solving for I_2 yields

$$I_2 = \frac{2.5}{(S+1)\,(S+3)} \tag{13.38}$$

By taking partial fraction expansion gives

$$I_2 = \frac{1.25}{S+1} - \frac{1.25}{S+3} \tag{13.39}$$

By taking inverse transform of Eq. (13.39) gives

$$i_2 = (1.25\, e^{-t} - 1.25\, e^{-3t}) \text{ A} \tag{13.40}$$

The Use of Superposition

Consider a circuit shown in Fig. 13.19 having two sources and the inductor is carrying and initial current $i_L(0)$ amperes and the capacitor is carrying an initial voltage of $v_c(0)$ volts. The desired response of the circuit is the voltage across the resistor R_2, labeled v_2.

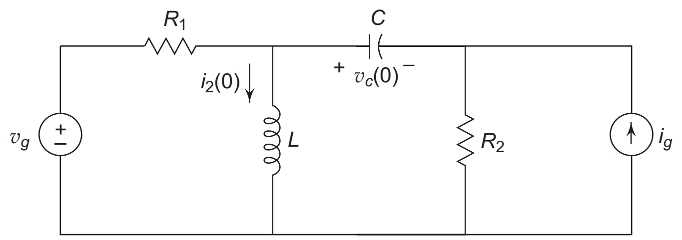


Fig. 13.19

The s-domain equivalent circuit for the Fig. 13.19 is shown in Fig. 13.20. Here, we have taken parallel equivalents for L and C into consideration. Now we find V_2 using node-voltage method.

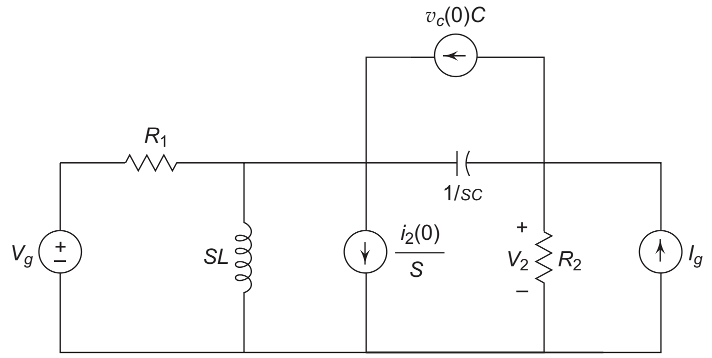


Fig. 13.20

To find V_2 by superposition, we calculate the voltage V_2 resulting from each source acting alone, and then we sum the voltages. We consider with V_g acting alone by setting other three current sources set equal to zero. Figure 13.21 shows the resulting circuit.

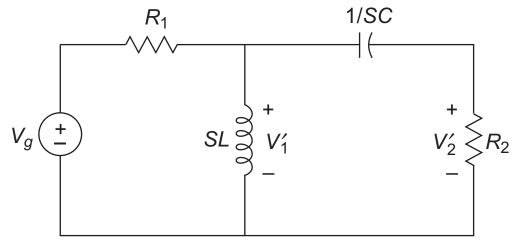


Fig. 13.21

V_1' and V_2' are the voltages across inductor and resistor when V_g acting alone. The two equations the describe the circuit in Fig. 13.21 are

$$\left(\frac{1}{R_1} + \frac{1}{SL} + SC\right) V_1' - SCV_2' = \frac{V_g}{R_1} \tag{13.41}$$

$$-SC\, V_1' + \left(\frac{1}{R_2} + SC\right) V_2' = 0 \tag{13.42}$$

The above equations can be written as

$$Y_{11}\, V_1' + Y_{12}\, V_2' = \frac{V_g}{R_1} \tag{13.43}$$

$$Y_{12}\, V_1' + Y_{22}\, V_2' = 0 \tag{13.44}$$

where
$$Y_{11} = \frac{1}{R_1} + \frac{1}{SL} + SC$$

$$Y_{12} = -SC$$

$$Y_{22} = \frac{1}{R_2} + SC$$

Solving Eqs (13.43) and (13.44) for V_2' gives

$$V_2' = \frac{-Y_{12}/R_1}{Y_{11}\, Y_{22} - Y_{12}^2}\, V_g \tag{13.45}$$

With the current source I_g acting alone, the circuit shown in Fig. 13.20 reduces to the one shown in Fig. 13.22.

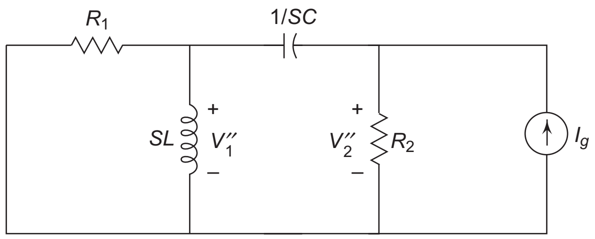


Fig. 13.22

The two node voltages, equations are given by

$$Y_{11}\, V_1'' + Y_{12}\, V_2'' = 0 \tag{13.46}$$

$$Y_{12}\, V_1'' + Y_{22}\, V_2'' = I_g \tag{13.47}$$

Solving for V_2'' yields

$$V_2'' = \frac{Y_{11}}{Y_{11}\, Y_{22} - Y_{12}^2}\, I_g \tag{13.48}$$

The circuit shown in Fig. 13.23 gives when the energized inductor acting alone on the circuit of Fig. 13.20.

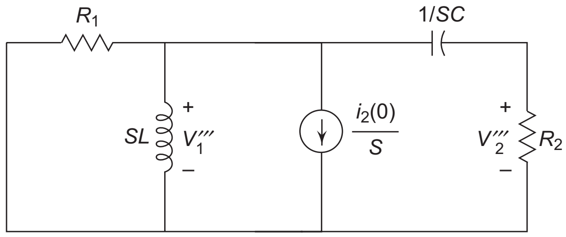


Fig. 13.23

The two node voltage equations are given by

$$Y_{11}\,V_1''' + Y_{12}\,V_2''' = -\frac{i_L(0)}{S} \tag{13.49}$$

$$Y_{12}\,V_1''' + Y_{22}\,V_2''' = 0 \tag{13.50}$$

Thus

$$V_2''' = \frac{Y_{12}/S}{Y_{11}\,Y_{22} - Y_{12}^2} - i_L(0) \tag{13.51}$$

The circuit shown in Fig. 13.24 gives when the energy stored in the capacitor acting alone.

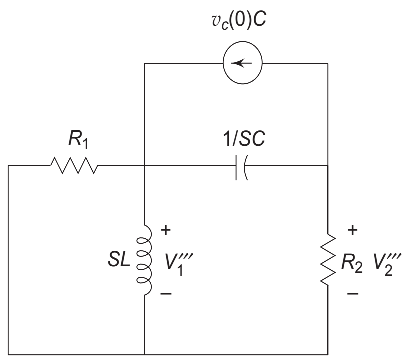


Fig. 13.24

The node-voltage equations describing this circuit are

$$Y_{11}\,V_1''' + Y_{12}\,V_2''' = v_c(0)C \tag{13.52}$$

$$Y_{12}\,V_1''' + Y_{22}\,V_2''' = -\,v_c(0)C \tag{13.53}$$

Solving for V_2''' yields

$$V_2''' = \frac{-(Y_{11} + Y_{12})C}{Y_{11}\,Y_{22} - Y_{12}^2}\,v_{c(0)} \tag{13.54}$$

The expression for V_2 is

$$V_2 = V_2' + V_2'' + V_2''' + V_2''''$$

$$= \frac{-(Y_{12}/R_1)}{Y_{11}\,Y_{22} - Y_{12}^2} V_g + \frac{Y_{11}}{Y_{11}\,Y_{22} - Y_{12}^2} I_g$$

$$+ \frac{Y_{12}/S}{Y_{11}\,Y_{22} - Y_{12}^2} i_L(0) + \frac{-C(Y_{11} + Y_{12})}{Y_{11}\,Y_{22} - Y_{12}^2} v_c(0) \qquad (13.55)$$

By taking inverse transform, we can obtain time domain voltage across resistor R_2.

13.3 THE TRANSFER FUNCTION

The transfer function is defined as the *s*-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input (source). In computing the transfer function, we restrict our attention to circuits where all initial conditions are zero. If a circuit has a multiple independent sources, we can find the transfer function for each source and use superposition to find the response to all sources.

The transfer function is

$$H(S) = \frac{Y(S)}{X(S)} \qquad (13.56)$$

where $Y(S)$ is the Laplace transform of the output signal, and $X(S)$ is the Laplace transform of the input signal. Note that the transfer function depends on what is defined as the output signal. Consider a series circuit shown in Fig. 13.25.

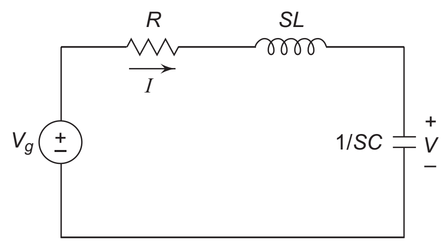


Fig. 13.25

If the current is defined as the response signal of the circuit, then the transfer function

$$H(S) = \frac{I}{V_g} = \frac{1}{R + SL + \dfrac{1}{SC}} = \frac{SC}{S^2 LC + RCS + 1} \qquad (13.57)$$

In the above equation, we recognized that I corresponds to the output $Y(S)$ and V_g corresponds to the input $X(S)$. If the voltage across the capacitor is defined as the output signal of the circuit in Fig. 13.25, the transfer function is

$$H(S) = \frac{V}{V_g} = \frac{1/SC}{R + SL + \dfrac{1}{SC}} = \frac{1}{S^2 LC + RCS + 1} \qquad (13.58)$$

Thus, because circuits may have multiple sources and because the definition of the output signal of interest can vary, a single circuit can generate many transfer functions. When multiple sources are involved, no single transfer function represent the total output-transfer functions associated with each source must be combined using superposition to yield the total response. We can write the circuit output as the product of the transfer function and the driving function

$$Y(S) = H(S) \times (S) \qquad (13.59)$$

$H(S)$ is a rational function of S and $X(S)$ is also a rational function of S for the excitation functions of most interest in circuit analysis. We can expand the right hand side of Eq. (13.58) into a sum of partial fractions.

13.4 USE OF TRANSFER FUNCTION IN CIRCUIT ANALYSIS

Consider the response of the circuit to a delayed input. If the input is delayed by a seconds

$$\mathscr{L}\,[x(t-a)\,u(t-a)] = e^{-as} \times (S) \qquad (13.60)$$

The response becomes

$$Y(S) = H(S) \times (S)e^{-as} \qquad (13.61)$$

If $y(t)= L^{-1}\,[H(S) \times (S)]$, then from Eq. (13.61),

$$y(t-a))\,u(t-a) = \mathscr{L}^{-1}\,[H(S) \times (S)\,e^{-as}] \qquad (13.62)$$

Therefore, delaying the input by a seconds simply delays the response function by a seconds. A circuit the exhibits this characteristic is said to be time invariant.

If a unit impulse source drives the circuit, the response of the circuit equals the inverse transform of the transfer function. Thus if

$$x(t) = \delta(t), \text{ then } X(S) = 1$$

and $$Y(S) = H(S) \qquad (13.63)$$

Hence $$y(t) = h(t) \qquad (13.64)$$

where the inverse transform of the transfer function equals the unit impulse response of the circuit. The unit impulse response of the circuit $h(t)$ contains enough information to compute the response to any source that drives the circuit.

13.5 THE TRANSFER FUNCTION AND THE CONVOLUTION INTEGRAL

The convolution integral relates the output $y(t)$ of a linear time invariant circuit to the input $x(t)$ of the circuit and the circuits impulse response $h(t)$. The convolution integral is defined as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau \qquad (13.65)$$

The above equation is based on the assumption that the circuit is linear and time invariant. Because the circuit is linear, the principle of superposition is valid, and because it is time invariant, the amount of the response delay is exactly the same as that of the input delay. Consider block diagram of a general circuit shown in Fig. 13.26 in which $h(t)$ represents any linear time-invariant circuit whose impulse response is known, $x(t)$ represents the excitation signal and $y(t)$ represents the desired output signal.

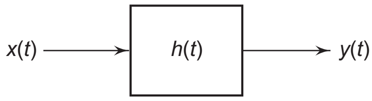


Fig. 13.26

We assume that $x(t)$ is the general excitation signal shown in Fig. 13.27(a). Also assume that $x(t) = 0$ for $t < 0$.

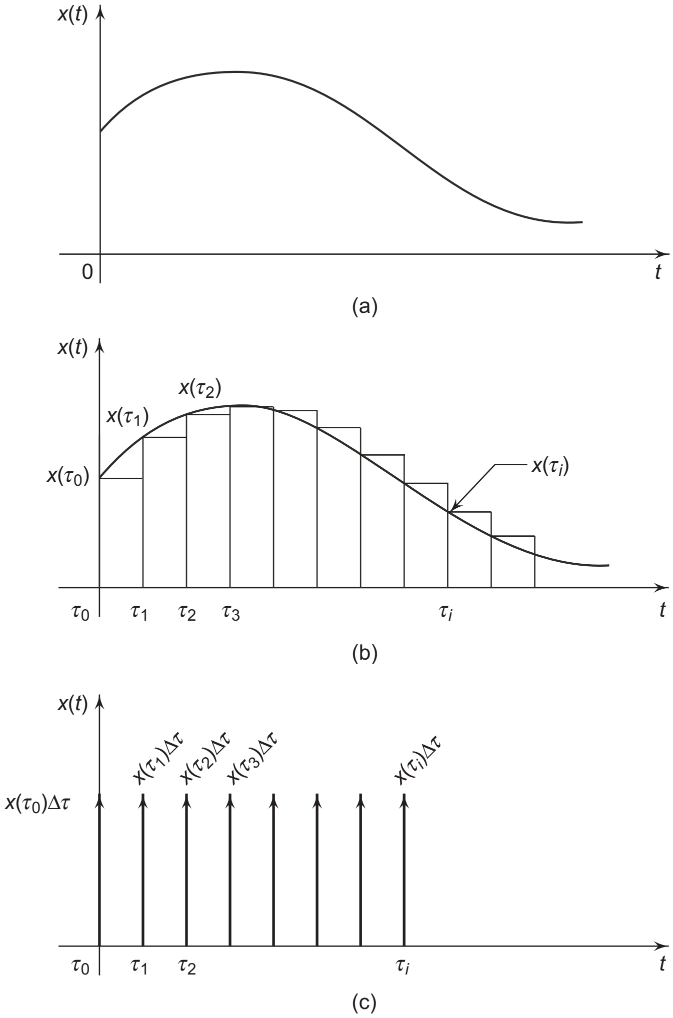


Fig. 13.27

Now we see approximate $x(t)$ by a series of rectangular pulses of uniform width $\Delta\tau$ as shown in Fig. 13.27 (b). Thus

$$x(t) = x_0(t) + x_1(t) + \dots + x_i(t) + \dots \tag{13.66}$$

where $x_i(t)$ is a rectangular pulse that equals $x(\tau_i)$ between τ_i and τ_{i+1} and is zero elsewhere. Note that the ith pulse can be expressed interms of step functions; that is

$$x_i(t) = x(\tau_i)\,[u(t-\tau_i) - u(t-(\tau_i+\Delta\tau))] \tag{13.67}$$

The next step in the approximation of $x(t)$ is to make $\Delta\tau$ small enough that the ith component can be approximated by an impulse function of strength $x(\tau_i)\,\Delta\tau$. Fig. 13.27 (c) shows the impulse representation, with the strength of each impulse shown in brackets beside each arrow. The impulse representation of $x(t)$ is

$$x(t) = x(\tau_0)\,\Delta\tau\,\delta(t-\tau_0) + x(\tau_1)\,\Delta\tau\,\delta(t-\tau_1) + \dots$$
$$+\, x(\tau_i)\,\Delta\tau\,\delta(t-\tau_i) + \dots \tag{13.68}$$

Now when $x(t)$ is represented by a series of impulse functions, the response function $y(t)$ consists of the sum of a series of uniformly delayed impulse responses. The strength of each response depends on the strength of the impulse driving the circuit. For example, let's assume that the unit impulse response of the circuit contained with in the box in Fig. 13.26 is the exponential decay function shown in Fig. 13.28 (a). Then the approximation of $y(t)$ is the sum of the impulse responses shown in Fig. 13.28(b).

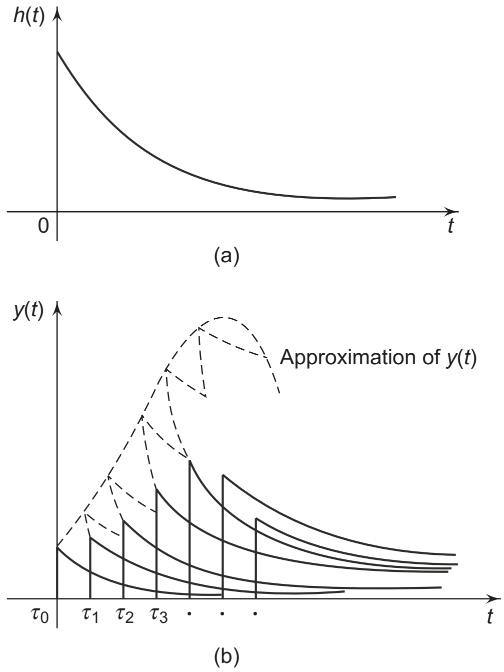


Fig. 13.28

Analytically, the expression for $y(t)$ is

$$y(t) = x(\tau_0)\,\Delta\tau\, h(t-\tau_0) + x(\tau_1)\,\Delta\tau\, h(t-\tau_1)$$
$$+ x(\tau_2)\,\Delta\tau\, h(t-\tau_2) + \dots$$
$$+ x(\tau_i)\,\Delta\tau\, h(t-\tau_i) + \dots \qquad (13.69)$$

As $\Delta\tau \rightarrow 0$, the summation in Eq. (13.69) approaches a continuous integration, or

$$\sum_{i=0}^{\infty} x(\tau_i)\, h(t-\tau_i)\,\Delta\tau \rightarrow \int_0^{\infty} x(\tau)\, h(t-\tau) d\tau \qquad (13.70)$$

Therefore,

$$y(t) = \int_0^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad (13.71)$$

If $x(t)$ exists over all time, then the lower limit on Eq. (13.71) becomes $-\infty$, thus, in general

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad (13.72)$$

The integral relation between $y(t)$, $h(t)$ and $x(t)$ is written in a shorthand notation

$$y(t) = h(t) * x(t) = x(t) * h(t) \qquad (13.73)$$

Thus $h(t) * x(t)$ is read as "$h(t)$ is convolved with $x(t)$" and implies that

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad (13.74)$$

The above integral gives the most general relation for the convolution of two functions. However, in our applications, we can change the lower limit to zero and the upper limit to t. Then the above equation can be written as

$$y(t) = \int_0^{t} x(\tau)\, h(t-\tau)\, d\tau = \int_0^{t} h(\tau)\, x(t-\tau)\, d\tau \qquad (13.75)$$

For physically realizable circuits, $h(t)$ is zero for $t < 0$. In other words, there can be no impulse response before an impulse is applied. We start measuring time at the instant the excitation $x(t)$ is turned on, therefore $x(t) = 0$ for $t < 0$.

A graphical interpretation of the convolution integrals contained in Eq. (13.75) is important in the use of integral as a computational tool. Consider the impulse response of our circuit is the exponential decay function shown in Fig. 13.29 (a) and the excitation function has the waveform shown in Fig. 13.29(b).

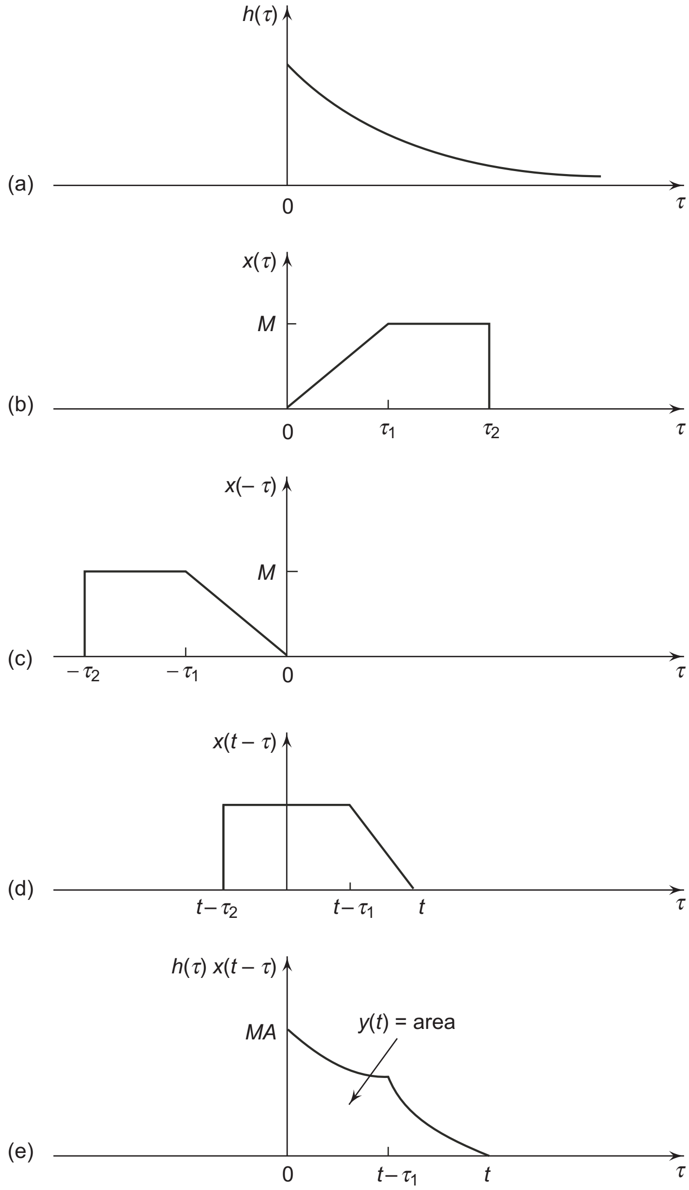


Fig. 13.29

Replacing τ with $-\tau$ simply folds the excitation function over the vertical axis and replacing $-\lambda$ with $\tau - \lambda$ slides the folded function to the right. This folding and sliding operation gives rise to the term convolution. At any specified value of t, the response function $y(t)$ is the area under the product function $\lambda(\tau)\, x(t - \tau)$ as shown in Fig. 13.29 (c). For $\tau < 0$, the product $\lambda(\tau)\, x(t - \tau)$ is zero because $h(\tau)$ is zero. For $\tau > t$, the product $h(\tau)\, x(t - \tau)$ is zero because $x(t - \tau)$ is zero.

13.6 THE TRANSFER FUNCTION AND THE STEADY STATE SINUSOIDAL RESPONSE

We use the transfer function to relate the steady state response to the excitation source. First we assume that

$$x(t) = A \cos (\omega t + \phi) \tag{13.76}$$

and we use the equation

$$Y(S) = H(S) \times (S) \tag{13.77}$$

to find the steady state solution of Y(t).

To find the Laplace transform of $x(t)$, we first write $x(t)$ as

$$x(t) = A \cos \omega t \cos \phi - A \sin \omega t \sin \phi \tag{13.78}$$

from which

$$X(S) = \frac{A\cos\phi \; S}{S^2 + \omega^2} - \frac{A \sin\phi \; \omega}{S^2 + \omega^2}$$

$$= \frac{A(S\cos\phi - \omega\sin\phi)}{S^2 + \omega^2} \tag{13.79}$$

Substituting Eq. (13.79) into Eq. (13.77) gives the s-domain expression for the response

$$Y(S) = H(s) \; \frac{A(S\cos\phi - \omega\sin\phi)}{S^2 + \omega^2} \tag{13.80}$$

By taking partial fractions

$$Y(S) = \frac{k_1}{S - j\omega} + \frac{k_1^*}{S + j\omega}$$

$$+ \Sigma \text{ terms generated by the poles of } H(S) \tag{13.81}$$

In Eq. (13.81), the first two terms result from the complex conjugate poles of the deriving source. However, the terms generated by the poles of $H(s)$ do not contribute to the steady-state response of $y(t)$, because all these poles lie in the left half of the s plane, consequently, the corresponding time-domain terms approach zero as t increases.

Thus the first two terms on the right hand side of Eq. (13.81) determine the steady-state response. Now K_1 can be determined

$$k_1 = \left. \frac{H(S)\; A(S\cos\phi - \omega\sin\phi)}{S + j\omega} \right|_{S = j\omega}$$

$$= \frac{H(j\omega)\; A(j\omega\cos\phi - \omega\sin\phi)}{2j\omega}$$

$$= H(j\omega)\,\frac{A(\cos\phi + j\sin\phi)}{2} = \frac{1}{2}\,H(j\omega)A\,e^{i\phi} \quad (13.82)$$

In general, $H(j\omega)$ is a complex quantity, thus

$$H(j\omega) = |H(j\omega)|\,e^{i\theta(\omega)} \quad (13.83)$$

where $|H(j\omega)|$ is the magnitude, and phase angle is (ω) of the transfer function vary with the frequency ω, the expression for K_1 becomes

$$K_1 = \frac{A}{2}\,|H(j\omega)|\,e^{i[\theta(\omega) + \phi]} \quad (13.84)$$

we obtain the steady state solution for $y(t)$ by taking inverse transform of Eq. (13.81) ignoring the terms generated by the poles of $H(S)$. Thus

$$y_{ss}(t) = A|H(j\omega)|\cos\,[\omega t + \phi + \theta(\omega)] \quad (13.85)$$

which indicates how to use the transfer function to find the steady state sinusoidal response of a circuit.

13.7 THE IMPULSE FUNCTION IN CIRCUIT ANALYSIS

Impulse functions occur in circuit analysis either because of a switching operation or because a circuit is excited by an impulse source. The Laplace transform can be used to predict the impulsive currents and voltages created during switching and the response of a circuit to an impulsive source.

Switching Operation

We use two different circuits to illustrate how an impulse function can be created with a switching operation. A capacitor circuit and a series inductor circuit.

Capacitor Circuit

In the circuit shown in Fig. 13.30, the capacitor C_1 is charged to an initial voltage of V_0 at the time the switch is closed.

In the circuit, the initial charge on C_2 is zero. Figure 13.31 shows the s-domain equivalent circuit.

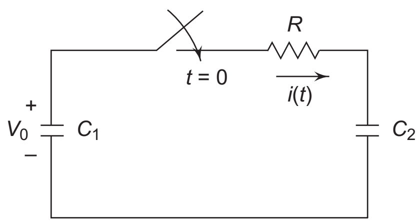


Fig. 13.30

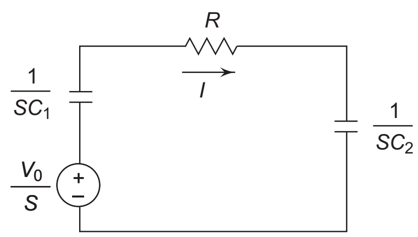


Fig. 13.31

From Fig. 13.31,

$$I = \frac{V_0/S}{R + \left(\frac{1}{SC_1}\right) + \left(\frac{1}{SC_2}\right)}$$

$$= \frac{V_0/R}{S + \left(\frac{1}{RC_e}\right)} \tag{13.86}$$

where the equivalent capacitance $\frac{C_1 C_2}{C_1 + C_2}$ is replaced by C_e.

By taking inverse transform of Eq. (13.86), we obtain

$$i = \frac{V_0}{R}\, e^{-t/RC_e} \tag{13.87}$$

which indicates that as R decreases, the initial current $\left(\frac{V_0}{R}\right)$ increases and the time constant (RC_e) decreases. Thus R gets smaller, the current starts from a larger initial value and then dropped off more rapidly. Figure 13.32 shows these characteristics of i.

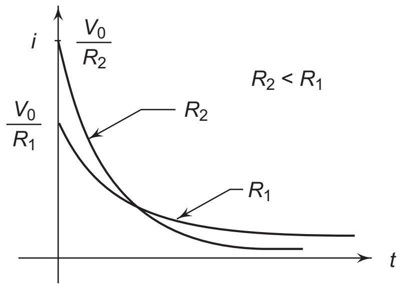


Fig. 13.32

The characteristics shows, i is approaching an impulse function as R approaching to zero because the initial value of i is approaching infinity and time duration of i is approaching zero. We still have to determine whether the area under the current function is independent of R. Physically the total area under the i versus t curve represents the total charge transferred to C_2 after the switch is closed. Thus

$$\text{Area} = q = \int_{0-}^{\infty} \frac{V_0}{R}\, e^{-t/RC_e}\, dt = V_0\, C_e \tag{13.88}$$

which says that the total charge transferred to C_2 is independent of time and equals $V_0 C_e$ coulombs. Thus, as R approaches zero, the current approaches an impulse strength $V_0 C_e$.

$$i \to V_0 C_e\, \delta(t) \tag{13.89}$$

when $R = 0$, a finite amount of charge is transferred to C_2 instantaneously. When the switch is closed, the voltage across C_2 does not jump to V_0 but its final value of

$$v_2 = \frac{C_1 V_0}{C_1 + C_2} \tag{13.90}$$

If we set R equal to zero, the Laplace transform analysis will predict the impulsive current response.

Thus,

$$I = \frac{V_0/S}{\left(\frac{1}{SC_1}\right)+\left(\frac{1}{SC_2}\right)} = \frac{C_1 C_2 V_0}{C_1 + C_2} = C_e V_0 \qquad (13.91)$$

The inverse transform of the above equation is

$$i = C_e V_0 \delta(t) \qquad (13.92)$$

Series Inductor Circuit

The circuit shown in Fig. 13.33 illustrates a second switching operation that produces an impulsive response. The problem is to find the time-domain expression for v_0– after the switch has been opened. Note that opening the switch forces an instantaneous change in the current of L_2, which courses v_0 to contain an impulsive component.

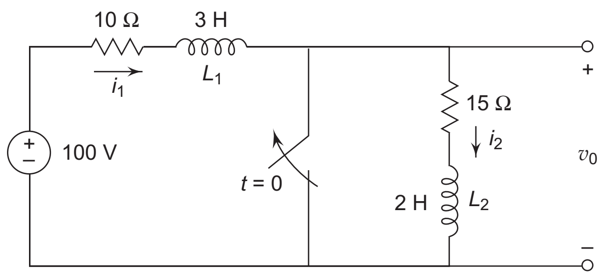


Fig. 13.33

Figure 13.34 shows the s-domain equivalent with the switch open. The current in the 3H inductor at $t = 0$ is 10 A, and the current in 2H inductor at $t = 0$ is zero.

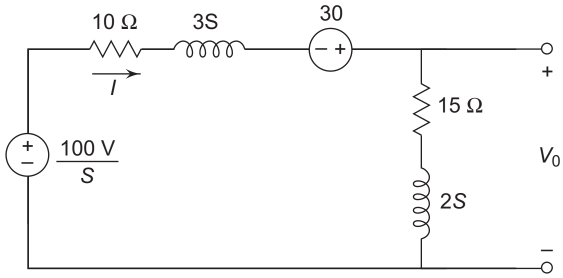


Fig. 13.34

Applying Kirchhoff's current law, we get

$$\frac{V_0}{2S+15} + \frac{V_0 - [(100/S)+30]}{3S+10} = 0 \qquad (13.93)$$

Solving for V_0 yields

$$V_0 = \frac{40(S+7.5)}{S(S+5)} + \frac{12(S+7.5)}{S+5} \tag{13.94}$$

By taking partial fractions, we get

$$V_0 = \frac{60}{S} - \frac{20}{S+5} + 12 + \frac{30}{S+5}$$

$$= 12 + \frac{60}{S} + \frac{10}{S+5} \tag{13.95}$$

By taking inverse transform, we have

$$v_0 = 12\ \delta(t) + (60 + 10e^{-5t})\ u(t) \text{ volts} \tag{13.96}$$

Lets derive the expression for the current when $t > 0$. After the switch has been opened, the current in L_1 is the same as the current in L_2. The current equation is

$$I = \frac{\left(\frac{100}{S}\right) + 30}{5S + 25} = \frac{20}{S(S+5)} + \frac{6}{S+5}$$

$$= \frac{4}{S} - \frac{4}{S+5} + \frac{6}{S+5}$$

$$= \frac{4}{S} + \frac{2}{S+5} \tag{13.97}$$

By taking inverse transform gives

$$i = (4 + 2e^{-5t})\ u(t) \tag{13.98}$$

Before the switch is opened, the current in L_1 is 10 A, and the current in L_2 is 0A. We know that at $t = 0$, the current in L_1 and in L_2 is 6A. Then, the current in L_1 changes instantaneously from 10 to 6A, while the current in L_2 changes instantaneously from 0 to 6A. From this value of 6A, the current decreases exponentially to a final value of 4A. Figure 13.35 shows these characteristics of i_1 and i_2.

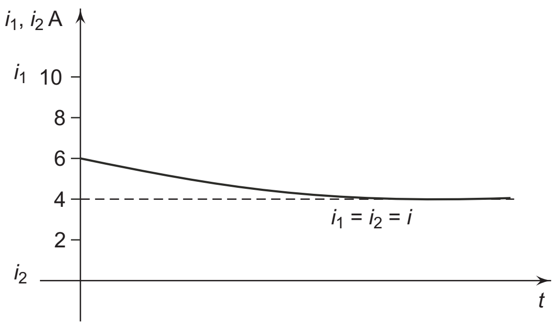


Fig. 13.35

Impulse Sources

Impulse functions can occur in sources as well as responses. Such sources are called impulsive sources. An impulse source driving a circuit imparts a finite amount of energy into the system instantaneously. In the circuit shown in Fig. 13.36a, an impulsive voltage source having a strength of V_0 volt-seconds is applied to a series connection of a resistor and an inductor. When the voltage source is applied, the initial energy in the inductor is zero, therefore the initial current is zero. There is no voltage drop across R. So the impulse voltage source appears directly across L. An impulse voltage at the terminals of an inductor establishes an instantaneous current. The current is

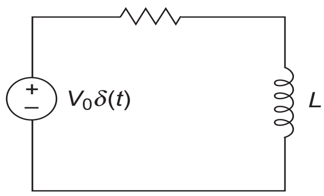


Fig. 13.36(a)

$$i = \frac{1}{L}\int_0^t V_0 \;\delta(x)\; dx \qquad (13.99)$$

The integral of $\delta(t)$ over any interval that includes zero is one, thus we have

$$i(0) = \frac{V_0}{L}\; A \qquad (13.100)$$

For an infinitesimal moment, the impulsive voltage source has stored

$$W = \frac{1}{2}\, L\left(\frac{V_0}{L}\right)^2 = \frac{1}{2}\frac{V_0^2}{L}\;\text{J} \qquad (13.101)$$

in the inductor.

The current $\frac{V_0}{L}$ decays to zero in accordance with the natural response of the circuit, that is,

$$i = \frac{V_0}{L}\; e^{-t/\tau}\, u(t) \qquad (13.102)$$

where $\tau = \frac{L}{R}$.

When a circuit is driven by only an impulsive source, the total response is completely defined by the natural response. The duration of the impulse source is so infinitesimal that it does not contribute to any forced response.

We may also obtain Eq. (13.102) by direct application of the Laplace transform method. Figure 13.37 shows the s-domain equivalent of the circuit in Fig. 13.36a.

The current I in the circuit is

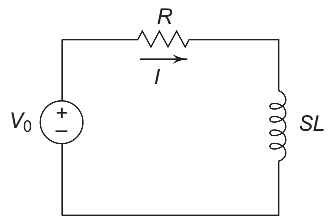


Fig. 13.36(b)

$$I = \frac{V_0}{R + SL} = \frac{V_0/L}{S + R/L} \qquad (13.103)$$

Taking inverse Laplace transform, we get

$$i = \frac{V_0}{L}\, e^{-\left(\frac{R}{L}\right)^t} = \frac{V_0}{L}\, e^{-t/\tau}\, u(t) \; (13.103a)$$

Thus the Laplace transform method gives the correct solution for $i \geq 0$.

Solved Problems

Problem 13.1 A 500 Ω resistor, a 16 mH inductor, and a 25 mF capacitor are connected in parallel. Express the admittance of this parallel combination of elements as a rational function of S.

Solution The circuit represented in s-domain of the above problem is shown in Fig. 13.37.

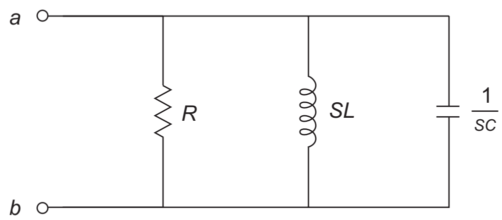


Fig. 13.37

The admittance of terminals ab is

$$Y(S) = \frac{1}{R} + \frac{1}{SL} + SC \tag{13.104}$$

Substituting the numerical values in the above equation

$$Y(S) = \frac{1}{500} + \frac{1}{S \times 16 \times 10^{-3}} + S \times 25 \times 10^{-9} \tag{13.105}$$

Simplifying the above equation, we have

$$Y(S) = \frac{25 \times 10^{-9}}{S}\,(S^2 + 80{,}000\,S + 25 \times 10^8) \tag{13.106}$$

Problem 13.2 The switch in the circuit shown has been in position a for a long time. At $t = 0$, the switch is thrown to position b. Find the current I as rational function of s. Find the time-domain expression for the current i.

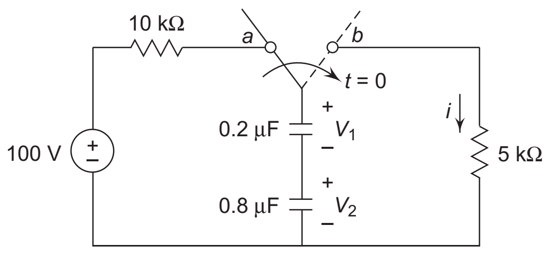


Fig. 13.38

Solution When the switch is at position for a long time both the capacitors are charged to 100 V.

When the switch is at position b, the s domain circuit is shown in Fig. 13.39.

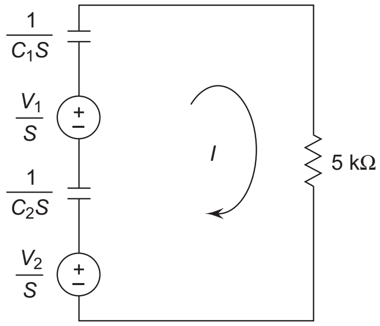


Fig. 13.39

By applying Kirchhoff's voltage law, we have

$$\frac{V_1}{S} + \frac{V_2}{S} = \frac{1}{C_1 S} I + \frac{1}{C_2 S} I + I\,(5K) \tag{13.107}$$

$$\frac{1}{S}(V_1 + V_2) = \frac{I}{S}\left[\frac{1}{0.2 \times 10^{-6}} + \frac{1}{0.8 \times 10^{-6}} + 5 \times 10^3\right]$$

$$\frac{I}{S} 100 = \frac{I}{S}\left[\frac{1}{0.16 \times 10^{-6}} + 5 \times 10^3\right]$$

$$I = \frac{0.02}{S + 1250} \tag{13.108}$$

By taking inverse transform, we get the time domain expression for i

$$i = 0.02\, e^{-1250\, t} \text{ A.} \tag{13.109}$$

Problem 13.3 Obtain the current s-domain expression for current I_L in the circuit shown in Fig. 13.40. Also obtain the time domain expression for inductor current. The switch is opened at $t = 0$. Assume initial energy stored in the circuit is zero.

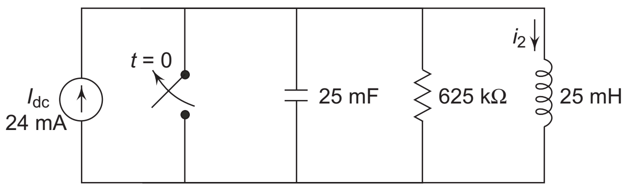


Fig. 13.40

Solution The s-domain equivalent circuit for the circuit shown in Fig. 13.40 is shown in Fig. 13.41.

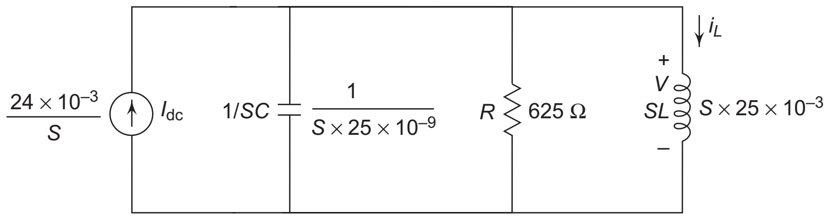


Fig. 13.41

By applying Kirchhoff's current law, we get

$$SCV + \frac{V}{R} + \frac{V}{SL} = \frac{I_{dc}}{S} \tag{13.110}$$

$$V = \frac{I_{dc}/C}{S^2 + \left(\frac{1}{RC}\right)S + \frac{1}{LC}} \tag{13.111}$$

We know $$I_L = \frac{V}{SL} \tag{13.112}$$

Substituting Eq. (13.111) into Eq. (13.112), we get

$$IL = \frac{I_{dc}/LC}{S\left[S^2 + \left(\frac{1}{RC}\right)S + \left(\frac{1}{LC}\right)\right]} \tag{13.113}$$

Substituting the numerical values yields

$$IL = \frac{384 \times 10^5}{S(S^2 + 64000S + 16 \times 10^8)} \tag{13.114}$$

By taking partial fractions, we get

$$I_L = \frac{384 \times 10^5}{S(S + 32000 - j24000)(S + 32000 + j24000)} \tag{13.115}$$

$$I_L = \frac{K_1}{S} + \frac{K_2}{S + 32000 - j24000} + \frac{K_2^*}{S + 32000 - j24000} \tag{13.116}$$

The partial fraction coefficients are

$$K_1 = \frac{384 \times 10^5}{16 \times 10^8} = 24 \times 10^{-3}$$

$$K_2 = \frac{384 \times 10^5}{(-32000 + j\,24000)(j\,48000)}$$

$$= 20 \times 10^{-3} \angle 126.87° \tag{13.117}$$

Substituting the numerical values of K_1 and K_2 into Eq. (13.116) and inverse transforming the resulting expression yields

$$i_L = [24 + 40\, e^{-32,000\, t} \cos (24000\, t + 126.87°)] \text{ mA} \tag{13.118}$$

Problem 13.4 Obtain the s-domain expression for the current I_L in the circuit shown in Fig. 13.40 when the dc current source is replaced by sinusoidal current source $i_g = I_m \cos \omega t$. Where $I_m = 24$ mA and $\omega = 40,000$ rod/s. Assume initial energy stored in the circuit is zero.

Solution The s-domain expression for the source current is

$$I_g = \frac{SI_m}{s^2 + \omega^2} \tag{13.119}$$

The voltage across the parallel elements is

$$V = \frac{(I_g / C)s}{s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC}} \tag{13.120}$$

Substituting Eq. (13.119) into Eq. (13.120) results in

$$V = \frac{(I_m / C)s^2}{(s^2 + \omega^2)\left(s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)\right)} \tag{13.121}$$

from which

$$I_L = \frac{V}{SL} = \frac{(I_m / LC)s}{(s^2 + \omega^2)\left[s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)\right]} \tag{13.122}$$

Substituting the numerical values of I_m, ω, R, L, and C in Eq. (13.122) gives

$$I_L = \frac{384 \times 10^5 s}{(s^2 + 16 \times 10^8)(s^2 + 64000s + 16 \times 10^8)} \tag{13.123}$$

By factoring the denominator, we get

$$I_2 = \frac{384 \times 10^5 s}{(s^2 - j\omega)(s + j\omega)(s + \alpha - j\beta)(s + \alpha + j\beta)} \tag{13.124}$$

where $\omega = 40000$, $\alpha = 32000$ and $\beta = 24000$

By taking partial fractions, we get

$$I_L = \frac{K_1}{s - j\,40000} + \frac{K_1^*}{s + j40000} + \frac{K_2}{s + 32000 - j24000} + \frac{K_2^*}{s + 32000 + j24000} \tag{13.125}$$

The coefficients K_1 and K_2 are

$$K_1 = \frac{384 \times 10^5\, (j\,40000)}{(j80000)(32000 + j\,16000)(32000 + j64000)}$$

$$= 7.5 \times 10^{-3} \angle -90° \qquad (13.126)$$

$$K_2 = \frac{384 \times 10^5 (-32000 + j24000)}{(-32000 - j\,16000)(-32000 + j64000)(j48000)}$$

$$= 12.5 \times 10^{-3} \angle 90° \qquad (13.127)$$

Substituting the numerical values from (13.126) and (13.127) into (13.125) and inverse-transforming the resulting expression yields

$$i_L = [15 \cos (40000\,t - 90°) + 25\,e^{-32000\,t} \cos (24000\,t + 90°)] \text{ mA}$$

$$i_L = [15 \sin 40000\,t - 25\,e^{-32000\,t} \sin 24000\,t] \text{ mA} \qquad (13.128)$$

Problem 13.5 Obtain the expression for i_1 and i_2 in the circuit shown in Fig. 13.42 when dc voltage source is applied suddenly. Assume that the initial energy stored in the circuit is zero.

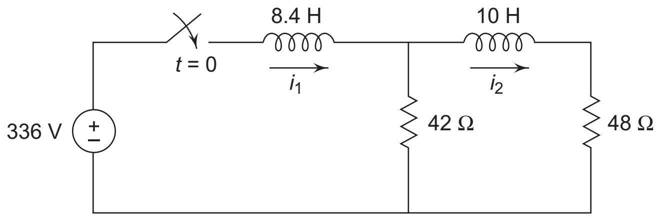


Fig. 13.42

Solution The Fig. 13.43 shows the s-domain equivalent circuit for the circuit shown in Fig. 13.42.

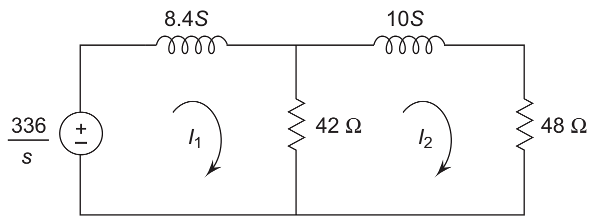


Fig. 13.43

The two mesh current equations are

$$\frac{336}{s} = (42 + 8.4\,s)\,I_1 - 42\,I_2 \qquad (13.129)$$

$$0 = -42\,I_1 + (90 + 10\,s)\,I_2 \qquad (13.130)$$

Using Cramer's method to solve for I_1 and I_2, we get

$$\Delta = \begin{vmatrix} 42 + 8.4s & -42 \\ -42 & 90 + 10s \end{vmatrix}$$

$$= 84\,(s^2 + 14\,s + 24)$$

$$= 84\,(s + 2)\,(s + 12) \tag{13.131}$$

$$N_1 = \begin{vmatrix} 336/s & -42 \\ 0 & 90 + 10s \end{vmatrix} = \frac{3360\,(s + 9)}{s} \tag{13.132}$$

$$N_2 = \begin{vmatrix} 42 + 8.4s & 336/s \\ -42 & 0 \end{vmatrix} = \frac{14112}{s} \tag{13.133}$$

Based on Eqs (13.131) to (13.133)

$$I_1 = \frac{N_1}{\Delta} = \frac{40(s+9)}{s(s+2)(s+12)} \tag{13.134}$$

$$I_2 = \frac{N_2}{\Delta} = \frac{168}{s(s+2)(s+12)} \tag{13.135}$$

Expanding I_1 and I_2 into a sum of partial fractions gives

$$I_1 = \frac{15}{s} - \frac{14}{s+2} - \frac{1}{s+12} \tag{13.136}$$

$$I_2 = \frac{7}{s} - \frac{8.4}{s+2} + \frac{1.4}{s+12} \tag{13.137}$$

We obtain the expressions for i_1 and i_2 by inverse transforming Eq. (13.136) and (13.137) respectively

$$i_1 = (15 - 14\,e^{-2t} - e^{-12t})\text{ A} \tag{13.138}$$

$$i_2 = (7 - 8.4\,e^{-2t} + 1.4\,e^{-12t})\text{ A} \tag{13.139}$$

Problem 13.6 Transform the circuit shown in Fig. 13.44 to the s domain and determine the Laplace impedance.

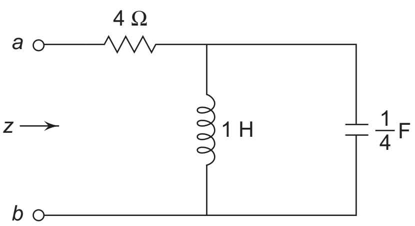


Fig. 13.44

Solution The transformed circuit for the above circuit is shown in Fig. 13.45.

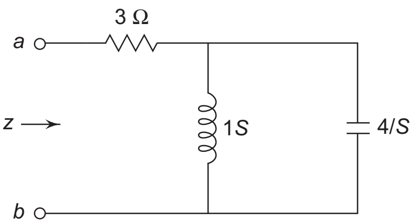


Fig. 13.45

The parallel combination of inductor and capacitor is in series with the resistor.

$$z = 3 + \left\{(s)/\left(\frac{4}{s}\right)\right\} = 3 + \frac{s\left(\frac{4}{5}\right)}{s + \frac{4}{s}}$$

$$z = \frac{3s^2 + 4s + 12}{s^2 + 4} \tag{13.140}$$

Problem 13.7 Determine the current i if the circuit is driven by a voltage source as shown in Fig. 13.46. The initial value of the voltage across the capacitor and the initial current through the inductor are both zero.

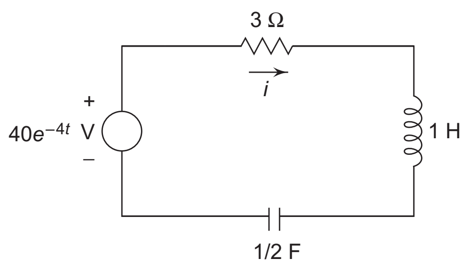


Fig. 13.46

Solution The transformed circuit is as shown in Fig. 13.47

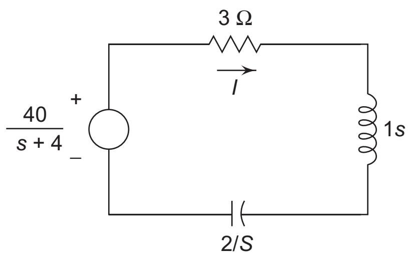


Fig. 13.47

Total Laplace impedance across the voltage source is

$$Z = 3 + s + \frac{2}{s}$$

$$Z = \frac{s^2 + 3s + 2}{s} \tag{13.141}$$

Thus, the current is

$$I = \frac{V}{Z} = \frac{40/(s+4)}{(s^2 + 3s + 2)/s} \tag{13.142}$$

$$I = \frac{40s}{(s+4)(s^2 + 3s + 2)} \tag{13.143}$$

By taking partial fractions

$$I = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+4} \tag{13.144}$$

The coefficients K_1, K_2 and K_3 are

$$K_1 = I \ (s+1)|_{s=-1} \frac{-40}{3}$$

$$K_2 = I \ (s+2)|_{s=-2} = 40$$

$$K_3 = I \times (s+4)|_{s=-4} = \frac{-80}{3}$$

Substituting the coefficients and taking inverse Laplace transform, we get

$$i = 40\, e^{-2t} - \frac{40}{3} e^{-t} - \frac{80}{3} e^{-4t} \tag{13.145}$$

Problem 13.8 Determine the current i for $t \geq 0$ if initial current $i(0) = 1$ for the circuit shown in Fig. 13.48.

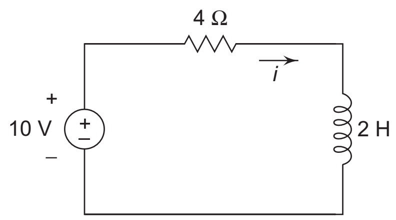


Fig. 13.48

Solution The s domain circuit with series initial current in the inductor is shown in Fig. 13.49.

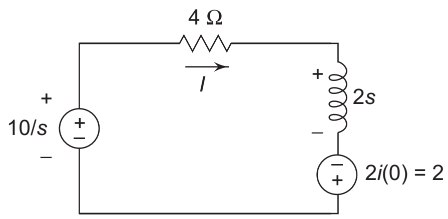


Fig. 13.49

Applying Kirchhoff's voltage law, results in

$$\frac{10}{s} - 4I - 2sI + 2 = 0 \tag{13.146}$$

$$10 - 4sI - 2s^2 I + 2s = 0$$

$$I = \frac{10 + 2s}{2s\,(s+2)}$$

$$I = \frac{5+s}{s(s+2)} \tag{13.147}$$

Taking partial fractions

$$I = \frac{K_1}{s} + \frac{K_2}{s+2} \tag{13.148}$$

The coefficients K_1 and K_2 are

$$K_1 = \frac{5}{2}; \quad K_2 = \frac{-3}{2}$$

Substituting coefficients and taking inverse Laplace transform of Eq. (13.148) gives

$$i = \frac{5}{2} - \frac{3}{2}e^{-2t} \tag{13.149}$$

Alternatively, the inductor initial condition can be represented parallelly as shown in Fig. 13.50.

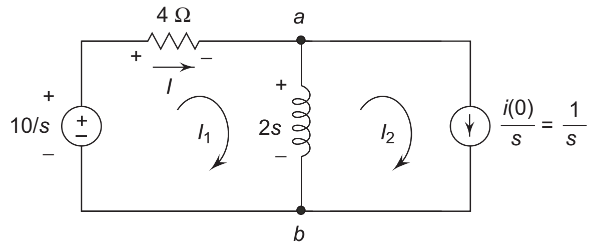


Fig. 13.50

By inspection,

$$I_2 = \frac{1}{s}$$

By applying Kirchhoff's voltage law to mesh results in

$$\frac{10}{s} - 4\,I_1 - 2sI_1 + 2sI_2 = 0 \tag{13.150}$$

From the figure $I_1 = I$ and $I_2 = \dfrac{1}{s}$

$$\frac{10}{s} - 4\,I - 2sI + 2s\left(\frac{1}{s}\right) = 0$$

$$I = \frac{s+4}{s(s+2)} \tag{13.151}$$

Taking partial fractions and inverse Laplace transform, we get

$$i = \frac{5}{2} - \frac{3}{2}e^{-2t} \tag{13.152}$$

Problem 13.9 Determine the current i for $t \geq 0$ if $V_c(0) = 4$ V for the circuit shown in Fig. 13.51.

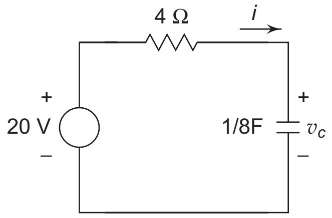


Fig. 13.51

Solution The transformed s domain circuit is shown in Fig. 13.52

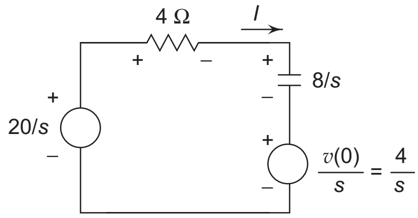


Fig. 13.52

Application of Kirchhoff's voltage law gives

$$\frac{20}{s} - 4\,I - \left(\frac{8}{5}\right) I - \frac{4}{s} = 0 \tag{13.153}$$

$$I = \frac{4}{s+2}$$

Taking inverse Laplace transform gives

$$i = 4\,e^{-2t} \tag{13.154}$$

Alternatively, the initial condition can be represented as shown in Fig. 13.53.

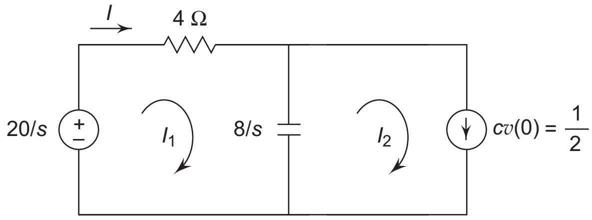


Fig. 13.53

By inspection, we have

$$I_2 = \frac{-1}{2}$$

Applying Kirchhoff's voltage law to mesh 1 results in

$$\frac{20}{s} - 4\,I_1 - \frac{8}{5}\,I_1 + \left(\frac{8}{s}\right)\left(\frac{-1}{2}\right) = 0 \tag{13.155}$$

Because $I_1 = I$ and $I_2 = \dfrac{-1}{2}$

$$\frac{20}{s} - 4\,I - \frac{8}{s}\,I + \left(\frac{8}{s}\right)\left(\frac{-1}{2}\right) = 0$$

$$I = \frac{4}{s+2} \tag{13.156}$$

Taking inverse transform, we get

$$i = 4\,e^{-2t} \tag{13.157}$$

Problem 13.10 Convert the current source in Fig. 13.54 to a voltage source in the s domain.

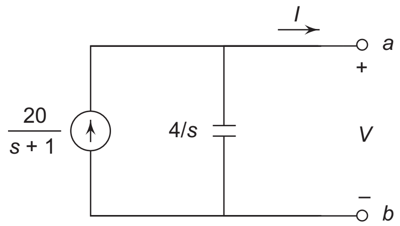


Fig. 13.54

Solution Converting the circuit Fig. 13.54 into the voltage source results in the circuit shown in Fig. 13.55.

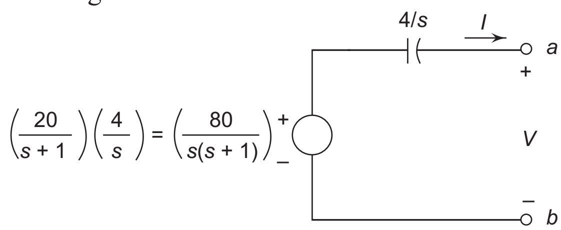


Fig. 13.55

Problem 13.11 Convert the voltage source in Fig. 13.56 to a current source in the s domain.

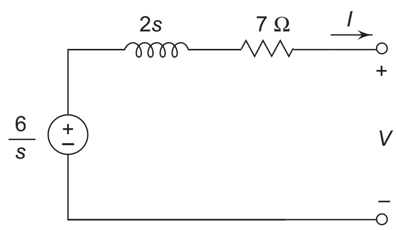


Fig. 13.56

Solution Converting Fig. 13.56 into a current source in the *s* domain results in the circuit shown in Fig. 13.57.

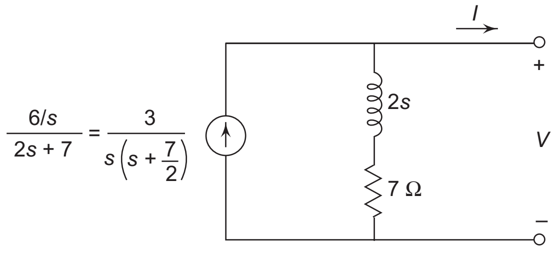


Fig. 13.57

Problem 13.12 Determine v_0 for the circuit shown in Fig. 13.58.

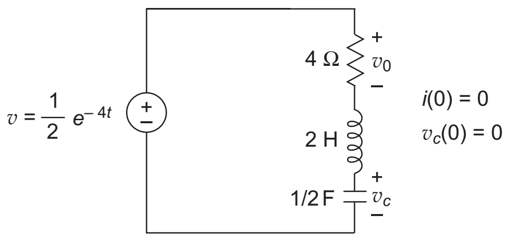


Fig. 13.58

Solution The circuit in Fig. 13.58 in the *s* domain is as shown in Fig. 13.59.

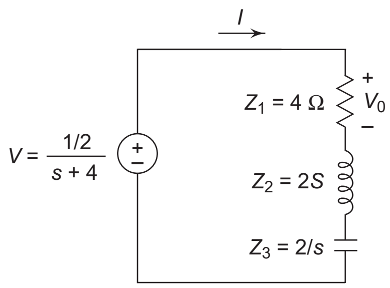


Fig. 13.59

Total impedance in the circuit

$$Z_{eq} = Z_1 + Z_2 + Z_3 \tag{13.158}$$

$$V_0 = \left(\frac{Z_1}{Z_1 + Z_2 + Z_3}\right) V$$

$$= \left\{ \frac{4}{2s + 4 + \dfrac{2}{s}} \right\} \left\{ \frac{\dfrac{1}{2}}{s+4} \right\} \tag{13.159}$$

$$= \frac{s}{(s+4)\,(s^2 + 2s + 1)}$$

$$= \frac{s}{(s+4)\,(s+1)^2} \tag{13.160}$$

Taking partial fractions

$$V_0 = \frac{K_1}{(s+1)^2} + \frac{K_2}{s+1} + \frac{K_3}{s+4} \tag{13.161}$$

The coefficients K_1, K_2 and K_3 are

$$K_1 = \frac{-1}{3},\; K_2 = \frac{4}{9},\; K_3 = \frac{-4}{9}$$

Thus

$$V_0 = \frac{(-1/3)}{(s+1)^2} + \frac{4/9}{s+1} + \frac{(-4/9)}{s+4}$$

Taking inverse transform both sides

$$v_0 = \frac{1}{3}\, t\, e^{-t} + \frac{4}{9} e^{-t} - \frac{4}{9}\, e^{-4t} \tag{13.162}$$

Problem 13.13 Determine i_1, i_2, V and V_1 for the circuit in Fig. 13.60.

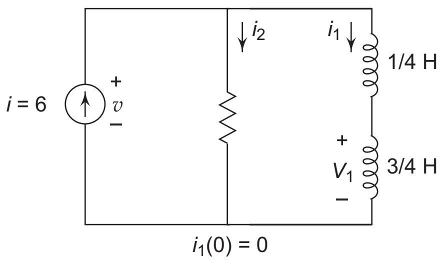


Fig. 13.60

Solution The transformed circuit in the s domain is shown in Fig. 13.61.

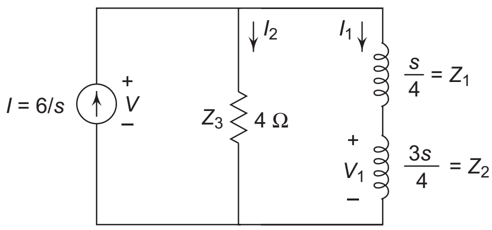


Fig. 13.61

Applying current division to the circuit in the s domain

$$Z_{eq} = 4 // \left(\frac{s}{4} + \frac{3s}{4}\right) \tag{13.163}$$

$$I_1 = \left\{\frac{Z_{eq}}{Z_1 + Z_2}\right\}\left(\frac{6}{s}\right)$$

$$= \frac{24}{s(s+4)} \tag{13.164}$$

By taking partial fraction expansion

$$I_1 = \frac{K_1}{s} + \frac{K_2}{s+4}$$

$$K_1 = 6; \quad K_2 = -6$$

$$I_1 = \frac{6}{s} + \frac{(-6)}{s+4} \tag{13.165}$$

Taking inverse transform, we get

$$i_1 = 6 - 6\, e^{-4t} \tag{13.166}$$

From Kirchhoff's current law

$$\frac{6}{s} = I_2 + I_1 \tag{13.167}$$

$$= I_2 + \frac{24}{s\,(s+4)}$$

$$I_2 = \frac{6}{s} - \frac{24}{s\,(s+4)}$$

$$I_2 = \frac{6}{s+4} \tag{13.168}$$

We know

$$V = 4\, I_2 = \frac{24}{s+4} \tag{13.169}$$

$$V_1 = \left\{\frac{3s}{4}\right\} I_1$$

$$= \left\{\frac{3s}{4}\right\}\left\{\frac{24}{s(s+4)}\right\} \tag{13.170}$$

$$V_1 = \frac{18}{s+4}$$

Taking inverse transforms

$$i_2 = \mathscr{L}^{-1}\,\{I_2\} = \mathscr{L}^{-1}\left\{\frac{6}{s+4}\right\} = 6\, e^{-4}t \tag{13.171}$$

$$v = \mathscr{L}^{-1}\{V\} = \mathscr{L}^{-1}\left\{\frac{24}{s+4}\right\} = 24\,e^{-4t} \tag{13.172}$$

$$v_1 = \mathscr{L}^{-1}\{V_1\} = \mathscr{L}^{-1}\left\{\frac{18}{s+4}\right\} = 18\,e^{-4t} \tag{13.173}$$

Problem 13.14 Determine the voltage v for the circuit shown in Fig. 13.62.

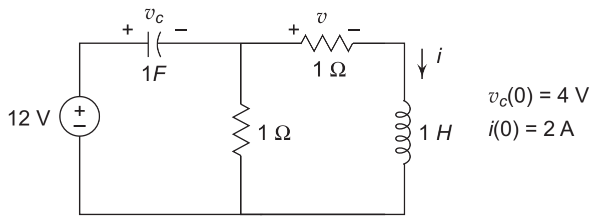


Fig. 13.62

Solution The circuit in Fig. 13.62 is transformed into s domain as shown in Fig. 13.63.

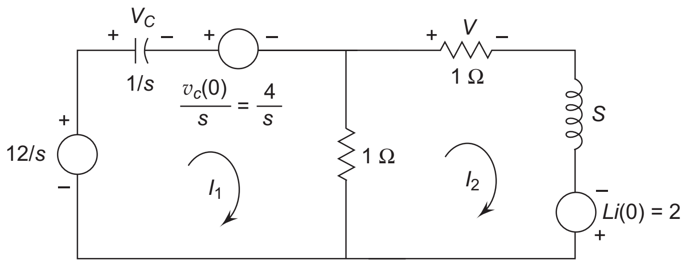


Fig. 13.63

By using mesh analysis, the current I_2 in the circuit is

$$I_2 = \frac{\begin{vmatrix} 1+\frac{1}{s} & \frac{12}{s}-\frac{4}{s} \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1+\frac{1}{s} & -1 \\ -1 & 1+1+s \end{vmatrix}} \tag{13.174}$$

$$I_2 = \frac{2\left(1+\frac{1}{s}\right)+\left(\frac{8}{s}\right)}{\left(1+\frac{1}{s}\right)(2+s)-1} = \frac{2s+10}{s^2+2s+2} \tag{13.175}$$

The voltage across 1 Ω resistor

$$V = R\,I_2 = \frac{2s+10}{s^2+2s+2} \tag{13.176}$$

The above equation can be written as

$$V = \frac{2s+2+8}{s^2+2s+2}$$

$$= 2\left\{\frac{s+1}{s^2+2s+2}\right\} + 8\left\{\frac{1}{s^2+2s+2}\right\}$$

$$= 2\left\{\frac{s+1}{(s+1)^2+1}\right\} + 8\left\{\frac{1}{(s+1)^2+1}\right\} \tag{13.177}$$

Taking inverse Laplace transform both sides

$$v = 2\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + 8\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$v = 2\,e^{-t}\cos t + 8\,e^{-t}\sin t \tag{13.178}$$

Problem 13.15 Determine the voltage v for the circuit in Fig. 13.64. Assume $v_c(0) = 0$.

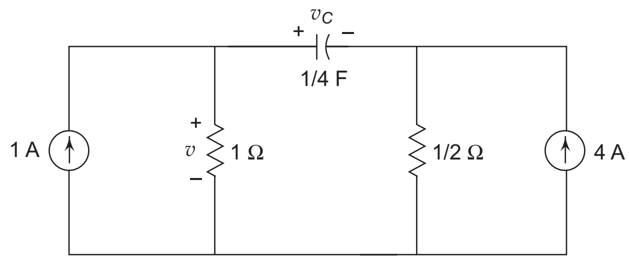


Fig. 13.64

Solution The circuit in Fig. 13.64 is transformed into the s domain results in the circuit in Fig. 13.65.

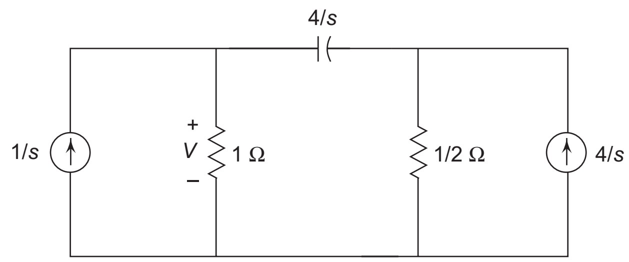


Fig. 13.65

Replacing the Laplace impedance for R and C with Laplace admittance, we get

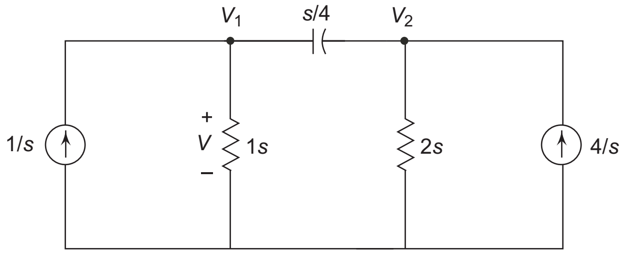


Fig. 13.66

By using modal analysis, we get

$$V_1 = \frac{\begin{vmatrix} \frac{1}{s} & \frac{-s}{4} \\ \frac{4}{s} & 2+\frac{s}{4} \end{vmatrix}}{\begin{vmatrix} 1+\frac{s}{4} & \frac{-s}{4} \\ \frac{-s}{4} & 2+\frac{s}{4} \end{vmatrix}} \tag{13.179}$$

$$V_1 = \frac{\left(\frac{1}{s}\right)\left\{2+\left(\frac{s}{4}\right)\right\} - \left(\frac{4}{s}\right)\left(\frac{-s}{4}\right)}{\left\{1+\left(\frac{s}{4}\right)\right\}\left\{2+\left(\frac{s}{4}\right)\right\} - \left(\frac{-s}{4}\right)\left(\frac{-s}{4}\right)} \tag{13.180}$$

$$= \frac{\left(\frac{5}{3}\right)s + \left(\frac{8}{3}\right)}{s\left(s+\frac{8}{3}\right)} \tag{13.181}$$

Taking partial fraction expansion

$$V_1 = \frac{K_1}{s} + \frac{K_2}{s+\frac{8}{3}}$$

The coefficients K_1 and K_2 are

$$K_1 = 1; \quad K_2 = \frac{2}{3}$$

$$V_1 = \frac{1}{s} + \frac{2/3}{s+\frac{8}{3}} \tag{13.182}$$

Taking the inverse Laplace transform of each side of the equation results in

$$\mathscr{L}^{-1}\{V_1\} = \mathscr{L}^{-1}\left\{\frac{1}{s} + \frac{(2/3)}{\left(s+\frac{8}{3}\right)}\right\} \tag{13.183}$$

$$v_1 = \mathscr{L}^{-1}\left(\frac{1}{s}\right) + \frac{2}{3}\,\mathscr{L}^{-1}\left\{\frac{1}{s+8/3}\right\}$$

$$v_1 = 1 + \frac{2}{3}e^{-\frac{8}{3}t}$$

and because $v = v_1$

$$v = 1 + \frac{2}{3}e^{\left(-\frac{8}{3}\right)t} \tag{13.184}$$

Problem 13.16 Determine the voltage v for the circuit shown in Fig. 13.67 using Thevemin's theorem.

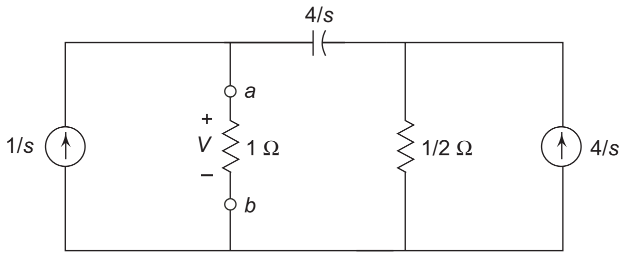


Fig. 13.67

Solution We have to find out open circuit voltage as shown in Fig. 13.68.

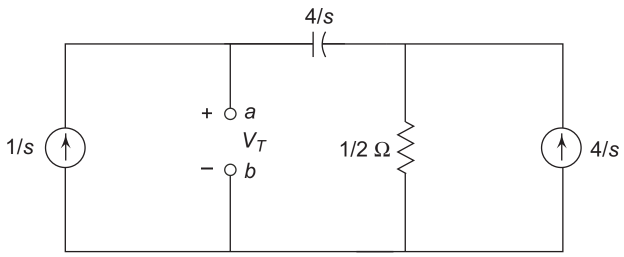


Fig. 13.68

Applying the superposition method results in the circuits in Figs 13.69 and 13.70, where V_T' and V_T'' are the contributions to V_T from the Laplace transformed sources $\left(\frac{1}{s}\right)$ and $\left(\frac{4}{s}\right)$ respectively

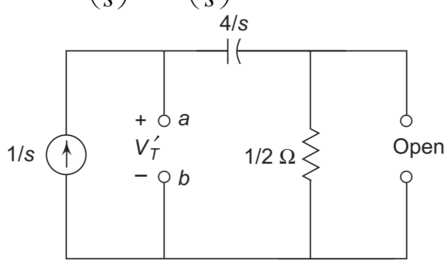


Fig. 13.69

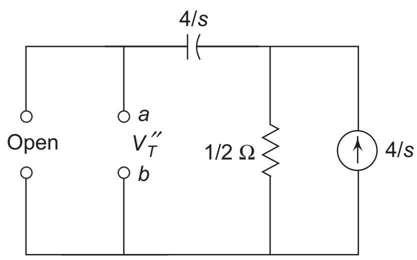


Fig. 13.70

From Fig. 13.69, the open circuit voltage

$$V'_T = IZ = \left(\frac{1}{s}\right)\left(\frac{4}{s}+\frac{1}{2}\right) = \frac{8+s}{2s^2} \tag{13.185}$$

From Fig. 13.70, the open circuit voltage

$$V''_T = IZ = \left(\frac{4}{s}\right)\left(\frac{1}{2}\right) = \frac{2}{s} \tag{13.186}$$

because no current flows through the capacitor
From the superposition method

$$V_T = V'_T + V''_T$$

$$= \frac{8+s}{2s^2} + \frac{2}{s} \tag{13.187}$$

$$= \frac{8+s+4s}{2s^2}$$

$$V_T = \frac{5s+8}{2s^2} \tag{13.188}$$

Replacing both convert sources by opens, as required Thevenin's theorem to determine the Thevenin impedance results in Fig. 13.71.

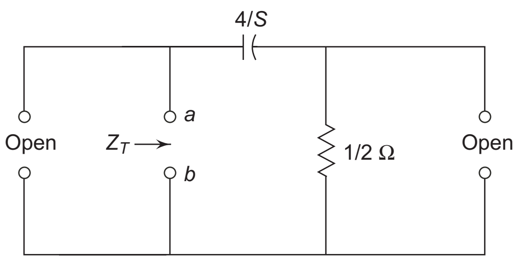


Fig. 13.71

The impedance seen into the terminals *ab*

$$Z_T = \frac{4}{s} + \frac{1}{2}$$

$$= \frac{s+8}{2s} \tag{13.189}$$

and the Thevenin equivalent circuit for terminals *a*–*b* is shown in Fig. 13.72.

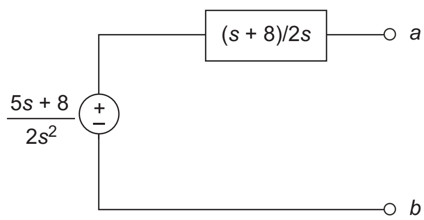


Fig. 13.72

If the 1 Ω resistor is reconnected across terminals *ab*, then *V* can be determined in Fig. 13.73.

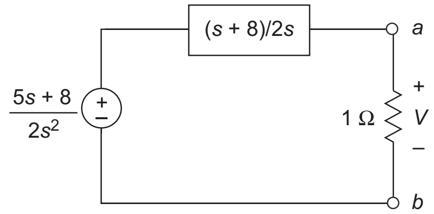


Fig. 13.73

$$V = \left\{ \frac{1}{1 + \frac{s+8}{2s}} \right\} \left\{ \frac{5s+8}{2s^2} \right\}$$

$$= \left\{ \frac{2s}{3s+8} \right\} \left\{ \frac{5s+8}{3s^2} \right\} \tag{13.190}$$

$$V = \frac{5s+8}{s\,(3s+8)} \tag{13.191}$$

The inverse Laplace transform of *V* is

$$v = 1 + \left(\frac{2}{3}\right) e^{-\left(\frac{8}{3}\right)t} \tag{13.192}$$

Problem 13.17 Determine the voltage *V* for the circuit shown in Fig. 13.72, using Norton's theorem.

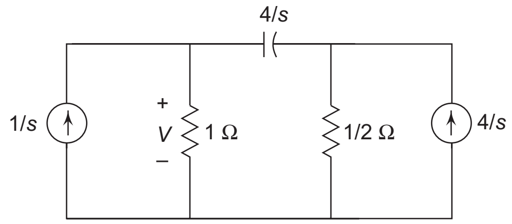


Fig. 13.74

Solution The application of Norton's theorem in the *s* domain requires the removal of the 1 Ω resistor as shown in Fig. 13.75 and the determination of resulting short-circuited current.

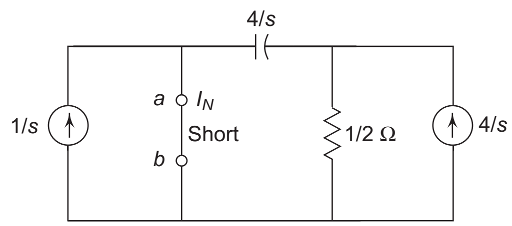


Fig. 13.75

Applying the superposition method results in the circuits in Figs. 13.76 and 13.77, where I_N' and I_N'' are the contributions to I_N from the Laplace transformed current sources $\left(\frac{1}{s}\right)$ and $\left(\frac{4}{s}\right)$ respectively

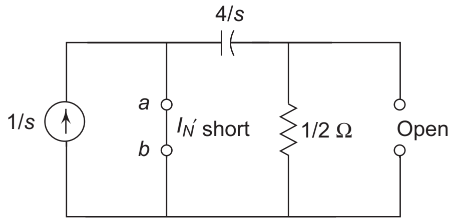


Fig. 13.76

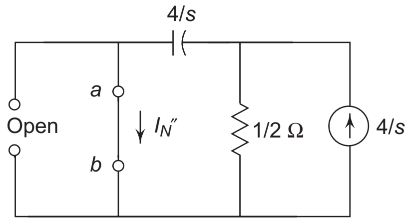


Fig. 13.77

By inspection of the circuit in Fig. 13.76

$$I_N' = \frac{1}{s} \tag{13.193}$$

Applying current division to the circuit in Fig. 13.77 results in

$$I_N'' = \left\{ \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right) + \left(\frac{4}{s}\right)} \right\} \left\{ \frac{4}{s} \right\} = \frac{4}{s+8} \tag{13.194}$$

From the superposition method

$$I_N = I_N' + I_N''$$

$$= \frac{1}{s} + \frac{4}{s+8}$$

$$I_N = \frac{5s+8}{s(s+8)} \tag{13.195}$$

Because Thevenin and Norton impedances are equal

$$Z_N = \frac{s+8}{2s} \tag{13.196}$$

The Norton equivalent circuit for terminals *a–b* is as shown in Fig. 13.78.

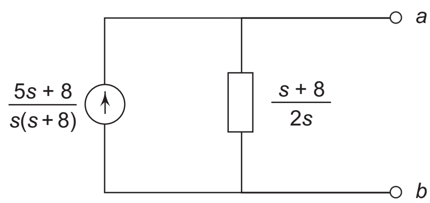


Fig. 13.78

If the 1 Ω resistor is reconnected across terminals *ab*, the voltage *V* can be determined in the circuit shown in Fig. 13.79.

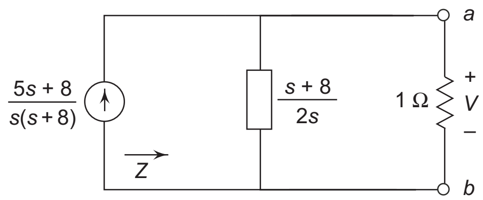


Fig. 13.79

$$Z = \left(\frac{s+8}{2s}\right) // (1) = \frac{\left(\frac{s+8}{2s}\right)\{1\}}{1+\frac{s+8}{2s}} \tag{13.197}$$

$$Z = \frac{s+8}{3s+8} \tag{13.198}$$

We know $V = ZI$

$$= \left\{\frac{s+8}{3s+8}\right\}\left\{\frac{5s+8}{s(s+8)}\right\}$$

$$V = \frac{5s+8}{s\,(3s+8)} \tag{13.199}$$

Taking inverse Laplace transform of V, we get

$$v = 1 + \left(\frac{2}{3}\right)e^{-\left(\frac{8}{3}\right)t} \tag{13.200}$$

Problem 13.18 The initial charge on the capacitor in the circuit shown in Fig. 13.80 is zero.

(a) Find the *s*-domain Thevenin equivalent circuit with respect to terminals *a* and *b*.

(b) Find the *s*-domain expression for the current that the circuit delivers to a load consisting of a 1 H inductor in series with a 2 Ω resistor.

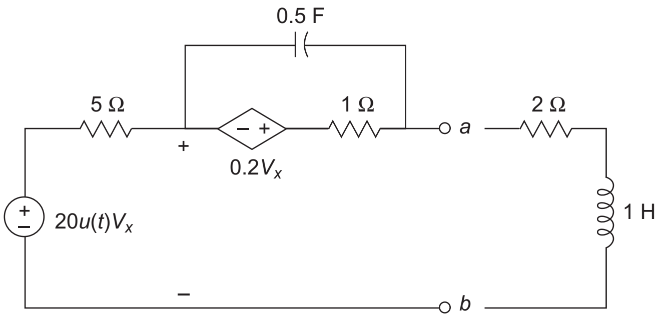


Fig. 13.80

Solution First we have to find out the Thevenins equivalent circuit from the *s*-domain circuit shown in Fig. 13.81.

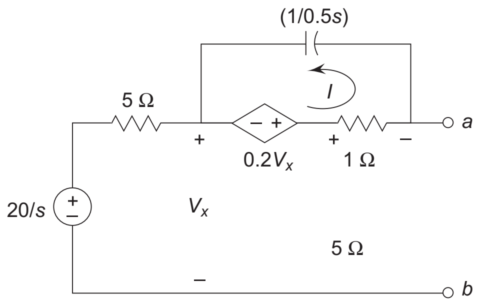


Fig. 13.81

Thevenins voltage across terminals *ab* is

$V_{ab} = V_x + 0.2\, V_x - I(1)$ (13.201)

By applying Kirchhoff's voltage law we can determine the current *I*

$$0.2\, V_x = I\left[1 + \frac{1}{0.5s}\right]$$

$$I = \frac{0.1s\, V_x}{(1 + 0.5s)} \tag{13.202}$$

Since no current is passing through 5Ω resistor.

The voltage $$V_x = \frac{20}{s} \tag{13.203}$$

Substituting V_x and I in Eq. (13.201), we get

$$V_{ab} = 1.2\left(\frac{20}{s}\right) - \frac{0.1s}{1 + 0.5s}\left(\frac{20}{s}\right) \tag{13.204}$$

$$V_{ab} = \frac{20}{s}\left[\frac{s + 2.4}{s + 2}\right] \tag{13.205}$$

The Thevenin's impedance after short circuiting the voltage sources shown in Fig. 13.82.

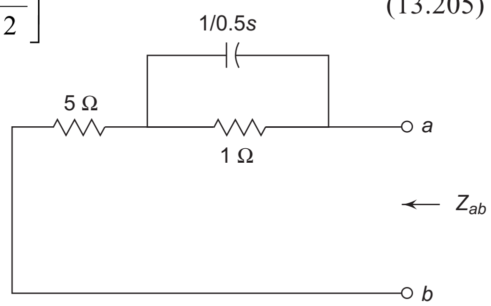


Fig. 13.82

$$Z_{ab} = \left\{\left(\frac{1}{0.5s}\right) // (1)\right\} + 5$$

$$Z_{ab} = \frac{5(s + 2.4)}{s + 2} \tag{13.206}$$

The Thevenin's equivalent circuit is shown in Fig. 13.83.

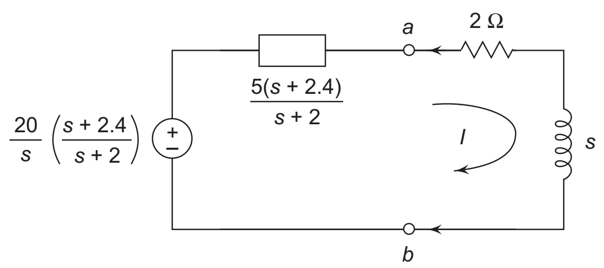


Fig. 13.83

The current I in the circuit of Fig. 13.83 is

$$I = \frac{\dfrac{20}{s}\left(\dfrac{s+2.4}{s+2}\right)}{\dfrac{5(s+2.4)}{s+2}+2+s} \tag{13.207}$$

$$= \frac{20}{s}\left[\frac{(s+2.4)}{5s+12+s^2+4s+4}\right] \tag{13.208}$$

$$I = \frac{20}{s}\left[\frac{s+2.4}{s^2+9s+16}\right] \tag{13.209}$$

Problem 13.19 The voltage source v_g drives the circuit shown in Fig. 13.84. The response signal is the voltage across the capacitor v_o.

(a) Calculate the numerical expression for the transfer function.

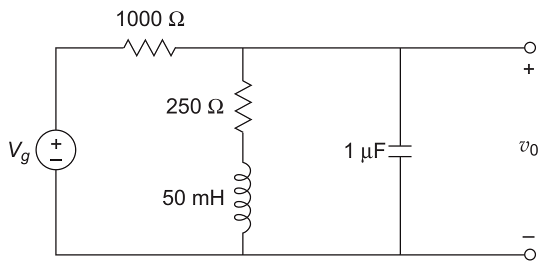


Fig. 13.84

Solution The s-domain equivalent circuit is shown in Fig. 13.85.

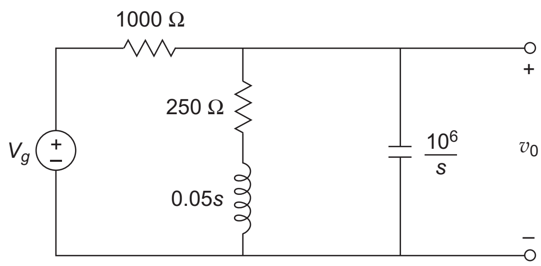


Fig. 13.85

By definition transfer function is the ratio v_o/v_g.

By applying Kirchhoff's current law, we get

$$\frac{V_0 - V_g}{1000} + \frac{V_0}{250+0.05s} + \frac{V_0 s}{10^6} = 0 \tag{13.210}$$

Solving for V_0 yields

$$V_0 = \frac{1000(s+5000)V_g}{s^2+6000s+25\times 10^6} \tag{13.211}$$

Hence the transfer function is

$$H(s) = \frac{V_0}{V_g} = \frac{1000(s+5000)}{s^2+6000s+25\times10^6} \tag{13.212}$$

Problem 13.20 The circuit shown in Fig. 13.86 is driven by a voltage source whose voltage increases linearly with time, namely $v_0 = 50\ t\ u(t)$.

(a) Use the transfer function to find v_0.

(b) Indentify by transient component of the response.

(c) Indentify the steady state component of the response.

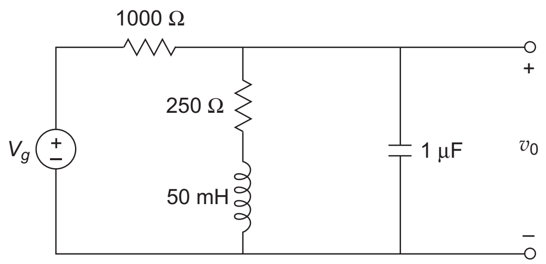


Fig. 13.86

Solution From the previous example

$$H(S) = \frac{1000(s+5000)}{s^2+6000s+25\times10^6} \tag{13.213}$$

The transform of the driving voltage is $50/s^2$, therefore, the s domain expression for the output voltage is

$$V_0 = \frac{1000(s+5000)}{(s^2+6000s+25\times10^6)}\,\frac{50}{s^2} \tag{13.214}$$

The partial fraction expansion of V_0 is

$$V_0 = \frac{K_1}{s+3000-j4000} + \frac{K_1^*}{s+3000+j4000} + \frac{K_2}{s^2} + \frac{K_3}{s} \tag{13.215}$$

The involves of coefficients are

$$K_1 = 5\sqrt{5}\times10^{-4}\angle 79.70°$$
$$K_1^* = 5\sqrt{5}\times10^{-4}\angle -79.70°$$
$$K_2 = 10$$
$$K_3 = -4\times10^{-4}$$

The time domain expression for v_0 is

$$v_0 = [10\sqrt{5}\times10^{-4}\,e^{-3000t}\cos(4000\,t + 79.70°) + 10\,t - 4\times10^{-4}]\ \text{V} \tag{13.216}$$

(b) The transient component of v_0 is

$$10\sqrt{5}\times10^{-4}\,e^{-3000t}\cos(4000t + 79.70°)\text{V} \tag{13.217}$$

(c) The steady state component of the response is

$$(10\,t - 4\times10^{-4})\ \text{V} \tag{13.218}$$

Problem 13.21 The excitation voltage v_i for the circuit shown in Fig. 13.87 is shown in Fig. 13.88.

(a) Use convolution integral to find v_0.

(b) Plot v_0 over the range of $0 \le t \le 15\ s$.

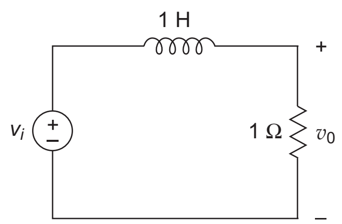


Fig. 13.87

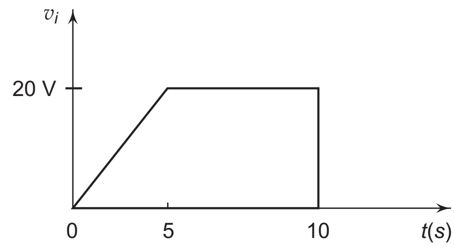


Fig. 13.88

Solution The first step in using the convolution integral is to find the unit impulse response of the circuit use obtain the expression for V_0 from the s-domain equivalent of the circuit in Fig. 13.87.

$$V_0 = \frac{V_i}{s+1}(1) \tag{13.219}$$

When v_i is a unit impulse function $\delta(t)$

$$\begin{aligned} v_o &= h(t) \\ &= e^{-t}\, u(t) \end{aligned} \tag{13.220}$$

from which

$$h(\lambda) = e^{-\lambda}\, u(\lambda) \tag{13.221}$$

The impulse response and the folded excitation function is shown in Fig. 13.89.

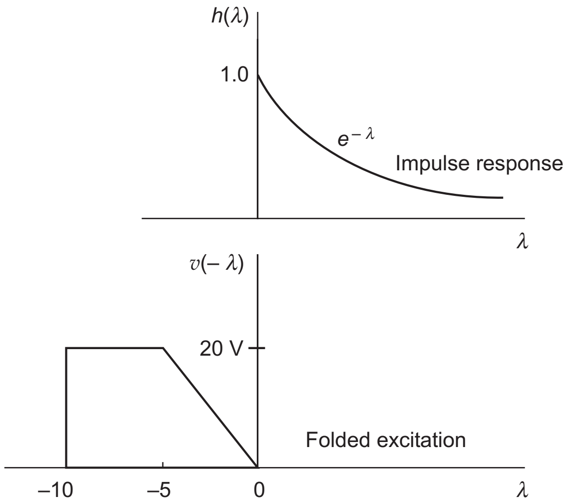


Fig. 13.89

Sliding the bolded excitation function to the right requires breaking the integration into intervals: $0 \le t \le 5$; $5 \le t \le 10$; and $10 \le t \le \infty$. The breaks in the excitation function at 0.5, and 10*s* dictate these break points. Figure 13.90 shows the positioning of the folded excitation for each of these intervals. The analytical expression for v_i in the time interval $0 \le t \le 5$ is

$$v_i = 4t;\ 0 \le t \le 5s \tag{13.222}$$

Hence, the analytical expression for the folded excitation function in the interval $t - 5 \le \lambda \le t$ is

$$v_i\,(t - \lambda) = 4\,(t - \lambda),\ t - 5 \le \lambda \le t \tag{13.223}$$

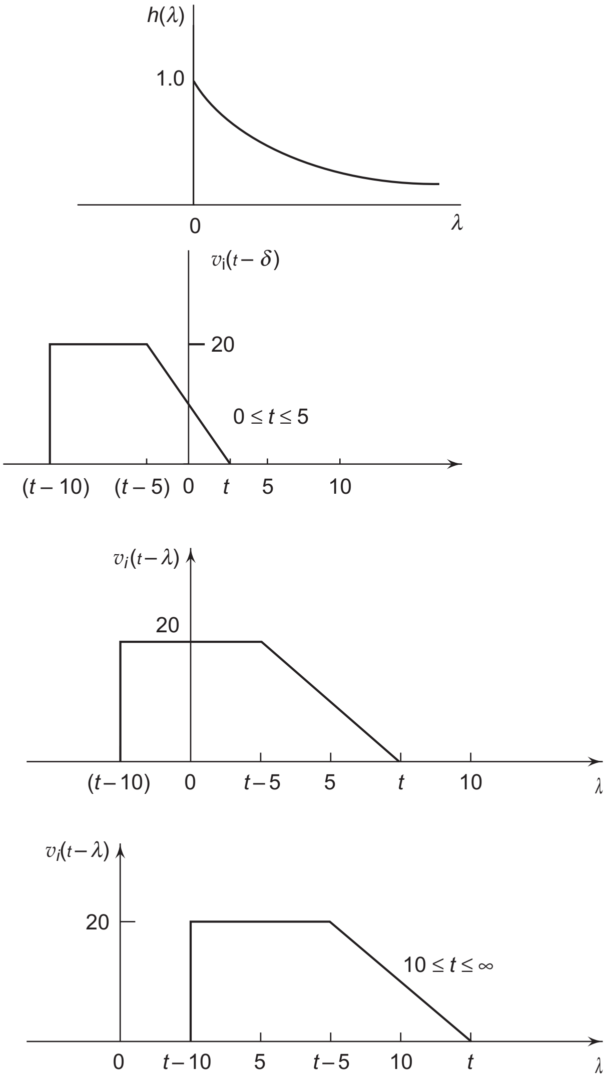


Fig. 13.90

We can now set up the three integral expression for v_0.
For $0 \le t \le 5s$,

$$v_0 = \int_0^t 4(t-\lambda)e^{-\lambda}\, d\lambda$$
$$= 4(e^{-t} + t - 1)\text{ V} \tag{13.224}$$

For $5 \le t \le 10s$,

$$v_0 = \int_0^{t-5} 20e^{-\lambda} d\lambda + \int_{t-5}^{t} 4(t-\lambda)e^{-\lambda} d\lambda$$
$$= 4\,(5 + e^{-t} - e^{-(t-5)})\text{ V} \tag{13.225}$$

For $0 \le t \le \infty\ s$

$$v_0 = \int_{t-10}^{t-5} 20e^{-\lambda} d\lambda + \int_{t-5}^{t} 4(t-\lambda)e^{-\lambda} d\lambda$$
$$= 4\,(e^{-t} - e^{-(t-5)} + 5e^{-(t-10)})\text{ V} \tag{13.226}$$

Table 13.1 Numerical Values of $v_0(t)$

t	v_0	t	v_0	t	v_0
1	1.47	6	18.54	11	7.35
2	4.54	7	19.56	12	2.70
3	8.20	8	19.8	13	0.99
4	12.07	9	19.93	14	0.37
5	16.03	10	19.97	15	0.13

The results are computed for v_0 and tabulated in Table 13.1. The voltage response is shown graphically in Fig. 13.91.

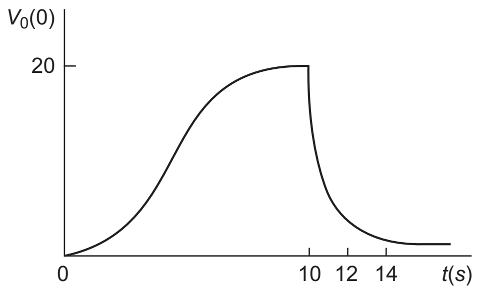


Fig. 13.91

Problem 13.22 For the circuit shown in Fig. 13.92, the sinusiodal source voltage is $v_g = 120\cos(5000\,t + 30°)$ V. Find the steady state expression for V_0.

Solution From the Problem 13.19:

$$H(S) = \frac{1000(s+5000)}{s^2 + 6000s + 25\times 10^6} \tag{13.227}$$

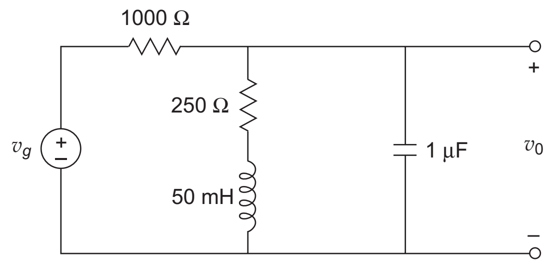


Fig. 13.92

The frequency of the voltage source is 5000 rod/s;
Hence we evaluate $H(S)$ at $H(j5000)$.

$$H(j5000) = \frac{1000(5000 + j5000)}{-25\times10^{-6} + j5000(6000) + 25\times10^{6}}$$

$$= \frac{1+j1}{j6} = \frac{1-j1}{6} = \frac{\sqrt{2}}{6} \angle -45^{\circ}$$

Then the steady state voltage is

$$v_{0ss} = \frac{120\sqrt{2}}{6} \cos(5000t + 30° - 45°)$$

$$= 20\sqrt{2}\cos(5000t - 15°)\text{ V.}$$

Practice Problems

13.1 A 500 Ω resistor, a 16 mH inductor, and a 25 nF capacitor are connected in parallel which is placed in series with a 2000 Ω resistor. Express the impedance of this series combination as a rational function of s.
Ans.: $2000(s + 50{,}000)^2/(s^2 + 80{,}000\, s + 25 \times 10^8)$

13.2 A 1 kΩ resistor is in series with a 500 mH inductor. This series combination is in parallel with a 0.4 μF capacitor. Express the equivalent s-domain impedance of these parallel branches as a rational function.

13.3 The energy stored in the circuit shown is zero at the time when the switch is closed.

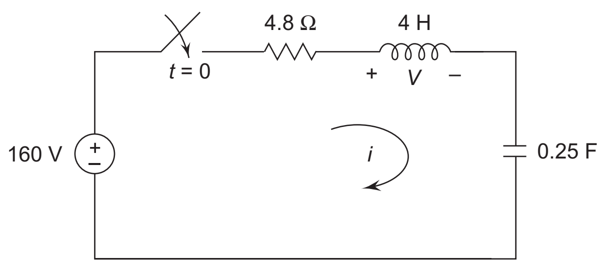


Fig. 13.93

(a) Find the s-domain expression for I.
(b) Find the s-domain expression for i when $t > 0$.
(c) Find the s-domain expression for V.
(d) Find the time-domain expression for v when $t > 0$.
Ans.: (a) $I = 40/(s^2 + 12\,s + 1)$ (b) $i = 50e^{-0.6t}$ sin $0.8t$ A
(c) $V = 16s/(s^2 + 1.2\,s + 1)$ (d) $v = 200\,e^{-0.6t}$ cos $(0.8\,t + 36.87°)$ V

13.4 The dc current and voltage sources are applied simultaneously to the circuit shown. No energy is stored in the circuit at the instant of application.
(a) Derive the s-domain expressions for V_1 and V_2.
(b) For $t > 0$, derive the time-domain expressions for v_1 and v_2.
(c) Calculate $v_1(0+)$ and $v_2(0+)$.
(d) Compute the steady state value of v_1 and v_2.

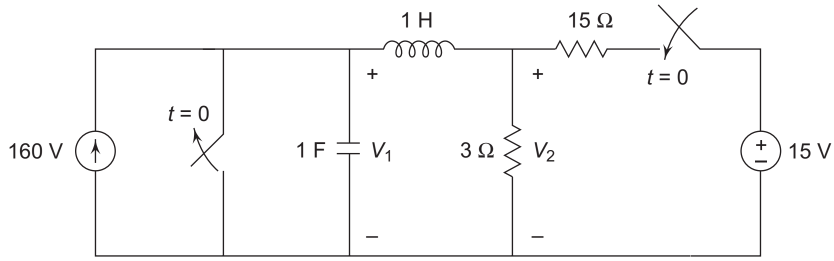


Fig. 13.94

13.5 The energy stored in the circuit shown is zero at the instant the two sources are turned on
(a) Find the component of v for $t > 0$ owing to the voltage source.
(b) Find the component of v for $t > 0$ owing to the current source.
(c) Find the expression for v when $t > 0$.

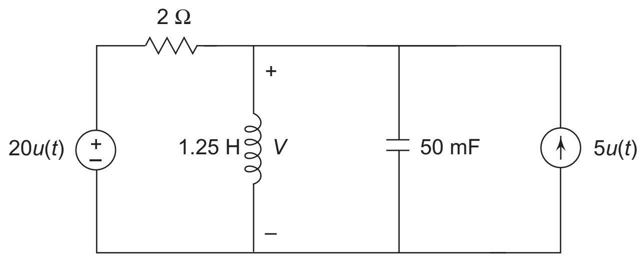


Fig. 13.95

Ans.: (a) $(100/3)\,e^{-2t} - \left(\dfrac{100}{3}\right) e^{-8t}$ V

(b) $\dfrac{50}{3}\,e^{-2t} - \dfrac{50}{3}\,e^{-8t}$ V
(c) $50e^{-2t} - 50\,e^{-8t}$ V.

13.6 In the circuit shown in Fig. 13.96, there is no energy stored at the time the current source turns on. Given that $i_g = 100\,u(t)$ A;

(a) Find $I_0(s)$
(b) Use the initial and final value theorem to find $i_0(0+)$ and $i_0(\infty)$.
(c) Determine if the results obtained in (b) agree with known circuit behaviour.
(d) Find $i_0(t)$.

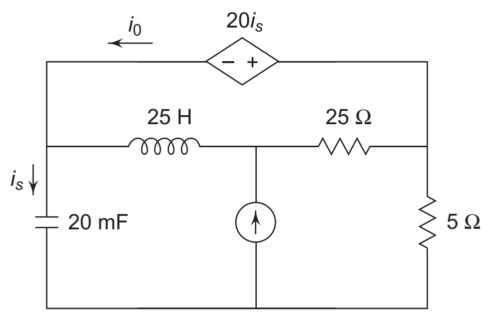


Fig. 13.96

13.7 Derive the numerical expression for the transfer function v_0/I_g for the circuit shown.

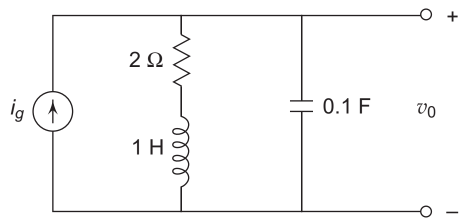


Fig. 13.97

Ans.: $H(S) = 10(s + 2)/s^2 + 2s + 10$

13.8 There is no energy stored in the circuit seen in Fig. 13.98 at the time the two sources are energized.
(a) Use the principle of superposition to find V_0.
(b) Find v_0 for $t > 0$.

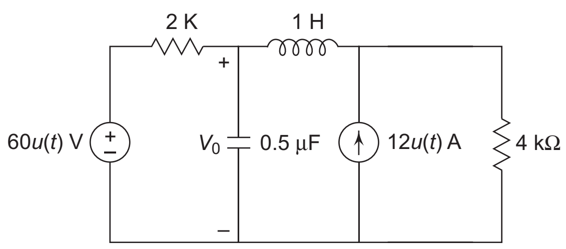


Fig. 13.98

13.9 Find (a) the unit step and (b) the unit impulse response of the circuit shown in Fig. 13.99.

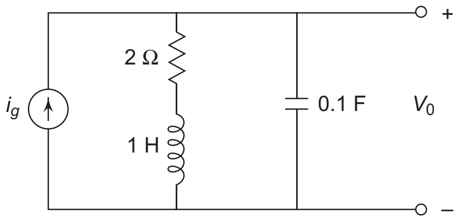


Fig. 13.99

Ans.: (a) $2 + \dfrac{10}{3} e^{-t} \cos(3t - 126.87°)$ V

(b) $10.54\, e^{-t} \cos(3t - 18.43°)$ V.

13.10 The unit impulse response of a circuit is

$$v_0(t) = 10{,}000\, e^{-70t} \cos(240\, t + \theta)\, u(t) \text{ V}$$

where $\tan\theta = \dfrac{7}{24}$

(a) Find the transfer function of the circuit.

(b) Find the unit step response of the circuit.

13.11 A rectengular voltage pulse $v_i = [u(t) - u(t-1)]$ V is applied to the circuit shown in Fig. 13.100. Use convolution to find v_0.

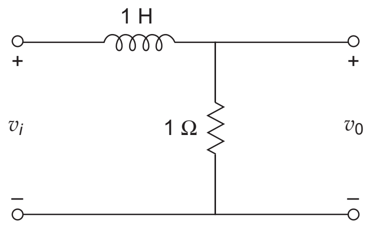


Fig. 13.100

Ans.: $v = (1 - e^{-t})$V $0 \le t \le 1$

$v_0 = (e - 1)e^{-t}$ V $1 \le t \le \infty$

13.12 Interchange the inductor and resistor in the Problem 13.12 and again use the convolution integral to find v_0.

13.13 The current source in the circuit shown is delivering 10 cos $4t$ A. Use the transfer function to compute the steady-state expression for v_0.

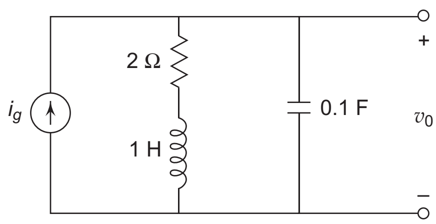


Fig. 13.101

Ans.: 44.7 cos $(4t - 63.43°)$ V.

13.14 There is no energy stored in the circuit shown in Fig. 13.102 at the time the impulse voltage is applied. Find $v_0(t)$ for $t \geq 0$.

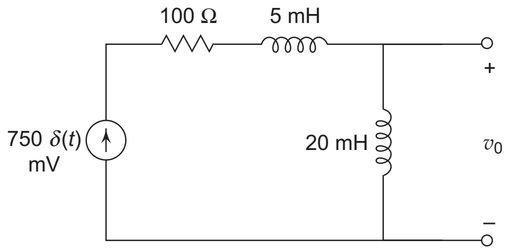


Fig. 13.102

13.15 The switch in the circuit shown in Fig. 13.103 has been in position *a* for a long time. At $t = 0$, the switch moves to position *b*. Compute (a) $v_1(0)$ (b) $v_1(0–)$ (c) $v_3(0–)$ (d) $i(t)$ (e) v_1 (0+) (f) v_2 (0+) (g) v_3 (0+).

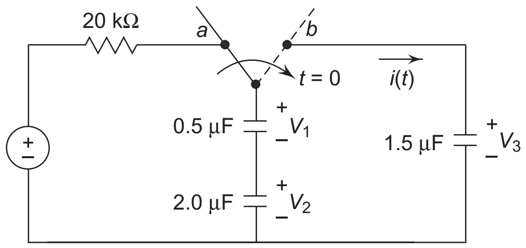


Fig. 13.103

Ans.: (a) 80 V (b) 20 V (c) 0 V (d) 32 $\delta(t)$ μA (e) 16 V (f) 4 V (g) 20 V.

Objective-type Questions

1. An inductor in the *s*-domain consists of
   (a) Current source in series with an inductor
   (b) Voltage source in parallel with an inductor
   (c) Voltage source of LI_0 in series with an inductor
   (d) Current source I_0/s in series with an inductor
2. A capacitor in the *s*-domain consists of
   (a) Current source CV_0 in parallel with capcitor
   (b) Current source in series with capacitor
   (c) Voltage source $\dfrac{V_0}{s}$ in parallel with capacitor
   (d) Voltage source CV_0 in parallel with capacitor
3. The current in the circuit when the switch is closed at $t = 0$.
   (a) $10\ e^{-100t}$ (b) $0.01\ e^{-1000\ t}$
   (c) $0.1\ e^{-1000\ t}$ (d) $10\ e^{-0.1\ t}$

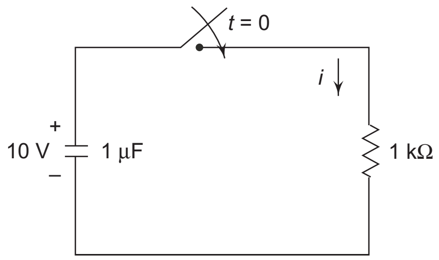


Fig. 13.104

4. The initial voltage across the capacitor when the switch s is opened at $t = 0$.

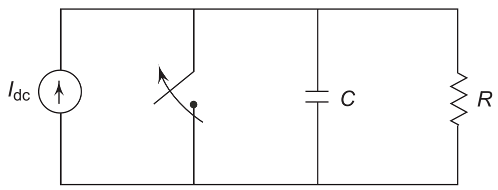


Fig. 13.105

(a) zero (b) $C.\frac{I_{\text{dc}}}{s}$ (d) $\frac{1}{CS}I_{\text{dc}}$ (d) $CS\ (I_{\text{dc}})$

5. Thevenins equivalent circuit across terminals ab.

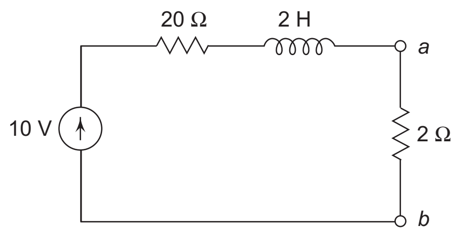


Fig. 13.106

(a) The voltage source 10 V with $(20 + 2s)$ impedance in series
(b) The voltage source 10 V in parallel with $(20 + 2s)\ \Omega$ in parallel
(c) The voltage source $\frac{10}{s}$ in series with an impedance of $(20 + 2s)\ \Omega$
(d) The voltage source $\frac{10}{s}$ in series with an impedance of $22\ \Omega$

6. The transfer function of multiple independent sources can easily be obtained by
(a) Superposition theorem
(b) Thevenin's theorem
(c) Norten's theorem
(d) Reciprocity

7. In the circuit shown in Fig.13.107, the current is defined as the response signal, then the transfer function

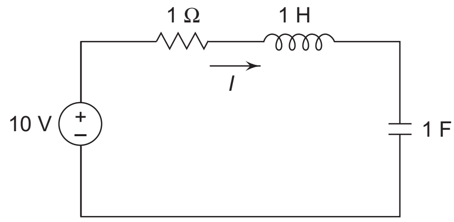


Fig. 13.107

(a) $\dfrac{10^{-6}s}{10^{-12}s^2+s+1}$ (b) $\dfrac{s}{s^2+s+1}$ (c) $\dfrac{s}{s^2+1}$ (d) $\dfrac{s}{s+1}$

8. The circuit is driven by an unit impulse source, then the response equals to
   (a) transfer function (b) one
   (c) zero (d) Inverse of transfer function
9. If the input of a circuit is represented by series of impulse functions, the response consists of
   (a) sum of the series of uniformly delayed impulse responses
   (b) sum of the series of responses
   (c) one
   (d) zero
10. For physically realizable circuit, impulse response is
    (a) zero for $t < 0$ (b) zero for $t > 0$
    (c) one for $t < 0$ (d) infinite for $t > 0$
11. The instantaneous current in an inductor when an impulse voltage V_0 applied to the terminals of an inductor

    (a) zero (b) unity (c) $\dfrac{V_0}{L}$ (d) $\dfrac{V_0}{L}\delta(t)$

Chapter 14

Network Functions

14.1 SINGULARITY FUNCTIONS

So far we have discussed the response of networks to simple waveforms, such as dc, exponential or sinusoidal. Another class of signals is defined by singularity functions. These are step, ramp and impulse functions. These functions are divided into the following two groups.

1. *Non-recurring type* These functions appear for a particular time interval and become zero for all other times, and
2. *Recurring type* These functions appear for all time, that is, the waveform exists for $t > 0$.

Singularity functions are continuous time functions, and their derivatives, except one, are also continuous. Singularity functions can be obtained from one another by successive differentiation or integration. Our analysis of general networks can be enhanced by the utilisation of singularity functions.

14.2 UNIT FUNCTIONS

(a) *Unit step function* This function has already been discussed in the preceding chapter. It is defined as one that has magnitude of one for time greater than zero, and has zero magnitude for time less than zero.

A unit step function is defined mathematically as

$$\begin{aligned} u(t) &= 0 \text{ for } t < 0 \\ &= 1 \text{ for } t > 0 \end{aligned}$$

The function is represented as shown in Fig. 14.1

The Laplace transform of the unit step function is

$$\mathscr{L}[f(t)] = \mathscr{L}[u(t)] = \int_0^{\infty} u(t)e^{-st}dt$$

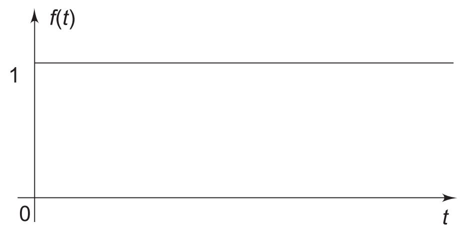


Fig. 14.1

$$= \int_0^{\infty} 1e^{-st}dt$$

$$= \left[-\frac{e^{-st}}{s}\right]_0^{\infty} = \frac{1}{s}$$

(b) *Unit ramp function* If the unit step function is integrated with respect to time t, then the unit ramp function results. It is symbolised by $r(t)$. A unit ramp function increases linearly with time. A unit ramp function may be defined mathematically as

$$r(t) = \int_{-\infty}^{t} u(t)\,dt$$

$$= \int_{-\infty}^{0} u(t) + \int_0^t u(t)\,dt$$

$$= 0 + \int_0^t u(t)\,dt = t$$

$\therefore$ $r(t) = 0$ for $t < 0$

$= t$ for $t > 0$

The function is represented as shown in Fig. 14.2.

The Laplace transform of the unit ramp function is

$$\mathscr{L}[f(t)] = \mathscr{L}[r(t)] = \mathscr{L}\left[\int_0^t u(t)\,dt\right]$$

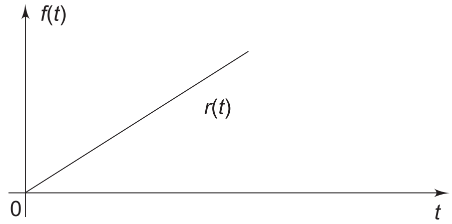


Fig. 14.2

$$= \int_0^\infty te^{-st}\, dt$$

$$\mathscr{L}[r(t)] = \frac{1}{s^2}$$

(c) *Unit impulse function* If a unit step function $u(t)$ is differentiated with respect to t, the derivative is zero for time t greater than zero, and is infinite for time t equal to zero. Mathematically, the function is defined as

$$\delta(t) = 0 \text{ for } t \neq 0$$

and
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

where the symbol $\delta(t)$ (delta) is used to represent the unit impulse. An impulse of unity amplitude occurring at $t = 0$ gives that it has an area 'δ' equal to unity. The unit impulse function is represented as shown in Fig. 14.3.

Fig. 14.3

The Laplace transform of the unit impulse function is

$$\mathscr{L}[f(t)] = \mathscr{L}[\delta(t)] = \mathscr{L}\left[\frac{d}{dt}u(t)\right] = s\,\mathscr{L}[u(t)] = s \times \frac{1}{s} = 1$$

Therefore $\mathscr{L}[\delta(t)] = 1$

(d) *Unit doublet function* If a unit impulse function $\delta(t)$ is differentiated with respect to t, we get

$$\delta'(t) = \frac{d}{dt}[\delta(t)] = +\infty \text{ and } -\infty \text{ for } t = 0$$

$$= 0 \text{ for } t \neq 0$$

This function is called unit doublet, where $\delta'(t)$ is the symbol used to represent the unit doublet.

The unit doublet is shown in Fig. 14.4.

The Laplace transform of the unit doublet is

$$\mathscr{L}[\delta'(t)] = \mathscr{L}\left[\frac{d}{dt}\delta(t)\right]$$

where $\delta(t)$ is a unit impulse occurring at $t = 0$.

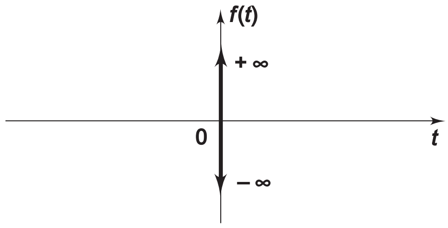


Fig. 14.4

$$\mathscr{L}\left[\frac{d}{dt}\,\delta(t)\right] = s\{\mathscr{L}\,[\delta(t)]\}$$

$$= s \times 1 = s$$

$$\therefore \qquad \mathscr{L}\,[\delta'(t)] = s$$

14.3 SHIFTER FUNCTIONS

Consider unit functions such as unit step, ramp and impulse functions as discussed in Section 14.2. If these functions are displaced by 'a' second or delayed by 'a' second then these functions are said to be delayed functions. These are represented as shown in Fig. 14.5.

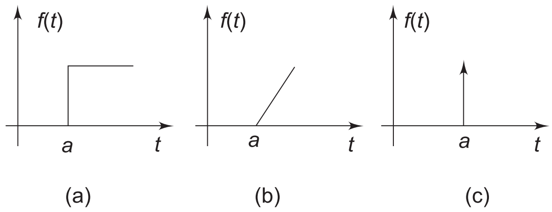


Fig. 14.5

The delayed unit step function shown in Fig. 14.5(a) is defined as

$$u\,(t - a) = 0 \text{ for } t < a$$

$$= 1 \text{ for } t > a$$

The delayed unit ramp function shown in Fig. 14.5(b) is defined as

$$r\,(t - a) = 0 \text{ for } t < a$$

$$= t \text{ for } t > a$$

The delayed unit impulse function is defined as

$$\delta\,(t - a) = 0 \text{ for } t \neq a$$

and $$\int_{-\infty}^{\infty} \delta(t - a)\,dt = 1$$

14.4 GATE FUNCTION

By the use of step functions, any pulse of unit height can be realised. The pulse of width a can be generated by combining unit step function $u(t)$ and delayed inverted unit step function by a time interval a as shown in Fig. 14.6.

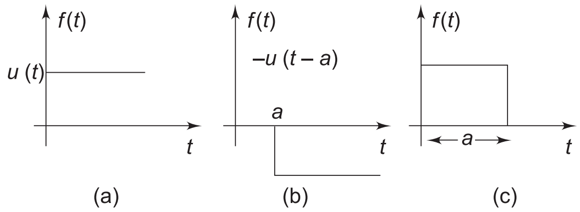


Fig. 14.6

In Fig. 14.6(a), the unit step function $u(t)$ combined with $-u(t-a)$, the inverted unit step function, delayed by a results in the waveform shown in Fig. 14.6(c).

$$G(T) = u(t) - u(t-a)$$

The gate function is only for $0 < t < a$.

A periodic pulse train with pulse width a and pulse repetition period T_1 may be generated by combining a sequence of positive unit step functions $u(t)$, $u(t-T_1)$, $u(t-2T_1)$..., with negative unit step functions $u(t-a)$, $u(t-T_1-a)$, $u(t-2T_1-a)$..., as shown in Fig. 14.7.

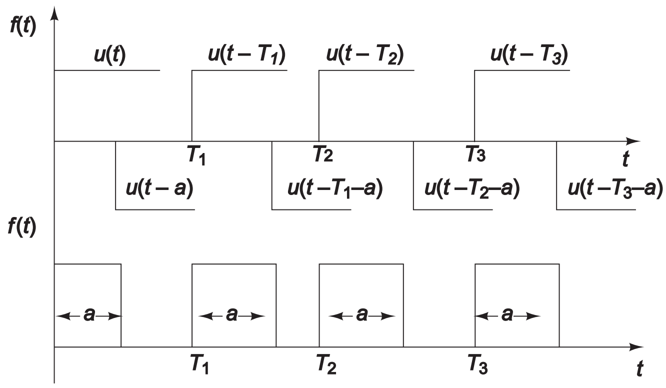


Fig. 14.7

Therefore, the periodic pulses may be defined as,

$$f(t) = u(t) - u(t-a) + u(t-T_1) - u(t-T_1-a) + \ldots$$

14.5 NETWORK FUNCTIONS

Network functions give the relation between the transform of the excitation to the transform of the response. Consider the network shown in Fig. 14.8.

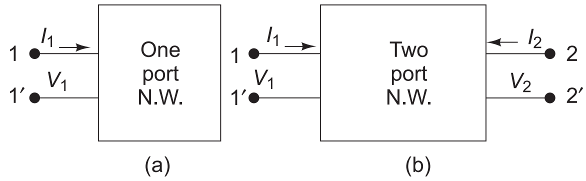


Fig. 14.8

For the network shown in Fig. 14.8(a), only one voltage and one current exist and only one network function is defined. It constitutes of one pair of terminals called a port. Generally, a driving source is connected to the pair of terminals. For the two terminal pair network shown in Fig. 14.8(b), two currents and two voltages must exist. Normally in Fig. 14.8(b), 1–1′ and 2–2′ are called ports. Hence, it is called two-port network. If the driving source is connected across 1–1′, the load is connected across 2–2′. Otherwise, if the source is connected across 2–2′, the output is taken across 1–1′.

14.6 TRANSFER FUNCTIONS OF TWO-PORT NETWORK

For a one-port network, the driving point impedance or impedance of the network is defined as

$$Z(s) = \frac{V(s)}{I(s)}$$

The reciprocal of the impedance function is the driving point admittance function, and is denoted by $Y(s)$.

For the two-port network without internal sources, the driving point impedance function at port 1–1′ is the ratio of the transform voltage at port 1–1′ to the transform current at the same port.

$$\therefore \qquad Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$$

Similarly, the driving point impedance at port 2–2′ is the ratio of transform voltage at port 2–2′ to the transform current at the same port.

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

For the two-port network, the driving point admittance is defined as the ratio of the transform current at any port to the transform voltage at the same port.

Therefore $$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$$

or $$Y_{22}(s) = \frac{I_2(s)}{V_2(s)},$$ which is the driving point admittance.

The four other network functions are called transfer functions. These functions give the relation between voltage or current at one port to the voltage or current at the other port as shown hereunder.

(i) *Voltage transfer ratio* This is the ratio of voltage transform at one port to the voltage transform at the other port, and is denoted by $G(s)$

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)}$$

and
$$G_{12}(s) = \frac{V_1(s)}{V_2(s)}$$

(ii) *Current transfer ratio* This is the ratio of current transform at one port to current transform at other port, and is denoted by $\alpha(s)$

$$\alpha_{12}(s) = \frac{I_1(s)}{I_2(s)}$$

and
$$\alpha_{21}(s) = \frac{I_2(s)}{I_1(s)}$$

(iii) *Transfer impedance* It is defined as the ratio of voltage transform at one port to the current transform at the other port, and is denoted by $Z(s)$.

$\therefore$
$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$$

and
$$Z_{12}(s) = \frac{V_1(s)}{I_2(s)}$$

(iv) *Transfer admittance* It is defined as the ratio of current transform at one port to the current transform at the other port, and is denoted by $Y(s)$.

$$Y_{21}(s) = \frac{I_2(s)}{V_1(s)}$$

and
$$Y_{12}(s) = \frac{I_1(s)}{V_2(s)}$$

The above network functions are found by forming the system of equations using node or mesh analysis, and taking the transforms of equations by setting the initial conditions to zero and solving for ratio of the response to excitation.

14.7 POLES AND ZEROS

In general, the network function $N(s)$ may be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are the coefficients of the polynomials $P(s)$ and $Q(s)$; they are real and positive for a passive network. If the numerator and denominator of polynomial $N(s)$ are factorised, the network function can be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 (s - z_1)(s - z_2) \ldots (s - z_n)}{b_0 (s - p_1)(s - p_2) \ldots (s - p_m)}$$

where $z_1, z_2, ..., z_n$ are the n roots for $P(s) = 0$
and $p_1, p_2, ..., P_m$ are the m roots for $Q(s) = 0$
and $a_0/b_0 = H$ is a constant called the *scale factor*.

$z_1, z_2,..., z_n$ in the transfer function are called zeros, and are denoted by 0. Similarly, $p_1, p_2,..., p_m$ are called poles, and are denoted by $\times$. The network function $N(s)$ becomes zero when s is equal to anyone of the zeros. $N(s)$ becomes infinite when s is equal to any one of the poles. The network function is completely defined by its poles and zeros. If the poles or zeros are not repeated, then the function is said to be having simple poles or simple zeros. If the poles or zeros are repeated, then the function is said to be having multiple poles multiple zeros. When $n > m$, then $(n - m)$ zeros are at $s = \infty$, and for $m > n$, $(m - n)$ poles are at $s = \infty$.

