NETWORK THEORY

About the Authors

P K Satpathy, obtained his PhD degree from IIT, Kharagpur in 2003 and M Tech degree from IIT, Delhi in 1996 (in the area of Electrical Engineering with specialization on Electric Power Systems). Being a graduate in Electrical Engineering from University College of Engineering, Burla (1989), he joined academics in the year 1990 and has acquired 19 years of experience in the field of Academics and Research pertaining to Electrical Engineering. Presently, he is positioned as Reader and Head of Electrical Engineering in College of Engineering and Technology, a premier constituent college of BPUT, Orissa. He has several international publications to his credit. Besides an author of a textbook, he is a Member/Fellow/Reviewer of various organizations of National and international repute (Elsevier, IE-India, ISTE, Inpowerg).

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Dedicated to my family

Wife, *Sasmita*

Daughters, *Sheetal* and *Komal*

and

My Parents and Parents-in-Law

P K Satpathy

Dedicated to my family

Wife, *Indrani*

Daughter, *Priyasha*

and

Son In-Law, *Shankar*

P Kabisatpathy

Dedicated to my family

Wife, *Lipika*

and

My Parents

S P Ghosh

Dedicated to my family

Wife, *Indira*

Daughters, *Amrita* and *Ananya*

A K Chakraborty

Contents

Foreword

It's indeed a pleasure to present before you a book on Network Theory authored by P K Satpathy, P Kabisatpathy, S P Ghosh and A K Chakraborty. The book has been developed carefully and the concepts have been clearly defined with the help of practical examples and exercise problems. Difficult concepts are explained and presented in a simple and understandable way using ample numbers of illustrations. There is an emphasis on the use of right notation and mathematics throughout the book. Every topic has been treated in its totality and practical applications are mentioned right through.

This book is an outcome of over twenty years of teaching and research experience of the authors. I am sure the readers will find the book extremely useful for enhancement of their learning and expertise in this field.

I wish the authors success and sincerely hope that more publications would emerge out of their vast experience of teaching and research in electrical sciences.

Sabyasachi Sengupta
Vice Chancellor (WBUT)

Preface

About the Subject

Network Theory is a foundation core course to many disciplines of Engineering and Technology such as Electrical Engineering (EE), Electrical and Electronics Engineering (EEE), Instrumentation and Electronics Engineering (I&EE), Electronics and Communication Engineering (ECE), Instrumentation and Control Engineering (ICE), Applied Electronics Engineering (AEE), Computer Science and Engineering (CSE) and Information Technology (IT). Besides these branches in particular, almost all other branches of Engineering and Technology may take this course as a professional elective or free elective subject. A deep understanding of any electrical circuit requires the fundamentals of Network Theory. Hence the course has been modeled to contain the basics of networks and important theorems related to network operation in the first place. In the second place, one would find the fundamentals of network analysis followed by network synthesis and realization in the third place.

About the Book

This book has been written to serve as a primary textbook for the revised course on Network Theory pertaining to Biju Patnaik University of Technology (BPUT), Orissa. This book is an outcome of the consistent work of the authors that imbibes their long-time experience in academics and research in this area. Although a handful of books are available, yet it has been felt that none of them go step-in-step with the requirements of this particular course. As a result, students face various difficulties in following the texts taught to them in the class with the support of the available texts in these books. This book aims at presenting the contents of the course in such a way that the students and faculty may rely on the book as a primary resource. Provision of well-documented text material, simple and lucid diagrams at appropriate locations of the text, chapter-end examples, solved problems and multiple/short answer questions make the book an effective and competitive tool for the course on Network Theory. Over and above, the book presents the concepts behind each model in a concise and self-explanatory way which would add extra advantage for the reference and understanding of the reader.

Scope

This book is mainly prepared to meet the credit requirements of the third and fourth semester courses on Network Theory related to the four year B Tech program of relevant disciplines of Biju Patnaik University of Technology (BPUT), Orissa. Besides this, it may be useful to students and faculty members of other universities in India and abroad. However, beyond the credit requirements of B Tech program, this book may be very helpful for the students aspiring and preparing for national-level competitive examinations such as GATE, NET, UPSC, PSC of various states, IES, etc.

Organization

This book has a total of nine chapters. The first chapter provides information about the basic characteristics of different types of systems. **Chapter 2** deals with the fundamental laws, and circuit components essential for network theory. **Chapter 3** discusses the application of graph theory concepts of circuit analysis. **Chapter 4** is devoted to explain the usefulness of various theorems essential for network analysis. **Chapter 5** presents the frequency domain analysis of networks by using a mathematical tool called Laplace Transform. **Chapter 6** encompasses the concepts of two-port networks which find vast application in other areas such as control and communication. **Chapter 7** focuses on the application of another powerful mathematical tool called Fourier series for network analysis. **Chapter 8** is devoted to operational amplifiers and filters. **Chapter 9** presents the basics of network synthesis through verification and possibility of realization of network functions into network elements.

Acknowledgements

The authors express their thankfulness to their colleagues in the department and beyond, who shared their valuable inputs in bringing the manuscript to this shape. Besides that, the authors would like to convey their appreciation to some students for their constructive feedback at various levels. Last but not the least, the authors owe a deep sense of respect and gratitude to their respective family members without whose contribution, in whatsoever manner, the project would not have been completed.

Feedback

Criticism and suggestions for improvement shall be gratefully acknowledged. Readers may contact P K Satpathy at satpathy_pk@yahoo.com, S P Ghosh at ghosh_shankar@rediffmail.com and A K Chakraborty at akcalll@yahoo.co.in.

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P KABISATPATHY
S P GHOSH
A K CHAKRABORTY

Note to the Students

This book can be used as a core course for many disciplines of Engineering and Technology. The contents of this course have been designed to make the students familiar with the basic laws and theorems governing the operation of electrical circuits subject to numerous practical situations. It is quite challenging and exciting to learn the fundamental concepts covered in this subject. This book has been so developed that a novice can go through its chapters comfortably. Students might well learn more efficient methods of handling the subject by using this textbook. The authors would like to remind the students about the following points and if you keep these ideas in mind you shall do very well in this course.

- This course is the foundation on which most other courses in the electrical engineering curriculum rest. For this reason, put in as much effort, time and at the same time motivate yourself to become a master in this subject by studying the course on a regular basis.
- Problem-solving methods are essential parts of the learning process. Hence, try to solve as many problems as you can. Begin by solving practice problems following each example, illustrations and then proceed to the chapter-end problems. The best way to learn is to solve a lot of problems.
- Attempt the multiple choice questions in each chapter. It will help you to discover some problem-solving tricks and analytical thoughts. This would strengthen your learning.
- Clearly, a lot of effort has gone into making the technical details in this book compatible for easy understanding. It also contains all the mathematics and physics necessary to understand the subject and the same would help in higher-semester courses of Electrical Sciences.
- A core textbook is an asset. It would be useful throughout your life. Hence take care that the book is not lost and remember that you are not selling out the book once your course is over.

Your success will surely add to the success of this book. May your success be electrifying!

P K Satpathy
P Kabisatpathy
S P Ghosh
A K Chakraborty

CHAPTER 1

Introduction to Different Types of Systems

1.1 INTRODUCTION

An electrical network is one of the many important physical systems. In order to understand the basic characteristics of an electric network, we must first know the different concepts of systems. In this chapter, the different types of systems have been discussed.

1.2 CONCEPTS OF SIGNALS AND SYSTEMS

1.2.1 Signals

A signal is defined as a function of one or more variables, which provides information on the nature of a physical phenomenon.

When the function depends on a single variable, the signal is said to be one-dimensional, for example, a speech signal whose amplitude varies with time, depending on the spoken word and who speaks it.

When the function depends on two or more variables, the signal is said to be multidimensional, for example, an image (2-D signal).

1.2.2 Systems

A system is an entity that takes an input signal and produces an output signal. It is a combination and interconnection of several components to perform a desired task.

The system responds to one or more input quantities, called input signals or excitation, to produce one or more output quantities, called output signals or response.

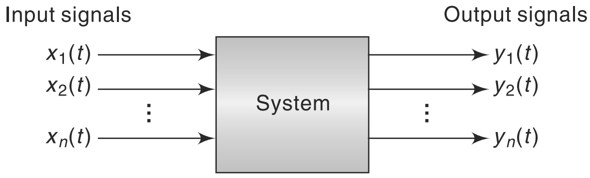


Figure 1.1 *Block diagram representation of a system*

1.3 DIFFERENT TYPES OF SYSTEMS

1. Continuous and Discrete Time Systems
2. Fixed and Time-varying Systems
3. Linear and Non-linear Systems
4. Lumped and Distributed Systems
5. Instantaneous and Dynamic Systems
6. Active and Passive Systems
7. Causal and Non-causal Systems
8. Stable and Unstable Systems
9. Invertible and Non-invertible Systems

1.3.1 Continuous and Discrete Time Systems

Signals are represented mathematically as functions of one or more independent variables. We classify signals as being either ***continuous-time*** (functions of a real-valued variable) or ***discrete-time*** (functions of an integer-valued variable).

In other words, a continuous-time signal has a value defined for each point in time and a discrete-time signal is defined only at discrete points in time.

To signify the difference, we (usually) use round parenthesis around the argument for continuous time signals, e.g., $x(t)$ and square brackets for discrete-time signals, e.g., $x[n]$. We will also use the notation x_n for discrete-time signals.

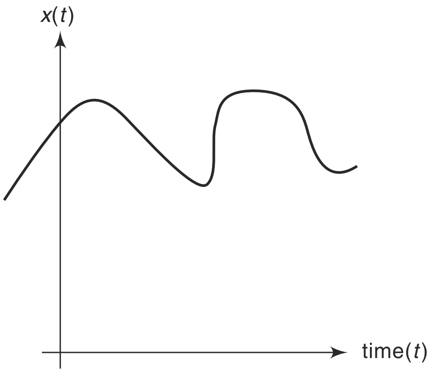


Figure 1.2(a) *Continuous-time signal*

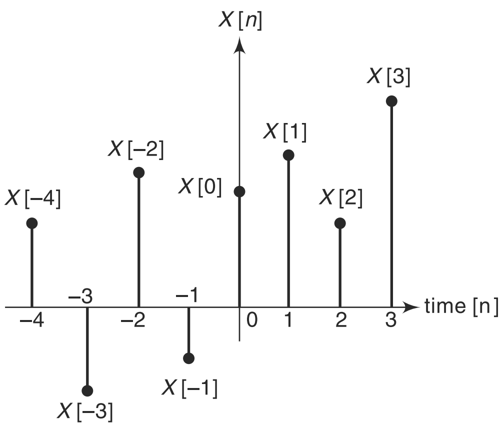


Figure 1.2(b) *Discrete-time signal*

A continuous-time system is a system which accepts only continuous-time signal to produce continuous-time internal and output signals. On the other hand, a discrete-time system is a system that transforms discrete-time input(s) into discrete-time output(s).

The examples given below are common in our daily life.

Continuous-time systems

(i) Atmospheric pressure as a function of altitude
(ii) Electric circuits composed of resistors, inductors, capacitors driven by continuous-time sources.

Discrete-time systems

(i) Weekly stock market index
(ii) Balance in a bank account from month to month.

The sequence of values of the discrete-time signal shown in Fig. 1.2(b) defined at discrete points in time are called ***samples*** and the spacing between them is called the ***sample spacing***. For equal sample spacing, the sequence of values are expressed as a function of the signed integer n as $x[n]$, where n is termed as a ***sequence of samples*** or ***sequence***, in short.

1.3.2 Time-Invariant (Fixed) and Time-Varying Systems

A system is time-invariant or fixed if the behaviour and characteristics of the system do not change with time. Otherwise, the system is time-varying.

Mathematically, if the input $x(t)$ gives the output $y(t)$, then the system is time-invariant if the input $x(t - T)$ gives the output $y(t - T)$ for any delay T. Hence, a time-shift of the input gives the same time-shift of the output.

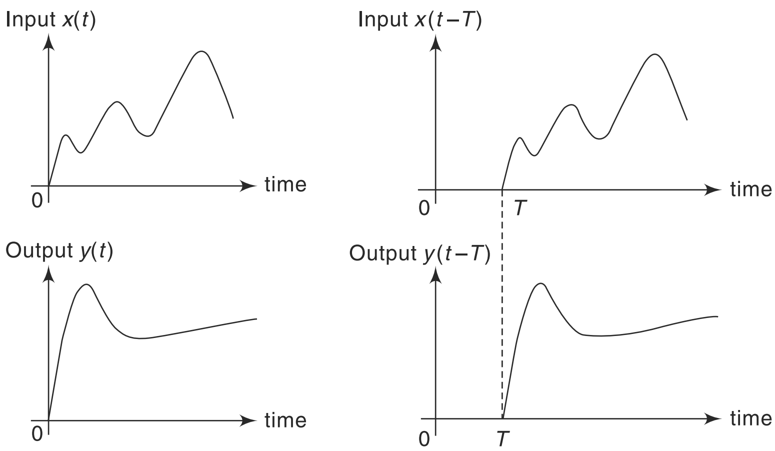


Figure 1.3 *Time-invariant system*

Whether a system is time-invariant or time-varying can be seen in the differential equation (or difference equation) describing it. ***Time-invariant systems are modeled with constant coefficient equations.*** A constant coefficient differential (or difference) equation means that the parameters of the system are not changing over time and an input now will give the same result as the input later.

Example 1.1 *A continuous system is modeled by the equation $y(t) = tx(t) + 4$, and a discrete-time system is modeled by $y[n] = x^2[n]$. Are these systems time-invariant?*

Solution *For continuous-time system:*

For input $x(t) = x_1(t)$, output $y_1(t) = tx_1(t) + 4$ (i)

For input $x(t) = x_1(t - T)$, output, $y_2(t) = tx_1(t - T) + 4$ (ii)

From the condition of time-invariance, the output should be,

$$y_1(t - T) = (t - T)x_1(t - T) + 4 \quad \text{(iii)}$$

From Eqs (ii) and (iii), $y_2(t) \neq y_1(t - T)$

Hence, the system is not time-invariant.

For discrete-time system:

For input $x_1[n]$, output $y_1[n] = x_1^2[n]$

For input $x_1[n - n_0]$, output $= x_1^2[n - n_0]$

From the condition of time-invariance, the shifted output $y_1[n - n_0] = x_1^2[n - n_0]$

Hence, the system is time-invariant.

1.3.3 Linear and Non-Linear Systems

A system, in continuous-time or discrete-time, is said to be linear, if it obeys the ***properties of superposition, i.e., additivity and homogeneity (or scaling),*** while a system is non-linear if it does not obey at least any one of these properties.

The superposition principle says that the output to a linear combination of input signals is the same linear combination of the corresponding output signals. Mathematically, the linearity condition is based on two properties.

1. *Additivity* If the input signals $x_1(t)$ and $x_2(t)$ correspond to the output signals $y_1(t)$ and $y_2(t)$, respectively, then the input signal $\{x_1(t) + x_2(t)\}$ should correspond to the output signal $\{y_1(t) + y_2(t)\}$.

2. *Homogeneity* If the input signal $x_1(t)$ corresponds to the output signal $y_1(t)$, then the input signal $a_1x_1(t)$ should correspond to the output signal $a_1y_1(t)$ for any constants a_1.

Combining these two properties, the condition for a linear system can be written as, if the input signals $x_1(t)$ and $x_2(t)$ correspond to the output signals $y_1(t)$ and $y_2(t)$, respectively, then the input signal $a_1x_1(t) + a_2x_2(t)$ should correspond to the output signal $a_1y_1(t) + a_2y_2(t)$ for any constants a_1 and a_2.

Example 1.2 Check whether the systems with the input-output relationship given below are linear.

(a) $y(t) = mx(t) + c$, (b) $y(t) = tx(t)$

Solution

(a) For an input $x_1(t)$, output, $y_1(t) = mx_1(t) + c$

For an input $x_2(t)$, output, $y_2(t) = mx_2(t) + c$

For an input $\{x_1(t) + x_2(t)\}$, output, $y_3(t) = m\{x_1(t) + x_2(t)\} + c$ (i)

From the condition of linearity, the output should be

$$\{y_1(t) + y_2(t)\} = m\{x_1(t) + x_2(t)\} + 2c \quad \text{(ii)}$$

From Eqs (i) and (ii), we conclude that the *system is non-linear.*

(b) For an input $x_1(t)$, output, $y_1(t) = tx_1(t)$

For an input $x_2(t)$, output, $y_2(t) = tx_2(t)$

For an input $\{k_1x_1(t) + k_2x_2(t)\}$, output, $y_3(t) = t\{k_1x_1(t) + k_2x_2(t)\}$ (i)
where, k_1 and k_2 are any arbitrary constants.
From the condition of linearity, the output should be

$$\{k_1y_1(t) + k_2y_2(t)\} = k_1tx_1(t) + k_2tx_2(t) = t\{k_1x_1(t) + k_2x_2(t)\} \quad \text{(ii)}$$

From Eqs (i) and (ii), we conclude that the *system is linear.*

1.3.4 Lumped and Distributed Systems

All physical systems contain distributed parameters because of the physical size of the system components. For example, the resistance of a resistor is distributed throughout its volume.

However, if the ***size of the system components is very small with respect to the wavelength*** of the highest frequency present in the signals associated with it, then the system components behave as if it all were occurring at a point. This system is said to be ***lumped-parameter system.***
Distributed parameter systems are modeled as given below.

1. By partial differential equations if they are continuous-time systems
2. By partial difference equations if they are discrete-time systems.

Lumped parameter systems are modeled with ordinary differential or difference equations.

Example 1.3 Consider an electric power system of frequency 50 Hz. The wavelength of the signal is obtained as,

$$n\lambda = C \Rightarrow \lambda = \frac{C}{n} = \frac{3\times10^5}{50} = 6000 \text{ km}$$

Thus, the electrical system inside a room can be treated as a lumped-parameter system, but will be treated as distributed system for a long-distance transmission line.

1.3.5 Instantaneous (Static or Memoryless) and Dynamic Systems

An instantaneous or static or memoryless system is a system where the output at any specific time depends on the input at that time only. On the other hand, a dynamic system is one whose output depends on the past or future values of the input in addition to the present time.

A static system has no memory. Physically, it contains no energy-storage elements, whereas a dynamic system has one or more energy-storage element(s).

Example 1.4 An electrical circuit containing resistance R has the v–i relationship as, $v(t) = Ri(t)$, and so the system is static. But an electrical circuit containing capacitor C has the v–i relationship as, $v(t) = \frac{1}{C}\int_0^t i(t)dt$, and so the system is dynamic system.

1.3.6 Active and Passive Systems

A system having no source of energy is known as a passive system, for example, electric circuits containing resistance, capacitance, inductance, diodes, etc.

A system having source of energy together with other passive elements is known as an active system, for example, electric circuits containing voltage source or current source or op-amp, etc.

1.3.7 Causal and Non-causal Systems

A system is said to be causal if the output of the system depends only on the input at the present time and/or in the past, but not the future value of the input. Thus, a causal system is ***nonanticipative,*** i.e., output cannot come before the input.

On the other hand, the output of a non-causal system depends on the future values of the input.

Example The moving-average system described by

$$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

is causal, but the moving-average system described by

$$y[n] = \frac{1}{3}\{x[n+1] + x[n] + x[n-1]\}$$

is non-causal, since the output depends on the future value of the input $x[n + 1]$.

It is obvious that the idea of future inputs does not have any physical meaning if we take time as our independent variable and for that reason all real-time systems are causal. However, for the case of image processing, the independent variable may be the pixels to the left and right (the "future") of the current position on the image, and thus, we can have a non-causal system.

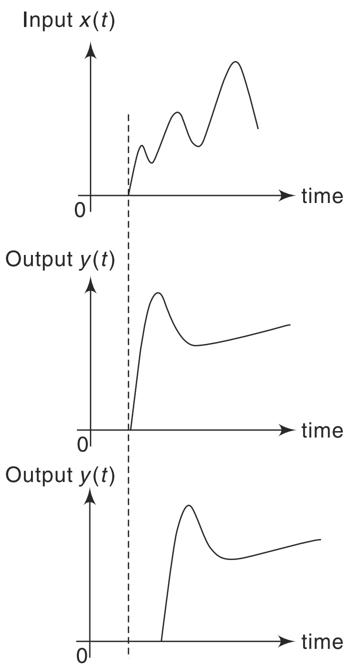


Figure 1.4(a) *Causal systems*

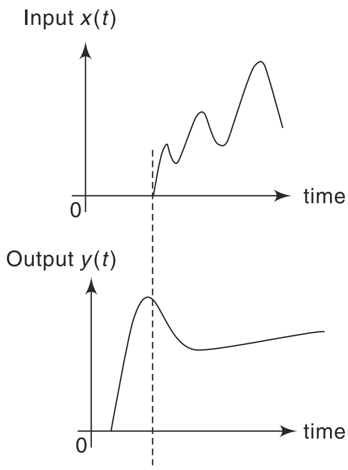


Figure 1.4(b) *Non-causal systems*

1.3.8 Stable and Unstable Systems

A stable system is one where the output does not diverge as long as the input does not diverge. A bounded input produces a bounded output. For this reason, this type of system is known as ***bounded input-bounded output (BIBO)*** stable system.

Mathematically, a stable system must have the following property:

If $x(t)$ be the input and $y(t)$ be the output, then the output must satisfy the condition.

$$|y(t)| \leq M_y < \infty; \quad \text{for all } t$$

whenever the input satisfy the condition

$$|x(t)| \leq M_x < \infty; \quad \text{for all } t$$

where, M_x and M_y both represent a set of finite positive numbers.

If these conditions are not met, i.e., the output of the system grows without limit (diverges) from a bounded input, then the system is **unstable**.

1.3.9 Invertible and Non-invertible Systems

A system is referred to as an invertible system if

(i) distinct inputs lead to distinct output, and
(ii) the input can be recovered from the output.

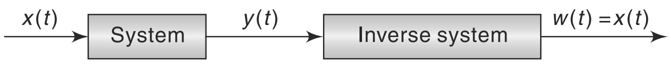


Figure. 1.5 *Invertible system*

The property of invariability is important in the design of communication systems. When a transmitted signal propagates through a communication channel, it becomes distorted due to the physical characteristics of the channel. An equalizer is connected in cascade with the channel in the receiver to compensate this distortion. By designing the equalizer to be inverse of the channel, the transmitted signal is restored.

MULTIPLE-CHOICE QUESTIONS

1.1 The output $y(t)$ and the input $x(t)$ of a system are related by the equation $y(t) = mx(t) + c$, where m and c are constants. The system is
(a) linear
(b) non-linear
(c) may be linear or non-linear depending on $y(t)$ and $x(t)$
(d) none of the above

1.2 If the impulse response is realizable by delaying it appropriately and is bounded for bounded excitation, then the system is said to be
(a) causal and stable
(b) causal but not stable
(c) non-causal but stable
(d) non-causal, not stable

1.3 In a linear circuit, when the ac input is doubled, the ac output becomes
(a) one fourth (b) half (c) two times (d) four times

1.4 A circuit having an e.m.f. source or any energy source is
(a) active circuit (b) passive circuit (c) unilateral circuit (d) bilateral circuit

1.5 A network is said to be linear if and only if
(a) a response is proportional to the excitation function
(b) the principle of superposition applies
(c) the principle of homogeneity applies
(d) both the principles (b) and (c).

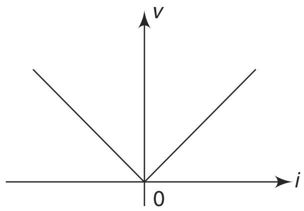
1.6 Consider the following data.
1. Input applied for $t < t_0$
2. Input applied for $t \geq t_0$
3. State of the network at $t = t_0$
4. State of the network at $t < t_0$

Among these, those needed for determining the response of a linear network for $t > t_0$ would include
(a) 1, 3 and 4 (b) 2, 3 and 4 (c) 2 and 3 (d) 2 and 4.

1.7 An excitation is applied to a system at $t = T$ and its response is zero for $-\infty < t < T$. Such a system is
(a) non-causal system (b) stable system (c) causal system (d) unstable system.

1.8 The elements which are not capable of delivering energy by its own are known as
(a) unilateral elements (b) non-linear elements
(c) passive elements (d) active elements.

1.9 The v–i characteristic of an element is shown in the given figure. The element is
(a) non-linear, active, non-bilateral
(b) linear, active, non-bilateral
(c) non-linear, passive, non-bilateral
(d) non-linear, active, bilateral



EXERCISES

1.1 A continuous system is modeled by the equation

$$y(t) = t\,x(t) + 4$$

and a discrete-time system is modeled by,

$$y[n] = x^2[n]$$

Are these systems time-invariant?

1.2 Consider the continuous-time system defined by,

$$y(t) = \sin[x(t)]$$

Check whether the system is time-invariant?

1.3 Consider a system S with input $x[n]$ and output $y[n]$ related by,

$$y[n] = x[n]\{g[n] + g[n-1]\}$$

(a) If $g[n] = 1$, for all n, show that S is time-invariant.
(b) If $g[n] = n$, show that S is not time-invariant.
(c) If $g[n] = 1 + (-1)^n$, show that S is time-invariant.

1.4 Consider the systems S whose input and output are related by,
  (a) $y(t) = t\,x(t)$
  (b) $y(t) = x(t)\,x(t-1)$
  (c) $y(t) = x^2(t)$
  (d) $y = mx + c$
  Check whether S is linear.

1.5 Consider the following discrete-time systems with input-output relationships as given
  (a) $y[n] = 2x[n] + 3$
  (b) $y[n] = \text{Re}\{x[n]\}$
  (c) $y[n] = n\,x[n]$
  Check whether the systems are linear.

SHORT-ANSWER TYPE QUESTIONS

1.1 What is a system? What are the different types of systems? Give their definitions.

1.2 Define the following and give examples of each.
  (a) Continuous and discrete system.
  (b) Time-invariant and time-varying system.
  (c) Lumped and distributed system.
  (d) Instantaneous (Static or Memoryless) and dynamic system.
  (e) Causal and non-causal system.
  (f) Active and passive system.

1.5 (a) What are the conditions for a system to be a linear system?
  (b) Give the conditions for a BIBO stability of a system.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1.1 (b)	1.2 (a)	1.3 (c)	1.4 (a)	1.5 (d)	1.6 (c)	1.7 (c)
1.8 (c)	1.9 (b)					

CHAPTER

2
Introduction to Circuit Theory Concepts

2.1 INTRODUCTION

The fundamental theory on which many branches of electrical engineering, such as electric power, electric machines, control, electronics, computers, communications and instrumentation are built is the electric circuit theory. Thus, it is essential to have a proper grounding with electric circuit theory as the base. An electric circuit is the interconnection of electrical elements.

2.2 TERMINOLOGY

In circuit analysis, we are concerned with the four basic manifestations of electricity, namely, electric charge [$q(t)$], magnetic flux [$\varphi(t)$], electric potential [$v(t)$] and electric current [$i(t)$].

The most basic quantity in an electric circuit is the *electric charge* $\boldsymbol{q}$. The law of *conservation of charge* states that charge can neither be created nor destroyed. Thus the algebraic sum of the charges in a system does not change. The charge on an electron is 1.602×10^{-19} C. [Unit of electric charge is the *coulomb* (*C*)].

The rate of flow of electric charges or electrons constitute an *electric current* $\boldsymbol{i}$. By convention (a standard way of describing an electric current), the electric current flows in the opposite direction to the electrons. [Unit of electric current is the *ampere* (*A*)].

$$i = \frac{dq}{dt}$$

and the charge transferred between time t_0 and t is given by

$$q = \int_{t_0}^{t} i\,dt \tag{2.1}$$

To move an electron in a conductor in a particular direction, or to create a current, requires some work or energy.

This work is done by the *electromotive force (emf)* of the source or the *potential difference*. This is also known as *voltage difference* or *voltage* (with reference to a selected point such as *earth*).

The voltage V_{ab} between two points *a* and *b* is the *energy* (or work) ***w*** required to move a unit positive charge from ***a*** to ***b*.** [Unit of voltage is the *volt (V)*]

$$V_{ab} = \frac{dw}{dq} \tag{2.2}$$

The potential difference between the terminals of a circuit element in a magnetic field is equal to the time derivative of the flux $\varphi(t)$, i.e., rate of change of flux linkages,

so, $$v(t) = \frac{d}{dt}\,\phi(t) \tag{2.3}$$

2.3 DIFFERENT NOTATIONS

C	Capacitance	Farad, F
E	Voltage source	Volt, V
e	Instantaneous **E**	Volt, V
G	Conductance	Siemens, S
I	Current	Ampere, A
i	Instantaneous **I**	Ampere, A
K	Coefficient	number
L	Inductance	Henry, H
M	Mutual inductance	Henry, H
N	Number of turns	number
P	Power	Watt, W
Q	Charge	Coulomb, C
q	Instantaneous **Q**	Coulomb, C
R	Resistance	Ohm, Ω
T	Time constant	second, s
t	Instantaneous time	Second, s
V	Voltage drop	Volt, V
v	Instantaneous **V**	Volt, V
W	Energy	Joule, J
F	Magnetic flux	Weber, Wb
Y	Magnetic linkage	Weber, Wb
y	Instantaneous **Y**	Weber, Wb

2.4 BASIC CIRCUIT ELEMENTS

(i) Active and Passive Elements Electric Circuits consist of two basic types of elements. These are the *active elements* and the *passive elements*.

An ***active element*** is capable of generating electrical energy. (In electrical engineering, generating or producing electrical energy actually refers to conversion of electrical energy from a non-electrical form to electrical form. Similarly, energy loss would mean that electrical energy is converted to a non-useful form of energy and not actually lost. *Principle of Conservation of Mass and Energy*).

Examples of active elements are *voltage source* (such as a battery or generator) and *current source*. Most sources are independent of other circuit variables, but some elements are *dependent* (modeling elements such as transistors and operational amplifiers would require dependent sources).

Active elements may be ***ideal*** voltage sources or current sources. In such cases, the particular generated voltage (or current) would be independent of the connected circuit.

A ***passive element*** is one which does not generate electricity but either consumes it or stores it. *Resistors, Inductors* and *Capacitors* are simple passive elements. Diodes, transistors etc. are also passive elements.

Passive elements may either be *linear* or *non-linear.* Linear elements obey a straight-line law. For example, a linear resistor has a linear *voltage* vs *current* relationship which passes through the origin ($V = R.I$). A linear inductor has a linear *flux* vs *current* relationship which passes through the origin ($\phi = k\ I$) and a linear capacitor has a linear *charge* vs *voltage* relationship which passes through the origin ($q = CV$). [R, k and C are constants].

Resistors, inductors and capacitors may be linear or non-linear, while diodes and transistors are always nonlinear.

(ii) Linear Element A circuit/network element is linear if the relation between current and voltage involves a constant coefficient.

Examples Voltage-current relationship of resistor, inductor and capacitor (both with zero initial conditions) are linear $\left(v = ri,\ v = L\dfrac{di}{dt},\ v = \dfrac{1}{c}\int idt\right)$ Hence, the elements are linear.

Diode and transistors are non-linear devices having non-linear characteristics.

(iii) Bilateral system In a bilateral system, the same relationship between current and voltage exists for current flowing in either direction. On the other hand, a unilateral system has different current-voltage relationships for the two possible directions of current, as in diode.

2.5 PASSIVE CIRCUIT ELEMENTS

2.5.1 Electrical Resistance

Electrical resistance is a measure of the degree to which an object opposes an electric current through it.

The SI unit of electrical resistance is ohm (Ω). Its reciprocal quantity is **electrical conductance** measured in Siemens. Electrical resistance shares some conceptual parallels with the mechanical notion of friction.

The resistance of an object determines the amount of current through the object for a given voltage across the object.

$$I = \frac{V}{R} \tag{2.4}$$

where, R is the resistance of the object, measured in ohm equivalent to J.s/C^2

V is the voltage across the object, measured in volt

I is the current through the object, measured in ampere

For a wide variety of materials and conditions, the electrical resistance does not depend on the amount of current through or the amount of voltage across the object, meaning that the resistance R is constant.

Resistance of a Conductor DC Resistance As long as the current density is totally uniform in the conductor, the DC resistance R of a conductor of regular cross section can be computed as

$$R = \rho \frac{l}{A} \tag{2.5}$$

where, l is the length of the conductor, measured in meter,

A is the cross-sectional area, measured in square meter,

ρ (Greek: rho) is the electrical resistivity (also called *specific electrical resistance*) of the material, measured in ohm metre. Resistivity is a measure of the material's ability to oppose the flow of electric current.

For practical reasons, almost any connections to a real conductor will almost certainly mean the current density is not totally uniform. However, this formula still provides a good approximation for long thin conductors such as wires.

AC Resistance If a wire conducts high-frequency alternating current then the effective cross-sectional area of the wire is reduced. This is because of the skin effect.

This formula applies to isolated conductors. In a conductor close to others, the actual resistance is higher because of the proximity effect.

Resistor A **resistor** is a two-terminal electrical or electronic component that resists an electric current by producing a voltage drop between its terminals in accordance with Ohm's law:

$$R = \frac{V}{I} \tag{2.6}$$

The *electrical resistance* is equal to the voltage drop across the resistor divided by the current through the resistor. Resistors are used as part of electrical networks and electronic circuits.

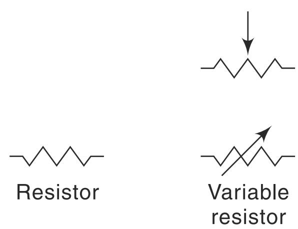


Figure 2.1 *Resistor symbols*

Energy in Resistor Instantaneous power absorbed in the resistor,

$$p = vi = iR \times i = i^2 R \text{ (in Watt)} \tag{2.7}$$

Therefore, the energy converted into heat energy is given by,

$$W = \int_0^t p\,dt = \int_0^t i^2 R\,dt = i^2 Rt \text{ (in Joule)} \tag{2.8}$$

Series and Parallel Arrangements of Resistors Resistors in a parallel configuration each have the same potential difference (voltage). To find their total equivalent resistance (R_{eq}):

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} \tag{2.9}$$

The parallel, property can be represented in equations by two vertical lines "||" (as in geometry) to simplify equations. For two resistors,

$$R_{eq} = R_1 || R_2 = \frac{R_1 R_2}{R_1 + R_2} \tag{2.10}$$

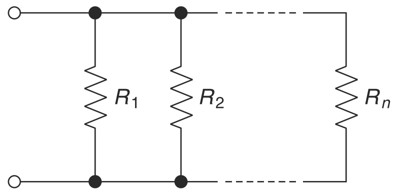


Figure 2.2 *Parallel arrangement of resistors*

The current through resistors in series stays the same, but the voltage across each resistor can be different. The sum of the potential differences (voltage) is equal to the total voltage. To find their total resistance:

$$R_{eq} = R_1 + R_2 + \ldots + R_n \tag{2.11}$$

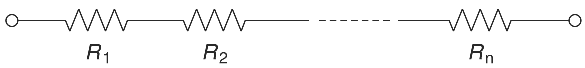


Figure 2.3 *Series arrangement of resistors*

A resistor network that is a combination of parallel and series can sometimes be broken up into smaller parts. For instance,

$$R_{eq} = (R_1 || R_2) + R_3 = \frac{R_1 R_2}{R_1 + R_2} + R_3 \tag{2.12}$$

However, many resistor networks cannot be split up in this way. Consider a cube, each edge of which has been replaced by a resistor. For example, determining the resistance between two opposite vertices requires matrix methods for the general case. However, if all twelve resistors are equal, the corner-to-corner resistance is 5/6 of any one of them.

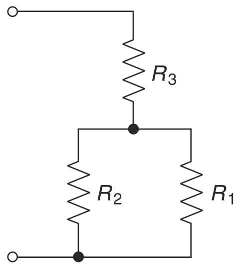


Figure 2.4 *Series-parallel arrangement of resistors*

Current Division by Parallel Resistances When a total current $\boldsymbol{I_P}$ is passed through parallel connected resistances $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$, the voltage $\boldsymbol{V_P}$ which appears across the parallel circuit is:

$$\boldsymbol{V_P = I_P R_P = I_P R_1 R_2/(R_1 + R_2)}$$

The currents $\boldsymbol{I_1}$ and $\boldsymbol{I_2}$ which pass through the respective resistances $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ are:

$$\boldsymbol{I_1 = V_P/R_1 = I_P R_P/R_1 = I_P R_2/(R_1 + R_2)}$$
$$\boldsymbol{I_2 = V_P/R_2 = I_P R_P/R_2 = I_P R_1/(R_1 + R_2)}$$

In general terms, for resistances $\boldsymbol{R_1}$, $\boldsymbol{R_2}$, $\boldsymbol{R_3}$, …, $\boldsymbol{R_n}$ (with conductances $\boldsymbol{G_1}$, $\boldsymbol{G_2}$, $\boldsymbol{G_3}$, …, $\boldsymbol{G_n}$) connected in parallel:

$$\boldsymbol{V_P = I_P R_P = I_P/G_P = I_P/(G_1 + G_2 + G_3 + \ldots)}$$
$$\boldsymbol{I_n = V_P/R_n = V_P G_n = I_P G_n/G_P = I_P G_n/(G_1 + G_2 + G_3 + \ldots)}$$

where $\boldsymbol{G_n = 1/R_n}$ and $\boldsymbol{I_n}$ is the current through n^{th} resistance R_n

Note that the highest current passes through the highest conductance (with the lowest resistance).

2.5.2 Capacitance

Capacitance is a measure of the amount of electric charge stored (or separated) for a given electric potential. The most common form of charge storage device is a two-plate capacitor. If the charges on the plates are $+Q$ and $-Q$, and V gives the voltage difference between the plates, then the capacitance is given by

$$C = \frac{O}{V} \tag{2.13}$$

The SI unit of capacitance is farad; 1 farad = 1 coulomb per volt.

The capacitance of the majority of capacitors used in electronic circuits is several orders of magnitude smaller than the farad. The most common units of capacitance in use today are milli-farad (mF), microfarad (μF), the nano-farad (nF) and the pico-farad (pF)

The capacitance can be calculated if the geometry of the conductors and the dielectric properties of the insulator between the conductors are known. For example, the capacitance of a *parallel-plate* capacitor constructed of two parallel plates of area A separated by a distance d is approximately equal to the following:

$$C = \varepsilon \frac{A}{d} \tag{2.14}$$

where

C is the capacitance in farad, F
ε is the permittivity of the insulator used (or ε_0 for a vacuum)
A is the area of each plate, measured in square meter
d is the separation between the plates, measured in meter

The equation is a good approximation if d is small compared to the other dimensions of the plates.

Capacitor A **capacitor** is an electrical device that can store energy in the electric field between a pair of closely-spaced conductors (called 'plates'). When current is applied to the capacitor, electric charges of equal magnitude, but opposite polarity, build up on each plate.

Capacitors are used in electrical circuits as energy-storage devices. They can also be used to differentiate between high-frequency and low-frequency signals and this makes them useful in electronic filters.

Capacitors are occasionally referred to as **condensers**. This is now considered an antiquated term.

Properties of Capacitance The relation between charge and voltage in a capacitor is written as,

$$Q = CV \tag{2.15}$$

The current, $$i = \frac{dQ}{dt} = C\frac{dV}{dt} + V\frac{dC}{dt}$$

In most physical cases, the capacitance is constant with time.

$$\therefore \quad i = C\frac{dW}{dt} \tag{2.16}$$

$$\therefore \quad dV = \frac{1}{C} idt$$

Taking integration on both sides,

$$\int_0^{v_c} dV = \frac{1}{C}\int_0^t idt$$

or $$v_c(t) = \frac{1}{C}\int_0^t i(t)dt + v_c(0)$$

where, $v_c(0)$ is the initial voltage across the capacitor. For zero initial voltage,

$$v_c = \frac{1}{C}\int_0^t idt \tag{2.17}$$

From equation (2.16), it is clear that for an abrupt change of voltage across the capacitor, the current becomes infinite. Also, from equation (2.17), it is observed that for a finite change of current in zero time the integral must be zero.

Therefore, *the voltage acorss a capacitor cannot change instantaneously.*

Explanation of Initial Voltage $v_c(0)$ It is possible that this capacitor might have been used in some other circuit earlier, where it absorbed some energy and then it was disconnected. Because of its non-dissipative nature, the energy was stored within the capacitor. Now, as this capacitor is connected to a circuit, it gets some path to release its stored energy. Here, this stored energy is represented by the initial voltage $v_c(0)$.

Energy Stored in Capacitors The energy (measured in joule) stored in a capacitor is equal to the *work* done to charge it. Consider a capacitance C, holding a charge $+q$ on one plate and $-q$ on the other. Moving a small element of charge dq from one plate to the other against the potential difference $V = q/C$ requires the work dW.

$$dW = \frac{q}{C}dq \tag{2.18}$$

where, W is the work measured in joule
q is the charge measured in coulomb
C is the capacitance, measured in farad

We can find the energy stored in a capacitance by integrating this equation. Starting with an uncharged capacitance ($q = 0$) and moving charge from one plate to the other until the plates have charge $+Q$ and $-Q$ requires the work W.

$$W_{\text{charging}} = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = W_{\text{stored}} \tag{2.19}$$

Combining this with the Eq. (2.14) for the capacitance of a flat-plate capacitor, we get

$$W_{\text{stored}} = \frac{1}{2}CV^2 = \frac{1}{2}\varepsilon\frac{A}{d}V^2 \tag{2.20}$$

where W is the energy measured in joule,
C is the capacitance, measured in farad,
V is the voltage measured in volt.

Series or Parallel Arrangements of Capacitors Capacitors in a parallel configuration each have the same potential difference (voltage). Their total capacitance (C_{eq}) is given by

$$C_{eq} = C_1 + C_2 + \ldots + C_n$$

The reason for putting capacitors in parallel is to increase the total amount of charge stored. In other words, increasing the capacitance also increases the amount of energy that can be stored. Its expression is

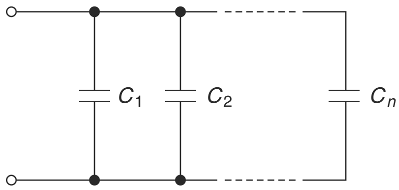


Figure 2.5 *Parallel arrangement of capacitors*

$$E_{\text{stored}} = \frac{1}{2}CV^2 \tag{2.21}$$

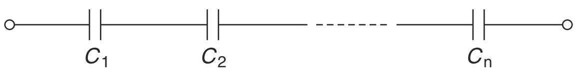


Figure 2.6 *Series arrangement of capacitors*

The current through capacitors in series stays the same, but the voltage across each capacitor can be different. The sum of the potential differences (voltage) is equal to the total voltage. Their total capacitance is given by

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n} \tag{2.22}$$

In parallel, the effective area of the combined capacitor has increased, increasing the overall capacitance. In series, the distance between the plates has effectively been increased, reducing the overall capacitance.

Voltage Division by Capacitances *In series connection* When a total voltage $\boldsymbol{E_S}$ is applied to series connected capacitances $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$, the charge $\boldsymbol{Q_S}$ which accumulates in the series circuit is:

$$\boldsymbol{Q_S = i_S\, dt = E_S C_S = E_S C_1 C_2/(C_1 + C_2)}$$

The voltages $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ which appear across the respective capacitances $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ are

$$\boldsymbol{V_1 = i_S\, dt/C_1 = E_S C_S/C_1 = E_S C_2/(C_1 + C_2)}$$
$$\boldsymbol{V_2 = i_S\, dt/C_2 = E_S C_S/C_2 = E_S C_1/(C_1 + C_2)}$$

In general terms, for capacitances $\boldsymbol{C_1, C_2, C_3}$, ... connected in series

$$\boldsymbol{Q_S = i_S\, dt = E_S C_S = E_S/(1/C_S) = E_S/(1/C_1 + 1/C_2 + 1/C_3 + \ldots)}$$
$$\boldsymbol{V_n = i_S\, dt/C_n = E_S C_S/C_n = E_S/C_n(1/C_S) = E_S/C_n(1/C_1 + 1/C_2 + 1/C_3 + \ldots)}$$

Note that the highest voltage appears across the lowest capacitance.

In parallel connection: When a voltage $\boldsymbol{E_P}$ is applied to parallel connected capacitances $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$, the charge $\boldsymbol{Q_P}$ which accumulates in the parallel circuit is

$$\boldsymbol{Q_P = i_P dt = E_P C_P = E_P(C_1 + C_2)}$$

The charges $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$ which accumulate in the respective capacitances $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ are:

$$\boldsymbol{Q_1 = i_1 dt = E_P C_1 = Q_P C_1/C_P = Q_P C_1/(C_1 + C_2)}$$
$$\boldsymbol{Q_2 = i_2 dt = E_P C_2 = Q_P C_2/C_P = Q_P C_2/(C_1 + C_2)}$$

In general terms, for capacitances $\boldsymbol{C_1}$, $\boldsymbol{C_2}$, $\boldsymbol{C_3}$, … connected in parallel:

$$\boldsymbol{Q_P = i_P dt = E_P C_P = E_P(C_1 + C_2 + C_3 + \dots)}$$
$$\boldsymbol{Q_n = i_n dt = E_P C_n = Q_P C_n / C_P = Q_P C_n / (C_1 + C_2 + C_3 + \dots)}$$

Note that the highest charge accumulates in the highest capacitance.

2.5.3 Inductance

An electric current i flowing round a circuit produces a magnetic field and hence a magnetic flux Φ through the circuit. The ratio of the magnetic flux to the current is called the **inductance**, or more accurately **self-inductance** of the circuit. It is customary to use the symbol L for inductance, possibly in honour of the physicist Heinrich Lenz. The quantitative definition of the inductance is, therefore,

$$L = \frac{\phi}{i} \tag{2.23}$$

It follows that the SI unit for inductance is Webbers per ampere. In honour of Joseph Henry, the unit of inductance has been given the name **Henry (H)**: 1 H = 1 Wb/A.

In the above definition, the magnetic flux φ is that caused by the current flowing through the circuit concerned. There may, however, be contributions from other circuits. Consider, for example, two circuits C_1, C_2, carrying the currents i_1, i_2. The magnetic fluxes Φ_1 and Φ_2 in C_1 and C_2, respectively, are given by

$$\Phi_1 = L_{11}i_1 + L_{12}i_2,$$
$$\Phi_2 = L_{21}i_1 + L_{22}i_2$$

According to the above definition, L_{11} an L_{22} are the self-inductances of C_1 and C_2, respectively. It can be shown (see below) that the other two coefficients are equal: $L_{12} = L_{21} = M$, where M is called the **mutual inductance** of the pair of circuits.

Inductor An **inductor** is a passive electrical device employed in electrical circuits for its property of inductance.

Properties of Inductance The equation relating inductance and flux linkages can be rearranged as follows.

$$\lambda = Li \tag{2.24}$$

Taking the time derivative of both sides of the equation yields

$$\frac{d\lambda}{dt} = L\frac{di}{dt} + i\frac{dL}{dt}$$

In most physical cases, the inductance is constant with time and so

$$\frac{d\lambda}{dt} = L\frac{di}{dt}$$

By Faraday's Law of Induction,we have

$$\frac{d\lambda}{dt} = -E = v$$

where E is the Electromotive force (emf) and v is the induced voltage. Note that the emf is opposite to the induced voltage. Thus

$$v = L\frac{di}{dt} \tag{2.25}$$

or $$i(t) = \frac{1}{L}\int_0^t v(t)dt + i(0)$$

where $i(0)$ is the initial current. When initial current is zero,

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt \tag{2.26}$$

These equations together state that, for a steady applied voltage v, the current changes in a linear manner, at a *rate* proportional to the applied voltage, but inversely proportional to the inductance. Conversely, if the current through the inductor is changing at a constant rate, the induced voltage is constant.

From equation (2.25), it is clear that for an abrupt change in current, the voltage across the inductor becomes infinite. Also, from equation (2.26), it is observed that for a finite change in voltage in zero time the integral must be zero.

Therefore, *the current through an inductor cannot change instantaneously.*

Explanation of Initial Current i(0) It is possible that this inductor might have been used in some other circuit earlier, where it absorbed some energy and then it was disconnected. Because of its non-dissipative nature, the energy was stored within the inductor core. Now, as this inductor is connected to a circuit, it gets some path to release its stored energy. Here, this stored energy is represented by the initial current $i(0)$.

Series and Parallel Arrangement of Inductors Inductors *in a parallel* configuration each have the same potential difference (voltage). To find their total equivalent inductance (L_{eq}):

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n} \tag{2.27}$$

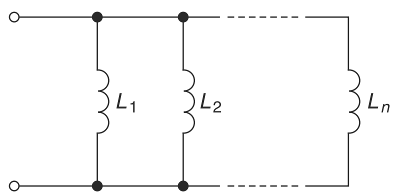


Figure 2.7 *Parallel arrangement of inductors*

The current through inductors *in series* stays the same, but the voltage across each inductor can be different. The sum of the potential differences (voltage) is equal to the total voltage. To find their total inductance:

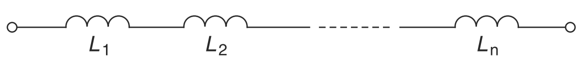


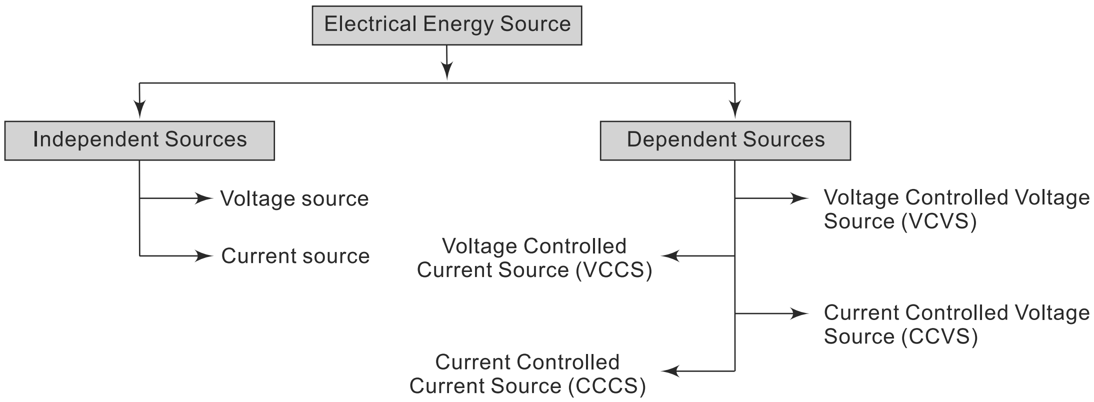
Figure 2.8 *Series arrangement of inductors*

$$L_{eq} = L_1 + L_2 + \ldots + L_n \tag{2.28}$$

These simple relationships hold true only when there is no mutual coupling of magnetic fields between individual inductors.

2.6 TYPES OF ELECTRICAL ENERGY SOURCES

Energy source is defined as the device that generates electrical energy. They are classified according to the current voltage characteristics. The classification is given below.



Independent Voltage Source An ideal voltage source has the following features.

(i) It is a voltage generator whose output voltage remains absolutely constant whatever be the value of the output current.

(ii) It has zero internal resistance so that voltage drop in the source is zero.

(iii) The power drawn by the source is zero.

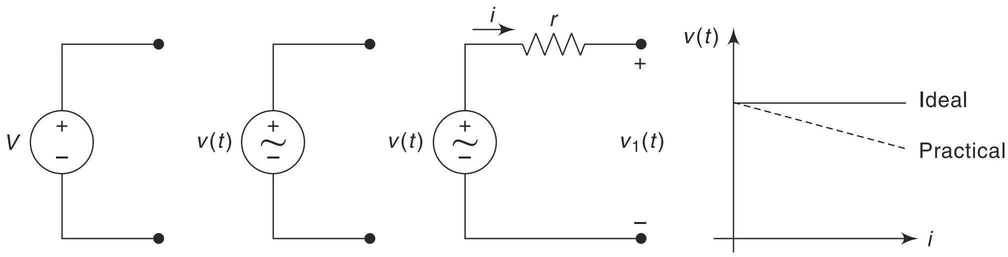


Figure 2.9 *Independent voltage sources and their characteristics*

In practical, the voltage does not remain constant, but falls slightly. This is taken care of by connecting a small resistance (r) in series with the ideal source. In this case, the terminal voltage will be,

$$v_1(t) = v(t) - ir$$

i.e., it will decrease with increase in current i.

Independent Current Source An ideal current source has the following features.

(i) It produces a constant current irrespective of the value of the voltage across it.

(ii) It has infinity resistance.

(iii) It is capable of supplying infinity power.

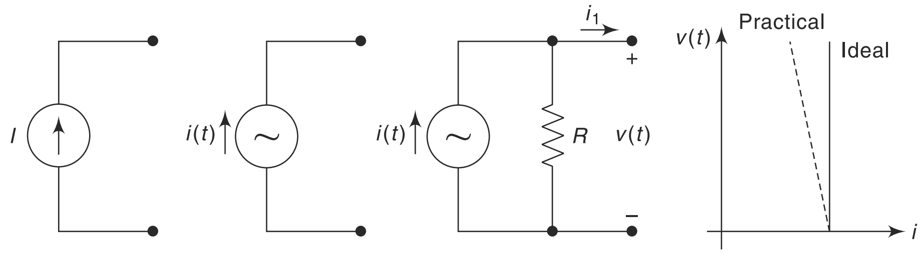


Figure 2.10 *Independent current sources and their characteristics*

In practice, the output current does not remain constant but decreases with increase in voltage. So, a practical current source is represented by an ideal current source in parallel with a high resistance (R) and the output current becomes,

$$i_1(t) = i(t) - \frac{v(t)}{R}$$

Dependent Sources In dependent sources (also referred as controlled sources), the source voltage or current is not fixed, but is dependent on a voltage or current at some other location in the circuit. Thus, there are four types of dependent sources.

(a) Voltage Controlled Voltage Source (VCVS)
(b) Current Controlled Voltage Source (CCVS)
(c) Voltage Controlled Current Source (VCCS)
(d) Current Controlled Current Source (CCCS)

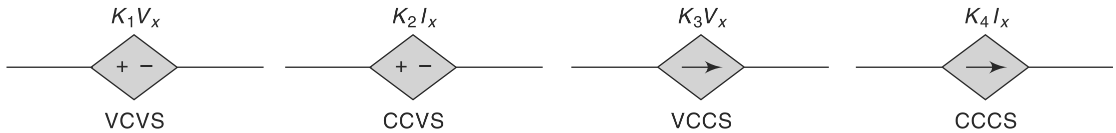


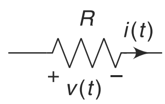
Figure 2.11 *Symbols of dependent sources*

2.7 FUNDAMENTAL LAWS

The fundamental laws that govern electric circuits are the Ohm's Law and the Kirchhoff's Laws.

2.7.1 Ohm's Law

Ohm's Law states that the voltage $v(t)$ across a resistor R is directly proportional to the current $i(t)$ flowing through it.



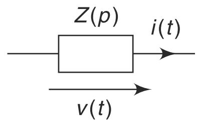
$$v(t) \propto i(t)$$

or $$v(t) = R \cdot i(t)$$

This general statement of Ohm's Law can be extended to cover inductances and capacitors as well under alternating current conditions and transient conditions. This is then known as the Generalized Ohm's Law. This may be stated as

$$v(t) = Z(p) \cdot i(t) \text{ , where } p = d/dt = \text{differential operator}$$

$Z(p)$ is known as the impedance function of the circuit, and the above equation is the differential equation governing the behaviour of the circuit.



For a resistor, $Z(p) = R$

For an inductor $Z(p) = L\,p$

For a capacitor, $Z(p) = \dfrac{1}{Cp}$

In the particular case of alternating current, $p = j\omega$, so that the equation governing circuit behaviour may be written as

$$V = Z(j\omega)\,.\,I, \text{ and}$$

For a resistor, $Z(j\omega)) = R$

For an inductor, $Z(j\omega) = j\omega L$

For a capacitor, $Z(j\omega) = \dfrac{1}{j\omega C}$

2.7.2 Kirchhoff's Current Law (KCL)

Kirchhoff's current law is based on the principle of conservation of charge. This requires that the algebraic sum of the charges within a system cannot change. Thus, the total rate of change of charge must add up to zero. Rate of change of charge is current.

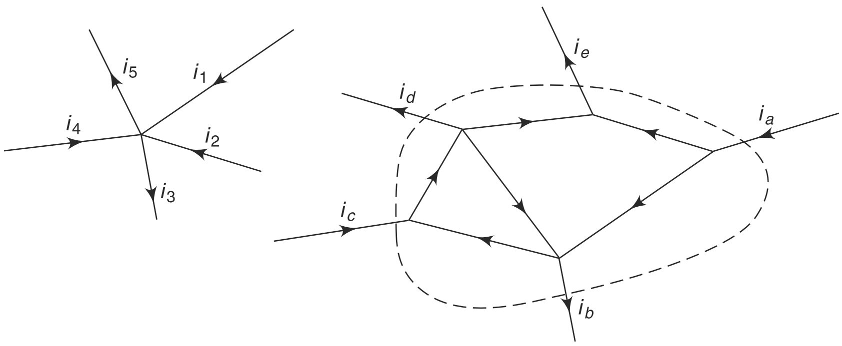


Figure 2.12 *Illustration of KCL*

This gives us our basic Kirchhoff's current law as the algebraic sum of the currents meeting at a point is zero,. i.e., at a node, $\sum I_n = 0$, where I_n are the currents in the branches meeting at the node.

This is also sometimes stated as the sum of the currents entering a node is equal to the sum of the currents leaving the node.

The theorem is applicable not only to a node, but to a closed system.

$i_1 + i_2 - i_3 + i_4 - i_5 = 0$. Also for the closed boundary, $i_a - i_b + i_c - i_d - i_e = 0$.

2.7.3 Kirchhoff's Voltage Law (KVL)

Kirchhoff's voltage law is based on the principle of conservation of energy. This requires that the total work done in taking a unit positive charge around a closed path and ending up at the original point is zero.

This gives us our basic Kirchhoff's law voltage as the algebraic sum of the potential differences taken round a closed loop is zero.
i. e., around a loop, $\Sigma V_n = 0$, where V_n are the voltages across the branches in the loop.

$$v_a + v_b + v_c + v_d - v_e = 0$$

This is also sometimes stated as the sum of the emfs taken around a closed loop is equal to the sum of the voltage drops around the loop.

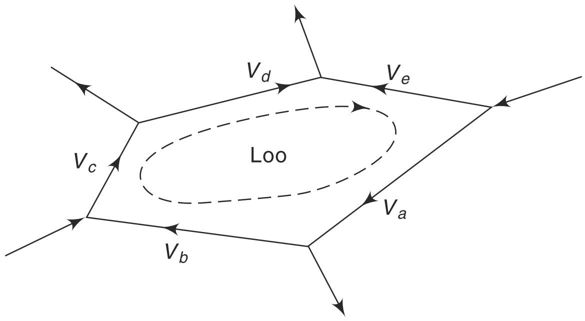


Figure 2.13 *Illustration of KVL*

Although all circuits could be solved using only Ohm's Law and Kirchhoff's laws, the calculations would be tedious. Various network theorems have been formulated to simplify these calculations.

- ***Sign Conventions for applying Kirchhoff's Laws***
  1. When tracing through a voltage source from positive to negative terminal, the voltage should be given a positive sign.
  2. When tracing through a voltage source from negative to positive terminal, the voltage should be given a negative sign.
  3. When tracing through a resistance in the direction of current flow, the voltage should be given a positive sign.
  4. When tracing through a resistance in a direction opposite to the direction of current flow, the voltage should be given a negative sign.

2.8 SOURCE TRANSFORMATION

Transformation of several voltage (or current) sources into a single voltage (or current) source and a voltage source into a current source or vice-versa is known as source transformation. This makes circuit analysis easier.

There are some rules of source transformation.

Rule (1) Several voltage sources $\{V_1(t), V_2(t), \ldots, V_n(t)\}$ connected in series will be replaced by a single voltage source of value $V = V_1(t) + V_2(t) + \ldots + V_n(t)$. Similarly, a number of current sources

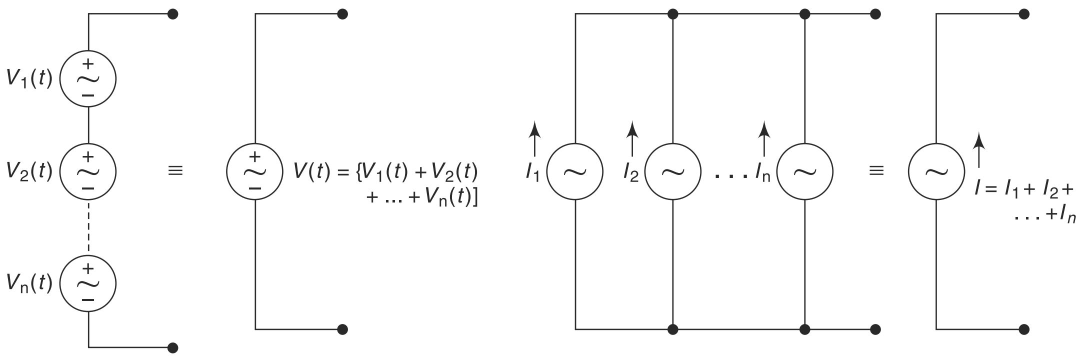


Figure 2.14 *Source transformation technique: Rule (1)*

$\{I_1(t), I_2(t), \ldots, I_n(t)\}$ connected in parallel is replaced by a single current source of value $I(t) = I_1(t) + I_2(t) + \ldots + I_n(t)$.

Rule (2) A number of voltage sources $V_1(t), V_2(t), \ldots, V_n(t)$ in parallel will result in a single voltage source, $V(t) = V_1(t) = V_2(t) = \ldots = V_n(t)$.

Therefore, *voltage sources should not be connected in parallel unless they have identical potential,* as paralleling of sources with non-similar potential waveforms will result in heavy current, which may damage the equipment.

Similarly, a number of current sources $I_1(t), I_2(t), \ldots, I_n(t)$ in series will result in a single current source of value $I(t) = I_1(t) = I_2(t) = \ldots = I_n(t)$ and thus, *current sources cannot be connected in series if they are not identical*.

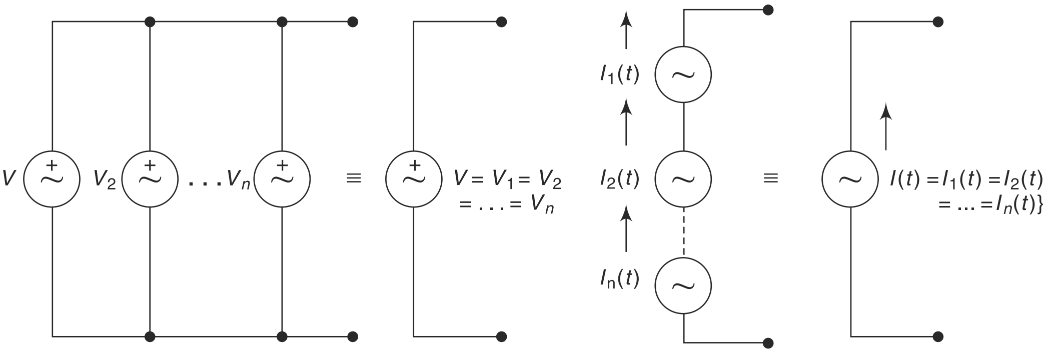


Figure 2.15 *Source transformation technique: Rule (2)*

Rule (3) As far as the computations in the remainder of the network are concerned, a resistor in parallel with an ideal voltage source and a resistor in series with an ideal current source may be ignored.

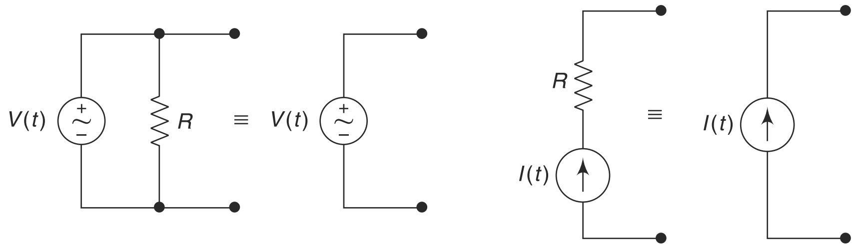


Figure 2.16 *Source transformation technique: Rule (3)*

Rule (4) A voltage source $V(t)$ in series with a resistor R can be converted into a current source $I(t)$ in parallel with the same resistor R, where, $I(t) = \dfrac{V(t)}{R}$.

Similarly, a voltage source $V(t)$ in series with a capacitor C may be converted into a current source $I(t)$ in parallel with C, where, $I(t) = C\dfrac{dV(t)}{dt}$; and a voltage source $V(t)$ in series with an inductor L may be converted into a current source $I(t)$ in parallel with L, where, $I(t) = \dfrac{1}{L}\int V(t)dt$

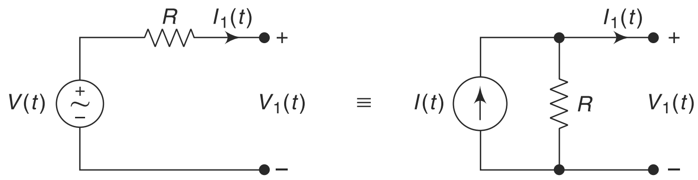


Figure 2.17 *Source transformation technique: Rule (4)*

2.9 NETWORK ANALYSIS TECHNIQUES

Network analysis is the determination of the response output of a network when the input excitation is given. There are two techniques of network analysis.

1. Nodal Analysis
2. Loop or Mesh Analysis

Nodal Analysis It is based on Kirchhoff's current law (KCL). In this method, the unknown variables are the node voltages. It is generally used when the circuit contains several current sources.

Steps

- If there is N number of nodes in a network, all nodes are labeled. One node is treated as *datum or reference node* (zero potential) and the other node voltages are treated as unknowns to be determined with respect to this reference.
- KCL is written at each node in terms of node voltages.
  - KCL is applied at $N - 1$ of the N nodes of the circuit using assumed current directions, as necessary. This will create $N - 1$ linearly independent equations, known as *node equations*.
  - In a circuit with independent voltage sources, if two nodes of interest are separated by a voltage source instead of a resistor or current source, then the concept of *supernode* is used that creates constraint equations.
  - The current is computed based on voltage difference between two nodes. The current in any branch is obtained via ohm's law as,

$$i = \frac{V_{mm}}{R} = \frac{V_m - V_m}{R}\text{, for D.C.}$$

$$l = \frac{V_{mm}}{Z} = \frac{V_m - V_m}{Z}\text{, for A.C.}$$

where, $V_m > V_n$ and current flows from node m to n.

- Solution of the $N - 1$ simultaneous equations (by Gaussian elimination or matrix method) gives the unknown node voltages.

Example 2.1 Let node voltages are E_1, E_2 and E_3 at node-1, 2 and 3 respectively.

At node –1,

$$I_1 = I_3 + I_4$$

$$I_1 = \frac{E_1}{R_1} + \frac{(E_1 - E_2)}{R_2}$$

$$I_1 = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)E_1 - \frac{E_2}{R_2} \qquad \text{(i)}$$

At node-2,

$I_4 = I_5 + I_6$

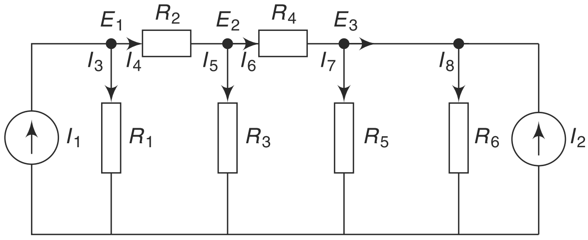


Figure 2.18 *Network explaining node analysis technique*

$$\frac{(E_1 - E_2)}{R_2} = \frac{(E_2 - E_3)}{R_4} + \frac{E_2}{R_3}$$

$$0 = -\frac{E_1}{R_2} + E_2\left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) - \frac{E_3}{R_4} \qquad \text{(ii)}$$

At node-3,

$$I_6 = I_7 + I_8 - I_2$$

$$\frac{(E_2 - E_3)}{R_4} = \frac{E_3}{R_5} + \frac{E_3}{R_6} - I_2$$

$$I_2 = -\frac{E_2}{R_4} + E_3\left(\frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right) \qquad \text{(iii)}$$

Given the other values, solution of Equations (i), (ii), and (iii) gives the values of E_1, E_2 and E_3.

Concept of Supernode This concept is used when a circuit contains voltage sources. A supernode is formed by enclosing a dependent or independent voltage source connected between two non-reference nodes and any elements connected in parallel with it. This concept is necessary for nodal analysis with voltage source, because the current through a voltage source is unknown. We consider the following two cases.

Case 1 *When a voltage source is connected between the reference node and a non-reference node:* In this case, the voltage of the non-reference node is taken equal to the voltage of the voltage source. For the circuit shown in Fig. 2.19(a),

$$V_1 = 5 \text{ V} \qquad \text{(i)}$$

Case 2 *When a voltage source is connected between two non-reference nodes:*
In this case, a supernode is considered enclosing the non-reference nodes. Both KCL and KVL is written for the supernode.

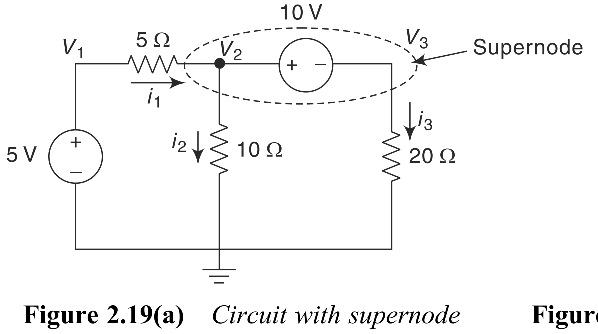


Figure 2.19(a) *Circuit with supernode*

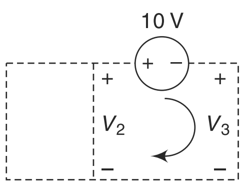


Figure 2.19(b) *KVL with supernode*

For this example, nodes 2 and 3 are forming the supernode.

By KCL at the supernode, $i_1 = i_2 + i_3$

or $$\frac{V_1 - V_2}{5} = \frac{V_2 - 0}{10} + \frac{V_3 - 0}{20} \quad \text{(ii)}$$

To apply KVL to the supernode, the circuit is drawn as shown in Fig. 2.19(b). By KVL,

$$10 + V_3 - V_2 = 0 \quad \text{(iii)}$$

Solving equations (i), (ii) and (iii), the node voltages are obtained, $V_1 = 5$ V, $V_2 = 4.2857$ V, $V_3 = -5.7143$ V.