Consider, the network function

$$N(s) = \frac{(s+1)^2\,(s+5)}{(s+2)\,(s+3+j2)\,(s+3-j2)}$$

that has double zeros at $s = -1$ and a zero at $s = -5$; and three finite poles at $s = -2$, $s = -3 + j2$, and $s = -3 - j2$ as shown in Fig. 14.9.

The network function is said to be stable when the real parts of the poles and zeros are negative. Otherwise, the poles and zeros must lie within the left half of the s-plane.

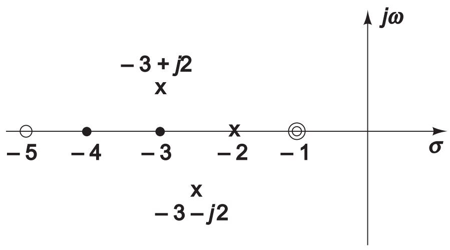


Fig. 14.9

14.8 NECESSARY CONDITIONS FOR DRIVING POINT FUNCTION

The restrictions on pole and zero locations in the driving point function with common factors in $P(s)$ and $Q(s)$ cancelled are listed below.

1. The coefficients in the polynomials $P(s)$ and $Q(s)$ of network function $N(s) = P(s)/Q(s)$ must be real and positive.
2. Complex or imaginary poles and zeros must occur in conjugate pairs.
3. (a) The real parts of all poles and zeros must be zero, or negative.
   (b) If the real part is zero, then the pole and zero must be simple.
4. The polynomials $P(s)$ and $Q(s)$ may not have any missing terms between the highest and the lowest degrees, unless all even or all odd terms are missing.
5. The degree of $P(s)$ and $Q(s)$ may differ by zero, or one only.
6. The lowest degree in $P(s)$ and $Q(s)$ may differ in degree by at the most one.

14.9 NECESSARY CONDITIONS FOR TRANSFER FUNCTIONS

The restrictions on pole and zero location in transfer functions with common factors in $P(s)$ and $Q(s)$ cancelled are listed below.

1. (a) The coefficients in the polynomials $P(s)$ and $Q(s)$ of $N(s) = P(s)/Q(s)$ must be real.
   (b) The coefficients in $Q(s)$ must be positive, but some of the coefficients in $P(s)$ may be negative.
2. Complex or imaginary poles and zeros must occur in conjugate pairs.
3. The real part of poles must be negative, or zero. If the real part is zero, then the pole must be simple.
4. The polynomial $Q(s)$ may not have any missing terms between the highest and the lowest degree, unless all even or all odd terms are missing.
5. The polynomial $P(s)$ may have missing terms between the lowest and the highest degree.
6. The degree of $P(s)$ may be as small as zero, independent of the degree of $Q(s)$.
7. (a) For the voltage transfer ratio and the current transfer ratio, the maximum degree of $P(s)$ must equal the degree of $Q(s)$.
   (b) For transfer impedance and transfer admittance, the maximum degree of $P(s)$ must equal the degree of $Q(s)$ plus one.

14.10 TIME DOMAIN RESPONSE FROM POLE ZERO PLOT

For the given network function, a pole zero plot can be drawn which gives useful information regarding the critical frequencies. The time domain response can also be obtained from pole zero plot of a network function. Consider an array of poles shown in Fig. 14.10.

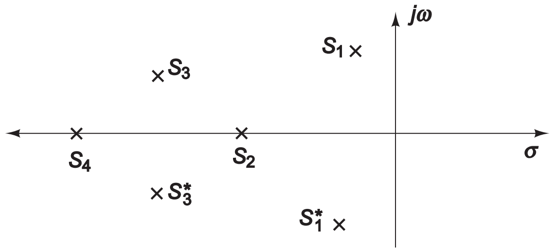


Fig. 14.10

In Fig. 14.10 s_1 and s_3 are complex conjugate poles, whereas s_2 and s_4 are real poles. If the poles are real, the quadratic function is

$$s^2 + 2\delta\omega_n s + \omega_n^2 \text{ for } \delta > 1$$

where δ is the damping ratio and ω_n is the undamped natural frequency.

The roots of the equation are

$$s_2, s_4 = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}\ ;\ \delta > 1$$

For these poles, the time domain response is given by

$$i(t) = k_2 e^{s_2 t} + k_4 e^{s_4 t}$$

The response due to pole s_4 dies faster compared to that of s_2 as shown in Fig. 14.11.

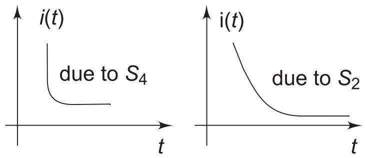


Fig. 14.11

s_1 and s_3 constitute complex conjugate poles. If the poles are complex conjugate, then the quadratic function is

$$s^2 + 2\delta\omega_n s + \omega_n^2 \text{ for } \delta < 1$$

The roots are $s_1, s_1^* = -\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2}$ for $\delta < 1$

For these poles, the time domain response is given by

$$i(t) = k_1 e^{-\delta\omega_n t + j\left(\omega_n\sqrt{1-\delta^2}\right)t} + k_1^* e^{-\delta\omega_n t - j\left(\omega_n\sqrt{1-\delta^2}\right)t}$$

$$= ke^{-\delta\omega_n t}\sin\left(\omega_n\sqrt{1-\delta^2}\right)t$$

From the above equation, we can conclude that the response for the conjugate poles is damped sinusoid. Similarly, s_3, s_3^* are also a complex conjugate pair. Here the response due to s_3 dies down faster than that due to s_1 as shown in Fig. 14.12.

Consider a network having transfer admittance $Y(s)$. If the input voltage $V(s)$ is applied to the network, the corresponding current is given by

$$I(s) = V(s)\,Y(s) = \frac{P(s)}{Q(s)}$$

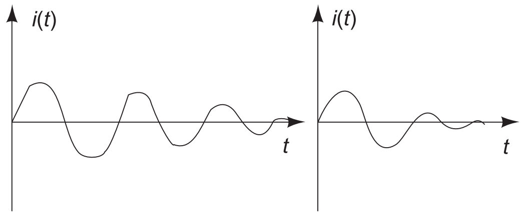


Fig. 14.12

This may be taken as

$$I(s) = H\,\frac{(s - s_a)\,(s - s_b)\ldots(s - s_n)}{(s - s_1)\,(s - s_2)\ldots(s - s_m)}$$

where H is the scale factor.

By taking the partial fractions, we get

$$I(s) = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2} + \ldots + \frac{k_m}{s - s_m}$$

The time domain response can be obtained by taking the inverse transform

$$i(t) = \mathscr{L}^{-1}\left[\frac{k_1}{s - s_1} + \frac{k_2}{s - s_2} + \ldots + \frac{k_m}{s - s_m}\right]$$

Any of the above coefficients can be obtained by using Heavisides method.

To find the coefficient k_l

$$k_l = H\left[\frac{(s - s_a)\,(s - s_b)\ldots(s - s_n)}{(s - s_1)\,(s - s_2)\ldots(s - s_m)}\right](s - s_l)\Bigg|_{s = s_l}$$

Here s_l, s_m, s_n are all complex numbers, the difference of $(s_l - s_n)$ is also a complex number.

$$\therefore \qquad (s_l - s_n) = M_{ln}\, e^{j\phi_{ln}}$$

Hence $k_l = H\,\dfrac{M_{la}\; M_{lb}\,\ldots M_{ln}}{M_{l1}\; M_{l2}\ldots M_{lm}} \times e^{j(\phi_{la} + \phi_{lb} + \ldots + \phi_{ln}) - (\phi_{l1} + \phi_{l2} + \ldots + \phi_{lm})}$

Similarly, all coefficients k_1, k_2, …, k_m may be obtained, which constitute the magnitude and phase angle.

The residues may also be obtained by pole zero plot in the following way.

1. Obtain the pole zero plot for the given network function.
2. Measure the distances M_{la}, M_{lb}, …, M_{ln} of a given pole from each of the other zeros.
3. Measure the distances M_{l1}, M_{l2}, …, M_{lm} of a given pole from each of the other poles.
4. Measure the angle ϕ_{la}, ϕ_{lb}, ..., ϕ_{ln} of the line joining that pole to each of the other zeros.
5. Measure the angle ϕ_{l1}, ϕ_{l2}, ..., ϕ_{lm} of the line joining that pole to each of the other poles.
6. Substitute these values in required residue equation.

14.11 AMPLITUDE AND PHASE RESPONSE FROM POLE ZERO PLOT

The steady state response can be obtained from the pole zero plot, and it is given by

$$N(j\omega) = M(\omega)e^{j\phi(\omega)}$$

where $M(\omega)$ is the amplitude

$\phi(\omega)$ is the phase

These amplitude and phase responses are useful in the design and analysis of network functions. For different values of ω, corresponding values of $M(\omega)$ and $\phi(\omega)$ can be obtained and these are plotted to get amplitude and phase response of the given network.

14.12 STABILITY CRITERION FOR ACTIVE NETWORK

Passive networks are said to be stable only when all the poles lie in the left half of the s-plane. Active networks (containing controlled sources) are not always stable. Consider transformed active network shown in Fig. 14.13.

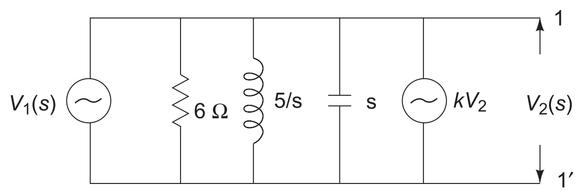


Fig. 14.13

By applying Millman Theorem, we get

$$V_2(s) = \frac{V_1(s) + kV_2(s)}{6 + 5/s + s}$$

$$= \frac{s\,[V_1(s) + kV_2(s)]}{s^2 + 6s + 5}$$

$$V_2(s)\,[s^2 + 6s + 5] - ksV_2(s) = sV_1(s)$$

$$V_2(s)\,[s^2 + (6 - k)s + 5] = sV_1(s)$$

$$\therefore \quad \frac{V_2(s)}{V_1(s)} = \frac{s}{s^2 + (6 - k)s + 5}$$

From the above transformed equation, the poles are dependent upon the value of k.

The roots of the equation are

$$s = \frac{-(6 - k) \pm \sqrt{(6 - k)^2 - 4 \times 5}}{2}$$

For $k = 0$, the poles are at $-1, -5$, which lie on the left half of the s-plane. As k increases, the poles move towards each other and meet at a point $\sqrt{(6 - k)^2 - 20} = 0$, when $k = 1.53$ or 10.47. The root locus plot is shown in Fig. 14.14.

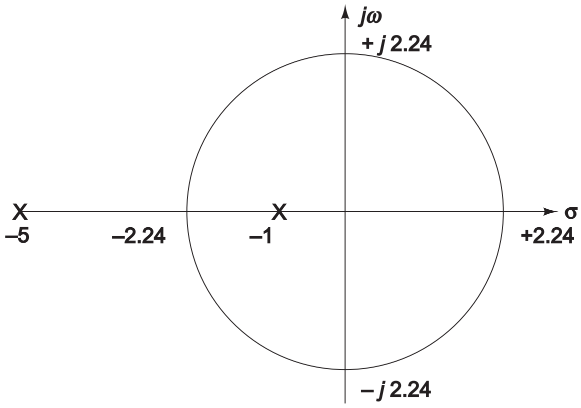


Fig. 14.14

The root locus is obtained from the characteristic equation $s^2 + (6 - k)s + 5 = 0$. As the value of k increases beyond 1.53, the locus of root is a circle. The poles are located on the imaginary axis at $\pm j2.24$ for $k = 6$. At -2.24, poles are coincident for $k = 1.53$ while at $+2.24$, poles are coincident for $k = 10.47$. When $k > 10.47$, the poles again lie on the real axis but remain on the right half of the s-plane, one pole moving towards the origin and the other moving towards infinity. From this we can conclude, as long as k is less than 6, the poles lie on the left half of the s-plane and the system is said to be stable. For $k = 6$, the poles lie on the imaginary axis and the system is oscillatory in nature. For values of k greater than 6, the poles lie on the right half of the s-plane. Then the system is said to be unstable.

14.13 ROUTH CRITERIA

The locations of the poles gives us an idea about stability of the active network. Consider the denominator polynomial

$$Q(s) = b_0 s^m + b_1 s^{m-1} + \ldots + b_m \tag{14.1}$$

To get a stable system, all the roots must have negative real parts. There should not be any positive or zero real parts. This condition is not sufficient.

Let us consider the polynomial

$$s^3 + 4s^2 + 15s + 100 = (s + 5)(s^2 - s + 20)$$

In the above polynomial, though the coefficients are positive and real, the two roots have positive real parts. From this we conclude that the coefficients of $Q(s)$ being positive and real is not a sufficient condition to get a stable system. Therefore, we have to seek the condition for stability which is necessary and sufficient.

Consider the polynomial $Q(s) = 0$. After factorisation, we get

$$b_0 (s - s_1)(s - s_2) \ldots (s - s_m) = 0 \tag{14.2}$$

On multiplication of these factors, we get

$$\begin{aligned} Q(s) &= b_0 s^m - b_0(s_1 + s_2 + \ldots + s_m)s^{m-1} \\ &\quad + b_0(s_1 s_2 + s_2 s_3 + \ldots)\, s^{m-2} \\ &\quad + b_0\,(-1)^m\,(s_1 s_2 \ldots s_m) = 0 \end{aligned} \tag{14.3}$$

Equating the coefficients of Eqs 14.1 and 14.3, we have

$$\frac{b_1}{b_0} = -\,(s_1 + s_2 + \ldots + s_m) \tag{14.4}$$

$$= -\text{ sum of the roots}$$

$$\frac{b_2}{b_0} = 1(s_1 s_2 + s_2 s_3 + \ldots) \tag{14.5}$$

$$= \text{sum of the products of the roots taken two at a time}$$

$$\frac{b_3}{b_0} = -\,(s_1\, s_2\, s_3 + s_2\, s_3\, s_4 + \ldots) \tag{14.6}$$

$$= -\text{ sum of the products of the roots taken three at a time.}$$

$$(-1)^m\,\frac{b_m}{b_0} = (s_1\, s_2\, s_3 \ldots s_m) = \text{product of the roots} \tag{14.7}$$

If all the roots have negative real parts, then from the above equations it is clear that all the coefficients must have the same sign. This condition is not sufficient due to the fact that the zero value of a coefficient involves cancellation, which requires some root to have positive real parts.

The Routh criterion for stability is discussed below. Consider a polynomial

$$Q(s) = b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ldots + b_m$$

Taking first row coefficients and second row coefficients separately, we have

$$\begin{array}{cccc} b_0 & b_2 & b_4 & \ldots \\ b_1 & b_3 & b_5 & \ldots \end{array}$$

Now we complete the Routh array as follows.

For $m = 5$

$$\begin{array}{c|ccc} s^5 & b_0 & b_2 & b_4 \\ s^4 & b_1 & b_3 & b_5 \\ s^3 & c_1 & c_2 & \\ s^2 & d_1 & d_2 & \\ s^1 & e_1 & & \\ s^0 & f_1 & & \end{array}$$

where $c_1, c_2, d_1, d_2, e_1, f_1$ are determined by the algorithm given below.

$$c_1 = \frac{\begin{matrix} b_0 & b_2 \\ b_1 & b_3 \end{matrix}}{b_1} = \frac{b_1 b_2 - b_0 b_3}{b_1}$$

$$c_2 = \frac{\begin{matrix} b_0 & b_4 \\ b_1 & b_5 \end{matrix}}{b_1} = \frac{b_1 b_4 - b_0 b_5}{b_1}$$

$$d_1 = \frac{\begin{matrix} b_1 & b_3 \\ c_1 & c_2 \end{matrix}}{c_1} = \frac{c_1 b_3 - b_1 c_2}{c_1}$$

$$d_2 = \frac{\begin{matrix} b_1 & b_5 \\ c_1 & 0 \end{matrix}}{c_1} = \frac{b_5 c_1 - 0}{c_1}$$

$$e_1 = \frac{\begin{matrix} c_1 & c_2 \\ d_1 & d_2 \end{matrix}}{d_1} = \frac{c_2 d_1 - c_1 d_2}{d_1}$$

$$f_1 = \frac{\begin{matrix} d_1 & d_2 \\ e_1 & 0 \end{matrix}}{e_1} = \frac{d_2 e_1 - 0}{e_1}$$

In order to find out the element in kth row and jth column, it is required to know the four elements. These elements in the row $(k-1)$ and row $(k-2)$ just above the elements are in column 1 of the array and $(J+1)$ column of the array. The product of the elements joined by a line with positive slope has positive sign while the product of elements joined with a line with negative slope has a negative sign. The difference of these products is divided by the element of column 1 and row $(k-1)$. The above process is repeated till $m+1$ rows are found in the Routh array.

According to the Routh-Hurwitz theorem, the number of changes in the sign of the first column to the right of the vertical line in an array moving from top to bottom is equal to the number of roots of $Q(s) = 0$ with positive real parts. To get a stable system, the roots must have negative real parts.

According to the Routh-Hurwitz criterion, the system is stable, if and only if, there are no changes in signs of the first column of the array. This requirement is, both the necessary and sufficient condition for stability.

Additional Solved Problems

Problem 14.1 For the circuit shown in Fig. 14.15, determine the curent $i(t)$ when the switch is closed at $t = 0$. Assume that the initial current in the inductor is zero.

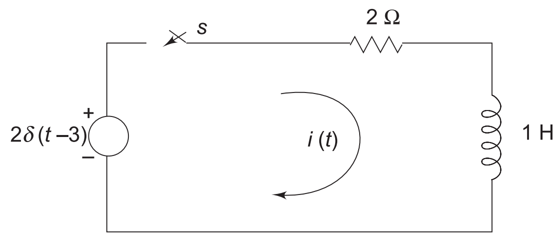


Fig. 14.15

Solution By applying Kirchhoff's laws to the circuit

$$2i(t) + 1\,\frac{di}{dt} = 2\delta(t-3)$$

Taking Laplace transform on both sides, we get

$$2I(s) + 1[sI(s) - i(0)] = 2e^{-3s}$$

Since the initial current through inductor is zero,

$$i(0) = 0$$

The equation becomes

$$2I(s) + 2I(s) = 2e^{-3s}$$

$$I(s)\,[s+2] = 2e^{-3s}$$

$$\therefore \qquad I(s) = \frac{2e^{-3s}}{s+2}$$

Taking inverse transform, we get

$$i(t) = 2e^{-2(t-3)}\,u(t-3)$$

Problem 14.2 For the circuit shown in Fig. 14.16, determine the current $i(t)$ when the switch is closed at $t = 0$. Assume that the initial charge on the capacitor is zero.

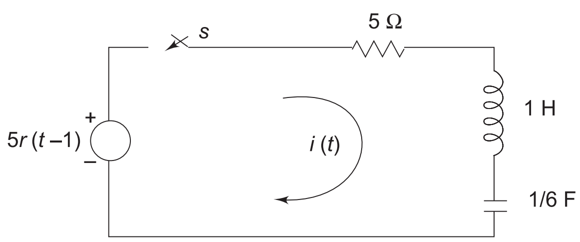


Fig. 14.16

Solution By applying Kirchhoff's law to the circuit, we have

$$5i(t) + 1\frac{di}{dt} + 6\int i\,dt = 5r\,(t-1)$$

Taking Laplace transforms on both sides, we get

$$5I(s) + 1[sI(s) - i(0)] + 6\left[\frac{I(s)}{s} + \frac{q(0)}{s}\right] = \frac{5e^{-s}}{s^2}$$

Since the initial current in the inductor and initial charge on the capacitor is zero

$$i(0) = 0,\ q(0) = 0$$

Therefore, the above equation becomes

$$I(s)\left[s + 5 + \frac{6}{s}\right] = \frac{5e^{-s}}{s^2}$$

$$I(s) = \frac{5e^{-s}}{s(s^2 + 5s + 6)} = \frac{5e^{-s}}{s(s+3)(s+2)}$$

By taking partial fraction, we have

$$\frac{1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

Applying Heavyside rule, we get the coefficients

$$\therefore \qquad I(s) = 5e^{-s}\left[\frac{1}{6s} + \frac{1}{3(s+3)} - \frac{1}{2(s+2)}\right]$$

$$I(s) = 5\left[\frac{e^{-s}}{6s} + \frac{e^{-s}}{3(s+3)} - \frac{e^{-s}}{2(s+2)}\right]$$

Taking inverse transform on both sides, we have

$$i(t) = \left[\frac{5}{6}u(t-1) + \frac{5}{3}e^{-3(t-1)}\,u(t-1) - \frac{5}{2}e^{-2(t-1)}\,u(t-1)\right] A$$

Problem 14.3 A rectangular voltage pulse of unit height and T seconds duration is applied to a series R-C combination at $t = 0$, as shown in Fig. 14.11. Determine the current in the capacitor as a function of time. Assume the capacitor to be initially uncharged.

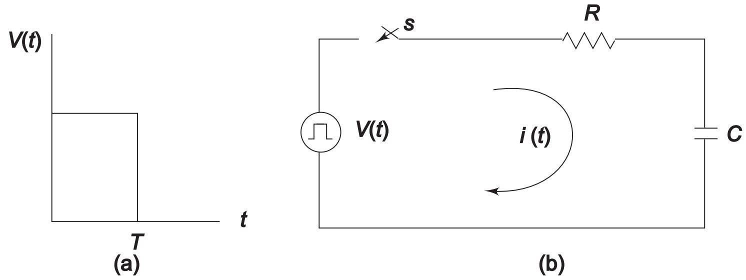


Fig. 14.17

Solution The input voltage can be written as a combination of two steps, i.e.

$$v(t) = u(t) - u(t - T)$$

Applying Kirchhoff's law to the circuit, we get

$$Ri(t) + \frac{1}{C}\int i(t)\,dt = [u(t) - u(t-T)]$$

Taking Laplace transforms on both sides, we get

$$RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0)}{s}\right] = \frac{1}{s}\,(1 - e^{-sT})$$

Since the initial charge on the capacitor is zero

$$q(0) = 0$$

Therefore, $$I(s)\left[R + \frac{1}{Cs}\right] = \frac{1}{s}\,(1 - e^{-sT})$$

or $$I(s) = \frac{1 - e^{-sT}}{R\left(s + \dfrac{1}{RC}\right)}$$

$$= \frac{1}{R}\left[\frac{1}{s + 1/RC} - \frac{e^{-sT}}{s + 1/RC}\right]$$

Taking inverse transform on both sides, we get

$$i(t) = \frac{1}{R}\,\{u(t)e^{-t/RC} - u(t-T)\,e^{-(1/RC)(t-T)}\}$$

Problem 14.4 For the network shown in Fig. 14.18, determine the transform impedance $Z(s)$.

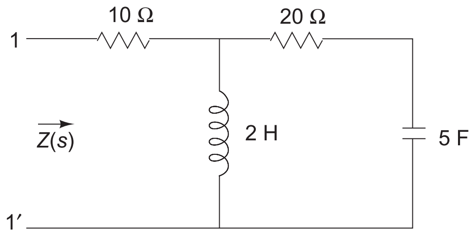


Fig. 14.18

Solution The transform network for the network shown in Fig. 14.18 is shown in Fig. 14.19.

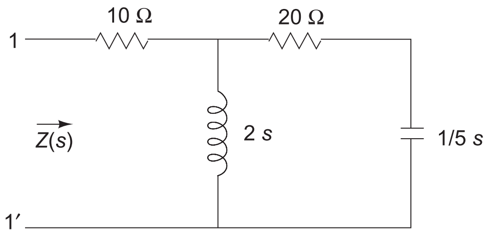


Fig. 14.19

From Fig. 14.19, the equivalent impedance at port 1–1′ is

$$Z(s) = \left\{10 + \left[2s \,||\left(20 + \frac{1}{5s}\right)\right]\right\}$$

$$= 10 + \frac{2s(20 + 1/5s)}{2s + 20 + 1/5s}$$

$$= \frac{20s + 200 + 2/s + 40s + 2/5}{\dfrac{10s^2 + 100s + 1}{5s}}$$

$$= \frac{100s^2 + 1000s + 10 + 200s^2 + 2s}{10s^2 + 100s + 1}$$

Therefore, the network transform impedance is

$$Z(s) = \frac{300s^2 + 1002s + 10}{10s^2 + 100s + 1}$$

Problem 14.5 For the two port network shown in Fig. 14.20, determine the driving point impedance $Z_{11}(s)$ and the driving point admittance $Y_{11}(s)$. Also find the transfer impedance $Z_{21}(s)$.

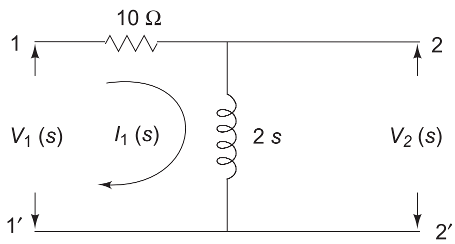


Fig. 14.20

Solution By applying Kirchhoff's law to the circuit, we have

$$V_1(s) = 10I_1(s) + 2s\, I_1(s) \tag{14.8}$$

The voltage across port 2–2′ is

$$V_2(s) = I_1(s) \times (2s) \tag{14.9}$$

From Eq. 14.8, the driving point impedance is

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = (2s + 10)$$

Similarly, the driving point admittance is

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)} = \frac{1}{2s + 10}$$

From Eq. 14.9, the transfer impedance is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = 2s$$

Problem 14.6 For the network shown in Fig. 14.21, determine the transfer functions $G_{21}(s)$ and $Z_{21}(s)$ and the driving point admittance $Y_{11}(s)$.

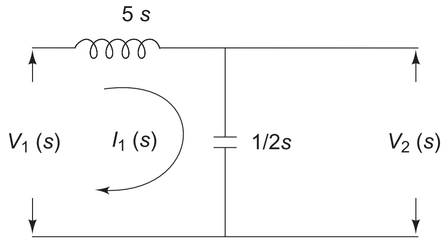


Fig. 14.21

Solution By applying Kirchhoff's voltage law at the ports, we get

$$V_1(s) = I_1(s)\left[5s + \frac{1}{2s}\right]$$

$$V_2(s) = \frac{1}{2s}\; I_1(s)$$

Therefore, the voltage transfer ratio

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{2s(5s + 1/2s)}$$

$$G_{21}(s) = \frac{1}{10s^2 + 1}$$

The transform impedance is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{1}{2s}$$

The driving point admittance is

$$Y_{11}(s) = \frac{I_2(s)}{V_1(s)} = \frac{1}{5s + 1/2s}$$

$$\therefore \qquad Y_{11}(s) = \frac{2s}{(10s^2 + 1)}$$

Problem 14.7 For the network shown in Fig. 14.22, determine the transfer functions $G_{21}(s)$ and $Z_{21}(s)$. Also find the driving point impedance $Z_{11}(s)$.

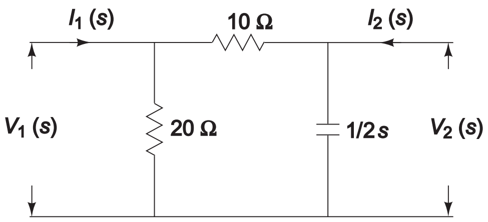


Fig. 14.22

Solution From Fig. 14.23, by application of Kirchhoff's laws, we get the following equations

The driving point impedance

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = [20 \parallel (10 + 1/2s)] = \frac{20 \times (10 + 1/2s)}{20 + 10 + 1/2s}$$

$$Z_{11}(s) = \frac{20(10 + 1/2s)}{30 + 1/2s}$$

$$Z_{11}(s) = \frac{400s + 20}{60s + 1}$$

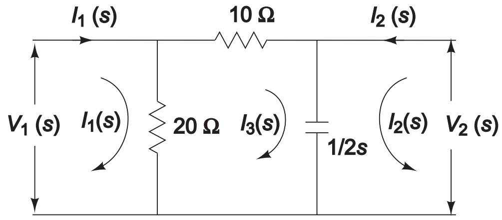


Fig. 14.23

From the above figure, by application of Kirchhoff's laws, we get

$$V_1(s) = 20I_1(s) - 20I_3(s) \tag{14.10}$$

$$10I_3(s) + 20[I_3(s) - I_1(s)] + \frac{1}{2s}[I_3(s) + I_2(s)] = 0 \tag{14.11}$$

$$V_2(s) = [I_2(s) + I_3(s)]\frac{1}{2s} \tag{14.12}$$

From Eq. 14.11, we get

$$\left(30 + \frac{1}{2s}\right)I_3(s) - 20\, I_1(s) = 0$$

$$I_3(s) = \frac{40s}{60s + 1}\, I_1(s) \tag{14.13}$$

From Eq. 14.12, since $I_2 = 0$ we get

$$V_2(s) = +\, I_3(s)\left(\frac{1}{2s}\right) \tag{14.14}$$

The transfer impedance at port 2 is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{40s}{(60s + 1)} \times \frac{1}{2s} = \frac{20}{(60s + 1)}$$

The voltage transfer ratio

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{I_3(s)(1/2s)}{20I_1(s) - 20I_3(s)} = \frac{(1/2s)}{\frac{60s + 1 - 40s}{2s}} = \frac{1}{20s + 1}$$

Problem 14.8 Draw the pole zero diagram for the given network function $I(s)$ and hence obtain $i(t)$.

$$I(s) = \frac{20s}{(s+5)(s+2)}$$

Solution In the network function

$$P(s) = 20s$$

and $$Q(s) = (s+2)(s+5) = 0$$

By taking partial fractions, $I(s)$ can be written as

$$I(s) = \frac{k_1}{s+2} + \frac{k_2}{s+5}$$

Therefore, the time domain response is

$$i(t) = k_1 e^{-2t} + k_2 e^{-5t}$$

Here, the coefficients k_1 and k_2 are determined by using the pole zero plot as shown in Fig. 14.24.

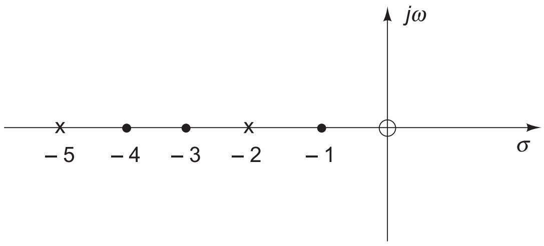


Fig. 14.24

Consider a pole at – 2

The distance between zero to pole at – 2 is

$$M_{02} = 2$$

The angle between the line joining to the pole at – 2 to the zero is

$$\phi_{02} = 180°$$

Similarly, the distance between pole at – 5 to pole at – 2 is

$$M_{52} = 3$$

The angle between the line joining the pole at – 2 to the pole at – 5 is

$$\phi_{52} = 0°$$

Hence $$k_1 = H\frac{M_{02}\, e^{j\phi_{02}}}{M_{52}\, e^{j\phi_{52}}}$$

$$= 20 \times \frac{2e^{j180}}{3e^{j0}} = 13.33\, e^{j180} = -13.33$$

Similarly, $$k_2 = H\frac{M_{05}\, e^{j\phi_{05}}}{M_{25}\, e^{j\phi_{25}}}$$

where $$M_{05} = 5,\ \phi_{05} = 180°$$
$$M_{25} = 3,\ \phi_{25} = 180°$$

Hence $$k_2 = \frac{20 \times 5}{3}\ e^{j(180 - 180)}$$
$$= \frac{100}{3} = 33.3$$

Substituting these values, we get
$$i(t) = (-\ 13.33e^{-2t} + 33.3\ e^{-5t})\ \text{A}$$

Problem 14.9 Draw the pole zero diagram for the given network function and hence obtain $v(t)$

$$V(s) = \frac{4(s+2)s}{(s+1)\,(s+3)}$$

Solution In the network function
$$p(s) = 4s(s + 2)$$
and $$Q(s) = (s + 1)\ (s + 3) = 0$$

By taking partial fractions, we have
$$V(s) = \frac{k_1}{s+1} + \frac{k_2}{s+3}$$

The time domain response can be obtained by taking the inverse transform
$$v(t) = k_1\ e^{-t} + k_2\ e^{-3t}$$

Here, the coefficients k_1 and k_2 may be determined by using the pole zero plot as shown in Fig. 14.25.

To determine k_1, we have to find out the distances and phase angles from other zeros and poles to that particular pole.

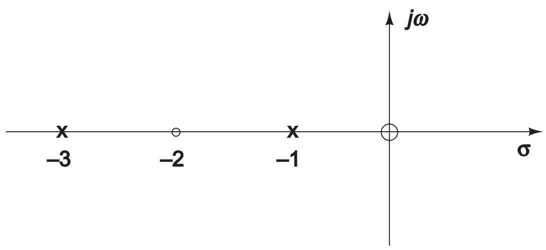


Fig. 14.25

Hence $$k_1 = H\ \frac{M_{01}M_{21}\ e^{j(\phi_{01} + \phi_{21})}}{M_{31}\ e^{j(\phi_{31})}}$$

where M_{01} and M_{21} are the distances between the zeros at 0 and – 2 to the pole at – 1, ϕ_{01}, ϕ_{21} are the phase angle between the corresponding zeros to the pole.

Similarly, M_{31} and ϕ_{31} are the distance and phase angle, respectively, from pole at – 3 to pole at – 1.

$\therefore$ $$M_{01} = 1;\ \phi_{01} = 180°$$
$$M_{21} = 1;\ \phi_{21} = 0$$
$$M_{31} = 2;\ \phi_{31} = 0°$$

$\therefore$ $$k_1 = 4 \times \frac{1 \times 1}{2} e^{j(180°)}$$
$$k_1 = -2$$

Similarly,

$$k_2 = H \frac{M_{03}\,M_{23}}{M_{13}} e^{+j(\phi_{03} + \phi_{23} - \phi_{13})}$$

where $$M_{03} = 3,\ \phi_{03} = 180°$$
$$M_{23} = 1,\ \phi_{23} = 180°$$
$$M_{13} = 2,\ \phi_{13} = 180°$$

$\therefore$ $$k_2 = \frac{4 \times 3 \times 1}{2} e^{j(180 + 180 - 180)}$$
$$k_2 = -6$$

Substituting the values, we get

$$v(t) = (-2e^{-t} - 6e^{-3t})\text{V}$$

Problem 14.10 For the given network function, draw the pole zero diagram and hence obtain the time domain response $i(t)$.

$$I(s) = \frac{5s}{(s+1)\,(s^2 + 4s + 8)}$$

Solution In the network function

$$P(s) = 5s$$
$$Q(s) = (s + 1)\,(s^2 + 4s + 8) = 0$$

By taking the partial fraction expansion of $I(s)$, we get

$$I(s) = \frac{k_1}{s+1} + \frac{k_2}{(s + 2 + j2)} + \frac{k_3}{(s + 2 - j2)} \tag{14.15}$$

The time domain response can be obtained by taking the inverse transform as under,

$$i(t) = k_1 e^{-t} + k_2\, e^{-(2 + j2)t} + k_2\, e^{-(2 - j2)t} \tag{14.16}$$

To find the value of k_1, we have to find out the distances, and phase angles from other zeros and poles to that particular pole as shown in Fig. 14.26.

Hence $$k_1 = \frac{H\,M_{01}\,e^{j(\phi_{01})}}{M_{p11}\,M_{p21}\,e^{j[\phi_{p11} + \phi_{p21}]}}$$
$$M_{01} = 1;\ \phi_{01} = 180°$$
$$M_{p11} = \sqrt{5};\ \phi_{p11} = -63.44°$$
$$M_{p21} = \sqrt{5};\ \phi_{p21} = 63.44°$$

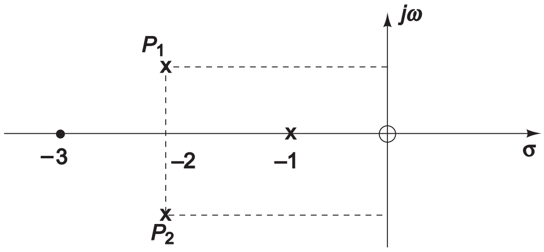


Fig. 14.26

$$\therefore \qquad k_1 = \frac{5 \times 1 e^{j180°}}{\sqrt{5} \times \sqrt{5}\, e^{j(-63.44° + 63.44°)}}$$

$$k_1 = -1$$

Similarly
$$k_2 = \frac{H M_{0p_1} e^{j\phi_{0p_1}}}{M_{1p_1} M_{p_2 p_1} e^{j(\phi_{1p_1} + \phi_{p_2 p_1})}}$$

$$M_{0p_1} = \sqrt{8}\,;\ \phi_{0p_1} = 135°$$

$$M_{1p_1} = \sqrt{5}\,;\ \phi_{1p_1} = 116.56°$$

$$M_{p_1 p_2} = 4; \phi_{p_2 p_1} = 90°$$

Hence
$$k_2 = \frac{5 \times \sqrt{8}}{\sqrt{5} \times 4} e^{j(135° - 116.56° - 90°)}$$

$$= 1.58\, e^{-j(71.56°)}$$

$$k_2^* = \frac{H M_{0p_2} e^{j\phi_{0p_2}}}{M_{1p_2} M_{p_1 p_2} e^{j(\phi_{1p_2} + \phi_{p_1 p_2})}}$$

$$= \frac{5 \times \sqrt{8}\, e^{-j(135°)}}{\sqrt{5} \times 4 e^{j(-116.56° - 90°)}}$$

$$= 1.58 e^{j71.56°}$$

If we substitute the values in Eq. 14.16, we get

$$i(t) = [-1e^{-t} + 1.58\, e^{-j(71.56°)}\, e^{-(2+j2)t} + 1.58 e^{j(71.56°)}\, e^{-(2-j2)t}] \text{A}$$

Problem 14.11 For the given denominator polynomial of a network function, verify the stability of the network by using the Routh criterion.

$$Q(s) = s^3 + 2s^2 + 8s + 10$$

Solution Routh array for this polynomial is given below

s^3	1	8
s^2	2	10
s^1	3	
s^0	10	

There is no change in sign in the first column of the array. Hence, there are no roots with positive real parts. Therefore, the network is stable.

Problem 14.12 For the given denominator polynomial of a network function, verify the stability of the network using the Routh criterion.

$$Q(s) = s^3 + s^2 + 3s + 8$$

Solution Routh array for this polynomial is given below.

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 1 & 8 \\ s^1 & -5 & \\ s^0 & +8 & \end{array}$$

There are two changes in sign of the first column, one from 1 to – 5 and the other from – 5 to + 8. Therefore, the two roots have positive real parts. Hence the network is not stable.

Problem 14.13 For the given denominator polynomial of a network function, determine the value of k for which the network to stable.

$$Q(s) = s^3 + 2s^2 + 4s + k$$

Solution Routh array for the given polynomial is given below.

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & k \\ s^1 & \dfrac{8-k}{2} & \\ s^0 & k & \end{array}$$

When $k < 8$, all the terms in the first column are positive. Therefore, there is no sign change in the first column. Hence, the network is stable. When $k > 8$, the $8 - k/2$ is negative. Therefore, there are two sign changes in the first column. There are two roots which have positive real parts. Hence, the network is unstable.

When $k = 8$, the Routh array becomes

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & 8 \\ s^1 & \alpha & \\ s^0 & 8 & \end{array}$$

The element in the first column and third row is zero. But we can take it as a small number. In this case there are no changes in the sign of the first column. Hence, the network is stable.

Problem 14.14 Apply Routh criterion to the given polynomial and determine the number of roots (i) with positive real parts (ii) with zero real parts (iii) with negative real parts.

$$Q(s) = s^4 + 4s^3 + 8s^2 + 12s + 15$$

Solution The Routh array for the polynomial is

$$\begin{array}{c|ccc} s^4 & 1 & 8 & 15 \\ s^3 & 4 & 12 & \\ s^2 & 5 & 15 & \\ s^1 & 0 & 0 & \\ s^0 & ? & ? & \end{array}$$

In this case, all the elements in the 4th row have become zero and the array cannot be completed.

The given equation is reduced by taking the new polynomial from the 3rd row

$$5s^2 + 15 = 0$$
$$5(s^2 + 3) = 0$$

Hence the other polynomial

$$Q_2(s) = \frac{s^4 + 4s^3 + 8s^2 + 12s + 15}{5(s^2 + 3)}$$

The equation reduces to the following polynomial

$$(s^2 + 3)\,(s^2 + 4s + 5) = 0$$

The roots of the equation $s^2 + 3 = 0$ are $s = \pm j\sqrt{3}$

There two roots have zero real parts.

Again forming Routh array for the polynomial

$$s^2 + 4s + 5 = 0$$

$$\begin{array}{c|cc} s^2 & 1 & 5 \\ s^1 & 4 & 0 \\ s^0 & 5 & \end{array}$$

There are no changes in the sign of the first column. Hence, all the two roots have negative real parts. Therefore, out of four roots, two roots have negative real parts and two roots have zero real parts.

Practice Problems

14.1 For the circuit shown in Fig. 14.27, determine the current $i(t)$, when the switch is closed at $t = 0$. Assume that there is no initial charge on the capacitor.

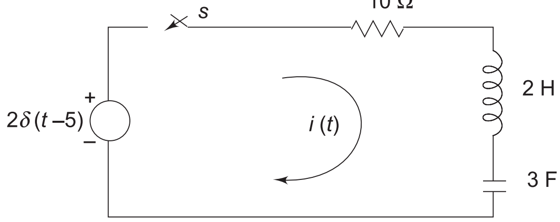


Fig. 14.27

14.2 For the circuit shown in Fig. 14.28, determine the voltage across capacitor, when the switch is closed at $t = 0$. Assume that there is no initial charge on the capacitor.

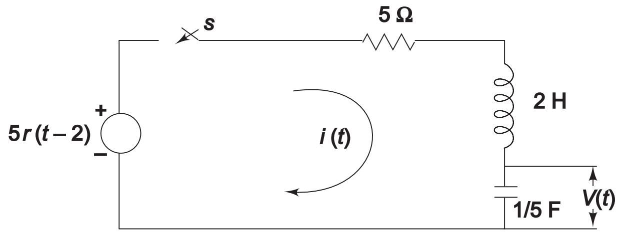


Fig. 14.28

14.3 For the circuit shown in Fig. 14.29(b), determine the current when the switch is closed at $t = 0$. The waveform shown in Fig. 14.29(a) is applied to the circuit. Assume that there is no initial charge on the capacitor.

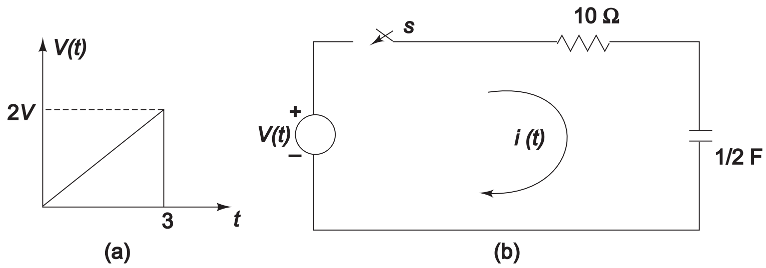


Fig. 14.29

14.4 The waveform shown in Fig. 14.30(a) is applied to the circuit in Fig. 14.30(b) when the switch is closed at $t = 0$. Assume no initial current in the circuit. Determine the current $i(t)$ in the circuit.

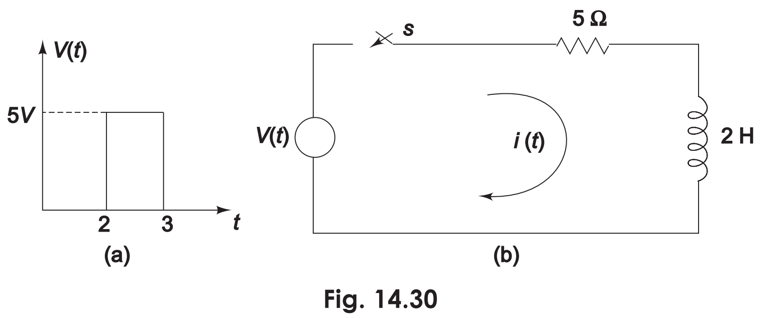


Fig. 14.30

14.5 For the two-port network shown in Fig. 14.31, determine the driving point impedance $Z_{11}(s)$, the transfer impedance $Z_{21}(s)$ and the voltage transfer ratio $G_{21}(s)$.

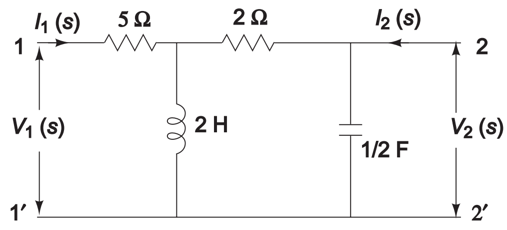


Fig. 14.31

14.6 For the network shown in Fig. 14.32, determine the following transfer functions. (a) $G_{21}(s)$, (b) $Y_{21}(s)$ and (c) $\alpha_{21}(s)$.

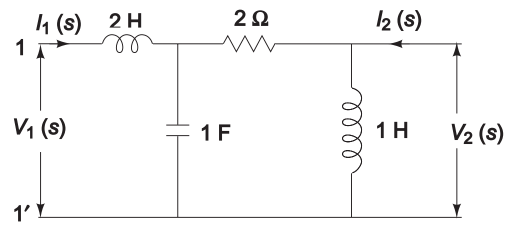


Fig. 14.32

14.7 For the network shown in Fig. 14.33, determine the following transfer functions (a) $G_{21}(s)$, (b) $Z_{21}(s)$.

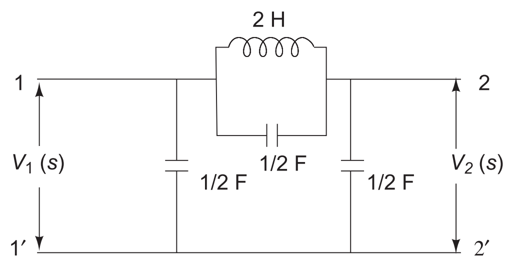


Fig. 14.33

14.8 For the network shown in Fig. 14.34, determine the following functions (a) $Z_{11}(s)$, (b) $Y_{11}(s)$, (c) $G_{21}(s)$ and (d) $\alpha_{21}(s)$.

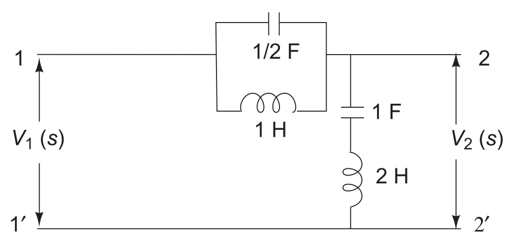


Fig. 14.34

14.9 For the network shown in Fig. 14.35, determine transfer impedance $Z_{21}(s)$ and $Y_{21}(s)$. Also find the transfer voltage ratio $G_{21}(s)$ and the transfer current ratio $\alpha_{21}(s)$.

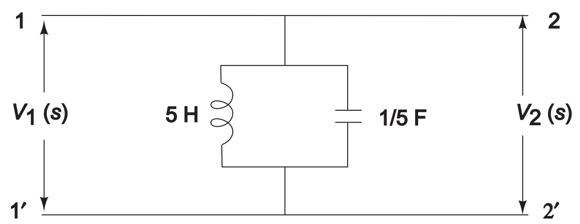


Fig. 14.35

14.10 For the given network function, draw the pole zero diagram and hence obtain the time domain response. Verify the result analytically.

$$V(s) = \frac{5(s+5)}{(s+2)(s+7)}$$

14.11 For the given network function draw the pole zero diagram and hence obtain the time domain response. Verify this result analytically.

$$I(s) = \frac{3s}{(s+1)(s+3)}$$

14.12 For the given network function, draw the pole zero diagram and hence obtain the time domain response. Verify the result analytically.

$$I(s) = \frac{5s}{(s+3)(s^2+2s+2)}$$

14.13 For the given denominator polynomial of a network function, verify the stability of the network using Routh criteria.

$$Q(s) = s^5 + 3s^4 + 4s^3 + 5s^2 + 6s + 1$$

14.14 For the given denominator polynomial of a network function, verify the stability of the network using Routh criteria.

$$Q(s) = s^4 + s^3 + 2s^2 + 2s + 12$$

14.15 Apply Routh criterion to the following equations and determine the number of roots (i) with positive real parts (ii) with zero real parts (iii) with negative real parts

(a) $6s^3 + 2s^2 + 5s + 2 = 0$
(b) $s^6 + 5s^5 + 13s^4 + 21s^3 + 20s^2 + 16s + 8 = 0$
(c) $s^6 - s^5 - 2s^4 + 4s^3 - 5s^2 + 21s + 30 = 0$

Objective-type Questions

1. The function is said to be non-recurring when it
   (a) appears for a particular time interval
   (b) appears for all time
   (c) both *a* and *b*
   (d) neither of the two
2. The inverse transform of $1/S$ is
   (a) $\delta(t)$ (b) $u(t)$
   (c) $u(t-a)$ (d) t
3. The Laplace transform of a ramp function is
   (a) 1 (b) $1/s$
   (c) $1/s^2$ (d) $1/s^3$
4. The inverse transform of S is
   (a) impulse (b) ramp
   (c) step (d) unit doublet
5. The driving point impedance is defined as
   (a) the ratio of transform voltage to transform current at the same port
   (b) the ratio of transform voltage at one port to the transform current at the other port
   (c) both (a) and (b)
   (d) none of the above
6. The transfer impedance is defined as
   (a) the ratio of transform voltage to transform current at the same port
   (b) the ratio of transform voltage at one port to the current transform at the other port
   (c) both (a) and (b)
   (d) none of the above
7. The function is said to be having simple poles and zeros and only if
   (a) the poles are not repeated
   (b) the zeros are not repeated
   (c) both poles and zeros are not repeated
   (d) none of the above
8. The necessary condition for a driving point function is
   (a) the real part of all poles and zeros must not be zero or negative
   (b) the polynomials $P(s)$ and $Q(s)$ may not have any missing terms between the highest and lowest degree unless all even or all odd terms are missing.
   (c) the degree of $P(s)$ and $Q(s)$ may differ by more than one
   (d) the lowest degree in $P(s)$ and $Q(s)$ may differ in degree by more than two.

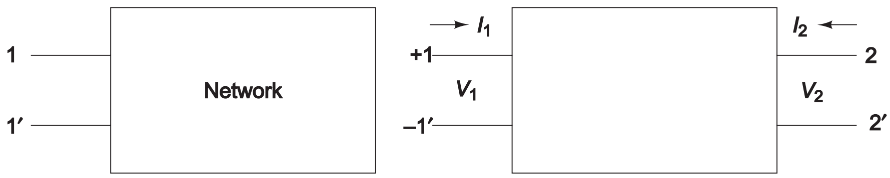
9. The necessary condition for the transfer functions is that
   (a) the coefficients in the polynomials $P(s)$ and $Q(s)$ must be real
   (b) coefficients in $Q(s)$ may be negative
   (c) complex or imaginary poles and zeros may not conjugate
   (d) if the real part of pole is zero, then that pole must be multiple
10. The system is said to be stable, if and only if
    (a) all the poles lie on right half of the s-plane
    (b) some poles lie on the right half of the s-plane
    (c) all the poles does not lie on the right half of the s-plane
    (d) none of the above.

Chapter 15

Two-Port Networks

15.1 TWO-PORT NETWORK

Generally any network may be represented schematically by a rectangular box. A network may be used for representing either source or load, or for a variety of purposes. A pair of terminals at which a signal may enter or leave a network is called a port. A *port* is defined as any pair of terminals into which energy is supplied, or from which energy is withdrawn, or where the network variables may be measured. One such network having only one pair of terminals (1–1′) is shown in Fig. 15.1.



(a) (b)

Fig. 15.1

A two-port network is simply a network inside a black box, and the network has only two pairs of accessible terminals; usually one pair represents the input and the other represents the output. Such a building block is very common in electronic systems, communication systems, transmission and distribution systems. Figure 15.1 (b) shows a two-port network, or two terminal pair network, in which the four terminals have been paired into ports 1–1′ and 2–2′. The terminals 1–1′ together constitute a port. Similarly, the terminals 2–2′ constitute another port. Two ports containing no sources in their branches are called *passive ports*; among them are power transmission lines and transformers. Two ports containing sources in their branches are called *active ports*. A voltage and current assigned to each of the two ports. The voltage and current at the input terminals are V_1 and I_1; whereas V_2 and I_2 are specified at the output port. It is also assumed that the currents I_1 and I_2 are entering into the network at the upper terminals 1 and 2, respectively. The variables of the two-port network are V_1, V_2, and I_1, I_2. Two of these are dependent variables, the other two are independent variables. The number of possible combinations generated by the four variables, taken two at a time, is six. Thus, there are six possible sets of equations describing a two-port network.

15.2 OPEN CIRCUIT IMPEDANCE (Z) PARAMETERS

A general linear two-port network defined in Section 15.1 which does not contain any independent sources is shown in Fig. 15.2.

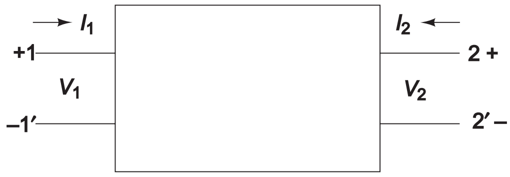


Fig. 15.2

The Z parameters of a two-port for the positive directions of voltages and currents may be defined by expressing the port voltages V_1 and V_2 in terms of the currents I_1 and I_2. Here V_1 and V_2 are dependent variables, and I_1, I_2 are independent variables. The voltage at port 1–1′ is the response produced by the two currents I_1 and I_2. Thus

$$V_1 = Z_{11} I_1 + Z_{12} I_2 \tag{15.1}$$

Similarly, $$V_2 = Z_{21} I_1 + Z_{22} I_2 \tag{15.2}$$

Z_{11}, Z_{12}, Z_{21} and Z_{22} are the network functions, and are called impedance (Z) parameters, and are defined by Eqs. 15.1 and 15.2. These parameters can be represented by matrices.

We may write the matrix equation $[V] = [Z]\,[I]$

where V is the column matrix $= \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

Z is the square matrix $= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$

and we may write $|I|$ in the column matrix $= \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

Thus, $$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The individual Z parameters for a given network can be defined by setting each of the port currents equal to zero. Suppose port 2–2′ is left open-circuited, then $I_2 = 0$

Thus $$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0}$$

where Z_{11} is the driving-point impedance at port 1–1′ with port 2–2′ open circuited. It is called the open circuit input impedance

Similarly, $$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

where Z_{21} is the transfer impedance at port 1–1′ with port 2–2′ open circuited. It is also called the open circuit forward transfer impedance. Suppose port 1–1′ is left open circuited, then $I_1 = 0$

Thus, $$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

where Z_{12} is the transfer impedance at port 2–2′, with port 1–1′ open circuited. It is also called the open circuit reverse transfer impedance.

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$

where Z_{22} is the open circuit driving point impedance at port 2–2′ with port 1–1′ open circuited. It is also called the open circuit output impedance. The equivalent circuit of the two-port networks governed by the Eqs. 15.1 and 15.2, i.e. open circuit impedance parameters is shown in Fig. 15.3.

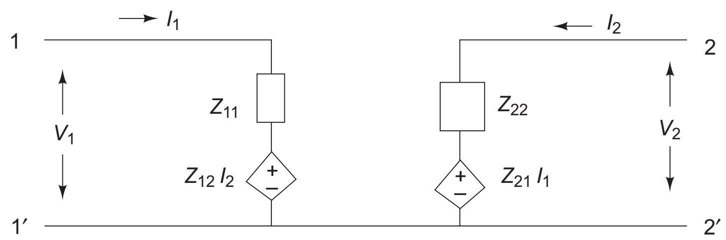


Fig. 15.3

If the network under study is reciprocal or bilateral, then in accordance with the reciprocity principle

$$\left.\frac{V_2}{I_1}\right|_{I_2 = 0} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

or $$Z_{21} = Z_{12}$$

It is observed that all the parameters have the dimensions of impedance. Moreover, individual parameters are specified only when the current in one of the ports is zero. This corresponds to one of the ports being open circuited from which the Z parameters also derive the name *open circuit impedance parameters*.

Example 15.1 Find the Z parameters for the circuit shown in Fig. 15.4.

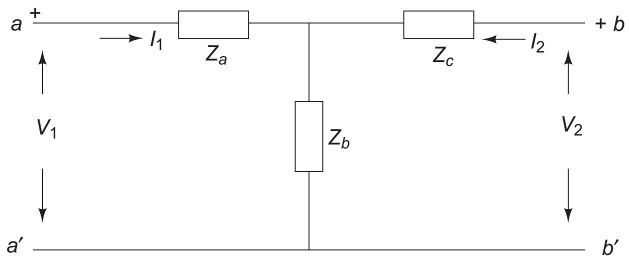


Fig. 15.4

Solution The circuit in the problem is a T network. From Eqs. 15.1 and 15.2 we have

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

When port b-b' is open circuited, $Z_{11} = \dfrac{V_1}{I_1}$

where $\quad V_1 = I_1(Z_a + Z_b)$

$\therefore \quad Z_{11} = (Z_a + Z_b)$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

where $\quad V_2 = I_1 Z_b$

$\therefore \quad Z_{21} = Z_b$

When port a-a' is open circuited, $I_1 = 0$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$

where $\quad V_2 = I_2(Z_b + Z_c)$

$\therefore \quad Z_{22} = (Z_b + Z_c)$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

where $V_1 = I_2 Z_b$

$\therefore \quad Z_{12} = Z_b$

It can be observed that $Z_{12} = Z_{21}$, so the network is a bilateral network which satisfies the principle of reciprocity.

15.3 SHORT CIRCUIT ADMITTANCE (Y) PARAMETERS

A general two-port network which is considered in Section 15.2 is shown in Fig. 15.5.

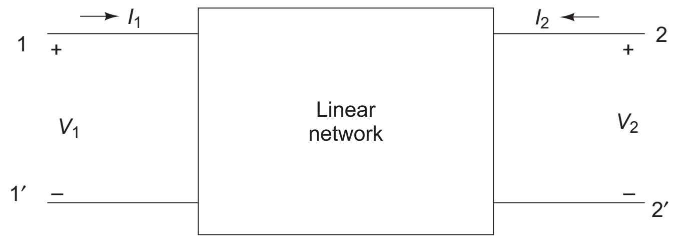


Fig. 15.5

The Y parameters of a two-port for the positive directions of voltages and currents may be defined by expressing the port currents I_1 and I_2 in terms of the voltages V_1 and V_2. Here I_1, I_2 are dependent variables and V_1 and V_2 are independent variables. I_1 may be considered to be the superposition of two components, one caused by V_1 and the other by V_2. Thus,
Thus,

$$I_1 = Y_{11} V_1 + Y_{12} V_2 \tag{15.3}$$

Similarly,

$$I_2 = Y_{21} V_1 + Y_{22} V_2 \tag{15.4}$$

Y_{11}, Y_{12}, Y_{21} and Y_{22} are the network functions and are also called the admittance (Y) parameters. They are defined by Eqs 15.3 and 15.4. These parameters can be represented by matrices as follows

$$[I] = [Y]\,[V]$$

where

$$I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix};\ Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

and

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The individual Y parameters for a given network can be defined by setting each port voltage to zero. If we let V_2 be zero by short circuiting port 2–2′, then

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0}$$

Y_{11} is the driving point admittance at port 1–1′, with port 2–2′ short circuited. It is also called the short circuit input admittance.

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0}$$

Y_{21} is the transfer admittance at port 1–1 with port 2–2′ short circuited. It is also called short circuited forward transfer admittance. If we let V_1 be zero by short circuiting port 1–1′, then

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0}$$

Y_{12} is the transfer admittance at port 2–2′ with port 1–1′ short circuited. It is also called the short circuit reverse transfer admittance.

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0}$$

Y_{22} is the short circuit driving point admittance at port 2–2′ with port 1–1′ short circuited. It is also called the short circuit output admittance. The equivalent circuit of the network governed by Eqs. 15.3 and 15.4 is shown in Fig. 15.6.

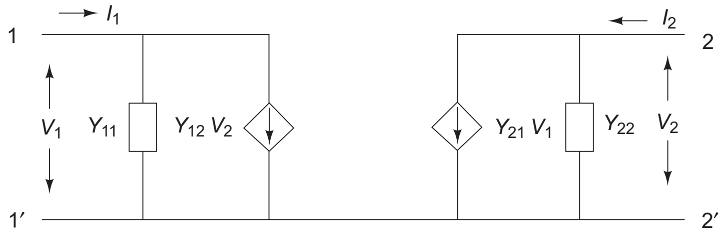


Fig. 15.6

If the network under study is reciprocal, or bilateral, then

$$\left.\frac{I_1}{V_2}\right|_{V_1 = 0} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0}$$

or $$Y_{12} = Y_{21}$$

It is observed that all the parameters have the dimensions of admittance which are obtained by short circuiting either the output or the input port from which the parameters also derive their name, i.e. the *short circuit admittance parameters*.

Example 15.2 Find the *Y* parameters for the network shown in Fig. 15.7.

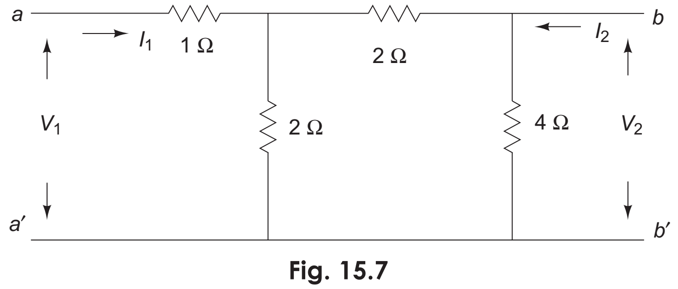


Fig. 15.7

Solution $$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0}$$

When b-b' is short circuited, $V_2 = 0$ and the network looks as shown in Fig. 15.8(a)

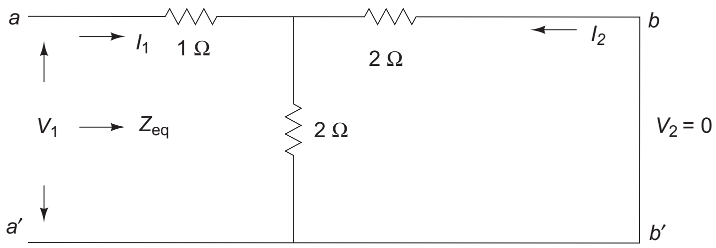


Fig. 15.8(a)

$$V_1 = I_1 Z_{eq}$$

$$Z_{eq} = 2\ \Omega$$

$\therefore$ $$V_1 = I_1 2$$

$$Y_{11} = \frac{I_1}{V_1} = \frac{1}{2}\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_2}\right|_{V_2 = 0}$$

With port b-b' short circuited, $-I_2 = I_1 \times \frac{2}{4} = \frac{I_1}{2}$

$\therefore$ $$-I_2 = \frac{V_1}{4}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0} = -\frac{1}{4}\ \mho$$

Similarly, when port a-a' is short circuited, $V_1 = 0$ and the network looks as shown in Fig. 15.8(b).

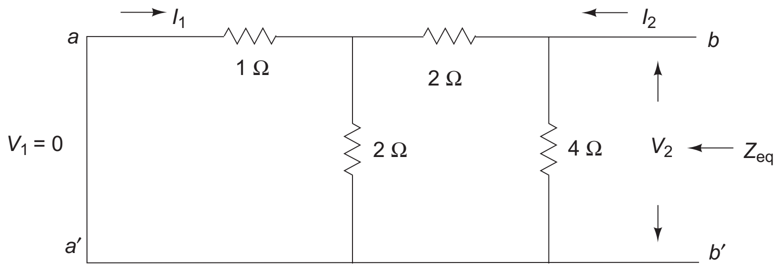


Fig. 15.8(b)

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0}$$

$$V_2 = I_2\, Z_{eq}$$

where Z_{eq} is the equivalent impedance as viewed from b-b'.

$$Z_{eq} = \frac{8}{5}\ \Omega$$

$$V_2 = I_2 \times \frac{8}{5}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} = \frac{5}{8}\ \mho$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0}$$

With a-a' short circuited, $-I_1 = \frac{2}{5} I_2$

Since $$I_2 = \frac{5V_2}{8}$$

$$-I_1 = \frac{2}{5} \times \frac{5}{8} V_2 = \frac{V_2}{4}$$

$\therefore$ $$Y_{12} = \frac{I_1}{V_2} = -\frac{1}{4}\ \mho$$

The describing equations in terms of the admittance parameters are

$$I_1 = 0.5\ V_1 - 0.25\ V_2$$
$$I_2 = -0.25\ V_1 + 0.625\ V_2$$

15.4 TRANSMISSION (ABCD) PARAMETERS

Transmission parameters, or *ABCD* parameters, are widely used in transmission line theory and cascade networks. In describing the transmission parameters, the input variables V_1 and I_1 at port 1-1′, usually called the *sending end*, are expressed in terms of the output variables V_2 and I_2 at port 2-2′, called the *receiving end*. The transmission parameters provide a direct relationship between input and output. Transmission parameters are also called general circuit parameters, or chain parameters. They are defined by

$$V_1 = AV_2 - BI_2 \tag{15.5}$$
$$I_1 = CV_2 - DI_2 \tag{15.6}$$

The negative sign is used with I_2, and not for the parameter B and D. Both the port currents I_1 and $-I_2$ are directed to the right, i.e. with a negative sign in Eqs 15.5 and 15.6 the current at port 2-2′ which leaves the port is designated as positive. The parameters A, B, C and D are called the *transmission parameters*. In the matrix form, Eqs 15.5 and 15.6 are expressed as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is called the *transmission matrix*.

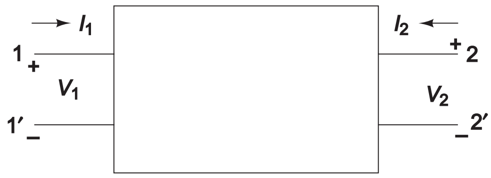


Fig. 15.9

For a given network, these parameters can be determined as follows. With port 2-2′ open, i.e. $I_2 = 0$; applying a voltage V_1 at the port 1-1′, using Eq. 15.5, we have

$$A = \left.\frac{V_1}{V_2}\right|_{I_2 = 0} \quad \text{and } C = \left.\frac{I_1}{V_2}\right|_{I_2 = 0}$$

$$\frac{1}{A} = \left.\frac{V_2}{V_1}\right|_{I_2 = 0} = \left. g_{21}\right|_{I_2 = 0}$$

$1/A$ is called the open circuit voltage gain, a dimensionless parameter. And $\dfrac{1}{C}$ $= \left.\dfrac{V_2}{I_1}\right|_{I_2 = 0} = Z_{21}$, which is the open circuit transfer impedance. With port 2-2′ short circuited, i.e. with $V_2 = 0$, applying voltage V_1 at port 1-1′, from Eq. 15.6, we have

$$-B = \left.\frac{V_1}{I_2}\right|_{V_2 = 0} \quad \text{and} \quad -D = \left.\frac{I_1}{I_2}\right|_{V_2 = 0}$$

$-\dfrac{1}{B} = \left.\dfrac{I_2}{V_1}\right|_{V_2 = 0} = Y_{21}$, which is the short circuit transfer admittance

$-\dfrac{1}{D} = \dfrac{I_2}{I_1}\bigg|_{V_2=0} = \alpha_{21}\,|_{V_2=0}$, which is the short circuit current gain, a dimensionless parameter.

15.4.1 Cascade Connection

The main use of the transmission matrix is in dealing with a cascade connection of two-port networks as shown in Fig. 15.10.

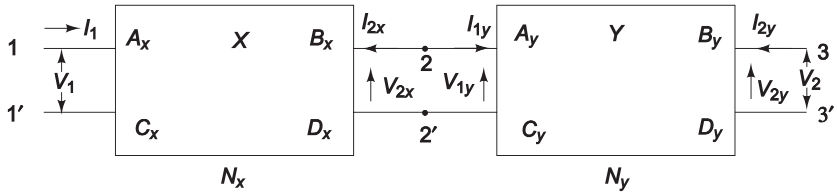


Fig. 15.10

Let us consider two two-port networks N_x and N_y connected in cascade with port voltages and currents as indicated in Fig. 15.10. The matrix representation of *ABCD* parameters for the network *X* is as under.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} V_{2x} \\ -I_{2x} \end{bmatrix}$$

And for the network *Y*, the matrix representation is

$$\begin{bmatrix} V_{1y} \\ I_{1y} \end{bmatrix} = \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix} \begin{bmatrix} V_{2y} \\ -I_{2y} \end{bmatrix}$$

It can also be observed that at for 2–2′

$$V_{2x} = V_{1y} \text{ and } I_{2x} = -I_{1y}.$$

Combining the results, we have

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix} \begin{bmatrix} V_2 \\ -I_1 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the transmission parameters matrix for the overall network.

Thus, the transmission matrix of a cascade of a two-port networks is the product of transmission matrices of the individual two-port networks. This property is used in the design of telephone systems, microwave networks, radars, etc.

Example 15.3 Find the transmission or general circuit parameters for the circuit shown in Fig. 15.11.

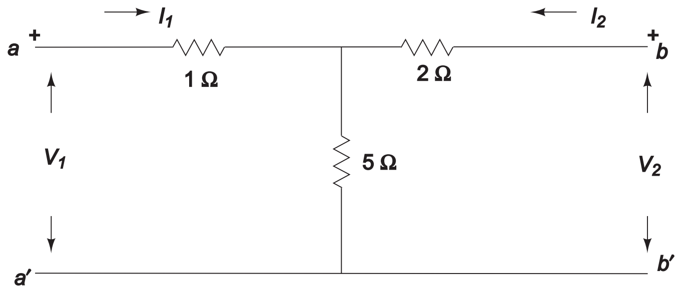


Fig. 15.11

Solution From Eqs 15.5 and 15.6 in Section 15.4, we have

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

When b-b' is open, $I_2 = 0$; $A = \left.\dfrac{V_1}{V_2}\right|_{I_2=0}$

where $V_1 = 6I_1$ and $V_2 = 5I_1$

$\therefore$ $$A = \frac{6}{5}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{5}\ \mho$$

When b-b' is short circuited; $V_2 = 0$ (See Fig. 15.12)

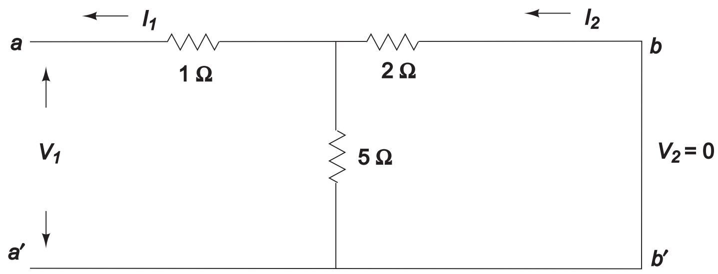


Fig. 15.12

$$B = \left.\frac{-V_1}{I_2}\right|_{V_2=0};\ D = \left.\frac{-I_1}{I_2}\right|_{V_2=0}$$

In the circuit, $$-I_2 = \frac{5}{17}V_1$$

$\therefore$ $$B = \frac{17}{5}\ \Omega$$

Similarly, $I_1 = \frac{7}{17} V_1$ and $-I_2 = \frac{5}{17} V_1$

$\therefore$ $D = \frac{7}{5}$

15.5 INVERSE TRANSMISSION (A′ B′ C′ D′) PARAMETERS

In the preceding section, the input port voltage and current are expressed in terms of output port voltage and current to describe the transmission parameters. While defining the transmission parameters, it is customary to designate the input port as the sending end and output port as receiving end. The voltage and current at the receiving end can also be expressed in terms of the sending end voltage and current. If the voltage and current at port 2-2′ is expressed in terms of voltage and current at port 1-1′, we may write the following equations.

$$V_2 = A'V_1 - B'I_1 \tag{15.7}$$

$$I_2 = C'V_1 - D'I_1 \tag{15.8}$$

The coefficients A', B', C' and D' in the above equations are called inverse transmission parameters. Because of the similarities of Eqs. 15.7 and 15.8 with Eqs. 15.5 and 15.6 in Section 15.4, the A', B', C', D' parameters have properties similar to $ABCD$ parameters. Thus when port 1-1′ is open, $I_1 = 0$.

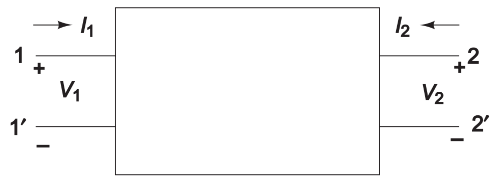


Fig. 15.13

$$A' = \left.\frac{V_2}{V_1}\right|_{I_1 = 0} ; C' = \left.\frac{I_2}{V_1}\right|_{I_1 = 0}$$

If port 1-1′ is short circuited, $V_1 = 0$

$$B' = \left.\frac{-V_2}{I_1}\right|_{V_1 = 0} ; D = \left.\frac{-I_2}{I_1}\right|_{V_1 = 0}$$

15.6 HYBRID (h) PARAMETERS

Hybrid parameters, or h parameters find extensive use in transistor circuits. They are well suited to transistor circuits as these parameters can be most conveniently measured. The hybrid matrices describe a two-port, when the voltage of one port and the current of other port are taken as the independent variables. Consider the network in Fig. 15.14.

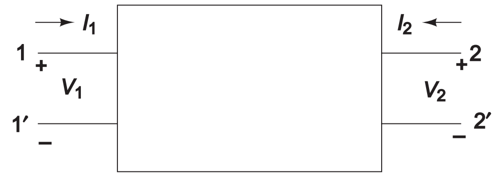


Fig. 15.14

If the voltage at port 1-1′ and current at port 2-2′ are taken as dependent variables, we can express them in terms of I_1 and V_2.

$$V_1 = h_{11}\, I_1 + h_{12}\, V_2 \tag{15.9}$$

$$I_2 = h_{21}\, I_1 + h_{22}\, V_2 \tag{15.10}$$

The coefficients in the above equations are called hybrid parameters. In matrix notation

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

From Eqs. 15.9 and 15.10, the individual h parameters may be defined by letting $I_1 = 0$ and $V_2 = 0$.

When $V_2 = 0$, the port 2-2′ is short circuited.

Then $h_{11} = \left.\dfrac{V_1}{I_1}\right|_{V_2 = 0}$ Short circuit input impedance $\left(\dfrac{1}{Y_{11}}\right)$

$h_{21} = \left.\dfrac{I_2}{I_1}\right|_{V_2 = 0}$ Short circuit forward current gain $\left(\dfrac{Y_{21}}{Y_{11}}\right)$

Similarly, by letting port 1-1′ open, $I_1 = 0$

$h_{12} = \left.\dfrac{V_1}{V_2}\right|_{I_1 = 0}$ Open circuit reverse voltage gain $\left(\dfrac{Z_{12}}{Z_{22}}\right)$

$h_{22} = \left.\dfrac{I_2}{V_2}\right|_{I_1 = 0}$ Open circuit output admittance $\left(\dfrac{1}{Z_{22}}\right)$

Since the h parameters represent dimensionally an impedance, an admittance, a voltage gain and a current gain, these are called hybrid parameters. An equivalent circuit of a two-port network in terms of hybrid parameters is shown in Fig. 15.15.

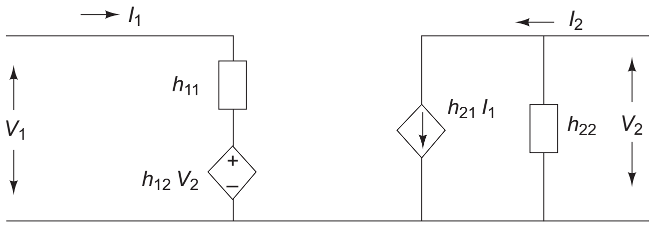


Fig. 15.15

Example 15.4 Find the h parameters of the network shown in Fig. 15.16.

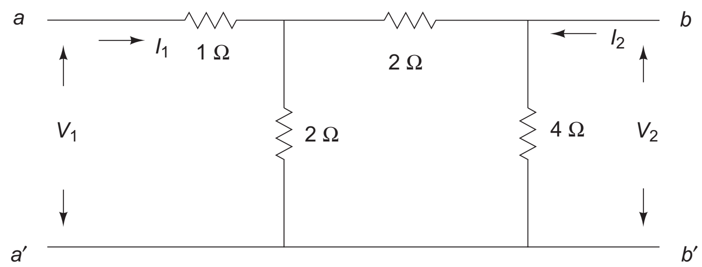


Fig. 15.16

Solution From Eqs. 15.9 and 15.10, we have

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}; \; h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}; \; h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}; \; h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

If port b-b' is short circuited, $V_2 = 0$. The circuit is shown in Fig. 15.17(a).

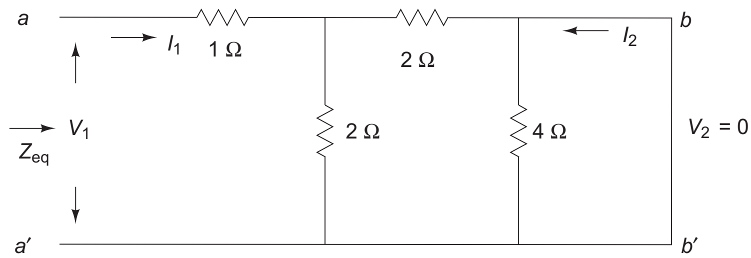


Fig. 15.17(a)

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}; \; V_1 = I_1 Z_{eq}$$

Z_{eq} the equivalent impedance as viewed from the port a-a' is 2 Ω

$$\therefore \qquad V_1 = I_1 \, 2\text{V}$$

$$h_{11} = \frac{V_1}{I_1} = 2\ \Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} \text{ when } V_2 = 0; -I_2 = \frac{I_1}{2}$$

$$\therefore \qquad h_{21} = -\frac{1}{2}$$

If port a-a' is let open, $I_1 = 0$. The circuit is shown in Fig. 15.17(b). Then

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

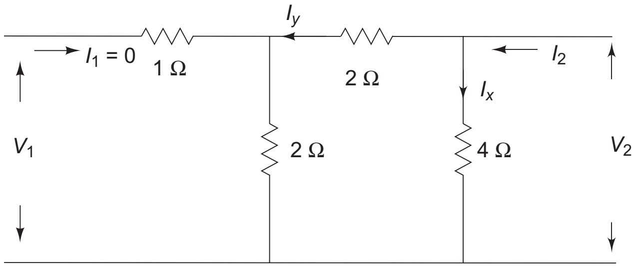


Fig. 15.17(b)

$$V_1 = I_Y 2;\; I_Y = \frac{I_2}{2}$$

$$V_2 = I_X 4;\; I_X = \frac{I_2}{2}$$

$$\therefore \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1 = 0} = \frac{1}{2}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1 = 0} = \frac{1}{2}\ \mho$$

15.7 INVERSE HYBRID (g) PARAMETERS

Another set of hybrid matrix parameters can be defined in a similar way as was done in Section 15.6. This time the current at the input port I_1 and the voltage at the output port V_2 can be expressed in terms of I_2 and V_1. The equations are as follows.

$$I_1 = g_{11} V_1 + g_{12} I_2 \tag{15.11}$$

$$V_2 = g_{21} V_1 + g_{22} I_2 \tag{15.12}$$

The coefficients in the above equations are called the inverse hybrid parameters. In matrix notation

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

It can be verified that $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}^{-1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$

The individual g parameters may be defined by letting $I_2 = 0$ and $V_1 = 0$ in Eqs 15.11 and 15.12.

Thus, when $I_2 = 0$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2 = 0} = \text{Open circuit input admittance} \left(\frac{1}{Z_{11}}\right)$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2 = 0} = \text{Open circuit voltage gain}$$

When $V_1 = 0$

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1 = 0} = \text{Short circuit reverse current gain}$$

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1 = 0} = \text{Short circuit output impedance}\left(\frac{1}{Y_{22}}\right)$$

15.8 INTER RELATIONSHIPS OF DIFFERENT PARAMETERS

15.8.1 Expression of Z-parameters in Terms of y-parameters and Vice-versa

From Eqs 15.1, 15.2, 15.3 and 15.4, it is easy to derive the relation between the open circuit impedance parameters and the short circuit admittance parameters by means of two matrix equations of the respective parameters. By solving Eqs 15.1 and 15.2 for I_1 and I_2, we get

$$I_1 = \begin{vmatrix} V_1 & Z_{12} \\ V_2 & Z_{22} \end{vmatrix} \Big/ \Delta_z \text{; and } I_2 = \begin{vmatrix} Z_{11} & V_1 \\ V_{21} & V_2 \end{vmatrix} \Big/ \Delta_z$$

where Δ_z is the determinant of Z matrix

$$\Delta_z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$I_1 = \frac{Z_{22}}{\Delta_z} V_1 - \frac{Z_{12}}{\Delta_z} V_2 \tag{15.13}$$

$$I_2 = \frac{-Z_{21}}{\Delta_z} V_1 + \frac{Z_{11}}{\Delta_z} V_2 \tag{15.14}$$

Comparing Eqs. 15.13 and 15.14 with Eqs. 15.3 and 15.4 we have

$$Y_{11} = \frac{Z_{22}}{\Delta_z};\ Y_{12} = \frac{-Z_{12}}{\Delta_z}$$

$$Y_{21} = \frac{Z_{21}}{\Delta_z};\ Y_{22} = \frac{Z_{11}}{\Delta_z}$$

In a similar manner, the Z parameters may be expressed in terms of the admittance parameters by solving Eqs. 15.3 and 15.4 for V_1 and V_2

$$V_1 = \begin{vmatrix} I_1 & Y_{12} \\ I_2 & Y_{22} \end{vmatrix} \Big/ \Delta_y \text{ and } V_2 = \begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & I_2 \end{vmatrix} \Big/ \Delta_y$$

where Δ_y is the determinant of the Y matrix

$$\Delta_y = \begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix}$$

$$V_1 = \frac{Y_{22}}{\Delta_y} I_1 - \frac{Y_{12}}{\Delta_y} I_2 \tag{15.15}$$

$$V_2 = \frac{-Y_{21}}{\Delta_y} I_1 + \frac{Y_{11}}{\Delta_y} I_2 \tag{15.16}$$

Comparing Eqs. 15.15 and 15.16 with Eqs. 15.1 and 15.2, we obtain

$$Z_{11} = \frac{Y_{22}}{\Delta_y};\ Z_{12} = \frac{-Y_{12}}{\Delta_y}$$

$$Z_{21} = \frac{-Y_{21}}{\Delta_y};\ Z_{22} = \frac{Y_{11}}{\Delta_y}$$

Example 15.5 For a given, $Z_{11} = 3\ \Omega$, $Z_{12} = 1\ \Omega$; $Z_{21} = 2\ \Omega$ and $Z_{22} = 1\ \Omega$, find the admittance matrix, and the product of Δ_y and Δ_z.

Solution The admittance matrix $= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{Z_{22}}{\Delta_z} & \dfrac{-Z_{12}}{\Delta_z} \\ \dfrac{-Z_{21}}{\Delta_z} & \dfrac{Z_{11}}{\Delta_z} \end{bmatrix}$

given $Z = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$\therefore$ $$\Delta_z = 3 - 2 = 1$$

$\therefore$ $$\Delta_y = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} = 1$$

$$(\Delta_y)\ (\Delta_z) = 1$$

15.8.2 General Circuit Parameters or ABCD Parameters in Terms of *Z* Parameters and *Y* Parameters

We know that

$$V_1 = AV_2 - BI_2;\ V_1 = Z_{11}\, I_1 + Z_{12}\, I_2;\ I_1 = Y_{11}\, V_1 + Y_{12}\, V_2$$
$$I_1 = CV_2 - DI_2;\ V_2 = Z_{21}\, I_1 + Z_{22}\, I_2;\ I_2 = Y_{21}\, V_1 + Y_{22}\, V_2$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2 = 0};\ C = \left.\frac{I_1}{V_2}\right|_{I_2 = 0};\ B = \left.\frac{-V_1}{I_2}\right|_{V_2 = 0};\ D = \left.\frac{-I_1}{I_2}\right|_{V_2 = 0}$$

Substituting the condition $I_2 = 0$ in Eqs 15.1 and 15.2 we get

$$\left.\frac{V_1}{V_2}\right|_{I_2 = 0} = \frac{Z_{11}}{Z_{21}} = A$$

Substituting the condition $I_2 = 0$ in Eq. 15.4 we get,

$$\left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{-Y_{22}}{Y_{21}} = A$$

Substituting the condition $I_2 = 0$ in Eq. 15.2

we get $$\left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{Z_{21}} = C$$

Substituting the condition $I_2 = 0$ in Eqs 15.3 and 15.4, and solving for V_2 gives $\dfrac{-I_1 Y_{21}}{\Delta y}$

where Δy is the determinant of the admittance matrix

$$\left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{-\Delta y}{Y_{21}} = C$$

Substituting the condition $V_2 = 0$ in Eq. 15.4, we get

$$\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{1}{Y_{21}} = B$$

Substituting the condition $V_2 = 0$ in Eqs. 15.1 and 15.2 and solving for $I_2 = \dfrac{-V_1 Z_{21}}{\Delta_z}$

$$-\left.\frac{V_1}{I_2}\right|_{V_2=0} = \frac{\Delta_z}{Z_{21}} = B$$

where Δ_z is the determinant of the impedance matrix.

Substituting $V_2 = 0$ in Eq. 15.2

we get $$-\left.\frac{I_1}{I_2}\right|_{V_2=0} = \frac{Z_{22}}{Z_{21}} = D$$

Substituting $V_2 = 0$ in Eqs. 15.3 and 15.4, we get

$$\left.\frac{-I_1}{I_2}\right|_{V_2=0} = \frac{-Y_{11}}{Y_{21}} = D$$

The determinant of the transmission matrix is given by

$$-AD + BC$$

Substituting the impedance parameters in A, B, C and D, we have

$$BC - AD = \frac{\Delta z}{Z_{21}}\,\frac{1}{Z_{21}} - \frac{Z_{11}}{Z_{21}}\,\frac{Z_{22}}{Z_{21}}$$

$$= \frac{\Delta z}{(Z_{21})^2} - \frac{Z_{11}\,Z_{22}}{(Z_{21})^2}$$

$$BC - AD = \frac{-Z_{12}}{Z_{21}}$$

For a bilateral network, $Z_{12} = Z_{21}$

$\therefore \qquad BC - AD = -1$

or $\qquad AD - BC = 1$

Therefore, in a two-port bilateral network, if three transmission parameters are known, the fourth may be found from equation $AD - BC = 1$.

In a similar manner the h parameters may be expressed in terms of the admittance parameters, impedance parameters or transmission parameters. Transformations of this nature are possible between any of the various parameters. Each parameters has its own utility. However, we often find that it is necessary to convert from one set of parameters to another. Transformations between different parameters, and the condition under which the two-port network is reciprocal are given in Table 15.1.

Example 15.6 The impedance parameters of a two port network are $Z_{11} = 6\,\Omega$; $Z_{22} = 4\,\Omega$; $Z_{12} = Z_{21} = 3\,\Omega$. Compute the Y parameters and $ABCD$ parameters and write the describing equations.

Solution $ABCD$ parameters are given by

$$A = \frac{Z_{11}}{Z_{21}} = \frac{6}{3} = 2; \quad B = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}} = 5\,\Omega$$

$$C = \frac{1}{Z_{21}} = \frac{1}{3}\ \mho; \quad D = \frac{Z_{22}}{Z_{21}} = \frac{4}{3}$$

Y parameters are given by

$$Y_{11} = \frac{Z_{22}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{4}{15}\ \mho; \quad Y_{12} = \frac{-Z_{12}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{-1}{5}\ \mho$$

$$Y_{21} = Y_{12} = \frac{-Z_{12}}{\Delta z} = \frac{-1}{5}\ \mho; \quad Y_{22} = \frac{Z_{11}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{2}{5}\ \mho$$

The equations, using Z parameters are

$$V_1 = 6I_1 + 3I_2$$
$$V_2 = 3I_1 + 4I_2$$

Using Y parameters

$$I_1 = \frac{4}{15}V_1 - \frac{1}{5}V_2$$

$$I_2 = \frac{-1}{5}V_1 + \frac{2}{5}V_2$$

Table 15.1

	Z	Y	$ABCD$	$A'B'C'D'$	h	g
Z	$\begin{matrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{matrix}$	$\begin{matrix} \frac{Y_{22}}{\Delta_y} & \frac{-Y_{12}}{\Delta_y} \\ \frac{-Y_{21}}{\Delta_y} & \frac{Y_{11}}{\Delta_y} \end{matrix}$	$\begin{matrix} \frac{A}{C} & \frac{\Delta_T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$	$\begin{matrix} \frac{D'}{C'} & \frac{1}{C'} \\ \frac{\Delta_{T'}}{C'} & \frac{A'}{C'} \end{matrix}$	$\begin{matrix} \frac{\Delta_h}{h_{22}} & \frac{h_{22}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{11}} & \frac{-g_{12}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{\Delta_g}{g_{11}} \end{matrix}$
Y	$\begin{matrix} \frac{Z_{22}}{\Delta_z} & \frac{-Z_{12}}{\Delta_z} \\ \frac{-Z_{21}}{\Delta z} & \frac{Z_{11}}{\Delta z} \end{matrix}$	$\begin{matrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{matrix}$	$\begin{matrix} \frac{D}{B} & \frac{-\Delta_T}{B} \\ \frac{-1}{B} & \frac{A}{B} \end{matrix}$	$\begin{matrix} \frac{A'}{B'} & \frac{-1}{B'} \\ \frac{-\Delta_{T'}}{B'} & \frac{D'}{B'} \end{matrix}$	$\begin{matrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_h}{h_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta_g}{g_{22}} & \frac{g_{12}}{g_{22}} \\ \frac{-g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{matrix}$
AB CD	$\begin{matrix} \frac{Z_{11}}{Z_{21}} & \frac{\Delta z}{Z_{21}} \\ \frac{1}{Z_{21}} & \frac{Z_{22}}{Z_{21}} \end{matrix}$	$\begin{matrix} \frac{-Y_{22}}{Y_{21}} & \frac{-1}{Y_{21}} \\ \frac{\Delta Y}{Y_{21}} & \frac{-Y_{11}}{Y_{21}} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} \frac{D'}{\Delta_{T'}} & \frac{B'}{\Delta_{T'}} \\ \frac{C'}{\Delta_{T'}} & \frac{A'}{\Delta_{T'}} \end{matrix}$	$\begin{matrix} \frac{\Delta_h}{h_{21}} & \frac{h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{21}} & \frac{g_{22}}{g_{21}} \\ \frac{g_{11}}{g_{21}} & \frac{\Delta_g}{g_{21}} \end{matrix}$
$A'\,B'$ $C'\,D'$	$\begin{matrix} \frac{Z_{22}}{Z_{12}} & \frac{\Delta z}{Z_{12}} \\ \frac{1}{Z_{12}} & \frac{Z_{11}}{Z_{12}} \end{matrix}$	$\begin{matrix} \frac{-Y_{11}}{Y_{12}} & \frac{-1}{Y_{12}} \\ \frac{-\Delta_Y}{Y_{12}} & \frac{-Y_{22}}{Y_{12}} \end{matrix}$	$\begin{matrix} \frac{D}{\Delta_T} & \frac{B}{\Delta_T} \\ \frac{C}{\Delta_T} & \frac{A}{\Delta_T} \end{matrix}$	$\begin{matrix} A' & B' \\ C' & D' \end{matrix}$	$\begin{matrix} \frac{1}{h_{12}} & \frac{h_{11}}{h_{12}} \\ \frac{h_{22}}{h_{12}} & \frac{\Delta_h}{h_{12}} \end{matrix}$	$\begin{matrix} \frac{-\Delta_g}{g_{12}} & \frac{-g_{22}}{g_{12}} \\ \frac{-g_{11}}{g_{12}} & \frac{-1}{g_{12}} \end{matrix}$
h	$\begin{matrix} \frac{\Delta_z}{Z_{22}} & \frac{Z_{12}}{Z_{22}} \\ \frac{-Z_{21}}{Z_{22}} & \frac{1}{Z_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{Y_{11}} & \frac{-Y_{12}}{Y_{11}} \\ \frac{Y_{21}}{Y_{11}} & \frac{\Delta_Y}{Y_{11}} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta_T}{D} \\ \frac{-1}{D} & \frac{C}{D} \end{matrix}$	$\begin{matrix} \frac{B'}{A'} & \frac{1}{A'} \\ \frac{\Delta_{T'}}{A'} & \frac{C'}{A'} \end{matrix}$	$\begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix}$	$\begin{matrix} \frac{g_{22}}{\Delta_g} & \frac{-g_{12}}{\Delta_g} \\ \frac{-g_{21}}{\Delta_g} & \frac{g_{11}}{\Delta_g} \end{matrix}$
g	$\begin{matrix} \frac{1}{Z_{11}} & \frac{-Z_{12}}{Z_{11}} \\ \frac{Z_{21}}{Z_{11}} & \frac{\Delta_Z}{Z_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta_Y}{Y_{22}} & \frac{Y_{12}}{Y_{22}} \\ \frac{-Y_{21}}{Y_{22}} & \frac{1}{Y_{22}} \end{matrix}$	$\begin{matrix} \frac{C}{A} & \frac{-\Delta_T}{A} \\ \frac{1}{A} & \frac{B}{A} \end{matrix}$	$\begin{matrix} \frac{C'}{D'} & \frac{-1}{D'} \\ \frac{\Delta_{T'}}{D'} & \frac{B'}{D'} \end{matrix}$	$\begin{matrix} \frac{h_{22}}{\Delta_h} & \frac{-h_{12}}{\Delta_h} \\ \frac{-h_{21}}{\Delta_h} & \frac{h_{11}}{\Delta_h} \end{matrix}$	$\begin{matrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{matrix}$
The two port is reciprocal If	$Z_{12} = Z_{21}$	$Y_{12} = Y_{21}$	The determinant of the transmission matrix = 1 ($\Delta_T = 1$)	The determinant of the inverse transmission matrix = 1	$h_{12} = -h_{21}$	$g_{12} = -g_{21}$

Using $ABCD$ parameters

$$V_1 = 2V_2 - 5I_2$$

$$I_1 = \frac{1}{3}V_2 - \frac{4}{3}I_2$$

15.9 INTER CONNECTION OF TWO-PORT NETWORKS

15.9.1 Series Connection of Two-port Network

It has already been shown in Section 15.4.1 that when two-port networks are connected in cascade, the parameters of the interconnected network can be conveniently expressed with the help of *ABCD* parameters. In a similar way, the *Z*-parameters can be used to describe the parameters of series connected two-port networks; and *Y* parameters can be used to describe parameters of parallel connected two-port networks. A series connection of two-port networks is shown in Fig. 15.18.

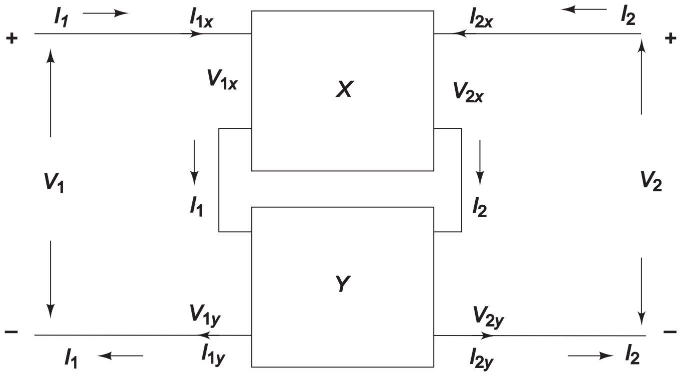


Fig. 15.18

Let us consider two two-port networks, connected in series as shown. If each port has a common reference node for its input and output, and if these references are connected together then the equations of the networks *X* and *Y* in terms of *Z* parameters are

$$V_{1X} = Z_{11X} I_{1X} + Z_{12X} I_{2X}$$
$$V_{2X} = Z_{21X} I_{1X} + Z_{22X} I_{2X}$$
$$V_{1Y} = Z_{11Y} I_{1Y} + Z_{12Y} I_{2Y}$$
$$V_{2Y} = Z_{21Y} I_{1Y} + Z_{22Y} I_{2Y}$$

From the inter-connection of the networks, it is clear that

$$I_1 = I_{1X} = I_{1Y};\ I_2 = I_{2X} = I_{2Y}$$

and $$V_1 = V_{1X} + V_{1Y};\ V_2 = V_{2X} + V_{2Y}$$

$\therefore$ $$V_1 = Z_{11X} I_1 + Z_{12X} I_2 + Z_{11Y} I_1 + Z_{12Y} I_2$$
$$= (Z_{11X} + Z_{11Y})I_1 + (Z_{12X} + Z_{12Y})\, I_2$$
$$V_2 = Z_{21X} I_1 + Z_{22X} I_2 + Z_{21Y} I_1 + Z_{22Y} I_2$$
$$= (Z_{21X} + X_{21Y})I_1 + (Z_{22X} + Z_{22Y})I_2$$

The describing equations for the series connected two-port network are

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

where $\quad Z_{11} = Z_{11X} + Z_{11Y};\ Z_{12} = Z_{12X} + Z_{12Y}$

$$Z_{21} = Z_{21X} + Z_{21Y};\ Z_{22} = Z_{22X} + Z_{22Y}$$

Thus, we see that each Z parameter of the series network is given as the sum of the corresponding parameters of the individual networks.

15.9.2 Parallel Connection of Two Two-port Networks

Let us consider two two-port networks connected in parallel as shown in Fig. 15.19. If each two-port has a reference node that is common to its input and output port, and if the two ports are connected so that they have a common reference node, then the equations of the networks X and Y in terms of Y parameters are given by

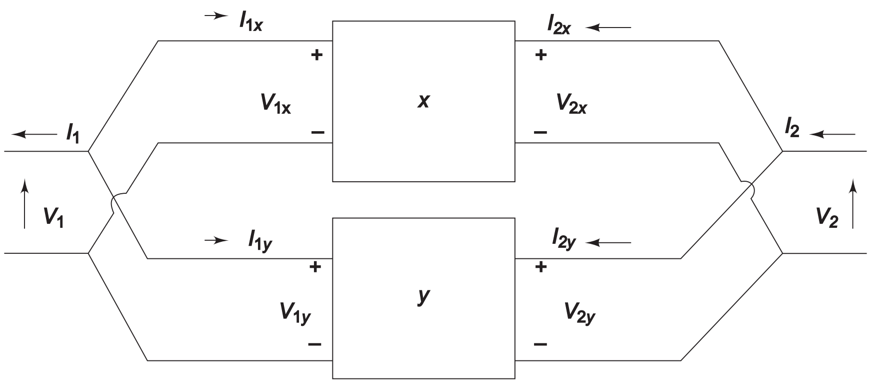


Fig. 15.19

$$I_{1X} = Y_{11X}\, V_{1X} + Y_{12X}\, V_{2X}$$
$$I_{2X} = Y_{21X}\, V_{1X} + Y_{22X}\, V_{2X}$$
$$I_{1Y} = Y_{11Y}\, V_{1Y} + Y_{12Y}\, V_{2Y}$$
$$I_{2Y} = Y_{21Y}\, V_{1Y} + Y_{22Y}\, V_{2Y}$$

From the interconnection of the networks, it is clear that

$$V_1 = V_{1X} = V_{1Y};\ V_2 = V_{2X} = V_{2Y}$$

and $\quad I_1 = I_{1X} + I_{1Y};\ I_2 = I_{2X} + I_{2Y}$

$\therefore \quad I_1 = Y_{11X}\, V_1 + Y_{12X}\, V_2 + Y_{11Y}\, V_1 + Y_{12Y}\, V_2$

$$= (Y_{11X} + Y_{11Y})\, V_1 + (Y_{12X} + Y_{12Y})\, V_2$$

$$I_2 = Y_{21X}\, V_1 + Y_{22X}\, V_2 + Y_{21Y}\, V_1 + Y_{22Y}\, V_2$$
$$= (Y_{21X} + Y_{21Y})\, V_1 + (Y_{22X} + Y_{22Y})\, V_2$$

The describing equations for the parallel connected two-port networks are

$$I_1 = Y_{11}\, V_1 + Y_{12}\, V_2$$
$$I_2 = Y_{21}\, V_1 + Y_{22}\, V_2$$

where $\quad Y_{11} = Y_{11X} + Y_{11Y};\ Y_{12} = Y_{12X} + Y_{12Y}$

$$Y_{21} = Y_{21X} + Y_{21Y};\ Y_{22} = Y_{22X} + Y_{22Y}$$

Thus we see that each Y parameter of the parallel network is given as the sum of the corresponding parameters of the individual networks.

Example 15.7 Two networks shown in Figs. 15.20(a) and (b) are connected in series. Obtain the Z parameters of the combination. Also verify by direct calculation.

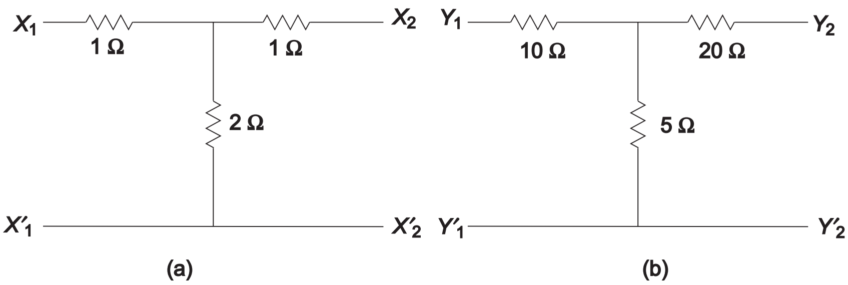


Fig. 15.20

Solution The Z parameters of the network in Fig. 15.20(a) are

$$Z_{11X} = 3\ \Omega\ Z_{12X} = Z_{21X} = 2\ \Omega\ Z_{22X} = 3\ \Omega$$

The Z parameters of the network in Fig. 15.20 (b) are

$$Z_{11Y} = 15\ \Omega\ Z_{21Y} = 5\ \Omega\ Z_{22Y} = 25\ \Omega\ Z_{12Y} = 5\ \Omega$$

The Z parameters of the combined network are

$$Z_{11} = Z_{11X} + Z_{11Y} = 18\ \Omega$$
$$Z_{12} = Z_{12X} + Z_{12Y} = 7\ \Omega$$
$$Z_{21} = Z_{21X} + Z_{21Y} = 7\ \Omega$$
$$Z_{22} = Z_{22X} + Z_{22Y} = 28\ \Omega$$

Check If the two networks are connected in series as shown in Fig. 15.20(c), the Z parameters are

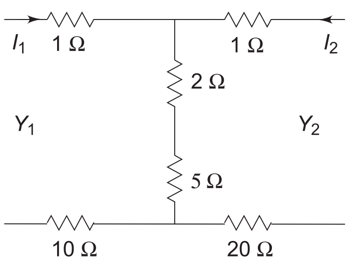


Fig. 15.20(c)

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = 18\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0} = 7\ \Omega$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = 28\ \Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} = 7\ \Omega$$

Example 15.8 Two identical sections of the network shown in Fig. 15.21 are connected in parallel. Obtain the Y parameters of the combination.

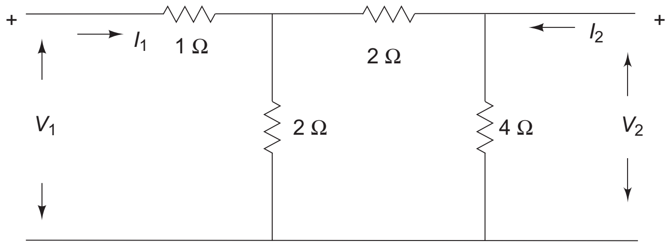


Fig. 15.21

Solution The Y parameters of the network in Fig. 15.21 are (See Ex. 15.2).

$$Y_{11} = \frac{1}{2}\ \mho \quad Y_{21} = \frac{-1}{4}\ \mho \quad Y_{22} = \frac{5}{8}\ \mho \quad Y_{12} = \frac{-1}{4}\ \mho$$

If two such networks are connected in parallel then the Y parameters of the combined network are

$$Y_{11} = \frac{1}{2} + \frac{1}{2} = 1\ \mho \qquad Y_{21} = \frac{-1}{4} \times 2 = \frac{-1}{2}\ \mho$$

$$Y_{22} = \frac{5}{8} \times 2 = \frac{5}{4}\ \mho \qquad Y_{12} = \frac{-1}{4} \times 2 = \frac{-1}{2}\ \mho$$

15.10 T AND Π REPRESENTATION

A two-port network with any number of elements may be converted into a two-port three-element network. Thus, a two-port network may be represented by an equivalent T network, i.e. three impedances are connected together in the form of a T as shown in Fig. 15.22.

It is possible to express the elements of the T-network in terms of Z parameters, or $ABCD$ parameters as explained below.

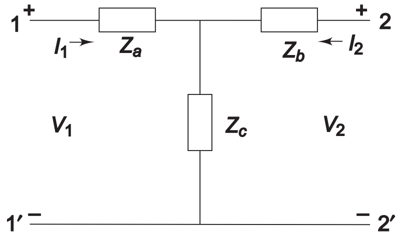


Fig. 15.22

Z parameters of the network

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = Z_a + Z_c$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0} = Z_c$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = Z_b + Z_c$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} = Z_c$$

From the above relations, it is clear that

$$Z_a = Z_{11} - Z_{21}$$
$$Z_b = Z_{22} - Z_{12}$$
$$Z_c = Z_{12} = Z_{21}$$

ABCD parameters of the network

$$A = \left.\frac{V_1}{V_2}\right|_{I_2 = 0} = \frac{Z_a + Z_c}{Z_c}$$

$$B = \left.\frac{-V_1}{I_2}\right|_{V_2 = 0}$$

When 2-2′ is short circuited

$$-I_2 = \frac{V_1 Z_c}{Z_b Z_c + Z_a (Z_b + Z_c)}$$

$$B = (Z_a + Z_b) + \frac{Z_a Z_b}{Z_c}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2 = 0} = \frac{1}{Z_c}$$

$$D = \left.\frac{-I_1}{I_2}\right|_{V_2 = 0}$$

When 2-2′ is short circuited

$$-I_2 = I_1 \frac{Z_c}{Z_b + Z_c}$$

$$D = \frac{Z_b + Z_c}{Z_c}$$

From the above relations we can obtain

$$Z_a = \frac{A-1}{C}; \quad Z_b = \frac{D-1}{C}; \quad Z_c = \frac{1}{C}$$

Example 15.9 The Z parameters of a two-port network are $Z_{11} = 10\ \Omega$; $Z_{22} = 15\ \Omega$; $Z_{12} = Z_{21} = 5\ \Omega$. Find the equivalent T network and $ABCD$ parameters.

Solution The equivalent T network is shown in Fig. 15.23,

where $\quad Z_a = Z_{11} - Z_{21} = 5\ \Omega$

$\quad Z_b = Z_{22} - Z_{12} = 10\ \Omega$

and $\quad Z_c = 5\ \Omega$

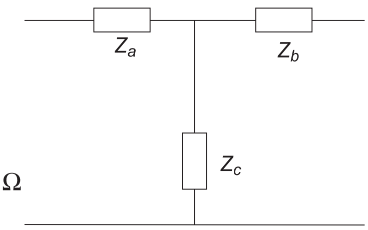


Fig. 15.23

The $ABCD$ parameters of the network are

$$A = \frac{Z_a}{Z_c} + 1 = 2; \; B = (Z_a + Z_b) + \frac{Z_a Z_b}{Z_c} = 25\ \Omega$$

$$C = \frac{1}{Z_c} = 0.2\ \mho \quad D = 1 + \frac{Z_b}{Z_c} = 3$$

In a similar way, a two-port network may be represented by an equivalent π-network, i.e. three impedances or admittances are connected together in the form of π as shown in Fig. 15.24.

It is possible to express the elements of the π-network in terms of Y parameters or $ABCD$ parameters as explained below.

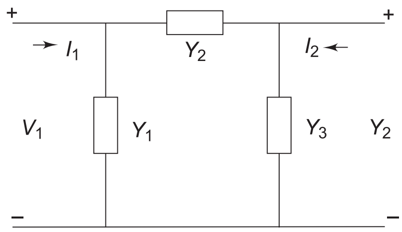


Fig. 15.24

Y parameters of the network

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0} = Y_1 + Y_2$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0} = -\,Y_2$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} = Y_3 + Y_2$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0} = -\,Y_2$$

From the above relations, it is clear that

$$Y_1 = Y_{11} + Y_{21}$$

$$Y_2 = -\,Y_{12}$$

$$Y_3 = Y_{22} + Y_{21}$$

Writing *ABCD* parameters in terms of *Y* parameters yields the following results.

$$A = \frac{-Y_{22}}{Y_{21}} = \frac{Y_3 + Y_2}{Y_2}$$

$$B = \frac{-1}{Y_{21}} = \frac{1}{Y_2}$$

$$C = \frac{-\Delta y}{Y_{21}} = Y_1 + Y_3 + \frac{Y_1 Y_3}{Y_2}$$

$$D = \frac{-Y_{11}}{Y_{21}} = \frac{Y_1 + Y_2}{Y_2}$$

From the above results, we can obtain

$$Y_1 = \frac{D-1}{B}$$

$$Y_2 = \frac{1}{B}$$

$$Y_3 = \frac{A-1}{B}$$

Example 15.10 The port currents of a two-port network are given by

$$I_1 = 2.5V_1 - V_2$$
$$I_2 = -V_1 + 5V_2$$

Find the equivalent π-network.

Solution Let us first find the *Y* parameters of the network

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 2.5\ \mho;\ Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -1\ \mho$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -1\ \mho;\ Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 5\ \mho$$

The equivalent π-network is shown in Fig. 15.25.

where $Y_1 = Y_{11} + Y_{21} = 1.5\ \mho$;

$Y_2 = -Y_{12} = -1\ \mho$

and $Y_3 = Y_{22} + Y_{12} = 4\ \mho$

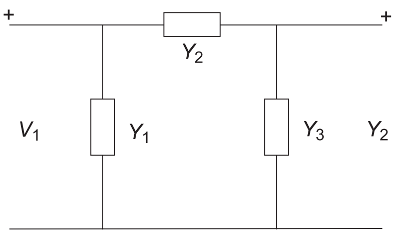


Fig. 15.25

15.11 TERMINATED TWO-PORT NETWORK

15.11.1 Driving Point Impedance at the Input Port of a Load Terminated Network

Figure 15.26 shows a two-port network connected to an ideal generator at the input port and to a load impedance at the output port. The input impedance of this network can be expressed in terms of parameters of the two port network.

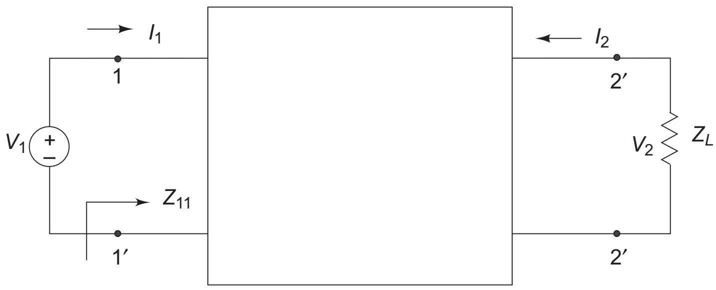


Fig. 15.26

(i) In Terms of Z Parameters

The load at the output port 2-2′ impose the following constraint on the port voltage and current,

i.e., $$V_2 = -Z_L I_2$$

Recalling Eqs 15.1 and 15.2, we have

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Substituting the value of V_2 in Eq. 15.2, we have

$$-Z_L I_2 = Z_{21} I_1 + Z_{22} I_2$$

from which $$I_2 = \frac{-I_1 Z_{21}}{Z_L + Z_{22}}$$

Substituting the value of I_2 in Eq. 15.1 gives

$$V_1 = Z_{11} I_1 - \frac{Z_{12} Z_{21} I_1}{Z_L + Z_{22}}$$

$$V_1 = I_1 \left(Z_{11} - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}} \right)$$

Hence the driving point impedance at 1-1′ is

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}}$$

If the output port is open, i.e. $Z_L \to \infty$, the input impedance is given by $V_1/I_1 = Z_{11}$

If the output port is short circuited, i.e. $Z_L \to 0$, The short circuit driving point impedance is given by

$$\frac{Z_{11}\ Z_{22} - Z_{12}\ Z_{21}}{Z_{22}} = \frac{1}{Y_{11}}$$

(ii) In Terms of Y Parameters

If a load admittance Y_L is connected across the output port. The constraint imposed on the output port voltage and current is

$$-I_2 = V_2\ Y_L, \text{ where } Y_L = \frac{1}{Z_L}$$

Recalling Eqs 15.3 and 15.4 we have

$$I_1 = Y_{11}\ V_1 + Y_{12}\ V_2$$

$$I_2 = Y_{21}\ V_1 + Y_{22}\ V_2$$

Substituting the value of I_2 in Eq. 15.4, we have

$$-V_2\ Y_L = Y_{21}\ V_1 + Y_{22}\ V_2$$

$$V_2 = -\left(\frac{Y_{21}}{Y_L + Y_{22}}\right) V_1$$

Substituting V_2 value in Eq. 15.3, we have

$$I_1 = Y_{11}\ V_1 - \frac{Y_{12}\ Y_{21}\ V_1}{Y_L + Y_{22}}$$

From which $$\frac{I_1}{V_1} = Y_{11} - \frac{Y_{12}\ Y_{21}}{Y_L + Y_{22}}$$

Hence the driving point impedance is given by

$$\frac{V_1}{I_1} = \frac{Y_{22} + Y_L}{Y_{11}(Y_1 + Y_{22}) - Y_{12}\ Y_{21}}$$

If the output port is open, i.e., $Y_L \to 0$

$$\frac{V_1}{I_1} = \frac{Y_{22}}{\Delta_y} = Z_{11}$$

If the output port is short circuited, i.e. $Y_L \to \infty$

Then $$Y_{in} = Y_{11}$$

In a similar way, the input impedance of the load terminated two port network may be expressed in terms of other parameters by simple mathematical manipulations. The results are given in Table 15.2.

15.11.2 Driving Point Impedance at the Output Port with Source Impedance at the Input Port

Let us consider a two-port network connected to a generator at input port with a source impedance Z_s as shown in Fig. 15.27. The output impedance, or the driving point impedance, at the output port can be evaluated in terms of the parameters of two-port network.

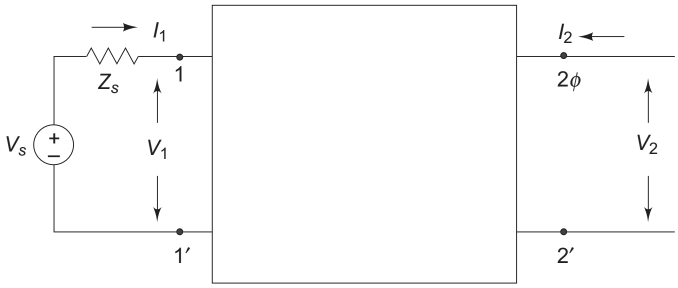


Fig. 15.27

(i) In terms of Z parameters

If I_1 is the current due to V_s at port 1-1′

From Eqs. 15.1 and 15.2, we have

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

$$V_1 = V_s - I_1Z_s$$

$$= Z_{11}\,I_1 + Z_{12}I_2 - (I_1)\,(Z_s + Z_{11}) = Z_{12}I_2 - V_s$$

$$-I_1 = \frac{Z_{12}\,I_2 - V_s}{Z_s + Z_{11}}$$

Substituting I_1 in Eq. 15.2, we get

$$V_2 = -Z_{21}\,\frac{(Z_{12}\,I_2 - V_s)}{Z_s + Z_{11}} + Z_{22}\,I_2$$

With no source voltage at port 1-1′, i.e. if the source V_s is short circuited

$$V_2 = \frac{-Z_{21}\,Z_{12}}{Z_s + Z_{11}}\,I_2 + Z_{22}I_2$$

Hence the driving point impedance at port 2-2′ $= \dfrac{V_2}{I_2}$

$$\frac{V_2}{I_2} = \frac{Z_{22}\,Z_s + Z_{22}\,Z_{11} - Z_{21}\,Z_{12}}{Z_s + Z_{11}} \text{ or } \frac{\Delta_z + Z_{22}\,Z_s}{Z_s + Z_{11}}$$

If the input port is open, i.e. $Z_s \to \infty$

Then $$\frac{V_2}{I_2} = \left[\frac{\dfrac{\Delta_Z}{Z_s} + Z_{22}}{1 + \dfrac{Z_{11}}{Z_s}}\right]_{Z_s = \infty} = Z_{22}$$

If the source impedance is zero with a short circuited input port, the driving point impedance at output port is given by

$$\frac{V_2}{I_2} = \frac{\Delta_Z}{Z_{11}} = \frac{1}{Y_{22}}$$

(ii) In terms of Y parameters

Let us consider a two-port network connected to a current source at input port with a source admittance Y_s as shown in Fig. 15.28.

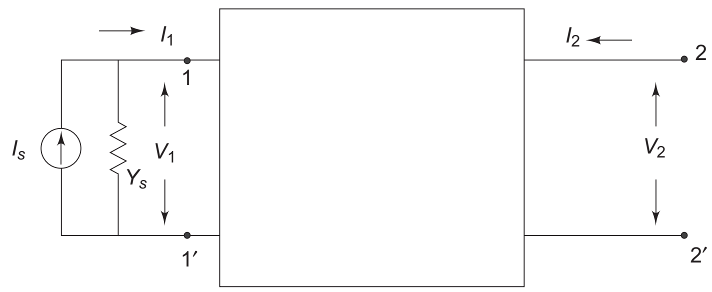


Fig. 15.28

At port 1-1′ $\quad I_1 = I_s - V_1\, Y_s$

Recalling Eqs. 15.3 and 15.4, we have

$$I_1 = Y_{11}\, V_1 + Y_{12}\, V_2$$
$$I_2 = Y_{21}\, V_1 + Y_{22}\, V_2$$

Substituting I_1 in Eq. 15.3, we get

$$I_s - V_1 Y_s = Y_{11} V_1 + Y_{12} V_2$$
$$-V_1(Y_s + Y_{11}) = Y_{12}\, V_2 - I_s$$
$$-V_1 = \frac{Y_{12}\, V_2 - I_s}{Y_s + Y_{11}}$$

Substituting V_1 in Eq. 15.4, we get

$$I_2 = -Y_{21}\left(\frac{Y_{12}\, V_2 - I_s}{Y_s + Y_{11}}\right) + Y_{22}\, V_2$$

With no source current at 1-1′, i.e. if the current source is open circuited

$$I_2 = \frac{-Y_{21}\, Y_{12}\, V_2}{Y_s + Y_{11}} + Y_{22}\, V_2$$

Hence the driving point admittance at the output port is given by

$$\frac{I_2}{V_2} = \frac{Y_{22}\, Y_s + Y_{22}\, Y_{11} - Y_{21}\, Y_{12}}{Y_s + Y_{11}} \text{ or } \frac{\Delta_y + Y_{22}\, Y_s}{Y_s + Y_{11}}$$

If the source admittance is zero, with an open circuited input port, the driving point admittance at the output port is given by

$$\frac{I_2}{V_2} = \frac{\Delta_y}{Y_{11}} = \frac{1}{Z_{22}} = Y_{22}$$

In a similar way, the output impedance may be expressed in terms of the other two port parameters by simple mathematical manipulations. The results are given in Table 15.2.

Table 15.2

			In terms of			
Driving point impedance at input port, or input impedance $\left(\frac{V_1}{I_1}\right)$	Z parameters $\frac{\Delta_z + Z_{11} Z_L}{Z_{22} + Z_L}$	Y parameters $\frac{Y_{22} + Y_L}{\Delta_y + Y_{11} Y_L}$	$ABCD$ $\frac{AZ_L + B}{CZ_L + D}$	$A'B'C'D'$ $\frac{B' - D'Z_L}{C'Z_L - A'}$	h parameter $\frac{\Delta_h Z_L + h_{11}}{1 + h_{22} Z_L}$	g parameter $\frac{1 + g_{22} Y_L}{\Delta_{gYL} + g_{11}}$
Driving point impedance at output port, or output impedance $\left(\frac{V_2}{I_2}\right)$	$\frac{\Delta_z + Z_{22} Z_s}{Z_1 + Z_{11}}$	$\frac{Y_{11} + Y_s}{\Delta_y + Y_s Y_{22}}$	$\frac{DZ_s + B}{CZ_s + A}$	$\frac{A'Z_s + B'}{C'Z_s + D'}$	$\frac{h_{11} + Z_s}{\Delta_h + h_{22} Z_s}$	$\frac{g_{22} + \Delta_s}{1 + g_{11} Z_s}$

Note The above relations are obtained, when $V_s = 0$ and $I_s = 0$ at the input port.

Example 15.11 Calculate the input impedance of the network shown in Fig. 15.29.

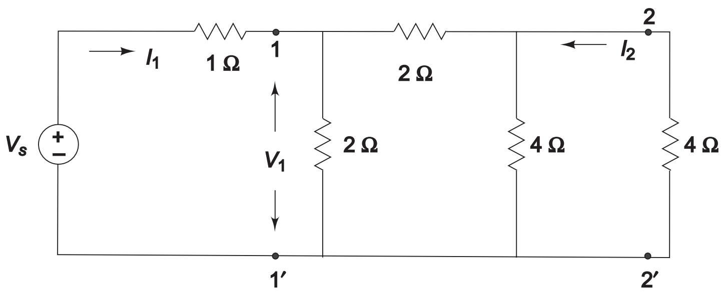


Fig. 15.29

Solution Let us calculate the input impedance in terms of Z parameters. The Z parameters of the given network (see Solved Problem 15.1) are $Z_{11} = 2.5\ \Omega$; $Z_{21} = 1\ \Omega$; $Z_{22} = 2\ \Omega$; $Z_{12} = 1\ \Omega$

From Section 15.11.1 we have the relation

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12}\,Z_{21}}{Z_L + Z_{22}}$$

where Z_L is the load impedance = 2 Ω

$$\frac{V_1}{I_1} = 2.5 - \frac{1}{2+2} = 2.25\ \Omega$$

The source resistance is 1 Ω

$\therefore$ $$Z_{in} = 1 + 2.25 = 3.25\ \Omega$$

Example 15.12 Calculate the output impedance of the network shown in Fig. 15.30 with a source admittance of 1 ℧ at the input port.

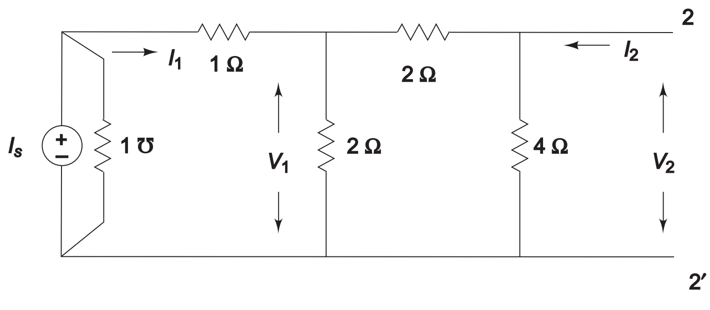


Fig. 15.30

Solution Let us calculate the output impedance in terms of Y parameters. The Y parameters of the given network (see Ex. 15.2) are

$$Y_{11} = \frac{1}{2}\ ℧;\ Y_{22} = \frac{5}{8}\ ℧,\ Y_{21} = Y_{12} = \frac{-1}{4}\ ℧$$

From Section 15.11.2, we have the relation

$$\frac{I_2}{V_2} = \frac{Y_{22}Y_s + Y_{22}Y_{11} - Y_{21}Y_{12}}{Y_s + Y_{11}}$$

where Y_s is the source admittance = 1 mho

$$Y_{22} = \frac{I_2}{V_2} = \frac{\frac{5}{8} \times 1 + \frac{5}{8} \times \frac{1}{2} - \frac{1}{16}}{1 + \frac{1}{2}} = \frac{7}{12}\ \mho$$

or $$Z_{22} = \frac{12}{7}\ \mho$$

15.12 LATTICE NETWORKS

One of the common four-terminal two-port network is the lattice, or bridge network shown in Fig. 15.31(a). Lattice networks are used in filter sections and are also used as attenuaters. Lattice structures are sometimes used in preference to ladder structures in some special applications. Z_a and Z_d are called series arms, Z_b and Z_c are called the diagonal arms. It can be observed that, if Z_d is zero, the lattice structure becomes a π-section. The lattice network is redrawn as a bridge network as shown in Fig. 15.31(b).

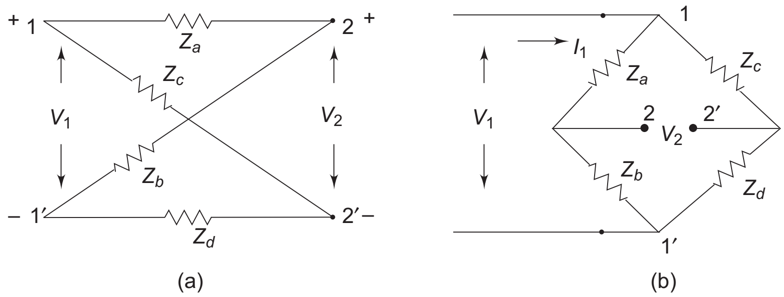


Fig. 15.31

Z Parameters

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0}$$

When $I_2 = 0$; $$V_1 = I_1 \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d} \tag{15.17}$$

$\therefore$ $$Z_{11} = \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric, then $Z_a = Z_d$ and $Z_b = Z_c$

$$\therefore \qquad Z_{11} = \frac{Z_a + Z_b}{2}$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

When $I_2 = 0$, V_2 is the voltage across 2–2′

$$V_2 = V_1\left[\frac{Z_b}{Z_a + Z_b} - \frac{Z_d}{Z_c + Z_d}\right]$$

Substituting the value of V_1 from Eq. 15.17, we have

$$V_2 = \left[\frac{I_1(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d}\right]\left[\frac{Z_b(Z_c + Z_d) - Z_d(Z_a + Z_b)}{(Z_a + Z_b)(Z_c + Z_d)}\right]$$

$$\frac{V_2}{I_1} = \frac{Z_b(Z_c + Z_d) - Z_d(Z_a + Z_b)}{Z_a + Z_b + Z_c + Z_d} = \frac{Z_bZ_c - Z_aZ_d}{Z_a + Z_b + Z_c + Z_d}$$

$$\therefore \qquad Z_{21} = \frac{Z_bZ_c - Z_aZ_d}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric, $Z_a = Z_d$, $Z_b = Z_c$

$$Z_{21} = \frac{Z_b - Z_a}{2}$$

When the input port is open, $I_1 = 0$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

The network can be redrawn as shown in Fig. 15.31(c).

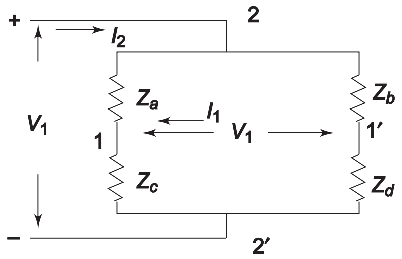


Fig. 15.31(c)

$$V_1 = V_2\left[\frac{Z_c}{Z_a + Z_c} - \frac{Z_d}{Z_b + Z_d}\right] \tag{15.18}$$

$$V_2 = I_2\left[\frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d}\right] \tag{15.19}$$

Substituting the value of V_2 in Eq. 15.18, we get

$$V_1 = I_2\left[\frac{Z_c(Z_b + Z_d) - Z_d(Z_a + Z_c)}{Z_a + Z_b + Z_c + Z_d}\right]$$

$$\frac{V_1}{I_2} = \frac{Z_c Z_b - Z_a Z_d}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric, $Z_a = Z_d$; $Z_b = Z_c$

$$\frac{V_1}{I_2} = \frac{Z_b^2 - Z_a^2}{2(Z_a + Z_b)}$$

$$\therefore \qquad Z_{12} = \frac{Z_b - Z_a}{2}$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_2 = 0}$$

From Eq. 15.19, we have

$$\frac{V_2}{I_2} = \frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric,

$$Z_a = Z_d;\ Z_b = Z_c$$

$$Z_{22} = \frac{Z_a + Z_b}{2} = Z_{11}$$

From the above equations, $Z_{11} = Z_{22} = \dfrac{Z_a + Z_b}{2}$

and $\qquad Z_{12} = Z_{21} = \dfrac{Z_b - Z_a}{2}$

$$\therefore \qquad Z_b = Z_{11} + Z_{12}$$

$$Z_a = Z_{11} - Z_{12}.$$

Example 15.13 Obtain the lattice equivalent of a symmetrical *T* network shown in Fig. 15.32.

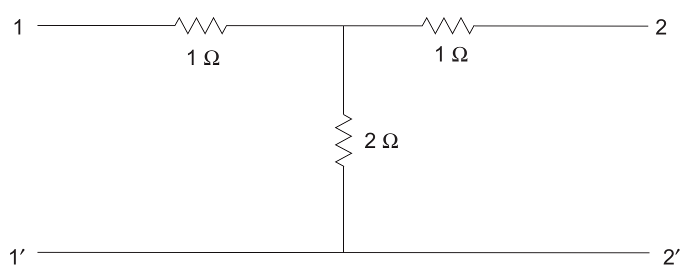


Fig. 15.32

Solution A two two-port network can be realised as a symmetric lattice if it is reciprocal and symmetric. The Z parameters of the network are (see Ex. 15.1). $Z_{11} = 3\ \Omega$; $Z_{12} = Z_{21} = 2\ \Omega$; $Z_{22} = 3\ \Omega$.

Since $Z_{11} = Z_{22}$; $Z_{12} = Z_{21}$, the given network is symmetrical and reciprocal

$\therefore$ The parameters of the lattice network are

$$Z_a = Z_{11} - Z_{12} = 1\ \Omega$$

$$Z_b = Z_{11} + Z_{12} = 5\ \Omega$$

The lattice network is shown in Fig. 15.33.

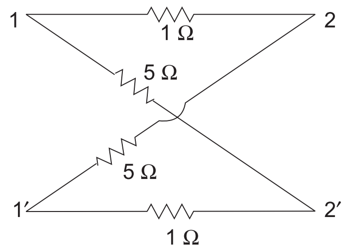


Fig. 15.33

Example 15.14 Obtain the lattice equivalent of a symmetric π-network shown in Fig. 15.34.

Solution The Z parameters of the given network are

$$Z_{11} = 6\ \Omega = Z_{22};\ Z_{12} = Z_{21} = 4\ \Omega$$

Hence the parameters of the lattice network are

$$Z_a = Z_{11} - Z_{12} = 2\ \Omega$$

$$Z_b = Z_{11} + Z_{12} = 10\ \Omega$$

The lattice network is shown in Fig.15.35

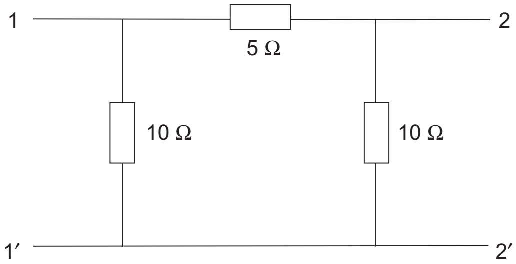


Fig. 15.34

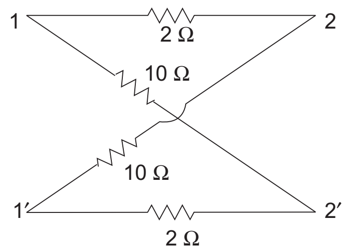


Fig. 15.35

15.13 IMAGE PARAMETERS

The image impedance Z_{I1} and Z_{I2} of a two-port network shown in Fig. 15.36 are two values of impedance such that, if port 1–1′ of the network is terminated in Z_{I1}, the input impedance of port 2-2′ is Z_{I2}; and if port 2-2′ is terminated in Z_{I2}, the input impedance at port 1-1′ is Z_{I1}.

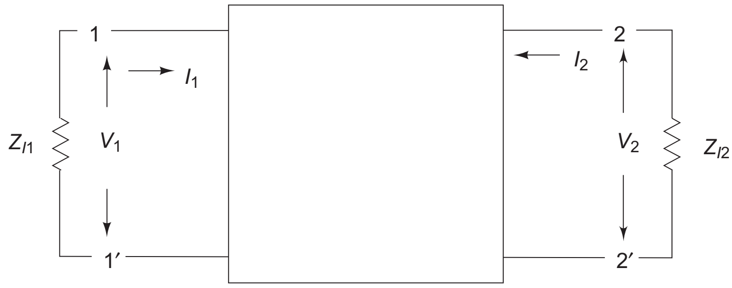


Fig. 15.36

Then, Z_{I1} and Z_{I2} are called image impedances of the two port network shown in Fig. 15.36. These parameters can be obtained in terms of two-port parameters. Recalling Eqs 15.5 and 15.6 in Section 15.4, we have

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

If the network is terminated in Z_{I2} at 2-2′ as shown in Fig. 15.37.

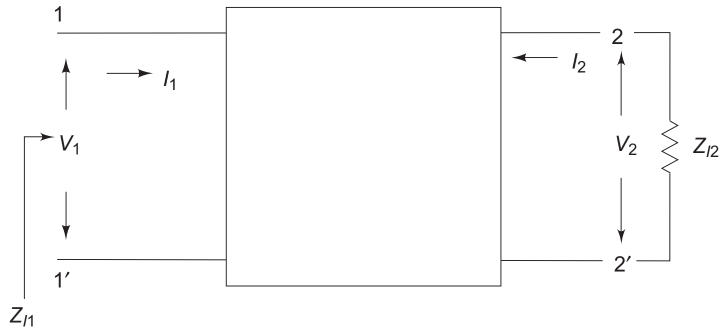


Fig. 15.37

$$V_2 = -I_2 Z_{I2}$$

$$\frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2} = Z_{I1}$$

$$Z_{I1} = \frac{-AI_2 Z_{I2} - BI_2}{-CI_2 Z_{I2} - DI_2}$$

$$Z_{I1} = \frac{-AZ_{I2} - B}{-CZ_{I2} - D}$$

or
$$Z_{I1} = \frac{AZ_{I2} + B}{CZ_{I2} + D}$$

Similarly, if the network is terminated in Z_{I1} at port 1-1′ as shown in Fig. 15.38, then

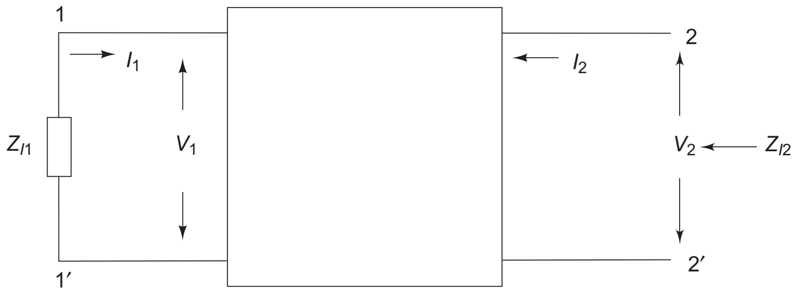


Fig. 15.38

$$V_1 = -I_1 Z_{I1}$$

$$\frac{V_2}{I_2} = Z_{I2}$$

$$\therefore \quad -Z_{I1} = \frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$

$$-Z_{I1} = \frac{AI_2 Z_{I2} - BI_2}{CI_2 Z_{I2} - DI_2}$$

$$-Z_{I1} = \frac{AZ_{I2} - B}{CZ_{I2} - D}$$

From which $\quad Z_{I2} = \dfrac{DZ_{I1} + B}{CZ_{I1} + A}$

Substituting the value of Z_{I1} in the above equation

$$Z_{I2}\left[C\frac{(-AZ_{I2} + B)}{(CZ_{I2} - D)} + A\right] = D\left[\frac{-AZ_{I2} + B}{CZ_{I2} - D}\right] + B$$

From which $Z_{I2} = \sqrt{\dfrac{BD}{AC}}$

Similarly, we can find $Z_{I1} = \sqrt{\dfrac{AB}{CD}}$

If the network is symmetrical, then $A = D$

$$\therefore \quad Z_{I1} = Z_{I2} = \sqrt{\frac{B}{C}}$$

If the network is symmetrical, the image impedances Z_{I1} and Z_{I2} are equal to each other; the image impedance is then called the *characteristic* impedance, or the *iterative* impedance, i.e. if a symmetrical network is terminated in Z_L, its input impedance will also be Z_L, or its impedance transformation ratio is unity. Since a reciprocal symmetric network can be described by two independent parameters, the image parameters Z_{I1} and Z_{I2} are sufficient to characterise reciprocal symmetric networks. Z_{I1} and Z_{I2} the two image parameters do not completely define a network. A third parameter called *image transfer constant* ϕ is also used to describe reciprocal networks. This parameter may be obtained from the voltage and current ratios.

If the image impedance Z_{I2} is connected across port 2-2′, then

$$V_1 = AV_2 - BI_2 \tag{15.20}$$

$$V_2 = -I_2 Z_{I2} \tag{15.21}$$

$$\therefore \quad V_1 = \left[A + \frac{B}{Z_{I2}}\right] V_2 \tag{15.22}$$

$$I_1 = CV_2 - DI_2 \tag{15.23}$$

$$I_1 = -[CZ_{I2} + D]I_2 \tag{15.24}$$

From Eq. 15.22

$$\frac{V_1}{V_2} = \left[A + \frac{B}{Z_{I2}}\right] = A + B\sqrt{\frac{AC}{BD}}$$

$$\frac{V_1}{V_2} = A + \sqrt{\frac{ABCD}{D}} \tag{15.25}$$

From Eq. 15.24

$$\frac{-I_1}{I_2} = [CZ_{I2} + D] = D + C\sqrt{\frac{BD}{AC}}$$

$$\frac{-I_1}{I_2} = D + \sqrt{\frac{ABCD}{A}} \tag{15.26}$$

Multiplying Eqs. 15.25 and 15.26 we have

$$\frac{-V_1}{V_2} \times \frac{I_1}{I_2} = \left(\frac{AD + \sqrt{ABCD}}{D}\right)\left(\frac{AD + \sqrt{ABCD}}{A}\right)$$

$$\frac{-V_1}{V_2} \times \frac{I_1}{I_2} = \left(\sqrt{AD} + \sqrt{BC}\right)^2$$

or $$\sqrt{AD} + \sqrt{BC} = \sqrt{\frac{-V_1}{V_2} \times \frac{I_1}{I_2}}$$

$$\sqrt{AD} + \sqrt{AD - 1} = \sqrt{\frac{-V_1}{V_2} \times \frac{I_1}{I_2}} \qquad (\because AD - BC = 1)$$

Let $$\cos h\,\phi = \sqrt{AD}\,;\ \sin h\,\phi = \sqrt{AD - 1}$$

$$\tan h\,\phi = \frac{\sqrt{AD - 1}}{\sqrt{AD}} = \sqrt{\frac{BC}{AD}}$$

$$\therefore \quad \phi = \tan h^{-1}\sqrt{\frac{BC}{AD}}$$

Also $$e^{\phi} = \cos h\ \phi + \sin h\ \phi = \sqrt{-\frac{V_1 I_1}{V_2 I_2}}$$

$$\phi = \log_e \sqrt{\left(-\frac{V_1 I_1}{V_2 I_2}\right)} = \frac{1}{2}\log_e\left(\frac{V_1}{V_2}\ \frac{I_1}{I_2}\right)$$

Since $$V_1 = Z_{I1} I_1;\ V_2 = -I_2 Z_{I2}$$

$$\phi = \frac{1}{2}\log_e\left[\frac{Z_{I1}}{Z_{I2}}\right] + \log\left[\frac{I_1}{I_2}\right]$$

For symmetrical reciprocal networks, $Z_{I1} = Z_{I2}$

$$\phi = \log_e\left[\frac{I_1}{I_2}\right] = \gamma$$

where γ is called the *propagation constant*.

Example 15.15 Determine the image parameters of the T network shown in Fig.15.39.

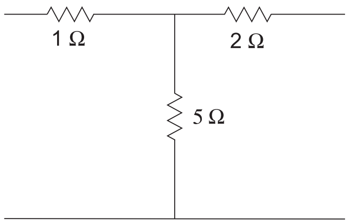


Fig. 15.39

Solution The *ABCD* parameters of the network are

$$A = \frac{6}{5};\ B = \frac{17}{5};\ C = \frac{1}{5};\ D = \frac{7}{5}\ \text{(See Ex. 15.3)}$$

Since the network is not symmetrical, ϕ, Z_{I1} and Z_{I2} are to be evaluated to describe the network.

$$Z_{I1} = \sqrt{\frac{AB}{CD}} = \sqrt{\frac{\frac{6}{5}\times\frac{17}{5}}{\frac{1}{5}\times\frac{7}{5}}} = 3.817\ \Omega$$

$$Z_{I2} = \sqrt{\frac{BC}{AC}} = \sqrt{\frac{\frac{17}{5}\times\frac{7}{5}}{\frac{6}{5}\times\frac{1}{5}}} = 4.453\ \Omega$$

$$\phi = \tan h^{-1}\sqrt{\frac{BC}{AD}} = \tan h^{-1}\sqrt{\frac{17}{42}}$$

or $$\phi = \ln\left[\sqrt{AD} + \sqrt{AD-1}\right]$$
$$\phi = 0.75$$

Additional Solved Problems

Problem 15.1 Find the Z parameters for the circuit shown in Fig. 15.40.

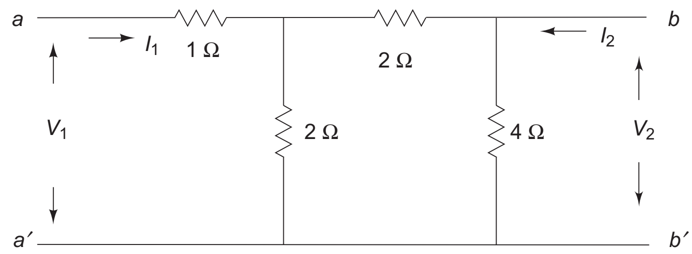


Fig. 15.40

Solution $$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

When $I_2 = 0$; V_1 can be expressed in terms of I_1 and the equivalent impedance of the circuit looking from the terminal a-a' as shown in Fig. 15.41(a).

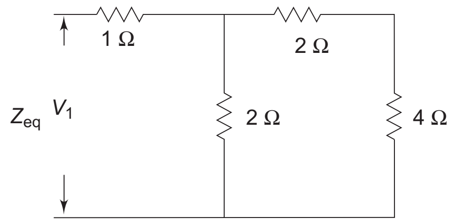


Fig. 15.41(a)

$$Z_{eq} = 1 + \frac{6\times 2}{6+2} = 2.5\ \Omega$$
$$V_1 = I_1 Z_{eq} = I_1\, 2.5$$
$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 2.5\ \Omega$$
$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

V_2 is the voltage across the 4 Ω impedance as shown in Fig. 15.41(b).

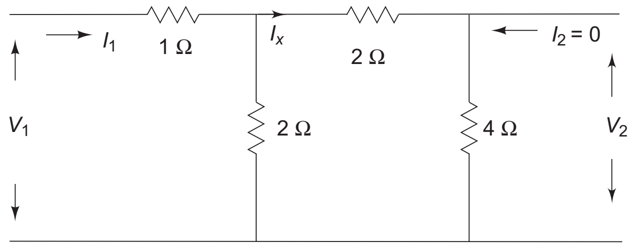


Fig. 15.41(b)

Let the current in the 4 Ω impedance be I_x

$$I_x = I_1 \times \frac{2}{8} = \frac{I_1}{4}$$

$$V_2 = I_x 4 = \frac{I_1}{4} \times 4 = I_1$$

$$Z_{21} = \left.\frac{V_2}{I_2}\right|_{I_2 = 0} = 1\ \Omega$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$

When port *a-a′* is open circuited the voltage at port *b-b′* can be expressed in terms of I_2, and the equivalent impedance of the circuit viewed from *b-b′* as shown in Fig. 15.41(c).

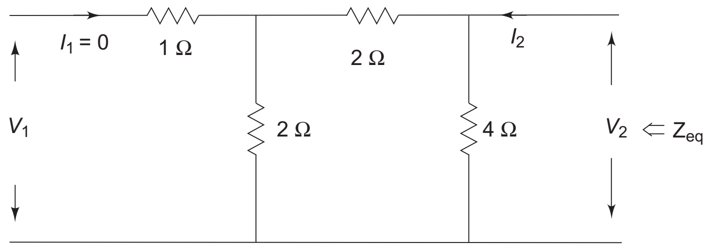


Fig. 15.41(c)

$$V_2 = I_2 \times 2$$

$$\therefore \quad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = 2\ \Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

V_1 is the voltage across the 2 Ω (parallel) impedance, let the current in the 2 Ω impedance is I_Y as shown in Fig. 15.41(d).

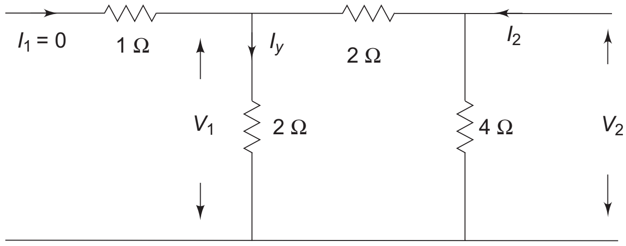


Fig. 15.41(d)

$$I_Y = \frac{I_2}{2}$$
$$V_1 = 2\,I_Y$$
$$V_1 = 2\,\frac{I_2}{2}$$
$$\therefore \qquad Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} = 1\ \Omega$$

Here $Z_{12} = Z_{21}$, which indicates the bilateral property of the network. The describing equations for this two-port network in terms of impedance parameters are

$$V_1 = 2.5I_1 + I_2$$
$$V_2 = I_1 + 2I_2$$

Problem 15.2 Find the short circuit admittance parameters for the circuit shown in Fig. 15.42.

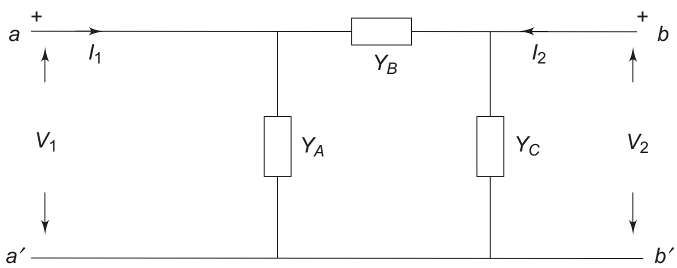


Fig. 15.42

Solution The elements in the branches of the given two-port network are admittances. The admittance parameters can be determined by short circuiting the two-ports.

When port b-b' is short circuited, $V_2 = 0$. This circuit is shown in Fig. 15.43(a).

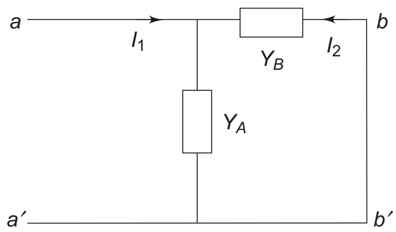


Fig. 15.43(a)

$$V_1 = I_1 Z_{eq}$$

where Z_{eq} is the equivalent impedance as viewed from a-a'.

$$Z_{eq} = \frac{1}{Y_{eq}}$$

$$Y_{eq} = Y_A + Y_B$$

$$V_1 = \frac{I_1}{Y_A + Y_B}$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = (Y_A + Y_B)$$

With port b-b' short circuited, the nodal equation at node 1 gives

$$-I_2 = V_1 Y_B$$

$$\therefore \qquad Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -Y_B$$

when port a-a' is short circuited; $V_1 = 0$ this circuit is shown in Fig. 15.43(b).

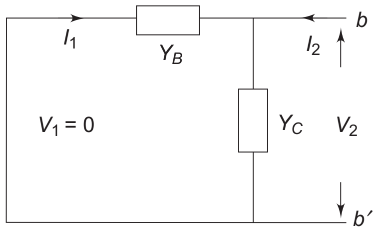


Fig. 15.43(b)

$$V_2 = I_2 Z_{eq}$$

where Z_{eq} is the equivalent impedance as viewed from b-b'

$$Z_{eq} = \frac{1}{Y_{eq}}$$

$$Y_{eq} = Y_b + Y_c$$

$$\therefore \qquad V_2 = \frac{I_2}{Y_B + Y_C}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} = (Y_B + Y_C)$$

With port a-a' short circuited, the nodal equation at node 2 gives

$$-I_1 = V_2\, Y_B$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0} = -\,Y_B$$

The describing equations in terms of the admittance parameters are

$$I_1 = (Y_A + Y_B)V_1 - Y_B V_2$$
$$I_2 = -\,Y_B V_1 + (Y_C + Y_B)V_2$$

Problem 15.3 Find the Z parameters of the RC ladder network shown in Fig. 15.44.

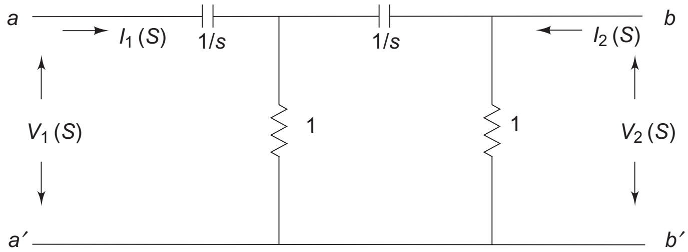


Fig. 15.44

Solution With port b-b' open circuited and assuming mesh currents with $V_1(S)$ as the voltage at a-a', the corresponding network is shown in Fig. 15.45(a).

The KVL equations are as follows

$$V_2(S) = I_3(S) \tag{15.27}$$

$$I_3(S) \times \left(2 + \frac{1}{S}\right) = I_1(S) \tag{15.28}$$

$$\left(1 + \frac{1}{S}\right) I_1(S) - I_3(S) = V_1(S) \tag{15.29}$$

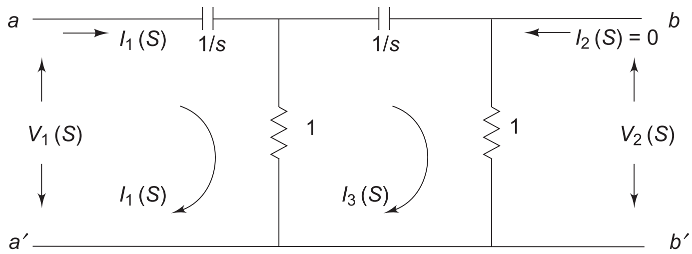


Fig. 15.45(a)

From Eq. 15.28, $I_3(S) = I_1(S)\left(\dfrac{S}{1+2S}\right)$

From Eq. 15.29 $\left(\dfrac{S+1}{S}\right) I_1(S) - I_1(S)\,\dfrac{S}{1+2S} = V_1(S)$

$$I_1(S)\left(\frac{1+S}{S} - \frac{S}{1+2S}\right) = V_1(S)$$

$$I_1(S)\left(\frac{S^2+3S+1}{S(1+2S)}\right) = V_1(S)$$

$$Z_{11} = \left.\frac{V_1(S)}{I_1(S)}\right|_{I_2=0} = \frac{(S^2+3S+1)}{S(1+2S)}$$

Also $V_2(S) = I_3(S) = I_1(S)\,\dfrac{S}{1+2S}$

$$Z_{21} = \left.\frac{V_2(S)}{I_1(S)}\right|_{I_2=0} = \frac{S}{1+2S}$$

With port *a*-*a′* open circuited and assuming mesh currents with $V_2(S)$ as the voltage as *b*-*b′*, the corresponding network is shown in Fig. 15.45(b).

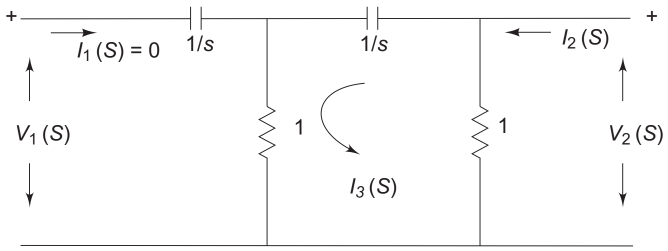


Fig. 15.45(b)

The KVL equations are as follows

$$V_1(S) = I_3(S) \tag{15.30}$$

$$\left(2+\frac{1}{S}\right) I_3(S) = I_2(S) \tag{15.31}$$

$$V_2(S) = I_2(S) - I_3(S) \tag{15.32}$$

From Eq. 15.31 $I_3(S) = I_2(S)\left(\dfrac{S}{2S+1}\right)$

From Eq. 15.32 $V_2(S) = I_2(S) - I_2(S)\left(\dfrac{S}{2S+1}\right)$

$$V_2(S) = I_2(S)\left(1 - \frac{S}{2S+1}\right)$$

$$Z_{22} = \left.\frac{V_2(S)}{I_2(S)}\right|_{I_1(S)=0} = \frac{S+1}{2S+1}$$

Also $$V_1(S) = I_3(S) = I_2(S)\left(\frac{S}{2S+1}\right)$$

$$Z_{12} = \left.\frac{V_1(S)}{I_2(S)}\right|_{I_1(S)=0} = \left(\frac{S}{2S+1}\right)$$

The describing equations are

$$V_1(S) = \left[\frac{S^2+3S+1}{3(2S+1)}\right] I_1 + \left[\frac{S}{2S+1}\right] I_2$$

$$V_2(S) = \left[\frac{S}{2S+1}\right] I_1 + \left[\frac{S+1}{2S+1}\right] I_2$$

Problem 15.4 Find the transmission parameters for the circuit shown in Fig. 15.46.

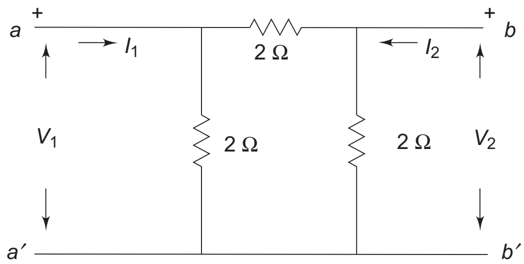


Fig. 15.46

Solution Recalling Eqs 15.5 and 15.6, we have

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

When port b-b' is short circuited with V_1 across a-a', $V_2 = 0$ $B = \frac{-V_1}{I_2}$ and the circuit is as shown in Fig. 15.47(a)

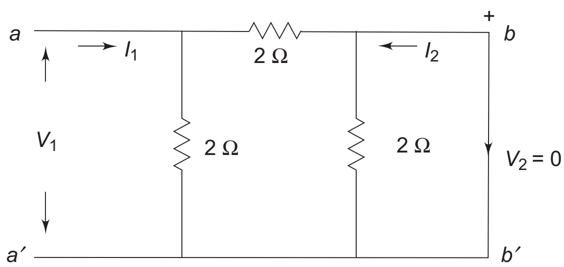


Fig. 15.47(a)

$$-I_2 = \frac{V_1}{2}\ I_1 = V_1$$

$\therefore$ $$B = 2\ \Omega$$

$$D = \frac{-I_1}{I_2} = 2$$

When port b-b' is open with V_1 across a-a', $I_2 = 0$

$A = V_1/V_2$ and the circuit is as shown in Fig. 15.47(b), where V_1 is the voltage across the 2 Ω resistor across port a-a' and V_2 is the voltage across the 2 Ω resistor across port b-b' when $I_2 = 0$.

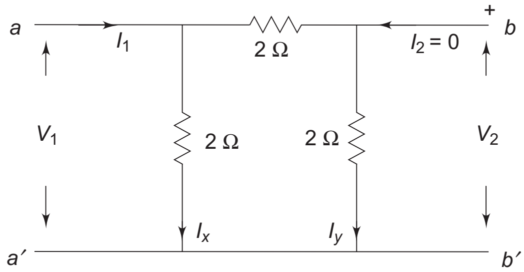


Fig. 15.47(b)

From Fig. 15.47(b), $I_Y = \dfrac{V_1}{4}$

$$V_2 = 2 \times I_Y = \frac{V_1}{2}$$

$$A = 2$$

From Fig. 15.47(b) $I_x = \dfrac{V_1}{2}$

$$C = \frac{I_1}{V_2}$$

where $$I_1 = \frac{3V_1}{4}$$

Therefore $$C = \frac{3}{2}\ \mho$$

Problem 15.5 Find h parameters for the network in Fig. 15.48.

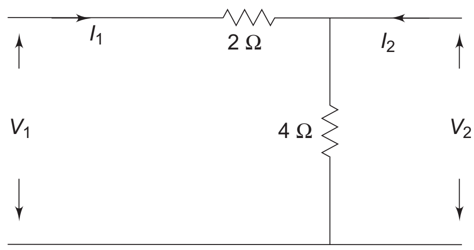


Fig. 15.48

Solution When $V_2 = 0$ the network is as shown in Fig. 15.49.

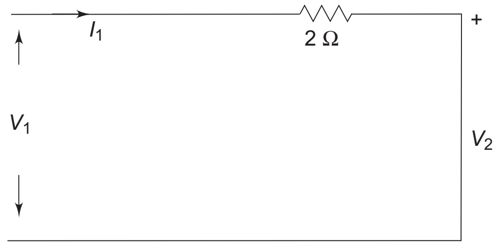


Fig. 15.49

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2 = 0} = 2\ \Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2 = 0} ;\ I_2 = -I_1$$

$\therefore$ $$h_{21} = -1$$

When $$I_1 = 0;\ h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1 = 0} ;\ h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1 = 0}$$

$$V_1 = I_2\,4,\ V_2 = I_2\,4$$

$\therefore$ $$h_{12} = 1\ h_{22} = \frac{1}{4}\ \mho$$

Problem 15.6 For the hybrid equivalent circuit shown in Fig. 15.50, (a) determine the current gain, and (b) determine the voltage gain.

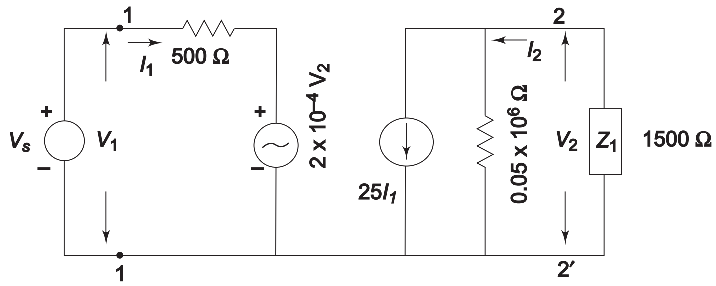


Fig. 15.50

Solution From port 2-2′ we can find

$$I_2 = \frac{(25I_1)(0.05 \times 10^6)}{(1500 + 0.05 \times 10^6)}$$

(a) current gain $\dfrac{I_2}{I_1} = \dfrac{1.25 \times 10^6}{0.0515 \times 10^6} = 24.3$

(b) applying KVL at port 1-1′

$$V_1 = 500\, I_1 + 2 \times 10^{-4}\, V_2$$

$$I_1 = \frac{V_1 - 2\times 10^{-4}\, V_2}{500} \qquad (15.33)$$

Applying KCL at port 2-2′

$$I_2 = 25 I_1 + \frac{V_2}{0.05} \times 10^{-6}$$

also $$I_2 = \frac{-V_2}{1500}$$

$$\frac{-V_2}{1500} = 25 I_1 + \frac{V_2}{0.05} \times 10^{-6}$$

Substituting the value of I_1 from Eq. 15.33, in the above equation, we get

$$\frac{-V_2}{1500} = 25\left(\frac{V_1 - 2\times 10^{-4}\, V_2}{500}\right) + \frac{V_2}{0.05} \times 10^{-6}$$

$$-6.6 \times 10^{-4}\, V_2 = 0.05 V_1 - 0.1 \times 10^{-4}\, V_2 + 0.2 \times 10^{-4}\, V_2$$

$$\therefore \qquad \frac{V_2}{V_1} = -73.89$$

The negative sign indicates that there is a 180° phase shift between input and output voltage.

Problem 15.7 The hybrid parameters of a two-port network shown in Fig. 15.51 are $h_{11} = 1$ K; $h_{12} = 0.003$; $h_{21} = 100$; $h_{22} = 50\ \mu℧$. Find V_2 and Z parameters of the network.

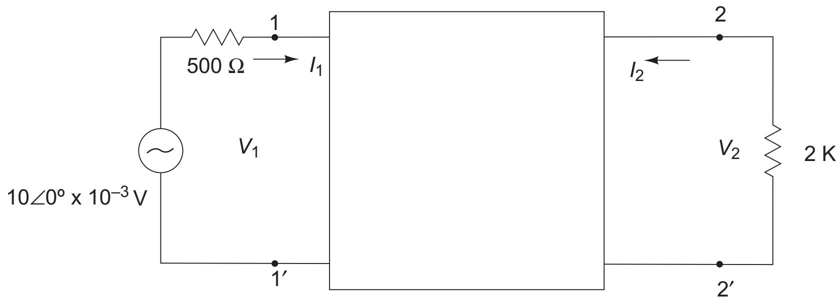


Fig. 15.51

Solution $$V_1 = h_{11}\, I_1 + h_{12}\, V_2 \qquad (15.34)$$

$$I_2 = h_{21}\, I_1 + h_{22}\, V_2 \qquad (15.35)$$

At port 2-2′ $\quad V_2 = -I_2\, 2000$

Substituting in Eq. 15.35, we have

$$I_2 = h_{21} I_1 - h_{22} I_2\, 2000$$

$$I_2\,(1 + h_{22}\, 2000) = h_{21}\, I_1$$

$$I_2 (1 + 50 \times 10^{-6} \times 2000) = 100\, I_1$$

$$I_2 = \frac{100 I_1}{1.1}$$

Substituting the value of V_2 in Eq. 15.34, we have

$$V_1 = h_{11} I_1 - h_{12} I_2 \, 2000$$

Also at port 1-1′, $\quad V_1 = V_S - I_1 \, 500$

$$\therefore \quad V_S - I_1 \, 500 = h_{11} I_1 - h_{12} \frac{100 I_1}{1.1} \times 2000$$

$$(10 \times 10^{-3}) - 500 \, I_1 = 1000 \, I_1 - 0.003 \times \frac{100}{1.1} I_1 \times 2000$$

$$954.54 I_1 = 10 \times 10^{-3}$$

$$I_1 = 10.05 \times 10^{-6} \text{ A}$$

$$V_1 = V_S - I_1 \times 500$$

$$= 10 \times 10^{-3} - 10.5 \times 10^{-6} \times 500 = 4.75 \times 10^{-3} \text{ V}$$

$$V_2 = \frac{V_1 - h_{11} I_1}{h_{12}}$$

$$V_2 = \frac{4.75 \times 10^{-3} - 1000 \times 10.5 \times 10^{-6}}{0.003} = -1.916 \text{ V}$$

(b) Z parameters of the network can be found from Table 15.1.

$$Z_{11} = \frac{\Delta_h}{h_{22}} = \frac{h_{11}h_{22} - h_{21}h_{12}}{h_{22}} = \frac{1 \times 10^3 \times 50 \times 10^{-6} - 100 \times 0.003}{50 \times 10^{-6}}$$

$$= -5000 \; \Omega$$

$$Z_{12} = \frac{h_{12}}{h_{22}} = \frac{0.003}{50 \times 10^{-6}} = 60 \; \Omega$$

$$Z_{21} = \frac{-h_{21}}{h_{22}} = \frac{-100}{50 \times 10^{-6}} = -2 \times 10^6 \; \Omega$$

$$Z_{22} = \frac{1}{h_{22}} = 20 \times 10^3 \; \Omega$$

Problem 15.8 The Z parameters of a two port network shown in Fig. 15.52 are $Z_{11} = Z_{22} = 10 \; \Omega$; $Z_{21} = Z_{12} = 4 \; \Omega$. If the source voltage is 20 V, determine I_1, V_2, I_2 and input impedance.

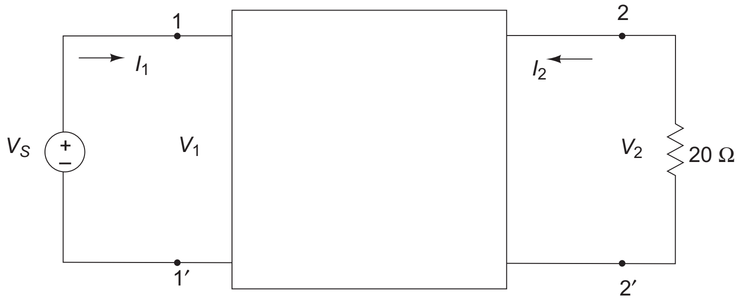


Fig. 15.52

Solution Given $V_1 = V_S = 20$ V

From Section 15.11.1, $V_1 = I_1\left(Z_{11} - \dfrac{Z_{12}\,Z_{21}}{Z_L + Z_{22}}\right)$

where $Z_L = 20\ \Omega$

$$\therefore \qquad 20 = I_1\left(10 - \frac{4\times 4}{20+10}\right)$$

$$I_1 = 2.11 \text{ A}$$

$$I_2 = -I_1\,\frac{Z_{21}}{Z_L + Z_{22}} = -2.11 \times \frac{4}{20+10} = -0.281 \text{ A}$$

At port 2-2′

$$V_2 = -I_2 \times 20 = 0.281 \times 20 = 5.626 \text{ V}$$

Input impedance $= \dfrac{V_1}{I_1} = \dfrac{20}{2.11} = 9.478\ \Omega$

Problem 15.9 The Y parameters of the two-port network shown in Fig. 15.53 are $Y_{11} = Y_{22} = 6$ ℧; $Y_{12} = Y_{21} = 4$ ℧

(a) determine the driving point admittance at port 2-2′ if the source voltage is 100 V and has an impedance of 1 ohm.

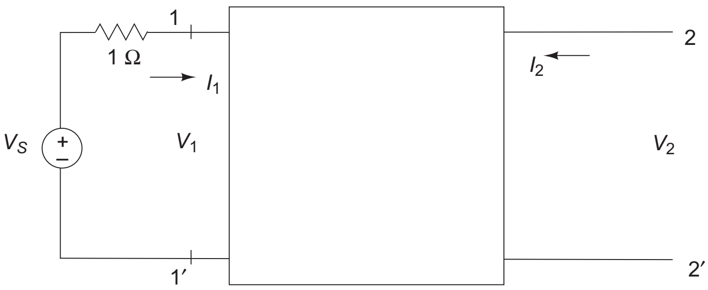


Fig. 15.53

Solution From Section 15.11.2,

$$\frac{I_2}{V_2} = \frac{Y_{22}\,Y_S + Y_{22}Y_{11} - Y_{21}Y_{12}}{Y_S + Y_{11}}$$

where Y_S is the source admittance = 1 ℧

$\therefore$ The driving point admittance $= \dfrac{6\times 1 + 6\times 6 - 4\times 4}{1+6} = 3.714$ ℧

Or the driving point impedance at port 2-2′ $= \dfrac{1}{3.714}\ \Omega$

Problem 15.10 Obtain the Z parameters for the two-port unsymmetrical lattice network shown in Fig. 15.54.

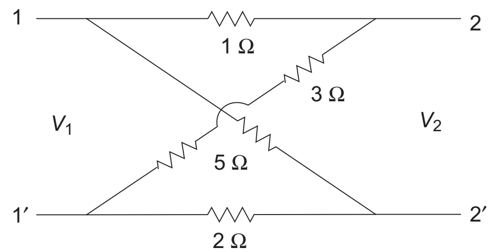


Fig. 15.54

Solution From Section 15.12, we have

$$Z_{11} = \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d} = \frac{(1+3)(2+5)}{1+3+5+2} = 2.545\ \Omega$$

$$Z_{21} = \frac{Z_b Z_c - Z_a Z_d}{Z_a + Z_b + Z_c + Z_d} = \frac{3\times 5 - 1\times 2}{11} = 1.181\ \Omega$$

$$Z_{21} = Z_{12}$$

$$Z_{22} = \frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d} = \frac{(1+5)\,(2+3)}{11} = 2.727\ \Omega$$

Problem 15.11 For the ladder two-port network shown in Fig. 15.55, find the open circuit driving point impedance at port 1-2.

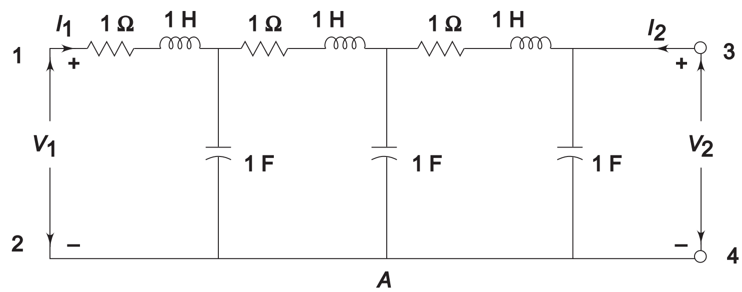


Fig. 15.55

Solution The Laplace transform of the given network is shown in Fig. 15.56.

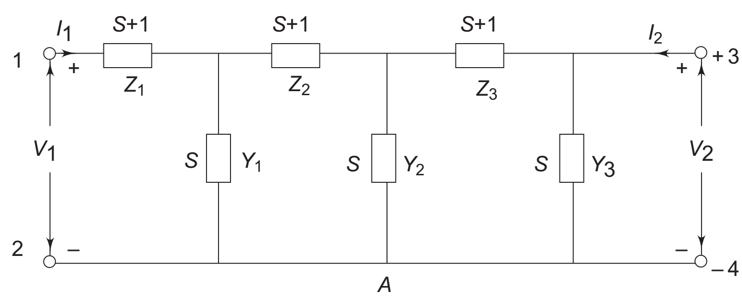


Fig. 15.56

Then the open circuit driving point impedance at port 1-2 is given by

$$Z_{11} = (s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s}}}}}$$

$$= \frac{s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1}{s^5 + 2s^4 + 5s^3 + 4s^2 + 3s}$$

Problem 15.12 For the bridged T network shown in Fig. 15.57, find the driving point admittance y_{11} and transfer admittance y_{21} with a 2 Ω load resistor connected across port 2.

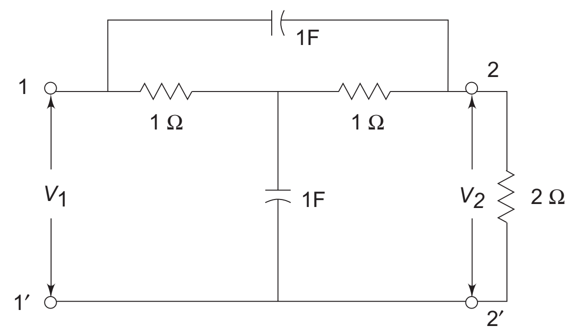


Fig. 15.57

Solution The corresponding Laplace transform network is shown in Fig. 15.58.

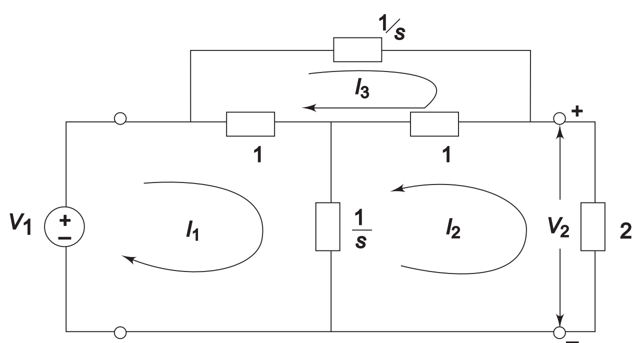


Fig. 15.58

The loop equations are

$$I_1\left(1+\frac{1}{s}\right) + I_2\left(\frac{1}{s}\right) - I_3 = V_1$$

$$I_1\left(\frac{1}{s}\right) + I_2\left(1+\frac{1}{s}\right) + I_3 = 0$$

$$I_1(-1) + I_2 + I_3\left(2 + \frac{1}{s}\right) = 0$$

Therefore,

$$\Delta = \begin{vmatrix} \left(1 + \frac{1}{s}\right) & \frac{1}{s} & -1 \\ \frac{1}{s} & 1 + \frac{1}{s} & 1 \\ -1 & 1 & 2 + \frac{1}{s} \end{vmatrix} = \frac{s+2}{s^2}$$

Similarly, $$\Delta_{11} = \begin{vmatrix} \left(1 + \frac{1}{s}\right) & \frac{1}{s} \\ 1 & \left(2 + \frac{1}{s}\right) \end{vmatrix} = \frac{s^2 + 3s + 1}{s^2}$$

and $$\Delta_{12} = \begin{vmatrix} \frac{1}{s} & +1 \\ +1 & \left(2 + \frac{1}{s}\right) \end{vmatrix} = \frac{s^2 + 2s + 1}{s^2}$$

Hence, $$y_{11} = \frac{\Delta_{11}}{\Delta} = \frac{s^2 + 3s + 1}{s + 2}$$

and $$y_{21} = \frac{\Delta_{12}}{\Delta} = \frac{-\left(s^2 + 2s + 1\right)}{s + 2}$$

Problem 15.13 For the two port network shown in Fig. 15.59, determine the h-parameters. Using these parameters calculate the output (Port 2) voltage, V_2, when the output port is terminated in a 3 Ω resistance and a 1V (dc) is applied at the input port (V_1 = 1 V).

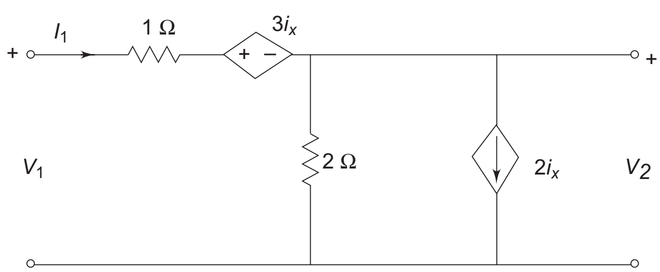


Fig. 15.59

Solution The h parameters are defined as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For $V_2 = 0$, the circuit is redrawn as shown in Fig. 15.60(a)

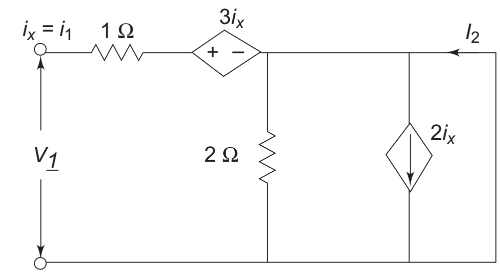


Fig. 15.60(a)

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{i_1 \times 1 + 3i_1}{i_1} = 4$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{i_2}{i_1} = \frac{2i_1 - i_1}{i_1} = 1$$

For $I_1 = 0$, the circuit is redrawn as shown in Fig. 15.60(b).

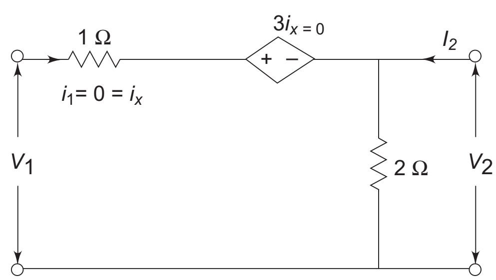


Fig. 15.60(b)

$$h_{12} = \frac{V_1}{V_2} = 1;\ h_{22} = \frac{I_2}{V_2} = \frac{1}{2} = 0.5$$

Hence,
$$h = \begin{bmatrix} 4 & 1 \\ 1 & 0.5 \end{bmatrix}$$

$$V_1 = 1 \text{ V}$$
$$V_1 = 4I_1 + V_2$$
$$I_2 = I_1 + 0.5\ V_2$$

Eliminating I_1 from the above equations and putting

$$V_1 = 1 \text{ and } I_2 = \frac{-V_2}{3} \text{ we get, } V_2 = \frac{-3}{7} \text{ V}$$

Problem 15.14 Find the current transfer ratio $\frac{I_2}{I_1}$ for the network shown in Fig. 15.61.

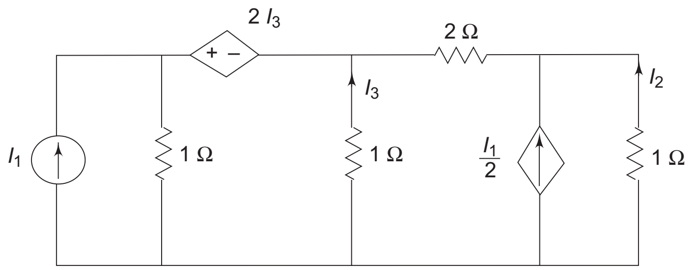


Fig. 15.61

Solution By transforming the current source into voltage source, the given circuit can be redrawn as shown in Fig. 15.62.

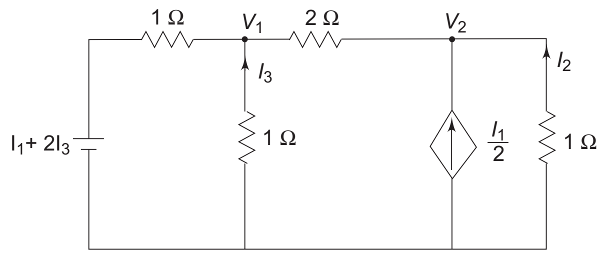


Fig. 15.62

Applying Kirchhoff's nodal analysis

$$\frac{V_1 - (I_1 + 2I_3)}{1} + \frac{V_1}{1} + \frac{V_1 - V_2}{2} = 0$$

and
$$\frac{V_2 - V_1}{2} - \frac{I_1}{2} - I_2 = 0$$

Putting $V_1 = -I_3$ and $V_2 = -I_2$

The above equations become

$$-I_3 - I_1 - 2I_3 - I_3 + \frac{I_2 - I_3}{2} = 0$$

and
$$\frac{I_2 - I_3}{2} - \frac{I_1}{2} - I_2 = 0$$

or
$$I_1\ 0.5I_2 - 4.5\ I_3 = 0$$

and
$$-0.5\ I_1 - 1.5I_2 + 0.5I_3 = 0$$

By eliminating I_3, we get

$$\frac{I_2}{I_1} = \frac{-5.5}{13} = -0.42$$

Practice Problems

15.1 Find the Z parameters of the network shown in Fig. 15.63.

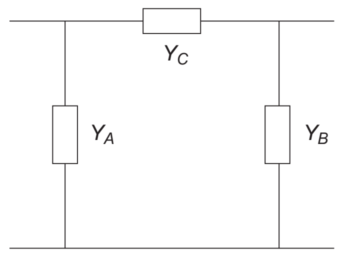


Fig. 15.63

15.2 Find the transmission parameters for the R–C network shown in Fig. 15.64.

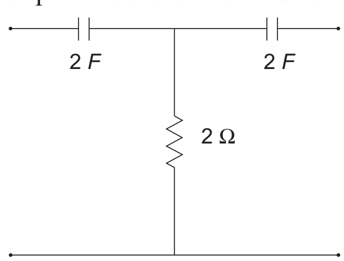


Fig. 15.64

15.3 Find the inverse transmission parameters for the network in Fig. 15.65.

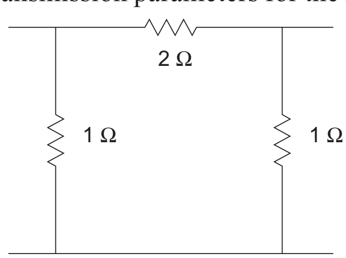


Fig. 15.65

15.4 Calculate the overall transmission parameters for the cascaded network shown in Fig. 15.66.

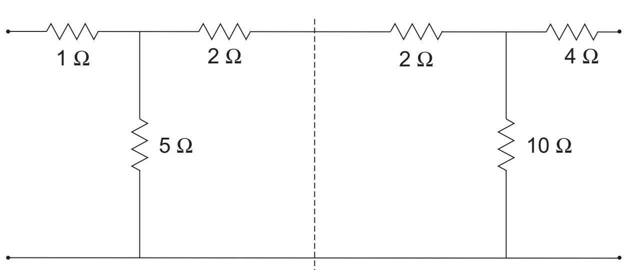


Fig. 15.66

15.5 For the two-port network shown in Fig. 15.67, find the h parameters and the inverse h parameters.

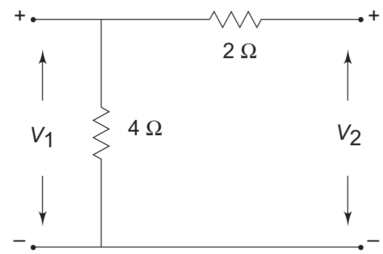


Fig. 15.67

15.6 Determine the impedance parameters for the T network shown in Fig. 15.68 and draw the Z parameter equivalent circuit.

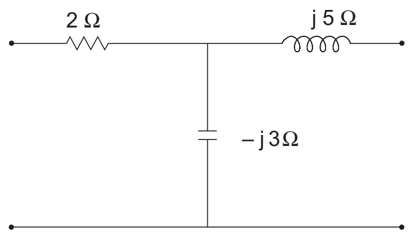


Fig. 15.68

15.7 Determine the admittance parameters for the π-network shown in Fig. 15.69 and draw the Y parameter equivalent circuit.

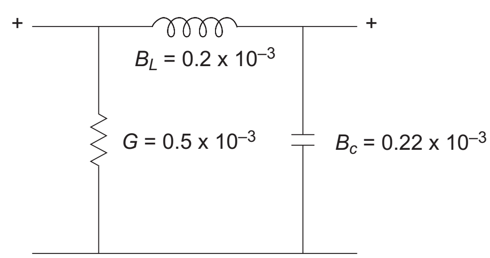


Fig. 15.69

15.8 Determine the impedance parameters and the transmission parameters for the network in Fig. 15.70.

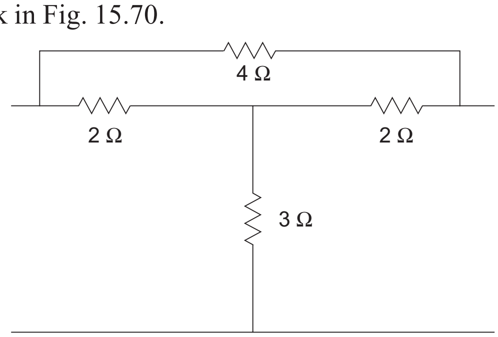


Fig. 15.70

15.9 For the hybrid equivalent circuit shown in Fig. 15.71, determine (a) the input impedance (b) the output impedance.

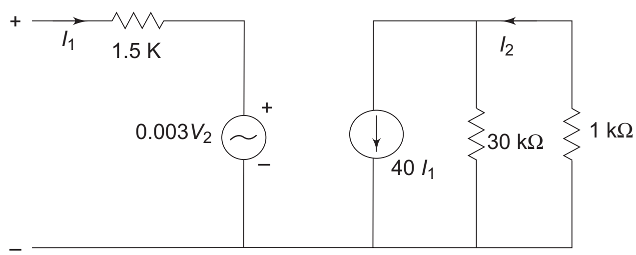


Fig. 15.71

15.10 Determine the input and output impedances for the Z parameter equivalent circuit shown in Fig. 15.72

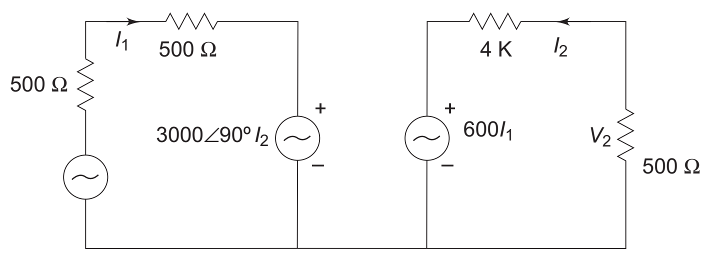


Fig. 15.72

15.11 The hybrid parameters of a two-port network shown in Fig. 15.73 are h_{11} = 1.5 K; $h_{12} = 2 \times 10^{-3}$; $h_{21} = 250$; $h_{22} = 150 \times 10^{-6}$ ℧ (a) Find V_2 (b). Draw the Z parameter equivalent circuit.

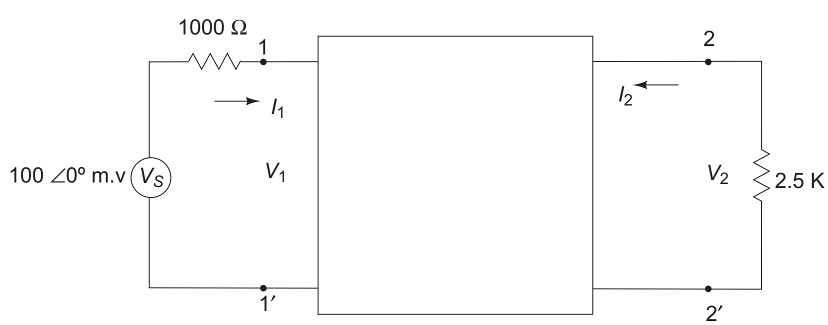


Fig. 15.73

15.12 The Z parameters of a two-port network shown in Fig. 15.74 are $Z_{11} = 5\ \Omega$; $Z_{12} = 4\ \Omega$; $Z_{22} = 10\ \Omega$; $Z_{21} = 5\ \Omega$. If the source voltage is 25 V, determine I_1, V_2 I_2, and the driving point impedance at the input port.

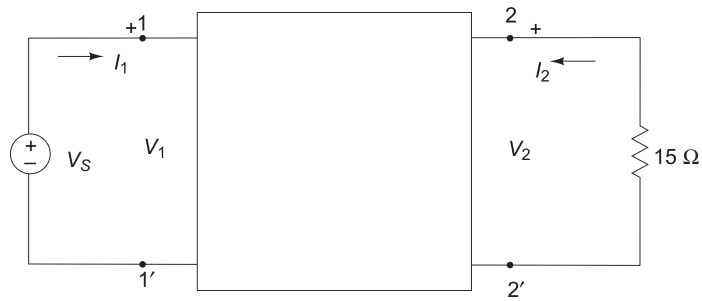


Fig. 15.74

15.13 Obtain the image parameters of the symmetric lattice network given in Fig. 15.75.

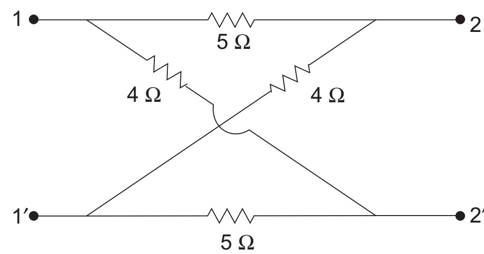


Fig. 15.75

15.14 Determine the Z parameters and image parameters of a symmetric lattice network whose series arm impedance is 10 Ω and diagonal arm impedance is 20 Ω.

15.15 For the network shown in Fig. 15.76, determine all four open circuit impedance parameters.

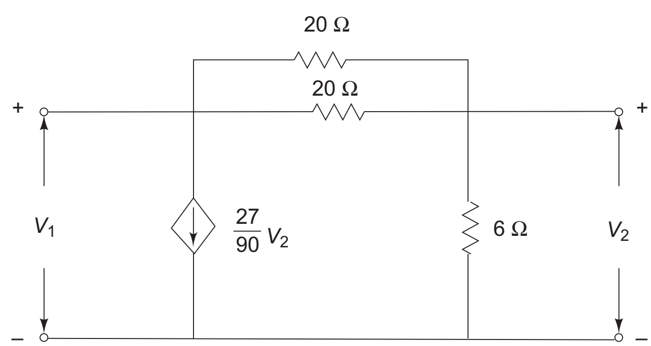


Fig. 15.76

15.16 For the network shown in Fig. 15.77, determine y_{12} and y_{21}

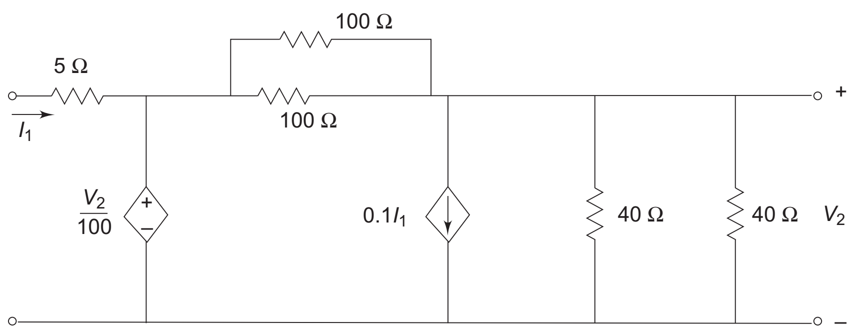


Fig. 15.77

15.17 For the network shown in Fig.15.78, determine h parameters at $\omega = 10^8$ rad/sec.

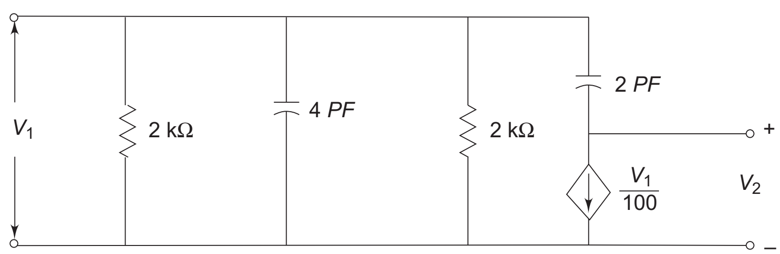


Fig. 15.78

15.18 For the network shown in Fig. 15.79, determine y parameters.

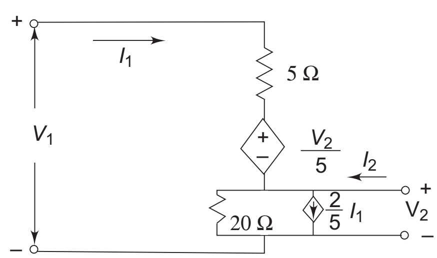


Fig. 15.79

Objective-type Questions

1. A two-port network is simply a network inside a black box, and the network has only
   (a) two terminals
   (b) two pairs of accessible terminals
   (c) two pairs of ports

2. The number of possible combinations generated by four variables taken two at a time in a two-port network is
   (a) four (b) two (c) six
3. What is the driving-point impedance at port one with port two open circuited for the network in Fig. 15.80?

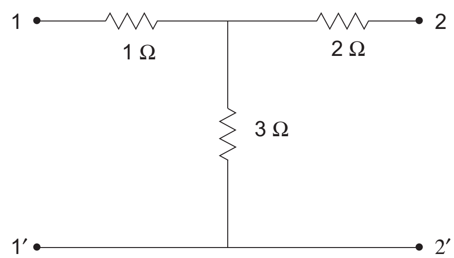


Fig. 15.80

   (a) 4 Ω (b) 5 Ω (c) 3 Ω
4. What is the transfer impedance of the two-port network shown in Fig. 15.80?
   (a) 1 Ω (b) 2 Ω (c) 3 Ω
5. If the two-port network in Fig. 15.80 is reciprocal or bilateral then
   (a) $Z_{11} = Z_{22}$ (b) $Z_{12} = Z_{21}$ (c) $Z_{11} = Z_{12}$
6. What is the transfer admittance of the network shown in Fig. 15.81.

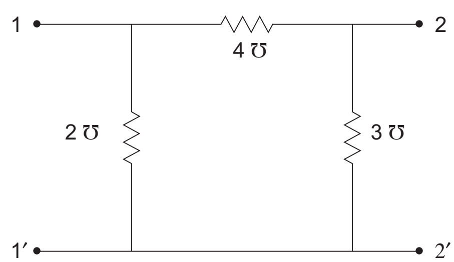


Fig. 15.81

   (a) – 2 ℧ (b) – 3 ℧ (c) – 4 ℧
7. If the two-port network in Fig. 15.81 is reciprocal then
   (a) $Y_{11} = Y_{22}$ (b) $Y_{12} = Y_{22}$ (c) $Y_{12} = Y_{11}$
8. In describing the transmission parameters
   (a) the input voltage and current are expressed in terms of output voltage and current
   (b) the input voltage and output voltage are expressed in terms of output current and input current.
   (c) the input voltage and output current are expressed in terms of input current and output voltage.
9. If $Z_{11} = 2\ \Omega$; $Z_{12} = 1\ \Omega$; $Z_{21} = 1\ \Omega$ and $Z_{22} = 3\ \Omega$, what is the determinant of admittance matrix.
   (a) 5 (b) 1/5 (c) 1

10. For a two-port bilateral network, the three transmission parameters are given by $A = \frac{6}{5}$; $B = \frac{17}{5}$ and $C = \frac{1}{5}$, what is the value of D?

(a) 1 (b) $\frac{1}{5}$ (c) $\frac{7}{5}$

11. The impedance matrices of two, two-port networks are given by $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 15 & 5 \\ 5 & 25 \end{bmatrix}$. If the two networks are connected in series. What is the impedance matrix of the combination?

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$ (c) $\begin{bmatrix} 15 & 2 \\ 5 & 3 \end{bmatrix}$

12. The admittance matrices of two two-port networks are given by $\begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 5/8 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$. If the two networks are connected in parallel, what is the admittance matrix of the combination?

(a) $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ -1 & 5/2 \end{bmatrix}$ (c) $\begin{bmatrix} 3/2 & -3/4 \\ -3/4 & 15/8 \end{bmatrix}$

13. If the Z parameters of a two-port network are $Z_{11} = 5\ \Omega$ $Z_{22} = 7\ \Omega$; $Z_{12} = Z_{21} = 3\ \Omega$ then the A, B, C, D parameters are respectively given by

(a) $\frac{5}{3}; \frac{26}{3}; \frac{1}{3}; \frac{7}{3}$ (b) $\frac{10}{3}; \frac{52}{3}; \frac{2}{3}; \frac{14}{3}$ (c) $\frac{15}{3}; \frac{78}{3}; \frac{3}{3}; \frac{21}{3}$

14. For a symmetric lattice network the value of the series impedance is 3 Ω and that of the diagonal impedance is 5 Ω, then the Z parameters of the network are given by

(a) $Z_{11} = Z_{22} = 2\ \Omega$
$Z_{12} = Z_{21} = 1/2\ \Omega$

(b) $Z_{11} = Z_{22} = 4\ \Omega$
$Z_{12} = Z_{21} = 1\ \Omega$

(c) $Z_{11} = Z_{22} = 8\ \Omega$
$Z_{12} = Z_{21} = 2\ \Omega$

15. For a two-port network to be reciprocal.

(a) $Z_{11} = Z_{22}$ (b) $y_{21} = y_{22}$
(c) $h_{21} = -h_{12}$ (d) $AD - BC = 0$

16. Two-port networks are connected in cascade. The combination is to be represented as a single two port network. The parameters of the network are obtained by adding the individual

(a) Z parameter matrix (b) h parameter matrix
(c) $A^1\ B^1\ C^1\ D^1$ matrix (d) $ABCD$ parameter matrix

17. The h parameters h_{11} and h_{12} are obtained
    (a) By shorting output terminals
    (b) By opening input terminals
    (c) By shorting input terminals
    (d) By opening output terminals
18. Which parameters are widely used in transmission line theory
    (a) Z parameters
    (b) Y parameters
    (c) $ABCD$ parameters
    (d) h parameters

Chapter **16**

S-Domain Analysis

16.1 THE CONCEPT OF COMPLEX FREQUENCY

The solution of differential equations for networks is of the form

$$i(t) = K_n\, e^{S_n t} \tag{16.1}$$

where S_n is a complex number which is a root of the characteristic equation and may therefore be expressed as

$$S_n = \sigma_n + j\omega_n \tag{16.2}$$

The complex number consists of two parts, the real part σ_n and the imaginary part ω_n. The real part of the complex frequency σ_n is *neper frequency*, while imaginary part ω_n is the *radian frequency*. The neper frequency has dimensions neper per second. In the time domain equations, ω_n is in the form of sin $\omega_n t$ or cos $\omega_n t$. The radian frequency ω_n is expressed in radian/sec and is related to the frequency f_n in cycles/sec or the periodic time T (in seconds) by the relation.

$$\omega_n = 2\pi f_n = \frac{2\pi}{T} \tag{16.3}$$

From the Equation (16.2), we see that the real part σ_n and the imaginary part ω_n must have identical dimensions. Radian frequency ω_n is $\frac{2\pi}{T}$ has dimensions $(\text{time})^{-1}$. Therefore, the dimensions of σ_n must also be $(\text{time})^{-1}$ or the unit of σ_n must be "Something per unit time". Since σ_n appears as an exponential factor

$$I = I_o\, e^{\sigma_n t} \tag{16.4}$$

Such that

$$\sigma_n = \frac{1}{t}\ln\left(\frac{I}{I_o}\right) \tag{16.5}$$

It is fact that "something per unit time" should be nondimensional logarithmic unit. The usual unit for the natural logarithm is the neper, making the dimensions for σ the neper per second. The complex quantity

$$S_n = \sigma_n + j\omega_n \tag{16.6}$$

is defined as the complex frequency. Thus, complex frequency consists of a real part σ_n called the neper frequency and an imaginary part ω_n is called radian frequency (or *real frequency*).

16.2 PHYSICAL INTERPRETATION OF COMPLEX FREQUENCY

The complex frequency appears in the exponential form $e^{S_n t}$. Let us consider the physical significance of complex frequency and a number of special cases for the values of S_n.

Case (i): Let $S_n = \sigma_n + jo$ having zero radian frequency. The exponential function becomes

$$K_n\, e^{S_n t} = K_n e^{\sigma_n t} \tag{16.7}$$

The above exponential function increases exponentially for $\sigma_n > 0$ and decreases exponentially for $\sigma_n < 0$. For $\sigma_n = 0$, the exponential function reduces to K_n and it is a time-invariant resulting current $i(t)$ which is a dc current. Figure 16.1 shows the variation of exponential term $K_n\, e^{\sigma_n t}$ with time t for the cases of $\sigma_n > 0$, $\sigma_n < 0$ and $\sigma_n = 0$.

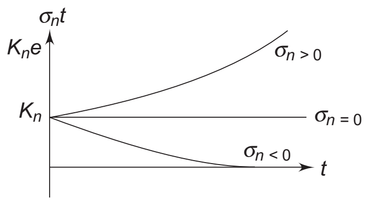


Fig. 16.1

Case (ii): Let $S_n = 0 \pm j\omega_n$ having radian frequency and zero neper frequency. The exponential becomes

$$i(t) = K_n\, e^{\pm j\omega_n t}$$
$$= K_n\,(\cos \omega_n\, t \pm j \sin \omega_n\, t) \tag{16.8}$$

The exponential $e^{\pm j\omega_n t}$ may be interpreted in terms of the physical model of a rotating phasor of unit length. A positive sign of exponential $e^{\pm j\omega_n t}$ implies counter clockwise or positive rotation, while a negative sign $e^{-j\omega_n t}$ implies clockwise or negative rotation.

For positive or counter-clockwise rotation, the real part of $e^{+j\omega_n t}$ or the projection on the real axis equals cos $\omega_n t$, while the imaginary part of $e^{+j\omega_n t}$ or the projection on the imaginary axis equals sin $\omega_n t$ (Fig. 16.2).

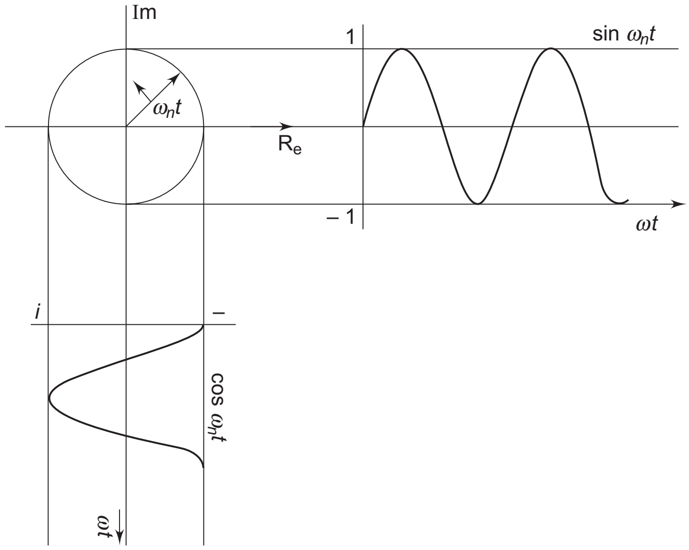


Fig. 16.2

The variation of exponential function $e^{+j\omega_n t}$ with time is thus sinusoidal and hence constitutes the case of sinusoidal steady state.

Case (iii): Let $S_n = \sigma_n + j\omega_n$ is the general case and the frequency is complex and the exponential is given by

$$i(t) = K_n\, e^{S_n t} = K_n\, e^{(\sigma_n + j\omega_n)t}$$

$$i(t) = K_n\, e^{\sigma_n t} \cdot e^{j\omega_n t} \qquad (16.9)$$

Equation (16.9) shows that with complex frequency, the time variation of response $i(t)$ is the product of the response for $S_n = \sigma_n + j_o$ and the response for $S_n = 0 + j\omega_n$. The response $e^{\sigma_n t}$ for the case of neper frequency alone, $S_n = \sigma_n + j_o$ is an exponentially increasing or decaying function. The response $e^{j\omega_n t}$ for the case of radian frequency alone $S_n = 0 + j\omega_n$ may be represented by a rotating phasor. The product $e^{\sigma_n t} \cdot e^{j\omega_n t}$ may then be visualized as a rotating phasor whose magnitude is not constant at unity but changes continuously with time. Such phasors are illustrated in Fig. 16.3.

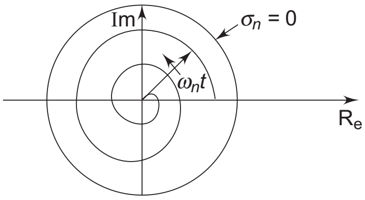


Fig. 16.3(a)

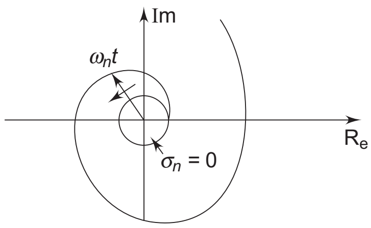


Fig. 16.3(b)

The real and imaginary projections of this phasor are

$$\text{Re}(e^{S_n t}) = e^{\sigma_n t} \cos \omega_n t \tag{16.10}$$

and $$\text{Im}(e^{S_n t}) = e^{\sigma_n t} \sin \omega_n t$$

Consider the projections of this rotating phasor on the real and imaginary axes for the two cases. These projections for the case $\sigma_n < 0$ are known as a *damped sinusoid* and for $\sigma_n > 0$, the increasing oscillations are shown in Figs. 16.4 (a) and (b) respectively.

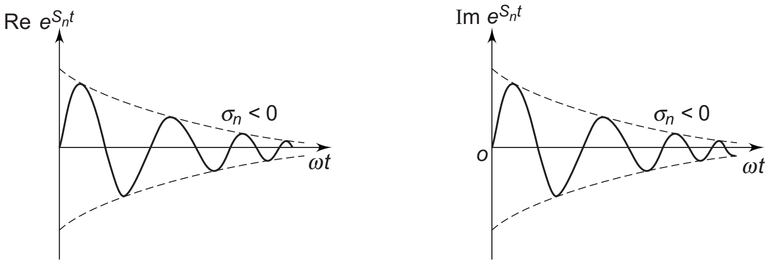


Fig. 16.4(a)

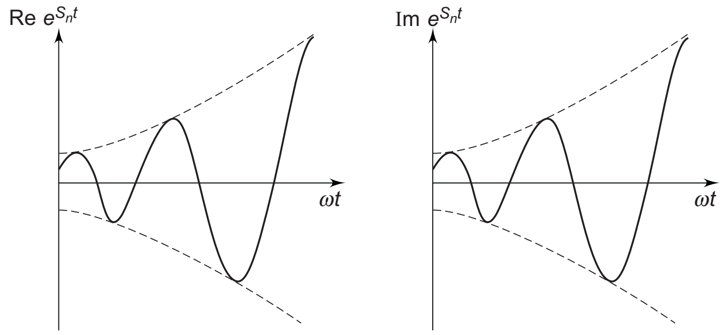


Fig. 16.4(b)

From the above discussion it is clear that the imaginary part of complex frequency governs sinusoidal oscillations and the real part of complex frequency governs the exponential decay or rise.

The roles of two types of frequency are similar even though the variations caused by them are different. This is the justification of unifying the two concepts under the name of complex frequency.

16.3 TRANSFORM IMPEDANCE AND TRANSFORM CIRCUITS

In this section, we determine the transform impedance and admittance representations for each of the elements and initial condition sources.

Resistance For a resistance, the voltage and current are related in the time domain by ohm's law.

$$V_R(t) = Ri_R(t) \quad \text{or} \quad i_R(t) = GV_R(t);\ G = \frac{1}{R} \tag{16.11}$$

The corresponding transform equations are

$$V_R(S) = RI_R(S)$$
$$I_R(S) = GV_R(S) \tag{16.12}$$

The ratio of transform voltage $V_R(S)$ to the transform current $I_R(S)$ is defined as the transform impedance of the resistor, expressed as

$$Z_R(S) = \frac{V_R(S)}{I_R(S)} = R \tag{16.13}$$

Similarly, the reciprocal of this ratio is the transform admittance for the resistor, expressed as

$$Y_R(S) = \frac{I_R(S)}{V_R(S)} = G \tag{16.14}$$

From the above results, we can say that the resistor is frequency insensitive to the complex frequency.

Figure 16.5 (a) shows a network representing resistor R current $i_R(t)$ and voltage $V_R(t)$ in time domain. Fig. 16.5 (b) gives the network representation of the same resistor and also transform current $I_R(S)$ and voltage $V_R(S)$.

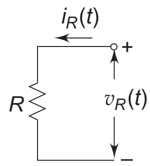


Fig. 16.5 (a)

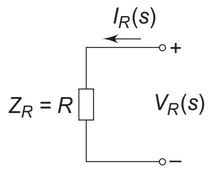


Fig. 16.5 (b)

Inductance For inductance, the time domain relation between the current in inductance $i_L(t)$ and the voltage $V_L(t)$ across it is expressed as

$$v_L(t) = L\,\frac{di_L(t)}{dt}$$

and
$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)\,dt \tag{16.15}$$

The equivalent transform equation for the voltage expression is

$$V_L(s) = L\,[SI_L(S) - i_L(0\,+)] \tag{16.16}$$

which, on rearranging, results

$$L_S\,I_L\,(S) = V_L(S) + Li_L(0\,+) \tag{16.17}$$

In the Eqs (16.15) and (16.16), $V_L(S)$ is the transform of the applied voltage $v_L(t)$ and $Li_L(0+)$ is the transform voltage caused by the initial current $i_L(0+)$ present in the inductor at time $t = 0+$.

Considering the sum of the transform voltage and the initial current voltage as $V_1(S)$ we have the transform impedance for the inductor.

$$Z_L(S) = \frac{V_1(S)}{I_L(S)} = L_S \tag{16.18}$$

Figure 16.6(a) shows the time domain network representation of inductor L, current $i_L(t)$ through it and applied voltage $V_L(t)$. Figure 16.6(b) gives the transform representation of same inductor in terms of impedance using Eq. (16.16).

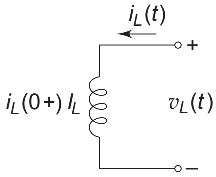


Fig. 16.6 (a)

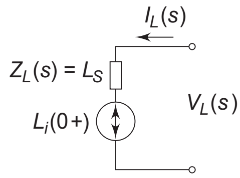


Fig. 16.6 (b)

The transform equation for the current expression of Eq. (16.17) is

$$I_L(S) = \left[\frac{V_L(S)}{S} + \frac{\int_{-\infty}^{0+} v_L(t)dt}{S}\right]\frac{1}{L} \tag{16.19}$$

But $\displaystyle\int_{-\infty}^{0+} v_L(t)\,dt = Li_L(0+)$ (16.20)

Hence, Eq. (16.19) becomes

$$I_L(S) = \frac{1}{L}\cdot\frac{V_L(S)}{S} + \frac{i_L(0+)}{S} \tag{16.21}$$

or

$$\frac{1}{LS}V_L(S) = I_L(S) - \frac{i_L(0+)}{S} \tag{16.22}$$

In the above equation $i_L(0+)/S$ is the transform caused by the initial current $i_L(0+)$ in the inductor.

Let $\displaystyle I_1(S) = I_L(S) - \frac{i_L(0+)}{S}$ (16.23)

Then the Eq. (16.22) becomes

$$\frac{1}{LS}V_L(S) = I_1(S) \tag{16.24}$$

where $I_1(S)$ is the total transform current through the inductor L. The transform admittance becomes

$$Y_L(S) = \frac{I_1(S)}{V_L(S)} = \frac{1}{LS} \tag{16.25}$$

Figure 16.7(a) shows the time domain representation of inductor L with initial current $i_L(0+)$. Figure 16.7(b) shows equivalent transform circuit thus contains an admittance of value $\frac{1}{LS}$ and equivalent transform current source.

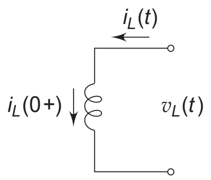


Fig. 16.7 (a)

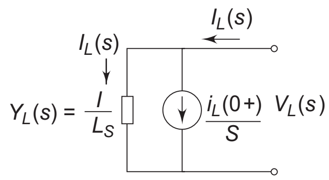


Fig. 16.7 (b)

Capacitance For capacitance, the time domain relation between voltage and current is expressed as

$$i_c(t) = c\,\frac{dv_c(t)}{dt}$$

and $$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(t)\,dt \tag{16.26}$$

The equivalent transform equation for the voltage expression is

$$V_C(S) = \frac{1}{C}\left[\frac{I_C(S)}{S} + \frac{q(0+)}{S}\right] \tag{16.27}$$

where $\frac{q(0+)}{C} = v_C(0+)$ is the initial voltage across the capacitor.

The above equation becomes

$$\frac{1}{CS} I_C(S) = V_C(S) - \frac{v_C(0+)}{S} \tag{16.28}$$

Considering the total transform voltage across the capacitor as $V_1(S)$.

$$V_1(S) = V_C(S) - \frac{v_C(0+)}{S} \tag{16.29}$$

Then, the Eq. (16.28) becomes

$$\frac{1}{CS} I_C(S) = V_1(S) \tag{16.30}$$

The transform impedance of the capacitor is the ratio of transform voltage $V_1(S)$ to the transform current $I_C(S)$ and is

$$Z_C(S) = \frac{V_1(S)}{I_C(S)} = \frac{1}{CS} \tag{16.31}$$

Figure 16.8(a) shows the time domain representation of capacitor C with initial voltage $V_C(0+)$ across it. Voltage $V_1(S)$ includes the initial voltage $V_C(0+)$. Figure 16.8(b) gives the transform representation of the same capacitor in terms of transform impedance.

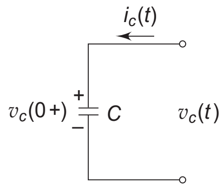


Fig. 16.8 (a)

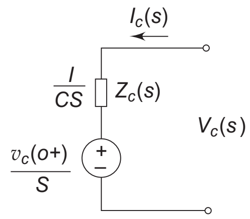


Fig. 16.8 (b)

Considering the current expression, the transform equation corresponding to Eq. (16.26) is

$$I_C(S) = C\,[SV_C(S) - v_C(0+)] \tag{16.32}$$

on rearranging

$$CSV_C(S) = I_C(S) + CV_C(0+) \tag{16.33}$$

Considering the transform current through $Y_C(S)$ as $I_1(S)$.

Equation (16.32) may be put as

$$CSV_C(S) = I_1(S) \tag{16.34}$$

Then the transform admittance of the capacitor C is the ratio of transform current $l_1(S)$ to transform voltage $V_C(S)$ is

$$Y_C(S) = \frac{I_1(S)}{V_C(S)} = CS \tag{16.35}$$

Figure 16.9 (a) shows the time domain representation of capacitor C with initial voltage $V_C(0+)$ across it. Figure 16.9(b) gives the transform representation of the same capacitor in terms of admittance.

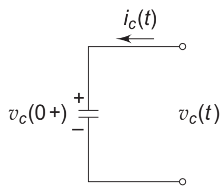


Fig. 16.9 (a)

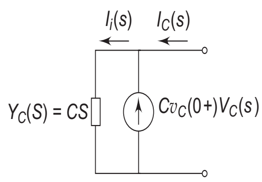


Fig. 16.9 (b)

16.4 SERIES AND PARALLEL COMBINATION OF ELEMENTS

In general, any network can be connected in series, parallel and series parallel combination. Consider the series combination of elements shown in Fig. 16.10.

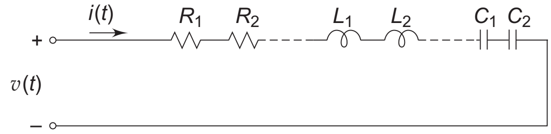


Fig. 16.10

Assuming that all initial conditions are zero, i.e. the current in all inductors is zero and that the initial voltage of all capacitors is also zero. By application of Kirchhoff's voltage law, we get

$$v(t) = v_{R_1} + v_{R_2} + \dots + v_{L_1} + v_{L_2} + \dots + v_{C_1} + v_{C_2} + \dots \tag{16.36}$$

Taking transform for the above equation, we get

$$V(S) = V_{R_1}(S) + \dots + V_{L_1}(S) + \dots + V_{C_1}(S) + \dots \tag{16.37}$$

Dividing the equation by $I(S)$, the transform current through the series circuit, we get

$$Z(S) = Z_{R_1}(S) + \dots + Z_{L_1}(S) + \dots + Z_{C_1}(S) \tag{16.38}$$

or

$$Z(S) = \sum_{k=1}^{n} Z_K(S) \tag{16.39}$$

where n is the total number of elements of all kinds in series.

Figure 16.11 shows the transform representation of the series circuit and represents Eq. (16.38).

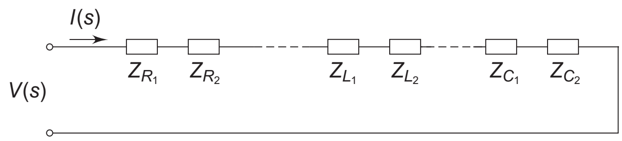


Fig. 16.11

Consider parallel combination of resistors, inductors and capacitors as shown in Fig. 16.12. Here, we assume that the inductors have zero initial currents and capacitors have zero initial voltages. Let $v(t)$ is the common voltage applied to all the elements in the circuit.

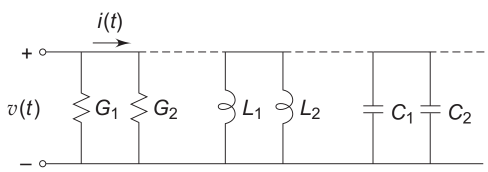


Fig. 16.12

Applying Kirchhoff's current law to the above circuit yields.

$$i(t) = i_{G_1}(t) + i_{G_2}(t) + \ldots + i_{L_1}(t) + i_{L_2}(t) + \ldots + i_{C_1}(t) + i_{C_2}(t) + \ldots \quad (16.40)$$

and the corresponding transform equation is

$$I(S) = I_{G_1}(S) + I_{G_2}(S) + \ldots + I_{L_1}(S) + I_{L_2}(S) + \ldots + I_{C_1}(S) + I_{C_2}(S) + \ldots \quad (16.41)$$

If this equation is divided by $V(S)$, we get transform admittance which is the ratio of the current transform to the voltage transform and is

$$Y(S) = Y_{G_1}(S) + Y_{G_2}(S) + \ldots + Y_{L_1}(S) + Y_{L_2}(S) + \ldots + Y_{C_1}(S) + Y_{C_2}(S) + \ldots \quad (16.42)$$

or

$$Y(S) = \sum_{k=1}^{n} Y_K(S) \quad (16.43)$$

where n is the total number of all kinds of elements in parallel.

Figure 16.13 gives the transform representation of the parallel network and represents the Eq. 16.42.

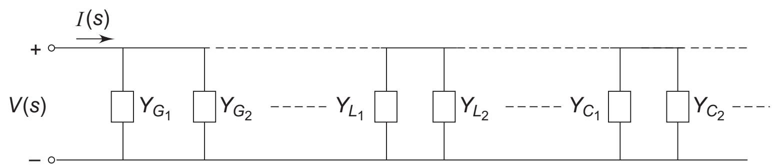


Fig. 16.13

For a series-parallel, rules for the combination of impedance and of admittance can be used to reduce a network to a single equivalent impedance or admittance.

Example 16.1 In the circuit shown, switch K is moved from position 1 to position 2 at time $t = 0$. At time $t = 0^-$, the current through inductor L is I_0 and the voltage across capacitor is V_0. Find the transform current $I(S)$.

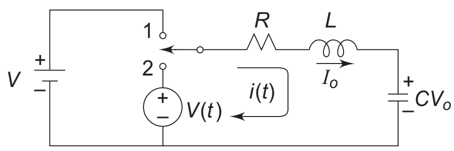


Fig. 16.14

Solution The inductor has an initial current of I_0. It is represented by a transform impedance LS in series with a voltage source LI_0 as shown in Fig. 16.15. The capacitor has an initial voltage V_0 across it. It is represented by a transform impedance of $\frac{1}{CS}$ with an initial voltage $\frac{V_0}{S}$. The transform circuit derived from the circuit of Fig. 16.14 is shown in Fig. 16.15.

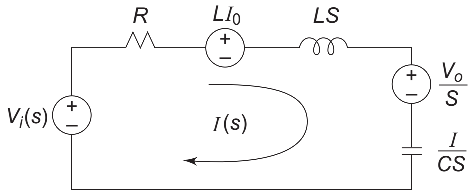


Fig. 16.15

The current $I(S)$ is given as the total transform voltage in the circuit divided by the total transform impedance. Then

$$I(S) = \frac{V(S)}{Z(S)} = \frac{V_1(S) + LI_0 - \frac{V_0}{S}}{R + LS + \frac{1}{CS}} = \frac{SV_1(S) + LI_0 - V_0}{LS^2 + RS + \frac{1}{C}} \qquad (16.44)$$

Example 16.2 The network shown in Fig. 16.16 is a parallel combination of L, R and C connected across a current source I. At time $t = 0^-$, the current through inductor L is I_0 and the voltage across capacitor C is V_0. At time $t = 0^+$, the current source $I_1(t)$ is connected to the parallel RLC circuit. Find the transform voltage $V(S)$.

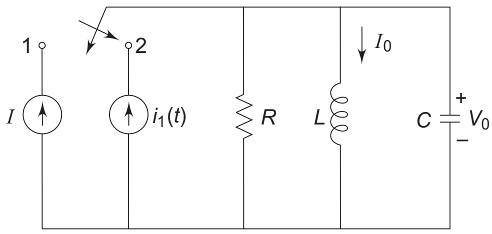


Fig. 16.16

Solution Figure 16.17 gives the transform network corresponding to the given network with switch K moved to position 2.

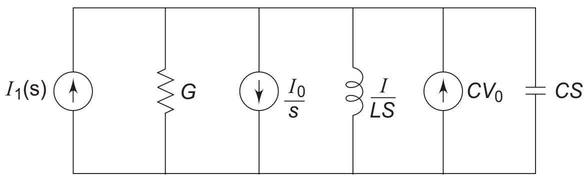


Fig. 16.17

From the above transform circuit, the transform voltage $V(S)$ may be obtained by taking the ratio of the total transform current to the total transform admittance.

The total transform current in the network is given by

$$I(S) = I_1(S) - \frac{I_0}{S} + CV_0$$

Total transform admittance is given by

$$Y(S) = G + \frac{1}{LS} + CS$$

Hence, the transform voltage is given by

$$V(S) = \frac{I(S)}{Y(S)} = \frac{I_1(s) + CV_0 - \frac{I_0}{S}}{G + \frac{1}{LS} + CS}$$

Example 16.3 Obtain the transform impedance of the network shown in Fig. 16.18.

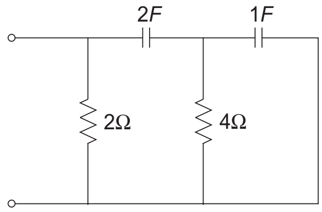


Fig. 16.18

Solution The transform network of the Fig. 16.18 is shown in Fig. 16.19.

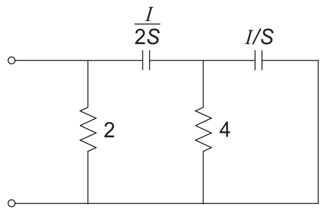


Fig. 16.19

The admittance of the last two elements is the parallel combination.

$$Y_1(S) = 4 + S$$

Therefore impedance is $Z_1(S) = \dfrac{1}{S+4}$

Series combination of last elements

$$Z_2(S) = \frac{1}{2S} + \frac{1}{S+4} = \frac{S+4+2S}{2S(S+4)} = \frac{3S+4}{2S(S+4)}$$

Parallel combination of elements

$$Y_2(S) = \frac{1}{2} + \frac{2S(S+4)}{3S+4} = \frac{(3S+4)+4s(S+4)}{6S+8} = \frac{4S^2+19S+4}{6S+8}$$

Hence, the impedance $Z(S) = \dfrac{1}{Y_2(S)} = \dfrac{6s+8}{4S^2+19S+4}$

Example 16.4 In the given network in Fig. 16.20, switch S is opened at $t = 0$, the steady state having established previously. With switch S open, draw the transform network for analysis on the loop basis representing all elements and all initial conditions. Write transform equation for current in the loop.

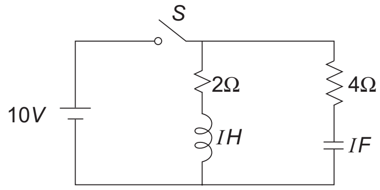


Fig. 16.20

Solution Under steady state conditions, the capacitor is open circuited and inductor is short circuited. The current through the inductor is $i_0 = \dfrac{10}{2} = 5\text{A}$. The voltage across the capacitor is $V_0 = 10$ V. Hence, the corresponding transform network is shown in Fig. 16.21.

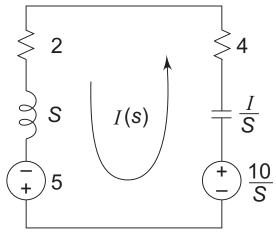


Fig. 16.21

Hence, $$I(S) = \frac{V(S)}{Z(S)} = \frac{5 + \dfrac{10}{S}}{2 + S + 4 + \dfrac{1}{S}} = \frac{5(S+2)}{S^2 + 6S + 1}$$

16.5 TERMINAL PAIRS OR PORTS

Consider an arbitrary network made up of passive elements. It can be represented by a rectangular box shown in Fig. 16.22.

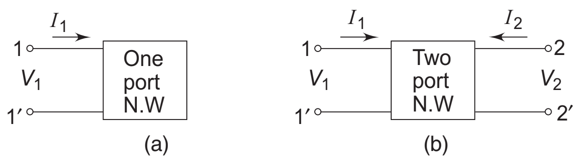


Fig. 16.22

For the network shown in Fig. 16.22(a) only one voltage and one current exist and only one network function is defined. It constitutes one pair of terminals called a **port**. Generally, a driving source is connected to the pair of terminals.

For the two terminal pair network shown in Fig. 16.22(b), two currents and two voltages must exists. Normally in Fig. 16.22(b), 1-1′ and 2-2′ are called ports. Hence, it is called **two port network**. If the driving source is connected across 1-1′, the load is connected across 2-2′. Otherwise, if the source is connected across 2-2′, the output is taken across 1-1′.

16.6 NETWORK FUNCTIONS FOR THE ONE-PORT AND TWO-PORT

For a one-port network, the driving point impedance or impedance of the network is defined as

$$Z(s) = \frac{V(s)}{I(s)} \tag{16.45}$$

The reciprocal of the impedance function is the driving point admittance function, and is denoted by $Y(s)$.

For the two-port network without internal sources, the driving point impedance function at port 1–1′ is the ratio of the transform voltage at port 1–1′ to the transform current at the same port.

$$\therefore \qquad Z_{11}(s) = \frac{V_1(s)}{I_1(s)} \tag{16.46}$$

Similarly, the driving point impedance at port 2–2′ is the ratio of transform voltage at port 2–2′ to the transform current at the same port.

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)} \tag{16.47}$$

For the two-port network, the driving point admittance is defined as the ratio of the transform current at any port to the transform voltage at the same port.

Therefore
$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)} \tag{16.48}$$

or
$$Y_{22}(s) = \frac{I_2(s)}{V_2(s)}, \text{ which is the driving point admittance.}$$

The four other network functions are called transfer functions. These functions give the relation between voltage or current at one port to the voltage or current at the other port as shown hereunder.

(i) ***Voltage Transfer Ratio*** This is the ratio of voltage transform at one port to the voltage transform at the other port, and is denoted by $G(s)$

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)}$$

and
$$G_{12}(s) = \frac{V_1(s)}{V_2(s)} \tag{16.49}$$

(ii) ***Current Transfer Ratio*** This is the ratio of current transform at one port to current transform at other port, and is denoted by $\alpha(s)$

$$\alpha_{12}(s) = \frac{I_1(s)}{I_2(s)}$$

and
$$\alpha_{21}(s) = \frac{I_2(s)}{I_1(s)} \tag{16.50}$$

(iii) ***Transfer Impedance*** It is defined as the ratio of voltage transform at one port to the current transform at the other port, and is denoted by $Z(s)$.

$\therefore$
$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$$

and
$$Z_{12}(s) = \frac{V_1(s)}{I_2(s)} \tag{16.51}$$

(iv) ***Transfer Admittance*** It is defined as the ratio of current transform at one port to the current transform at the other port, and is denoted by $Y(s)$.

$$Y_{21}(s) = \frac{I_2(s)}{V_1(s)}$$

and
$$Y_{12}(s) = \frac{I_1(s)}{V_2(s)} \tag{16.52}$$

The above network functions are found by forming the system of equations using node or mesh analysis, and taking the transforms of equations by setting the initial conditions to zero and solving for ratio of the response to excitation.

Example 16.5 For the network shown in Fig. 16.23, obtain the driving point impedance.

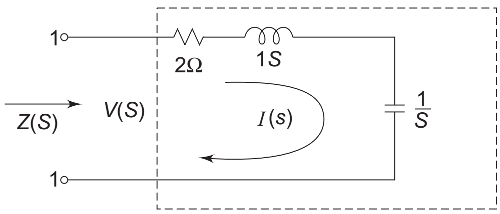


Fig. 16.23

Solution Applying Kirchhoff's law at port 1-1′

$$Z(S) = \frac{V(S)}{I(S)}$$

where $V(S)$ is applied at port 1-1′ and $I(S)$ is the current flowing through the network. Then

$$Z(S) = \frac{V(S)}{I(S)} = 2 + S + \frac{1}{S}$$

$$Z(S) = \frac{S^2 + 2S + 1}{S}$$

Example 16.6 For the network shown in Fig. 16.24, obtain the transfer functions $G_{21}(S)$ and $Z_{21}(S)$ and the driving point impedance $Z_{11}(S)$.

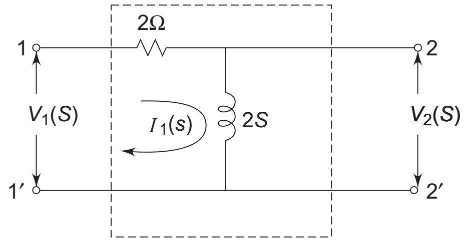


Fig. 16.24

Solution Applying Kirchhoff's law

$$V_1(S) = 2I_1(S) + 2SI_1(S)$$

$$V_2(S) = I_1(S) \times 2S$$

Hence

$$G_{21}(S) = \frac{V_2(S)}{V_1(S)} = \frac{2S}{2+2S} = \frac{S}{S+1}$$

$$Z_{21}(S) = \frac{V_2(S)}{I_1(S)} = 2S$$

$$Z_{11}(S) = \frac{V_1(S)}{I_1(S)} = 2(S+2)$$

Example 16.7 For the network shown in Fig. 16.25, obtain the transfer functions $G_{21}(S)$, $Z_{21}(S)$ and driving point impedance $Z_{11}(S)$.

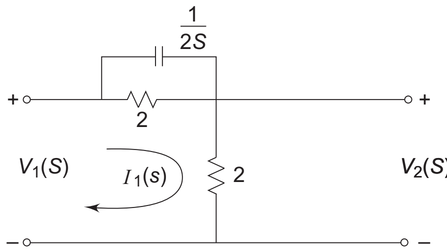


Fig. 16.25

Solution From the circuit, the parallel combination of resistance and capacitance can be combined into equivalent impedance.

$$Z_{eq}(S) = \frac{1}{2S + \frac{1}{2}} = \frac{2}{4S+1}$$

Applying Kirchhoff's laws, we have

$$V_2(S) = 2I_1(S)$$

and $$V_1(S) = I_1(S)\left[\frac{2}{4S+1}+2\right]$$

$$= I_1(S)\left[\frac{8S+4}{4S+1}\right]$$

The transfer functions

$$G_{21}(S) = \frac{V_2(S)}{V_1(S)} = \frac{2I_1(S)}{\left(\frac{8S+4}{4S+1}\right)I_1(S)} = \frac{8S+2}{8S+4}$$

$$Z_{21}(S) = \frac{V_2(S)}{I_1(S)} = 2$$

The driving point function is

$$Z_{11}(S) = \frac{V_1(s)}{I_1(S)} = \frac{8S+4}{4S+1}$$

16.7 POLES AND ZEROS OF NETWORK FUNCTIONS

In general, the network function $N(s)$ may be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m} \tag{16.53}$$

where $a_0, a_1,..., a_n$ and $b_0, b_1, ..., b_m$ are the coefficients of the polynomials $P(s)$ and $Q(s)$; they are real and positive for a passive network. If the numerator and denominator of polynomial $N(s)$ are factorised, the network function can be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0(s-z_1)(s-z_2)\ldots(s-z_n)}{b_0(s-p_1)(s-p_2)\ldots(s-p_m)} \tag{16.54}$$

where $z_1, z_2, ..., z_n$ are the n roots for $P(s) = 0$
and $p_1, p_2, ..., P_m$ are the m roots for $Q(s) = 0$
and $a_0/b_0 = H$ is a constant called the *scale factor*.

$z_1, z_2,..., z_n$ in the transfer function are called zeros, and are denoted by 0. Similarly, $p_1, p_2,..., p_m$ are called poles, and are denoted by ×. The network function $N(s)$ becomes zero when s is equal to anyone of the zeros. $N(s)$ becomes infinite when s is equal to any one of the poles. The network function is completely defined by its poles and zeros. If the poles or zeros are not repeated, then the function is said to be having simple poles or simple zeros. If the poles or zeros are repeated, then the function is said to be having multiple poles multiple zeros. When $n > m$, then $(n - m)$ zeros are at $s = \infty$, and for $m > n$, $(m - n)$ poles are at $s = \infty$.

Consider, the network function

$$N(s) = \frac{(s+1)^2\,(s+5)}{(s+2)\,(s+3+j2)\,(s+3-j2)} \tag{16.55}$$

that has double zeros at $s = -1$ and a zero at $s = -5$; and three finite poles at $s = -2$, $s = -3 + j2$, and $s = -3 - j2$ as shown in Fig. 16.26.

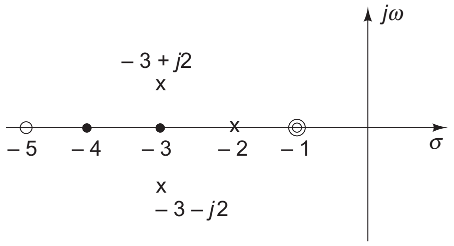


Fig. 16.26

The network function is said to be stable when the real parts of the poles and zeros are negative. Otherwise, the poles and zeros must lie within the left half of the s-plane.

16.8 SIGNIFICANCE OF POLES AND ZEROS

Poles and zeros are critical frequencies. At poles, the network function become infinite, while at zeros, the network function becomes zero. At other complex frequencies, the network function has a finite non-zero value.

Poles and zeros provide useful information in network functions. Consider the following cases.

(a) Driving Point Impedance

$$Z(S) = \frac{V(S)}{I(S)} \tag{16.56}$$

A pole of $Z(S)$ implies zero current for a finite voltage which means an open circuit. A zero of $Z(S)$ implies no voltage for a finite current or a short circuit.

Consider $Z(S) = \dfrac{1}{CS}$ (16.57)

The above function has a pole at $S = 0$ and zero at $S = \infty$.

Therefore, the above function represented by capacitor acts an open circuit at pole frequency and acts as short circuit at zero frequency.

(b) Driving Point Admittance

$$Y(S) = \frac{I(S)}{V(S)} \tag{16.58}$$

A pole of $Y(S)$ implies zero voltage for a finite value of current which gives a short circuit. A zero of $Y(S)$ implies zero current for a finite value of voltage which gives an open circuit.

(c) Voltage Transform Ratio

$$G_{21}(S) = \frac{V_2(S)}{V_1(S)} \tag{16.59}$$

$$V_2(S) = G_{21}(S)\ V_1(S)$$

To obtain output voltage, we require the product of input and transfer function. The expression for $G_{21}(S)$. $V_1(S)$ is obtained in the form of a ratio of polynomials in S.

By making use of partial fractions, we can obtain a pole of either $G_{21}(S)$ or $V_1(S)$ and no repeated roots.

$$G_{21}(S)\ V_1(S) = \sum_{i=1}^{n} \frac{A}{S - a_i} + \sum_{j=1}^{m} \frac{A}{S - a_j} \tag{16.60}$$

where n and m are the number of poles of $G_{21}(S)$ and $V_1(S)$ respectively. The frequencies a_i from the natural complex frequencies corresponding to free oscillations and depend on the network function $G_{21}(S)$. While frequencies a_j constitute the complex frequencies corresponding to the forced oscillations and depend on the driving force $V_1(S)$. From the above discussion, we can say that the poles determine the time variation of the response whereas the zeros determine the magnitude response.

(d) Other Network Functions

Significance of poles and zeros in other transfer functions is the same as discussed above. In each of the cases, poles determine the time domain behaviour and zeros determine the magnitude of each of the terms of the response.

16.9 PROPERTIES OF DRIVING POINT FUNCTIONS

(a) The driving point function is a ratio of polynomials in S. Polynomials are obtained from the transform impedances of the elements and their combinations.

Let $P(S) = a_0 S^n + a_1 S^{n-1} + \dots + a_{n-1} S + a_n$
and $Q(S) = b_0 S^m + b_1 S^{m-1} + \dots + b_{m-1} S + b_m$ (16.61)

be the numerator and the denominator polynomials respectively. The above equations can be factorized and therefore written as

$$\left.\begin{aligned} P(S) &= (S - Z_1)(S - Z_2) \dots (S - Z_n) \\ Q(S) &= (S - P_1)(S - P_2) \dots (S - P_m) \end{aligned}\right\} \tag{16.62}$$

The driving point function $N(S)$ may be written as

$$N(S) = \frac{P(S)}{Q(S)} = \frac{(S-Z_1)(S-Z_2)\dots(S-Z_n)}{(S-P_1)(S-P_2)\dots(S-P_m)} \quad (16.63)$$

The quantities $Z_1, Z_2 \dots Z_n$ are called zeros of $N(S)$ as $N(Z_1) = N(Z_2) = \dots N(Z_n) = 0$.

The quantities $P_1, P_2 \dots P_m$ are called poles of $N(S)$ as $N(P_1) = N(P_2) = \dots = N(P_m) = \infty$

That is, pole is that finite value of S for which $N(S)$ becomes infinity.

If the zeros and poles are not repeated then the poles or zeros are said to be **distinct** or **simple**.

A zero or a pole is said to be of **multiplicity** 'r' if $(S-Z)^r$ or $(S-P)^r$ is a factor of $P(S)$ or $Q(S)$. A function $N(S)$ is said to have a pole (or zero) at infinity, if the function $N\left(\frac{1}{S}\right)$ has a pole (or zero) at $S = 0$.

Consider the function

$$N(S) = \frac{S+1}{(S+2)(S+4)} \quad (16.64)$$

$$N\left(\frac{1}{S}\right) = \frac{\frac{1}{S}+1}{\left(\frac{1}{S}+2\right)\left(\frac{1}{S}+4\right)} = \frac{S(S+1)}{(1+2S)(1+4S)} \quad (16.65)$$

i.e. $N\left(\frac{1}{S}\right)$ has a zero at $S = 0$

$N(S)$ has a zero at $S = \infty$

From the above example, we say that the number of zeros including zeros at ∞ equals the number of poles including poles at ∞.

(b) (i) $N(S)$ be a driving point impedance, i.e. $Z(S)$

$$Z(S) = \frac{V(S)}{I(S)} \quad (16.66)$$

A zero of $N(S)$ is a zero of $V(S)$, it signifies a short circuit. Similarly, a pole of $Z(S)$ is a zero of $I(S)$. The poles of $Z(S)$ are those frequencies corresponding to open circuit conditions.

(ii) Consider a driving point admittance function

$$Y(S) = \frac{I(S)}{V(S)} \quad (16.67)$$

A zero of $Y(S)$ means a zero of $I(S)$, i.e. the open circuit condition and a pole of $Y(S)$ means a zero of $V(S)$ signifies a short circuit.

(c) Since all the elements in the circuit are real positive quantities the coefficients $a_0, a_1, a_2, \dots a_n$ and $b_0, b_1, b_2 \dots b_m$ are real and positive. Therefore, any zeros or poles, if complex, must occur in conjugate pairs.

(d) The real parts of all zeros and poles must be negative or zero. Consider a pole 'P' of $N(S)$, i.e. $(S - P)$ is a factor of the denominator of $N(S)$. Using partial fractions, we know that this gives rise to a term of the form $\dfrac{A}{S-P}$ whose inverse Laplace transform contains the term $e^{pt.}$ The real part of e^{pt} tends to zero as t tends to infinity if the real part P is negative. Therefore, for a finite input the response is finite as t tends to infinity if the real part of P is negative. A network function whose response is finite for all t, for a given finite input is said to be stable. Thus, a driving point impedance $Z(S)$ is stable if all the poles lie in the negative half of the S-plane.

Since $Y(S) = \dfrac{1}{Z(S)}$, poles of $Y(S)$ are zeros of $Z(S)$. Therefore $Y(S)$ is stable if all the zeros of $Z(S)$ also lie in the negative half of the S-plane. Thus, the real parts of all zeros and poles of a driving point function must be negative or zero.

(e) Poles or zeros lying on the $j\omega$-axis must be simple. Consider a pole 'P' lying on the $j\omega$-axis. If it is not simple then in the time response of the function of which it is a pole contains the term $t^k e^{j\omega t}$ which tends to infinity as t tends to infinite. Therefore, the function becomes unstable. Since, zeros of one function will be poles of the other. Therefore, the zeros of driving point function should also satisfy this condition.

(f) The degree of $P(S)$ and $Q(S)$ may differ by zero or one only.
At very high frequencies, the term $a_0 S^n$ dominates over the other terms in the numerator and the term $b_0 S^m$ dominates over other terms in the denominator.

$$\underset{S\to\infty}{\text{Lt}}\ N(S) = \underset{S\to\infty}{\text{Lt}}\ \frac{a_0}{b_0}\ S^{n-m} \tag{16.68}$$

Consider the network elements R, L, C and M. R is independent of frequency.

$\therefore$ If $n = m$, then the function behaves as a resistance $R = \dfrac{a_0}{b_0}$ at high frequency. The impedance LS of an inductor increases linearly with the complex frequency S and therefore is an open circuit at $S = \infty$. Thus if $n = m + 1$, the function $N(S)$ behaves as an inductance as S approaches infinity. A capacitor is a short circuit at infinite frequencies. Thus, $N(S)$ behaves as a capacitance if $m = n + 16$.
Now, consider the driving point impedance $Z(S)$. $Z(S)$ will behave as an inductor as LS increases with increasing S while $\dfrac{1}{CS}$ decreases with increasing S and therefore the impedance of an inductance dominates over the capacitive impedance. If inductors are not present in the circuit, then R

dominates over $\frac{1}{CS}$ as S tends to infinity. Thus, $Z(S) = LS$ or R as S tends infinity.

$\therefore \quad n - m = 0 \quad \text{or} \quad 1$

If $N(S)$ is a driving point impedance.

On the other hand the admittance of an inductance $\frac{1}{LS}$ tends to zero as S tends to infinity. Similarly, the admittance of a capacitance CS tends to infinity as S tends to infinity. Therefore, CS dominates over $\frac{1}{LS}$ as S tends to infinity. If the network does not contain capacitors then the resistance R dominates over $\frac{1}{LS}$ at higher frequencies.

$\therefore$ If $N(S)$ is a driving point admittance function $m - n = 0$ or 1.

Therefore, $|n - m| = 0$ or 1

(g) The lowest degree terms in $P(S)$ and $Q(S)$ may differ in degree by zero or one only.

As S approaches zero, the higher power of S tends to zero faster than S, therefore $N(S)$ can be approximated by

$$N(S) = \frac{a_{n-1}\,S + a_n}{b_{m-n}\,S + b_m} \tag{16.69}$$

The impedance of an inductance 'LS' approaches zero as S tends to zero while that of a capacitor $\frac{1}{CS}$ approaches infinity as S tends to zero. $\frac{1}{CS}$ dominates over LS as S tends to zero. Therefore, for $Z(S)$, the capacitance dominates over an inductance as S tends to zero. If the network does not contain capacitors then R dominates over LS and S tends to zero. Thus, the network can be replaced by R or $\frac{1}{CS}$ if $Z(S)$ is of interest. Similarly, for $Y(S)$; $\frac{1}{R}, \frac{1}{LS}$ dominate over CS as S tends to zero. Therefore, for purposes of $Y(S)$, the network is just a conductance or an inductance. Thus, the network is just one inductor or one capacitance or one resistance as S tends to zero.

$\therefore$ $N(S)$ is of the form K_1 or $K_2\,S$ or $\frac{K_3}{S}$ where K_1, K_2, K_3 are constants.

Hence, the lowest degree of $P(S)$ and $Q(S)$ can differ at most in one degree.

(h) $P(S)$ and $Q(S)$ cannot have missing terms unless all even or all odd degree terms are absent.
We know that

$$P(S) = a_0 S^n + a_1 S^{n-1} + \ldots + a_{i+1} S^{n-i-1} + a_i S^{n-i} + \ldots + a_{n-1} S + a_n$$

and $$Q(S) = b_0 S^m + b_1 S^{m-1} + \ldots + b_{i+1} S^{m-i-1} + b_i S^{m-i} + \ldots + b_{m-1} S + b_m \quad (16.70)$$

The above requirement means that for any i, a_i or b_i cannot be zero unless $a_j = 0$ or $b_j = 0$ for all $i \geq j$. The only exception to this rule is when all even or all odd powers of S are missing. To understand this, consider the network under study contains only elements like R, L, C, M whose transfer impedances are R, LS, $\frac{1}{CS}$, MS respectively.

Also, R, L, C, M are positive quantities. A combination of RL or RC will give rise to a term of the form $(aS + b)$ or $a + \frac{b}{S}$. Since $\left(a + \frac{b}{S}\right) = \frac{aS + b}{S}$ means $(aS + b)$ in the numerator and S in the denominator. Similarly, R, L, C give rise terms of the form $(aS^2 + bS + C)$ and a combination of only L and C gives rise to a term of the form $(aS^2 + b)$. Therefore, when two such factors are multiplied, since all the coefficients in each term are positive, in the expansion of the product no term can be zero. If all the terms are of the form $(aS^2 + b)$, then their product contains only even powers of S. If this is multiplied by S, the resulting function contains only odd powers of S. Given a ratio of polynomials $N(S)$, these properties can therefore be used to find out if $N(S)$ represents a driving point function of a network.

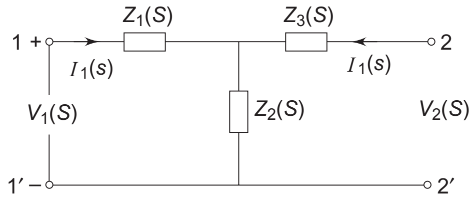
16.10 PROPERTIES OF TRANSFER FUNCTIONS

(a) The transfer function is a ratio of polynomials in S.
(b) The coefficients of $P(S)$, the numerators polynomial and of $Q(S)$, the denominator polynomial must be real. Therefore, all poles and zeros, if complex, must occur in conjugate pairs.
(c) The real parts of all poles must be negative and any pole on the $j\omega$-axis must be simple. As in the case of driving point functions, this follows from the stability considerations.
(d) Since poles of the transfer function are zeroes of $Q(S)$, it follows that the zeroes of $Q(S)$ must lie in the negative half plane and any zero lying on the $j\omega$-axis must be simple.
Let $P_1, P_2, \ldots P_m$ be the zeros of $Q(S)$
Then $Q(S) = K \cdot (S - P_1)(S - P_2)(S - P_3) \ldots (S - P_m)$
Since all poles have negative real parts and complex poles occur in conjugate pair the product of these factors contains all powers of S whose coefficients are positive. Therefore, $Q(S)$ does not have missing terms unless

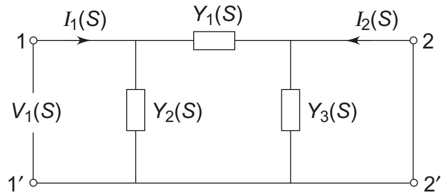
all even or all odd powers are missing. Since there are no restrictions on the zeroes of the transfer function, $P(S)$ can have missing terms. Also co-efficients of powers of S in $P(S)$ can be negative.

(e) For $G(S)$ and $\alpha(S)$, the degree of the numerator polynomial $P(S)$ is less than or equal to the degree of $Q(S)$.

To prove this, we use the fact that a two port can be represented by an equivalent T(star) or Π (delta) shown in Fig. 16.27.



(a) T-equivalent network



(b) π-equivalent network

Fig. 16.27

Let a source of known voltage $V_1(S)$ be applied to a T-network of the port 11′. Let a source of known current $I_1(S)$ be applied to the π-network at 11^1 and 22^1 be short circuited, then assuming $I_2(S) = 0$.

$$G(S) = \frac{V_2(S)}{V_1(S)} = \frac{Z_2(S)}{Z_1(S) + Z_2(S)} \tag{16.71}$$

and
$$\alpha(S) = \frac{I_2(S)}{I_1(S)} = \frac{Y_2(S)}{Y_1(S) + Y_2(S)} \tag{16.72}$$

Since $Z_1(S)$, $Z_2(S)$, $Z_3(S)$; $Y_1(S)$, $Y_2(S)$ and $Y_3(S)$ can be thought off as the driving point functions of some one ports, they have to satisfy the properties of driving point immittance functions.

Since $Z_1(S)$ and $Z_2(S)$ are ratio of polynomials.

Let
$$Z_1(S) = K\,\frac{(S+\alpha_1)(S+\alpha_2)\ldots(S+\alpha_{n1})}{(S+\beta_1)(S+\beta_2)\ldots(S+\beta_{m1})} \tag{16.73}$$

$$Z_2(S) = \frac{K_2\,(S+r_1)(S+r_2)\ldots(S+r_{n2})}{(S+S_1)(S+\delta_2)\ldots(S+\delta_{m2})} \tag{16.74}$$

Substituting these expression in $G(S)$

$$G(S) = \frac{Z_2(S)}{Z_1(S) + Z_2(S)} \tag{16.75}$$

$$= \frac{K_1(S+\alpha_1)(S+\alpha_2)\ldots(S+\alpha_{n1})(S+\delta_1)\ldots(S+\delta_{m2})}{(S+\alpha_1)(S+\alpha_2)\ldots(S+\alpha_{n1})(S+\delta_1)\ldots(S+\delta_{m2})(S+\gamma_1) \\ (S+\gamma_2)(S+\gamma_3)\ldots(S+\gamma_{n2})(S+\beta_1)\ldots(S+\beta_{m1})}$$

Let $P(S)$ denote the numerator polynomial of $G(S)$ and $Q(S)$ the denominator polynomial of $G(S)$.

Then degree of $P(S) = n_1 + m_2$ and degree of $Q(S) = n_1 + m_2$ or $n_2 + m_1$, whichever is greater.

Thus if $n_1 + m_2 > n_2 + m_1$, the degree of $P(S)$ equals the degree of $Q(S)$.

If $n_1 + m_2 < n_2 + m_1$, degree of $Q(S) = n_2 + m_1$ and the degree of $P(S)$ is less than the degree of $Q(S)$.