Properties of Supernode

(i) It provides the constraint equations.
(ii) Both KCL and KVL are written for supernode.
(iii) A supernode does not have any voltage of its own.

Loop or Mesh Analysis It is based on Kirchhoff's voltage law (KVL). In this method, the unknown variables are the loop currents. It is generally used when the circuit contains several voltage sources.

Steps

- If there is 'N' number of loops/meshes in a network, all loops are labeled.
- KVL is written at each loop/mesh in terms of loop/mesh currents. *Loop currents* are those currents flowing in a loop; they are used to define *branch currents*.
  - For N independent loops, total N equations are written using KVL around each loop. These equations are known as *loop/mesh equations*.
  - The concept of *supermesh* is used in case a circuit contains current source that provides the constraint equations.
- Solution of the N simultaneous equations gives the required loop/mesh currents.

Example 2.2 Two meshes are labeled as mesh-1 and mesh-2.

Applying KVL for mesh-1,

$$V_s = R_1 I_1 + R_2(I_1 - I_2) \quad \text{(i)}$$

By constraint equation,

$$I_2 = -I_s \quad \text{(ii)}$$

Solving the equations, we get I_1 and I_2.

Concept of Supermesh This concept is used when a circuit contains current sources. A supermesh is formed by excluding the branch containing a dependent or independent current source connected in common to two meshes and any elements connected in series with it. This concept is necessary for loop analysis with current source, because the voltage drop across a current source is unknown. We consider the following two cases.

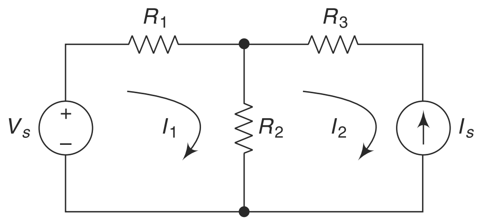


Figure 2.20 *Circuit explaining loop analysis technique*

Case-1 *When a current source is in one mesh:*

In this case, the mesh current is taken equal to the current of the current source. For example, for the circuit shown in Fig. 2.21,

$$i_2 = -10 \text{ A}$$

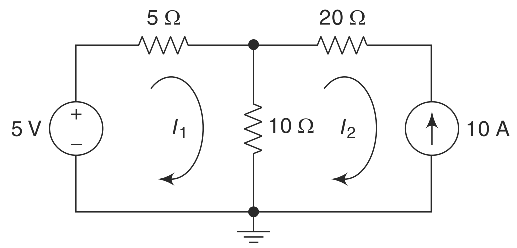


Figure 2.21 *Current source in one mesh*

Case-2 *When a current source is connected between two meshes:*

In this case, a supermesh is considered excluding the branch with the current source and any elements connected in series with it. Both KCL and KVL is written for the supermesh. For example, consider the circuit shown in Fig. 2.22. Supermesh is formed by excluding the branch with 3A current source.

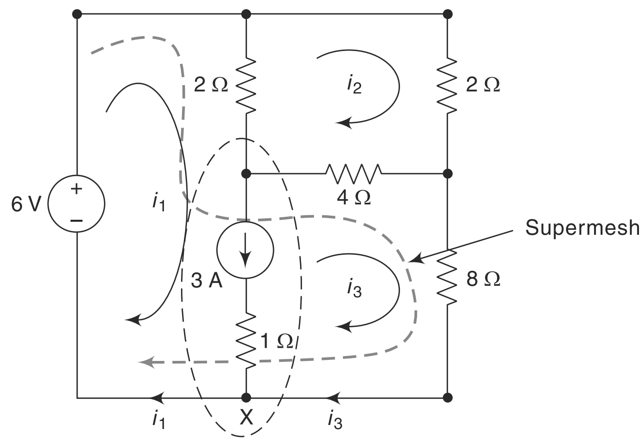


Figure 2.22 *Current source connected between two meswhes*

By KVL for the supermesh,

$$2(i_1 - i_2) + 4(i_3 - i_2) + 8i_3 = 6 \quad \text{(i)}$$

By KCL at any one node of the omitted branch (say, *X*),

$$i_1 = 3 + i_3 \quad \text{(ii)}$$

Also by KVL for second mesh,

$$2i_2 + 4(i_2 - i_3) + 2(i_2 - i_1) = 0 \quad \text{(iii)}$$

Solving equations (i), (ii) and (iii), the mesh currents are obtained, i_1 = 3.437A, i_2 = 1.1052A, i_3 = 0.4737A.

Properties of Supermesh

(i) It provides the constraint equations.

(ii) Both KCL and KVL are written for supermesh.

(iii) A supermesh does not have any current of its own.

Comparison of Loop and Node Analysis In any network having *N* nodes and *B* branches, there are 2*B* unknowns, i.e., *B*–branch currents and *B*–branch voltages. These unknowns can be determined either by loop analysis or nodal analysis.

The choice of the method depends on two factors given below.

1. *Nature of the network* The mesh-method is generally used for circuits having many series-connected elements, voltage sources, or supermeshes. On the other hand, nodal analysis is more suitable for circuits having many parallel-connected elements, current sources, or supernodes.

   The main factor for selecting any one method is the *minimum number of equations*. If a circuit is having fewer nodes than meshes, then nodal analysis is used, while if a circuit with fewer meshes than nodes, then loop method is used.

2. *Requirement of the problem* If node voltages are required, nodal analysis is used. If branch/ mesh currents are required, loop analysis is used.

However, there are some particular circuits, where only one method can be applied. For example, in analyzing transistor circuits, mesh method is the only possible method; while for op-amp circuits and for non-planar networks, node method is the only possible method.

2.10 DUALITY

Duality is a transformation in which currents and voltages are interchanged. Two phenomena are said to be dual if they are described by equations of the same mathematical form.

There are a number of similarities and analogies between the two circuit analysis techniques based on loop-current method and node voltage method. The principal quantities and concepts involved in these two methods based on KVL and KCL are dual of each other with voltage variables substituted by current variables, independent loop by independent node-pair, etc.

This similarity is termed as 'principle of duality'.

Some dual relations are:

$$v = Ri \qquad i = Gv$$

$$v = L\frac{di}{dt} \qquad i = C\frac{dv}{dt}$$

$$v = \frac{1}{C}\int idt \qquad i = \frac{1}{L}\int vdt$$

Thus, the circuit elements (R, L, C) have some dual relationship. Duality also appears as relation between two networks. For example, an RLC series circuit with voltage excitation is dual of an RLC parallel circuit with current excitation.

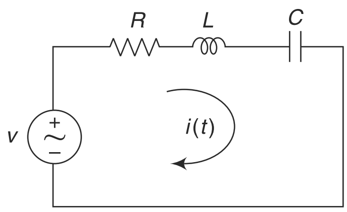


Figure.23(a) *Series RLC Circuit*

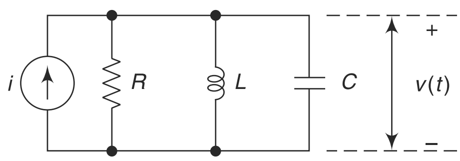


Figure 2.23(b) *Parallel RLC Circuit*

For series circuit, $v = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$

For parallel circuit, $i = Gv + C\frac{dv}{dt} + \frac{1}{L}\int vdt$

Dual Quantities and Concepts

Sl No.	*Quantity/Concept*	*Dual*
1	Current	Voltage
2	Resistance	Conductance
3	Inductance	Capacitance
4	Impedance	Admittance
5	Reactance	Susceptance
6	Branch current	Branch voltage
7	Mesh or Loop	Node or Node-pair
8	Mesh Current or Loop Current	Node Voltage or Node-pair Voltage
9	Link	Tree Branch
10	Link Current	Tree Branch Voltage
11	Tree Branch Current	Link Voltage
12	Tie-set	Cut-set
13	Short-circuit	Open-circuit
14	Parallel Paths	Series Paths

Construction of Dual of a Network

1. A dot is placed inside each independent loop of the given network. These dots correspond to the non-reference nodes of the dual network.
2. A dot is placed outside the network. This dot corresponds to the datum node.
3. All internal dots are connected by dashed lines crossing the common branches and placing the elements which are duals of the elements the original network.
4. All internal dots are connected to the external dot by dashed lines crossing all external branches and placing dual elements of the external branch.

Conventions for Reference Polarities of Voltage Source and Reference Directions of Current Source

(i) A clockwise current in a loop corresponds to a positive polarity (with respect to reference node) at the dual independent node.

(ii) A voltage rise in the direction of a clockwise loop current corresponds to a current flowing towards the dual independent nodes.

Finally, the dual construction can be checked by writing mesh equations and node equations of two networks.

Example 2.3 Draw the dual of the network shown in figure.

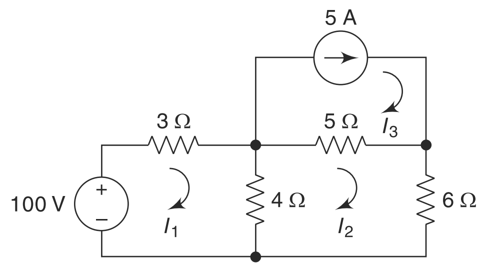


Figure 2.24 *Circuit of example*

Solution Following the steps, dual network is drawn.
Therefore, the dual network becomes as shown in Fig. 2.26.
By KVL to the original network,

$$I_1(3 + 4) - I_2(4) = 100$$

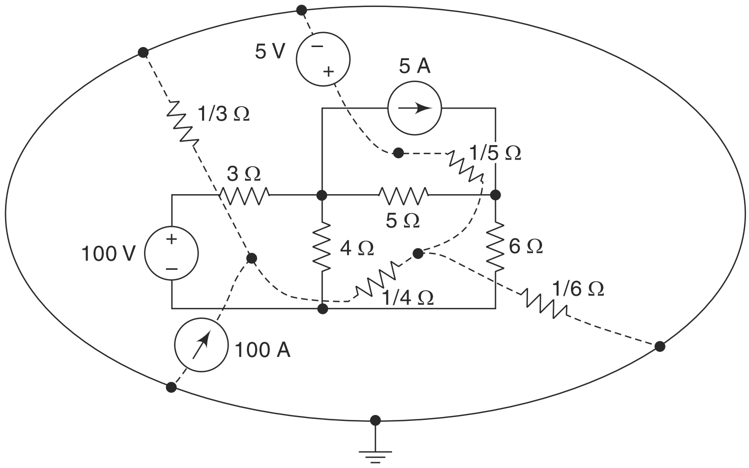


Figure 2.25 *Figure explaining drawing dual of network of Fig. 2.24*

$-I_1(4) + I_2(4 + 5 + 6) - 5I_3 = 0$
$I_3 = 5$

The dual equations will be,

$V_1(3 + 4) - V_2(4) = 100$
$-V_1I(4) + V_2(4 + 5 + 6) - 5V_3 = 0$
$V_3 = 5$

These equations satisfy the dual network.

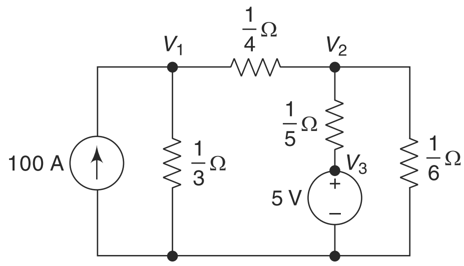


Figure 2.26 *Dual of network of Fig. 2.24*

2.11 STAR-DELTA CONVERSION TECHNIQUE

The ***Y*-Δ transform**, also written ***Y*-delta**, **Wye-delta**, Kennelly's **delta-star transformation**, **star-mesh transformation**, ***T*-Π** or ***T*-pi transform**, is a mathematical technique to simplify the analysis

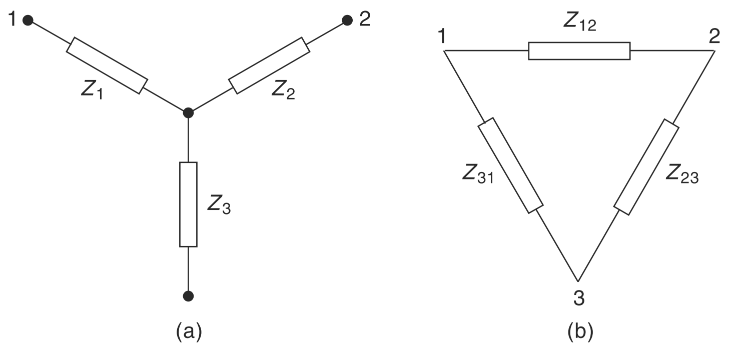


Figure 2.27 *(a) Star connection (b) Delta connection*

of an electrical network. The name derives from the shapes of the circuit diagrams, which look respectively like the letter Y and the Greek capital letter Δ.

The transformation is used to establish equivalence for networks with three terminals. For equivalence, the impedance between any pair of terminals must be the same for both networks.

For the star connection, the impedance between terminals 1 and 2 is $Z_1 + Z_2$.

For delta connection, the the impedance between terminals 1 and 2 is

$$Z_{12} \,||\, (Z_{23} + Z_{31}) = \frac{Z_{12}(Z_{23} + Z_{31})}{Z_{12} + Z_{23} + Z_{31}}$$

As the impedance between terminals 1 and 2 should be same, therefore,

$$Z_1 + Z_2 = \frac{Z_{12}(Z_{23} + Z_{31})}{Z_{12} + Z_{23} + Z_{31}} \tag{i}$$

Similarly, for terminals 2 and 3 we get,

$$Z_2 + Z_3 = \frac{Z_{23}(Z_{31} + Z_{12})}{Z_{23} + Z_{31} + Z_{12}} \tag{ii}$$

$$Z_3 + Z_1 = \frac{Z_{31}(Z_{12} + Z_{23})}{Z_{31} + Z_{12} + Z_{23}} \tag{iii}$$

Delta to Star Conversion

In this case, Z_1, Z_2, and Z_3 are to be written in terms of Z_{12}, Z_{23}, and Z_{31}.

By (i) – (ii) + (iii), we get $$\boxed{Z_1 = \frac{Z_{12}Z_{31}}{Z_{12} + Z_{23} + Z_{31}}} \tag{iv}$$

Similarly we get, $$\boxed{Z_2 = \frac{Z_{23}Z_{12}}{Z_{12} + Z_{23} + Z_{31}}} \tag{v}$$

and $$\boxed{Z_3 = \frac{Z_{31}Z_{23}}{Z_{12} + Z_{23} + Z_{31}}} \tag{vi}$$

Star to Delta Conversion

In this case, Z_{12}, Z_{23}, and Z_{31} are to be written in terms of Z_1, Z_2, and Z_3.

Let $Z = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$. Then from Eq. (iv) to Eq. (vi), we get

$$Z = \frac{Z_{12}Z_{23}^2Z_{31}}{(Z_{12} + Z_{23} + Z_{31})^2} + \frac{Z_{12}Z_{23}Z_{31}^2}{(Z_{12} + Z_{23} + Z_{31})^2} + \frac{Z_{12}^2Z_{23}Z_{31}}{(Z_{12} + Z_{23} + Z_{31})^2} = \frac{Z_{12}Z_{23}Z_{31}}{Z_{12} + Z_{23} + Z_{31}} \tag{vii}$$

From Eq. (vii) and Eq. (iv), we get $Z = Z_{12}Z_3 \quad \Rightarrow Z_{12} = \dfrac{Z}{Z_3}$

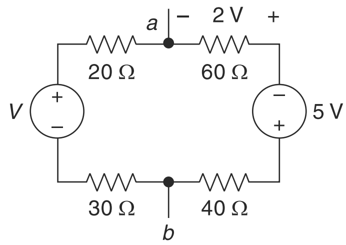
Therefore, $$\boxed{Z_{12} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3} = Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}}$$

Similarly, $$\boxed{Z_{23} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1} = Z_2 + Z_3 + \frac{Z_2Z_3}{Z_1}}$$

and $$\boxed{Z_{31} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2} = Z_3 + Z_1 + \frac{Z_3Z_1}{Z_2}}$$

SOLVED EXERCISES

2.1 Find the values of V, V_{ab} and the power delivered by the 5V source. All values of resistances are in ohm.

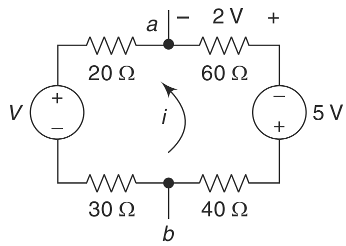


Solution

Current, $$i = \frac{2}{60} = \frac{1}{30} \text{ A}$$

By KVL,
$$20i + 2 + 5 + v + 70i = 0$$

$$v = -7 - 90i = -7 - 90 \times \frac{1}{30} = -10 \text{ V}$$

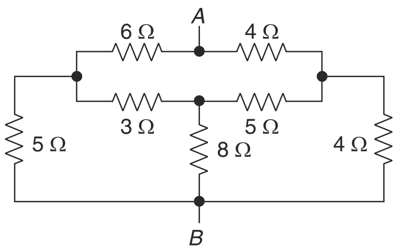


∴ $$v_{ab} = 20i + v + 30i = 50i - 10$$

$$= 50 \times \frac{1}{30} - 10 = -8.33 \text{ V}$$

Power drawn by the 5V source = – (Power taken source) = $-5 \times \frac{1}{30}$ = –0.166 W

2.2 Find the equivalent resistance between the terminals A and B of the circuit shown below.

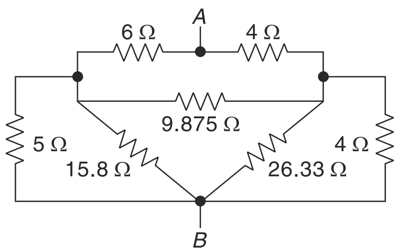


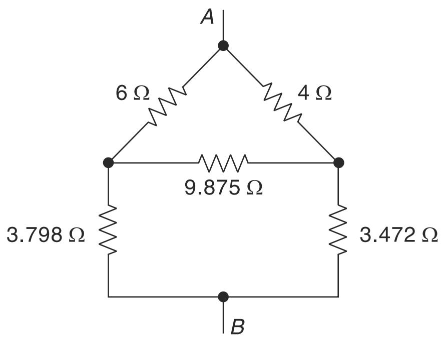
Solution Converting star into delta,

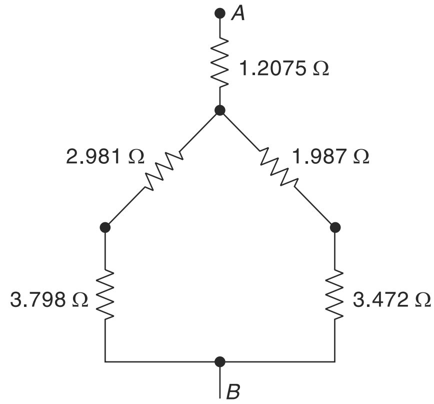
$$r_{12} = \left(r_1 + r_2 + \frac{r_1r_2}{r_3}\right) = 8 + \frac{15}{8} = 9.875\ \Omega$$

$$r_{23} = \left(r_2 + r_3 + \frac{r_2r_3}{r_1}\right) = 13 + \frac{40}{3} = 26.33\ \Omega$$

$$r_{31} = \left(r_3 + r_1 + \frac{r_3r_1}{r_2}\right) = 11 + \frac{24}{5} = 15.8\ \Omega$$







Combining the parallel connections of 5 Ω and 15.8 Ω and 4 Ω and 26.33 Ω, we have the reduced circuit.

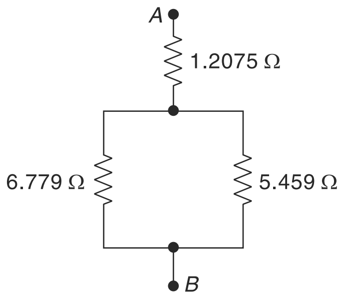
Again, converting the delta made of 6 Ω, 4 Ω and 9.875 Ω into equivalent star,

$$r_1 = \frac{r_{12} r_{31}}{r_1 + r_2 + r_3}$$

$$= \frac{6 \times 4}{19.875} = 1.2075\ \Omega$$

$$r_2 = \frac{4 \times 9.875}{19.875} = 1.987\ \Omega$$

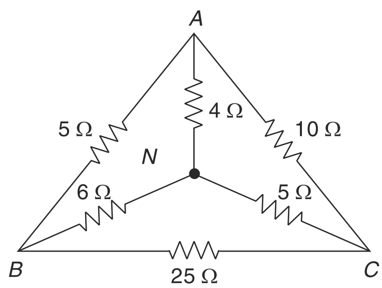
$$r_3 = \frac{6 \times 9.875}{19.875} = 2.981\ \Omega$$



So, the given circuit becomes as shown in figure.

$$\therefore \qquad R_{AB} = 1.2075 + \frac{6.779 \times 5.459}{6.779 + 5.459} = 4.23\ \Omega \qquad \textit{Ans.}$$

2.3 Find the equivalent resistance between
- (i) *A* and *B*,
- (ii) *B* and *C*,
- (iii) *C* and *A*, and
- (iv) *A* and *N* of the circuit shown.

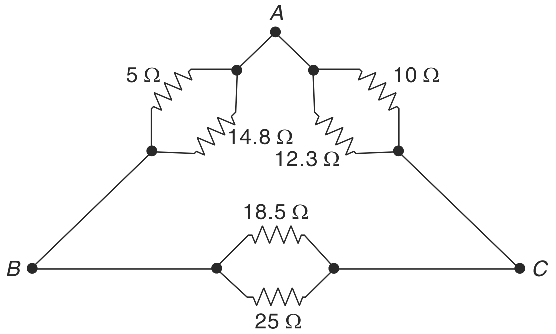


Solution Converting the star into delta,

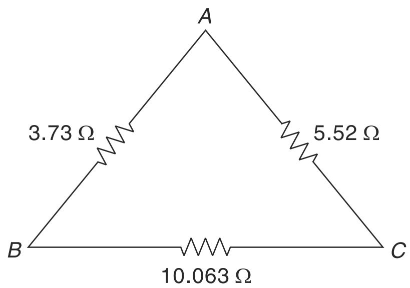
$$r_{12} = \left(r_1 + r_2 + \frac{r_1 r_2}{r_3} \right) = 4 + 5 + \frac{5 \times 4}{6} = 12.33\ \Omega$$

$$r_{23} = \left(r_2 + r_3 + \frac{r_2 r_3}{r_1} \right) = 5 + 6 + \frac{5 \times 6}{4} = 18.5\ \Omega$$

$$r_{31} = \left(r_3 + r_1 + \frac{r_3 r_1}{r_2} \right) = 6 + 4 + \frac{6 \times 4}{5} = 14.8\ \Omega$$



The circuit becomes, as shown in below figure.



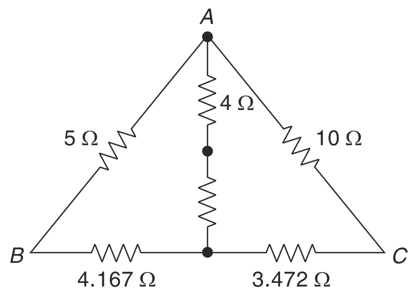
(i) Equivalent resistance between A and B,

$$R_{AB} = \frac{3.73 \times (10.06 + 5.52)}{3.70 + 10.06 + 5.52} = 3.035\ \Omega. \qquad \textit{Ans.}$$

(ii) $$R_{BC} = \frac{10.06 \times (3.73 + 5.52)}{10.06 + 3.73 + 5.52} = 4.82\ \Omega \qquad \textit{Ans.}$$

(iii) $$R_{CA} = \frac{5.52 \times (10.06 + 3.73)}{6.52 + 10.06 + 3.73} = 3.94\ \Omega \qquad \textit{Ans.}$$

(iv) Converting the delta into equivalent star,

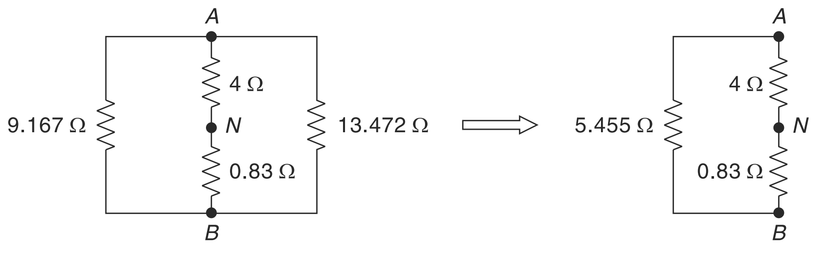


$$r_1 = \frac{5 \times 6}{5 + 6 + 25} = 0.83\ \Omega$$

$$r_2 = \frac{25 \times 6}{5 + 6 + 25} = 4.167\ \Omega$$

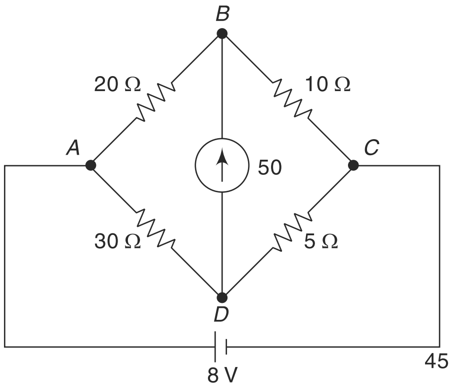
$$r_3 = \frac{25 \times 5}{5 + 6 + 25} = 3.472\ \Omega$$

The circuit becomes:

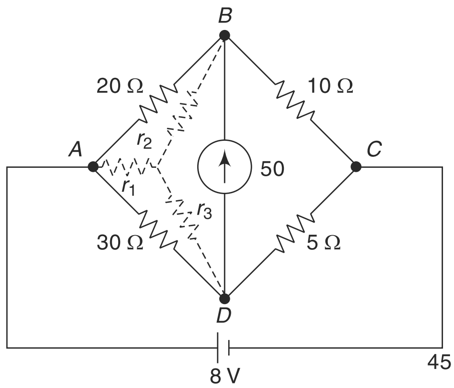


$$\therefore \qquad R_{AN} = \frac{4 \times 6.288}{4 + 6.288} = 2.4448\ \Omega \qquad \textit{Ans.}$$

2.4 Find the current through the galvanometer using delta-star conversion.



Solution Converting the delta consisting of 20 Ω, 30 Ω and 50 Ω, we get,

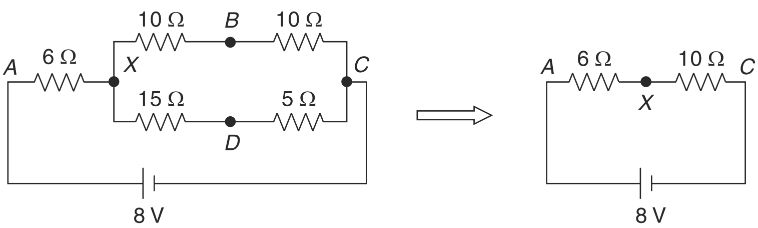


$$r_1 = \frac{20 \times 30}{20 + 30 + 50} = 6\ \Omega$$

$$r_2 = \frac{20 \times 50}{20 + 30 + 50} = 10\ \Omega$$

$$r_3 = \frac{30 \times 50}{20 + 30 + 50} = 15\ \Omega$$

$\therefore \qquad R_{AC} = 16\ \Omega$



Main current $i = \dfrac{8}{16} = 0.5$ A

Now, to calculate potential difference between the points *B* and *D* ;

$$V_{XC} = 10 \times 0.5 = 5 \text{ V}$$

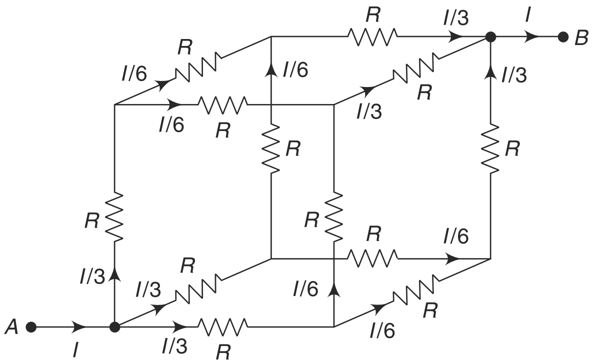
$\therefore \qquad V_{BD} = (10 \times 0.25 - 5 \times 0.25) = 1.25$ V

$\therefore$ Currant through the galvanometer, (50 (Ω)

$$i_G = \frac{1.25}{50} = 0.025 \text{ A}$$ *Ans.*

2.5 Twelve similar conductors each of *R* resistance form a cubical frame. Find the resistance across two opposite corners of the cube.

Solution The configuration is shown in figure.



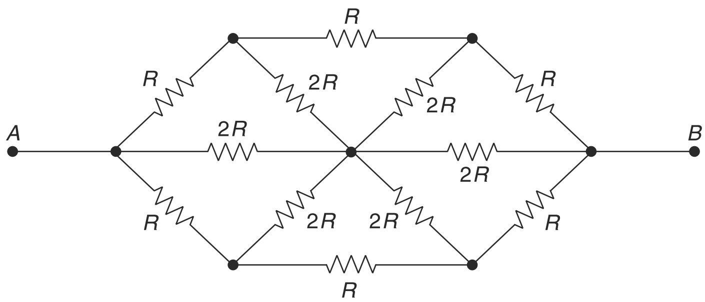
The current distribution is shown.

So, the total voltage drop between two opposite corners A and B for a total current of I is,

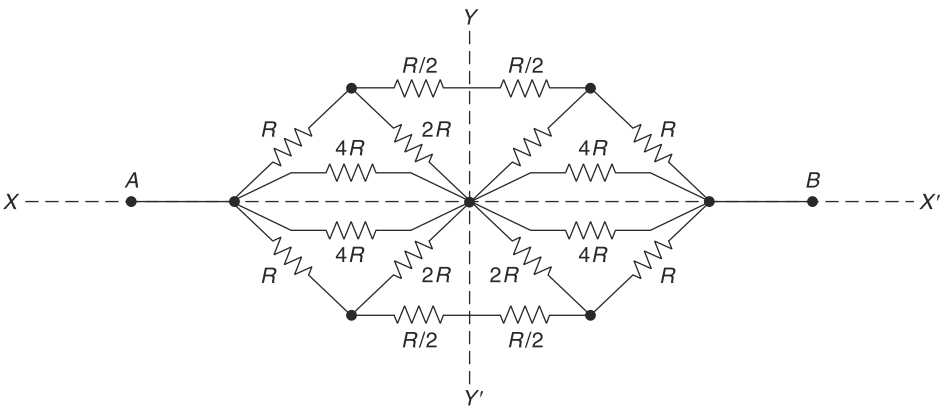
$$V_{AB} = R.\frac{I}{3} + R.\frac{I}{6} + R.\frac{I}{3} = R \times \frac{5}{6} I$$

Equivalent resistance, $R_{AC} = \frac{V_{AB}}{I} = \frac{5}{6} R$ *Ans.*

2.6 A regular hexagon is formed from 6 wires of R ohm each. The corners are joined to the centre by six more wires of $2R$ ohm each. Calculate the resistance of the hexagon between any two nodes diametrically opposite.



Solution The hexagon can be redrawn as shown.

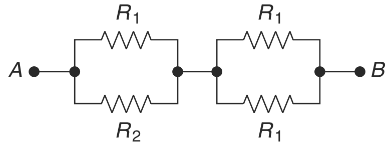


The hexagon is symmetrical about XX'

Equivalent resistance of the second quadrant,

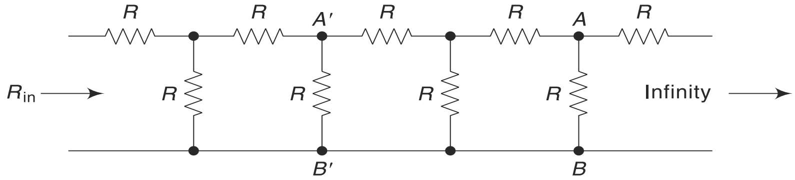
$$R_1 = (2R \,||\, R/2 + R) \,||\, 4R = \frac{28}{27}R$$

So, the figure is modified as,



$$\therefore \qquad R_{AB} = (R_1 \,||\, R_1) + (R_1 \,||\, R_1) = R_1 = \frac{28}{27}R \qquad \textit{Ans.}$$

2.7 Find the input resistance of the infinite section resistive network shown below.



Solution Let the equivalent resistance be R_{in}.

The network can be terminated at $A'\,B'$ instead of AB.

$$R_{A'B'} = [R + (R_{in}) \,||\, (R)]$$

By assumption,

$$R_{in} = R + \frac{R_{in}R}{R + R_{in}} = \frac{2R_{in}R + R^2}{R + R_{in}}$$

$$\Rightarrow \qquad RR_{in} + R_{in}^2 = 2RR_{in} + R^2$$

$$\Rightarrow \qquad R_{in}^2 - RR_{in} - R^2 = 0$$

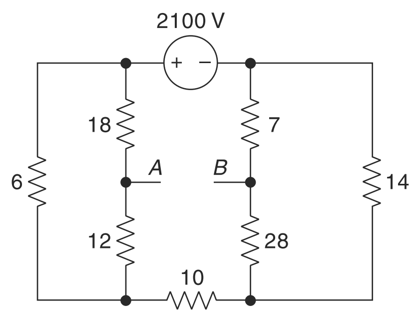
$$\Rightarrow \qquad R_{in} = \frac{R \pm \sqrt{R^2 + 4R^2}}{2} = \frac{R}{2}[1 \pm \sqrt{5}]$$

Taking positive sign, $R_{in} = \left(\frac{\sqrt{5}+1}{2}\right)R$

2.8 In the network shown, calculate the power input to each of the following elements when it is connected across A and B.

(a) a resistance R_{AB} of 59 Ω.

(b) a voltage source of –160 V.



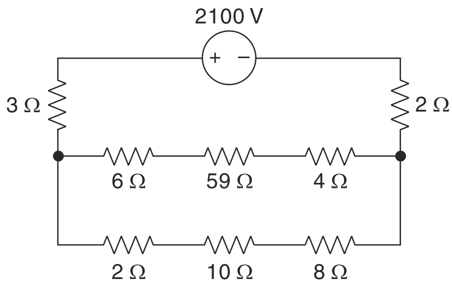
Solution

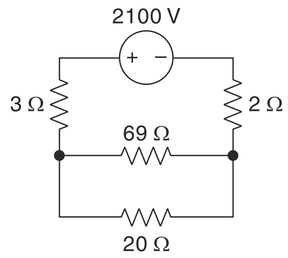
(a) Converting the two deltas into star,

$$r_1 = \frac{18 \times 6}{18 + 12 + 6} = 3\,\Omega, \quad r_2 = \frac{6 \times 12}{36} = 2\,\Omega,$$

$$r_3 = \frac{18 \times 12}{36} = 6\,\Omega$$

and $\quad r_1^1 = \dfrac{14 \times 7}{49} = 2\,\Omega,\ r_2^1 = \dfrac{28 \times 14}{49} = 8\,\Omega,\ r_3^1 = 4\,\Omega$





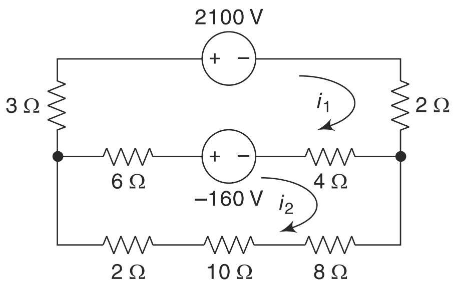
$$\therefore \qquad R_{eq} = \left(3 + \frac{69 \times 20}{69 + 20} + 2\right) = 20.5\ \Omega$$

Main current, $i = \dfrac{2100}{20.5} = 102.41$ A

Current in 59Ω resistor, $i_{59} = 102.41 \times \dfrac{20}{89} = 23.01$ A

Power input, $P_i = (i_{59})^2 \times 59 = (23 \cdot 01)^2 \times 59 = 31248 \cdot 189$ W $= 31$ kW *Ans.*

(b)



By KVL,

$$15i_1 - 10i_2 + 2260 = 0$$

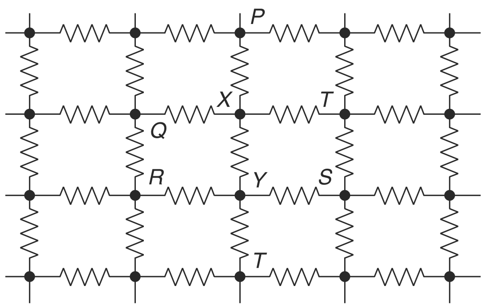
and $\qquad 30i_2 - 10i_1 - 160 = 0$

Solving, $\qquad i_1 = -206.285$ A

$$i_2 = -63.43 \text{ A}$$

$\therefore$ Power input, $P_i = v \times i = -160 \times (i_1 - i_2) = -160 \times (-206 \cdot 285 + 63 \cdot 43) = 17.37$ kW

2.9 The two-dimensional network of the figure consists of an infinite number of square meshes, each side of which has a resistance of R. Find the effective resistance between two adjacent nodes such as X and Y.



Solution Let the current flowing into the circuit at node X be I. Since the infinite network is symmetrical about X, the current I in going from X to infinity, is divided equally along the branches XQ, XT, XP and XY.

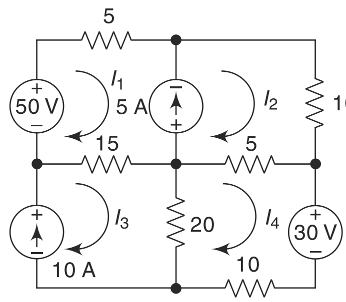
The current I then returns from infinity and is taken from the network at node Y. Again, by symmetry, the currents flowing along RY, XY, SY and TY are each $I/4$.

Hence, the total current flowing along XY is $\frac{I}{4}+\frac{I}{4}=\frac{I}{2}$. So, the voltage between X and Y,

$$V_{XY}=\frac{I}{2}\times R$$

So, the effective resistance between X and Y, $R_{XY}=\frac{V_{XY}}{I}=\frac{R}{2}$ *Ans.*

2.10 Use loop current analysis to find currents in all branches of the network of figure. Also, find the power delivered by 5A current source. All resistances are in ohm.



Solution By KVL,

$$5i_1+10i_2+5(i_2-i_4)+15(i_1-i_3)=50$$

or, $20i_1+15i_2-15i_3-5i_4=50$ (i)

and, $5(i_4-i_2)+30+10i_4+20(i_4-i_3)=0$

or, $-5i_2-20i_3+35i_4=-30$ (ii)

By constraint equations,

$(i_2-i_1)=5$ (iii)

and $i_3=10$ (iv)

From Equation (i) and Equation (ii),

$$20(i_2-5)+15i_2-15\times10-5i_4=50$$

or, $35i_2-5i_4=300 \Rightarrow 7i_2-i_4=60$

and, $-5i_2+35i_4=170 \Rightarrow -i_2+7i_4=34$

Solving $i_4=6.02083$ A

$i_2=4.4583$ A, $i_2=9.4583$ A

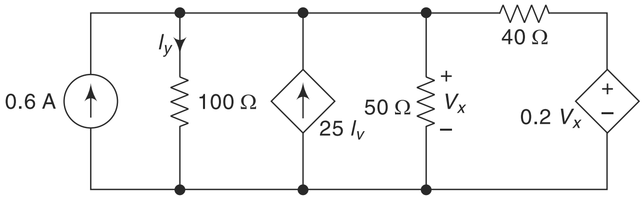
and $i_3=10A$

Power delivered by 5A current source = $v\times i=110.83\times5=554.16$ W

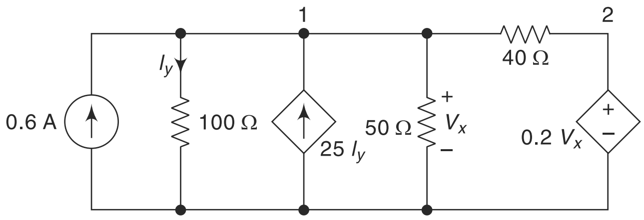
[To calculate the voltage across the 5A current source, v, writing KVL for Mesh (1),

$5i_1+v+15(i_1-i_3)=50 \Rightarrow v=50-20i_1+15i_3=200-20\times4.4583=110.83$ V]

2.11 For the circuit, find the voltage V_x using nodal analysis.



Solution



By KCL at node (1),

$$-0.6 + I_y + \frac{V_x}{50} - 25I_y + \frac{v_1 - v_2}{40} = 0 \quad \text{(i)}$$

By KCL at node (2),

$$v_2 = 0.2\, V_x \quad \text{(ii)}$$

and other constraint equation,

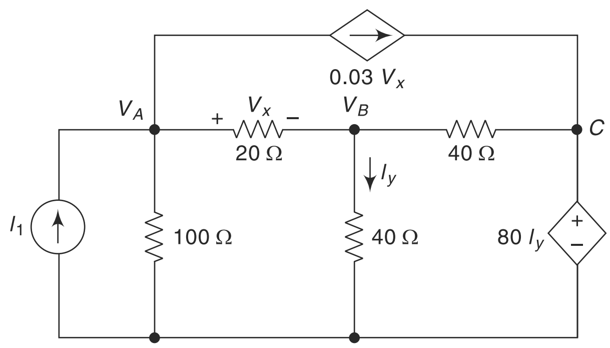
$$I_y = \frac{V_x}{100} \text{ and } v_1 = V_x \quad \text{(iii)}$$

From Equation (i), $-0\cdot 6 + \frac{V_x}{100} + \frac{V_x}{50} - 25\frac{V_x}{100} + \frac{v_1 - v_2}{40} = 0$

$$\Rightarrow \quad -120 + 2V_x + 4V_x - 50V_x + 5V_x - 5 \times 0.2V_x = 0$$

$$\Rightarrow \quad V_x = \frac{120}{-40} = -3 \text{ V}$$ *Ans.*

2.12 Use nodal analysis to find the voltages V_A, V_B and V_x in the circuit, in which $I_1 = 0.4$ A.



Solution By KCL at node (A),

$$-0.4 + \frac{V_A}{100} + \frac{V_A - V_B}{20} + 0.03V_x = 0 \quad \text{(i)}$$

By KCL at node (B),

$$\frac{V_B - V_A}{20} + \frac{V_B}{40} + \frac{V_B - V_C}{40} = 0 \quad \text{(ii)}$$

Constraint equations,

$$I_y = \frac{V_B}{40} \quad \text{(iii)}$$

and $V_C = 80I_y$ (iv)

and $(V_A - V_B) = V_x$ (v)

From Equation (i),

$$-0.4 + \frac{V_A}{100} + \frac{V_A}{20} - \frac{V_B}{20} + 0.03V_A - 0.03V_B = 0$$

$\Rightarrow$ $(9V_A - 8V_B) = 40$ (vi)

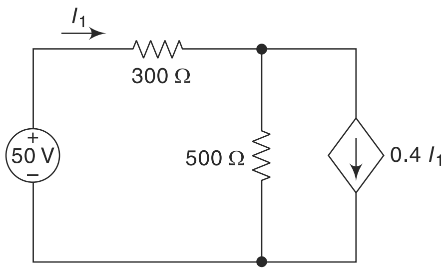
From Equation (ii),

$V_A = V_B$ [by Eq. (iii) and Eq. (iv)]

Thus, solving Equation (vi) $V_A = V_B = 40$ V

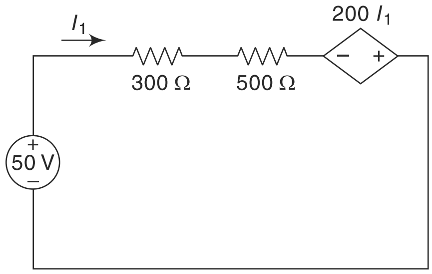
$\therefore$ $V_x = (V_A - V_B) = 0$ *Ans.*

2.13 For the circuit, use loop analysis to find I_1 and the power absorbed by the 500 Ω resistor.



Solution Converting the dependent current source into dependent voltage source,

By KVL,



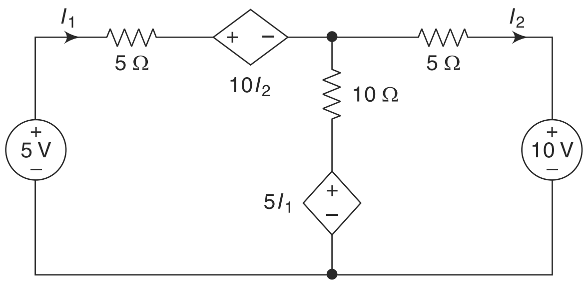
$800I_1 - 200I_1 = 50 \Rightarrow I_1 = \frac{50}{600} = 0.083$ A *Ans.*

Power absorbed by the 500 resistor

$$= I_1^2 R = \left(\frac{50}{600}\right)^2 \times 500$$

$$= \frac{500}{144} = 3.47 \text{ W}$$ *Ans.*

2.14 Determine the currents in all the branches of the network.



Solution By KVL, for Mesh (1)

$$5I_1 + 10I_2 + 10(I_1 - I_2) + 5I_1 = 5$$

$\Rightarrow$ $$20I_1 = 5 \quad \Rightarrow \quad I_1 = 0.25 \text{ A}$$

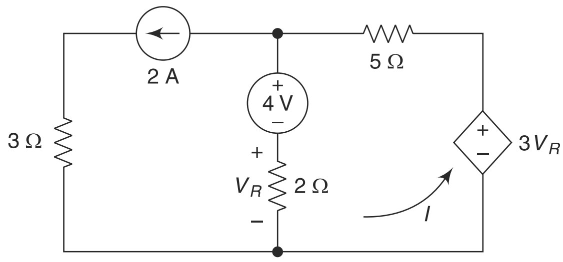
By KVL for Mesh (2),

$$5I_2 + 10 - 5I_1 + (I_2 - I_1) \times 10 = 0$$

$$15I_2 = 15I_1 - 10 = (3.75 - 10) = -6.25$$

$\therefore$ $$I_2 = -0.416 \text{ A} \qquad \textit{Ans.}$$

2.15 Obtain the current I in the network shown.



Solution By KVL for the second mesh

$$-3V_R + 5I + 4 + V_R = 0$$

or, $$-2V_R + 5I + 4 = 0 \tag{i}$$

Also, $V_R = 2 \times (I - 2)$, putting this in Equation (i),

$$-2 \times 2(I - 2) + 5I + 4 = 0$$

$\Rightarrow$ $$-4I + 8 + 5I + 4 = 0$$

$\Rightarrow$ $$I = -12 \text{ A}$$ *Ans.*

MULTIPLE-CHOICE QUESTIONS

2.1 Find the odd one from the following elements.
(a) Inductor (b) Capacitor (c) Resistor (d) Transistor

2.2 Kirchhoff's laws are valid for
(a) linear circuits only. (b) passive time-invariant circuits.
(c) non-linear circuits only. (d) both linear and non-linear circuits.

2.3 Kirchhoff's laws are applicable to
(a) d.c. circuits.
(b) circuits with sinusoidal excitation only.
(c) circuits with d.c. and sinusoidal excitation only
(d) circuits with any excitation.

2.4 Kirchhoff's law fails in case of
(a) linear networks. (b) non-linear networks.
(c) dual networks. (d) distributed parameter networks.

2.5 KCL is a consequence of law of conservation of
(a) energy (b) charge (c) flux (d) all of the above.

2.6 A component that opposes the change in circuit current is
(a) resistance (b) capacitance (c) inductance (d) conductance.

2.7 A component that opposes the change in circuit voltage is
(a) resistance (b) capacitance (c) inductance (d) conductance

2.8 For a d.c. voltage, an inductor
(a) is virtually a short-circuit. (b) is an open-circuit.
(c) depends on polarity. (d) depends on voltage value.

2.9 A network N' is a dual of a network N if
(a) both of them have same mesh equations.
(b) both of them have same node equations.
(c) mesh equations of one of them are node equations of the other.
(d) none of the above.

2.10 A connected planar network has 4 nodes and 5 elements. The number of meshes in its dual network is
(a) 4 (b) 3 (c) 2 (d) 1.

2.11 Two networks can be dual when
(a) their nodal equations are the same.
(b) the loop equations of one network are the nodal equations of the other.
(c) their loop equations are the same.
(d) none of these.

2.12 The internal impedance of an ideal current source is
(a) zero (b) infinite (c) both (a) and (b) (d) none of these.

2.13 The internal impedance of an ideal voltage source is
(a) zero (b) infinite (c) both (a) and (b) (d) none of these.

2.14 The internal impedance of a dependent voltage source is
(a) zero (b) infinity (c) fraction of ohm (d) any unknown value.

2.15 An ideal voltage source will charge an ideal capacitor
(a) in infinite time (b) exponentially (c) instantaneously (d) none of the above.

2.16 A practical current source is usually represented by
(a) a resistance in series with an ideal current source.
(b) a resistance in parallel with an ideal current source.
(c) a resistance in series with an ideal voltage source.
(d) none of the above.

2.17 Energy stored in a capacitor is
(a) $\frac{1}{4}CV^2$ (b) $\frac{1}{2}CV^2$ (c) $\int_0^\infty \frac{1}{2}C$ (d) 0

2.18 The node method of circuit analysis is based on
(a) KVL and Ohm's law (b) KCL and KVL
(c) KCL, KVL and Ohm's law (d) KCL and Ohm's law

2.19 The loop method of circuit analysis is based on
(a) KVL and Ohm's law. (b) KCL and KVL.
(c) KCL, KVL and Ohm's law. (d) KCL and Ohm's law.

2.20 Two wires A and B of the same material and length L and $2L$ have radius r and $2r$, respectively. The ratio of their specific resistance will be

(a) 1 : 1 (b) 1 : 2 (b) 1 : 4 (d) 1 : 8

2.21 There are two wires A and B. A is 20 times longer than B and has half the cross-section of that of B. If the resistance of B is 1 Ω, the resistance of A will be

(a) 40 Ω (b) $\frac{1}{40}$ Ω (c) 20 Ω (d) 10 Ω

2.22 The resistance between the opposite faces of 1 m cube is found to be 1 Ω. If its length is increased to 2 m, with its volume remaining the same, then its resistance between the opposite faces along its length is

(a) 2 Ω (b) 4 Ω (c) 1 Ω (d) 8 Ω

(e) $\frac{1}{2}$ Ω

2.23 A wire of length l and of circular cross-section of radius r has a resistance R ohm. Another wire of same material and cross-sectional radius $2r$ will have the same resistance R if the length is

(a) $2l$ (b) $l/2$ (c) $4l$ (d) l^2

2.24 Two resistances of equal value, when connected in parallel give an equivalent resistance of R. If these resistances are connected in series, the equivalent resistance will be

(a) R (b) $4R$ (c) $2R$ (d) $\frac{R}{2}$

2.25 A series arrangement of 'n' identical resistances is changed into a parallel arrangement. The new total resistance will become____times the original resistance.

(a) $\frac{1}{n}$ (b) $\frac{1}{n^2}$ (c) $\frac{1}{n^3}$ (d) $\frac{1}{n^4}$

2.26 If a two-terminal network element in a circuit has voltage and current variables that follow the associated reference directions and its power is negative, which of the following is true?

(a) The element is supplying energy to the rest of the circuit.
(b) The element is receiving energy from the rest of the circuit.
(c) Either (a) or (b) could be true.

2.27 If an ideal voltage source and an ideal current source are connected in parallel, what are the properties of the combination?

(a) The same as a voltage source.
(b) The same as a current source.
(c) Different from either a voltage source or a current source.

2.28 If an ideal voltage source and an ideal current source are connected in series, what are the properties of the combination?

(a) The same as a voltage source.
(b) The same as a current source.
(c) Different from either a voltage source or a current source.

2.29 When ideal voltage sources are connected in series, which of the following is true?

(a) The voltages add, independent of whether the individual sources are constant valued or have outputs that are functions of time.
(b) The connection violates KVL; thus it is not permitted.
(c) Neither is true.

2.30 When ideal arbitrary voltage sources are connected in parallel, which of the following is true?
 (a) The voltages add, independent of whether the individual sources are constant valued or have outputs that are functions of time.
 (b) The connection violates KVL; thus it is not permitted.
 (c) Neither is true.

2.31 When ideal arbitrary current sources are connected in series, which of the following is true?
 (a) The currents add, independent of whether the individual sources are constant valued or have outputs that are functions of time.
 (b) The connection violates KCL; thus it is not permitted.
 (c) Neither is true.

2.32 When ideal current sources are connected in parallel, which of the following is true?
 (a) The currents add, independent of whether the individual sources are constant valued or have outputs that are functions of time.
 (b) The connection violates KCL; thus it is not permitted.
 (c) Neither is true.

2.33 In a network containing only independent current sources and resistors, if the values of all resistors are doubled, the values of the node voltages
 (a) are doubled. (b) remain the same.
 (c) are halved. (d) change in some other way.

2.34 In a network containing only independent current sources and resistors, if the values of all the current sources are doubled, the values of the node voltages
 (a) are doubled. (b) remain the same.
 (c) are halved. (d) change in some other way.

2.35 In a network containing only independent voltage sources and resistors, if the values of all the voltage sources are doubled, the values of the mesh currents
 (a) are doubled. (b) remain the same.
 (c) are halved. (d) change in some other way.

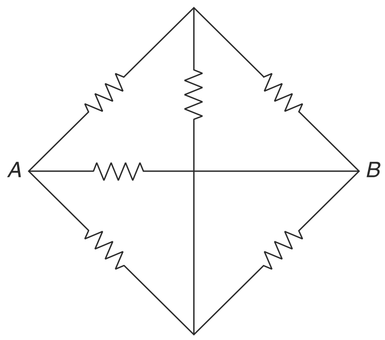
2.36 In a network containing only independent voltage sources and resistors, if the values of all the resistors are doubled, the values of the mesh currents
 (a) are doubled. (b) remain the same.
 (c) are halved. (d) change in some other way

2.37 If the same constant value of current is added to all the independent current sources in a network, the node voltages
 (a) will all have a constant value added. (b) will remain the same.
 (c) will all have a constant value subtracted. (d) will change in some other way.

2.38 If the same constant value of voltage is added to each of the independent voltage sources in an arbitrary network containing only resistors are independent voltage sources, the mesh currents
 (a) will all have a constant value added. (b) will remain the same.
 (c) will all have a constant value subtracted. (d) will change in some other way.

2.39 Two resistors R_1 and R_2 give combined resistance of 4.5 Ω when in series and 1 Ω when in parallel, the resistances are
 (a) 2 Ω and 2.5 Ω (b) 1 Ω and 3.5 Ω (c) 1.5 Ω and 3.5 Ω (d) 4 Ω and 0.5 Ω.

2.40 When all the resistances in the circuit are of 1Ω each, the equivalent resistance across the points A and B will be



(a) 1 Ω (b) 0.5 Ω (c) 2 Ω (d) 1.5 Ω.

2.41 Energy expanded or heat generated in Joules when a current of I flows through a conductor R for t second is given by

(a) I^2Rt (b) IRt (c) IR^2t (d) IRt^2

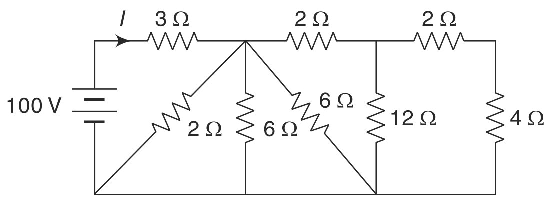
2.42 A 2 Ω resistance having a current of 2A will dissipate the power of

(a) 2 W (b) 4 W (c) 8 W (d) 8 J

2.43 The ratio of resistances of a 100 W, 220 V lamp to that of a 100 W, 110 V lamp will be, at the respective voltages

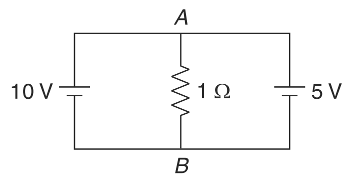
(a) 4 (b) 2 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

2.44 For the circuit shown in figure, the value of current I is



(a) 10 A (b) 15 A
(c) 20 A (d) 25 A

2.45. The current in the 1 Ω resistor is

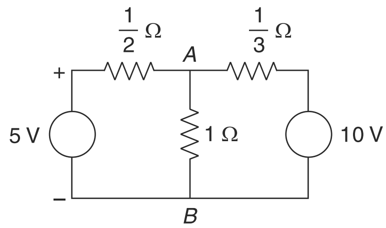


(a) 5 A (b) 10 A
(c) 15 A (d) zero.

2.46 The current in a 5 Ω resistor branch in a linear network is 5 A. If this branch is replaced by a resistor of 10 Ω, the current in this branch will be

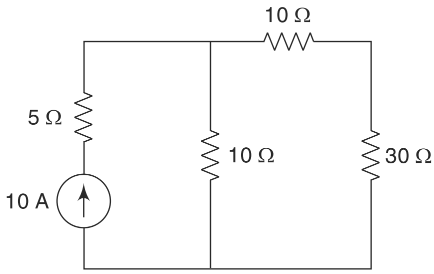
(a) 5 A (b) 10 A
(c) less than 4 A (d) none of these.

2.47 The potential of the point A in the given network is



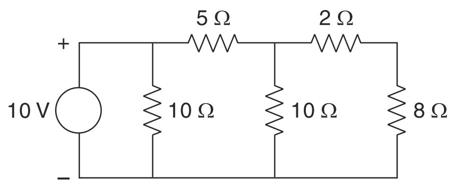
(a) 6 V
(b) 7 V
(c) 8 V
(d) none of these.

2.48 The current through 30 Ω branch in the given circuit is



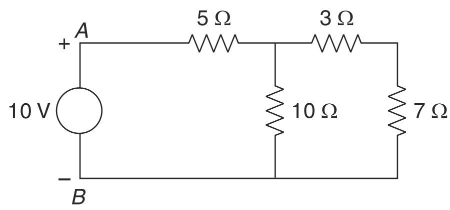
(a) 2.5 A (b) 2.25 A (c) 2 A (d) 10 A

2.49 The current through 8 Ω branch is



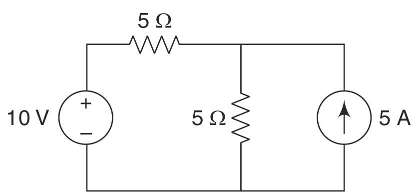
(a) 1 A
(b) 0.5 A
(c) 1.5 A
(d) none of these.

2.50 If the current in the 7 Ω resistor branch is 0.5 A as shown in the figure and now if the source is connected in series with 7 Ω branch and the terminals *AB* are shorted, the current in the 5 Ω resistor is



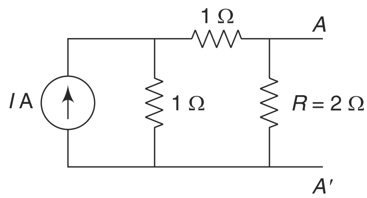
(a) 1 A (b) 0.5 A
(c) 9.75 A (d) none of these.

2.51 The voltage across 5 A source in the given circuit is



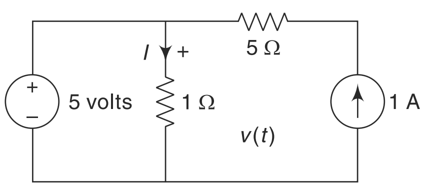
(a) 25 V (b) 15 V (c) 17.5 V (d) 20 V.

2.52 In the circuit shown in figure, current *I* flows through the resistance *R*. If a battery of e.m.f. 2 V and internal resistance of 1 Ω is connected between the terminals *A* and *A′* with positive terminal connected to *A′*, the current through *R* would be



(a) 2 A (b) 1.66 A
(c) 1 A (d) 1.5 A.

2.53 The circuit shown in the figure is linear and time-invariant. The sources are ideal. The voltage across the 1 Ω resistor and the current through it will be

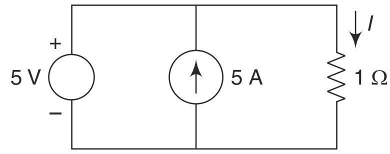


(a) –5 V and –5 A (b) 1 V and 1 A
(c) 1 1 and 6 A (d) 5 V and 5 A.

2.54 The number of 2 μF, 400 V capacitors needed to obtain a capacitance value of 1.5 μF rated for 1600 V is

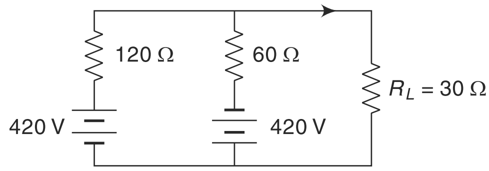
(a) 12 (b) 8 (c) 6 (d) 4.

2.55 The value of the current I flowing in the 1 Ω resistor in the circuit, shown in figure will be



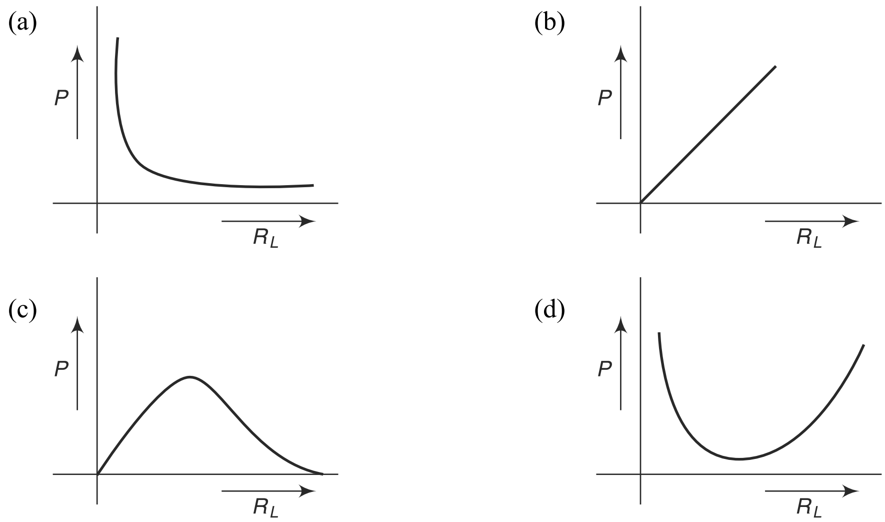
(a) 10 A (b) 6 A (c) 5 A (d) zero.

2.56 In the circuit shown in figure, the current I through R_L is

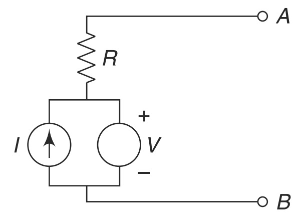


(a) 2 A (b) zero (c) –2 A (d) –6 A.

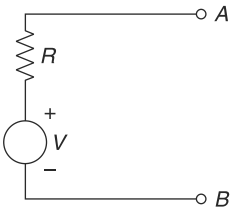
2.57 A voltage source with an internal resistance R_S, supplies power to a load R_L. The power delivered to the load varies with R_L as



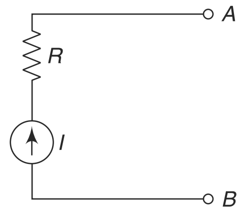
2.58 A simple equivalent circuit of the 2-terminal network shown in figure is



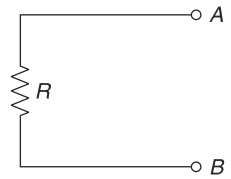
(a)



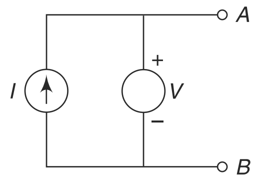
(b)



(c)



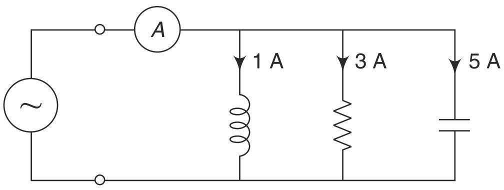
(d)



2.59 Two condensers of 20 μF and 40 μF capacitances are connected in series across a 90 V supply. After charging, they are removed from the supply and are connected in parallel with positive terminals connected together and similarly the negative terminals. Then the voltage across them will be

(a) 90 V (b) 60 V (c) 40 V (d) 20 V.

2.60 The current read by the ammeter *A* in the a.c. circuit shown in the given figure is

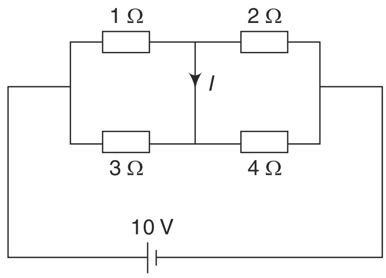


(a) 9 A (b) 5 A (c) 3 A (d) 1 A

2.61 In the circuit shown in the given figure, current *I* is

(a) $-\frac{2}{5}$ A (b) $\frac{24}{5}$ A

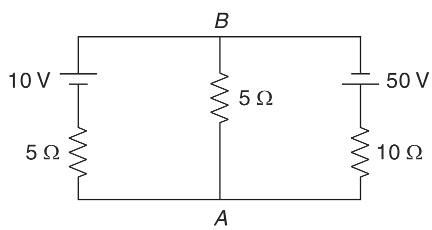
(c) $\frac{18}{5}$ A (d) $\frac{2}{5}$ A



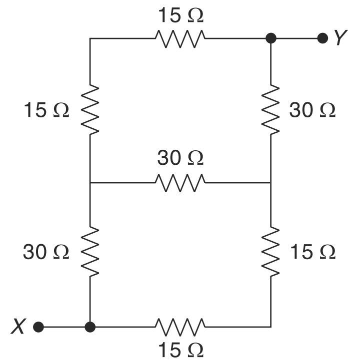
2.62 For the circuit shown in the given figure, the voltage V_{AB} is

(a) 6 V (b) 10 V

(c) 25 V (d) 40 V

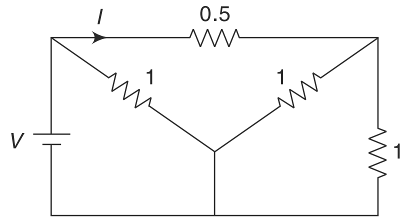


2.63 The equivalent resistance between the terminal points X and Y in the circuit shown is



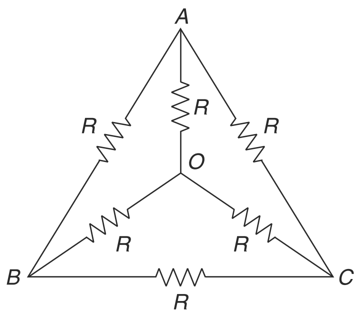
(a) 15 Ω (b) 45 Ω (c) 55 Ω (d) 30 Ω

2.64 In the circuit shown in the figure, if $I = 2$, then the value of the battery voltage V will be



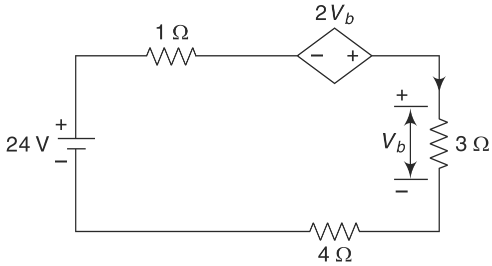
(a) 5 V (b) 3 V (c) 2 V (d) 1 V

2.65 The effective resistance between the terminals A and B in the circuit shown in the figure is



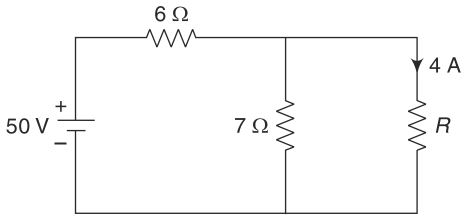
(a) R (b) $R - 1$ (c) $R/2$ (d) $\frac{6}{11}R$

2.66 The current in the given circuit with a dependent source is



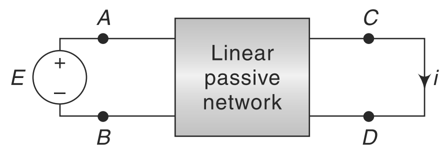
(a) 10 A (b) 12 A (c) 14 A (d) 16 A

2.67 The value of the resistance R shown in the given figure is



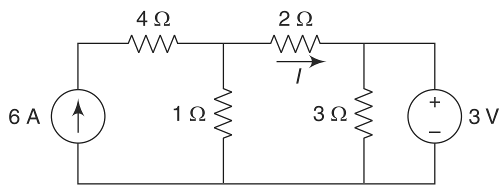
(a) 3.5 Ω (b) 2.5 Ω (c) 1 Ω (d) 4.5 Ω.

2.68 For the circuit shown in the given figure, when the voltage E is 10 V, the current i is 1 A. If the applied voltage across terminal C–D is 100 V, the short circuit current flowing through the terminal A–B will be



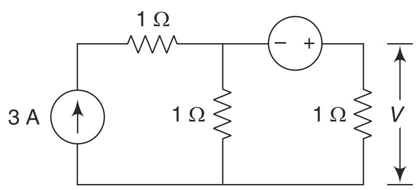
(a) 0.1 A (b) 1 A
(c) 10 A (d) 100 A.

2.69 For the circuit shown in the given figure, the current I is given by



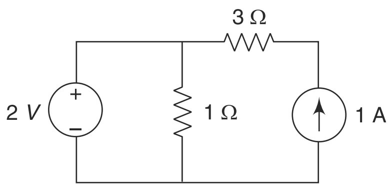
(a) 3 A (b) 2 A
(c) 1 A (d) zero.