Similarly assuming $Y_1(S)$ and $Y_2(S)$ as ratios of polynomials and substituting those expressions in $\alpha(S)$, it can be shown that the degree of the numerator of $\alpha(S)$ is less than or equal to the degree of the denominator.

(f) The degree of the numerator polynomial of $Z_{21}(S)$ or $Y_{21}(S)$ is less than or equal to the degree of the denominator polynomial plus one.
Referring to the T and π equivalent networks of two port network shown in Fig. 16.27

$$Z_{21}(S) = \left.\frac{V_2(S)}{I_1(S)}\right|_{I_2=0} = \frac{Z_2(S)\,I_1(S)}{I_1(S)} = Z_2(S) \tag{16.76}$$

and $$Y_{21}(S) = \left.\frac{I_2(S)}{V_1(S)}\right|_{V_2=0} = \frac{-V_2(S)\,I_1(S)}{I_1(S)} = -Y_2(S) \tag{16.77}$$

Thus, the highest degree of the numerator of $Z_{21}(S)$ equals the highest degree of the numerator of $Z_2(S)$. But as $Z_2(S)$ is a driving point impedance, the highest degree of the numerator of $Z_2(S)$ is the degree of denominator plus one. Therefore, the highest degree of the numerator of $Z_{12}(S)$ is the degree of its denominator plus one. Similarly, since $Y_2(S)$ is a driving point admittance, the highest degree of the numerator or $Y_{21}(S)$, which is also the numerator of $Y_2(S)$ is equal to the degree of the denominator plus one.

16.11 NECESSARY CONDITIONS FOR DRIVING POINT FUNCTION

The restrictions on pole and zero locations in the driving point function with common factors in $P(s)$ and $Q(s)$ cancelled are listed below.

1. The coefficients in the polynomials $P(s)$ and $Q(s)$ of network function $N(s) = P(s)/Q(s)$ must be real and positive.
2. Complex or imaginary poles and zeros must occur in conjugate pairs.
3. (a) The real parts of all poles and zeros must be zero, or negative.
   (b) If the real part is zero, then the pole and zero must be simple.
4. The polynomials $P(s)$ and $Q(s)$ may not have any missing terms between the highest and the lowest degrees, unless all even or all odd terms are missing.

5. The degree of $P(s)$ and $Q(s)$ may differ by zero, or one only.
6. The lowest degree in $P(s)$ and $Q(s)$ may differ in degree by at the most one.

16.12 NECESSARY CONDITIONS FOR TRANSFER FUNCTIONS

The restrictions on pole and zero location in transfer functions with common factors in $P(s)$ and $Q(s)$ cancelled are listed below.

1. (a) The coefficients in the polynomials $P(s)$ and $Q(s)$ of $N(s) = P(s)/Q(s)$ must be real.
   (b) The coefficients in $Q(s)$ must be positive, but some of the coefficients in $P(s)$ may be negative.
2. Complex or imaginary poles and zeros must occur in conjugate pairs.
3. The real part of poles must be negative, or zero. If the real part is zero, then the pole must be simple.
4. The polynomial $Q(s)$ may not have any missing terms between the highest and the lowest degree, unless all even or all odd terms are missing.
5. The polynomial $P(s)$ may have missing terms between the lowest and the highest degree.
6. The degree of $P(s)$ may be as small as zero, independent of the degree of $Q(s)$.
7. (a) For the voltage transfer ratio and the current transfer ratio, the maximum degree of $P(s)$ must equal the degree of $Q(s)$.
   (b) For transfer impedance and transfer admittance, the maximum degree of $P(s)$ must equal the degree of $Q(s)$ plus one.

16.13 TIME DOMAIN RESPONSE FROM POLE ZERO PLOT

For the given network function, a pole zero plot can be drawn which gives useful information regarding the critical frequencies. The time domain response can also be obtained from pole zero plot of a network function. Consider an array of poles shown in Fig. 16.28.

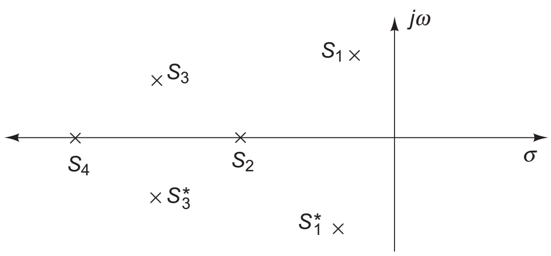


Fig. 16.28

In Fig. 16.28 s_1 and s_3 are complex conjugate poles, whereas s_2 and s_4 are real poles. If the poles are real, the quadratic function is

$$s^2 + 2\delta\omega_n s + \omega_n^2 \text{ for } \delta > 1$$

where δ is the damping ratio and ω_n is the undamped natural frequency.

The roots of the equation are

$$s_2, s_4 = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}\ ;\ \delta > 1$$

For these poles, the time domain response is given by

$$i(t) = k_2 e^{s_2 t} + k_4 e^{s_4 t}$$

The response due to pole s_4 dies faster compared to that of s_2 as shown in Fig. 16.29.

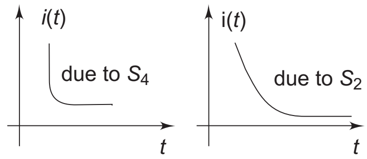


Fig. 16.29

s_1 and s_3 constitute complex conjugate poles. If the poles are complex conjugate, then the quadratic function is

$$s^2 + 2\delta\omega_n s + \omega_n^2 \text{ for } \delta < 1$$

The roots are $s_1, s_1^* = -\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2}$ for $\delta < 1$

For these poles, the time domain response is given by

$$i(t) = k_1 e^{-\delta\omega_n t + j\left(\omega_n\sqrt{1-\delta^2}\right)t} + k_1^* e^{-\delta\omega_n t - j\left(\omega_n\sqrt{1-\delta^2}\right)t}$$

$$= ke^{-\delta\omega_n t}\sin\left(\omega_n\sqrt{1-\delta^2}\right)t$$

From the above equation, we can conclude that the response for the conjugate poles is damped sinusoid. Similarly, s_3, s_3^* are also a complex conjugate pair. Here the response due to s_3 dies down faster than that due to s_1 as shown in Fig. 16.30.

Consider a network having transfer admittance $Y(s)$. If the input voltage $V(s)$ is applied to the network, the corresponding current is given by

$$I(s) = V(s)\,Y(s) = \frac{P(s)}{Q(s)}$$

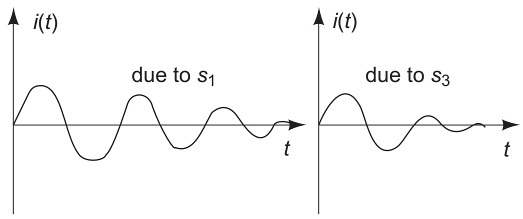


Fig. 16.30

This may be taken as

$$I(s) = H\,\frac{(s-s_a)(s-s_b)\ldots(s-s_n)}{(s-s_1)(s-s_2)\ldots(s-s_m)}$$

where H is the scale factor.

By taking the partial fractions, we get

$$I(s) = \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \ldots + \frac{k_m}{s-s_m}$$

The time domain response can be obtained by taking the inverse transform

$$i(t) = \mathscr{L}^{-1}\left[\frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \ldots + \frac{k_m}{s-s_m}\right]$$

Any of the above coefficients can be obtained by using Heavisides method.

To find the coefficient k_l

$$k_l = H\left[\frac{(s-s_a)(s-s_b)\ldots(s-s_n)}{(s-s_1)(s-s_2)\ldots(s-s_m)}\right](s-s_l)\Big|_{s=s_l}$$

Here s_l, s_m, s_n are all complex numbers, the difference of $(s_l - s_n)$ is also a complex number.

$$\therefore \qquad (s_l - s_n) = M_{ln}\,e^{j\phi_{ln}}$$

Hence $k_l = H\,\dfrac{M_{la}\;M_{lb}\;\ldots\;M_{ln}}{M_{l1}\;M_{l2}\;\ldots\;M_{lm}} \times e^{j(\phi_{la}+\phi_{lb}+\ldots+\phi_{ln})-(\phi_{l1}+\phi_{l2}+\ldots+\phi_{lm})}$

Similarly, all coefficients $k_1, k_2, \ldots, k_m$ may be obtained, which constitute the magnitude and phase angle.

The residues may also be obtained by pole zero plot in the following way.

1. Obtain the pole zero plot for the given network function.
2. Measure the distances $M_{la}, M_{lb}, \ldots, M_{ln}$ of a given pole from each of the other zeros.
3. Measure the distances $M_{l1}, M_{l2}, \ldots, M_{lm}$ of a given pole from each of the other poles.
4. Measure the angle $\phi_{la}, \phi_{lb}, \ldots, \phi_{ln}$ of the line joining that pole to each of the other zeros.
5. Measure the angle $\phi_{l1}, \phi_{l2}, \ldots, \phi_{lm}$ of the line joining that pole to each of the other poles.
6. Substitute these values in required residue equation.

16.14 AMPLITUDE AND PHASE RESPONSE FROM POLE ZERO PLOT

The steady state response can be obtained from the pole zero plot, and it is given by

$$N(j\omega) = M(\omega)e^{j\phi(\omega)}$$

where $M(\omega)$ is the amplitude

$\phi(\omega)$ is the phase

These amplitude and phase responses are useful in the design and analysis of network functions. For different values of ω, corresponding values of $M(\omega)$ and $\phi(\omega)$ can be obtained and these are plotted to get amplitude and phase response of the given network.

16.15 STABILITY CRITERION FOR ACTIVE NETWORK

Passive networks are said to be stable only when all the poles lie in the left half of the s-plane. Active networks (containing controlled sources) are not always stable. Consider transformed active network shown in Fig. 16.31.

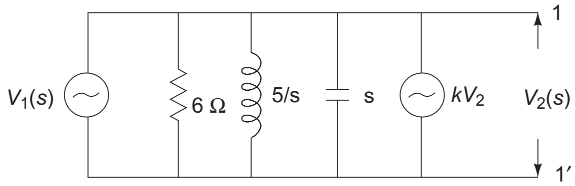


Fig. 16.31

By applying Millman Theorem, we get

$$V_2(s) = \frac{V_1(s) + kV_2(s)}{6 + 5/s + s}$$

$$= \frac{s\,[V_1(s) + kV_2(s)]}{s^2 + 6s + 5}$$

$$V_2(s)\,[s^2 + 6s + 5] - ksV_2(s) = sV_1(s)$$

$$V_2(s)\,[s^2 + (6 - k)s + 5] = sV_1(s)$$

$$\therefore \quad \frac{V_2(s)}{V_1(s)} = \frac{s}{s^2 + (6 - k)s + 5}$$

From the above transformed equation, the poles are dependent upon the value of k.

The roots of the equation are

$$s = \frac{-(6 - k) \pm \sqrt{(6 - k)^2 - 4 \times 5}}{2}$$

For $k = 0$, the poles are at $-1, -5$, which lie on the left half of the s-plane. As k increases, the poles move towards each other and meet at a point

$\sqrt{(6 - k)^2 - 20} = 0$, when $k = 1.53$ or 10.47. The root locus plot is shown in Fig. 16.32.

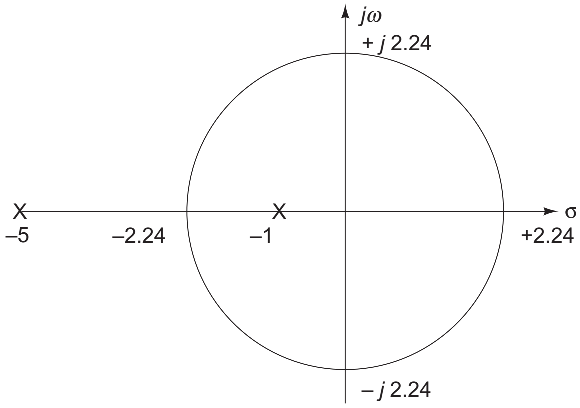


Fig. 16.32

The root locus is obtained from the characteristic equation $s^2 + (6 - k)s + 5 = 0$. As the value of k increases beyond 1.53, the locus of root is a circle. The poles are located on the imaginary axis at $\pm j2.24$ for $k = 6$. At -2.24, poles are coincident for $k = 1.53$ while at $+2.24$, poles are coincident for $k = 10.47$. When $k > 10.47$, the poles again lie on the real axis but remain on the right half of the s-plane, one pole moving towards the origin and the other moving towards infinity. From this we can conclude, as long as k is less than 6, the poles lie on the left half of the s-plane and the system is said to be stable. For $k = 6$, the poles lie on the imaginary axis and the system is oscillatory in nature. For values of k greater than 6, the poles lie on the right half of the s-plane. Then the system is said to be unstable.

16.16 ROUTH CRITERIA

The locations of the poles gives us an idea about stability of the active network. Consider the denominator polynomial

$$Q(s) = b_0 s^m + b_1 s^{m-1} + \ldots + b_m \tag{16.78}$$

To get a stable system, all the roots must have negative real parts. There should not be any positive or zero real parts. This condition is not sufficient.

Let us consider the polynomial

$$s^3 + 4s^2 + 15s + 100 = (s + 5)(s^2 - s + 20)$$

In the above polynomial, though the coefficients are positive and real, the two roots have positive real parts. From this we conclude that the coefficients of $Q(s)$ being positive and real is not a sufficient condition to get a stable system. Therefore, we have to seek the condition for stability which is necessary and sufficient.

Consider the polynomial $Q(s) = 0$. After factorisation, we get

$$b_0 (s - s_1)(s - s_2) \ldots (s - s_m) = 0 \tag{16.79}$$

On multiplication of these factors, we get

$$\begin{aligned} Q(s) = b_0 s^m - b_0(s_1 + s_2 + \dots + s_m)s^{m-1} \\ + b_0(s_1 s_2 + s_2 s_3 + \dots)\, s^{m-2} \\ + b_0\,(-1)^m\,(s_1 s_2 \dots s_m) = 0 \end{aligned} \tag{16.80}$$

Equating the coefficients of Eqs 16.78 and 16.80, we have

$$\frac{b_1}{b_0} = -(s_1 + s_2 + \dots + s_m) \tag{16.81}$$

$= -$ sum of the roots

$$\frac{b_2}{b_0} = 1(s_1 s_2 + s_2 s_3 + \dots) \tag{16.82}$$

$=$ sum of the products of the roots taken two at a time

$$\frac{b_3}{b_0} = -(s_1\,s_2\,s_3 + s_2\,s_3\,s_4 + \dots) \tag{16.83}$$

$= -$ sum of the products of the roots taken three at a time.

$$(-1)^m\,\frac{b_m}{b_0} = (s_1\,s_2\,s_3 \dots s_m) = \text{product of the roots} \tag{16.84}$$

If all the roots have negative real parts, then from the above equations it is clear that all the coefficients must have the same sign. This condition is not sufficient due to the fact that the zero value of a coefficient involves cancellation, which requires some root to have positive real parts.

The Routh criterion for stability is discussed below. Consider a polynomial

$$Q(s) = b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_m$$

Taking first row coefficients and second row coefficients separately, we have

$$\begin{matrix} b_0 & b_2 & b_4 & \dots \\ b_1 & b_3 & b_5 & \dots \end{matrix}$$

Now we complete the Routh array as follows.

For $m = 5$

$$\begin{array}{c|ccc} s^5 & b_0 & b_2 & b_4 \\ s^4 & b_1 & b_3 & b_5 \\ s^3 & c_1 & c_2 & \\ s^2 & d_1 & d_2 & \\ s^1 & e_1 & & \\ s^0 & f_1 & & \end{array}$$

where $c_1, c_2, d_1, d_2, e_1, f_1$ are determined by the algorithm given below.

$$c_1 = \frac{\begin{matrix} b_0 & b_2 \\ b_1 & b_3 \end{matrix}}{b_1} = \frac{b_1 b_2 - b_0 b_3}{b_1}$$

$$c_2 = \frac{\begin{matrix} b_0 & b_4 \\ b_1 & b_5 \end{matrix}}{b_1} = \frac{b_1 b_4 - b_0 b_5}{b_1}$$

$$d_1 = \frac{\begin{matrix} b_1 & b_3 \\ c_1 & c_2 \end{matrix}}{c_1} = \frac{c_1 b_3 - b_1 c_2}{c_1}$$

$$d_2 = \frac{\begin{matrix} b_1 & b_5 \\ c_1 & 0 \end{matrix}}{c_1} = \frac{b_5 c_1 - 0}{c_1}$$

$$e_1 = \frac{\begin{matrix} c_1 & c_2 \\ d_1 & d_2 \end{matrix}}{d_1} = \frac{c_2 d_1 - c_1 d_2}{d_1}$$

$$f_1 = \frac{\begin{matrix} d_1 & d_2 \\ e_1 & 0 \end{matrix}}{e_1} = \frac{d_2 e_1 - 0}{e_1}$$

In order to find out the element in kth row and jth column, it is required to know the four elements. These elements in the row $(k-1)$ and row $(k-2)$ just above the elements are in column 1 of the array and $(J+1)$ column of the array. The product of the elements joined by a line with positive slope has positive sign while the product of elements joined with a line with negative slope has a negative sign. The difference of these products is divided by the element of column 1 and row $(k-1)$. The above process is repeated till $m+1$ rows are found in the Routh array.

According to the Routh-Hurwitz theorem, the number of changes in the sign of the first column to the right of the vertical line in an array moving from top to bottom is equal to the number of roots of $Q(s) = 0$ with positive real parts. To get a stable system, the roots must have negative real parts.

According to the Routh-Hurwitz criterion, the system is stable, if and only if, there are no changes in signs of the first column of the array. This requirement is, both the necessary and sufficient condition for stability.

Solved Problems

Problem 16.1 For the network shown in Fig. 16.33, determine the transform impedance $Z(s)$.

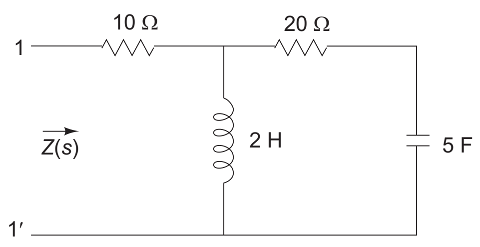


Fig. 16.33

Solution The transform network for the network shown in Fig. 16.33 is shown in Fig. 16.34.

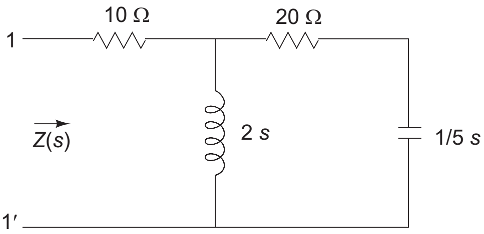


Fig. 16.34

From Fig. 16.34, the equivalent impedance at port 1–1′ is

$$Z(s) = \left\{10 + \left[2s \,||\left(20 + \frac{1}{5s}\right)\right]\right\}$$

$$= 10 + \frac{2s(20 + 1/5s)}{2s + 20 + 1/5s}$$

$$= \frac{20s + 200 + 2/s + 40s + 2/5}{\dfrac{10s^2 + 100s + 1}{5s}}$$

$$= \frac{100s^2 + 1000s + 10 + 200s^2 + 2s}{10s^2 + 100s + 1}$$

Therefore, the network transform impedance is

$$Z(s) = \frac{300s^2 + 1002s + 10}{10s^2 + 100s + 1}$$

Problem 16.2 For the two port network shown in Fig. 16.35, determine the driving point impedance $Z_{11}(s)$ and the driving point admittance $Y_{11}(s)$. Also find the transfer impedance $Z_{21}(s)$.

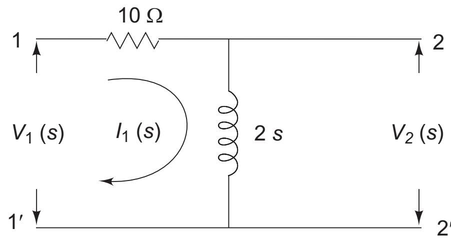


Fig. 16.35

Solution By applying Kirchhoff's law to the circuit, we have

$$V_1(s) = 10I_1(s) + 2s\, I_1(s) \tag{16.85}$$

The voltage across port 2–2′ is

$$V_2(s) = I_1(s) \times (2s) \tag{16.86}$$

From Eq. 16.85, the driving point impedance is

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = (2s + 10)$$

Similarly, the driving point admittance is

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)} = \frac{1}{2s+10}$$

From Eq. 16.86, the transfer impedance is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = 2s$$

Problem 16.3 For the network shown in Fig. 16.36, determine the transfer functions $G_{21}(s)$ and $Z_{21}(s)$ and the driving point admittance $Y_{11}(s)$.

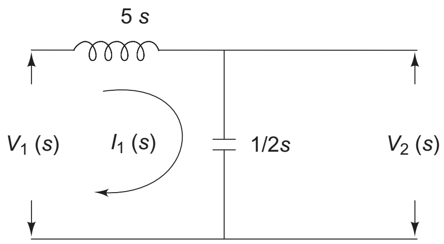


Fig. 16.36

Solution By applying Kirchhoff's voltage law at the ports, we get

$$V_1(s) = I_1(s)\left[5s + \frac{1}{2s}\right]$$

$$V_2(s) = \frac{1}{2s}\ I_1(s)$$

Therefore, the voltage transfer ratio

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{2s(5s+1/2s)}$$

$$G_{21}(s) = \frac{1}{10s^2+1}$$

The transform impedance is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{1}{2s}$$

The driving point admittance is

$$Y_{11}(s) = \frac{I_2(s)}{V_1(s)} = \frac{1}{5s+1/2s}$$

$$\therefore \qquad Y_{11}(s) = \frac{2s}{(10s^2 + 1)}$$

Problem 16.4 For the network shown in Fig. 16.37, determine the transfer functions $G_{21}(s)$ and $Z_{21}(s)$. Also find the driving point impedance $Z_{11}(s)$.

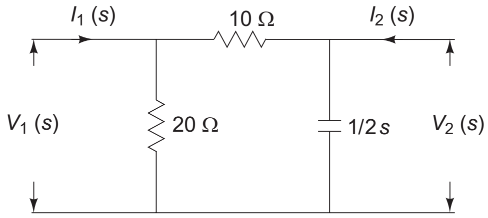


Fig. 16.37

Solution From Fig. 16.38, by application of Kirchhoff's laws, we get the following equations

The driving point impedance

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = [20 \parallel (10 + 1/2s)] = \frac{20 \times (10 + 1/2s)}{20 + 10 + 1/2s}$$

$$Z_{11}(s) = \frac{20(10 + 1/2s)}{30 + 1/2s}$$

$$Z_{11}(s) = \frac{400s + 20}{60s + 1}$$

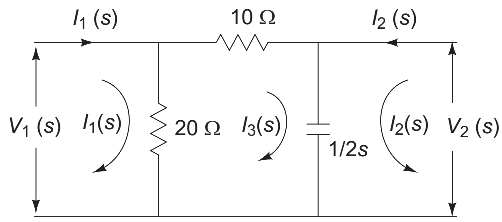


Fig. 16.38

From the above figure, by application of Kirchhoff's laws, we get

$$V_1(s) = 20I_1(s) - 20I_3(s) \tag{16.87}$$

$$10I_3(s) + 20[I_3(s) - I_1(s)] + \frac{1}{2s}[I_3(s) + I_2(s)] = 0 \tag{16.88}$$

$$V_2(s) = [I_2(s) + I_3(s)]\frac{1}{2s} \tag{16.89}$$

From Eq. 16.88, we get

$$\left(30 + \frac{1}{2s}\right)I_3(s) - 20\, I_1(s) = 0$$

$$I_3(s) = \frac{40s}{60s+1}\, I_1(s) \tag{16.90}$$

From Eq. 16.89, since $I_2 = 0$ we get

$$V_2(s) = +\, I_3(s)\left(\frac{1}{2s}\right) \tag{16.91}$$

The transfer impedance at port 2 is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{40s}{(60s+1)} \times \frac{1}{2s} = \frac{20}{(60s+1)}$$

The voltage transfer ratio

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{I_3(s)\,(1/2s)}{20I_1(s) - 20I_3(s)} = \frac{(1/2s)}{\dfrac{60s+1-40s}{2s}} = \frac{1}{20s+1}$$

Problem 16.5 Draw the pole zero diagram for the given network function $I(s)$ and hence obtain $i(t)$.

$$I(s) = \frac{20s}{(s+5)\,(s+2)}$$

Solution In the network function

$$P(s) = 20s$$

and $$Q(s) = (s+2)(s+5) = 0$$

By taking partial fractions, $I(s)$ can be written as

$$I(s) = \frac{k_1}{s+2} + \frac{k_2}{s+5}$$

Therefore, the time domain response is

$$i(t) = k_1 e^{-2t} + k_2 e^{-5t}$$

Here, the coefficients k_1 and k_2 are determined by using the pole zero plot as shown in Fig. 16.39.

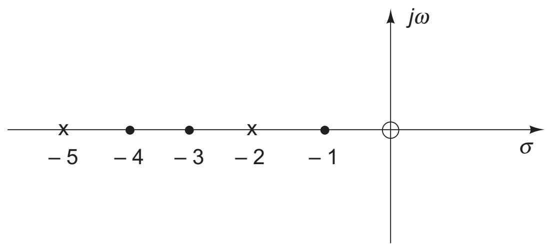


Fig. 16.39

Consider a pole at – 2

The distance between zero to pole at – 2 is

$$M_{02} = 2$$

The angle between the line joining to the pole at – 2 to the zero is

$$\phi_{02} = 180°$$

Similarly, the distance between pole at – 5 to pole at – 2 is

$$M_{52} = 3$$

The angle between the line joining the pole at – 2 to the pole at – 5 is

$$\phi_{52} = 0°$$

Hence $$k_1 = H\,\frac{M_{02}\,e^{j\phi_{02}}}{M_{52}\,e^{j\phi_{52}}}$$

$$= 20 \times \frac{2e^{j180}}{3e^{j0}} = 13.33\,e^{j180} = -\,13.33$$

Similarly, $$k_2 = H\,\frac{M_{05}\,e^{j\phi_{05}}}{M_{25}\,e^{j\phi_{25}}}$$

where $$M_{05} = 5,\ \phi_{05} = 180°$$
$$M_{25} = 3,\ \phi_{25} = 180°$$

Hence $$k_2 = \frac{20 \times 5}{3}\,e^{j(180\,-\,180)}$$

$$= \frac{100}{3} = 33.3$$

Substituting these values, we get

$$i(t) = (-\,13.33e^{-2t} + 33.3\,e^{-5t})\text{ A}$$

Problem 16.6 Draw the pole zero diagram for the given network function and hence obtain $v(t)$

$$V(s) = \frac{4(s+2)s}{(s+1)\,(s+3)}$$

Solution In the network function

$$p(s) = 4s(s+2)$$

and $$Q(s) = (s+1)\,(s+3) = 0$$

By taking partial fractions, we have

$$V(s) = \frac{k_1}{s+1} + \frac{k_2}{s+3}$$

The time domain response can be obtained by taking the inverse transform

$$v(t) = k_1\,e^{-t} + k_2\,e^{-3t}$$

Here, the coefficients k_1 and k_2 may be determined by using the pole zero plot as shown in Fig. 16.40.

To determine k_1, we have to find out the distances and phase angles from other zeros and poles to that particular pole.

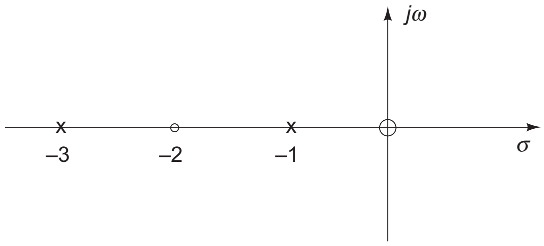


Fig. 16.40

Hence $$k_1 = H\,\frac{M_{01}M_{21}\,e^{j(\phi_{01}+\phi_{21})}}{M_{31}\,e^{j(\phi_{31})}}$$

where M_{01} and M_{21} are the distances between the zeros at 0 and – 2 to the pole at – 1, ϕ_{01}, ϕ_{21} are the phase angle between the corresponding zeros to the pole.

Similarly, M_{31} and ϕ_{31} are the distance and phase angle, respectively, from pole at – 3 to pole at – 1.

$$\therefore \quad M_{01} = 1;\ \phi_{01} = 180°$$
$$M_{21} = 1;\ \phi_{21} = 0$$
$$M_{31} = 2;\ \phi_{31} = 0°$$

$$\therefore \quad k_1 = 4 \times \frac{1\times 1}{2}\,e^{\,j(180°)}$$
$$k_1 = -2$$

Similarly,

$$k_2 = H\,\frac{M_{03}\,M_{23}}{M_{13}}\;e^{+j(\phi_{03}+\phi_{23}-\phi_{13})}$$

where $M_{03} = 3,\ \phi_{03} = 180°$
$$M_{23} = 1,\ \phi_{23} = 180°$$
$$M_{13} = 2,\ \phi_{13} = 180°$$

$$\therefore \quad k_2 = \frac{4\times 3\times 1}{2}\;e^{\,j(180\,+\,180\,-\,180)}$$
$$k_2 = -6$$

Substituting the values, we get
$$v(t) = (-\,2e^{-t} - 6e^{-3t})\text{V}$$

Problem 16.7 For the given network function, draw the pole zero diagram and hence obtain the time domain response $i(t)$.

$$I(s) = \frac{5s}{(s+1)\,(s^2 + 4s + 8)}$$

Solution In the network function
$$P(s) = 5s$$

$$Q(s) = (s + 1)\,(s^2 + 4s + 8) = 0$$

By taking the partial fraction expansion of $I(s)$, we get

$$I(s) = \frac{k_1}{s+1} + \frac{k_2}{(s+2+j2)} + \frac{k_3}{(s+2-j2)} \tag{16.92}$$

The time domain response can be obtained by taking the inverse transform as under,

$$i(t) = k_1 e^{-t} + k_2\, e^{-(2+j2)t} + k_2\, e^{-(2-j2)t} \tag{16.93}$$

To find the value of k_1, we have to find out the distances, and phase angles from other zeros and poles to that particular pole as shown in Fig. 16.41.

Hence $$k_1 = \frac{H M_{01}\, e^{j(\phi_{01})}}{M_{p11}\, M_{p21}\, e^{j[\phi_{p11} + \phi_{p21}]}}$$

$$M_{01} = 1;\ \phi_{01} = 180°$$

$$M_{p11} = \sqrt{5}\,;\ \phi_{p11} = -\,63.44°$$

$$M_{p21} = \sqrt{5}\,;\ \phi_{p21} = 63.44°$$

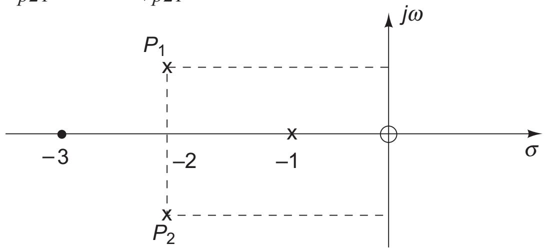


Fig. 16.41

$\therefore$ $$k_1 = \frac{5 \times 1 e^{j180°}}{\sqrt{5} \times \sqrt{5}\, e^{j(-63.44° + 63.44°)}}$$

$$k_1 = -\,1$$

Similarly $$k_2 = \frac{H M_{0p_1}\, e^{j\phi_{0p1}}}{M_{1p_1}\, M_{p_2 p_1}\, e^{j(\phi_{1p1} + \phi_{p2p1})}}$$

$$M_{0p_1} = \sqrt{8}\,;\ \phi_{0p_1} = 135°$$

$$M_{1p_1} = \sqrt{5}\,;\ \phi_{1p_1} = 116.56°$$

$$M_{p_1 p_2} = 4; \phi_{p_2 p_1} = 90°$$

Hence $$k_2 = \frac{5 \times \sqrt{8}}{\sqrt{5} \times 4}\ e^{j(135° - 116.56° - 90°)}$$

$$= 1.58\ e^{-j(71.56°)}$$

$$k_2^* = \frac{H M_{0p_2}\, e^{j\phi_{0p2}}}{M_{1p_2}\, M_{p_1 p_2}\, e^{j(\phi_{1p2} + \phi_{p1p2})}}$$

$$= \frac{5 \times \sqrt{8}\, e^{-j(135°)}}{\sqrt{5} \times 4 e^{j(-116.56° - 90°)}}$$

$= 1.58e^{j71.56°}$

If we substitute the values in Eq. 16.93, we get

$$i(t) = [-1e^{-t} + 1.58\, e^{-j(71.56°)}\, e^{-(2+j2)t} + 1.58e^{j(71.56°)}\, e^{-(2-j2)t}]\text{A}$$

Problem 16.8 For the given denominator polynomial of a network function, verify the stability of the network by using the Routh criterion.

$$Q(s) = s^3 + 2s^2 + 8s + 10$$

Solution Routh array for this polynomial is given below

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 2 & 10 \\ s^1 & 3 & \\ s^0 & 10 & \end{array}$$

There is no change in sign in the first column of the array. Hence, there are no roots with positive real parts. Therefore, the network is stable.

Problem 16.9 For the given denominator polynomial of a network function, verify the stability of the network using the Routh criterion.

$$Q(s) = s^3 + s^2 + 3s + 8$$

Solution Routh array for this polynomial is given below.

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 1 & 8 \\ s^1 & -5 & \\ s^0 & +8 & \end{array}$$

There are two changes in sign of the first column, one from 1 to – 5 and the other from – 5 to + 8. Therefore, the two roots have positive real parts. Hence the network is not stable.

Problem 16.10 For the given denominator polynomial of a network function, determine the value of k for which the network to stable.

$$Q(s) = s^3 + 2s^2 + 4s + k$$

Solution Routh array for the given polynomial is given below.

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & k \\ s^1 & \dfrac{8-k}{2} & \\ s^0 & k & \end{array}$$

When $k < 8$, all the terms in the first column are positive. Therefore, there is no sign change in the first column. Hence, the network is stable. When $k > 8$, the $8 - k/2$ is negative. Therefore, there are two sign changes in the first column. There are two roots which have positive real parts. Hence, the network is unstable.

When $k = 8$, the Routh array becomes

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & 8 \\ s^1 & \alpha & \\ s^0 & 8 & \end{array}$$

The element in the first column and third row is zero. But we can take it as a small number. In this case there are no changes in the sign of the first column. Hence, the network is stable.

Problem 16.11 Apply Routh criterion to the given polynomial and determine the number of roots (i) with positive real parts (ii) with zero real parts (iii) with negative real parts.

$$Q(s) = s^4 + 4s^3 + 8s^2 + 12s + 15$$

Solution The Routh array for the polynomial is

$$\begin{array}{c|ccc} s^4 & 1 & 8 & 15 \\ s^3 & 4 & 12 & \\ s^2 & 5 & 15 & \\ s^1 & 0 & 0 & \\ s^0 & ? & ? & \end{array}$$

In this case, all the elements in the 4th row have become zero and the array cannot be completed.

The given equation is reduced by taking the new polynomial from the 3rd row

$$5s^2 + 15 = 0$$
$$5(s^2 + 3) = 0$$

Hence the other polynomial

$$Q_2(s) = \frac{s^4 + 4s^3 + 8s^2 + 12s + 15}{5(s^2 + 3)}$$

The equation reduces to the following polynomial

$$(s^2 + 3)\,(s^2 + 4s + 5) = 0$$

The roots of the equation $s^2 + 3 = 0$ are $s = \pm j\sqrt{3}$

There two roots have zero real parts.

Again forming Routh array for the polynomial

$$s^2 + 4s + 5 = 0$$

$$\begin{array}{c|cc} s^2 & 1 & 5 \\ s^1 & 4 & 0 \\ s^0 & 5 & \end{array}$$

There are no changes in the sign of the first column. Hence, all the two roots have negative real parts. Therefore, out of four roots, two roots have negative real parts and two roots have zero real parts.

Practice Problems

16.1 For the two-port network shown in Fig. 16.42, determine the driving point impedance $Z_{11}(s)$, the transfer impedance $Z_{21}(s)$ and the voltage transfer ratio $G_{21}(s)$.

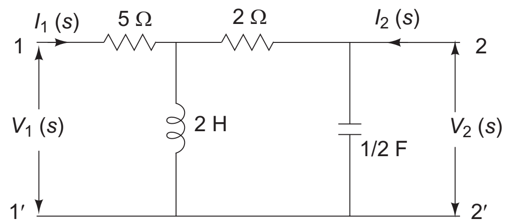


Fig. 16.42

16.2 For the network shown in Fig. 16.43, determine the following transfer functions. (a) G_{21} (s), (b) Y_{21} (s) and (c) $\alpha_{21}(s)$.

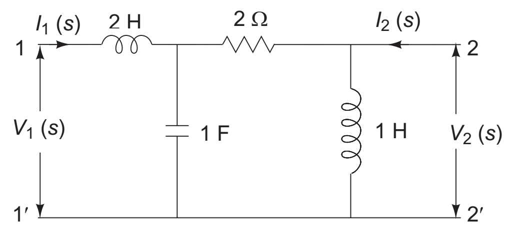


Fig. 16.43

16.3 For the network shown in Fig. 16.44, determine the following transfer functions (a) $G_{21}(s)$, (b) $Z_{21}(s)$.

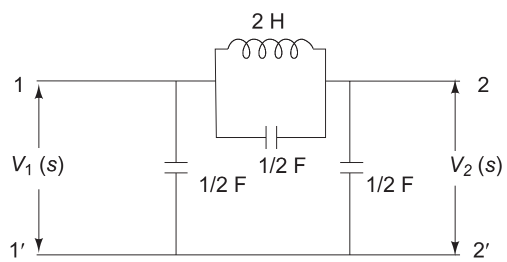


Fig. 16.44

16.4 For the network shown in Fig. 16.45, determine the following functions (a) $Z_{11}(s)$, (b) $Y_{11}(s)$, (c) $G_{21}(s)$ and (d) $\alpha_{21}(s)$.

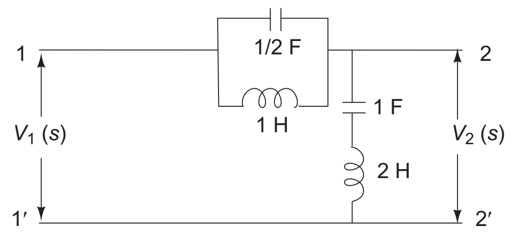


Fig. 16.45

16.5 For the network shown in Fig. 16.46, determine transfer impedance $Z_{21}(s)$ and $Y_{21}(s)$. Also find the transfer voltage ratio $G_{21}(s)$ and the transfer current ratio $\alpha_{21}(s)$.

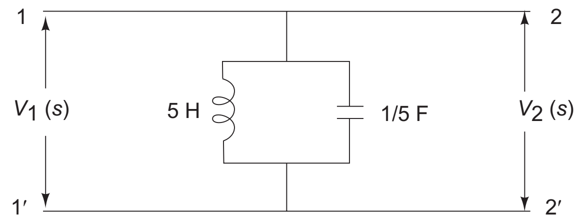


Fig. 16.46

16.6 For the given network function, draw the pole zero diagram and hence obtain the time domain response. Verify the result analytically.

$$V(s) = \frac{5(s+5)}{(s+2)(s+7)}$$

16.7 For the given network function draw the pole zero diagram and hence obtain the time domain response. Verify this result analytically.

$$I(s) = \frac{3s}{(s+1)(s+3)}$$

16.8 For the given network function, draw the pole zero diagram and hence obtain the time domain response. Verify the result analytically.

$$I(s) = \frac{5s}{(s+3)(s^2+2s+2)}$$

16.9 For the given denominator polynomial of a network function, verify the stability of the network using Routh criteria.

$$Q(s) = s^5 + 3s^4 + 4s^3 + 5s^2 + 6s + 1$$

16.10 For the given denominator polynomial of a network function, verify the stability of the network using Routh criteria.

$$Q(s) = s^4 + s^3 + 2s^2 + 2s + 12$$

16.11 Apply Routh criterion to the following equations and determine the number of roots (i) with positive real parts (ii) with zero real parts (iii) with negative real parts

(a) $6s^3 + 2s^2 + 5s + 2 = 0$

(b) $s^6 + 5s^5 + 13s^4 + 21s^3 + 20s^2 + 16s + 8 = 0$

(c) $s^6 - s^5 - 2s^4 + 4s^3 - 5s^2 + 21s + 30 = 0$

Objective-type Questions

1. The driving point impedance is defined as
   (a) the ratio of transform voltage to transform current at the same port
   (b) the ratio of transform voltage at one port to the transform current at the other port
   (c) both (a) and (b)
   (d) none of the above
2. The transfer impedance is defined as
   (a) the ratio of transform voltage to transform current at the same port
   (b) the ratio of transform voltage at one port to the current transform at the other port
   (c) both (a) and (b)
   (d) none of the above
3. The function is said to be having simple poles and zeros and only if
   (a) the poles are not repeated
   (b) the zeros are not repeated
   (c) both poles and zeros are not repeated
   (d) none of the above
4. The necessary condition for a driving point function is
   (a) the real part of all poles and zeros must not be zero or negative
   (b) the polynomials $P(s)$ and $Q(s)$ may not have any missing terms between the highest and lowest degree unless all even or all odd terms are missing.
   (c) the degree of $P(s)$ and $Q(s)$ may differ by more than one
   (d) the lowest degree in $P(s)$ and $Q(s)$ may differ in degree by more than two.
5. The necessary condition for the transfer functions is that
   (a) the coefficients in the polynomials $P(s)$ and $Q(s)$ must be real
   (b) coefficients in $Q(s)$ may be negative
   (c) complex or imaginary poles and zeros may not conjugate
   (d) if the real part of pole is zero, then that pole must be multiple
6. The system is said to be stable, if and only if
   (a) all the poles lie on right half of the s-plane
   (b) some poles lie on the right half of the s-plane
   (c) all the poles does not lie on the right half of the s-plane
   (d) none of the above.

Chapter 17

Filters and Attenuators

17.1 CLASSIFICATION OF FILTERS

Wave filters were first invented by G.A. Campbell and O.I. Lobel of the Bell Telephone Laboratories. A filter is a reactive network that freely passes the desired bands of frequencies while almost totally suppressing all other bands. A filter is constructed from purely reactive elements, for otherwise the attenuation would never becomes zero in the pass band of the filter network. Filters differ from simple resonant circuits in providing a substantially constant transmission over the band which they accept; this band may lie between any limits depending on the design. Ideally, filters should produce no attenuation in the desired band, called the *transmission* band or *pass* band, and should provide total or infinite attenuation at all other frequencies, called *attenuation* band or *stop* band. The frequency which separates the transmission band and the attenuation band is defined as the cut-off frequency of the wave filters, and is designated by f_c.

Filter networks are widely used in communication systems to separate various voice channels in carrier frequency telephone circuits. Filters also find applications in instrumentation, telemetering equipment, etc. where it is necessary to transmit or attenuate a limited range of frequencies.

A filter may, in principle, have any number of pass bands separated by attenuation bands. However, they are classified into four common types, viz. low pass, high pass, band pass and band elimination.

Decibel and Neper

The attenuation of a wave filter can be expressed in decibels or nepers. Neper is defined as the natural logarithm of the ratio of input voltage (or current) to the output voltage (or current), provided that the network is properly terminated in its characteristic impedance Z_0.

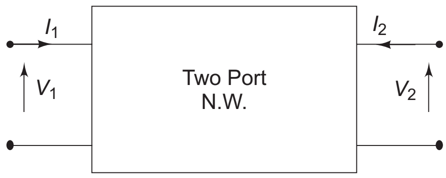


Fig. 17.1 (a)

From Fig. 17.1(a) the number of nepers, $N = \log_e \left|\frac{V_1}{V_2}\right|$ or $\log_e \left|\frac{I_1}{I_2}\right|$.

A neper can also be expressed in terms of input power, P_1 and the output power P_2 as $N = 1/2 \log_e P_1/P_2$.

A decibel is defined as ten times the common logarithms of the ratio of the input power to the output power.

$$\therefore \qquad \text{Decibel } D = 10 \log_{10} \frac{P_1}{P_2}$$

The decibel can be expressed in terms of the ratio of input voltage (or current) and the output voltage (or current.)

$$D = 20 \log_{10} \left|\frac{V_1}{V_2}\right| = 20 \log_{10} \left|\frac{I_1}{I_2}\right|$$

$\therefore$ One decibel is equal to 0.115 N.

Low Pass Filter

By definition, a low pass (LP) filter is one which passes without attenuation all frequencies up to the cut-off frequency f_c, and attenuates all other frequencies greater than f_c. The attenuation characteristic of an ideal LP filter is shown in Fig. 17.1(b). This transmits currents of all frequencies from zero up to the cut-off frequency. The band is called pass band or transmission band. Thus, the pass band for the LP filter is the frequency range 0 to f_c. The frequency range over which transmission does not take place is called the stop band or attenuation band. The stop band for a LP filter is the frequency range above $f_{c.}$

High Pass Filter

A high pass (HP) filter attenuates all frequencies below a designated cut-off frequency, f_c, and passes all frequencies above f_c. Thus the pass band of this filter is the frequency range above f_c, and the stop band is the frequency range below f_c. The attenuation characteristic of a HP filter is shown in Fig. 17.1(b).

Band Pass Filter

A band pass filter passes frequencies between two designated cut-off frequencies and attenuates all other frequencies. It is abbreviated as *BP filter*. As shown in Fig. 17.1(b), a BP filter has two cut-off frequencies and will have the pass band

$f_2 - f_1$ f_1 is called the lower cut-off frequency, while f_2 is called the upper cut-off frequency.

Band Elimination Filter

A band elimination filter passes all frequencies lying outside a certain range, while it attenuates all frequencies between the two designated frequencies. It is also referred as band stop filter. The characteristic of an ideal band elimination filter is shown in Fig. 17.1(b).

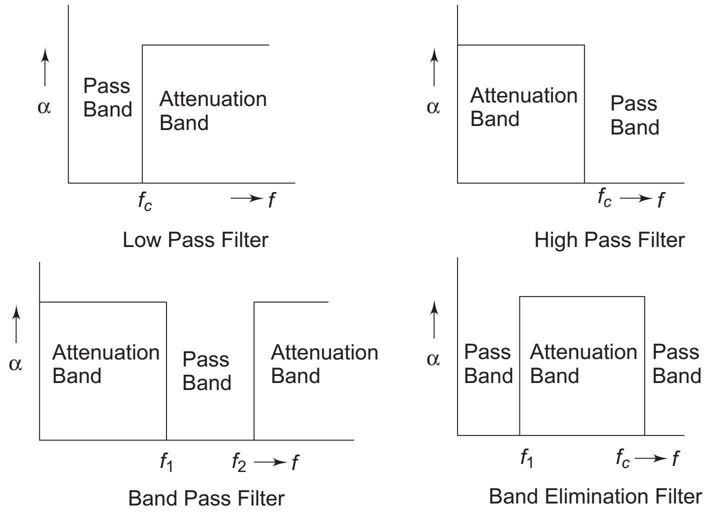
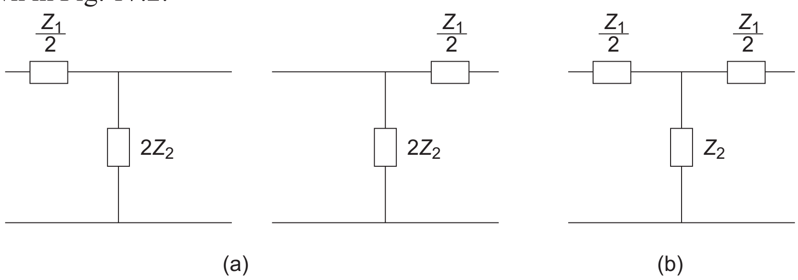


Fig. 17.1(b)

All frequencies between f_1 and f_2 will be attenuated while frequencies below f_1 and above f_2 will be passed.

17.2 FILTER NETWORKS

Ideally a filter should have zero attenuation in the pass band. This condition can only be satisfied if the elements of the filter are dissipationless, which cannot be realized in practice. Filters are designed with an assumption that the elements of the filters are purely reactive. Filters are made of symmetrical T, or π sections. T and π sections can be considered as combinations of unsymmetrical L sections as shown in Fig. 17.2.



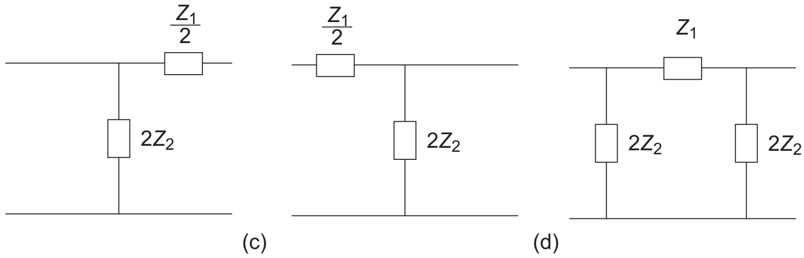


Fig. 17.2

The ladder structure is one of the commonest forms of filter network. A cascade connection of several T and π sections constitutes a ladder network. A common form of the ladder network is shown in Fig. 17.3.

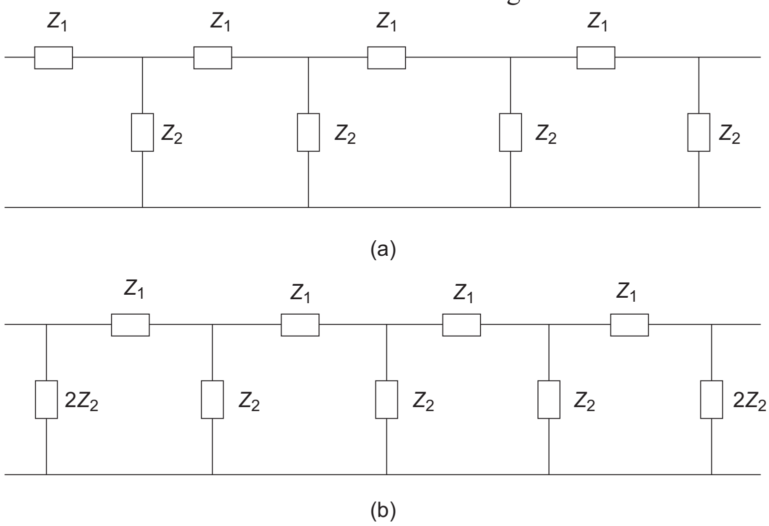


Fig. 17.3

Figure 17.3(a) represents a T section ladder network, whereas Fig. 17.3(b) represents the π section ladder network. It can be observed that both networks are identical except at the ends.

17.3 EQUATIONS OF FILTER NETWORKS

The study of the behaviour of any filter requires the calculation of its propagation constant γ, attenuation α, phase shift β and its characteristic impedance Z_0.

T-Network

Consider a symmetrical T-network as shown in Fig. 17.4.

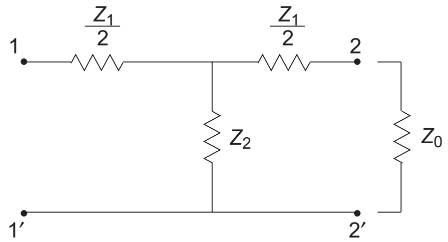


Fig. 17.4

As has already been mentioned in Section 15.13, if the image impedances at port 1-1′ and port 2-2′ are equal to each other, the image impedance is then called the characteristic, or the iterative impedance, Z_0. Thus, if the network in Fig. 17.4 is terminated in Z_0, its input impedance will also be Z_0. The value of input impedance for the T-network when it is terminated in Z_0 is given by

$$Z_{\text{in}} = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_0\right)}{\frac{Z_1}{2} + Z_2 + Z_0}$$

also $$Z_{\text{in}} = Z_0$$

$\therefore$ $$Z_0 = \frac{Z_1}{2} + \frac{2Z_2\left(\frac{Z_1}{2} + Z_0\right)}{Z_1 + 2Z_2 + 2Z_0}$$

$$Z_0 = \frac{Z_1}{2} + \frac{(Z_1Z_2 + 2Z_2Z_0)}{Z_1 + 2Z_2 + 2Z_0}$$

$$Z_0 = \frac{Z_1^2 + 2Z_1Z_2 + 2Z_1Z_0 + 2Z_1Z_2 + 4Z_0Z_2}{2(Z_1 + 2Z_2 + 2Z_0)}$$

$$4Z_0^2 = Z_1^2 + 4Z_1Z_2$$

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1Z_2$$

The characteristic impedance of a symmetrical T-section is

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1Z_2} \qquad (17.1)$$

Z_{0T} can also be expressed in terms of open circuit impedance Z_{0c} and short circuit impedance Z_{sc} of the T-network. From Fig. 17.4, the open circuit impedance Z_{0c} $= \frac{Z_1}{2} + Z_2$ and

$$Z_{sc} = \frac{Z_1}{2} + \frac{\frac{Z_1}{2} \times Z_2}{\frac{Z_1}{2} + Z_2}$$

$$Z_{sc} = \frac{Z_1^2 + 4Z_1Z_2}{2Z_1 + 4Z_2}$$

$$Z_{0c} \times Z_{sc} = Z_1Z_2 + \frac{Z_1^2}{4}$$

$$= Z_0^2\, T \;\text{ or }\; Z_{0T} = \sqrt{Z_{0c}\, Z_{sc}} \tag{17.2}$$

Propagation Constant of T-Network

By definition the propagation constant γ of the network in Fig. 17.5 is given by $\gamma = \log_e I_1/I_2$

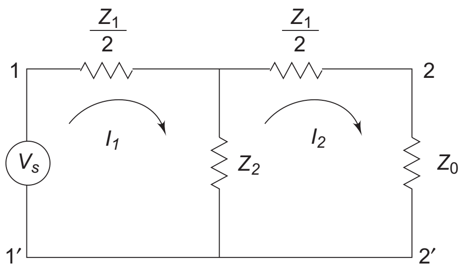


Fig. 17.5

Writing the mesh equation for the 2nd mesh, we get

$$I_1Z_2 = I_2\left(\frac{Z_1}{2} + Z_2 + Z_0\right)$$

$$\frac{I_1}{I_2} = \frac{\frac{Z_1}{2} + Z_2 + Z_0}{Z_2} = e^{\gamma}$$

$\therefore$
$$\frac{Z_1}{2} + Z_2 + Z_0 = Z_2\, e^{\gamma}$$

$$Z_0 = Z_2\,(e^{\gamma} - 1) - \frac{Z_1}{2} \tag{17.3}$$

The characteristic impedance of a T-network is given by

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1\, Z_2} \tag{17.4}$$

Squaring Eqs. 17.3 and 17.4 and subtracting Eq. 17.4 from Eq. 17.3, we get

$$Z_2^2\,(e^{\gamma} - 1)^2 + \frac{Z_1^2}{4} - Z_1Z_2\,(e^{\gamma} - 1) - \frac{Z_1^2}{4} - Z_1Z_2 = 0$$

$$Z_2^2\,(e^{\gamma} - 1)^2 - Z_1Z_2\,(1 + e^{\gamma} - 1) = 0$$

$$Z_2^2\,(e^{\gamma} - 1)^2 - Z_1Z_2\, e^{\gamma} = 0$$

$$Z_2(e^{\gamma}-1)^2 - Z_1 e^{\gamma} = 0$$

$$(e^{\gamma}-1)^2 = \frac{Z_1 e^{\gamma}}{Z_2}$$

$$e^{2\gamma} + 1 - 2e^{\gamma} = \frac{Z_1}{Z_2\, e^{-\gamma}}$$

Rearranging the above equation, we have

$$e^{-\gamma}(e^{2\gamma} + 1 - 2e^{\gamma}) = \frac{Z_1}{Z_2}$$

$$(e^{\gamma} + e^{-\gamma} - 2) = \frac{Z_1}{Z_2}$$

Dividing both sides by 2, we have

$$\frac{e^{\gamma} + e^{-\gamma}}{2} = 1 + \frac{Z_1}{2Z_2}$$

$$\cosh\gamma = 1 + \frac{Z_1}{2Z_2} \tag{17.5}$$

Still another expression may be obtained for the complex propagation constant in terms of the hyperbolic tangent rather than hyperbolic cosine.

$$\sinh\gamma = \sqrt{\cos h^2\gamma - 1}$$

$$= \sqrt{\left(1 + \frac{Z_1}{2Z_2}\right)^2 - 1} = \sqrt{\frac{Z_1}{Z_1} + \left(\frac{Z_1}{2Z_2}\right)^2}$$

$$\sinh\gamma = \frac{1}{Z_2}\sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} = \frac{Z_{0T}}{Z_2} \tag{17.6}$$

Dividing Eq. 17.6 by Eq. 17.5, we get

$$\tanh\gamma = \frac{Z_{0T}}{Z_2 + \dfrac{Z_1}{2}}$$

But $$Z_2 + \frac{Z_1}{2} = Z_{0c}$$

Also from Eq. 17.2, $Z_{0T} = \sqrt{Z_{0c}\, Z_{sc}}$

$$\tanh\gamma = \sqrt{\frac{Z_{sc}}{Z_{0c}}}$$

Also
$$\sinh\frac{\gamma}{2} = \sqrt{\frac{1/2\,(\cosh\gamma - 1)}{1/2\,(1 + Z_1/2Z_2 - 1)}}$$
$$= \sqrt{\frac{Z_1}{4Z_2}} \qquad (17.7)$$

π-Network

Consider asymmetrical π-section shown in Fig. 17.6. When the network

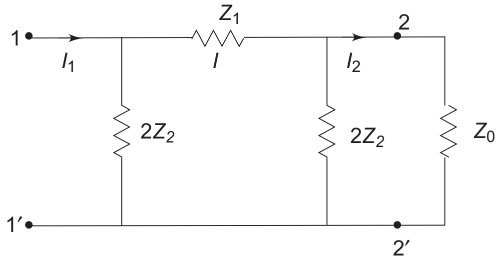


Fig. 17.6

is terminated in Z_0 at port 2-2′, its input impedance is given by

$$Z_{\text{in}} = \frac{2Z_2\left[Z_1 + \dfrac{2Z_2\,Z_0}{2Z_2 + Z_0}\right]}{Z_1 + \dfrac{2Z_2\,Z_0}{2Z_2 + Z_0} + 2Z_2}$$

By definition of characteristic impedance, $Z_{\text{in}} = Z_0$

$$Z_0 = \frac{2Z_2\left[Z_1 + \dfrac{2Z_2\,Z_0}{2Z_2 + Z_0}\right]}{Z_1 + \dfrac{2Z_2\,Z_0}{2Z_2 + Z_0} + 2Z_2}$$

$$Z_0Z_1 + \frac{2Z_2\,Z_0^2}{2Z_2 + Z_0} + 2Z_0Z_2 = \frac{2Z_2\,(2Z_1\,Z_2 + Z_0Z_1 + 2Z_0\,Z_2)}{(2Z_2 + Z_0)}$$

$$2Z_0Z_1Z_2 + Z_1Z_0^2 + 2Z_0Z_2^2 + 4Z_2Z_0^2 + 2Z_2Z_0^2$$
$$= 4Z_1Z_2^2 + 2Z_0Z_1Z_2 + 4Z_0Z_2^2$$
$$Z_1Z_0^2 + 4Z_2Z_0^2 = 4Z_1Z_2^2$$
$$Z_0^2(Z_1 + 4Z_2) = 4Z_1Z_2^2$$
$$Z_0^2 = \frac{4Z_1Z_2^2}{Z_1 + 4Z_2}$$

Rearranging the above equation leads to

$$Z_0 = \sqrt{\frac{Z_1 Z_2}{1 + Z_1/4Z_2}} \tag{17.8}$$

which is the characteristic impedance of a symmetrical π-network,

$$Z_{0\pi} = \frac{Z_1 Z_2}{\sqrt{Z_1 Z_2 + Z_1^2/4}}$$

From Eq. 17.1 $\qquad Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$

$\therefore$
$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} \tag{17.9}$$

$Z_{0\pi}$ can be expressed in terms of the open circuit impedance Z_{0c} and short circuit impedance Z_{sc} of the π network shown in Fig. 17.6 exclusive of the load Z_0.

From Fig. 17.6, the input impedance at port 1-1′ when port 2-2′ is open is given by $Z_{0c} = \frac{2Z_2 (Z_1 + 2Z_2)}{Z_1 + 4Z_2}$

Similarly, the input impedance at port 1-1′ when port 2-2′ is short circuited is given by $Z_{sc} = \frac{2Z_1 Z_2}{2Z_2 + Z_1}$

Hence $\qquad Z_{0c} \times Z_{sc} = \frac{4Z_1 Z_2^2}{Z_1 + 4Z_2} = \frac{Z_1 Z_2}{1 + Z_1/4Z_2}$

Thus from Eq. 17.8

$$Z_{0\pi} = \sqrt{Z_{0c} \times Z_{sc}} \tag{17.10}$$

Propagation Constant of π-Network

The propagation constant of a symmetrical π-section is the same as that for a symmetrical T-section.

i.e. $\qquad \cosh \gamma = 1 + \frac{Z_1}{2Z_2}$

17.4 CLASSIFICATION OF PASS BAND AND STOP BAND

It is possible to verify the characteristics of filters from the propagation constant of the network. The propagation constant γ, being a function of frequency, the pass band, stop band and the cut-off point, i.e. the point of separation between the two bands, can be identified. For symmetrical T or π-section, the expression for propagation constant γ in terms of the hyperbolic functions is given by Eqs. 17.5 and 17.7 in Section 17.3. From Eq. 17.7, $\sinh \dfrac{\gamma}{2} = \sqrt{\dfrac{Z_1}{4Z_2}}$.

If Z_1 and Z_2 are both pure imaginary values, their ratio, and hence $Z_1/4Z_2$, will be a pure real number. Since Z_1 and Z_2 may be anywhere in the range from $-j_\infty$ t_0 $+j_\infty$, $Z_1/4Z_2$ may also have any real value between the infinite limits. Then sinh $\gamma/2 = \sqrt{Z_1}/\sqrt{4Z_2}$ will also have infinite limits, but may be either real or imaginary depending upon whether $Z_1/4Z_2$ is positive or negative.

We know that the propagation constant is a complex function $\gamma = \alpha + j\beta$, the real part of the complex propagation constant α, is a measure of the change in magnitude of the current or voltage in the network, known as the attenuation constant. β is a measure of the difference in phase between the input and output currents or voltages, known as phase shift constant. Therefore α and β take on different values depending upon the range of $Z_1/4Z_2$. From Eq. 17.7, we have

$$\sinh \frac{\gamma}{2} = \sinh\left(\frac{\alpha}{2} + \frac{j\beta}{2}\right) = \sinh\frac{\alpha}{2}\cos\frac{\beta}{2} + j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2}$$

$$= \sqrt{\frac{Z_1}{4Z_2}} \tag{17.11}$$

Case A If Z_1 and Z_2 are the same type of reactances, then $\left|\dfrac{Z_1}{4Z_2}\right|$ is real and equal to say $+x$.

The imaginary part of the Eq. 17.11 must be zero.

$$\therefore \qquad \cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = 0 \tag{17.12}$$

$$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = x \tag{17.13}$$

α and β must satisfy both the above equations.

Equation 17.12 can be satisfied if $\beta/2 = 0$, or $n\pi$ where $n = 0, 1, 2, \ldots$, then $\cos \beta/2 = 1$ and $\sinh \alpha/2 = x = \sqrt{\dfrac{Z_1}{4Z_2}}$.

That x should be always positive implies that

$$\left|\frac{Z_1}{4Z_2}\right| > 0 \text{ and } \alpha = 2\sinh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{17.14}$$

Since $\alpha \neq 0$, it indicates that the attenuation exists.

Case B Consider the case of Z_1 and Z_2 being opposite type of reactances, i.e. $Z_1/4Z_2$ is negative, making $\sqrt{Z_1/4Z_2}$ imaginary and equal to say Jx

$\therefore$ The real part of the Eq. 17.11 must be zero.

$$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = 0 \tag{17.15}$$

$$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = x \tag{17.16}$$

Both the above equations must be satisfied simultaneously by α and β. Equation 17.15 may be satisfied when $\alpha = 0$, or when $\beta = \pi$. These conditions are considered separately hereunder.

(i) When $\alpha = 0$; from Eq. 17.15, $\sinh \alpha/2 = 0$. And from Eq 17.16 $\sin \beta/2 = x = \sqrt{Z_1/4Z_2}$. But the sine can have a maximum value of 1. Therefore, the above solution is valid only for negative $Z_1/4Z_2$, and having maximum value of unity. It indicates the condition of pass band with zero attenuation and follows the condition as

$$-1 \le \frac{Z_1}{4Z_2} \le 0$$

$$\beta = 2\sin^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{17.17}$$

(ii) When $\beta = \pi$, from Eq. 17.15, $\cos \beta/2 = 0$. And from Eq. 17.16, $\sin \beta/2 = \pm 1$; $\cosh \alpha/2 = x = \sqrt{Z_1/4Z_2}$.

Since $\cosh \alpha/2 \ge 1$, this solution is valid for negative $Z_1/4Z_2$, and having magnitude greater than, or equal to unity. It indicates the condition of stop band since $\alpha \ne 0$.

$$-\alpha \le \frac{Z_1}{4Z_2} \le -1$$

$$\alpha = 2\cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{17.18}$$

It can be observed that there are three limits for case *A* and *B*. Knowing the values of Z_1 and Z_2, it is possible to determine the case to be applied to the filter. Z_1 and Z_2 are made of different types of reactances, or combinations of reactances, so that, as the frequency changes, a filter may pass from one case to another. Case A and (ii) in case B are attenuation bands, whereas (i) in case B is the transmission band.

The frequency which separates the attenuation band from pass band or vice versa is called cut-off frequency. The cut-off frequency is denoted by f_c, and is also termed as nominal frequency. Since Z_0 is real in the pass band and imaginary in an attenuation band, f_c is the frequency at which Z_0 changes from being real to being imaginary. These frequencies occur at

$$\left.\begin{aligned} \frac{Z_1}{4Z_2} &= 0 \text{ or } Z_1 = 0 \\ \frac{Z_1}{4Z_2} &= -1 \text{ or } Z_1 + 4Z_2 = 0 \end{aligned}\right\} \quad (17.18\,(a))$$

The above conditions can be represented graphically, as in Fig. 17.7.

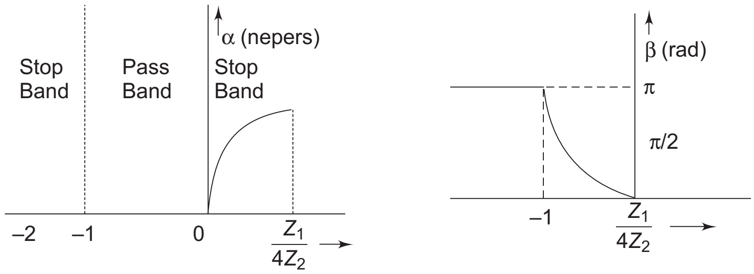


Fig. 17.7

17.5 CHARACTERISTIC IMPEDANCE IN THE PASS AND STOP BANDS

Referring to the characteristic impedance of a symmetrical *T*-network, from Eq. 17.1 we have

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)}$$

If Z_1 and Z_2 are purely reactive, let $Z_1 = jx_1$ and $Z_2 = jx_2$, then

$$Z_{0T} = \sqrt{-x_1 x_2 \left(1 + \frac{x_1}{4x_2}\right)} \quad (17.19)$$

A pass band exists when x_1 and x_2 are of opposite reactances and

$$-1 < \frac{x_1}{4x_2} < 0$$

Substituting these conditions in Eq. 17.19, we find that Z_{0T} is positive and real. Now consider the stop band. A stop band exists when x_1 and x_2 are of the same type of reactances; then $x_1/4x_2 > 0$. Substituting these conditions in Eq. 17.19, we find that Z_{0T} is purely imaginary in this attenuation region. Another stop band exists when x_1 and x_2 are of the same type of reactances, but with $x_1/4x_2 < -1$. Then from Eq. 17.19, Z_{0T} is again purely imaginary in the attenuation region.

Thus, in a pass band if a network is terminated in a pure resistance $R_0 (Z_{0T} = R_0)$, the input impedance is R_0 and the network transmits the power received from the source to the R_0 without any attenuation. In a stop band Z_{0T} is reactive.

Therefore, if the network is terminated in a pure reactance (Z_0 = pure reactance), the input impedance is reactive, and cannot receive or transmit power. However, the network transmits voltage and current with 90° phase difference and with attenuation. It has already been shown that the characteristic impedance of a symmetrical π-section can be expressed in terms of T. Thus, from Eq. 17.9, $Z_{0\pi} = Z_1Z_2/Z_{0T}$.

Since Z_1 and Z_2 are purely reactive, $Z_{0\pi}$ is real if Z_{0T} is real, and Z_{0x} is imaginary if Z_{0T} is imaginary. Thus the conditions developed for T-sections are valid for π sections.

17.6 CONSTANT—*K* LOW PASS FILTER

A network, either T or π, is said to be of the constant-k type if Z_1 and Z_2 of the network satisfy the relation

$$Z_1Z_2 = k^2 \tag{17.20}$$

where Z_1 and Z_2 are impedances in the T and π sections as shown in Fig. 17.8. Equation 17.20 states that Z_1 and Z_2 are inverse if their product is a constant, independent of frequency. k is a real constant, that is the resistance. k is often termed as design impedance or nominal impedance of the constant k-filter.

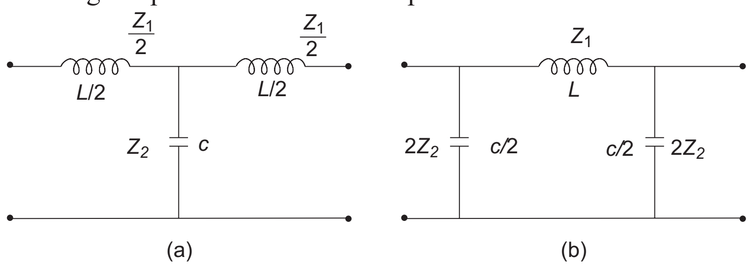


Fig. 17.8

The constant k, T or π type filter is also known as the *prototype* because other more complex networks can be derived from it. A prototype T and π-sections are shown in Fig. 17.8(a) and (b), where $Z_1 = j\omega_L$ and $Z_2 = 1/j\omega_C$. Hence $Z_1Z_2 = \dfrac{L}{C}$ $= k^2$ which is independent of frequency.

$$Z_1Z_2 = k^2 = \frac{L}{C} \text{ or } k = \sqrt{\frac{L}{C}} \tag{17.21}$$

Since the product Z_1 and Z_2 is constant, the filter is a constant-k type. From Eq. 17.18(a) the cut-off frequencies are $Z_1/4Z_2 = 0$,

i.e. $$\frac{-\omega^2 LC}{4} = 0$$

i.e. $$f = 0 \text{ and } \frac{Z_1}{4Z_2} = -1$$

$$\frac{-\omega^2 LC}{4} = -1$$

or $$f_c = \frac{1}{\pi\sqrt{LC}} \tag{17.22}$$

The pass band can be determined graphically. The reactances of Z_1 and $4Z_2$ will vary with frequency as drawn in Fig. 17.9. The cut-off frequency at the intersection of the curves Z_1 and $-4Z_2$ is indicated as f_c. On the X-axis as $Z_1 = -4Z_2$ at cut-off frequency, the pass band lies between the frequencies at which $Z_1 = 0$, and $Z_1 = -4Z_2$. All the frequencies above f_c lie in a stop or attenuation band. Thus, the network is called a low-pass filter. We also have from Eq. 17.7 that

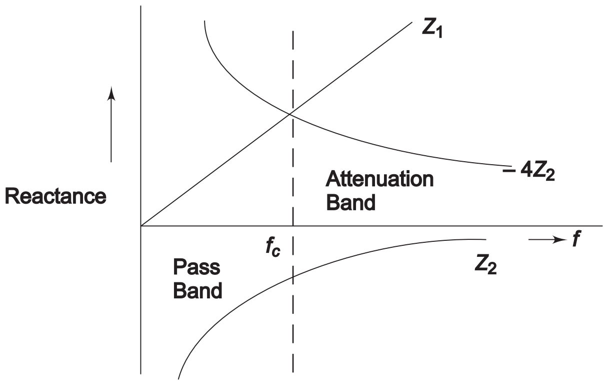


Fig. 17.9

$$\sin h\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{\frac{-\omega^2 LC}{4}} = \frac{J\omega\sqrt{LC}}{2}$$

From Eq. 17.22 $$\sqrt{LC} = \frac{1}{f_c \pi}$$

$\therefore$ $$\sinh\frac{\gamma}{2} = \frac{j2\pi f}{2\pi f_c} = j\frac{f}{f_c}$$

We also know that in the pass band

$$-1 < \frac{Z_1}{4Z_2} < 0$$

$$-1 < \frac{-\omega^2 LC}{4} < 0$$

$$-1 < -\left(\frac{f}{f_c}\right)^2 < 0$$

or $$\frac{f}{f_c} < 1$$

and $$\beta = 2\sin^{-1}\left(\frac{f}{f_c}\right);\ \alpha = 0$$

In the attenuation band,

$$\frac{Z_1}{4Z_2} < -1, \text{ i.e. } \frac{f}{f_c} < 1$$

$$\alpha = 2\cosh^{-1}\left[\frac{Z_1}{4Z_2}\right] = 2\cos h^{-1}\left(\frac{f}{f_c}\right);\ \beta = \pi$$

The plots of α and β for pass and stop bands are shown in Fig. 17.10.

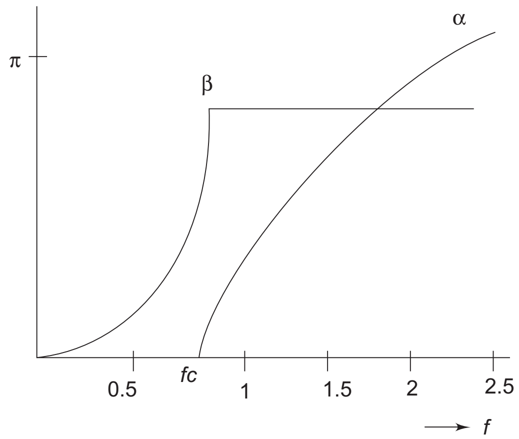


Fig. 17.10

Thus, from Fig. 17.10, $\alpha = 0$, $\beta = 2\sinh^{-1}\left(\frac{f}{f_c}\right)$ for $f < f_c$

$$\alpha = 2\cos h^{-1}\left(\frac{f}{f_c}\right);\ \beta = \pi \text{ for } f > f_c$$

The characteristic impedance can be calculated as shown below.

$$Z_{0T} = \sqrt{Z_1 Z_2\left(1 + \frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)}$$

$$Z_{0T} = k\sqrt{1-\left(\frac{f}{f_c}\right)^2} \tag{17.23}$$

From Eq. 17.23, Z_{0T} is real when $f < f_c$, i.e. in the pass band at $f = f_c$, $Z_{0T} = 0$; and for $f > f_c$, Z_{0T} is imaginary in the attenuation band, rising to infinite reactance at infinite frequency. The variation of Z_{0T} with frequency is shown in Fig. 17.11.

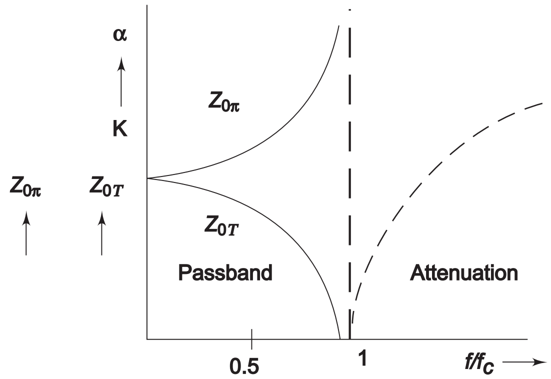


Fig. 17.11

Similarly, the characteristic impedance of a π-network is given by

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} = \frac{k}{\sqrt{1-\left(\frac{f}{f_c}\right)^2}} \tag{17.24}$$

The variation of $Z_{0\pi}$ with frequency is shown in Fig. 17.11. For $f < f_c$, $Z_{0\pi}$ is real; at $f = f_c$, $Z_{0\pi}$ is infinite, and for $f > f_c$, $Z_{0\pi}$ is imaginary. A low pass filter can be designed from the specifications of cut-off frequency and load resistance.

At cut-off frequency, $Z_1 = -4Z_2$

$$j\omega_c L = \frac{-4}{j\omega_c C}$$

$$\pi^2 f_c^2 LC = 1$$

Also we know that $k = \sqrt{L/C}$ is called the design impedance or the load resistance

$$\therefore \qquad k^2 = \frac{L}{C}$$

$$\pi^2 f_c^2 k^2 C^2 = 1$$

$C = \dfrac{1}{\pi f_c k}$ gives the value of the *shunt capacitance*.

and $L = k^2 C = \dfrac{k}{\pi f_c}$ gives the value of the series inductance.

Example 17.1 Design a low pass filter (both π and T-sections) having a cut-off frequency of 2 KHz to operate with a terminated load resistance of 500 Ω.

Solution It is given that $k = \sqrt{\dfrac{L}{C}} = 500\ \Omega$, and $f_c = 2000$ Hz

We know that $$L = \frac{k}{\pi f_c} = \frac{500}{3.14 \times 2000} = 79.6 \text{ mH}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{3.14 \times 2000 \times 500} = 0.318\ \mu\text{F}$$

The T and π-sections of this filter are shown in Fig. 17.12(a) and (b) respectively.

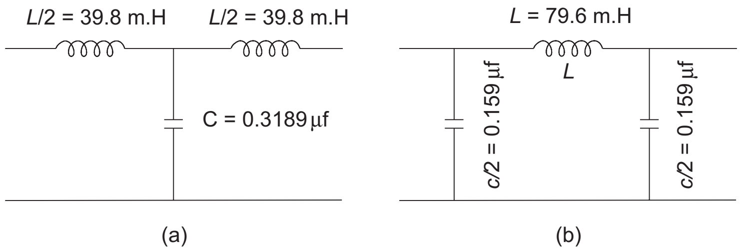


Fig. 17.12

17.7 CONSTANT *K*-HIGH PASS FILTER

Constant K-high pass filter can be obtained by changing the positions of series and shunt arms of the networks shown in Fig. 17.8. The prototype high pass filters are shown in Fig. 17.13, where $Z_1 = -j/\omega_C$ and $Z_2 = j\omega L$.

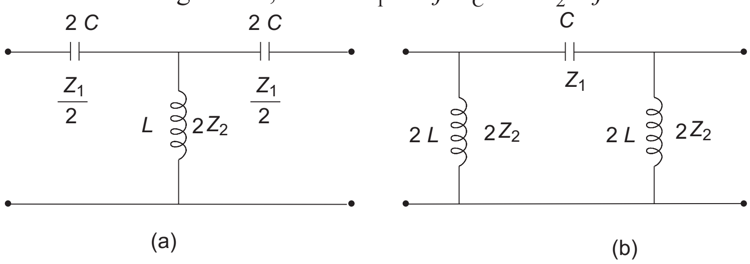


Fig. 17.13

Again, it can be observed that the product of Z_1 and Z_2 is independent of frequency, and the filter design obtained will be of the constant k type. Thus, Z_1Z_2 are given by

$$Z_1Z_2 = \frac{-J}{\omega C}\, j\omega L = \frac{L}{C} = k^2$$

$$k = \sqrt{\frac{L}{C}}$$

The cut-off frequencies are given by $Z_1 = 0$ and $Z_1 = -4Z_2$.

$$Z_1 = 0 \text{ indicates } \frac{-j}{\omega C} = 0, \text{ or } \omega \to \infty$$

From $\quad Z_1 = -4Z_2$

$$\frac{-j}{\omega C} = -4j\omega L$$

$$\omega^2 LC = \frac{1}{4}$$

or $\quad f_c = \dfrac{1}{4\pi\sqrt{LC}} \quad (17.25)$

The reactances of Z_1 and Z_2 are sketched as functions of frequency as shown in Fig. 17.14.

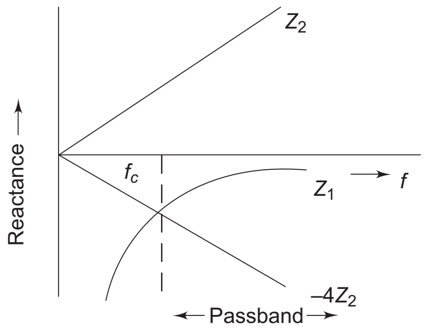


Fig. 17.14

As seen from Fig. 17.14, the filter transmits all frequencies between $f = f_c$ and $f = \infty$. The point f_c from the graph is a point at which $Z_1 = -4Z_2$.

From Eq. 17.7,

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{\frac{-1}{4\omega^2 LC}}$$

From Eq. 17.25, $\quad f_c = \dfrac{1}{4\pi\sqrt{LC}}$

$$\therefore \quad \sqrt{LC} = \frac{1}{4\pi f_c}$$

$$\therefore \quad \sinh\frac{\gamma}{2} = \sqrt{\frac{-(4\pi)^2 (f_c)^2}{4\omega^2}} = j\frac{f_c}{f}$$

In the pass band, $-1 < \dfrac{Z_1}{4Z_2} < 0$, $\alpha = 0$ or the region in which $\dfrac{f_c}{f} < 1$ is a pass band $\beta = 2\sin^{-1}\left(\dfrac{f_c}{f}\right)$

In the attenuation band $\dfrac{Z_1}{4Z_2} < -1$, i.e. $\dfrac{f_c}{f} > 1$

$$\alpha = 2\cosh^{-1}\left[\frac{Z_1}{4Z_2}\right] = 2\cos^{-1}\left(\frac{f_c}{f}\right);\ \beta = -\pi$$

The plots of α and β for pass and stop bands of a high pass filter network are shown in Fig. 17.15.

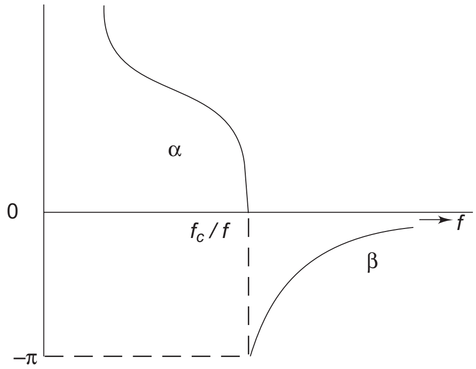


Fig. 17.15

A high pass filter may be designed similar to the low pass filter by choosing a resistive load r equal to the constant k, such that $R = k = \sqrt{L/C}$

$$f_c = \frac{1}{4\pi\sqrt{L/C}}$$

$$f_c = \frac{k}{4\pi L} = \frac{1}{4\pi Ck}$$

Since $$\sqrt{C} = \frac{L}{k},$$

$$L = \frac{k}{4\pi f_c} \text{ and } C = \frac{1}{4\pi f_c k}$$

The characteristic impedance can be calculated using the relation

$$Z_{0T} = \sqrt{Z_1\ Z_2\left(1 + \frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{L}{C}\left(1 - \frac{1}{4\omega^2 LC}\right)}$$

$$Z_{0T} = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Similarly, the characteristic impedance of a π-network is given by

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} = \frac{k^2}{Z_{0T}} = \frac{k}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad (17.26)$$

The plot of characteristic impedances with respect to frequency is shown in Fig. 17.16.

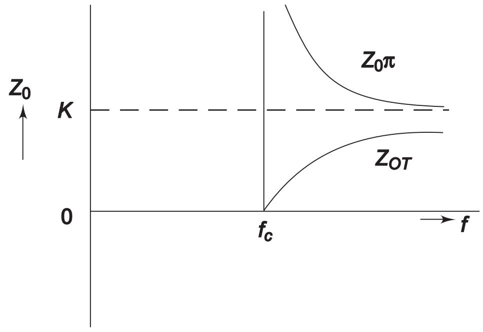


Fig. 17.16

Example 17.2 Design a high pass filter having a cut-off frequency of 1 KHz with a load resistance of 600 Ω.

Solution It is given that $R_L = K = 600\ \Omega$ and $f_c = 1000$ Hz

$$\therefore \quad L = \frac{K}{4\pi f_c} = \frac{600}{4 \times \pi \times 1000} = 47.74 \text{ mH}$$

$$C = \frac{1}{4\pi K f_c} = \frac{1}{4\pi \times 600 \times 1000} = 0.133\ \mu\text{F}$$

The T and π-sections of the filter are shown in Fig. 17.17.

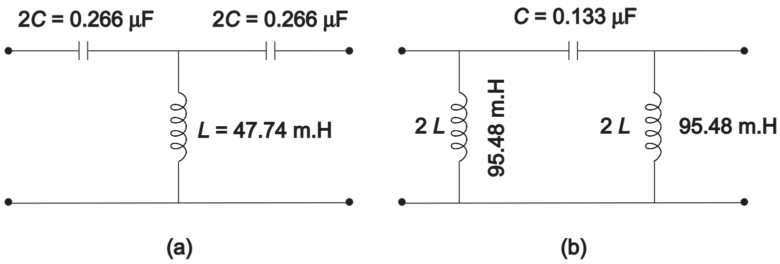


Fig. 17.17

17.8 *m*-DERIVED T-SECTION

It is clear from Figs 17.10 and 17.15 that the attenuation is not sharp in the stop band for k-type filters. The characteristic impedance, Z_0 is a function of frequency and varies widely in the transmission band. Attenuation can be increased in the stop band by using ladder section, i.e. by connecting two or more identical sections. In order to join the filter sections, it would be necessary that their characteristic impedances be equal to each other at all frequencies. If their characteristic impedances match at all frequencies, they would also have the same pass band. However, cascading is not a proper solution from a practical point of view. This is because practical elements have a certain resistance, which gives rise to attenuation in the pass band also. Therefore, any attempt to increase attenuation in stop band by cascading also results in an increase of 'α' in the pass band. If the constant k section is regarded as the prototype, it is possible to design a filter to have rapid attenuation in the stop band, and the same characteristic impedance as the prototype at all frequencies. Such a filter is called *m-derived filter*. Suppose a prototype T-network shown in Fig. 17.18(a) has the series arm modified as shown in Fig. 17.18(b), where m is a constant. Equating the characteristic impedance of the networks in Fig. 17.18, we have

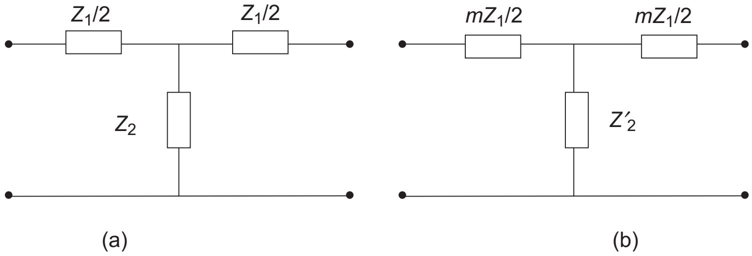


Fig. 17.18

where Z'_{0l} is the characteristic impedance of the modified (*m-derived*) T-network.

$$Z_{0T} = Z_{0T'}$$

$$\sqrt{\frac{Z_1^2}{4} + Z_1\, Z_2} = \sqrt{\frac{m^2 Z_1^2}{4} + mZ_1\, Z'_2}$$

$$\frac{Z_1^2}{4} + Z_1 Z_2 = \frac{m^2\, Z_1^2}{4} + mZ_1 Z'_2$$

$$mZ_1 Z'_2 = \frac{Z_1^2}{4}\,(1 - m^2) + Z_1 Z_2$$

$$Z'_2 = \frac{Z_1}{4m}\,(1 - m^2) + \frac{Z_2}{m} \tag{17.27}$$

It appears that the shunt arm Z'_2 consists of two impedances in series as shown in Fig. 17.19.

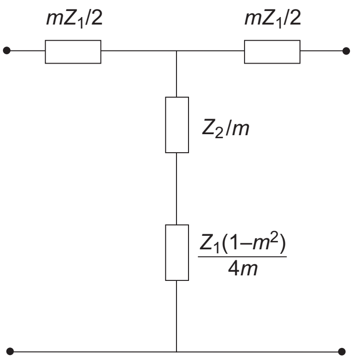


Fig. 17.19

From Eq. 17.27, $\dfrac{1-m^2}{4m}$ should be positive to realize the impedance Z_2' physically, i.e. $0 < m < 1$. Thus m-derived section can be obtained from the prototype by modifying its series and shunt arms. The same technique can be applied to π section network. Suppose a prototype π-network shown in Fig. 17.20(a) has the shunt arm modified as shown in Fig. 17.20(b).

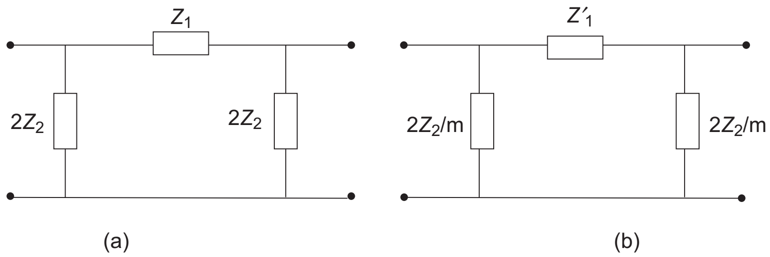


Fig. 17.20

The characteristic impedances of the prototype and its modified sections have to be equal for matching.

$$Z_{0\pi} = Z'_{0\pi}$$

where $Z'_{0\pi}$ is the characteristic impedance of the modified (m-derived) π-network.

$$\therefore \qquad \sqrt{\frac{Z_1 Z_2}{1+\dfrac{Z_1}{4Z_2}}} = \sqrt{\frac{Z_1' \dfrac{Z_2}{m}}{1+\dfrac{Z_1'}{4\times Z_2/m}}}$$

Squaring and cross multiplying the above equation results as under.

$$(4Z_1Z_2 + mZ_1'\,Z_1) = \frac{4Z_1'Z_2 + Z_1 Z_1'}{m}$$

$$Z_1'\left(\frac{Z_1}{m} + \frac{4Z_2}{m} - mZ_1\right) = 4Z_1Z_2$$

or
$$Z_1' = \frac{Z_1\,Z_2}{\dfrac{Z_1}{4m} + \dfrac{Z_2}{m} - \dfrac{mZ_1}{4}}$$

$$= \frac{Z_1\,Z_2}{\dfrac{Z_2}{m} + \dfrac{Z_1}{4m}\left(1-m^2\right)}$$

$$Z_1' = \frac{Z_1 Z_2 \times \dfrac{4m^2}{\left(1-m^2\right)}}{\dfrac{Z_2 4m^2}{m\left(1-m^2\right)} + Z_1 m} = \frac{mZ_1\dfrac{Z_2 4m}{\left(1-m^2\right)}}{mZ_1 + \dfrac{Z_2 4m}{\left(1-m^2\right)}} \qquad (17.28)$$

It appears that the series arm of the m-derived π section is a parallel combination of mZ_1 and $4mZ_2/1 - m^2$. The derived m section is shown in Fig. 17.21.

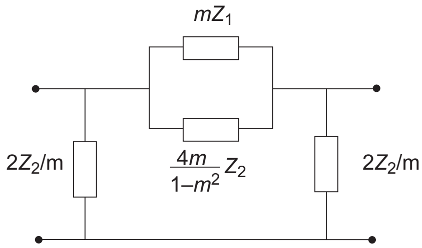


Fig. 17.21

(i) *m*-Derived Low Pass Filter

In Fig. 17.22, both m-derived low pass T and π filter sections are shown. For the T-section shown in Fig. 17.22(a), the shunt arm is to be chosen so that it is resonant at some frequency f_∞ above cut-off frequency f_c.

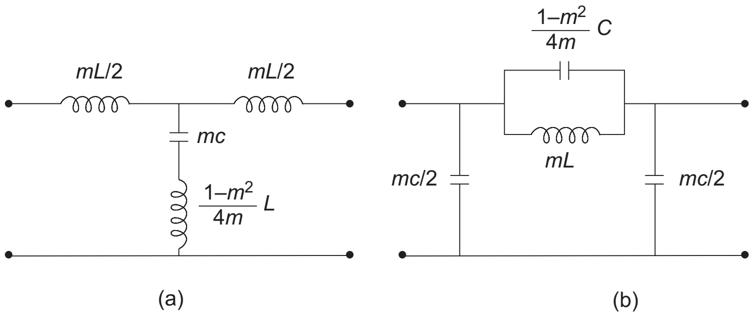


Fig. 17.22

If the shunt arm is series resonant, its impedance will be minimum or zero. Therefore, the output is zero and will correspond to infinite attenuation at this particular frequency. Thus, at f_∞

$$\frac{1}{m\omega_r C} = \frac{1-m^2}{4m} \omega_r L, \text{ where } \omega_r \text{ is the resonant frequency}$$

$$\omega_r^2 = \frac{4}{\left(1-m^2\right) LC}$$

$$f_r = \frac{1}{\pi\sqrt{LC\left(1-m^2\right)}} = f_\infty$$

Since the cut-off frequency for the low pass filter is $f_c = \dfrac{1}{\pi\sqrt{LC}}$

$$f_\infty = \frac{f_c}{\sqrt{1-m^2}} \tag{17.29}$$

or
$$m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2} \tag{17.30}$$

If a sharp cut-off is desired, f_∞ should be near to f_c. From Eq. 17.29, it is clear that for the smaller the value of m, f_∞ comes close to f_c. Equation 17.30 shows that if f_c and f_∞ are specified, the necessary value of m may then be calculated. Similarly, for m-derived π section, the inductance and capacitance in the series arm constitute a resonant circuit. Thus, at f_∞ a frequency corresponds to infinite attenuation, i.e. at f_∞

$$m\omega_r L = \frac{1}{\left(\frac{1-m^2}{4m}\right)\omega_r C}$$

$$\omega_r^2 = \frac{4}{LC\left(1-m^2\right)}$$

$$f_r = \frac{1}{\pi\sqrt{LC\left(1-m^2\right)}}$$

Since,
$$f_c = \frac{1}{\pi\sqrt{LC}}$$

$$f_r = \frac{f_c}{\sqrt{1-m^2}} = f_\infty \tag{17.31}$$

Thus for both m-derived low pass networks for a positive value of m $(0 < m < 1)$, $f_\infty > f_c$. Equations 17.30 or 17.31 can be used to choose the value of m, knowing f_c and f_r. After the value of m is evaluated, the elements of the T or π-networks can be found from Fig. 17.22. The variation of attenuation for a low pass m-derived section can be verified from $\alpha = 2\cosh^{-1}\sqrt{Z_1/4Z_2}$ for $f_c < f < f_\infty$. For $Z_1 = j\omega L$ and $Z_2 = -j/\omega C$ for the prototype.

$\therefore$
$$\alpha = 2\cosh^{-1}\frac{m\dfrac{f}{f_c}}{\sqrt{1-\left(\dfrac{f}{f_\infty}\right)^2}}$$

and
$$\beta = 2\sin^{-1}\sqrt{\left|\frac{Z_1}{4Z_1}\right|} = 2\sin^{-1}\frac{m\dfrac{f}{f_c}}{\sqrt{1-\left(\dfrac{f}{f_c}\right)^2(1-m)^2}}$$

Figure 17.23 shows the variation of α, β and Z_0 with respect to frequency for an m-derived low pass filter.

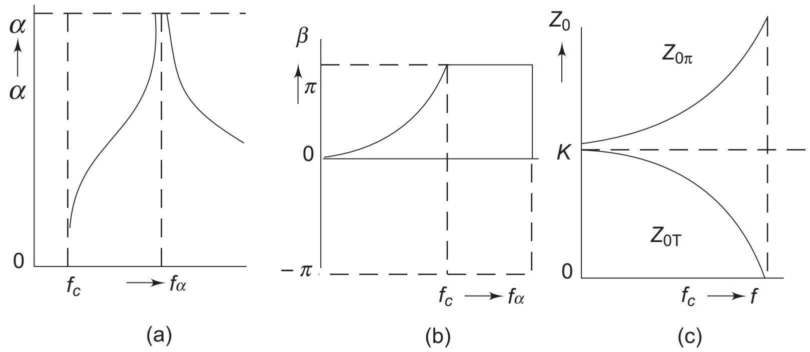


Fig. 17.23

Example 17.3 Design a m-derived low pass filter having cut-off frequency of 1 kHz, design impedance of 400 Ω, and the resonant frequency 1100 Hz.

Solution $k = 400\ \Omega$, $f_c = 1000$ Hz; $f_\infty = 1100$ Hz

From Eq. 17.30

$$m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1-\left(\frac{1000}{1100}\right)^2} = 0.416$$

Let us design the values of L and C for a low pass, K-type filter (prototype filter). Thus,

$$L = \frac{k}{\pi f_c} = \frac{400}{\pi \times 1000} = 127.32 \text{ mH}$$

$$C = \frac{1}{\pi k f_c} = \frac{1}{\pi \times 400 \times 1000} = 0.795\ \mu\text{F}$$

The elements of m-derived low pass sections can be obtained with reference to Fig. 17.22.

Thus the T-section elements are

$$\frac{mL}{2} = \frac{0.416 \times 127.32 \times 10^{-3}}{2} = 26.48 \text{ mH}$$

$$mC = 0.416 \times 0.795 \times 10^{-6} = 0.33\ \mu\text{F}$$

$$\frac{1-m^2}{4m}\ L = \frac{1-(0.416)^2}{4 \times 0.416} \times 127.32 \times 10^{-3} = 63.27 \text{ mH}$$

The π-section elements are

$$\frac{mC}{2} = \frac{0.416 \times 0.795 \times 10^{-6}}{2} = 0.165\ \mu F$$

$$\frac{1-m^2}{4m}\times C = \frac{1-(0.416)^2}{4\times 0.416}\times 0.795\times 10^{-6} = 0.395\ \mu\text{F}$$

$$mL = 0.416\times 127.32\times 10^{-3} = 52.965\ \text{mH}$$

The m-derived LP filter sections are shown in Fig. 17.24.

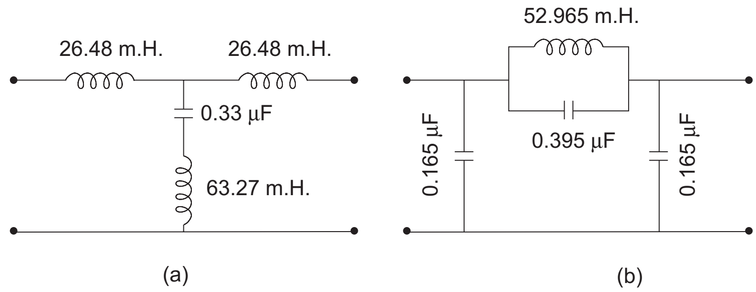


Fig. 17.24

(ii) *m*-derived high pass filter

In Fig. 17.25 both m-derived high pass T and π-sections are shown.

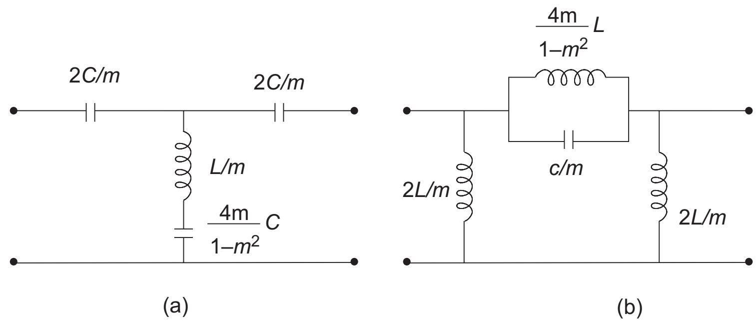


Fig. 17.25

If the shunt arm in T-section is series resonant, it offers minimum or zero impedance. Therefore, the output is zero and, thus, at resonance frequency, or the frequency corresponds to infinite attenuation.

$$\omega_r\frac{L}{m} = \frac{1}{\omega_r\dfrac{4m}{1-m^2}C}$$

$$\omega_r^2 = \omega_\infty^2 = \frac{1}{\dfrac{L}{m}\dfrac{4m}{1-m^2}C} = \frac{1-m^2}{4LC}$$

$$\omega_\infty = \frac{\sqrt{1-m^2}}{2\sqrt{LC}}\quad\text{or } f_\infty = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}}$$

From Eq. 17.25, the cut-off frequency f_c of a high pass prototype filter is given by

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

$$f_\infty = f_c\sqrt{1-m^2} \tag{17.32}$$

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2} \tag{17.33}$$

Similarly, for the m-drived π-section, the resonant circuit is constituted by the series arm inductance and capacitance. Thus, at f_∞

$$\frac{4m}{1-m^2}\ \omega_r L = \frac{1}{\dfrac{\omega_r}{m}C}$$

$$\omega_r^2 = \omega_\infty^2 = \frac{1-m^2}{4LC}$$

$$\omega_\infty = \frac{\sqrt{1-m^2}}{2\sqrt{LC}} \qquad \text{or} \qquad f_\infty = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}}$$

Thus, the frequency corresponding to infinite attenuation is the same for both sections.

Equation 17.33 may be used to determine m for a given f_∞ and f_c. The elements of the m-derived high pass T or π-sections can be found from Fig. 17.25. The variation of α, β and Z_0 with frequency is shown in Fig. 17.26.

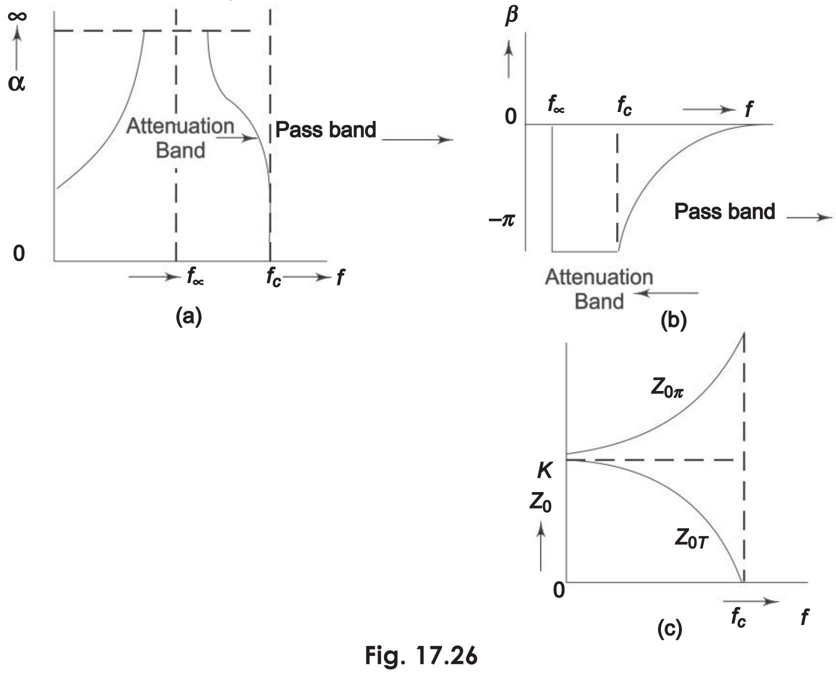


Fig. 17.26

Example 17.4 Design a m-derived highpass filter with a cut-off frequency of 10 kHz; design impedance of 5 Ω and $m = 0.4$.

Solution For the prototype high pass filter,

$$L = \frac{k}{4\pi f_c} = \frac{500}{4 \times \pi \times 10000} = 3.978 \text{ mH}$$

$$C = \frac{1}{4\pi k f_c} = \frac{1}{4\pi \times 500 \times 10000} = 0.0159\ \mu\text{F}$$

The elements of m-derived high pass sections can be obtained with reference to Fig. 17.25. Thus, the T-section elements are

$$\frac{2C}{m} = \frac{2 \times 0.0159 \times 10^{-6}}{0.4} = 0.0795\ \mu\text{F}$$

$$\frac{L}{m} = \frac{3.978 \times 10^{-3}}{0.4} = 9.945 \text{ mH}$$

$$\frac{4m}{1 - m^2}\ C = \frac{4 \times 0.4}{1 - (0.4)^2} \times 0.0159 \times 10^{-6} = 0.0302\ \mu\text{F}$$

The π-section elements are

$$\frac{2L}{m} = \frac{2 \times 0.0159 \times 10^{-3}}{0.4} = 19.89 \text{ mH}$$

$$\frac{4m}{1 - m^2} \times L = \frac{4 \times 0.4}{1 - (0.4)^2} \times 3.978 \times 10^{-3} = 7.577 \text{ mH}$$

$$\frac{C}{m} = \frac{0.0159}{0.4} \times 10^{-6} = 0.0397\ \mu\text{F}$$

T and π sections of the m-derived highpass filter are shown in Fig. 17.27.

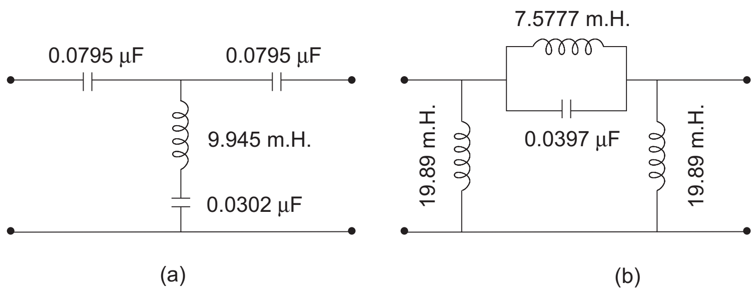


Fig. 17.27

17.9 BAND PASS FILTER

As already explained in Section 17.1, a band pass filter is one which attenuates all frequencies below a lower cut-off frequency f_1 and above an upper cut-off frequency f_2. Frequencies lying between f_1 and f_2 comprise the pass band, and are transmitted with zero attenuation. A band pass filter may be obtained by using a low pass filter followed by a high pass filter in which the cut-off frequency of the LP filter is above the cut-off frequency of the HP filter, the over lap thus allowing only a band of frequencies to pass. This is not economical in practice; it is more economical to combine the low and high pass functions into a single filter section.

Consider the circuit in Fig. 17.28, each arm has a resonant circuit with same resonant frequency, i.e. the resonant frequency of the series arm and the resonant frequency of the shunt arm are made equal to obtain the band pass characteristic.

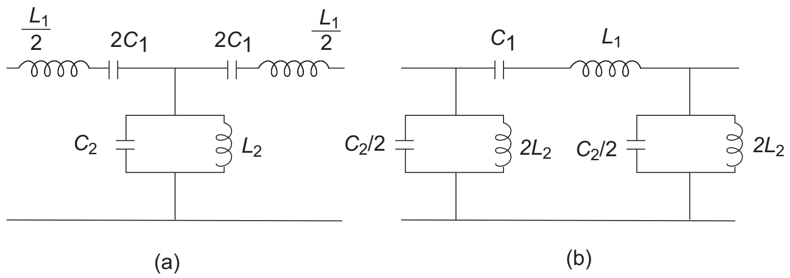


Fig. 17.28

For this condition of equal resonant frequencies.

$$\omega_0 \frac{L_1}{2} = \frac{1}{2\omega_0 C_1} \text{ for the series arm}$$

from which, $\omega_0^2 L_1 C_1 = 1$ (17.34)

And $\dfrac{1}{\omega_0 C_2} = \omega_0 L_2$ for the shunt arm

from which, $\omega_0^2 L_2 C_2 = 1$ (17.35)

$$\omega_0^2 L_1 C_1 = 1 = \omega_0^2 L_2 C_2$$

$$L_1 C_1 = L_2 C_2 \tag{17.36}$$

The impedance of the series arm, Z_1 is given by

$$Z_1 = \left(j\omega L_1 - \frac{j}{\omega C_1} \right) = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1} \right)$$

The impedance of the shunt arm, Z_2 is given by

$$Z_2 = \frac{j\omega L_2 \dfrac{1}{j\omega C_2}}{j\omega L_2 + \dfrac{1}{j\omega C_2}} = \frac{j\omega L_2}{1 - \omega^2 L_2 C_2}$$

$$Z_1Z_2 = j\left(\frac{\omega^2 L_1C_1 - 1}{\omega C_1}\right)\left(\frac{j\omega L_2}{1-\omega^2 L_2C_2}\right)$$

$$= \frac{-L_2}{C_1}\left(\frac{\omega^2 L_1C_1 - 1}{1-\omega^2 L_2C_2}\right)$$

From Eq. 17.36, $\quad L_1C_1 = L_2C_2$

$$Z_1Z_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = k^2$$

where k is constant. Thus, the filter is a constant k-type. Therefore, for a constant k-type in the pass band.

$$-1 < \frac{Z_1}{4Z_2} < 0, \text{ and at cut-off frequency}$$

$$Z_1 = -4Z_2$$

$$Z_1^2 = -4Z_1Z_2 = -4k^2$$

$\therefore \quad Z_1 = \pm j2k$

i.e. the value of Z_1 at lower cut-off frequency is equal to the negative of the value of Z_1 at the upper cut-off frequency.

$\therefore$
$$\left(\frac{1}{j\omega_1C_1} + j\omega_1L_1\right) = -\left(\frac{1}{j\omega_2C_1} + j\omega_2L_1\right)$$

or
$$\left(\omega_1L_1 - \frac{1}{\omega_1C_1}\right) = \left(\frac{1}{\omega_2C_1} - \omega_2L_1\right)$$

$$(1-\omega_1^2L_1C_1) = \frac{\omega_1}{\omega_2}\,(\omega_2^2L_1C_1 - 1) \tag{17.37}$$

From Eq. 17.34, $L_1C_1 = \dfrac{1}{\omega_0^2}$

Hence Eq. 17.37 may be written as

$$\left(1-\frac{\omega_1^2}{\omega_0^2}\right) = \frac{\omega_1}{\omega_2}\left(\frac{\omega_2^2}{\omega_0^2} - 1\right)$$

$$(\omega_0^2 - \omega_1^2)\omega_2 = \omega_1\left(\omega_2^2 - \omega_0^2\right)$$

$$\omega_0^2\,\omega_2 - \omega_1^2\,\omega_2 = \omega_1\omega_2^2 - \omega_1\omega_0^2$$

$$\omega_0^2(\omega_1 + \omega_2) = \omega_1\omega_2(\omega_2 + \omega_1)$$

$$\omega_0^2 = \omega_1\omega_2$$

$$f_0 = \sqrt{f_1\,f_2} \tag{17.38}$$

Thus, the resonant frequency is the geometric mean of the cut-off frequencies. The variation of the reactances with respect to frequency is shown in Fig. 17.29.

Design If the filter is terminated in a load resistance $R = K$, then at the lower cut-off frequency

$$Z_1 = -\,2jk$$

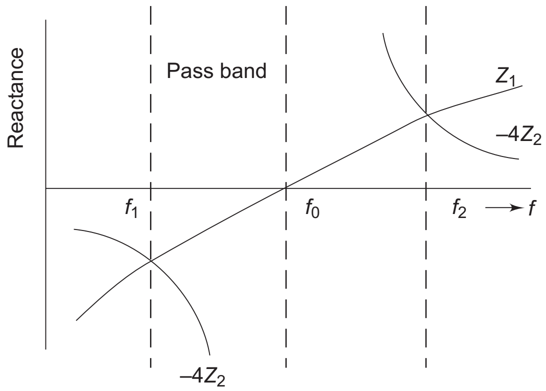


Fig. 17.29

$$\left(\frac{1}{j\omega_1 C_1} + j\omega_1 L_1\right) = -\,2jk$$

$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = 2k$$

$$1 - \omega_1^2 C_1 L_1 = 2k\omega_1 C_1$$

Since $$L_1 C_1 = \frac{1}{\omega_0^2}$$

$$1 - \frac{\omega_1^2}{\omega_0^2} = 2\,k\omega_1 C_1$$

or $$1 - \left(\frac{f_1}{f_0}\right)^2 = 4\pi k f_1 C_1$$

$$1 - \frac{f_1^2}{f_1\,f_2} = 4\pi k f_1 C_1 \qquad (\because f_0 = \sqrt{f_1\,f_2}\,)$$

$$f_2 - f_1 = 4\pi k f_1 f_2 C_1$$

$$C_1 = \frac{f_2 - f_1}{4\pi k f_1\ f_2} \tag{17.39}$$

Since $$L_1 C_1 = \frac{1}{\omega_0^2}$$

$$L_1 = \frac{1}{\omega_0^2\,C_1} = \frac{4\pi k f_1 f_2}{\omega_0^2\,(f_2 - f_1)}$$

$$L_1 = \frac{k}{\pi\,(f_2 - f_1)} \tag{17.40}$$

To evaluate the values for the shunt arm, consider the equation

$$Z_1 Z_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = k^2$$

$\therefore$
$$L_2 = C_1 k^2 = \frac{(f_2 - f_1)\,k}{4\pi f_1 f_2} \tag{17.41}$$

and
$$C_2 = \frac{L_1}{k^2} = \frac{1}{\pi\,(f_2 - f_1)\,k} \tag{17.42}$$

Equations 17.39 through 17.42 are the design equations of a prototype band pass filter. The variation of α, β with respect to frequency is shown in Fig. 17.30.

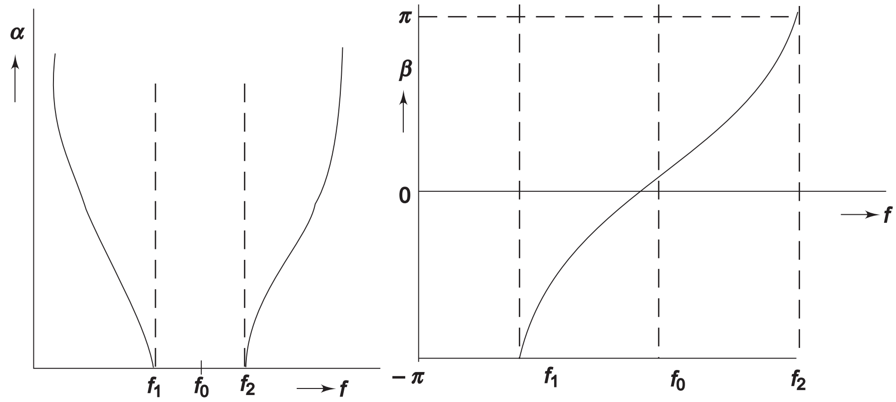


Fig. 17.30

Example 17.5 Design k-type band pass filter having a design impedance of 500 Ω and cut-off frequencies 1 kHz and 10 kHz.

Solution $k = 500\ \Omega$; $f_1 = 1000$ Hz; $f_2 = 10000$ Hz

From Eq. 17.40,

$$L_1 = \frac{k}{\pi(f_2 - f_1)} = \frac{500}{\pi\, 9000} = \frac{55.55}{\pi}\ \text{mH} = 17.68\ \text{mH}$$

From Eq. 17.39,

$$C_1 = \frac{f_2 - f_1}{4\pi k f_1 f_2} = \frac{9000}{4 \times \pi \times 500 \times 1000 \times 10000} = 0.143\ \mu\text{F}$$

From Eq. 17.41,

$$L_2 = C_1 k^2 = 3.57\ \text{mH}$$

From Eq. 17.42,

$$C_2 = \frac{L_1}{k^2} = 0.0707\ \mu\text{F}$$

Each of the two series arms of the constant k, T-section filter is given by

$$\frac{L_1}{2} = \frac{17.68}{2} = 8.84 \text{ mH}$$

$$2C_1 = 2 \times 0.143 = 0.286\ \mu\text{F}$$

And the shunt arm elements of the network are given by

$$C_2 = 0.0707\ \mu\text{F and } L_2 = 3.57 \text{ mH}$$

For the constant-k, π section filter the elements of the series arm are

$$C_1 = 0.143\ \mu\text{F and } L_1 = 17.68 \text{ mH}$$

The elements of the shunt arms are

$$\frac{C_2}{2} = \frac{0.0707}{2} = 0.035 \text{ mF}$$

$$2L_2 = 2 \times 0.0358 = 0.0716 \text{ H}$$

17.10 BAND ELIMINATION FILTER

A band elimination filter is one which passes without attenuation all frequencies less than the lower cut-off frequency f_1, and greater than the upper cut-off frequency f_2. Frequencies lying between f_1 and f_2 are attenuated. It is also known as band stop filter. Therefore, a band stop filter can be realized by connecting a low pass filter in parallel with a highpass section, in which the cut-off frequency of low pass filter is below that of a high pass filter. The configurations of T and π constant k band stop sections are shown in Fig. 17.31. The band elimination filter is designed in the same manner as is the band pass filter.

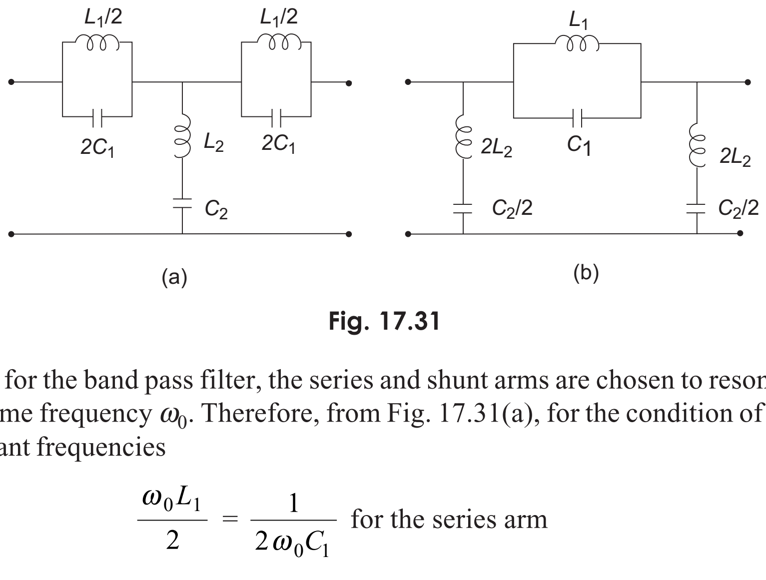


Fig. 17.31

As for the band pass filter, the series and shunt arms are chosen to resonate at the same frequency ω_0. Therefore, from Fig. 17.31(a), for the condition of equal resonant frequencies

$$\frac{\omega_0 L_1}{2} = \frac{1}{2\omega_0 C_1} \quad \text{for the series arm}$$

or $$\omega_0^2 = \frac{1}{L_1 C_1} \tag{17.43}$$

$$\omega_0 L_2 = \frac{1}{\omega_0 C_2} \text{ for the shunt arm}$$

$$\omega_0^2 = \frac{1}{L_2 C_2} \tag{17.44}$$

$$\frac{1}{L_1 C_1} = \frac{1}{L_2 C_2} = k$$

Thus $$L_1 C_1 = L_2 C_2 \tag{17.45}$$

It can be also verified that

$$Z_1 Z_2 = \frac{L_1}{C_2} = \frac{L_2}{C_1} = k^2 \tag{17.46}$$

and $$f_0 = \sqrt{f_1 f_2} \tag{17.47}$$

At cut-off frequencies, $Z_1 = -4Z_2$

Multiplying both sides with Z_2, we get

$$Z_1 Z_2 = -4Z_2^2 = k^2$$

$$Z_2 = \pm j \frac{k}{2} \tag{17.48}$$

If the load is terminated in a load resistance, $R = k$, then at lower cut-off frequency

$$Z_2 = j\left(\frac{1}{\omega_1 C_2} - \omega_1 L_2\right) = j\,\frac{k}{2}$$

$$\frac{1}{\omega_1 C_2} - \omega_1 L_2 = \frac{k}{2}$$

$$1 - \omega_1^2 C_2 L_2 = \omega_1 C_2\,\frac{k}{2}$$

From Eq. 17.44, $$L_2 C_2 = \frac{1}{\omega_0^2}$$

$$1 - \frac{\omega_1^2}{\omega_0^2} = \frac{k}{2}\,\omega_1 C_2$$

$$1 - \left(\frac{f_1}{f_0}\right)^2 = k\pi f_1 C_2$$

$$C_2 = \frac{1}{k\pi f_1}\left[1 - \left(\frac{f_1}{f_0}\right)^2\right]$$

Since $$f_0 = \sqrt{f_1 f_2}$$

$$C_2 = \frac{1}{k\pi}\left[\frac{1}{f_1} - \frac{1}{f_2}\right]$$

$$C_2 = \frac{1}{k\pi}\left[\frac{f_2 - f_1}{f_1 f_2}\right] \tag{17.49}$$

From Eq. 17.44, $$\omega_0^2 = \frac{1}{L_2 C_2}$$

$$L_2 = \frac{1}{\omega_0^2\, C_2} = \frac{\pi k f_1 f_2}{\omega_0^2\,(f_2 - f_1)}$$

Since $$f_0 = \sqrt{f_1 f_2}$$

$$L_2 = \frac{k}{4\pi\,(f_2 - f_1)} \tag{17.50}$$

Also from Eq. 17.46,

$$k^2 = \frac{L_1}{C_2} = \frac{L_2}{C_1}$$

$\therefore$ $$L_1 = k^2 C_2 = \frac{k}{\pi}\left(\frac{f_2 - f_1}{f_1\, f_2}\right) \tag{17.51}$$

and $$C_1 = \frac{L_2}{k^2} = \frac{1}{4\pi k\,(f_2 - f_1)} \tag{17.52}$$

The variation of the reactances with respect to frequency is shown in Fig. 17.32.

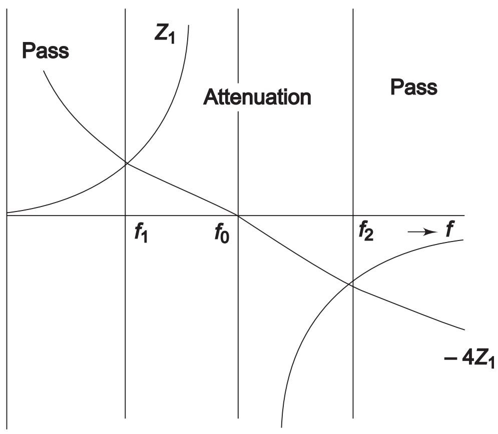


Fig. 17.32

Equation 17.49 through Eq. 17.52 are the design equations of a prototype band elimination filter. The variation of α, β with respect to frequency is shown in Fig. 17.33.

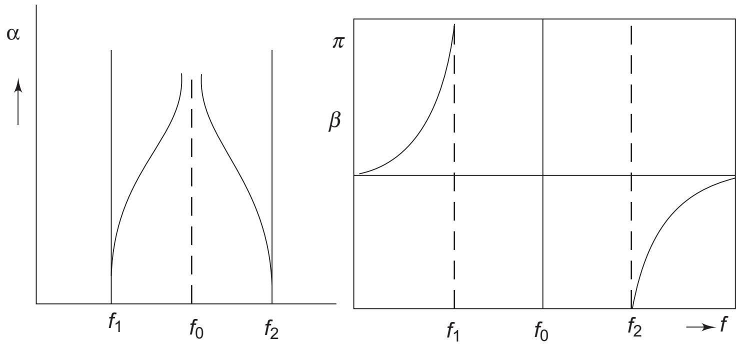


Fig. 17.33

Example 17.6 Design a band elimination filter having a design impedance of 600 Ω and cut-off frequencies f_1 = 2 kHz and f_2 = 6 kHz.

Solution $(f_2 - f_1) = 4$ kHz

Making use of the Eqs 17.49 through 17.52 in Section 17.10, we have

$$L_1 = \frac{k}{\pi}\left(\frac{f_2 - f_1}{f_2 f_1}\right) = \frac{600 \times 4000}{\pi \times 2000 \times 6000} = 63 \text{ mH}$$

$$C_1 = \frac{1}{4\pi k\,(f_2 - f_1)} = \frac{1}{4 \times \pi \times 600(4000)} = 0.033\ \mu\text{F}$$

$$L_2 = \frac{1}{4\pi k\,(f_2 - f_1)} = \frac{600}{4\pi(4000)} = 12 \text{ mH}$$

$$C_2 = \frac{1}{k\pi}\left[\frac{f_2 - f_1}{f_1 f_2}\right] = \frac{1}{600 \times \pi}\left[\frac{4000}{2000 \times 6000}\right] = 0.176\ \mu\text{F}$$

Each of the two series arms of the constant k, T-section filter is given by

$$\frac{L_1}{2} = 31.5 \text{ mH}$$

$$2C_1 = 0.066\ \mu\text{F}$$

And the shunt arm elements of the network are

$$L_2 = 12 \text{ mH and } C_2 = 0.176\ \mu\text{F}$$

For the constant k, π-section filter the elements of the series arm are

$$L_1 = 63 \text{ mH}, \ C_1 = 0.033\ \mu\text{F}$$

and the elements of the shunt arms are

$$2L_2 = 24 \text{ mH and } \frac{C_2}{2} = 0.088\ \mu\text{F}$$

17.11 ATTENUATORS

An attenuator is a two-port resistive network and is used to reduce the signal level by a given amount. In a number of applications, it is necessary to introduce a specified loss between source and a matched load without altering the impedance relationship. Attenuators may be used for this purpose. Attenuators may be symmetrical or asymmetrical, and can be either fixed or variable. A fixed attenuator with constant attenuation is called a *pad*. Variable attenuators are used as volume controls in radio broadcasting sections. Attenuators are also used in laboratory to obtain small value of voltage or current for testing circuits.

The increase or decrease in power due to insertion or substitution of a new element in a network can be conveniently expressed in decibels (dB), or in nepers. In other words, attenuation is expressed either in decibels (dB) or in nepers. Accordingly, the attenuation offered by a network in decibels is

$$\text{Attenuation in dB} = 10 \log_{10}\left(\frac{P_1}{P_2}\right) \tag{17.53}$$

where P_1 is the input power and P_2 is the output power

For a properly matched network, both terminal pairs are matched to the characteristic resistance, R_0 of the attenuator.

Hence,
$$\frac{P_1}{P_2} = \frac{I_1^2 R_0}{I_2^2 R_0} = \frac{I_1^2}{I_2^2} \tag{17.54}$$

where I_1 is the input current and I_2 is the output current leaving the port.

or
$$\frac{P_1}{P_2} = \frac{V_1^2}{V_2^2} \tag{17.55}$$

where V_1 is the voltage at port 1 and V_2 is the voltage at port 2

$$\text{Hence, attenuation in dB} = 20 \log_{10}\left(\frac{V_1}{V_2}\right) \tag{17.56}$$

$$= 20 \log_{10}\left(\frac{I_1}{I_2}\right) \tag{17.57}$$

If
$$\frac{V_1}{V_2} = \frac{I_1}{I_2} = N \tag{17.58}$$

then
$$\frac{P_1}{P_2} = N^2$$

and
$$\text{dB} = 20 \log_{10} N \tag{17.59}$$

or
$$N = \text{antilog}\left(\frac{\text{dB}}{20}\right) \tag{17.60}$$

17.12 *T*-TYPE ATTENUATOR

Basically, there are four types of attenuators, T, π, lattice and bridged T-type. The basic design principles are discussed in the following Sections. Figure 17.34 shows the symmetrical T-attenuator. An attenuator is to be designed for desired values of characteristic resistance, R_0 and attenuation.

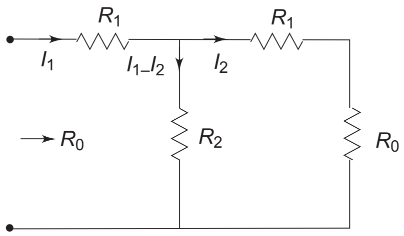


Fig. 17.34

The values of the arms of the network can be specified in terms of characteristic impedance, Z_0, and propagation constant, γ, of the network. The network in the figure is a symmetrical resistive circuit; hence $Z_0 = R_0$ and $\gamma = \alpha$. The design equations can be obtained by applying Kirchhoff's law to the network in Fig. 17.34.

$$R_2 (I_1 - I_2) = I_2 (R_1 + R_0)$$

$$I_2 (R_2 + R_1 + R_0) = I_1 R_2$$

$$\frac{I_1}{I_2} = \frac{R_1 + R_0 + R_2}{R_2} = N \tag{17.61}$$

The characteristic impedance of the attenuator is R_0 when it is terminated in a load of R_0

Hence, $$R_0 = R_1 + \frac{R_2 (R_1 + R_0)}{R_1 + R_0 + R_2}$$

Substituting Eq. 17.61, we have

$$R_0 = R_1 + \frac{(R_1 + R_0)}{N}$$

$$NR_0 = NR_1 + R_1 + R_0$$

$$R_0(N - 1) = R_1 (N + 1)$$

$$R_1 = \frac{R_0 (N - 1)}{N + 1} \tag{17.62}$$

From Eq. 17.61, we have

$$NR_2 = R_1 + R_0 + R_2$$

$$(N - 1)R_2 = (R_1 + R_0)$$

Substituting the value of R_1 from Eq. 17.62, we have

$$(N-1)\,R_2 = R_0\,\frac{(N-1)}{(N+1)} + R_0$$

$$(N-1)\,R_2 = \frac{2\,NR_0}{(N+1)}$$

$$R_2 = \frac{2\,NR_0}{N^2-1} \tag{17.63}$$

Equations 17.62 and 17.63 are the design equations for the symmetrical T-attenuator.

Example 17.7 Design a T-pad attenuator to give an attenuation of 60 dB and to work in a line of 500 Ω impedance.

Solution

$$N = \frac{I_1}{I_2} = \text{antilog}\ \frac{D}{20}$$

$$= \text{antilog}\ \frac{60}{20} = 1000$$

Each of the series arm is given by

$$R_1 = \frac{R_0\,(N-1)}{N+1} = 500\,\frac{(1000-1)}{(1000+1)} = 499\ \Omega$$

The shunt arm resistor R_2 is given by

$$R_2 = \frac{2N}{N^2-1}\,R_0 = \frac{2\times 1000}{(1000)^2-1}\times 500 = 1\ \Omega$$

17.13 π-TYPE ATTENUATOR

Figure 17.35 shows symmetrical attenuator. The series and shunt arm of the attenuator can be specified in terms of Z_0 and propagation constant γ. In this case also, the network is formed by resistors and is symmetrical, hence $Z_0 = R_0$ and $\gamma = \alpha$. From the fundamental equations, we have

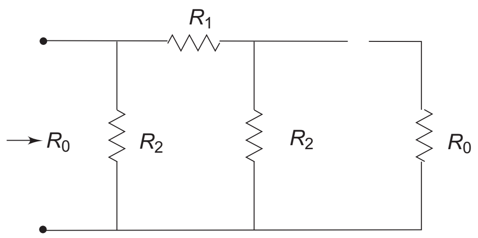


Fig. 17.35

$$R_1 = R_0 \sinh\alpha \tag{17.64}$$

$$R_2 = R_0 \coth \alpha/2 \tag{17.65}$$

$$\therefore \qquad R_1 = R_0 \frac{e^{\alpha} - e^{-\alpha}}{2} \tag{17.66}$$

By definition of propagation constant

$$e^{\gamma} = \frac{I_1}{I_2} = N$$

Here $\gamma = \alpha$ and $e^{\alpha} = N$

Therefore, Eq. 17.66 can be written as

$$R_1 = R_0 \frac{N - \dfrac{1}{N}}{2} = R_0 \frac{N^2 - 1}{2N} \tag{17.67}$$

Similarly, from Eq. 17.65,

$$R_2 = R_0 \frac{\cosh \alpha/2}{\sinh \alpha/2} = R_0 \frac{e^{\alpha/2} + e^{-\alpha/2}}{e^{\alpha/2} - e^{-\alpha/2}}$$

$$R_2 = R_0 \frac{e^{\alpha} + 1}{e^{\alpha} - 1} = R_0 \frac{(N+1)}{(N-1)} \tag{17.68}$$

Equations 17.67 and 17.68 are the design equations for the symmetrical π-attenuator.

Example 17.8 Design a π-type attenuator to give 20 dB attenuation and to have a characteristic impedance of 100 Ω.

Solution Given R_0 = 100 Ω, D = 20 dB.

$$N = \text{Antilog}\ \frac{D}{20} = 10$$

$$R_1 = R_0 \frac{\left(N^2 - 1\right)}{2N} = 100 \frac{\left(10^2 - 1\right)}{2 \times 10} = 495\ \Omega$$

$$R_2 = R_0 \frac{(N+1)}{(N-1)} = 100 \left(\frac{10+1}{10-1}\right) = 122.22\ \Omega$$

17.14 LATTICE ATTENUATOR

A symmetrical resistance lattice is shown in Fig. 17.36. The series and the diagonal arm of the network can be specified in terms of the characteristic impedance Z_0 and propagation constant γ.

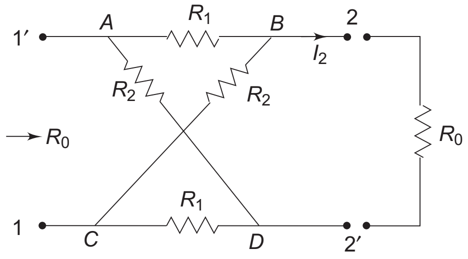


Fig. 17.36

It has already been stated and proved that characteristic impedance of symmetrical network is the geometric mean of the open and short circuit impedance. The circuit in Fig. 17.36 is redrawn as in Fig. 17.37 to calculate the open and short circuit impedances.

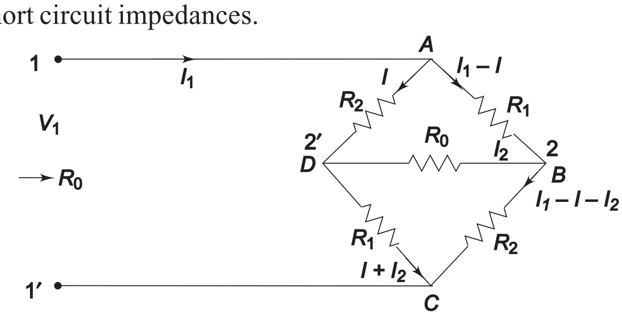


Fig. 17.37

Thus,
$$Z_{sc} = \frac{2\,R_1\,R_2}{R_1 + R_2}$$

$$Z_{0c} = \frac{R_1 + R_2}{2}$$

Hence,
$$Z_0 = R_0 = \sqrt{Z_{0c}\,Z_{sc}}$$

$$R_0 = \sqrt{R_1\,R_2}$$

In Fig. 17.37 the input impedance at 1-1′ is R_0 when the network is terminated in R_0 at 2-2′. By applying Kirchhoff's voltage law, we get

$$V_1 = I_1R_0 = (I_1 - I)R_1 + I_2R_0 + (1 + I_2)R_1$$

$$I_1R_0 = R_1(I_1 + I_2) + I_2R_0$$

$$I_1(R_0 - R_1) = I_2(R_1 + R_0)$$

$$\frac{I_1}{I_2} = \frac{R_1 + R_0}{R_0 - R_1} = \frac{1 + \dfrac{R_1}{R_0}}{1 - \dfrac{R_1}{R_0}} \tag{17.69}$$

$$N = e^{\alpha} = \frac{I_1}{I_2} = \frac{1 + \dfrac{R_1}{R_0}}{1 - \dfrac{R_1}{R_0}} \tag{17.70}$$

$$e^{\alpha} = \frac{1 + \sqrt{R_1/R_2}}{1 - \sqrt{R_1/R_2}}$$

The propagation constant $\alpha = \log \left[\dfrac{1 + \sqrt{\dfrac{R_1}{R_2}}}{1 - \sqrt{\dfrac{R_1}{R_2}}} \right]$ (17.71)

From Eq. 17.70

$$N\left(1 - \frac{R_1}{R_0}\right) = \left(1 + \frac{R_1}{R_0}\right)$$

$$R_1 = R_0 \frac{(N-1)}{(N+1)} \tag{17.72}$$

Similarly, we can express $R_2 = R_0 \dfrac{(N+1)}{(N-1)}$ (17.73)

Equations 17.72 and 17.73 are the design equations for lattice attenuator.

Example 17.9 Design a symmetrical lattice attenuator to have characteristic impedance of 800 Ω and attenuation of 20 dB.

Solution Given $R_0 = 800\ \Omega$ and $D = 20$ dB

$$N = \text{Antilog } \frac{D}{20} = \text{Antilog } \frac{20}{20} = 10$$

From the design equations of lattice attenuator

Series arm resistance $R_1 = R_0 \dfrac{(N-1)}{(N+1)}$

$$= 800 \frac{(10-1)}{(10+1)} = 654.545\ \Omega$$

Diagonal arm resistance $R_2 = R_0 \dfrac{(N+1)}{(N-1)}$

$$= 800 \frac{(10+1)}{(10-1)} = 977.777\ \Omega$$

The resulting lattice attenuator is shown in Fig. 17.38.

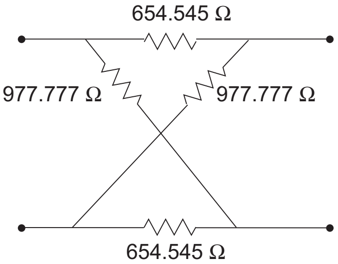


Fig. 17.38

17.15 BRIDGED-*T* ATTENUATOR

A bridged-T attenuator is shown in Fig. 17.39. In this case also since the attenuator is formed by resistors only, $Z_0 = R_0$ and $\gamma = \alpha$. The bridged-T network may be designed to have any characteristic resistance R_0 and desired attenuation by making $R_A\ R_B = R_0^2$. Here R_A and R_B are variable resistances and all other resistances are equal to the characteristic resistance R_0 of the network.

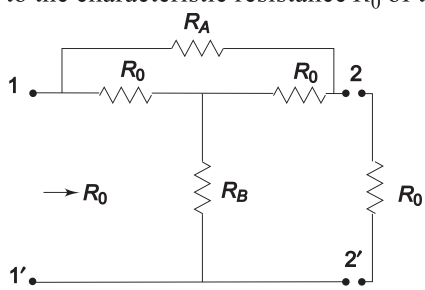


Fig. 17.39

A symmetrical resistance lattice network can be converted into an equivalent T, π or bridged-T resistance network using the bisection theorem. We can obtain the design equations of the bridged-T attenuator by bisection theorem. A bisected half sections is shown in Fig. 17.40. According to the bisection theorem, a network having mirror image symmetry can be reduced to an equivalent lattice structure. The series arm of the equivalent lattice is found by bisecting the given network into two parts, short circuiting all the cut wires and equating the series impedance of the lattice to the input impedance of the bisected network; the diagonal arm is equal to the input impedance of the bisected network when cut wires are open circuited.

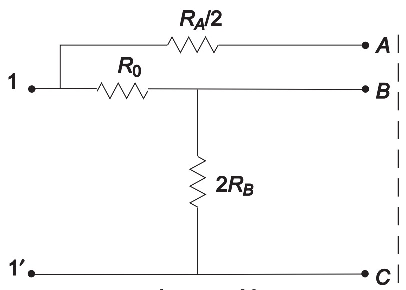


Fig. 17.40

From Fig. 17.40, when the cut wires A, B, C are shorted, the input resistance of the network is given by

$$R_{sc} = \frac{R_0 \times R_{A/2}}{R_0 + R_{A/2}} = \frac{R_0 \; R_A}{2\,R_0 + R_A} \tag{17.74}$$

This resistance is equal to the series arm resistance of the lattice network shown in Fig. 17.36.

$$\therefore \qquad \frac{R_0 \; R_A}{2\,R_0 + R_A} = R_1 \tag{17.75}$$

From Eq. 17.72, we have

$$R_1 = R_0 \, \frac{(N-1)}{(N+1)}$$

Hence,
$$\frac{R_0 \; R_A}{(2\,R_0 + R_A)} = R_0 \, \frac{(N-1)}{(N+1)}$$

From which
$$R_A = R_0 \, (N-1) \tag{17.76}$$

From Fig. 17.40, when the cut wires A, B, C are open, the input resistance of the network is given by

$$R_{0c} = (R_0 + 2R_B) \tag{17.77}$$

This resistance is equal to the diagonal arm resistance of the lattice network shown in Fig. 17.36.

$$\therefore \qquad R_0 + 2R_B = R_2 \tag{17.78}$$

From Eq. 17.73, we have

$$R_2 = R_0 \, \frac{(N+1)}{(N-1)}$$

Hence
$$(R_0 + 2R_B) = R_0 \, \frac{(N-1)}{(N+1)}$$

From which
$$R_B = \frac{R_0}{N-1} \tag{17.79}$$

Equations 17.76 and 17.79 are the design equations for bridged-T attenuator.

Example 17.10 Design a symmetrical bridged-T attenuator with an attenuation of 20 dB and terminated into a load of 500 Ω.

Solution $D = 20$ dB; $R_0 = 500\ \Omega$

$$N = \text{Antilog}\ \frac{D}{20} = \text{Antilog}\ \frac{20}{20} = 10$$

$$R_A = R_0\,(N-1) = 500\,(10-1) = 4500\ \Omega$$

$$R_B = \frac{R_0}{(N-1)} = \frac{500}{(10-1)} = 55.555\ \Omega$$

The desired configuration of the attenuator is shown in Fig.17.41.

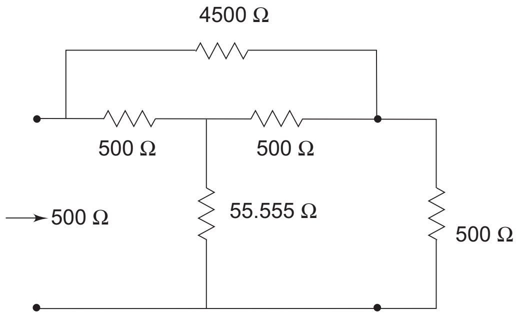


Fig. 17.41

17.16 L-TYPE ATTENUATOR

An L-type asymmetrical attenuator is shown in Fig. 17.42. The attenuator is connected between a source with source resistance $R_s = R_0$ and load resistance $R_L = R_0$.

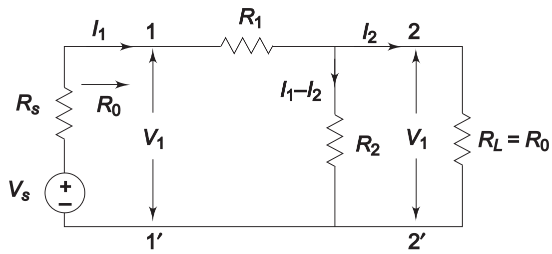


Fig. 17.42

The design equations can be obtained by applying simple laws.

$$V_2 = (I_1 - I_2)R_2 = I_2 R_L$$

or
$$I_1 R_2 = I_2(R_2 + R_L)$$

$$\frac{I_1}{I_2} = \frac{R_2 + R_L}{R_2} = N \tag{17.80}$$

$$1 + \frac{R_L}{R_2} = N$$

$$R_2 = \frac{R_L}{N-1} \tag{17.81}$$

As $R_L = R_0$, Eq. 17.81 can be written as

$$R_2 = \frac{R_0}{N-1} \tag{17.82}$$

The resistance of the network as viewed from 1-1′ into the network is

$$R_0 = R_1 + \frac{R_2 \, R_0}{R_2 + R_0}$$

$$R_1 = \frac{R_0^2}{R_2 + R_0} \tag{17.83}$$

Substituting the value of R_2 from Eq. 17.82, we have

$$R_1 = \frac{R_0^2}{\dfrac{R_0}{N-1} + R_0} = \frac{R_0^2\,(N-1)}{R_0 + R_0\,(N-1)}$$

$$R_1 = R_0\,\frac{(N-1)}{N} \tag{17.84}$$

Equations 17.82 and 17.84 are the design equations. Attenuation N of the network can be varied by varying the values of R_1 and R_2.

Example 17.11 Design a L-type attenuator to operate into a load resistance of 600 Ω with an attenuation of 20 dB.

Solution $N = \text{Antilog}\,\dfrac{\text{dB}}{20} = \text{Antilog}\,\dfrac{20}{20} = 10$

The series arm of the attenuator is given by

$$R_1 = R_0\left(\frac{N-1}{N}\right) = 600\left(\frac{10-1}{10}\right) = 540\ \Omega$$

The shunt arm of the attenuator is given by

$$R_2 = \frac{R_0}{N-1} = \frac{600}{9} = 66.66\ \Omega$$

The desired configuration of the network is shown in Fig. 17.43.

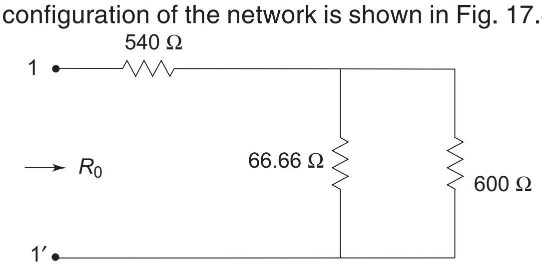


Fig. 17.43

17.17 EQUALIZERS

Equalizers are networks designed to provide compensation against distortions that occur in a signal while passing through an electrical network. In general, any electrical network has attenuation distortion and phase distortion. Attenuation distortion occurs due to non-uniform attenuation against frequency characteristics. Phase distortion occurs due to phase delay against frequency characteristics. An attenuation equalizer is used to compensate attenuation distortion in any network. These equalizers are used in medium to high frequency carrier telephone systems, amplifiers, transmission lines and speech reproduction, etc. A phase equalizer is used to compensate phase distortion in any network. These equalizers are used in TV signal transmission lines and in facsimile.

17.18 INVERSE NETWORK

The geometrical mean of two impedances Z_1 and Z_2 is a real number and they are said to be inverse if

$$Z_1 Z_2 = R_0^2$$

where R_0 is a resistance

Consider $Z_1 = R_1$ and $Z_2 = R_2$

The product $Z_1\, Z_2$ is a real number

Therefore, the two impedances are said to be inverse if they satisfy the relation $Z_1 Z_2 = R_1 R_2 = R_0^2$.

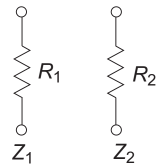


Fig. 17.44

In another case, consider $Z_1 = j\omega L$ and $Z_2 = \dfrac{1}{j\omega C}$

$$Z_1 Z_2 = \frac{j\omega L}{j\omega C} = \frac{L}{C}$$

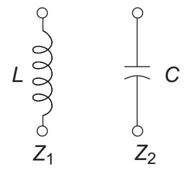


Fig. 17.45

The product $Z_1 Z_2$ is a real number

Therefore, the two impedances are inverse.

Similarly,

Let $$Z_1 = R_1 + j\omega L \tag{17.85}$$

and $$Z_2 = \frac{R_2\,\dfrac{1}{j\omega C}}{R_2 + \dfrac{1}{j\omega C}} = \frac{-\,jR_2}{\omega C R_2 - j}\cdot\frac{\omega C R_2 + j}{\omega C R_2 + j} \tag{17.86}$$

$$= \frac{R_2 - j\omega C R_2^2}{1 + \omega^2 C^2 R_2^2}$$

$$Z_1 Z_2 = (R_1 + j\omega L)\left(\frac{R_2 - j\omega C R_2^2}{1 + \omega^2 C^2 R_2^2}\right)$$

$$= \frac{R_1\, R_2 + \omega^2 R_2^2 LC + j\left(\omega L R_2 - \omega C R_1 R_2^2\right)}{1 + \omega^2 C^2 R_2^2} \tag{17.87}$$

The imaginary part of the above equation must be zero.

Therefore, we get $\omega LR_2 = \omega CR_1R_2^2$

$$\frac{L}{C} = R_1R_2 = R_0^2 \qquad (17.88)$$

The two impedances Z_1 and Z_2 are inverse, when the above condition is satisfied.

An inverse network may be obtained by

(i) Converting each series branch into parallel branch and vice-versa.

(ii) Converting each resistance element R into a corresponding resistive element $\frac{R_0^2}{R}$.

(iii) Converting each inductance L into capacitance $C^1 = \frac{L}{R_0^2}$.

(iv) Converting each capacitance C into inductance $L^1 = CR_0^2$.

Example 17.12 Obtain the inverse network of the network shown in Fig. 17.46.

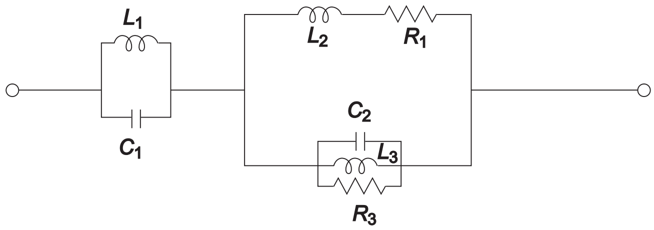


Fig. 17.46

Solution The parallel branch is converted into a series branch and vice-versa. The capacitance is replaced by inductance and vice-versa. The resistance is replaced by another resistance as shown in Fig. 17.47.

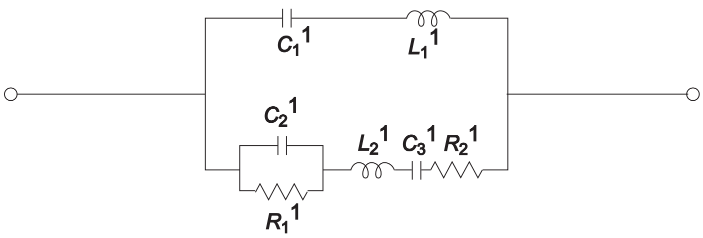


Fig. 17.47

Where $$C_1^1 = \frac{L_1}{R_0^2},\ iL_1^1 = C_1R_0^2,\ R_1^1 = \frac{R_0^2}{R_1}$$

$$C_2^1 = \frac{L_2}{R_0^2},\ L_2^1 = C_2R_0^2,\ R_2^1 = \frac{R_0^2}{R_2}$$

$$C_3^1 = \frac{L_3}{R_0^2} \text{ and } R_0 = \text{design impedance.}$$

17.19 SERIES EQUALIZER

The series equalizer is a two terminal network connected in series with a network to be corrected. (see Fig. 17.48)

Let N = Input to output power ratio of the load.
D = Attenuation in decibels.
R_0 = Resistance of the load as well as source.
P_1 = Input power.
P_1 = Load power.
$2X_1$ = Reactance of the equalizer.
V_{max} = Voltage applied to the network.

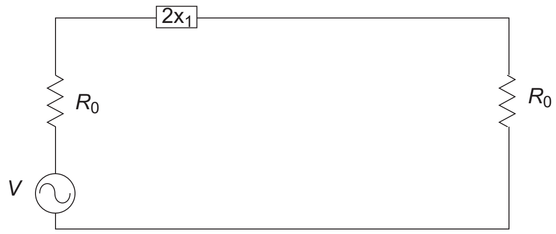


Fig. 17.48

Attenuation $D = \log_{10} N$

or $$N = \text{antilog}\left(\frac{D}{10}\right) \tag{17.89}$$

$$N = \frac{\text{Maximum power delivered to the load when equlizer is not present}}{\text{Power delivered to the load when equalizer is present}}$$

$$N = \frac{P_i}{P_l}$$

$$P_i = \left(\frac{V_{max}}{2R_0}\right)^2 R_0 = \frac{V_{max}^2}{4R_0}$$

When the equalizer is connected,

$$l_1 = \frac{V_{max}}{\sqrt{(2\,R_0)^2 + (2X_1)^2}}$$

$$P_l = \left[\frac{V_{\max}}{\sqrt{(2R_0)^2 + (2X_1)^2}}\right]^2 R_0$$

$$= \left[\frac{V_{\max}^2}{4\left(R_0^2 + X_1^2\right)}\right] R_0 \tag{17.90}$$

Therefore,
$$N = \frac{P_i}{P_l} = \frac{V_{\max}^2 / 4R_0}{\dfrac{V_{\max}^2\ R_0}{4\left(R_0^2 + X_1^2\right)}} = 1 + \frac{X_1^2}{R_0^2} \tag{17.91}$$

By knowing the values of R_0 and N, X_1 can be determined.

17.20 FULL SERIES EQUALIZER

Figure 17.49 shows the configuration of full series equalizer.

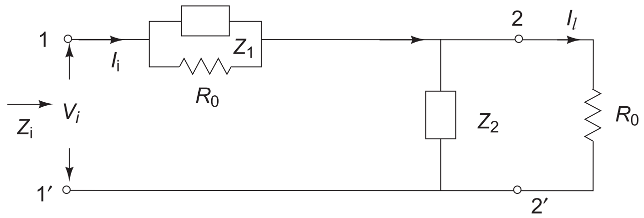


Fig. 17.49

The circuit is a constant resistance equalizer satisfying the relation $Z_1Z_2 = R_0^2$. The input impedance is given by

$$Z_i = \frac{R_0Z_1}{R_0 + Z_1} + \frac{R_0Z_2}{R_0 + Z_2}$$

$$= \frac{R_0\left[2Z_1\ Z_2 + R_0\ (Z_1 + Z_2)\right]}{R_0^2 + R_0\ (Z_1 + Z_2) + Z_1Z_2} \tag{17.92}$$

If we substitute $Z_1Z_2 = R_0^2$ in the above equation

$$Z_i = R_0$$

$$|V_i| = I_iZ_i = I_i\ R_0$$

$$|V_l| = I_iZ_i = I_i\ \frac{R_0Z_2}{R_0 + Z_2} \tag{17.93}$$

$$N = \left|\frac{V_i}{V_l}\right|^2 = \left|\frac{R_0 + Z_2}{Z_2}\right|^2 = 1 + \frac{R_0^2}{X_2^2} \tag{17.94}$$

$$= 1 + \frac{X_1^2}{R_0^2}$$

Since Z_1 and Z_2 are pure reactances and $X_1X_2 = R_0^2$

(i) When $X_1 = \omega L$,

$$X_2 = \frac{1}{\omega C_1} \text{ since both are inverse}$$

The full series equalizer is shown in Fig. 17.50.

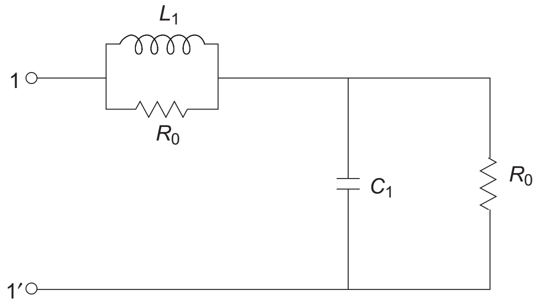


Fig. 17.50

Where $\dfrac{L_1}{C_1} = R_0^2$

From the equation $N = 1 + \dfrac{X_1^2}{R_0^2}$

$$= 1 + \frac{\omega^2 L_1^2}{R_0^2}$$

By knowing the values of N and R_0, the elemental values of L_1, C_1 may be obtained.

(ii) When $X_1 = \dfrac{1}{\omega C_1}$,

$$X_2 = \omega L_1$$

The full series equalizer is shown in Fig. 17.51.

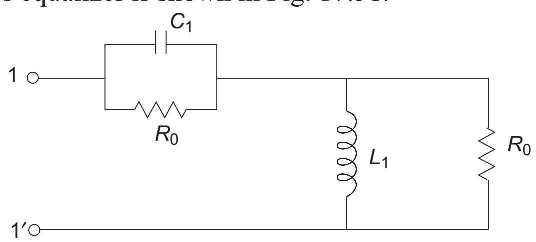


Fig. 17.51

Here $$\frac{L_1}{C_1} = R_0^2$$

From the equation $$N = 1 + \frac{R_0^2}{X_2^2} = 1 + \frac{R_0^2}{\omega^2 L_1^2}$$

By knowing the values of N and R_0, the values of L_1, C_1 may be obtained.

17.21 SHUNT EQUALIZER

The shunt equalizer is a two terminal network connected in shunt with a network to be corrected.

Let
N = Input to output power ratio.
D = Attenuation in decibels.
R_0 = Source resistance/load resistance.
I_s = Source current.
I_l = Load current.
P_i = Input power.
P_l = Load power.
$\frac{X_1}{2}$ = Reactance of shunt equalizer

The shunt equalizer connected to the network is shown in Fig. 17.52.

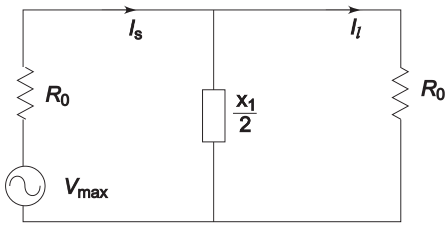


Fig. 17.52

Source current $$I_s = \frac{V_{max}}{R_0 + \left(R_0 // \frac{jX_1}{2}\right)} \tag{17.95}$$

$$= \frac{V_{max}}{R_0 + \left[\frac{jX_1 R_0}{2R_0 + jX_1}\right]}$$

$$= \frac{V_{max}[2R_0 + jX_1]}{2R_0(R_0 + jX_1)}$$

Load current $$I_l = I_s \frac{jX_1/2}{R_0 + \dfrac{jX_1}{2}} = I_s \frac{jX_1}{2R_0 + jX_1} \qquad (19.96)$$

Substituting I_s in the above equation

$$I_l = \frac{V_{\max} \cdot jX_1}{2R_0 (R_0 + jX_1)} \qquad (17.97)$$

Power delivered to the load

$$P_l = |I_l|^2 R_0 = \frac{V_{\max}^2 X_1^2}{4R_0 \left(R_0^2 + X_1^2\right)} \qquad (17.98)$$

and $$P_i = V_{\max}^2/4R_0$$

Therefore, $$N = \frac{P_i}{P_l} = \frac{\dfrac{V_{\max}^2}{4R_0}}{\dfrac{V_{\max}^2 X_1^2}{4R_0 \left(R_0^2 + X_1^2\right)}}$$

∴ $$N = 1 + \left(\frac{R_0}{X_1}\right)^2 \qquad (17.99)$$

By knowing the values of R_0 and N, X_1 can be determined.

17.22 FULL SHUNT EQUALIZER

Figure 17.53 shows the full shunt equalizer. It is also a constant resistance equalizer which satisfied the equation $Z_1 Z_2 = R_0^2$.

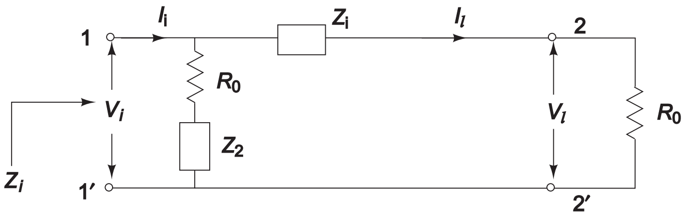


Fig. 17.53

The input impedance is given by

$$Z_i = \frac{(R_0 + Z_2)(R_0 + Z_1)}{2R_0 + Z_1 + Z_2} \qquad (17.100)$$

$$= \frac{Z_1 Z_2 + R_0^2 + R_0 (Z_1 + Z_2)}{2R_0 + Z_1 + Z_2}$$

$$Z_1 = R_0$$

Since $$Z_1 Z_2 = R_0^2,$$

$$V_i = I_i Z_i = I_i R_0$$

$$V_l = I_l R_0$$

$$\frac{V_i}{V_l} = \frac{I_i}{I_l}$$

But $$I_l = I_i \frac{(R_0 + Z_2)}{2R_0 + Z_1 + Z_2} \tag{17.101}$$

$$\frac{I_i}{I_l} = \frac{Z_1 + Z_2 + 2R_0}{R_0 + Z_2} \tag{17.102}$$

Multiplying both numerator and denominator by Z_1, we get

$$\frac{I_i}{I_l} = \frac{Z_1^2 + Z_1 Z_2 + 2R_0 Z_1}{Z_1 R_0 + Z_1 Z_2}$$

$$\frac{I_i}{I_l} = \frac{(Z_1 + R_0)^2}{R_0 (Z_1 + R_0)} = \frac{Z_1 + R_0}{R_0}$$

Therefore, $$N = \left|\frac{V_i}{V_l}\right|^2 = \left|\frac{I_i}{I_l}\right|^2 = \left|\frac{R_0 + Z_1}{R_0}\right|^2$$

$$N = 1 + \frac{X_1^2}{R_0^2} = 1 + \frac{R_0^2}{X_2^2} \tag{17.103}$$

since Z_1 and Z_2 are pure reactances and are equal to X_1 and X_2 respectively.

By knowing the values or R_0 and N_1 the elemental values X_1 and X_2 can be obtained.

(i) When $X_1 = \omega L_1$

X_2 becomes $\dfrac{1}{\omega C_1}$

The circuit is shown in Fig. 17.54.

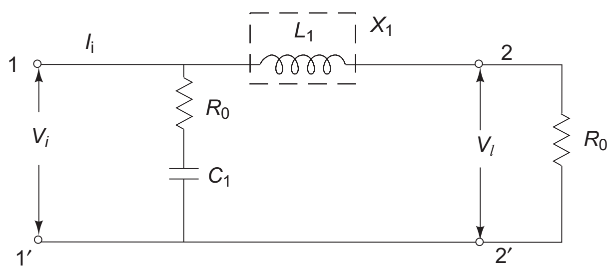


Fig. 17.54

(ii) When $X_1 = \dfrac{1}{\omega C_1}$

X_2 becomes ωL_1

The circuit is shown in Fig. 17.55

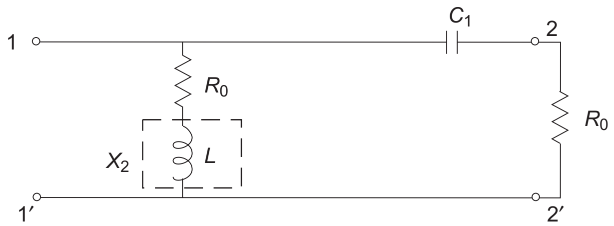


Fig. 17.55

17.23 CONSTANT RESISTANCE EQUALIZER

The disadvantage of a reactance equalizer either in a shunt equalizer or a series equalizer, the variation of impedance with frequency causes impedance mismatch which results in reflection losses. A four terminal equalizer which offers a constant resistance at all frequencies avoids reflection loss when terminated in its design impedance. Constant resistance equalizer is a four terminal network which can be T, π, lattice and bridged-T network type. All these types have characteristic impedance satisfying the relation $Z_1Z_2 = R_0^2$.

17.24 BRIDGE-*T* ATTENUATION EQUALIZER

The network shown in Fig. 17.56 is a bridged-T attenuation equalizer. Let Z_1 be a parallel combination of resistor R_1 and inductance L_1. To provide a constant resistance the impedance Z_2 must be an inverse of Z_1 which is a series combination of R_2 and a capacitor C_1. Let R_0 be the design resistance.

Then, $$Z_1Z_2 = R_0^2$$

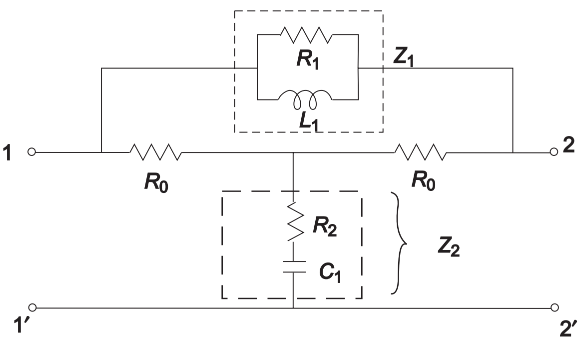


Fig. 17.56 *Bridged-T attenuation equalizer*

The propagation constant for a bridged-T network is given by

$$\gamma = \ln\left[1+\frac{Z_1}{Z_0}\right] = \ln\left[1+\frac{Z_0}{Z_2}\right] \tag{17.104}$$

But $\quad Z_0 = R_0$

And $$Z_1 = \frac{jR_1\ \omega L_1}{R_1 + \omega L_1} \tag{17.105}$$

Therefore, the propagation constant

$$\gamma = \ln\left[1+\frac{jR_1\omega L_1}{R_0(R_1 + j\omega L_1)}\right] \tag{17.106}$$

$$\alpha + j\beta = \ln\left[\frac{R_0R_1 + j\omega L_1\,(R_0 + R_1)}{R_0R_1 + j\omega L_1 R_0}\right] \tag{17.107}$$

Equating real parts on both sides

$$\alpha = \ln\left[\frac{(R_0R_1)^2 + \omega^2L_1^2R_0^2 + \omega^2L_1^2R_1^2 + 2\,\omega^2L_1^2R_0R_1}{R_0^2\ R_1^2 + \omega^2L_1^2R_0^2}\right]^{1/2}$$

$$= \frac{1}{2}\ln\left[1+\frac{\omega^2L_1^2R_1\,(2R_0 + R_1)}{R_0^2\left(R_1^2 + \omega^2L_1^2\right)}\right] \tag{17.108}$$

and $$R_1R_2 = R_0^2 = \frac{L_1}{C_1} \tag{17.109}$$

The elements may be calculated from the above design Eqs. (17.108) and (17.109).

17.25 BRIDGED-*T* PHASE EQUALIZER

A bridged-*T* phase equalizer is shown in Fig. 17.57.

It consists of only pure reactances.

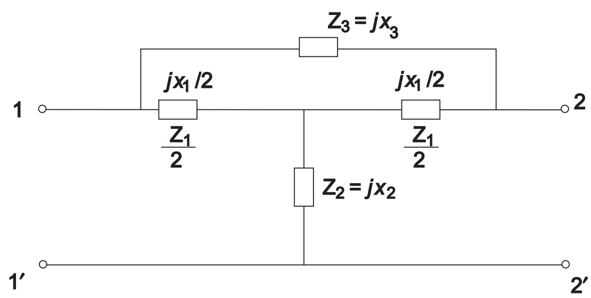


Fig. 17.57

The characteristics impedance is given by

$$Z_0 = \left[\frac{Z_1 Z_3 \,(Z_1 + 4Z_2)}{4\,(Z_1 + Z_3)} \right]^{1/2} \tag{17.110}$$

From the Fig. 17.57, $Z_3 = jX_3$, $\frac{Z_1}{2} = jX_1$, $Z_2 = jX_2$ and $Z_0 = R_0$.

$$R_0^2 = \frac{2jX_1 \cdot jX_3 \,(2jX_1 + 4jX_2)}{4\,(2jX_1 + 4jX_3)} \tag{17.111}$$

$$= \frac{-\,X_1 X_3 \,(X_1 + 2X_2)}{2X_1 + X_3}$$

Let X_1 and X_3 be made inverse

$$jX_1 .\, jX_3 = R_0^2$$

$$-\,X_1 X_3 = R_0^2 \tag{17.112}$$

Substituting this in the above equation, we get

$$X_2 = \frac{X_1 + X_3}{2} \tag{17.113}$$

The propagation constant is given by

$$e^{\gamma} = \frac{Z_0\,(Z_1 + Z_3) + (Z_1 Z_3/2)}{Z_0\,(Z_1 + Z_3) - (Z_1 Z_3/2)} \tag{17.114}$$

$$e^{\gamma} - 1 = \frac{Z_1 Z_3}{Z_0\,(Z_1 + Z_3) - (Z_1 Z_3/2)}$$

and similarly,

$$e^{\gamma} + 1 = \frac{2Z_0\,(Z_1 + Z_3)}{Z_0\,(Z_1 + Z_3) - \left(\frac{Z_1 Z_3}{2}\right)}$$

From the above equations

$$\frac{e^{\gamma} - 1}{e^{\gamma} + 1} = \tanh\frac{\gamma}{2} = \frac{Z_1 Z_3}{2Z_0\,(Z_1 + Z_3)} = \frac{2jX_1 \cdot jX_3}{2R_0\,(2jX_1 + jX_3)}$$

$$= \frac{2R_0^2}{2R_0\, j\,(2X_1 + X_3)} \tag{17.115}$$

$$\tanh\frac{\gamma}{2} = \frac{2R_0^2}{2R_0 j\left(2X_1 - \frac{R_0^2}{X_1}\right)}$$

$$\frac{\gamma}{2} = \tanh^{-1} \frac{jR_0 X_1}{R_0^2 - 2X_1^2}$$

$$\therefore \qquad \alpha + j\beta = 2j \tanh^{-1} \frac{R_0 X_1}{R_0^2 - 2X_1^2}$$

Equating the real and imaginary parts, we get

$$\alpha = 0$$

$$\beta = 2 \tan^{-1}\left(\frac{R_0 X_1}{R_0^2 - 2X_1^2}\right) \tag{17.116}$$

Equations 17.112, 17.113 and 17.116 are the design equations of a bridged-T phase equalizer.

17.26 LATTICE ATTENUATION EQUALIZER

The constant resistance lattice attenuation equalizer is shown in Fig. 17.58. The element Z_1 represents series arm and Z_2 represents diagonal arm as shown in Fig. 17.58. The equalizer is a constant resistance equalizer such that Z_1 must be inverse of Z_2 to the design resistance R_0.

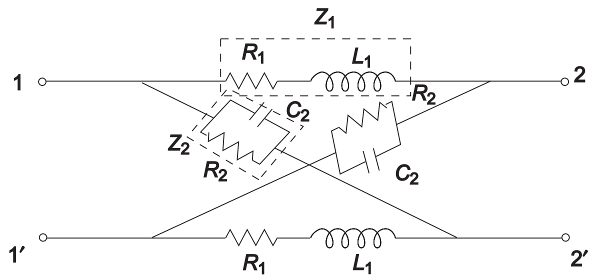


Fig. 17.58

$$Z_1 Z_2 = R_0^2$$

$$R_1 R_2 = \frac{L_1}{C_1} R_0^2 \tag{17.117}$$

The propagation constant of a lattice network is given by

$$\gamma = \ln\left(\frac{1 + \dfrac{Z_1}{R_0}}{1 - \dfrac{Z_1}{R_0}}\right) = \ln\left(\frac{1 + \dfrac{Z_2}{R_0}}{\dfrac{Z_2}{R_0} - 1}\right) \tag{17.118}$$

$$\alpha + j\beta = \ln\left(\frac{1 + \dfrac{R_1 + j\omega L_1}{R_0}}{1 - \dfrac{R_1 + j\omega L_1}{R_0}}\right) \tag{17.119}$$

$$\alpha + j\beta = \ln\left[\frac{(R_0 + R_1) + j\omega L_1}{(R_0 - R_1) - j\omega L_1}\right]$$

Equating real parts on both sides

$$\alpha = \ln\left[\frac{(R_0 + R_1)^2 + \omega^2 L_1^2}{(R_0 - R_1)^2 + \omega^2 L_1^2}\right]^{1/2}$$

$$N = e^{\alpha} = \left[\frac{(R_0 + R_1)^2 + \omega^2 L_1^2}{(R_0 - R_1)^2 + \omega^2 L_1^2}\right]^{1/2} \tag{17.120}$$

On the other hand if $X_1 = \dfrac{1}{\omega C_1}$

$$N = e^{\alpha} = \left[\frac{(R_0 + R_1)^2 + \dfrac{1}{\omega^2 C_1^2}}{(R_0 - R_1)^2 + \dfrac{1}{\omega^2 C_1^2}}\right]^{1/2} \tag{17.121}$$

Equations (17.117) and (17.121) are called design equations for the lattice attenuator network.

17.27 LATTICE PHASE EQUALIZER

The lattice phase equalizer is shown in Fig. 17.59. It consists of only reactive components. This is also a constant resistance equalizer which satisfies the equation $Z_1 Z_2 = R_0^2$.

Z_1 is the series arm impedance and Z_2 is the shunt arm impedance as shown in Fig. 17.59.

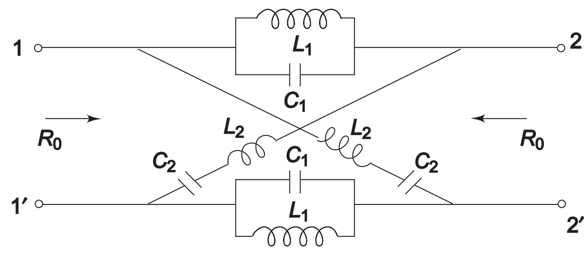


Fig. 17.59

The propagation constant is given by

$$\tanh\left(\frac{\gamma}{2}\right) = \left(\frac{Z_1}{R_0}\right) = \sqrt{\frac{Z_1}{Z_2}}$$

$$\therefore \qquad \tanh\left(\frac{\gamma}{2}\right) = \frac{j\omega L_1 / j\omega C_1}{R_0\left(j\omega L_1 + \dfrac{1}{j\omega C_1}\right)}$$

$$\tanh\left(\frac{\gamma}{2}\right) = \frac{j\omega L_1}{R_0\left(1-\omega^2 L_1 C_1\right)}$$

$$\gamma = 2\tanh^{-1}\left[\frac{j\omega L_1}{R_0\left(1-\omega^2 L_1\, C_1\right)}\right]$$

$$= 2j\tan^{-1}\left[\frac{\omega L_1}{R_0\left(1-\omega^2 L_1\, C_1\right)}\right]$$

$$\alpha + j\beta = 2j\tan^{-1}\left[\frac{\omega L_1}{R_0\left(1-\omega^2 L_1\, C_1\right)}\right]$$

Here $\alpha = 0$

$$\beta = 2\tan^{-1}\left[\frac{\omega L_1}{R_0\left(1-\omega^2 L_1\, C_1\right)}\right]$$

The above expression gives the phase delay in a lattice phase equalizer.

Additional Solved Problems

Problem 17.1 Determine the cut-off frequency for the low pass filter shown in Fig. 17.60.

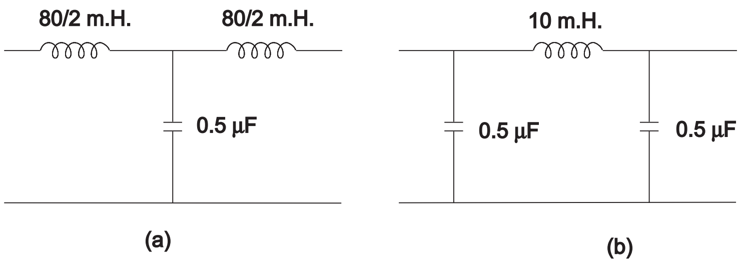


Fig. 17.60

Solution (a) For the T-network given $L/2 = 40$ mH, $C = 0.5\ \mu$F

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{80\times 10^{-3}}{0.5\times 10^{-6}}} = 400\ \Omega$$

Cut-off frequency $$f_c = \frac{k}{L\pi} = \frac{400}{80\times 10^{-3}\times\pi} = 1591\text{ Hz}$$

or $$f_c = \frac{1}{\pi k C} = \frac{1}{\pi\times 400\times 0.5\times 10^{-6}} = 1591\text{ Hz}$$

(b) For the π-network, given $C/2 = 0.5\ \mu$F, $L = 10$ mH

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{10 \times 10^{-3}}{1 \times 10^{-6}}} = 100\ \Omega$$

Cut-off frequency $$f_c = \frac{k}{L\pi} = \frac{100}{10 \times 10^{-3} \times \pi} = 3183 \text{ Hz}$$

or $$f_c = \frac{1}{\pi k C} = \frac{1}{\pi \times 100 \times 1 \times 10^{-6}} = 3183 \text{ Hz}$$

Problem 17.2 Determine the cut-off frequency for the high pass filter shown in Fig. 17.61.

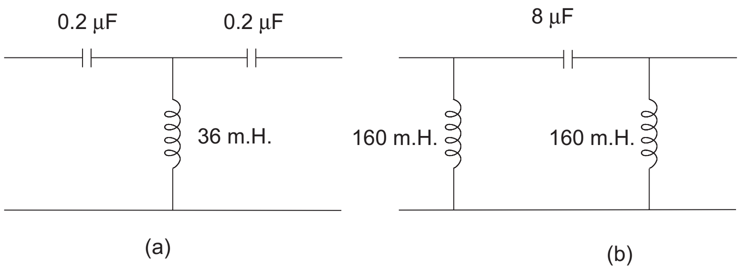


Fig. 17.61

Solution (a) For the T-network, given $2C = 0.2\ \mu$F, $L = 36$ mH

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{36 \times 10^{-3}}{0.1 \times 10^{-6}}} = 600\ \Omega$$

Cut-off frequency $$f_c = \frac{1}{4\pi\sqrt{LC}} = \frac{k}{4\pi L}$$

$$= \frac{600}{4\pi \times 36 \times 10^{-3}} = 1326 \text{ Hz}$$

or $$f_c = \frac{1}{4\pi k C} = \frac{1}{4\pi \times 600 \times 0.1 \times 10^{-6}} = 1326 \text{ Hz}$$

(b) For the π-section, given $2L = 160$ mH, $C = 8\ \mu$F

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{80 \times 10^{-3}}{8 \times 10^{-6}}} = 100\ \Omega$$

Cut-off frequency $$f_c = \frac{1}{4\pi k C} = \frac{1}{4\pi \times 100 \times 8 \times 10^{-6}} = 99.47 \text{ Hz}$$

or $$f_c = \frac{k}{4\pi L} = \frac{1}{4\pi \times 80 \times 10^{-3}} = 99.47 \text{ Hz}$$

Problem 17.3 For the low pass filter shown in Fig. 17.60, find the m-derived section to have a resonant frequencies of 1700 Hz and 3300 Hz for T and π networks respectively.

Solution For the T-network the value of $m = \sqrt{1-\left(\dfrac{f_c}{f_r}\right)^2}$

$$= \sqrt{1-\left(\frac{1591}{1700}\right)^2} = 0.352$$

For the π network the value of $m = \sqrt{1-\left(\dfrac{f_c}{f_r}\right)^2}$

$$= \sqrt{1-\left(\frac{3183}{3300}\right)^2} = 0.2639$$

Corresponding m-derived sections are shown in Fig. 17.62(a) and (b).

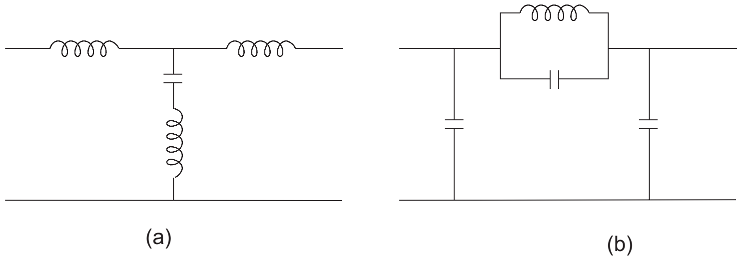


Fig. 17.62

The value of the series element for the m-derived T-section is given by

$$m\frac{L}{2} = 0.352 \times 40 \times 10^{-3} = 14.08 \text{ mH}$$

The values of the shunt elements are given by $mC = 0.352 \times 0.5 \times 10^{-6} = 0.176\ \mu\text{F}$

and $$\frac{1-m^2}{4m} \times L = 49.778 \text{ mH}$$

The values of the series elements for the m-derived π section are given by

$$mL = 0.2639 \times 10 \times 10^{-3} = 2.639 \text{ mH}$$

and $$\frac{1-m^2}{4m} \times C = 0.88\ \mu\text{F}$$

The value of the shunt element is given by

$$\frac{mC}{2} = 0.13195\ \mu\text{F}$$

The corresponding m-derived sections are shown in Fig. 17.63(c) and (d).

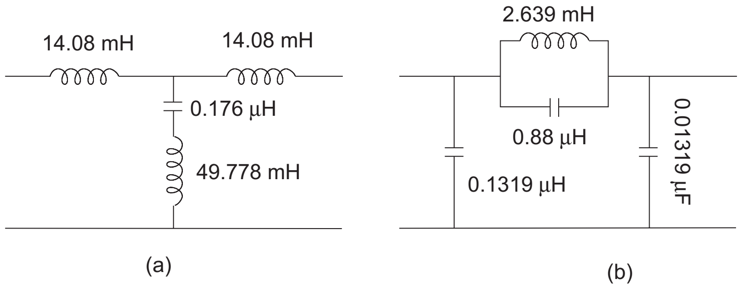


Fig. 17.63

Problem 17.4 Design an m-derived T-section filter (high pass) with a cut-off frequency 10 kHz, design impedance of 200 Ω and $m = 0.4$.

Solution It is given that $f_c = 10 \times 10^3$ Hz; $k = 200\ \Omega$, $m = 0.4$

For a prototype highpass

$$L = \frac{1}{4\pi f_c} = \frac{200}{4\pi \times 10 \times 10^3} = 1.59 \text{ mH}$$

$$C = \frac{1}{4\pi k f_c} = \frac{1}{4\pi \times 200 \times 10 \times 10^3} = 0.0397\ \mu\text{F}$$

The series element of the m-derived highpass T-section is given by

$$\frac{2C}{m} = \frac{2 \times 0.0397 \times 10^6}{0.4} = 0.1985\ \mu\text{F}$$

The elements in the shunt arm are given by

$$\frac{L}{m} = \frac{1.59 \times 10^{-3}}{0.4} = 3.975 \text{ mH}$$

$$\frac{4m}{1-m^2}\,C = \frac{4 \times 0.4}{1-(0.4)^2} \times 0.0397 \times 10^{-6} = 0.0756\ \mu\text{F}$$

The m-derived highpass T-section is shown in Fig. 17.64,

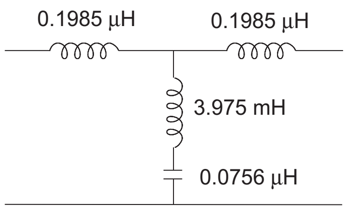


Fig. 17.64

Problem 17.5 Find the frequency at which a prototype T-section lowpass filter having a cut-off frequency f_c have an attenuation of 15 dB.

Solution We have $\alpha = 15$ dB

$$= \frac{15}{8.696} \text{ nepers} = 1.724 \text{ nepers}$$

If f is the desired frequency for 15 dB attenuation, then $\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$

$$1.724 = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$$

$$\frac{f}{f_c} = \cosh(0.862)$$

$$f = 1.395 f_c$$

The frequency at which lowpass T section filter has an attenuation of 15 dB will be 1.395 times the cut-off frequency.

Problem 17.6 Design an m-derived LPF (T-section) having a cut-off frequency of 6 KHz and a design impedance of 500 Ω. The frequency of infinite attenuation should be 1.75 times the cut-off frequency.

Solution We have $f_c = 6000$ Hz; $k = 500\ \Omega$, and $f_\infty = 1.75 f_c$.

For the prototype lowpass section $L = \frac{k}{\pi f_c}$

$$= \frac{500}{\pi \times 6000} = 26.525 \text{ mH}$$

and $C = \frac{1}{\pi k f_c} = \frac{1}{\pi \times 500 \times 6000} = 0.106\ \mu\text{F}$

For an m-derived section the value of $m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2}$

$$= \sqrt{1 - \left(\frac{6000}{1.75 \times 6000}\right)^2} = 0.820$$

Now each of the series element of lowpass T-section is given by

$$m\frac{L}{2} = \frac{0.820 \times 26.525 \times 10^{-3}}{2} = 10.68 \text{ mH}$$

The shunt arm elements are $mC = 0.82 \times 0.106 \times 10^{-6} = 0.087\ \mu\text{F}$

and $$\frac{1-m^2}{4m} \times L = \frac{1-(0.82)^2}{4 \times 0.82} \times (26.525 \times 10^{-3}) = 2.65 \text{ mH}$$

The required m-derived network is shown in Fig. 17.65

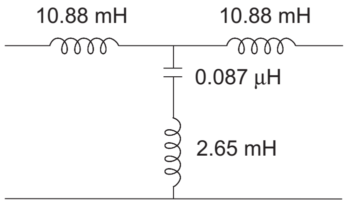
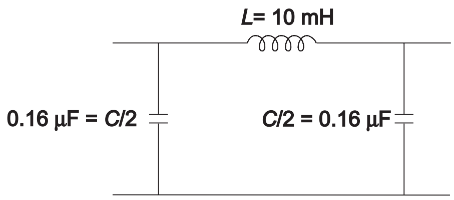


Fig. 17.65

Problem 17.7 A π-section filter network consists of a series arm inductance of 10 mH and two shunt arm capacitances of 0.16 μF each. Calculate the cut-off frequency and attenuation and phase shift at 12 kHz. What is the value of nominal impedance in the pass band.

Solution The given filter is shown in Fig. 17.66, it is a low pass filter given L = 10 mH; $C/2$ = 0.16 μF; C = 0.32 μF.

For π-section lowpass filter



(a)

Fig. 17.66

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

$$= \frac{1}{\pi\sqrt{10 \times 10^{-3} \times 0.32 \times 10^{-6}}}$$

$$= 5.627 \text{ kHz}$$

Nominal terminating impedance is given by

$$k = \sqrt{\frac{L}{C}}$$

$$= \sqrt{\frac{10 \times 10^{-3}}{0.32 \times 10^{-6}}} = 176.77\ \Omega$$

The attenuation constant = $2\cosh^{-1}\left(\frac{\omega}{\omega_c}\right)$ nepers

$$= 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{12 \times 10^3}{5.627 \times 10^3}\right) = 2.78 \text{ nepers}$$

The phase shift introduced by the LPF will be π rad in the attenuation band.

Problem 17.8 Each of the two series elements of a T-type low pass filter consists of an inductance of 30 mH having negligible resistance and a shunt element having capacitance of 0.16 μF. Calculate the value of cut-off frequency and determine the iterative impedance and the phase shift of the network at 2 kHz.

Solution We have $L/2 = 30$ mH $\Rightarrow L = 60$ mH, $C = 0.16\ \mu$F

The cut-off frequency $f_c = \dfrac{1}{\pi\sqrt{LC}}$

$$= \frac{1}{\pi\sqrt{60\times10^{-3}\times0.16\times10^{-6}}}$$

$$f_c = 3.24 \text{ kHz}$$

The characteristic impedance is given by

$$Z_{0T} = \sqrt{\frac{L}{C}}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$$

$$= \sqrt{\frac{L}{C}}\sqrt{1-\left(\frac{f}{f_c}\right)^2}$$

$$= \sqrt{\frac{60\times10^{-3}}{0.16\times10^{-6}}}\sqrt{1-\frac{2\times10^3}{3.248\times10^3}} = (612)\ (0.619) = 379.05\ \Omega$$

Since $f < f_c$ the attenuation $\alpha = 0$ and the phase shift in the pass band is given by

$$\beta = 2\sin^{-1}\left(\frac{\omega}{\omega_c}\right) = 2\sin^{-1}\left(\frac{2}{3.248}\right) = 76°$$

Problem 17.9 Design the full series equalizer shown in Fig. 17.67. The design resistance $R_0 = 600\ \Omega$ and attenuation of 12 dB at 800 Hz. Compute the elemental values.

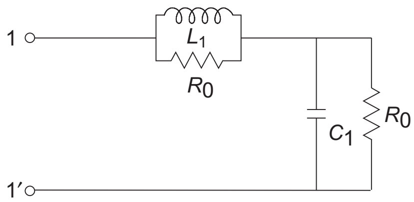


Fig. 17.67

Solution

$$D = 10 \log N$$

$$12 = 10 \log N$$

$$N = \text{Antilog}\left(\frac{12}{10}\right)$$

$$= 15.85$$

We know that $N = 1 + \dfrac{\omega^2 L_1^2}{R_0^2}$

$$L_1 = \frac{R_0\sqrt{N-1}}{\omega}$$

$$L_1 = \frac{600\times\sqrt{15.58-1}}{2\pi\times800} = 0.46 \text{ henry}$$

$$\frac{L_1}{C_1} = R_0^2$$

$$C_1 = \frac{L_1}{R_0^2} = \frac{0.46}{600 \times 600} = 1.28\ \mu\text{F}$$

Problem 17.10 Design the full shunt equalizer shown in Fig. 17.68 for a design resistance $R_0 = 600\ \Omega$ and attenuation of 10 dB at 600 Hz. Calculate the elemental values.

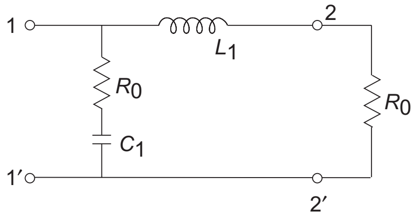


Fig. 17.68

Solution

$$D = 10 \log N$$

$$D = 10\text{ dB}$$

$$N = \text{Antilog } 1 = 10$$

$$N = 1 + \frac{X_1^2}{R_0^2} = 1 + \frac{R_0^2}{X_1^2}$$

$$X_1 = R_0 \sqrt{N-1}$$

$$\omega L_1 = R_0 \sqrt{N-1}$$

$$L_1 = \frac{R_0 \sqrt{N-1}}{2\pi f} = \frac{600\sqrt{10-1}}{2\pi \times 600}$$

$$L_1 = 0.48\text{ H}$$

$$X_2 = \frac{R_0}{\sqrt{N-1}} = \frac{600}{3}$$

$$\frac{1}{\omega C_1} = \frac{600}{3}$$

$$C_1 = \frac{3}{2\pi \times 600 \times 600} = 1.33\ \mu\text{F}$$

Problem 17.11 Design a constant resistance lattice attenuation equalizer shown in Fig. 17.69. The series arm consists of $R_1 = 2\text{ k}\Omega$ in series with $L_1 = 30$ mH. If $R_2 = 300\ \Omega$, calculate the values of R_0 and capacitance C_1 of the shunt arm.

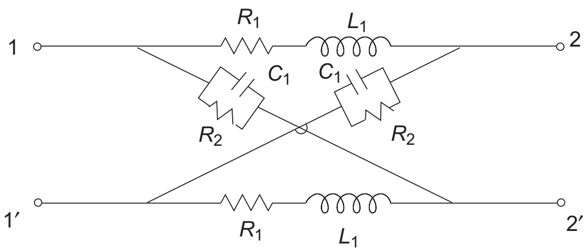


Fig. 17.69

Solution

$$R_1 = 2000\ \Omega \qquad L_1 = 30\text{ mH}$$
$$R_2 = 300\ \Omega$$
$$R_1 R_2 = R_0^2$$
$$R = \sqrt{R_1\ R_2} = 774.6\ \Omega$$
$$C_1 = \frac{L_1}{R_0^2} = \frac{0.03}{(774.6)^2} = 0.049\ \mu\text{F}$$

Problem 17.12 Determine the series arm of a constant resistance lattice attenuation equalizer shown in Fig. 17.70 having design impedance of 2 Ω, the shunt arm consists of R_2 = 2 Ω in series with a capacitor C_2 = 0.1 F.

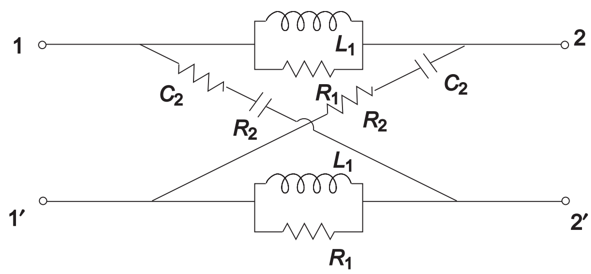


Fig. 17.70

Solution The shunt arm values are given as follows

$$R_2 = 2\ \Omega$$
$$C_2 = 0.1\text{ F}$$
$$R_0 = 2\ \Omega$$
$$R_1\ R_2 = \frac{L_1}{C_2} = R_0^2$$
$$R_1 = \frac{4}{2} = 2\ \Omega$$
$$L_1 = C_2\ R_0^2$$
$$= (0.1)\ (2)^2 = 0.4\text{ H}$$

Problem 17.13 Obtain the inverse network for the network shown in Fig. 17.71.

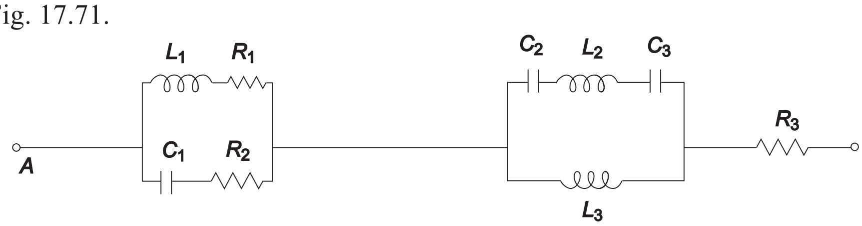


Fig. 17.71

Solution The elements of the inverse network are given by

$$C_1^1 = \frac{L_1}{R_0^2} \quad C_3^1 = \frac{L_3}{R_0^2} \quad L_2^1 = C_2\,R_0^2 \quad R_1^1 = \frac{R_0^2}{R_1}$$

$$C_2^1 = \frac{L_2}{R_0^2} \quad L_1^1 = C_1\,R_0^2 \quad L_3^1 = C_3\,R_0^2 \quad R_2^1 = \frac{R_0^2}{R_2} \quad R_3^1 = \frac{R_0^2}{R_3}$$

The inverse network is shown in Fig. 17.72.

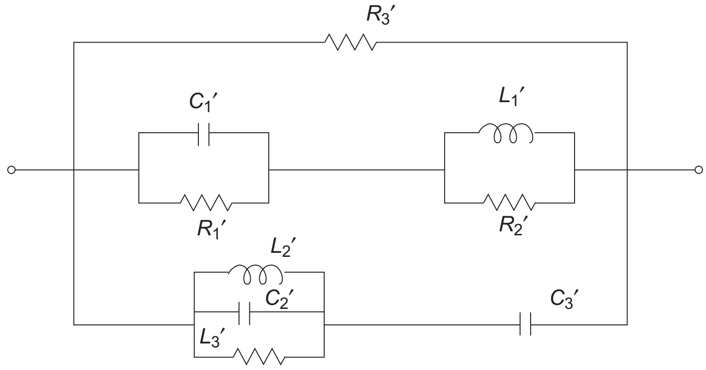


Fig. 17.72

Practice Problems

17.1 Design a low pass *T*-section filter having a cut-off frequency of 1.5 kHz to operate with a terminated load resistance of 600 Ω.

17.2 Design a low pass π-section filter with a cut-off frequency of 2 kHz to operate with a load resistance of 400 Ω.

17.3 Design a high pass filter with a cut-off frequency of 1 kHz with a terminated design impedance of 800 Ω.

17.4 Design a *m*-derived low pass filter having cut-off frequency of 1.5 kHz with a nominal impedance of 500 Ω, and resonant frequency is 1600 Hz.

17.5 Design a *m*-derived high pass filter with a cut-off frequency of 10 kHz, design impedance of 600 Ω and $m = 0.3$.

17.6 For a π section filter network shown in Fig. 17.73, calculate the cut-off frequency and the value of nominal impedance in the pass band.

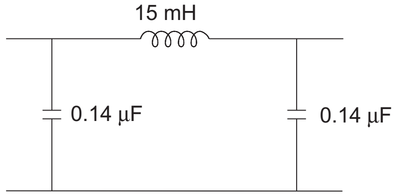


Fig. 17.73

17.7 Determine the cut-off frequency and design impedance for the *T*-section shown in Fig. 17.74.

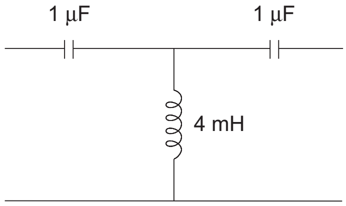


Fig. 17.74

17.8 Determine the band width and cut-off frequency for the filter shown in Fig. 17.75.

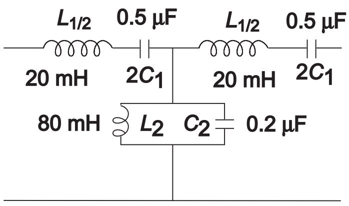


Fig. 17.75

17.9 Design a band elimination filter having a design impedance of 500 Ω and cut-off frequencies f_1 = 1 kHz and f_2 = 5 kHz.

17.10 A π-section filter network is shown in Fig. 17.76. Calculate the cut-off frequency and phase shift at 10 kHz. What is the value of nominal impedance in the pass band.

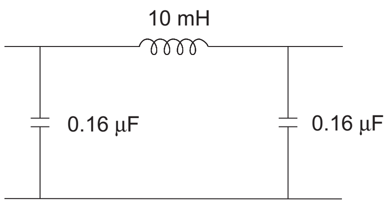


Fig. 17.76

17.11 A *T*-section filter is shown in Fig. 17.77. Calculate the value of cut-off frequency and determine the iterative impedance and the phase shift of the network at 1.5 kHz.

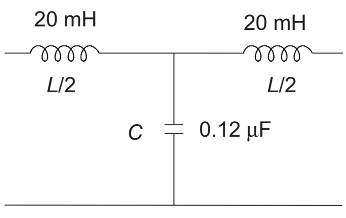


Fig. 17.77

17.12 Find the frequency at which a prototype π-section low pass filter having a cut-off frequency f_c has an attenuation of 20 dB.

17.13 Design full series equalizer for a design resistance $R_0 = 600\ \Omega$ and attenuation of 20 dB at 400 Hz. Calculate the attenuation M at 1000 MHz.

17.14 Design the full shunt equalizer, for design resistance $R_0 = 600\ \Omega$ and attenuation at frequencies of 600 Hz and 1200 Hz.

17.15 Design a constant resistance lattice attenuation equalizer to produce an attenuation of 20 dB at 50 Hz and 3 dB at 3000 Hz. Calculate its loss at 500 Hz. The equalizer is working between two impedances of 500 Ω each.

Objective-type Questions

1. A low pass filter is one which
   (a) passes all low frequencies
   (b) attenuates all high frequencies
   (c) passes all frequencies up to cut-off frequency, and attenuates all other frequencies.
2. A high pass filter is one which
   (a) passes all high frequencies
   (b) attenuates all low frequencies
   (c) attenuates all frequencies below a designated cut-off frequency, and passes all frequencies above cut-off.
3. A band pass filter is one which
   (a) attenuates frequencies between two designated cut-off frequencies and passes all other frequencies
   (b) passes frequencies between two designated cut-off frequencies, and attenuates all other frequencies.
   (c) passes all frequencies
4. An ideal filter should have
   (a) zero attenuation in the pass band
   (b) infinite attenuation in the pass band
   (c) zero attenuation in the attenuation band
5. The propagation constant of a symmetrical T-section and π-section are the same.
   (a) true (b) false
6. The values of L and C for a low pass filter with cut-off frequency of 2.5 kHz to operate with a terminated load resistance of 450 ohms are given by
   (a) 57.32 mH; 0.283 μF (b) 28.66 mH; 0.14 μF
   (c) 114.64 μH; 0.566 μF
7. The attenuation is sharp in the stop band for K-type filter.
   (a) true (b) false
8. The attenuation is not sharp in the stop band for an m-derived filter.
   (a) true (b) false

9. In the m-derived low pass filters, the resonant frequency is to be chosen so that it is
   (a) above the cut-off frequency (b) below the cut-off frequency
   (c) none of the above
10. In the m-derived high pass filters, the resonant frequency is to be chosen so that it is
    (a) above the cut-off frequency (b) below the cut-off frequency
    (c) none of the above
11. A band pass filter may be obtained by using a high pass filter followed by a low pass filter
    (a) true (b) false
12. A band elimination filter is one
    (a) which attenuates all frequencies less than lower cut-off frequency f_1
    (b) which attenuates all frequencies greater than upper cut-off frequency f_2
    (c) frequencies lying between f_1 and f_2 are attenuated and all other frequencies are passed.

Appendix A

Fourier Series

A.1 INTRODUCTION

In most of the cases, the response of linear circuits to sinusoidal excitations can be found easily. A function $f(t)$ is said to be periodic, if the process repeats itself every T sec, so that we have

$$f(t+T) = f(t)$$

If a periodic function $f(t)$ is to have a Fourier series, it must satisfy the following Dirichlet conditions.

(i) $f(t)$ must be bounded and possess a finite number of discontinuities.

(ii) $f(t)$ must have a finite number of maxima and minima, and

(iii) $f(t)$ must have a finite average value.

The function $f(t)$ can be represented over a complete period from $t = -\infty$ to $t = +\infty$, except at the discontinuities, by a series of simple harmonic functions, the frequencies of which are integral multiples of the fundamental frequency. A series in this form is called a Fourier Series.

A.2 DEFINITIONS AND DERIVATIONS

A periodic function $f(t)$ can be expressed in the complex form

$$\begin{aligned} f(t) = {} & a_0 + a_1\, e^{j\omega t} + a_2\, e^{2j\omega t} + \cdots + a_n\, e^{nj\omega t} + \cdots \\ & + a_{-1}\, e^{-j\omega t} + a_{-2}\, e^{-2j\omega t} + \cdots + a_{-n}\, e^{-nj\omega t} + \cdots \end{aligned}$$

or

$$f(t) = \sum_{n=-\infty}^{\infty} a_n\, e^{jn\omega t} \tag{1}$$

where

$$\omega = \frac{2\pi}{T}$$

To determine a_0, integrating both sides of Eq. 1 over one complete period, we get

$$\int_0^{2\pi/\omega} f(t)dt = \int_0^{2\pi/\omega} \left(\sum_{n=-\infty}^{\infty} a_n\, e^{jn\omega t} \right) dt$$

$$= \sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi/\omega} e^{jn\omega t}\, dt \tag{2}$$

$$\int_0^{2\pi/\omega} f(t)dt = \int_0^{2\pi/\omega} a_0\, dt = a_0 \frac{2\pi}{\omega} = a_0 T \tag{3}$$

or
$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt \tag{4}$$

To determine the other term, a_n, we multiply both sides of Eq. 1 by $e^{-jn\omega t}$, and integrate from 0 to $2\pi/\omega$ to obtain

$$\int_0^T f(t)\, e^{-jn\omega t}\, dt = a_n\, T \tag{5}$$

$$a_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega t}\, dt \tag{6}$$

Similarly, we have from Eq. 6, the relation

$$a_{-n} = \frac{1}{T}\int_0^T f(t)\, e^{jn\omega t}\, dt \tag{7}$$

Then Eq. 1 may be written in the form,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n\, e^{jn\omega t} + a_{-n}\, e^{-jn\omega t}) \tag{8}$$

By using Euler's relation, the function $f(t)$ may be written in the form

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n + a_{-n}) \cos n\omega t + \sum_{n=1}^{\infty} j(a_n - a_{-n}) \sin n\omega t \tag{9}$$

Now let

$$A_n = a_n + a_{-n}\ ; \quad B_n = j(a_n - a_{-n})\ ; \quad \frac{A_0}{2} = a_0$$

We get

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega t + \sum_{n=1}^{\infty} B_n \sin n\omega t \tag{10}$$

Therefore, we have

$$A_n = a_n + a_{-n} = \frac{1}{T}\int_0^T f(t)\, (e^{jn\omega t} + e^{-jn\omega t})\, dt$$

$$= \frac{2}{T} \int_0^T f(t) \cos n\omega t \, dt \tag{11}$$

$$B_n = j(a_n - a_{-n})$$

$$= \frac{1}{T} \int_0^T f(t)\, j(e^{jn\omega t} - e^{-jn\omega t})\, dt$$

$$= \frac{2}{T} \int_0^T f(t) \sin n\omega t \, dt \tag{12}$$

Example of Fourier Series

To determine the Fourier series for the square wave shown in Fig. A.1.

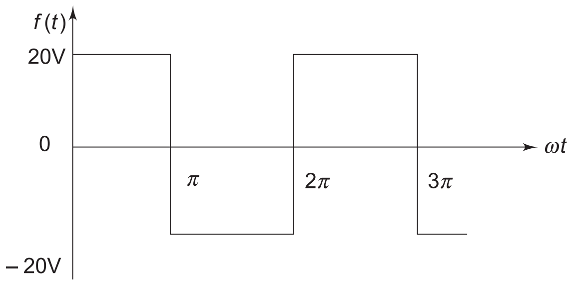


Fig. A.1

The function $f(t)$ is represented as

$$f(t) = 20, \; 0 < \omega t < \pi$$
$$= -20, \; \pi < \omega t < 2\pi$$

Since the average value of the wave is zero, the term $A_0/2 = 0$
The cosine coefficients are obtained as follows.

$$a_n = \frac{1}{\pi} \left\{ \int_0^{\pi} 20 \cos n\omega t \; d(\omega t) + \int_{\pi}^{2\pi} (-20) \cos n\omega t \; d(\omega t) \right.$$

$$= \frac{20}{\pi} \left\{ \left[\frac{1}{n} \sin n\omega t \right]_0^{\pi} - \left[\frac{1}{n} \sin n\omega t \right]_{\pi}^{2\pi} \right\} = 0 \text{ for all } n$$

Thus, the series contains no cosine terms.
To determine the sine terms

$$b_n = \frac{1}{\pi} \left\{ \int_0^{\pi} 20 \cos n\omega t \; d(\omega t) + \int_{\pi}^{2\pi} (-20) \cos n\omega t \; d(\omega t) \right.$$

$$= \frac{20}{\pi} \left\{ \left[\frac{-1}{n} \cos n\omega t \right]_0^{\pi} + \left[\frac{1}{n} \cos n\omega t \right]_{\pi}^{2\pi} \right\}$$

$$= \frac{20}{\pi n}\,[-\cos n\pi + \cos 0 + \cos n2\pi - \cos n\pi] = \frac{40}{\pi n}\,(1 - \cos n\pi)$$

Then

$$b_n = \frac{80}{\pi n} \text{ for } n = 1, 3, 5, \cdots$$
$$= 0 \text{ for } n = 2, 4, 6, \cdots$$

The series for the square wave is

$$f(t) = \frac{80}{\pi}\sin \omega t + \frac{80}{3\pi}\sin 3\omega t + \frac{80}{5}\sin 5\omega t + \cdots$$

The Fourier series contains only odd harmonic sine terms.

Appendix B

Fourier Transforms

B.1 FOURIER INTEGRAL

In this section, the limiting form of the Fourier series as the period T is made to approach infinity. Then the resulting function is called the Fourier integral representation, or simply, the Fourier integral of $f(t)$.

Consider the complex Fourier-series expansion of the periodic function $f(t)$;

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega t} \quad T = \frac{2\pi}{\omega} \tag{1}$$

where $$a_n = \frac{1}{T}\int_0^T f(x)\, e^{-jn\omega x}\, dx$$

or $$a_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-jn\omega x}\, dx, \quad \omega = \frac{2\pi}{T} \tag{2}$$

Substituting this into Eq. 1, we get

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-jn\omega x}\, dx\right] e^{jn\omega t}$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(x)\exp\left[\left(\frac{2\pi n j}{T}\right)(t-x)\right] dx\right] \tag{3}$$

Let $1/T = \Delta s$

Then

$$f(t) = \sum_{n=-\infty}^{\infty} \Delta s \int_{-T/2}^{T/2} f(x)\, e^{2\pi n j(t-x)\Delta s}\, dx \tag{4}$$

Now the definite integral $\int_0^\infty \tau(s)\, ds$ may be defined as the limit, as Δs approaches zero, of the sum.

$$\sum_{n=0}^{\infty} \tau(n\,\Delta s)\,\Delta s \tag{5}$$

Also we have

$$\int_{-\infty}^{\infty} \tau(s)\, ds = \int_{-\infty}^{0} \tau(s)\, ds + \int_{0}^{\infty} \tau(s)\, ds$$

$$= \lim_{\Delta s \to 0} \sum_{n=-\infty}^{\infty} \tau(n\,\Delta s)\,\Delta s \qquad (6)$$

From this it follows that as T grows beyond all bounds, the expression in Eq. 4 passes over into the Fourier integral, or

$$f(t) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} f(x)\, e^{2\pi js\,(t-x)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{2\pi jst}\, ds \int_{-\infty}^{\infty} f(x)\, e^{-2\pi jsx}\, dx \qquad (7)$$

This is the general Fourier integral representation. Another form of the Fourier integral may be obtained from Eq. 7 by using Euler's relation on the complex exponentials. We thus obtain the real form of the Fourier integral.

$$f(t) = 2\int_{0}^{\infty} ds \int_{-\infty}^{\infty} f(x) \cos 2\pi s\,(t-x)\, dx \qquad (8)$$

B.2 FOURIER TRANSFORMS

Equation 7 can be written in slightly different form. Let us introduce another variable

$$\omega = 2\pi s$$

In terms of the variable ω, Eq. 7 is transformed to

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega \int_{-\infty}^{\infty} f(x)\, e^{-j\omega x}\, dx \qquad (1)$$

If we write

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-j\omega x}\, dx \qquad (2)$$

Then Eq. 1 can be written as

$$f(t) = \int_{-\infty}^{\infty} g(\omega)\, e^{j\omega t}\, d\omega \qquad (3)$$

The relations in Eqs. 2 and 3 are known as Fourier transforms. The expression $g(\omega)$ in Eq. 2 is usually called the Fourier transform of the function $f(t)$.

Appendix C

The j Factor

C.1 DEFINITION OF j FACTOR

j is used in all electrical circuits to denote imaginary numbers. Alternate symbol for j is $\sqrt{-1}$, and is known as j factor or j operator.

Thus

$$\sqrt{-1} = \sqrt{(-1)\,(1)} = j(1)$$

$$\sqrt{-2} = \sqrt{(-1)\,2} = j\sqrt{2}$$

$$\sqrt{-4} = \sqrt{(-1)\,4} = j2$$

$$\sqrt{-5} = \sqrt{(-1)\,5} = j\sqrt{5}$$

Since j is defined as $\sqrt{-1}$, it follows that $(j)\,(j) = j^2 = \left(\sqrt{-1}\right)\left(\sqrt{-1}\right) = -1$

$\therefore$ $(j3)\,(j3) = j^2 3^2$

Since $j^2 = -1$

$(j3)\,(j3) = -9$

(i.e.) the square root of -9 is $j3$

Therefore $j3$ is a square root of -9

The use of j factor provides a solution to an equation of the form $x^2 = -4$

Thus $x = \sqrt{-4} = \sqrt{(-1)4}$

$x = \left(\sqrt{-1}\right)2$

With $j = \sqrt{-1}$, $x = j2$

The real number 9 when multiplied three times by j becomes $-j9$.

$$(j)\,(j)\,(j) = (j)^2\, j = (-1)j = -j$$

Finally when real number 10 is multiplied four times by j, it becomes 10

$$j = +j$$

$$j^2 = (j)\,(j) = -1$$

$$j^3 = (j^2)\,(j) = (-1)j = -j$$

$$j^4 = (j^2)\,(j)^2 = (-1)\,(-1) = +1$$

Example C.1 Express the following imaginary numbers using the *j* factor

(a) $\sqrt{-13}$ (b) $\sqrt{-9}$ (c) $\sqrt{-29}$ (d) $\sqrt{-49}$

Solution

(a) $\sqrt{-13} = \sqrt{(-1)\,(13)} = j\sqrt{13}$

(b) $\sqrt{-9} = \sqrt{(-1)9} = j3$

(c) $\sqrt{-29} = \sqrt{(-1)29} = j\sqrt{29}$

(d) $\sqrt{-49} = \sqrt{(-1)\,(49)} = j7$

C.2 RECTANGULAR AND POLAR FORMS

A complex number $(a + jb)$ can be represented by a point whose coordinates are (a, b). Thus, the complex number $3 + j4$ is located on the complex plane at a point having rectangular coordinates (3, 4).

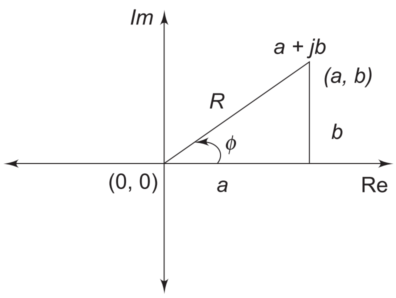


Fig. C.1

This method of representing complex numbers is known as the rectangular form. In ac analysis, impedances, currents and voltages are commonly represented by complex numbers that may be either in the rectangular form or in the polar form. In Fig. C.1 the complex number in the polar form is represented. Here R is the magnitude of the complex number and ϕ is the angle of the complex number. Thus, the polar form of the complex number is $R \angle\phi$. If the rectangular coordinates (a, b) are known, they can be converted into polar form. Similarly, if the polar coordinates (R, ϕ) are known, they can be converted into rectangular form.

In Fig. C.1, a and b are the horizontal and vertical components of the vector R, respectively. From Fig. C.1, R can be found as $R = \sqrt{a^2 + b^2}$.

Also from Fig. C.1,

$$\sin\phi = \frac{b}{R}$$

$$\cos \phi = \frac{a}{R}$$

$$\tan \phi = \frac{b}{a}$$

$$\phi = \tan^{-1} \frac{b}{a}$$

$$R = \sqrt{a^2 + b^2}$$

Example C.2 Express 10 $\angle 53.1°$ in rectangular form.

Solution

$$a + jb = R(\cos \phi + j \sin \phi)$$

$$R = 10 \; \angle\phi = 53.1°$$

$$a + jb = R \cos \phi + jR \sin \phi$$

$$R \cos \phi = 10 \cos 53.1° = 6$$

$$R \sin \phi = 10 \sin 53.1° = 8$$

$$a + jb = 6 + j8$$

Example C.3 Express $3 + j4$ in polar form

Solution

$$R \cos \phi = 3 \tag{1}$$

$$R \sin \phi = 4 \tag{2}$$

Squaring and adding the above equations, we get

$$R^2 = 3^2 + 4^2$$

$$R = \sqrt{3^2 + 4^2} = 5$$

From (1) and (2), $\tan \phi = 4/3$

$$\phi = \tan^{-1} \frac{4}{3} = 53.13°$$

Hence the polar form is 5 $\angle 53.13°$

C.3 OPERATIONS WITH COMPLEX NUMBERS

The basic operations such as addition, subtraction, multiplication and division can be performed using complex numbers.

Addition It is very easy to add two complex numbers in the rectangular form. The real parts of the two complex numbers are added and the imaginary parts of the two complex numbers are added. For example,

$$(3 + j4) + (4 + j5) = (3 + 4) + j(4 + 5)$$
$$= 7 + j9$$

Subtraction Subtraction can also be performed by using the rectangular form. To subtract, the sign of the subtrahand is changed and the components are added. For example, subtract $5 + j3$ from $10 + j6$:

$$10 + j6 - 5 - j3 = 5 + j3$$

Multiplication To multiply two complex numbers, it is easy to operate in polar form. Here we multiply the magnitudes of the two numbers and add the angles algebraically. For example, when we multiply $3 \angle 30°$ with $4 \angle 20°$, it becomes $(3)\ (4) \angle 30° + 20° = 12 \angle 50°$.

Division To divide two complex numbers, it is easy to operate in polar form. Here we divide the magnitudes of the two numbers and subtract the angles. For example, the division of

$$9 \angle 50° \text{ by } 3 \angle 15° = \frac{9 \angle 50°}{3 \angle 15°} = 3 \angle 50° - 15° = 3 \angle 35°$$

Appendix D

Answers

ANSWERS TO OBJECTIVE-TYPE QUESTIONS

Chapter 1

1. (a)	2. (c)	3. (d)	4. (b)	5. (a)
6. (c)	7. (b)	8. (d)	9. (a)	10. (c)
11. (a)	12. (b)	13. (a)	14. (c)	15. (a)
16. (a)	17. (d)	18. (a)	19. (d)	20. (c)
21. (b)	22. (a)	23. (b)	24. (c)	25. (c)
26. (b)	27. (a)	28. (a)	29. (b)	30. (a)
31. (d)	32. (a)			

Chapter 2

1. (b)	2. (a)	3. (a)	4. (b)	5. (b)
6. (a)	7. (c)	8. (a)	9. (b)	10. (c)
11. (c)	12. (b)	13. (c)	14. (a)	15. (a), (d)

Chapter 3

1. (c)	2. (b)	3. (a)	4. (c)	5. (c)
6. (a)	7. (d)	8. (c)	9. (c)	10. (a)
11. (c)	12. (c)	13. (c)	14. (a)	15. (b)
16. (a)	17. (d)			

Chapter 4

1. (a)	2. (c)	3. (a)	4. (b)	5. (c)

6. (a)	7. (d)	8. (b)	9. (b)	10. (c)
11. (c)	12. (a)	13. (a)	14. (b)	15. (c)
16. (d)	17. (d)	18. (c)	19. (c)	20. (b)
21. (c)	22. (c)			

Chapter 5

1. (a)	2. (b)	3. (d)	4. (c)	5. (a)
6. (c)	7. (b)	8. (d)	9. (a)	10. (b)
11. (a)	12. (c)	13. (a)	14. (b)	15. (a)
16. (a)	17. (c)	18. (c)	19. (c)	20. (d)
21. (c)				

Chapter 6

1. (b)	2. (a)	3. (a)	4. (b)	5. (c)
6. (b)	7. (a)	8. (c)	9. (a)	10. (b)
11. (c)	12. (d)	13. (b)	14. (c)	

Chapter 7

1. (c)	2. (a)	3. (b)	4. (d)	5. (b)
6. (c)	7. (c)	8. (a)	9. (b)	10. (d)
11. (d)	12. (c)	13. (a)		

Chapter 8

1. (d)	2. (a)	3. (c)	4. (a)	5. (b)
6. (b)	7. (a)	8. (b)	9. (a)	10. (d)
11. (a)	12. (c)			

Chapter 9

1. (c)	2. (b)	3. (a)	4. (b)	5. (a)
6. (a)	7. (c)	8. (b)	9. (c)	10. (a)
11. (b)				

Chapter 10

1. (a)	2. (b)	3. (b)	4. (a)	5. (c)
6. (d)	7. (b)	8. (c)	9. (b)	10. (d)

Chapter 11

1. (d)	2. (d)	3. (a)	4. (c)	5. (b)
6. (b)	7. (c)	8. (c)	9. (a)	10. (d)
11. (a)	12. (b)	13. (a)	14. (a)	

Chapter 12

1. (a)	2. (a)	3. (b)	4. (a)	5. (d)
6. (a)	7. (b)	8. (b)	9. (a)	10. (a)
11. (c)	12. (d)	13. (b)	14. (b)	15. (a)

Chapter 13

1. (c)	2. (a)	3. (b)	4. (a)	5. (c)
6. (a)	7. (b)	8. (d)	9. (a)	10. (a)
11. (c)				

Chapter 14

1. (a)	2. (b)	3. (c)	4. (d)	5. (a)
6. (b)	7. (c)	8. (b)	9. (a)	10. (c)

Chapter 15

1. (b)	2. (c)	3. (a)	4. (c)	5. (b)
6. (c)	7. (b)	8. (a)	9. (c)	10. (c)
11. (b)	12. (c)	13. (a)	14. (b)	15. (c)
16. (a)	17. (a)	18. (c)		

Chapter 16

1. (a)	2. (b)	3. (c)	4. (d)	5. (a)
6. (b)	7. (c)	8. (b)	9. (a)	10. (c)

Chapter 17

1. (c)	2. (c)	3. (b)	4. (a)	5. (a)
6. (a)	7. (b)	8. (b)	9. (a)	10. (b)
11. (b)	12. (c)			

ANSWERS TO SELECTED PRACTICE PROBLEMS

Chapter 1

1.1	(a) 75 A	(b) 20 A	(c) 2.5 A; 2 S
1.3	3.33 V		
1.5	1.5 mF		
1.7	10 V; 30 V		
1.9	0.3×10^{-2} J		
1.11	25 V; 5 V		
1.13	$V_1 = V_2 = V_3 = 100$ V		

1.15 0.682 A; 4.092 A

1.17 150 Ω

1.19 0.7 A; 67.3 V

1.21 – 4 V, 12 V, 192 W

1.23 $P_{0.2}$ = – 148.8 W, P_{20} = – 1090.9 W, P_4 = 743.8 W, P_6 = 495.9 V

Chapter 2

2.1 2580 W; – 32 V

2.3 – 60.9 V; 195.7 W

2.5 $I_2 = I_4 = 6.25$ A; $I_3 = 0$; $I_1 = I_5 = 1.25$ A; $I = 7.5$ A

2.9 1.2 A; 4.2 A; 2 A; 3.2 A

2.9 2.65 V

2.11 $1.25\ V_1 - 0.72\ V_2 = -12.5$

$-0.75\ V_1 + 1.75\ V_2 = -2.5 + 4\ V_3$

36.8 W

2.13 18.5 V

Chapter 3

3.1 1.182 Ω

3.3 0.82 A

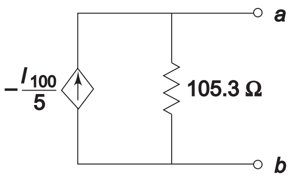
3.5 $I_1 = 4.6$ a; $I_2 = 2.6$ A; $I_3 = 2$ A

3.7 32 V, 4 W, 8 A

3.9 12 Ω, 0.75 W, 6 V, 0.5 A

3.11 4 A

3.13 0.5 A

3.15 

Chapter 4

4.1 5 Hz; 20 Hz; 2 KHz; 100 KHz

4.3 15.4 V; 26.57 V; 16.22 V; – 16.22 V

4.5 12.99 V; 12.99 V; 14.49 V; – 7.5 V; – 7.5 V

4.7 7.07 mA; 6.37 mA; 10 mA; 20 mA

4.9 V_{RL} is 300 V peak to peak sine wave riding on a 200 V dc level.

$I_{max} = 3.5$ A, $V_{av} = 200$ V

4.11 2.82 cos 100 πt; 20 A; – 20 A; 1/300 sec
4.13 106.06
4.15 27.57
4.17 55.25°

Chapter 5

5.1 157.4 $\angle$ – 17.6°; 17.6° lead, 0.635 A
5.3 55.85 $\angle$ – 57.5°; 57.5°
5.5 0.074 A; 41.9°; increases by 19°
5.7 944.2 Ω; 0.053 A; 3.67°; 16.3 V; 30.7 V
5.9 (0.3 – j3.15) A; (0.48 + j3.1) A; (0.044 – j0.66) A
V_3 = 9.5 V; V_5 = 15.7 V; V_{10} = 6.61 V; $V_{0.1\text{H}}$ = 99.35 V;
$V_{100\,\mu\text{F}}$ = 99.8 V; $V_{0.5\text{H}}$ = 103.93 V $V_{500\,\mu\text{F}}$ = 4.21 V

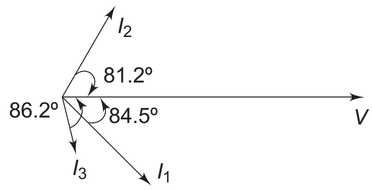


Fig. 5.9

5.11 1.44 A; 7.05°; $V_{100\,\mu\text{F}}$ = 22.9 V; $V_{10\,\Omega}$ = 14.4 V
$V_{30\,\Omega}$ = 38.93 V; $V_{0.1\text{H}}$ = 38.93 V

5.13 $V_T = \sqrt{R^2 + (\omega L)^2}\; I_m \sin\left(\omega t + \tan^{-1}\dfrac{\omega L}{R}\right)$

$\theta = \tan^{-1}\dfrac{\omega L}{R}$, where ω = 200 rad/sec

5.15 L = 6.67 mH; C = 3.33 μF
5.17 i_T = 1.74 sin (100t + 67.4°) A
θ = 67.4°; Z = 115 Ω

Chapter 6

6.1 0.97
6.3 3.12 Ω, 9.93 H
6.5 (0.28 + j0.78) Ω; 282.7 VA
6.7 486.5; 0.27
6.9 0.891; 1587.7 W; 806.2 VAR; 1781.9 VA
6.11 1136.36 VA; 529.6 VAR; 0.88

6.13 15.396 kW; 3944 VAR; 15.87 KVA; 0.97

6.15 0.0812 mW

6.17 – 0.114 W

Chapter 7

7.1 $3.39 \angle -97.3°$

7.3 $(3.82 - j1.03)\ \Omega$; 15.11 W

7.5 4.37 A

7.7 2.69 W

7.9 $(20 - j5)$ V in series with $(2 - j)\ \Omega$
$(8.99 + j2)$ A in parallel with $(2 - j)\ \Omega$

7.11 $I_{10} = 7.34 \angle -21.84°$; $I_5 = 1.65 \angle 33.69°$; $I_3 = 8.39 \angle -12.5°$

7.13 $(1.1 + j4.7)$ V in series with $(0.93 + j0.75)\ \Omega$
$(3.2 + j2.4)$ A in parallel with $(0.93 + j0.75)\ \Omega$

7.15 1874.9 W

7.17 $(-0.18 - j0.6)V_1$ volts in series with $(100 - j30)\ \Omega$

7.19 $0.894 \angle -63.4°$ in series with $(0.4 + j1.25)\ \Omega$

Chapter 8

8.1 50.3 Hz; 63.2 V; 3 (approx.)

8.3 $2.07\ \Omega$

8.5 875.35 Hz; 914.42 Hz; 836.28 Hz; 0.2H; 0.165 μF

8.7 1.77

8.9 $Q = 1$; $R = 60\ \Omega$; $C = 50\ \mu$F

Chapter 9

9.1 $v_1 = L_1 \dfrac{di_1}{dt} + M \dfrac{di_2}{dt}$; $v_2 = L_2 \dfrac{di_2}{dt} + M \dfrac{di_1}{dt}$

9.3 $\begin{bmatrix} 2 & 5 & -2 \\ 5 & 4 & 0 \\ -2 & 0 & 6 \end{bmatrix}$

9.5 $v_1 = 181.44 \cos(40t - 30°)$
$v_2 = 202.88 \cos(40t - 30°)$

9.7 $L = 13$ H

9.9 $L = \frac{2}{3}$ M

9.11 $1 \angle -90$ V

Chapter 11

11.1 $i(t) = (2 + 10t)e^{-5t}$

11.3 $i_1(t) = 9.99 - 8.49\ e^{-5 \times 10^4\, t}$; $i_2(t) = 5e^{-5 \times 10^4\, t}$

11.5 $i(t) = 101.2 + 30.9\ e^{-0.1t} - 52.11\ e^{-4.94t}$

11.7 $i(t) = 5.06\ [e^{-0.033t} - e^{-4.966t}]$

11.9 $i(t) = 3.8 + e^{-0.05t} + 0.12\ e^{-0.31t}$

11.11 $i(t) = -0.35\ e^{-500t}$

11.13 $5e^{-5.71t}$

11.15 $V_1(t) = -4e^{-0.4t} + 4e^{-4999.8t}$

Chapter 14

14.1 $i(t) = e^{-2.5(t-5)}$ [cos h 2.46 $(t-5)$ – 1.01 sin h 2.46 $(t-5)$]

14.2 $1 - \frac{1}{3} e^{-t/3} + \frac{2}{15} e^{\overline{-(t-3)/5}^{\,-1}}$

14.5 $Z_{11}(s) = \frac{7s^2 + 7s + 5}{s^2 + s + 1}$; $Z_{12}(s) = \frac{2s}{s^2 + s + 1}$;

$G_{21}(s) = \frac{2s}{7s^2 + 7s + 5}$

14.7 $G_{12}(s) = \frac{s^2 + 1}{2s^2 + 1}$; $Z_{12}(s) = \frac{s^2 + 1}{s(3s^2 + 1)}$

14.9 $Z_{12}(s) = \frac{5s}{s^2 + 1}$; $Y_{12}(s) = \frac{s^2 + 1}{5s}$; $G_{12}(s) = G_{12}(s) = 1$

14.11 $i(t) = 4.5e^{-3t} - 1.5e^{-t}$

14.13 Unstable

14.15 (a) 2, 0, 1 (b) 0, 2, 4 (c) 2, 2, 2

Chapter 15

15.1 $Z_{11} = \frac{Y_B + Y_C}{\Delta Y}$; $Z_{12} = Z_{21} = \frac{Y_C}{\Delta Y}$; $Z_{22} = \frac{Y_A + Y_C}{\Delta Y}$

$\Delta_Y = Y_A Y_B + Y_B Y_C + Y_C Y_A$

15.3 $A' = 3; B' = 2; C' = 4; D' = 3$

15.5 $h_{11} = \frac{4}{3}; h_{21} = \frac{-2}{3}; h_{22} = \frac{1}{6}; h_{12} = \frac{2}{3}$

$g_{11} = \frac{1}{4}; g_{12} = -1; g_{21} = 1; g_{22} = 2$

15.7 $Y_{11} = (0.5 - j0.2)10^{-3}; Y_{12} = Y_{21} = (j0.2 \times 10^{-3})$
$Y_{22} = j(0.02 \times 10^{-3})$

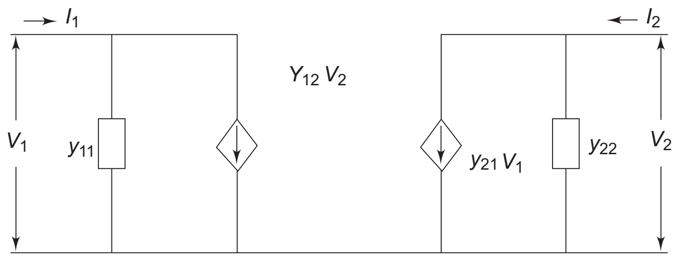


Fig. 15.7

15.9 $Z_i = 1.5\ \text{k}\Omega; Z_0 = 0.033 \times 10^{-3}\ \Omega$

15.15 $\begin{bmatrix} 5.71 & -4.29 \\ 2.14 & 2.14 \end{bmatrix}$

15.17 $\begin{bmatrix} 0.857 \angle -31°\text{k}\Omega & 0.17 \angle 59° \\ 8.58 \angle -32.1° & 1.89 \angle 61.1°\text{m} \end{bmatrix}$

Chapter 16

16.1 $i(t) = e^{-2.5(t-5)}$ [cos h 2.46 $(t-5)$ – 1.01 sin h 2.46 $(t-5)$]

16.2 $1 - \frac{1}{3} e^{-t/3} + \frac{2}{15} e^{-(t-3)/5}$

16.5 $Z_{11}(s) = \frac{7s^2 + 7s + 5}{s^2 + s + 1}; Z_{12}(s) = \frac{2s}{s^2 + s + 1};$

$G_{21}(s) = \frac{2s}{7s^2 + 7s + 5}$

16.7 $G_{12}(s) = \frac{s^2 + 1}{2s^2 + 1}; Z_{12}(s) = \frac{s^2 + 1}{s\left(\frac{3s^2}{2} + 1\right)}$

16.9 $Z_{12}(s) = \frac{5s}{s^2 + 1}; Y_{12}(s) = \frac{s^2 + 1}{5s}; G_{12}(s) = G_{12}(s) = 1$

16.11 $i(t) = 4.5e^{-3t} - 1.5e^{-t}$

16.13 Unstable

16.15 (a) 2, 0, 1 (b) 0, 2, 4 (c) 2, 2, 2

Chapter 17

17.1 $L = 0.127$ H; $C = 0.35\ \mu$F

17.3 0.09 μF; 0.06 H

17.5 For T section; series arm component 6.66×10^{-9} F; shunt arm 0.015 mH, 1.3×10^{-9} F.

For π section series arm 6.19 mH: 3.33×10^{-9} F; shunt arm 0.031 H.

17.7 1.779 kHz; 89.44 Ω

17.9 For the T-section, series arm components are 63.5 mH; 0.08 μF and shunt arm components are 9.9 mH; 0.5 μF.

For the π-section; series arm components are 0.04 μF; 127 mH and shunt arm components are 19.8 mH, 0.25 μF.

17.11 4.6 Hz; 545.5 Ω; 38°.

Appendix **E**

Model Question Papers

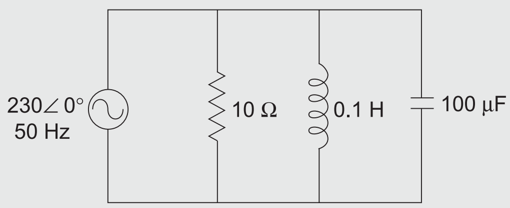
PAPER 1

1. (a) Obtain the response of R-L-C series circuit for impulse excitations.
   (b) Define reluctance of a magnetic circuit and derive an expression for reluctance.

Solution Refer section 10.11 in the textbook.

2. In an electrical circuit R, L and C are connected in parallel. $R = 10\ \Omega$, $L = 0.1\text{H}$, $C=100\ \mu\text{F}$. The circuit is energized with a supply at 230 V, 50 Hz. Calculate
   (a) Impedance
   (b) Current taken from supply
   (c) p.f. of the circuit
   (d) Power consumed by the circuit

Solution The circuit is as shown in figure.



The impedance of 3 branches are

$Z_1 = 10\ \Omega$

$Z_2 = j2\ \pi f L = 2 \times 50 \times 0.1 = j31.41\ \Omega$

$$Z_3 = \frac{-j}{2\pi\, fc} = \frac{-j}{2\times 50\times 100\mu} = -j31.84\,\Omega$$

(a) Impedance of circuit $Z = \left[\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right]^{-1}$

$$= \left[\frac{1}{10} + \frac{1}{j31.41} + \frac{1}{-j31.84}\right]^{-1}$$

$$\approx 10\Omega$$

(b) Current taken from supply $I = \frac{V}{Z} = \frac{230\angle 0^\circ}{10} = 23A$. i.e. $23\angle 0^\circ$ A

(c) p.f. of the circuit = $\cos\theta = 1$

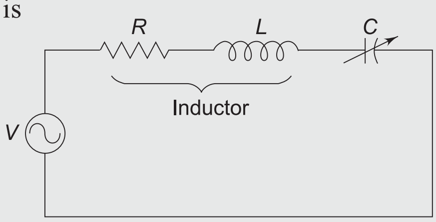
(d) Power consumed by the circuit
Real power consumed = $I^2R = 23^2 \times 10 = 5.3$ kW
Reactive power consumed = 0 KVAR

3. A constant voltage at a frequency of 1 MHz is applied to an inductor in series with a variable capacitor when the capacitor is set to 500 PF, the current has the max. value, while it is reduced to one half when capacitance (i) 600 PF, find (i) resistance (ii) inductance (iii) Q factor of inductor.

Solution Given f = 1 MHz
Let the max. current be $I_{max.}$
Given at 1 MHz, for C = 500 Pf

$$I = I_{max}$$

$\therefore$ Imaginary part of impedance is zero, i.e. $X_L = X_C$



$$2\pi fL = \frac{1}{2\pi fc}$$

$$6.283 \times 10^6 \times L = 318.31$$

$$L = 50.66\ \mu\text{H}$$

Now also given $I = \frac{I_{max}}{2}$ at C = 600 PF

$$I = \frac{I_{max}}{2} = \frac{V}{R + j\left(6.283\times 10^6 L - 265.25\right)} \tag{1}$$

$$\left(\because X_C = \frac{1}{2\pi fc} = \frac{1}{2\pi\times 10^6 \times 600\times 10^{-12}} = 265.25\right)$$

$$\text{and } I_{max} = \frac{V}{R} \tag{2}$$

Dividing Equation (2) by Equation (1)

$$Z = \frac{R + j\left(6.283 \times 10^6 L - 265.25\right)}{R}$$

$\Rightarrow \quad 2R = R + j\,(6.283 \times 10^6\, L - 265.25)$

$R = j(318.31 - 265.25)$

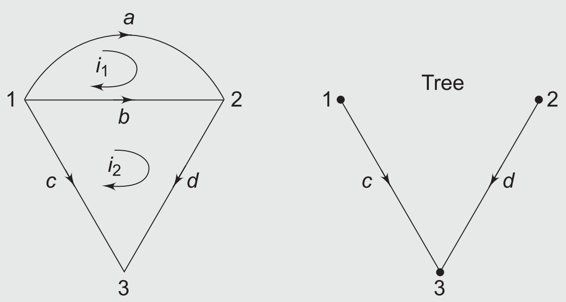
$R = 53.06\ \Omega$

$\therefore$ (i) $R = 53.06\ \Omega$

(ii) $L = 50.66\ \mu\text{H}$

(iii) $G = \dfrac{\omega L}{R} = 5.999 \approx 6$

4. For the given graph and tree shown in the figure, write the tie-set matrix and obtain the relation between branch currents and link currents.



Solution Number of link branches = $b - (n - 1)$

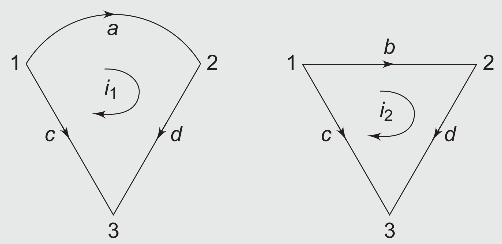
Where b is number of branches and n is number of nodes

$\therefore$ Link branches = $4 - (3 - 1) = 2$

The link branches are a and b.

Let the branch currents are i_a, i_b, i_c and i_d

The two link currents are i_1 and i_2 as shown in the figure.



There are two fundamental loops corresponding to the link branches a and b. If V_a and V_b are branch voltages, the KVL equations for the two f-loops can be written as

$$V_a + V_d - V_c = 0$$

$$V_b + V_d - V_c = 0$$

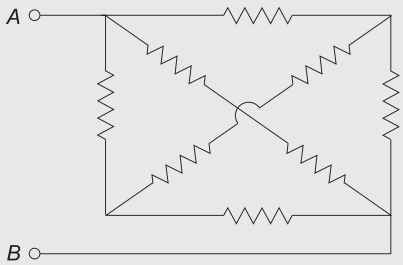
The above equation can be written in matrix form as

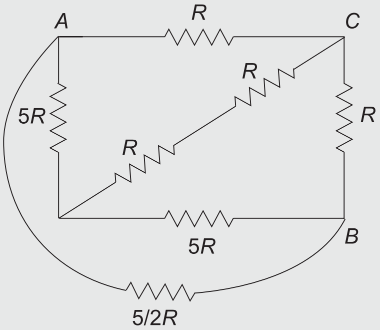
Loop Currents $\downarrow$ branches $\rightarrow$

$$\begin{array}{c} \\ i_1 \\ i_2 \end{array} \quad \begin{array}{cccc} a & b & c & d \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & +1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \end{bmatrix} = 0$$

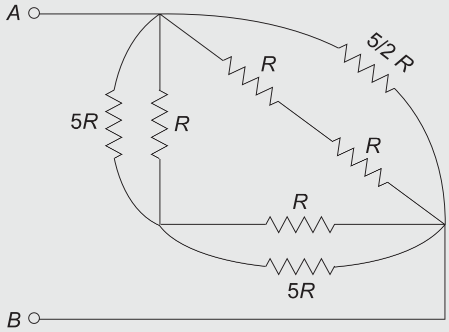
5. Find the equivalent resistance between AB in the circuit shown in the figure. All resistances are equal to R.

Solution Converting the Y point C into Δ

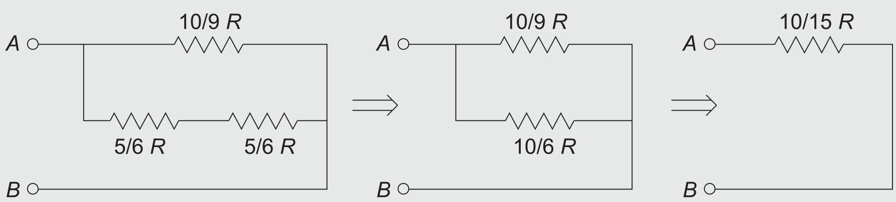




$\rightarrow$ The equivalent circuit now is

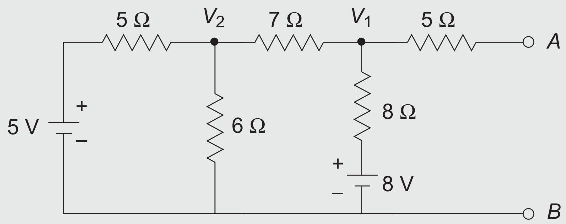


The equivalent circuit for this is as shown below.

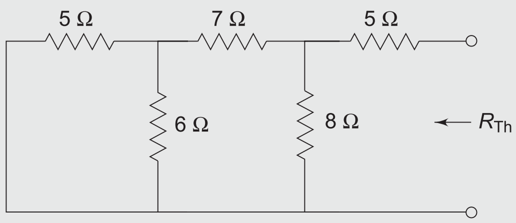


$\therefore$ Resistance between AB is $R = \dfrac{10}{15} R = 0.666\,R$

6. Find the Thevenins equivalent for the circuit in figure



Solution The Thevenins equivalent resistance is calculated assuming all voltage sources shorted and as seen from *AB*, the circuit will be as shown below:



$$R_{Th} = [\{(5//6) - 7\}//8] + 5$$

$$\left[\left\{\frac{30}{11}+7\right\}//8\right]+5=\left[\frac{\frac{107}{11}\times 8}{\frac{107}{11}+8}\right]+5 = 4.389+5 = 9.389\ \Omega$$

Let us assure voltages at nodes (1) and (2) be V_1 and V_2.
Now writing node equations.

$$\frac{V_1-8}{8}+\frac{V_1-V_2}{7}=0$$

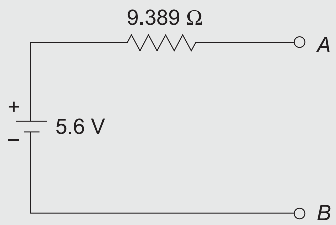
$$7V_1 - 56 + 8V_1 - 8V_2 = 0 \Rightarrow 15\ V_1 - 8V_2 = 56 \tag{1}$$

$$\frac{V_2}{6}+\frac{V_2-V_1}{7}+\frac{V_2-5}{5}=0 \quad \Rightarrow -30V_1+107V_2=210 \tag{2}$$

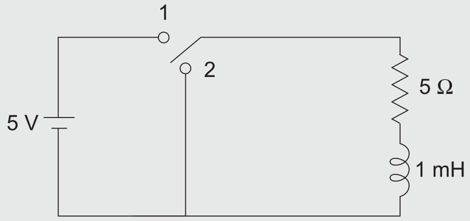
on solving equations (1) and (2) we get

$V_1 = 5.6$ V $\quad\Rightarrow\quad V_{OC} = 5.6$

$\therefore$ Thevenins equivalent circuit is



7. The switch in the circuit shown in figure is in position (1) for two time constants and then charged to position (2) find transient response.



Solution When the switch is in position (1)
Convert equation in laplace transform is given as

$$I(S) = \frac{V(S)}{R + LS} = \frac{5/S}{5 + 0.001S} = \frac{5000}{S(5000 + 3)}$$

Assuming initial conditions be zero.

$$I(S) = \frac{1}{S} - \frac{1}{S + 5000}$$

Taking inverse Laplace transform

$$i(t) = 1 - e^{-5000t}$$

the switch is cosed for two time constants
$\therefore$ $i(t)$ after two time constants i

$$i = 1 - e^{-2} = 0.864 \text{ A}$$

Now when switch is moved to position (2) the mesh equation is given by

$$L\frac{di}{dt} + Ri(t) = 0$$

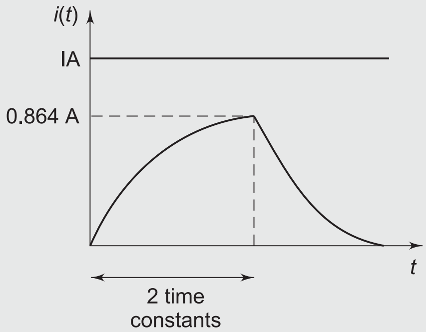
$$\Rightarrow i(t) = C_1 e^{-5000t}$$

initially $i(o) = 0.864$ A

$$C_1 = 0.864 \text{ A}$$

$$\therefore \quad i(t) = 0.864\, e^{-5000t}$$

The response can be plotted as



8. Derive phase and line voltage, current relations in a balanced star and delta connected loads.

Solution Refer Sections 9.7.1, 9.7.2, 9.7.3, and 9.8.1, 9.8.2, and 9.8.3.

PAPER 2

1. (a) Discuss Kirchhoff's Laws.

Solution Refer Sections 1.9 and 1.12.

(b) Derive the expression for self, mutual inductance and coefficient of coupling.

Solution Refer Sections 10.3. and 10.5.

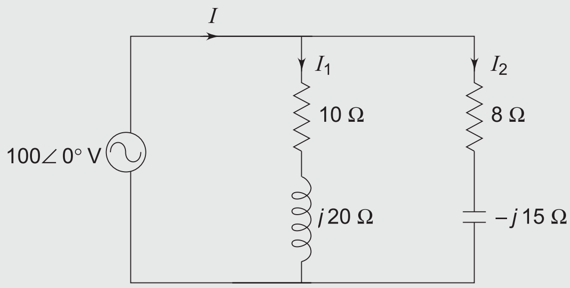
(c) Explain source transformation with example.

Solution Refer Section 2.15.

2. (a) What is the use of operator j?

Solution Refer Appendix C.

(b) For the circuit shown in figure, find the current I drawn from the source.



The impedance as seen by the source is

$$Z = (10 + j20) \,//\, (8 - j15)$$

$$= \frac{380 + j10}{18 + 5j} = 19.742 - j4.928$$

$$\therefore \text{ Current drawn from source } I = \frac{V}{Z} = \frac{100}{19.742 - j4.928}$$

$$= 4.768 + j1.19$$

$$= 4.914 \underline{|14.01^\circ}$$

or

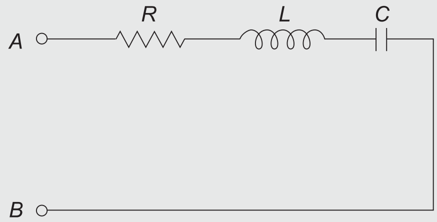
$$I_1 = \frac{100}{10 + 20j} = 2 - 4j$$

$$I_2 = \frac{100}{8 - 15j} = 2.768 + 5.1903j$$

$$I = I_1 + I_2 = 4.768 + j1.19$$

$$= 4.914 \underline{|14.01^\circ}$$

3. (a) A series RLC circuit with $Q = 250$ is resonant at 1.5 MHZ. Find the frequencies at half power points and also bandwidth.



Solution Given $Q = 250$

$$Q = \frac{\omega_o L}{R}$$

$$250 = \frac{2\pi \times f_o \times L}{R} \Rightarrow \frac{R}{L} = \frac{2\pi \times 1.5 \times 10^6}{250} = 37.7 \times 10^3$$

Lower half power frequency $f_1 = f_r - \dfrac{R}{4\pi L}$

$$= 1.5 \times 10^6 - \frac{37.7 \times 10^3}{\Delta\pi}$$
$$= 1.5 \times 10^6 - 3 \times 10^3$$
$$= 1.496 \text{ MHz}$$

Upper half power frequency $f_2 = f_r + \dfrac{R}{4\pi L}$

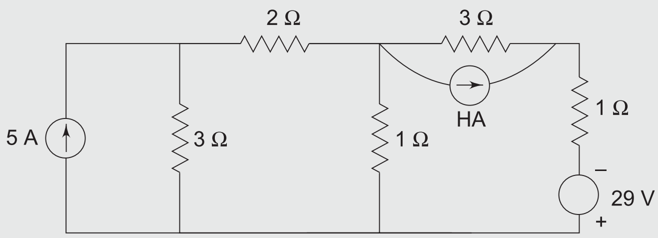
$$= 1.5 \times 10^6 + \frac{37.7 \times 10^3}{4\pi}$$
$$= 1.5\text{M} + 3k = 1.53 \text{ MHz}$$

Bandwidth $= f_2 - f_1 = 1.53$ M – 1.496 M = 6 kHz

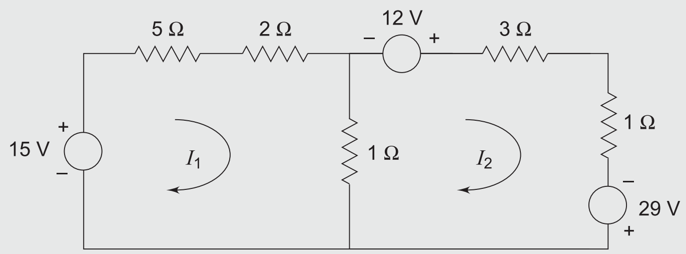
(b) Distinguish between the average value and rms value of an alternating current.

Solution Refer Section 4.4.