2.70 The value of V in the circuit shown in the given figure is



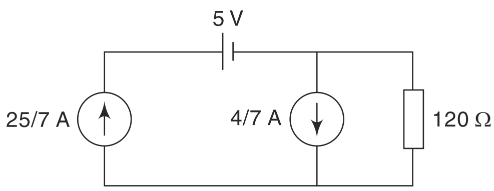
(a) 1 V (b) 2 V (c) 3 V (d) 4 V

2.71 For the circuit given in figure, the power delivered by the 2 volt source is given by



(a) 4 W (b) 2 W
(c) –2 W (d) –4 W

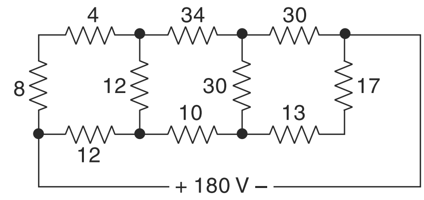
2.72 The current through 120 Ω resistor in the circuit shown in the figure is



(a) 1 A (b) 2 A (c) 3 A (d) 4 A

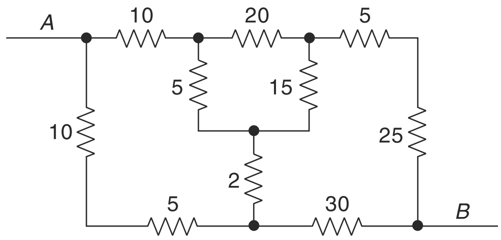
EXERCISES

2.1 Find the current in the $10\ \Omega$ resistor in the network shown. All values are in ohm.



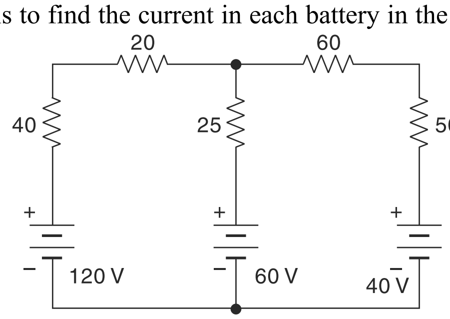
[4 A]

2.2 Find R_{AB} in the network shown in figure below



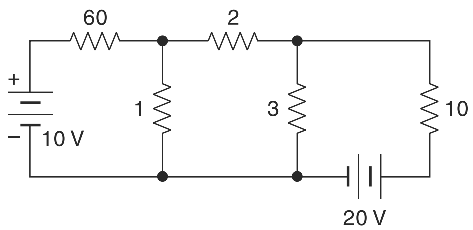
[23.52 Ω]

2.3 Use loop current analysis to find the current in each battery in the network shown in figure below.



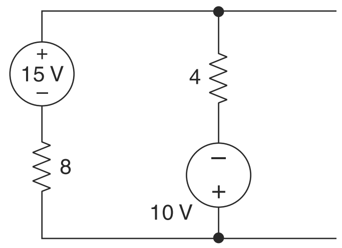
[0.793 A, 0.408 A, 0.295 A]

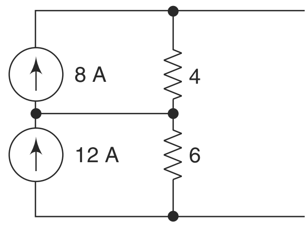
2.4 Find the current through $2\ \Omega$ resistance in the network shown below. Use loop current method.



[–0.841 A]

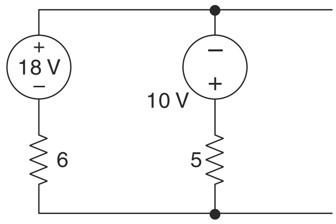
2.5 Convert the circuit shown in figure to a single voltage source in series with a single resistor.





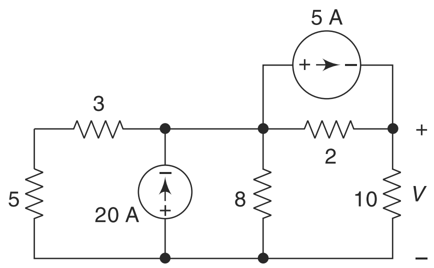
$\left[V = \frac{5}{3}\text{ V}, R = \frac{8}{3}\,\Omega\right]$ [V = 104 V, R = 10 Ω]

2.6 Convert the circuit shown in figure to a single current source in parallel in with a single resistor.



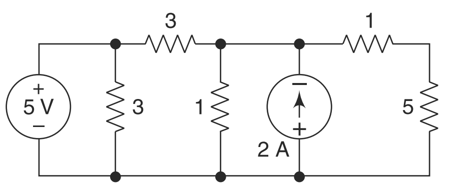
[I = 1 A, R = 2.73 Ω]

2.7 Determine the voltage V in the circuit, using the source transformation technique and/or any other method.



[V = 56.25 V]

2.8 Find the current flowing through 5Ω resistor using source transformation technique.



$\left[\frac{11}{27}\text{ A}\right]$

2.9 Reduce the network shown in Fig. (a) to a form shown in Fig. (b) using successive source transformations.

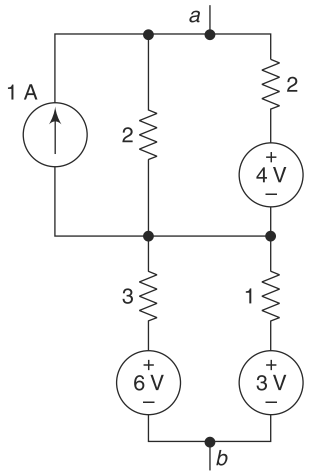


Figure (a)

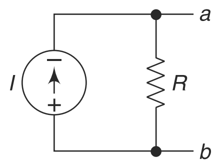
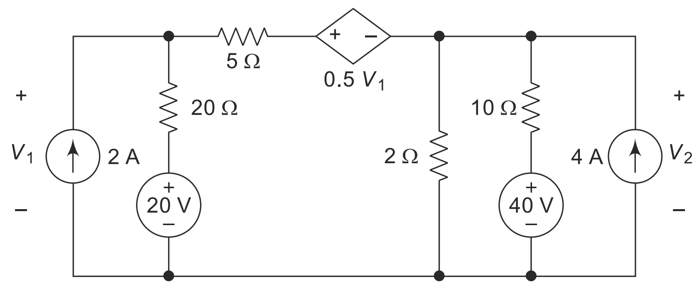


Figure (b)

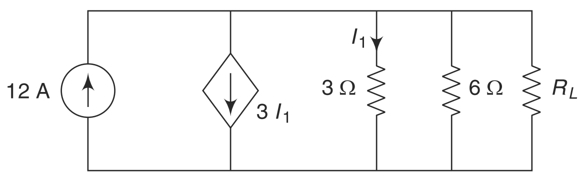
[I = 2.14 A, R = 1.75 Ω]

2.10 For the circuit of figure, apply source transformation and then find V_1 and V_2 by nodal analysis.

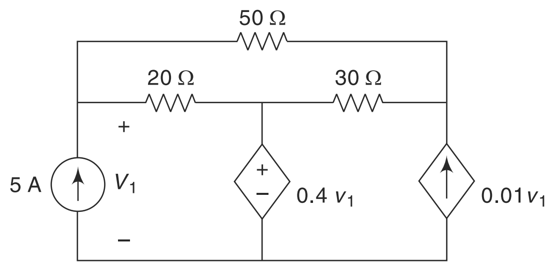


[V_1 = 40 V, V_2 = 15 V]

2.11 In the circuit shown in figure if I_1 = 2 A, determine R_L and the power delivered in it. [2 Ω; 18 W]

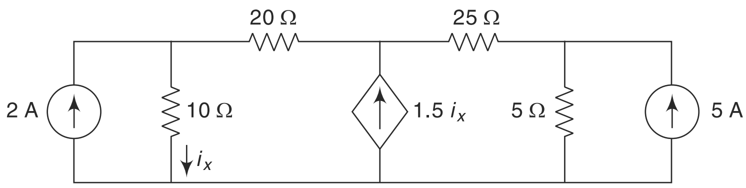


2.12 Use nodal analysis to determine v_1 and power being supplied by the dependent current source in the circuit shown in figure.



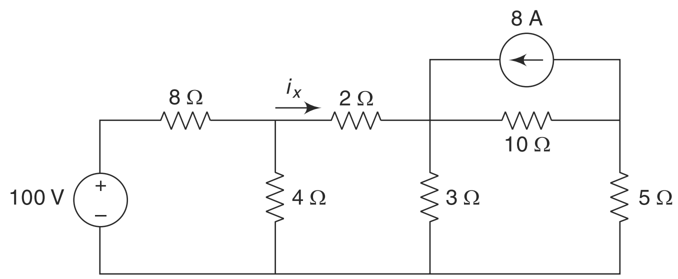
[148.1 V; 178.2 W]

2.13 Use mesh analysis to find the current i_x in the circuit shown in figure.



[8.33 A]

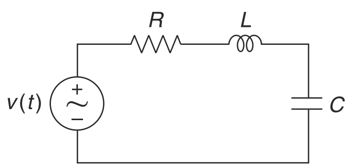
2.14 Use mesh analysis to find the current i_x in the circuit shown in figure.



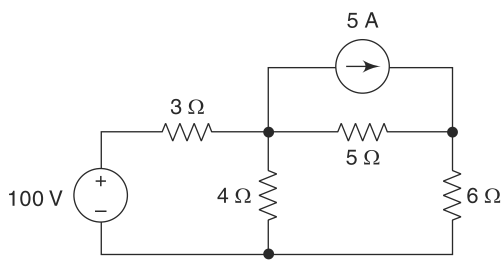
[2.79 A]

2.15 Construct the dual of a network shown below.

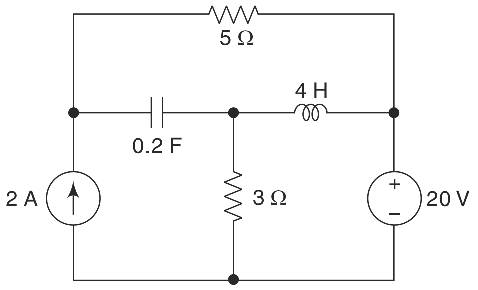
(a)



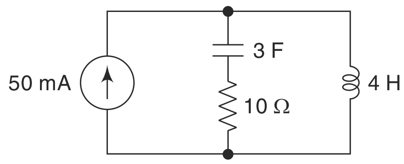
(b)



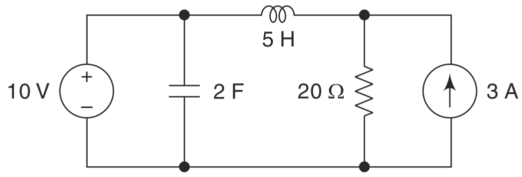
(c)



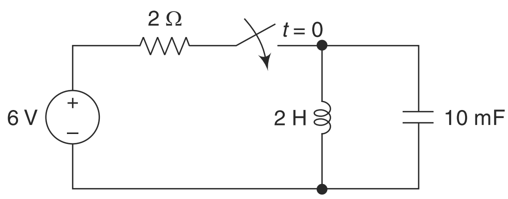
(d)



(e)



(f)



2.16 Draw a circuit and its dual if the mesh equations of the circuit are

(a) $8i_1 - 2i_2 - 4i_3 = 5;\ 14i_2 - 6i_3 = 3;\ -4i_1 - 6i_2 + 15i_3 = 6$

(b) $8i_1 - 2i_2 - 4i_3 = 6;\ 7i_2 - 5i_3 = -3;\ -4i_1 - 5i_2 + 9i_3 = 5$

(c) $4i_1 - i_2 - i_3 = -4;\ -i_1 + 6i_2 - 5i_3 = 6;\ -i_1 - 5i_2 + 8i_3 = 2$

2.17 Draw a circuit and its dual if the node equations of the circuit are

(a) $4v_1 - v_2 - v_3 = -4;\ -v_1 + 6v_2 - 5v_3 = 6;\ -5v_2 + 8v_3 = 2$

(b) $4v_1 - v_2 - v_3 = 5;\ 3v_2 - v_3 = -3;\ -v_1 - v_2 + 2v_3 = 6$

(c) $6v_1 - 2v_2 - v_3 = 5;\ -2v_1 + 8v_2 - 3v_3 = -4;\ -v_1 - 3v_2 + 9v_3 = 0$

SHORT-ANSWER TYPE QUESTIONS

2.1 Define an Electrical Network. "All circuits are networks, but all networks are not circuits." Justify this statement.

2.2 (a) State the basic assumptions for circuit analysis.
(b) Briefly mention the different source transformation techniques.
(c) Discuss the properties of an ideal current source and ideal voltage source.
(d) Explain how a voltage source can be converted into an equivalent current source and vice-versa.

2.3 (a) State and explain Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL).
(b) Give a brief comparison of the loop method and node method of circuit analysis.

2.4 Explain 'duality' in electrical engineering. How can you draw the dual of a network?

2.5 Establish the $\pi - T$ transformation relations. Show that the dual of a T network is a π-network.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

2.1 (d)	2.2 (d)	2.3 (d)	2.4 (d)	2.5 (b)	2.6 (c)	2.7 (b)
2.8 (a)	2.9 (c)	2.10 (b)	2.11 (b)	2.12 (b)	2.13 (a)	2.14 (d)
2.15 (c)	2.16 (b)	2.17 (b)	2.18 (d)	2.19 (a)	2.20 (a)	2.21 (a)
2.22 (b)	2.23 (c)	2.24 (b)	2.25 (b)	2.26 (a)	2.27 (a)	2.28 (a)
2.29 (a)	2.30 (b)	2.31 (b)	2.32 (a)	2.33 (a)	2.34 (a)	2.35 (a)
2.36 (c)	2.37 (d)	2.38 (d)	2.39 (c)	2.40 (b)	2.41 (a)	2.42 (c)
2.43 (a)	2.44 (d)	2.45 (d)	2.46 (c)	2.47 (c)	2.48 (c)	2.49 (b)
2.50 (b)	2.51 (c)	2.52 (d)	2.53 (d)	2.54 (a)	2.55 (c)	2.56 (c)
2.57 (c)	2.58 (a)	2.59 (c)	2.60 (b)	2.61 (b)	2.62 (a)	2.63 (d)
2.64 (c)	2.65 (c)	2.66 (b)	2.67 (a)	2.68 (c)	2.69 (c)	2.70 (c)
2.71 (b)	2.72 (c)					

CHAPTER 3
Network Topology (Graph Theory)

3.1 INTRODUCTION

The word ***topology*** refers to the science of place. In mathematics, ***topology is a branch of geometry in which figures are considered perfectly elastic***.

Network Topology refers to the properties that relate to the ***geometry of the network*** (circuit). These properties remain unchanged even if the circuit is bent into any other shape provided that ***no parts are cut*** and ***no new connections*** are made.

The first step in studying the topological properties of a network is to suppress the nature of the circuit elements that make up the network. We do so by constructing a ***graph*** of the circuit.

3.2 GRAPH OF A NETWORK

A linear graph (or simply a graph) is defined as a collection of points called nodes, and line segment called branches, the nodes being joined together by the branches.

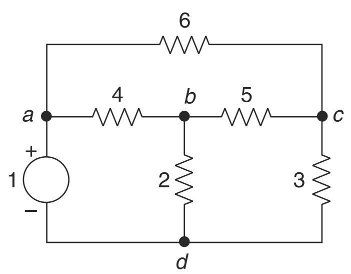


Figure 3.1(a) *Circuit*

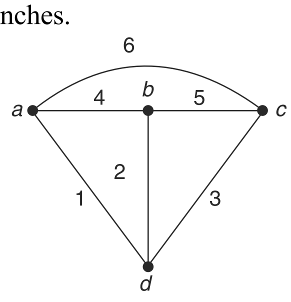


Figure 3.1(b) *Graph of the Circuit*

While drawing graph of a given network, the following rules are to be noted.

(i) All passive elements between the nodes are represented by lines.

(ii) The independent current sources and voltage sources are represented by their internal impedances (i.e., current sources by open circuit and voltage sources by short circuit) if they are accompanied by passive element, viz., a shunt admittance in a current source and a series impedance in a voltage source.

(iii) If the sources are not accompanied by passive elements, an arbitrary impedance (say resistance R) or admittance is assumed to accompany the sources and finally, we find the results by letting the impedance $R \to 0$ or $R \to \propto$ as the case may be for the current or voltage sources.

3.3 TERMINOLOGY

In order to discuss the more involved methods of circuit analysis, we must define a few basic terms necessary for a clear, concise description of important circuit features.

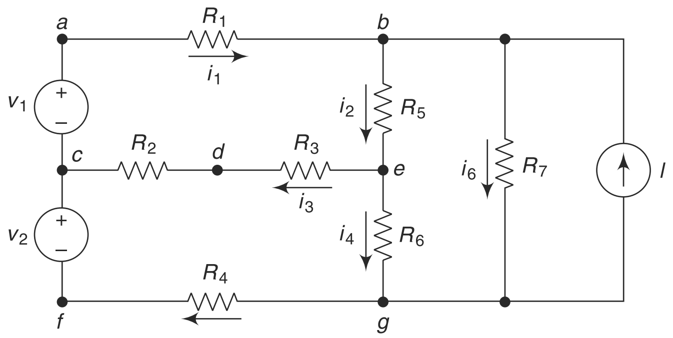


Figure 3.2 *Circuit illustrating terminologies*

(a) ***Node*** A node is a point in a circuit where two or more circuit elements join.

Example a, b, c, d, e, f and g

(b) ***Essential Node*** A node that joins three or more elements.

Example b, c, e and g

(c) ***Branch*** A branch is a path that connects two nodes.

Example v_1, R_1, R_2, R_3, v_2, R_4, R_5, R_6, R_7 and I

(d) ***Essential branch*** Those paths that connect essential nodes without passing through an essential node.

Example c–a–b, c–d–e, c–f–g, b–e, e–g, b–g (through R_7), and b–g (through I)

(e) ***Loop*** A loop is a complete path, i.e., it starts at a selected node, traces a set of connected basic circuit elements and returns to the original starting node without passing through any intermediate node more than once.

Example $abedca$, $abegfca$, $cdebgfc$, etc.

(f) ***Mesh*** A mesh is a special type of loop, i.e., it does not contain any other loops within it.

Example $abedca$, $cdegfc$, $gebg$ (through R_7) and $gebg$ (through I)

(g) ***Oriented Graph*** A graph whose branches are oriented is called a directed or oriented graph.

(h) ***Rank of Graph*** The rank of a graph is (n–1) where n is the number of nodes or vertices of the graph.

(i) ***Planar and Non-planar Graph*** A graph is planar if it can be drawn in a plane such that no two branches intersect at a point which is not a node.

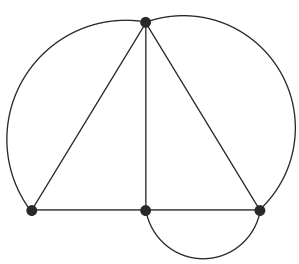


Figure 3.3(a) *Planar graph*

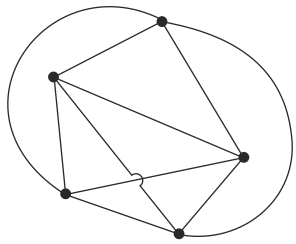


Figure 3.3(b) *Non-planar graph*

(j) ***Subgraph*** A subgraph is a subset of the branches and nodes of a graph. The subgraph is said to be proper if it consists of strictly less than all the branches and nodes of the graph.

(k) ***Path*** A path is a particular sub graph where only two branches are incident at every node except the internal nodes (i.e., starting and finishing nodes). At the internal nodes, only one branch is incident.

In the example in the Fig. 3.3 (c), branches 2, 3, and 4, together with all the four nodes, constitute a path. A graph is connected if there exists a path between any pair of vertices. Otherwise, the graph is disconnected.

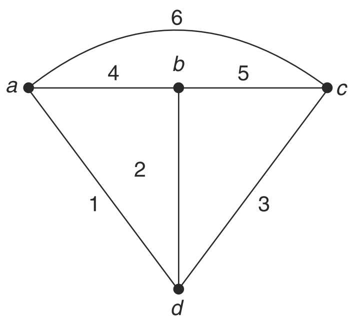


Figure 3.3(c)

3.4 CONCEPT OF TREE

For a given connected graph of a network, a connected subgraph is known as a tree of the graph if the subgraph has all the nodes of the graph without containing any loop.

Twigs The branches of tree are called twigs or tree-branches. The number of branches or twigs, in any selected tree is always one less than the number of nodes, i.e.,

Twigs = (n – 1), where n is the number of nodes of the graph.

For this case, twigs = (4 – 1) = 3 twigs. These are shown by solid lines in Fig. 3.4 (b).

Links and Co-tree If a graph for a network is known and a particular tree is specified, ***the remaining branches*** are referred to as the ***links***. The ***collection of links*** is called a ***co-tree***. So, co-tree is the complement of a tree. These are shown by dotted lines in Fig. 3.4(b).

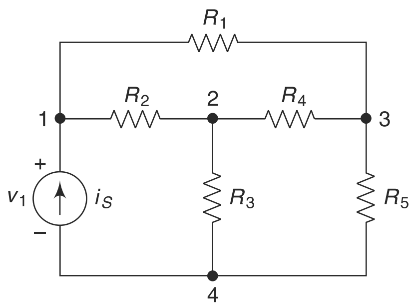


Figure 3.4(a) *Circuit*

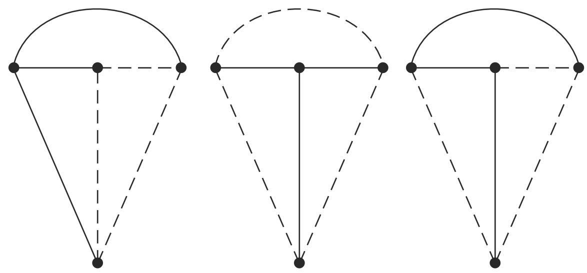


Figure 3.4(b) *Trees and links of circuit of Fig. 3.4(a)*

The branches of a co-tree may or may not be connected, whereas the branches of a tree are always connected.

To summarize,

Number of nodes in a graph = n

Number of independent voltages = $n - 1$

Number of tree-branches = $n - 1$

Number of links = L = (Total number of branches) – (Number of tree-branches)

$= b - (n - 1) = b - n + 1$

Total number of branches = $b = L + (n - 1)$

Properties of a Tree

1. In a tree, there exists one and only one path between any pairs of nodes.
2. Every connected graph has at least one tree.
3. A tree contains all the nodes of the graph.
4. There is no closed path in a tree and hence, tree is circuitless.
5. The rank of a tree is $(n - 1)$.

3.5 INCIDENCE MATRIX [A_a]

The incidence matrix symbolically describes a network. It also facilitates the testing and identification of the independent variables. Incidence matrix is a matrix which represents a graph **uniquely**.

For a given graph with n nodes and b branches, the complete incidence matrix A_a is a rectangular matrix of order $n \times b$, whose elements have the following values.

Number of columns in [A_a] = Number of branches = b

Number of rows in [A_a] = Number of nodes = n

A_{ij} = 1, if branch j is associated with node i and oriented away from node j.

= –1, if branch j is associated with node i and oriented towards node j.

= 0, if branch j is not associated with node i.

This matrix tells us which branches are incident at which nodes and what are the orientations relative to the nodes.

Example

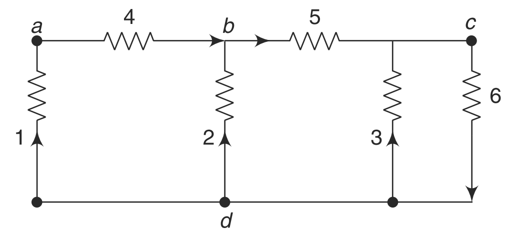


Figure 3.5(a) *Network*

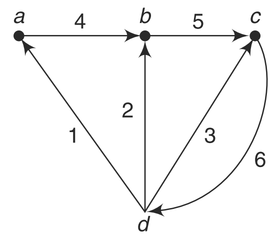


Figure 3.5(b) *Graph of network*

Incidence matrix A

		Branches 1	2	3	4	5	6	
	a	1	0	0	–1	0	0	Reduced
Nodes	b	0	1	0	1	–1	0	Incidence
	c	0	0	1	0	1	–1	Matrix A
Reference Node	d	–1	–1	–1	0	0	1	

3.5.1 Incidence Matrix and KCL

For the above graph, Kirchhoff's current law for the branch currents ($i_1, i_2, \ldots, i_6$) gives the equations,

$$i_1 + i_2 + i_6 = 0$$
$$-i_1 + i_3 + i_5 = 0$$
$$-i_2 - i_3 + i_4 = 0$$
$$-i_4 + i_5 - i_6 = 0$$

In matrix form, these equations can be represented as,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_{b1} \\ i_{b2} \\ i_{b3} \\ i_{b4} \\ i_{b5} \\ i_{b6} \end{bmatrix} = 0$$

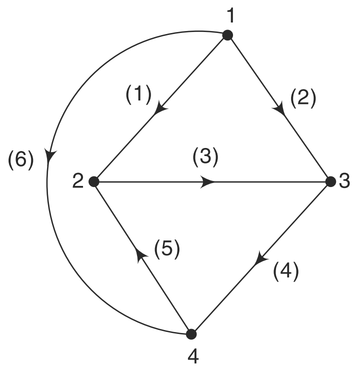


Figure 3.6 *Graph illustrating incidence matrix and KLC*

or $\boxed{A_a I_b = 0}$

where, A_a is the complete incidence matrix of the graph.

Reduced Incidence Matrix [A] The matrix obtained from A_a by eliminating one of the rows is called Reduced Incidence Matrix. In other words, suppression of the datum node (reference node) from the incidence matrix results in reduced incidence matrix.

For the graph shown in Fig. 3.6, reduced incidence matrix is given as,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

3.5.2 Incidence Matrix and KVL

For the graph shown in Fig. 3.6, the branch voltages (v_{b1}, v_{b2}, $\ldots v_{b6}$) can be represented in terms of the node voltages (v_{n1}, v_{n2}, v_{n3}, v_{n4}) as,

$$v_{b1} = (v_{n1} - v_{n2}), \quad v_{b2} = (v_{n1} - v_{n3}), \quad v_{b3} = (v_{n2} - v_{n3}), \quad v_{b4} = (v_{n3} - v_{n4}),$$
$$v_{b5} = (-v_{n1} + v_{n4}), \quad v_{b6} = (v_{n1} - v_{n4}),$$

Thus, the Kirchhoff's voltage law in matrix form can be written as,

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_{n_1} \\ V_{n_2} \\ V_{n_3} \\ V_{n_4} \end{bmatrix} = \begin{bmatrix} v_{b1} \\ v_{b2} \\ v_{b3} \\ v_{b4} \\ v_{b5} \\ v_{b6} \end{bmatrix}$$

or $$\boxed{A_a^T V_n = V_b}$$

3.5.3 Properties of Complete Incidence Matrix

(i) The sum of the entries in any column is zero.
(ii) The determinant of the incidence matrix of a closed loop is zero.
(iii) The rank of incidence matrix of a connected graph is (n–1).

3.6 NUMBER OF POSSIBLE TREES OF A GRAPH

The number of possible trees of a graph, = det $\{[A] \times [A]^T\}$
where, A is the reduced incidence matrix obtained by eliminating any one row of the complete incidence matrix A_a, and $[A]^T$ is the transpose of the matrix $[A]$.

Example For the graph shown in Fig. 3.6, the complete incidence matrix is,

$$A_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

So, reduced incidence matrix is,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Thus, the number of possible trees of the graph of Fig. 3.6

$$= \det\left\{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right\}$$

$$= \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16$$

3.7 TIE-SET MATRIX AND LOOP CURRENTS

Tie-Set A ***tie-set*** is a set of branches contained in a loop such that each loop contains one link or chord and the remainder are tree branches.

Consider the graph and the tree as shown. This selected tree will result in ***three fundamental loops*** as we connect each link, in turn to the tree.

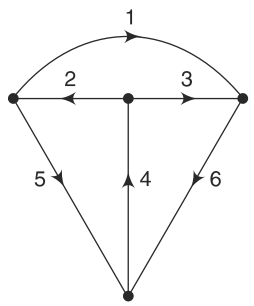


Figure 3.7(a) *Graph*

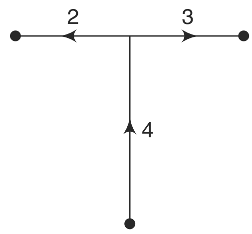


Figure 3.7(b) *Tree of the graph*

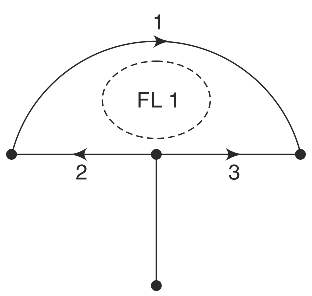


Figure 3.7(c) *Loop-1*

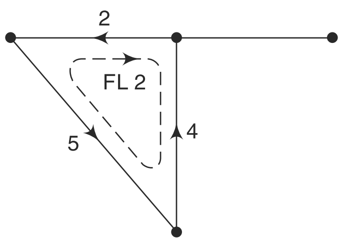


Figure 3.7(d) *Loop-2*

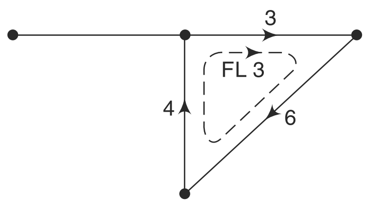


Figure 3.7(e) *Loop-3*

Fundamental Loop 1 (FL1): Connecting link 1 to the tree.
Fundamental Loop 2 (FL2): Connecting link 5 to the tree.
Fundamental Loop 3 (FL3): Connecting link 6 to the tree.
These sets of branches (1, 2, 3), (2, 4, 5) and (3, 4, 6) form three tie-sets.

3.7.1 Tie-Set Matrix or Loop Incidence Matrix or Circuit Matrix (B_a)

For a given graph having n nodes and b branches, tie-set matrix is a rectangular matrix with b columns and as many rows as there are loops. Its elements have the following values:

B_{ij} = 1, if branch j is in loop i and their orientations coincide (i.e., loop current and branch current flow in the same direction);
= –1, if branch j is in loop i and their orientations do not coincide;
= 0, if branch j is not in loop i.

Example For the graph shown in Fig. 3.8(a) and tree selected in Fig. 3.8(b), the tie-set matrix is written as follows. The entries in the Tie-set schedule are given as +1 or –1 if the branch current is in the same direction as the link current or not. If the branch current does not depend on the link current, then entry is zero.

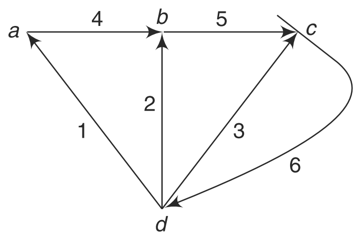


Figure 3.8(a) *Graph*

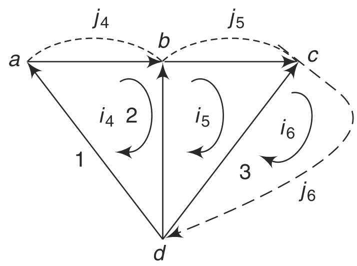


Figure 3.8(b) *Formation of loops*

Branch no. (i)

Links (j)	1	2	3	4	5	6
4	1	–1	0	1	0	0
5	0	1	–1	0	1	0
6	0	0	1	0	0	1

3.7.2 Tie-Set Matrix and KVL

For the graph shown in Fig. 3.7(a) and three loops shown in Fig. 3.7(c), (d) and (e), three fundamental mesh KVL equations can be written as follows.
For Fundamental Loop 1 (FL1): $v_{b1} - v_{b3} + v_{b2} = 0$
For Fundamental Loop 2 (FL2): $-v_{b2} - v_{b4} - v_{b5} = 0$
For Fundamental Loop 3 (FL3): $v_{b3} + v_{b6} + v_{b4} = 0$

These equations in matrix form is written as,

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{b1} \\ v_{b2} \\ v_{b3} \\ v_{b4} \\ v_{b5} \\ v_{b6} \end{bmatrix} = 0$$

or $\boxed{B_a V_b = 0}$

3.7.3 Tie-Set Matrix and KCL

For the graph shown in Fig. 3.7(a) and three loops shown in Fig. 3.7(c), (d) and (e), the branch currents ($i_{b1}, i_{b2}, \ldots, i_{b6}$) can be represented in terms of the loop currents (I_{L1}, I_{L2}, I_{L3}) as,

$$i_{b1} = I_{L1}, \quad i_{b2} = (I_{L1} - I_{L2}), \quad i_{b3} = (-I_{L1} + I_{L3}), \quad i_{b4} = (-I_{L2} + I_{L3}), \quad i_{b5} = I_{L2}, \; i_{b6} = I_{L3}$$

In matrix form, these equations can be written as,

$$\begin{bmatrix} i_{b1} \\ i_{b2} \\ i_{b3} \\ i_{b4} \\ i_{b5} \\ i_{b6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{L1} \\ I_{L2} \\ I_{L3} \end{bmatrix}$$

or $\boxed{I_b = B_a^T I_L}$

3.8 CUT-SET MATRIX AND NODE-PAIR POTENTIAL

Cut-Set A cut-set is a minimum set of elements that when cut, or removed, separates the graph into two groups of nodes. A cut-set is a **minimum set of branches** of a connected graph, such that the removal of these branches from the graph **reduces the rank of the graph by one.**

In other words, for a given connected graph (G), a set of branches (C) is defined as a cut-set if and only if:

(i) the removal of all the branches of C results in an unconnected graph.

(ii) the removal of all but one of the branches of C leaves the graph still connected.

Example Consider the graph shown in Fig. 3.9(a). The rank of the graph is 3.
The removal of branches 1 and 3 reduces the graph into two connected subgraphs as shown in Fig. 3.9(b).

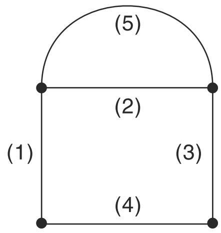


Figure 3.9(a) *Graph*

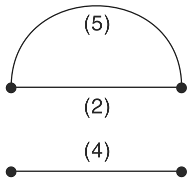


Figure 3.9(b) *Subgraphs with removal of 1 and 3*

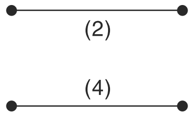


Figure 3.9(c) *Subgraphs with removal of 1, 3 and 5*

The rank of the graph of Fig. 3.9(a) = (4 – 1) = 3
The rank of the graph of Fig. 3.9(b) = Addition of the ranks of the subgraphs = (1 + 1) = 2
So, branches [1, 3] may be a cut-set.
Also, removal of the branches 1, 3 and 5 reduces the graph into two connected subgraphs as shown in Fig. 3.9(c) and the rank becomes 2. So, [1, 3, 5] may also be a cut-set.
As cut-set is the minimum set of branches and [1, 3] is a subset of [1, 3, 5], so [1, 3] is the cut-set, [1, 3, 5] is not a cut-set.

3.8.1 Fundamental Cut-Set

A fundamental cut-set (FCS) is a cut-set that cuts or contains **one and only one tree branch**. Therefore, for a given tree, the number of fundamental cut-sets will be equal to the number of twigs.

3.8.2 Procedure for Finding the Fundamental Cut-sets

1. First, select a tree of the given graph.
2. Focus on a tree branch (b_k).
3. Check whether removing this tree branch (b_k) from the tree disconnects the tree into two separate parts.
4. All the links which go from one part of this disconnected tree to the other, together with the tree branch (b_k) forms a fundamental cut-set.

Following this procedure, the fundamental cut-sets for the graph of Fig. 3.10 will be

f-cut-set – 1: [1, 2, 6];
f-cut-set – 2: [2, 3, 5, 6];
f-cut-set – 3: [4, 5, 6]

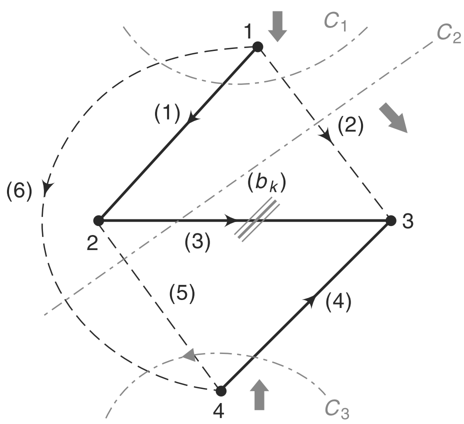


Figure 3.10 *Graph illustrating fundamental cut-set*

3.8.3 Properties of Cut-Set

1. A cut-set divides the set of nodes into two subsets.
2. Each fundamental cut-set contains one tree-branch, the remaining elements being links.
3. Each branch of the cut-set has one of its terminals incident at a node in one subset and its other terminal at a node in the other subset.

4. A cut-set is oriented by selecting an orientation from one of the two parts to the other. Generally, the direction of cut-set is chosen same as the direction of the tree branch.

3.8.4 Cut-Set Matrix (Q_C)

For a given graph, a cut-set matrix (Q_C) is defined as a rectangular matrix whose rows correspond to cut-sets and columns correspond to the branches of the graph. Its elements have the following values:

Q_{ij} = 1, if branch j is in the cut-set i and the orientations coincide.
= –1, if branch j is in the cut-set i and the orientations do not coincide.
= 0, if branch j is not in the cut-set i.

Example For the graph shown in Fig. 3.10, fundamental cut-sets have been identified as follows.

f-cut-set – 1: [1, 2, 6];
f-cut-set – 2: [2, 3, 5, 6];
f-cut-set – 3: [4, 5, 6]

So, the cut-set matrix is written as,

Branch no:

f-cut-sets	1	2	3	4	5	6
1	1	1	0	0	0	1
2	0	1	1	0	1	1
3	0	0	0	1	–1	–1

3.8.5 Cut-Set Matrix and KVL

By cut-set schedule, the branch voltages can be expressed in terms of the tree-branch voltages.

A cut-set consists of *one and only one* branch of the tree together with any links which must be cut to divide the network into two parts. A set of fundamental cut-sets includes those cut-sets which are obtained by applying cut-set division for each of the branches of the network tree.

Consider the following graph.

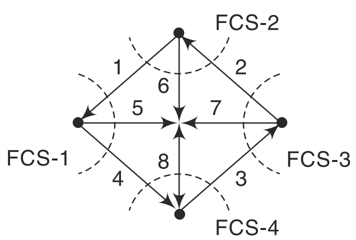


Figure 3.11(a) *Graph*

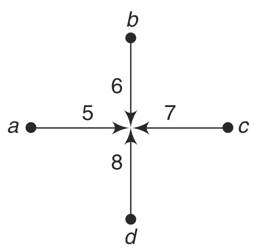


Figure 3.11(b) *Tree*

Applying cut-sets at nodes *a*, *b*, *c*, *d*, which are the fundamental cut-sets (FCS), we can write the cut-set schedule as follows.

		1	2	3	4	5	6	7	8
FCS-1 →	*a*	–1	0	0	1	1	0	0	0
FCS-2 →	*b*	1	–1	0	0	0	1	0	0
FCS-3 →	*c*	0	1	1	0	0	0	1	0
FCS-4 →	*d*	0	0	–1	–1	0	0	0	1

The tree-branch voltages are $[v_{t5}, v_{t6}, v_{t7}, v_{t8}]$ and the branch voltages are $[V_{b1}, V_{b2}, \ldots V_{b8}]$ and the relationship between tree-branch voltages and branch voltages are:

$$\begin{aligned} V_{b1} &= -v_{t5} + v_{t6} & V_{b5} &= v_{t5} \\ V_{b2} &= -v_{t6} + v_{t7} & V_{b6} &= v_{t6} \\ V_{b3} &= v_{t7} - v_{t8} & V_{b7} &= v_{t7} \\ V_{b4} &= v_{t5} - v_{t8} & V_{b8} &= v_{t8} \end{aligned}$$

The above equations can be related by using the cut-set schedule as:

$$\begin{bmatrix} V_{b1} \\ V_{b2} \\ V_{b3} \\ V_{b4} \\ V_{b5} \\ V_{b6} \\ V_{b7} \\ V_{b8} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{t5} \\ v_{t6} \\ v_{t7} \\ v_{t8} \end{bmatrix}$$

or $$\boxed{V_b = Q_c^T V_t}$$

3.8.6 Cut-Set Matrix and KCL

For the graph of Fig. 3.11, writing Kirchhoff's current laws for the nodes, the branch currents can be expressed as,

$$\begin{aligned} &\text{Node } a: & -i_{b1} + i_{b4} + i_{b5} &= 0 \\ &\text{Node } b: & i_{b1} - i_{b2} + i_{b6} &= 0 \\ &\text{Node } c: & i_{b2} + i_{b3} + i_{b7} &= 0 \\ &\text{Node } d: & -i_{b3} - i_{b4} + i_{b8} &= 0 \end{aligned}$$

In matrix form, they can be written as,

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_{b1} \\ i_{b2} \\ i_{b3} \\ i_{b4} \\ i_{b5} \\ i_{b6} \\ i_{b7} \\ i_{b8} \end{bmatrix} = 0$$

or $$\boxed{Q_c I_b = 0}$$

There is a cut-set matrix for a given tree. If a graph contains more than one tree, there will be as many numbers of cut-set matrices as the number of tree of the graph.

To summarize, KVL and KCL equations in three matrix forms are given below.

Matrix	*KCL*	*KVL*
Incidence Matrix (A_a)	$A_a \times I_b = 0$	$V_b = A_a^T \times V_n$
Tie-Set Matrix (B_a)	$I_b = B_a^T \times I_L$	$B_a \times V_b = 0$
Cut-Set Matrix (Q_C)	$Q_C \times I_b = 0$	$V_b = Q_C^T \times V_t$

3.9 FORMULATION OF NETWORK EQUILIBRIUM EQUATIONS

The network equilibrium equations are a set of equations that completely and uniquely determine the state of a network at any instant of time. These equations are written in terms of suitably chosen current variables or voltage variables.

These equations will be unique if the number of independent variables be equal to the number of independent equations.

Number of Independent Variables or Equations $= b - (n - 1)$; for loop method of analysis
$= (n - 1)$; for node method of analysis.

The equations for a network can be formed in either of the two methods as given below.

1. Through a set of voltage law equations in which the currents are the independent variables (Loop-Basis Method);
2. Through a set of current law equations in which the node-pair voltages are the independent variables (Node-Basis Method).

3.9.1 Formulation of Network Equations on Loop Basis

Steps

1. Draw the ***directed graph*** of the network selecting the direction of assumed current flow to coincide for current sources.
2. Select a tree of the graph.
3. Place all ***voltage sources in the tree*** and all ***current sources in the co-tree***.
4. Place all control-voltage branches for voltage-controlled dependent sources in the tree and all control-current branches for current-controlled dependent sources in the co-tree, if possible.
5. Add one link to the tree, creating a fundamental loop, and write a KVL equation for this fundamental loop (FL). Repeat for each additional link until L $(= b - n + 1)$ mesh equations are obtained in the form $B_a \times V_b = 0$.
6. The current sources in the cotree, if present, will provide the constraint equations.
7. The KCL equations are obtained by representing the branch currents in terms of loop currents in the form $I_b = B_a^T \times I_L$.
8. For each branch, the relationship between the voltage and current is obtained from Ohm's law ($V = RI$).
9. Finally, the equilibrium equations are obtained in terms of loop currents by suitable substitution of the equations obtained in steps 5 to 8.

3.9.2 Formulation of Network Equations on Node Basis

Steps

1. Draw a directed graph of the circuit under considerations, selecting the directions of assumed current flow to coincide for current sources.
2. Select the tree of the graph so that current ***sources are in the co-tree and the voltage sources are within the tree, if possible***. Also, if possible, select the tree so that ***at least two branches of the tree are incident at the reference node***.
3. Identify (n – 1) fundamental cut-sets (FCS) and draw the FCS lines.
4. Write the (n – 1) FCS KCL equations in the form $\boldsymbol{A_a \times I_b = 0}$ **or** $\boldsymbol{Q_C \times I_b = 0}$.
5. Obtain each of the branch currents in terms of node voltages in the form $\boldsymbol{V_b = A_a^T \times V_n}$ or, $\boldsymbol{V_b = Q_C^T \times V_t}$.
6. For each branch, the relationship between the voltage and current is obtained from Ohm's law ($\boldsymbol{V = RI}$).
7. Substitute the equations of step 6 into the KVL equations of step 5 and finally into the KCL equations of step 4, thus obtaining the (n – 1) independent node voltage equations.

3.9.3 Generalized Equations in Matrix Forms for Circuits having Sources

A general branch consisting of a voltage source V_s and a current source I_s is shown in Fig. 3.12.

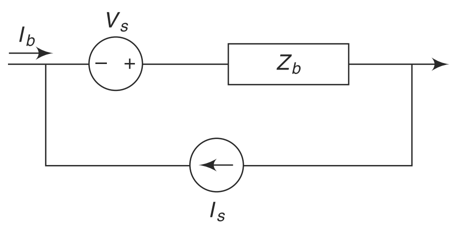


Figure 3.12

Here, the branch current is (I_b + I_s) and the branch voltage is $V_b = Z_b(I_b + I_s) - V_s$.

Without sources, the KCL and KVL equations are:

$$\left.\begin{aligned} &A_a \times I_b = 0 && (3.1)\\ &I_b = B_a^T \times I_L && (3.2)\\ &Q_C \times I_b = 0 && (3.3)\end{aligned}\right\}\text{KCL}$$

and

$$\left.\begin{aligned} &V_b = A_a^T \times V_n && (3.4)\\ &B_a \times V_b = 0 && (3.5)\\ &V_b = Q_C^T \times V_t && (3.6)\end{aligned}\right\}\text{KVL}$$

With the sources, the KCL and KVL equations are modified as,

$$A_a I_b + A_a I_s = 0 \tag{3.7}$$

$$I_b + I_s = B_a^T I_L \tag{3.8}$$

$$Q_c I_b + Q_c I_s = 0 \tag{3.9}$$

and

$$V_b + V_s = A_a^T V_n \tag{3.10}$$

$$B_a V_b + B_a V_s = 0 \tag{3.11}$$

$$V_b + V_s = Q_c^T V_t \tag{3.12}$$

The branch voltage-current relations for the passive network elements are written in matrix form as,

$$V_b = Z_b I_b \tag{3.13}$$

and

$$I_b = Y_b V_b \tag{3.14}$$

where, $\boldsymbol{Z_b}$ is the *branch impedance matrix* and $\boldsymbol{Y_b}$ is the *branch admittance matrix*, both of the order $b \times b$. On the basis of these equations the general equations can be written in terms of three matrices as follows.

Node Equations From equation (3.7),

$$A_a I_s = -A_a I_b = -A_a Y_b V_b = -A_a Y_b (A_a^T V_n - V_s) \qquad \text{\{by equation (3.10)\}}$$

or $$A_a Y_b A_a^T V_n = A_a Y_b V_s - A_a I_s = A_a [Y_b V_s - I_s]$$

or $$\boxed{YV_n = A_a[Y_b V_s - I_s]}$$

where, $\boldsymbol{Y}$ is called the *nodal admittance matrix* of the order of $(n - 1) \times (n - 1)$. The above equation represents a set of $(n - 1)$ number of equations, known as *node equations*.

Mesh Equations From equation (3.11),

$$B_a V_s = -B_a V_b = -B_a Z_b I_b = -B_a Z_b (B_a^T I_L - I_s) \qquad \text{\{by equation (3.8)\}}$$

or $$B_a Z_b B_a^T I_L = B_a [Z_b I_s - V_s]$$

or $$\boxed{ZL_L = B_a[Z_b I_s - V_s]}$$

where, $\boldsymbol{Z}$ is the *loop-impedance matrix* of the order of $(b - n + 1) \times (b - n + 1)$. The above equation represents a set of $(b - n + 1)$ number of equations, known as *mesh* or *loop equations*.

Cut-set Equations From equation (3.8),

$$Q_c I_s = -Q_c I_b = -Q_c Y_b V_b = -Q_c Y_b (Q_c^T V_t - V_s) \qquad \text{\{by equation (3.12)\}}$$

or $$Q_c Y_b Q_c^T V_t = Q_c [Y_b V_s - I_s]$$

or $$\boxed{Y_c V_t = Q_c[Y_b V_s - I_s]}$$

where, $\boldsymbol{Y_c}$ is the *cut-set admittance matrix* of the order of $(n - 1) \times (n - 1)$ and the set of $(n - 1)$ equations represented by the above equation is known as *cut-set equations*.

3.10 SOLUTION OF EQUILIBRIUM EQUATIONS

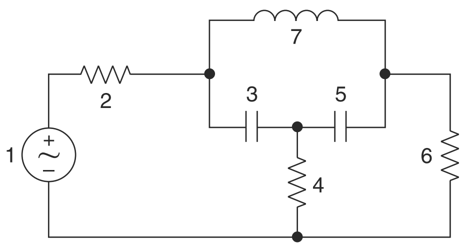
There are two methods of solving equilibrium equations given as follows.

(i) *Elimination method*: by eliminating variables until an equation with a single variable is achieved, and then by the method of substitution.

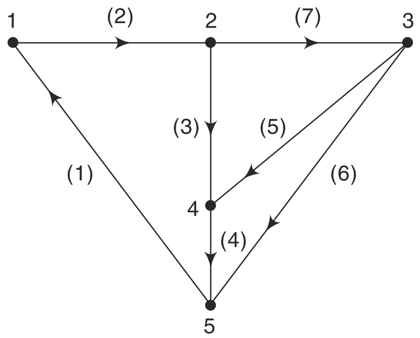
(ii) *Determinant method*: by the method known as Cramer's rule.

SOLVED EXERCISES

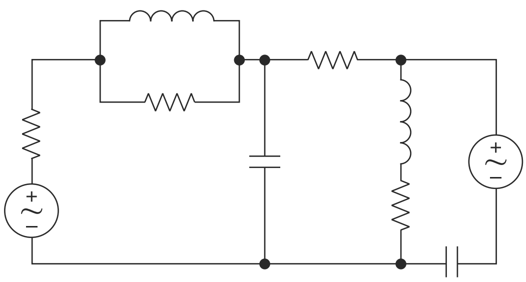
3.1 Draw the graph of the network shown in the figure.



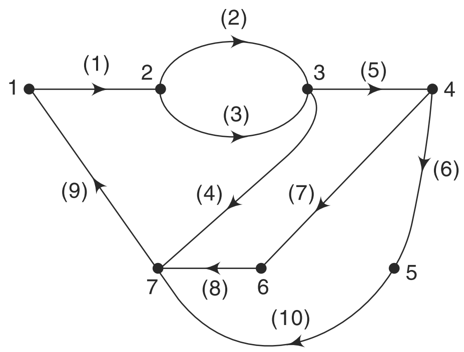
Solution The graph of the network is shown below.



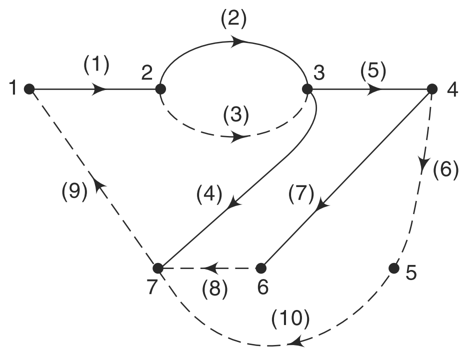
3.2 From the figure, make the graph and find one tree. How many mesh currents are required for solving the network? Find the number of possible trees.



Solution The graph of the network is shown below. One tree of the graph is shown.



Graph of the network



Tree of the graph

The complete incidence matrix is obtained as,

A_a =

Nodes	*Branches*									
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	1	0	0	0	0	0	0	0	–1	0
2	–1	1	1	0	0	0	0	0	0	0
3	0	–1	–1	1	1	0	0	0	0	0
4	0	0	0	0	–1	1	1	0	0	0
5	0	0	0	0	0	–1	0	0	0	1
6	0	0	0	0	0	0	–1	1	0	0
7	0	0	0	–1	0	0	0	–1	1	–1

Reduced incidence matrix becomes,

A =

Nodes	*Branches*									
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	1	0	0	0	0	0	0	0	–1	0
2	–1	1	1	0	0	0	0	0	0	0
3	0	–1	–1	1	1	0	0	0	0	0
4	0	0	0	0	–1	1	1	0	0	0
5	0	0	0	0	0	–1	0	0	0	1
6	0	0	0	0	0	0	–1	1	0	0

Hence the number of possible trees is,

$$n = \det\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\right\}$$

$$= \det\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -2 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$\Rightarrow \quad n = 12 \quad$ *Ans.*

3.3 Branch current and loop current relations are expressed in matrix form as,

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}$$

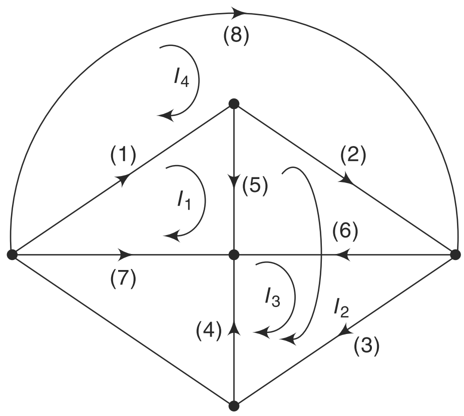
Draw the oriented graph.

Solution We know that, $[I_b] = [B_a]^T [I_L]$. So, the tie-set matrix, here, is,

$B_a =$

Loop or Link Currents	*Branches* 1	2	3	4	5	6	7	8
1	1	0	0	0	1	0	–1	0
2	0	1	1	1	–1	0	0	0
3	0	0	1	1	0	–1	0	0
4	–1	–1	0	0	0	0	0	1

So, the graph consists of four loops and eight branches. Loop 1 consists of branch 1, 5 and 7. The orientations are given following the sign +1 or –1. Following the procedure, the complete oriented graph is shown below.



3.4 The fundamental cut-set matrix is given as,

Twigs				*Links*		
1	2	3	4	5	6	7
1	0	0	0	–1	0	0
0	1	0	0	1	0	1
0	0	1	0	0	1	1
0	0	0	1	0	1	0

Draw the oriented graph of the network.

Solution The graph has seven branches and four fundamental cut-sets:

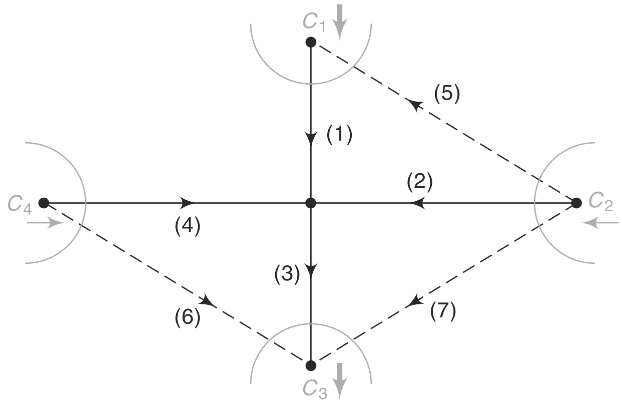
Cut-set-1: [1, 5]

Cut-set-2: [2, 5, 7]

Cut-set-3: [3, 6, 7]

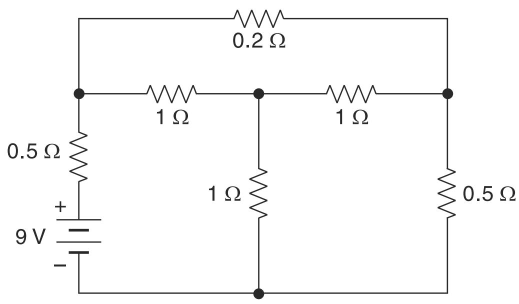
Cut-set-4: [4, 6]

So, the oriented graph is as shown in figure.

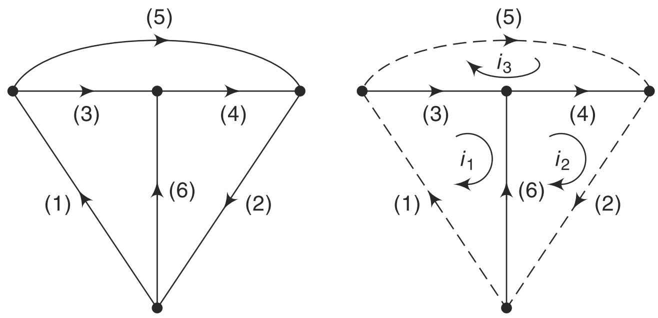


3.5 (a) For the network of the figure, draw the graph and write a tie-set schedule. Using the tie-set schedule obtain the loop equations and find the currents in all branches.

(b) For the network of (a), write a cut-set schedule, obtain nodal equations and find branch currents.



Solution The graph and one tree are shown in figure.



The tie-set matrix,

$$B_a = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}$$

Branch impedance matrix is,

$$Z_b = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$[B_a][Z_b] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0.5 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0.2 & 0 \end{bmatrix}$$

$\therefore$

$$[B_a][Z_b][B_a]^T = \begin{bmatrix} 0.5 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0.5 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2.5 & -1 & -1 \\ -1 & 2.5 & -1 \\ -1 & -1 & 2.2 \end{bmatrix}$$

Now, $$-[B_a][V_s] = -\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$$

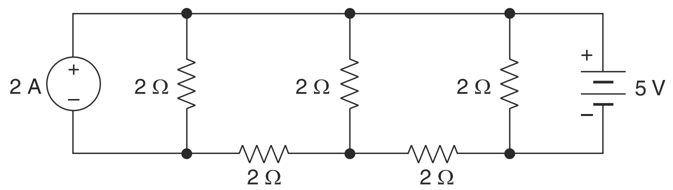
So, the loop equations are,

$$\begin{bmatrix} 2.5 & -1 & -1 \\ -1 & 2.5 & -1 \\ -1 & -1 & 2.2 \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$$

Solving three equations,

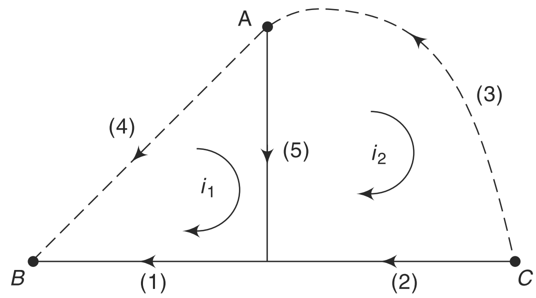
$$i_1 = 8.9 \text{ A}, \ i_2 = 6.33 \text{ A}, \ i_3 = 6.92 \text{ A} \qquad \textit{Ans.}$$

3.6 Figure shows a d.c. network. (a) Draw a graph of the network. Which elements are not included in the graph and why? (b) Write a loop incidence matrix and use it to obtain loop equations. (c) Find branch currents.



Solution

(a) The graph is shown below.
The 2 Ω resistor in parallel with voltage source and the 2 A current source have not been included in the graph. This is because of the reason that passive elements in parallel with a voltage source are not included in graph and the current source in parallel with a passive element is open-circuited while drawing graph.



(b) The tie-set matrix for the tree chosen is,

$$B_a = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}$$

Branch impedance matrix is,

$$Z_b = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$B_a Z_b B_a^T = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & -2 & 2 \\ 0 & 2 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$$

Now,

$$B_a Z_b I_s - B_a V_s = \begin{bmatrix} 2 & 0 & 0 & -2 & 2 \\ 0 & 2 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

So, the loop equations are,

$$\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Solving these equations, $i_1 = 0.3$ A, $i_2 = -1.1$ A *Ans.*

(c) Putting these values, the branch voltages are

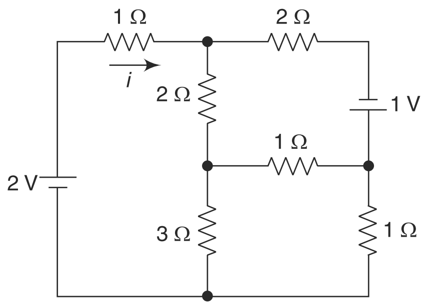
$V_1 = 2 \times i_1 = 0.6$ V, $V_2 = 2 \times i_2 = -2.2$ V, $V_3 = -5$ V, $V_4 = -2 \times i_1 + 4 = 3.4$ V, $V_5 = 2.8$ V *Ans.*

Thus, the branch currents are

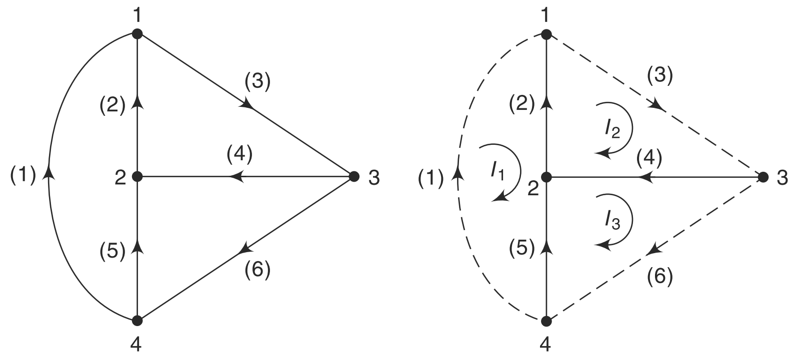
$$I_{AB} = \frac{3.4}{2} = 1.7 \text{ A}, \; I_{AD} = \frac{2.8}{2} = 1.4 \text{ A}, \; I_{AC} = \frac{5}{2} = 2.5 \text{ A}, \; I_{DB} = \frac{0.6}{2} = 0.3 \text{ A}, \; I_{DC} = \frac{2.2}{2} = 1.1 \text{ A}$$

So, the current supplied by the battery = (1.7 + 1.4 + 2.5 – 2) = 3.6 A *Ans.*

3.7 For the network shown in figure, draw the oriented graph and obtain the tie-set matrix. Use this matrix to calculate *i*.



Solution The oriented graph and any one tree are shown.



The tie-set matrix is given as,

$$B_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

The branch impedance matrix,

$$Z_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & -2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b B_a^T = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & -2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{bmatrix}$$

$$\text{Now,} \qquad -B_a V_s = -\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

So, the loop equations become,

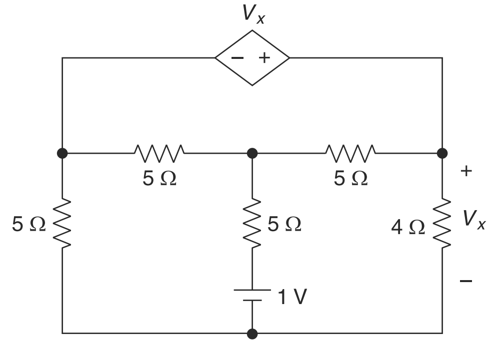
$$\begin{bmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Solving for I_1,

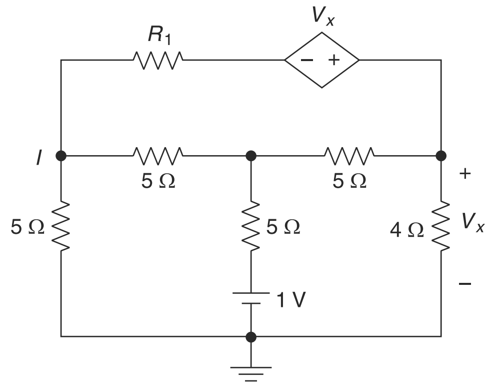
$$I_1 = \frac{\begin{vmatrix} 2 & -2 & -3 \\ 1 & 5 & -1 \\ 0 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{vmatrix}} = 0.91 \text{ A}$$

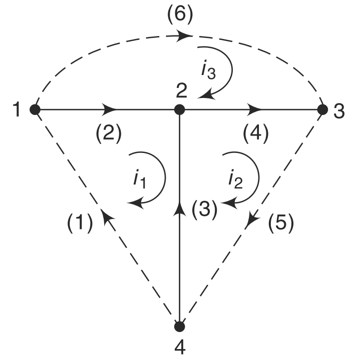
$$\therefore \qquad i_1 = 0.91 \text{ A}$$

3.8 The circuit of figure contains a voltage controlled voltage source. For this circuit, draw the oriented graph. By selecting a proper tree obtain the tie-set matrix and hence calculate the voltage, V_x.



Solution Since the controlled voltage source is not accompanied by any passive element, we will consider a resistance R_1 in series with the controlled voltage source, and finally let $R_1 \to 0$.





The graph of the network is shown with one tree.
The tie-set matrix is,

$$B_a = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The branch impedance matrix,

$$Z_b = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_1 \end{bmatrix} = \begin{bmatrix} 5 & 5 & -5 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 & 4 & 0 \\ 0 & -5 & 0 & 5 & 0 & R_1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b B_a^T = \begin{bmatrix} 5 & 5 & -5 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 & 4 & 0 \\ 0 & -5 & 0 & 5 & 0 & R_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -5 & -5 \\ -5 & 14 & -5 \\ -5 & -5 & (10+R_1) \end{bmatrix}$$

$$\text{Now,} \qquad -B_a V_s = -\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -V_x \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \\ -V_x \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ V_x \end{bmatrix}$$

So, the loop equations become,

$$\begin{bmatrix} 15 & -5 & -5 \\ -5 & 14 & -5 \\ -5 & -5 & (10+R_1) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ V_x \end{bmatrix}$$

With $R_1 \to 0$ and $V_x = 4I_2$, the equations reduce to,

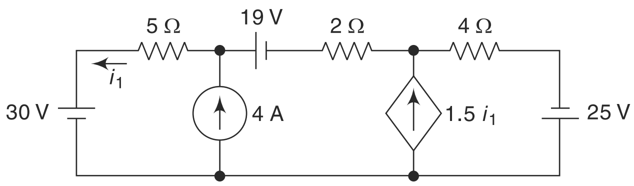
$$\begin{bmatrix} 15 & -5 & -5 \\ -5 & 14 & -5 \\ -5 & -9 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Solving for I_2,

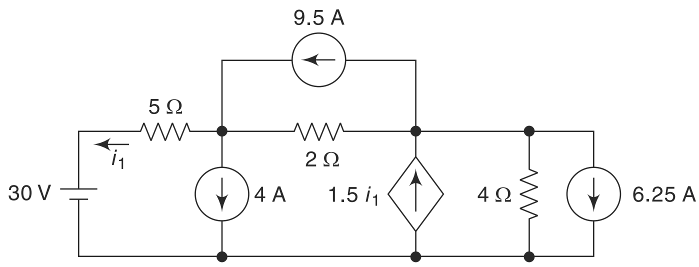
$$I_2 = \frac{\begin{vmatrix} 15 & -1 & -5 \\ -5 & 1 & -5 \\ -5 & 0 & 10 \end{vmatrix}}{\begin{vmatrix} 15 & -5 & -5 \\ -5 & 14 & -5 \\ -5 & -9 & 10 \end{vmatrix}} = \frac{1}{19}\text{ A}$$

$$\therefore \qquad V_x = 4 \times I_2 = 4 \times \frac{1}{19} = \frac{4}{19}\text{ V} \qquad \textit{Ans.}$$

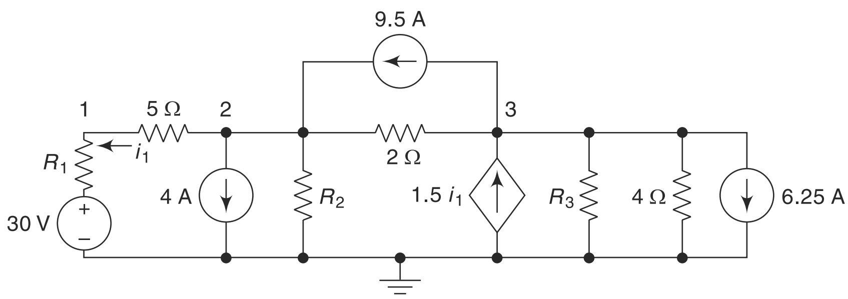
3.9 Determine the current i_1 in the circuit using nodal analysis method and graph theory concepts.



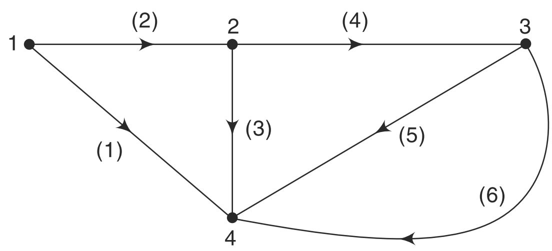
Solution By source transformation technique, we convert the 19 V and 25 V voltage sources into current sources.



Since the 30 V voltage source, the 4 A current source, and controlled current source are not accompanied by the passive elements, we consider three resistors R_1, R_2 and R_3 and finally let, $R_1 \to 0$, $R_2 \to \propto$, and $R_3 \to \propto$.



The graph of the network is shown.



The complete incidence matrix is,

$$A_a = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Reduced Incidence matrix is,

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

Branch admittance matrix is,

$$Y_b = \begin{bmatrix} G_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \quad \text{where, } G_1 = \frac{1}{R_1}, G_2 = \frac{1}{R_2}, G_3 = \frac{1}{R_3}$$

$$\therefore \qquad AY_b = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} G_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} G_1 & -\frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & G_2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{2} & G_3 & \frac{1}{4} \end{bmatrix}$$

$$\therefore \qquad AY_bA^T = \begin{bmatrix} G_1 & -\frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & G_2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{2} & G_3 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(G_1 + \frac{1}{5}\right) & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \left(G_1 + \frac{1}{5} + \frac{1}{2}\right) & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \left(G_3 + \frac{1}{2} + \frac{1}{4}\right) \end{bmatrix}$$

$$AY_bV_s - AI_s = -AI_s = -\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -30G_1 \\ 0 \\ 4 \\ -9.5 \\ -1.5i_1 \\ 6.25 \end{bmatrix} \quad \{\because \text{We made } V_s = 0\}$$

Now,

$$= -\begin{bmatrix} -30G_1 \\ -5.5 \\ (15.75 - 1.5\, i_1) \end{bmatrix}$$

Thus, node equations are,

$$\begin{bmatrix} \left(G_1 + \frac{1}{5}\right) & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \left(G_2 + \frac{7}{10}\right) & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \left(G_3 + \frac{3}{4}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 30G_1 \\ 5.5 \\ 1.5\, i_1 - 15.75 \end{bmatrix}$$

With $R_1 \to 0$, $G_1 \to \propto$, $R_2 \to \propto$, $G_2 \to 0$, $R_3 \to \propto$, $G_3 \to 0$ the equations become:

$$\left(G_1 + \frac{1}{5}\right)V_1 - \frac{1}{5}V_2 = 30G_1$$

$$-\frac{1}{5}V_1 + \left(G_2 + \frac{7}{10}\right)V_2 - \frac{1}{2}V_3 = 5.5$$

$$-\frac{1}{2}V_2 + \frac{3}{4}V_3 = (1.5i_1 - 15.75)$$

or

$$V_1 = 30 \tag{i}$$

$$-\frac{1}{5}V_1 + \frac{7}{10}V_2 - \frac{1}{2}V_3 = 5.5 \quad \Rightarrow \quad 7V_2 - 5V_3 = 115 \tag{ii}$$

$$-\frac{1}{2}V_2 + \frac{3}{4}V_3 = \left[1.5\left(\frac{V_2 - V_1}{5}\right) - 15.75\right] \quad \Rightarrow \quad 16V_2 - 15V_3 = 495 \tag{iii}$$

Solving equations (i), (ii), and (iii), we get,

$$V_2 = -30 \text{ V}, \quad V_3 = 65 \text{ V}$$

Hence, the current, $i_1 = \left(\dfrac{V_2 - V_1}{5}\right) = \dfrac{-30 - 30}{5} = -12 \text{ A}$ *Ans.*

MULTIPLE-CHOICE QUESTIONS

3.1 The number of links for a graph having n nodes and b branches are
(a) $b - n + 1$ (b) $n - b + 1$ (c) $b + n - 1$ (d) $b + n$

3.2 The tree branches of a graph are called
(a) chords (b) links (c) twigs (d) co-tree

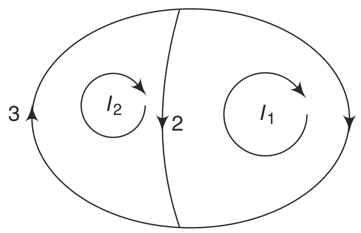
3.3 The tie-set matrix gives the relation between
(a) branch currents and link currents. (b) branch voltages and link currents.
(c) branch currents and link voltages. (d) none of these.

3.4 The graph of a network has six branches with three tree branches. The minimum number of equations required for the solution of the network is
(a) 2 (b) 3 (c) 4 (d) 5

3.5 For a connected planar graph of v vertices and e edges, the number of meshes is
(a) $(e - v + 1)$ (b) $(e + v + 1)$ (c) $(e + v - 1)$

3.6 The number of chords of a tree of a connected graph G of v vertices and e edges is
(a) $(v - 1)$ (b) $(e - v + 1)$ (c) $(e - v - 1)$

3.7 The table meant for the oriented graph represents



Link or Loop Current	← Branch →		
	1	2	3
i_1	+1	–1	0
i_2	0	+1	+1

(a) tie-set matrix (b) cut-set matrix (c) incidence matrix (d) none of the above.

3.8 The reduced incidence matrix of a circuit is given by

$$A_i = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The set of branches forming a tree are
(a) 1, 2 and 3 (b) 2, 3 and 5 (c) 1, 2 and 4 (d) 1, 2 and 6.

3.9 Relative to a given fixed tree of a network
1. link currents form an independent set.

2. branch currents form an independent set.
3. link voltages form an independent set.
4. branch voltages form an independent set.

Of these statements

(a) 1, 2, 3 and 4 are correct. (b) 1, 2 and 3 are correct.
(c) 2, 3 and 4 are correct (d) 1, 3 and 4 are correct

3.10 For a given network the incidence matrix is given as

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The series branches in the graph are

(a) 3 and 4 (b) 6 and 7 (c) 2 and 3 (d) none of the above.

3.11 For a given network the incidence matrix is given as

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The parallel branches in the graph are

(a) 1 and 2 (b) 2 and 3 (c) 6 and 7 (d) none of the above.

3.12 For a given network the incidence matrix is given as

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The series branches in the graph are

(a) 3 and 4 (b) 3 and 5 (c) 3 and 6 (d) none of the above.

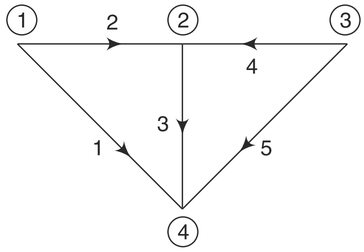
3.13 For a given network the incidence matrix is given as

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The parallel branches in the graph are

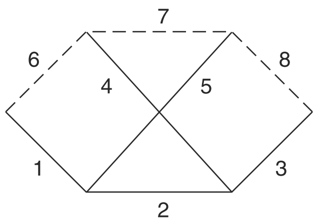
(a) 3 and 5 (b) 4 and 5 (c) 3 and 6 (d) none of the above.

3.14 Which one of the following represents the total number of trees in the graph given in the figure?



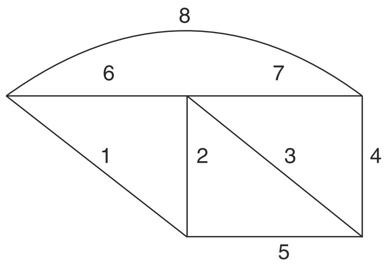
(a) 4 (b) 5 (c) 6 (d) 8

3.15 In the graph and the tree shown in the given figure, the fundamental cut-set for the branch 2 is



(a) 2, 1, 5 (b) 2, 6, 7, 8 (c) 2, 1, 3, 4, 5 (d) 2, 3, 4

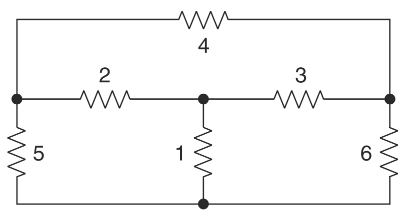
3.16 In the graph shown in the figure, one possible tree is formed by the branches 4, 5, 6, 7. Then one possible fundamental cut set is



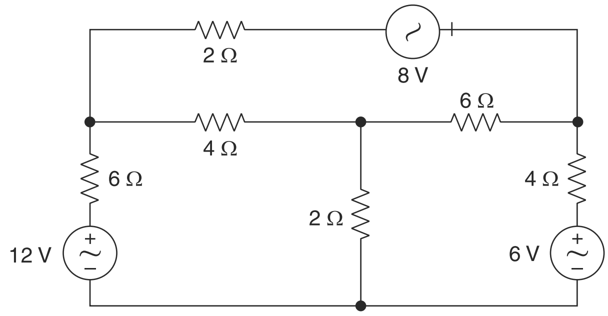
(a) 1, 2, 3, 8 (b) 1, 2, 5, 6 (c) 1, 5, 6, 8 (d) 1, 2, 3, 7, 8

EXERCISES

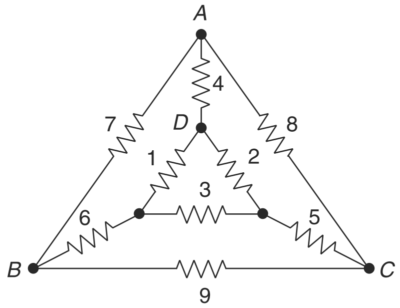
3.1 For the network shown in the figure, draw the graph and a possible tree. Show the links and write the tie-set matrix. Write the equations of the branch currents in terms of loop currents.



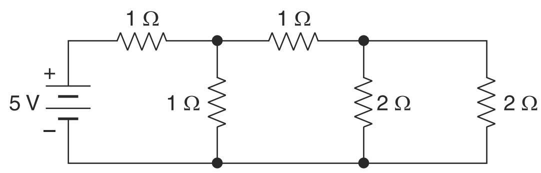
3.2 Find out the currents through and voltage across all branches of the network shown in the figure with the help of its tie-set schedule.



3.3 Find a tree from the graph of the network shown in the figure. Make the tie-set matrix and write the equations containing branch currents and loop currents. All the values are in ohm.

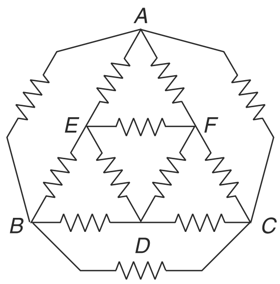


3.4 Draw the graph of the circuit shown in the figure and select a suitable tree to write tie set matrix. Then find the three loop currents.

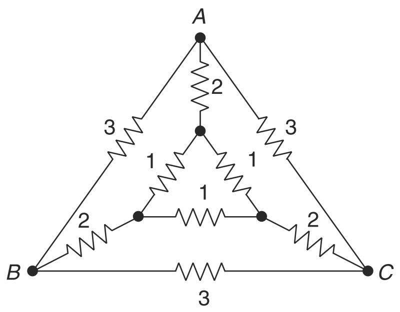


[$i_1 = 3$ A, $i_2 = 1$ A, $i_3 = 0.5$ A]

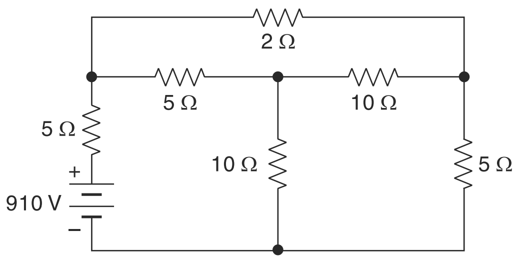
3.5 For the given network of the figure draw the graph and a tree. Write the cut-sets and the cut-set matrix of the tree. Write the equations of link branch voltages in terms of tree branch voltages.



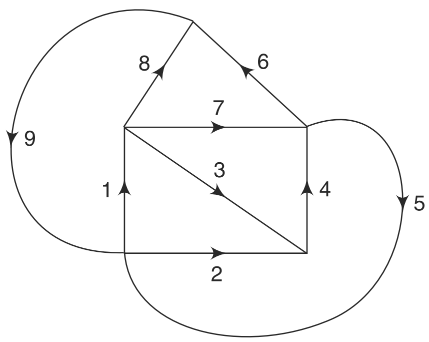
3.6 For the given network of the figure draw the graph and a tree. Write the cut-sets and the cut-set matrix of the tree. Write the equations of link branch voltages in terms of tree branch voltages. All the values are in ohm.



3.7 Calculate the branch voltage and branch currents, using the node basis method for the network given in the figure.



3.8 The linear oriented graph is given in the figure. Considering a tree, mark all the fundamental cut-sets and form the cut-set matrix.



SHORT-ANSWER TYPE QUESTIONS

3.1 (a) Define the following terms.

(i) Graph of a network
(ii) Oriented graph
(iii) Rank of graph
(iv) Planar and Non-planar graph
(v) Subgraph
(vi) Path

(b) State the advantages offered by the graph theory as applied to electric circuit problems.

3.2 (a) Define a Tree of a graph of a network. Mention some basic properties of a 'Tree'. How can you calculate the number of possible trees of a given graph?
(b) Define the following
(i) Twigs (ii) Co-tree (iii) Links or Chords

3.3 (a) Explain what is meant by incidence matrix of a graph and indicate how the values of the incidence matrix elements are obtained.
(b) What are the properties of an incidence matrix?
(c) How can you determine the number of possible trees of a graph with this matrix?

3.4 (a) Explain the term 'tie-set' and 'tie-set matrix' of a network with an illustrative example.
(b) Show that the matrix equation,

$$I_b = B^T I_L$$

where, B is the tie-set matrix and I_b and I_L represent branch current and loop current matrix respectively.

3.5 (a) Define cut-set in a network graph. How can you find out a fundamental cut-set? Mention some properties of a cut-set.
(b) Define cut-set matrix with an illustrative example and show that the matrix equation $QI_b = 0$, where Q is the cut-set matrix and I_b represents the branch current matrix of the graph.
(c) Briefly discuss the relation between branch voltage matrix and node voltage matrix in terms of cut-sets.

3.6 (a) Write notes on: Network Equilibrium Equation.
(b) Establish that the independent loop equations of a network can be formulated from the tie-set matrix of its graph, with illustrative example.
(c) Establish the formulation of node equation of a network from the cut-set matrix.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

3.1 (a)	3.2 (c)	3.3 (a)	3.4 (b)	3.5 (a)	3.6 (b)	3.7 (a)
3.8 (a)	3.9 (b)	3.10 (d)	3.11 (c)	3.12 (c)	3.13 (b)	3.14 (d)
3.15 (b)	3.16 (d)					

CHAPTER

4
Network Theorems

4.1 INTRODUCTION

A *theorem* is a relatively simple rule used to solve a problem, derived from a more intensive analysis using fundamental rules of mathematics. At least hypothetically, any problem in mathematics can be solved just by using the simple rules of arithmetic, but human beings are not as consistent or as fast as a digital computer. We need some shortcut methods in order to avoid procedural errors.

In electric network analysis, the fundamental rules are Ohm's Law and Kirchhoff's Laws. While these humble laws may be applied to analyse any circuit configuration, for complex circuits, it is sometimes necessary to simplify the network to find current or voltage in a particular branch without solving the entire circuit. For this purpose, there are some 'shortcut' methods of analysis, known as *Network Theorem*. As with any theorem of geometry or algebra, the network theorems are also derived from fundamental rules.

4.2 NETWORK THEOREMS

In this chapter, we will discuss the following network theorems:

1. Substitution Theorem
2. Superposition Theorem
3. Reciprocity Theorem
4. Thevenin's Theorem
5. Norton's Theorem
6. Maximum Power Transfer Theorem
7. Tellegen's Theorem
8. Millman's Theorem
9. Compensation Theorem

4.2.1 Substitution Theorem

Statement Any branch in a network may be substituted by a different branch without disturbing the voltages and currents in the entire network, provided the new branch has the same set of terminal voltage and current as the original branch.

Proof In a network *N*, let the number of branches be *b*. The branch method requires the solution of 2*b* equations. Now, after substitution, 2(*b*–1) branch equations remain unaltered. However, as the branch voltage and current of the replaced branch remain unaltered, it implies that the set of 2*b* simultaneous equations will still be satisfied with the same voltages and currents as before. This proves the substitution theorem.

Points to be noted

(i) The substitution theorem is a general theorem and is applicable for any arbitrary network.
(ii) The modified network must have a unique solution.
(iii) This theorem is very useful in circuit analysis of network having non-linear elements.

Example 4.1 We consider the branch *xy* of the circuit shown. Branch voltage V_{xy} = 50 V, and branch current I_{xy} = 5 A. This branch can be substituted by any other branch as shown in Figure 4.1, without altering the voltage and current in the branch.

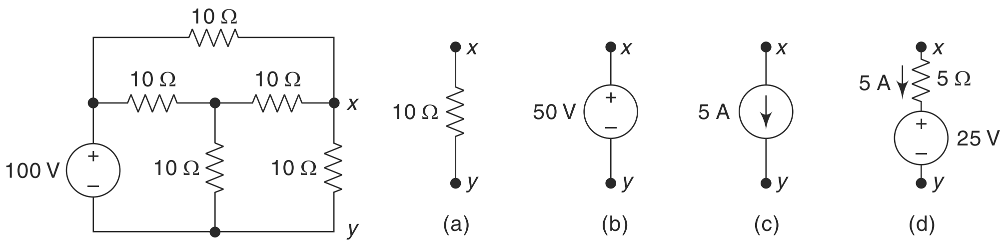


Figure 4.1 *Illustration of Substitution Theorem*

The branch can be substituted using the relation as given below.

$$V_{xy} = Z_{xy} I_{xy} + E, \text{ before substitution}$$

$$V_{xy} = Z'_{xy} I_{xy} + E', \text{ after substitution}$$

4.2.2 Superposition Theorem

Statement This theorem states that in a linear bilateral network, the current at any point (or voltage between any two points) due to the simultaneous action of a number of independent sources in the

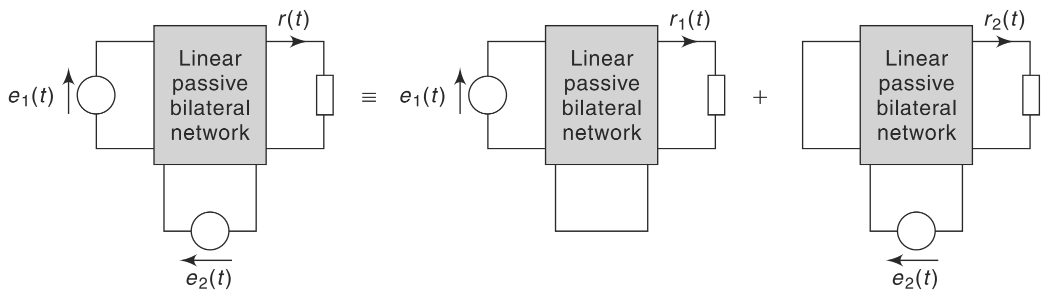


Figure 4.2 *Illustration of Superposition Theorem*

network is equal to the summation of the component currents (or voltages). A component current (or voltage) is defined as that due to one source acting alone in the network with all the remaining sources removed.

Proof

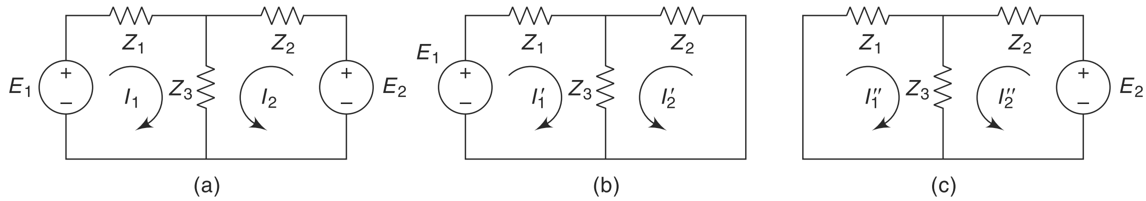


Figure 4.3 *Proof of Superposition Theorem*

Using KVL for the above network, as shown in Fig. 4.3(a)

$$E_1 = I_1(Z_1 + Z_3) + I_2 Z_3$$

$$E_2 = I_1 Z_3 + I_2(Z_2 + Z_3)$$

Solving above two equations,

$$I_1 = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1 - \frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2$$

$$I_1 = \frac{-Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1 + \frac{Z_1 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2$$

Making E_2 inoperative then the circuit diagram becomes as shown in Fig. 4.3 (b)
Then the KVL equations are,

$$E_1 = I_1'(Z_1 + Z_3) + I_2' Z_3$$

$$0 = I_1' Z_3 + I_2'(Z_2 + Z_3)$$

Solving above two equations,

$$I_1' = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1$$

$$I_2' = \frac{-Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1$$

Making E_1 inoperative then the circuit diagram becomes as shown in Fig. 4.3(c)
Then the KVL equations are,

$$0 = I_1''(Z_1 + Z_3) + I_2'' Z_3$$

$$E_2 = I_1'' Z_3 + I_2''(Z_2 + Z_3)$$

Solving above two equations,

$$I_1'' = \frac{-Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2$$

$$I_2'' = \frac{Z_2 + Z_3}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} E_2$$

So, $I_1 = I_1' + I_1''$, $I_2 = I_2' + I_2''$ (Proved)

If an excitation $e_1(t)$ alone gives a response $r_1(t)$, and an excitation $e_2(t)$ alone gives a response $r_2(t)$, then, by superposition theorem, the excitation $e_1(t)$ and the excitation $e_2(t)$ together would give a response $r(t) = r_1(t) + r_2(t)$.

The superposition theorem can even be stated in a more general manner, where the superposition occurs with scaling.

Thus an excitation of $k_1\, e_1(t)$ and an excitation of $k_2\, e_2(t)$ occurring together would give a response of $k_1\, r_1(t) + k_2\, r_2(t)$.

Points to be noted

(i) This theorem is valid for all types of linear circuits.
(ii) This theorem is not valid for power relationship.
(iii) This theorem in not applicable to circuits containing only dependent sources. With dependent sources, superposition can be used only when the controlling functions are external to the network containing sources, so that the controls are unchanged when the sources act at a time.

4.2.3 Reciprocity Theorem

Statement In any linear time-varying network, the ratio of response to excitation remains same for an interchange of the position of excitation and response in the network.

Proof Let us consider a network N having only one driving voltage source $E = E_i$ in loop i and the current source I_j in loop j, Then $I_j = Y_{ji}E_i$; where Y_{ij} is the admittance matrix.

Next, interchanging the positions of cause and effect, i.e., placing the same voltage source $E = E_j$ in loop j, we get the current response I_i in loop i as, $I_i = Y_{ij} E_j$.

Then I_i will be equal to I_j provided, $Y_{ij} = Y_{ji}$.

This is the condition for reciprocity, $Y_{ij} = Y_{ji}$ for all j and i, it signifies that the admittance matrix Y is symmetric.

Points to be noted

(i) This theorem is applicable to the networks comprising of linear, time-varying, bilateral, passive elements, such as ordinary resistors, inductors, capacitors and transformers.
(ii) This theorem is inapplicable to unilateral networks, such as networks comprising of electron tubes or other control devices.
(iii) Both dependent and independent sources are not permissible.
(iv) We have to consider only the zero-state response by taking all the initial conditions to be zero.

4.2.4 Thevenin's Theorem

Statement A linear active bilateral network can be replaced at any two of its terminals, by an equivalent voltage source (Thevenin's Voltage source), V_{oc}, in series with an equivalent Impedance (Thevenin's impedance), Z_{th}.

Here, V_{oc} is the open circuit voltage between the two terminals under the action of all sources and initial conditions, and Z_{th} is the impedance obtained across the terminals with all sources removed by their internal impedance and initial conditions reduced to zero.

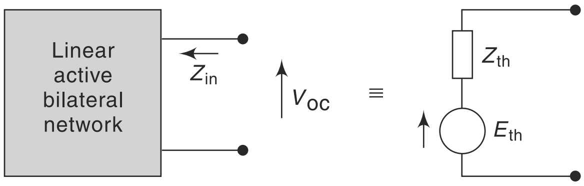
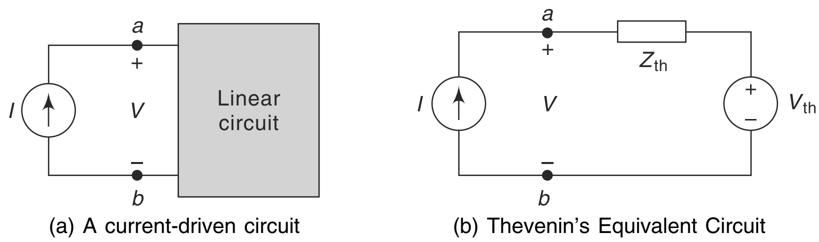


Figure 4.4 *Illustration of Thevenin's Theorem*

Proof We consider a linear active circuit of Fig. 4.5(a). An external current source is applied through the terminals *a–b* where we have access to the circuit.



(a) A current-driven circuit (b) Thevenin's Equivalent Circuit

Figure 4.5

We have to prove that the *v–i* relation at terminals *a–b* of Fig. 4.5(a) is identical with that of the Thevenin's Equivalent circuit of Fig. 4.5(b).

For simplicity, we assume that the circuit contains two independent voltage sources V_{s1} and V_{s2} and two independent current sources I_{s1} and I_{s2}.

Considering the contribution due to each independent source including the external one, the voltage at *a–b*, *V*, is, by Superposition theorem,

$$V = K_0\, I + K_1\, V_{s1} + K_2\, V_{s2} + K_3\, I_{s1} + K_4\, I_{s2}$$

where, K_0, K_1, K_2, K_3, K_4 are constants.

or $$V = K_0\, I + P_0 \tag{4.1}$$

where, $P_0 = K_1\, V_{s1} + K_2\, V_{s2} + K_3\, I_{s1} + K_4\, I_{s2}$ = Total Contribution due to internal independent sources

To evaluate the constants K_0 and P_0 of equation (4.1), two conditions are to be noted.

(i) *When the terminals a and b are open-circuited*

$$I = 0, \text{ and } V = V_{oc} = V_{th}$$

From equation (4.1), $V_{th} = V_{oc} = P_0 \;\Rightarrow\; \boxed{V_{th} = P_0}$

(ii) *When all the internal sources are turned off*

$P_0 = 0$ and the equivalent impedance is Z_{th}.

From equation (4.1), $V = K_0 I$

or $$\frac{V}{I} = K_0 = Z_{th} \quad \Rightarrow \quad \boxed{K_0 = Z_{th}}$$

Thus, substituting the values of K_0 and P_0, the *v–i* relation becomes,

$$\boxed{V = Z_{th} I + V_{th}}$$

This represents the *v–i* relationship of Fig. 4.2(b). So, Thevenin's theorem is proved.

Points to be noted

(i) This theorem is applicable to any linear, bilateral, active network. The only restriction imposed is that the network is not magnetically coupled to external circuits.

(ii) This theorem is inapplicable for non-linear and unilateral networks.

4.2.5 Norton's Theorem

Statement A linear active bilateral network can be replaced at any two of its terminals, by an equivalent current source (Norton's current source), I_{sc}, in parallel with an equivalent admittance (Norton's admittance), Y_N.

Here, I_{sc} is the short circuit current flowing from one terminal to the other under the action of all sources and initial conditions, and Y_N is the admittance obtained across the terminals with all sources removed by their internal impedance and initial conditions reduced to zero.

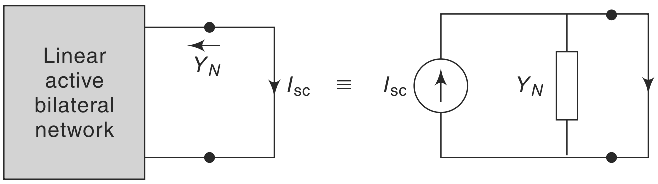


Figure 4.6 *Illustration of Norton's Theorem*

Proof We consider a linear active circuit of Fig. 4.7(a). An external voltage source is applied through the terminals *a–b* where we have access to the circuit.

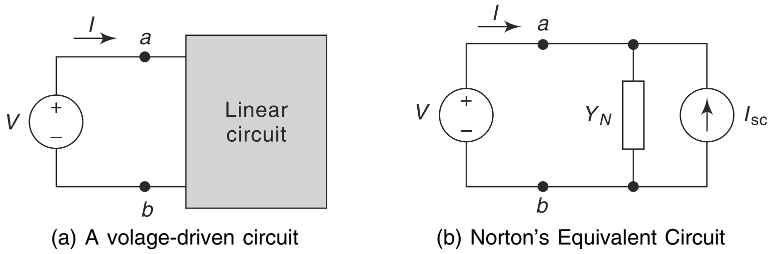


Figure 4.7 *(a) A Voltage-driven Circuit (b) Norton's Equivalent Circuit*

We have to prove that the *v–i* relation at terminals *a-b* of Fig. 4.7(a) is identical with that of the Norton's Equivalent circuit of Fig. 4.7(b).

For simplicity, we assume that the circuit contains two independent voltage sources V_{s1} and V_{s2} and two independent current sources I_{s1} and I_{s2}.

Considering the contribution due to each independent source including the external one, the current entering at *a*, *I*, is, by Superposition theorem,

$$I = K_0\,V + K_1\,V_{s1} + K_2\,V_{s2} + K_3\,I_{s1}I_{s1} + K_4\,I_{s2}$$

where, K_0, K_1, K_2, K_3, K_4 are constants.

or $$I = K_0\,V + P_0 \tag{4.2}$$

where, $P_0 = K_1\,V_{s1} + K_2\,V_{s2} + K_3\,I_{s1} + K_4\,I_{s2}$

= Total contribution due to internal independent sources

To evaluate the constants K_0 and P_0 of equation (4.2), two conditions are:

(iii) *When the terminals a and b are short-circuited*

$$V = 0, \text{ and } I = -I_{sc} = I_N$$

From equation (4.2), $-I_{sc} = P_0 \quad \Rightarrow \quad \boxed{I_{sc} = -P_0}$

(iv) *When all the internal sources are turned off*

$P_0 = 0$ and the equivalent admittance is Y_N.

From equation (4.2), $I = K_0 V$

or $$\frac{I}{V} = K_0 = Y_N \quad \Rightarrow \quad \boxed{K_0 = Y_N}$$

Thus, substituting the values of K_0 and P_0, the *v–i* relation becomes,

$$\boxed{I = VY_N - I_N}$$

This represents the *v–i* relationship of Fig. 4.7(b). So, Norton's theorem is proved.

Points to be noted

(i) This theorem is applicable to any linear, bilateral, active network. The only restriction imposed is that the network is not magnetically coupled to external circuits.

(ii) This theorem is inapplicable for non-linear and unilateral networks.

Steps for Determination of Thevenin's/Norton's Equivalent Circuit

1. The portion of the network across which the Thevenin's or Norton's equivalent circuit is to be found out is removed from the network.
2. (a) The open circuit voltage (V_{oc} or V_{th}) is calculated keeping all the sources at their normal values.

   (b) The short circuit current (I_{sc} or I_N) flowing from one terminal to the other is calculated keeping all the sources at their normal values.
3. Calculation of Z_{th} or Y_N.

Case I. *When circuit contains only independent sources, the following points are to be noted.*

All voltage sources are short-circuited.

All current sources are open-circuited.

Equivalent impedance or admittance is calculated looking back to the circuit with respect to the two terminals.

Case II. *When circuit contains both dependent and independent sources,the following points are to be noted.*

Open circuit voltage (V_{oc}) is calculated with all sources alive.

Short circuit current (I_{sc}) is calculated with all sources alive.

Thevenin's impedance is obtained as, $Z_{th} = \dfrac{V_{oc}}{I_{sc}} = \dfrac{1}{Y_N}$

Case III. *When circuit contains only dependent sources the following points are to be noted.*

In this case, $V_{oc} = 0$.

We connect a test voltage (or current) source at the terminals *a* and *b* and the current flowing through *a–b* (voltage drop between the terminals *a–b*) is calculated.

Thevenin's impedance is obtained as, $Z_{th} = \dfrac{V_{test}}{I_{test}} = \dfrac{1}{Y_N}$

4. Finally, Thevenin's equivalent circuit is obtained by placing V_{oc} in series with Z_{th} and Norton's equivalent is obtained by placing I_{sc} in parallel with Y_N.

4.2.6 Maximum Power Transfer Theorem

Statement Maximum power is absorbed by one network from another connected to it at two terminals, when the impedance of one is the complex conjugate of the other.

This means that for maximum active power to be delivered to the load, load impedance must correspond to the conjugate of the source impedance (or in the case of direct quantities, be equal to the source impedance).

Proof Let E be the voltage source, $(R + jX)$ the internal impedance of the source and $(R_L + jX_L)$ the load impedance.

$$E = \frac{E}{Z + Z_L} = \frac{E}{(R + R_L) + j(X + X_L)} \quad (4.3)$$

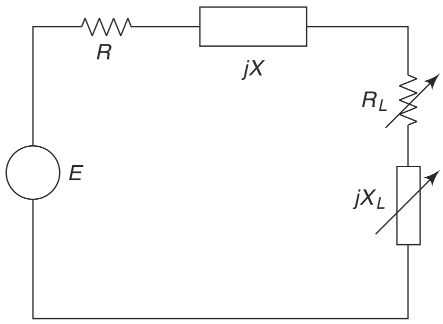


Figure 4.8 *Circuit for Explaining Maximum Power Transfer Theorem*

Power delivered to the load is,

$$P = |I|^2 R_L = \frac{E^2 R_L}{(R + R_L)^2 + (X + X_L)^2} \quad (4.4)$$

where, $Z = R + jX$, $Z_L = R_L + jX_L$

For maximum power, $\frac{\partial P}{\partial X_L}$ must be zero.

Now, $$\frac{\partial P}{\partial X_L} = \frac{-2(E)^2 R_L (X_L + X)}{\left[(R_L + R)^2 + (X_L + X)^2\right]^2} = 0$$

From which, $X_L + X = 0$ or $X_L = -X$

i.e.,the reactance of the load impedance is of opposite sign to the reactance of the source impedance.

Putting $X_L = -X$ in equation 4.4, $P = \frac{E^2 R_L}{(R_L + R)^2}$

For maximum power, $$\frac{\partial p}{\partial R_L} = \frac{E^2 (R_L + R)^2 - 2E^2 R_L (R_L + R)}{(R_L + R)^4} = 0$$

or $E^2(R_L + R) - 2E^2 R_L = 0$ or $R_L = R$

The maximum power transferred will be $P_{max} = \frac{E^2}{4R_L} = \frac{(E/2)^2}{R_L}$ and thus, the efficiency will be 50%.

Points to be noted

(i) It is to be noted that when maximum power is being transferred, only half the applied voltage is available to the load and the other half drops across the source. Also, under these conditions, half the power supplied is wasted as dissipation in the source.

Thus, the useful maximum power will be less than the theoretical maximum power derived and will depend on the voltage required to be maintained at the load.

(ii) For circuits having a resistive load being supplied from a source with only an internal resistance (the case for d.c.), the maximum power will be transferred to the load when the load resistance is equal to the source resistance.

Concept of Internal Resistance of Voltage and Current Sources A **voltage source** is any device or system that produces an electromotive force between its terminals. An example of a primary source is a common battery. Similarly, a **current source** is an electrical or electronic device that delivers electric current. Examples of current sources are a large voltage source in series with a large resistor (however, this type of current source has very poor efficiency), an active current source involving transistors, high voltage current source like Van de Graff generator, etc. A current source is the dual of a voltage source.

In circuit theory, an **ideal voltage source** is a circuit element where the voltage across it is independent of the current through it. It only exists in mathematical models of circuits. The internal resistance of an ideal voltage source is zero; it is able to supply any amount of current. The current through an ideal voltage source is completely determined by the external circuit. When connected to an open circuit, there is zero current and thus zero power. When connected to a load resistance, the current through the source approaches infinity as the load resistance approaches zero (a short circuit). Thus, an ideal voltage source can supply unlimited power.

Similarly, an **independent current source** with zero current is identical to an ideal open circuit. For this reason, the internal resistance of an ideal current source is infinite. The voltage across an ideal current source is completely determined by the circuit it is connected to. When connected to a short circuit, there is zero voltage and thus zero power delivered. When connected to a load resistance, the voltage across the source approaches infinity as the load resistance approaches infinity (an open circuit). Thus, an ideal current source can supply unlimited power forever and so represents an unlimited source of energy. Connecting an ideal open circuit to an ideal non-zero current source is not valid in circuit analysis as the circuit equation would be paradoxical, e.g., 3 = 0.

However, no real voltage source is ideal; all have a non-zero effective internal resistance, and none can supply unlimited current. *The internal resistance of a real voltage source is effectively modeled in linear circuit analysis by combining a non-zero resistance in series with an ideal voltage source*. Similarly, no real current source is ideal (no unlimited energy sources exist) and all have a finite internal resistance (none can supply unlimited voltage). *The internal resistance of a physical current source is effectively modeled in circuit analysis by combining a non-zero resistance in parallel with an ideal current source.*

4.2.7 Tellegen's Theorem

Statement Consider an arbitrary lumped network whose graph G has b branches and n nodes. Let the associated reference polarities and directions be chosen for the branch voltages v_1, v_2, v_3, $\ldots v_b$ and the branch currents i_1, i_2, i_3. $\ldots i_b$, which satisfy all the constraints imposed by KVL and KCL, respectively.

Then, the summation of instantaneous power delivered to all branches is zero.

i.e. $$\sum_{k=1}^{b} v_k i_k = 0$$

Proof We have to prove that,

$$\sum_{k=1}^{b} v_k i_k = 0$$

or
$$\begin{bmatrix} v_1 & v_2 & \dots & v_b \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ i_b \end{bmatrix} = 0$$

or
$$[V_b]^T\,[I_b] = 0 \tag{4.5}$$

Now, by KCL and KVL using Complete Incidence Matrix, we have,

$$[A_a]\,[I_b] = 0 \tag{4.6}$$

and
$$[V_b] = [A_a]^T\,[V_n] \tag{4.7}$$

So,
$$\begin{aligned} [V_b]^T\,[I_b] &= [[A_a]^T\,[V_n]]^T\,[I_b] && \text{\{by equation (4.7)\}} \\ &= [A_a]\,[I_b]\,[V_n]^T \\ &= 0.[V_n]^T && \text{\{by equation (4.6)\}} \end{aligned}$$

$$\Rightarrow \quad \boxed{[V_b]^T\,[I_b] = 0}$$

Thus, Tellegen's Theorem is proved.

Points to be noted

(i) This theorem is applicable for any lumped network having elements which are linear or non-linear, active or passive, time-varying or time-invariant.

(ii) This theorem is completely independent of the nature of the elements and is only concerned with the graph of the network.

(iii) This theorem is based on two Kirchhoff's laws, i.e., KVL and KCL.

(iv) This theorem implies that the power delivered by independent sources of the network must be equal to the sum of the power absorbed (dissipated or stored) in all other elements in the network.

(v) If the network is in sinusoidally steady-state (a.c. circuits), then Tellegen's theorem is given as,

$$\sum_{k=1}^{b} V_k I_k^* = 0$$

where, V_k are the phasor voltages, I_k are the phasor currents and I_k^* is the complex conjugate of I_k.

(vi) If t_1 and t_2 refer to two different instants of observations, it still follows form Tellegen's theorem that,

$$\sum_{k=1}^{b} v_k(t_1) \cdot i_k(t_2) = 0$$

(vii) If N_1 and N_2 refer to two different circuits having the same graph, with the same reference directions assigned to the branches in the two circuits, then by Tellegen's theorem,

$$\sum_{k=1}^{b} v_{1k} . i_{2k} = 0 \quad \text{and} \quad \sum_{k=1}^{b} v_{2k} . i_{1k} = 0$$

where v_{1k} and i_{1k} are the voltages and currents in N_1 and v_{2k} and i_{2k} are the voltages and currents in N_2, all satisfying the Kirchhoff's laws.

4.2.8 Millman's Theorem

Consider a number of admittances Y_1, Y_2, Y_3 $\ldots Y_p \ldots$ $Y_q, \ldots Y_n$ are connected together at a common point S. If the voltages of the free ends of the admittances with respect to a common reference N are known to be V_{1N}, V_{2N}, V_{3N} $\ldots V_{pN} \ldots V_{qN}, \ldots V_{nN}$, then Millman's theorem gives the voltage of the common point S with respect to the reference N, as follows.

Applying Kirchhoff's Current law at node S,

$$\sum_{p=1}^{n} I_p = 0, I_p = Y_p (V_{pN} - V_{sN})$$

or
$$\sum_{p=1}^{n} Y_p (V_{pN} - V_{sN}) = 0$$

or
$$\sum_{p=1}^{n} Y_p V_{pN} = V_{sN} \sum_{p=1}^{n} Y_p$$

$$\boxed{\Rightarrow \quad V_{sN} = \frac{\sum_{p=1}^{n} Y_p V_{pN}}{\sum_{p=1}^{n} Y_p}}$$

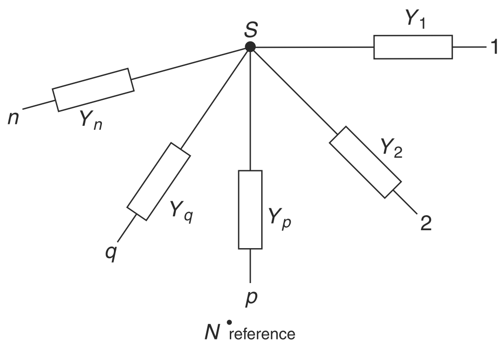


Figure 4.9 *Illustration of Millman's Theorem*

An extension of the Millman's theorem is the *equivalent generator theorem.*

Statement

(I) This theorem states that if several ideal voltage sources (V_1, V_2, …) in series with impedances (Z_1, Z_2,…) are connected in parallel , then the circuit may be replaced by a single ideal voltage source (V) in series with an impedance (Z) such that

$$V = \frac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i} \quad \text{and} \quad Z = \frac{1}{\sum_{i=1}^{n} Y_i}$$

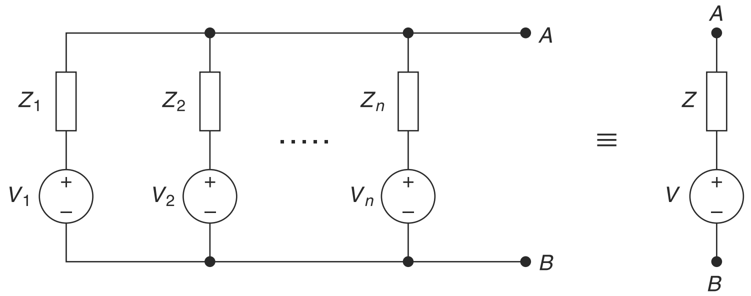


Figure 4.10 *Voltage Source Equivalent using Millman's Theorem*

(II) If several ideal current sources (I_1, I_2,…) in parallel with impedances (Z_1, Z_2…) are connected in series, they may be replaced by a single ideal current source (I) in parallel with an impedance (Z) such that

$$I = \frac{\sum_{i=1}^{n} I_i / Y_i}{\sum_{i=1}^{n} 1/Y_i} \quad \text{and,} \quad Y = \frac{1}{\sum_{i=1}^{n} 1/Y_i} \quad \text{or,} \quad Z = \sum_{i=1}^{n} Z_i$$

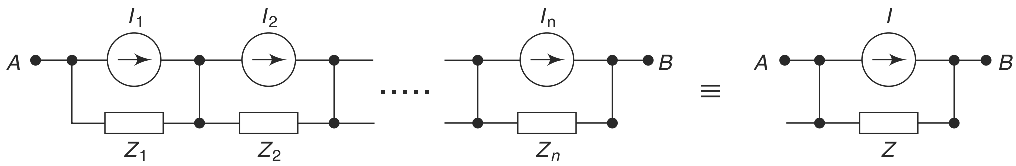


Figure 4.11 *Current Source Equivalent using Millman's Theorem*

Proof

(I) Using Superposition theorem, the short circuit current through A–B considering only one source acting alone and replacing other sources by their internal impedances, (i.e., short circuit for ideal voltage sources),

$$I_{sc1} = V_1 Y_1$$

$$I_{sc2} = V_2 Y_2$$

$$I_{scn} = V_n Y_n$$

Total short circuit current through A–B, $I_{sc} = (I_{sc1} + I_{sc2} + \dots + I_{sc\,n})$

$$= V_1\, Y_1 + V_2\, Y_2 + \dots + V_n\, Y_n$$

$$= \sum_{i=1}^{n} V_i Y_i \qquad (4.8)$$

Impedance looking back from A–B with all the sources removed,

$$Z = \frac{1}{Y_1 + Y_2 + \dots + Y_n} = \frac{1}{\sum_{i=1}^{n} Y_i} \qquad (4.9)$$

Thus, by Thevenin's theorem, the equivalent voltage is,

$$V = I_{sc} \cdot Z = \frac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i} \tag{4.10}$$

Form equation (4.8), (4.9) and (4.10), Millman's Theorem is proved.

(II) Using Superposition theorem, the short circuit current through *A–B* considering only one source acting alone and replacing other sources by their internal impedances, (i.e., open circuit for ideal current sources),

$$I_{sc1} = \frac{I_1 Z_1}{\sum_{i=1}^{n} Z_i}; I_{sc2} = \frac{I_2 Z_2}{\sum_{i=1}^{n} Z_i}; \ldots I_{sc\,n} = \frac{I_n Z_n}{\sum_{i=1}^{n} Z_i}$$

Total short circuit current, $I_{sc} = I = (I_{sc1} + I_{sc2} + \ldots + I_{sc\,n})$

$$I = \frac{\sum_{i=1}^{n} I_i Z_i}{\sum_{i=1}^{n} Z_i} \tag{4.11}$$

Impedance looking back from *A–B* with all the sources removed,

$$Z = \sum_{i=1}^{n} Z_i \tag{4.12}$$

From equation (4.11) and (4.12), Millman's theorem is proved.

4.2.9 Compensation Theorem

In many circuits, after the circuit is analysed, it is realised that only a small change needs to be made to a component to get a desired result. In such a case, we would normally have to recalculate. The compensation theorem allows us to compensate properly for such changes without sacrificing accuracy.

Statement In any linear bilateral active network, if any branch carrying a current I has its impedance Z changed by an amount δZ, the resulting changes that occur in the other branches are the same as those which would have been caused by the injection of a voltage source of ($-I\delta Z$) in the modified branch.

In other words, in a linear network N, if the current in a branch is I and the impedance Z of the branch is increased by δZ, then the increment of voltage and current in each branch of the network is that voltage or current that would be produced by an opposite voltage source of value v_c(= $I\delta Z$) introduced into the altered branch after the modification.

Proof Consider the network N, having branch impedance Z.
Let the current through Z be I and its voltage be V.

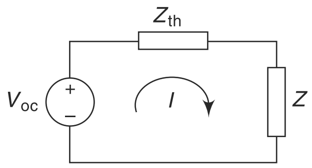


Figure 4.12(a) *Circuit for Explaining Compensation Theorem*

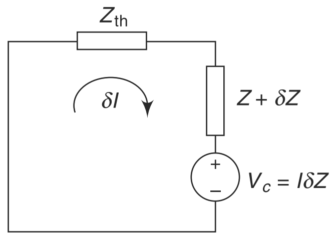


Figure 4.12(b) *Equivalent Circuit using Compensation Theorem*

Let δZ be the change in Z.
Then, I' (the new current) can be written as,

$$I' = \frac{V_{oc}}{Z + \delta Z + Z_{th}};$$

$$\delta I = I' - I = \frac{V_{oc}}{Z + \delta Z + Z_{th}} - \frac{V_{oc}}{Z + Z_{th}} = -\left(\frac{V_{oc}}{Z + Z_{th}}\right)\left(\frac{\delta Z}{Z + \delta Z + Z_{th}}\right)$$

or
$$\delta I = -\frac{I\delta Z}{Z + \delta Z + Z_{th}} = -\frac{V_c}{Z + \delta Z + Z_{th}} \quad \text{where } V_c = I\delta Z$$

How to find δI

(i) Find the product $I\delta Z$, where I is the current through the branch before changing the impedance.
(ii) Remove all the independent sources.
(iii) Connect a voltage source of magnitude $V_c = I\delta Z$, in series with the branch. The polarity of V_c is such as to oppose the direction of current I.
(iv) Solve the network assuming current flowing to be δI and thus the value of δI.

4.3 COUPLED CIRCUITS

Coupling is an electric/magnetic phenomenon which sets up direct/mutual interaction between two or more passive elements of a network in such a way that changes in one element or circuit affect the performance of other elements in the same circuit or neighbouring circuits. Depending upon the situation, coupled circuits may be classified as (i) conductively coupled circuits, and (ii) magnetically coupled circuits. In a conductively coupled circuit, variation in one loop of a given circuit would bring about changes in neighbouring loops of the same circuit through current conduction. However, in case of magnetically coupled circuits, variation in one loop of a given circuit might affect the performance of other loops in the same circuit or even in the neighbouring circuits although the circuits remain electrically isolated. Therefore, it may be inferred that electric current plays the key role in conductively coupled circuits through conduction of current, whereas magnetic flux plays the key function in magnetically coupled circuits. As per the fundamentals of electromagnetism, magnetic flux is the result of electric current flow in a conductor or a coil and it is the inductance of the conductor or the coil in question that governs the relationship between electric current and magnetic flux. Further, it is also known to us that according to the principles of electromagnetic induction, a time-varying magnetic flux of a particular coil may induce voltage in the form of emf in the same coil or neighbouring coils irrespective of the fact that these coils remain electrically connected or isolated. The inherent property of a coil by virtue of which a time-varying magnetic flux of the coil on linking the coil itself induces an emf across the same coil is known as *self-inductance* and is designated by the symbol L (SI unit henry, H). On the contrary, the mutual property of two coils by virtue of which a time-varying flux of one coil on linking a neighbouring coil induces an emf in the second coil is known as mutual inductance of the second coil with respect to the first coil and is designated by the symbol M_{21} (SI unit henry, H). Similarly, mutual inductance of the first coil with respect to the second coil may be referred as M_{12}, which may be responsible for inducing an emf in the first coil due to linkage of a common magnetic flux produced by an exciting current flowing in the second coil. The simplest example of magnetic coupling due to self inductance is the choke coil used as a ballast in fluorescent tube operation and that of mutual inductance is the practical transformer used in power systems.

4.3.1 Self-inductance

Self-inductance of a coil is defined as an inherent property of a coil which opposes any change in flux linkage with the coil itself or any change in the current flow in the coil itself by inducing an emf across the coil. The principles governing this are better known as *Faraday's laws of electromagnetic induction* and *Lenz's law*. Accordingly, the expressions for the self-induced emf in the physical arrangement of Fig. 4.13 consisting of a coil of N_1 turns and excited by a time-varying voltage $v1(t)$ may be given as

$$e_1(t) = -N_1 \frac{d\phi_1(t)}{dt} \tag{4.13}$$

Also,

$$e_1(t) = -L_1 \frac{di_1(t)}{dt} \tag{4.14}$$

On comparison of the above expressions, we may find an expression for the self-inductance of the coil given as

$$L_1 = N_1 \frac{d\phi_1(t)}{di_1(t)} = N_1 \frac{d\phi_1}{di_1} \tag{4.15}$$

where, L_1 represents the self-inductance of coil-1, i_1 represents the excitation current flowing in coil-1, ϕ_1 represents the magnetic flux linking coil-1, and e_1 is the self-induced emf in coil-1.

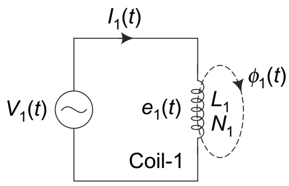


Figure 4.13 *Self-induced emf and self-inductance*

4.3.2 Mutual Inductance

Mutual inductance of a coil with respect to another coil is defined as an inherent property of the coils which would tend to oppose any change in flux linkage with the second coil due to the excitation of the first coil by inducing an emf across the second coil. The opposition would be viable when the second coil would provide a closed loop for a current to flow in it and the flux developed due to this current would oppose the main flux in question. In another way, mutual inductance may also be defined as the ability of a coil or an inductor to induce an emf across a neighbouring coil or inductor with the help of magnetic coupling between the two coils. The principles governing this are better known as Faraday's laws of electromagnetic induction and Lenz's law. Accordingly, the expressions for the mutually induced emf in the physical arrangement of Fig. 4.14 consisting of two coils (coil-1 having N_1 turns and coil-2 having N_2 turns, which are electrically isolated but magnetically coupled) may be given under two categories as shown below.

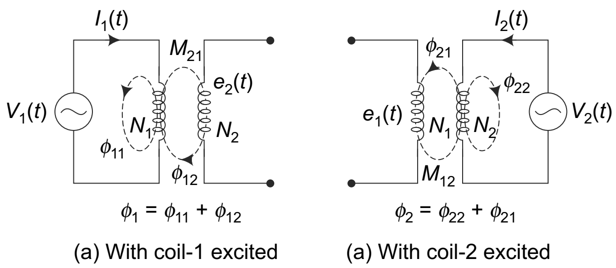


Figure 4.14 *Mutually induced emf and mutual inductance*

Case I Coil-1 is excited and mutually induced emf appears in coil-2

In view of Fig. 4.14 (a), it may be found that

$$e_2(t) = -N_2\frac{d\phi_{12}(t)}{dt} \tag{4.16}$$

Also,
$$e_2(t) = -M_{21}\frac{di_1(t)}{dt} \tag{4.17}$$

On comparison of the above expressions, we may find an expression for the mutual inductance of coil-2 with respect to coil-1 given as

$$M_{21} = N_2\frac{d\phi_{12}(t)}{di_1(t)} = N_2\frac{d\phi_{12}}{di_1} \tag{4.18}$$

where, i_1 represents the excitation current flowing in coil-1, ϕ_1 represents the total magnetic flux produced by coil-1, ϕ_{11} represents the magnetic flux that does not link coil-2 which may be treated as a leakage flux, ϕ_{12} represents the common magnetic flux linking both coil-1 and coil-2 which may be treated as the useful flux, and e_2 is the mutually induced emf in coil-2. It may be noted that the total flux is an algebraic sum of the leakage flux and useful flux, i.e., $\phi_1 = \phi_{11} + \phi_{12}$.

Case II Coil-2 is excited and mutually induced emf appears in coil-1

In view of Fig. 4.14 (b), it may be found that

$$e_1(t) = -N_1 \frac{d\phi_{21}(t)}{dt} \tag{4.19}$$

Also,

$$e_1(t) = -M_{12} \frac{di_2(t)}{dt} \tag{4.20}$$

On comparison of the above expressions, we may find an expression for the mutual inductance of coil-1 with respect to coil-2 given as

$$M_{12} = N_1 \frac{d\phi_{21}(t)}{di_2(t)} = N_1 \frac{d\phi_{21}}{di_2} \tag{4.21}$$

where, i_2 represents the excitation current flowing in coil-2, ϕ_2 represents the total magnetic flux produced by coil-2, ϕ_{22} represents the magnetic flux that does not link coil-1 which may be treated as a leakage flux, ϕ_{21} represents the common magnetic flux linking both coil-1 and coil-2 which may be treated as the useful flux, and e_1 is the mutually induced emf in coil-1. It may be noted that the total flux is an algebraic sum of the leakage flux and useful flux, i.e., $\phi_2 = \phi_{22} + \phi_{21}$. For a given physical set-up, it may be verified that mutual inductances remain same for both the coils irrespective of the order of their excitation. This might lead to the simplification that mutual inductance of a combination of two neighbouring coils may be assigned one value as shown in Equation 4.22.

$$M_{12} = M_{21} = M \tag{4.22}$$

4.4 DOT CONVENTION FOR REPRESENTING COUPLED CIRCUITS

In the previous section, we have come across the fundamental concepts behind the operation and utility of coupled circuits and the key parameters governing them. It may be observed that two coils coupled to each other magnetically develop a mutual property for the common set-up which is characterized as the mutual inductance (M) for the coupled coils. The most striking effect of mutual inductance is reportedly the development of a mutually induced emf in one of the coils of the coupled set-up subject to a time-varying excitation resulting from an alternating current flow in the other coil. Unlike the self-induced emf which has an inherent property of opposition towards the cause of its production, a mutually induced emf may exhibit dual characteristics. It would be interesting to note something in the course of this section that sometimes the mutually induced emf may oppose the source voltage while at other times, it might assist the source voltage depending on the physical structure of the existing windings in the coupled set-up and the mode of coupling described by the existing and/or desirable current flow in the individual loops of the coupled set-up. Since the polarity of the mutually induced emf dictates the formation of KVL expression for a loop, the reader must remain careful about the sign convention of the emfs induced in the elements of the coupled circuits due to coupling effect. It may be well appreciated that the polarity of the mutually induced emf in the coils depend on many factors such as (i) direction of the current flow in the coil, (ii) development and progressive growth of the winding over the core, (iii) orientation of the coils in the space and their proximity in the neighbourhood, and finally (iv) their connectivity in the circuit. Therefore, unless clearly implemented otherwise, the reader may face several difficulties in finding the accurate polarities of mutually induced emf. It is thus mandatory on the part of the manufacturers and users as well that they should follow a unified code for the polarity conventions of coils or inductors as regards to the mutually induced emfs to be developed there in when used in coupled circuits.

If we look into the practical situation of a coupled circuit having two coils (coil-1 and coil-2) placed in the neighbourhood of each other as shown in Fig.4.15, each of them carrying their own loop currents which are time varying, then the KVL expressions for individual loops for the instant depicted in the figure may have the structure of Equation 4.23 (for loop-1) and Equation 4.24 (for loop-2) respectively.

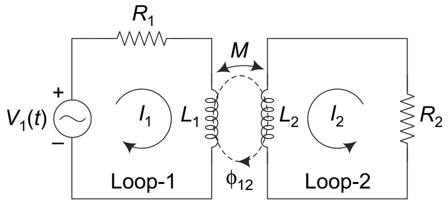


Figure 4.15

KVL for loop-1 of Fig.4.15:

$$v_1(t) = I_1R_1 + L_1\frac{dI_1}{dt} \pm M\frac{dI_2}{dt} \tag{4.23}$$

KVL for loop-2 of Fig.4.15:

$$0 = I_2R_2 + L_2\frac{dI_2}{dt} \pm M\frac{dI_1}{dt} \tag{4.24}$$

Though both the signs (positive and negative) appear in the above equations as regards to the mutually induced emf, yet any one of them would be valid for the particular situation. The reader may note here that Fig.4.15 is not presented in sufficient details to explore the correct signs for the emf terms induced in the coils under the action of mutual coupling between the two coils. While making an attempt to clarify this situation, it may be essential to provide more information about the factors responsible for the mutually induced emf.

In view of this, let us draw our attention to review the physical arrangements of Fig. 4.16 (*a* to *h*). In this figure we have considered eight different situations for two magnetically coupled coils such that each situation is unique and differs from the rest as regards to the physical get-up of the winding structure developed over the core and direction of current flow in the loops of the two coils. With the given allocation of loop currents it may be possible to identify the direction of magnetic flux for each coil by the help of Ampere's right-hand thumb rule. Then it may be observed that for the four cases (i.e., *a*, *d*, *f*, and *g* of Fig. 4.16), the magnetic flux of coil-1 and that of coil-2 maintain same direction of flow in the core. This would necessarily mean that the polarity of a mutually induced emf appearing in Equations 4.23 and 4.24 for these four cases would be positive. Thus the correct form of KVL expressions for these four cases would be as per Equations 4.25 and 4.26.

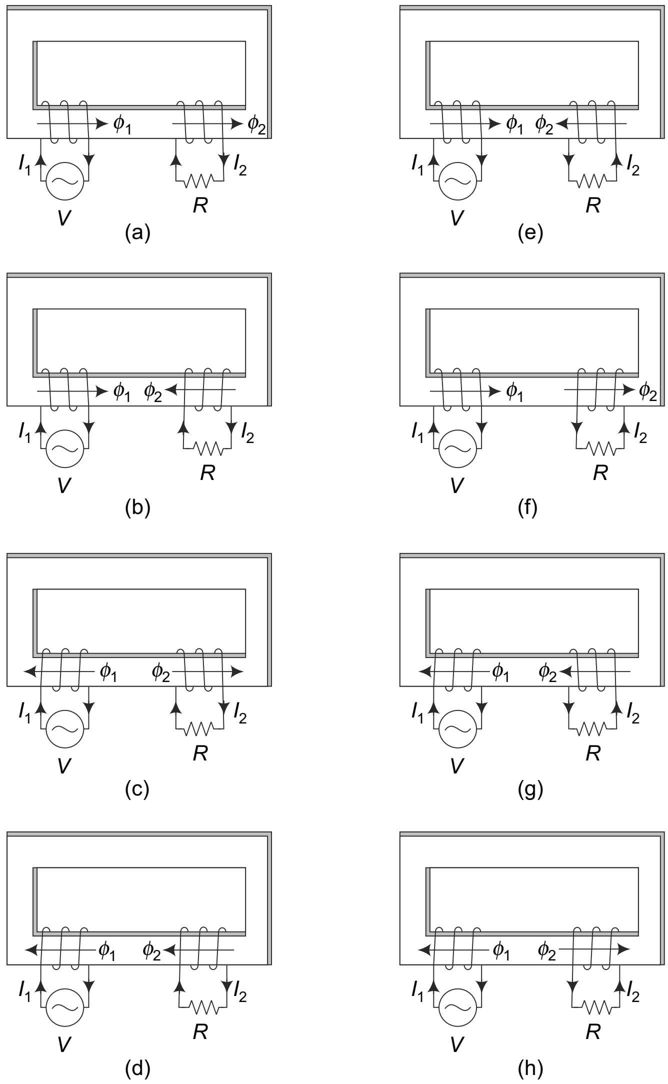


Fig. 4.16

KVL for loop-1 (for the cases *a*, *d*, *f*, and *g* of Fig.4.16):

$$v_1(t) = I_1R_1 + L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} \tag{4.25}$$

KVL for loop-2 (for the cases *a*, *d*, *f*, and *g* of Fig.4.16):

$$0 = I_2R_2 + L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} \tag{4.26}$$

However, for the remaining four cases (i.e., *b*, *c*, *e*, and *h* of Fig. 4.16), it may be observed that the magnetic flux produced by coil-1 is seemingly opposite to that of the flux of coil-2, thus leading to the inference that the polarity of mutually induced emf appearing in Equations 4.23 and 4.24 for these four cases would be negative. Thus the correct form of KVL expressions for these four cases would be as per Equations 4.27 and 4.28.

KVL for loop-1 (for the cases *b*, *c*, *e*, and *h* of Fig. 4.16):

$$v_1(t) = I_1R_1 + L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} \tag{4.27}$$

KVL for loop-2 (for the cases *b*, *c*, *e*, and *h* of Fig. 4.16):

$$0 = I_2R_2 + L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} \tag{4.28}$$

This much of clarity in finding the polarity of the mutually induced emf could be possible only due to a real isometric presentation of the core showing the actual progress of the winding layout over it that facilitated the prediction of the direction of flow of magnetic flux in the core due to an impressed current as per the schedule shown in each case of Fig. 4.16.

On the contrary, had it been a situation happening normally in most of the diagrams, where neither the core nor the actual progress of the winding layout over it is presented clearly, the analysis would not be so easier to predict the polarity of the mutually induced emf in the coupled coils correctly by knowing only the current allotment in the coils. Hence, it is recommended that some practical and easy-to-handle convention may be adopted for fulfillment of the requirements of coupled coils as regards to identification of polarity of a mutually induced emf in them. Such a convention is known as DOT convention.

In order to bring the dot convention into force and implement it correctly to the case of coupled coils, let us redraw the eight cases of Fig. 4.16 in another figure (Fig. 4.17) where the core has been removed and the coils have been marked with a dot at one end. In order to resemble the transformer structure, the coils have been put vertically in the diagram. According to this convention, the dot would convey the direction of the resulting magnetic flux in such a way that, "*polarity of the emf induced mutually across a neighbouring coil would be positive at the dotted end of the coil subject to entry of a time-varying exciting current at the dotted end of the other coil*". Although identifying the polarity of the mutually induced emf in the coils is a foolproof method, yet sometimes, the interpretation of dot convention becomes comfortable and implementation of the same for formation of KVL becomes easier by assigning a polarity for the mutual inductance ($\pm M$) instead. A thorough examination of the eight cases presented in Fig. 4.17 would reveal another modified terminology for the linguistic presentation of the dot convention, which states that, "*polarity of mutual inductance of a coupled coil set-up may be treated as positive only if the loop currents make their entry into the respective coils at the dotted ends of respective coils, failing which, mutual inductance may be treated as negative*".

Thus, any coupled circuit may be viewed correctly in the absence of a core in the diagram if dot convention is followed properly and the mutual inductance polarity is marked correctly. In view of this, the diagram shown in Fig. 4.18 may be treated as a general diagram for representing coupled circuits with the help of dot convention.

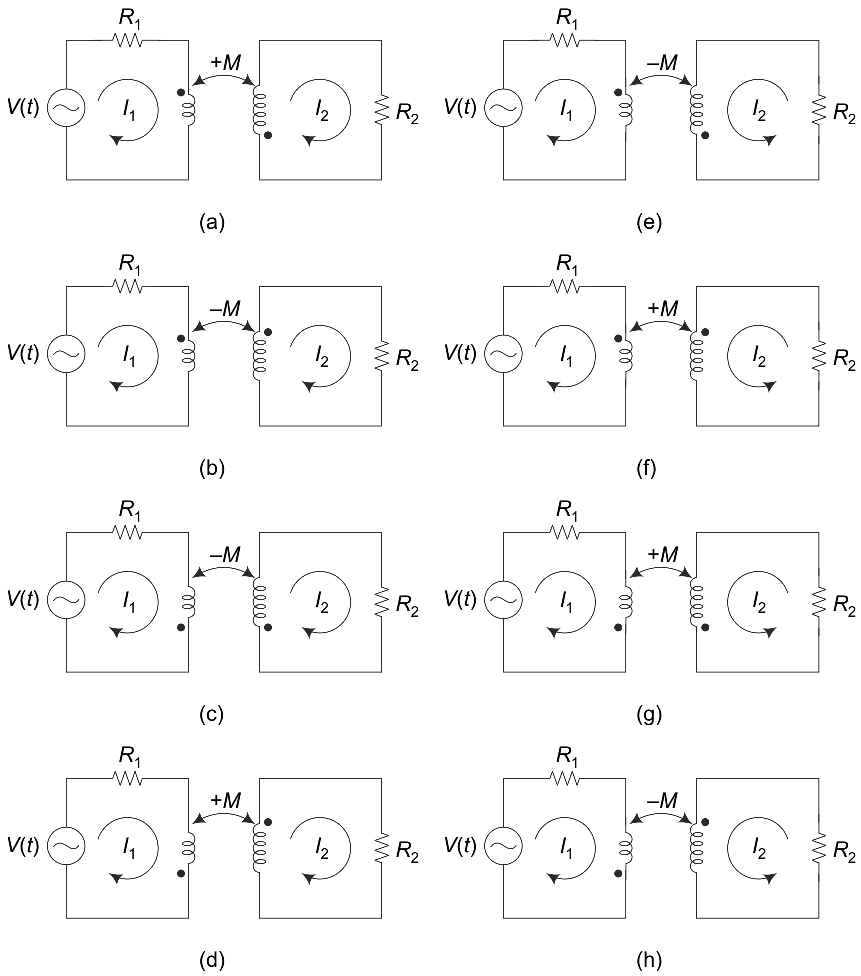


Fig. 4.17

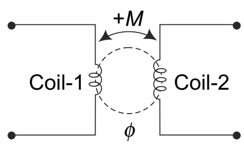


Fig. 4.18

4.5 COEFFICIENT OF COUPLING

While dealing with coupled circuits, it is very important to see the aspects of self inductance and mutual inductance and the magnetic coupling between them. The degree or order or effectiveness of this coupling is otherwise known as *coefficient of coupling*. This may be derived as follows.

In view of Fig. 4.14, the self-inductance and mutual inductance values of respective coils and their coupled combinations may be given by

Self inductance of coil-1: $$L_1 = N_1 \frac{d\phi_1}{di_1} \tag{4.29}$$

Self inductance of coil-2: $$L_2 = N_2 \frac{d\phi_2}{di_2} \tag{4.30}$$

Mutual inductance of coil-2 with respect to coil-1:

$$M_{21} = N_2 \frac{d\phi_{12}}{di_1} \tag{4.31}$$

Mutual inductance of coil-1 with respect to coil-2:

$$M_{12} = N_1 \frac{d\phi_{21}}{di_2} \tag{4.32}$$

Taking the product of the two sides of Equations 4.31 and 4.32, we may get

$$M_{21}\ M_{12} = N_2 \frac{d\phi_{12}}{di_1}\ N_1 \frac{d\phi_{21}}{di_2} \tag{4.33}$$

In view of Equation 4.22, we may simplify the above equation as shown in Equation 4.34.

$$M^2 = N_2 \frac{d\phi_{12}}{di_1}\ N_1 \frac{d\phi_{21}}{di_2} = N_1 \frac{d\phi_{12}}{di_1}\ N_2 \frac{d\phi_{21}}{di_2} \tag{4.34}$$

It is also possible to express the flux distributions in respective coils as a fraction of the total flux such that the useful flux in each coil becomes

$$\phi_{12} = k\phi_1 \tag{4.35}$$

And $$\phi_{21} = k\phi_2 \tag{4.36}$$

where, k is a fractional constant usually less than unity or ideally equal to unity. While substituting these values in Equation 4.34 we may get

$$M^2 = N_1 \frac{d\phi_{12}}{di_1}\ N_2 \frac{d\phi_{21}}{di_2} = N_1 \frac{d(k\phi_1)}{di_1}\ N_2 \frac{d(k\phi_2)}{di_2}$$

$$= k^2\left(N_1 \frac{dk\phi_1}{di_1} N_2 \frac{d\phi_2}{di_2}\right) = k^2 (L_1 L_2) \tag{4.37}$$

Thus, the mutual inductance of the set-up may be found by taking the square root of Equation 4.37, which results in

$$M = k\sqrt{L_1 L_2} \tag{4.38}$$

The fractional constant (k) appearing in the above expressions for flux and inductance is called coefficient of coupling which may be written as

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2} \tag{4.39}$$

For the case of two coupled coils if $k = 1$, then the coupling is ideal, which is never possible in practice. However, for the range $0.5 < k < 1$, the coupling may be viewed as strong and the coils are said to be tightly coupled. On the other hand, the coupling is regarded as weak or the coils may be treated as loosely coupled if k is found to be less than 0.5 for the given set-up. In order to improve the coupling, one would have to be careful for enhancement in the coupling coefficient by providing highly permeable magnetic materials in the core, arranging the coils in the close vicinity of each other and providing the best possible orientation of the coils to facilitate flux linkage.

4.6 RESONANCE IN COUPLED CIRCUITS

Resonance is a typical state or condition of a system during which the frequency of oscillation produced by an external forcing function matches with the natural frequency of the system thereby causing a response of maximum amplitude. In fact, resonance does not take place in the steady state and for its occurrence in a system, there must be some external periodic forcing function in the form of a disturbance from outside the system that introduces some kind of forced oscillations into the system. As the system continues with these oscillations, interchange of energy takes place between two independent energy-storing components present within the system. Depending on the initial condition of the system and frequency of the forcing function, system oscillations may grow or diminish in scale as compared to the oscillations of the immediate previous state. At a particular stage it may so happen that the frequency of the forcing function may go step-in-step with the natural frequency of the system. This kind of a situation is certainly a rare and typical occurrence and it may lead the system oscillations into a state of resonance thereby exhibiting a response of highest magnitude. In mechanical systems, resonance may take place in springs, cantilevers, beams, columns, bridges, and so on, leading to ultimate collapse if sustained for long. That is perhaps why a group of soldiers marching ahead over a river bridge are often advised not to march along rhythmically. In electrical networks for the resonance to take place, it is mostly desirable that the network should have inductors (coupled coils) and capacitors representing the two kinds of energy-storing elements which actively take part in exchanging the energy stored in them in their magnetic and electric fields respectively. In addition to this, there must also be a supply source that inputs a periodic forcing function into the system in the form of an external disturbance. Depending upon the type of passive elements present in the electric network and their connectivity, resonance in coupled circuits may be categorized as *series resonance* and *parallel resonance*. Both of these resonance cases have been dealt separately in the following sections. Other salient features of resonance such as bandwidth, quality factor and selectivity have also been covered in a concise manner.

4.6.1 Bandwidth and Q-factor During Series Resonance

When an electric network containing the passive elements (such as resistance R, inductance L and capacitance C) in series is excited from an external supply source having a periodic function of frequency f, as shown in Fig. 4.19, then it is expected that with the variation in the supply frequency resonance may occur in the circuit as and when the frequency of the periodic forcing function matches with the natural frequency of oscillation of the system. Such a type of resonance is known as series resonance and the frequency at which resonance occurs is called *resonant frequency*, given by f_0.

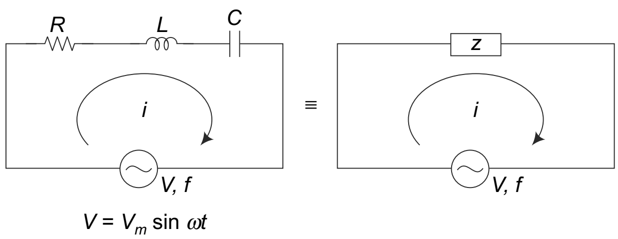


Fig. 4.19 *Network for study of series resonance*

During resonance it is expected that the response of the system attains highest amplitude, which for the given network of Fig. 4.19 may be viewed as the magnitude of the circuit current attaining its highest possible peak for a particular frequency of the supply source and the given network conditions. In other words, the network becomes frequency selective during resonance and the frequency of the source for which the circuit resonates is called the *series resonant frequency* (f_0). Since the circuit current is mostly regulated by the circuit impedance ($I = V/Z$), during resonance it may be required that impedance should pass through its series lowest value in order to ensure a highest possible peak for the current. Therefore, the expression for complex impedance of the circuit is being presented in Equation 4.40 for a better analysis of resonant condition.