4. Write and solve the equation for Mesh Current in the network shown.



Solution By source transformation technique transform 5A and 4A current sources into voltage sources.
5A current source in parallel with 3 Ω can be transformed to 15V in series with 3 Ω and 4A current source in parallel with 3 Ω can be transformed to 12 volts in series with 3 Ω. The equivalent circuit is as shown below:



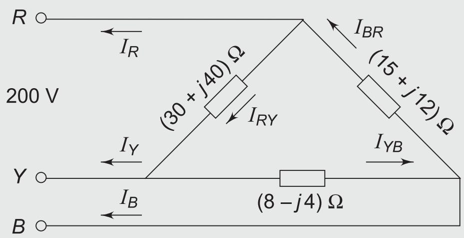
The mesh equations are

$$5I_1 + 1(I_1 - I_2) = 15$$
$$1(I_2 - I_1) + 4I_2 = 41$$
$$\Rightarrow \qquad -I_1 + 5I_2 = 41 \qquad (1)$$
$$6I_1 - I_2 = 15 \qquad (2)$$

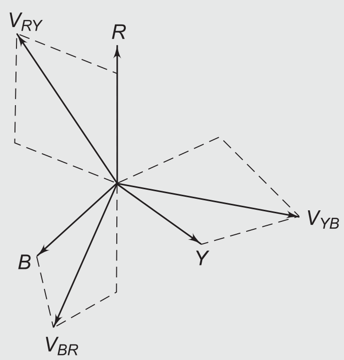
on solving equations (1) and (2) we get

$$I_1 = 4 \text{ Amps}$$
$$I_2 = 9 \text{ Amps}$$

5. Determine the line currents for the unbalanced delta connected load of the figure given. Assume phase sequence *RYB*.



Solution The phaser diagram will be as shown:



$$V_{RY} = 200\angle 0°$$

$$V_{YB} = 200\angle -120°$$

$$V_{BR} = 200\angle -240°$$

$$I_{RY} = \frac{200\angle 0°}{30 + j40} \angle -53.13°$$

$$I_{YB} = \frac{200\angle -120^\circ}{8 + j4} = 22.36\angle -93.43^\circ$$

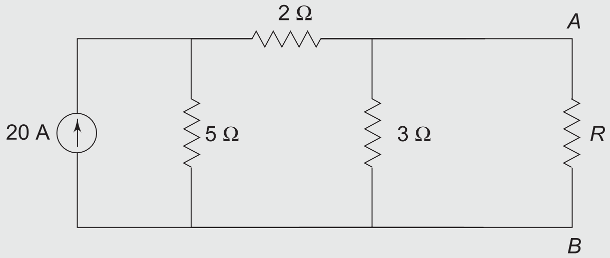
$$I_{BR} = \frac{200\angle -240^\circ}{15 + j12} = 10.41\angle -278.65^\circ$$

The line currents are $I_R = I_{BR} - I_{RY}$

$I_Y = I_{RY} - I_{YB}$

$I_B = I_{YB} - I_{BR}$

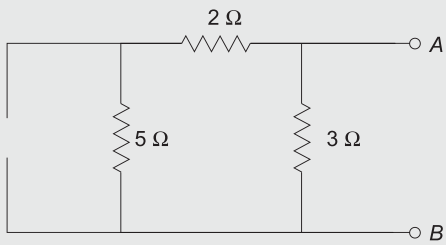
6. The circuit shown in the figure below has resistance R which absorbs maximum power. Compute the value of R and maximum power.



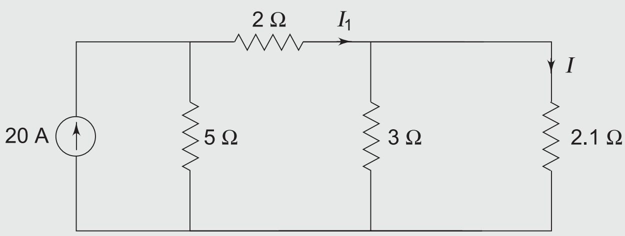
Solution According to maximum power transfer theorem, maximum power can be transferred when load resistance is equal to the interval resistance of the source which can be calculated as the resistance seen from AB with source open.

$\therefore R_{th} = 3//7$

$= \frac{21}{10} = 2.1\ \Omega$



Now the circuit can be drawn as



According to current dividing rule

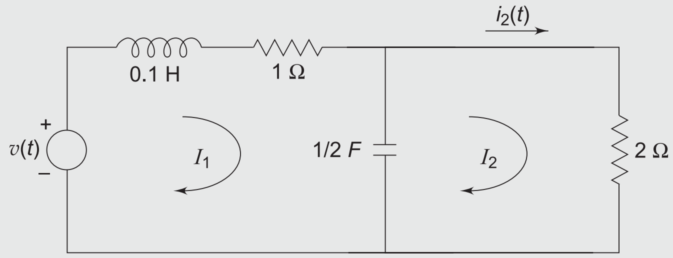
$$I_1 = \frac{20 \times 5}{(5+3.235)} = 12.14 \text{ A}$$

$$I_2 = \frac{I_1 \times 3}{5.1} = \frac{12.14 \times 3}{5} = 7.14 \text{ A}$$

So the maximum power that can be delivered to resistor R is

$$I^2R = (7.14)^2 \times 2.1 = 107 \text{ watts.}$$

7. In the figure shown below $v(t)$ = 10 V, find $i_2(t)$. Assume all initial conditions to be zero. Use Laplace transform technique.



Solution Writing mesh equation

$$\frac{10}{S} = 0.1S\, I_1(s) + I_1(s) + \frac{2}{s}(I_1(s) - I_2(s))$$

$$0 = \frac{2}{S}(I_2(s) - I_1(s)) + 2\, I_2(s) = 0$$

$$\left(\frac{2}{s} + 0.1s + 1\right) I_1(s) - \frac{2}{s} I_2(s) = \frac{10}{s} \tag{1}$$

$$-\frac{2}{s} I_1(s) + \left(\frac{2}{s} + 2\right) I_2(s) = 0 \tag{2}$$

on solving equations (1) and (2) we get

$$I_2(s) = \frac{100}{s(s^2 + 11s + 60)}$$

$$\frac{1}{S(S^2 + 11S + 60)} = \frac{AS + B}{S^2 + 11S + 60} + \frac{C}{S}$$

$$A = \frac{-1}{60};\ \ B = \frac{-11}{60};\ \ C = \frac{1}{60}$$

$$\therefore L^{-1}\left[\frac{1}{S\left(S^2 + 11S + 60\right)}\right] = L^{-1}\left[\frac{1}{60}\frac{1}{S} - \frac{1}{60}\frac{S}{S^2 + 11S + 60} - \frac{11}{60}\cdot\frac{1}{S^2 + 11S + 60}\right]$$

$$= \frac{1}{60} L^{-1}\left[\frac{1}{S} - \frac{\left(S + \frac{11}{2}\right)}{\left(S + \frac{11}{2}\right)^2 + \left(\frac{\sqrt{199}}{2}\right)^2} + \frac{319}{\left(S + \frac{11}{2}\right)^2 + \left(\frac{\sqrt{119}}{2}\right)^2}\right]$$

$$\therefore i_2(t) = \frac{100}{60}\left[1 - e^{\frac{11}{2}t}\cos\frac{\sqrt{119}}{2}t - 319 \times \frac{2}{\sqrt{119}}e^{\frac{11}{2}t}.\sin\frac{\sqrt{119}}{2}t\right]$$

$$\therefore i_2(t) = \left[1.667 - 1.667e^{-5.5t}\cos 5.45t - 97.47e^{5.5t}\sin 5.45t\right]$$

8. (a) In a two-port bilateral network show that $AD - BC = 1$.

Solution Refer Section 15.8.2.

(b) Derive an expression for DC response in an RC circuit.

Solution Refer Section 12.3.

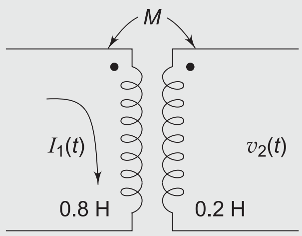
PAPER 3

1. (a) State the voltage current relationships for (i) resistance (ii) inductance and (iii) capacitance.

Solution Refer Sections 1.5, 1.6 and 1.7.

(b) Two coupled coils with self inductances L_1=0.8H and L_2=0.2H have a coupling coefficient of 0.6 has 500 turns. If the current in coil 1 is $I_1(t) = 10 \sin 200t$; determine the voltage at coil 2 and the maximum flux set up by the coil 1.

Solution



$$M = K\sqrt{L_1 L_2}$$
$$= 0.6\sqrt{0.8 \times 0.2}$$
$$= 240 \text{ mH}$$

The voltage across the coil 2 $v_2(t) = \pm M \dfrac{di_1(t)}{dt}$

$$v_2(t) = \frac{d}{dt}(10 \sin 200t)$$
$$v_2(t) = 2000\,\text{C is } 200t \text{ volts}$$

(c) A torroid is made of steel rod of 2 cm diameter. The mean radius of torroid is 20 cm relative permeability of steel is 2000. Compute the current required to produce 1 m web of flux and 1000 turns in the torroid.

Solution Length of the flux path = $\pi D = \pi \times 20$ = 62.83 cm = 0.6283 m

Area of flux path $= \dfrac{\pi}{4} d^2 = \dfrac{\pi}{4}(2)^2 = 3.141 \text{ cm}^2$

Magnetic field intensity $H = \dfrac{B}{\mu_o \mu_r}$

$$B = \frac{\phi}{\text{Area}} = \frac{10^{-3}}{3.141 \times 10^{-4}} 3.1 \text{web} / \text{m}^2$$

$$H = \frac{3.1}{4\pi \times 10^{-7} \times 2000} = 1233.45 \text{AT} / \text{m}$$

$$mmf = H \times l = 1233.45 \times 0.6283$$
$$= 775 \text{ A.T.}$$

$$\text{Exciting current} = \frac{mmf}{T}$$

$$= \frac{775}{1000} = 0.775 \text{ A}$$

2. (a) If $I_1 = 10\angle 0°$, $I_2 = 20\angle 60°$ and $I_3 = 12\angle -30°$ find $I_1 + I_2 + I_3$.

Solution $\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 = 10\angle 0° + 20\angle 60° + 12\angle -30°$

$$= 30.392 + j11.32$$

$$= 32.432 \angle 20.429$$

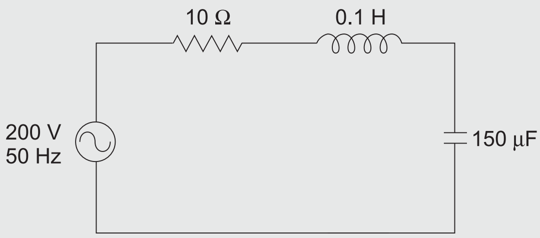
(b) Prove that the form factor for a sinusoidal current wave form is 1.11.

Solution Refer Section 4.4.7.

3. (a) Derive an expression for resonance frequency of a series R-L-C circuit.

Solution Refer Section 8.1.

(a) A coil of resistance $10l$ and an inductance of 0.1 H is connected in series with a capacitor of capacitance 150 μF a cross at 200V, 50Hz supply. Calculate (i) Impedance (ii) Current (iii) Power and power factor of the circuit



Solution (i) Total impedance

$$Z = R + j\omega L - \frac{j}{\omega c}$$

$$= 10 + j31.45 - j21.22$$

$$= 10 + j10.194$$

$$= 14.279 \angle 45.55$$

(ii) Current $I = \dfrac{V}{Z}$

$$= \frac{200\angle 0°}{14.279\angle 45.55°}$$

$$= 14\angle -45.55°$$

(iii) Power factor $= \cos(45.55^{o})$

$$= 0.7 \text{ lagging}$$

Real power $= VI\cos\phi$

$= 200 \times 14 \times 0.7$

$= 1.9$ kW

Reactive power $= VI\sin\phi$

$= 200 \times 14 \times \sin(-45.55)$

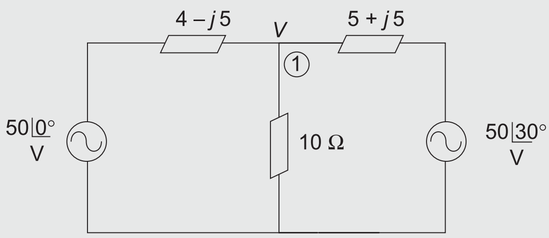
$= -1.998$ KVAR

"–1" Sign indicates that it absorbs the reactive power.

4. (a) Define cut set and tie set

Solution Refer Sections 2.7 and 2.8.

(b) Determine the current in the 10 Ω resistor in the circuit shown in the figure below.



Solution Apply nodal analysis at point (1), we get

$$\frac{V-50\angle 0^\circ}{4-j5}+\frac{V}{10}+\frac{V-50\angle 30^\circ}{5+j5}=0$$

$$V\left[\frac{1}{4-j5}+\frac{1}{10}+\frac{1}{5+j5}\right]=\frac{50\angle 0^\circ}{4-j5}+\frac{50\angle 30^\circ}{5+j5}$$

$V\,[0.297 + j0.0219] = 11.708 + j4.267$

$V[0.298\angle 4.219^\circ]=12.46\angle 20.02^\circ$

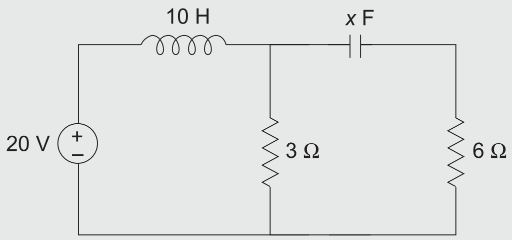
$\Rightarrow V = 41.812\ \angle 15.801^\circ$

Current through the 10 Ω resistor $I_{10} = \dfrac{V}{R}$

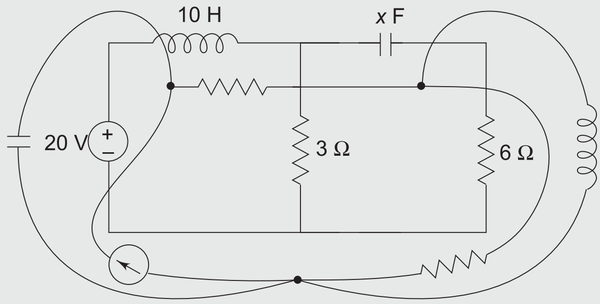
$$= \frac{41.812\ \angle 15.801^\circ}{10}$$

$= 4.1812\ \angle 15.80^\circ$ Amp

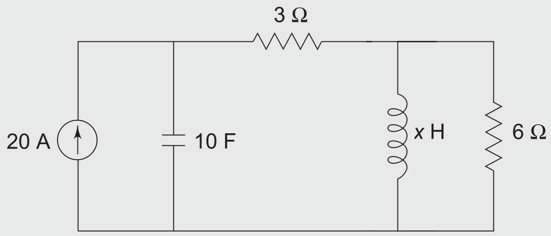
5. (a) Draw the dual network for the given network as in the following figure.



Solution



Dual network

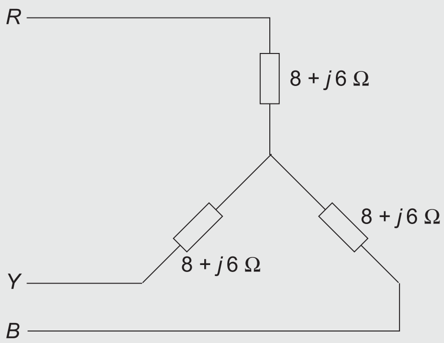


(b) A balanced star connected load of 8 + 6*j* Ω/phase is connected to a 3ϕ 230V, 50Hz supply. Find the line current, power factor, total Active and Reactive powers.

Solution $V_{RY} = 230\angle 0°$

$V_{YB} = 230\angle -120°$

$V_{BR} = 230\angle -240°$



$$I_R = \frac{V_{RY}}{\sqrt{3}\times Z}$$
$$= \frac{230}{\sqrt{3}\times 10\angle 36.86°}$$
$$= 10.623\angle -36.86°$$

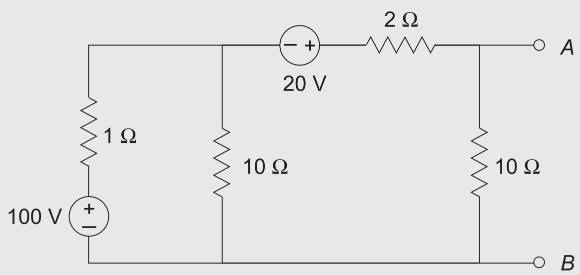
$$I_Y = \frac{230\angle -120°}{\sqrt{3}\times 10\angle 36.86°}$$
$$= 10.623\angle -156.86°$$
$$I_B = \frac{230\angle -240°}{\sqrt{3}\times 10\ \angle 36.86°}$$
$$= 10.623\angle 276.86°$$

P.f. = cos ϕ = cos (36.86°) = 0.8 (lagging)

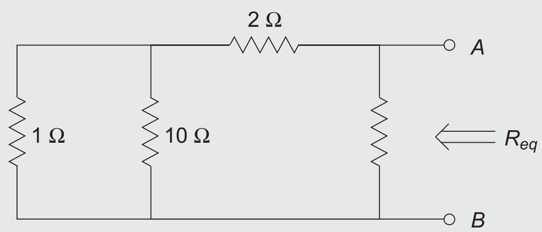
Active power = $3I^2R$
= $3(10.623)^2$ 8
= 2.708 kW

Reactive power = $3VI \sin\phi$
= 3(320) (10.623) sin (–36.86)
= 4.396 KVAR

6. (a) Obtain the Norton's equivalent circuit at the terminals A, B for the following figure.



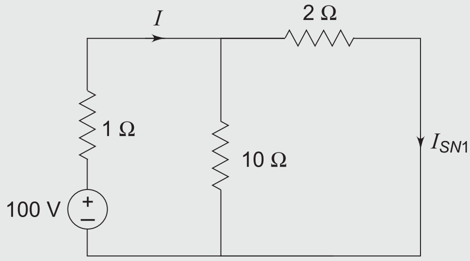
For finding the Nortons resistance, replace the voltage sources by the short circuit.



$$R_{eq} = \{[(1 \,||\, 10)+2] \,||\, 10\}$$
$$= 2.253\ \Omega$$

For finding the I_N short the terminals A and B and find current I_N. Apply superposition

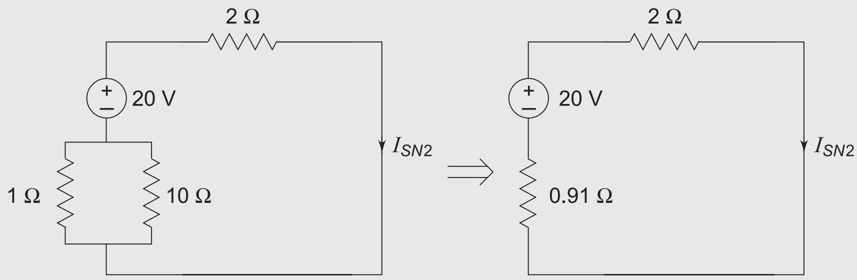
(i) with 100 V source



$$Z = [(2\|10)+1] = 2.67$$

$$\text{Total current } I = \frac{100}{Z}$$
$$= \frac{100}{2.67} = 37.45\text{ A}$$

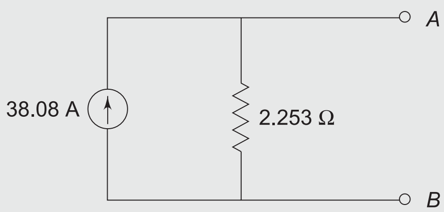
(ii) With 20 V source



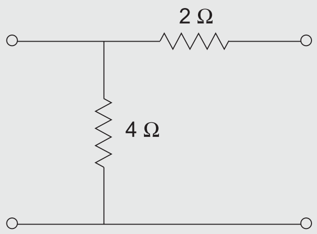
$$I_{SN2} = \frac{20}{2.91} = 6.872\text{ A}$$

$$\therefore I_{SN} = I_{SN_1} + I_{SN_2}$$
$$= 31.21 + 6.872$$
$$= 38.08\text{ A}$$

$\therefore$ Nortons equivalent circuit is given by



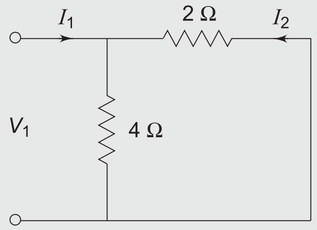
(b)



For the 2 port network, find h-parameters.

Solution We know that

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$



$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

$$\therefore \quad h_{11} = \frac{1.33 I_1}{I_1} = 1.33\Omega$$

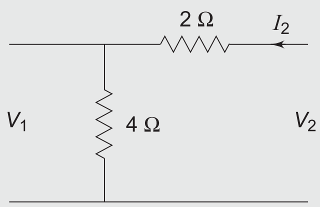
$$h_{21} = -\frac{4}{6}\frac{I_1}{I_1} = -0.66$$

$$V_1 = (2||4)I_1$$
$$= 1.33\ \mathrm{I}_1$$
$$I_2 = -I_1 \times \frac{4}{6}$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{12} = \left.\frac{I_{21}}{V_2}\right|_{I_1=0}$$

If $I_1 = 0$

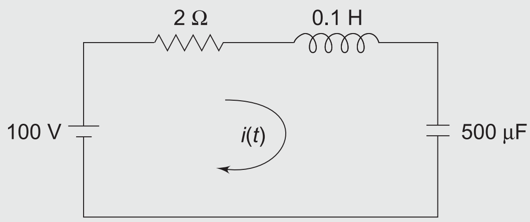


$$h_{22} = \frac{I_2}{6I_2}$$
$$= 0.166v$$

$$V_1 = 4I_2$$

$$h_{12} = \frac{4I_2}{6I_2} \qquad V_2 = 6I_2$$
$$= 0.66$$

7. A series RLC circuit with $R = 5\ \Omega$, $L = 0.1\mathrm{H}$ and $C = 500\ \mu\mathrm{F}$ has a D.C. voltage of 100V applied at $t = 0$ through a switch. Find the resulting current transient.



Solution $5i(t) + 0.1\,\mathrm{L}\,\dfrac{di(t)}{dt} + \dfrac{1}{500 \times 10^{-6}} \int i(t)dt = 100$

$$\frac{d^2 i}{dt^2} + 50\frac{di}{dt} + 2000 = 0$$

$D^2 + 50D + 2000 = 0$

$D = -25 \pm j37.08$

$\therefore i(t) = e^{-25t}\,[K_1 \cos 37.08t + K_2 \sin 37.08t]$ (1)

1st initial condition is that current through the inductor cannot change instantaneously.

Also voltage drop across capacitor cannot change instantaneously

Hence at $t = 0^+$

$$\frac{di}{dt}(0^+) = \frac{V_o}{L} = \frac{100}{0.1} = 1000$$

Substituting initial conditions $i(0^+) = 0$

$\therefore \quad 0 = K_3$

On differentiating equation (1), we get

$$\frac{di}{dt} = e^{-25t}\left[-37.08K_1 \sin 37.08t + 37.08K_2 \cos 37.08\right]$$

$$-25e^{-25t}(K_1 \cos 37.08t + K_2 \sin 37.08)$$

$$= e^{-25t}\left[\sin 37.08t\,(-37.08K_1 - 25K_2) + \cos 37.08t(37.08K_2 - 25K_1)\right]$$

$$\frac{di}{dt}(0^+) = 37.08K_2 - 25K_1$$

$$\Rightarrow K_2 = \frac{1000}{37.08} = 26.96$$

$\therefore\ i(t) = e^{-25t}\,(26.96 \sin 37.08t)$

8. (a) Explain Dot convention.

Solution Refer Section 10.4.

8. (b) Explain briefly about the locus diagrams.

Solution Refer Section 8.13.

PAPER 4

1. (a) Explain about dot convention.

Solution Refer Section 10.4.

(b) An iron ring of mean length 50 cm has an air gap of 1 mm and a winding of 200 turns. If the permeability of iron is 400 when a current of 1.25 A flows through the coil. Find the flux density.

Solution AT_1 required for iron path in the ring $= H_i \times l_i = \dfrac{B}{\mu_o \mu_r} \times l_i$

$$= \frac{B}{4\pi \times 10^{-7} \times 400} \times 0.5$$

AT_2 required for air gap of 1 mm $= H_g l_g = \dfrac{B}{\mu_o} \times l_g$

$$= \frac{B}{4\pi \times 10^{-7}} \times 1 \times 10^{-3}$$

Total ampere turns $= AT_1 + AT_2$

$$200 \times 1.25 = \left[\frac{B \times 0.5}{4\pi \times 10^{-7} \times 400} + \frac{B}{4\pi \times 10^{-7}} \times 10^{-3}\right]$$

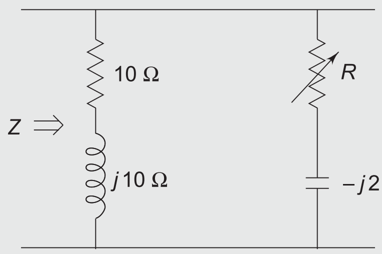
$$250 = \frac{B}{4\pi \times 10^{-7}}\left[1.25 \times 10^{-3} + 10^{-3}\right]$$

$B = 0.314$ web/m^2.

2. Derive the expressions for half power frequencies, Q factor ϕ Bandwidth of a series resonant circuit.

Solution Refer Sections 8.4. and 8.5.

3. (a) For the parallel network shown below, determine the value of R at 10 Ω resonance.



Solution $Z = (10 + j10) \,||\, (R - j2)$

$$= \frac{(10+j10)(R-j2)}{10+j10+R-j2}$$

$$= \frac{10R-j20+j10R+4}{10+R+8j}$$

$$= \frac{10R+4+j(10R-20)}{10+R+8j}$$

$$= \frac{[(10R+4)+j(10R-20)][10+R-j8]}{(10+R)^2+64}$$

$$= [(10R+4)(10+R)+8(10R-20)-j8(10R+4)+j(10+R)(10R-20)]\frac{1}{(10+R)^2+64}$$

At resonance imaginary part = 0

$\Rightarrow\quad 8(10R+4)-(10+R)(10R-20)$

$\Rightarrow\quad 100R-200+10R^2-20R=80R+32$

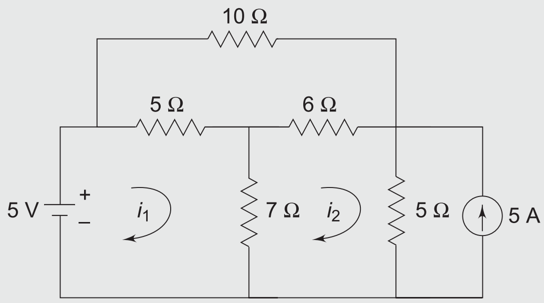
$$10R^2 = 232$$

$$R = 4.8166\ \Omega$$

(b) Define average value, rms value and form factor in a circuit.

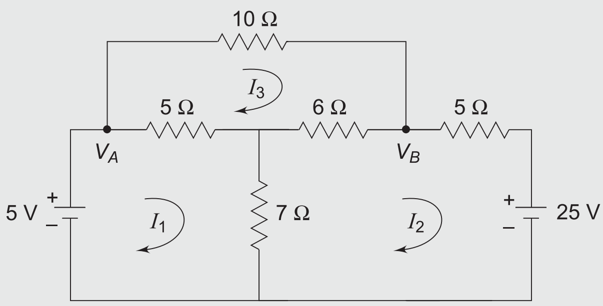
Solution Refer Section 4.4.

4. Determine the current in all branches of the following network and the voltage across for resistor using loop method.



Solution Applying mesh equation to the loops (1), (2) and (3)

We get



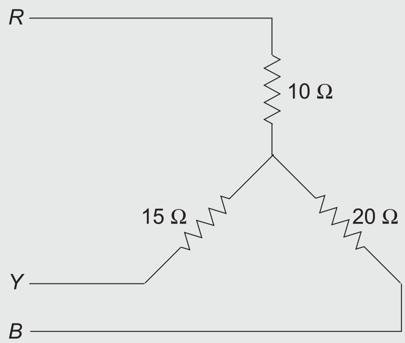
$$5(I_1 - I_3) + 7(I_1 - I_2) = 5$$
$$12I_1 - 7I_2 - 5I_3 = 5 \quad (1)$$
$$7(I_2 - I_1) + 6\,(I_2 - I_3) + 5I_2 = -25$$
$$-7I_1 + 18I_2 - 6I_3 = -25 \quad (2)$$
$$10I_3 + 5(I_3 - I_1) + 6(I_3 - I_2) = 0$$
$$-5I_1 - 6I_2 + 21I_3 = 0 \quad (3)$$

By solving above 3 equations, we get

$$I_1 = -1.231 \text{ A}$$
$$I_2 = -2.172 \text{ A}$$
$$I_3 = -0.9138 \text{ A}$$

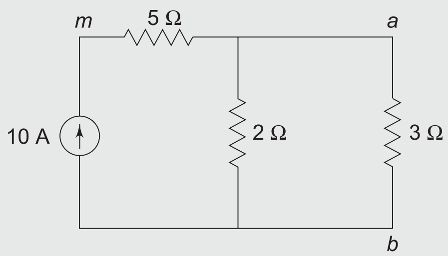
Current in 5 Ω resistor is –0.3172 A
7 Ω resistor is –1.231 A
6 Ω resistor is –1.2882 A
10 Ω resistor is –0.9138 A
5 Ω resistor is –2.172 A

5. A 440V, 3ϕ, 3-wire system is connected to an unbalanced star connected load shown in the figure. Determine the line currents and power I/P to the network.



$$V_{RY} = 440$$
$$I_R = \frac{440}{\sqrt{3}\times 10} = 25.4 \text{ A}$$
$$I_Y = \frac{440\ \angle -120^\circ}{\sqrt{3}\times 15}$$
$$= 16.93\ \angle -120^\circ \text{ A}$$
$$I_B = \frac{440\ \angle -240^\circ}{\sqrt{3}\times 20}$$
$$= 12.7\ \angle -240^\circ \text{ A}$$
$$P = I_R^2 R_R + I_Y^2 R_Y + I_B^2 R_B$$
$$= 13.976 \text{ kW}$$

6. (a) Verify reciprocity theorem in circuit shown in the following figure.

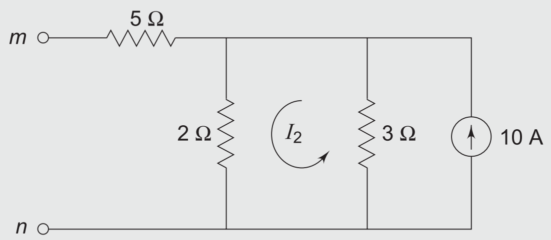


Solution Let us find current in 3 Ω resistor.

$$I_3 = 10 \times \frac{2}{2+3}$$
$$= 4\text{ A}$$
$$V_{ab} = 3 \times 4 = 12$$

According to reciprocity theorem the voltage across ab $V_{ab} = 12$

Now connect the current source across *ab* and find the voltage across *m* and *n*.



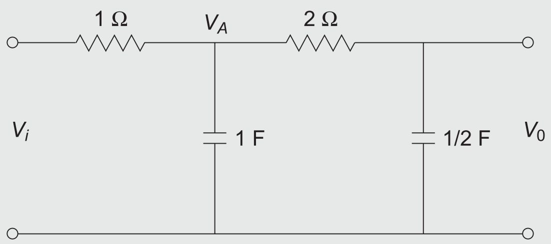
$$I_2 = 10 \times \frac{3}{5} = 6\text{ A}$$

The voltage across *mn* = 2 × 6 = 12 volts, same as V_{ab}. Hence, the reciprocity theorem is proved.

(b) State and explain compensation theorem.

Solution Refer Section 3.6.

7. Find transfer function $\dfrac{V_o(S)}{V_i(S)}$ for the circuit shown in the following figure.



Also $\dfrac{V_A - V_i}{1} + V_A\,S + \dfrac{V_A - V_O}{2} = 0$

$$V_A(1.5 + S) = V_i + \frac{V_o}{2}$$

$$V_A = \frac{V_i + \dfrac{V_o}{2}}{S + 1.5}$$

Also $V_o = \dfrac{V_A \times \dfrac{2}{5}}{2 + \dfrac{2}{5}}$

$$= \frac{V_A}{S+1}$$

$$V_o = \frac{2V_i + V_o}{(2S+3)(S+1)}$$

$$V_o\left[1 - \frac{1}{(2S+3)(S+1)}\right] = \frac{2V_i}{(2S+3)(S+1)}$$

$$V_o[(2S+3)(S+1) - 1] = 2V_i$$

$$\frac{V_o}{V_i} = \frac{2}{2S^2 + 5S + 2}$$

8. (a) In the circuit shown find the expression for transient current.

Solution $\quad 5i(t) + 3\dfrac{di(t)}{dt} = 100$

$$5I(S) + 3[SI(S) - i(0)] = \frac{100}{S}$$

$$i(0) = -6$$

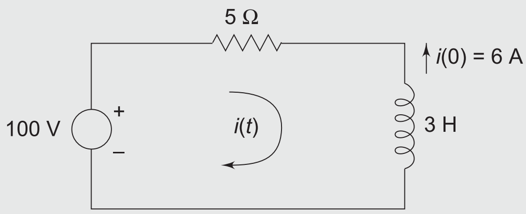
$$100 = 5I + 3\frac{di}{dt} + 1°(0)$$

$$\frac{100}{S} = (5+3S)\,I(S) + 18$$

$$I(S) = \frac{100 - 18S}{S(3S+5)}$$

$$= \frac{20}{S} - \frac{78}{3S+5}$$

$$= \frac{20}{S} - \frac{26}{S + 5/3}$$



$$i(t) = 20 - 26\,e^{5/3t}$$

8. (b) Obtain the lattice equivalent of a symmetrical T-network.

Solution $\quad$ Refer Example 15.13.

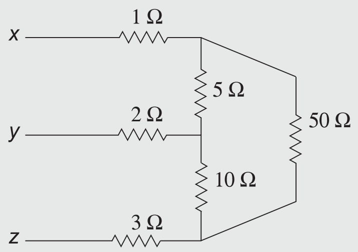
PAPER 5

1. (a) Obtain the expressions for star-delta equivalence of resistive networks.

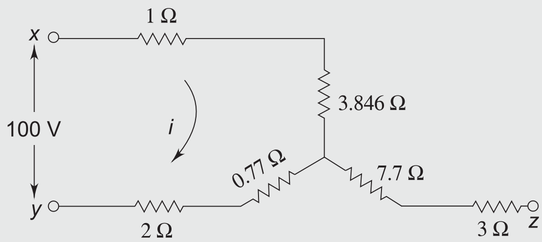
Solution

<u>Refer Section 3.1 (Chapter 3).</u>

(b) Determine the voltage appearing across terminals *y–z*, if a d.c. voltage of 100V is applied across *x–y* terminals in the figure below.



Solution Converting delta network to star network



Current, $i = \dfrac{100}{1 + 3.846 + 0.77 + 2} = \dfrac{100}{7.616} = 13.13\text{A}$

Voltage across $y_z{}^N, V_z$ $= -13.13 \times (2 + 0.77)$

$= -36.37$ V

2. (a) State and explain Faraday's law of electromagnetic induction. What are statically and dynamically induced emfs.

Solution

<u>Refer Section 1.6 (Chapter 1).</u>

First law : It states that whenever the magnetic flux linked with a circuit changes an emf is always induced in it.

Second law : It states that the magnitude of the induced emf is equal to the rate of change of flux linkage.

Explanation : Suppose a coil with 100 turns undergoes a change of flux from zero refers to 2 mwb in one millisecond.

Initial flux linkages = 0
Final flux linkages = $100 \times 2 \times 10^{-3}$ wb.T

$$\text{Induced emf} = \frac{100 \times 2 \times 10^{-3} - 0}{1 \times 10^{-3}} = 2000 \text{ V}$$

Induced emf can be expressed as $e = \frac{d}{dt}(NQ) = N\frac{dQ}{dt}$ v

Generally, a minus sign is associated with the $N\frac{dQ}{dt}$ to signify the fact that the induced emf sets up current in a such a direction that the magnetic effect produced by it opposes the very cause producing it. It is called Lenz's law

$$\therefore \qquad e = -N\frac{dQ}{dt}$$

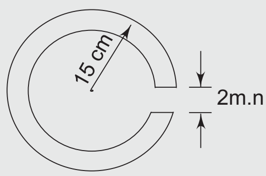
Statically induced EMF
EMF induced in a coil due to the change of its own flux linked with it or emf induced in one coil by the influence of the other coil is known as statically induced emf.
Dynamically Induced EMF:
When a coil with certain number of turns or a conductor is rotated in a magnetic filed (as in d.c. generator's), an emf is induced in it which is known as dynamically induced emf.

2. (b) An iron ring 15 cms in diameter and 10 cm^2 in area of cross section is wound with a coil of 200 turns. Determine the current in the coil to establish a flux density of 1 wb/m^2 if the relative permeability of iron is 500. In case if an air gap of 2 mm is cut in the ring, what is the current in the coil to establish the same flux density.

Solution

<u>Refer Example 10.12, Chapter 10: Refer Problem 10.13 Chapter 10</u>

(i) *Without air gap*
Diameter of Iron ring = 15 (cm) = 15×10^{-2} m
Area of Iron ring = 10 cm^2 = 10×10^{-4} m^2
Number of turns (N) = 200



Reluctance of Iron ring ($\Re_i$) $= \frac{l_i}{\mu_0 \, \mu_r . A}$

Length of Iron path (l_i) $= \pi . d$
$= \pi \times 15 \times 10^{-2}$ m

$$\Re_i = \frac{15\pi \times 10^{-2}}{4\pi \times 10^{-7} \times 500 \times 10 \times 10^{-4}} = 7.5 \times 105 \text{ AT/Wb}$$

mmf = Flux $\times$ reluctance
$I \times 200 = B.A.\ \Re$i

$$I = \frac{1\times 10\times 10^{-4}\times 7.5\times 10^{5}}{200} = 3.75 \text{ A}$$

(ii) With 2 mm air gap cut in the iron ring reluctance of air gap

$$(\Re_g) = \frac{2\times 10^{-3}}{4\pi\times 10^{-7}\times 10\times 10^{-4}}$$

$$= 15.915 \times 10^5 \text{ AT/Wb}$$

With 2 mm air gap the length of the Iron path is reduced by 2 mm.

$\therefore \quad l_i = 15\pi \times 10^{-2} - 2 \times 10^{-3}$

But this is negligibly small.

$\therefore$ Total reluctance = $\Re_i + \Re_g = 23.415 \times 10^5$ AT/Wb

$$\therefore \quad I = \frac{\phi.\Re}{N} = \frac{B.A.\Re}{N}$$

$$= \frac{1\times 10\times 10^{-4}\times 23.415\times 10^{5}}{200}$$

Required current (I) = 11.707A

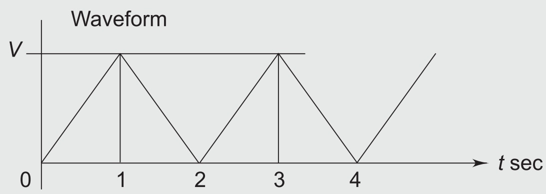
If the gap length is taken into consideration:

$$\text{Total emf} = \frac{B_i\, l_i}{\mu_0\, \mu_r} + \frac{B_i\, l_g}{\mu_0}$$

$$= \frac{1(\pi\times 15\times 10^{-2} - 2\times 10^{-3})}{4\pi\times 10^{-7}\times 500} + \frac{1\times 2\times 10^{-3}}{4\pi\times 10^{-7}} \quad 2338.35 \text{ AT}$$

$$\therefore \quad I = \frac{2338.35}{200} = 11.691\text{A}$$

3. (a) Find the form factor for the following waveform.



Solution

<u>Refer Section 4.4.7 (Chapter 4)</u>

$$\text{Form factor} = \frac{\text{R.M.S. value}}{\text{Average value}}$$

Average value of the triangular waveform 0 to 2 sec

$$\text{Vav} = \frac{1}{2}\left[\int_0^1 V.t\,dt + \int_1^2 -V(t-2)\,dt\right]$$

$$= \frac{1}{2}\left[V\frac{t^2}{2}\bigg|_0^1 + -V\frac{t^2}{2}\bigg|_1^2 + 2V.t|_1^2\right]$$

$$= \frac{1}{2}\left[\frac{V}{2} - \frac{3}{2}V + 2V\right] = V/2$$

$$\text{R.M.S. value, } (V_{\text{r.m.s.}}) = \left[\frac{1}{2}\left\{\int_0^1 V^2 t^2\,dt + \int_1^2 V^2 (t-2)^2\,dt\right\}\right]^{1/2}$$

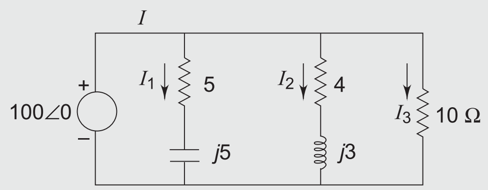
$$= \left[\frac{1}{2}\left\{V^2\frac{t^3}{3}\bigg|_0^1 + V^2\frac{t^3}{3}\bigg|_1^2 + 4V^2\,t|_1^2 - 4V^2\frac{t^2}{2}\bigg|_1^2\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\frac{V^2}{3} + \frac{7V^2}{3} - 2V^2\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\frac{8V^2 - 6V^2}{3}\right\}\right]^{1/2} = \frac{V}{\sqrt{3}}$$

$$\text{Form factor} = V/\sqrt{3}\,/V/2 = \frac{2}{\sqrt{3}} = 1.155$$

3. (b) Find the branch currents, total current and the total power in the circuit shown below:



Solution

$$\text{Branch currents } I_1 = \frac{100 + j0}{5 - j5} = 10 + j10$$

$$I_2 = \frac{100 + j0}{4 - j3} = 16 - j12$$

$$I_3 = \frac{100 + j0}{10} = 10 + j0$$

$$\text{Total current } (I) = I_1 + I_2 + I_3$$

$$= 36 - J2$$

$$= 36.055 \angle -3.179°$$

$$\text{Total power} = VI \times \cos Q$$

$$= 100 \times 36.055 \times \cos 3.179°$$

$$= 3599.95 \text{ watts.}$$

4. (a) Obtain the expression for the frequency at which maximum voltage occurs across the capacitance in series resonance circuit in terms of the Q-factor and resonance frequency.

Solution

Refer Section 8.3 (Chapter 8)

From Section 8.3 we know that

The frequency at which V_c is maximum is given by

$$f_c = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

$$f_c = \frac{1}{2\pi}\left[\sqrt{\frac{1}{LC}\left[1 - \frac{R^2C}{2L}\right]}\right]$$

$$= \frac{1}{2\pi}\left[\sqrt{\frac{R^2}{LC}\left(\frac{1}{R^2} - \frac{C}{2L}\right)}\right]$$

$$= \frac{1}{2\pi}\frac{R}{\sqrt{LC}}\left[\sqrt{\frac{1}{R^2} - \frac{C}{2L}}\right]$$

$$= \frac{1}{2\pi}\frac{R}{\sqrt{LC}}\left[\sqrt{\frac{C}{L}\left[\frac{L}{CR^2} - \frac{1}{2}\right]}\right]$$

$$= \frac{1}{2\pi\sqrt{LC}} \cdot R\sqrt{\frac{C}{L}}\left[\frac{L}{CR^2} - \frac{1}{2}\right]^{1/2}$$

$$f_o = \frac{1}{2\pi\sqrt{LC}}; \; Q = \frac{1}{R}\sqrt{\frac{L}{C}} \Rightarrow \frac{1}{Q} = R\sqrt{\frac{C}{L}}$$

$$\therefore \qquad f_c = \frac{f_o}{Q}\left[\frac{L}{CR^2} - \frac{1}{2}\right]^{1/2}$$

4. (b) In a series RLC circuit if the applied voltage is 10V, and resonance frequency is 1 kHz, and Q factor is 10, what is the maximum voltage across the inductance.

Solution

$$\text{Resonance freq } (fr) = \frac{1}{2\pi\sqrt{LC}} = 1000 \quad (1)$$

$$\text{Quality factor } (Q) = \frac{1}{R}\sqrt{\frac{L}{C}} = 10 \quad (2)$$

$$\sqrt{LC} = \frac{1}{2\pi \times 1000} = 6283.18$$

$$LC = 39.47 \times 10^6$$

From 1,
$$\frac{1}{2\pi} = \sqrt{LC}\ 1000 \quad (3)$$

From 2,
$$\frac{1}{R} = \sqrt{\frac{C}{L}}\ 10 \quad (4)$$

From 3 and 4

$$\frac{1}{2\pi R} = 10^4\ \sqrt{LC}\ \sqrt{\frac{C}{L}}$$

$$\frac{1}{2\pi\ RC} = 10000$$

$$RC = 1.59154 \times 10^{-5} \simeq 1.6 \times 10^{-5}.$$

The maximum voltage across the inductance occurs at frequency greater than the resonance frequency which is given by

$$f_L = \frac{1}{2\pi\sqrt{LC - \frac{(RC)^2}{2}}}$$

$$f_L = \frac{1}{2\pi\sqrt{39.47 \times 10^6 - \frac{(1.6 \times 10^{-5})^2}{2}}} = 1002.5$$

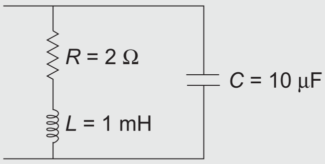
It can be observed that, the above frequency is approximately equal to resonance frequency,

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{39.47 \times 10^6}}$$

Hence we can take the voltage across the inductor

$$= Q \times V$$
$$= 10 \times 10$$
$$= 100 \text{ volts}$$

(c) In a parallel resonance circuit shown in figure find the resonance frequency, dynamic resistance and bandwidth.



Solution

The circuit shown in the above figure is the most common form of parallel resonant circuit in practical use and is also called the tank circuit.

The admittance of the circuit is

$$Y = \frac{1}{z} = \frac{1}{z_C} + \frac{1}{Z_L}$$

$$Y = \frac{1}{-jX_C} + \frac{1}{R + jX_L}$$

$$= j\omega C + \frac{1}{R + j\omega L}$$

$$= j\omega C + \frac{R - j\omega L}{R^2 + \omega^2 L^2}$$

$$= \frac{R}{R^2 + \omega^2 L^2} + j\omega\left(C - \frac{L}{R^2 + \omega^2 L^2}\right)$$

At resonance the susceptance part is zero.

Hence at $\omega = \omega_r,\ C = \dfrac{L}{R^2 + \omega_r^2 L^2} = 0$

$$R^2 + \omega_r^2 L^2 = \frac{L}{C}$$

$$\omega_r^2 L^2 = \frac{L}{C} - R^2 \Rightarrow \omega_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \qquad (1)$$

Resonance frequency, $f_r = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}}$ (2)

$\therefore$

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \frac{1}{2\pi L}\sqrt{\frac{L}{C} - R^2}$$

$$= \frac{1}{2\pi \times 1 \times 10^{-3}}\sqrt{\frac{1 \times 10^{-3}}{10 \times 10^{-6}} - 4}$$

$$= 1559.4 \text{ Hz}$$

Dynamic impedance:
The input admittance at resonance is given by

$$Y_r = \frac{R}{R^2 + \omega_r^2 L^2}$$

The impedance at resonance is

$$Z_r = \frac{1}{y_r} = \frac{R^2 + \omega_r^2 L^2}{R} = R + \frac{\omega_r^2 L^2}{R}$$

Substituting $\omega r^2 L^2$ from Eq. 1 gives,

$$Z_r = R + \frac{\frac{L}{C} - R^2}{R} = R + \frac{L}{CR} - R$$

$Z_r = \dfrac{L}{CR}$ which is called dynamic impedance.

This is a pure resistance because it is independent of the frequency.

$$\text{Here, dynamic resistance} = \frac{1 \times 10^{-3}}{10 \times 10^{-6} \times 2}$$

$$= 50\,\Omega$$

Bandwidth of the parallel resonance circuit $= \dfrac{\omega_r}{Q}$

$$\omega_r = \frac{1}{L}\sqrt{\frac{L}{C} - R^2}$$

$$= 9797.95$$

$$Q_o = \frac{\omega o L}{R} = \frac{9797.95 \times 1 \times 10^{-3}}{2} = 4.898$$

$$\text{Bandwidth} = \frac{1559.4}{4.898} = 318.311 + \mathrm{Z}$$

5. (a) A symmetrical 440V, 3 phase system supplies a star connected load with the following branch impedances: $Z_r = 10\Omega$ $Z_y = j5\Omega$ $Z_B = j5\Omega$. Calculate voltage drop across each branch and the potential difference between neutral and star point. The phase sequence is RYB. Draw phasor diagram.

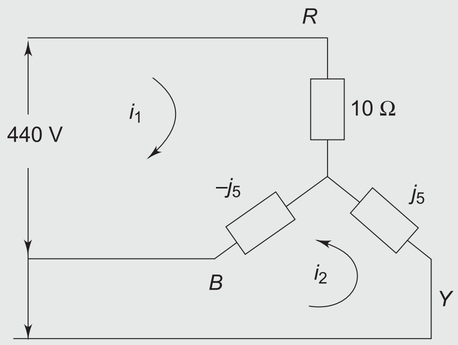
Solution

<u>Refer Problem 9.12 (Chapter 9)</u>

Applying KVL for the two loops

$$V_{\mathrm{RY}} = 440\angle 0 \text{ V}$$

$$V_{\mathrm{YB}} = 440\ \angle{-120}\text{V}$$



$$10I_1 - J_5I_2 = 440\angle 0 \qquad (1)$$

$$j_5I_2 + (I_1 + I_2)(-J_5) = 440\ \angle -120°$$

$$-j_5I_1 + (j_5 - j_5)I_2 = -220 - J\ 381.05 \qquad (2)$$

$$I_1 = \frac{-220 - j\,381.05}{-j\,5}$$

$$I_1 = (76.21 - j44)$$

Substituting the value of I_1 in Eq.1

$$10\ [76.21 - j44] - j5\ I_2 = 440$$

$$-j5\ I_2 = -322.1 + j440$$

$$I_2 = \frac{-322.1 + j\,440}{-j\,5}$$

$$I_2 = [-88 - j64.42]$$

Drop in the R-phase = 10 [76.21 – j44]

$$V_{RO'} = 880\ \angle -30°$$

Drop in the Y-phase = j5 [–88 – j64.42]

$$V_{YO'} = 545.3\ \angle -53.7°$$

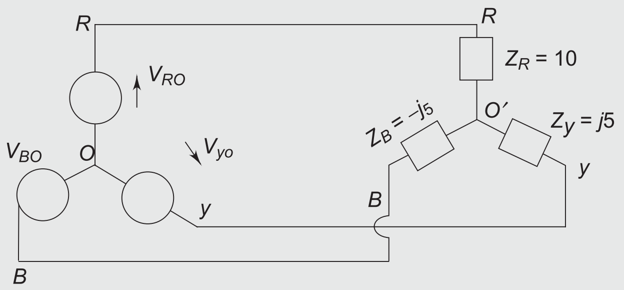
Drop in the B-Phase = $j5[I_1 + I_2]$

$$= j5\ (-11.79 - J108.42]$$

$$= 542.1 - J58.98$$

$$V_{BO'} = 545.3\ \angle\text{-}6.2$$

Neutral shift can be found by Millman's theorem



Taking V_{RY} as reference,

$$V_{RY} = 440\angle 0$$

$$V_{RO} = \frac{440}{\sqrt{30}}\ \angle{-30°} = 254\ \angle{-30°}$$

$$V_{YO} = 254\angle{-150°};\ V_{BO} = 254\angle 90°$$

$$Y_R = \frac{1}{Z_R} = \frac{1}{10\angle 0} = 0.1\angle 0°,$$

$$Y_Y = \frac{1}{Z_Y} = \frac{1}{j5} = 0.2\ \angle{-90°}$$

$$Y_B = \frac{1}{Z_B} = \frac{1}{-j5} = 0.2\ \angle 90°.$$

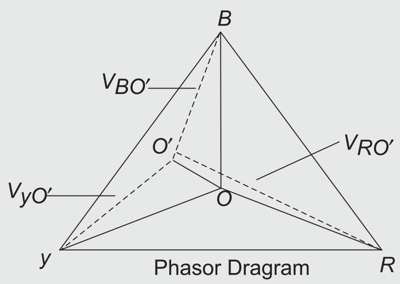
Neutral to star point voltage $V_{o'o} = \dfrac{V_{RO}\ Y_R + V_{YO}\,Y_y + V_{BO}\,Y_B}{Y_R + Y_Y + Y_B}$

$$V_{o'o} = 254\ \frac{0.1\angle -30° + 0.2\angle -240° + 0.2\angle 180°}{0.1\angle 0° + 0.2\angle -90° + 0.2\angle 90°}$$

$$= 625.8\ \angle 150°$$

$$V_{o'o} = 625.8\ \angle 150°.$$

The phasor diagram follows



Phasor Diagram

5. (b) A balanced star connected load is supplied from a symmetrical 3 phase, 440V, 50Hz supply. The current in each phase is 20A and lags behind its phase voltage by an angle of 40°. Calculate (i) load parameters (ii) total power and (iii) readings of two wattmeters connected in the load circuit to measure total power.

(i) Let the phase sequence be *RYB*.

The line voltage, $V_{RY} = 440\angle 0°$ V

Phase voltage, $V_R = \dfrac{440\angle 0}{\sqrt{3}} = 254\angle 0°$

$$Z_R = \frac{254\angle 0}{20\angle -40} = 12.7\angle 4°$$

$$= (9.72 + j8.16)\ \Omega$$

Load parameters are $R = 9.72\ \Omega$

$$x_L = 8.16; f = 50\text{ H}; L = \frac{8.16}{2\times\pi\times 50} = 25.9\text{ mH}$$

(ii) Total active power $= \sqrt{3}\ V_L I_L \cos\phi$

$$= \sqrt{3}\times 440\times 20\times\cos 40°$$

$$= 11676.08\text{ watts}$$

(iii) Reading of first watt meter $W_1 = V_L I_L \cos(30 + \phi)$

$$= 440\times 20\cos(30 + 40) = 3009.777\ W$$

Reading of second watt meter, $W_z = V_L I_L \cos(30 - \phi)$

$$= 440\times 20\cos(30 - 40) = 8666.308\ W$$

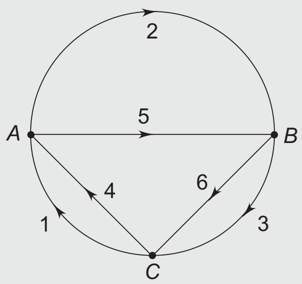
Total power, $w_1 + w_2 = 11676.08$ watts

(a) Define the following
 (i) Oriented graph
 (ii) Tree of a graph
 (iii) Cut set and basic cut set
 (iv) Tie set and Basic Tie set

Solution

Refer Sections 2.1, 2.2, 2.7 and 2.8 (Chapter 2)

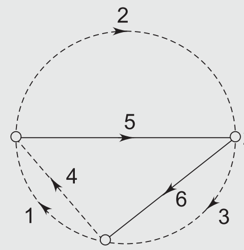
6. (b) For the topological graph shown in figure, obtain the fundamental Tie set matrix choosing the tree containing two elements 5 and 6.



Solution

Refer Section 2.7 (Chapter 2)

The tree of the graph is shown with solid lines (5 and 6) and the links are shown with dashed lines (1, 2, 3, 4).



For a given tree of a graph, addition of each link between any two nodes forms a loop called the fundamental loop. In a loop there exists a closed path and a circulating current, which is called the link current.

The fundamental loop formed by one link at a time, has a unique path in the tree rolling the two nodes of the link. This loop is also called *f*-loop or a tie-set. Every link defines a fundamental loop of the network.

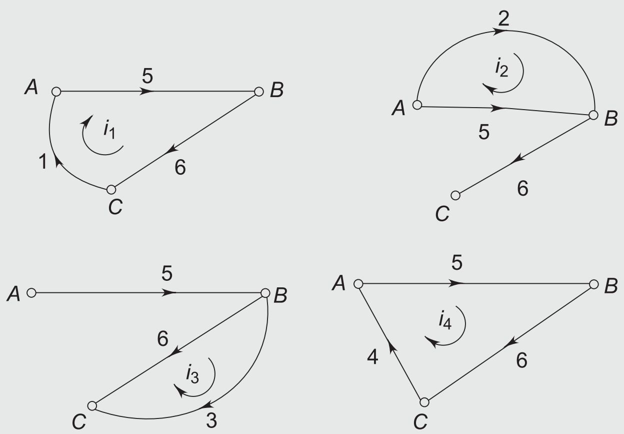
No. of nodes in the graph $n = 3 = (A, B, C)$

No. of branches, $b = 6 = (1, 2, 3, 4, 5, 6)$

No. of tree branches or twigs $= n - 1 = 2 = (5, 6)$

No. of link branches, $l = b - (n - 1) = 4$ (1, 2, 3, 4)

The following are the figures of the Tie-sets.



Tie set Matrix can be formed by considering the four fundamental loops. Corresponding to the link branches 1, 2, 3, 4.

If V_1, V_2, V_3, V_4, V_5 and V_6 are the respective branch voltages.

The KVL equations for the three *f*-loops can be written as

$$V_1 + V_5 + V_6 = 0$$
$$V_2 - V_5 = 0$$
$$V_3 - V_6 = 0$$
$$V_4 + V_5 + V_6 = 0$$

In order to apply KVL to each loop, we take the reference direction of the loop which coincides with the reference direction of the link defining the loop.

The above equations can be written in matrix form as

$[B][V_b] = 0$, where B is a 4×6 Tie-set matrix.

Loops Branches $\longrightarrow$

$$\begin{array}{c} \downarrow \\ l_1 \\ l_2 \\ l_3 \\ l_4 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, Tie-set Matrix, $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

7. (a) State and explain the superposition theorem.

Solution

Refer Section 3.2 (Chapter 3)

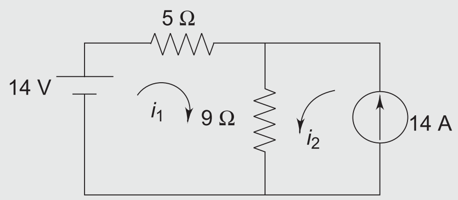
7. (b) Is superposition valid for power? Explain.

Solution

Superposition theorem is valid only for linear systems.

Superposition cannot be applied for power because the equation for power is non linear.

Let us consider a network with a voltage source and current source as shown below and find the power consumed in 9Ω resistor by super position.



When 14V source is acting the current in 9Ω is 1A

$$\text{The power} = i^2 \times 9 = 9 \text{ watts}$$

When 14A source is acting the current in 9Ω is 5A

$$\text{The power} = i^2 \times 9 = 225 \text{ watts}$$

$$\text{Total power} = 225 + 9 = 234 \text{ Watts}$$

When both are acting the KVL for loop 1 and 2

are $\quad 14 = 5i_1 + 9(i_1 + i_2)$

$$14i_1 = -112$$

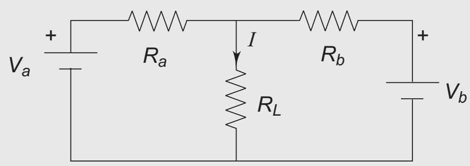
$$i_1 = -8\text{A}; i_2 = 14\text{A}$$

Current in 9Ω resistor is $i_1 + i_2 = 6\text{A}$

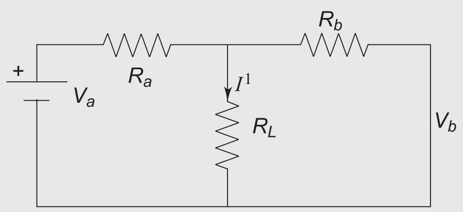
$$\text{Power} = (6)^2 \times 9 = 324 \text{ watts}$$

Since power is not the same in both the cases, the superposition theorem does not hold true.

Consider the circuit shown below.

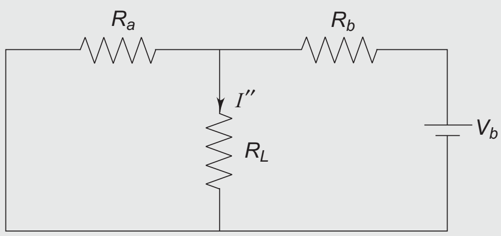


When V_a is acting.



I^1 be the current through R_L; and Power = $(I)^2R_L$

When V_b is acting I'' be the current



through B_L and Power = $(I'')^2R_L$

Total current through R_L by superposition

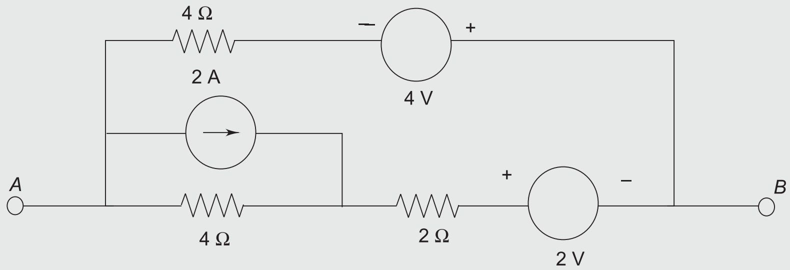
$I = I' + I''$, and power = $I^2 R_L$

$(I')^2 R_L + (I'')^2 R_L \neq I^2 R_L$

because $I^2 = (I' + I'')^2 = (I')^2 + (I'')^2 + 2I' I''$

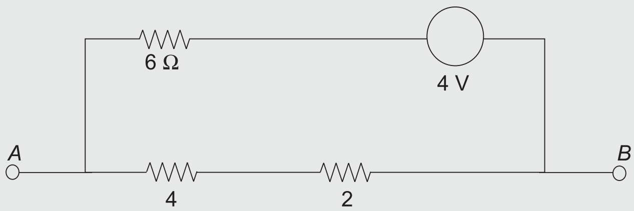
Hence, $(I')^2 + (I'')^2 \neq I^2$ and therefore superposition theorem is not valid for power.

7. (c) Using superposition theorem, find V_{AB}.



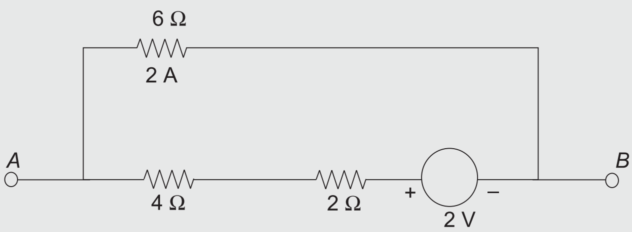
Solution

When 4V source is acting



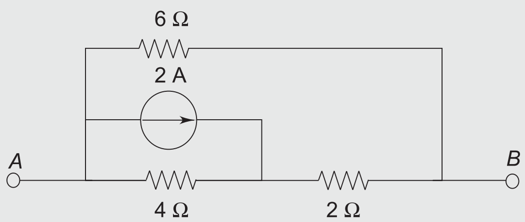
$V_{AB1} = -2$ V

When 2 V source is acting.



$V_{AB2} = 1$ V

When 2A source is acting



$V_{AB3} = -4$ V

Voltage across $AB = V_{AB1} + V_{AB2} + V_{AB2}$

$= -2 + 1 - 4$

$= -5$ volts

8. (a) Explain why the voltage across capacitor cannot change instantaneously?

Solution

Refer Section 1.7 (Chapter 1)

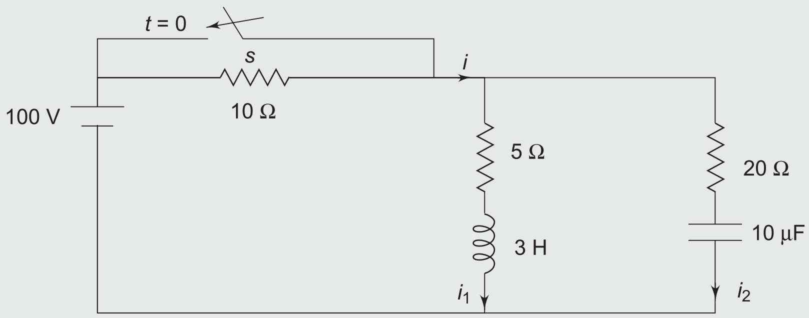
8. (b) What is the significance of time constant for R-L circuit? What are the difficult ways of defining time constant?

Solution

Refer Section 12.2 (Chapter 12)

8. (c) Switch S is closed at $t = 0$. Find initial conditions for voltage across capacitor.

$$i,\ i_2,\ \frac{di_1}{dt} \text{ and } \frac{di_2}{dt}$$



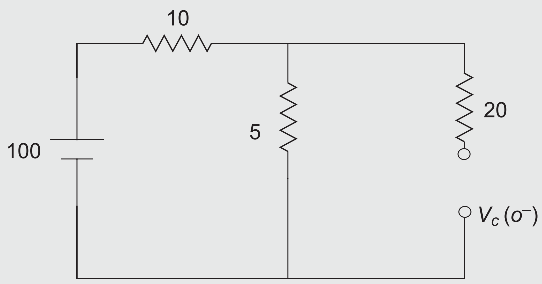
Solution

At $\quad t = \bar{0}\,;\ i = i_1 + i_2$

Since $\quad i_2 = 0,\ i = i_1\,(0^+) = \dfrac{V}{R_1 + R_2} = \dfrac{100}{15} = 6.67\text{A}$

$$i_1(0^+) = i_L(0^-) = i_L(0^+) = 6.667\text{A}$$

$$V_C(0^+) = V_C(0^+) = \frac{100}{10 + 5} \times 5 = 33.33\text{V}$$

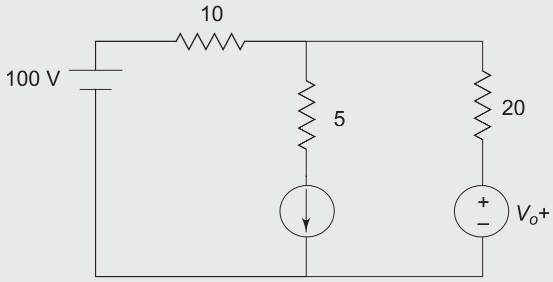


At $\quad t = 0^+;$

$$20i_2 + V_c(0^+) = 100$$

$$i_2(0^+) = \frac{100 - 33.33}{20}$$

$$i_2(0^+) = 3.33\text{A}$$



Applying KVL for the loops at $t = 0^+$

$$5i_1 + 3\frac{di_1}{dt} = 100$$

$$3\frac{di_1}{dt} = 100 - 5i_1$$

$$\left.\frac{di_1}{dt}\right|_{t=0^+} = \frac{100 - 5i_1(0^+)}{3} = \frac{100 - 5 \times 6.667}{3}$$

$$\frac{di_1}{dt}(0^+) = 22.21 \text{ A/sec.}$$

$$20i_2 + \frac{1}{C}\int i_2\, dt = 100$$

$$20\frac{di_2}{dt} + \frac{i_2}{C} = 0$$

$$\left.\frac{di_2}{dt}\right|_{t=0^+} = \frac{-i_2(0^+)}{20 \times 10 \times 10^{-6}} = \frac{-3.33}{2 \times 10^{-4}}$$

$$\frac{di_2}{dt}(0^+) = -16.65 \times 10^3 \text{ A/sec.}$$

PAPER 6

1. (a) Explain KCL and KVL.

Solution

Refer Sections 1.12 and 1.9 (Chapter 1)

1. (b) A capacitor is charged to 1 volt at $t = 0$. A resistor of 1 ohm is connected across its terminals. The current is known to be of the form $i(t) = e^{-t}$ amperes for $t > 0$. At a particular time the current drops to 0.37A at that instant determine.

(i) At what rate is the voltage across the capacitor changing?

(ii) What is the value of the charge on the capacitor?

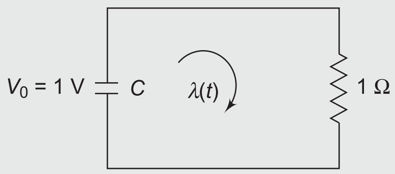
(iii) What is the voltage across the capacitor?

(iv) How much energy is stored in the electric field of the capacitor?

(v) What is the voltage across the resistor?

Solution

Refer Problem 12.3 (Chapter 12).



The current equation is

given as $i(t) = i(0^+)\, e^{-t|RC}$; given $i(t) = e^{-t|RC}$

$i(0^+) = 1\text{A}$; $RC=1$; $C=1\text{F}$

When $i(t)=0.37$ amperes

$$i(t) = 0.37 = e^{-t/1}$$

$$-t \log_e e = \log_e 0.37$$

$$t = 0.9942 \text{ sec}$$

$$i(t) = C\frac{dV(t)}{dt} \Rightarrow \frac{dV(t)}{dt} = \frac{i(t)}{C} = \frac{0.37}{1} = 0.37 \text{ V/sec}$$

or $$V_i(t) = \frac{1}{C}\int_0^t i(t)\,dt + V_0$$

$$= -\frac{1}{C}\int_0^t e^{-t}\,dt + V_0 \qquad [\therefore\ i(t) = -i(t)]$$

$$= \frac{-1}{1}\,\frac{e^{-t}}{(-1)} + 1 = e^{-t}$$

$$V_c(t) = e^{-t} \text{ for } t > 0$$

$$\therefore \quad \frac{dV_C(t)}{dt} = -e^{-t} = -e^{-0.9942} = -0.37 \text{ V/sec}$$

(ii) Charge on the capacitor

$$Q = C\,V_c = 1.e^{-t} = 0.37 \text{ coulombs}$$

(iii) Voltage across the capacitor

$$V_C(t) = e^{-t} = 0.37 \text{ volts}$$

(iv) Energy stored in the capacitor

$$W_C = \frac{1}{2}\,C\,V_c^2 = \frac{1}{2}1(e^{-t})^2 = \frac{e^{-2t}}{2} = 0.06845 \text{ joules}$$

(v) Voltage across the resistor at $t = 0.9942$ sec

$$V_R = i(t).R = e^{-t} = 0.37 \text{ V}$$

2. (a) Define Magneto Motive Force (MMF); reluctance, and flux density in a magnetic circuit. Specify the units of each of the above quantities.

Solution

<u>Refer Section 10.11 (Chapter 10).</u>

2. (b) Explain “dot convention” for a set of magnetically coupled coils. A cast steel electromagnet has an air gap of length 2 mm and an iron path of length 30 cms. Find the MMF needed to produce a flux density of 0.8T in the air gap. The relative permeability of the steel core at this flux density is 1000. Neglect leakage and fringing.

Solution

<u>For “dot convention” refer Section 10.4 (Chapter 10).</u>

Refer Example 10.2 (Chapter 10).

Air-gap length $l_g = 2 \text{ mm} = 2 \times 10^{-3}$ m

Iron path length $l_i = 30 \text{ cm} = 30 \times 10^{-2}$ m

Flux density $B = 0.8\text{T} = 0.8 \text{ Wb/m}^2$

$$\mu_r = 1000$$

$$\text{Total A.T} = \text{mmf} = H_i l_i + H_g l_g$$

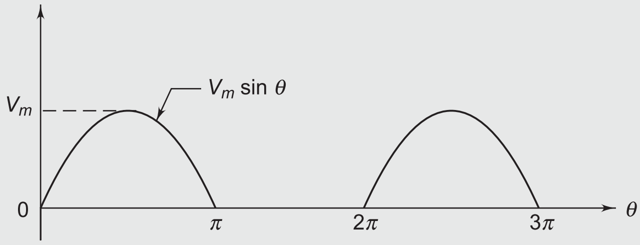
$$\frac{B \times l_i}{\mu_0\,\mu_g} + \frac{B}{\mu_0} l_g$$

$$= \frac{0.8 \times 30 \times 10^{-2}}{4\pi \times 10^{-7} \times 1000} + \frac{0.8 \times 2 \times 10^{-3}}{4\pi \times 10^{-7}}$$

$$= 1464 \text{ A.T.}$$

Hence, total MMF required to produce a flux density of 0.8T = 1464 AT.

3. (a) Find R.M.S. and average value of the following waveform.



Solution

Refer problem 4.13 (Chapter 4)

R.M.S. value, $$V_{\text{r.ms.}} = \sqrt{\frac{1}{2\pi}\int_0^{\pi} V_m^2 \sin^2\theta \, d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi} \frac{(1-\cos 2\theta)}{2}\, d\theta}$$

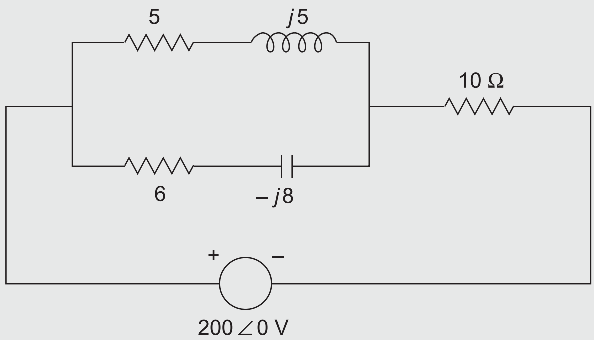
$$= \sqrt{\frac{V_m^2}{4\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}}$$

$$= \frac{V_m}{2}$$

Average value, $$V_{\text{ave}} = \frac{1}{2\pi}\int_0^{2\pi} V_m \sin\theta \, d\theta = \frac{V_m}{2\pi}\left[-\cos\theta\right]_0^{2\pi}$$

$$= \frac{V_m}{\pi}$$

3. (b) Find the total current and the power consumed by the circuit.



Solution

Total impedance of the circuit,

$$Z_T = (5 + j5) \parallel (t - j8) + 10$$
$$Z_T = 16.15 + j0.769$$

$$I = \frac{V}{Z} = \frac{200\angle 0}{16+5+j0.769} = 12.35 - j0.588\text{A}$$

$$= 12.36\angle{-2.72°}$$

Power consumed $= I^2R$

$$= (12.36)^2 \times 16.15 = 2467\text{W}$$

or $VI\cos\theta = 200 \times 12.36 \times \cos(-2.72)$

$$= 2467 \text{ W.}$$

4. (a) For a series RL circuit obtain the locus of current as inductance is changed from 0 to ∞ when the applied voltage is constant.

Solution

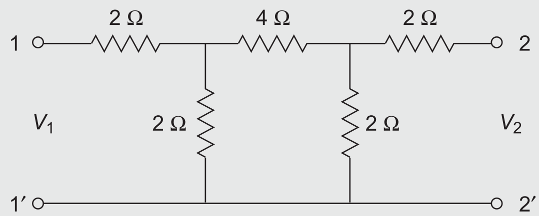
Refer Section 13.1(b) (Chapter 13)

4. (b) Show that for a series resonant circuit $f_1f_2 = f_r^2$ where f_1 and f_2 are half power frequencies and f_r is the resonance frequency.

Solution

Refer Section 8.4 (Chapter 8)

4. (c) Obtain the z-parameters of the following two-parts network.
Two-port network



Solution

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{6\times 2}{6+2} + 2 = 3.5\ \Omega$$

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{6\times 2}{6+2} + 2 = 3.5\ \Omega$$

$$z_{12} = \left.\frac{V_1}{I_1}\right|_{I_1=0} = \frac{\frac{I_2\times 2}{6+2}\times 2}{I_2} = 0.5\ \Omega$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{\frac{I_1\times 2}{6+2}\times 2}{I_2} = 0.5\ \Omega$$

The parameters of the network are

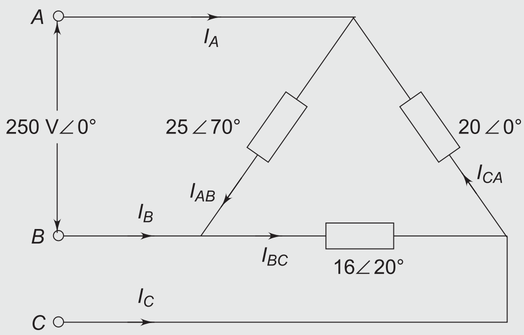
$$z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3.5 & 0.5 \\ 0.5 & 3.5 \end{bmatrix}$$

5. (a) Derive the relationship between phase quantities and line quantities in a 3 phase balanced (i) star connected system and (ii) Delta connected system. Draw phasor diagrams showing voltages and currents.

Solution

Refer Sections 9.7 and 9.8 (Chapter 9).

5. (b) A 3 phase supply with line voltage of 250V, has an unbalanced delta connected load as shown in figure. Determine the line currents, total active and reactive powers if the phase sequence is *ABC*.



Solution

Refer Problem 9.9 (Chapter 9).

$$I_{AB} = \frac{V_{AB}}{Z_{AB}} = \frac{250\ \angle 0°}{25\ \angle 90°} = 10\ \angle -90°$$

$$I_{BC} = \frac{V_{BC}}{Z_{BC}} = \frac{250\ \angle -120°}{16\ \angle 20°} = 15.625\ \angle -140°$$

$$I_{CA} = \frac{V_{\mathrm{CA}}}{Z_{\mathrm{CA}}} = \frac{250\ \angle 120°}{20\ \angle 0°} = 12.5\ \angle 120°$$

The line currents are

$$I_A = I_{AB} - I_{CA} = 10\angle -90° - 12.5\angle 120°$$

$$I_B = I_{BC} - I_{AB} = 15.625\ \angle -140° - 10\angle -90°$$

$$I_C = I_{CA} - I_{BC} = 12.5\angle 120° - 15.625\ \angle -140°$$

$$Z_{AB} = 0 + j25;\ Z_{BC} = 15.03 + j5.47;\ Z_{\mathrm{CA}} = 20 + j0$$

Active Powers $P_{AB} = I_{AB}^2\, R_{AB} = 10^2 \times 0 = 0$

$$P_{BC} = I_{BC}^2\, R_{BC} = (15.625)^2 \times 15.03 = 3669.4\text{W}$$

$$P_{CA} = I_{CA}^2 \times R_{CA} = (12.5)^2 \times 20 = 3125\text{W}$$

Total active power = $P_{AB} + P_{BC} + P_{CA} = 6795\text{W}$

Reactive powers

$$Q_{AB} = I_{AB}^2 \times \chi_{AB} = (10^2) \times 25 = 2500 \text{ VAR}$$

$$Q_{BC} = I_{BC}^2 \times \chi_{BC} = (15.625)^2 \times 5.47 = 1335 \text{ VAR}$$

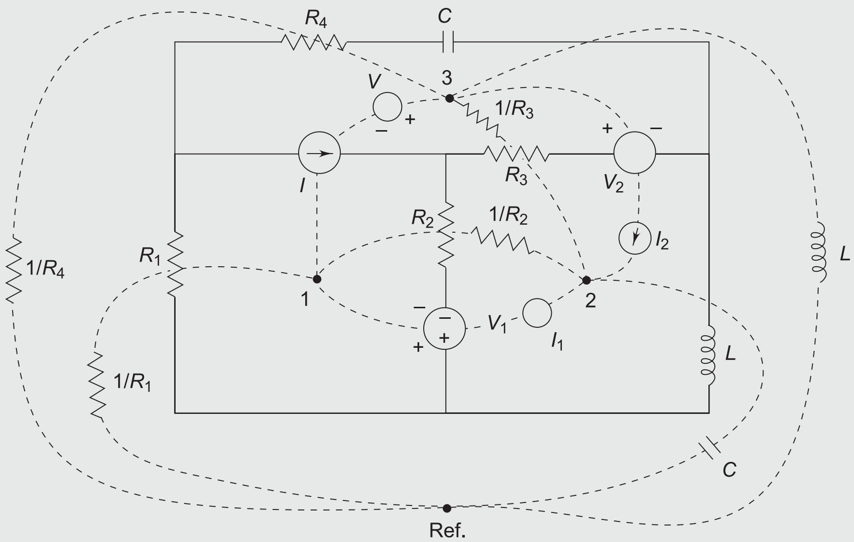
$$Q_{CA} = I_{CA}^2\, \chi_{CA} = (12.5)^2 \times 0 = 0$$

Total reactive power = $Q_{AB} + Q_{BC} + Q_{CA} = 3835$ VAR

Complex power, $S = P + jQ$

$= 6795 + j3835$

6. (a) What is duality? Explain the procedure for obtaining the dual of the given planar network shown below.

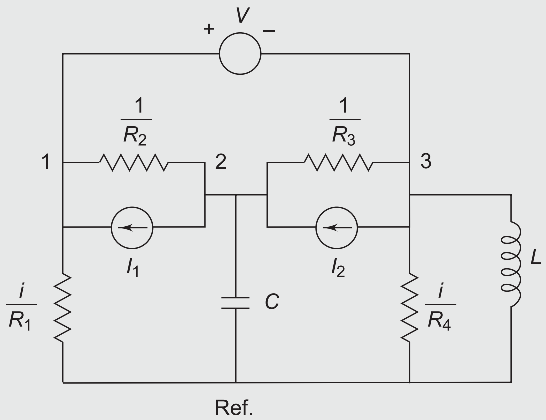


Solution

Refer Section 3.8 (Chapter 3)

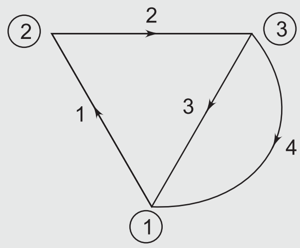
Rule 1 If a voltage source in the original network produces a c.w current in the mesh, the corresponding dual element is a current source whose direction is towards node representing the corresponding mesh.

Rule 2 If a current source in the original network produces a current in clockwise direction in the mesh, the voltage source in the dual network will have a polarity such that the node representing the corresponding mesh is positive.



Dual of the planar circuit given in 6(a).

6. (b) Construct the incident matrix for the graph show in figure.



Solution

<u>Refer Section 2.4 (Chapter 2)</u>

The dimensions of incidence matrix ‘A’ is $n \times b$ where n is number of nodes and b is number of branches, hence the dimensions of the incidence matrix for the above graph is 3×4.

Incidence matrix

n — nodes

b — branches

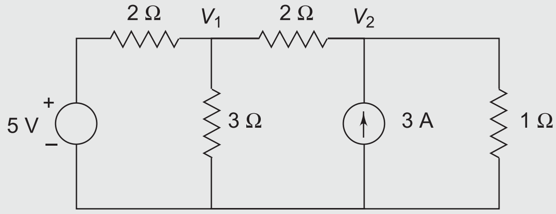
$A =$

$n \backslash p$	1	2	3	4
1	1	0	–1	–1
2	–1	1	0	0
3	0	–1	1	1

The incidence matrix is given by

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

6. (c) Use nodal analysis, to determine the voltage V_1 and V_2 in the circuit shown.



Solution

Refer Section 2.12 (Chapter 2).

The nodal equation for the two nodes are

$$\frac{V_1 - 5}{2} + \frac{V_1}{3} + \frac{V_1 - V_2}{2} = 0 \qquad ...1$$

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} = 3 \qquad ...2$$

From 1 $\quad 1.333\ V_1 - 0.5\ V_2 = 2.5$

From 2 $\quad -0.5\ V_1 + 1.5\ V_2 = 3$

Solving the above equations for V_1 and V_2 yields

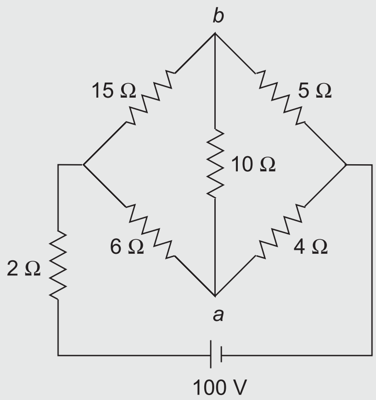
$V_1 = 3$ V and $V_2 = 3$ V.

7. (a) State and explain the Thevenin's theorem? State for what type problems this theorem is useful.

Solution

Refer Section 3.3 (Chapter 3).

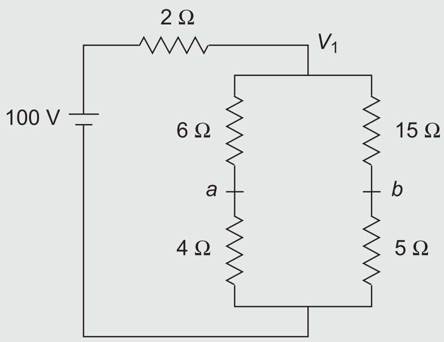
7. (b) Find the current through 10 Ω resistor using Thevenin's theorem.



Solution

Refer Problem 3.6 (Chapter 3).

Let us redraw the circuit by removing 10Ω.



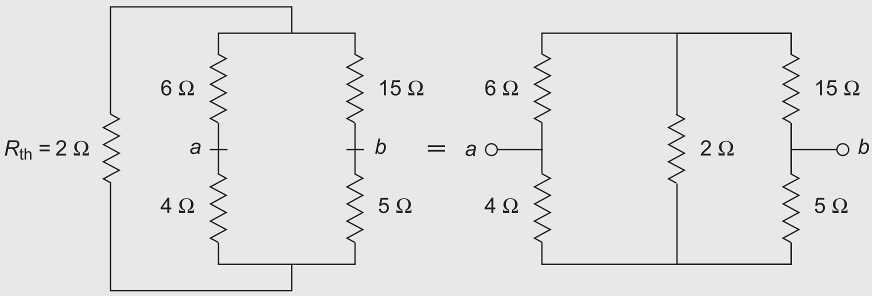
Applying KCL at V_1

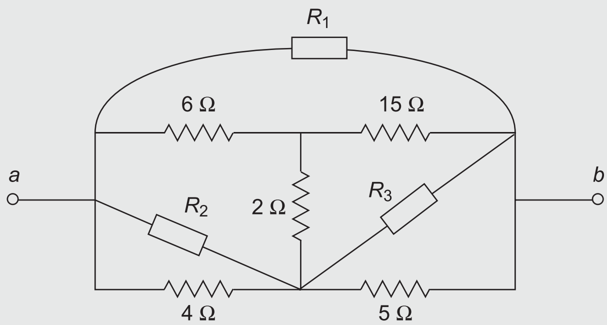
$$\frac{V_1 - 100}{2} + \frac{V_1}{10} + \frac{V_1}{20} = 0$$

from which $V_1 = 76.92$ V

$$V_{th} = V_a - V_b$$

$$= \frac{V_1}{6+4} \times 4 - \frac{V_1}{15+5} \times 5 = 11.538 \text{ V}$$

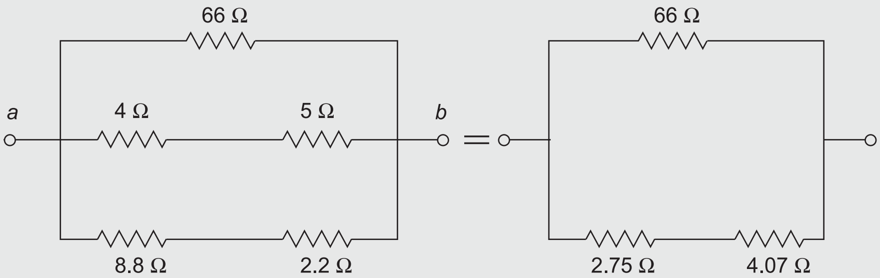




$$R_1 = \frac{6 \times 15 + 15 \times 2 + 2 \times 6}{2} = 66$$

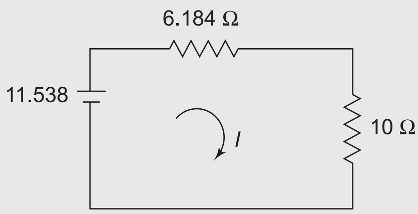
$$R_2 = \frac{132}{15} = 8.8$$

$$R_3 = \frac{132}{6} = 22$$



$$R_{ab} = R_{th} = \frac{66 \times 6.82}{72.82} = 6.184\ \Omega$$

Thevenin's equivalent circuit is given by



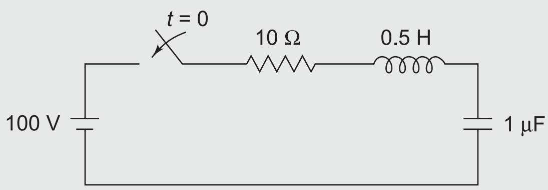
where $I = \dfrac{11.538}{16.184} = 0.7129\text{A}$

8. (a) For R-L-C series circuit with d.c. excitation discuss the under-damped, over-damped and critically damped cases.

Solution

Refer Section 12.4 (Chapter 12).

8. (b) Obtain the current $i(t)$ for $t \geq 0$ using tune domain approach.



Solution

Refer Example 12.3 (Chapter 12).

Writing KVL for the above circuit.

$$100 = 10i + 0.5\,\frac{di}{dt} + \frac{1}{1\times 10^{-6}}\int i\,dt$$

Differentiating w.r.t. t

$$0 = 10\,\frac{di}{dt} + 0.5\,\frac{d^2 i}{dt^2} + 10^6 i$$

$$\frac{d^2 i}{dt^2} + 20\,\frac{di}{dt} + 2\times 10^6 i = 0$$

$$(D^2 + 20D + 2\times 10^{-6})i = 0 \text{ where } D = \frac{di}{dt}$$

$$D_1, D_2 = \frac{-20 \pm \sqrt{400 - 4\times 2\times 10^6}}{2}$$

$$D_1 = -10 + j1414;\ D_2 = -10 - j1414$$

The roots are in the form of $-K_1 \pm jK_2$

Therefore the solution for the current is given by

$$i(t) = e^{-k1t}\,[C_1 \cos k_2 t + C_2 \sin k_2\, t]$$

$$i(t) = e^{-10t}\,[C_1 \cos 1414\, t + C_2 \sin 1414\, t]$$

Substitute the initial conditions to find C_1 and C_2

At $t = 0$; the current following through the circuit is zero.

$$i = 0 = 1\,[C_1 \cos 0 + C_2 \sin 0]$$

$$C_1 = 0$$

$$i(t) = e^{-10t}\, C_2 \sin 1414t.$$

$$\frac{di(t)}{dt} = C_2\,[e^{-10t}\, 1414 \cos 1414\, t + e^{-10t}\,(-10) \sin 1414t]$$

At $t = 0$, the voltage across the inductor

$$L\,\frac{di(t)}{dt} = 100$$

$$\frac{di(t)}{dt} = 200$$

$$200 = C_2\, e^{-(10\times 0)}\, 1414$$

$$C_2 = 0.1414$$

The equation for current is given by

$$i(t) = e^{-10t}\,(0.1414 \sin 1414\, t)$$

$$i(t) = 0.1414\, e^{-10t} \sin 1414t$$

PAPER 7

1. (a) Explain how source transformation is achieved.

Solution

Refer Section 2.15 (Chapter 2).

(b) A current of 0.5A is supplied by a source to an inductor of 1H. Calculate the energy stored in the inductor. What happens to this energy if the source is short circuited?

Solution

$$\text{Energy stored } \frac{1}{2} \text{ L I}^2 = \frac{1}{2}\ 1 \times 1^2 = 0.5 \text{ Joules}$$

If the inductor has an internal resistance, the stored energy is dissipated in the resistance after the short circuit as per the time constant (1/r) of the coil.

If the coil is a perfect inductor, the current would circulate through the shorted coil continuously.

(c) A current source $i = I_m \sin \omega t$ is applied across (i) a 1F capacitor (ii) 1H inductor. Assume initial conditions to be zero, show the voltage waveforms in the above two cases.

Solution

Refer Sections 4.6 and 4.7 (Chapter 4).

$$V_c(t) = \frac{1}{C}\int i\, dt = \frac{1}{C}\int I_m \sin \omega t\, dt$$

$$= \frac{1}{\omega C} I_m [-\cos \omega t]$$

[$\therefore$ initial conditions assumed to be zero]

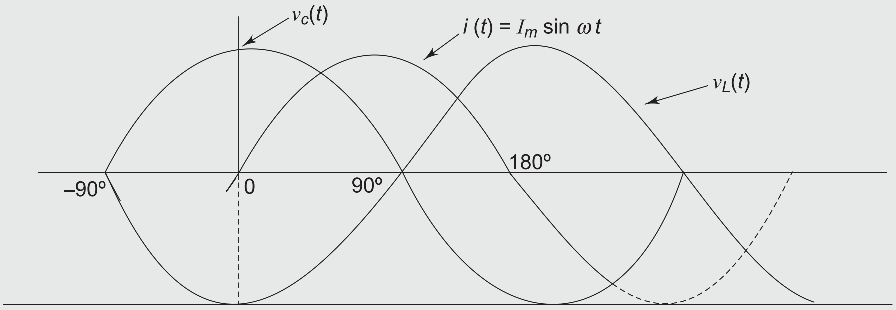
$$= \frac{I_m}{\omega C} \sin(\omega t - 90)$$

$$v(t) = V_n \sin(\omega t - 90°)$$

$$v(t) = IM\,[V_m\, e^{j(\omega t - 90°)}] \text{ or } V_m \angle{-90°}.$$

where $V_m = I_m/wC$.

The waveform is given by:



$$V_L(t) = L\frac{di(t)}{dt}$$

$$= L\cdot\frac{d}{dt}(I_m \sin \omega t)$$

$$= LW\, I_M \cos \omega t = WL\, I_m \cos \omega t$$

$$V_L(t) = V_m \cos \omega t \text{ or } V_m \sin(\omega t + 90°)$$

$$= IM\,[V_m e^{t(\omega + 90°)}] \text{ or } V_m \angle 90°$$

where $V_m = WLI_m$

The waveform is shown in the figure above.

2. (a) Define MMF, flux density, magnetising force and permeability specify the merits for each of the above quantities.

Solution

<u>Refer Section 10.11 (Chapter 10).</u>

2. (b) Two coupled coils have self induction cos $L_1 = 50$ mH and $L_2 = 200$ mH and a coefficient of coupling of 0.7. If coil 2 has 1000 turns and $i_1 = 5.0 \sin 400t$. Determine the voltage across coil 2.

Solution

<u>Refer Problem 10.3 (Chapter 10)</u>.

$$L_1 = 50 \text{ mH}, L_2 = 200 \text{ mH}; K = 0.7$$

$$M = K\sqrt{L_1\, L_2} = 0.7\sqrt{50 \times 200} \text{ mH}$$

$$= 70 \text{ mH}$$

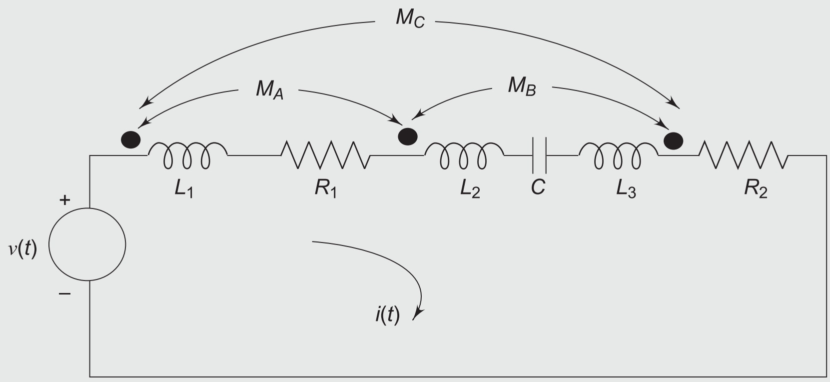
$$V_2 = M\frac{di_1}{dt} \text{ (Voltage induced in coil 2)}$$

$$= 70 \times 10^{-3}\frac{d}{dt}(5 \sin 400\, t)$$

$$= 70 \times 10^{-3} \times 2000 \cos 400\, t$$

Total voltage induced in coil 2 is = 140 cos 400 t volts.

2. (c) Write the voltage equation for the following circuit shown.



Solution

Applying KVL around the loop is given by

$$V(t) = L_1 \frac{di(t)}{dt} + M_A \frac{di(t)}{dt} - M_C \frac{di(t)}{dt} + R_1 i(t)$$

$$+ L_2 \frac{di(t)}{dt} + M_A \frac{di(t)}{dt} - M_B \frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

$$+ L_3 \frac{di(t)}{dt} - M_C \frac{di(t)}{dt} - M_B \frac{di(t)}{dt} + R_2\, i(t)$$

$$v(t) = (L_1 + L_2 + L_3)\frac{di(t)}{dt} + (R_1 + R_2)\, i(t) + \frac{1}{C}\int i(t)\,dt$$

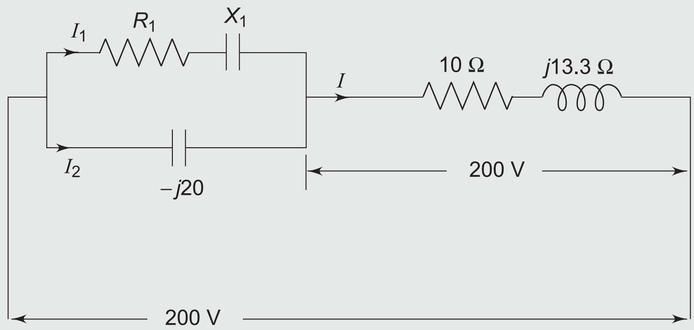
$$+ (2M_A - 2M_B - 2M_C)\frac{di(t)}{dt}$$

3. (a) Define rms value, average value, form factor and peak factor.

Solution

Refer Section 4.4 (Chapter 4).

3. (b) Find the value of R_1 and X_1 when a lagging current in the circuit gives a power of 2kW.



Solution

Let us take the voltage across ($10 + j13.3\Omega$) impedance as reference and calculate the total current I.

$$I = \frac{200\angle 0}{10 + j13.3} = 7.223 - j9.606 = 12.02\ \angle{-53.06°}\text{A}$$

Let us assume the phase angle between supply voltage and total current as ϕ which is equal to (θ + 53.06°).
Hence, real power in the circuit 2000 = 200 × 12.02 cos (θ + 53.06)
Therefore, θ = –19.5° and source voltage $V = 200\ \angle{-19.5°}$

Voltage across $R_1 + jX_1 = 200\ \angle{-195°} - 200\ \angle 0°$

$= -11.47 - j66.76$

$$I_2 = \frac{-11.47 - j66.76}{-j20} = 3.338 - j\,0.5\,735$$

$I_1 = I - I_2$

$= 7.223 - J9.606 - 3.338 + J0.5735$

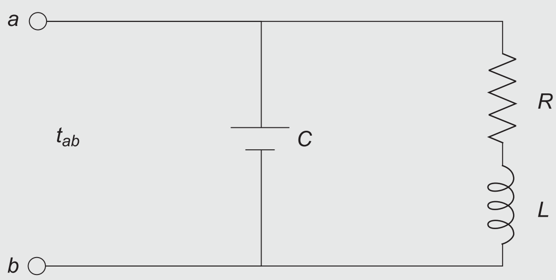
$= 9.8325\ \angle{-66.72°}$

$$Z_1 = \frac{V}{I_1} = \frac{-11.47 - j66.76}{9.8325\angle -66.72}$$

$= 5.776 - j3.7543$

Thus, $R_1 = 5.776\Omega$ and $x_1 = 3.7543\Omega$.

4. (a) For the parallel resonant circuit shown in the figure find the value of capacitance at which maximum impedance occurs at a given frequency.

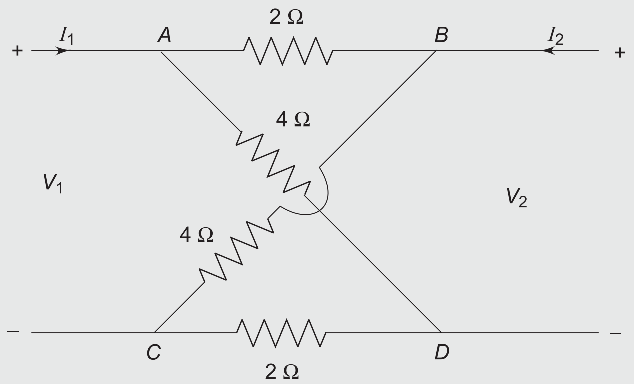


Solution

Refer Section 8.8 (Chapter 8)

The parallel resonant circuit shown is generally called a tank circuit. The impedance of the parallel resonant circuit is maximum at the resonance frequency.

4. (b) Determine the admittance parameters of the symmetrical lattice shown in the figure.

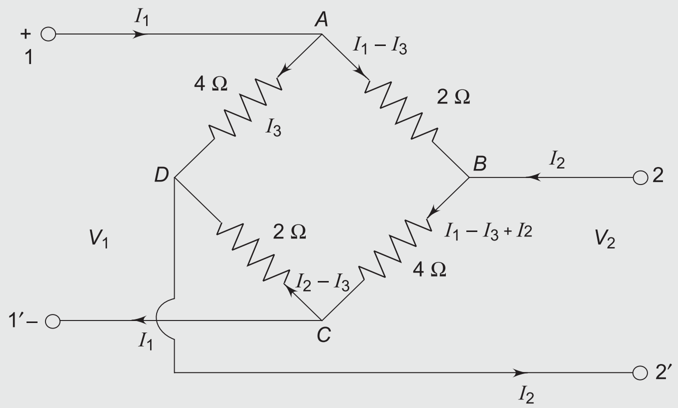


Solution

Refer Section 15.12 (Chapter 15)

The lattice network can be redrawn as a bridge network as shown.

Assume I_3 in AD as indicated.



Writing mesh equation $ADC\ 1^{'}1$

$$-4\ I_3 + 2\ (I_2 - I_3) + V_1 = 0$$

$$V_1 = -2\ I_2 + 6\ I_3 \tag{1}$$

Writing mesh equation $BC\ D_2\ 2'$

$$-4(I_1 - I_3 + I_2) - 2\ (I_2 - I_3) + V_2 = 0$$

$$V_2 = 4I_1 + 6I_2 - 6I_3 \tag{2}$$

Writing mesh equation $ABCDA$

$$-2(I_1 - I_3) - 4(I_1 - I_3 + I_2) - 2\ (I_2 - I_3) + 4I_3 = 0$$

$$I_3 = \frac{1}{2}\ (I_1 + I_2) \tag{3}$$

Substituting equation 3 in 1

$$V_1 = -2\ I_2 + 3\ (I_1 + I_2)$$

$$V_1 = 3\ I_1 + I_2 \tag{4}$$

Substituting equation 3 in 2

$$V_2 = 4\ I_1 + 6\ I_2 - 3\ (I_1 + I_2)$$

$$V_2 = I_1 + 3\ I_2 \qquad (5)$$

From equation 4 $\quad I_2 = V_1 - 3I_1$

Substituting in equation 5

$$V_2 = I_1 + 3\ (V_1 - 3I_1)$$

$$V_2 = -8I_1 + 3V_1$$

or $\quad I_1 = \frac{3}{8}\ V_1 - \frac{V_2}{8} \qquad (6)$

From equations 4 and 5

$$V_1 - 3V_2 = -\ 8I_2$$

or, $\quad I_2 = -\frac{V_1}{8} + \frac{3}{8}V_2 \qquad (7)$

Equation 6 and 7 are of the form

$$I_1 = Y_{11}\ V_1 + Y_{12}\ V_2$$

$$I_2 = Y_{21}\ V_1 + Y_{22}\ V_2$$

Therefore, $\quad Y_{11} = Y_{22} = \frac{3}{8};\ y_{12} = y_{21} = -\frac{1}{8}$

Also equation 4 and 5 are of the form

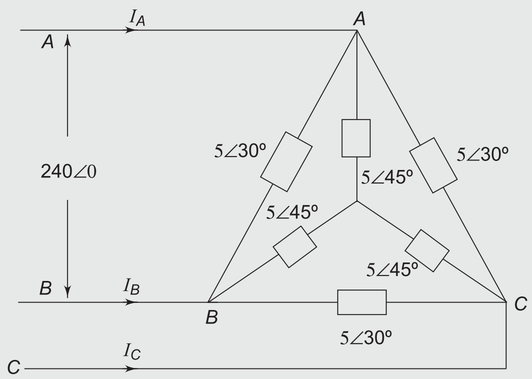
$$V_1 = Z_{11}\ I_1 + Z_{12}\ I_2$$

$$V_2 = Z_{21}\ I_1 + Z_{22}\ I_2$$

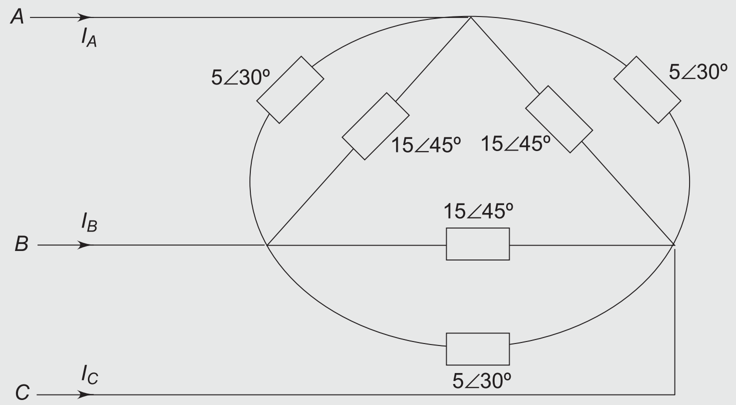
Therefore $\quad Z_{11} = Z_{22} = 3;\ Z_{12} = Z_{21} = 1.$

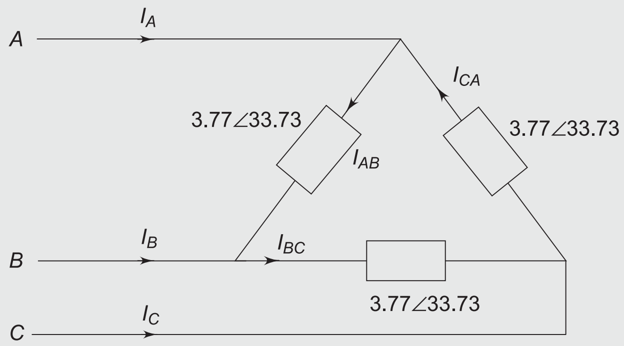
5. (a) A balanced delta connected load of 5.0 ∠30° Ω and a balanced star connected load of 5.0 ∠45° Ω are supplied by the same balanced 240V, 3 phase *ABC* system. Obtain line currents I_A, I_B and I_C.

Solution



The two loads are connected parallel across a 240V 3 phase system. Let us convert the star connected load into delta and redraw the circuit is as shown below.





The phase currents are given by

$$I_{AB} = \frac{V_{AB}}{Z_{AB}} = \frac{240\ \angle 0}{3.77\ \angle 33.73} = 63.5584\ \angle{-33.73}$$

$$I_{BC} = 63.5584\ \angle{-153.73}$$

$$I_{CA} = 6.35584\ \angle{-273.73}$$

The line currents are $\sqrt{3}$ times the phase currents and lag 30° behind their respective phase currents.

Therefore, $\quad I_A = \sqrt{3}\ \times 63.5584\ \angle{-33.73-30°}$

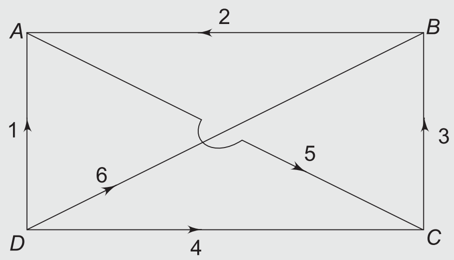
$= 110\ \angle{-63.73°}$

Similarly, $\quad I_B = 110\ -183.73°$ and $I_C = 110\ \angle{-303.73°}$

5. (b) Derive phase and line relations in a balanced delta connected load.

Solution <u>Refer Sections 9.8.1, 9.8.2 and 9.8.3.</u>

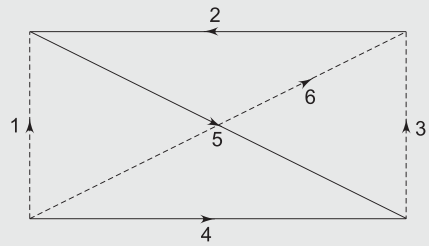
6. (a) For the given network graph shown below, write down the basic Tie set matrix, taking the tree consisting of edges 2, 4 and 5. Write down the KVL network equations from the matrix.



Solution

Refer Section 2.7 and Example 2.4 (Chapter 2)

The twigs of the tree are 2, 4 and 5. The links corresponding to the tree are 1, 3 and 6 as shown in the figure.



Number of nodes, $n = 4$

Number of branches, $b = 6$

Number of tree branches or twigs $= n - 1 = 3$

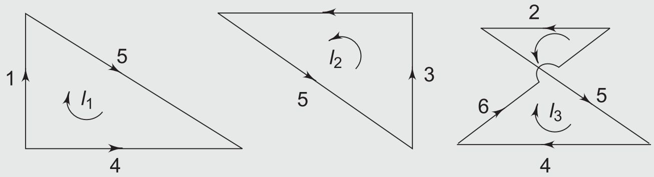
Number of link branches $l = b - (n - 1) = 3$

For writing the tie-set matrix consider the three links one at a time, the tie-set matrix B or fundamental loop matrix is given by.

$$\begin{array}{cc} \text{lops} \downarrow & \text{Branches} \longrightarrow \\ & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \end{array}$$

$$B = \begin{array}{c} l_1 \\ l_2 \\ l_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

There are three fundamental loops l_1, l_2 and l_3 as shown by the tie sets.



Link branch –1 Link branch 3 Link branch 6

From the tie-set matrix we can write KVL network equations as

$$[B][V_b] = 0$$

where B is an $l \times b$ tie-set matrix or fundamental loop matrix and V_b is a column vector of branch voltages of 1, 2, 3, 4, 5 and 6 respectively.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = 0$$

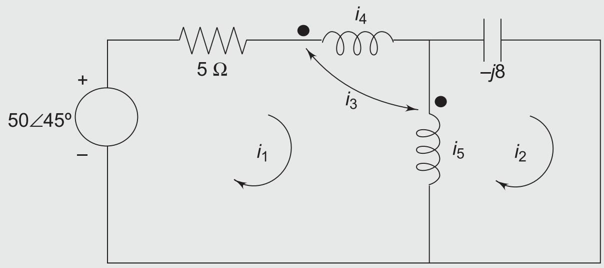
The KVL network Equation for the Three Tie-sets are

$$V_1 - V_4 + V_5 = 0 \tag{1}$$

$$V_2 + V_3 + V_5 = 0 \tag{2}$$

$$V_2 + V_4 + V_5 + V_6 = 0 \tag{3}$$

6. (b) Find the voltage across the 5Ω resistor for the coupled network shown in figure.



Solution

Refer Problem 10.10 (Chapter 10)

Applying KVL for loop 1

$$50\angle 45° = 5i_1 + j4i_1 + j3\,(i_1 - i_2) + j5(i_1 - i_2) + j3i_1 \tag{1}$$

Simplifying and rearranging the above equation yields to

$$50\angle 45° = (5 + j15)\, i_1 - j8i_2 \tag{2}$$

Applying KVL for loop 2

$$0 = -j8i_2 + j5\,(i_2 - i_1) - j3i_1 \tag{3}$$

Simplifying the above equation yields to

$$j8i_1 = -j3i_2 \text{ or } i_2 = -\frac{8}{3}\, i_1 \tag{4}$$

Substituting equation 4 in 2

$$50\angle 45° = (5 + j15)\, i_1 + j8\left(\frac{8}{3}\right) i_1$$

From which $i_1 = \dfrac{150\ \angle 45°}{15 + j109} = 1.363\ \angle{-37.165}°\text{A}$

Therefore voltage across 5Ω resistor is $5i$,

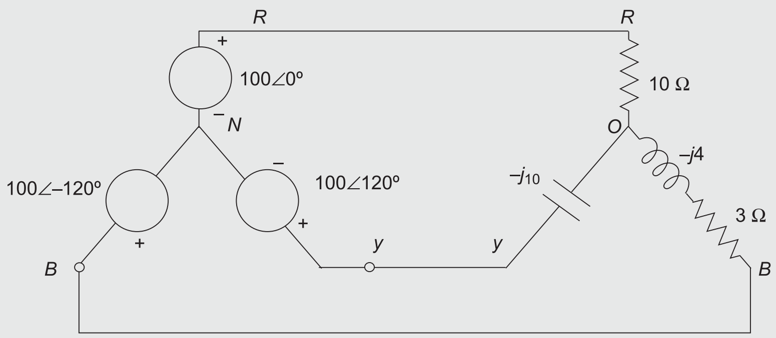
$= 5 \times 1.363 = 6.815$ volts

7. (a) State and explain Millman's theorem.

Solution

Refer Section 3.10 (Chapter 3)

(b) Using Millman's theorem find the neutral shift voltage V_{ON}.



Solution

Refer Example 9.21 (Chapter 9)

Converting load impedances into admittances

$$Y_R = \frac{1}{10}\ \Omega;\ Y_y = \frac{j}{10};\ Y_B = \frac{1}{3 + j4}$$

According to Millmans theorem the neutral shift voltage V_{ON} due to unbalanced load is given by

$$V_{ON} = \frac{V_{RN}\ Y_N + V_{YN}\ Y_Y + V_{BN}\ Y_B}{Y_R + Y_B + Y_Y}$$

$$V_{ON} = \frac{100\ \angle 0°\left(\dfrac{1}{10}\right) + 100\ \angle 120°\left(\dfrac{j}{10}\right) + 100\ \angle -120°\left(\dfrac{1}{3 + j4}\right)}{\dfrac{1}{10} + \dfrac{j}{10} + \dfrac{1}{3 + j4}}$$

$$V_{ON} = \frac{10 + 10\ \angle 210° - 19.856 - j\,2.392}{0.22 - j\,0.06}$$

$$= \frac{-18.5166 - j\,7.3923}{0.22 - j\,0.06} = -69.81 - j52.64$$

$V_{ON} = 87.43 \angle -142.98$ V

8. (a) Explain initial value theorem of Laplace transform.

For $\quad I(s) = \dfrac{s+4}{(s+2)\,(s+3)}$, find $I(0)$

Solution

<u>Refer Section 13.4(g) (Chapter 13)</u>

From initial value theorem

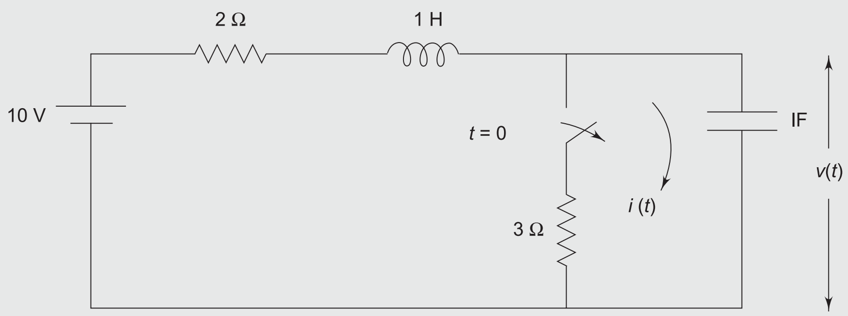
$$I(0) = \underset{s\to\infty}{\text{lt}}\; sI\,(s)$$

$$I(0) = \underset{s\to\infty}{\text{lt}}\; \frac{s(s+4)}{(s+2)\,(s+3)}$$

$$= \underset{s\to\infty}{\text{lt}}\; \frac{s^2\,(1+4/s)}{s^2\left(1+\dfrac{2}{s}\right)\left(1+\dfrac{3}{s}\right)}$$

$$I(\infty) = \frac{(1+4/\infty)}{(1+2/\infty)\,(1+3/\infty)} = 1$$

8. (b) Draw the network in Laplace domain and find $I(s)$.



Solution

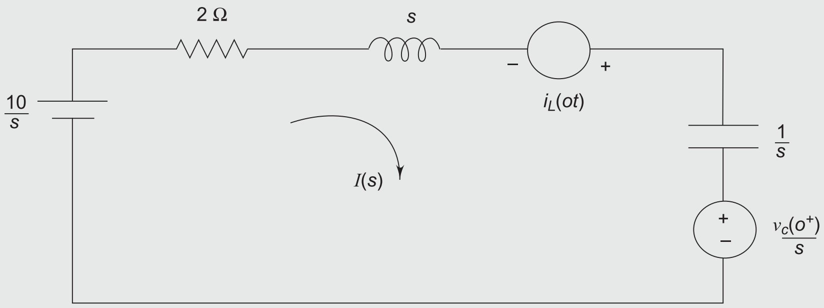
<u>Refer Problems 13.18 and 13.21 (Chapter 13).</u>

Before the switch is opened, the voltage across the capacitor is = voltage drop across $3\Omega = 10 \times \dfrac{3}{3+2} = 6$V

Therefore, $\qquad v_C(0^+) = 6$V

Initial current in the inductor before the opening of switch is $i_L(0^+) = \dfrac{10}{5} = 2$A

The transformed circuit in <s-domain



Applying KVL for the loop.

$$\frac{10}{s} = 2I(s) + sI(s) - i_L(0^+) + \frac{I(s)}{s} + \frac{v_C(0^+)}{s}$$

$$\frac{10}{s} = I(s)\left[s + \frac{1}{s} + 2\right] - 2 + \frac{6}{s}$$

$$I(s) = \frac{2(s+2)}{(s+1)^2}$$

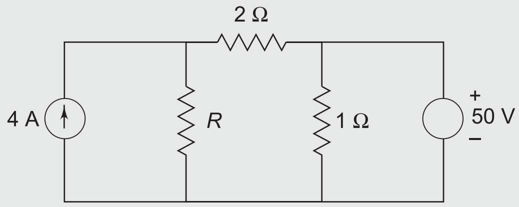
PAPER 8

1. (a) Differentiate between independent and dependent sources. What is their circuit representation.

Solution

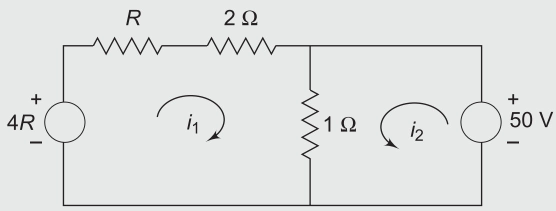
Refer Section 1.8 (Chapter 1)

1. (b) What is the value of R such that the powers supplied by both the sources are equal?



Solution

Converting current source into voltage source



Applying KVL for both the meshes

$$4R = (R + 3)\, i_1 + i_2 \tag{1}$$

$$50 = i_1 + i_2 \tag{2}$$

The power supplied by both the sources are equal

$$\therefore \quad 4Ri_1 = 50i_2$$

$$R = 12.5\, \frac{i_2}{i_1} \tag{3}$$

From eq 1

$$4R - i_1R - 3i_1 - i_2 = 0$$

$$R(4 - i_1) - 3i_1 - i_2 = 0 \tag{4}$$

Substituting equation 3 in 4

$$12.5\, \frac{i_2}{i_1}\, (4 - i_1) - 3\, i_1 - i_2 = 0 \tag{5}$$

$$50\frac{i_2}{i_1} - 13.5\, i_2 - 3\, i_1 = 0 \tag{6}$$

From equation 2, $i_2 = 50 - i_1$ (7)

Substituting equation 7 in 6

$$50\left(\frac{50 - i_1}{i_1}\right) - 13.5(50 - i_1) - 3i_1 = 0 \quad (8)$$

$$10.5i_1 - 725\, i_1 + 2500 = 0 \quad (9)$$

from which $i_1 = \dfrac{725 \pm 648.556}{21} = 65.407$ or 3.6402 A

If $i_1 = 65.407\text{A};$

from equation $2\ i_2 = -15.407\text{A}$

and $R = 12.5\ \dfrac{(-15.407)}{65.407} = -2.945\ \Omega$

If $i_1 = 3.6402$ A,

$i_2 = 46.3598\text{A}$

and $R = 12.5 \times \dfrac{46.3598}{3.6402} = 159.194\ \Omega$

Considering positive value of $R = 159.194$ W

Power supplied by current source

$$= 4 \times 159.194 \times 3.6402 = 2317.99 \text{ W}$$

Power supplied by voltage source

$$= 50 \times 46.3598 = 2317.99\ \Omega$$

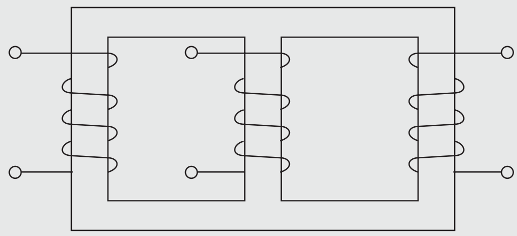
$\therefore$ The value of $R = 159.194\ \Omega$

2. (a) State and explain Faraday's law of electromagnetic induction. Distinguish between self and mutual induced voltages.

Solution

Refer Section 1.6 (Chapter 1).

2. (b) Explain "Dot convention" and determine the dotted ends of the set of coils shown in figure.



Solution

Refer Section 10.4 (Chapter 10).

2. (c) A circular iron ring having a cross section area of 5 cm^2 and a length of 4π cm in iron has an air gap of 0.1π cm made as a saw cut. The relative permeability of iron is 800. The ring is wound with a coil of

2000 turns and carries a current of 100 mA. Determine the air gap flux. Neglect leakage and fringing.

Solution

Refer Example 10.12 (Chapter 10)

Cross section area of Iron ring, $I_i = 5 \times 10^{-4} \text{m}^2$

Length of iron ring, $l_i = 4\pi \times 10^{-2}\text{m}$

Length of air gap, $l_g = 0.1\pi \times 10^{-2}\text{m}$

$\mu_r = 800$

No. of turns, $N = 2000$

$i = 100 \text{ mA}$

Total ampere turns (MMF) $= N \times i$

$= 2000 \times 100 \times 10^{-3}$

$= 200$ AT

Total reluctance $R = \dfrac{l_i}{a_i \mu_0 \mu_j} + \dfrac{l_g}{a_g . \mu_0}$

$$= \frac{4\pi \times 10^{-2}}{5 \times 10^{-4} \times 4\pi \times 10^{-7} \times 800} + \frac{0.1\pi \times 10^{-2}}{5 \times 10^{-4} \times 4\pi \times 10^{-7}}$$

$= 5.25 \times 10^6$ AT/wb

Air gap flux $= \dfrac{\text{Total MMF}}{\text{Reluctance}} = \dfrac{200}{5.25 \times 10^6}$

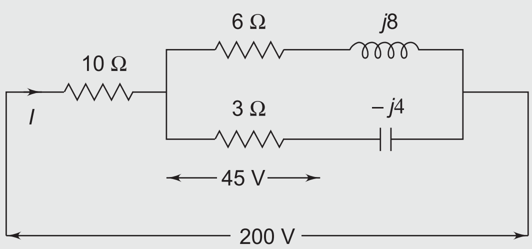
$\phi_g = 3\delta\mu w\text{b}$

3. (a) Define power factor, apparent power, active power and reactive power.

Solution

Refer Sections 6.2 and 6.3 (Chapter 6).

3. (b) Find complex power in the following circuit.



Solution

Taking the source voltage as reference

$V = 200 \angle 0\text{V}$

$$I = \frac{200\angle 0}{10 + \dfrac{(6+i8)(3-j4)}{(9+j4)}} = 13.396 + j1.886$$

$$= 13.52\angle 8^\circ$$

$$\text{Complex power} = V I^*$$

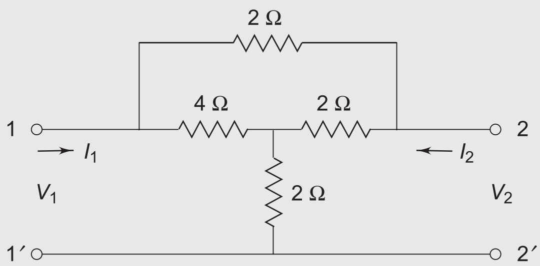
$$= (200\angle 0)(13.52\angle -8^\circ)$$

$$S = VI^* = 2704\angle -8^\circ \text{ VA}$$

Complex Power $(P + jQ) = 2704\angle{-80} = (2677.68 - j376.32)$

$$P = 2677.68\ W;\ Q = 376.32 \text{ VAR leading.}$$

4. (a) Obtain the y-parameters of the following bridged T-networks.



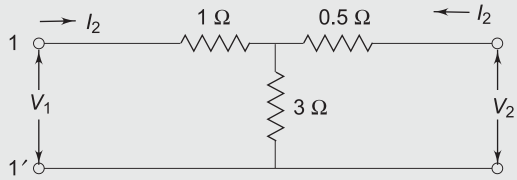
Solution

Refer Problem 15.12 (Chapter 15).

$$I_1 = y_{11} V_1 + y_{12} V_2$$

$$I_2 = y_{21} V_1 + y_{22} V_2$$

Convert delta to star and redraw the circuit.



$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{I_1}{\left(\dfrac{3\times 0.5}{3.5} + 1\right) I_1} = 0.7$$

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{-I_2 \times \dfrac{3}{4}}{\left(\dfrac{1\times 0.3}{4} + 0.5\right)} I_2 = -0.6$$

$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-I_1 \times \dfrac{3}{3.5}}{\left(\dfrac{3 \times 0.5}{3.5} + 1\right) I_1} = -0.6$$

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{I_2}{\left(\dfrac{3 \times 1}{3+1} + 0.5\right) I_2} = 0.8$$

$$y = \begin{bmatrix} y_{11} & y_{12} \\ y_{22} & y_{21} \end{bmatrix} = \begin{bmatrix} 0.7 & -0.6 \\ -0.6 & 0.8 \end{bmatrix}$$

4. (b) Obtain the expression for Y-parameter in terms of transmission parameters.

Solution

Refer Section 15.8.2 (Chapter 15)

4. (c) For a series resonance circuit obtain the expression for bandwidth in terms of resonance frequency and Q-factor.

Solution

Refer Section 8.4 (Chapter 8)

5. (a) Each phase of a balanced star connected load consists of $R = 10\ \Omega$ and $C = 10\mu$F. Calculate the line currents and total real and reactive powers when a symmetrical 400V, 50Hz, 3 phase supply is applied to it. If two wattmeters are employed to measure total power, find W_1 and W_2.