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{4.40}$$

Now it may be seen clearly that the impedance is also a function of supply frequency as Z is a function of R, and $\omega = 2\pi f$. Thus it is permissible that the impedance may attain a value which becomes the least in the series over a range of f, only when the term within the parentheses of Equation 4.40 reduces zero. The frequency of the supply for which the imaginary part of Equation 4.40 becomes zero is therefore treated as resonant frequency. A detailed workout for the calculation of resonant frequency is performed in the following paragraph with the clue that, $\omega L - \left.\frac{1}{\omega C}\right|_{f=f_0} = 0.$

This condition leads to $\omega L = \frac{1}{\omega C}$.

$$\Rightarrow \omega^2 = \frac{1}{LC} \quad \Rightarrow \omega = \frac{1}{\sqrt{LC}} \quad \Rightarrow 2\pi f_0 = \frac{1}{LC} \quad \Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}} \qquad (4.41)$$

The condition of resonance also reveals many other facts about the circuit performance, which are stated categorically hereafter. These facts may be useful for finding the bandwidth and quality factor of coupled circuits.

1. Resonant frequency is given by $\omega_0 = \frac{1}{\sqrt{LC}}$ or $f_0 = \frac{1}{2\pi\sqrt{LC}}$.
2. At resonance, the circuit impedance is the least and is numerically equal to the resistance of the network, given by, $Z_0 = R$. Hence the circuit behaves as a purely resistive circuit.
3. At resonance, the power factor of the network is the highest and becomes unity.
4. At resonance, current in the circuit is the highest, and it becomes, $I_0 = V/R$, which remains in the same phase of the voltage.
5. At resonance, the average power consumed by the circuit is the highest, and it becomes, $P_0 = \frac{1}{2}\left(\frac{V^2}{R}\right)$.

Bandwidth (B_w) for series resonance

The concept of bandwidth requires the understanding of frequency response of current in the circuit over the stretch of frequency variation from zero to infinity passing through resonant frequency. Such a plot is shown in Fig. 4.20.

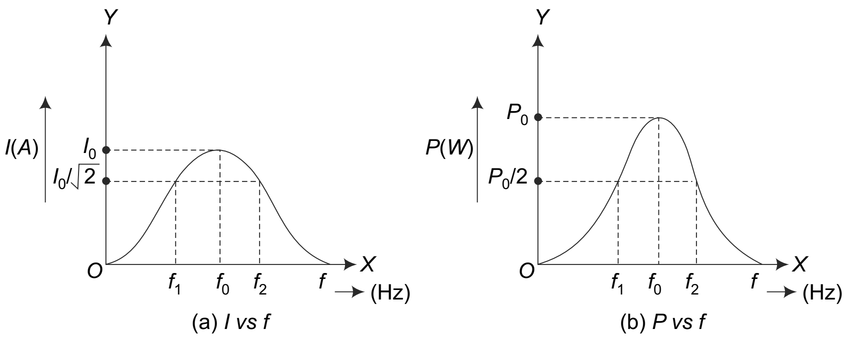


Fig. 4.20 *Frequency response of current and power for the circuit of Fig.4.19*

As revealed from the plot of Fig. 4.20, the performance of the circuit as regards to amplitude of current and the power consumption is highest at resonance. This property is very useful for frequency selectivity of circuits generally used for communication purpose. However, it may be difficult in practice to obtain this condition for all purposes. Therefore, a range of frequency on either side of resonant frequency for which the power consumption may reduce up to 50% of the power consumed during resonance may be considered as a reasonable band of frequency satisfying the frequency selectivity criterion. The corresponding power range is thus treated as the half-power zone during which the current in the circuit may reduce up to 70% of the current at resonance. In Fig. 4.20, extreme value of frequency on the left side of resonant frequency satisfying this half-power limit is designated as f_1 and may be referred as *lower half-power frequency*. Similarly, the extreme value of frequency on the right side of resonant frequency satisfying this half-power limit is designated as f_2 which may be referred as *upper half-power frequency*. The frequency range between these two extreme points expressed as a frequency band is referred as band-width. So, numerically band-width is given by

$$B_w = f_2 - f_1 \tag{4.42}$$

A detailed analysis for the series circuit of Fig.4.19 would result with the following value for bandwidth for the case of series resonance.

$$B_w = f_2 - f_1 = \frac{1}{2\pi}\left(\frac{R}{L}\right) \tag{4.43}$$

Quality Factor (*Q*) For Series Resonance

The concept of quality factor is also derived from the frequency response of Fig. 4.20 and it defines the sharpness of resonance for a particular network. It is often expressed as a ratio of the maximum energy stored in the energy-storing devices to that of the energy dissipated through the resistor in a given cycle during resonance. So, in view of the inductor L as the energy storing device,

$$Q = \frac{\text{Energy stored in } L \text{ per cycle}}{\text{Energy dissipated per cycle}} = \frac{\frac{1}{2}LI^2}{\frac{1}{\omega_0}\left(\frac{1}{2}RI^2\right)} = \frac{\omega_0 L}{R} = \frac{2\pi f_0 L}{R} \tag{4.44}$$

Since, $\omega_0 = \frac{1}{\sqrt{LC}}$, replacing the value of L in Equation 4.44 with $L = \frac{1}{\omega_0^2 C}$, we may find that

$$Q = \frac{\omega_0 L}{R} = \frac{\omega_0\left(\frac{1}{\omega_0^2 C}\right)}{R} = \frac{1}{\omega_0 CR} = \frac{1}{2\pi f_0 CR} \tag{4.45}$$

Few important relationships may also be established as shown below, which may be easily verified by the reader. Equation 4.46 is valid for series and parallel resonance as well.

$$f_0 = \sqrt{f_1 f_2} = Q.B_w \tag{4.46}$$

4.6.2 BANDWIDTH AND Q-FACTOR DURING PARALLEL RESONANCE

When an electric network containing the passive elements (such as resistance R, inductance L and capacitance C) in parallel is excited from an external supply source having a periodic function of frequency f, as shown in Fig.4.21 then it is expected that with the variation in the supply frequency, resonance may occur in the circuit as and when the frequency of the periodic forcing function matches with the natural frequency of oscillation of the system. Such a type of resonance is known as parallel resonance and the frequency at which resonance occurs is called resonant frequency, given by f_0. During resonance it is expected that the response of the system attains highest amplitude, which for the given network of Fig.4.21 may be viewed as the magnitude of the circuit current attaining its highest possible peak for a particular frequency of the supply source and the given network conditions.

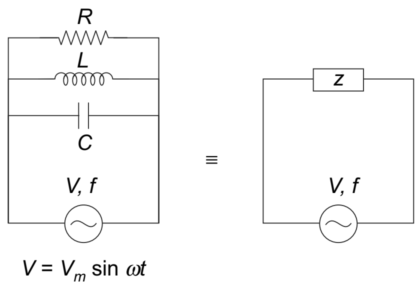


Fig. 4.21 *Network for study of parallel resonance*

In other words, the network becomes frequency selective during resonance and the frequency of the source for which the circuit resonates is called the parallel resonant frequency (f_0). Since the circuit current is mostly regulated by the circuit admittance ($I = Y.V$), during resonance it may be required that admittance should pass through its series highest value in order to ensure a highest possible peak for the current. For the parallel circuit of Fig. 4.21, the expression for complex admittance of the circuit is being presented in Equation 4.41 for a better analysis of resonant condition.

$$Y = Y_1 + Y_2 + Y_3 = \frac{1}{R} - j\left(\frac{1}{\omega L} - \omega C\right) \tag{4.47}$$

Now it may be seen clearly that the admittance is also a function of supply frequency as Y is a function of R, and $\omega = 2\pi f$. Thus it is permissible that the admittance may attain a value which becomes the highest in the series over a range of f, only when the term within the parentheses of Equation 4.47 reduces to zero. The frequency of the supply for which the imaginary part of Equation 4.47 becomes zero is therefore treated as resonant frequency. A detailed workout for the calculation of resonant frequency is performed in the following paragraph with the clue that

$\left.\frac{1}{\omega L} - \omega C\right|_{f=f_0} = 0$. This condition leads to $\omega C = \frac{1}{\omega L}$.

$$\Rightarrow \qquad \omega^2 = \frac{1}{LC} \Rightarrow \omega = \frac{1}{\sqrt{LC}} \Rightarrow 2\pi f_0 = \frac{1}{LC} \Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{4.48}$$

The condition of resonance also reveals many other facts about the circuit performance, which are stated categorically hereafter. These facts may be useful for finding the bandwidth and quality factor of coupled circuits.

1. Resonant frequency is given by $\omega_0 = \dfrac{1}{\sqrt{LC}}$ or $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$.
2. At resonance, the circuit impedance is the least and is numerically equal to the resistance of the network, given by $Z_0 = R$. Hence the circuit behaves as a purely resistive circuit.
3. At resonance, the power factor of the network is the highest and becomes unity.
4. At resonance, current in the circuit is the highest, and it becomes, $I_0 = V/R$, which remains in the same phase of the voltage.
5. At resonance, the average power consumed by the circuit is the highest, and it becomes, $P_0 = \dfrac{1}{2}\left(\dfrac{V^2}{R}\right)$.

Bandwidth (B_w) For Parallel Resonance

The concept of bandwidth requires the understanding of frequency response of current in the circuit over the stretch of frequency variation from zero to infinity passing through resonant frequency. Such a plot is shown in Fig. 4.22.

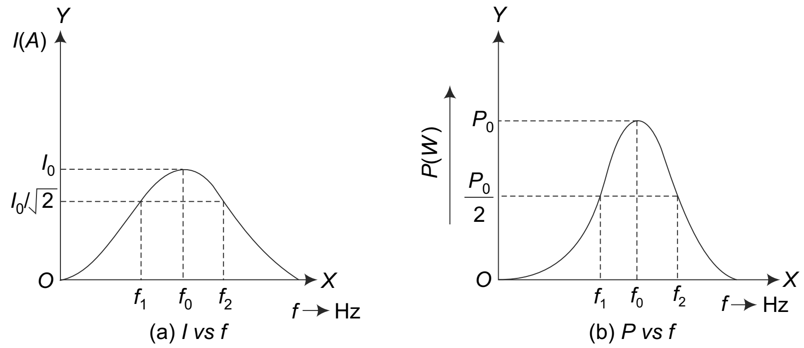


Fig. 4.22 *Frequency response of current and power for the circuit of Fig. 4.21*

As revealed from the plot of Fig. 4.22, the performance of the circuit as regards to amplitude of current and the power consumption is highest at resonance. This property is very useful for frequency selectivity of circuits generally used for communication purpose. However, it may be difficult in practice to obtain this condition for all purposes. Therefore, a range of frequency on either side of the resonant frequency for which the power consumption may reduce up to 50% of the power consumed during resonance may be considered as a reasonable band of frequency satisfying the frequency selectivity criterion. The corresponding power range is thus treated as the half-power zone during which the current in the circuit may reduce up to 70% of the current at resonance. In Fig. 4.22, extreme value of frequency on the left side of the resonant frequency satisfying this half power limit is designated as f_1 and may be referred as lower half-power frequency. Similarly, the extreme value of frequency on the right side of the resonant frequency satisfying this half power limit is designated as f_2 which may be referred as upper half-power frequency. The frequency range between these two extreme points expressed as a frequency band is referred as bandwidth. So, numerically bandwidth is given by

$$B_w = f_2 - f_1 \tag{4.49}$$

A detailed analysis for the series circuit of Fig. 4.21 would result with the following value for bandwidth for the case of parallel resonance.

$$B_w = f_2 - f_1 = \frac{1}{2\pi RC} \tag{4.50}$$

Quality Factor (Q) For Parallel Resonance

The concept of quality factor is also derived from the frequency response of Fig. 4.22 and it defines the sharpness of resonance for a particular network. In view of the finding of Equation 4.46, the quality factor for the case of parallel resonance may be derives as follows.

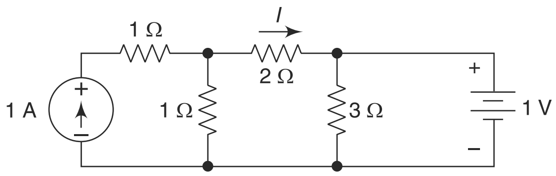
$$Q = \frac{f_0}{B_\omega} = \frac{f_0}{\dfrac{1}{2\pi RC}} = 2\pi f_0 RC = \omega_0 RC \tag{4.51}$$

Since, $\omega_0 = \dfrac{1}{\sqrt{LC}}$, replacing the value of C in Equation 4.51 with $C = \dfrac{1}{\omega_0^2 L}$, we may find that

$$Q = 2\pi f_0 RC = 2\pi f_0 R\frac{1}{\omega_0^2 L} = \frac{R}{2\pi f_0 L} = \frac{R}{\omega_0 L} \tag{4.52}$$

SOLVED EXERCISES

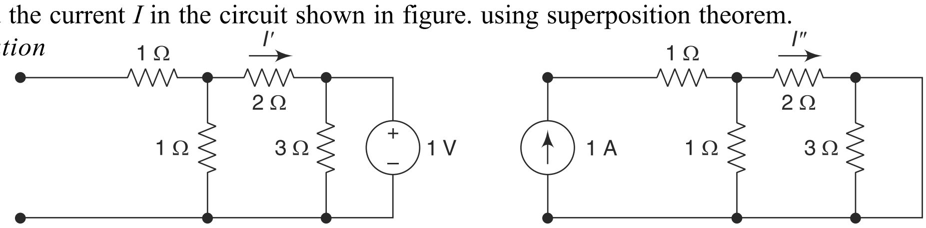
Superposition Theorem



4.1

Find the current I in the circuit shown in figure. using superposition theorem.

Solution



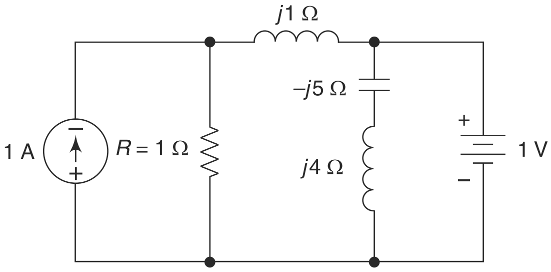
(i) Voltage Source acting alone *(ii) Current Source acting alone*

For Figure (i) $I' = -\frac{1}{3}$ A

For Figure (ii) $I'' = 1 \times \frac{1}{1+2} = \frac{1}{3}$ A

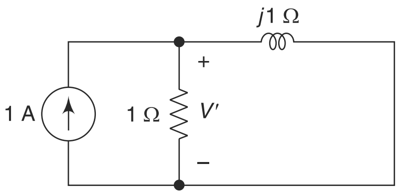
By superposition, $I = (I' + I'') = -\frac{1}{3} + \frac{1}{3} = 0$ *Ans.*

4.2

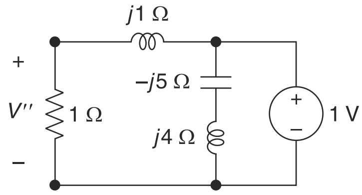


Calculate the voltage V across the resistor R by using superposition theorem.

Solution



(i) Circuit with current source acting alone



(ii) Circuit with voltage source acting alone

For Figure (i), $V' = \frac{j}{1+j}$

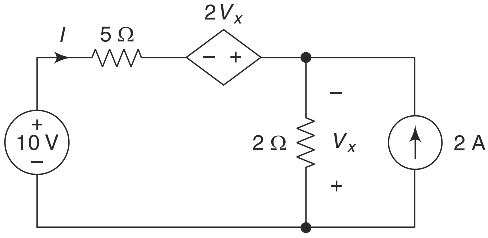
For Figure (ii),current through the resistor $I'' = \frac{1}{1+j}$

$\therefore \quad V'' = I'' \times 1 = \frac{1}{1+j}$

So, by superposition theorem

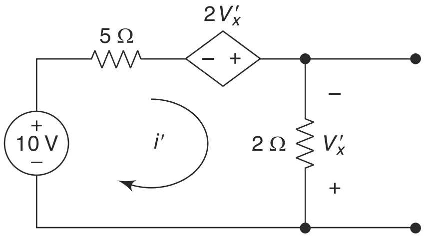
$$V = (V' + V'') = \frac{j}{1+j} + \frac{1}{1+j} = 1\text{ V}$$

4.3

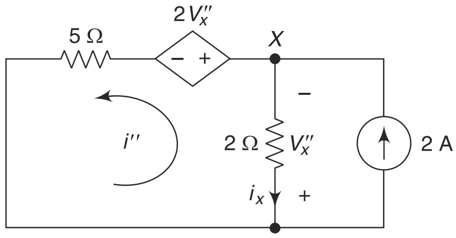


Use superposition theorem on the circuit shown in figure to find I.

Solution



(i) Voltage source acting alone



(ii) Current source acting alone

For Fig. (i), by KVL, $5i' - 2vx' + 2i' = 10$ with $v_x' = -2i'$

$\Rightarrow \qquad 7i' + 4i' = 10$

$\Rightarrow \qquad i' = 10/11 \text{ A}$

For Fig (ii), by KCL at node (x)

$$2 = i_x + i'' = -\frac{v_x'}{2} + i'' \qquad \text{(i)}$$

But loop analysis in the left loop gives,

$$5i'' + 3v_x'' = 0$$

or $\qquad i'' = -\frac{3}{5}v_x''$

From (i), $2 = -\frac{v_x''}{2} - \frac{3}{5}v_x''$

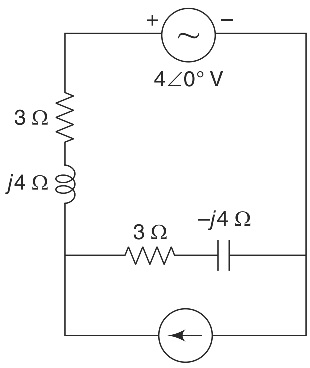
$\Rightarrow \qquad v_x'' = -\frac{20}{11}$

$\therefore \qquad i'' = -\frac{3}{5} \times \left(-\frac{20}{11}\right) = \frac{12}{11} \text{ A}$

So, by superposition theorem total current

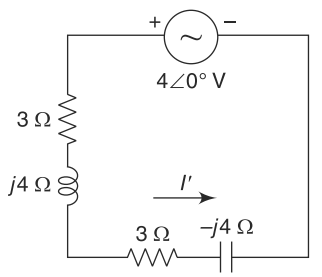
$$I = (i' - i'') = \left(\frac{10}{11} - \frac{12}{11}\right) = -\frac{2}{11} \text{ A}$$

4.4 *Determine the current in the capacitor branch by superposition theorem.*

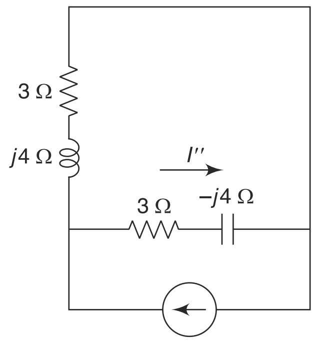


Solution *When the voltage source is acting alone:*
Here, the current in the capacitor branch is,

$$I' = \frac{4\angle 0°}{(3+j4)+(3-j4)} = \frac{2}{3}\angle 0° \text{ A}$$



(i) When voltage source acting alone



(ii) When curren source acting alone

When the current source is acting alone:
Here, the current in the capacitor branch is,

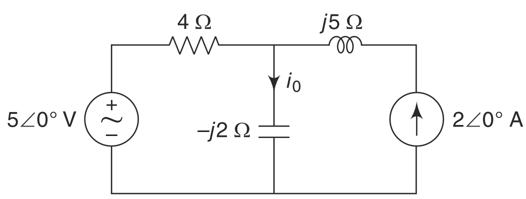
$$I'' = 2\angle 90° \times \frac{(3+j4)}{(3+j4)+(3-j4)} = \left(-\frac{4}{3}+j1\right) \text{A}$$

∴ Total current when both the sources are acting simultaneously, is

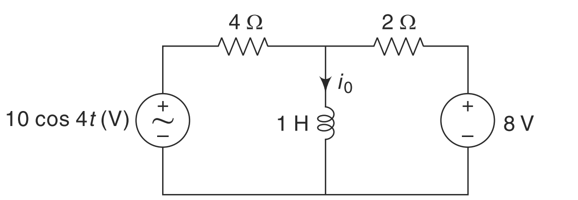
$$I = (I' + I'') = \left(\frac{2}{3}-\frac{4}{3}+j1\right) = \left(-\frac{2}{3}+j1\right) = 1.2\angle 123.7° \text{ A} \qquad \textit{Ans.}$$

4.5 Find the current i_0 using superposition theorem.

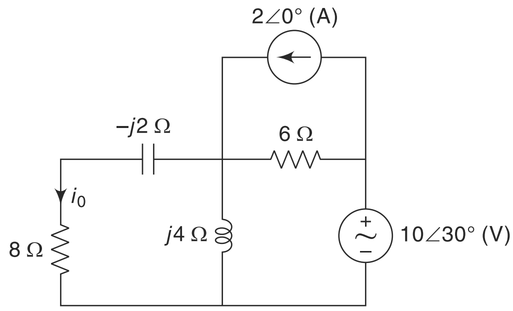
(a)



(b)



(c)

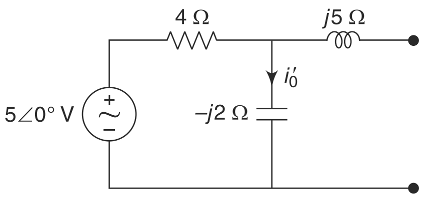


Solution

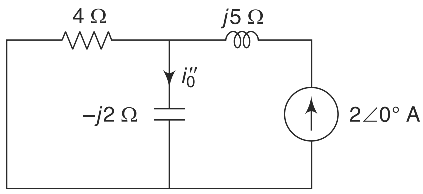
(a) *When voltage source is acting alone*

The current in this case is, $i_0' = \dfrac{5}{4 - j2} = \left(1 + j\dfrac{1}{2}\right)$ A

When current source is acting alone



(i) Voltage source acting alone



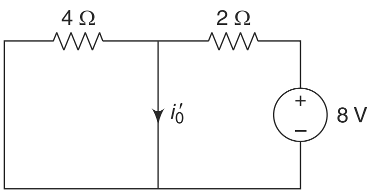
(ii) Current source acting alone

In this case, the current is, $i_0'' = 2\angle 0° \times \dfrac{4}{4 - j2} = \left(\dfrac{8}{5} + j\dfrac{14}{5}\right)$ A

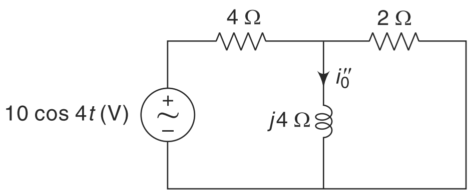
∴ By superposition theorem, total current is,

$$i_0 = \left(i_0' + i_0''\right) = \left(1 + \frac{8}{5}\right) + j\left(\frac{1}{2} + \frac{4}{5}\right) = 2.9\angle 26.56° \text{ A} \qquad \textit{Ans.}$$

(b) *When DC source is acting alone*



(i) DC source acting alone



(ii) AC source acting alone

$\therefore$ The current, $i_0' = \dfrac{8}{2} = 4$ A

When AC source is acting alone

Equivalent impedance, $Z = 4 + \left(\dfrac{j4 \times 2}{2 + j4}\right) = \dfrac{4 + j6}{1 + j2}$

$\therefore$ Main current, $I = \dfrac{10\angle 0^\circ}{Z} = 10\angle 0^\circ \dfrac{(1 + j2)}{4 + j6} = \dfrac{10 + j20}{4 + j6}$

$\therefore$ The current, $i_0'' = I \times \dfrac{2}{2 + j4} = \dfrac{10(1 + j2)}{4 + j6} \times \dfrac{1}{1 + j2} = \left(\dfrac{10}{13} - j\dfrac{15}{13}\right)$ A

$\therefore$ By superposition theorem, total current is,

$$i_o = (i_0' + i_0'') = 4 + \left(\frac{10}{3} - j\frac{15}{3}\right) = 4.9\angle{-13.6^\circ} \text{ A} \qquad \textit{Ans.}$$

(c) *When the voltage source is acting alone*

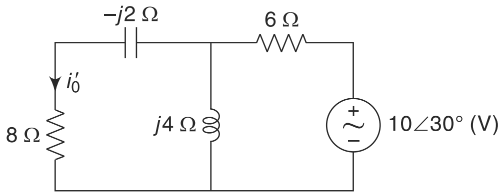
Equivalent impedance,

$$Z = \frac{j4(8 - j2)}{8 + j2} + 6 = \frac{28 + j22}{4 + j}$$

$\therefore$ Main current,

$$I = \frac{10\angle 30^\circ (4 + j)}{28 + j22}$$

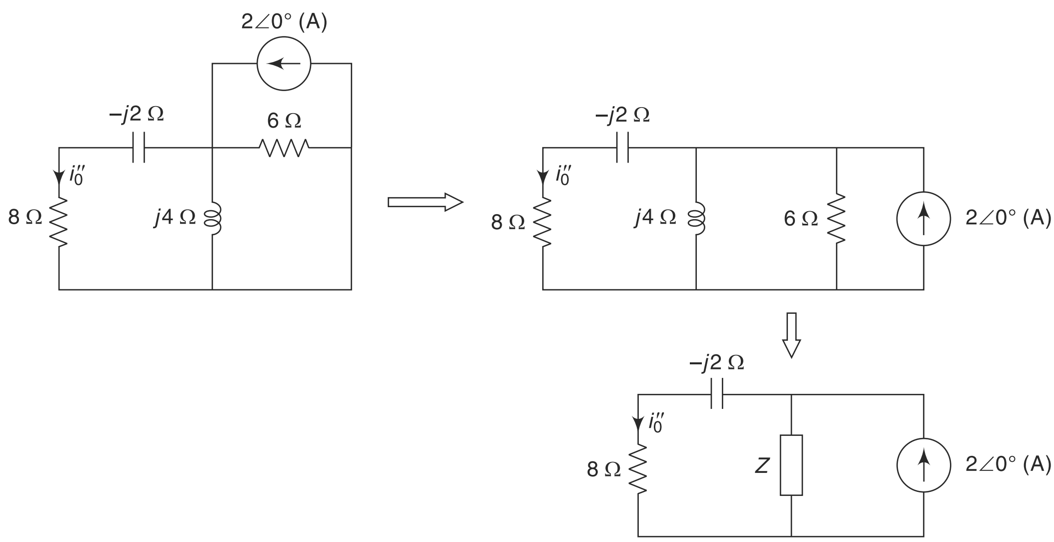
$$= \frac{(8.66 + j5)\,(4 + j)}{28 + j22}$$



(i) Voltage source acting alone

$\therefore$ The current, $i_0' = I \times \dfrac{8 - j2}{8 + j2} = \dfrac{8.66 + j5}{56 + j44} = 0.14\angle{-8.16^\circ}$ A

When current source is acting alone



Since $Z = \dfrac{j4 \times 6}{6 + j4} = \dfrac{j12}{3 + j2}$

$\therefore$ The current, $i_0'' = 2\angle 0° \times \dfrac{Z}{8 - j2 + Z}$

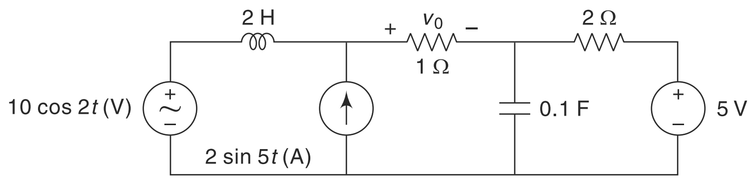
$$= \frac{j12}{12 + j11} = 0.73\angle 47.49° \text{ A}$$

$\therefore$ By superposition theorem, total current is,

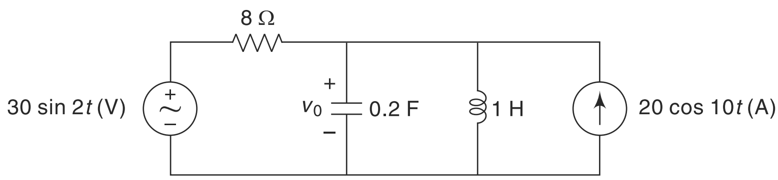
$$i_0 = (i_0' + i_0'') = (0.14\angle -8.16° + 0.73\angle 47.49°) = (0.631 + j0.518) = 0.81\angle 39.38° \text{ A} \quad \textit{Ans.}$$

4.6 Find v_0 using Superposition Theorem.

(a)



(b)



Solution

(a) *When 5 V DC source is acting alone:*

$$v_0' = -1 \text{ V}$$

When ac voltage source is acting alone,

$$v_0'' = 2.498\angle -30.79° \text{ (V)}$$

When ac current source is acting alone,

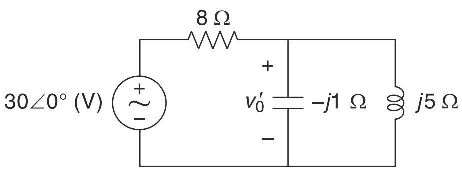
$$v_0''' = 2.328\angle -80° \text{ (V)}$$

By superposition theorem, when all sources are acting simultaneously, the voltage is,

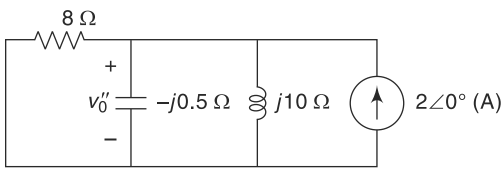
$$v_0 = (v_0' + v_0'' + v_0''') = -1 + 2.498\cos(2t - 30.79°) + 2.328\sin(5t + 10°) \text{ V} \quad \textit{Ans.}$$

(b) *When voltage source is acting alone:*

Here, $X_C = \dfrac{-j}{5 \times 0.2} = -j1\,\Omega$ and $X_L = j \times 5 \times 1 = j5\,\Omega$



(i) Voltage source acting alone



(ii) Current soure acting alone

By KCL,

$$-\frac{30-v_0'}{8}+\frac{v_0'}{-j1}+\frac{v_0'}{j5}=0 \Rightarrow v_0'=\frac{30}{8(0.125+j0.8)}=4.631\angle-81.12°\text{ (V)}$$

When current source is acting alone,

$$X_C=\frac{-j}{10\times 0.2}=-j0.5\ \Omega \quad \text{and} \quad X_L=j\times 10\times 1=j10\ \Omega$$

By KCL,

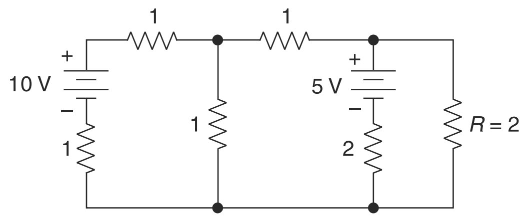
$$2=v_0''\left(\frac{1}{8}+\frac{1}{j10}+\frac{1}{-j0.5}\right)\Rightarrow v_0''=\frac{2}{0.125+j1.9}=1.051\angle-86.24°\text{ (V)}$$

By superposition theorem, when all sources are acting simultaneously, the voltage is,

$$v_0=(v_0'+v_0'')=4.631\sin(5t-81.12°)+1.051\cos(10t-86.24°)\text{ (V)} \qquad \textit{Ans.}$$

Thevenin's and Norton's Theorems

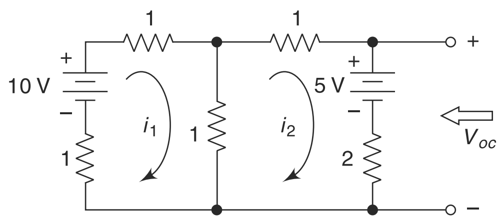
4.7



Draw the Thevenin's equivalent of the circuit in figure and find the load current, *i*. All values are in ohm.

Solution Open circuiting the terminals,

By KVL for two meshes,

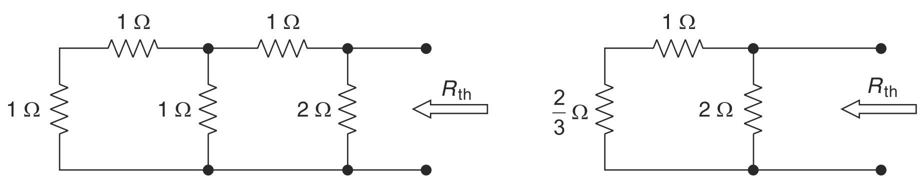


$$3i_1-i_2=10$$

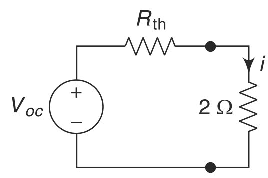
and $\quad -i_1+4i_2=-5$

Solving, $\quad i_1=5/11 \quad$ and $\quad i_2=-5/11$

$$\therefore \quad V_{oc}=(5+2i_2)=\left(5-\frac{10}{11}\right)=\frac{45}{11}\text{ V}$$

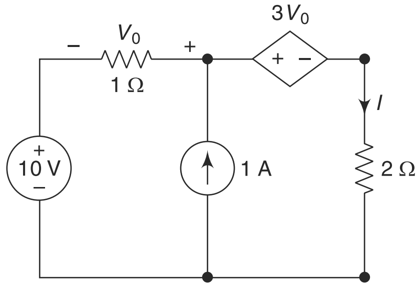


Equivalent resistance, $R_{th}=\dfrac{\frac{5}{3}\times 2}{5/3+2}=\dfrac{10}{11}\ \Omega$



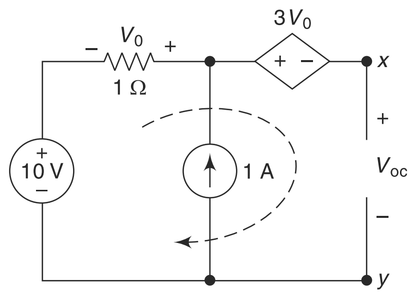
So, the load current is, $i=\dfrac{V_{oc}}{R_{th}+2}=\dfrac{45/11}{10/11+2}=\dfrac{45}{32}=1.40625\text{ A}$ *Ans.*

4.8



Find I in the given figure, using Thevenin's theorem.

Solution Removing the 2 Ω resistor,



By KVL for the supermesh,

$$-10 - v_0 + 3v_0 + v_{0c} = 0$$

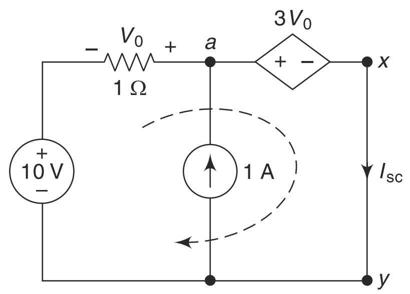
$$\Rightarrow \qquad v_{0c} = 10 - 2v_0$$

But, due to open-circuit, 1A source will circulate through 1 Ω resistor.

$$\therefore \qquad v_0 = 1 \times 1 = 1 \text{ V}$$

$$\therefore \qquad V_{0c} = (10 - 2) = 8 \text{ V}$$

Let's short circuit the terminals x-y,



By KVL,

$$-10 - v_0 + 3v_0 = 0$$

or $\qquad v_0 = 5$

But, by KCL at node (a),

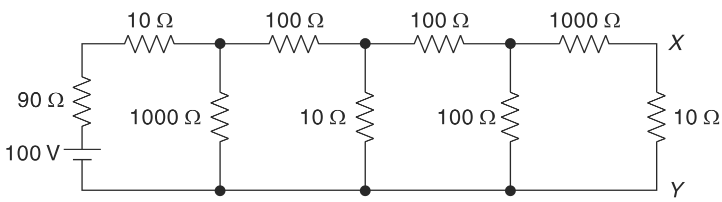
$$\frac{v_0}{1} = 1 - I_{sc}$$

$$I_{sc} = (1 - v_0) = -4 \text{ A (e.g. current is flowing from } y \text{ to } x)$$

$$\therefore \qquad R_{th} = \frac{V_{oc}}{I_{sc}} = \frac{8}{4} = 2\ \Omega$$

So, the current through 2 Ω resistor, $I = \dfrac{8}{2+2} = 2$ A $\qquad$ *Ans.*

4.9

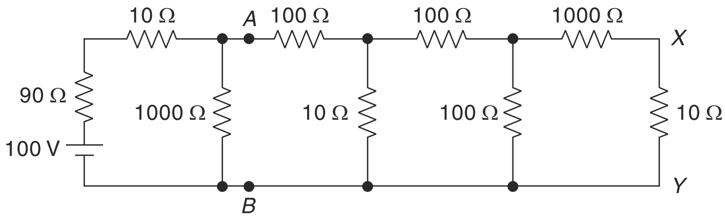


By the iterative use of Thevenin's theorem, reduce the circuit shown in figure to a single emf acting in series with a single resistor. Hence, calculate the current in the 10 Ω resistor connected across XY.

Solution Consider the section of the network to the left of *A–B*. By use of Thevenin's theorem, this portion is reduced to the form of Fig. (ii).

$$\therefore \qquad R_{\text{th}} = \frac{1000 \times 100}{1000 + 100} = \frac{1000}{11}\ \Omega$$

$$\therefore \qquad V_{\text{th}} = \frac{100 \times 1000}{1100} = \frac{1000}{11}\ \text{V}$$

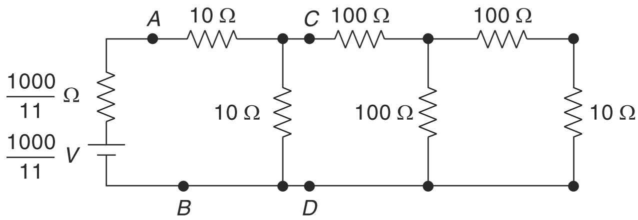


(i)

Applying Thevenin's Theorem to the section left of *CD* of Fig. (ii),

$$\therefore \qquad R_{th} = \frac{(2100/11) \times 10}{(2100/11) + 10} = \frac{2100}{221}\ \Omega$$

$$\therefore \qquad V_{th} = \frac{(1000/11) \times 10}{(2100/11) + 10} = \frac{1000}{221}\ \text{V}$$

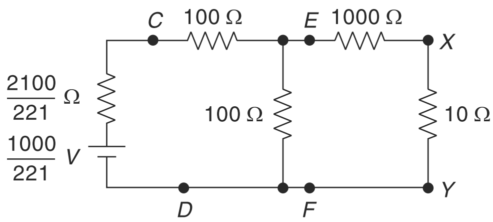


(ii)

Applying Thevenin's Theorem to the section left of *EF* of Fig. (iii),

$$\therefore \qquad R_{\text{th}} = \frac{(24200/221) \times 100}{(24200/221) + 100} = \frac{24200}{463}\ \Omega$$

$$\therefore \qquad R_{\text{th}} = \frac{(1000/221) \times 100}{(24200/463) + 100} = \frac{1000}{463}\ \text{V}$$

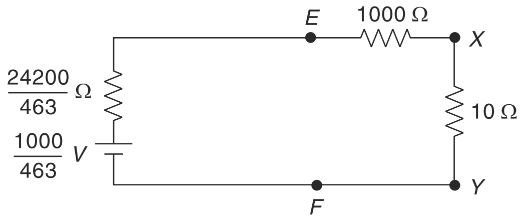


(iii)

Section left to XY is put as in Fig. (iv).

$$\therefore \quad R_{th} = \frac{24200}{463} + 1000 = \frac{487200}{463}\ \Omega$$

$$\therefore \quad V_{th} = \frac{(1000/463)\times 1000}{(24200/463)+1000} = \frac{1000}{4872}\ \text{V}$$

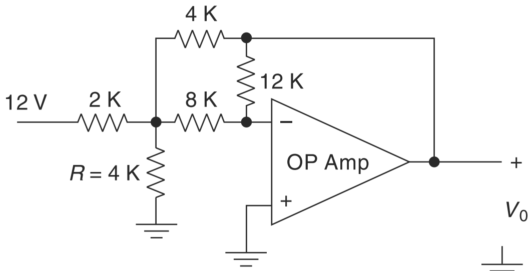


(iv)

Hence, the current in 10 Ω resistor is,

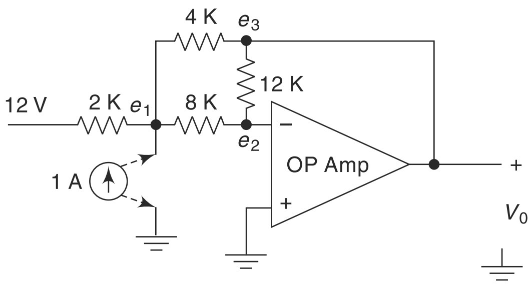
$$I = \frac{(1000/487.2)}{(487200/436)+10} = 0.0193\ \text{A} \qquad \textit{Ans.}$$

4.10



In the Operational-amplifier circuit shown in figure, find I, in the $R = 4$ kΩ resistor, using Thevenin's theorem.

Solution Open-circuiting the 4 kΩ resistor,



Here, $e_2 = 0, e_3 = V_0$

$$\frac{e_1 - 12}{2\times 10^3} + \frac{e_1 - V_0}{4\times 10^3} + \frac{e_1}{8\times 10^3} = 0$$

$$\Rightarrow \quad 7e_1 = (48 + 2V_0) \qquad \text{(i)}$$

$$\frac{0 - e_1}{8\times 10^3} + \frac{0 - V_0}{12\times 10^3} = 0$$

$$\Rightarrow \quad V_0 = -\frac{3}{2}e_1 \qquad \text{(ii)}$$

From equation (i) and equation (ii),

$$\Rightarrow \quad e_1 = 4.8\ \text{V} = e_{oc}$$

Now, we connect a 1 A current source at the place of 4 kΩ resistor.

By KCL at node (1),

$$\frac{e_1}{2\times 10^3} + \frac{e_1 - V_0}{4\times 10^3} + \frac{e_1}{8\times 10^3} = 1$$

$$\Rightarrow \quad 7e_1 = 8000 + 2V_0$$

By KCL at node (2),

$$V_0 = -\frac{3}{2}e_1$$

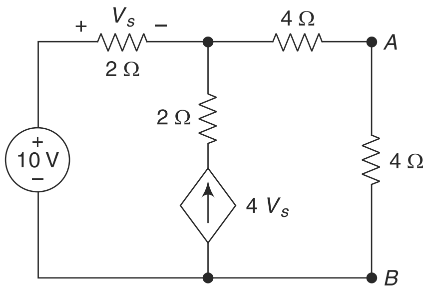
$$\Rightarrow \qquad 7e_1 = 8000 + 2\left(-\frac{3}{2}e_1\right)$$

$$\Rightarrow \qquad e_1 = 800 \text{ V}$$

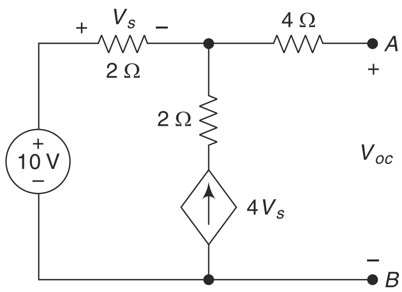
$$\therefore \qquad R_{th} = \frac{e_1}{1} = 800\ \Omega$$

$$\therefore \qquad i = \frac{4.8}{4000+800} = \frac{4.8}{4.8\times 10^3} = 1 \text{ mA} \qquad \textit{Ans.}$$

4.11



Find Thevenin's equivalent about *AB* for the circuit shown in figure.

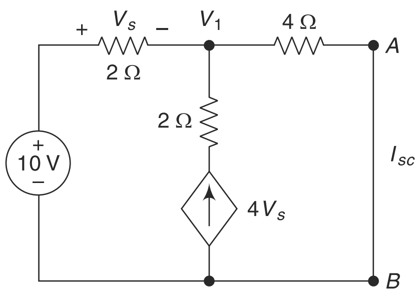


Solution Open-circuiting The 4 Ω resistor, by KCL,

$$\frac{V_{oc}-10}{2} = 4v_s = 4(10 - V_{oc})$$

$$\Rightarrow \qquad V_{oc} = 10 \text{ V}$$

Short-circuiting the terminals *AB*, by KCL,



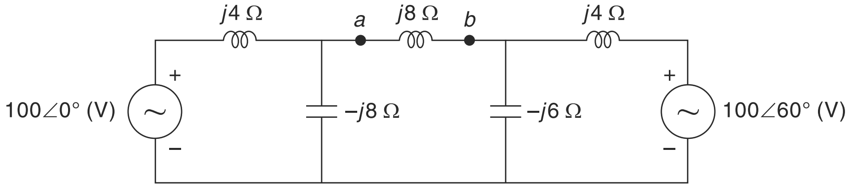
$$\frac{V_1-10}{2} + \frac{V_1}{4} = 4v_s = 4(10 - V_1)$$

$$V_1 = \frac{180}{19} = 9.47 \text{ V}$$

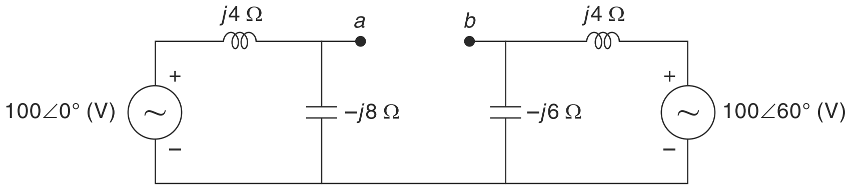
$$\therefore \qquad I_{sc} = \frac{9.47}{4} = 2.368 \text{ A}$$

$$\therefore \qquad R_{th} = \frac{V_{th}}{I_{sc}} = 4.22\ \Omega \qquad \textit{Ans.}$$

4.12 In the network, determine the steady current in the 8 Ω inductor using Thevenin's theorem.



Solution With *a–b* open-circuited,



$$V_a = \frac{100\angle 0^\circ}{j4 - j8}(-j8) = 200\angle 0^\circ \text{ V}$$

$$V_b = \frac{100\angle 60^\circ}{j4 - j6}(-j6) = 300\angle 60^\circ \text{ V}$$

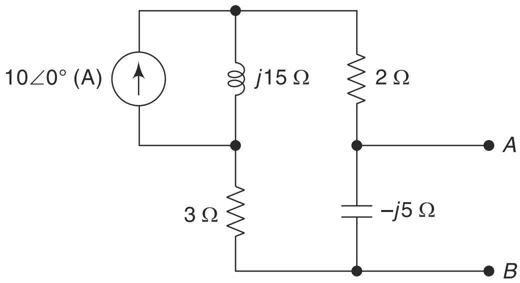
$\therefore \quad V_{\text{th}} = (V_a - V_b) = 200\angle 0^\circ - 300\angle 60^\circ = (50 - j259.81) \text{ V}$

$\therefore \quad Z_{\text{th}} = \dfrac{(j4)(-j8)}{j4 - j8} + \dfrac{(j4)(-j6)}{j4 - j6} = j20\ \Omega$

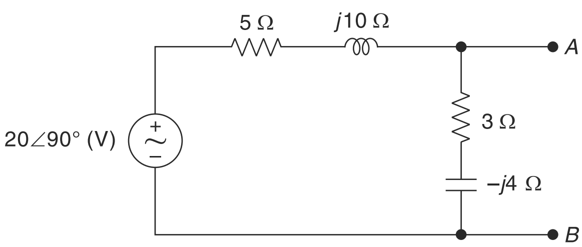
$\therefore$ Current in the 8 Ω inductor, $i = \dfrac{V_{\text{th}}}{Z_{\text{th}} + Z_L} = \dfrac{(50 - j259.81)}{j20 + j8} = 9.45\angle{-169.1^\circ} \text{ A}$ *Ans.*

4.13 Obtain Thevenin's equivalent circuit with respect to terminals *A–B* in the networks shown below.

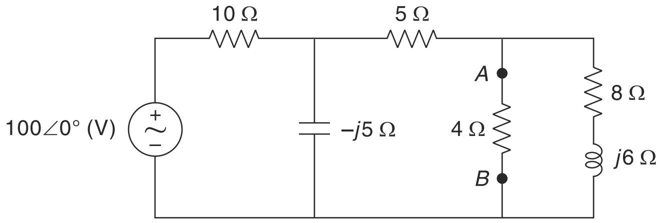
(a)



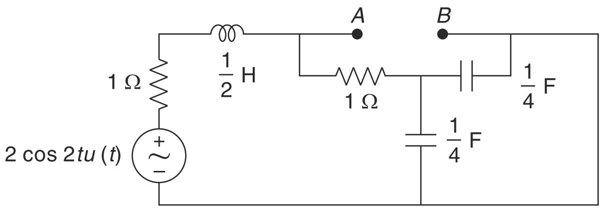
(b)



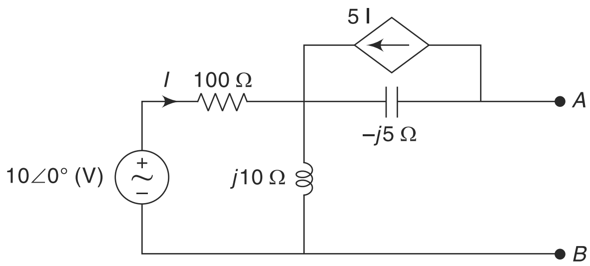
(c)



(d)



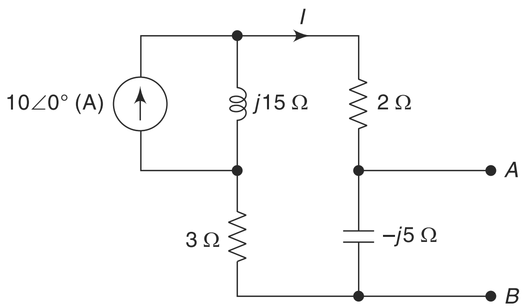
(e)



Solution

(a) With *A–B* open, the current is,

$$I = \frac{10\angle 0^\circ}{5 - j5 + j15} \times j15 = \frac{150\angle 90^\circ}{5 + j10}$$



Thevenin voltage,

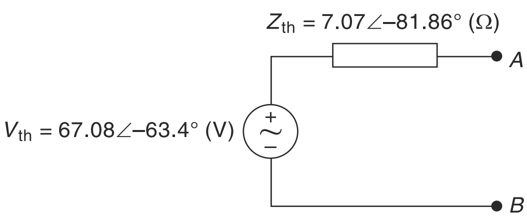
$$V_{\text{th}} = V_{AB} = I \times (-j5)$$

$$= \frac{150\angle 90^\circ}{5 + j10} \times (5\angle -90^\circ)$$

$$= 67.08\angle -63.4^\circ \text{ V} \qquad \textit{Ans.}$$

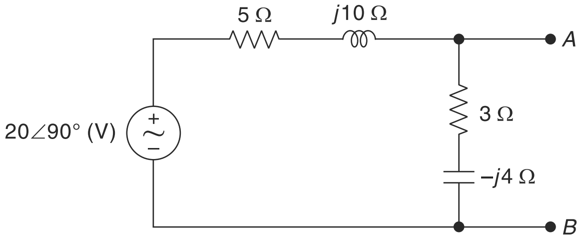
Thevenin impedance,

$$Z_{\text{th}} = Z_{AB} = \frac{-j5 \times (5 + j15)}{-j5 + 5 + j15} = 7.07\angle -81.86^\circ\ \Omega \qquad \textit{Ans.}$$



Thus, the Thevenin's equivalent circuit is shown in figure.

(b) Here, Thevenin voltage,

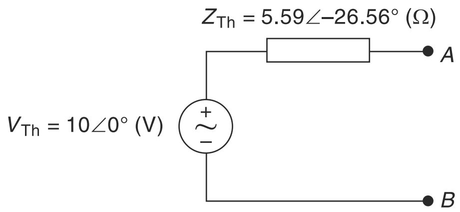


$$V_{th} = \frac{20\angle 90^\circ}{5 + j10 + 3 - j4} \times (3 - j4)$$

$$= \frac{j120(3 - j4)}{8 + j6}$$

$$V_{th} = \frac{50\angle 36.87^\circ}{5\angle 36.87^\circ} = 10\angle 0^\circ \text{ (V)}$$

Thevenin impedance,



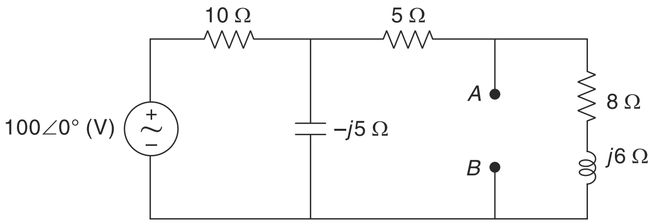
$$Z_{th} = \frac{(5 + j10) \times (3 - j4)}{(5 + j10) + (3 - j4)}$$

$$= \frac{11.8\angle 63.43^\circ \times 5\angle -53.13^\circ}{10\angle 36.87^\circ}$$

$$= 5.59\angle -26.56 \text{ } (\Omega)$$

Thus, the Thevenin's equivalent circuit is shown in figure. *Ans.*

(c)



Here, with A–B open, equivalent impedance,

$$Z = 10 + \frac{-j5 \times (13 + j6)}{-j5 + (13 + j6)} = \frac{160 - j55}{13 + j1}\ \Omega = 12.98\angle -23.37^\circ \text{ } (\Omega)$$

∴ Main current, $I = \dfrac{100\angle 0^\circ}{Z} = \dfrac{100\angle 0^\circ}{12.98\angle -23.37^\circ} = 7.7\angle 23.37^\circ \text{ (A)}$

∴ Thevenin voltage,

$$V_{th} = I \times \left(\frac{-j5}{-j5 + 5 + 8 + j6} \right) \times (8 + j6)$$

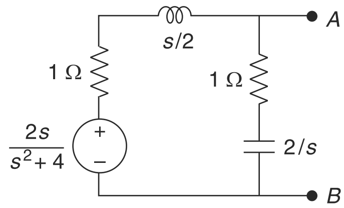
$$= 7.7\angle 23.37^\circ \times \left(\frac{-j5}{13 + j1} \right) \times (8 + j6) = 29.553\angle -34.16^\circ \text{ (V)} \qquad \textit{Ans.}$$

∴ Thevenin impedance, $Z_{th} = \left[\dfrac{10 \times (-j5)}{10 - j5} + 5 \right] || (8 + j6) = 5.33\angle -0.5^\circ \text{ } (\Omega)$ *Ans.*

(d) The circuit is redrawn as shown considering two capacitors in parallel.

∵ $$C_{eq} = (C_1 + C_2) = \left(\frac{1}{4} + \frac{1}{4} \right) = \frac{1}{2} \text{ F}$$

Thevenin voltage is given as,



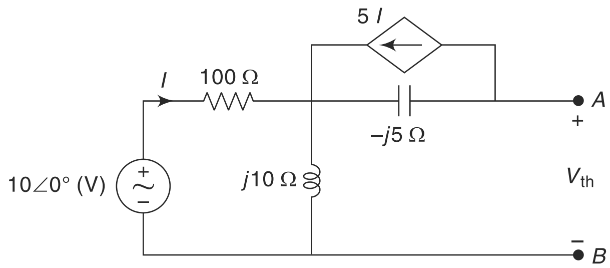
$$V_{th}(s) = \frac{2s}{s^2+4} \times \frac{(1+2/s)}{(1+2/s+1+s/2)}$$

$$= \frac{4s}{(s^2+4)(s+2)} \text{ (V)}$$

$\therefore$ Thevenin impedance, $Z_{th}(s) = (1+2/s) \,||\, (1+s/2) = 1\,\Omega$ *Ans.*

(e) *To find* V_{th}

With *A–B* open, current of the dependent source can flow through the capacitor only.

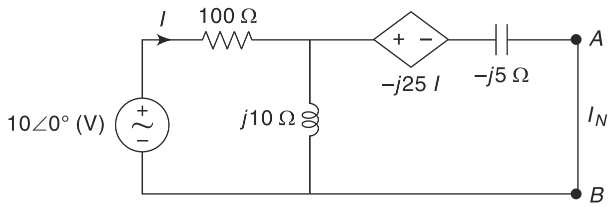


$$\therefore\ I = \frac{10\angle 0^\circ}{100+j10} = 0.09995\angle -5.7^\circ \text{ (A)}$$

$\therefore$ Thevenin voltage,

$$V_{th} = V_{AB} = (I \times j10) - \{5I \times (-j5)\} = j35I = j35 \times 0.09995\angle -5.7^\circ = 3.48\angle 84.3^\circ \text{ (V)} \quad \textit{Ans.}$$

To find I_N



Converting the dependent current source into voltage source, by KVL,

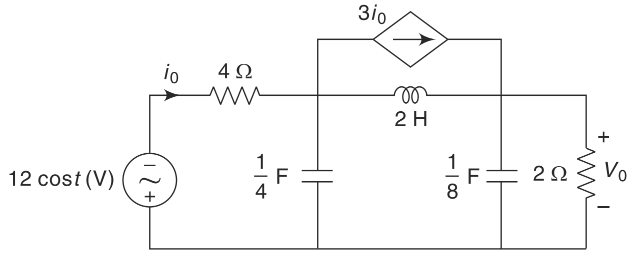
$$10\angle 0^\circ = (100+j10)\,I - j10I_N$$

and $\quad -(-j25I) = -j10I + I_N\,(j10 - j5)$

Solving for I_N, $I_N = 0.6\angle 31^\circ$ (A) *Ans.*

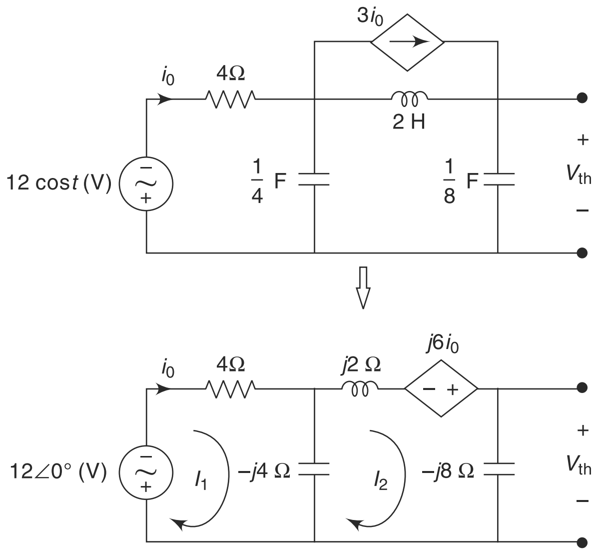
$$\therefore \text{ Thevenin impedance, } Z_{th} = \frac{V_{th}}{I_N} = \frac{3.48\angle 84.3^\circ}{0.6\angle 31^\circ} = 5.8\angle 53.3^\circ\ (\Omega) \quad \textit{Ans.}$$

4.14 Find V_0 using Thevenin's Theorem.



Solution *To find* V_{th}

Removing the 2 Ω resistor and open circuiting the terminals and then converting the dependent current source into dependent voltage source, we redraw the circuit as follows.



By KVL for the two loops, (here $i_0 = I_1$)

$$(4 - j4)\, I_1 + j4 I_2 = -12$$

$$-j2 I_1 + (-j6)\, I_2 = 0$$

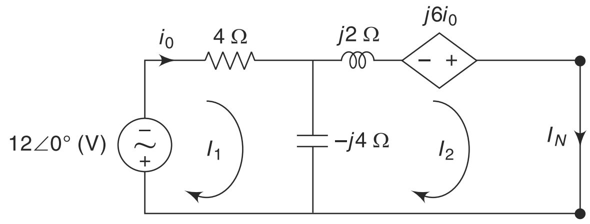
Solving for I_2,

$$I_2 = \frac{\begin{vmatrix} (4-j4) & -12 \\ -j2 & 0 \end{vmatrix}}{\begin{vmatrix} (4-j4) & j4 \\ -j2 & -j6 \end{vmatrix}} = \frac{-j24}{-j24-24-8} = \frac{j3}{4+j3} = 0.6\angle 53.13^\circ\ \text{(A)}$$

Therefore, Thevenin voltage is, $V_{th} = I_2 \times (-j8) = \dfrac{24}{4+j3} = 4.8\angle -36.87^\circ\ \text{(V)}$

To find I_N

Removing the 2 Ω resistor and short circuiting the terminals and then converting the dependent current source into dependent voltage source, we redraw the circuit as follows.



By KVL for the two loops,

$$(4-j4)\,I_1 + j4I_2 = -12$$

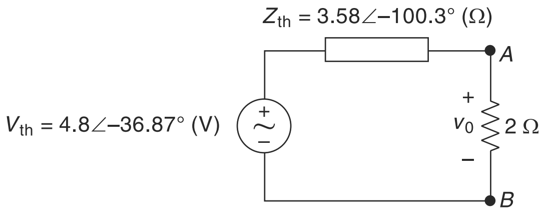
$$-j2I_1 + (-j2)\,I_2 = 0$$

Solving for I_2,

$$I_2 = I_N = \frac{\begin{vmatrix} (4-j4) & -12 \\ -j2 & 0 \end{vmatrix}}{\begin{vmatrix} (4-j4) & j4 \\ -j2 & -j2 \end{vmatrix}} = \frac{-j24}{-8-j8-8} = \frac{j3}{2+j} = 1.341\angle 63.435°\ \text{(A)}$$

Therefore, Thevenin impedance is, $Z_{\text{th}} = \dfrac{V_{\text{th}}}{I_N} = \dfrac{4.8\angle -36.87°}{1.341\angle 63.435°} = 3.58\angle -100.3°\ (\Omega)$

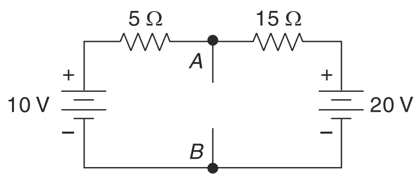
Thus, Thevenin's equivalent circuit becomes as shown.



Thus, the required voltage,

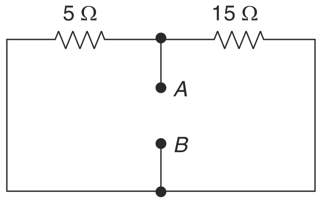
$$v_0 = \left(\frac{V_{\text{th}}}{Z_{\text{th}}+2}\right) \times 2 = \left(\frac{4.8\angle -36.87°}{3.58\angle -100.3° + 2}\right) \times 2 = 1.27\angle 32°\ \text{(V)} \qquad \textit{Ans.}$$

4.15



Obtain the Norton's equivalent circuit with respect to the terminals *AB* for the network shown in figure.

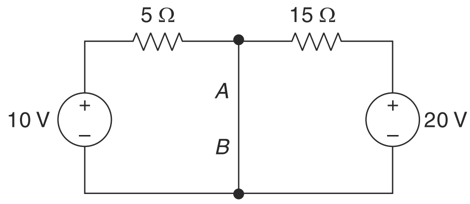
Solution Removing the sources,



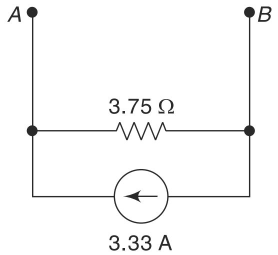
$$\therefore \qquad Z_{eq} = \frac{5 \times 15}{5 + 15} = \frac{75}{20} = 3.75\ \Omega$$

Short-circuiting *AB*,

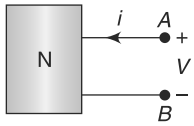
$$I_{sc} = \frac{10}{5} + \frac{20}{15} = 3.33\ \text{A}$$



So, Norton's equivalent circuit is shown.



4.16 For the one port shown in figure determine the Norton's equivalent at the terminals *AB*, if the *v–i* characteristic is given by, $16v = 80 - 2i$.



Solution *v–i* characteristic is given as,

$$16v = 80 - 2i \Rightarrow \frac{V}{5} + \frac{i}{40} = 1$$

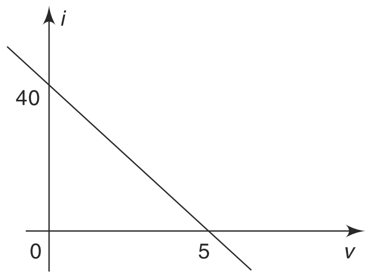
Thus, short circuit current,

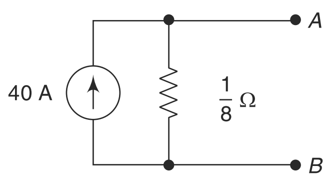
$$I_{sc} = 40\ \text{A}\ \ (\text{where } v = 0)$$

and open-circuit voltage,

$$V_{oc} = 5\ \text{V}\ \ (\text{whare } i = 0)$$

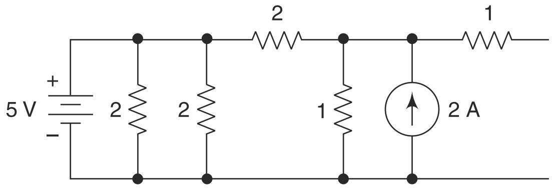
$$\therefore \qquad R_n = \frac{V_{oc}}{I_{sc}} = \frac{5}{40} = \frac{1}{8}\ \Omega$$





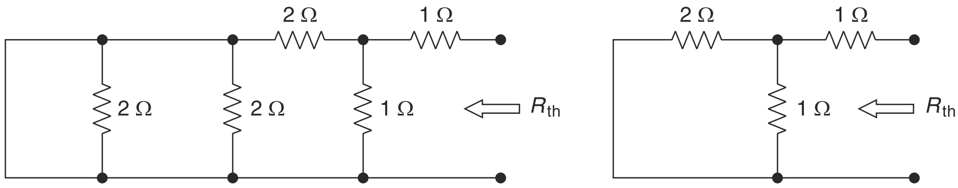
Norton's equivalent circuit is shown accordingly.

4.17



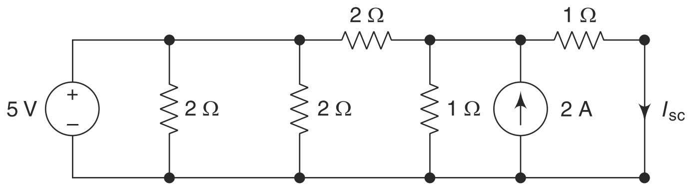
Find both Thevenin's and Norton's equivalent circuit for the network shown in figure. All values are in ohm.

Solution Removing the sources,



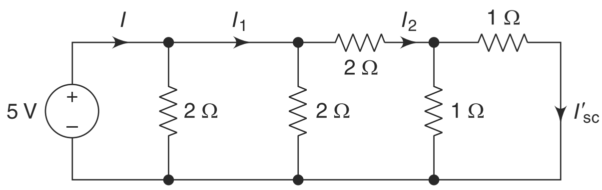
$$\Rightarrow \qquad R_N = R_{th} = \left(\frac{2}{3}+1\right) = \frac{5}{3}\ \Omega$$

Short-circuiting the terminals,



By superposition theorem, when 5 V source is acting alone,

$$I = \frac{5}{5/7} = 7\text{ A}$$

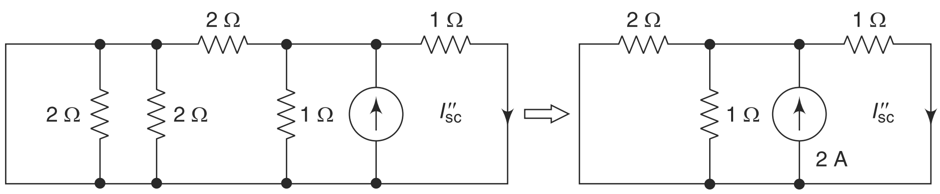


$$\therefore \qquad I_1 = 4.5\text{ A}$$

$$\therefore \qquad I_2 = 2A$$

$$\therefore \qquad I'_{sc} = 1\text{ A}$$

and when 2 A source is acting alone,



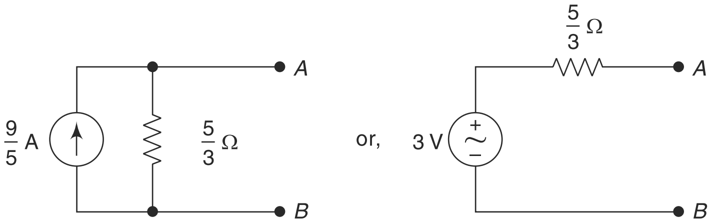
$$\therefore \qquad I''_{sc} = 2 \times \frac{2/3}{2/3+1} = \frac{4}{5}\text{ A}$$

$\therefore$ Total $I_{sc} = (I'_{sc} + I''_{sc}) = \left(1 + \frac{4}{5}\right) = \frac{9}{5}$ A

$\therefore$

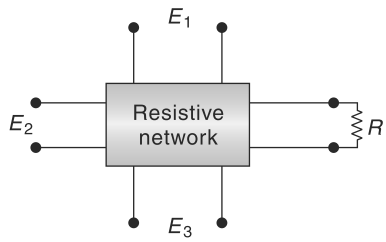
$$V_{th} = I_{sc} \times R_{th} = \frac{9}{5} \times \frac{5}{3} = 3 \text{ V}$$

The circuits are shown accordingly.



Maximum Power Transfer Theorem

4.18 In the network shown, the power dissipated in R when E_1, E_2 or E_3 acting alone is
(a) 20 W, 80 W, and 5 W respectively.
(b) 30 W, 270 W, and 120 W respectively.



Calculate the maximum power that R can dissipate due to the simultaneous action of all the sources. Calculate both for (a) and (b).

What will be the minimum power dissipated in R when all the sources are acting simultaneously?

Solution Current for E_1 at R, $i_1 = \pm\sqrt{\frac{P_1}{R}}$

Current for E_2 at R, $i_2 = \pm\sqrt{\frac{P_2}{R}}$

Current for E_3 at R, $i_3 = \pm\sqrt{\frac{P_3}{R}}$

$\therefore$ Total current flow for simultaneous action of all the three sources is,

$$i = \pm i_1 \pm i_2 \pm i_3 = \pm\sqrt{\frac{P_1}{R}} \pm \sqrt{\frac{P_2}{R}} \pm \sqrt{\frac{P_3}{R}}$$

$\therefore$

$$\text{Power, } P = i^2 R = \left[\pm\sqrt{\frac{P_1}{R}} \pm \sqrt{\frac{P_2}{R}} \pm \sqrt{\frac{P_3}{R}}\right]^2 R = [\pm\sqrt{P_1} \pm \sqrt{P_2} \pm \sqrt{P_3}]^2$$

- For maximum power,

$$P_{max} = [\sqrt{P_1} + \sqrt{P_2} + \sqrt{P_3}]^2$$

(a) $P_{max} = [\sqrt{20} + \sqrt{80} + \sqrt{5}]^2 = [2\sqrt{5} + 4\sqrt{5} + \sqrt{5}]^2 = 49 \times 5 = 245$ W *Ans.*

(b) $P_{max} = [\sqrt{30} + \sqrt{270} + \sqrt{120}]^2 = [4\sqrt{5} - 3\sqrt{5}]^2 = 1080$ W *Ans.*

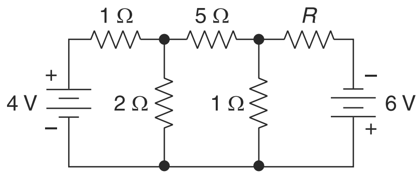
- For minimum power,

(a) $P_{min} = [-\sqrt{20} + \sqrt{80} - \sqrt{5}]^2 = [4\sqrt{5} - 3\sqrt{5}]^2 = 5\text{ W}$ *Ans.*

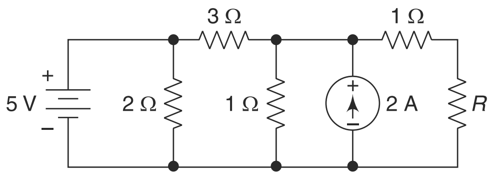
(b) $P_{min} = [-\sqrt{30} + \sqrt{270} - \sqrt{120}]^2 = [-\sqrt{30} + 3\sqrt{30} - 2\sqrt{3}]^2 = 0\text{ W}$ *Ans.*

4.19 Find the value of R in the circuit of the figure such that maximum power transfer takes place. What is the amount of this power?

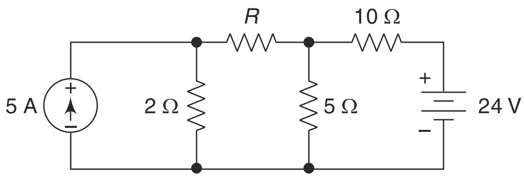
(a)



(b)

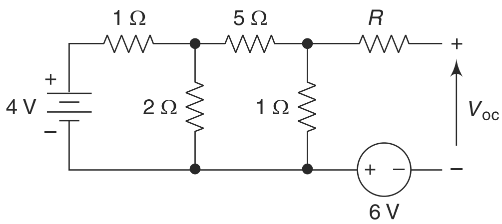


(c)



Solution

(a) Removing the résistance R,



$\therefore \quad 3i_1 - 2i_2 = 4$

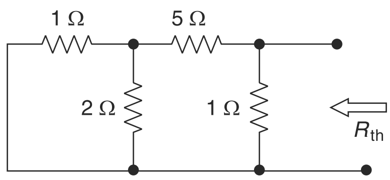
and $\quad -2i_1 + 8i_2 = 0$

Solving, $\quad i_2 = \frac{2}{5}\text{ A}$

$\therefore \quad 1 \times i_2 + 6 = V_{oc}$

$\Rightarrow \quad V_{oc} = \left(6 + \frac{2}{5}\right) = \frac{32}{5}\text{ V}$

Also, to find the R_{th},

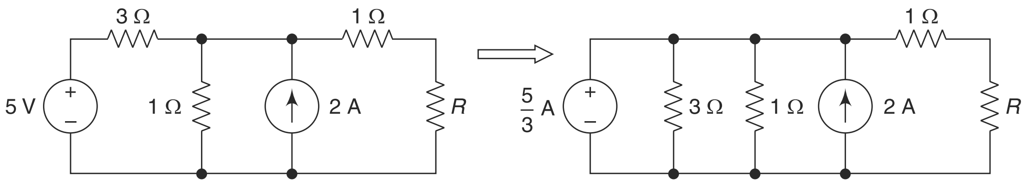


$$R_{th} = \left[\left(\frac{1\times 2}{1+2}\right)+5\right] \,||\, [1] = \left(\frac{2}{3}+5\right) \,||\, 1 = \frac{\frac{17}{3}\times 1}{\frac{17}{3}+1} = \frac{17}{20}\ \Omega$$

$\therefore$ For maximum power transfer, $R = R_{th} = \dfrac{17}{20} = 0.85\ \Omega$ *Ans.*

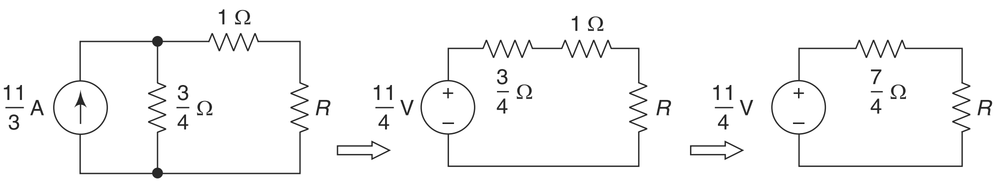
$\therefore$ Maximum power $P_{max} = \dfrac{V_{oc}^2}{4R} = 12\ \text{W}$ *Ans.*

(b) In the network, 2 Ω resistor is connected in parallel with an ideal voltage source of 5 V; hence this resistance can be removed without affecting the current flows in the other branches.



Converting the voltage source into current source,

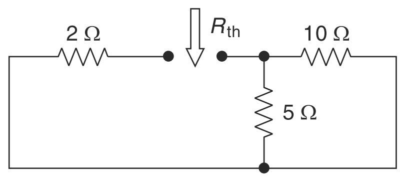
$$\left(\frac{5}{3}+2\right)\text{A} = \frac{11}{3}\ \text{A}$$



For maximum power transfer, $R = \dfrac{7}{4}\ \Omega$

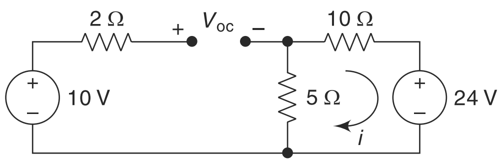
$$\text{Maximum Power, } P_{max} = \frac{\left(\frac{11}{4}\right)^2}{4\times 7/4} = 1.08\ \text{W} \qquad \textit{Ans.}$$

(c) *To find R_{th}*



$$R_{th} = \frac{10\times 5}{10+5} + 2 = 5.33\ \Omega \qquad \textit{Ans.}$$

To find V_{oc}

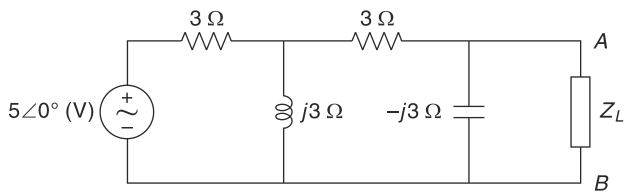


$$i = -\frac{24}{15} = -1.6\ \text{A}$$

$\therefore$ $V_{oc} = 5i + 10 = -8 + 10 = 2\ \text{V}$

$\therefore$ $P_{max} = \dfrac{4}{4\times 5.33} = 0.188\ \text{W}$ *Ans.*

4.20 In the network shown, find the value of Z_L to which the maximum power can be delivered. Hence, find the value of the maximum power.



Solution With respect to terminals A and B, the Thevenin voltage is,

$$V_{\text{th}} = \frac{5\angle 0^\circ}{3 + \dfrac{j3(3-j3)}{3-j3+j3}} \times \left(\frac{j3}{3+j3-j3}\right) = \frac{45\angle 0^\circ}{18+j9} = 2.236\angle -26.56^\circ \text{ (V)}$$

and Thevenin impedance,

$$Z_{\text{th}} = \frac{\left(3 + \dfrac{3\times j3}{3+j3}\right)\times(-j3)}{3+\dfrac{3\times j3}{3+j3} - j3} = 3\angle -53.12^\circ\ \Omega = (1.8 - j2.4)\ \Omega$$

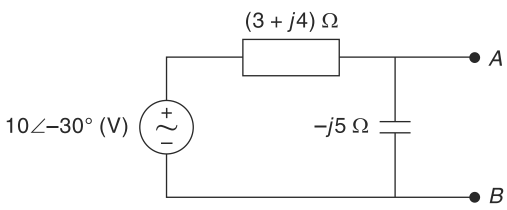
For maximum power transfer, $Z_L = Z_{\text{th}}^* = (1.8 + j2.4)\ \Omega$ *Ans.*

$\therefore$ Current, $I = \dfrac{2.236\angle -26.56^\circ}{1.8\times 2} = 0.621\angle -26.56^\circ$ A

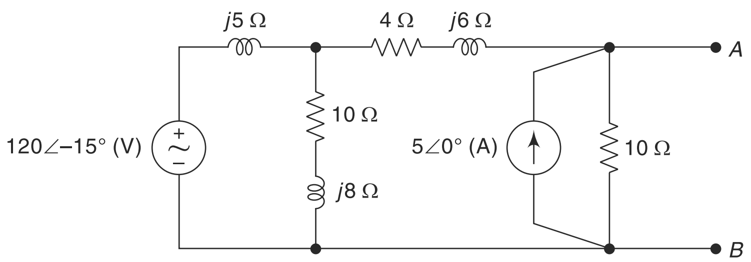
The value of the maximum power is, $P_{\max} = \dfrac{(V_{\text{th}})^2}{4R} = \dfrac{(2.236)^2}{4\times 1.8} = 0.694$ W *Ans.*

4.21 A loudspeaker is connected across terminals A and B of the network. What should its impedance be to obtain maximum power dissipation in it?

(a)



(b)



Solution
(a) Equivalent impedance with respect to the terminals A and B is,

$$Z_{th} = \frac{(3+j4)(-j5)}{3+j4-j5} = 7.9\angle -18.43°\ \Omega = (7.5 - j2.5)\ \Omega$$

For maximum power transfer, $Z_L = Z_{th}^* = (7.5 + j2.4)\ \Omega$ *Ans.*

(b) Equivalent impedance with respect to the terminals A and B is,

$$Z_{th} = \left[\frac{(10+j8)j5}{10+j8+j5} + 4 + j6\right] \| \ 10 = \left(\frac{-40+j50+40+j52+j60-78}{10+j13}\right) \| 10$$

$$= 6.14\angle 30°\ \Omega = (5.316 + j3.07)\ \Omega$$

For maximum power transfer, $Z_L = Z_{th}^* = 6.14\angle -30°\ \Omega = (5.316 - j3.07)\ \Omega$

4.22 Two inductors each of 1 Ω reactance and negligible resistance are connected in series across a 2 V a. c. source. Find the value of resistance which should be connected across one of the inductors for maximum power dissipation. Also, find the maximum power.

Solution Here, $Z = \dfrac{R \times j1}{R + j1} + j1 = \dfrac{-1 + j2R}{R + j1}$

$\therefore$ Current $I = \dfrac{2\angle 0°}{Z} = \dfrac{2\angle 0° \times (R + j1)}{-1 + j2R}$

$\therefore$ Current through the resistance, $I_R = I \times \dfrac{j1}{R + j1} = \dfrac{j2}{-1 + j2R}$

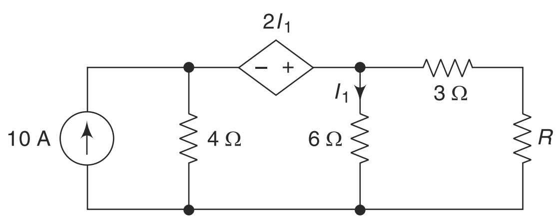
$\therefore$ Power, $P = |I|^2\ R = \dfrac{4R}{1 + 4R^2}$

For maximum power, $\dfrac{dP}{dR} = 0 \quad \Rightarrow \quad \dfrac{(1 + 4R^2) \times 4 - 4R \times 8R}{(1+4R^2)^2} = 0$

$\Rightarrow \qquad R = 0.5\ \Omega$ *Ans.*

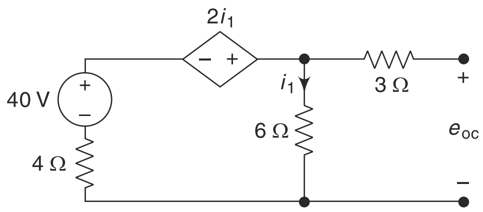
$\therefore$ Maximum Power, $P_{max} = \dfrac{4 \times 0.4}{1 + 4 \times (0.5)^2} = 1\ \text{W}$ *Ans.*

4.23



In the network shown, calculate the maximum power that may be dissipated in the external resistor R.
Solution Transforming the current source into voltage source,

By KVL,



$$6i_1 + 4i_1 - 40 - 2i_1 = 0$$

$\Rightarrow$ $i_1 = 5\text{ A}$

$\therefore$ $e_{oc} = 6i_1 = 30\text{ V}$

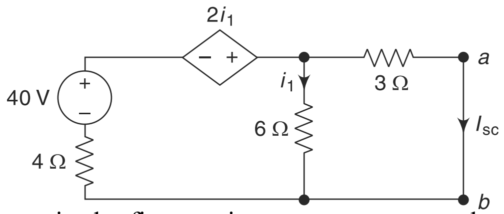
For maximum power, $R = R_{eq}$

Shorting the terminals a–b and solving by loop method,

$$I_{sc} = 5\text{ A}$$

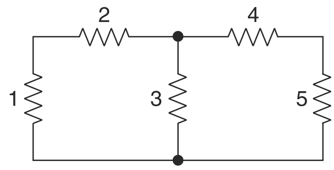
$\therefore$ $R_{th} = \dfrac{30}{5} = 6\ \Omega$

$\therefore$ $P_{max} = \dfrac{(30)^2}{4 \times 6} = \dfrac{900}{24} = 37.5\text{ W}$



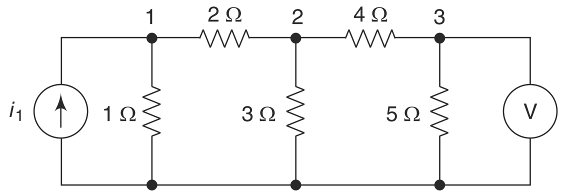
Reciprocity Theorem

4.24 Verify the Reciprocity Theorem for the network shown in the figure using current source and a voltmeter. All the values are in ohm.



Solution Using a current source and a voltmeter,

Let, e_1, e_2 be node voltages, v_1 be the voltmeter reading.



By KCL,

At node (1) $\Rightarrow$ $3e_1 - e_2 - 2i_1 = 0$ (i)

At node (2) $\Rightarrow$ $-6e_1 + 13e_2 - 3v_1 = 0$ (ii)

At node (3) $9v_1 = 5e_2$ (iii)

From (ii) $\Rightarrow$ $-6e_1 + 13 \times \dfrac{9}{5} v_1 - 3v_1 = 0$

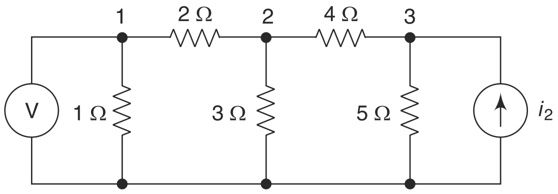
$\Rightarrow$ $-6e_1 + \left(\dfrac{117}{5} - 3\right) v_1 = 0$

$\Rightarrow$ $6e_1 + \dfrac{102}{5} v_1 \Rightarrow e_1 = \dfrac{17}{5} v_1$

From (i) $\Rightarrow$ $3 \times \frac{17}{5} v_1 - \frac{9}{5} v_1 = 2i$

$$\Rightarrow \quad \left(\frac{i_1}{v_1}\right) = \left(\frac{21}{5}\right) \qquad \text{(A)}$$

Interchanging the positions of the current source and the voltmeter,
Now, let v_2 be the voltmeter reading



By KCL,

At node (1) $\Rightarrow$ $3v_2 = e_2$ (iv)

At node (2) $\Rightarrow$ $-6v_2 + 13e_2 - 3e_3 = 0$

$\Rightarrow$ $-6v_2 + 13 \times 3v_2 - 3e_3 = 0$

$\Rightarrow$ $e_3 = 11v_2$ (v)

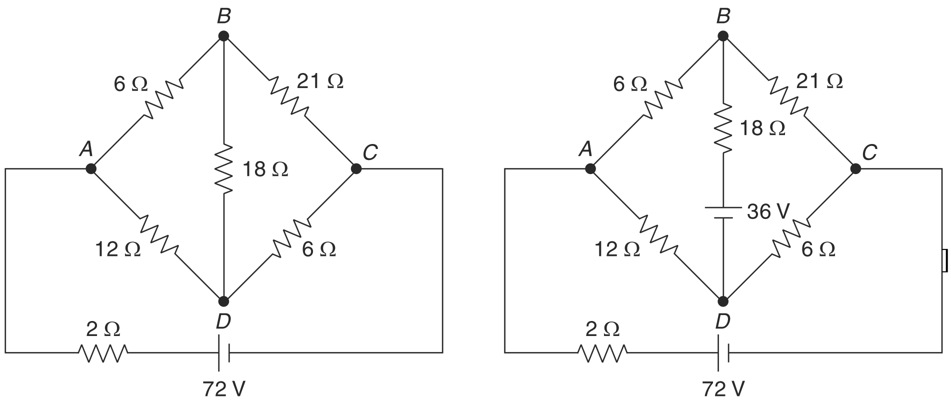
At node (3) $\Rightarrow$ $5e_3 - 5e_2 + 4e_3 - 20i_2 = 0$

$\Rightarrow$ $20i_2 = 9e_3 - 5e_2 = 9 \times 11v_2 - 5 \times 3v_2 = 84v_2$

$$\Rightarrow \quad \left(\frac{i_2}{v_2}\right) = \left(\frac{21}{5}\right) \qquad \text{(B)}$$

From equation (A) and (B), Reciprocity theorem is proved.

4.25 Solve the network shown in Figure (a) and hence find the current in the 2 Ω resistor in Figure (b) when an emf of 36 V is added in the branch *BD* as shown in Figure 7(b). All values are in ohm.

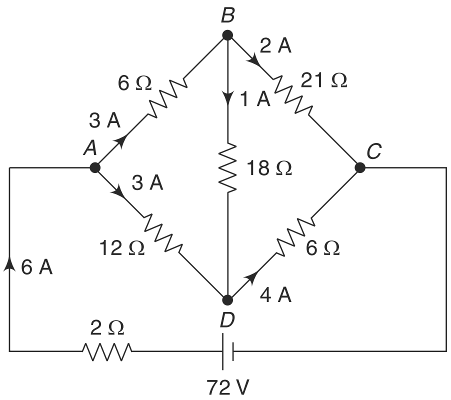


(a) *(b)*

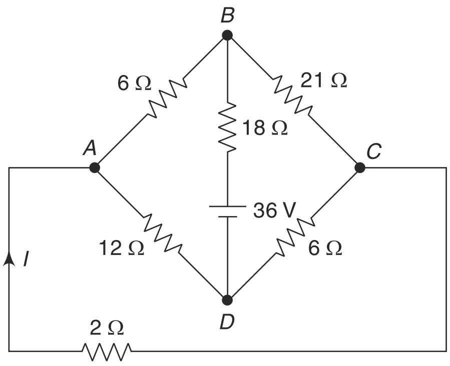
Solution

- Solve by any method of network analysis.
- We consider the 36 V source acting alone.

When 72 V sourer is acting alone, by network analysis,
The current in 2 Ω resistor = 6 A and in 18 Ω resistor = 1 A



(a)



(b)

By Reciprocity theorem,

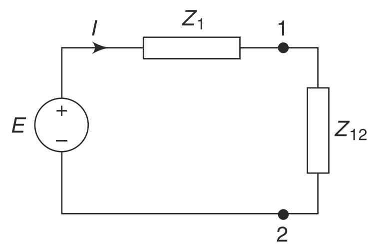
$$\frac{72}{1}=\frac{36}{I} \quad \Rightarrow \quad I = 0\cdot 5 \text{ A}$$

[Here, I = Current in 2 Ω resistor when 36 V source is acting alone]

∴ Current in 2Ω resistor for simultaneous action of two sources

$$I = (6-0.5) = 5.5 \text{ A}$$

4.26 An e.m.f. source E, having negligible internal impedance is connected in series with an impedance Z_1 to the input terminals 1–2 of a linear, bilateral four terminal network. It produces a current I_2 in impedance Z_L connected across the output terminals 3–4. The emf source is now transferred so as to act, in series with Z_2, between terminal 3–4. Z_1 is disconnected and the input terminals 1–2 are short circuited. The short-circuited current traversing terminals 1–2 is then I_1. Prove that the impedance looking into terminals 1–2 under the first condition is,

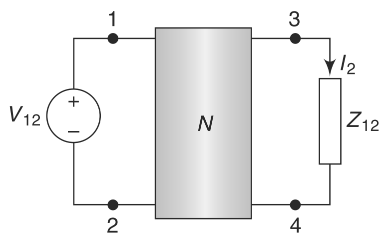


$$Z_{12} = \frac{Z_1 I_2}{I_1 - I_2}$$

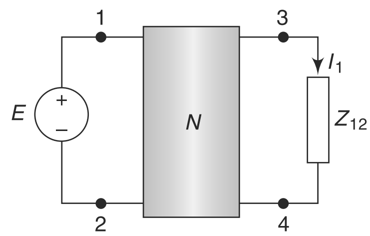
Solution Let the impedance looking into terminals 1–2 be Z_{12}. Thus the network becomes:

$$\therefore \qquad I = \frac{E}{Z_1 + Z_{12}}$$

$$\therefore \text{ Voltage across 1–2, } V_{12} = \frac{E \times Z_{12}}{Z_1 + Z_{12}}$$



So, the circuit becomes as shown.
The given network is linear and bilateral and according to the reciprocity theorem, if the source E is put across terminals 1–2, the response current flowing through Z_2 will be I_1 as shown.
Now, if a voltage equal to V_{12} is applied instead of E, the current flowing through Z_2 will be,



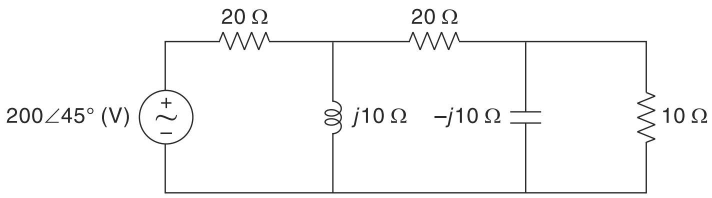
$$\frac{I_1}{E}\times V_{12}=\frac{I_1}{E}\times\frac{E\times Z_{12}}{Z_1+Z_{12}}=I_1\times\frac{Z_{12}}{Z_1+Z_{12}}$$

But, this current is equal to I_2.

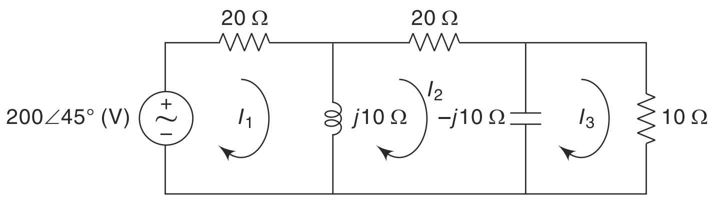
$\therefore$ $$I_2=I_1\frac{Z_{12}}{Z_1+Z_{12}}$$

$\Rightarrow$ $$Z_{12}=\left(\frac{Z_1I_2}{I_1-I_2}\right)\quad\text{(Proved)}$$

4.27 Verify the reciprocity theorem for the ladder network shown in figure.



Solution Let, the three loop currents be I_1, I_2, and I_3. By KVL for the three loops,



$$(20+j10)I_1-j10I_2=200\angle 45^\circ$$

$$-j10I_1+20I_2+j10I_3=0$$

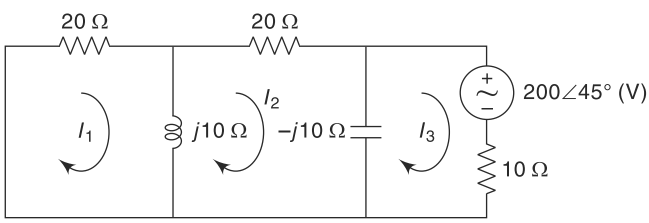
$$j10I_2+(10-j10)I_3=0$$

Solving for I_3,

$$I_3=\frac{\begin{vmatrix}(20+j10) & -j10 & 200\angle 45^\circ\\ -j10 & 20 & 0\\ 0 & j10 & 0\end{vmatrix}}{\begin{vmatrix}(20+j10) & -j10 & 0\\ -j10 & 20 & j10\\ 0 & j10 & (10-j10)\end{vmatrix}}=\frac{200\angle 45^\circ\times 100}{(20+j10)(200-j200+100)-j10(j100+100)}$$

$$=2.169\angle 57.53^\circ\text{ (A)}$$

Now by interchanging the positions of the voltage source and the response current, we get,



By KVL,

$$(20+j10)I_1-j10I_2=0$$

$$-j10I_1 + 20I_2 + j10I_3 = 0$$

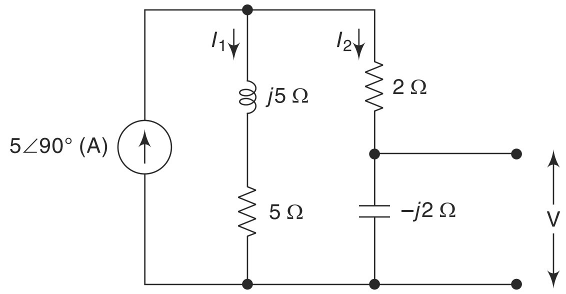
$$j10I_2 + (10 - j10)I_3 = 200\angle 45^\circ$$

Solving for I_1,

$$I_1 = \frac{\begin{vmatrix} 0 & -j10 & 0 \\ 0 & 20 & 0 \\ 200\angle 45^\circ & j10 & 0 \end{vmatrix}}{\begin{vmatrix} (20+j10) & -j10 & 0 \\ -j10 & 20 & j10 \\ 0 & j10 & (10-j10) \end{vmatrix}} = 2.169\angle 57.53^\circ \text{ (A)}$$

Since the currents in both the cases are the same, reciprocity theorem is verified.

4.28 In this circuit, find voltage V. Interchange the current source and resulting voltage V and show that the reciprocity theorem is verified.



Solution Here, the current $I_2 = 5\angle 90^\circ \times \dfrac{5 + j5}{5 + j5 + 2 - j2} = 4.64\angle 111.8^\circ \text{ (A)}$

$\therefore$ The voltage, $V = I_2 \times Z_C = 4.64\angle 111.8^\circ \times (-j2) = 9.28\angle 21.8^\circ \text{ (V)}$

Now, interchanging the positions of the current source and the finding the resulting voltage, we get,

$$I_1 = 5\angle 90^\circ \times \frac{-j2}{-j2 + 5 + 2 + j5}$$

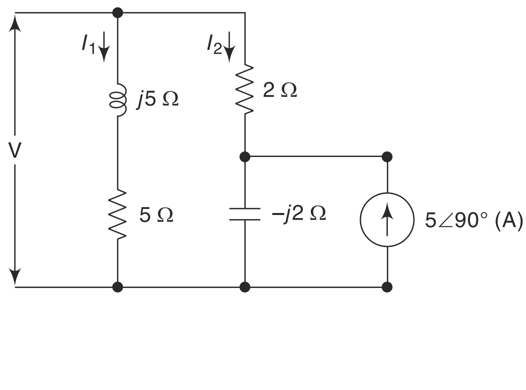
$$= 1.31\angle -23.2^\circ \text{ (A)}$$

$\therefore$ The voltage,

$$V = 1.31\angle -23.2^\circ \times (5 + j5)$$

$$= 1.31\angle -23.2^\circ \times 7.075\angle 45^\circ$$

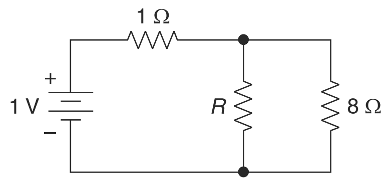
$$= 9.28\angle 21.8^\circ \text{ (V)}$$



As V is same as obtained before interchanging the position of the current source, reciprocity theorem is verified.

Compensation Theorem

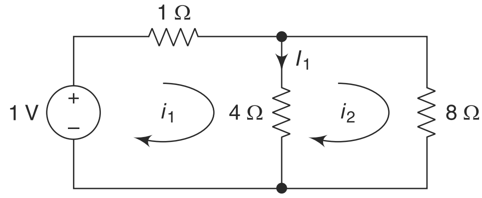
4.29 In the network shown, the resistance R is changed from 4Ω to 2Ω. Verify compensation theorem.



Solution By KCL,

$$5i_1 - 4i_2 = 1$$

and $$-4i_1 + 12i_2 = 0$$



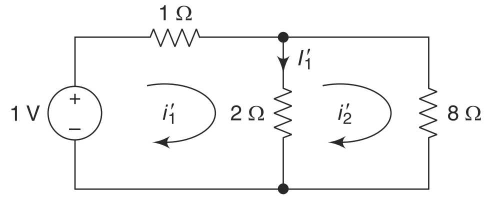
Solving $\Rightarrow i_1 = \frac{3}{11}$ A and $i_2 = \frac{1}{11}$ A

$\therefore$ $$I_1 = (i_1 - i_2) = \frac{2}{11} \text{ A}$$

After changing the value of the resistance from 4Ω to 2Ω, by KCL

$$3i_1' - 2i_2' = 1$$

and $$-2i_1' + 10i_2' = 0$$

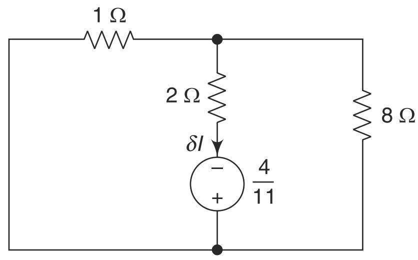


Solving $\Rightarrow i_1' = \frac{5}{13}$ A and $i_2' = \frac{1}{13}$ A

$\therefore$ $$I_1' = \frac{4}{13} \text{ A}$$

$\therefore$ Change in current, $\delta I = (I_1' - I_1) = \left(\frac{4}{13} - \frac{2}{11}\right) = \frac{18}{143}$ A (I)

Using Compensation Theorem,

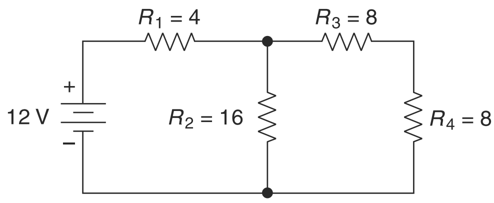


$$V_c = I_1 \times \delta Z = \frac{2}{11}(-2) = -\frac{4}{11} \text{ V}$$

$$\delta I = \frac{\frac{4}{11}}{2 + \frac{8}{9}} = \frac{18}{143} \text{ A} \qquad \text{(II)}$$

From (I) and (II), the compensation theorem is proved.

4.30 Find the current flowing in the resistor R_4 of the network shown in figure If a resistance of 0.5 Ω is inserted in series with R_4, find, using Compensation theorem, the current that will flow through R_4. All values are in ohm.

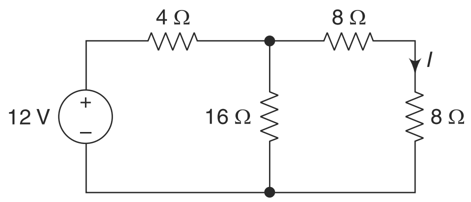


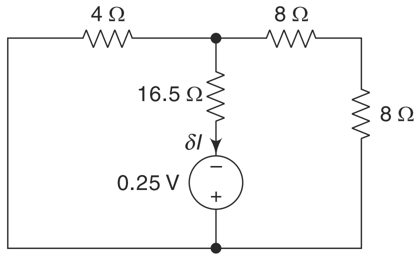
Solution Solving the network by any method of network analysis, $I = 0.5$ A

Now $\delta Z = 0.5$ Ω

$\therefore$ $$V_c = I.\delta Z = 0.5 \times 0.5 = 0.25 \text{ V}$$

$\Rightarrow$ $$\delta I = \frac{0.25}{19.7} \text{ A} = 0.01269 \text{ A}$$

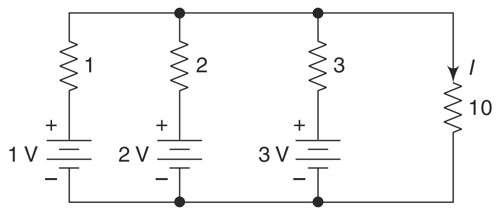




∴ $$I' = (I - \delta I) = (0.5 - 0.01269)\text{ A} = 0.4873\text{ A} \qquad \textit{Ans.}$$

Millman's Theorem

4.31 Find the load current using Millman's theorem. All values are in ohm.



Solution Here, $E_1 = 1$ V, $E_2 = 2$ V, $E_3 = 3$ V

$$Z_1 = 1\ \mho,\ Z_2 = 2\ \mho,\ Z_3 = 3\ \mho$$

∴ $$Y_1 = 1\ \mho,\ Y_2 = 0.5\ \mho,\ Y_3 = \frac{1}{3}\ \mho$$

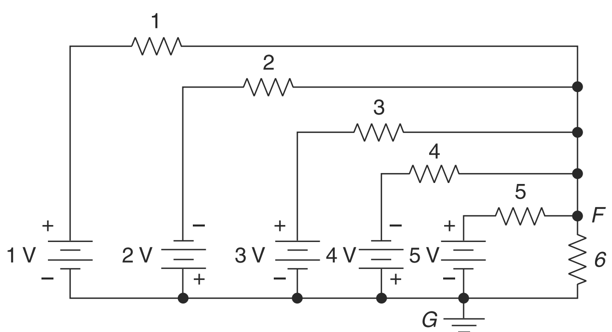
By Millman's theorem, the equivalent circuit is shown.

∴ $$E = \frac{\sum_{i=1}^{3} E_i Y_i}{\sum_{i=1}^{3} Y_i} = \frac{1 \times 1 + 2 \times 0.5 + 3 \times \frac{1}{3}}{1 + 0.5 + \frac{1}{3}} = \frac{3}{\frac{11}{6}} = \frac{18}{11}\text{ V}$$

and $$Z = \frac{1}{\sum_{i=1}^{3} Y_i} = \frac{6}{11}\ \Omega$$

∴ $$I = \frac{E}{Z + 10} = \frac{\frac{18}{11}}{\frac{6}{11} + 10} = \frac{18}{116} = \frac{9}{58}\text{ A} \qquad \textit{Ans.}$$

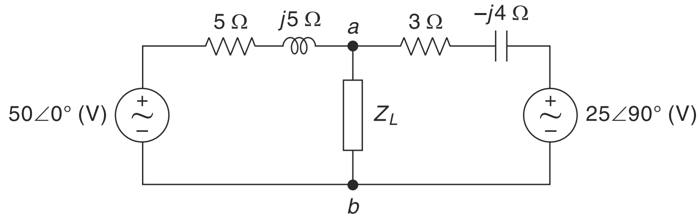
4.32 Obtain the potential of node F with respect to node G in the circuit of the figure. All values are in ohm.



Solution By Millman's Theorem,

$$V_{FG} = \frac{\sum_{i=1}^{5} E_i Y_i}{\sum_{i=1}^{5} Y_i} = \frac{1\times 1 - 2\times\frac{1}{2} + 3\times\frac{1}{3} - 4\times\frac{1}{4} + 5\times\frac{1}{5}}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}} = \frac{60}{137} \text{ Volt} \qquad \textit{Ans.}$$

4.33 In the network, two voltage sources act on the load impedance connected to terminals *a*, *b*. If the load is variable in both reactance and resistance, what load Z_L will receive the maximum power? What is the value of the maximum power? Use Millman's theorem.



Solution $V_1 = 50\angle 0° = 50$ V; $Z_1 = (5 + j5)\ \Omega$; $Y_1 = \frac{1}{Z_1} = \frac{1}{(5+j5)} = (0.1 - j0.1)$ ℧

$V_2 = 25\angle 90° = j25$ V; $Z_2 = (3 - j4)\ \Omega$; $Y_2 = \frac{1}{Z_2} = \frac{1}{(3-j4)} = (0.12 + j0.16)$ ℧

∴ Millman voltage source,

$$V_m = \frac{V_1Y_1 + V_2Y_2}{Y_1 + Y_2} = \frac{50(0.1 - j0.1) + j25(0.12 + j\,0.16)}{(0.1 - j0.1) + (0.12 + j\,0.16)} = 9.807\angle{-78.65°}\ (\text{V})$$

∴ Millman impedance,

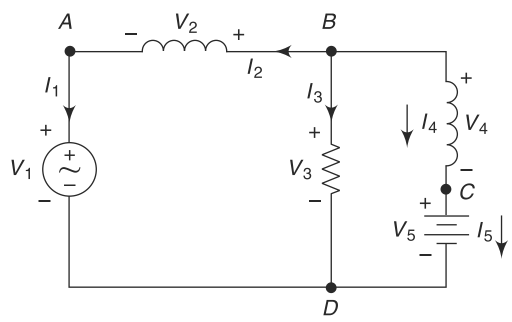
$$Z_m = \frac{1}{Y_1 + Y_2} = \frac{1}{0.22 - j0.06} = 4.385\angle{-15.25°} = (4.23 - j1.15)\ \Omega$$

For maximum power transfer to the load, $Z_L = Z_m^* = (4.23 + j1.15)\ \Omega$ *Ans.*

∴ Maximum power, $P_{\max} = \frac{V_m^2}{4R_L} = \frac{(9.807)^2}{4\times 4.23} = 5.68$ W *Ans.*

Tellegen's Theorem

4.34 Verify Tellegen's theorem for the network shown in the figure.



It is given that $V_1 = 4$ V, $V_2 = -2$ V, $V_3 = 2$ V, $V_4 = 8$ V, $V_5 = -6$ V, and $I_1 = 2$ A, $I_2 = 2$ A, $I_3 = -6$ A, $I_4 = 4$ A, $I_5 = 4$ A.

Solution Before verifying Tellegen's theorem, we have to check whether the voltage and current values satisfy the KVL and KCL, respectively.

At node (A), $(i_1 - i_2) = (2 - 2) = 0$

At node (B), $(i_2 + i_3 + i_4) = (2 + 2 - 4) = 0$

At node (C), $(i_5 - i_4) = (4 - 4) = 0$

At node (D), $(-i_1 - i_3 - i_5) = (-2 + 6 - 4) = 0$

Thus, the currents satisfy KCL.

For Loop *ABDA*, $(-v_2 + v_3 - v_1) = (2 + 2 - 4) = 0$

For Loop *ABCDA*, $(-v_2 + v_4 + v_5 - v_1) = (2 + 8 - 6 - 4) = 0$

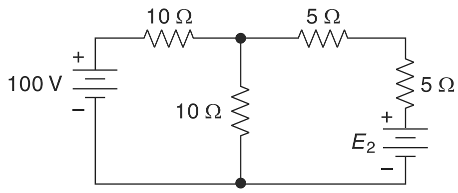
For Loop *BCDB*, $(v_4 + v_5 - v_3) = (8 - 6 - 2) = 0$

Thus, the voltages satisfy KVL

So, by Tellegen's theorem,

$$\sum_{k=1}^{5} V_k i_k = (4 \times 2) + (-2 \times 2) + (2 \times -6) + (8 \times 4) + (-6 \times 4) = 0 \qquad \text{[Proved]}$$

4.35 Find the value of source E_2 using Tellegen's theorem if the power absorbed by E_2 is 20 W.

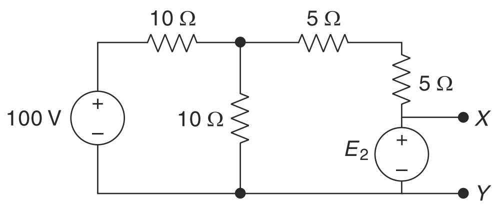


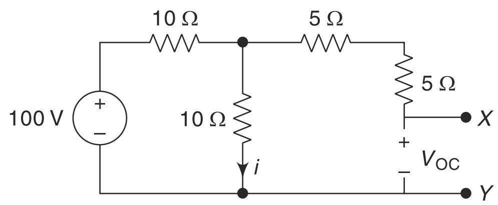
Solution We have to find out the Thevenin's equivalent across *XY*.

Here, $\therefore \quad i = \dfrac{100}{20} = 5 \text{ A}$

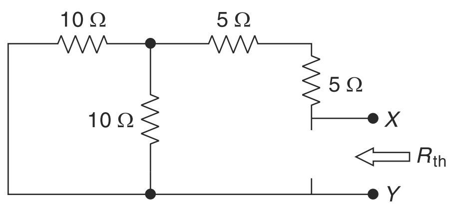
$$\therefore \quad V_{oc} = i \times 10 = 50 \text{ V}$$

$$R_{th} = (10 \| 10) + 10 = (5 + 10) = 15\ \Omega$$





Now Applying Tellegen's Theorem to the equivalent circuit,

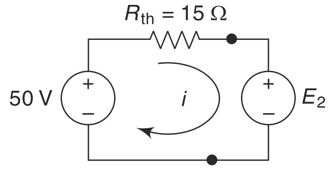


$$-50I + 15I^2 + E_2 I = 0$$

But it is given that

$$E_2 I = 20 \quad \Rightarrow \quad 15I^2 - 50I + 20 = 0$$

$$\Rightarrow \qquad 3I^2 - 10I + 4 = 0$$



$$\therefore \qquad I = \frac{10 \pm \sqrt{100 - 4 \times 3 \times 4}}{6} = \frac{10 \pm \sqrt{5^2}}{6} = \frac{5 \pm \sqrt{13}}{3}$$

$$= 2.8685 \text{ A} \quad \text{or} \quad 0.4648 \text{ A}$$

So, the value of $E_2 = \frac{20}{I} = 6.97$ V or, 43.03 V *Ans.*

4.36 A set of measurements is made on a linear time-invariant resistive circuit as shown in the figure (a). The circuit is then reconnected as shown in figure (b). Find the current through the 5 Ω resistance.

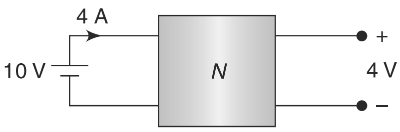


Figure (a)

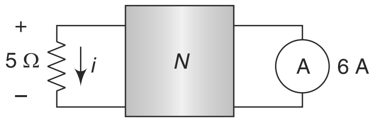
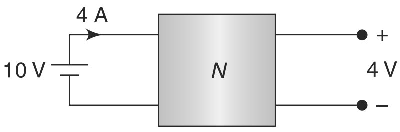


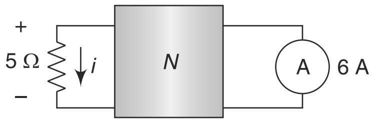
Figure (b)

Solution By Tellegen's theorem, if the set of voltages and currents is taken corresponding to two different instants of time $t_1$and t_2, then

$$\sum_{b=1}^{b} v_b(t_1) i_b(t_2) = \sum_{b=1}^{b} v_b(t_2) i_b(t_1) = 0$$

Here, the circuits for two different instants of time are as shown below





By Tellegen's theorem,

$$\sum_{b=1}^{2} v_k(t_1) i_k(t_2) = \sum_{b=1}^{2} v_k(t_2) i_k(t_1)$$

$$\Rightarrow \qquad v_1(t_1) i_1(t_2) + v_2(t_1) i_2(t_2) = v_1(t_2) i_1(t_1) + v_2(t_2) i_2(t_1) \qquad (1)$$

Here, $v_1(t_1) = 10$ V; $i_1(t_1) = -4$ A; and $v_2(t_1) = 4$ V; $i_2(t_1) = 0$

$v_1(t_2) = 5i$; $i_1(t_2) = i$ and $v_2(t_2) = 0$; $i_2(t_2) = 6$ A

So, from (1) we get

$$(10 \times i) + (4 \times 6) = (5i \times -4) + (0 \times 0)$$

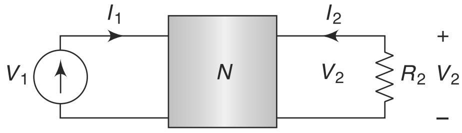
$$10i + 24 = -20i$$

$$\Rightarrow \qquad i = -\frac{24}{30} = -0.8 \text{ A} \qquad \textit{Ans.}$$

4.37 Two sets of measurements are taken on a resistive network shown. Find V_2.

(a) $R_2 = 1$ Ω, $V_1 = 5$ V, $I_1 = 2$ A, $V_2 = 1$ V

(b) $R_2 = 10$ Ω, $V_1 = 6$ V, $I_1 = 6$ A



Solution Here,

$$v_1(t_1)i_1(t_2) + v_2(t_1)i_2(t_2) = v_1(t_2)i_1(t_1) + v_2(t_2)i_2(t_2)$$

$$\Rightarrow \quad (5\times 6) + 1\times\left[-\frac{v_2(t_2)}{10}\right] = (6\times 2) + v_2(t_2)\times\left(-\frac{1}{1}\right)$$

$$\Rightarrow \quad 30 - \frac{v_2(t_2)}{10} = 12 - v_2(t_2) \quad \left\{\because i_2(t_2) = -\frac{v_2(t_2)}{R_2(t_2)} \text{ and } i_2(t_1) = -\frac{v_2(t_1)}{R_2(t_1)}\right\}$$

$$\Rightarrow \quad v_2(t_2) = -\frac{18}{\frac{9}{10}} = -20 \text{ V} \qquad \textit{Ans.}$$

4.38 A set of coupled coils (coil-1 and coil-2) having self-inductance $L_1 = 0.5$ H and $L_2 = 2$ H have number of turns as $N_1 = 200$ and $N_2 = 400$ respectively. If coil-1 is excited with a current of 4 A, what would be the flux distribution for a coupling coefficient of 0.9? Also find mutual inductance for the set-up.

Solution Since, $L_1 = \dfrac{N_1\phi_1}{I_1}$, we may find $\phi_1 = \dfrac{L_1 I_1}{N_1} = \dfrac{0.5\times 4}{200} = 0.01$ Wb.

So total flux produced by coil-1 is 0.01 Wb.

Since, coupling coefficient is given as $k = \dfrac{\phi_{12}}{\phi_1}$, we may find,

$$\phi_{12} = k.\ \phi_1 = 0.9 \times 0.01 = 0.009 \text{ Wb. or } 9.0 \text{ mWb.}$$

Hence, useful flux is found to be 9.0 mWb.

Now, the mutual inductance may be found as

$$M = k\sqrt{L_1 L_2} = 0.90\sqrt{0.5\times 2} = 0.9 \text{ H.}$$

4.39 Two coils having a coupling coefficient of 0.8 experience an induced emf of 100 V in coil-2 when coil-1 is excited with a current that is reduced linearly from 4 A to zero in just 5.0 ms. If coil-1 has 200 turns and total flux is 2.0 mWb, how many turns are present in coil-2? Also, find the self and mutual inductances for the set-up.

Solution Self-inductance of coil-1 may be found as $L_1 = \dfrac{N_1\phi_1}{I_1} = \dfrac{200(2\times 10^{-8})}{4} = 0.1$ H.

Since, $e_2 = M\dfrac{dI_1}{dt}$, we may find $M = \dfrac{100(5\times 10^{-8})}{4} = 0.125$ H.

Since, $M = k\sqrt{L_1 L_2}$, we may find $L_2 = \dfrac{M^2}{k^2 L_1} = \dfrac{0.125^2}{0.8^2\times 0.1} = 0.243$ H.

Useful flux may be found as $\phi_{12} = k.\ \phi_1 = 0.8 \times 0.002 = 1.6 \times 10^{-3}$ Wb. or 1.6 mWb.

Since, $e_2 = N_2\dfrac{d\phi_{12}}{dt}$, we may find $N_2 = \dfrac{100(5\times 10^{-8})}{1.6\times 10^{-8}} = 313$ turns (approx).

4.40 Two inductor coils, coupled to each other, have self-inductances of 30 mH and 30 mH respectively. If the mutual inductance of the combination is found to be 15 mH, find the coefficient of coupling. Also find the total flux and useful flux if coil-1 has 100 turns and is excited with a current of 5 A.

Solution Since, $M = k\sqrt{L_1 L_2}$, we may find that, $k = \dfrac{M}{\sqrt{L_1 L_2}} = \dfrac{15}{\sqrt{20 \times 30}} = 0.61$

Hence, the value of the coefficient of coupling is found to be 0.61, which means that only 61% of the total flux may be available as useful flux for the mutual coupling of the two coils.

Since, $L_1 = \dfrac{N_1 \phi_1}{I_1}$,

we may find the total flux as $\phi_1 = \dfrac{L_1 I_1}{N_1} = \dfrac{5(20 \times 10^{-8})}{100} = 0.001$ Wb. or 1.0 mWb.

Useful flux may be found as $\phi_{12} = k\phi_1 = 0.61 \times 1 = 0.61$ mWb.

MULTIPLE-CHOICE QUESTIONS

4.1 Which one of the following theorems is a manifestation of the law of conservation of energy?
(a) Tellegen's Theorem (b) Reciprocity Theorem
(c) Thevenin's Theorem (d) Norton's Theorem

4.2 Tellegen's theorem is applicable to
(a) circuits having passive elements.
(b) circuits having time-invariant elements only.
(c) circuits with linear elements only.
(d) circuits with active or passive, linear or non-linear and time-invariant or time-varying elements.

4.3 In any lumped network with elements in b branches, $\sum_{k=1}^{b} v_k(t) \cdot i_k(t) = 0$ for all t, holds good according to
(a) Norton's theorem. (b) Thevenin's theorem.
(c) Millman's theorem. (d) Tellegen's theorem.

4.4 Millman's theorem yields
(a) equivalent voltage source. (b) equivalent voltage or current source.
(c) equivalent resistance. (d) equivalent impedance.

4.5 The superposition theorem is applicable to
(a) current only. (b) voltage only.
(c) both current and voltage. (d)current, voltage and power.

4.6 Superposition theorem is not applicable for
(a) voltage calculations. (b) bilateral elements
(c) power calculations. (d) passive elements.

4.7 Thevenin's theorem can be applied to calculate the current in
(a) any load. (b) a passive load only.
(c) a linear load only. (d) a bilateral load only.

4.8 Norton's equivalent circuit consists of
(a) voltage source in parallel with impedance.
(b) voltage source in series with impedance.

(c) current source in parallel with impedance.
(d) current source in series with impedance.

4.9 The superposition theorem is applicable to
(a) linear responses only. (b) linear and non-linear responses.
(c) linear, non-linear and time-variant responses.

4.10 When a source is delivering maximum power to a load, the efficiency of the circuit
(a) is always 50%. (b) depends on the circuit parameters.
(c) is always 75%. (d) none of these.

4.11 Maximum power transfer occurs at a
(a) 100% efficiency. (b) 50% efficiency.
(c) 25% efficiency. (d) 75% efficiency.

4.12 Which of the following statements is true?
(a) A Norton's equivalent is a series circuit.
(b) A Thevenin's equivalent circuit is a parallel circuit.
(c) R-L circuit is dual pair.
(d) L-C circuit is a dual pair.

4.13 For a linear network containing generators and impedances, the ratio of the voltage to the current produced in other loop is the same as the ratio of voltage and current obtained if the position of the voltage source and the ammeter measuring the current are interchanged. This network theorem is known as
(a) Millman's theorem. (b) Norton's theorem.
(c) Tellegen's theorem. (d) Reciprocity theorem.

4.14 Under conditions of maximum power transfer from an ac source to a variable load
(a) the load impedance must also be inductive, if the generator impedance is inductive.
(b) the sum of the source and load impedance is zero.
(c) the sum of the source reactance and load reactance is zero.
(d) the load impedance has the same phase angle as the generator impedance.

4.15 Consider the following statements
The transfer impedances and admittances of a network remain constant when the position of excitation and response are interchanged if the network
1. is linear
2. consists of bilateral elements
3. has high impedance or admittance as the case may be.
4. is resonant.
Out of above these statements
(a) 1 and 2 are correct. (b) 1, 3 and 4 are correct.
(c) 2 and 4 are correct. (d) 1, 2, 3 and 4 are correct.

4.16 In a linear network, the ratio of voltage excitation to current response is unaltered when the position of excitation and response are interchanged. This assumption stems from the
(a) principle of duality. (b) reciprocity theorem.
(c) principle of superposition. (d) equivalence theorem.

4.17 If all the elements in a particular network are linear, then the superposition theorem hold when the excitation is
(a) dc only (b) ac only (c) either ac or dc (d) an impulse.

4.18 An a.c source of voltage E_s and an internal impedance of $Z_s = (R_s + jX_s)$ is connected to a load of impedance $Z_L = (R_L + jX_L)$. Consider the following conditions in this regard

1. $X_L = X_s$, if only X_L is varied.
2. $X_L = X_s$, if only X_S is varied.
3. $R_L = \sqrt{R_S^2 + (X_S + X_L)^2}$, if only R_L is varied.
4. $|Z_L| = |Z_S|$ if the magnitude of Z_L is varied, keeping the phase angle fixed.

Among these conditions, those which are to be satisfied for maximum power transfer from the source to the load would include

(a) 2 and 3 (b) 1 and 3 (c) 1, 2 and 4 (d) 2, 3 and 4

4.19 Reciprocity theorem is applicable to a network

1. which contains R, L and C as elements.
2. which is initially relaxed system.
3. which has both independent and dependent sources.

Tick out the correct combination from the combination given above

(a) 1 and 2 (b) 1 and 3 (c) 2 and 3 (d) 1, 2 and 3.

4.20 Reciprocity theorem is applicable to

(a) circuits with one independent source
(b) circuits with only one independent source and no dependent source
(c) circuits with any number of independent sources
(d) circuits with any number of sources.

4.21 Substitution theorem is applicable for a network which has

1. unique solution.
2. one or two non-linear elements.
3. one non-linear or time-varying element.

Choose the correct combination from the combination given above

(a) 1 and 2 (b) 1 and 3 (c) 2 and 3 (d) 1, 2 and 3.

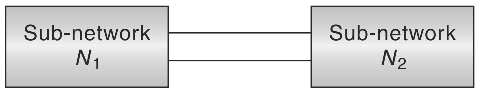
4.22 Substitution theorem applies to

(a) linear networks. (b) non-linear networks.
(c) linear time-invariant networks. (d) any networks.

4.23 Which of the following theorems is applicable for both linear and non-linear circuits?

(a) Superposition (b) Thevenin (c) Norton (d) none of these.

4.24 A network is composed of two sub-networks N_1 and N_2 as shown in the given figure. If the sub-network N_1 contains only linear, bilateral, time-invariant elements, then it can be replaced by its Thevenin equivalent even if the sub-network N_2 contains



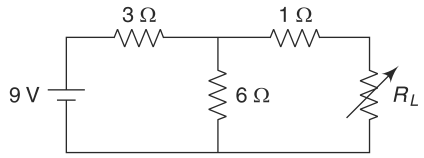
(a) a two-terminal element which is non-linear
(b) a non-linear inductance mutually coupled to an element in N_1
(c) an element which is linear, but mutually coupled to some element in N_1
(d) a dependent source the value of which depends upon the voltage or current in some element in N_1.

4.25 A certain network consists of two ideal identical voltage sources and a large number of ideal resistors. The power consumed in one of the resistors is 4W when either of the two sources is active and

the other is replaced by a short-circuit. The power consumed by the same resistor when both the sources are active would be
(a) zero or 16 W (b) 4 W or 8 W (c) zero or 8 W (d) 8 W or 16 W.

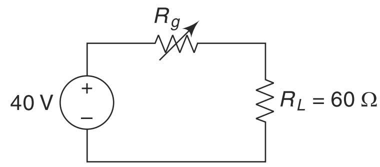
4.26 If a network has all linear elements except for a few non-linear ones, then superposition theorem
(a) cannot hold at all.
(b) always holds.
(c) may hold on careful selection of element values, source waveform and response.
(d) holds in case of direct current excitations.

4.27 The maximum power that can be dissipated in the load in the circuit shown in figure is

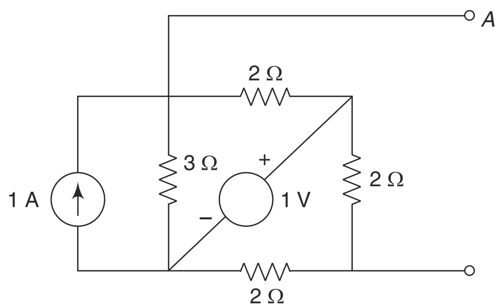


(a) 3 W (b) 6 W (c) 6.75 W (d) 13.5 W

4.28 If R_g in the circuit shown in figure is variable between 20 Ω and 80 Ω, then the maximum power transferred to the load R_L will be
(a) 15 W (b) 13.33 W
(c) 6.67 W (d) 2.4 W

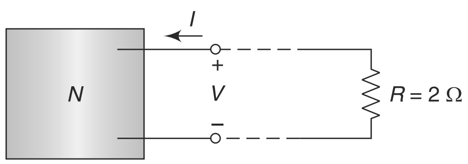


4.29 Thevenin impedance across the terminals *AB* of the given network is



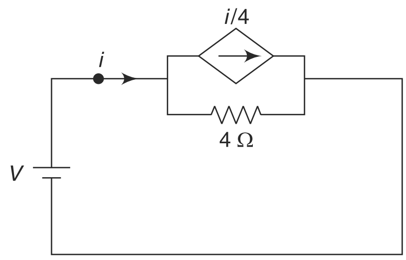
(a) $\frac{10}{3}$ Ω (b) $\frac{20}{9}$ Ω (c) $\frac{13}{4}$ Ω (d) $\frac{11}{5}$ Ω

4.30 The *V–I* relation for the network shown in the given box is $V = 4I - 9$.
If now a resistor $R = 2$ Ω is connected across it, then the value of I will be



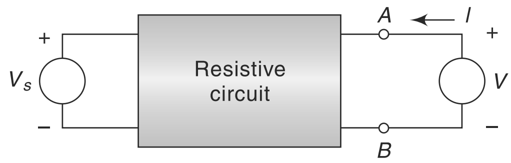
(a) –4.5 A (b) –1.5 A (c) 1.5 A (d) 4.5 A

4.31 In the network shown in the figure, the effective resistance faced by the voltage source is



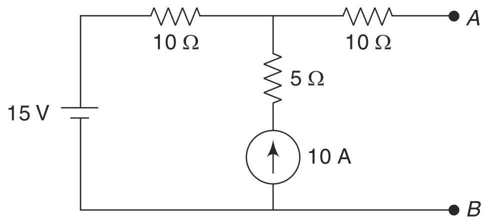
(a) 4 Ω (b) 3 Ω (c) 2 Ω (d) 1 Ω

4.32 For the network shown in the figure, if $Vs = V_1$ and $V = 0$, then $I = -5$ A and if $V_s = 0$ then $I = \frac{1}{2}$ A. The values of I_{SC} and R_1 of the Norton's equivalent across AB would be respectively



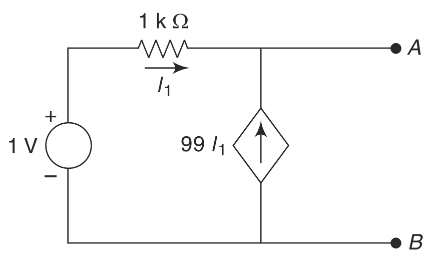
(a) –5 A and 2 Ω (b) 10 A and 0.5 Ω (c) 5 A and 2 Ω (d) 2.5 A and 5 Ω

4.33 In the network shown in the given figure, the Thevenin source and the impedance across terminals A–B will be respectively



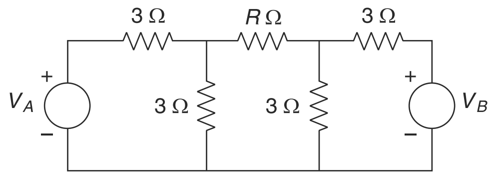
(a) 15 V and 13.33 Ω (b) 50 V and 15 Ω (c) 115 V and 20 Ω (d) 100 V and 25 Ω

4.34 Which one of the following combination of open-circuit voltage and Thevenin's equivalent resistance represents the Thevenin's equivalent of the circuit shown in the given figure?



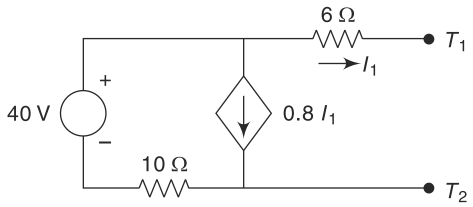
(a) 1 V, 10 Ω (b) 1 V, 1 kΩ (c) 1 mV, 1 kΩ (d) 1 mV, 10 Ω

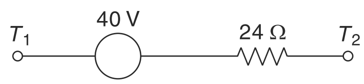
4.35 For the circuit shown in the given figure, the current through R, when $V_A = 0$ and $V_B = 15$ V is I ampere. Now, if both V_A and V_A are increased by 15 V, then the current through R will be



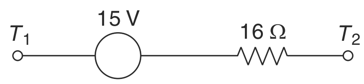
(a) I ampere (b) $\frac{I}{2}$ ampere (c) $3I$ ampere (d) $\frac{I}{3}$ ampere

4.36 Thevenin's equivalent circuit of the network shown in the given figure, between terminals T_1 and T_2 is



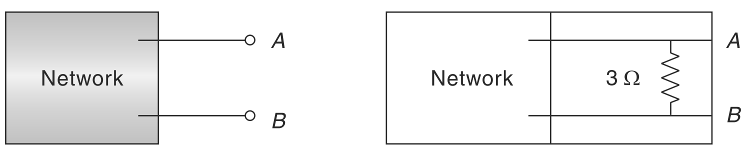
(a)


(b)


(c)


(d)

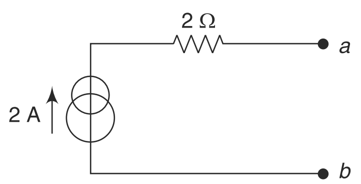

4.37 The Thevenin equivalent of the network shown in Figure (a) is 10 V in series with a resistance of 2 Ω. If now, resistance of 3 Ω is connected across AB in Figure (b), the Thevenin equivalent of the modified network across AB will be



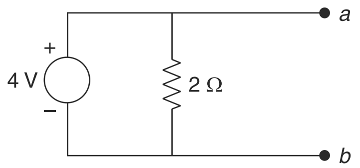
(a) *(b)*

(a) 10 V in series with 1.2 Ω resistance (b) 6 V in series with 1.2 Ω resistance
(c) 10 V in series with 5 Ω resistance (d) 6 V in series with 5 Ω resistance

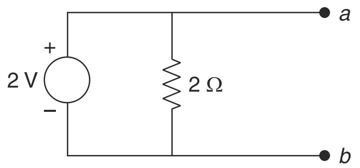
4.38 A d.c. current source is connected as shown in Figure below.



The Thevenin's equivalent of the network at terminals *a–b* will be

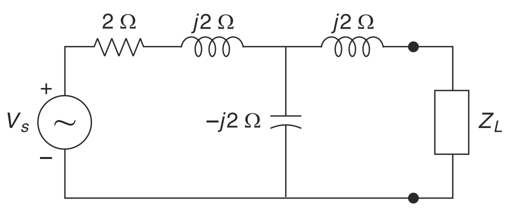
(a) 

(b) 

(c) 

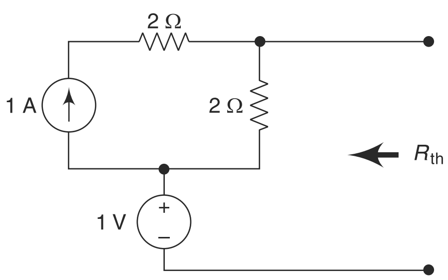
(d) is NOT feasible

4.39 Which one of the following impedance values of load will cause maximum power to be transferred to the load for the network shown in the given figure?



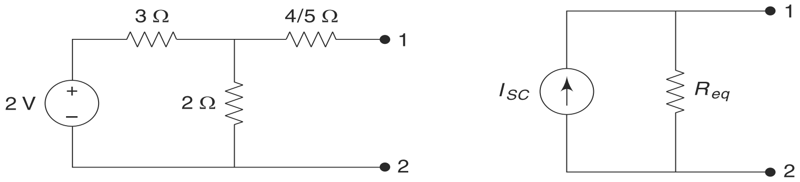
(a) $(2 + j2)$ (b) $(2 - j2)$ (c) $-j2$ (d) 2

4.40 The Thevenin's equivalent resistance R_{th} for the given network is



(a) 1 Ω (b) 2 Ω (c) 4 Ω (d) infinity

441 The Norton's equivalent of circuit shown in Figure (a) is drawn in the circuit shown in Figure (b). The values of I_{SC} and R_{eq} in Figure (b) are respectively



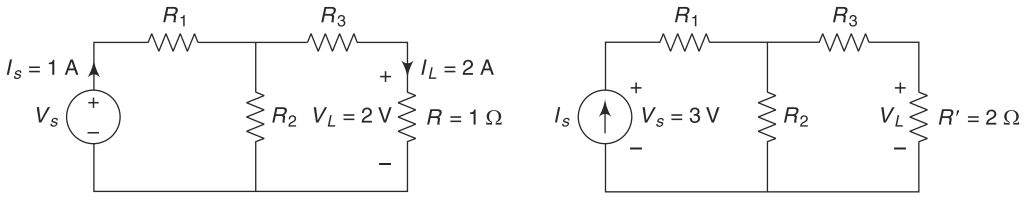
(a) (b)

(a) $\frac{5}{2}$ A and 2 Ω (b) $\frac{2}{5}$ A and 1 Ω (c) $\frac{4}{5}$ A and $\frac{12}{5}$ Ω (d) $\frac{2}{5}$ A and 2 Ω

EXERCISES

Tellegen's Theorem

4.1 The circuit of Fig. (a) is reconnected as of Fig. (b).



(a) *(b)*

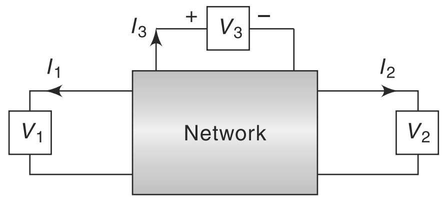
If $V_s = 2$ V and $I_s = 1$ A, find the voltage V_L of Fig (b). Use Tellegen's Theorem. [1 V]

4.2 Following readings were taken at a frequency of 50 Hz in a linear RLC network shown in figure.

$V_1 = 5e^{j5°}$ $I_1 = 12e^{j40°}$

$V_2 = 15e^{-j20°}$ $I_2 = 8e^{j10°}$

$V_3 = ?$ $I_3 = 10e^{j15°}$



At a frequency of 100 Hz, the readings are:

$V_1' = 10e^{j20°}$ $I_1' = 2^{j25°}$

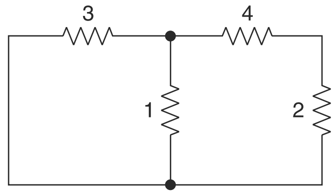
$V_2' = 12e^{j35}$ $I_2' = 10e^{-j10°}$

$V_3' = 5e^{j15°}$ $I_3' = 14.93e^{j68°}$

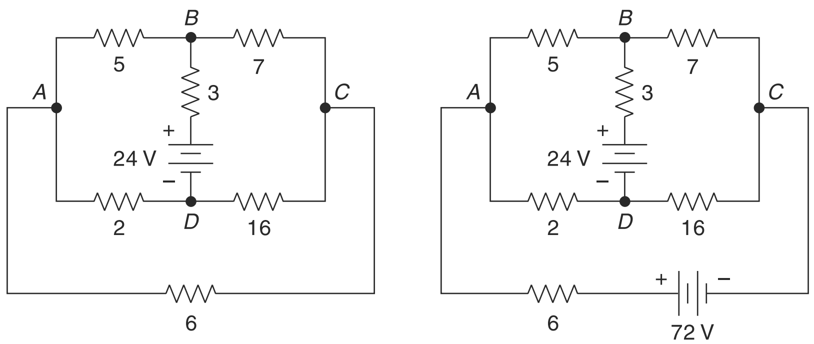
The reading of V_3 was missed. Calculate V_3 using Tellegen's Theorem. [$18e^{j15°}$]

Reciprocity Theorem

4.3 In the network shown in figure below, verify the Reciprocity Theorem using a voltage source and an ammeter. What are the methods of verifying the Reciprocity Theorem? All values are in ohm.

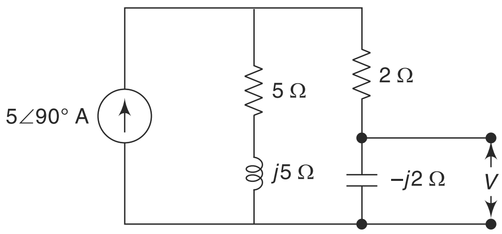


4.4 Find the current in the 6 Ω resistor and the source current in Figure (a). Hence, determine the current in the 3 Ω resistor when an emf of 72 V is added in series with the 6 Ω resistor as shown in Figure (b). [0.5 A, 6 A]

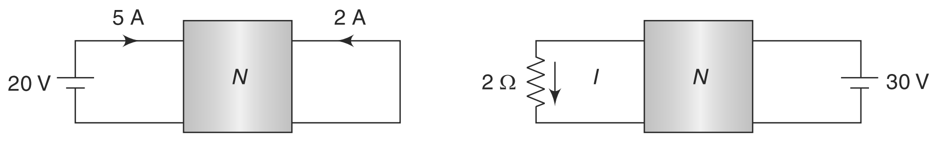


(a) (b)

4.5 In this circuit, find the voltage V. Interchange the current source and resulting voltage V and show that the reciprocity theorem is verified. [9.28∠21.8° (V)]



4.6. Two sets of measurements are made on a linear passive resistive network in Figure (a) and (b). Find the current through the 2Ω resistor. [2 A]

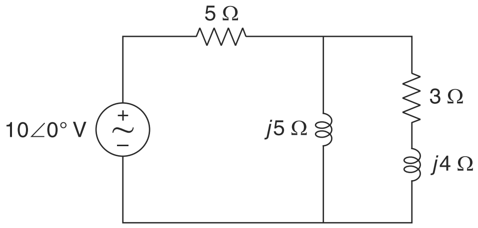


(a) (b)

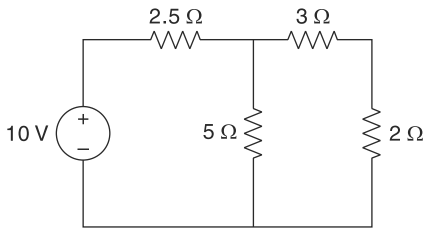
Compensation Theorem

4.7 The 5 Ω resistor has been changed to an 8 Ω resistor in the circuit. Determine the compensation source V_C and calculate the change in current through the 3Ω resistor.