Solution

$$R = 10\Omega;\ C = 10\ \mu\text{F};\ f = 50\text{Hz}\ V_L = 400\text{V}$$

$$Z = (R \pm jX)$$

$$\chi_c = \frac{1}{2\pi\ f.C} = \frac{10^6}{2\pi \times 50 \times 10} = 318.3\Omega$$

$$\therefore \quad Z = 10 - j318.3\Omega = 318.466\ \angle{-88.Z}°$$

Power factor = cos(–88.2) = 0.0314 leading.

The line currents which are also equal to phase currents are

$$I_R = \frac{\dfrac{400\ \angle 0}{\sqrt{3}}}{318.466 - 88.20}$$

$$= 0.725\ \angle 88.2°$$

Similarly we can write

$$I_Y = \frac{\dfrac{400\ \angle -120°}{\sqrt{3}}}{318{\cdot}466\ \angle -88{\cdot}2^{\circ}}$$

$$= 0.725\ \angle{-31.8}°$$

$$I_B = \frac{\dfrac{400\angle -240°}{\sqrt{3}}}{318{\cdot}466\angle -88{\cdot}2°}$$

$$= 0.725\angle{-151.8°}$$

Readings of the two wattmeters

$$W_1 = V_L I_L \cos(30 + \phi)$$
$$= 400 \times 0.721 \cos(30 + 88.2°)$$
$$= -136.28\ W$$
$$W_2 = V_L I_L \cos(30 - \phi)$$
$$= 400 \times 0.721 \cos(30 - 88.2°)$$
$$= 151.97\ W$$

Total active power = $W_1 + W_2$

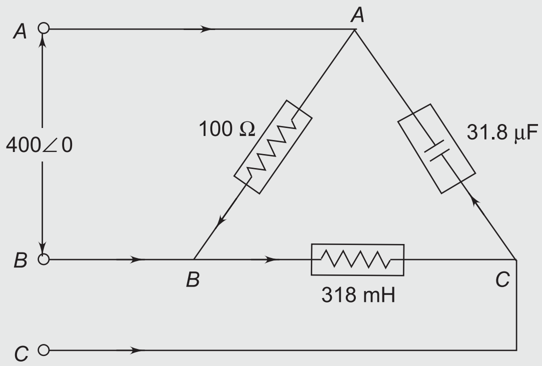
$$P_T = 15.69\mu w$$

Total reactive power = $\sqrt{3}\ (W_1 - W_2)$

$$Q = \sqrt{3}\ (-136.28 - 151.97)$$
$$= -500\ \text{VAR}$$

or $$Q = \sqrt{3}\ V_L I_L \sin\phi$$
$$= \sqrt{3} \times 400 \times 0.725 \sin(88.2°).$$

5. (b) A 400V, 50 Hz, 3 phase supply of phase sequence ABC is applied to a delta connected load consisting of 100 Ω between lines A & B, 318 mH inductance between lines B&C and 31.8μF capacitance between lines C&A. Determine phase and line currents.



From the given data

$$R = 100\Omega,\ X_L = j100\Omega;\ X_c = -100\Omega$$
$$Z_{ab} = 100\angle 0;\ Z_{bc} = j100\Omega;\ Z_{ca} = -j100\Omega$$

Phase currents

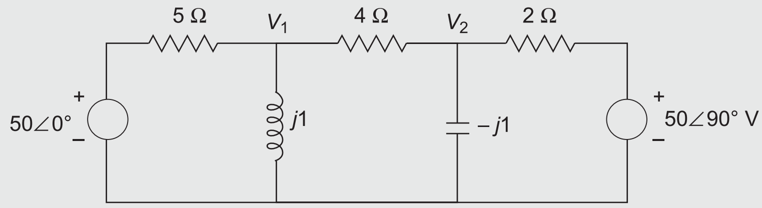
$$I_{AB} = 400\angle 0/100\angle 0 = 4\angle 0$$

$$I_{BC} = 400\ \angle 120°/100\ \angle 90° = 4\angle{-210}^{\circ}$$
$$I_{CA} = 400\ \angle{-240}°/100\angle{-90}° = 4\angle{-210}$$

Line currents

$$I_A = I_{AB} - I_{CA} = 7.72\ \angle 15°\text{A}$$
$$I_B = I_{BC} - I_{AB} = 7.727\ \angle 165°\text{A}$$
$$I_C = I_{CA} - I_{BC} = 4\ \angle{-90}°\text{A}$$

6. (a) In the network shown below find current I using nodal analysis.



Solution

<u>Refer Example 7.3 (Chapter 7).</u>

Writing node equations at node 1 and 2

$$\frac{V_1 - 50}{5} + \frac{V}{j_1} + \frac{V_1 - V_2}{4} = 0 \tag{1}$$

$$\frac{V_2 - V_1}{4} + \frac{V_2}{j_1} + \frac{V_2 - 50\ \angle 90°}{2} = 0 \tag{2}$$

Simplifying equation 1 leads to

$$(0.45 - j)\ V_1 - 0.25\ V_2 = 10 \tag{3}$$

Simplifying equation 2 leads to

$$-0.25\ V_1 + (0.75 + j)\ V_2 = 25\angle 90° \tag{4}$$

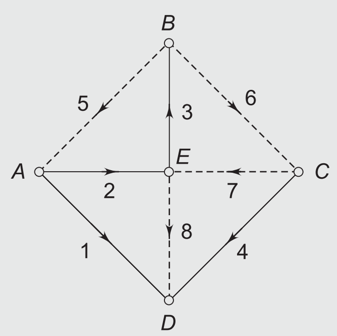
Solving equations 3 and 4

$$V_1 = 2.732 + j13.28$$
$$V_2 = 18.43 + j13.156.$$

Current $I = \dfrac{V_1 - V_2}{4}$

$$= -3.9245 - J0.056$$
$$I = 3.92\ \angle{-179.18}°$$

6. (b) Obtain the basic cut-set matrix for the given oriented graph, taking 1, 2, 3, 4 as tree branches. Write down KCL network equations from the matrix.



Solution

Refer Section 2.8.2 (Chapter2).

The fundamental cut-set or basic cut-set matrix are defined for a given tree of the graph. The procedure is to select a tree and then a twig is selected removing this twig from the tree separates the tree into two-parts. All the links which go from one part of the disconnected tree to the other, together with the twig of the selected tree will costitute a cut-set. The fundamental cut-set matrix Q_f is one in which each row represents a cut-set with respect to a given tree of the graph, and the columns correspond to the branches of the graph.

For each twig there will be a basic cut-set therefore for a network graph with n nodes and b branches, there will be $(n - 1)$ number of basic cut-sets.

From the given graph the number of nodes are 5. The twigs of the tree are 1, 2, 3, 4 and the links are 5, 6, 7, 8.

The number of basic cut-sets = (5 – 1) = 4.

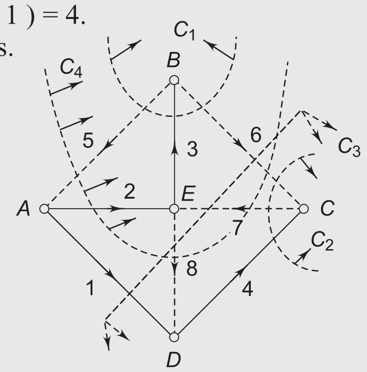
The tree is represented by solid lines.

Consider twig. 3 Corresponding to twig 3. The f-cut set is {3, 5,6} which is cut-set C_1. Its orientation coincides with the defining twig 3.

Corresponding to twig 4, the *f*-cut set is {4, 6, 7}

Which is cut-set C_2 having the same orientation as 4.

Corresponding to twig 1 the *f*-cut set is {1, 6, 7, 8}



Which is cut-set C_3. The orientation of C_3 is coincident with the direction of twig 1.

Corresponding to twig 2, the *f*-cut set is {2, 5, 6, 7, 8} which is cut-set C_4.

The *f*-cut-set matrix is written as follows:

f-cut-sets branches $\longrightarrow$

$\downarrow$

$$Q_f = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix} \end{array}$$

The basic property of the fundamental cut-sets is that they give linearly independent KCL equations.

Applying KCL to the *f*-cut-sets of the graph,

$$C_1 : i_3 - i_5 - i_6 = 0$$

$$C_2 : i_4 + i_6 - i_7 = 0$$

$$C_3 : i_1 + i_6 - i_7 + i_8 = 0$$
$$C_4 : i_2 - i_5 - i_6 + i_7 - i_8 = 0$$

In the matrix form

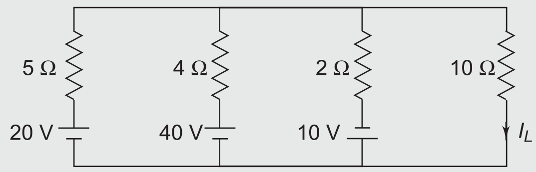
$$\begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

7. (a) State and explain Millman's theorem.

Solution

Refer Section 3.10 (Chapter 3).

7. (b) Find the current I_L. Use Millman's theorem.



Solution

Refer Example 3.10 (Chapter 3)

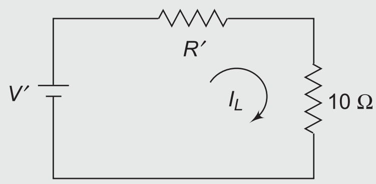
Millman's theorem states that

$$V' = \frac{V_1 G_1 + V_2 G_2 + \cdots + V_n G_n}{G_1 + G_2 + G_3 + \cdots + G_n}$$

$$R' = \frac{1}{G_1 + G_2 + \cdots + G_n}$$

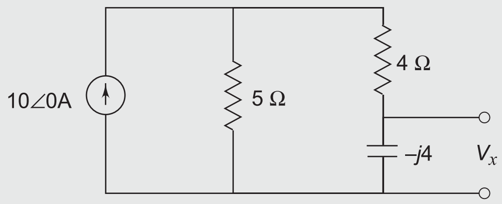
$$\therefore \qquad V' = \frac{20 \times \frac{1}{5} + 40 \times \frac{1}{4} + \left(-10 \times \frac{1}{2}\right)}{\frac{1}{5} + \frac{1}{4} + \frac{1}{2}} = 9.47736$$

$$R' = \frac{1}{\frac{1}{5} + \frac{1}{4} + \frac{1}{2}} = 1.0526$$



$$I_L = \frac{9.4736}{1.0526 + 10} = 0.857\text{A}$$

7. (c) Verify the reciprocity theorem for the network shown.



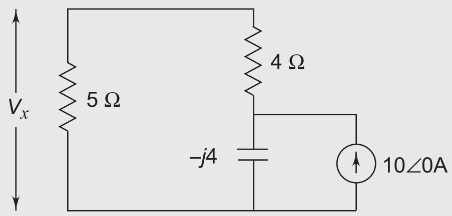
Solution

$$V_x = \frac{10 \times 5}{5 + 4 - j4}(-j4) = 8.24 - j18.556$$

$$= 20.3\ \angle{-66°}$$

$$\text{Output/input} = \frac{20.3\ \angle -66°}{10\ \angle 0°} = 2.03\ \angle{-66°}.$$

Exchanging the excitation and response



$$V_x = \frac{10\ \angle 0°\ (-j4)}{5 + 4 - j4} \times 5 = 20.3\ \angle{-66°}$$

$$\text{Output/input} = \frac{20.3\ \angle -66°}{10\ \angle 0°}$$

$$= 2.03\ \angle{-66°}$$

8. (a) Explain the final value theorem of Laplace transform.

Solution

Refer Section 13.4 (h) Chapter 13.

8. (b) Find $V_{(\infty)}$ given $V(s) = \dfrac{S^2 + 2S + 3}{S(S+1)(S^2 + 2S + 2)}$.

Final value $V_\infty = \underset{\cancel{S} \to 0}{\cancel{\text{Lt}}}\ \cancel{S}\, V(\cancel{S}) = \underset{\cancel{S} \to 0}{\cancel{\text{Lt}}} \dfrac{\cancel{S}(\cancel{S}^2 + 2\cancel{S} + 3)}{\cancel{S}(\cancel{S} + 1)(\cancel{S}^2 + 2\cancel{S} + 2)}$

Therefore, $V_\infty = \dfrac{3}{2}$ V.

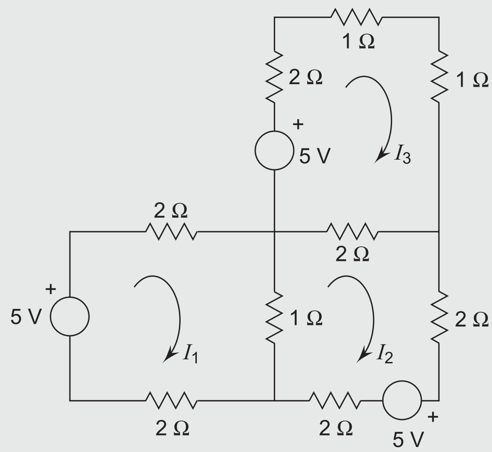
PAPER 9

1. (a) Write a note on source transformation.

Solution

Refer Section 2.15.

(b) Using KCL and KVL, find the currents in all the sources of the circuit of the following figure.



Solution:

Using KVL, the loop equations can be written as

$$5 = 5I_1 - I_2 \quad (1)$$

$$-5 = 7I_2 - I_1 - 2I_3 \quad (2)$$

$$5 = 6I_3 - 2I_2 \quad (3)$$

Solving Eq.(1), Eq. (2), and Eq. (3), we get

$$I_1 = 0.92 \text{ A}$$

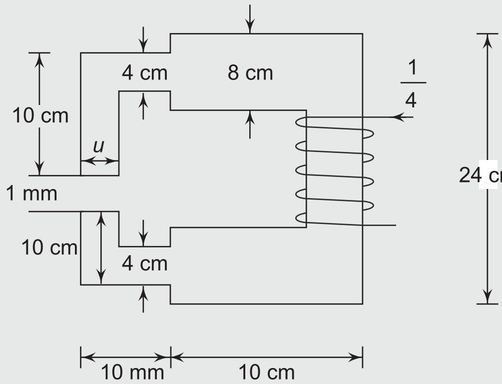
$$I_2 = -0.38 \text{ A}$$

$$I_3 = 0.706 \text{ A}$$

2. (a) Calculate the current to be passed through the coil so that a flux of 1 mwb is produced in the air gap (as shown in the following figure) the case of square cross section over its entire length and has permeability of 800.

$$\phi = 1 \text{ mwb}$$

$$u_r = 800$$



Assuming no. of turns = 500
Flux produced is given by

$$\phi = \frac{\text{mmf}}{\text{total reluctance}}$$

$$\phi = \left[\frac{\text{mmf}}{\frac{l_1}{u_1 A_1} + \frac{l_2}{u_1 A_2} + \frac{l_3}{u_3 A_3}}\right]$$

By dividing the given fig. in no. of section

$$1 \times 10^{-3} = \frac{I \times 500}{\frac{(20+20)\,10^{-2}}{800 \times 4\pi \times 10^{-7} \times 16 \times 10^{-4}} + \frac{(20+8)\,10^{-2}}{800 + 4\pi \times 10^{-7} \times 64 \times 10^{-4}} + \frac{1 \times 10^{-3}}{4\pi \times 10^{-7} \times 4 \times 10^{-5}}}$$

$$1 \times 10^{-3} = \frac{1 \times 500 \times 4\pi \times 10^{-7}}{0.3125 + 0.05459 + 25}$$

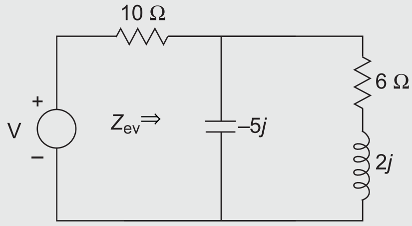
$$I = 40.37 \text{ A}$$

(b) Define the following terms
 (i) Co-efficient of coupling in coupled coils
 (ii) Magnetic flux density
 (iii) Reluctance of magnetic path
 (iv) Permeability

Solution:

Refer Sections 10.5; 10.11

3. (a) A series R-C circuit is excited by sinusoidal voltage find the expression for impedance using phasor diagram.



Solution: Refer Section 5.1

(b) Determine the supply voltage and the power factor in the following figure network if the total power delivered is 200 W.

Solution:

$$z_{\text{eq}} = 10 + (-5\,j||(6 + 2\,j)$$
$$= 13.33 - 3.33\,j$$

$\therefore$ $$z_{\text{eq}} = 13.33 - 3.33\,j = 13.74\ \angle{-14.02}$$

but given that total power delivered is 200 W

$\therefore$ $$200 = I^2\,(13.33)$$
$$I = 3.87\text{A}$$

$\therefore$ $$V = IZ = 3.87(13.33 - 3.33\,j)$$
$$= 51.63 - 12.89\,j$$
$$= 53.22\ \angle{-14.02}$$

$\therefore$ $$V = 53.22\ \angle{-14.02}\ \text{V}$$

$$\text{Power factor} = \frac{R}{Z} = \frac{13.33}{13.74},\ 0.97$$

$\therefore$ Supply voltage is 53.22 $\angle$–14.02 V

and power factor is 0.97

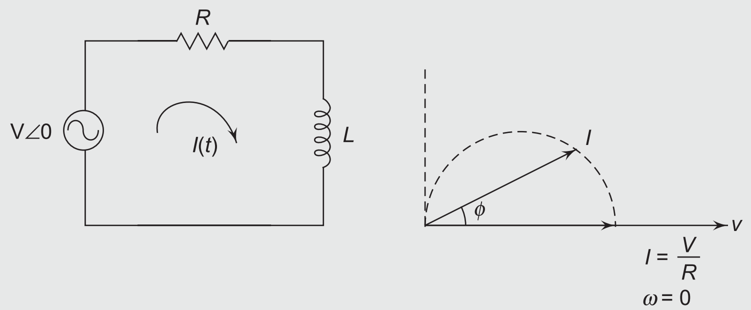
4. (a) For a series RL circuit obtain the locus of current as inductance is changed from 0 to ∞ when the applied voltage is constant.

Solution:

Refer Section 8 :13.1

$$\omega = 0\ J\omega L = 0 = I = \frac{V}{R}\quad \phi = Ta^{-1}\left(\frac{\omega L}{R}\right)$$

$$\omega = x\ J\omega L = \alpha, = I = 0$$

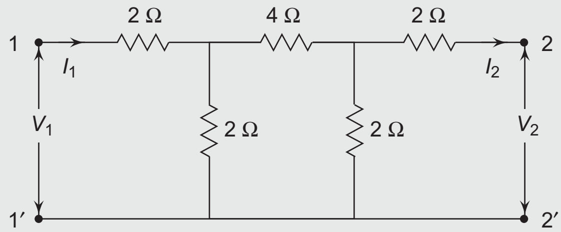


(b) Show that for a series resonant circuit $f_1 f_2 = f_r^2$ where f_1 and f_2 are half power frequencies and f_r is the resonance frequency.

Solution:

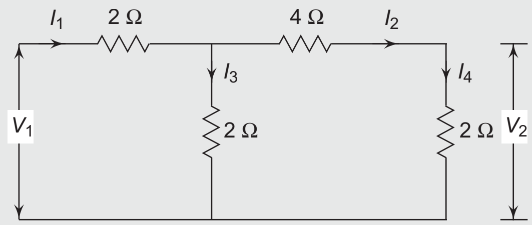
Refer Section 8.4

(c) Obtain the z-parameters of the following two-port Networks.



Solution:

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$



when $I_2 = 0$

$$V_1 = I_1 (3.5)$$

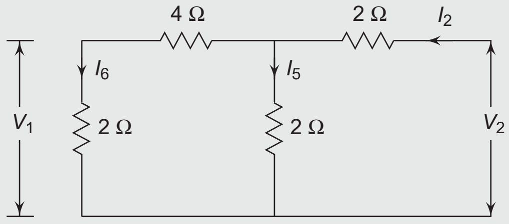
$$\therefore \quad Z_{11} = \frac{V_1}{I_1} = 3.5$$

$$Z_{21} = \frac{V_2}{I_1}$$

$$V_2 = I_4 2 \text{ but } I_4 = \frac{I_1(2)}{4+2+2} = \frac{I_1(\cancel{2})}{\cancel{8}_4} = \frac{I_1}{4}$$

$$\therefore \quad V_2 = \frac{I_1}{\cancel{4}_2}(\cancel{2}) = \frac{I_1}{2}$$

$$Z_{21} = \frac{V_2}{V_1} = \frac{1}{2} = 0.5$$



when $I_1 = 0$

$V_2 = I_2\,(3.5)$

$\therefore \quad Z_{22} = \dfrac{V_2}{I_2} = 3.5$

and

$$Z_{12} = \frac{V_1}{I_2}$$

but $\quad V_1 = I_6\,2 \quad$ but $\quad I_6 = \dfrac{I_2(2)}{8} = \dfrac{I_2}{4}$

$$V_1 = \frac{I_2}{4}(2) = \frac{I_2}{2}$$

$$Z_{12} = \frac{V_1}{I_2} = \frac{I}{2} = 0.5$$

$\therefore$ Z-parameters are

$Z_{11} = 3.5 \quad Z_{12} = 0.5$

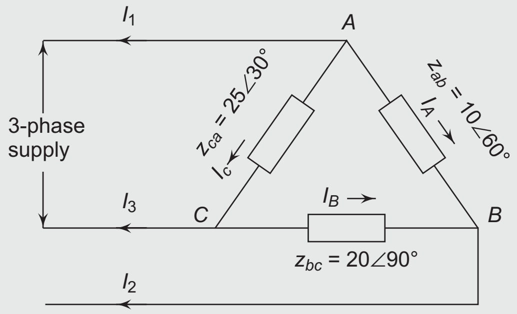
$Z_{21} = 0.5 \quad Z_{22} = 3.5$

5. (a) Determine the line currents and total power supplied to a delta connected load of $z_{ab} = 10\angle 60°$, $z_{bc} = 20\angle 90°$ and $z_{ca} = 25\angle 30°$. Assume a 3-phase, 400 V, ABC system.

Solution: $V_{AB} = 400\angle 0° \qquad Z_{ab} = 10\angle 60° = 5 + 8.66j$

$V_{BC} = 400\angle -120° \qquad Z_{bc} = 20\angle 90° = 0 + 20j$

$V_{CA} = 400\angle -240° \qquad Z_{ca} = 25\angle 30° = 21.65 + 12.5j$



$$I_A = \frac{V_{AB}}{Z_{ab}} = \frac{400\angle 0°}{10\angle 60°} = 20 - 34.64j = 40\angle -60°$$

$$I_B = \frac{V_{BC}}{Z_{bc}} = \frac{400\angle -120}{20\angle 90°} = 20\angle 150°$$

$$I_C = \frac{V_{CA}}{Z_{CA}} = \frac{400\angle{-240}}{25\angle 30^\circ} = 16\angle 90^\circ$$

Line currents

$$I_1 = I_A - I_C = (40\ \angle{-60}) - (16\ \angle 90) = 54.44\ \angle{-68.44}\text{ A}$$
$$I_2 = I_B - I_A = (20\ \angle 150) - (40\ \angle{-60}) = 58.18\ \angle 129.89\text{ A}$$
$$I_3 = I_C - I_B = (16\ \angle 90) - (20\ \angle 150) = 18.33\ \angle 19.10\text{ A}$$

Power

Power in A phase $= I_A^2\, R_A = (40)^2\,(5) = 8000$ W

Power in B phase $= I_B^2\, R_B = (20)^2\,(0) = 0$

Power in C phase $= I_C^2\, R_C = (16)^2\,(21.65) = 5542.4$ W

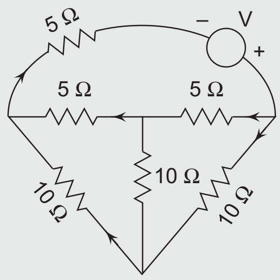
Total power consumed by load is 8000 + 5542.4 = 13542.4 W

(b) Derive the Relationship between line and phase voltages in a balanced three phase star connected load.

Solution:

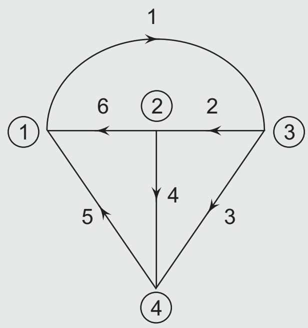
Refer Sections 9.7.1; 9.7.2 and 9.7.3

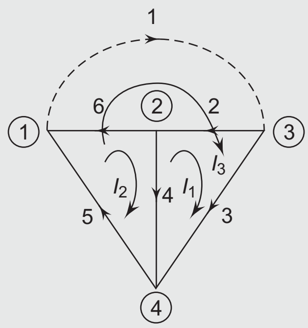
6. (a) Explain clearly what you understand by a cutset and tieset. Write down the basic tieset schedule for the network shown in the figure by taking 10 Ω resistor branches as free branches.



Solution: From the N/w the graph is to be drawn.

Now select 10Ω-resistor branches as free branches then tieset matrix.

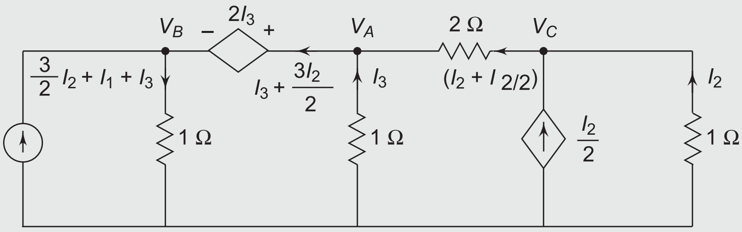




Tie set (loop) Branches →

$$\begin{array}{c} \downarrow \\ I_1 \\ I_2 \\ I_3 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b) For the N/W shown in figure determine the ratio of I_2/I_1.



Solution:

$$V_A = -I_3 \times 1 = -I_3$$

$$V_B = \left(I_1 + \frac{3}{2}I_2 + I_3\right) \times 1$$

$$V_A - V_B = 2I_3$$

$$\Rightarrow \quad -I_3 - I_1 - \frac{3}{2}I_2 - I_3 = 2\,I_3$$

$$\Rightarrow \quad 4I_3 = -I_1 - \frac{3}{2}\,I_2 \tag{1}$$

$$V_C = -I_2 \times 1 = -I_2$$

$$V_C - V_A = -I_2 + I_3 = \frac{3I_2}{2} \tag{2}$$

$$I_3 = \frac{5}{2}\,I_2 = 2.5I_2 \tag{3}$$

From Eqs (1) and (2)

$$4(2.5I_2) = -I_1 - \frac{3}{2}\,I_2$$

$$\left(10 + \frac{3}{2}\right)\,I_2 = -I_1$$

$$\boxed{\frac{I_2}{I_1} = \frac{-2}{23}}$$

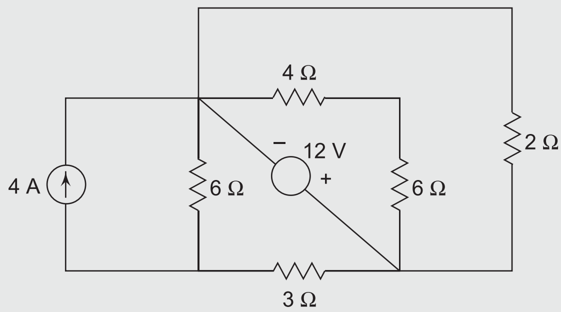
7. (a) State the explain the superposition theorem?

Solution:

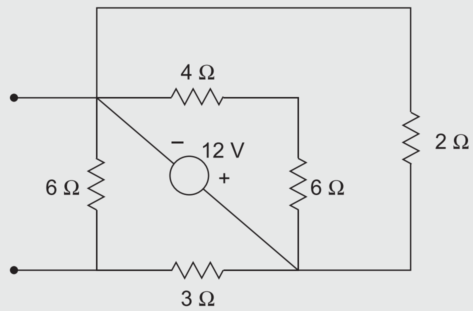
Refer Section 3.2

(b) Using superposition theorem find the current in 2Ω. Verify your result by any other method.

To know the current in the 2Ω resistor

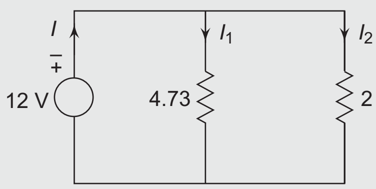


(i) Only having voltage source i.e. current source is replaced by a open circuit.



$$(6 + 3)||(4 + 6) \Rightarrow 9 \,||\, 10$$
$$= 4.736\ \Omega$$

the circuit can be drawn as

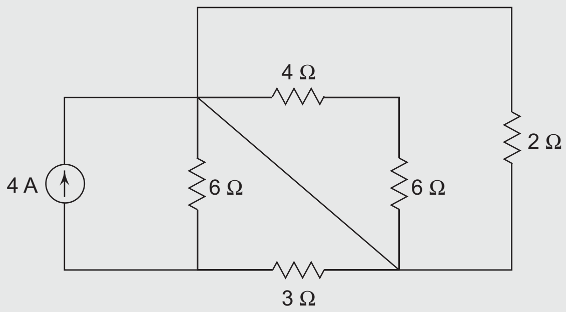


$$I_T = \frac{-12}{(4.73||2)} = \frac{-12}{1.406}$$
$$= -8.533$$

then $$I_2 = \frac{4.73\, I_T}{4.73+2} = \frac{4.73\,(-8.533)}{4.73+2}$$
$$= -5.997$$
$$\approx -6.00$$

(ii) Only current source is present, and voltage source is replaced by short circuit.
By this short circuit, the current flowing through the 2Ω resistance is zero.

$\therefore$ $I = 0$ A



$\therefore$ total current flowing through 2Ω resistor is

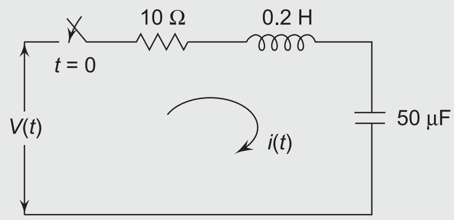
$-6 + 0 = -6$ Ar

8. (a) Derive the expression for $i(t)$ for R–L series circuit when excited by a sinusoidal source.

Solution:

<u>Refer Sections 4.5, 4.6 and 5.1</u>

(b) For R-L-C series circuit with $R = 10\ \Omega$, $L = 0.2$ H, $C = 50\ \mu$F determine the current $i(t)$ when the switch is closed at $t = 0$. Applied voltage is $V(t) = 100 \cos(1000t + 60°)$



$$V(t) = 100 \cos(1000t + 60)$$

Loop equation is

$$V(t) = 10\, i(t) + 0.2\, \frac{di(t)}{dt} + \frac{1}{50 \times 10^{-6}} \Big| idt$$

$$10i(t) + 0.2\, \frac{di}{dt^2} + \frac{1}{50 \times 10^{-6}} \Big| idt \;= 100 \cos(1000t + 60)$$

$$10 \frac{di}{dt} + 0.2 \frac{di}{dt^2} \;+ 2 \times 10^5\, i(t) = -100 \sin(1000t + 60) \times 1000\ (2)$$

$$[0.20^2 + 100 + 20000]i = -10^5 \sin(1000t + 60)$$

Characteristic equation

$$0.2D^2 + 10\, D + 20000 = 0$$

$$D = -25 \pm 315.23j$$

complementary solution:

$$i_c = e^{-25t}\,[c_1 \cos 315.23t + c_2 \sin 315.23t]$$

Assume particular solution

$$i_p = A \cos (1000t + 60) + B \sin (1000t + 60)$$

$$\frac{di_p}{dt} = -\,1000\text{ A sin }(1000\text{t} + 60) + 1000 \cos (1000\text{t} + 60)$$

$$\frac{d^2ip}{dt^2} = -(1000)^2 A \cos (1000t + 60) - (1000)^2 \sin (1000t + 60)$$

Substituting these values in Eq. (2)

$10[-1000\,A \sin (1000t + 60) + 1000 \cos (1000t + 60)]$
$+ 0.2\,[-(1000)^2 A \cos (1000t + 60) - (1000)^2 \sin (1000t + 60)] + 2 \times 10^5$
$[A \cos (1000t + 60° + B \sin (1000t + 60)] = -100 \sin (1000t + 60)\ 1000$

$$-B(1000)^2 - \frac{A(1000)(10)}{0.2} + \frac{B}{0.2\,(50 \times 10^{-6})} = \frac{-100(1000)}{(0.2)}$$

$$\Rightarrow \qquad A(50.000) + B[900.000] = 500000 \qquad (3)$$

$$-A(1000)^2 + \frac{B\,(1000)(10)}{0.2} + \frac{A}{0.2(50 \times 10^{-6})} = 0$$

$$A[-900{,}000] + \text{B } 50000 = 0 \qquad (4)$$

Solving for (3) and (4)

$$A = 0.03$$

$$B = 0.55$$

$$i_p = 0.03 \cos (1000t + 60) + 0.55 \sin (1000t + 60)$$

solution is

$$i = e^{-25t}\,[c_1 \cos 315.23t + c_2 \sin 315.23t]$$
$$+ [0.03 \cos (1000t + 60) + 0.55 \sin (1000t + 60)]$$

to evaluate, c_1 and c_2

$$i = 0 \text{ when } t = 0$$

$$O = (1)\,[c_1 + 0] + 0.03 \cos (60) + 0.56 \sin (60)$$

$$C_1 = -[0.03 \cos 60 + 0.55 \sin 60] = -0.491$$

$$\frac{di}{dt} = e^{-25t}\,(-25)\,[c_1 \cos 315.23t + c_2 \sin 315.23t]$$
$$+ e^{-25t}\,[-c_1 \sin 315.23t\,(315.23) + c_2 \cos 315.23t\,(315.23)]$$
$$-0.03 \sin (1000t + 60)\,(1000) + 0.55 \cos (1000t + 60)\ 1000$$

when $t = 0, \dfrac{di}{dt} = 250$

$$\therefore \quad 250 = (-25)\,[c_1] + [c_2\,(315.23)] - 0.03 \times 1000 \sin 60 + 0.55 \times 1000 \cos (60)$$

$$250 = -25\,(-0.491) + c_2\,(315.23) - 0.03 \times 1000 \sin 60 + 0.55 \times 1000 \cos 60$$

$$c_2 = 0.035$$

$\therefore$ Solution is

$$i(t) = e^{-25t}\,[-0.491 \cos 315.23t + 0.035 \sin 315.23t] + 0.03 \cos (1000t + 60) + 0.55 \sin (1000t + 60)$$

PAPER 10

1. (a) What are passive and active circuit elements? Explain the voltage-current relationships of passive elements with examples.

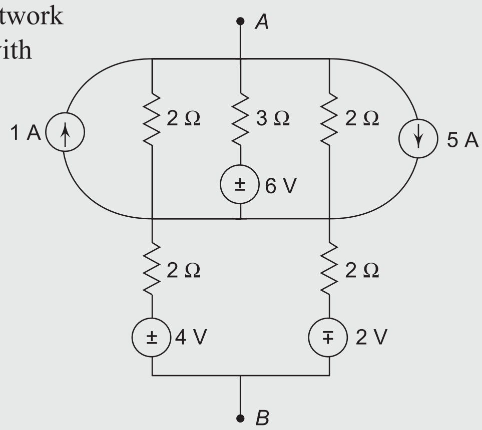
Solution:

Refer Sections 1.4.1; 1.5, 1.6 and 1.7

(b) Reduce the network of figure below into an equivalent network across terminals *A* and *B* with

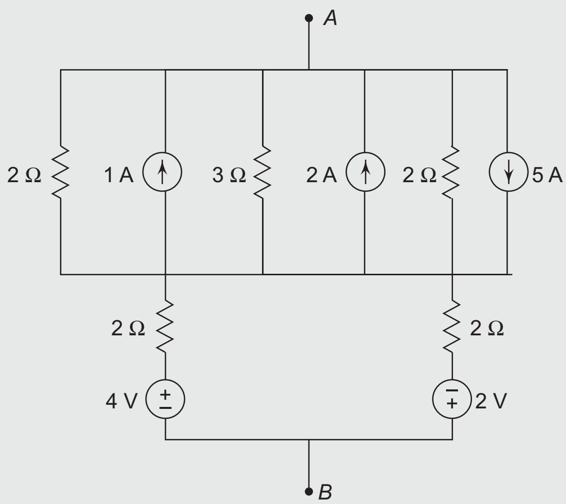
(i) one equivalent voltage source

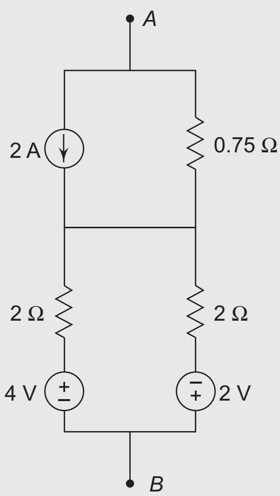
(ii) one equivalent current source

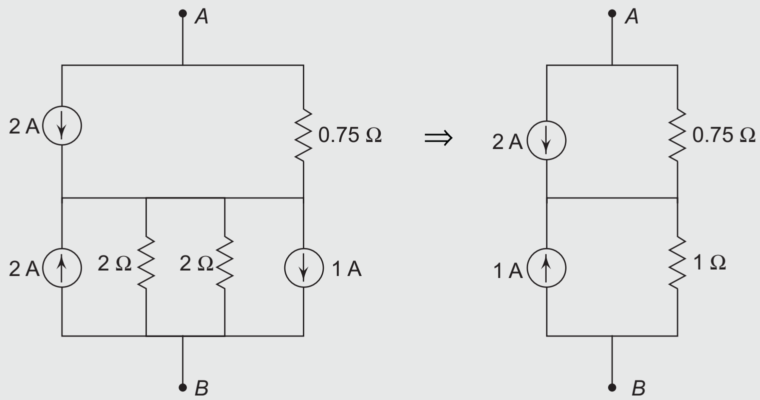


Solution:

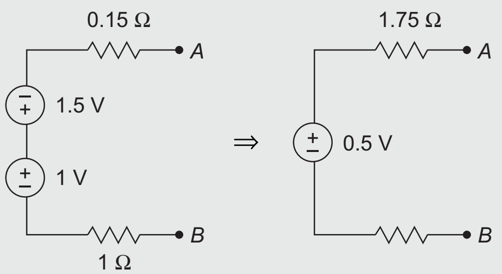
Using the source transformation, we get the N/W



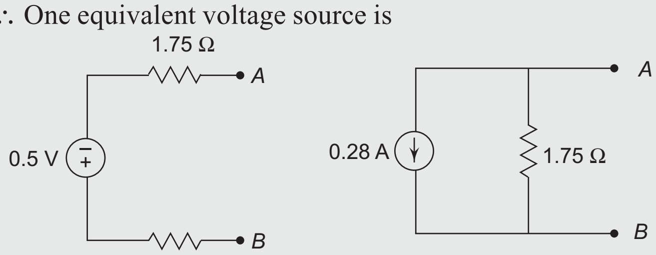




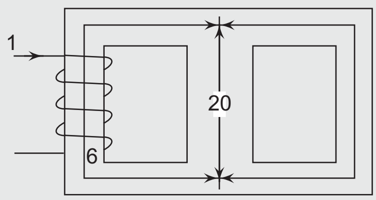
By again converting current source, into voltage source



$\therefore$ One equivalent voltage source is



One equivalent current source is



2. (a) A cast steal iron core has a square cross section of side 3 cm. Assuming the permeability of steel to be 800, find the mmf required to produce a flux $\phi = 0.2$ mwb 105 the right limb as shown in the figure.

Solution:

$$\phi = 0.2 \text{ mwb}$$
$$\mu_r = 800$$
$$H_g = 3 \times 3 \times 10^{-4} \text{ m}^2$$
$$\phi = 0.2 \times 10^{-3} \text{ wb}$$

$$B = \frac{\phi}{A} = \frac{0.2 \times 10^{-3}}{9 \times 10^{-4}} = 0.22 \text{ wb/m}^2$$

$$H = \frac{B}{\mu_o \mu_r} = \frac{0.22}{4\pi \times 10^{-7} \times 800} = 221.04 \text{ AT/m}$$

mmf required is given by Hl = $60 \times 10^{-2} \times 221.04$

$\therefore$ (ϕ = NI = Hl) = 132.62 AT

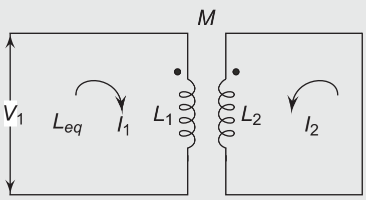
$\therefore$ mmf required to produce 0.2 mwb in right limb is 132.62 AT

(b) Define self and mutual inductances. Establish the polarity of two mutually coupled coils on a single magnetic core.

Solution:

Refer Section 10.3

3. (c) Find the equivalent inductance of the following circuit figure.



Solution: $\quad V_1 = L_{eq} \dfrac{di_1}{dt}$ (1)

but by applying mesh analysis

$$V_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$O = L_1 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$\therefore \quad \frac{di_2}{dt} = \frac{-M}{L_2} \frac{di_1}{dt}$$

$$\therefore \quad V_1 = L_1 \frac{di_1}{dt} + M \left[\frac{-M}{L_2} \frac{di_1}{dt} \right]$$

$$\Rightarrow \quad L_1 \frac{di_1}{dt} - \frac{M^2}{L_2} \frac{di_1}{dt} \Rightarrow \left(L_1 - \frac{M^2}{t_2} \right) \frac{di_1}{dt} \qquad (2)$$

Compare Eq. (1) with Eq. (2)

$$\boxed{L_{eq} = L_1 - \frac{M^2}{L_2}}$$

$\therefore L_{eq}$ for the magnetic circuit is obtained.

3. (a) Explain about active, reactive and apparent powers. Give expression for the above. Draw the power triangle.

Solution:

Refer Sections 6.3; 6.4 and 6.5

(b) Given $i = 50 \sin(\omega t + 60)$

$V = 200 \sin(\omega t + 30)$ find the elements of the network with their values active, reactive and apparent power.

Solution: $i = 50 \sin(\omega t + 60)$

$V = 200 \sin(\omega t + 30)$

Here the current leads the voltage by 30°

$\therefore$ the elements of the network are resistance and capacitance.

and the power factor of the network is, cos 30° = 0.866 (leading)

Active Power:

$$P_{\text{active}} = V_{\text{eff}} I_{\text{eff}} \cos\theta$$

$$= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos\theta$$

$$= \frac{\sqrt{50}}{\sqrt{2}} \cdot \frac{200}{\sqrt{2}} \cos 30 = 4330.12 \text{ W}$$

Reactive Power,

$$P_{\text{reactive}} = V_{\text{eff}} I_{\text{eff}} \sin\theta$$

$$= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \sin\theta$$

$$= -\frac{50}{\sqrt{2}} \cdot \frac{200}{\sqrt{2}} \sin 30 = -2500 \text{ VAR}$$

Apparent Power

$$P_{\text{apparent } P} = \sim V_{\text{eff}} I_{\text{eff}}$$

$$= \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} = \frac{50}{\sqrt{2}} \cdot \frac{200}{\sqrt{2}} = 5000 \text{ VA}$$

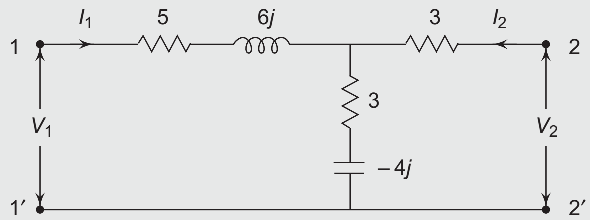
	Component in n.w. R.c.
$\therefore$ Active Power = 433012 W	$z = \frac{V}{I} = \frac{200\angle 30}{50\angle 60}$
Reactive Power = –2500 VAR	$= 3.464 - 2i$
Apparent Power = 5000 VA	$\boxed{\theta = 3.464\ \Omega}$
$R = 3.464\ \Omega$; $C = \frac{1}{2\omega}$	$\boxed{C = \frac{1}{2\omega}}\ \frac{11}{\omega c} = +2J$

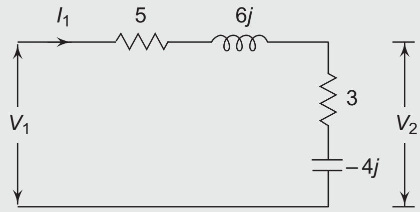
4. (a) Obtain the expression for frequency at which the voltage across the inductance becomes a maximum in a series RLC circuit. Explain what is meant by voltage magnification factor.

Solution:

Refer Sections 8.3; 8.6

(b) Obtain the transmission parameters for the following figure || circuit. Verify your result for reciprocity condition.





$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

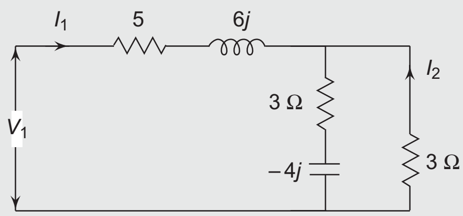
When $I_2 = 0$

$$V_1 = I_1(8 + 2j)$$

and $V_2 = I_1(3 - 4j)$

$$A = \frac{V_1}{V_2} = \frac{I_1(8+2j)}{I_1(3-4j)} = \frac{8+2j}{3-4j} = 0.64 + 1.52j$$

$$C = \frac{I_1}{V_2} = \frac{I_1}{I_1(3-4j)} = \frac{1}{3-4j} = 0.12 + 0.16j\ \mho$$



When $V_2 = 0$

$$B = \frac{-V_1}{I_2}$$

$$-I_2 = \frac{I_1(3-4j)}{6-4j}$$

$$-I_2 = +I_1\,(0.65 - 0.23j)$$

$$V_1 = I_1\,[(5 + 6j) + [(3 - 4j) \parallel 3]$$

$$= I_1\,[6.96 + 5.3j]$$

$$B = -\frac{V_1}{I_2} = \frac{I_1(6.96 + 5.3j)}{I_1(0.65 - 0.23j)} = 6.95 + 10.61j$$

$\therefore$ $B = 6.95 + 10.61j\ \Omega$

$$D = -\frac{I_1}{I_2} = \frac{I_1}{I_1(0.65 - 0.23j)} = 1.367 + 0.48j$$

$\therefore$ $A = 0.64 + 1.52j$

$B = 6.95 + 10.61j\ \Omega$

$C = 0.12 + 0.16j\ \mho$

$D = 1.367 + 0.48j$

Reciprocity condition:

$$AD - BC = 1$$

$$(0.64 + 1.52j)\,(1.367 + 0.48j) - (6.95 + 10.61j)\,(0.12 + 0.16j)$$

$$= 1.00 - 1.6 \times 10^{-4}j$$

$$= 1.008\ \angle -0.009 \approx 1$$

$\therefore$ Reciprocity condition is satisfied.

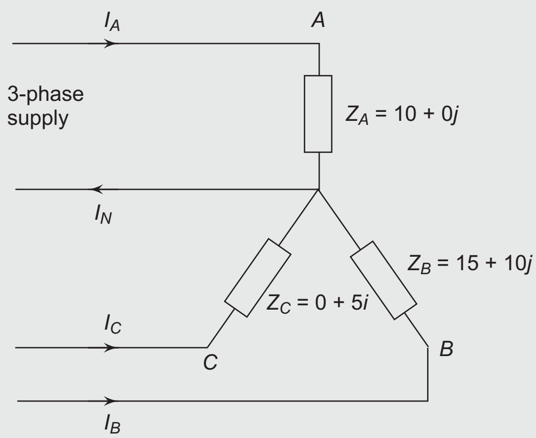
5. (a) Derive the relationship between line and phase voltages in a balanced three phase delta connected load

Solution:

<u>Refer Sections 9.8.1; 9.8.2 and 9.8.3</u>

(b) A 3 phase 400 V, 4 wire system has a star connected load with $z_A = (10 + j0)\ \Omega$, $z_B = (15 + j10)\ \Omega$, $z_C = (0 + 5j)\ \Omega$. Find the line currents and current through neutral conductor. Draw the phasor diagram.

Solution:



$$V_{\text{Ph}} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94$$

$$V_{AN} = 230.94 \angle 0 \text{ V}$$

$$V_{BN} = 230.94 \angle -120 \text{ V}$$

$$V_{CN} = 230.94 \angle -240 \text{ V}$$

$$I_A = \frac{V_{AN}}{Z_A} = \frac{230.94\angle 0}{(10+0j)}$$

$$= 23.09 + 0j \text{ A}$$

$$I_B = \frac{V_{BN}}{Z_B} = \frac{230.94\angle -120}{(15+10j)}$$

$$= -11.48 - 5.67j \text{ A}$$

$$I_C = \frac{V_{CN}}{Z_C} = \frac{230.94\angle -240}{(0+5j)}$$

$$= 39.99 + 23.094j \text{ A}$$

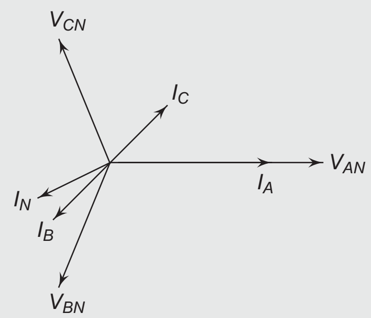
$$I_N = -(I_A + I_B + I_C)$$

$$= -[23.094 - 11.48 - 5.67j + 39.99 + 23.094j]$$

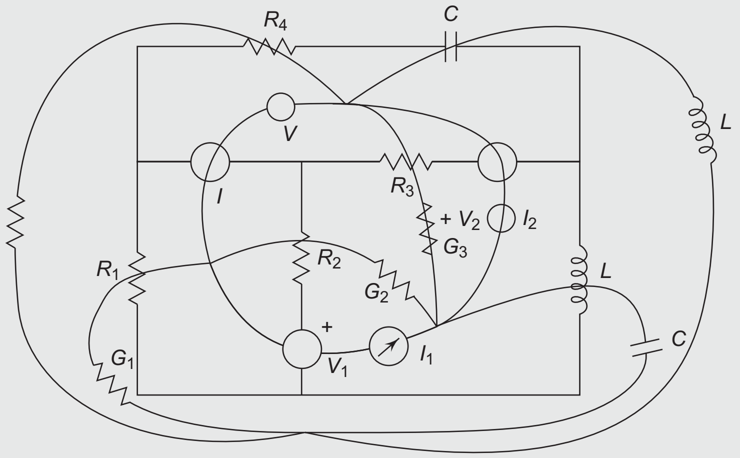
$$= -[51.604 + 17.424j]$$

$$= 54.46 \angle -161.34 \text{ A}$$

I_N phase with respect to V_{AN} is -161.34 phasor diagram is

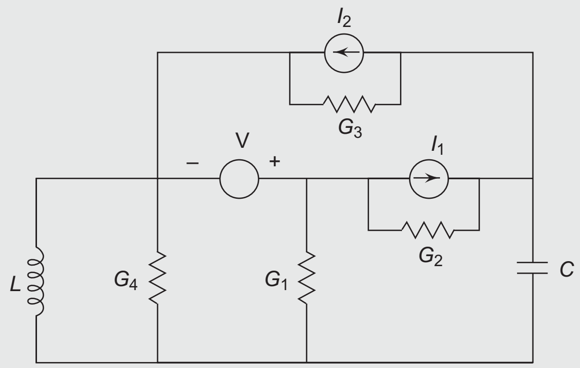


6. (a) What is duality? Explain the procedure for obtaining the dual of the given planar network shown below figure below.



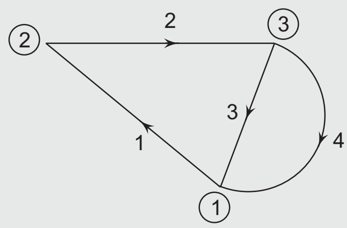
Solution:

Redrawing the N/W. I_2



(b) Construct the incidence matrix for the graph shown in figure below.

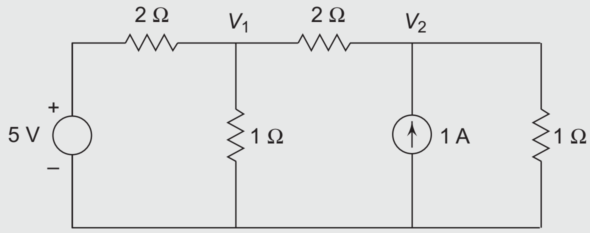
Solution:



$$\begin{array}{c} \\ \text{Nodes} \downarrow \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \text{Branches} \rightarrow \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \end{array} = [A_i]$$

$\therefore$ Incidence matrix for the given graph is constructed.

(c) Use nodal analysis, to determine the voltages V_1 and V_2 in the circuit shown in figure below.



Apply nodal analysis,

$$\frac{V_1 - 5}{2} + \frac{V_1}{1} + \frac{V_1 - V_2}{2} = 0 \tag{1}$$

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} = 1 \tag{2}$$

$\Rightarrow$ $\quad V_1 - 5 + 2V_1 + V_1 - V_2 = 5$

$$4V_1 - V_2 = 5 \tag{3}$$

$$V_2 - V_1 + 2V_1 = 2$$

$$3V_2 - V_1 = 2 \tag{4}$$

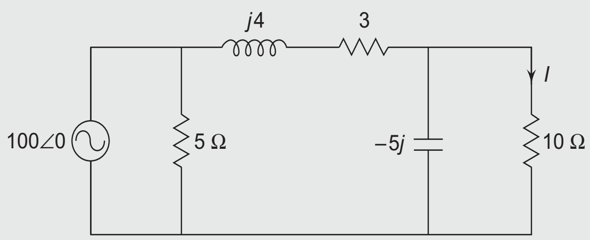
Solving equations (3) and (4) we get

$$V_1 = 1.545 \text{ V}$$

$$V_2 = 1.181 \text{ V}$$

$\therefore \quad V_1 = 1.545$ V and $V_2 = 1.181$ V

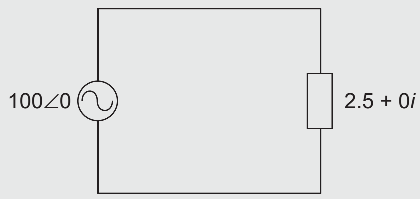
7. (a) State and explain the Reciprocity theorem? Is this theorem valid for N/W with two sources? Subtantiate your answers.



Solution:

Refer Section 3.5

(b) Verify the reciprocity theorem using the N/W given below. The N/W can be reduce to

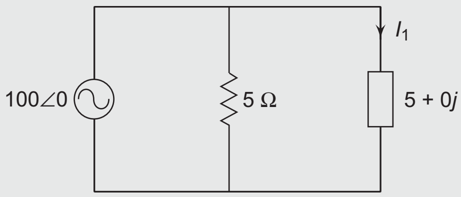


$$z_{eq} = 2.5 + 0j$$

$$I_T = \frac{100\angle 0}{25} = 40\angle 0$$

By using the current division the value of I can be obtained.

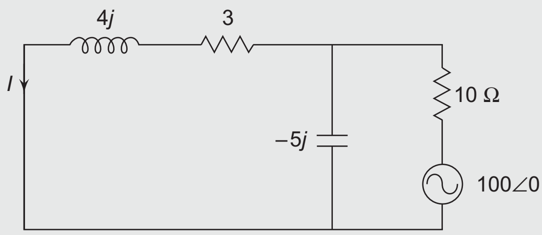
$$\therefore \qquad I_1 = \frac{I_T}{2} = \frac{40}{2} = 20 \text{ A}$$

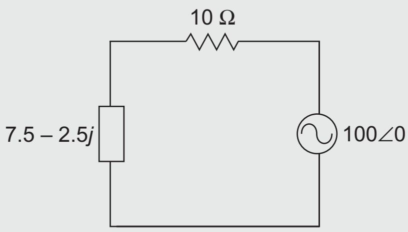


$$\text{then} \qquad I = \frac{-5j\,I_1}{(10-5j)} = \frac{-5j(20)}{(10-5j)} = 4 - 8i$$

Response to excitation is $\dfrac{V}{I} = \dfrac{100}{4-8i} = 5 + 10j$

the N/W can be reduced





$$I_T = \frac{I_T(-5j)}{(3+4j-5j)} = \frac{(5.6+0.8j)\,(-5j)}{(3-j)} = 5.6 + 0.8j$$

By using the current division,

$$I = \frac{I_T(-5j)}{(3+4j-5j)} = \frac{(5.6+0.8j)(-5j)}{(3-j)} = 4 - 8j$$

Response to excitation is $\dfrac{V}{I} = \dfrac{100}{4-8j} = 5 + 10j$

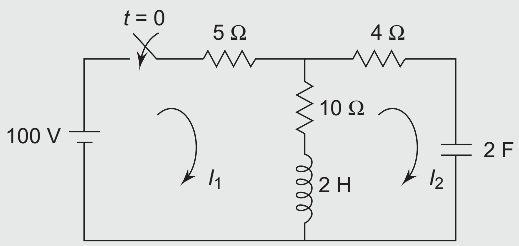
$\therefore$ Reciprocity theorem is verified.

8. (a) Compare the classical and Laplace transform method of solution of the network.

Solution:

Refer Chapter 13

(b) Draw the network in Laplace domain and find $i_1(t)$ and $i_2(t)$ the following figure.



By applying mesh analysis,

$$100 = 15i_1(t) + 2\frac{di_1}{dt} - 10i_2(t) - \frac{2di_2}{dt} \qquad (1)$$

$$0 = 14i_2(t) + \frac{2di_2}{dt} + \frac{1}{2}\int i_2 dt - 10i_1(t) - \frac{2di_1}{dt} \qquad (2)$$

Applying Laplace transform on both sides, for the two equations

$$15I_1(s) + 2SI_1(s) - 2SI_2(s) - 10I_2(s) = \frac{100}{s}$$

$$-10I_1(s) - 2SI_1(s) + 14I_2(s) + 2SI_2(s) + \frac{I_2(s)}{2s} = 0$$

$$\therefore \qquad I_2(s)\left[14 + 2s + \frac{1}{2s}\right] = I_1[10 + 2s] \qquad (1)$$

and

$$I_1(s)\,[15 + 2s] - I_2(s)\,[10 + 2s] = \frac{100}{s} \qquad (2)$$

$$I_1(s)\,[15 + 2s] - I_1(s)\frac{(10+2s)^2}{\left(14 + 2s + \frac{1}{2s}\right)} = \frac{100}{s}$$

$$I_1(s)\left[\frac{38s + 111}{14 + 2s + \frac{1}{2s}}\right] = \frac{100}{s}$$

$$I_1(s) = \frac{50}{s^2}\frac{(28s+4s^2+1)}{(38s+111)}$$

$$= \frac{50\,(s+0.038)\,(s+6.96)}{s^2(38s+111)}$$

Taking the partial fractions

$$\frac{(28s+4s^2+1)}{s^2(38s+111)} = \frac{A}{s}+\frac{B}{s^2}+\frac{c}{38s+111}$$

$$28s + 4s^2 + 1 = \text{As}\,(38s + 111) + B\,(38s + 111) + cs^2$$

Compare co-efficients of s^2, s, $s°$

$$38A + C = 4$$

$$111A + 38B = 28$$

$$B111 = 1$$

Solving these three equation:

$$A = 0.249$$

$$B = 0.009$$

$$C = –5.468$$

$$\therefore \qquad I_1(s) = \frac{50(0.249)}{s}+\frac{50(0.009)}{s^2}-\frac{50(5.468)}{38s+111}$$

$$I_1(s) = \frac{12.45}{s}+\frac{0.45}{s^2}-\frac{273.4}{383+111}$$

Applying inverse. Laplace transform

$$I_1(t) = 12.45 + 0.45t - \frac{273.4}{38}\,e^{-\frac{111}{38}t}$$

$$= 12.45 + 0.45\text{t} - 7.19\,e^{-2.92t}$$

$$\|\text{ly } I_2(s) = I_1(s)\;\frac{10+2s}{14+2s+\dfrac{1}{2s}}$$

$$= \frac{50}{s^2}\frac{(28s+4s^2+1)}{(38s+111)}\frac{(10+2s)}{\left(14+2s+\dfrac{1}{2s}\right)}$$

$$= \frac{50}{s^2}\frac{(10+2s)}{(38s+111)}\cdot\frac{(28s+\cancel{4s^2}+1)}{\dfrac{(28s+\cancel{4s^2}+1)}{2s}}$$

$$= \frac{100}{s}\frac{(10+2s)}{(38s+111)}$$

Taking partial fractions.

$$\frac{(10+2s)}{s(38s+111)} = \frac{A}{s}+\frac{B}{38s+111}$$

$$10 + 2s = A(38s + 111) + Bs$$

$$38A + B = 2 \quad \Rightarrow A = 0.09$$

$$A\,111 = 10 \qquad B = -1.423$$

$$\therefore \qquad I_2(s) = \frac{100(0.09)}{s} - \frac{100(1.423)}{38s+111}$$

$$= \frac{9}{8} - \frac{142.3}{38\left[s+\frac{111}{38}\right]}$$

take inverse Laplace transform

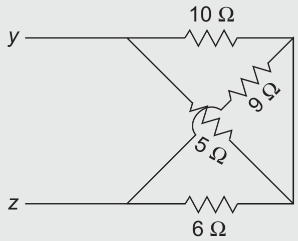
$$i_2(t) = 9 - 3.744\ e–^{2.92t}$$

$$\therefore \qquad i_1(t) = 12.45 + 0.45t - 7.19\ e^{-2.92t}$$

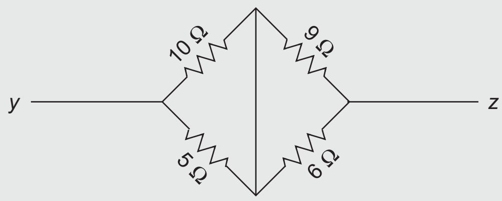
$$i_2(t) = 9 - 3.744\ e^{-2.92t}$$

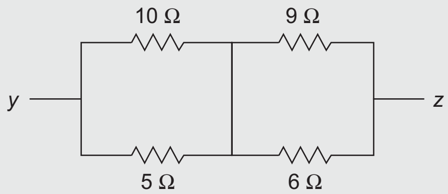
PAPER 11

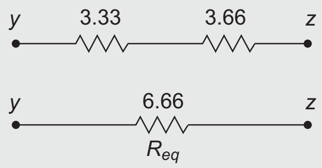
1. (a) Find the equivalent resistance between terminals y and z in the figure shown below.



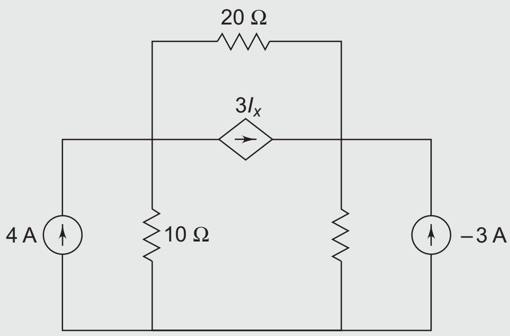
Solution: The above circuit can be represented as



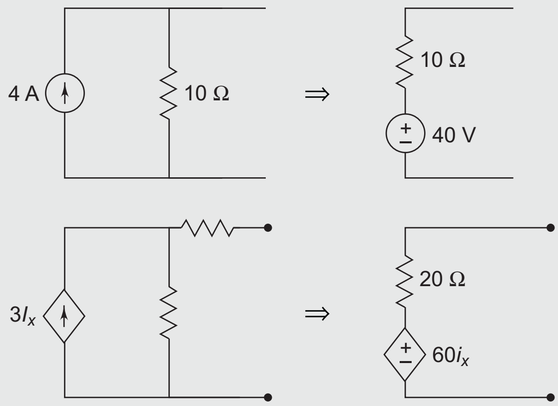


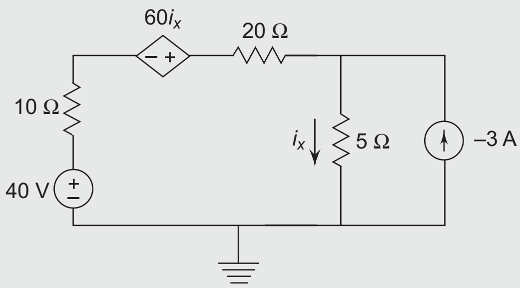


(b) In the network shown in figure below, determine i_x.

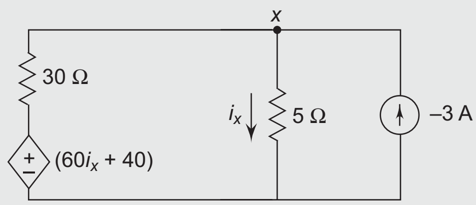


Solution: Apply source transformation for the current source (4A) is $3I_x$ current source.





In the above circuit voltage sources in series can be added and Eqn resistance is place.



Apply nodal analysis at note x.

$$\frac{(v_x - 0)}{5\Omega} + \frac{v_x - (60i_x + 40)}{30\Omega} + 3 = 0$$

But $\qquad v_x = 59_x$

replacing v_x by $5i_x$

$$i_x + \frac{5i_x - 60i_x - 40}{30} + 3 = 0$$

$$30i_x + 5i_x - 60i_x - 40 + 90 = 0$$

$$-25i_x + 50 = 0$$

$$i_x = \frac{50}{25} = 2\text{A}$$

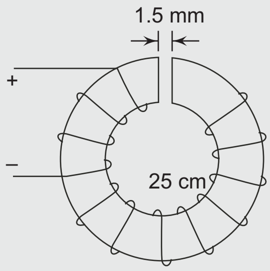
$\therefore$ The current i_x = **2A**

2. (a) Define mmf, for x and reluctance of ω magnetic circuit:

Solution:

Refer Section 10.11

(b) An iron ring has a mean diameter of 25 cms, and a cross-sectional area of 4 cms^2. It is wound with a coil of 1200 turns. An air gap of 1.0 mm width is cut in the ring. Determine the current required in the coil to produce a flux of 0.48 mwb in the air gap. The relative permeability of iron under the condition is 800. Neglect Leakage.



Solution: Given data:

$$\text{Mean diameter} = 25 \text{ cms} = D = 0.25 \text{ mb}$$

$$\text{Cross-sectional area, } A = 4 \text{ cm}^2 = 4 \times 10^{-4} \ M^2$$

$$\text{No. of turns} = N = 1200$$

$$\text{Relative permeability of iron} = 800$$

$$\phi = 0.48 \text{ mwb}$$

$$\text{Air gap reluctance } Rl_g = \frac{l_g}{\mu_0 A_C} = \frac{1.5 \times 10^{-3} \text{ mt}}{4\pi \times 10^{-7} \times \mu \times 10^{-4}}$$

$$= 10^8 \times 0.02981$$

$$= 2.9841 \times 10^6 \text{ AT/wb}$$

Iron core reluctance

$$R_{LC} = \frac{l_C}{\mu_0 \mu_r A_C} = \frac{(\pi D - l_g)}{4\pi \times 10^{-7} \times 800 \times 4 \times 10^{-4}}$$

$$= \frac{(\pi \times 0.25 - 1.5 \times 10^{-3})}{16\pi \times 8 \times 10^{-9}}$$

$$= 1.949 \times 10^{+6} \text{ AT/wb.}$$

$$\text{Total reluctance} = Rl_g + R_{lc}$$

$$= 4.933 \times 10^6 \text{ AT/wb}$$

$$\text{mmf} = \text{flux} \times \text{reluctance}$$

$$NI = \phi \cdot Rl$$

$$1200 \times i = 0.48 \times 10^{-3} \times 4.9331 \times 10^6$$

$$i = \frac{0.48 \times 10^{-3} \times 4.9331 \times 10^6}{1200}$$

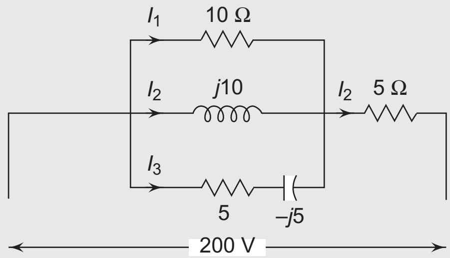
$\therefore$ current required = **1.973 Amp**

3. (a) Get the expression for complex power and sign of the active power.

Solution:

Refer Chapter 6.

(b) Find I_1, I_2, I_3 and I find also the power consumed. Draw the phasor diagram (Fig. 19)



Solution: Equivalent Impedance

$$= 5 + 10\Omega \,||\, j10 \,||\, 5 - j5$$

$$= 5 + \frac{\cancel{10} \times 10j}{\cancel{10} + 10j} \,||\, 5 - j5$$

$$= \frac{5 + \dfrac{10j}{(1+j)} \times 5 - j5}{\dfrac{10j}{1+j} + 5 - j5} = \frac{5 + \dfrac{\cancel{(1+j)}\,50}{\cancel{1+j}}}{\dfrac{10j}{1+j} + 5 - j5}$$

$$= 5 + \frac{50(1+j)}{10j + (1-j)(1+j)5}$$

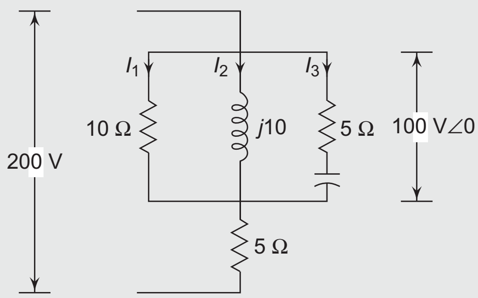
$$= 5 + \frac{50(1+j)}{10j + 10} = 5 + \frac{5(1+j)}{(1+j)}$$

$$Z_{eq} = 10\ \Omega$$

$\therefore$ $$I = \frac{200}{10} = 20\ \angle 0$$

$\therefore$ Voltage across parallel I_{MD}

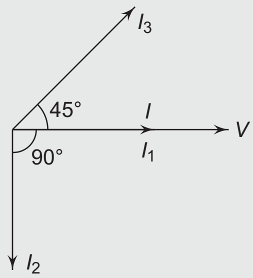
$$200 - I \times 5$$
$$= 200 - 20 \times 5$$
$$= 100\text{V}$$



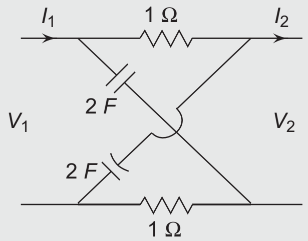
$$I_1 = \frac{100\text{V}\angle 0}{10} = 10 \text{ Amp}$$

$$I_2 = \frac{100\text{V}}{j10} = -j10 \text{ A} = 10 \angle -90^\circ$$

$$I_3 = \frac{100\text{V}}{5-j5} = \frac{20}{2}(1+j) = 10 + j10$$

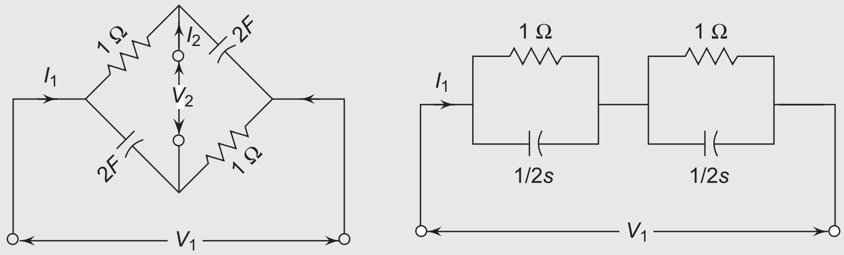


4. (a) Obtain the Y parameters for the following figure and network in Laplace transform variable.



Solution:

Y parameter Equations



$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

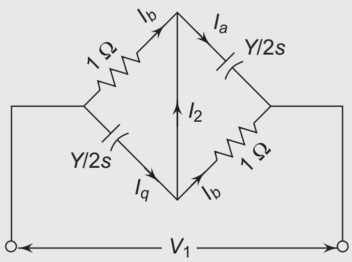
$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

$$Y_1 = I_1 \times 2\left(\frac{1\times 1/2s}{1+1/2s}\right)$$

$$V_1 = \frac{2I_1}{2s+1} \qquad Z_{eq} = 2\times\frac{\left(1\times\frac{1}{2s}\right)}{\left(1+\frac{1}{2s}\right)}$$

$$Y_{11} = \frac{I_1}{V_1} = \frac{(2s+1)}{2}$$

$$Y_{21} = \frac{I_2}{V_1}$$



$$I_1 = \frac{V_1(2s+1)}{2}$$

$$I_L = I_\omega - I_b$$

$$I_\omega = \frac{I_1 \times 1}{1 + \frac{1}{2s}} = \frac{2sJ_1}{1+2s}$$

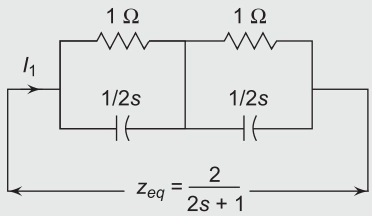
$$I_b = \frac{I_1 \times 1/2s}{1+1/2s} = \frac{I_1}{1+2s}$$

$$I_\omega - I_b = I_2 = \frac{I_1}{1+2s}\,(2s-1)$$

$$= \frac{(2s-1)}{\cancel{(1+2s)}} \times \frac{1}{2}\,\cancel{(2s+1)}V_1$$

$$I_2 = \left(\frac{2s-1}{2}\right) V_1$$

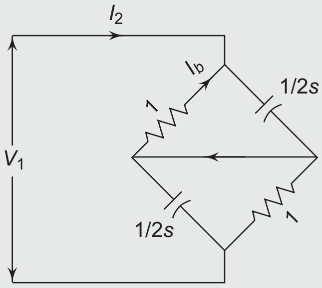
$$\therefore \qquad \left.\frac{V_2}{V_1}\right|_{V_L=0} = \frac{(2s-1)}{2}$$



$$V_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 0$$

$$V_2 = I_2 \times \frac{2}{2s+1}$$

$$\frac{I_2}{V_2} = \frac{(2s+1)}{2} = Y_{22}$$



and $$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

From figure $$I_1 = \mathrm{I}_\omega - I_b$$

$$I_\omega = \frac{I_2 \times 1}{1+1/2s} = \frac{2sI_L}{1+2s}$$

$$I_b = \frac{I_2 \times 1/2s}{1+1/2s} = \frac{I_L}{1+2s}$$

$$I_1 = \frac{(2s-1)I_L}{1+2s} = \frac{(2s-1)}{\cancel{(1+2s)}} \times \frac{\cancel{(2s+1)}V_2}{2}$$

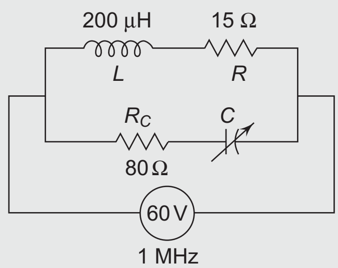
$$I_1 = \frac{2s-1}{2} V_2$$

$$\frac{I_1}{V_2} = \frac{2s-1}{2} = 1/12$$

$\therefore$ Y matrix

$$Y = \begin{bmatrix} \frac{2s+1}{2} & \frac{2s-1}{2} \\ \frac{2s-1}{2} & \frac{2s+1}{2} \end{bmatrix}$$

(b) A tuned circuit consists of a coil having an inductance of 200 μH and a resistance of 15 Ω in parallel with a series combination of a variable capacitance and resistor of 80 Ω. It is supplied by a 60 V source. If the supply frequency is 1 MHz what is the value of C to give resonance.



Solution: Total admittance, $Y = \dfrac{1}{R_L + j\omega L} + \dfrac{1}{R_C - (j/\omega C)}$

$$Y = \frac{R_L - j\omega L}{R_L^2 + \omega^2 L^2} + \frac{R_C + j/\omega C}{R_C^2 + \frac{1}{\omega^2 C^2}}$$

$$= \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_C}{R_C^2 + \frac{1}{\omega^2 C^2}} + j\left[\frac{1/\omega C}{R_C^2 + \frac{1}{\omega^2 C^2}} - \frac{\omega L}{R_L^2 + \omega^2 L^2}\right]$$

at resonance, susceptance part becomes zero

$$\frac{\omega_r L}{R_L^2 + \omega_r^2 L^2} = \frac{\frac{1}{\omega_r C}}{R_C^2 + \frac{1}{\omega_r^2 C^2}}$$

$$\omega_r L\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{\omega_r C}\,[R_L^2 + \omega_r^2 L^2]$$

$$\omega_r^2\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{LC}\,[R_L^2 + \omega_r^2 L^2]$$

$$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2 - 4C}{R_C^2 - 4C}} \qquad \omega_r = \frac{1}{\sqrt{LC}}\ (R_L = R_C)$$

$$\left[\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2 - 4C}{R_L^2 - 4C}} \quad \text{resonant frequency}\right.$$

$$R_C^2 - \frac{L}{C} = R_L^2 - \frac{L}{C}$$

$$\left. R_C = R_L \right]$$

$$(15 + j1256)\left(80 - j\frac{1}{2\pi C \times 10^6}\right)$$

Imaginary part = 0 at resonance

$$1256 \times 80 = \frac{15}{2\pi C \times 10^6}$$

$$C = 23.76 \text{ pF}$$

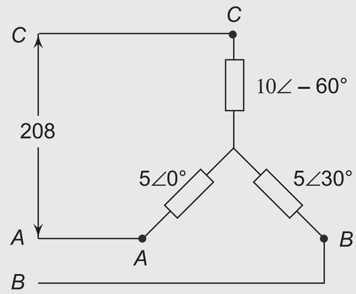
5. (a) Show that a balanced star connected load can be transformed in to an equivalent delta connected load and vice-versa.

Solution:

Refer Section 3.1

(b) 3ϕ, 3 wire, 208 V, CBA, has Y load
$Z_A = 5\angle 0$, $Z_B = 5\angle 30°$, $Z_C = 10\angle -68\Omega$, find true current voltage on across each Load Impedance.

By converter $Y\,N\,W$ into Δ form



$$Z_{BC} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_A}$$

$$= \frac{105.85\angle -31.81}{5\angle 0}$$

$$= 21.17\angle -31.81° \ \Omega$$

$$Z_{CA} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_B}$$

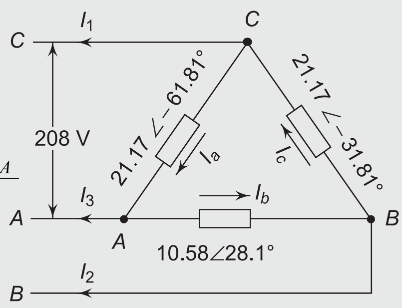
$$= \frac{105.85\angle -31.81}{5\angle 30°}$$

$$= 21.17\angle -61.81° \ \Omega$$

$$Z_{BA} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_C}$$

$$= \frac{105.85\angle -31.81}{10\angle -60°}$$

$$= 10.58\angle 28.19$$



$\because$ Phase sequence h CBA

$$V_{CB} = 208\angle 0°,\ V_{AC}$$

$$V_{BA} = 208\angle -120°$$

$V_{AC} = 208 \angle -240^o$

$$I_C = \frac{V_{CB}}{Z_{BC}} = \frac{208\angle 0}{21.17\angle -31.81} = 9.82 \angle 31.81 \text{ A}$$

$$I_B = \frac{V_{BA}}{Z_{AB}} = \frac{208\angle -120^o}{10.58\angle 28.19} = 19.65 \angle -148.19^o$$

$$I_A = \frac{V_{AC}}{Z_{AC}} = \frac{208\angle -240}{21.17\angle -61.81} = 9.825 \angle -179.19^o$$

Line currents are

$I_1 = I_c - I_a = 9.82 \angle 31.81 - 9.825 \angle -179.19^o$

$\boxed{I_1 = I_{CL} = 18.93 \angle 16.30}$

$I_2 = I_b - I_C = 19.65 \angle -148.19 - 9.82 \angle 31.81$

$\boxed{I_2 = I_{BL} = 29.47 \angle -148.19}$

$I_3 = I_a - I_B = 9.825 \angle -179.19 - 19.65 \angle -148.19$

$\boxed{I_3 = I_{AL} = 12.31 \angle 56.06}$

Voltage across each load Impedance are

$V_{ZC} = (I_{CL})Z_C = (18.93 \angle 16.30)\,(10 \angle -60^o)$

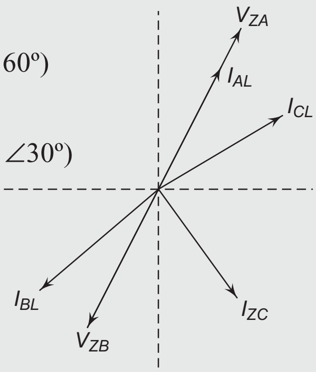
$\boxed{V_{ZC} = 189.3 \angle -43.7^o}$

$V_{ZB} = (I_{B_L})\,(Z_B) = (29.47 \angle -148.19)\,(5 \angle 30^o)$

$\boxed{V_{ZB} = 147.35 \angle -118.19^o}$

$V_{ZA} = I_{AL}\, Z_A = I\,(12.31 \angle 56.06)\,(5 \angle 0^o)$

$\boxed{V_{ZA} = 61.55 \angle 56.06}$

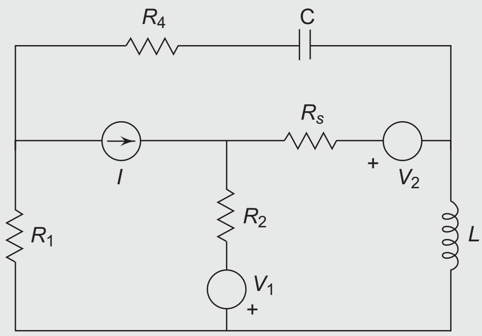


Phasor diagram

6. (a) What is duality? Explain the procedure for obtaining the dual of the given planar network shown below in the figure.

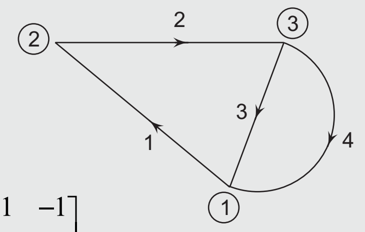
Solution:

<u>Refer 3.8</u>



(b) Construct the incidence matrix for the graph shown in the figure.

Solution: Let i_1, i_2, i_3, i_4 be the current in the branches 1, 2, 3, 4.



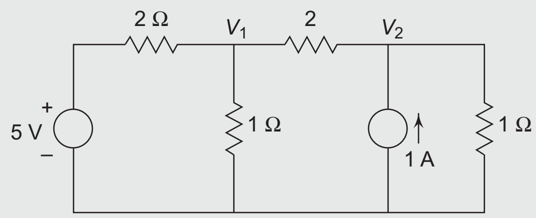
$$- i_1 + i_2 = 0$$
$$- i_4 - i_3 + i_1 = 0$$
$$- i_2 + i_3 + i_4 = 0$$

The incidence matrix is

Nodes branches

$$[A_i] = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

(c) Use nodal analysis, to determine the voltage $V_1 \leftarrow V_L$ in the circuit shown in figure below.



Solution: $\dfrac{(V_1 - 5)}{2} + \dfrac{V_1 - 0}{1} + \dfrac{V_1 - V_2}{2} = 0$

$$4V_1 - V_2 - 5 = 0 \qquad (1)$$

$$1 + \frac{V_1 - V_2}{2} + \frac{-V_2}{1} = 0$$

$$V_1 - 3V_2 + 2 = 0 \qquad (2)$$

Solving (1) and (2) $V_1 = 1.545$ V

$V_L = - 1.18$ V

7. (a) State and Explain the superposition theorem.

Solution:

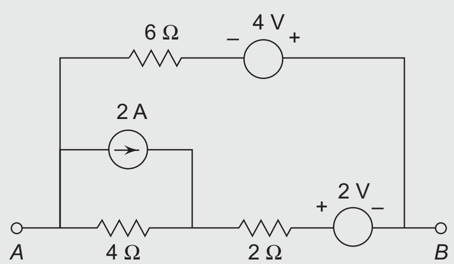
<u>Refer Section 3.2</u>

(b) Is superposition valied for power? Substantiate your answer.

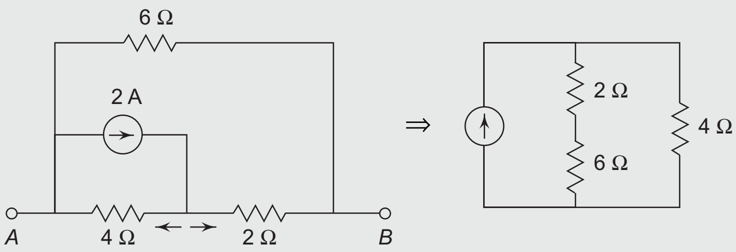
Solution:

<u>Refer section 3.2</u>

(c) Using superposition theorem find V_{ab} volts shown in figure below

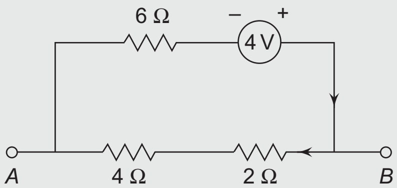


Solution: 2A current source alive.



Current through 2Ω

$$\frac{4}{8+4}\times 2 = \frac{2}{3}\text{A}$$



and current through 4 Ω

$$\frac{8}{8+4}\times 2 = \frac{4}{3}\text{A}$$

$$V_{a_1} + \frac{4}{3}\times 4 - 2\times\frac{2}{3} = V_{b_1}$$

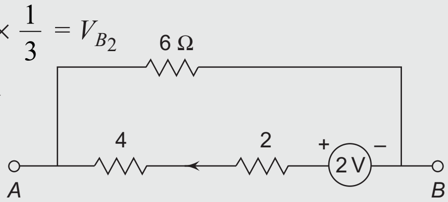
$$V_{a_1} - V_{b_1} = V_{ab_1} = \frac{4}{3} - \frac{16}{3} = \frac{-12}{3} = -4\text{V}$$

Within 4V voltage source alone.
Current

$$2 = \frac{4\text{V}}{6+4+2} = \frac{4}{12} = \frac{1}{3}\text{A}$$

$$V_{A_2} + 4\times\frac{1}{3} + 2\times\frac{1}{3} = V_{B_2}$$

$$V_{A_2} - V_{B_2} = -\frac{6}{3} = -2\text{V}$$



Within 2V voltage source alone.

Current $\qquad 2 = \dfrac{2}{6+4+2} = \dfrac{2}{12} = \dfrac{1}{6}\text{A}$

$$V_{A_3} + (4+2)\times\frac{1}{6} = V_{B_3}$$

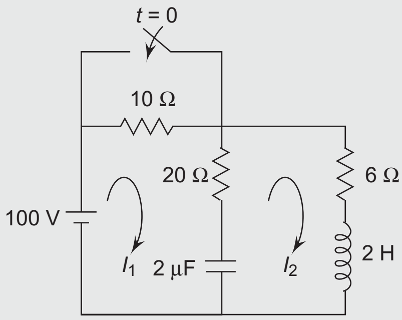
$\Rightarrow \qquad V_{A_3} - V_{B_3} = V_{ab_3} = -1\text{V}$

$\therefore$ By superposition $V_{ab} = V_{ab_1} + V_{ab_2} + V_{ab_3}$

$$= -4 + -2 + -1$$

$$V_{ab} = \mathbf{-7V}$$

8. (a) For the ckt shown below find the inerted condition of $q_1, p_2, \dfrac{di_1}{dt}, \dfrac{dq_2}{dt}$ and voltage across capacitor the ckt was in steady state before $t = 0$.



Solution: $t = 0^-$

$$i = \frac{100}{10+6} = 6.25\text{A}$$

$$V_C(0^-) = \frac{6\times 100}{16} = 37.5 \text{ V} \quad i_1(0^-) = i_2(0^-) = 6.25\text{A}$$

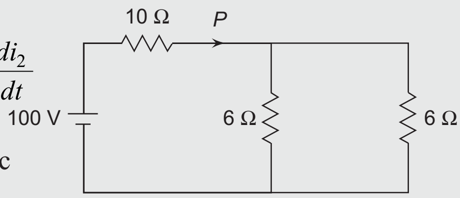
at $t = 0,\ i_2(0^-) = 6.25 \text{ A},\ i_1(0^-) = 6.25\text{A}$

$$100 = (i_1 - i_2)\ 20\ \frac{1}{2\mu\text{F}}\ (i_1 - i_2)\ dt \tag{1}$$

$$100 = 6i_2(t) + 2\frac{di_2}{dt} \tag{2}$$

$$i_1 = i_2 = 6.25$$

from Eq. (2) $100 = 6(6.25) + 2\ \dfrac{di_2}{dt}$



$$\frac{di_2(0^+)}{dt} = 31.25 \text{ A sec}$$

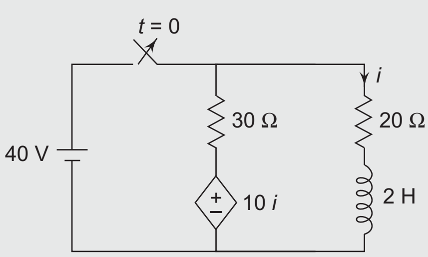
Taking derivative eq. (1)

$$0 = \left(\frac{di_1}{dt} - \frac{di_2}{dt}\right)20 + \frac{1}{2\times 10^{-6}}\ (i_1 - i_2)$$

$$i_1 = i_2 = 0$$

$$\frac{di_1(0^+)}{dt} = \frac{di_2(0^+)}{dt} = 31.25 \text{ A}$$

(b) Switch is opened at $t = 0$ find the current $i(t)$ for $t \geq 0$ in the following figure.



at $t = 0^-$

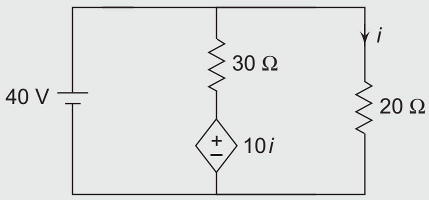
$$i = \frac{40}{20} = 2\text{A}$$

$$i_L\ (0) = 2\text{A}$$

$$\therefore \quad \boxed{i_L(0^+) = 2\text{A}}$$

at $t = 0^+$

By applying K.V.L to loop

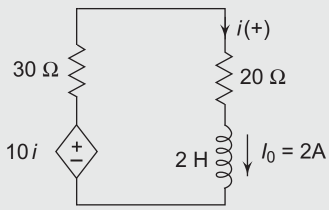


So $$10i(t) = 30i(t) + 20i(t) + \frac{2di(t)}{dt}$$

$$\frac{2di(t)}{dt} + 40i(t) = 0$$

$$(D + 20)\, i(t) = 0$$

$$i(t) = ke^{-20t}$$



at $t = 0 \quad i(t) = 2A$

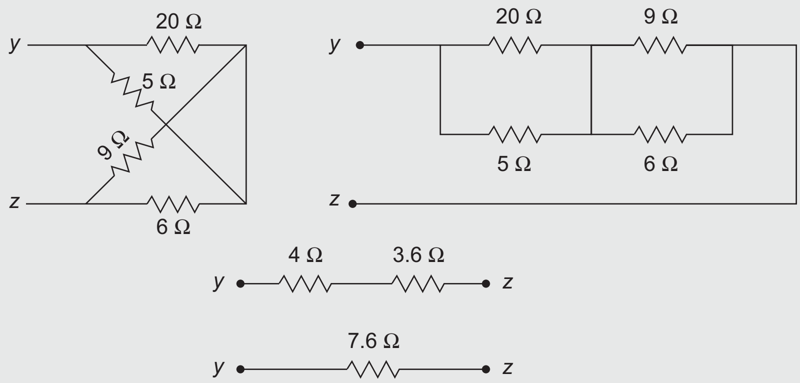
∴ $k = 2$

∴ $$\boxed{i(t) = 2e^{-20t}\text{ A}}$$

PAPER 12

1. (a) Find the equivalent resistance between terminals. y and z in the figure given below.

Solution:

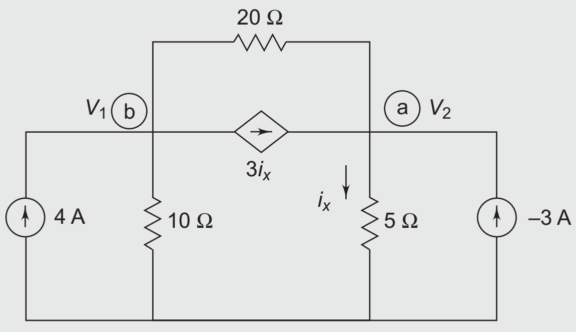


(b) In the network shown in the following, determine i_x.

Solution: At node (b)

$$\frac{V_1}{10}+\frac{V_1-V_2}{20}+3i_x=4$$

At node (a)



$$\frac{V_1}{10}+\frac{V_1-V_2}{20}=3i_x-3$$

and $i_x = \dfrac{V_2}{5}$

$$\frac{V_1}{10} + \frac{V_1 - V_2}{20} + 3\left(\frac{V_2}{5}\right) = 4$$

$$V_1\left(\frac{1}{10} + \frac{1}{20}\right) + v_2\left[\frac{-1}{20} + \frac{3}{5}\right] = 4 \quad (1)$$

$$\frac{V_2 - V_1}{20} + \frac{V_2}{5} = 3\left[\frac{V_2}{5}\right] - 3$$

$$\frac{-V_1}{20} + V_2\left[\frac{1}{20} + \frac{1}{5} - \frac{-3}{5}\right] = -3 \quad (2)$$

$0.15V_1 + 0.55\ V_2 = 4$ (A)

$-0.05V_1 - 0.35\ V_2 = -3$ (B)

Solving A and B

$V_1 = -10$ V

$V_2 = 10$ V

$\therefore \quad i_x = \dfrac{V_2}{5}$

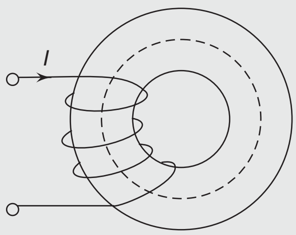
$= 2$A

(a) State and explain Faradays law of Electromagnetic induction. What are statically and dynamically induced EMFs.

Solution:

<u>Refer Section 1.6</u>

(b) An iron ring 15 cms in diameter and 10 cm^2 in area cross section. A wand with a coil of 800 kms. Determine the current in the coil to establish a fix density of 1 wb/m^2 of relative permeable *w* 500. In case if an air gap of 2 mm is cut in the ring what is the current in the coil to establish the same feet density.



Solution: Given data:

$$\beta = 1 \text{ wb/m}^2$$

$$\text{Diameter} = 150 \text{ cm} = 0.15 \text{ m}$$

$$\text{Core area } AC = 10 \text{ cm}^2 = 10 \times 10^{-4} \text{ m}^2$$

$$l_c = \pi D = \pi(0.15) \text{ mb}$$

$$\text{B, magnetic flux density} = \frac{\text{IN} \cdot \mu_0 \mu_v}{l_c}$$

$$1 = \frac{I \times 200 \times 4\pi \times 10^{-7} \times 500}{\pi \times 0.15}$$

$$\frac{100 \times 0.15}{4} = I \rightarrow I = 3.75 \text{ Amp.}$$

$$B = \frac{\text{mm}f}{\text{reluctance} \times \text{area}} = \frac{\text{NI}}{\text{reluctance} \times \text{area}}$$

If 2 mm is cut the reluctance will be sum of reluctance of air gap of core.

$$\text{e.g. (air gap flux)} = \frac{l_g}{\mu_0 AC}$$

$$= \frac{2 \times 10^{-3}}{4\pi \times 10^{-7} \times 10 \times 10^{-4}}$$

$$= 0.159 \times 10^7$$

$$= 1.59 \times 10^6 \text{ AT/wb}$$

$$R_{lc} \text{ (reluctance of core)} = \frac{l_c}{\mu_0 \mu_r AC}$$

$$= \frac{(\pi D - l_g)}{4\pi \times 10^{-7} \times 500 \times 10 \times 10^{-4}}$$

$$= \frac{(\pi \times 0.15 - 2 \times 10^{-3})}{4\pi \times 10^{-7} \times 0.5}$$

$$= 0.746 \times 10^6$$

$R_{lc} + R_g = 2.336 \times 10^6$ AT/wb

$$B = 1 = \frac{200 \times 1}{2.336 \times 10^6 \times 10 \times 10^{-4}} = \frac{NI}{Rl \times AC}$$

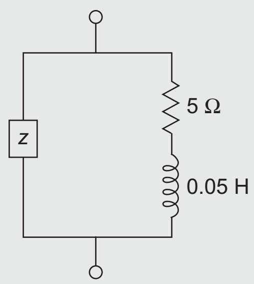
$$I = 11.684 \text{ Amp}$$

3. (a) Explain the significance of J-operator? What are the different forms of expressing the sinusoidal quarter in complex form?

Solution:

<u>Refer Appendix C</u>

(b) Find the components of Z such that the current drawn quantity by the circuit same at all frequencies the following figure.



$R_L = R_C = 5\ \Omega$

$$R_L = R_C = \sqrt{\frac{L}{C}}$$

$$5 = \sqrt{\frac{0.05}{C}}$$

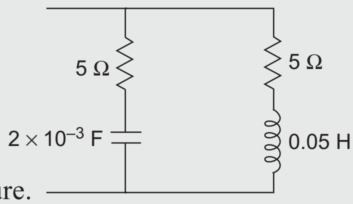
$$C = \frac{0.05}{25} = 2 \times 10^{-3} \text{ F} = 2 \text{ mF}$$

Solution: (c) The condition is that $R_L = R_C = \sqrt{\dfrac{L}{C}}$.

$\therefore$ $R_L = R_C = 5\,\Omega$

from which $5 = \sqrt{\dfrac{0.05}{C}}$

$C = 2 \times 10^{-3}$ F



The components of Z are shown in figure.

4. (a) Define the following terms

(i) Bandwidth

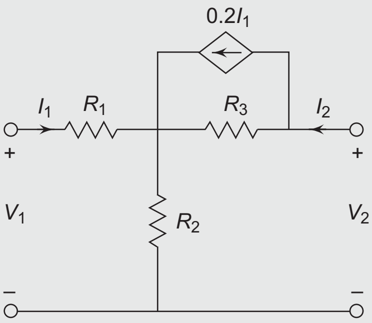
(ii) Q-factor

(iii) half power frequencies

Solution:

Refer Sections 8.4, 8.5

(b) Obtain a π-equivalent circuit for the following figure of 2 port network.



Solution:

$-V_1 + I_1R_1 + (I_1 + I_2)\, R_L = 0$

$$(R_1 + R_2)I_1 + R_2I_2 = V_1 \quad (1)$$

$$R_L(I_2 + I_1) + R_3(I_2 - 0.2\, I_1) - V_1 = 0$$

$$(R_2 - 0.2\, R_3)\, I_1 + (R_3 + R_2)I_2 = V_2 \quad (2)$$

w.r.t. $V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$

$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$

So $Z_{11} = R_1 + R_2$

$Z_{12} = R_2$

$Z_{21} = R_2 - 0.2\, R_3$

$Z_{22} = R_3 + R_2$

$$\Delta t = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} = \begin{vmatrix} R_1 + R_2 & R_2 \\ R_2 - 0.2R_3 & R_3 + R_2 \end{vmatrix}$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = -\frac{R_3R_2}{\Delta Z}$$

$$Y_{12} = \frac{-Z_{12}}{\Delta Z} = -\frac{R_2}{\Delta Z}$$

$$Y_{21} = \frac{-Z_{21}}{\Delta t} = \frac{R_2 - 0.2R_3}{\Delta Z}$$

$$Y_{22} = \frac{Z_{11}}{\Delta t} = \frac{R_1 + R_2}{\Delta Z}$$

$$Y_1 = Y_{11} + Y_{21}$$

$$= \frac{R_3 + R_2 + R_2 - 0.2\,R_3}{\Delta Z}$$

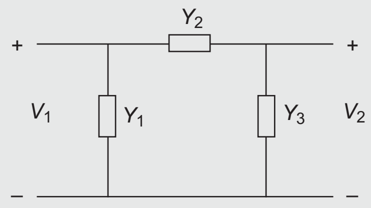
$$= \frac{2R_2 - 0.8\,R_3}{\Delta Z}$$

$$Y_2 = -Y_{12} = \frac{-R_2}{\Delta Z}$$

$$Y_3 = Y_{22} + Y_{12}$$

$$= \frac{R_1 + \cancel{R_2} - \cancel{R_2}}{\Delta Z}$$

$$= \frac{R_1}{\Delta Z}$$



5. (a) Derive the relationship between and phase quantities in a balanced star connected system.

Solution:

Refer Sections 9.7.1, 9.7.2 and 9.7.3

(b) A 3 phase 4-wire CBA system of phase sequence, with effective line voltage of 100 V has a star-connected impedance given by

$Z_A = 3.0\ \angle 0°\ \Omega$, $Z_B = 4.5\ \angle 56.31°\ \Omega$

$Z_C = 2.24\ \angle{-26.57°}\ \Omega$, obtain the line currents and the current in neutral wire draw the phasor diagram.

Solution:

$$V_{ph} = \frac{100}{\sqrt{3}} = 57.735\ \text{V}$$

$$V_{CN} = 57.735\ \angle 0°$$

$$V_{BN} = 57.735\ \angle{-120°}$$

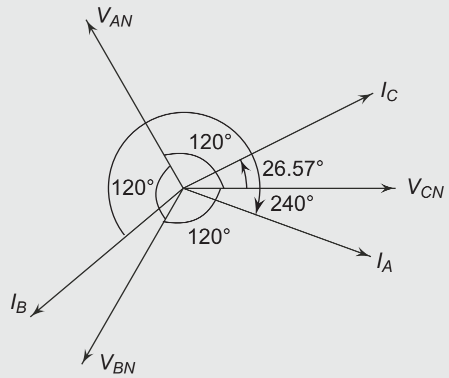
$$V_{AN} = 57.735\ \angle{-240°}$$

$$I_C = \frac{VCN}{Z_C} = \frac{57.735}{2.24\angle -26.57°} = 25.77\ \angle 26.57° = 23.048 + 11.52j$$

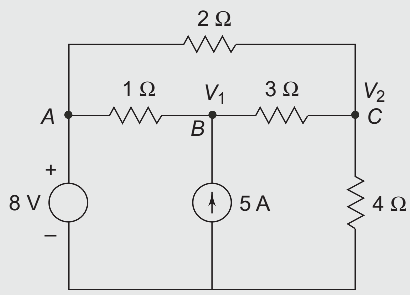
$$I_B = \frac{V_{BN}}{Z_B} = \frac{57.735\angle -120°}{4.5\angle 56.31°} = 12.83\ \angle{-176.31°} = -12.8 - 0.825j$$

$$I_A = \frac{V_{AN}}{Z_A} = \frac{57.735\angle -240°}{3°} = 19.245\ \angle{-240°} = -9.62 + 16.66j$$

$$I_{\text{Neutral}} = -(I_A + I_B + I_C) = -(0.628 + 27.355j)$$



6. (a) For the circuit shown below in Fig. 31. Find the currents of voltages in all the branches of circuit. Use node voltage method.



Solution: Let V_1 be the voltage as shown in figure

At V_1

$$\frac{8-V}{1}+5+\frac{V_2-V_1}{3}=0$$

$$39-4V_1+V_2=0 \tag{1}$$

At V_2

$$\frac{V_2-8}{2}+\frac{V_2-V_1}{3}+4V_2=0$$

$$-2V_1+29V_2-24=0 \tag{2}$$

From Eq. (1) and Eq. (2) $V_1 = 10.13$; $V_2 = 1.52$

$$\text{Current in } 2\Omega = \frac{8-1.52}{2} = 3.24 \text{ A from } A \text{ to } C$$

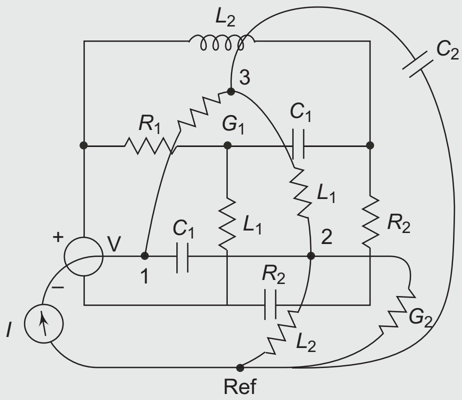
$$\text{Current in } 1\Omega = \frac{8-10.13}{1} = -2.13 \text{ A from } A \text{ to } B$$

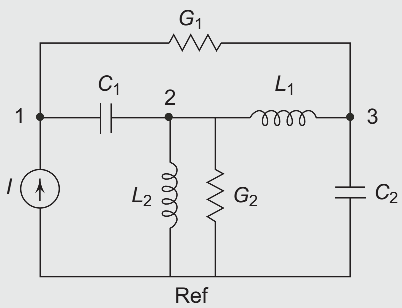
$$\text{Current in } 3\Omega = \frac{10.13-1.52}{3} = 2.87 \text{ A from } B \text{ to } C$$

Current in 4 ℧ = 1.52 × 4 = 6.08 A downwards

(b) Draw the dual of the network shown in the following figure. Explain the procedure employed.

Solution: For procedure refer Section 3.8





(c) Obtain the Expression for characteristic impedance of symmetrical T network.

Solution:

Refer Section 15.13.

7. (a) State and explain superposition theorem.

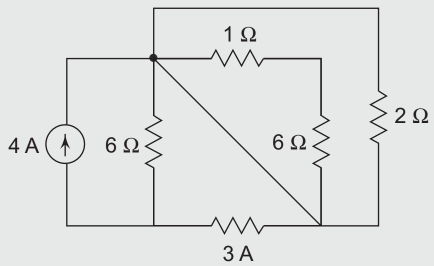
Solution:

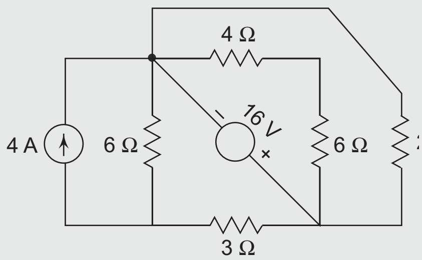
Refer Section 3.2

(b) Using superposition theorem find the current in 2 Ω resistor. Verify the result by any other method in following figure.

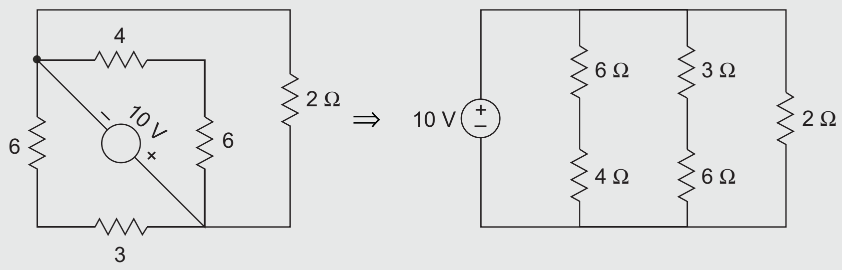
Solution:

Consider source (1) current source on line.





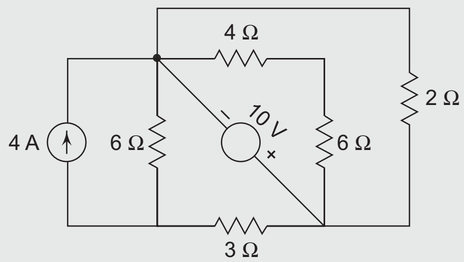
Current through 2Ω = 0
as two point ends are shorted.
Consider voltage source alive.



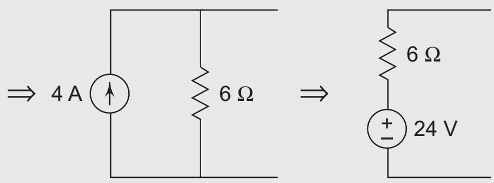
Current through 2Ω resistor = $\dfrac{10\text{V}}{2}$ = 5A

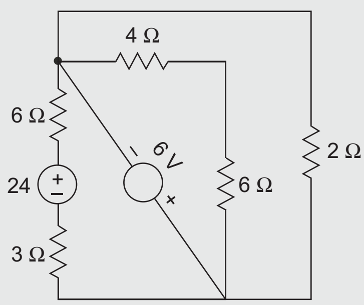
by superposition current = 0 + 5 = 5A

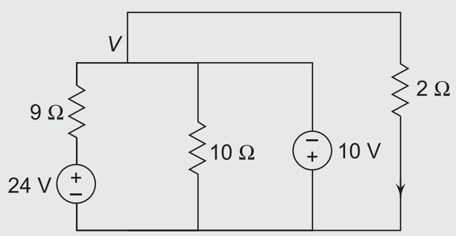
Verification: Consider the entire network.



By source transfer







Current through 2 Ω = $\dfrac{V}{2}$

$$= \frac{-10}{2} = -54 \text{ Downward}$$

I = 54 upward. Hence proved.

8. (a) What are the Initial conditions? How do you need them?

Solution:

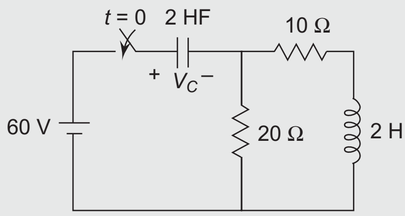
Refer Chapters 12 and 13

(b) Explain why the current in a pure Inductance cannot change in zero time?

Solution:

Refer Section 1.6

(c) Switch is closed at $t = 0$. Find initial conditions at $t(0^+)$ for i, i_2, V_C $\frac{di_1}{dt}, \frac{di_2}{dt}, \frac{d^2i_1}{dt^2}$ and $\frac{d^2i_1}{dt^2}$ in the following figure.

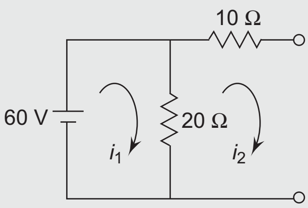


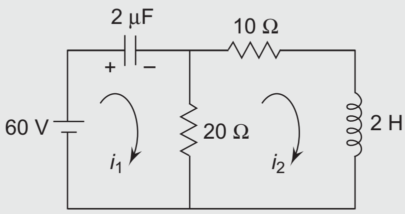
Solution:

At $t = 0^-$ the circuit in un energised so all initial condition are zero.

at $t = 0^+$

$$i_1(0^+) = \frac{60}{20} = 3\text{A}, \quad i_2(0^+) = 0\text{A}$$





By writing KVL to loops

$$60 = \frac{1}{2 \times 10^{-6}} \int i_1(t) + 20[i_1(t) - i_2(t)] \quad (1)$$

$$20\,(i_2(t) - i_1(t)) + 10i_2(t) + \frac{2di_2(t)}{dt} = 0 \quad (2)$$

By substituting Initial Coils (2)

$$20(0-3) + 30 + 2\frac{di_2(0^+)}{dt} = 0 \quad \boxed{\frac{di_2(0^+)}{dt} = 30}$$

Differentiate (1)

$$0 = \frac{1}{2\times10^{-6}}\ i_1(t) + 20\left(\frac{di_1(t)}{dt} - \frac{di_2(t)}{dt}\right) \qquad (3)$$

$$\left(\frac{di_1}{dt} - 30\right) = -\frac{3}{40} \times 10^6$$

$$\frac{di_1}{dt} = 30 - \frac{3}{40} \times 10^5 = -74.97 \times 10^3$$

$$\boxed{\frac{di_1(0^+)}{dt} = -74.97\times10^3}$$

Differentiate (2)

$$20\left(\frac{di_2(t)}{dt} - \frac{di_1}{dt}\right) + 10\frac{di_2}{dt} + 2\frac{d^v i_2}{dt^v}$$

$$20\,(30 + 74.97 \times 10^3) + 10 \times 30 + 2\frac{d^v i_2}{dt^v} = 0$$

$$\frac{d^v i_2(0^+)}{dt^v} = -760.15 \times 10^3$$

Differentiate Eq. (3)

$$0 = \frac{1}{2\times10^{-6}}\frac{di}{dt} + 20\left(\frac{d^v i_1}{dt^v} - \frac{d^v i_2}{dt^v}\right)$$

$$0 = \frac{1}{2\times10^{-6}}\ (-74.97 \times 10^3) + 20\left(\frac{d^v i}{dt^v} + 750.15\times10^3\right)$$

$$i_1(0^+) = 3\text{A}$$

$$i_2(0^+) = 0\text{A} \qquad \frac{d^v i_i}{dt^v}(0^+) = 1.873 \times 10^\circ$$

$$\frac{di_1}{dt}(0^+) = -74.97 \times 10^3 \qquad \frac{d^v i_2(0^+)}{dt^v} = -750.15 \times 10^3$$

$$\frac{di_2}{dt}(0^+) = 30$$

Appendix F

Solved Question Papers

Network Analysis, May/June 2006

SET 1

1. (a) Describe the Volt-ampere relations for R, L and C Parameters.

Solution: *Volt-ampere Relations for R, L and C Parameters*

The passive elements R, L, C are defined by the way in which the current and voltage are related for individual element.

(i) If the current 'I' and voltage 'V' are related by a constant for a single element then the element is a resistance 'R'. The Resistance 'R' represents the constant of proportionality.

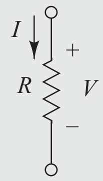


Fig. Set 1.1

$\therefore$ Voltage, $V = RI$ (ohms law)

Current, $I = \dfrac{V}{R}$

Power, $P = VI = I^2R$

The units of resistance 'R' is ohms (Ω).

(ii) If the current and voltage are related such that the voltage is the time derivative of current, then the element is an inductance 'L'. The inductance 'L' represents the constant of proportionality

$\therefore$ Voltage, $V = L\dfrac{dI}{dt}$

Current, $I = \dfrac{1}{L}\int V\,dt + K_1$ [K_1 = constant]

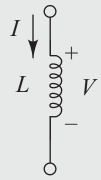


Fig. Set 1.2

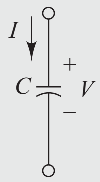
Power, $P = VI = LI\dfrac{dI}{dt}$

The units of inductance 'L' is Henry (H).

(iii) If the voltage and current are related such that the current is the time derivative of the voltage, then the element is a capacitance 'C'. The capacitance 'C' is the constant of proportionality

$\therefore$ Current, $I = C\dfrac{dV}{dt}$

Voltage, $V = \dfrac{1}{C}\int I\,dt + K_2$ [K_2 = constant]



Power, $P = VI = VC\dfrac{dV}{dt}$

Fig. Set 1.3

The units of capacitance 'C' is Farads (F).

1. (b) Derive the expression for the energy stored in an ideal inductor?

Solution: *Expression for Energy Stored in an ideal inductor*

Let 'L' be the co-efficient of self inductance and i be the current flowing through it.

Let 'dw' be the small amount of work to be expended to over come self induced emf.

$$\therefore \quad dw = Ei\,dt$$

$$dw = \frac{di}{dt}\,i\,dt \qquad \left[\because E = L\frac{di}{dt}\right]$$

from lenz law

$$dw = Li\,di \tag{1}$$

Hence total work to be done in establishing a maximum current i_0 is given by integrating (1) from 0 to i_0.

$$\therefore \quad w = \int_0^{i_0} dw = \int_0^{i_0} Li\,d\,i = L\int_0^{i_0} i\,d\,i$$

$$= L\left[\frac{1}{2}\frac{i_0^2}{1}\right]$$

$$w = \frac{1}{2}\ Li_0^2$$

$\therefore$ Energy stored in an inductor $w = \dfrac{1}{2}\ Li_0^2$

1. (c) Find the Current I_1 and I_2 using Nodal Analysis (Fig. Set 1.4)

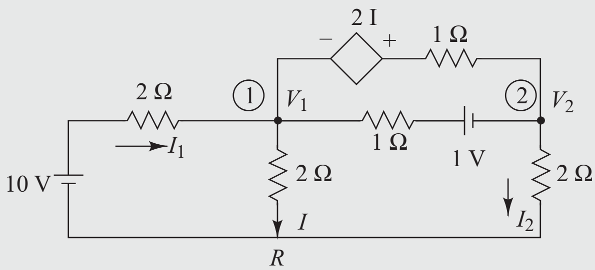


Fig. Set 1.4

Solution: At node (1):

$$\frac{V_1 - 10}{2} + \frac{V_1}{2} + \frac{V_1 - (1 + V_1)}{1} + \frac{(2I + V_1) - V_2}{1} = 0$$

$$\Rightarrow \quad V_1\left(\frac{1}{2} + \frac{1}{2} + 1 + 1\right) + (-1 - 1)V_2 = \frac{10}{2} + 1 - 2I = 6 - 2I$$

$$\Rightarrow \quad 3V_1 - 2V_2 = 6 - 2I \qquad (1)$$

At node 2:

$$\frac{V_2}{2} + \frac{(1 + V_2) - V_1}{1} + \frac{V_2\,(2I + V_1)}{1} = 0$$

$$\Rightarrow \quad (-1 - 1)V_1 + \left(\frac{1}{2} + 1 + 1\right)\ V_2 = -1 + 2I$$

$$\Rightarrow \quad -2V_1 + \frac{5}{2}\ V_2 = 2I - 1 \qquad (2)$$

But $\qquad I = \dfrac{V_1}{2} \qquad (3)$

From (3) $\qquad$ (1) $\Rightarrow$ $4V_1 - 2V_2 = 6$

$\qquad$ (2) $\Rightarrow$ $3V_1 - \dfrac{5}{2}\ V_2 = 1$

Solving,

$$V_1 = 3.25 \text{ V}$$
$$V_2 = 3.5 \text{ V}$$

$$\therefore \quad I_1 = \frac{10 - V_1}{2} = 3.375 \text{ A}; \qquad I_2 = \frac{V_2}{2} = 1.75 \text{ A.}$$

2. (a) Define Magneto Motive Force, Magnetic Flux, and Reluctance of a Magnetic circuit. Specify the unit for the above quantities, state the relation between the above quantities.

Solution: *Magneto Motive Force* (MMF)

Magneto Motive Force (MMF) is the measure of the ability of a coil to produce a flux.

As EMF is considered to be an electric pressure, MMF is also considered to be a magnetic pressure. A coil with N turns carrying a current of 'I' amperes represents a magnetic circuit producing an MMF of 'NI' ampere turns.

$\therefore$ MMF = NI Ampere Turns.

The MMF is the source of flux (ϕ) in the magnetic circuit. The length of the circuit and the MMF determines the amount of flux produced in the circuit.

Units of MMF = Ampere Turns (AT)

<u>Reluctance (S)</u>

It is the property of the medium which opposes the passage of magnetic flux. The reluctance in the Magnetic circuit is similar to the resistance in the electric circuit.

$$\therefore \quad \text{Reluctance} = \frac{\text{MMF}}{\text{flux}}$$

$$\therefore \quad S = \frac{\text{MMF}}{\phi}$$

Units of Reluctance is AT/wb.

The reluctance is the measure of the opposing offered to the set up of the flux by a magnetic circuit.

$$\therefore \quad S = \frac{\text{MMF}}{\phi} = \frac{NI}{\phi} \qquad [\because \phi = B \times A]$$

$$\therefore \quad S = \frac{NI}{B \times A} = \frac{NI}{\mu_o \mu_r + 1 \times A} \qquad [\because B = \mu_0 \mu_r H]$$

$$\therefore \quad S = \frac{NI}{\mu_o \mu_r \dfrac{NI}{l} \times A} \qquad \left[\because H = \frac{NI}{L}\right]$$

$$\therefore \qquad S = \frac{l}{\mu_o \, \mu_r \, A} \text{ AT/wb}$$

$$\therefore \qquad S = \frac{L}{\mu A} \text{ AT/wb}$$

where l = length of Magnetic Path; A = Area of cross section of magnetic circuit; and $\mu = \mu_0 \, \mu_r$ = Permeability of Medium.

<u>Magnetic Flux</u> (ϕ)

The total number of lines of induction passing normally through a surface is called Magnetic flux (ϕ).

Flux does not actually flow in a magnetic circuit.

Magnetic flux is directly proportional to the pole strength of the magnet.

i.e. $\phi \, \alpha \, m$

(or) $\phi = \mu m$

where μ = Permeability of Medium.

Units of magnetic flux is weber (wb).

<u>Relation between MMF, S and ϕ</u>

The Relation between MMF, Magnetic flux and Reluctance of a magnetic circuit is given as

$$\text{Magnetic flux} = \frac{\text{Maneto Motive Force}}{\text{Reluctance}}$$

i.e. $$\phi = \frac{\text{MMF}}{S}$$

i.e. $$\boxed{\phi = \frac{NI}{\frac{L}{MA}}}$$

2. (b) Write down the voltage equation for the following Fig. Set 1.5, and determine the effective inductance.

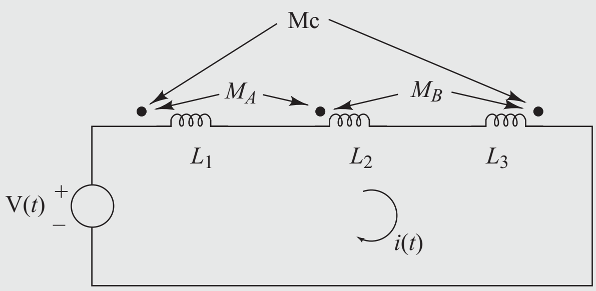


Fig. Set 1.5

Solution: Apply KVL in the given loop

$$V(t) = L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + M_A \frac{di(t)}{dt} + M_A \frac{di(t)}{dt} + L_3 \frac{di(t)}{dt} - M_B \frac{di(t)}{dt} - M_B \frac{di(t)}{dt} - M_C \frac{di(t)}{dt} - M_C \frac{di(t)}{dt}$$

$$\therefore \quad V(t) = [L_1 + L_2 + L_3 + 2M_A - 2M_B - 2M_C] \frac{di(t)}{dt}$$

is the required voltage equation.

We have $V(t) = L \dfrac{di(t)}{dt}$

$$L \frac{di(t)}{dt} = [L_1 + L_2 + L_3 + 2M_A - 2M_B - 2M_C] \frac{di(t)}{dt}$$

$\therefore$ $\boxed{L = L_1 + L_2 + L_3 + 2M_A - 2M_B - 2M_C}$ is the equivalent inductance.

2. (c) Two identical coils connected in series gave an inductance of 800 mH and when one of the coils is reversed gave an inductance of 400 mH. Determine self-inductance, mutual inductance between the coils and the co-efficient of coupling.

Solution: Let 'L' be the self inductance of the coils and M be the Mutual inductance between the coils.

<u>Given Data</u>

Two identical coils connected in series gave an inductance of 800 mH

i.e. $L + L + 2M = 800$ $\quad [\because$ identical coils $L_1 = L_2 = L]$

$$2L + 2M = 800 \quad (1)$$

When one of the coils is reversed gave an inductance of 400 mH

i.e. $L + L - 2M = 400$

$$2L - 2M = 400 \quad (2)$$

Add (1) and (2) we get $4L = 1200$

$$\boxed{L = 300 \text{ mH}}$$

Subtracting (2) from (1) we get $4M = 400$ mH

$$\boxed{M = 100 \text{ mH}}$$

$\therefore$ Self inductance of each coil = L = 300 mH

Mutual inductance between the coils = M = 100 mH

Co-efficient of coupling = $K = \dfrac{M}{\sqrt{L_1 L_2}}$

$\therefore \quad K = \dfrac{M}{\sqrt{LL}}$ $\quad [\because L_1 = L_2 = L]$

$\therefore \quad K = \dfrac{M}{\sqrt{L^2}} = \dfrac{M}{L}$

$\therefore \quad K = \dfrac{100\,\text{mH}}{300\,\text{mH}}$

$\therefore \quad \boxed{K = 1/3}$

$\therefore$ co-efficient of coupling = 1/3.

3. (a) Derive the expression for $i(t)$ when the switch is moved from position 1 to position 2 at $t = 0$ in the circuit (Fig. Set 1.6) shown. The switch was in position 1 for a long time. Sketch the variation of $i(t)$. Also determine $V_C(t)$.