[4.74∠–23.23° V; 0.271∠159.5° A]

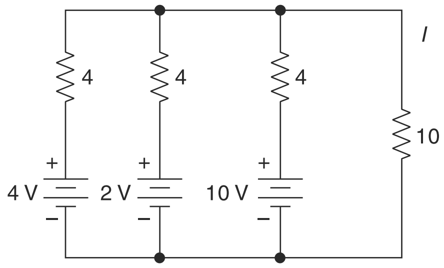


4.8 If the resistance 5 Ω increases to 6 Ω, determine the compensation source and find the current through the 6 Ω resistance. $\left[1\text{ V}; \dfrac{20}{23}\text{ A}\right]$

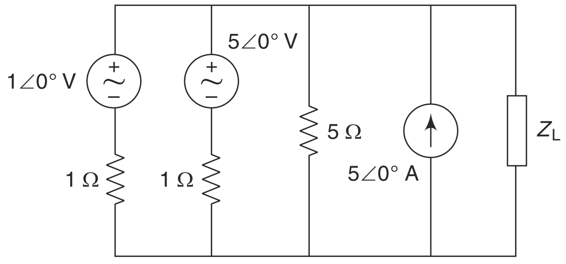


Millman's Theorem

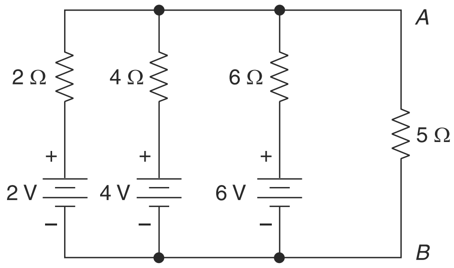
4.9 Find the load current using Millman's theorem. All values are in ohm. [1.176 A]



4.10 Using Millman's theorem, find the current in the load impedance, $Z_L = (2 + j4)\ \Omega$ [1.06∠–58.46° (A)]

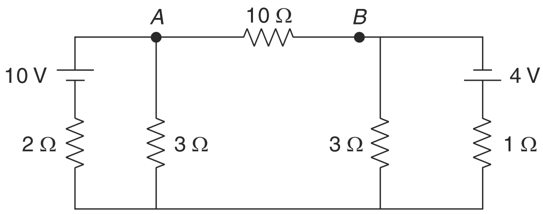


4.11 Determine the current through the branch *AB* using Millman's theorem. $\left[\dfrac{36}{67}\text{ A}\right]$



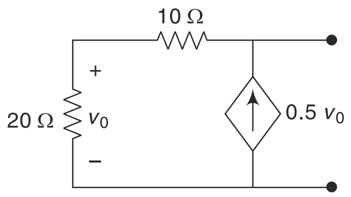
Thevenin's and Norton's Theorems

4.12 Determine the Thevenin equivalent circuit with respect to the terminals *A* and *B* for the circuit shown in the figure and hence the current flowing through 10 Ω resistor. [0.193 A]

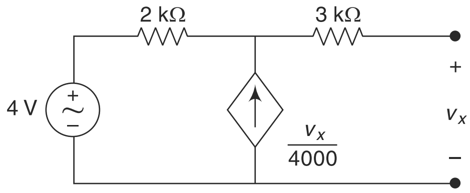


4.13 Find the Thevenin equivalent circuit for the following networks

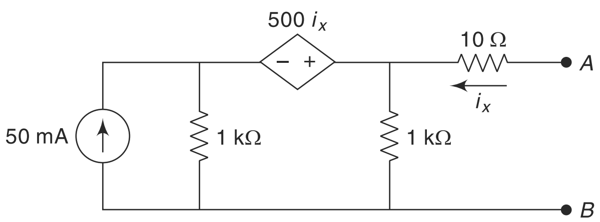
(i)



(ii)

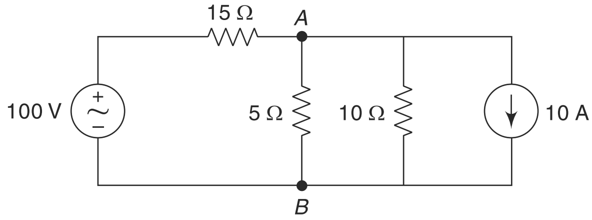


(iii)

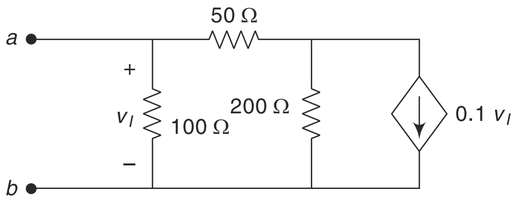


[(i) 0; – 0.33 Ω (ii) 8 V; 10 kΩ (iii) 25 V; 350 Ω]

4.14 Determine the current in the branch *AB* for the circuit shown in figure by using Thevenin's theorem. [1.818 A]

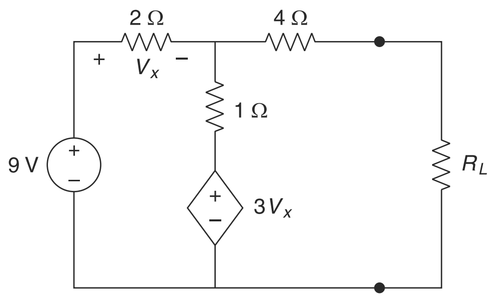


4.15 Find Norton's equivalent at terminals *a–b*. [0 A; 10.64 Ω]

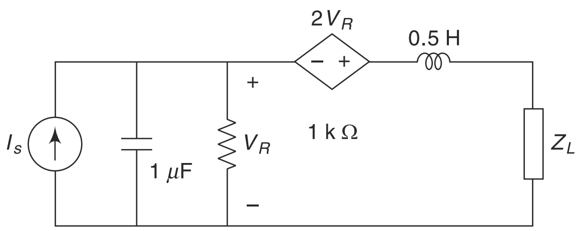


Maximum Power Transfer Theorem

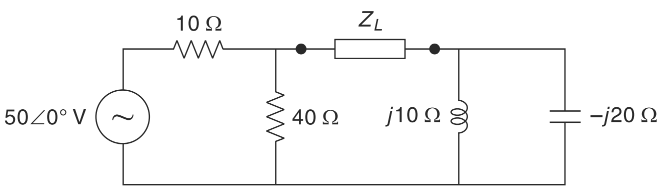
4.16 Determine the value of the resistor R_L that will draw maximum power from the rest of the circuit. What is the maximum power? [4.22 Ω, 2.901 W]



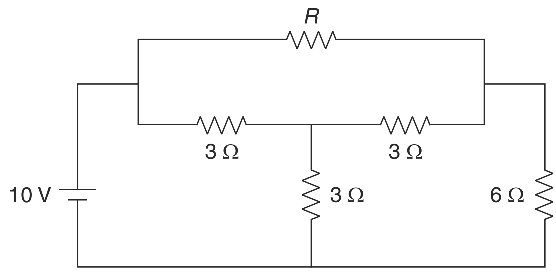
4.17 The circuit operates in the sinusoidal steady state with $\omega = 1000$ rad/s and $I_s = 1\angle 0°$ A(rms). Find the value of the load impedance for maximum average power transfer. Also, find the average power absorbed by the load under this condition. [$(1500 + j1000)$ Ω; 83.33 W]



4.18 Determine Z_L so that the maximum power is absorbed by it. [$40\angle 0°$ V; $(8 - j20)$ Ω, 50 W]

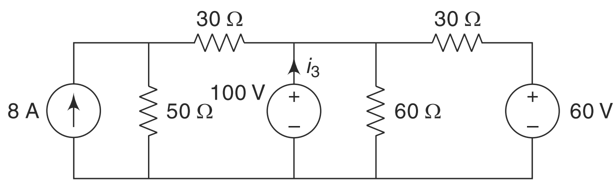


4.19 Determine the value of R such that the 6Ω resistor consumes the maximum power. [$R = -18\ \Omega$]

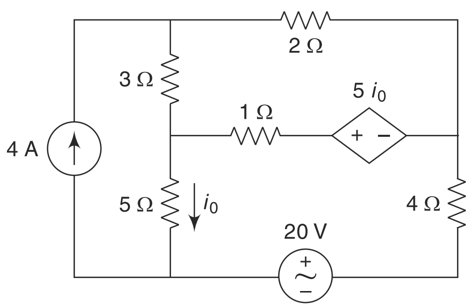


Superposition Theorem

4.20 Apply superposition theorem to the circuit to find i_3. [–0.75 A]



4.21 Find the current i_0 using superposition theorem. [–0.4706 A]



SHORT-ANSWER TYPE QUESTIONS

4.1 State and explain substitution theorem.

4.2 State and explain superposition theorem. Give a proof for a general n-mesh network indicating the conditions under which it is applicable.

4.3 State reciprocity theorem as applied to a network and give a proof of the same for a general network. Mention two networks where this theorem is not applicable.

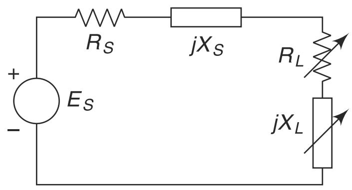
4.4 State Thevenin's theorem and give a proof of the same. Mention one example of a network where this network is not applicable.

4.5 (a) State Norton's theorem as applied to a network and give a proof of the same.
(b) What is 'Dual Network'? Mention the procedure for drawing the dual of a given network.

4.6 State and prove maximum power transfer theorem.

or

In the circuit, the source emf E_S, resistance R_S and reactance jX_S are fixed but both the load resistance R_L and reactance jX_L are variable. Show that maximum power is consumed in the load when $X_L = -X_S$ and $R_L = R_S$.



Prove that the load impedance which absorbs the maximum power from a source is the conjugate of the impedance of the source.

4.7 State and prove the following theorem
(a) Tellegen's theorem.
(b) Millman's theorem.
(c) Compensation theorem.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

4.1 (a)	4.2 (d)	4.3 (d)	4.4 (b)	4.5 (c)	4.6 (c)	4.7 (a)
4.8 (c)	4.9 (a)	4.10 (a)	4.11 (b)	4.12 (d)	4.13 (d)	4.14 (c)
4.15 (a)	4.16 (b)	4.17 (c)	4.18 (d)	4.19 (a)	4.20 (b)	4.21 (b)
4.22 (d)	4.23 (d)	4.24 (a)	4.25 (a)	4.26 (a)	4.27 (a)	4.28 (c)
4.29 (d)	4.30 (b)	4.31 (d)	4.32 (c)	4.33 (c)	4.34 (b)	4.35 (a)
4.36 (a)	4.37 (b)	4.38 (d)	4.39 (d)	4.40 (b)	4.41 (d)	

CHAPTER

5

Laplace Transform and its Applications

5.1 INTRODUCTION

Classical methods of solving differential equations become quite cumbersome when used for network involving higher order differential equations. In such cases, Laplace Transform method is used.

The classical methods consist of three steps, as given below.

(i) Determination of complementary function

(ii) Determination of particular integral

(iii) Determination of arbitrary constants.

But these methods become difficult for the equations containing derivatives, and transform methods prove to be superior.

The Laplace transform is an integral that transforms a time function into a new function of a complex variable.

5.2 ADVANTAGES OF LAPLACE TRANSFORM METHOD

Laplace transforms methods offer the following advantages over the classical methods.

1. It gives complete solution.
2. Initial conditions are automatically considered in the transformed equations.
3. Much less time is involved in solving differential equations.
4. It gives systematic and routine solutions for differential equations.

5.3 DEFINITION OF LAPLACE TRANSFORM

Let $f(t)$ be a function of time which is zero for $t < 0$ and which is arbitrarily defined for $t > 0$, subject to some mild conditions. Then the Laplace Transform of the function $f(t)$, denoted by $F(s)$ is defined as,

$$\mathcal{L}[f(t)] = F(s) = \int_{0_-}^{\infty} f(t)e^{-st}\,dt$$

Thus, the operator $\mathcal{L}[\ ]$ transforms $f(t)$, which is in time domain, into $F(s)$, which is in the complex frequency domain, or simply the s-domain, where,

$$s = \text{Complex frequency (unit is in Hz)} = (\sigma + j\omega)$$

where, σ = Real part of s = neper frequency and ω = Imaginary part of s = radian frequency.

NB: The *lower limit of the integration should be 0– instead of 0_+ or simple 0*. If $f(t)$ is continuous at $t = 0$, then the value of $f(0)$ is well-defined. But, if $f(t)$ is not continuous at $t = 0$, then the meaning of $f(0)$ becomes ambiguous. To consider the effect of "instantaneous energy transfer" we must use 0- as the lower limit to include the impulses at $t = 0$. The use of 0 will exclude the existence of any impulses at the origin.

So, we use 0– as the lower limit.

5.4 BASIC THEOREMS OF LAPLACE TRANSFORM

1. Linearity Theorem If Laplace transform of the functions $f_1(t)$ and $f_2(t)$ are $F_1(s)$ and $F_2(s)$ respectively, then Laplace transform of the functions $[K_1 f_1(t) + K_2 f_2(t)]$ will be $[K_1 F_1(s) + K_2 F_2(s)]$.

$$\mathcal{L}[K_1 f_1(t) + K_2 f_2(t)] = [K_1 F_1(s) + K_2 F_2(s)]$$

where, K_1 and K_2 are constants.

2. Scaling Theorem If Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L}[f(Kt)] = \frac{1}{K} F\left(\frac{s}{K}\right), \text{ where } K \text{ is a constant and } K > 0.$$

3. Time Differentiation Theorem If Laplace transform of $f(t)$ is $F(s)$, then,

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0_-)$$

4. Frequency Differentiation Theorem If Laplace transform of $f(t)$ is $F(s)$, then,

$$L[tf(t)] = -\frac{dF(s)}{ds}$$

5. Time Integration Theorem If Laplace transform of $f(t)$ is $F(s)$, then,

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

In general, for n^{th} order integration,

$$\mathcal{L}\left[\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n} f(t)\,dt_1\,dt_2\ldots dt_n\right] = \frac{F(s)}{s^n}$$

6. Shifting Theorem The shifting may be done with respect to time or frequency.

(a) Time Shifting Theorem

If Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L}[f(t \pm a)] = e^{\pm as} F(s)$$

(b) Frequency Shifting Theorem

If Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L}[e^{\mp at} f(t)] = F(s \pm a)$$

7. Initial Value Theorem If the Laplace Transform of $f(t)$ is $F(s)$ and the first derivative of $f(t)$ is Laplace transformable, then, the initial value of $f(t)$ is,

$$f(0^+) = \operatorname*{Lt}_{t \to 0} f(t) = \operatorname*{Lt}_{s \to \infty} [sF(s)]$$

Proof

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \int_{0_-}^{\infty} \left[\frac{df(t)}{dt}\right] e^{-st} dt$$

or

$$sF(s) - f(0_-) = \int_{0_-}^{\infty} \left[\frac{df(t)}{dt}\right] e^{-st} dt \qquad \text{[by time differentiation theorem]}$$

Taking limit $s \to \infty$,

$$\operatorname*{Lt}_{s \to \infty} [sF(s) - f(0_-)] = \operatorname*{Lt}_{s \to \infty} \int_{0_-}^{\infty} \left[\frac{df(t)}{dt}\right] e^{-st} dt$$

or

$$\operatorname*{Lt}_{s \to \infty} [sF(s)] - f(0_-) = \operatorname*{Lt}_{s \to \infty} \left[\int_{0_-}^{0^+} e^0 \frac{df(t)}{dt} dt + \int_{0^+}^{\infty} e^{-st} \frac{df(t)}{dt} dt\right]$$

or

$$\operatorname*{Lt}_{s \to \infty} [sF(s)] - f(0_-) = \operatorname*{Lt}_{s \to \infty} \left[\int_{0_-}^{0^+} e^0 \frac{df(t)}{dt} dt\right] \qquad \text{[as } s \text{ is not a function of time } t\text{]}$$

or

$$\operatorname*{Lt}_{s \to \infty} [sF(s)] - f(0_-) = \operatorname*{Lt}_{s \to \infty} \int_{0_-}^{0^+} df(t) = f(0^+) - f(0_-)$$

or

$$\boxed{f(0^+) = \operatorname*{Lt}_{s \to \infty} [sF(s)]}$$

8. Final Value Theorem If a function $f(t)$ and its derivatives are Laplace transformable, then the final value of $f(t)$ is,

$$f(\infty) = \operatorname*{Lt}_{t \to \infty} f(t) = \operatorname*{Lt}_{s \to 0} [sF(s)]$$

Proof

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \int_{0_-}^{\infty} \left[\frac{df(t)}{dt}\right] e^{-st} dt$$

or $\quad sF(s)-f(0_-)=\int_{0_-}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}dt$ [by time differentiation theorem]

Taking limit $s \to 0$,

$$\underset{s\to 0}{\text{Lt}}\left[sF(s)-f(0_-)\right]=\underset{s\to 0}{\text{Lt}}\int_{0_-}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}dt=\int_{0_-}^{\infty}\left[\frac{df(t)}{dt}\right]dt=\underset{t\to\infty}{\text{Lt}}\int_{0_-}^{t}\left(\frac{df(t)}{dt}\right)dt$$

or $\quad \underset{s\to 0}{\text{Lt}}\left[sF(s)-f(0_-)\right]=\underset{t\to\infty}{\text{Lt}}\left[f(t)-f(0_-)\right]$

or $\quad \underset{s\to 0}{\text{Lt}}\left[sF(s)\right]-f(0_-)=\underset{t\to\infty}{\text{Lt}}\left[f(t)\right]-f(0_-)$

or $\quad \boxed{\underset{t\to\infty}{\text{Lt}}\left[f(t)\right]=\underset{s\to 0}{\text{Lt}}\left[sF(s)\right]}$

This theorem is only applicable if the value of the function $f(t)$ is finite as t becomes infinity, i.e., $F(s)$ has all poles lying in the left half of s-plane or at most one simple pole at the origin.

5.5 LAPLACE TRANSFORM OF SOME BASIC FUNCTIONS

1. Exponential function

$$f(t)=e^{at}$$

By definition of Laplace transform,

$$F(s)=L[f(t)]=\int_{0-}^{\infty}e^{at}\cdot e^{-st}dt=\int_{0-}^{\infty}e^{(a-s)t}dt=\left[\frac{e^{(a-s)t}}{(a-s)}\right]_{0-}^{\infty}=\left(0-\frac{1}{(a-s)}\right)=\frac{1}{(s-a)}$$

Similarly, for $f(t)=e^{-at}$, $F(s)=\dfrac{1}{s+a}$

2. Unit step function or, Heaviside unit function

$$f(t)=u(t)=1 \text{ for } t>0$$
$$=0 \text{ for } t<0$$

and is undefined for $t = 0$.

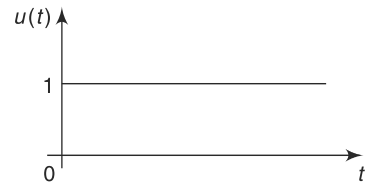


Figure 5.1(a) *Unit step function*

$$F(s)=L[f(t)]=\int_{0-}^{\infty}u(t)\cdot e^{-st}dt=\int_{0-}^{\infty}1\cdot e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_{0-}^{\infty}$$
$$=0-\frac{1}{-s}=\frac{1}{s}$$

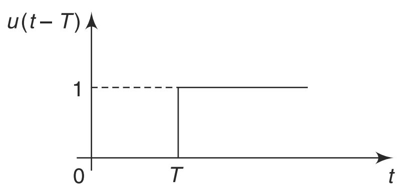


Figure 5.1(b) *Shifted unit step function*

Also, the Laplace transform of step function of magnitude K is

$$L[Ku(t)]=\frac{K}{s}$$

Similarly, the Laplace transform of the shifted unit step function $u(t - T)$ is,

$$\mathcal{L}[u(t-T)] = \frac{e^{-sT}}{s} \qquad \{\text{by differentiation theorem}\}$$

Another function, called *gate function* can be obtained from step function as follows.

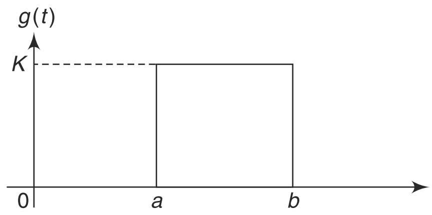


Figure 5.2 *Gate function*

Therefore, $g(t) = Ku(t-a) - Ku(t-b)$ and, $L[g(t)] = \frac{K}{s}(e^{-as} - e^{-bs})$

3. The sine function

$$f(t) = \sin \omega t = \frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty} \left[\frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]\right] \cdot e^{-st}\, dt = \frac{1}{2j}\int_{0-}^{\infty} [e^{(j\omega - s)t} - e^{-(j\omega + s)t}]\, dt$$

$$= \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right] = \frac{\omega}{s^2 + \omega^2}$$

4. The cosine function

$$f(t) = \cos \omega t = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty} \left[\frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]\right] \cdot e^{-st}\, dt$$

$$= \frac{1}{2}\int_{0-}^{\infty} [e^{(j\omega - s)t} + e^{-(j\omega + s)t}]\, dt = \frac{1}{2}\left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right] = \frac{s}{s^2 + \omega^2}$$

5. The hyperbolic sine function

$$f(t) = \sinh at = \frac{1}{2}[e^{at} - e^{-at}]$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty} \left[\frac{1}{2j}[e^{at} - e^{-at}]\right] \cdot e^{-st}\, dt$$

$$= \frac{1}{2}\int_{0-}^{\infty} [e^{(a-s)t} - e^{-(a+s)t}]\, dt = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 + a^2}$$

6. The hyperbolic cosine function

$$f(t) = \cosh at = \frac{1}{2}[e^{at} + e^{-at}]$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty}\left[\frac{1}{2}[e^{at} + e^{-at}]\right]\cdot e^{-st}\,dt$$

$$= \frac{1}{2}\int_{0-}^{\infty}[e^{(a-s)t} + e^{-(a+s)t}]\,dt = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}$$

7. The damped sinusoidal function

$$f(t) = e^{-at}\cdot\sin\omega t = e^{-at}\cdot\left\{\frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]\right\} = \left\{\frac{1}{2j}[e^{-(a-j\omega t)} - e^{-(a+j\omega t)}]\right\}$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty}\left[\frac{1}{2j}[e^{-(a-j\omega t)} - e^{-(a+j\omega t)}]\right]\cdot e^{-st}dt$$

$$= \frac{1}{2j}\int_{0-}^{\infty}[e^{-(s+a-j\omega)t} - e^{-(s+a+j\omega)t}]\cdot dt$$

$$= \frac{1}{2j}\left[\frac{1}{\{(s+a) - j\omega\}} - \frac{1}{\{(s+a) + j\omega\}}\right] = \frac{\omega}{(s+a)^2 + \omega^2}$$

8. The damped cosine function

$$f(t) = e^{-at}\cdot\cos\omega t = e^{-at}\cdot\left\{\frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]\right\} = \left\{\frac{1}{2}[e^{-(a-j\omega t)} + e^{-(a+j\omega t)}]\right\}$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty}\left[\frac{1}{2}[e^{-(a-j\omega t)} + e^{-(a+j\omega t)}]\right]\cdot e^{-st}dt$$

$$= \frac{1}{2}\int_{0-}^{\infty}[e^{-(s+a-j\omega)t} + e^{-(s+a+j\omega)t}]\cdot dt$$

$$= \frac{1}{2}\left[\frac{1}{\{(s+a) - j\omega\}} + \frac{1}{\{(s+a) + j\omega\}}\right] = \frac{(s+a)}{(s+a)^2 + \omega^2}$$

9. The ramp function

$$f(t) = t^n$$

$$F(s) = L[f(t)] = L[t^n] = \int_{0-}^{\infty} t^n\cdot e^{-st}dt$$

Integrating by parts, let,

$$u = t^n \quad \text{and} \quad dv = e^{-st}\,dt$$

then $\quad du = nt^{n-1} \quad$ and $\quad v = \int e^{-st}dt = -\dfrac{e^{-st}}{s}$

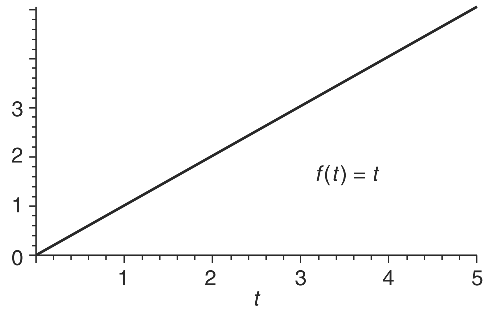


Figure 5.3 *Ramp function*

Now, $$F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} udv = uv|_{0-}^{\infty} - \int_{0-}^{\infty} vdu = -\frac{t^n}{s}[e^{-st}]_{0-}^{\infty} + \frac{n}{s}\int_{0-}^{\infty} t^{(n-1)}e^{-st}dt$$

$$= \frac{n}{s}\cdot\int t^{(n-1)}e^{-st}dt = \frac{n}{s}L[t^{(n-1)}] = \frac{n}{s}\cdot\frac{(n-1)}{s}L[t^{(n-2)}]$$

$$= \frac{n}{s}\cdot\frac{(n-1)}{s}\cdot\frac{(n-2)}{s}\ldots\ldots\frac{2}{s}\cdot\frac{1}{s}L[t^0] = \frac{\angle n}{s^n}L[u(t)]$$

$$= \frac{\angle n}{s^n}\cdot\frac{1}{s} = \frac{\angle n}{s^{n+1}}$$

For $n=1$, $\mathcal{L}[t] = \dfrac{1}{s^2}$

For $n=2$, $\mathcal{L}[t^2] = \dfrac{\angle 2}{s^3}$

10. Impulse Function or Dirac Delta Function [$\delta(t)$]

It is a function of a real variable t, such that the function is zero everywhere except at the instant $t = 0$. Physically, it is a very sharp pulse of infinitesimally small width and very large magnitude, the area under the curve being unity.

Consider a gate function as shown in Fig. 5.4.

The function is compressed along the time-axis and stretched along the y-axis, keeping area under the pulse unity. As $a \to 0$, the value of $\frac{1}{a} \to \infty$ and the resulting function is known as impulse.

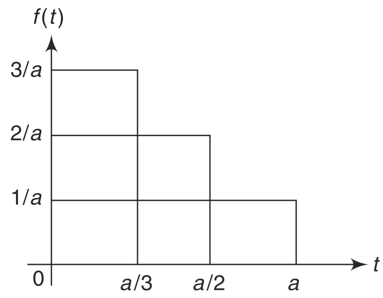


Figure 5.4 *Generation of impulse function from gate function*

It is defined as, $\delta(t) = 0$ for $t \neq 0$

and $$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Also, $$\delta(t) = \lim_{a\to 0}\frac{1}{a}[u(t) - u(t-a)]$$

The Laplace transform of the impulse function is obtained as,

$$\mathcal{L}[\delta(t)] = \lim_{a\to 0} L\left\{\frac{1}{a}[u(t) - u(t-a)]\right\}$$

$$= \lim_{a\to 0}\frac{1}{a}\left[\frac{1}{s} - \frac{e^{-as}}{s}\right]$$

$$= \lim_{a\to 0}\frac{1-e^{-as}}{as}$$

$$= \lim_{a\to 0}\frac{se^{-as}}{s} \qquad \text{[by L'Hospital's rule]}$$

$$= 1$$

5.6 LAPLACE TRANSFORM TABLE

Table 5.1 *Standard Laplace Transforms*

Sl. No.	*Functions [f(t)] in Time(t) Domain*	*Laplace Transform [F(s)] in Frequency(s) Domain*
Definition	If $f(t)$ is Laplace transformable	then $L[f(t)]=F(s)=\int_{0-}^{\infty} f(t)e^{-St}dt$
1	$U(t)$ (unit step function)	$\frac{1}{s}$
2	$U(t-T)$ (unit step function shifted/delayed by T)	$\frac{e^{-sT}}{s}$
3	$\delta(t)$ (unit impulse)	1
4	e^{at} (exponential function)	$\frac{1}{s-a}$
5	e^{-at} (exponential function)	$\frac{1}{s+a}$
6	$\sin \omega t$ (sine function)	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t$ (cosine function)	$\frac{s}{s^2+\omega^2}$
8	t^n (n=1, 2, 3, ……) (ramp function)	$\frac{n!}{s^{n+1}}$
9	t (unit ramp function)	$\frac{1}{s^2}$
10	$e^{-at} \sin \omega t$ (damped sine function)	$\frac{\omega}{(s+a)^2+\omega^2}$
11	$e^{-at} \cos \omega t$ (damped cosine function)	$\frac{(s+a)}{(s+a)^2+\omega^2}$
12	$e^{-at} t^n$ (damped ramp function)	$\frac{n!}{(s+a)^{n+1}}$
13	$\frac{d}{dt}f(t)$ (Differentiation theorem)	$sF(s)-f(0-)$
14	$\int_0^t f(t)dt$ (Integration theorem)	$\frac{F(s)}{s}+\frac{f(0-)}{s}$

(*Contd*)

(*Contd*)

15	sinh ωt (hyperbolic sine function)	$\dfrac{\omega}{s^2-\omega^2}$
16	cosh ωt (hyperbolic cosine function)	$\dfrac{s}{s^2-\omega^2}$
17	e^{-at} sinh ωt (damped hyperbolic sine function)	$\dfrac{\omega}{(s+a)^2-\omega^2}$
18	e^{-at} cosh ωt (damped hyperbolic cosine function)	$\dfrac{(s+a)}{(s+a)^2-\omega^2}$
19	Initial value theorem	$\underset{t\to 0}{\text{Lt}}\, f(t) = \underset{s\to\infty}{\text{Lt}}\, sF(s)$
20	Final value theorem	$\underset{t\to\infty}{\text{Lt}}\, f(t) = \underset{s\to 0}{\text{Lt}}\, sF(s)$
21	Shifting theorem $f(t \pm a)$	$e^{\pm as}F(s)$

5.6.1 Other Important Laplace Transforms

1	$\delta(t)$	1
2	$\delta(t-a)$	e^{-as}
3	$\delta(t-a)\, g(t)$	$e^{-as}\, g(a)$ *Note:* $g(a)$ NOT $G(a)$
4	$\dfrac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-z\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$	$\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\ (\zeta<1)$
5	$1-\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-z^2}\, t+\theta)$, where $\theta = \cos^{-1}\zeta$	$\dfrac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\ (\zeta<1)$

5.7 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

If $f(t)$ is periodic with time period T (> 0), so that $f(t + T) = f(t)$, then the Laplace transform of the function is equal to $\left(\dfrac{1}{1-e^{-Ts}}\right)$ times the Laplace transform of the first cycle.

$$\therefore \qquad \mathcal{L}[f(t)] = F(s) = F_1(s)\left[\frac{1}{1-e^{-Ts}}\right]$$

Proof

Let $f(t)$ be the periodic function, and

T be the time period,

Let $f_1(t), f_2(t), \dots, f_n(t)$ be the functions representing the first, second, …, n^{th} cycle, respectively

$$\therefore \quad f(t) = f_1(t) + f_2(t) + \dots + f_n(t) + \dots$$
$$= f_1(t) + f_1(t - T) + f_1(t - 2T) + \dots$$

Taking Laplace transform,

$$\mathcal{L}[f(t)] = F(s) = L[f_1(t)] + L[f_1(t-T)] + L[f_1(t-2T)] + \dots$$
$$= F_1(s) + e^{-Ts} F_1(s) + e^{-2Ts} F_1(s) + \dots$$
$$= F_1(s)[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \dots]$$

Therefore, $$\boxed{F(s) = F_1(s)\left[\frac{1}{1-e^{-Ts}}\right]}$$

Example 5.1 Find the Laplace transform of the square wave.

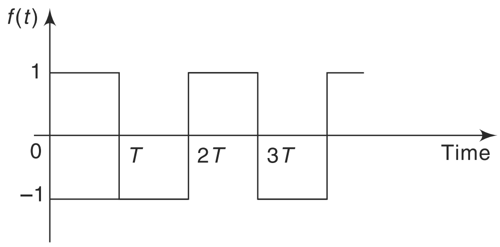


Figure 5.5(a) *Square wave of Example 5.1*

Solution The first cycle is shown below.
It can be written as,

$$f_1(t) = u(t) - 2u(t - T) + u(t - 2T)$$

Taking Laplace transform of the first cycle,

$$F_1(s) = \frac{1}{s} - \frac{2e^{-Ts}}{s} + \frac{e^{-2Ts}}{s} = \frac{1}{s}(1 - e^{-Ts})^2$$

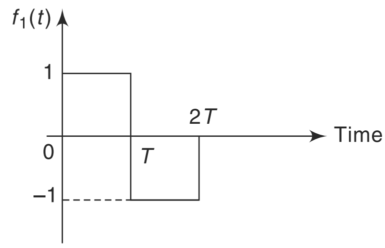


Figure 5.5(b) *First cycle of the square wave of Example 5.1*

By the theory of time periodicity, the Laplace transform of the square wave is given as,

$$F_1(s) = \frac{1}{s}(1 - e^{-Ts})^2 \times \frac{1}{1 - e^{-2Ts}}$$

(Since time period of the square wave is $2T$)

$$= \frac{1}{s}\left(\frac{1-e^{-Ts}}{1+e^{-Ts}}\right) = \frac{1}{s}\tanh\left(\frac{Ts}{2}\right)$$

5.8 INVERSE LAPLACE TRANSFORM

Let, $F(s)$ have the general form of

$$F(s) = \frac{N(s)}{D(s)}$$

where, $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial. The roots of $N(s) = 0$ are called the *zeros of F(s)* while the roots of $D(s) = 0$ are the *poles of F(s)*.

For example, for the function $F(s) = \dfrac{s-1}{s(s-2)(s-3)}$, the zero is at $s = 1$ and the poles are at $s = 0$, 2 and 3.

We use Partial Fraction Expansion to break $F(s)$ down into simple terms. Thus, there are two steps to find inverse Laplace transform as given below.

I. Decomposition of $F(s)$ into simple terms using Partial Fraction Expansion.
II. Evaluation of the inverse of each term comparing with the standard forms of Laplace transforms.

We consider the following three cases given below.

I. Simple Poles

Let $$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)(s+p_3)\dots(s+p_n)}$$

where, $s = -p_1, -p_2\ -p_3, \dots, -p_n$ are the simple poles, and $p_i \neq p_j$ for all $i \neq j$ (i.e. poles are distinct) Assuming that the degree of $N(s)$ is less than the degree of $D(s)$,

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \frac{k_3}{s+p_3} + \dots + \frac{k_n}{s+p_n} \tag{1}$$

where, expansion co-efficients $k_1, k_2, k_3, \dots, k_n$ are known as the residues of $F(s)$. These can be found out by Residue method explained below.

Multiplying both sides of Eq. (1), by $(s + p_1)$,

$$(s+p_1)F(s) = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \frac{(s+p_1)k_3}{s+p_3} + \dots + \frac{(s+p_1)k_n}{s+p_n}$$

Putting

$$s = -p_1 \quad \Rightarrow \quad (s+p_1)F(s)|_{s=p_i} = k_1$$

In general, $\boxed{k_i = (s+p_i)\,F(s)|_{s=p_i}}$. This is known as Heaviside's Theorem.

Once, the values of k_i are known, the inverse Laplace is obtained as,

$$f(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + k_3 e^{-p_3 t} + \dots + k_n e^{-p_n t})\,u(t)$$

Example 5.2 Find the inverse Laplace transform of the function,

$$F(s) = \frac{2s+1}{(s+1)(s+2)(s+3)}.$$

Solution Let $F(s) = \dfrac{2s+1}{(s+1)(s+2)(s+3)} = \dfrac{k_1}{s+1} + \dfrac{k_2}{s+2} + \dfrac{k_3}{s+3}$

$$\therefore \quad k_1 = (s+1)F(s)\big|_{s=-1} = \left.\frac{2s+1}{(s+2)(s+3)}\right|_{s=-1} = -\frac{1}{2}$$

$$\therefore \quad k_2 = (s+2)F(s)\big|_{s=-2} = \left.\frac{2s+1}{(s+1)(s+3)}\right|_{s=-2} = 3$$

$$\therefore \quad k_3 = (s+3)F(s)\big|_{s=-3} = \left.\frac{2s+1}{(s+1)(s+2)}\right|_{s=-3} = -\frac{5}{2}$$

$$\therefore \quad F(s) = -\frac{1}{2(s+1)} + \frac{3}{s+2} - \frac{5}{2(s+3)}$$

Thus, the inverse Laplace transform is given as,

$$f(t) = -\frac{1}{2}e^{-t} + 3e^{-2t} - \frac{5}{2}e^{-3t}$$

II. Repeated Poles

Suppose, $F(s)$ has n repeated poles at $s = -p$.

$$\therefore \quad F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \frac{k_{n-2}}{(s+p)^{n-2}} + \ldots + \frac{k_2}{(s+p)^2} + \frac{k_1}{(s+p)} + F_1(s)$$

where, $F_1(s)$ is the remaining part of $F(s)$ that does not have a pole at $s = -p$.
We find,

$$\therefore \quad k_n = (s+p)^n F(s)\big|_{s=-p}$$

To find $k_{n-1}, k_{n-2}, \ldots, k_{n-m}$, the procedure is,

$$k_{n-1} = \left.\frac{d}{ds}[(s+p)^n F(s)]\right|_{s=-p}$$

$$k_{n-1} = \left.\frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)]\right|_{s=-p}$$

In general, $k_{n-m} = \left.\dfrac{1}{m!}\dfrac{d^m}{ds^m}[(s+p)^n F(s)]\right|_{s=-p}$, where, $m = 1, 2, \ldots, (n-1)$.

Once, the values of $k_1, k_2, \ldots, k_n$ are known, the inverse Laplace is obtained as,

$$\boxed{f(t) = \left(k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{3!}t^2 e^{-pt} + \ldots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + f_1(t)}$$

Example 5.3 Find the inverse Laplace transform of the function $F(s) = \dfrac{12}{(s+2)^2\,(s+4)}$.

Solution Let $F(s) = \dfrac{12}{(s+2)^2\,(s+4)} = \dfrac{k_1}{(s+2)^2} + \dfrac{k_2}{s+2} + \dfrac{k_3}{s+4}$

By residue method,

$$k_1 = (s+2)^2 F(s)\big|_{s=-2} = \left.\frac{12}{(s+4)}\right|_{s=-2} = 6$$

$$\therefore \quad k_2 = \frac{d}{ds}[(s+2)^2 F(s)]\big|_{s=-2} = \left.\frac{d}{ds}\left[\frac{12}{(s+4)}\right]\right|_{s=-2} = -3$$

$$k_3 = (s+4)F(s)\big|_{s=-4} = \left.\frac{12}{(s+2)^2}\right|_{s=-4} = 3$$

Thus, $F(s) = \dfrac{6}{(s+2)^2} - \dfrac{3}{s+2} + \dfrac{3}{s+4}$

Taking inverse Laplace transform, $f(t) = 3e^{-4t} - 3e^{-2t} + 6te^{-2t}$

III. Complex Poles

Since $N(s)$ and $D(s)$ always have real co-efficients and as the complex roots of polynomials with real co-efficients occur in conjugate form, $F(s)$ may have the general form,

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s) = \frac{k_1}{s+\alpha - j\beta} + \frac{k_2}{s+\alpha + j\beta} + F_1(s)$$

where, $F_1(s)$ is the remaining part of $F(s)$ that does not have this pair of complex poles.

Let $\quad (s^2 + as + b) = (s^2 + 2\alpha s + \alpha^2 + \beta^2) = (s+\alpha)^2 + \beta^2$

$$\therefore \quad s_{1,2} = (-\alpha \pm j\beta) = -\frac{a}{2} \pm j\sqrt{b - \frac{a^2}{4}}$$

Thus, the coefficients are,

$$\boxed{k_1 = (s - s_1)F(s)\big|_{s=s_1} \quad \text{and} \quad k_2 = k_1^* = \text{Complex conjugate of } k_1}$$

Example 5.4 Find the inverse Laplace transform of the function $F(s) = \dfrac{2s+1}{(s+1)\,(s^2+2s+5)}$.

Solution Let $F(s) = \dfrac{2s+1}{(s+1)\,(s^2+2s+5)} = \dfrac{A}{s+1} + \dfrac{k_1}{s+1-j2} + \dfrac{k_2}{s+1+j2}$

$$\therefore \quad A = (s+1)F(s)\big|_{s=-1} = \left.\frac{2s+1}{s^2+2s+5}\right|_{s=-1} = -\frac{1}{4}$$

$$k_1 = (s+1-j2)F(s)|_{s=(-1+j2)} = \left.\frac{2s+1}{(s+1)(s+1+j2)}\right|_{s=(-1+j2)} = \left(\frac{1}{8} - j\frac{1}{2}\right)$$

$$\therefore \qquad k_2 = k_1^* = \left(\frac{1}{8} + j\frac{1}{2}\right)$$

$$\therefore \qquad F(s) = -\frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{\frac{1}{8} - j\frac{1}{2}}{s+1-j2} + \frac{\frac{1}{8} + j\frac{1}{2}}{s+1+j2}$$

Taking inverse Laplace transform,

$$f(t) = -\frac{1}{4}[e^{-t} - e^{-t}\cos 2t] + e^{-t}\sin 2t = -\frac{1}{2}e^{-t}\sin^2 t + e^{-t}\sin 2t$$

5.9 APPLICATIONS OF LAPLACE TRANSFORM

1. Solving Integro-Differential Equations and Simultaneous Differential Equations
2. Transient Analysis of Electrical Circuits.

5.9.1 Solving Integro Differential Equations and Simultaneous Differential Equations

An *integro-differential equation* is an integral equation in which various derivatives of the unknown function can also be present. Using the Laplace transform of integrals and derivatives, an integro-differential equation can be solved.

Similarly, it is easier with the Laplace transform method to solve *simultaneous differential equations* by transforming both equations and then solving the two equations in the s-domain and finally obtaining the inverse to get the solution in the time domain.

Example 5.5 (Integro-Differential Equation) Solve the equation for the response $i(t)$, given that

$$\frac{di}{dt} + 2i + 5\int_0^t i\,dt = u(t) \text{ and } i(0) = 0.$$

Solution Let $\mathcal{L}[i(t)] = I(s)$.

$$\therefore \qquad \mathcal{L}\left[\frac{di}{dt}\right] = sI(s) - i(0) = sI(s) - 0 = sI(s)$$

Taking Laplace transform on both sides of the given equation,

$$sI(s) + 2I(s) + 5\frac{I(s)}{s} = \frac{1}{s}$$

or

$$I(s) = \frac{1}{s^2+2s+5} = \frac{1}{2}\,\frac{2}{(s+1)^2+(2)^2}$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{1}{2} e^{-t} \sin 2t \ (A), t > 0$$

Example 5.6 (Integro-Differential Equation) Solve the initial value problem for *y(t)* when $\frac{d^2y}{dt^2} + y(t) = 3 \sin 2t$ and $y(0) = 1, \ y'(0) = -2$.

Solution Let $\mathcal{L}[y(t)] = Y(s)$.

$$\therefore \quad \mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 2$$

or $\quad Y(s) = \dfrac{s}{s^2+1} - \dfrac{2}{s^2+4}$

Taking inverse Laplace transform, we get, $y(t) = (\cos t - \sin 2t)$

Example 5.7 (Simultaneous Differential Equations) Find the solution of the system: $\frac{dx}{dt} - 6x + 3y = 8e^t$ and $\frac{dy}{dt} - 2x - y = 4e^t$ with initial conditions $x(0) = -1$, $y(0) = 0$.

Solution Taking Laplace transform,

$$(s-6)X + 3Y = \frac{-s+9}{s-1} \tag{i}$$

$$-2X + (s-1)Y = \frac{4}{s-1} \tag{ii}$$

Solving for X and Y,

$$X = \frac{-s+7}{(s-1)\,(s-4)} = -\frac{2}{s-1} + \frac{1}{s-4}$$

$$Y = \frac{2}{(s-1)\,(s-4)} = \frac{-2/3}{s-1} + \frac{2/3}{s-4}$$

Taking inverse Laplace transform,

$$x(t) = -2e^t + e^{4t} \quad \text{and} \quad y(t) = -\frac{2}{3}e^t + \frac{2}{3}e^{4t}$$

Example 5.8 (Simultaneous Differential Equations) Solve for $x(t)$ and $y(t)$, given that $x(0) = 4$, $y(0) = 3$ and

$$\frac{dx}{dt} + x + 4y = 10 \quad \text{and} \quad x - \frac{dy}{dt} - y = 0$$

Solution Following the same procedures, as in Ex (5.7), we get,

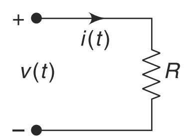
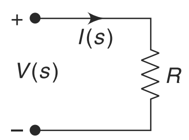
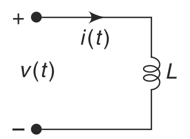
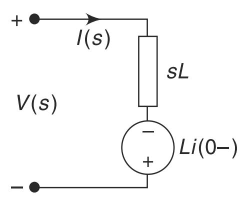
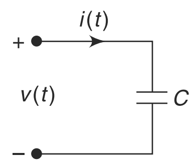
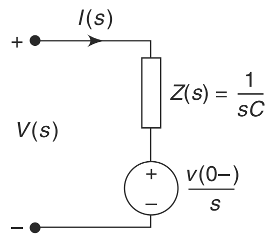
$$X = \frac{4s^2 + 2s + 10}{s(s^2 + 3)} \quad \text{and} \quad Y = \frac{3s^2 + s + 10}{s(s^2 + 3)}$$

Taking inverse Laplace transform, we get the desired results.

5.10 APPLICATION OF LAPLACE TRANSFORM METHOD TO CIRCUIT ANALYSIS

We now apply the mathematical tool for the analysis of electric circuits.

5.10.1 Transform Impedance of Network Elements

Element	*Time Domain*	*s-Domain*
1. Resistor (R)	$v(t) = Ri(t)$	$V(s) = RI(s)$
		
2. Inductor (L)	$v(t) = L\frac{di(t)}{dt}$	$V(s) = L[sI(s) - i(0\_)]$
	$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)dt$	$I(s) = \frac{1}{L}\left[\frac{V(s)}{s} + \frac{i(0-)}{s}\right]$
		
3. Capacitor (C)	$i(t) = C\frac{dv(t)}{dt}$	$I(s) = sCV(s) - Cv(0\_)$
	$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$	$V(s) = \frac{I(s)}{Cs} + \frac{v(0\_)}{s}$
		

5.11 TRANSIENT ANALYSIS OF ELECTRIC CIRCUITS USING LAPLACE TRANSFORM

In electrical engineering, a **transient response** or **natural response** is the electrical response of a system to a change from equilibrium.

The condition prevailing in an electric circuit between two steady-state conditions is known as the *transient state*; it lasts for a very short time. The currents and voltages during the transient state are called *transients*.

In general, transient phenomena occur whenever

(i) a circuit is suddenly connected or disconnected to/from the supply,
(ii) there is a sudden change in the applied voltage from one finite value to another,
(iii) a circuit is short-circuited.

A simple example would be the output of a 5 volt DC power supply when it is turned on: the transient response is from the time the switch is turned on and the output is a steady 5 volt. At this point the power supply reaches its steady-state response of a constant 5 volt.

The transient response is not necessarily tied to "on/off" events but to any event that affects the equilibrium of the system. If in an RC circuit the resistor or capacitor is replaced with a variable resistor or variable capacitor (or both) then the transient response is the response to a change in the resistor or capacitor.

The transient currents are not caused by any part of the supply voltage, but are entirely associated with the changes in the stored energy in capacitor and inductors. As there is no energy stored in resistors, there are *no transients in purely resistive circuits*.

Although transients last for a very short time, their study is very important because.

(i) They indicate what dangerous rises in voltage or current may happen in individual sections of a circuit.
(ii) They indicate how signals are distored in waveform or amplitude as they pass through amplifiers, filters, or other circuit elements.

We consider the transient analysis for the following circuits subject to step input, impulse input and sinusoidal input:

1. RL Series Circuit,
2. RC Series Circuit,
3. RLC Series Circuit, and
4. RLC Parallel Circuit.

5.11.1 RL Series Circuit

1. RL Series Circuit with Step Input We consider an RL series circuit as shown in figure.

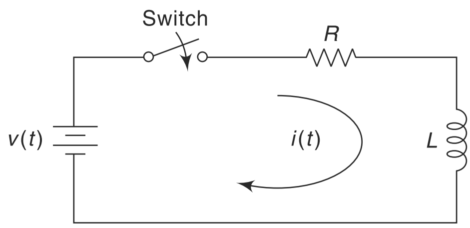


Figure 5.6 *R-L series circuit*

If the switch is closed at time $t = 0$, the voltage across the RL combination would be $v(t)$ which is a step of magnitude V [or $Vu(t)$] and not a constant as is the supply voltage V.

$$v(t) = 0, \text{ for } t \le 0$$
$$= V, \text{ for } t \ge 0$$

Thus the differential equation governing the behaviour of the circuit would be

$$Ri(t) + \mathcal{L}\frac{di(t)}{dt} = Vu(t)$$

Taking Laplace transform, we get

$$RI(s) + \mathcal{L}[sI(s) - i(0-)] = \frac{V}{s}$$

or,
$$I(s) = \frac{\frac{V}{L}}{s\left(s + \frac{R}{L}\right)} + \frac{i(0-)}{s + \frac{R}{L}} = \frac{V}{R}\left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right) + \frac{i(0-)}{s + \frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right) + i(0-)\,e^{-\left(\frac{R}{L}\right)t} = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right) \text{ with } i(0-) = 0.$$

The transient part of the current response, $i_{tr} = [i(t) - i_s] = -\frac{V}{R}e^{-\frac{R}{L}t}$

From the current equation at $t = \tau = \frac{L}{R}$, $i = \frac{V}{R}(1 - e^{-1}) = 0.63\frac{V}{R} = 0.63 i_s$

When the switch is first closed, the voltage across the inductor will immediately jump to battery voltage (acting as though it were an open-circuit) and decay down to zero over time (eventually acting as though it were a short-circuit). Voltage across the inductor is determined by calculating how much voltage is being dropped across R, given the current through the inductor, and subtracting that voltage value from the battery to see what's left. When the switch is first closed, the current is zero, then it increases over time until it is equal to the battery voltage divided by the series resistance. This behavior is precisely opposite that of the series resistor-capacitor circuit, where current started at a maximum and capacitor voltage at zero.

The steady state part of the current response, $i_s = \frac{V}{R}$

The variation of the current is shown in Figure 5.7.

The quantity $\tau = \frac{L}{R}$ is known as the Time-constant of the circuit and it is defined as follows.

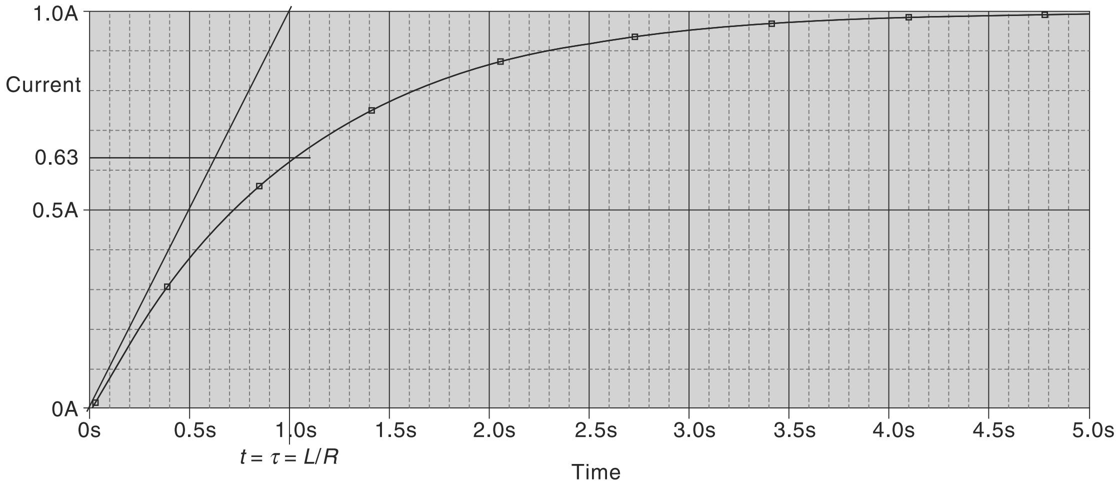


Figure 5.7 *Variation of current with time RL series circuit with step input*

Definitions of Time-constant (τ)

1. It is the time taken for the current to reach 63% of its final value. Thus, it is a measure of the rapidity with which the steady state is reached.
   Also, at $t = 5\tau$, $i = 0.993i_s$; the transient is therefore, said to be practically disappeared in five time constants.
2. The tangent to the equation $i = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ at $t = 0$, intersects the straight line, $i = \frac{V}{R}$ at $t = \tau = \frac{L}{R}$. Thus, time-constant is the time in which steady state would be reached if the current increases at the initial rate.

Physically, time-constant represents the speed of the response of a circuit. A low value of time-constant represents a fast response and a high value of time-constant represents a sluggish response.

Calculations of the Voltage Across Elements

Voltage across the resistor, $V_R = Ri(t) = V\left(1 - e^{-\frac{R}{L}t}\right)$

Voltage across the inductor, $V_L = L\frac{di(t)}{dt} = L\frac{d}{dt}\left[\frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right] = Ve^{-\frac{R}{L}t}$

Review

- A fully “discharged” inductor (no current through it) initially acts as an open circuit (voltage drop with no current) when faced with the sudden application of voltage. After “charging” fully to the final level of current, it acts as a short circuit (current with no voltage drop).
- In a resistor-inductor “charging” circuit, inductor current goes from nothing to full value while voltage goes from maximum to zero, both variables changing most rapidly at first, approaching their final values slower and slower as time goes on.

2. RL Series Circuit with Impulse Input By KVL, the mesh equation becomes,

$$Ri(t) + L\frac{di(t)}{dt} = V\delta(t)$$

Taking Laplace transform,

$$RI(s) + sLI(s) = V \text{ with } i(0-) = 0$$

or
$$I(s) = \frac{V}{L}\left(\frac{1}{s + R/L}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{L}e^{-\frac{R}{L}t}$$

Here, the plot of the current is shown in Figure 5.8.

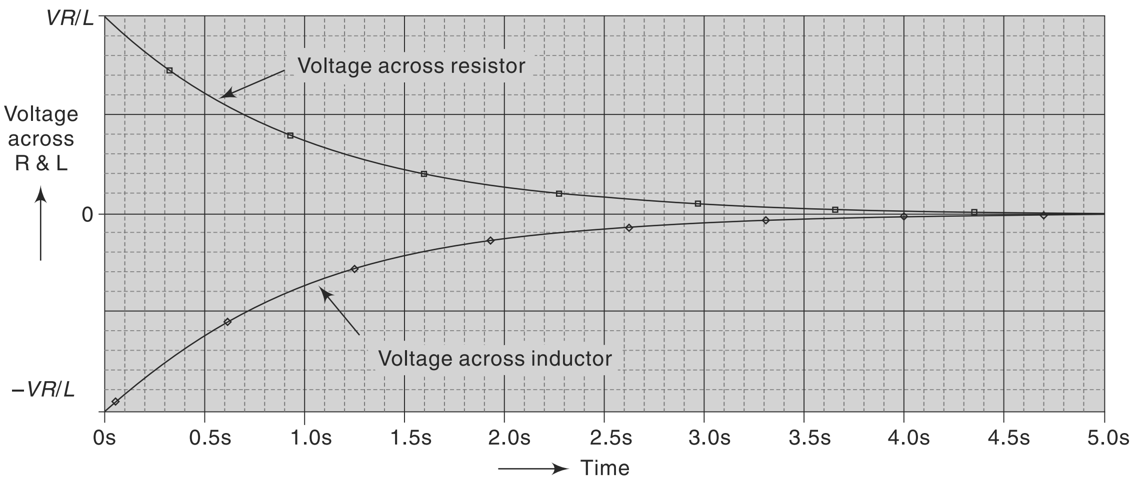


Figure 5.8 *Variation of voltages with time in RL series circuit with impulse input*

Voltage across the resistor, $V_R = Ri(t) = \frac{VR}{L}e^{-\frac{R}{L}t}$

Voltage across the inductor, $V_L = L\frac{di(t)}{dt} = L\frac{d}{dt}\left(\frac{V}{L}e^{-\frac{R}{L}t}\right) = -\frac{VR}{L}e^{-\frac{R}{L}t}$

3. RL Series Circuit with Sinusoidal Input Here, the input voltage is given as, $v(t) = V \sin \omega t$

By KVL,

$$Ri(t) + L\frac{di(t)}{dt} = V \sin \omega t, \text{ with } i(0-) = 0$$

or
$$I(s)[R + sL] = \frac{V\omega}{s^2 + \omega^2}$$

or $$I(s) = \frac{\frac{V\omega}{L}}{(s^2+\omega^2)\left(s+\frac{R}{L}\right)} = \frac{V\omega}{L}\left\{\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{R}{L}\right)}\right\}$$

$$= \frac{V\omega}{L}\left[\frac{A_1}{s-j\omega}+\frac{A_2}{s+j\omega}+\frac{A_3}{s+\frac{R}{L}}\right]$$

where, $$A_1 = \left\{(s-j\omega)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{R}{L}\right)}\right\}_{s=j\omega} = \frac{L}{2j\omega(R+j\omega L)}$$

$$A_2 = \left\{(s+j\omega)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{R}{L}\right)}\right\}_{s=-j\omega} = -\frac{L}{2j\omega(R-j\omega L)}$$

and $$A_3 = \left\{\left(s+\frac{R}{L}\right)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{R}{L}\right)}\right\}_{s=-\frac{R}{L}} = \frac{L^2}{(R^2+\omega^2L^2)}$$

$\therefore$ $$I(s) = \frac{V\omega}{L}\left[\frac{L}{2j\omega(R+j\omega L)(s-j\omega)}-\frac{L}{2j\omega(R-j\omega L)(s+j\omega)}+\frac{L^2}{(R^2+\omega^2L^2)\left(s+\frac{R}{L}\right)}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V\omega}{L}\left[\frac{Le^{j\omega t}}{2j\omega(R+j\omega L)}-\frac{Le^{-j\omega t}}{2j\omega(R-j\omega L)}+\frac{L^2e^{-\frac{R}{L}t}}{R^2+\omega^2L^2}\right]$$

$$= \frac{V}{2j}\left[\frac{e^{j\omega t}}{R+j\omega L}-\frac{e^{-j\omega t}}{R-j\omega L}\right]+V\omega L\frac{e^{-\frac{R}{L}t}}{R^2+\omega^2L^2}$$

Let, $(R+j\omega L)=Ze^{j\theta}$ and $(R-j\omega L)=Ze^{-j\theta}$ so that, $Z=\sqrt{(R^2+\omega^2L^2)}$ and $\theta=\tan^{-1}\left(\frac{\omega L}{R}\right)$

Putting these values,

$$i(t) = \frac{V}{2j}\left[\frac{e^{j\omega t}}{Ze^{j\theta}}-\frac{e^{-j\omega t}}{Ze^{-j\theta}}\right]+V\omega L\frac{e^{-\frac{R}{L}t}}{Z^2}$$

$$= \frac{V}{Z}\left[\frac{e^{j(\omega t-\theta)}-e^{-j(\omega t-\theta)}}{2j}\right]+\frac{V\omega L}{Z^2}e^{-\frac{R}{L}t}$$

or, finally, the current is,

$$\boxed{i(t) = \frac{V}{Z}\sin(\omega t - \theta) + \frac{V\omega L}{Z^2} e^{-\frac{R}{L}t}}$$

From this result, it is clear that the current in RL series circuit lags behind the voltage by an angle, $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$. If the resistance $R = 0$, then $\theta = 90°$ as is the case for a perfect inductor.

5.11.2 RC Series Circuit

1. RC Series Circuit with Step Input We consider an RC series circuit as shown in Figure 5.9.

By KVL, $Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt = Vu(t)$

Taking Laplace transform,

$$RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0-)}{s}\right] = \frac{V}{s}$$

or
$$I(s)\left[R + \frac{1}{Cs}\right] = \frac{V}{s} - \frac{q(0-)}{Cs}$$

or
$$I(s) = \frac{V - \frac{q(0-)}{C}}{s(R + 1/Cs)} = \frac{1}{R}\frac{V - \frac{q(0-)}{C}}{(s + 1/RC)}$$

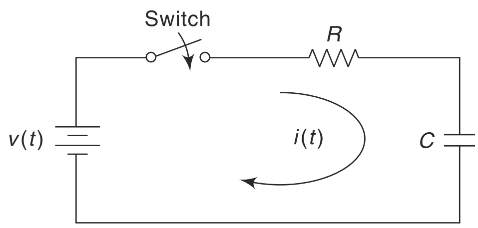


Figure 5.9 *RC series circuit*

Taking inverse Laplace transform,

$$i(t) = \left[\frac{V}{R} - \frac{q(0-)}{RC}\right]e^{-\frac{t}{RC}}\text{; for } t \geq 0$$

$$= \frac{V}{R}e^{-\frac{t}{RC}}\text{; if } q(0-) = 0$$

The steady state part of the current response, $i_s = 0$

The transient part of the current response, $i_{tr} = [i(t) - i_s] = \frac{V}{R}e^{-\frac{t}{RC}}$

From the current equation at $t = \tau = RC$, $i = \frac{V}{R}e^{-1} = 0.37\frac{V}{R}$

When the switch is first closed, the voltage across the capacitor (which we were told was fully discharged) is zero volt; thus, it first behaves as though it were a short-circuit. Over time, the capacitor voltage will rise to equal battery voltage, ending in a condition where the capacitor behaves as an open-circuit. Current through the circuit is determined by the difference in voltage between the battery and the capacitor, divided by the resistance. As the capacitor voltage approaches the battery voltage (V), the current approaches zero. Once the capacitor voltage has reached V, the current will be exactly zero.

The variation of current in the circuit is shown in Figure 5.10.

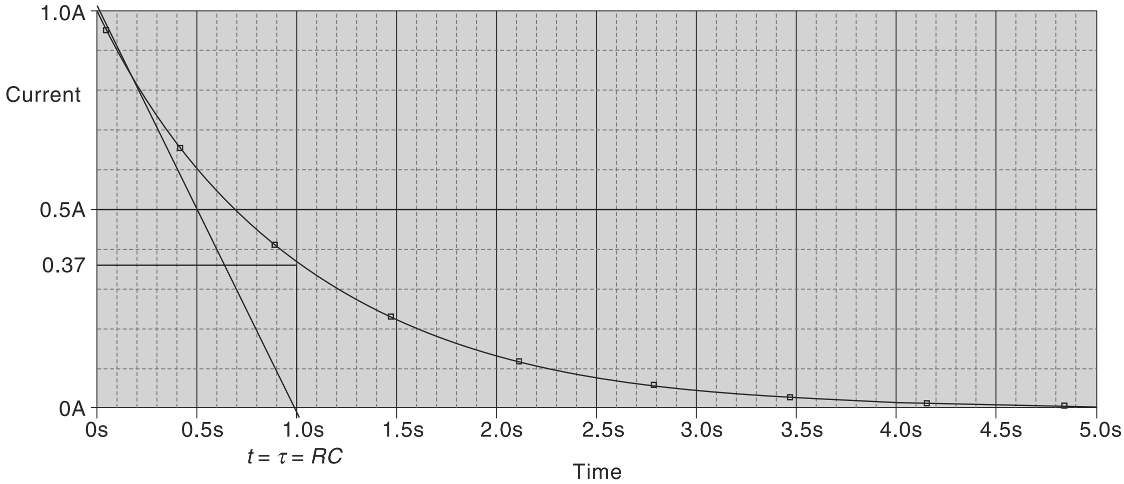


Figure 5.10 *Variation of current with time in RC series circuit with step input*

The quantity $\tau = RC$ is known as the Time-constant of the circuit and it is defined as follows.

Definitions of Time-constant (τ)

1. It is the time in which the current decays to 37% of its initial value.

   Also, at $t = 5\tau$, $i = 0.07\dfrac{V}{R}$; the transient is therefore, said to be practically disappeared in five time constants.

2. The tangent to the equation $i = \dfrac{V}{R}e^{-\frac{t}{RC}}$ at $t = 0$, intersects the time axis at $t = \tau = RC$.

Thus, time-constant is the time in which the current would reach the steady state zero value if the current decays at the initial rate.

Physically, time-constant represents the speed of the response of a circuit. A low value of time-constant represents a fast response and a high value of time-constant represents a sluggish response.

Calculations of the Voltage Across Elements

Voltage across the resistor, $V_R = Ri(t) = Ve^{-\frac{t}{RC}}$

Voltage across the capacitor, $V_C = \dfrac{1}{C}\int_0^t i(t)dt = \dfrac{1}{C}\int_0^t \dfrac{V}{R}e^{\frac{-t}{RC}}dt = V\left(1 - e^{\frac{-t}{RC}}\right)$

2. RC Series Circuit with Impulse Input With zero initial condition, $q(0–) = 0$, KVL equation becomes,

$$Ri(t) + \frac{1}{C}\int_0^t i(t)dt = V\delta(t)$$

$$RI(s) + \frac{I(s)}{Cs} = V$$

or, $$I(s)=\frac{V}{R+\frac{1}{Cs}}=\frac{V}{R}\left(\frac{s}{s+\frac{1}{RC}}\right)=\frac{V}{R}\left[1-\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right]$$

Taking inverse Laplace transform,

$$i(t)=\frac{V}{R}\left[\delta(t)-\frac{1}{RC}e^{\frac{-t}{RC}}\right];\text{ for } t\geq 0$$

Voltage across the resistor, $V_R=Ri(t)=V\left[\delta(t)-\frac{1}{RC}e^{\frac{-t}{RC}}\right]$

Voltage across the capacitor, $V_C=\{V\delta(t)-V_R\}=\frac{V}{RC}e^{\frac{-t}{RC}}$

These variations of the voltages are shown in Figure 5.11.

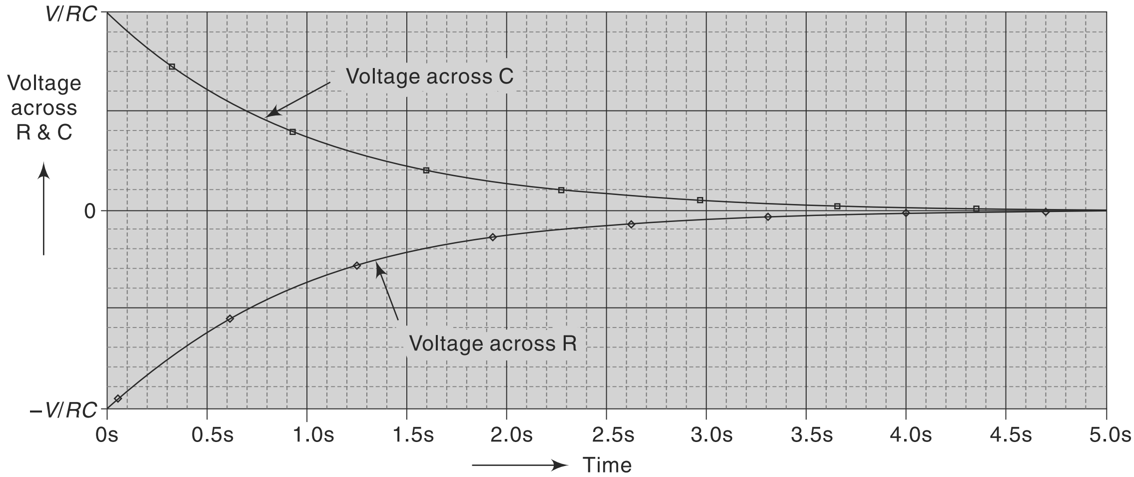


Figure 5.11 *Variation of voltages with time in RC series circuit with impulse input*

3. RC Series Circuit with Sinusoidal Input Here, the input voltage is given as, $v(t) = V\sin\omega t$

By KVL,

$$Ri(t)+\frac{1}{C}\int_0^t i(t)dt=V\sin\omega t\text{, with } q(0-)=0$$

or $$I(s)\left[R+\frac{1}{Cs}\right]=\frac{V\omega}{s^2+\omega^2}$$

or $$I(s)=\frac{V\omega Cs}{(s^2+\omega^2)(1+sRC)}=\frac{V\omega}{R}\left\{\frac{s}{(s+j\omega)(s-j\omega)\left(s+\frac{1}{RC}\right)}\right\}$$

$$=\frac{V\omega}{L}\left[\frac{A_1}{s-j\omega}+\frac{A_2}{s+j\omega}+\frac{A_3}{s+\frac{R}{L}}\right]$$

where, $$A_1 = \left\{(s-j\omega)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{1}{RC}\right)}\right\}_{s=j\omega} = \frac{RC}{2(1+j\omega RC)}$$

$$A_2 = \left\{(s+j\omega)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{1}{RC}\right)}\right\}_{s=-j\omega} = -\frac{RC}{2(1-j\omega RC)}$$

and $$A_3 = \left\{\left(s+\frac{1}{RC}\right)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{1}{RC}\right)}\right\}_{s=-\frac{1}{RC}} = \frac{-\frac{1}{RC}}{\left(\omega^2+\frac{1}{R^2C^2}\right)}$$

$\therefore$ $$I(s) = \frac{V\omega}{R}\left[\frac{RC}{2(1+j\omega RC)(s-j\omega)} - \frac{RC}{2(1-j\omega RC)(s+j\omega)} + \frac{-\frac{1}{RC}}{\left(\omega^2+\frac{1}{R^2C^2}\right)\left(s+\frac{1}{RC}\right)}\right]$$

Taking inverse Laplace transform,

$$i(t) = V\omega C\left[\frac{e^{j\omega t}}{2(1+j\omega RC)} - \frac{e^{-j\omega t}}{2(1-j\omega RC)}\right] - \frac{Ve^{\frac{-t}{RC}}}{\omega RC\left(R^2+\frac{1}{\omega^2C^2}\right)}$$

$$= \frac{V}{2j}\left[\frac{e^{j\omega t}}{R+\frac{1}{j\omega C}} - \frac{e^{-j\omega t}}{R-\frac{1}{j\omega C}}\right] - \frac{Ve^{\frac{-t}{RC}}}{\omega RC\left(R^2+\frac{1}{\omega^2C^2}\right)}$$

Let, $\left(R+\frac{1}{j\omega C}\right)=\left(R-\frac{j}{\omega C}\right)=Ze^{-j\theta}$ and $\left(R-\frac{1}{j\omega C}\right)=\left(R+\frac{j}{\omega C}\right)=Ze^{j\theta}$

so that, $Z=\sqrt{\left(R^2+\frac{1}{\omega^2C^2}\right)}$ and $\theta=\tan^{-1}\left(\frac{1}{\omega RC}\right)$

Putting these values,

$$i(t) = \frac{V}{2j}\left[\frac{e^{j\omega t}}{Ze^{-j\theta}} - \frac{e^{-j\omega t}}{Ze^{j\theta}}\right] - \frac{V}{\omega CZ^2}e^{\frac{-t}{RC}}$$

$$= \frac{V}{Z}\left[\frac{e^{j(\omega t+\theta)}-e^{-j(\omega t+\theta)}}{2j}\right] - \frac{V}{\omega CZ^2}e^{\frac{-t}{RC}}$$

or, finally, the current is,

$$\boxed{i(t) = \frac{V}{Z}\sin(\omega t + \theta) - \frac{V}{\omega C Z^2}e^{\frac{-t}{RC}}}$$

From this result, it is clear that the current in RC series circuit leads the voltage by an angle, $\theta = \tan^{-1}\left(\frac{1}{\omega RC}\right)$. If the resistance $R = 0$, then $\theta = 90°$ as is the case for a perfect capacitor.