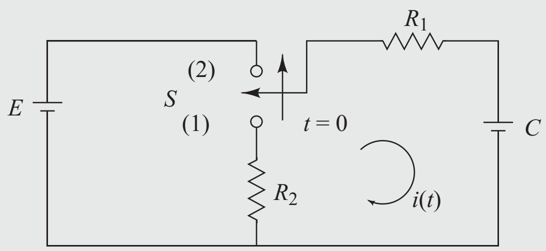


Fig. Set 1.6

Solution: When switch is in position 1 for a *long time* (steady state) capacitor is *not charged* to any voltage.

i.e., $V_C = 0$ when switch is at position (1)

Switch of position (2): Capacitor doesn't allow sudden change in voltage

$\therefore \quad V_C(t = 0^+) = 0$ (Here acting as short circuit initially)

$\therefore \quad i(t = 0^+) = \dfrac{\text{E}}{\text{R}_1}$ (Initial conditions)

Applying KVL, $E = i(t)\, R_1 + \dfrac{1}{C}\displaystyle\int_0^t i(t)\, dt$

Differentiating once, $O = R_1 \dfrac{d\,i(t)}{d\,t} + \dfrac{i(t)}{C}$ (1)

$$\Rightarrow \left(D + \frac{1}{R_1 C}\right) i = 0 \qquad \left(\text{where } D = \frac{d}{dt}\right)$$

$\therefore \quad i(t) = Ce^{-t/R_1C}$ is the solution of eqn. (1)

By initial condition $i(0) = \dfrac{E}{R_1}$

$$\therefore \quad \boxed{i(t) = \frac{E}{R_1} e^{-t/R_1C}}$$

$$V_C(t) = \frac{1}{C}\int_0^t i(t)\,dt = \frac{E}{R_1 C}\int_0^t e^{-t/R_1C}\,dt$$

$$= \frac{E}{R_1 C}(-R_1 C)(e^{-t/RC} - 1)$$

$$\therefore \quad \boxed{V_C(t) = E(1 - e^{-t/RC})}$$

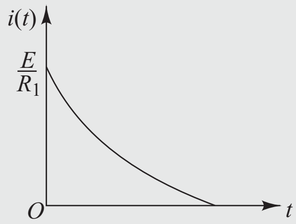


Fig. Set 1.7

3. (b) In the circuit (Fig. Set 1.8) shown, determine the voltage V_{AB} to be applied to the circuit if a current of 2.5 A is required to flow in the capacitor. Determine also total power factor and total active and reactive powers. Draw the phasor diagram.

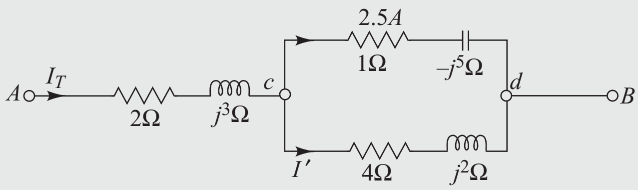


Fig. Set 1.8

Solution: $V_{cd} = 2.5\,(1 - j5) = I'(4 + j2)$

(Assuming "I'" is the current through '$4 + j2$' Ω)

$$\therefore \quad I' = \frac{2.5\,(1-5j)}{4+2j} = 2.85\,\angle{-105.25^\circ} = -\,0.75 - 2.75j$$

$$\therefore \quad I_T = 2.5 - 0.75 - 2.75j \qquad (\because\ I_T = 2.5 + I')$$

$$= 1.75 - 2.75j = 3.26\,\angle{-57.53^\circ}$$

$$\therefore \quad V_{AB} = I_T(2 + j3) + 2.5(1 - j5)$$

$$= (1.75 - 2.75j)(2 + j3) + 2.5(1 - j5)$$

$$= 14.25 - 12.75j = 19.12\,\angle{-41.82^\circ}$$

$$Z_{AB} = \frac{V_{AB}}{I_T} = \frac{19.12\,\angle -41.82^\circ}{3.26\,\angle -57.53^\circ} = 5.865\,\angle 15.71^\circ$$

$$\theta = 15.71^\circ$$

Total power factor = cos θ = cos 15.71° = 0.962

Total active power = $V_{AB}\, I_T \cos\,\theta$
(P_{arg})

$= 19.12 \times 3.26 \times 0.962 = 59.96$ W

Total reactive power = $V_{AB}\, I_T \sin\,\theta$
(P_o)

$= 19.12 \times 3.26 \times \sin 15.71° = 16.87$ VAR

Apparent power = $V_{AB}\, I_T\, (P_o) = 19.12 \times 3.26 = 62.3312\ V_A$

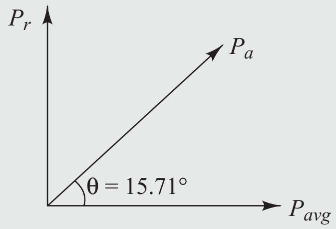


Fig. Set 1.9

4. (a) Obtain the response of *R-L-C* series circuit for impulse excitation. Use Laplace transform method

Solution:

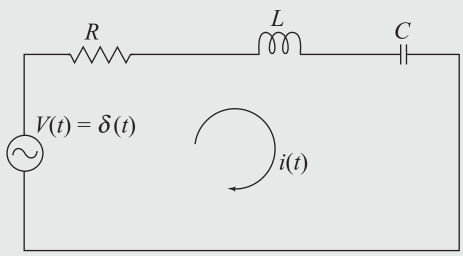


Fig. Set 1.10

Apply KVL to the *R-L-C* series circuit

$$\delta(t) = i(t)\,R + L\,\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

Apply Laplace transform

$$L\,\{\delta(t)\} = L\left\{i(t)\,R + L\,\frac{d\,i(t)}{d\,t} + \frac{1}{C}\int_0^t i(t)\,d\,t\right\}$$

$$1 = I(S)\,R + L\,[SI(S) - I(0)] + \frac{1}{C}\left[\left[\frac{I(S)}{S}\right] - \frac{I(0^-)}{S}\right]$$

$$1 = I(S)\left[R + LS + \frac{1}{CS}\right] \quad [\because \text{ Initial conditions are zero}]$$

$$I(S) = \frac{1}{R + LS + \frac{1}{CS}}$$

$$I(S) = \frac{1}{\frac{L}{S}\left[S^2 + S\frac{R}{L} + \frac{1}{LC}\right]}$$

$$I(S) = \frac{S}{L\left[S^2 + S\frac{R}{L} + \frac{1}{LC}\right]} \qquad (1)$$

Applying partial fraction

$$\frac{(S/L)}{S^2 + S\frac{R}{L} + \frac{1}{LC}} = \frac{1}{L}\left[\frac{S}{S^2 + \frac{R}{L}S + \frac{1}{LC}}\right] = \frac{1}{L}\left[\frac{S}{\left(S + \frac{R}{2L}\right)^2 - \left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right]$$

$$= \frac{1}{L}\left[\frac{S + \frac{R}{2L} - \frac{R}{2L}}{\left(S + \frac{R}{2L}\right)^2 - \left(\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right)}\right]$$

$$= \frac{1}{L}\left[\frac{S + \frac{R}{2L}}{\left(S + \frac{R}{2L}\right)^2 - \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]} - \frac{R}{2L}\frac{1}{\left(S + \frac{R}{2L}\right)^2 - \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]}\right]$$

Let $\quad K_1 = -\frac{R}{2L}$ and $K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

But $L\left(\frac{S + k}{(S + k)^2 + \omega^2}\right) = e^{-kt}\cos \omega t$ and $L\left(\frac{\theta}{S^2 + \theta^2}\right) = \sin \omega t]$

$$= \frac{1}{L}\left[\frac{S - K_1}{(S - k_1)^2 - K_2^2} + \frac{K_{1/k_2}.k_2}{(S - k_1)^2 - K_2^2}\right]$$

If $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}, \qquad k_2 > 0$

Applying inverse Laplace transform of (1)

$$i(t) = \frac{1}{L}\left[e^{k_1 t} \cosh k_2 t - \frac{k_1}{k_2} e^{k_1 t} \sinh k_2 t\right]$$

If $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$, k_2 imaginary $k_2^2 \to - k_2^2$

$$i(t) = \frac{1}{L}\ [e^{k_1 t} \cos k_2 t + \frac{k_1}{k_2}\ e^{k_1 t} \sin k_2 t]$$

If $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$, $k_2 = 0$

$$i(t) = \frac{1}{L}\ [e^{k_1 t} + k_1\ e^{k_1 t}\ t] = \frac{k_1}{L}\ e^{k_1 t}\ (1 + k_1 t)$$

4. (b) Obtain the S domain equivalent for the following network elements.

Solution:

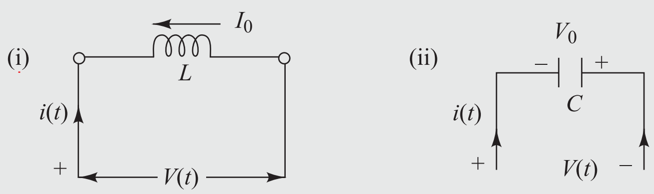


Fig. Set 1.11

(i) $$V(t) = L\ \frac{d\, i(t)}{dt} \qquad (1)$$

$$i(t) = \frac{1}{L}\int V(t)\ dt \qquad (2)$$

$V(S) = L[SI(S) - I(0^-)]$ Applying Laplace Transform to (1)

$I(S) = \frac{1}{SL}\ V(S) + \frac{i(0^-)}{S}$ Applying Laplace Transform to (2)

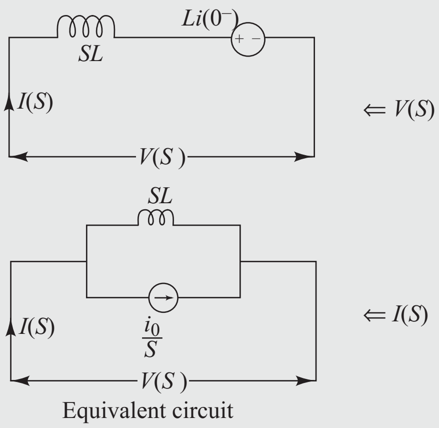


Fig. Set 1.12

(ii) We have

$$i(t) = C\frac{dV(t)}{dt}$$

Applying Laplace Transform

$$L\{i(t)\} = C\left\{\frac{dV(t)}{dt}\right\}$$

$$I(S) = C\,[SV(S) - V(0)] \qquad (1)$$

$$dV(t) = \frac{1}{C}\int i(t)\, dt$$

Integrating

$$V(t) = \frac{1}{C}\int i(t)\, dt$$

Apply laplace transform

$$L\{V(t)\} = \frac{1}{C}L\{i(t)\, dt\}$$

$$V(S) = \frac{1}{Cs}\; I(S) + \frac{V(o)}{S} \qquad (2)$$

For Eq. (2) circuit is

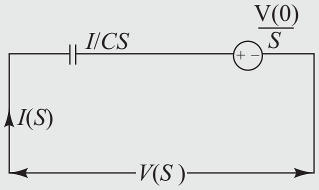


Fig. Set 1.13

For Eq. (1) circuit is

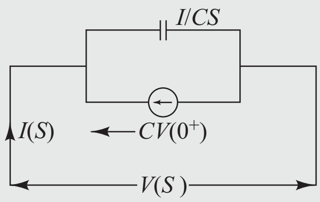


Fig. Set 1.14

4. (c) Define RMS Value, Average Value, form factor of a periodic quantity.

Solution: *Average Value*

For any alternating quantity $f(t)$ with time period 'T' average value f_{avg} is given by

$$f_{avg} = \frac{1}{T}\int_0^T f(t)\,dt$$

Waveforms with half-wave symmetry i.e.,

$f(t) = -f\left(t + \frac{T}{2}\right)$ have zero average values. For these waveforms f_{avg} is computed over the positive half of the period which is called as half-cycle average.

Average value is that value of direct current which gives same amount of the charge to the network in same amount of time as given by the alternating current to the same electrical network.

<u>RMS Value</u>

For any alternating quantity $f(t)$ with time period 'T', r.m.s. value $f_{r.m.s}$ is given by,

$$f_{r.m.s} = \sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt}$$

If $\quad f(t) = a_0 + (a_1 \cos \omega t + a_2 \cos 2\omega t \ldots) + (b_1 \sin \omega t + b_2 \sin 2\omega t \ldots)$

$$\text{then } f_{r.m.s.} = \left[a_0^2 + \frac{1}{2}\left(a_1^2 + a_2^2 \cdots\right) + \frac{1}{2}\left(b_1^2 + b_2^2 \cdots\right)\right]^{1/2}$$

R.M.S. value is equal to that direct current which when allowed to flow through a given circuit for a given time, produces same amount of heat as produced by the alternating current when allowed to flow through the same circuit for the same time.

From Factor [K_f]

The ratio of r.m.s value of an alternating quantity to its average value is called form factor.

$$K_f = \frac{f_{\text{r.m.s}}}{f_{\text{avg}}} = \frac{\sqrt{\frac{1}{T}\int_0^T t^2(t)\,dt}}{\frac{1}{T}\int_0^T f(t)\,dt}$$

5. (a) Draw the dual network for the following circuit. Shown in Fig. Set 1.15.

Solution:

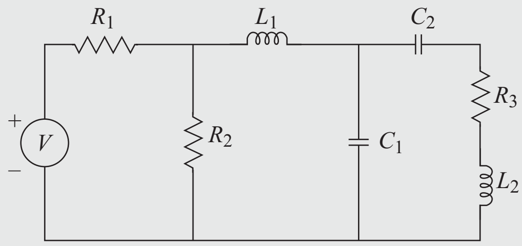


Fig. Set 1.15

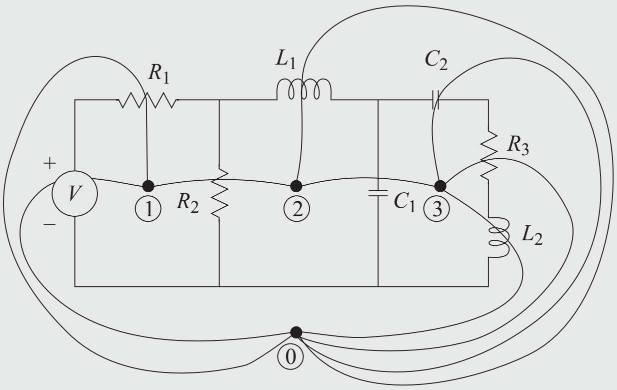


Fig. Set 1.16

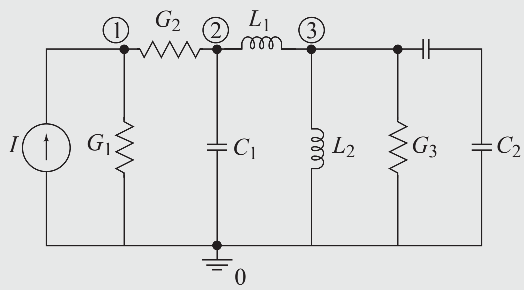


Fig. Set 1.17

5. (b) Explain, what are the dual quantities?

Solution:

Refer to Set No. 2 Questions 1 (c)

5. (c) Draw the phasor diagram of *R*, *L*, *C* elements connected parallel across a sinusoidal voltage source?

Solution

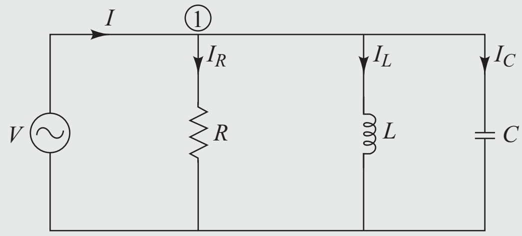


Fig. Set 1.18

Let *Z* be the equivalent impedance
Apply KCL at Node (1)

$$\bar{I} = \bar{I}_R + \bar{I}_L + \bar{I}_C$$

$$\frac{\bar{V}}{Z} = \frac{\bar{V}}{R} + \frac{\bar{V}}{jx_L} + \frac{\bar{V}}{-jx_C}$$

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{jx_L} - \frac{1}{jx_C}$$

$$\left|\frac{1}{Z}\right| = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{x_L} - \frac{1}{x_C}\right)^2}$$

$$|Y| = \left|\frac{1}{Z}\right| = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{x_L} - \frac{1}{x_C}\right)^2}$$

The term *Z* is known as complex impedance the term *Y* is known as complex admittance of the parallel RLC circuit and we have three cases.

Case (i)

if $x_C > x_L$ (low freq.) or $I_L > I_C$

Current triangle

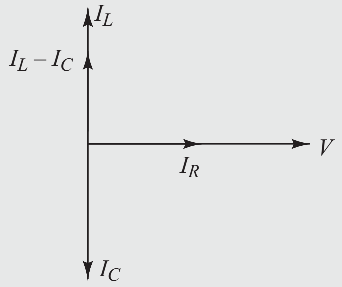


Fig. Set 1.19

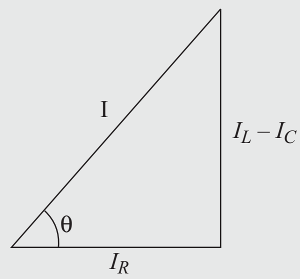


Fig. Set 1.20

$$\theta = \tan^{-1}\left(R\left(\frac{1}{x_L} - \frac{1}{x_C}\right)\right)$$

Impedence triangle

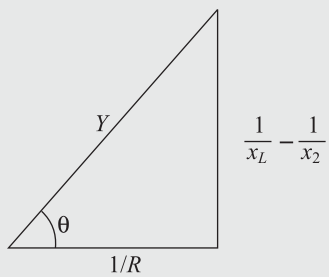


Fig. Set 1.21

Case (ii)

if $x_L > x_C$ (high freq.) of $I_C > I_L$

Impedence triangle

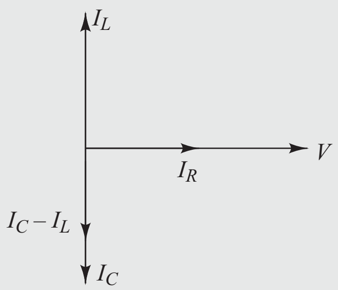


Fig. Set 1.22

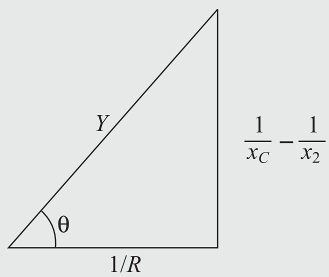


Fig. Set 1.23

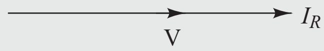
$$\theta = \tan^{-1}\left(R\left(\frac{1}{x_L} - \frac{1}{x_C}\right)\right)$$

Case (iii)

if $x_L = x_C$

then $|Z| = |R|$

The circuit is Purely Real circuit i.e. it contains only Resistive element.



6. (a) Write the standard Y-parameter equations. Obtain the Y-parameters in terms of Z-parameters.

Solution: <u>Y-parameter</u>

Y-parameters are also known as short circuit parameters.
The standard Y-parameter equations are

$$I_1 = Y_{11}\, V_1 + Y_{12}\, V_2 \tag{1}$$

$$I_2 = Y_{21}\, V_1 + Y_{22}\, V_2 \tag{2}$$

Multiply equation (1) with Y_{21} and equation (2) with Y_{11}

$$Y_{21}\, I_1 = Y_{11}\, Y_{21}\, V_1 + Y_{21}\, Y_{12}\, V_2$$

$$Y_{11}\, I_2 = Y_{11}\, Y_{21}\, V_1 + Y_{11}\, Y_{22}\, V_2$$

Subtracting we get

$$(Y_{21}\, Y_{12} - Y_{11}\, Y_{22})V_2 = Y_{21}\, I_1 - Y_{11}\, I_2 - \Delta Y V_2 = Y_{21}\, I_1 - Y_{11}\, I_2$$

$$V_2 = \frac{Y_{21}}{-\Delta Y}\, I_1 + \frac{Y_{11}}{\Delta Y}\, I_2$$

We know

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0} = \frac{-Y_{21}}{\Delta Y} \qquad [\text{where } \Delta Y = Y_{11}Y_{22} - Y_{21}Y_{12}]$$

$$Z_{22} = \left.\frac{V_2}{I_1}\right|_{I_1 = 0} = \frac{Y_{11}}{\Delta Y}$$

Multiply equation (1) with Y_{22} and equation (2) with Y_{12}.

$$Y_{22}\, I_1 = Y_{22}\, Y_{11}\, V_1 + Y_{22}\, Y_{12}\, V_2$$

$$Y_{12}\, I_2 = Y_{12}\, Y_{21}\, V_1 + Y_{12}\, Y_{22}\, V_2$$

Subtracting

$$V_1\, (Y_{11}\, Y_{22} - Y_{12}\, Y_{21}) = Y_{22}\, I_1 - Y_{12}\, I_2$$

$$V_1 = \frac{Y_{22}}{\Delta Y}\, I_1 - \frac{Y_{12}}{\Delta Y}\, I_2$$

and
We know

$$Z_{11} = \left.\frac{V_1}{I_2}\right|_{I_2 = 0} = \frac{Y_{22}}{\Delta Y}$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{-Y_{12}}{\Delta Y}$$

$\therefore$ Z-parameters interms of Y-parameters

$$Z_{11} = \frac{Y_{22}}{\Delta Y} \qquad Z_{12} = \frac{-Y_{12}}{\Delta Y}$$

$$Z_{21} = \frac{-Y_{21}}{\Delta Y} \qquad Z_{22} = \frac{Y_{11}}{\Delta Y}$$

Similarly we can get Y-parameters in terms of Z.

$$Y_{11} = \frac{Z_{22}}{\Delta Z} \qquad Y_{12} = \frac{-Z_{12}}{\Delta Z}$$

$$Y_{21} = \frac{-Z_{21}}{\Delta Z} \qquad Y_{22} = \frac{Z_{11}}{\Delta Z}$$

and $Z = Y^{-1}$ (or) $Y = Z^{-1}$.

6. (b) Obtain Z-parameters for the circuit shown in Figure 7 and there by obtain ABCD parameters.

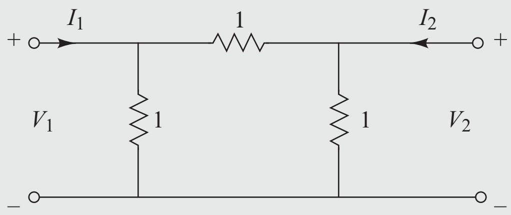


Fig. Set 1.24

Solution:

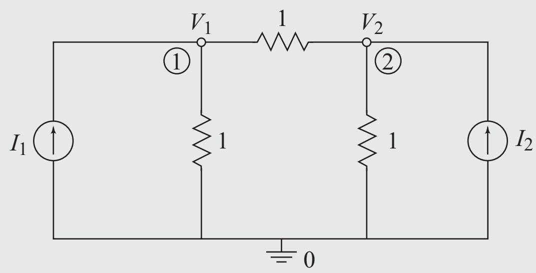


Fig. Set 1.25

Apply Nodal Analysis

At Node (1)

$$I_1 = \frac{V_1}{1} + \frac{V_1 + V_2}{1}$$

$$I_1 = 2V_1 - V_2 \tag{1}$$

At Node (2)

$$I_2 = \frac{V_2}{1} + \frac{V_2 - V_1}{1}$$

$$I_2 = 2V_2 - V_1$$

$$\text{We get } Y = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Z = Y^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We have

$$A = \frac{Z_{11}}{Z_{21}} = \frac{2/3}{1/3} = 2$$

$$B = \frac{\Delta Z}{Z21} = \frac{1/3}{1/3} = 1$$

$$C = \frac{1}{Z_{21}} = \frac{1}{1/3} = 3$$

$$D = \frac{Z_{22}}{Z_{21}} = \frac{2/3}{1/3} = 2$$

$$\therefore \qquad \boxed{A = 2 \quad B = 1 \quad C = 3 \quad D = 2}$$

7. (a) Figure Set 1.26 shows a resistive *T* network and a resistive *P* network connected in parallel. Find the overall *Y* parameters of the combination.

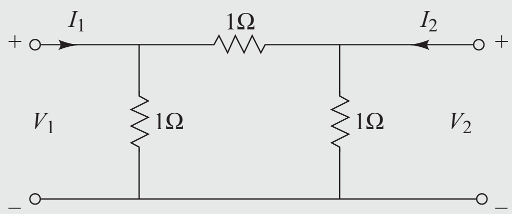


Fig. Set 1.26

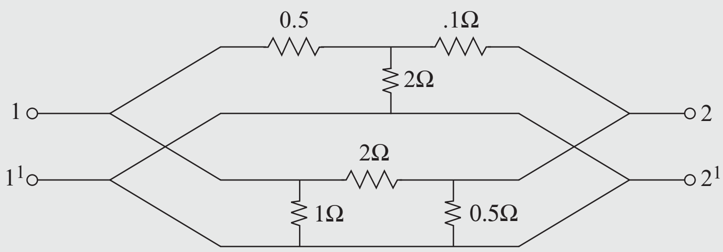


Fig. Set 1.27

Solution: For '*T*' network

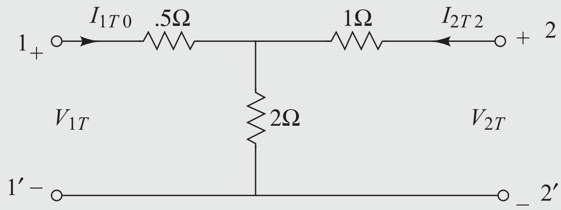


Fig. Set 1.28

$$I_{1T} = Y_{11T} V_{1T} + Y_{12T} V_{2T} \quad \therefore \; Y_{11T} = \left.\frac{I_{1T}}{V_{1T}}\right|_{V_{2T}=0} = 0.854$$

$$I_{2T} = Y_{21T} V_{1T} + Y_{22T} V_{2T}.$$

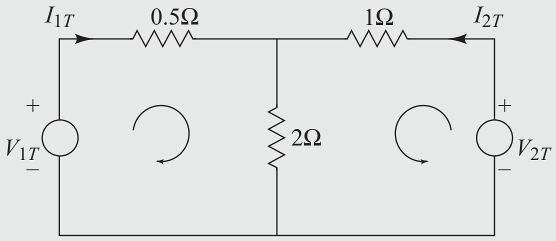


Fig. Set 1.29

$$V_{1T} = \frac{I_{1T}}{2} + 2(I_{1T} + I_{2T}) \quad \Rightarrow \quad V_{1T} = \frac{5}{2} I_{1T} + 2I_{2T}$$

$$V_{2T} = I_{2T} + 2(I_{1T} + I_{2T}) \quad \Rightarrow \quad V_{2T} = 2I_{1T} + 3I_{2T}$$

$$Z\text{-parameters} = \begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 3 \end{pmatrix} \quad Y\text{-parameters} = (Z)^{-1}$$

$$Y_T = \frac{1}{3.5}\begin{pmatrix} 3 & -2 \\ -2 & 5/2 \end{pmatrix} = \begin{pmatrix} 0.857 & -0.571 \\ -0.571 & 0.714 \end{pmatrix}$$

For π network

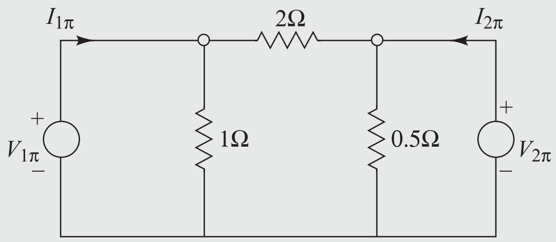


Fig. Set 1.30

$$I_{1\pi} = \frac{V_{1\pi}}{1} + \frac{V_{1\pi} - V_{2\pi}}{2}$$

$$I_{2\pi} = \frac{V_{2\pi}}{0.5} + \frac{V_{2\pi} - V_{1\pi}}{2}$$

$$\Rightarrow \qquad I_{1\pi} = \frac{3V_{1\pi}}{2} - \frac{V_{2\pi}}{2}$$

$$I_{2\pi} = \frac{\sqrt{V_{2\pi}}}{2} - \frac{V_{1\pi}}{2}$$

$$Y_{T1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 2.5 \end{pmatrix}$$

The overall Y-parameters of the combination is

$$Y_{OV} = Y_T + Y_\pi = \begin{pmatrix} \frac{3}{3.5} + \frac{3}{2} & \frac{-2}{3.5} - \frac{1}{2} \\ \frac{-2}{3.5} - \frac{1}{2} & \frac{5}{7} + \frac{5}{2} \end{pmatrix}$$

$$\therefore \qquad Y_{0V} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} 2.357 & -1.071 \\ -1.071 & 3.214 \end{pmatrix}$$

7. (b) Find the characteristic impedance of a symmetric T network.

Solution: <u>*T*-Network</u>

Consider a symmetrical T-network as shown in Fig. 1.

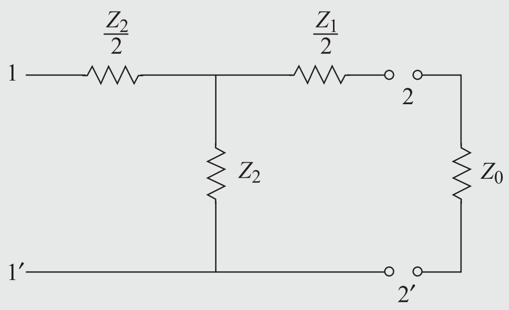


Fig. Set 1.31

If the image impedances at port 1–1′ and port 2–2′ are equal to each other, the image impedance is then called the characteristic or the iterative impedance, Z_0. Thus, if the network in figure is terminated in Z_0, its input impedance will be Z_0. The value of input impedance for T-network when it is terminated in Z_0 is given by

$$Z_{in} = \frac{Z_1}{2} + \frac{Z_2\left[\frac{Z_1}{2} + Z_0\right]}{\frac{Z_1}{2} + Z_2 + Z_0}$$

Also $\quad Z_{in} = Z_0$

$$\therefore \quad Z_0 = \frac{Z_1}{2} + \frac{2Z_2\left[\frac{Z_1}{2} + Z_0\right]}{Z_1 + 2Z_2 + 2Z_0}$$

$$Z_0 = \frac{Z_1}{2} + \frac{(Z_1Z_2 + 2Z_2Z_0)}{Z_1 + 2Z_2 + 2Z_0}$$

$$Z_0 = \frac{Z_1^2 + 2Z_1Z_2 + 2Z_1Z_0 + 2Z_1Z_2 + 4Z_0Z_2}{2(Z_1 + 2Z_2 + 2Z_0)}$$

$$4Z_0^2 = Z_1^2 + 4Z_1Z_2$$

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1Z_2$$

$$\therefore \quad Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1Z_2}$$

$\therefore$ The characteristic impedance of a symmetrical T-section is

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1Z_2}\ .$$

8. (a) What is constant k-filter? What is the difference between constant k-filter and m-derived filter? What are the limitations of constant k-filter?

Solution: Refer to Text Book (Chapter 6)

8. (b) Find the circuit parameters of a constant k-band pass filter having a pass band from 500 Hz and a characteristic resistance of 100 Ω.

Solution: k = 100 Ω, f_1 = 500 Hz, F_2 = ? (Not given)

Assume f_2 = 10000 Hz

for band pass filter.

$$L_1 = \frac{k}{\pi(f_2 - f_1)} = \frac{100}{\pi(9500)} = 3.35 \text{ mH}$$

$$C_1 = \frac{f_2 - f_1}{4\pi k\, f_1 f_2} = \frac{9500}{4 \times \pi \times 100 \times 500 \times 10000} = 1.512\ \mu\text{F}$$

$$L_2 = C_1\, k^2 = 15.12 \text{ mH}$$

$$C_2 = \frac{L_1}{k^2} = 0.335\ \mu\text{F}$$

Each of the two series erms of the constant h, T-section filter is given by

$$\frac{L_1}{2} = \frac{3.35}{2} = 1.675 \text{ mH}$$

$$2C_1 = 2 \times 1.512\ \mu\text{F} = 3.024\ \mu\text{F}$$

And the shunt erm elements of the network are given by

$$C_2 = 0.335\ \mu\text{F and } L_2 = 15.12 \text{ mH}$$

Constant-k T-section bendpass filter

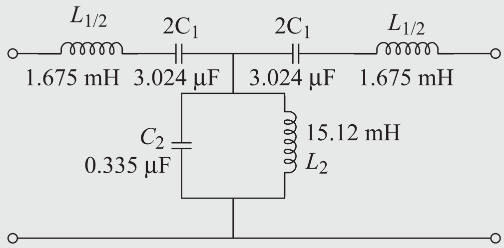


Fig. Set 1.32

Constant-k π-section bandpass filter

$$\frac{C_2}{2} = 0.1675\ \mu\text{F}$$

$$2L_2 = 30.24 \text{ mH}$$

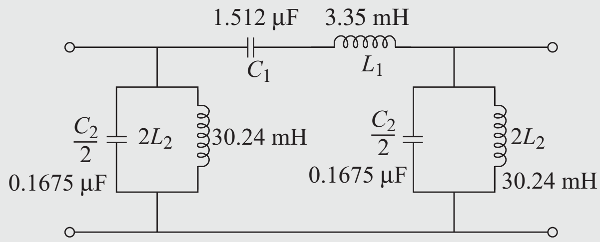


Fig. Set 1.33

Network Analysis, May/June 2006

SET 2

1. (a) For the given network (Fig. Set 2.1), draw the oriented graph and choose one possible tree and construct the basic cutest schedule. Write down the network Equations from the above matrix.

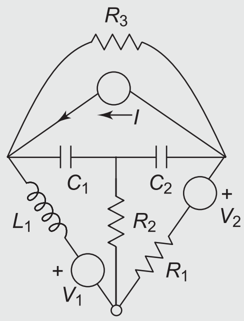


Fig. Set 2.1

Solution:

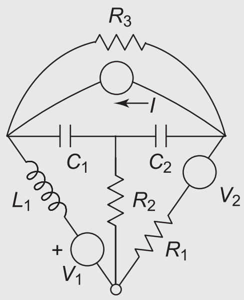


Fig. Set 2.2

The oriented graph for the given network can be as shown in Fig. Set 2.3

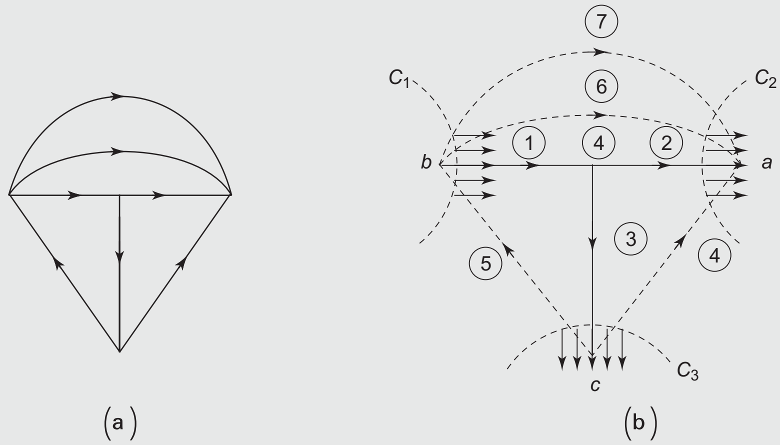


Fig. Set 2.3

$$C_1: i_1 - i_5 + i_6 + i_7 = 0$$

$$C_2: i_2 - i_4 + i_6 + i_7 = 0$$

$$C_3: i_3 + i_4 - i_5 = 0$$

$$\begin{array}{ll} & \text{branches} \\ f-\text{cutsets} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{rc} [7, 6, \bar{2}, 4] & a \\ [7, 6, \bar{1}, 4] & b \\ [3, 4, \bar{5}] & c \end{array} & \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

1. (b) For the network shown (Fig. Set 2.4), determine the node Voltages V_1 and V_2. Determine the power dissipated in each resistor.

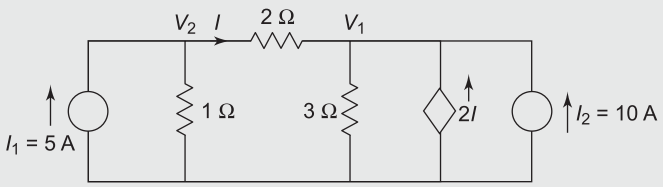


Fig. Set 2.4

Solution:

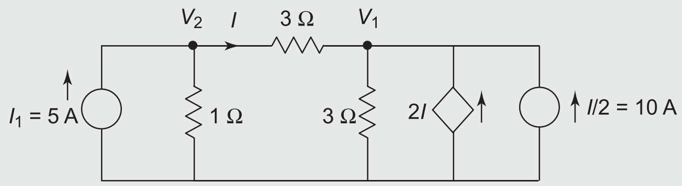


Fig. Set 2.5

Applying KCL

$$5 = \frac{V_2}{1} + \frac{V_2 - V_1}{2} \quad \Rightarrow \quad V_2\left(1 + \frac{1}{2}\right) - \frac{V_2}{2} = 5$$

$$3V_2 - V_1 = 10 \tag{1}$$

$$I = \frac{V_2 - V_1}{2} \tag{2}$$

$$\frac{V_1}{3} + \frac{V_1 - V_2}{2} = 10 + 2I = \frac{-V_2}{2} = 10 + 2\left(\frac{V_2 - V_1}{2}\right)$$

$$\Rightarrow \quad V_1\left(\frac{1}{2} + \frac{1}{3}\right) = 10 + V_2 - V_1$$

$$\therefore \quad 11V_1 - 9V_2 = 60 \tag{3}$$

Solving (1) and (3),

$$V_1 = 11.25 \text{ volts}$$
$$\text{and } V_2 = 7.083 \text{ volts}$$

$$\text{Power dissipated in } 1\ \Omega \text{ resistor} = VI = I^2R = \frac{V^2}{R} = \frac{V^2}{1} = (7.083)^2$$
$$= 50.17 \text{ watts}$$

$$\text{Power dissipated in } 2\ \Omega \text{ resistor} = \frac{V^2}{R} = \frac{(V_2 - V_1)^2}{2} = 8.682 \text{ watts}$$

$$\text{Power dissipated in } 3\ \Omega \text{ resistor} = \frac{V_1^2}{3} = \frac{(11.25)^2}{3} = 42.19 \text{ watts}$$

1. (c) Explain cleanly what you understand by "Duality" and "Dual network". Illustrate the procedure for drawing the dual of a given network.

Solution: Two circuits are duals, if the mesh equations that characterise one of them have the same mathematical form as the nodal equations that characterise other.

Then they are said to duals (OH) satisfy duality of property i.e., if each mesh equation of one circuit is numerically identical with the corresponding nodal equation of other.

Networks that satisfy duality property are called "Dual networks."

Dual pairs:

Resistance (R) $\rightarrow$ Conductance (G)
Inductance (L) $\rightarrow$ Capacitance (C)
Voltage (V) $\rightarrow$ Current (I)
Voltage Source $\rightarrow$ Current source
Node $\rightarrow$ Mesh
Series path $\rightarrow$ Parallel path
Open circuit $\rightarrow$ Short ckt
Thevenin $\rightarrow$ Norton

<u>Steps to construct a dual circuit:</u>

1. Place a node at the centre of each mesh of the given ckt. Place the reference node of the dual ckt outside the given ckt.
2. Draw dotted lines between the nodes such that each line crosses a network element by its dual.
3. A voltage source that produces a positive (clockwise) mesh current has it dual or current source whose reference direction is from ground to non-reference node.

∴ Two circuits are said to be dual if they are described by the same characterising equations with dual quantities interchanged.

2. (a) Explain the Dot Convention for mutually coupled coils.

Solution: *Dot Convention*

Mutual inductance is the ability of one inductor to induce voltage across the neighbouring inductor measured in Henrys (H).

The mutually induced emf $\frac{Mdi}{dt}$ may be positive (or) negative but M is always positive.

We apply dot convention to determine the polarity of the induced emf.

Place a dot at one end of coil (1) Assume that the current enters at the dotted end of the coil. Determine the direction of flux produced due to this current. Then place another dot at one of the ends of coil (2) such that the current entering at that dotted end in coil (2) produce flux in the same direction. Consider two coils (1) and (2) as shown.

1. Place a dot at one end of coil (1) and assume that the current enters at that dotted end in coil (1).
2. Place another dot at one of the ends of coil (2) such that the current entering at that end in coil (2) establishes magnetic flux in the same direction.

In order that the flux produced by I_2 flowing in coil (2) produce flux in the same upward direction it should enter at lower end of coil (2). Hence place a dot at that end of coil (2).

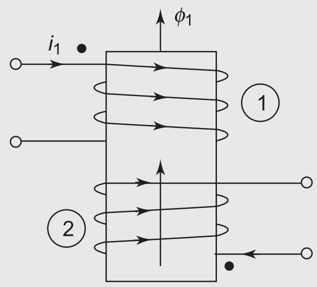


Fig. Set 2.6

2. (b) Derive the Expression for coefficient coupling between pair of magnetically coupled coils.

Solution: *Coefficient of Coupling*: It is a measure of the flux linkages between the two coils.

The coefficient of coupling is defined as the fraction of the total flux produced by one coil linking with another and it is denoted by 'k'.

Let $\phi_1 \rightarrow$ flux produced by coil–1

$\phi_2 \rightarrow$ flux produced by coil–2

$\phi_{12} \rightarrow$ flux produced by coil–1 linking with coil–2

$\phi_{21} \rightarrow$ flux produced by coil–2 linking with coil–1

∴ Coefficient of coupling $k = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2}$

k value lies between 0 and 1.

We know that $M_{12} = \dfrac{M_2\,\phi_{12}}{i_1}$, $M_{21} = \dfrac{M_1\,\phi_{21}}{i_2}$

$$M_{12} \times M_{21} = \frac{M_2\,\phi_{12} \times M_1\,\phi_{21}}{i_1\,i_2}$$

$$M^2 = \frac{M_2 \times k\phi_1}{i_1} \times \frac{M_1 \times k\,\phi_2}{i_2}$$

$$M^2 = k^2\;\frac{M_1\,\phi_1}{i_1} \times \frac{M_2\,\phi_2}{i_2} = k^2\,L_1\,L_2$$

$$\Rightarrow \qquad \boxed{k = \frac{M}{\sqrt{L_1 L_2}}}$$

2. (c) Write the Loop Equations for the Coupled circuit shown in Fig. Set 2.7.

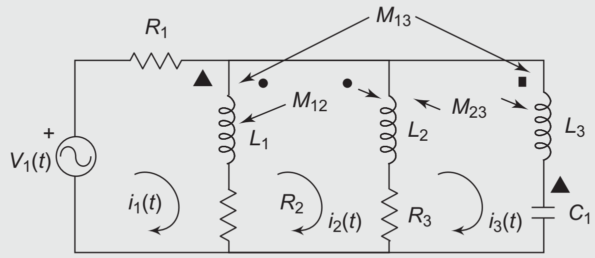


Fig. Set 2.7

Solution:

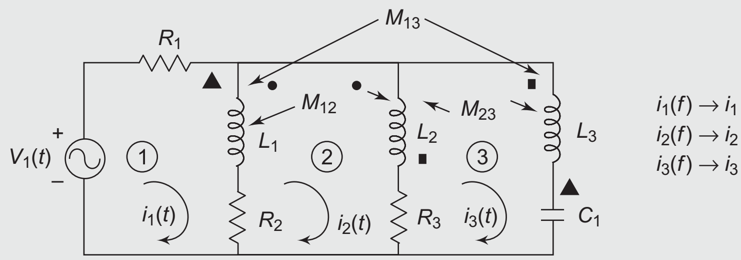


Fig. Set 2.8

Loop Equations: (By Dot Rule Convention)

(1) $\Rightarrow$ $V_1(t) = R_1\,i_1(t) + L_1 \left(\dfrac{d\,i_1(t)}{d\,t} - \dfrac{d\,i}{d\,t}\right) + M_{12}$

$$\left(\frac{d\,i_2(t)}{d\,t} - \frac{d\,i_3(t)}{d\,t} - M_{13}\,\frac{di_3(t)}{d\,t}\right) + R_2\,(i_1(t) - i_2(t) = 0$$

$$\Rightarrow \; V_1(t) = i_1(t)\,(R_1 + R_2) + L_1 \frac{d\,i_1(t)}{d\,t} - i_2(t)\,R_2 + M_{12} \frac{d\,i_2(t)}{d\,t} -$$

$$M_{13} \frac{d\,i_3(t)}{d\,t} - L_1 \frac{d\,i_2(t)}{d\,t} - M_{12} \frac{d\,i_3(t)}{d\,t}$$

(2) $$\Rightarrow \; R_2\,(i_2(t) - i_1(t)) + L_1 \left(\frac{d\,i_2(t)}{d\,t} - \frac{d\,i_1(t)}{d\,t}\right) - M_{12} \left(\frac{d\,i_2(t)}{d\,t} - \frac{d\,i_3}{d\,t}\right) +$$

$$M_{13} \frac{d\,i_3}{d\,t} + L_2 \left(\frac{d\,i_2}{d\,t} - \frac{d\,i_3}{d\,t}\right) - M_{12} \left(\frac{d\,i_2}{d\,t} - \frac{d\,i_1}{d\,t}\right) - M_{23} \frac{d\,i_3}{d\,t} +$$

$$R_3\,(i_2 - i_3) = 0$$

(3) $$\Rightarrow \; R_3\,(i_3 - i_2) + L_2 \left(\frac{d\,i_3}{d\,t} - \frac{d\,i_2}{d\,t}\right) - M_{12} \left(\frac{d\,i_1}{d\,t} - \frac{d\,i_2}{d\,t}\right) + M_{23} \frac{d\,i_3}{d\,t}$$

$$+ L_3 \frac{d\,i_3}{d\,t} - M_{13} \left(\frac{d\,i_1}{d\,t} - \frac{d\,i_2}{d\,t}\right) + M_{23} \left(\frac{d\,i_3}{d\,t} - \frac{d\,i_2}{d\,t}\right) + \frac{1}{C_1} \int i_3\, dt = 0.$$

3. (a) What are initial conditions? Why do you need them?

Solution: *Initial Conditions:*

Initial conditions *are those conditions that exist in the circuit immediately after switching operation.*

At $t = 0$, one-or more switches are operated which disturb the equilibrium of the circuit. We assume that the switch is operated in zero time. To distinguish time immediately before and immediately after the operation of the switch we use $t = 0^-$ and $t = 0^+$. *The initial conditions will depend on the post history of the network* before time instant $t = 0^-$. The initial conditions are given in terms of capacitor voltage and inductor current.

Necessity: After switching, $t = 0^+$, the new voltages and currents may appear in the network, as the result of initial capacitor voltages and inductor currents or because of the sources. The elevation of currents, voltages and their derivative at $t = 0^+$ constitutes the evaluation of initial conditions.

3. (b) The switch is closed at $t = 0$. Find the initial conditions at $t = 0^+$ for i_1, i_2, V_C, di_1/dt, di_2/dt. (Fig. Set 2.9)

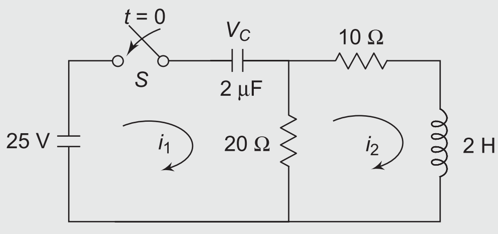


Fig. Set 2.10

Solution: Capacitor doesn't allow sudden changes in voltage.

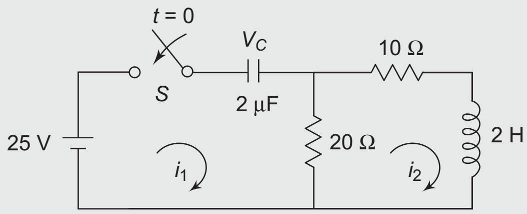


Fig. Set 2.11

Inductor doesn't allow sudden changes in current.

Since no sources are present initially (at $t = 0^-$) $VC = 0$ V and $i_2 = 0$A

$\therefore$ At $t = 0^+$, 'C' $\rightarrow$ Shoot circuit

'L' $\rightarrow$ Open circuit

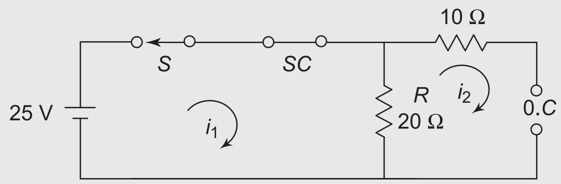


Fig. Set 2.12

At $t = 0^+$, $20i_1 = 25$ and $i_2|_{t=0^+} = 0$A

$$\Rightarrow \quad i_1\big|_{t=0^+} = 1.25 \text{ A}$$

$$V_C\big|_{t=0^+} = 0 \text{ But } V_C = \frac{1}{C}\int_0^t i_1\, dt$$

$$\Rightarrow \quad \frac{dV_C}{dt} = \frac{i_1}{C}$$

Applying KVL,

$$25 = V_C + 20i_R$$

Differentiating

$$0 = \frac{dV_C}{d\,t} + 20\,\frac{di_R}{d\,t}$$

$$\Rightarrow \quad 20\,\frac{di_R}{d\,t} + \frac{i_1}{C} = 0$$

$$\text{At } t = 0^+,\ 20\,\frac{di_R}{d\,t}\bigg|_{t=0^+} + \frac{i_1}{C}\bigg|_{t=0^+} = 0 \quad \Rightarrow \frac{di_R}{d\,t} = -\frac{1.25}{2\mu \times 20}$$

$$\therefore \quad \left.\frac{di_R}{dt}\right|_{t=0^+} = -31{,}250 \text{ A/S}$$

Also, $10i_2 + 2\dfrac{di_2}{dt} = 25 \qquad \Rightarrow 10i_2|_{t=0^+} + 2\left.\dfrac{di_2}{dt}\right|_{t=0^+} = 23$

$$\Rightarrow \quad \left.\frac{di_2}{dt}\right|_{t=0^+} = 12.5 \text{ A/S} \qquad (\text{Since } i_2|_{t=0^+} = 0)$$

$$\therefore \quad \left.\frac{di_1}{dt}\right|_{t=0^+} = \left.\frac{di_R}{dt}\right|_{t=0^+} + \frac{di_2(t)}{dt} = -31237.5 \text{ A/S}$$

3. (c) A current of 5A flows through a non-inductive resistance in series with a chocking coil when supplied at 250v, 50 Hz. If the voltage across the non inductive resistance is 125 V and that across that coil 200 V, calculate the Impedance, Reactance and Resistance of the coil, power absorbed by the coil and the total power draw the phasor diagram.

Solution:

Given $|V_R| = 125$ V

$|V_L| = 200$ V

$|I| = 5$ A

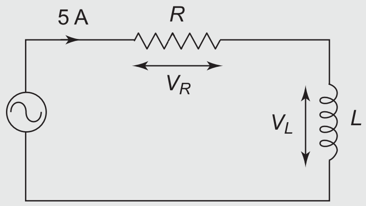


Fig. Set 2.13

$|V_R| = |I|R = 125$ V $\quad \Rightarrow \quad R = \dfrac{125}{5} = 25\ \Omega \quad (\because I = 5 \text{ A})$

$|V_L| = |I|X_L = |I|(j\omega L) \quad \therefore \quad |V_L| = 200$ V

$\Rightarrow \quad |X_L| = 40 \quad \Rightarrow \quad 5(2\pi \times 50)L = 200$

$$\Rightarrow \quad L = \frac{200}{500\pi} = 127.3 \text{ mH}$$

$$Z = 25 + j40 = 47.16\ \underline{|57.99^\circ}$$

Power absorbed by coil $= \dfrac{1}{2}LI^2$

$= \dfrac{1}{2} \times 0.1273 \times 25$

$= 1.59$ watts

True power $P_{av} = VI\cos\theta = 250 \times 5 \times \cos 57.99^\circ$

$= 662.58$ watts

Phasor diagram:

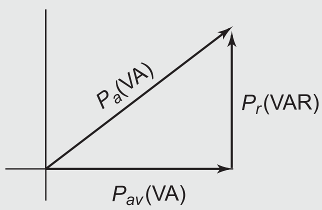


Fig. Set 2.14

$$\text{Reactive power, } P_r = I^2 X_L = 25 \times 40 = 1000 \text{ VAR}$$

$$\text{Apparent power, } P_a = I^2 Z = 25 \times 47.16 = 1179 \text{ VA}$$

4. (a) Determine $V_C(t)$ and $i_L(t)$ in the circuit shown in the Fig. Set 2.15. Assume Zero initial conditions. Use Laplace Transform method.

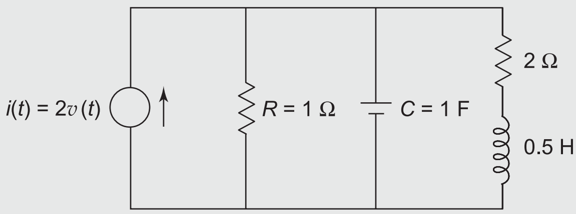


Fig. Set 2.15

Solution: (a) Applying Nodal analysis,

$$2u(t) = \frac{V_C}{R} + C\frac{dV_C}{dt} + i_L$$

$$\text{But } i_L = \frac{V_C}{2 + j0.5}$$

Applying Laplace Transform on both sides,

$$\frac{2}{5} = \frac{V_C(S)}{R} + \frac{1}{SC}(SV_C(S) - V_C(0)) + \frac{V_C(S)}{2 + 0.5S}$$

Assuming zero initial conditions, $V_C(0) = 0$

$$\Rightarrow \quad V_C(S)\left(1 + \frac{S}{S} + \frac{1}{2 + S(0.5)}\right) = \frac{2}{5}.$$

$$\Rightarrow \quad V_C(S)\left(2 + \frac{1}{2 + 5/2}\right) = \frac{2}{5}$$

$$\Rightarrow \quad V_C(S)\left(1 + \frac{1}{4 + 5}\right) = \frac{1}{5} \quad \Rightarrow \quad V_C(S) = \frac{S + 4}{S(S + 5)}$$

$$\therefore \quad i_L(S) = \frac{V_C(S)}{2 + 0.5S} = \frac{2}{S(S + 5)}$$

Applying inverse Laplace Transform for

$$V_C(S) = \frac{S + 4}{S(S + 5)} = \frac{4/5}{5} + \frac{1/5}{S + 5}$$

$$\therefore \quad V_C(t) = \frac{4}{5}u(t) + \frac{1}{5}e^{-5t}u(t)$$

$$\Rightarrow \qquad V_C(t) = \frac{1}{5}\ (4 + e^{-5t})\ u(t)$$

Similarly $$i_e(S) = \frac{2}{S\,(S+5)} - \frac{2/5}{5} - \frac{2/5}{S+5}$$

Applying inverse Laplace Transform

$$\boxed{i_L(t) = \frac{2}{5}\ u(t) - \frac{2}{5}\ e^{-5t}\ u(t)}$$

4. (b) Obtain the S-Domain Equivalent for the following elements
   i. Resistance R ii. Inductance with initial current-I_0
   iii. Capacitors
   iv. Capacitors with initial Voltage V_0 give the relevant equations.

Solution: *S-Domain Equivalent for the elements*

(i) Resistor, R

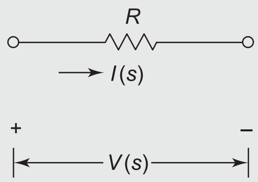


Fig. Set 2.16

$$V(S) = I(S)R$$

The ratio of $V(S)$ to $I(S)$ is called transform impedance, $Z(S)$.

$$Z(S) = \frac{V(S)}{I(S)} = R$$

$\therefore$ S-Domain equivalent is also = 'R'.

(ii)

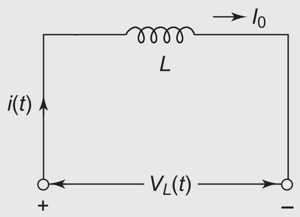


Fig. Set 2.17

$$V_i(t) = L\frac{di}{dt} \tag{1}$$

$$i(t) = \frac{1}{L}\int V_L(t)\ dt \tag{2}$$

Applying Laplace Transform to (1) and (2)

$$V_L(S) = L[SI(S) - I(0^-)] \qquad (1')$$

$$I(S) = \frac{1}{SL}\, V(S) + \frac{i(0^+)}{S} \qquad (2')$$

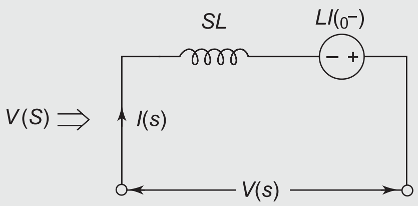


Fig. Set 2.18

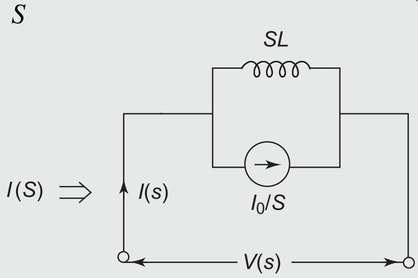


Fig. Set 2.19

(iii)

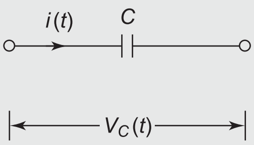


Fig. Set 2.20

$$V_C(t) = \frac{1}{C}\int i(t)\, dt \qquad (1)$$

$$i(t) = C\,\frac{dV_C}{dt} \qquad (2)$$

Apply Laplace Transform to (1)

$$V_C(S) = \frac{1}{SC}\, I(S) + \frac{V(0)}{S} \qquad (V(0) = 0)$$

Assuming initial conditions = 0

$$V_C(S) = \frac{1}{SC}\, I(S)$$

$$\therefore \quad Z(S) = \frac{1}{SC}$$

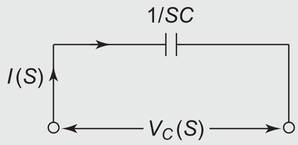


Fig. Set 2.21

S-domain equivalent of capacitor with no initial voltage.

(iv) Capacitor with initial voltage from (1) and (2) of the above problem.

$$V_C(S) = \frac{1}{C_S}\, I(S) + \frac{V(0)}{S} \qquad (1')$$

$$I(S) = C[SV_C(S) - V(0^-)] \qquad (2')$$

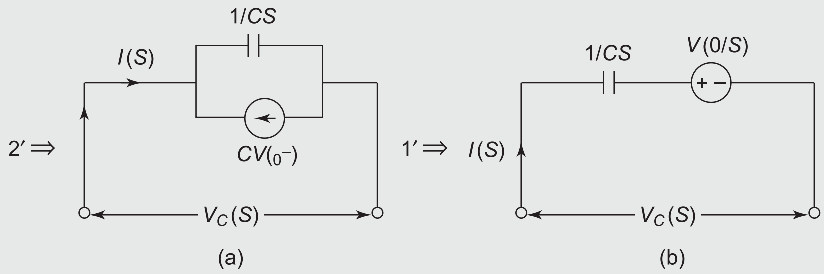


Fig. Set 2.22

5. (a) Verify Tellegen's theorem in the network shown in the Fig. Set 2.23.

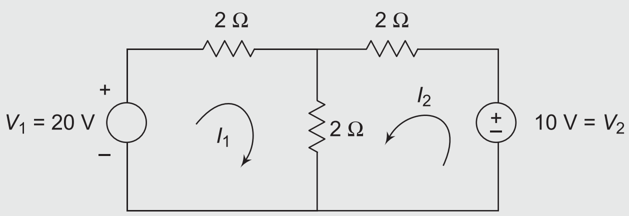


Fig. Set 2.23

Solution: Tellegens theorem states that in any arbitrary lumped network, the algebraic sum of the powers in all the branches at any instant is zero and all the branch currents and voltages must satisfy Kirchoff's law.

Verifying Tellegens theorem for the above ckt.

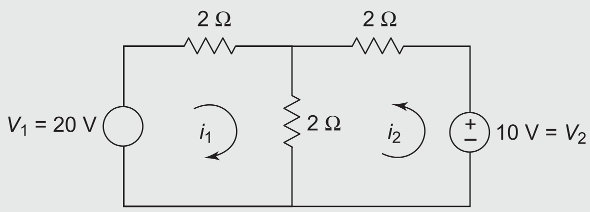


Fig. Set 2.24

There are 5 elements in the above circuit.
Applying mesh equations.

$$4i_1 + 2i_2 = 20$$

$$\Rightarrow 2i_1 + i_2 = 10 \quad (1)$$

$$2i_1 + 4i_2 = 10$$

$$i_1 + 2i_2 = 5 \quad (2)$$

Solving (1) and (2)

$$i_1 = 5, \qquad i_2 = 0$$

$$\sum_{k=1}^{5} V_k I_k \text{ for this circuit is}$$

$$-100 + 50 + 50 + (0)^2 (2) - (0)(10) = 0$$

Hence, verified.

5. (b) Verify reciprocity theorem for the network shown in Fig. Set 2.25.

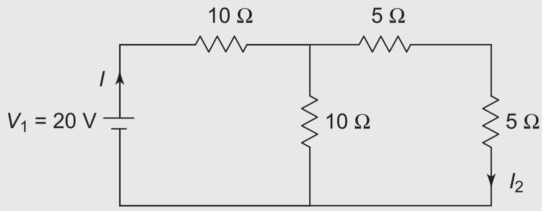


Fig. Set 2.25

Solution: Reciprocity theorem states that in any passive linear bilateral single source network interchanging the positions of ideal voltage source and an ideal ammeter does not after the ammeter reading (current) and interchanging then positions of current source and ideal voltmeter does not after voltmeter reading (voltage).

Verifying theorem for the above ckt

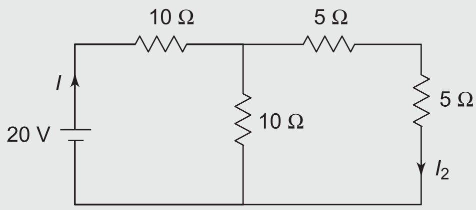


Fig. Set 2.26

$$I = \frac{20}{10+5} = \frac{4}{3} \qquad \therefore \quad I_2 = \frac{2}{3}$$

(Current divider rule)

Interchanging the voltage source.

$$I = \frac{20}{15} = \frac{4}{3}$$

$$\Rightarrow \qquad I_1 = \frac{2}{3}$$

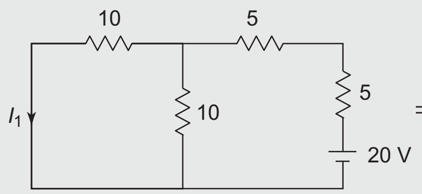


Fig. Set 2.27

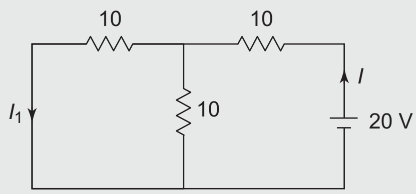


Fig. Set 2.28

$\therefore$ The ratio of excitation to response when only one excitation is applied is constant when positions of excitation and response are interchanged. Hence reciprocity theorem is verified.

6. (a) A typical two-port network is characterised by the equation $2V_1 + 4I_2 = I_1$ and $V_2 + 6V_1 = 8I_2$. Determine the values of

i. y_{11} ii. z_{21} and iii. h_{21}

Solution: A typical two-port network is characterised by the equation

$$2V_1 + 4I_2 = I_1 \quad (1) \text{ and}$$

$$V_2 + 6V_1 = 8I_2 \quad (2)$$

$$6V_1 + 12I_2 = 3I_1 \quad (1) \times 3$$

$$6V_1 + V_2 = 8I_2 \quad (2)$$

$$12I_2 - V_2 = 3I_1 - 8I_2$$

$$3I_1 - 20I_2 = -V_2 \quad (3)$$

$$V_2 = -3I_1 + 20I_2$$

$$V_1 = \frac{I_1}{2} - 2I_2 \quad \text{(from (2) and (3))}$$

$$\therefore \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -2 \\ -3 & 20 \end{pmatrix}$$

(i) $Y_{11} = \dfrac{Z_{22}}{\Delta Z}$ $(\Delta Z = 10 - 6 = 4) = \dfrac{-20}{4} = 5$

(ii) $Z_{21} = -3$

(iii) $h_{21} = \dfrac{-Z_{21}}{Z_{22}} = \dfrac{3}{20}$.

6. (b) Obtain the input and output impedances of an amplifier having $h_{11} = 2\Omega$; $h_{12} = 1\Omega$; $h_{21} = 5$ and $h_{22} = 2\Omega$, if it is driven by a source having an internal resistance of 4Ω and is terminated through a load which draws maximum power from the amplifier.

Solution:

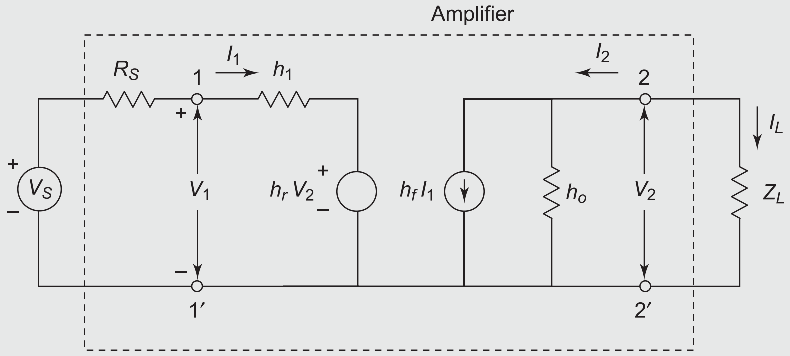


Fig. Set 2.29

Given $h_{11} = 2\ \Omega$, $h_{12} = 1\Omega$, $h_{21} = 5\ \Omega$, $h_{22} = 2\ \Omega$

(h_i) (h_r) (h_f) (h_o)

$R_S = 4\ \Omega$

Load draws maximum power when

$Z_L = Z_{TH}$. (More power Transform Theorem)

To find Z_{TH} (Remove Z_L) Output admittance

$$Z_L = \frac{V_2}{-I_2} = \frac{-1}{Y_o} \quad \Bigg| \quad Y_o = \frac{I_2}{V_2}.\ \text{with}\ V_S = and\ R_L = \infty$$

$$I_2 = h_f I_1 + h_0 V_2 = t$$

$$\Rightarrow \qquad y_0 = h_f \frac{I_1}{V_2} + h_0 \tag{1}$$

From Fig. $R_S I_1 + h_i I_1 + hr V_2 = 0$ (or) $\dfrac{I_1}{V_2} = -\dfrac{h_r}{h_i + R_S}$ (2)

$$(2) \text{ in } (1) \Rightarrow Y_0 = h_0 \frac{h_f h_r}{h_i + R_S}$$

$$\therefore \qquad Y_0 = 2 - \frac{5}{2+4} = 2 - \frac{5}{6} = \frac{7}{6} = 1.167\ \Omega$$

$$Z_{TH} = 0.857\ \Omega$$

$$Z_L Y_L = Z_{TH} = 0.857\ \Omega$$

$$Z_i = \frac{V_1}{I_1} \quad V_1 = h_i I_1 + h_r V_2$$

Hence

$$Z_i = \frac{hi\, I_1 + h_r V_2}{I_1} = h_i + h_r\, \frac{V_2}{I_1}$$

Substituting,

$$V_2 = -\, I_2\, Z_L = A_I\, I_1\, Z_L$$

$$A_I = \frac{I_2}{I_1} = \frac{-h_f}{1 + h_0 Z_L} \qquad \left(\therefore I_2 = h_f\, I_1 + h_0\, V_2 \text{ and } A_I - \frac{I_L}{I_1} \right)$$

$$Z_i = h_i + h_r\, A_I\, Z_L = h_i - \frac{h_f\, h_r}{Y_L + h_0}$$

$$= 2 - \frac{5}{1.167 + 2} = 0.421\ \Omega$$

$$\therefore \quad \text{O/P impedance} = \frac{1}{Y_0} \Bigg\| \ Z_L = \frac{0.857}{2} = 0.418\ \Omega$$

I/P impedance = 0.421 Ω

7\. (a) Draw the circuit of an asymmetrical L-attenuator working between tow equal impedances with a given loss. Derive the design equations for the circuit elements in terms of

i. the iterative resistance R_i, and

ii. the current ratio N.

Solution: *L-Type Attenuator:*

An L-type asymmetrical attenuator is connected between a source with source resistance $R_S = R_i$ and load resistance $R_L = R_i$ (P-707 in Network Theory) filters and Attenuators.

Simply replace $R_0 \rightarrow$ by 'R_i' (iterative resistance).

7\. (b) Design an asymmetrical L-attenuator to operate into a resistance of 300 Ω and to provide attenuation of 30 DB.

Solution: $N = \text{Antilog}\ \dfrac{dB}{20} = \text{Antilog}\ \dfrac{30}{20} = 31.62$

The series erm of the attenuator

$$R_1 = R_{\text{i}} \left(\frac{N - 1}{N} \right) = 300 \left(\frac{31.62 - 1}{31.62} \right) = 290.51\ \Omega$$

The shunt erm of the attenuator

$$R_2 = \frac{R_i}{N - 1} = \frac{300}{31.62 - 1} = 9.79\ \Omega$$

The desired configuration of L-attenuator is

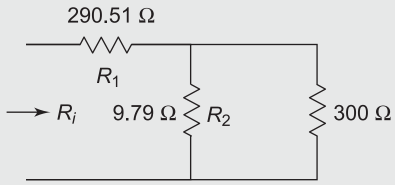


Fig. Set 2.30

8. (a) Explain the variation of Attenuation, phase shift and characteristic impedance of m derived high pass filter?

Solution: (a) In P-689 in Network Theory (Filters and Attenuators).

8. (b) Draw the circuit diagram for T and Π section of m-derived high pass filter.

Solution: With both T and Π sections shown in P-688 in Network Theory.

Network Analysis, May/June 2006

SET 3

1. (a) For the given network (Fig. Set 3.1) graph, Construct the Basic cutest incidence matrix, tracking elements 1,6,8,3 as tree branches. Express the link branch Voltage in terms of tree branch voltages.

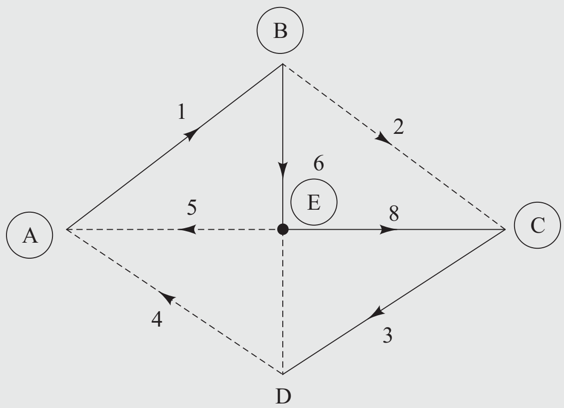


Fig. Set 3.1

Solution:

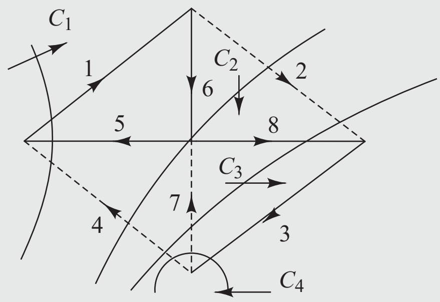


Fig. Set 3.2

Cut set incidence matrix is

$$
Q = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 0
\end{bmatrix}
\end{array}
$$

The link branch voltage in terms of tree branch voltages is given by

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

1. (b) Using source Transformation, reduce the network between *A* and *B* into an equivalent voltage source. (Fig. Set 3.3)

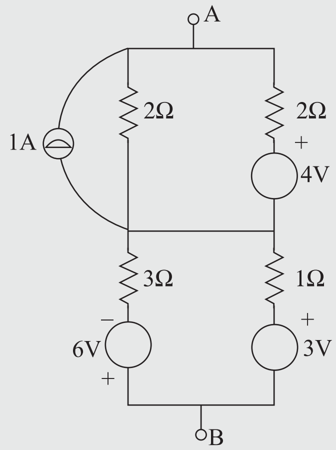


Fig. Set 3.3

Solution: Given circuit

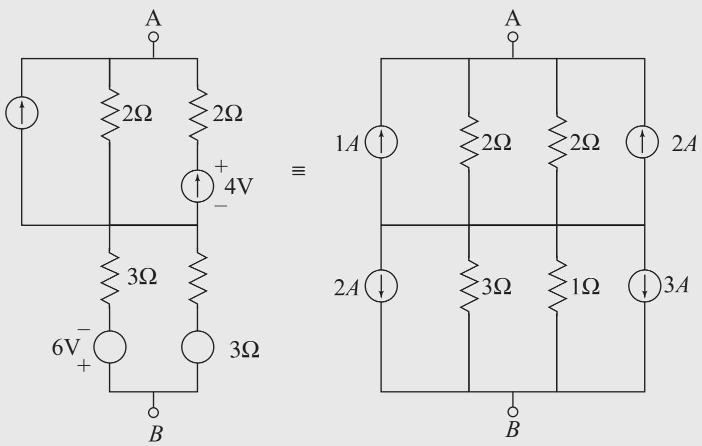


Fig. Set 3.4

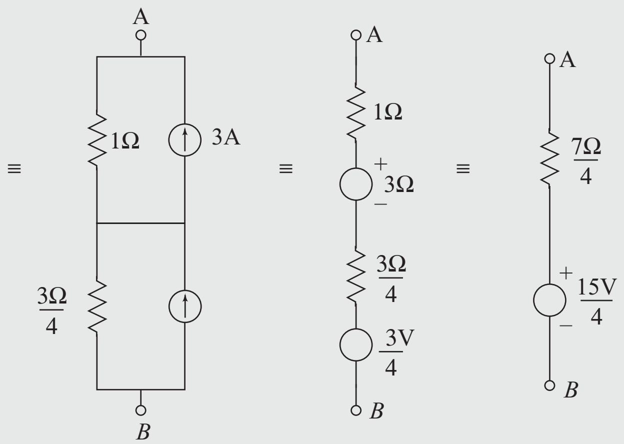


Fig. Set 3.5

1. (c) What is Duality? Explain the procedure for drawing the dual of given network with an example.

Solution: *Refer Duals and Duality; Section 3.16*

2. (a) Explain the Dot Convention for mutually coupled coils.

Solution *Refer Dot Convention; Section 9.14*

2. (b) Derive the Expression for coefficient coupling between pair of magnetically coupled coils.

Solution: *Refer coefficient of coupling; Section 9.5*

2. (c) Write the Loop Equations for the Coupled circuit shown in Fig. Set 3.6.

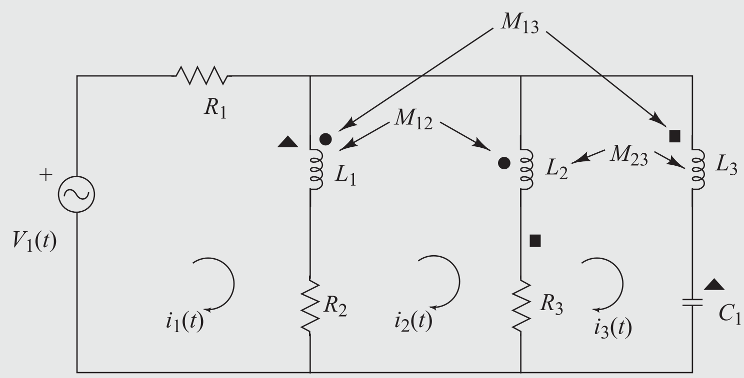


Fig. Set 3.6

Solution Given circuit is

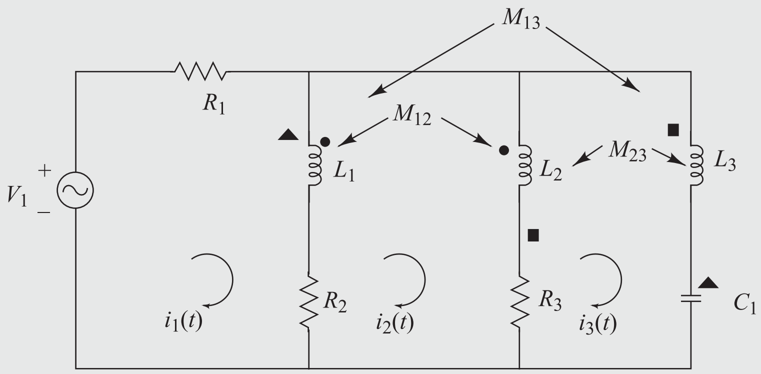


Fig. Set 3.7

The loop equations are

$$V_1(t) = R_1\, i_1(t) + L_1 \frac{d}{dt}[i_1(t) - i_2(t] - M_{12}\frac{d}{dt}[i_2(t) = i_3(t)]$$

$$- M_{13}\frac{d}{dt}[i_3(t)] + R_2[i_1(t) - i_2(t)] \qquad (1)$$

Loop 2

$$R_2\,[i_2(t) - i_1(t)] + L_1\left[\frac{di_2(t)}{dt} - \frac{di_i(t)}{dt}\right] - M_{12}\frac{d}{dt}[i_2(t) - i_3(t)]$$

$$+ M_{13}\frac{di_3(t)}{dt} + L_2\frac{d[i_2(t) - i_3(t)]}{dt} - M_{12}\left[\frac{di_2(t)}{dt} - \frac{di_i(t)}{dt}\right]$$

$$- M_{23}\frac{di_3(t)}{dt} + R_3(i_2 - i_3) = 0$$

Loop 3

$$R_3(i_3 - i_2) + L_2\frac{d(i_3 - i_2)}{dt} - M_{12}\frac{d(i_1 - i_2)}{dt} + M_{23}\frac{di_3}{dt}$$

$$+ L_3\frac{di_3}{dt} - M_{13}\frac{d}{dt} + M_{23}\frac{d(i_3 - i_2)}{dt} + \frac{1}{C_1}\int i_3\, dt = 0$$

3. (a) Explain clearly the significance of "Time Constant" in transient analysis of *R-L* and *R-C* Circuits.

Solution: Refer 11.2, 11.3

3. (b) In the following circuit (Fig. Set 3.8), when 220V A.C. is applied across A and B, Current drawn is 20 Amps and power input is 3000w. Find the value of Z and its parameters.

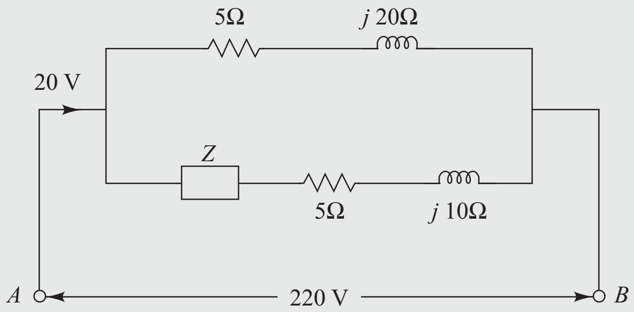


Fig. Set 3.8

Solution:

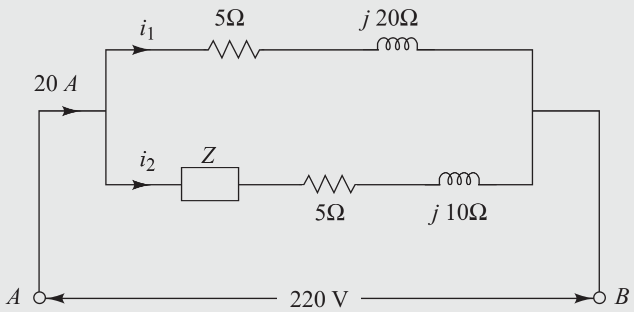


Fig. Set 3.9

$$i_1 = \frac{220}{5 + j20} \text{ A}$$

But $i_1 + i_2 = 20$ A

$$i_2 = 20 - \frac{220}{5 + j20} \tag{1}$$

$$\text{Also, } i_2 = \frac{220}{Z + 5 + j10} \tag{2}$$

From (1) and (2)

$$20 - \frac{220}{5 + j20} = \frac{220}{Z + 5 + j20}$$

$$\frac{-120 + j400}{5 + j20} = \frac{220}{5 + Z + j20}$$

$$Z = \frac{5700 + j3600}{-120 + j400}$$

$$Z = -4.33 + j15.55$$

$$Z = 16.14 \angle 105.56°$$

3. (c) Obtain the expression for resonant frequency for the circuit shown in Fig. Set 3.10.

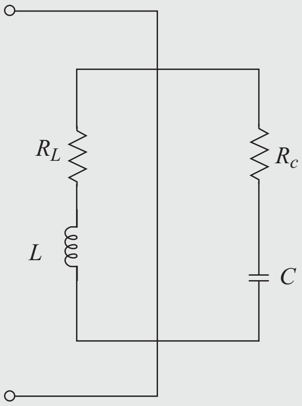


Fig. Set 3.10

Solution: *Refer Parallel Resonance* 8.10 *in Page* 8.34.

4. (a) Determine $V_C(t)$ and $i_L(t)$ in the circuit shown in the Fig. Set 3.6. Assume Zero initial conditions. Use Laplace Transform method.

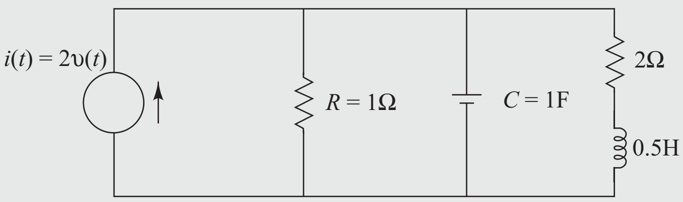


Fig. Set 3.11

Solution: Applying nodal analysis,

$$2u(t) = \frac{V_C}{R} + C\frac{dV_C}{dt} + i_L$$

$$\text{But, } i_L = \frac{V_C}{2 + j0.5}$$

Applying Laplace transform on both sides,

$$\frac{2}{S} = \frac{V_C(S)}{R} + \frac{1}{SC}\,(SV_C(S) - V_C(0)) + \frac{V_C(S)}{2+0.5S}$$

Assuming zero initial conditions, $V_C(0) = 0$

$$\Rightarrow \qquad V_C(S)\left[1 + \frac{S}{S} + \frac{1}{2+(0.5)S}\right] = \frac{2}{S}$$

$$\Rightarrow \qquad V_C(S)\left(2 + \frac{1}{2+\frac{S}{2}}\right) = \frac{2}{S}$$

$$\Rightarrow \qquad V_C(S) = \frac{S+4}{S(S+5)}$$

$$\therefore \qquad i_L(S) = \frac{V_C(S)}{2+(0.5)S} = \frac{2}{S(S+5)}$$

Applying inverse Laplace transform for $V_C(S)$;

$$V_C(S) = \frac{S+4}{S(S+5)} - \frac{(4/5)}{S} + \frac{(1/5)}{S+5}$$

$$\therefore \qquad V_C(t) = \frac{4}{5}\,u(t) + \frac{1}{5}\,e^{-5t}\,u(t)$$

$$\Rightarrow \qquad V_C(t) = \frac{1}{5}\,u(t)\,[4 + e^{-5t}]$$

Similarly, $\quad i_L(S) = \dfrac{2}{S(S+5)} = \dfrac{(2/5)}{S} - \dfrac{(2/5)}{(S+5)}$

Applying inverse Laplace transform for $i_L(S)$;

$$i_L(t) = \frac{2}{5}\,u(t)\,[1 - e^{-5t}]$$

4. (b) Obtain the S-Domain Equivalent for the following elements
   i. Resistance R
   ii. Inductance with initial current-I_0
   iii. Capacitors
   iv. Capacitors with initial Voltage V_0 give the relevant equations.

Solution *Refer circuit element in S-Domain* 13.1

5. (a) State and explain Nortion's theorem?

Solution *Refer Norton's theorem* 3.4

5. (b) Using Thevenin's theorem, find the current through 1 Ω resistor in the circuit shown in Fig. Set 3.12.

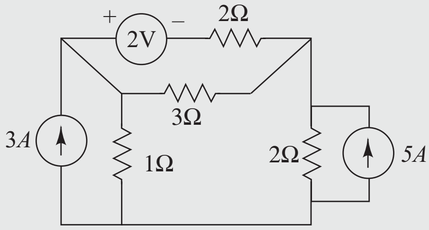


Fig. Set 3.12

Solution The given circuit is

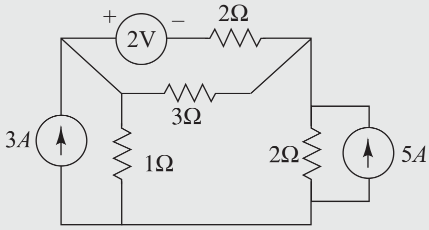


Fig. Set 3.13

To find R_{TH}

BY keeping all the series to zero the circuit reduces to

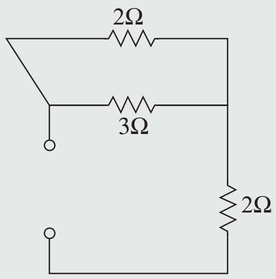


Fig. Set 3.14

$$R_{TH} = 2 \| 3 + 2$$

$$R_{TH} = \frac{6}{5} + 2$$

$$R_{TH} = \frac{16}{5}$$

To find V_{TH}

Transforming current source of 5A to voltage source the circuit reduces to

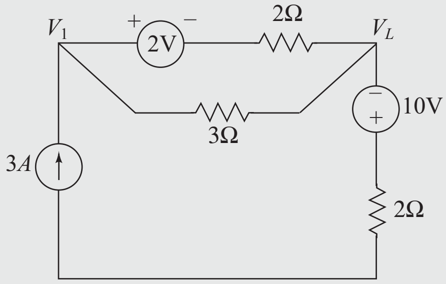


Fig. Set 3.15

Applying Nodal analysis

$$\frac{V_1 - V_2 - 2}{2} + \frac{V_1 - V_2}{3} = 3$$

$$V_1 - V_2 = \frac{24}{5} \tag{1}$$

$$\frac{V_2 - V_1 + 2}{2} + \frac{V_2 - V_1}{3} + \frac{V_2 + 10}{2} = 0$$

$$(V_2 - V_1) + \frac{5}{6} + \frac{V_2}{2} + 6 = 0 \tag{2}$$

From (1) and (2)

$$V_1 = -\frac{76}{5}\ \text{V}$$

$$V_2 = -20\ \text{V}$$

6. (a) Why h-parameters are called as hybrid parameters?

Solution: *Refer Hybrid Parameters* 15.6

6. (b) Obtain the condition for a given network to be reciprocal as well as symmetrical network in terms of h-parameters?

Solution: *Refer* 15.20

6. (c) Obtain the z-parameters of the network shown in Fig. Set 3.16.

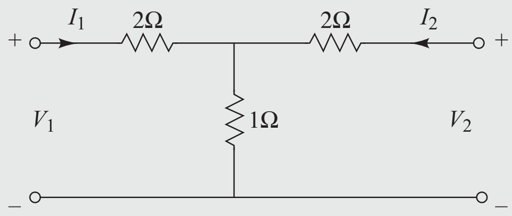


Fig. Set 3.16

Solution:

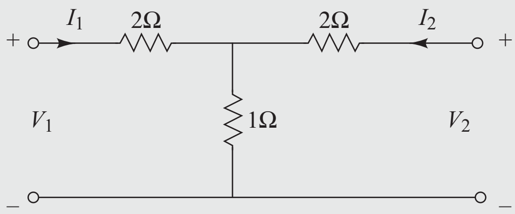


Fig. Set 3.17

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = 3\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0} = 1\ \Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} = 1\ \Omega$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = 3\ \Omega$$

7. (a) Fig. Set 3.18 shows a resistive T network and a resistive Π network connected in parallel. Find the overall y parameters of the combination.

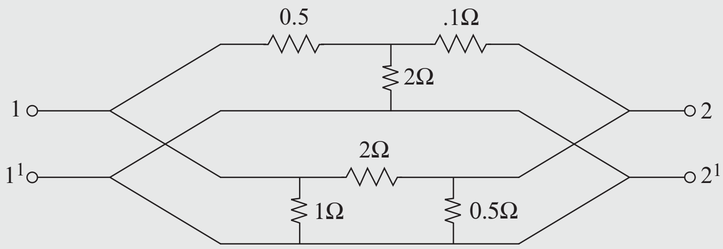


Fig. Set 3.18

Solution:

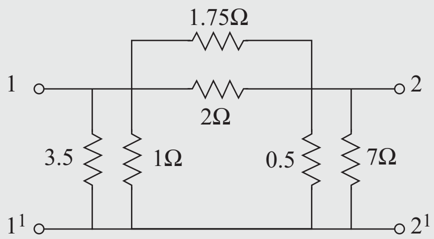


Fig. Set 3.19

The upper star connection is connected into π and the circuit is redrawn as follows

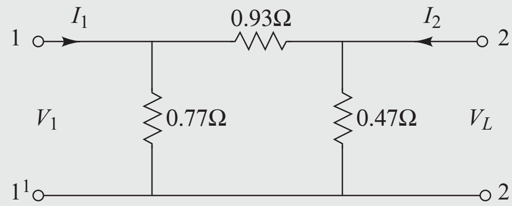


Fig. Set 3.20

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0} = 1.7\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0} = -\,0.93\ \mho$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} = 1.4\ \mho$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0} = -\,0.93\ \mho$$

(b) Find the characteristic impedance of a symmetrical *T* network.

Solution: Refer Section 15.10

8. What is composite filter? Draw its circuit diagram? Give a general procedure for its design?

Solution In the *m*-derived filter sections, the stop band attenuation drastically reduces after f_∞ in low pass section and before f_∞ in high pass section. This drawback of *m*-derived filter can be overcome by connecting number of sections including prototype sections and *m*-derived sections with terminating half sections. Such a combination of different sections is called COMPOSITE FILTER.

The block diagram of the composite filter is shown in Fig. Set 3.21.

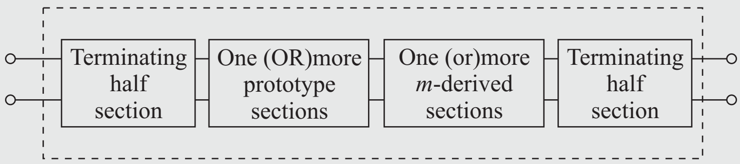


Fig. Set 3.21

In composite filter, cut off frequency and the design impedance are the two important design specifications. The number of various section in the composite filter totally depends on attenuation characteristics required. The typical value of *m* for attenuation at cut off is $m = 0.3$ to 0.35. If it is required to maintain the attenuation at a high value in attenuation band, we must connect either a prototype section in another *m*-derived section with comparatively larger value of *m*. To have proper impedance matching, and constant characteristic impedance throughout pass band, we must connect the terminating sections with $m = 0.6$.

Network Analysis, May/June 2006

SET 4

1. (a) The following current wave form $i(t)$ is passed through a series R-L circuit with $R = 2\ \Omega$ and $L = 2$ mH. Find the Voltage across each element and sketch the same. (Fig. Set 4.1)

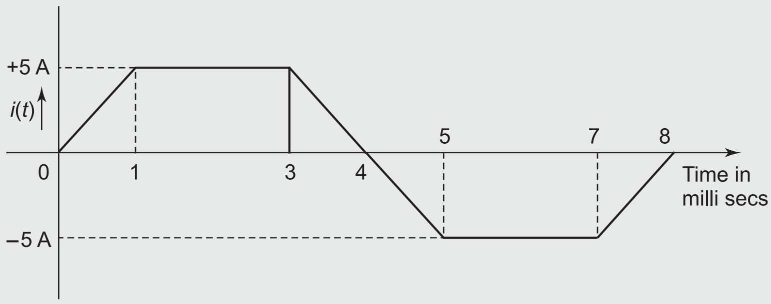


Fig. Set 4.1

Solution:

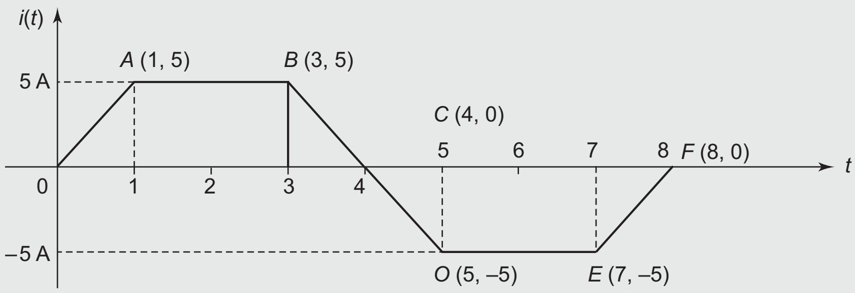


Fig. Set 4.2

For line OA, Slope $= \dfrac{5}{1} = 5$

line equation $i(t) - 0 = 5(t - 0)$

$\Rightarrow \quad i(t) = 5t \qquad [\because\ y = mx]$

For line AB, $\quad i(t) = 5$ (constant)

For line BD, $\quad i(t) - 0 = \dfrac{10}{-2}\,(t - 4)$

(From line equation: $y - y_t = m(x - x_1)$)

$\Rightarrow \quad i(t) = -\,5t + 20$

For line DE, $\quad i(t) = -\,5$ (constant)

For line EF, $i(t) = 5(t - 8)$

$\Rightarrow$ $i(t) = 5t - 40$

Voltage induced in the inductor

Along OA

$$V_{OA} = \frac{Ldi}{dt} = 2 \times 10^{-3} \times \frac{d}{dt}(5t) = 10\ \mu\text{V}$$

($\because$ 't' in m sec)

Along AB,

$$V_{AB} = \frac{Ldi}{dt} = 0 \qquad (\because\ i(t) = \text{cons t} = 5\text{A})$$

Along BD

$$V_{BD} = \frac{Ldi}{dt} = 2 \times 10^{-3} \times \frac{d}{dt}(-5t + 20) = -10\ \mu V$$

Along DE

$$V_{DE} = \frac{Ldi}{dt} = 0 \qquad (\because\ i(t) = \text{const} = -5\text{A})$$

Along EF

$$V_{EF} = \frac{Ldi}{dt} = 2 \times 10^{-3} \times \frac{d}{dt}(5t - 40) = 10\ \mu V$$

Waveform:

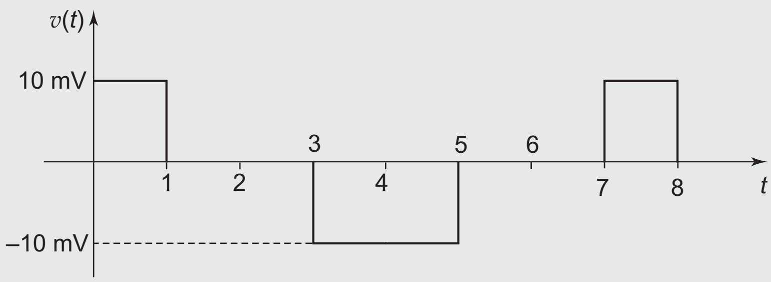


Fig. Set 4.3

Voltage across the resistor is same as current through the circuit multiplied by the resistance

$$N = IR = 2I$$

Waveform

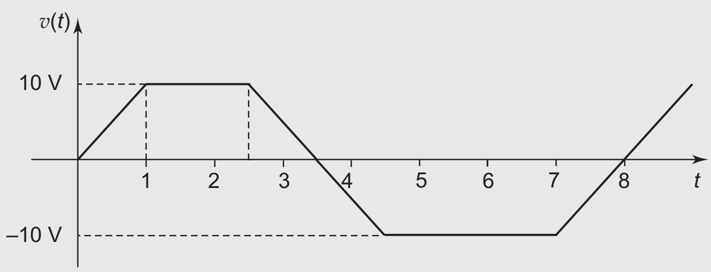


Fig. Set 4.4

1. (b) Using nodal analysis, determine the Power supplied by 8V Voltage source. (Fig. Set 4.5)

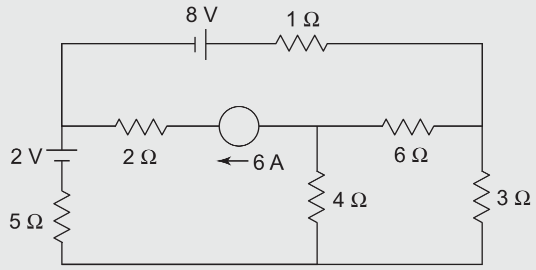


Fig. Set 4.5

Solution:

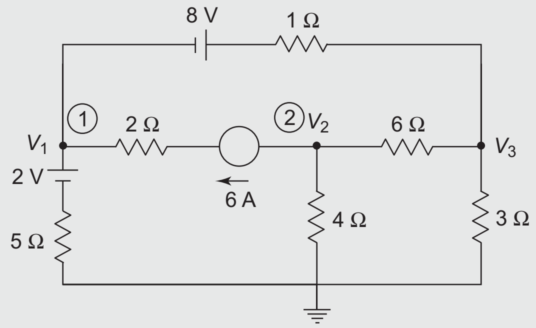


Fig. Set 4.6

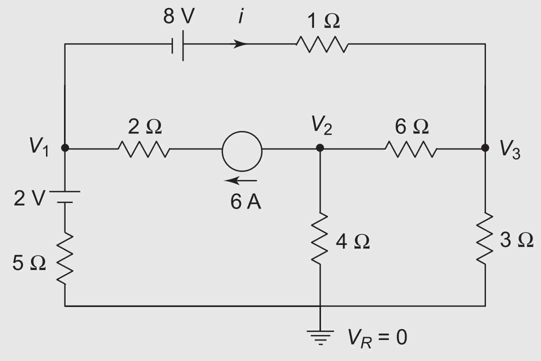


Fig. Set 4.7

Applying KCL at node (1);

$$\frac{V_1 - 2}{5} + \frac{V_1 - V_3 + 8}{1} = 6 \quad \Rightarrow \quad 5V_3 - 6V_1 = 8 \qquad (1)$$

Applying KCL at node (2);

$$6 + \frac{V_2}{4} + \frac{V_2 - V_3}{6} = 0 \quad \Rightarrow \quad 5V_2 - 2V_3 + 72 = 0 \qquad (2)$$

Applying KCL at node (3);

$$\frac{V_3 - V_2}{6} + \frac{V_3}{3} + \frac{V_3 - V_1 - 8}{1} = 0 \quad \Rightarrow \quad 9V_3 - V_2 - 6V_1 = 48$$

(3)

Solving (1), (2) and (3), we get

$$V_1 = -\ 4.593 \text{ volts}$$
$$V_2 = 11.56 \text{ volts}$$
$$V_3 = -\ 7.11 \text{ volts}$$

$$\text{From the circuit, } i = \frac{V_1 + 8 - V_3}{1} = 10.517 \text{ A}$$

Power supplied by 8 V source is (8 × 10.517)

$$= 84.136 \text{ Watts}$$

1. (c) Write the Tieset matrix for the graph shown in Fig. Set 4.8, taking the tree consisting of branches 2,3,4.

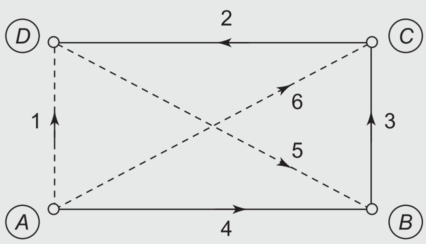


Fig. Set 4.8

Solution

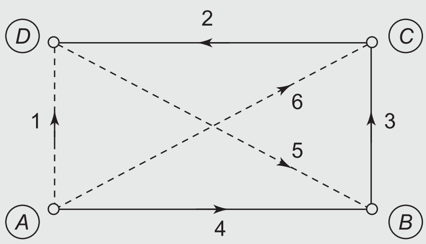


Fig. Set 4.9

Basictiesets \ e	1	2	3	4	5	6
(5, 3, 2)	0	–1	–1	0	1	0
(6, 3, 4)	0	0	–1	–1	0	1
(1, 2, 3, 4)	1	–1	–1	–1	0	0

2. (a) Obtain the Equivalent '*T*' for magnetically Coupled circuit shown in Fig. Set 4.10.

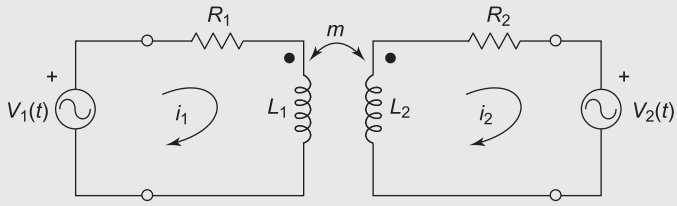


Fig. Set 4.10

Solution

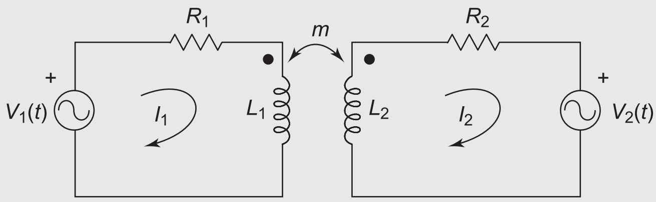


Fig. Set 4.11

$$V_1(t) = I_1F_2 + L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

$$V_2(t) = I_2F_2 + L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}$$

The equivalent '*T*' for magnetically coupled circuit is

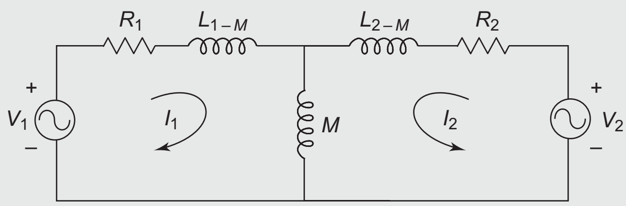


Fig. Set 4.12

2. (b) A coil of 500 turns is wound uniformly over a wooden ring having a mean circumference of 50 cms and a cross sectional area of 500 mm^2. If the current through the coil is 3 Amps, Calculate
   (i) The magnetic field strength
   (ii) The flux density and
   (iii) The total flux.

Solution Given $N = 500$, $I = 3\text{A}$

$$A = 500 \times 10^{-6} \text{ m}^2$$

Mean circumference (Magnetic path)

$$l = 50 \times 10^{-2} \text{ m}$$

(i) $H = \dfrac{mmt}{l}$

But mmf = NI = 1500 AT

and $l = 50 \times 10^{-2}$

Magnetic Field Strength, H = 3000 AT/m

(ii) $B = \mu_0\, m = 4\pi \times 10^{-7} \times 3000 = 3.769\ mwb/\text{m}^2$

$\therefore$ flux density (B) = 3.769 mwb/m^2

(iii) $\phi = B \times A = 3.769 \times 10^{-3} \times 500 \times 10^{-6}$

$= 1.8845 \times 10^{-6}\ wb$

$\therefore$ Total flux (f) = $1.8845 \times 10^{-6}\ wb$

2. (c) Write down the Loop Equations for the network shown in Fig. Set 4.13.

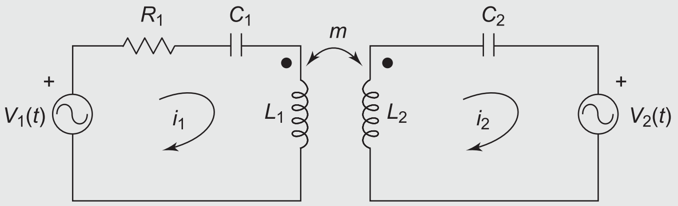


Fig. Set 4.13

Solution: As i_1 is entering at the dot terminal, and i_2 is leaving the dot terminal, sign of M (mutual inductance) is –ve

$$i_1(R_1 = j/\omega C_1 + jwL_1) - i_2\, jwM = V_1(t)$$

is loop equation for 1st mesh.

$$i_2(jwL_2 - j/wC_2) - i_1(jwM) = -V_2(t)$$

is loop equation for 2nd mesh.

3. (a) In the circuit (Fig. Set 4.14) shown, the switch is changes from position 1 to 2 at $t = 0$. Determine the initial conditions i, di/dt, d^2i/dt^2 at $t = 0^+$

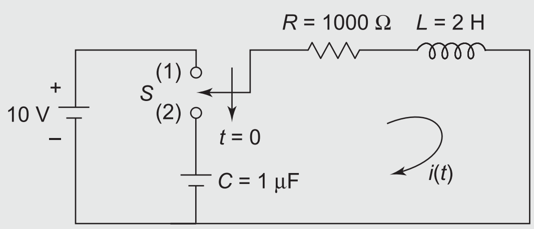


Fig. Set 4.14

Solution:

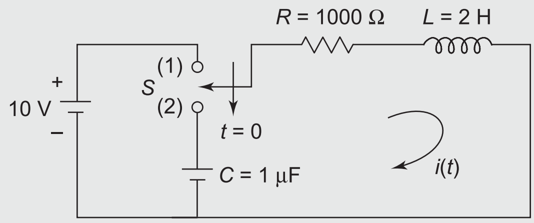


Fig. Set 4.15

Initially the voltage across 'C' = 0
Capacitor does not allow sudden charge in voltage.
Inductor does not allow sudden change in current
A position (2), using KVt

$$1000i(t) + \frac{1}{1\times10^{-6}}\int i dt + 2\frac{di}{dt} = 0 \qquad (1)$$

At $t = 0$

$$i(t)|_{t=0} = \frac{10}{1000} = 10 \text{ mA}$$

Since inductor is short circuit at steady state i.e., when switch is at position '1'
By inductor property

$$i(t)|_{t=0^-} = i(t)|_{t=0^+} = 10 \text{ mA}$$

At $t = 0^+$, $\int i(t)\, dt = 0$ (Since voltage across capacitor is zero)

$$(1) \Rightarrow 1000\, i(t)|_{t=0^+} + 2\left.\frac{di}{dt}\right|_{t=0^+} = 0 \qquad \text{(using 2)}$$

$$\left.\frac{di}{dt}\right|_{t=0^+} = -5 \text{ A/S} \qquad (3)$$

Diff (1) once,

$$1000\frac{di}{dt} + \frac{1}{1\times10^{-6}}\, i(t) + 2\frac{d^2i}{dt^2} = 0$$

$t = 0^+$

$$1000\left.\frac{di}{dt}\right|_{t=0^+} + \frac{1}{1\times10^{-6}}\, i(t)|_{t=0^+} + 2\left.\frac{d^2i}{dt^2}\right|_{t=0^+} = 0$$

$$\Rightarrow \qquad \left.\frac{d^2 i}{d t^2}\right|_{t=0^+} = \frac{1}{2}(+\,5000 - 10000) \qquad \text{(Using 3)}$$

$$\therefore \qquad \left.\frac{d^2 i}{d t^2}\right|_{t=0^+} = -\,2500 \ \text{A/S}^2$$

3. (b) In the parallel resonant circuit, determine the resonance frequency, dynamic resistance and Band width for the circuit (Fig. Set 4.16) shown.

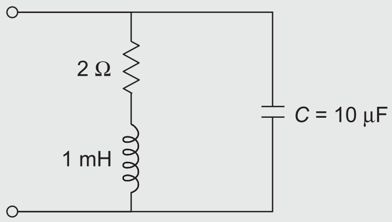


Fig. Set 4.16

Solution: Total admittance

(tan k ckt)

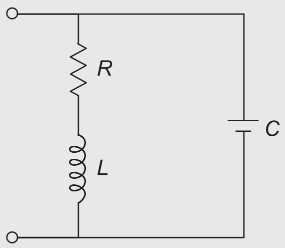


Fig. Set 4.17

$$Y = \frac{1}{R + jwL} + \frac{1}{-\,j/wc}$$

$$= \frac{R - jwL}{R^2 + w^2 L^2} + jwc$$

$$= \frac{R}{R^2 + w^2 C^2} + j\left(\omega C - \frac{wL}{R^2 + w^2 L^2}\right)$$

At resonance, the susceptance part (B) becomes zero.
Reactance

$$\underbrace{Y = G}_{\text{Conductance}} + \underbrace{jB}_{\text{Susceptance}} \qquad Z = \underbrace{R}_{\text{Resistance}} + jX$$

$$\therefore \qquad w_r\, C = \frac{w_r L}{R^2 t\, w_r^2\, L^2}$$

$$R^2 + w_r^2\, L^2 = \frac{L}{C} \qquad \Rightarrow \quad w_r^2 = \frac{1}{L^2}\left(\frac{L}{C} - R^2\right)$$

$$\Rightarrow \qquad w_r^2 = \frac{1}{LC} - \frac{R^2}{L^2} \qquad \Rightarrow \quad w_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

Here $R = 2\ \Omega$, $L = 1$ mH, $C = 10\ \mu F$

$$w_r = \sqrt{\frac{1}{10^{-8}} - \frac{4}{10^{-6}}} = \sqrt{10^6 \times 96}$$
$$= 9.79 \times 10^3 \text{ Hz}$$

$$\therefore \qquad \text{fr} = \frac{wr}{2\pi} = 1.559 \text{ kHz}$$

$$\text{Dynamic resistance } (R) = \frac{1}{6} = \frac{R^2 + w_r^2 L^2}{R}$$

$$= \left.\frac{R + w^2 L^2}{R}\right|_{w - wr} = 2 + \frac{96 \times 10^6 \times 10^{-6}}{2} = 50\ \Omega$$

$$\text{Bandwidth} = \frac{1}{RC} \quad \text{(for IId resonant ckt)}$$

$$= \frac{1}{50 \times 10 \mu f} = 2 \text{ kHz}$$

$$\text{(or)} \qquad \text{BW} = \frac{R}{L} = \frac{2}{1 mH} = 2 \text{ kHz}$$

3. (c) When an voltage of 220V A.C. supply connected across the AB terminals, the total power input is 3.25 kw and the current is 20 Amps. Find the current through Z_3. (Fig. Set 4.18)

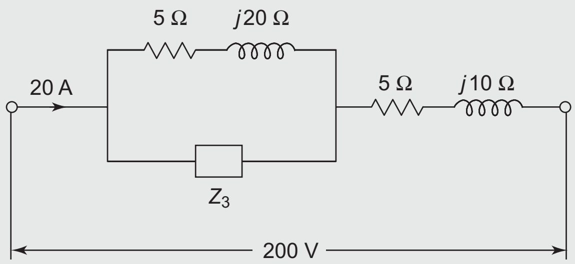


Fig. Set 4.18

Solution: Across IId branch

$$V = 20\ (5 + j10) = 223.6\ \angle 63.43° = 100 + j200$$
$$I(5 + j20) + 100 + j200 = 220.$$

(Let I be the current through $5 + j20\ \Omega$ branch)

$$I = \frac{120 - j200}{5 + j20} = -\ 8 - 8i$$

$$I_{Rs} = 20 - I = 28 + 8i = 29.12\ \angle 15.9°$$

4. (a) Find the Laplace Transform of single pulse shown in Fig. Set 4.19.

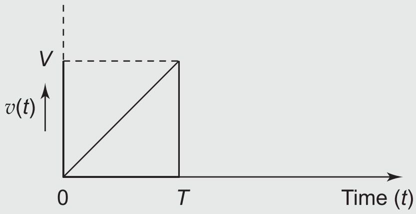


Fig. Set 4.19

Solution:

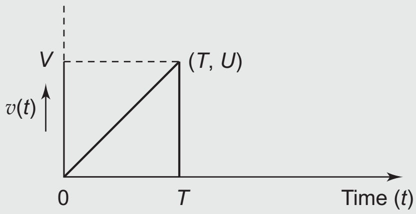


Fig. Set 4.20

$$f(t) = \begin{cases} \dfrac{V}{T}.t, 0 < t < T \\ 0, \text{elsewhere} \end{cases}$$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} \frac{V}{T} \, t\, dt$$

$$= \frac{V}{T}\int_0^T te^{-st}\, dt = \frac{V}{T}\left[\left.\frac{t\, e^{-st}}{-s}\right|_0^T - \left.\frac{e^{-st}}{s^2}\right|_0^T\right]$$

$$= \frac{V}{T}\left[\frac{-T_{e^{-sT}}}{s} - \left(\frac{e^{-sT}-1}{s^2}\right)\right] = \frac{V}{T}\left[\frac{1}{s^2} - e^{-sT}\left(\frac{1}{s^2}+\frac{T}{s}\right)\right]$$

$$= \frac{V}{T}\left[\left(\frac{1-e^{-sT}}{s^2}\right) - \frac{Te^{-sT}}{s}\right]$$

$$L[f(t)] = \frac{V}{T}\left[\frac{1-e^{-sT}(1+ST)}{s^2}\right]$$

4. (b) Define RMS value, Average value, Form factor of an alternating quantity. Also state the relationship between them.

Solution: Refer to Set-1 4(c) [AC is periodic]

4. (c) Find the RMS value of the voltage wave whose equation is $v(t) = 10 + 200 \sin (wt - 30°) + 100 \cos 3\, wt - 50 \sin (5wt + 60°)$.

Solution: $$V_{rms} = \sqrt{10^2 + \frac{(200)^2}{2} + \frac{(100)^2}{2} + \frac{(50)^2}{2}}$$

$$= \sqrt{100 + 20000 + 5000 + 1250}$$

$$= 162.327 \text{ V}$$

5. (a) What is complex power? Explain in detail.

Solution: *Complex power*

Active power (P):

The active power or real power in an a.c. circuit is given by the product of voltage, current and cosine of the phase angle. It is always positive

$$P = VI \cos \phi \text{ watts}$$

Reactive power (Q):

The reactive power in an a.c. circuit is given by the product of voltage, current and sine of the phase angle ϕ.
If ϕ is leading then reactive power is taken as +ve and it is capacitive.
If ϕ is lagging then reactive power is taken as −ve and it is inductive

$$Q = VI \sin \phi \text{ VARs.}$$

Apparent power:

The apparent power in an a.c. circuit is the product of voltage and current. It is measured in voltamps.

$$S = VI \text{ volt amps.}$$

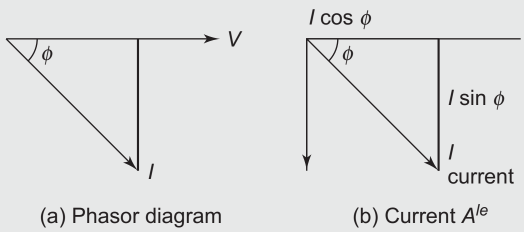


Fig. Set 4.21

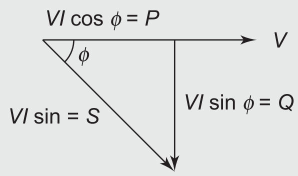


Fig. Set 4.22

The component $I \cos \phi$ = Active component or real component or in phase component of a current.
The product of voltage and the above component (active component) gives active power. The component $I \sin \phi$ = Reactive component or quadrature component of current)
The product of this component with voltage V gives the reactive power.

$$\text{Power factor } \cos \phi = \frac{\text{Read power}}{\text{Apparent power}}$$

The factor $\sin \phi$ is called the reactive factor.

Complex power = (Active power) + j (Reactive power)

5. (b) The current in a given circuit is $I = (12 - j5)$ A when the applied voltage is $V = (160 - j120)V$. Determine
   i. the complex expression for power
   ii. power factor of the circuit
   iii. the complex expression for impedance of the circuit
   iv. Draw the phasor diagram.

Solution: (i) $P_a = V_{\text{eff}} I_{\text{eff}}\ VA$

$$P_{ar} = V_{eff} I_{eff} \cos\theta \text{ watts}$$

$$P_r = V_{eff} I_{eff} \sin\theta \text{ VAR}$$

$$Z = \frac{V}{I} = \frac{160 - j120}{12 - j5} = \frac{14.91}{R} - \frac{3.786}{x} j$$

$$|I| = 13\ A$$

$$= 15.38 \angle -14.25^\circ$$

$$\therefore \quad P_{\text{avg}} = I^2 R = 2519.79 \text{ W}$$

$$P_r = I^2 X = 639.834\ VAR$$

$$P_a = I^2 Z = 2599.22 \text{ W}$$

Complex power = $2519.79 + j639.834$

(ii) $Pf = \cos\phi = \cos(-14.25^\circ) = 0.969$

(iii) $Z = 14.91 - 3.786j$

(iv) Power Δ^{le}

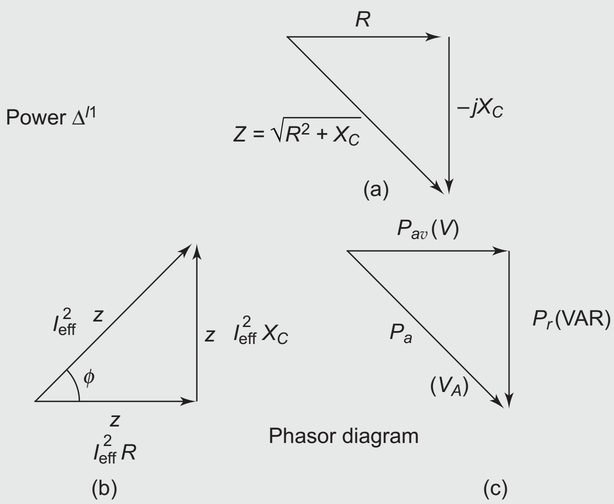


Fig. Set 4.23

6. (a) Why Z-parameters are known as open circuit parameters?

Solution:

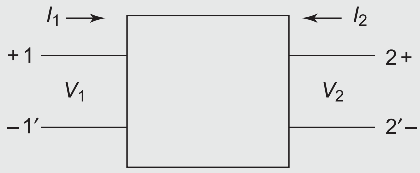


Fig. Set 4.24

The Z parameters of a two-port for the positive directions of voltages and currents may be defined by expressing the port voltages V_1 and V_2 in terms of the currents I_1 and I_2. Here V_1 and V_2 are dependent variables, and I_1, I_2 are independent variables. The voltages at port 1–1′ is the response produced by the two currents. I_1 and I_2

Thus

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

The individual Z-parameters for a given network can be defined by setting each of the port currents equal to zero. Suppose port 2–2′ is left open-circuited, then

$$I_2 = 0$$

Thus $Z_{11} = \left.\dfrac{V_1}{I_1}\right|_{I_2 = 0}$ driving point impedance at port 1–1′ with port 2–2′ open circuited. It is called open circuit input impedance

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

It is called open circuit forward transfer impedance

Suppose port 1–1′ is left open circuited then, $I_1 = 0$

Thus $Z_{12} = \left.\dfrac{V_1}{I_2}\right|_{I_1 = 0}$

It is called open circuit reverse transfer impedance

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$

It is called open circuit output impedance.

It is observed that the individual parameters are specified only when the current in one of the ports is zero. This corresponds to one of the ports being open circuited from which Z-parameters also derive the name open circuit impedance parameters.

6. (b) What is meant by port? Explain two port network?

Solution: Port: A port is defined as any pair of terminals into which energy is supplied, or from which energy is withdrawn, or where the network variables may be measured.

Two port network

A two-port network is simply a network inside a black box, and the network has only two pairs of accessible terminals; usually one pair represents input and the other represents the output.

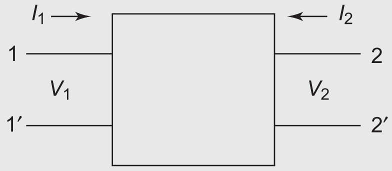


Fig. Set 4.25

Two ports containing no sources in their branches are called passive ports; among them are power transmission lines and transformers. Two ports containing sources in their branches are called active ports. Two of these are dependent variables and the other two are independent variables. The number of possible combinations generated by the four variables taken two at a time sin ($4C_2$). Thus, there are six possible sets of equations describing a two-port network.

6. (c) Find the Y-parameters for the network shown in Fig. Set 4.26.

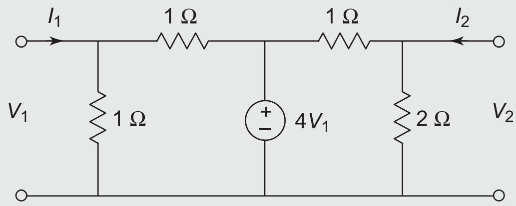


Fig. 4.26

Solution:

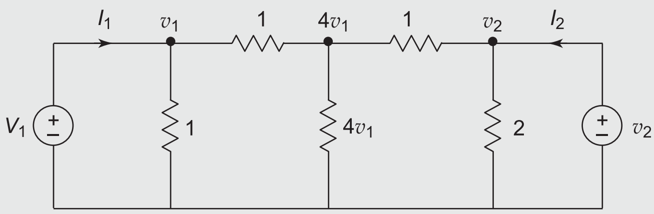


Fig. Set 4.27

Y-parameters are generally of the form

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

By nodal analysis

$$I_1 = \frac{V_1}{1} + \left(\frac{V_1 - 4V_1}{1}\right) = V_1 - 3V_1$$

$$I_1 = V_1 - 3V_1$$

$$\Rightarrow \quad I_1 = -2V_1 \tag{1}$$

$$I_2 = \frac{V_2}{2} + V_2 - 4V_1$$

$$\Rightarrow \quad I_2 = -4V_1 + \frac{3}{2}V_2 \tag{2}$$

$\therefore$ By comparing 1 and 2 with the above equations

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -4 & 3/2 \end{pmatrix}$$

7. (a) Show that the propagation constant for Π network is $\gamma_A = \cosh^{-1}\left(1 + \frac{Z_1}{2Z_2}\right)$

Solution: π*-type attenuator*

$\Rightarrow$

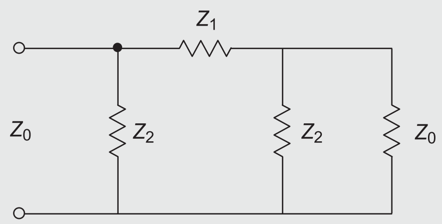


Fig. Set 4.28

From fundamental equations, we have

$$Z_1 = Z_0 \sinh \alpha$$

$$Z_2 = Z_0 \coth \alpha/2$$

$$\frac{Z_1}{Z_2} = \frac{\sinh \alpha}{\coth \alpha/2}$$

$$\frac{Z_1}{2Z_2} = \frac{\sinh \alpha \,.\, \sinh \alpha/2}{2\cosh \alpha/2} = \frac{2\sinh \alpha/2 \cosh \alpha/2 \,.\, \sinh \alpha/2}{2\cosh \alpha/2}$$

($\because$ sin 2h x = 2 sinh x cosh x)

$$\frac{Z_1}{2Z_2} = \sin 2\,h\ \alpha/2 \qquad \text{(Here } \gamma = \alpha\text{)}$$

since it is symmetric

$$\Rightarrow 1 + \frac{Z_1}{2Z_2} = (\cosh \alpha/2)^2 = \left(\frac{e^{\alpha/2} + e^{-\alpha/2}}{2}\right)^2$$

$$= \frac{e^{\alpha} + e^{-\alpha} + 2}{4} = \frac{e^{\alpha} + e^{-\alpha}}{4} + \frac{1}{2}$$

Multiplying by 2

$$2\left(1 + \frac{Z_1}{2Z_2}\right) = \frac{e^{\alpha} + e^{-\alpha}}{2} + 1$$

$$1 + \frac{Z_1}{Z_2} = \cosh\,\alpha \quad \Rightarrow \quad \gamma_A = \cosh^{-1}\left(1 + \frac{Z_1}{Z_2}\right)$$

If $Z_2 \longrightarrow 2Z_2$ (replaced then $\gamma_A = \cosh^{-1}\left(1 + \frac{Z_1}{2Z_2}\right)$

7. (b) Write short note on iterative and image impedances in symmetrical networks.

Solution: Two importance parameter for design of attenuators is image impedance for unsymmetrical attenuator and characteristic impedance for symmetrical attenuator and also attenuation constant for both types of attenuators.

Z_{11} and Z_{12} are two impedances such that when terminals 2–2′ are terminated in Z_{12} the input impedance at terminals marked 1–1′ is Z_{11}. Using ABCD parameters the two-terminal pair impedances and admittances and certain algebraic expression it can be shown that Z_{11}

$= \sqrt{Z_{OC_1}} Zsc_1$

where Z_{oc1} is the input impedance measured at terminals 1–1′ when the terminals 2–2′ are kept open circuited and $Z_{sc1}$is the short ckt impedance as measured at 1–1′

when terminals marked 22′ shorted

Similarly $Z_{12} = \sqrt{Z_{oc2}\,Z_{sc2}}$

If the network is symmetrical then

$$Z_{11} = Z_{12} = Z_0 = \sqrt{Z_{oc1}\,Z_{sc1}} = \sqrt{Z_{oc2}\,Z_{sc2}}$$

Where Z_0 is the characteristic impedance of the attenuating network.

The characteristic impedance or iterative impedance is defined as the impedance of a network with which a network must be terminated so that the input and terminating resistances are equal.

If the attenuation network is asymmetric the network will have two different characteristic impedances known as image impedances.

The values of impedances ($Z_{11} \neq Z_{12}$) are different depending on which end is used as the input.

8. What is a half section? What is its main characteristic? Why it is used? Derive expression for impedances as seen from the two ports of an *m*-derived half section.

Solution: Refer to textbook (Chapter 16).