5.11.3 RLC Series Circuit

1. RLC Series Circuit with Step Input With zero initial conditions, the Kirchhoff's voltage law equation becomes,

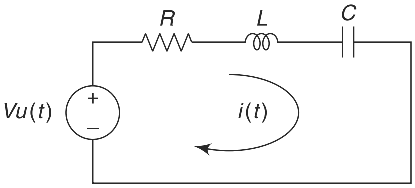


Figure 5.12 *RLC series circuit*

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt = Vu(t)$$

or $$RI(s) + sLI(s) + \frac{1}{Cs}I(s) = \frac{V}{s}$$

or $$I(s) = \frac{\frac{V}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{5.1}$$

The roots of the denominator polynomial of equation are,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

or $$s_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad \text{and,} \quad s_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Let $$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \xi\omega_0 = \frac{R}{2L} \quad \text{i.e.} \quad \xi = \frac{R}{2}\sqrt{\frac{C}{L}} = \text{Damping Ratio}$$

Then, $$s_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1} \quad \text{and} \quad s_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$$

So, $$I(s) = \frac{\frac{V}{L}}{(s - s_1)(s - s_2)} = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

$\therefore$ $$A = (s - s_1)\left.\frac{\frac{V}{L}}{(s - s_1)(s - s_2)}\right|_{s = s_1} = \frac{\frac{V}{L}}{(s_1 - s_2)} = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}$$

and, therefore $$B = (s - s_2)\left.\frac{\frac{V}{L}}{(s - s_1)(s - s_2)}\right|_{s = s_2} = \frac{\frac{V}{L}}{(s_2 - s_1)} = -\frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}$$

Putting these values of A and B, we get,

$$I(s) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}[e^{s_1 t} - e^{s_2 t}] = \frac{V}{2\,\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}]$$

Depending upon the values of R, L and C, three cases may appear:

(a) $\dfrac{R}{2L} > \dfrac{1}{\sqrt{LC}}$ (Overdamped condition)

(b) $\dfrac{R}{2L} < \dfrac{1}{\sqrt{LC}}$ (Underdamped condition)

(c) $\dfrac{R}{2L} = \dfrac{1}{\sqrt{LC}}$ (Critically Damped condition)

A. Overdamped Condition The condition is, $\dfrac{R}{2L} > \dfrac{1}{\sqrt{LC}}$ or, $\xi > 1$ or $Q < \dfrac{1}{2}$

$$\left(\text{Since, Quality Factor, } Q = \frac{\omega_0 L}{R} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}}\right)$$

Under this condition, the current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}] = \frac{V}{\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}\sinh(\omega_0\sqrt{\xi^2 - 1})t$$

The graphical plot for the current is shown in Figure 5.13.

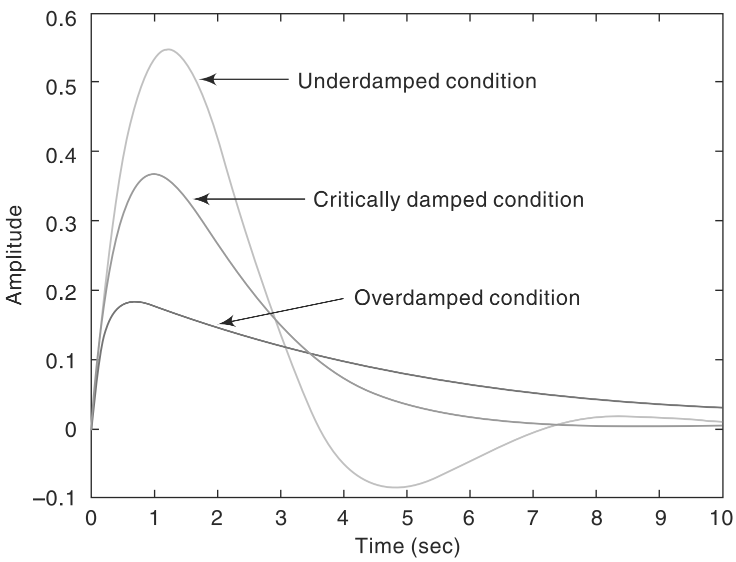


Figure 5.13 *Current response in RLC series circuit for three different damping conditions*

B. Critically Damped Condition The condition is, $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$ or, $\xi = 1$ or $Q = \frac{1}{2}$

From equation (5.1),

$$I(s) = \frac{\frac{V}{L}}{s^2 + 2\omega_0 s + \omega_0^2} = \frac{V}{L}\left(\frac{1}{(s+\omega_0)^2}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{L} t e^{-\omega_0 t}$$

The graphical plot for the current is shown in Figure 5.13.

C. Underdamped Condition The condition is, $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$ or, $\xi < 1$ or $Q > \frac{1}{2}$

So, the current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t} [e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}]$$

$$= \frac{V}{\omega_0 L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}\left[\frac{e^{\left(j\omega_0\sqrt{1-\xi^2}\right)t} - e^{-\left(j\omega_0\sqrt{1-\xi^2}\right)t}}{2j}\right]$$

$$= \frac{V}{\omega_0 L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t} \sin(\omega_0\sqrt{1-\xi^2})t$$

So, the circuit is oscillatory. When $R = 0$, $\xi = 0$, the oscillations are undamped or sustained. The frequency of the undamped oscillation (ω_0) is known as *undamped natural frequency*.

2. RLC Series Circuit with Impulse Input With zero initial conditions, the Kirchhoff's voltage law equation becomes,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt = V\delta(t)$$

or $$RI(s) + sLI(s) + \frac{1}{Cs}I(s) = V$$

or $$I(s) = \frac{\left(\frac{V}{L}\right)s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{5.2}$$

The roots of the denominator polynomial of equation are,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

or $\quad s_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ and, $s_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

Let $\quad \omega_0 = \frac{1}{\sqrt{LC}}$ and $\xi\omega_0 = \frac{R}{2L}$ i.e. $\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$ = Damping Ratio

Then, $\quad s_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1}$ and $s_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$

So, $\quad I(s) = \dfrac{\left(\frac{V}{L}\right)}{(s-s_1)(s-s_2)} = \dfrac{A}{s-s_1} + \dfrac{B}{s-s_2}$

$$\therefore \quad A = (s-s_1)\left.\frac{\left(\frac{V}{L}\right)s}{(s-s_1)(s-s_2)}\right|_{s=s_1} = \frac{\left(\frac{V}{L}\right)s_1}{(s_1-s_2)} = \frac{Vs_1}{2\omega_0 L\sqrt{\xi^2-1}}$$

and, therefore $\quad B = (s-s_2)\left.\dfrac{\left(\frac{V}{L}\right)s_2}{(s-s_1)(s-s_2)}\right|_{s=s_2} = \dfrac{\left(\frac{V}{L}\right)s_2}{(s_2-s_1)} = -\dfrac{Vs_2}{2\omega_0 L\sqrt{\xi^2-1}}$

Putting these values of A and B, we get,

$$I(s) = \frac{V}{2\omega_0 L\sqrt{\xi^2-1}}\left[\frac{s_1}{s-s_1} - \frac{s_2}{s-s_2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2-1}}[s_1 e^{s_1 t} - s_2 e^{s_2 t}] = \frac{V}{2\omega_0 L\sqrt{\xi^2-1}} e^{-\xi\omega_0 t}[s_1 e^{(\omega_0\sqrt{\xi^2-1})t} - s_2 e^{-(\omega_0\sqrt{\xi^2-1})t}]$$

$$= \frac{V}{2\omega_0 L\sqrt{\xi^2-1}} e^{-\xi\omega_0 t}[(-\xi\omega_0 + \omega_0\sqrt{\xi^2-1})e^{(\omega_0\sqrt{\xi^2-1})t}$$

$$- (-\xi\omega_0 - \omega_0\sqrt{\xi^2-1})e^{-(\omega_0\sqrt{\xi^2-1})t}]$$

Three cases are considered:

(A) $\dfrac{R}{2L} > \dfrac{1}{\sqrt{LC}}$ (Overdamped condition)

(B) $\dfrac{R}{2L} < \dfrac{1}{\sqrt{LC}}$ (Underdamped condition)

(C) $\dfrac{R}{2L} = \dfrac{1}{\sqrt{LC}}$ (Critically Damped condition)

A. Overdamped Condition Here, $\xi > 1$

The current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t} [(-\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1})e^{(\omega_0\sqrt{\xi^2 - 1})t}$$

$$- (-\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1})e^{-(\omega_0\sqrt{\xi^2 - 1})t}]$$

$$= \frac{V}{L\sqrt{\xi^2 - 1}} [\sqrt{\xi^2 - 1}\cosh(\omega_0\sqrt{\xi^2 - 1})t - \xi\sinh(\omega_0\sqrt{\xi^2 - 1})t]$$

B. Critically Damped Condition The condition is, $\xi = 1$

From equation (5.2),

$$I(s) = \frac{(V/L)s}{s^2 + 2\omega_0 s + \omega_0^2} = \frac{V}{L}\left(\frac{s}{(s+\omega_0)^2}\right) = \frac{V}{L}\left[\frac{A}{(s+\omega_0)^2} + \frac{B}{s+\omega_0}\right]$$

where $$A = (s+\omega_0)^2 \left.\frac{s}{(s+\omega_0)^2}\right|_{s=-\omega_0} = -\omega_0$$

and $$B = \frac{d}{ds}\left[(s+\omega_0)^2 \frac{s}{(s+\omega_0)^2}\right]_{s=-\omega_0} = 1$$

So, $$I(s) = \frac{V}{L}\left[\frac{1}{s+\omega_0} - \frac{\omega_0}{(s+\omega_0)^2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{L}[1 - \omega_0 t]e^{-\omega_0 t}$$

C. Underdamped Condition The condition is, $\xi < 1$

So, the current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[(\omega_0\sqrt{\xi^2 - 1})\{e^{(\omega_0\sqrt{\xi^2-1})t} + e^{-(\omega_0\sqrt{\xi^2-1})t}\} - \xi\omega_0\{e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}\}]$$

$$= \frac{V}{2\omega_0 Lj\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}[(j\omega_0\sqrt{1-\xi^2})\{e^{(j\omega_0\sqrt{1-\xi^2})t} + e^{-(j\omega_0\sqrt{1-\xi^2})t}\} - \xi\omega_0\{e^{(j\omega_0\sqrt{1-\xi^2})t} - e^{-(j\omega_0\sqrt{1-\xi^2})t}\}]$$

$$i(t) = \frac{V}{L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}[\sqrt{1-\xi^2}\cos(\omega_0\sqrt{1-\xi^2})t - \xi\sin(\omega_0\sqrt{1-\xi^2})t]$$

$$= \frac{V}{L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}\cos\{(\omega_0\sqrt{1-\xi^2})t + \theta\}, \text{ where } \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

3. RLC Series Circuit with Sinusoidal Input Sinusoidal voltage $v(t) = V_m \sin(\omega t + \theta)$ is applied to a series RLC circuit at time $t = 0$. We want to find the complete solution for the current $i(t)$ using Laplace transform method.

$$v(t) = V_m \sin(\omega t + \theta)$$

By KVL,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t)dt = V_m \sin(\omega t + \theta)$$

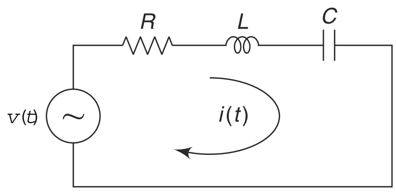


Figure 5.14 *RLC series circuit with sinusoidal input*

Taking Laplace transform with zero initial conditions,

$$I(s)\left[R + sL + \frac{1}{Cs}\right] = V_m\frac{(s\sin\theta + \omega\cos\theta)}{s^2 + \omega^2}$$

or $$I(s) = \frac{V_m s(s\sin\theta + \omega\cos\theta)}{L(s^2 + \omega^2)\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$$= \frac{V_m}{L}\frac{s(s\sin\theta + \omega\cos\theta)}{(s + j\omega)(s + j\omega)(s - s_1)(s - s_2)}$$

where, s_1, s_2 are the roots of the quadratic equation:

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

Thus, $s_1 = -\dfrac{R}{2L} + \sqrt{\dfrac{R^2}{4L^2} - \dfrac{1}{LC}}$ and, $s_2 = -\dfrac{R}{2L} - \sqrt{\dfrac{R^2}{4L^2} - \dfrac{1}{LC}}$

Now, let $\dfrac{s(s\sin\theta + \omega\cos\theta)}{(s+j\omega)(s-j\omega)(s-s_1)(s-s_2)} = \dfrac{K_1}{s-s_1} + \dfrac{K_2}{s-s_2} + \dfrac{K_3}{s+j\omega} + \dfrac{K_4}{s-j\omega}$

So, by residue method, multiplying by $(s - s_1)$ and putting $s = s_1$,

$$K_1 = \frac{s_1(s_1\sin\theta + \omega\cos\theta)}{(s_1+j\omega)(s_1-j\omega)(s_1-s_2)} \quad \text{and} \quad K_2 = \frac{s_2(s_2\sin\theta + \omega\cos\theta)}{(s_2+j\omega)(s_2-j\omega)(s_2-s_1)}$$

Similarly, multiplying by $(s + j\omega)$ and putting $s = -j\omega$,

$$K_3 = \frac{-j\omega(-j\omega\sin\theta + \omega\cos\theta)}{(-j\omega-j\omega)(-j\omega-s_1)(-j\omega-s_2)} = \frac{\omega(\cos\theta - j\sin\theta)}{2(s_1+j\omega)(s_2+j\omega)}$$

and, $$K_4 = \frac{j\omega(-\omega\sin\theta + \omega\cos\theta)}{(j\omega+j\omega)(j\omega-s_1)(j\omega-s_2)} = \frac{\omega(\cos\theta + j\sin\theta)}{2(s_1-j\omega)(s_2-j\omega)}$$

Hence the current response becomes,

$$i(t) = \frac{V_m}{L}[K_1e^{s_1t} + K_2e^{s_2t}] + \frac{V}{L}[K_3e^{-j\omega t} + K_4e^{j\omega t}] = I_{tr} + I_{ss}$$

Thus, the *transient part of the total current* is,

$$\boxed{I_{tr} = \frac{V_m}{L}\left[\frac{s_1(s_1\sin\theta + \omega\cos\theta)}{(s_2^2+\omega^2)\sqrt{\dfrac{R^2}{L^2} - \dfrac{4}{LC}}}e^{s_1t} - \frac{s_2(s_2\sin\theta + \omega\cos\theta)}{(s_2^2+\omega^2)\sqrt{\dfrac{R^2}{L^2} - \dfrac{4}{LC}}}e^{s_2t}\right]}$$

The steady-state part of the total current is obtained as follows.

$$I_{ss} = \frac{V_m}{2L}\left[\frac{\omega e^{-j\theta}e^{-j\omega t}}{(s_1+j\omega)(s_2+j\omega)} + \frac{\omega e^{j\theta}e^{j\omega t}}{(s_1-j\omega)(s_2-j\omega)}\right]$$

$$= \frac{V_m\omega}{2L}\left[\frac{e^{-j(\omega t+\theta)}}{(s_1+j\omega)(s_2+j\omega)} + \frac{e^{j(\omega t+\theta)}}{(s_1-j\omega)(s_2-j\omega)}\right]$$

$$I_{ss} = \frac{V_m\omega}{2L(s_1^2+\omega^2)(s_2^2+\omega^2)}[e^{-j(\omega t+\theta)}(s_1s_2 - \omega^2 - j\omega s_1 - j\omega s_2)]$$

or $$= \frac{V_m\omega}{2L(s_1^2+\omega^2)(s_1^2+\omega^2)}[(s_1s_2-\omega^2)\,2\cos(\omega t+\theta) - (\omega s_1 + \omega s_2)\,2\sin(\omega t+\theta)]$$

$$= \frac{V_m\omega}{L}\frac{1}{(s_1^2+\omega^2)(s_2^2+\omega^2)}\left[\left(\frac{1}{LC} - \omega^2\right)\cos(\omega t+\theta) - \left(-\frac{\omega R}{L}\right)\sin(\omega t+\theta)\right]$$

or $$I_{ss} = \frac{V_m\omega}{L}\frac{\left[\frac{\omega R}{L}\sin(\omega t+\theta)-\left(\omega^2-\frac{1}{LC}\right)\cos(\omega t+\theta)\right]}{(s_1^2+\omega^2)(s_2^2+\omega^2)}$$

$$= \frac{V_m\omega}{L}\frac{1}{(s_1^2+\omega^2)(s_2^2+\omega^2)}\sin\left\{\omega t+\theta-\tan^{-1}\left(\frac{\omega L-\frac{1}{\omega C}}{R}\right)\right\}$$

$$\times\frac{\omega}{L}\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$$

or, $$\boxed{I_{ss} = \frac{V_m}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}\sin\left\{\omega t+\theta-\tan^{-1}\left(\frac{\omega L-\frac{1}{\omega C}}{R}\right)\right\}}$$

This gives the steady-state current of the series RLC circuit to a sinusoidal voltage.

5.11.4 RLC Parallel Circuit

1. RLC Parallel Circuit with Step Current Input With zero initial conditions, the Kirchhoff's current law equation becomes,

$$\frac{v(t)}{R}+C\frac{dv(t)}{dt}+\frac{1}{L}\int_0^t v(t)dt = Iu(t)$$

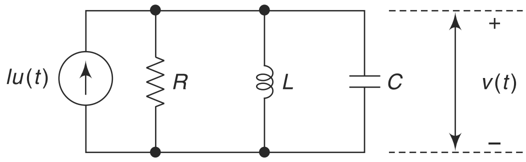


Figure 5.15 *RLC parallel circuit*

or $$\frac{V(s)}{R}+sCV(s)+\frac{1}{sL}V(s)=\frac{I}{s}$$

or $$V(s) = \frac{I/C}{s^2+\frac{1}{RC}s+\frac{1}{LC}} \tag{5.3}$$

The roots of the denominator polynomial of equation are,

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

or $$s_1 = -\frac{1}{2RC}+\sqrt{\frac{1}{4R^2C^2}-\frac{1}{LC}} \quad \text{and,} \quad s_2 = -\frac{1}{2RC}-\sqrt{\frac{1}{4R^2C^2}-\frac{1}{LC}}$$

Let $$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \xi\omega_0 = \frac{1}{2RC} \quad \text{i.e.} \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}} = \text{Damping Ratio}$$

Then, $s_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1}$ and $s_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$

So, $V(s) = \dfrac{I/C}{(s - s_1)(s - s_2)} = \dfrac{A}{s - s_1} + \dfrac{B}{s - s_2}$

$\therefore$ $A = (s - s_1)\left.\dfrac{I/C}{(s - s_1)(s - s_2)}\right|_{s = s_1} = \dfrac{I/C}{(s_1 - s_2)} = \dfrac{I}{2\omega_0 C\sqrt{\xi^2 - 1}}$

and, therefore, $B = (s - s_2)\left.\dfrac{I/C}{(s - s_2)(s - s_2)}\right|_{s = s_2} = \dfrac{I/C}{(s_2 - s_1)} = -\dfrac{I}{2\omega_0 C\sqrt{\xi^2 - 1}}$

Putting these values of A and B, we get,

$$V(s) = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking inverse Laplace transform,

$$v(t) = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}}[e^{s_1 t} - e^{s_2 t}] = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2 - 1})t} - e^{-(\omega_0\sqrt{\xi^2 - 1})t}]$$

Depending upon the values of R, L and C, three cases may appear:

(a) $\dfrac{1}{2RC} > \dfrac{1}{\sqrt{LC}}$ (Overdamped condition)

(b) $\dfrac{1}{2RC} < \dfrac{1}{\sqrt{LC}}$ (Underdamped condition)

(c) $\dfrac{1}{2RC} = \dfrac{1}{\sqrt{LC}}$ (Critically Damped condition)

A. Overdamped Condition The condition is, $\dfrac{1}{2RC} > \dfrac{1}{\sqrt{LC}}$ or, $\xi > 1$ or $Q < \dfrac{1}{2}$

$\left(\text{Since, Quality Factor, } Q = \dfrac{1}{\omega_0 RC} \text{ and } \omega_0 = \dfrac{1}{\sqrt{LC}}\right)$

Under this condition, the current becomes,

$$v(t) = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2 - 1})t} - e^{-(\omega_0\sqrt{\xi^2 - 1})t}] = \frac{I}{\omega_0 C\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}\sinh\left(\omega_0\sqrt{\xi^2 - 1}\right)t$$

The graphical plot for the voltage is shown in Figure 5.16.

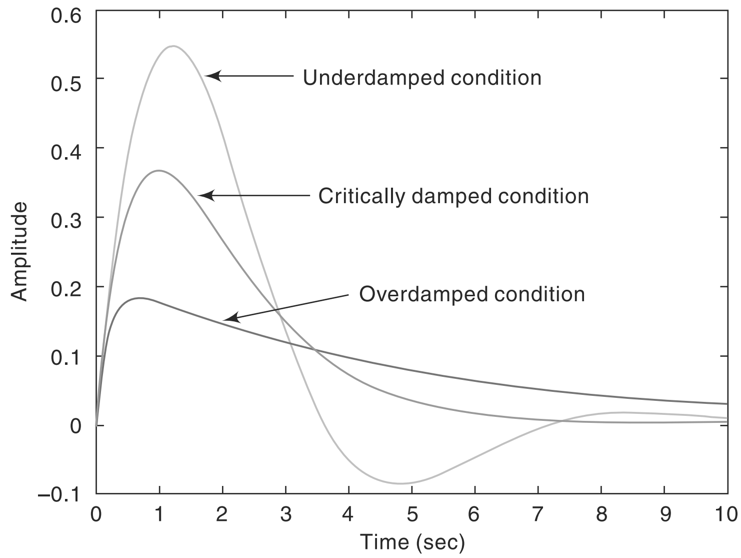


Figure 5.16 *Voltage response in RLC parallel circuit for three different damping conditions*

B. Critically Damped Condition The condition is, $\frac{1}{2RC} = \frac{1}{\sqrt{LC}}$ or, $\xi = 1$ or $Q = \frac{1}{2}$

From equation (5.3),

$$V(s) = \frac{I/C}{s^2 + 2\omega_0 s + \omega_0^2} = \frac{I}{C}\left(\frac{1}{(s+\omega_0)^2}\right)$$

Taking inverse Laplace transform,

$$v(t) = \frac{I}{C} te^{-\omega_0 t}$$

The graphical plot for the voltage is shown in Fig. 5.16.

C. Underdamped Condition The condition is, $\frac{1}{2RC} < \frac{1}{\sqrt{LC}}$ or, $\xi < 1$ or $Q > \frac{1}{2}$

So, the voltage becomes,

$$v(t) = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t} [e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}]$$

$$= \frac{I}{\omega_0 C\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}\left[\frac{e^{(j\omega_0\sqrt{1-\xi^2})t} - e^{-(j\omega_0\sqrt{1-\xi^2})t}}{2j}\right]$$

$$= \frac{I}{\omega_0 C\sqrt{1-\xi^2}} e^{-\xi\omega_0 t} \sin(\omega_0\sqrt{1-\xi^2})t$$

Similarly we can find out the impulse response and sinusoidal response of a parallel RLC circuit using Laplace transform method as for the series RLC circuit.

5.11.5 Response with Pulse Input Voltage

1. RC Series Circuit If a voltage pulse of width as shown in Fig. 5.17 is applied to an RC series circuit, then by KVL,

$$Ri(t) + \frac{1}{C}\int i(t)dt = v(t)$$

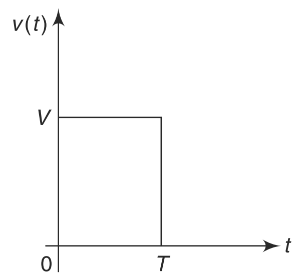


Figure 5.17 *Pulse voltage*

Taking Laplace transform with zero initial condition,

$$RI(s) + \frac{1}{Cs}I(s) = \frac{V}{s} - \frac{Ve^{-sT}}{s}$$

or $$I(s) = \frac{V}{R}\frac{1-e^{-sT}}{s+1/RC}$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R}[e^{-t/RC} - e^{-(t-T)/RC}]$$

Hence the voltage across the resistance is given as,

$$v_R(t) = Ri(t) = V[e^{-t/RC} - e^{-(t-T)/RC}]$$

and the voltage across the capacitor is given as,

$$v_c(t) = V - v_R(t) = V[e^{-t/RC} + e^{-(t-T)/RC}]$$

To plot the two voltages with varying time, we have the following observations:

(i) At $t = 0$, all the voltage appears across the resistance R and thus,

$$v_R = V \quad \text{and} \quad v_C = 0$$

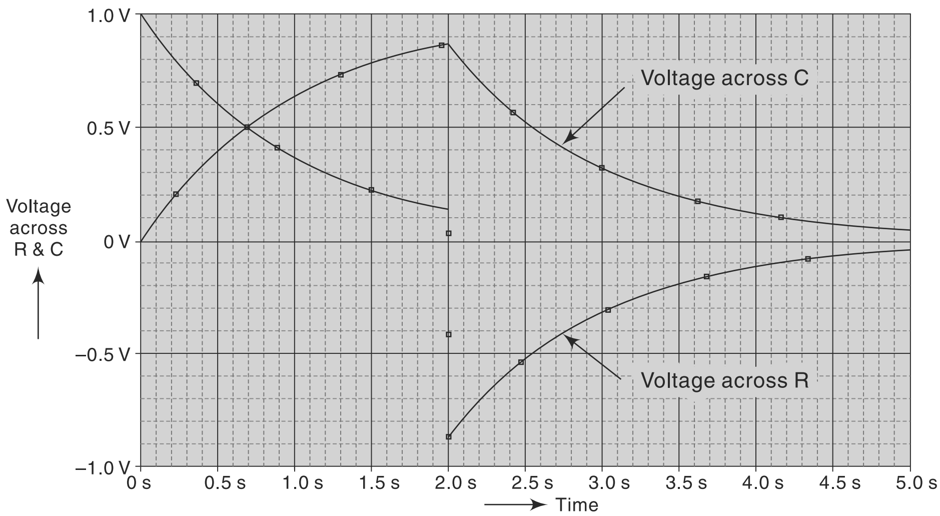


Figure 5.18 *Voltage response of RC series circuit with pulse input*

(ii) As the time increases, the voltage v_C grows and the voltage v_R decays exponentially, with time-constant $\tau = RC$.

(iii) At $t = T$, voltage across the network drops abruptly to zero from V. Again this entire drop is instantaneously felt across the resistance R.

(iv) For time $t > T$, total voltage across the circuit is zero. So, at any instant of time t, $v_R(t) + v_c(t) = 0$ and both v_R and v_C asymptotically approach zero.

Case (1) If Time-constant (τ = RC) << Pulse-width (T) The voltage across the resistance v_R will consist of two trigger pulses one positive and the other negative, of height V at the points where the voltage across the network changes abruptly (i.e., $t = 0$ and T).

In this case, the voltage across capacitor attains the steady state very quickly, i.e. $v_c = V$.

$$\therefore \qquad v_R = Ri = RC\frac{dv_C}{dt} \approx RC\frac{dV}{dt} \quad \text{or,} \quad \boxed{v_R = RC\frac{dV}{dt}}$$

Thus, the voltage v_R is the differentiation of the input voltage and hence the circuit acts as a *Differentiator*.

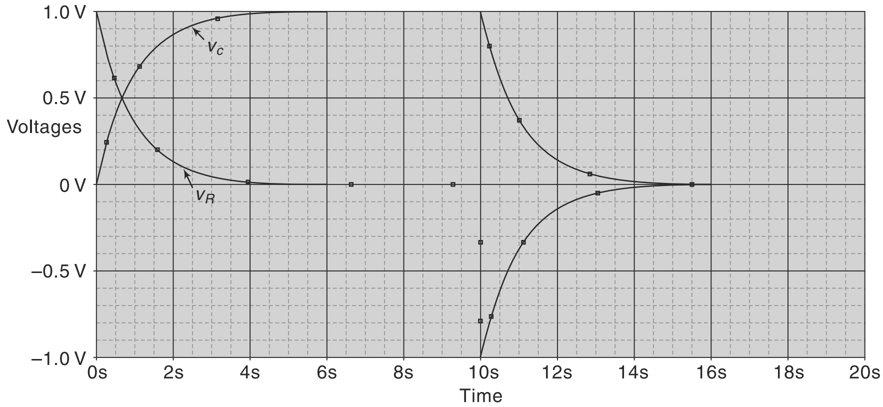


Figure 5.19 *Voltage response of RC series circuit (RC << T) with pulse input*

Case (2) If Time-constant (τ = RC) >> Pulse-width (T) In this case, the voltage across the capacitor varies with time almost linearly and the value is far from the steady state value V; i.e. $v_R = V$.

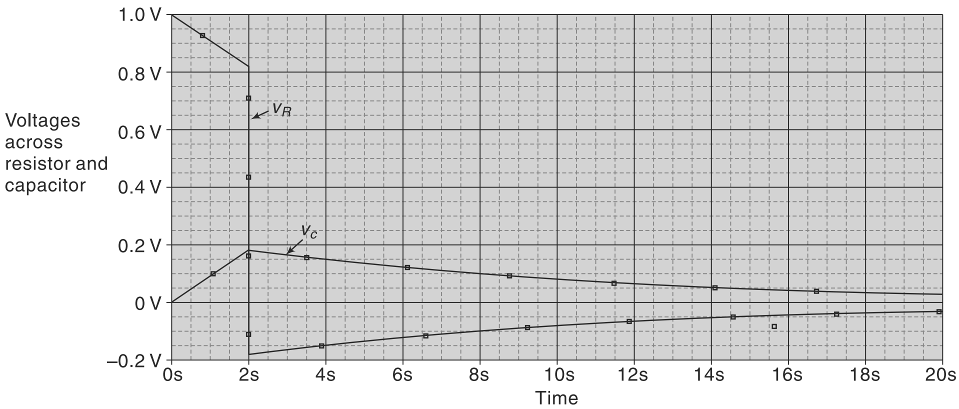


Figure 5.20 *Voltage response of RC series circuit (RC >> T) with pulse input*

$\therefore \qquad v_C = \dfrac{1}{C}\int_0^t i dt = \dfrac{1}{C}\int_0^t \dfrac{v_R}{R}\, dt \approx \dfrac{1}{RC}\int_0^t V dt$ or, $\boxed{\therefore\ v_C \approx \dfrac{1}{RC}\int_0^t V dt}$

Thus, the voltage v_C is the integration of the input voltage and hence the circuit acts as an *Integrator*.

2. RL Series Circuit If a similar pulse voltage is applied to an RL series circuit, then the KVL equation will be,

$$Ri(t) + L\frac{di}{dt} = v(t)$$

Taking Laplace transform with zero initial condition,

$$RI(s) + sLI(s) = \frac{V}{s} - \frac{Ve^{-sT}}{s}$$

or
$$I(s) = \frac{V}{L}\left[\frac{1}{s(s+R/L)} - \frac{e^{-sT}}{s(s+R/L)}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R}\left[\left(1-e^{-\frac{R}{L}t}\right)u(t) - \left(1-e^{-\frac{R}{L}(t-T)}\right)u(t-T)\right]$$

The variation of the two voltages is shown in figure.

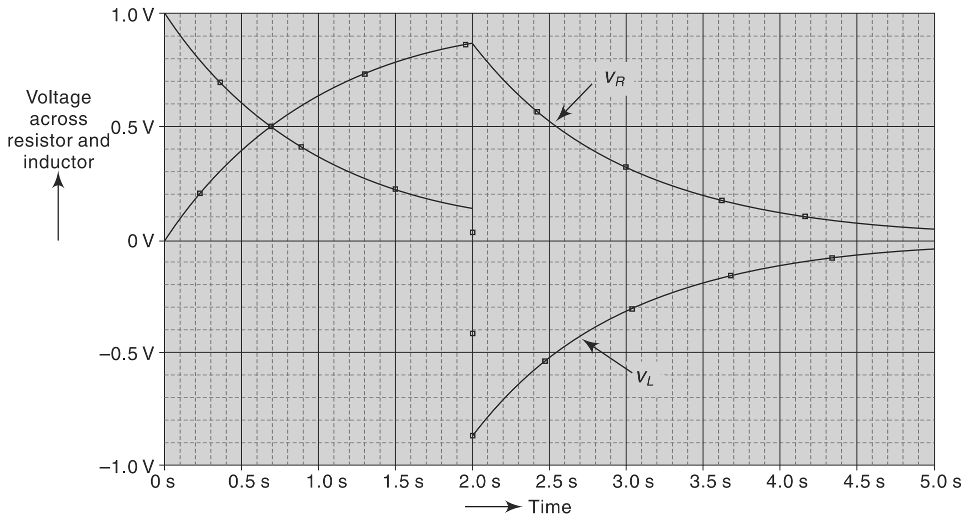


Figure 5.21 *Voltage response of RL series circuit with pulse input*

Case (1) If Time-constant (τ = L/R) << Pulse-width (T) In this case, the voltage across resistor attains the steady state very quickly, i.e. $v_R = V$.

$\therefore \qquad v_L = L\dfrac{di}{dt} = L\dfrac{d}{dt}\left(\dfrac{v_R}{R}\right) = L\dfrac{d}{dt}\left(\dfrac{V}{R}\right) \approx \dfrac{L}{R}\dfrac{dV}{dt}$ or, $\boxed{v_L = \dfrac{L}{R}\dfrac{dV}{dt}}$

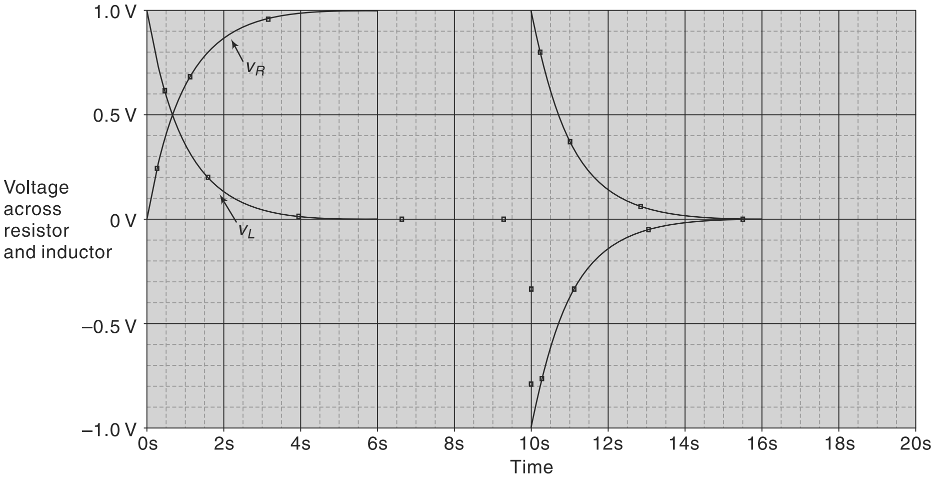


Figure 5.22 *Voltage response of RL series circuit ($L/R << T$) with pulse input*

Thus, the voltage v_L is the differentiation of the input voltage and hence the circuit acts as a *Differentiator.*

Case (2) If Time-constant ($\tau = L/R$) >> Pulse-width (T) In this case, the voltage across the resistor varies with time almost linearly and the value is far from the steady state value V; i.e. $v_L = V$.

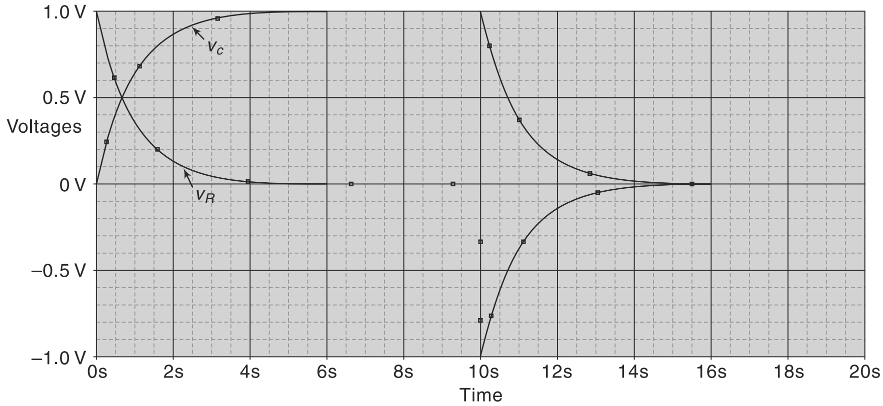


Figure 5.23 *Voltage response of RL series circuit ($L/R >> T$) with pulse input*

$$\therefore \qquad v_R = Ri = R\frac{1}{L}\int_0^t v_L dt \approx \frac{R}{L}\int_0^t V dt \quad \text{or,} \quad \boxed{\therefore v_R \approx \frac{R}{L}\int_0^t V dt}$$

Thus, the voltage v_R is the integration of the input voltage and hence the circuit acts as an *Integrator*.

5.12 STEPS FOR CIRCUIT ANALYSIS USING LAPLACE TRANSFORM METHOD

1. All circuit elements are transformed from time-domain to Laplace domain with initial conditions.
2. Excitation function is transformed into Laplace domain.
3. The circuit is solved using different circuit analysis techniques, such as, mesh analysis, node analysis, etc.
4. Time domain solution is obtained by taking inverse Laplace transform of the solution.

5.13 CONCEPT OF CONVOLUTION THEOREM

5.13.1 Convolution Integral

If $h(t)$ is the impulse response of a linear network, then the response of the same network $y(t)$ subject to any arbitrary input $w(t)$ is given by the convolution integral as,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)w(t-\tau)d\tau = \int_{-\infty}^{\infty} w(\tau)h(t-\tau)d\tau$$

Thus, if the impulse response of any linear time-invariant system is known, we can obtain the zero-state response of the system to any other type of input.

5.13.2 Convolution Theorem

If $f_1(t)$ and $f_2(t)$ are two functions of time which are zero for $t < 0$, and if their Laplace transforms are $F_1(s)$ and $F_2(s)$, respectively, then the convolution theorem states that the Laplace transform of the convolution of $f_1(t)$ and $f_2(t)$ is given by the product $F_1(s)\ F_2(s)$.

Mathematically, if the convolution of $f_1(t)$ and $f_2(t)$ is written as,

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau = f_2(t) * f_1(t)$$

where, τ is a dummy variable for time t, then the convolution theorem is written as,

$$L[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$

Proof By the definition of convolution,

$$L[f_1(t) * f_2(t)] = L\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right] = \int_0^{\infty}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right]e^{-st}dt \tag{i}$$

Also, by the definition of a shifted unit step function, using dummy variable,

$$u(t-\tau) = 1; \quad \text{for } \tau \le t$$
$$= 0; \quad \text{for } \tau > t$$

$$\therefore \qquad \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau = \int_0^{\infty} f_1(t-\tau)\,u(t-\tau)f_2(\tau)\,d\tau$$

Putting this in (i), we get,

$$L[f_1(t) * f_2(t)] = \int_0^{\infty}\left[\int_0^{\infty} f_1(t-\tau)\,u(t-\tau) f_2(\tau) d\tau\right] e^{-st} dt \tag{ii}$$

Now, let $(t - \tau) = x \therefore dt = dx$,

t	0	$\propto$
x	$-\tau$	$\propto$

From equation (ii), we get,

$$L[f_1(t) * f_2(t)] = \int_0^{\infty}\left[\int_{-\tau}^{\infty} f_1(x)\,u(x) f_2(\tau)\,d\tau\right] e^{-s(x+\tau)}\,dx$$

$$= \int_{-\tau}^{\infty} f_1(x)\,u(x) f_2(\tau)\,e^{-sx} dx \int_0^{\infty} f_2(\tau) d\tau e^{-s\tau}\,d\tau$$

$$= \int_0^{\infty} f_1(x)\,e^{-sx}\,dx \int_0^{\infty} f_2(\tau)\,e^{-s\tau}\,d\tau \quad \{\because u(x) = 0 \quad \text{for } x < 0\}$$

$$\boxed{\therefore L[[f_1(t) * f_2(t)]] = F_1(s) F_2(s)}$$

Thus, the convolution in time domain becomes multiplication in the frequency domain, and vise-versa.

5.13.3 Application of Convolution Theorem

The convolution theorem is used to find the response of a linear system to any arbitrary excitation if the impulse response of the system is known.

We know that the transfer function is defined as the ratio of response transform to excitation transform with zero initial conditions. Thus,

$$\text{Transfer Function} = \left.\frac{\text{Laplace transform of Response}}{\text{Laplace transform of Excitation}}\right|_{\text{all initial conditions reduced to zero}}$$

or $$H(s) = \left.\frac{Y(s)}{W(s)}\right|_{IC=0}$$

Thus, $$Y(s) = H(s)W(s)$$

Here, $W(s) = L[w(t)]$, is the input Laplace transform and $Y(s) = L[y(t)]$, is the output Laplace transform.

Now, if the input is an impulse function, then $w(t) = \delta(t)$ or $W(s) = 1$

$\therefore$ $$Y(s) = H(s)W(s) = H(s)$$

Taking inverse Laplace transform,

$$y(t) = h(t)$$

Thus, $h(t)$ is the impulse response of the system. If this impulse response of the system is known, we can find out the response of the system due to any arbitrary input *w(t)* from the following relation:

$$Y(s) = H(s)W(s)$$

or
$$y(t) = h(t) * w(t) = \int_0^t h(\tau)w(t-\tau)d\tau = \int_0^t h(t-\tau)w(\tau)d\tau$$

Example 5.9 Find the convolution integral when $f_1(t) = e^{-at}$ and $f_2(t) = t$.

Solution Here, the convolution integral is given as,

$$f_1(t) * f_2(t) = \int_0^t e^{-a(t-\tau)}\tau d\tau = e^{-at}\int_0^t \tau e^{a\tau} d\tau$$

$$= e^{-at}\left[\frac{\tau e^{a\tau}}{a} - \int 1\cdot\frac{e^{a\tau}}{a}d\tau\right]_0^t$$

$$= e^{-at}\left[\frac{\tau e^{a\tau}}{a} - \frac{\tau e^{a\tau}}{a^2}\right]_0^t$$

$$= e^{-at}\left[\frac{te^{at}}{a} - \frac{e^{at}}{a^2} + \frac{1}{a^2}\right]$$

$$= \frac{1}{a^2}[at - 1 + e^{-at}] \qquad \textit{Ans.}$$

SOLVED PROBLEMS

5.1 (a) Find the initial value of the function whose Laplace Transform is,

$$V(s) = A\cdot\frac{(s+a)\sin\theta + b\cos\theta}{(s+a)^2 + b^2}$$

Check the result by solving it for $v(t)$.

(b) Find the final value of the function whose Laplace Transform is, $I(s) = \dfrac{s+6}{s(s+3)}$

Solution

(a) By initial value theorem,

$$V(0+) = \lim_{s\to\infty} sV(s)$$

$$= \lim_{s\to\infty} SA\frac{(s+a)\sin\theta + b\cos\theta}{(s+a)^2 + b^2}$$

$$= \lim_{s\to\infty} A \frac{\left(1+\frac{a}{s}\right)\sin\theta + \frac{b}{s}\cos\theta}{\left(1+\frac{a}{s}\right)^2 + \left(\frac{b}{s}\right)^2}$$

$= A \sin\theta$ *Ans.*

In order to check this result, we find *v(t)* and then put $t = 0$.

$$v(t) = L^{-1}\left[A\frac{(s+a)\sin\theta + b\cos\theta}{(s+a)^2+b^2}\right]$$

$$= AL^{-1}\left[\frac{(s+a)\sin\theta}{(s+a)^2+b^2} + \frac{b\cos\theta}{(s+a)^2+b^2}\right]$$

$$= A[\sin\theta\, e^{-at}\cos bt + \cos\theta\, e^{-at}\sin bt]$$

$$= Ae^{-at}\sin(bt+\theta)$$

At $t = 0$, $v(0+) = Ae^0 \sin(0+\theta) = A\sin\theta$ [Checked]

(b) By final value theorem,

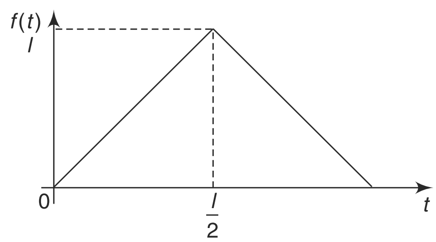
$$I(\infty) = \lim_{s\to 0} sI(s) = \lim_{s\to 0} s\frac{s+6}{s(s+3)} = \lim_{s\to 0}\frac{s+6}{(s+3)} = 2 \qquad Ans.$$

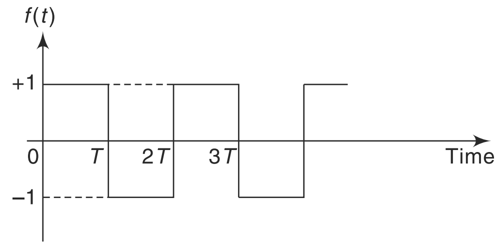
For checking it, $i(t) = L^{-1}\left[\frac{s+6}{s(s+3)}\right] = L^{-1}\left[\frac{2}{s} - \frac{1}{s+3}\right] = 2 - e^{-3t}$

At $t = \infty$, $i(\infty) = 2 - e^{-\infty} = 2$ [Checked]

5.2 (a) Obtain the Laplace Transform of square wave of unit amplitude and periodic time $2T$, as shown.

(b) Find the Laplace Transform of the following function:





Solution

(a) The equation of the square wave is,

$$f(t) = u(t) - u(t-T) - u(t-T) + u(t-2T) + u(t-2T) - u(t-3T) - \ldots$$

$$= u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + \ldots$$

Taking Laplace transform,

$$F(s) = \frac{1}{s} - \frac{2e^{-Ts}}{s} + \frac{2e^{-2Ts}}{s} - \frac{2e^{-3Ts}}{s} + \ldots$$

$$= \frac{1}{s}[1 - 2e^{-Ts}(1 - e^{-Ts} + e^{-2Ts} - e^{-3Ts} + \ldots)]$$

$$= \frac{1}{s}\left[1 - \frac{2e^{-Ts}}{1+e^{-Ts}}\right] \qquad \left\{\because \text{sum of G.P. series} = \frac{1}{1-e^{-Ts}}\right\}$$

$$= \frac{1}{s}\left[\frac{1-e^{-Ts}}{1+e^{-Ts}}\right]$$

$$F(s) = \frac{1}{s}\tanh\left(\frac{Ts}{2}\right) \qquad \textit{Ans.}$$

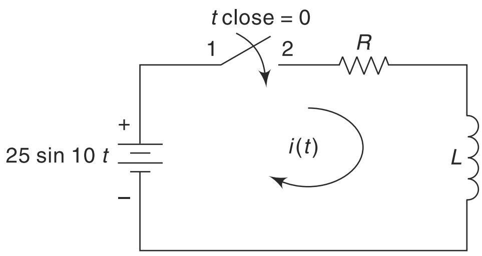
(b) The equation can be written as,

$$f(t) = 2r(t) - 4r\left(t - \frac{1}{2}\right) + 2r(t-1)$$

Taking Laplace transform,

$$F(s) = 2\frac{1}{s^2} - \frac{4e^{-\frac{1}{2}s}}{s^2} + \frac{2e^{-s}}{s^2} = \frac{2}{s^2}[1 - 2e^{-s/2} + e^{-s}] = \frac{2}{s^2}[1 - 2e^{-s/2}]^2$$

5.3 A sinusoidal voltage 25 sin 10t is applied at time $t = 0$ to a circuit as shown in the figure. Find the current $i(t)$, by Laplace transform method. $R = 5\Omega$ and $L = 1H$.



Solution By KVL, $RI(s) + sLI(s) = 25\dfrac{10}{s^2+100}$

with zero initial condition.

$$I(s) = \frac{250}{(s+5)(s^2+100)} = \frac{250}{(s+5)(s+j10)(s-j10)}$$

$$= 250\left[\frac{A_1}{s+5} + \frac{A_2}{s+j10} + \frac{A_3}{s-j10}\right]$$

$$A_1 = (s+5)\frac{1}{(s+5)(s^2+100)}\bigg|_{s=-5} = \frac{1}{125}$$

where,

$$A_2 = (s+j10)\frac{1}{(s+5)(s+j10)(s-j10)}\bigg|_{s=-j10} = -\frac{1}{j20(5-j10)} = -\frac{1}{100(2+j)}$$

$$A_3 = (s-j10)\frac{1}{(s+5)(s+j10)(s-j10)}\bigg|_{s=j10} = \frac{1}{100(-2+j)}$$

Substituting these,

$$I(s) = 250\left[\frac{A_1}{s+5} + \frac{A_2}{s+j10} + \frac{A_3}{s-j10}\right]$$

Taking inverse Laplace transform,

$$i(t) = 250[A_1 e^{-5t} + A_2 e^{-j10t} + A_3 e^{j10t}]$$

$$= 2e^{-5t} + 250\left\{-\frac{1}{100(2+j)}e^{-j10t} + \frac{1}{100(-2+j)}e^{j10t}\right\}$$

$$= 2e^{-5t} - \frac{5}{2}\left\{\frac{(2-j)e^{-j10t}}{5} - \frac{(-2-j)e^{j10t}}{5}\right\}$$

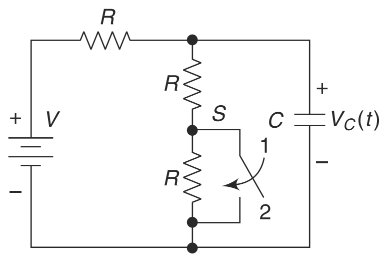
$$= 2e^{-5t} - \frac{1}{2}\{2e^{-j10t} - je^{-j10t} + 2e^{j10t} + je^{j10t}\}$$

or $$i(t) = 2e^{-5t} - 2\cos 10t + \sin 10t\ (A)$$

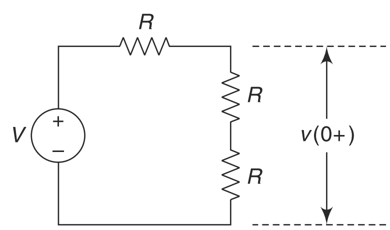
5.4 The circuit of the figure is initially in the steady state. The switch S is closed at $t = 0$.

(a) Find $V_c(t)$

(b) Determine the final value of $V_c(t)$ and verify it from the final value theorem of Laplace Transform.



Solution At steady-state before closing the switch, the capacitor becomes open-circuited. So, the circuit becomes as shown above.



$$v(0+) = \frac{2}{3}\text{ V}$$

For $t > 0$, by KVL,

$$RI_1 + R(I_1 - I_2) = \frac{V}{s} \Rightarrow 2RI_1 - RI_2 = \frac{V}{s} \quad \text{(i)}$$

and $$\frac{1}{Cs}I_2 + R(I_2 - I_1) = -\frac{2V}{3s} \Rightarrow -RI_1 + \left(R + \frac{1}{Cs}\right)I_2 = -\frac{2V}{3s} \quad \text{(ii)}$$

Solving equations (i) and (ii),

$$I_2 = \frac{\begin{vmatrix} 2R & V/s \\ -R & -2V/3s \end{vmatrix}}{\begin{vmatrix} 2R & -R \\ -R & (R+1/Cs) \end{vmatrix}} = \frac{-\dfrac{4VR}{3s} + \dfrac{VR}{s}}{2R(R+1/Cs) - R^2} = -\frac{V}{3s}\left(\frac{Cs}{2+RCs}\right)$$

$$\therefore \quad V_C(s) = I_2 \times \frac{1}{Cs} + \frac{2V}{3s} = -\frac{V}{3s\,(2+RCs)} + \frac{2V}{3s} = \frac{V}{3s}\left[2 - \frac{1}{RCs+2}\right]$$

$$= \frac{V}{2s} + \frac{V}{6}\left(\frac{1}{s+2/RC}\right)$$

Taking inverse Laplace transform,

$$v_C(s) = \frac{V}{2} + \frac{V}{6}e^{-2t/RC} \text{ (Volt)}, \quad t > 0 \qquad \textit{Ans.}$$

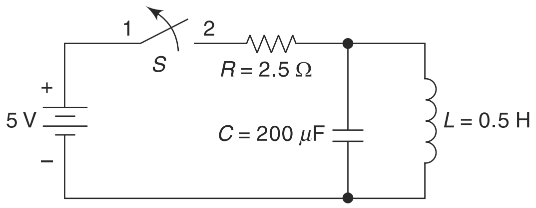
Thus, the final value of the voltage,

$$v_C(\infty) = \lim_{t\to\infty} v_C(t) = \frac{V}{2} \qquad \textit{Ans.}$$

By final value theorem,

$$v_C(\infty) = \lim_{s\to 0} SV_C(s) = \lim_{s\to 0}\left(\frac{V}{2} + \frac{Vs}{(s+2/RC)}\right) = \frac{V}{2} \qquad \text{(Proved)}$$

5.5 In the network shown in the figure, the switch S is closed and a steady state is attained. At $t = 0$, the switch is opened. Determine the current through the inductor for $t > 0$.



Solution When the switch S is closed and the steady-state exists, the current through the inductor is,

$$i(0-) = \frac{V}{R} = \frac{5}{2.5} = 2 \text{ A}$$

The voltage across the capacitor, $V_C(t) = 0$ as it is shorted.
For $t > 0$, the switch is opened. By KVL,

$$L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt = 0$$

Taking Laplace transform,

$$L[sI(s) - i(0-)] + \frac{I(s)}{Cs} = 0$$

or

$$I(s)\left[sL + \frac{1}{Cs}\right] = Li(0-)$$

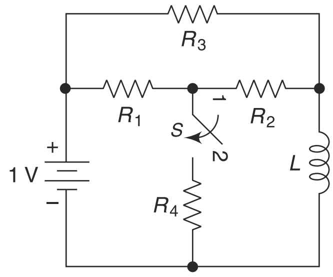
Putting the values,

$$I(s) = 2\frac{s}{s^2 + 10^4}$$

Taking inverse Laplace transform,

$$i(t) = 2\cos 100t \quad (A);\ t \ge 0 \qquad \textit{Ans.}$$

5.6 The circuit shown in the figure is initially in the steady state with the switch S open. At $t = 0$, the switch S is closed. Obtain the current through the inductor for $t > 0$. Take $R_1 = R_2 = R_4 = 1\Omega$ and $R_3 = 2\Omega$ and $L = 1H$.



Solution When the switch S is open and steady state exists, the current through the inductor is,

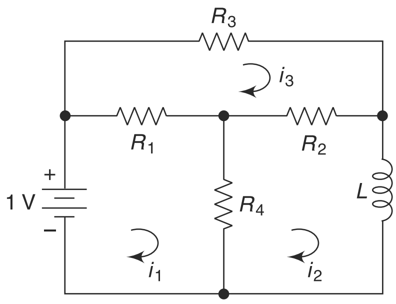
$$i_2(0-) = \frac{1}{R_3\,(R_1 + R_2)/R_3 + R_1 + R_2} = 1\text{ A}$$

After S is closed, for $t > 0$, by KVL,

$$2i_1 - i_2 - i_3 = 1$$

$$-i_1 + 2i_2 + \frac{di_2}{dt} - i_3 = 0$$

$$-i_1 - i_2 + 4i_3 = 0$$



Taking Laplace transform,

$$2I_1(s) - I_2(s) - I_3(s) = \frac{1}{s}$$

$$-I_1(s) + I_2(s)[s + 2] - I_3(s) = i_2(0-) = 1$$

$$-I_1(s) - I_2(s) + 4I_3(s) = 0$$

By Cramer's Rule,

$$I_2(s) = \frac{\begin{vmatrix} 2 & 1/s & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ -1 & (s+2) & -1 \\ -1 & -1 & 4 \end{vmatrix}} = \frac{\frac{5}{6}}{s} + \frac{\frac{1}{6}}{s + \frac{6}{7}}$$

Taking inverse Laplace transform,

$$i_2(t) = \frac{5}{6} + \frac{1}{6}e^{-6t/7} \quad (A); \quad t > 0 \qquad \textit{Ans.}$$

5.7 A series R-L-C circuit with $R = 3\Omega$, $L = 1H$ and $C = 0.5$ F is excited with a unit step voltage. Obtain an expression for the current, using Laplace transform. Assume that the circuit is relaxed initially.

Solution By KVL,

$$RI(s) + sLI(s) - Li(0-) + \frac{1}{sC}I(s) + \frac{Q(0-)}{sC} = \frac{1}{s}$$

Since the circuit is initially relaxed,

$$\therefore \qquad i(0-) = 0 \quad \text{and} \quad Q(0-) = 0$$

Putting the values,

$$I(s)\left[3 + s + \frac{2}{s}\right] = \frac{1}{s}$$

or $$I(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

where, $$A_1 = \left.\frac{1}{s+2}\right|_{s=-1} = 1 \quad \text{and} \quad A_2 = \left.\frac{1}{s+1}\right|_{s=-2} = -1$$

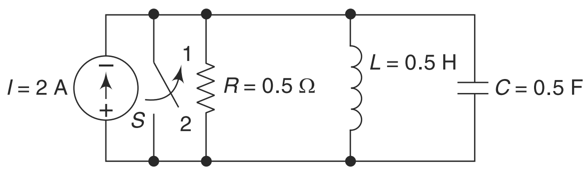
$\therefore$ $$I(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

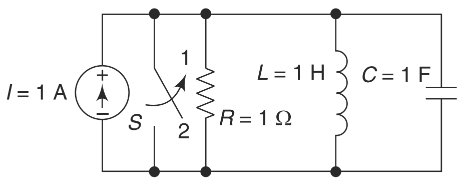
Taking inverse Laplace transform,

$$i(t) = e^{-t} + e^{-2t}\ (\text{A})$$

$$= 2e^{3t/2}\sinh\left(\frac{t}{2}\right)(\text{A}) \qquad \textit{Ans.}$$

5.8 The switch S in the figure is opened at $t = 0$. Determine the voltage $v(t)$, for $t > 0$. What is the nature of the response?

(a) 

(b) 

Solution

(a) By KVL,

$$\frac{v(t)}{R} + i(0-) + \frac{1}{L}\int_0^t v\,dt + C\frac{dv}{dt} = I$$

Taking Laplace transform,

$$V(s)\left[\frac{1}{R} + \frac{1}{sL} + sC\right] = \frac{I}{s}$$

Putting the values,

$$V(s)\left[2 + \frac{2}{s} + \frac{s}{2}\right] = \frac{2}{s}$$

or $$V(s) = \frac{4}{s^2+4s+4} = \frac{4}{(s+2)^2}$$

Taking inverse Laplace transform,

$$v(t) = 4te^{-2t}\ (V),\ t > 0 \qquad \textit{Ans.}$$

The response is *critically damped* ($\because\ \xi = 1$) *Ans.*

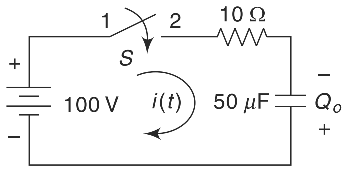
(b) Proceeding in the same way as Prob. 5.8(a),

$$V(s)=\frac{1}{s^2+s+1}=\frac{(\sqrt{3}/2)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\times\frac{2}{\sqrt{3}}$$

$$\Rightarrow \qquad v(t)=\frac{2}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\ \text{(V)};\quad t>0 \qquad \textit{Ans.}$$

The response is under-damped ($\because\ \xi<1$) *Ans.*

5.9 In the *R-C* series circuit of figure, the capacitor has an initial charge of 2.5 mC. At t = 0, the switch is closed and a constant voltage source of V = 100 V is applied. Use the Laplace transform method to find the current $i(t)$ in the circuit.



Solution By KVL, after the switch is closed,

$$Ri(t)+\frac{1}{C}\left[Q(0-)+\int_0^t i(t)dt\right]=V$$

Taking Laplace transform,

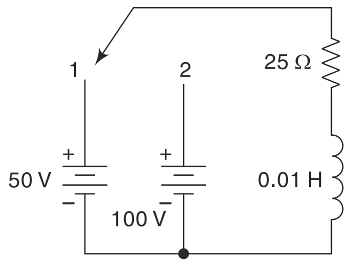
$$10I(s)+\frac{I(s)}{50\times10^{-6}s}-\frac{2.5\times10^{-3}}{50\times10^{-6}s}=\frac{100}{s}$$

or $$I(s)=\frac{15}{s+2\times10^3}$$

Taking inverse Laplace transform,

$$i(t)=15e^{-2\times10^3t}\ \text{(A)};\quad t>0 \qquad \textit{Ans.}$$

5.10 In the R-L circuit as shown, the switch is in position-1 long enough to establish steady state condition and at t = 0 it is switched to position-2. Find the resulting current, $i(t)$.



Solution When the switch is in position 1, steady-state exists and the initial current through the inductor is,

$$i(0-)=\frac{50}{25}=2\ \text{A}$$

After the switch is moved to position 2, the KVL gives, in Laplace transform,

$$25I(s)+0.01sI(s)-0.01\times2=\frac{100}{s}$$

or $$I(s)=\frac{10^4}{s(s+2500)}-\frac{2}{s+2500}=\frac{A_1}{s}+\frac{A_2}{s+2500}-\frac{2}{s+2500}$$

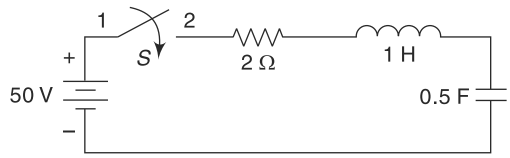
where, $$A_1=\left.\frac{10^4}{(s+2500)}\right|_{s=0}=4 \text{ and } A_2=\left.\frac{10^4}{s}\right|_{s=-2500}=-4$$

$$\therefore \qquad I(s) = \frac{4}{s} - \frac{4}{s+2500} - \frac{2}{s+2500} = \frac{4}{s} - \frac{6}{s+2500}$$

Taking inverse Laplace transform,

$$i(t) = 4 - 6e^{-2500t} \text{ (A)}; \quad t > 0 \qquad \textit{Ans.}$$

5.11 In the series R-L-C circuit as shown, there is no initial charge on the capacitor. If the switch is closed at $t = 0$, determine the resulting current at $i(t)$.



Solution By KVL, for $t > 0$,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt = V \quad [\because i(0-) = 0]$$

Taking Laplace transform,

$$RI(s) + sLI(s) + \frac{I(s)}{Cs} = \frac{V}{s}$$

Putting the values,

$$2I(s) + sI(s) + 2\frac{I(s)}{s} = \frac{50}{s}$$

or $$I(s) = \frac{50}{s^2 + 2s + s} = \frac{50}{(s+1+j)\,(s+1-j)} = \frac{50}{(s+1)^2 + 1}$$

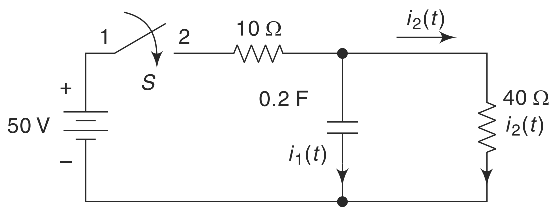
By Partial Fraction Expansion,

$$I(s) = \frac{j25}{s+1+j} - \frac{j25}{s+1-j}$$

Taking inverse Laplace transform,

$$i(t) = j25\,[e^{(-1-j)t} - e^{(-1+j)t}] = 50e^{-t}\sin t \text{ (A)}; \quad t > 0 \qquad \textit{Ans.}$$

5.12 In the two-mesh network shown in the figure, there is no initial charge on the capacitor. Find the loop currents $i_1(t)$ and $i_2(t)$ which result when the switch is closed at $t = 0$.



Solution Writing two mesh equations,

$$10i_1(t) + \frac{1}{0.2}\int_{0-}^{t} i_1(t)dt + 10i_2(t) = 50$$

and $$50i_2(t) + 10i_1(t) = 50$$

Taking Laplace transform,

$$10I_1(s)+\frac{I_1(s)}{0.2s}+10I_2(s)=\frac{50}{s}\Rightarrow I_1(s)\left[10+\frac{5}{s}\right]+10I_2(s)=\frac{50}{s}$$

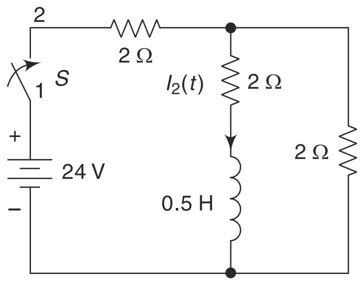
and $$10I_1(s)+50I_2(s)=\frac{50}{s}$$

Solving, $$I_1(s)=\frac{5}{s+0.625} \quad \text{and} \quad I_2(s)=\frac{1}{s}-\frac{1}{s+0.625}$$

Taking inverse Laplace transform,

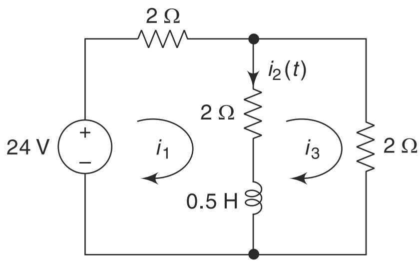
$$i_1(t)=5e^{-0.625t}\text{ (A)} \quad \text{and} \quad i_2(t)=1-e^{-0.625t}\text{ (A)},\ t>0$$

5.13 Find using Final value theorem, the steady state value of $I_2(t)$ in the circuit shown in figure below. Switch S is closed at $t = 0$. The inductor is initially de-energized.



Solution Circuit for $t > 0$ is,

By KVL, in Laplace transform,



$$I_1(s)[2+2+0.5s]-[2+0.5s]I_3(s)=\frac{24}{s}$$

or $$I_1(s)[s+8]-[s+4]I_3(s)=\frac{48}{s} \quad \text{(i)}$$

and $$-I_1(s)[2+0.5s]+[4+0.5s]I_3(s)=0$$

or $$-I_1(s)[s+4]+[s+8]I_3(s)=0 \quad \text{(ii)}$$

Solving equations (i) and (ii),

$$I_1(s)=\frac{\begin{vmatrix}48/s & -(s+4)\\ 0 & s+8\end{vmatrix}}{\begin{vmatrix}s+8 & -(s+4)\\ -(s+4) & s+8\end{vmatrix}}=\frac{6(s+8)}{s(s+8)}$$

and

$$I_3(s)=\frac{\begin{vmatrix}s+8 & 48/s\\ -(s+4) & 0\end{vmatrix}}{\begin{vmatrix}s+8 & -(s+4)\\ -(s+4) & s+8\end{vmatrix}}=\frac{6(s+4)}{s(s+6)}$$

$\therefore$ $$I_2(s) = I_1(s) - I_3(s) = \frac{6(s+8)}{s(s+6)} - \frac{6(s+4)}{s(s+6)} = \frac{24}{s(s+6)}$$

$\therefore$ Final value of the current, $i_2(\infty) = \lim_{s\to 0} sI_2(s) = \lim_{s\to 0} \frac{24}{s+6} = 4$ A *Ans.*

5.14 In a series LC circuit, the supply voltage being $v = V_m \cos(t)$, find $i(t)$ with zero initial conditions. Assume L = 1H, C = 1F.

Solution By KVL, for $t > 0$,

$$I(s)\left[sL + \frac{1}{Cs}\right] = \frac{sV_m}{s^2+1}$$

or $$I(s) = \frac{sV_m}{(s^2+1)\left(s+\frac{1}{s}\right)} = V_m\left[\frac{s^2}{(s^2+1)^2}\right] = V_m\left[\frac{s^2}{(s+j)(s-j)(s+j)(s-j)}\right]$$

$$= V_m\left[\frac{s^2}{(s+j)^2(s-j)^2}\right]$$

$$= V_m\left[\frac{K_1}{(s-j)^2} + \frac{K_1^*}{(s+j)^2} + \frac{K_2}{(s-j)} + \frac{K_2^*}{(s+j)}\right]$$

$$K_1 = I(s)\times(s-j)^2\Big|_{s=j} = \frac{1}{4}$$

where, $$K_2 = \frac{1}{(2-1)!}\frac{d}{ds}(s-j)^2 I(s)\Big|_{s=j} = \frac{(s+j)^2\, 2s - s^2\times 2(s+j)}{(s+j)^4} = -\frac{j}{4}$$

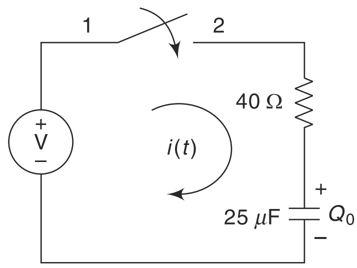
$\therefore$ $$K_1^* = \frac{1}{4}; \text{ and } K_2^* = \frac{j}{4}$$

Thus $$I(s) = \frac{V_m}{4}\left[\frac{1}{(s-j)^2} + \frac{1}{(s+j)^2} - \frac{j}{(s-j)} + \frac{j}{(s+j)}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V_m}{4}[te^{jt} + te^{-jt} - je^{jt} + je^{-jt}] = \frac{V_m}{4}[t\cos t + \sin t]\,(\text{A});\quad t > 0$$ *Ans.*

5.15 The series RC circuit of figure has a sinusoidal voltage source, $v = 180 \sin(2000t + \phi)$ (V) and an initial charge on the capacitor Q_0 = 1.25 mC with polarity as shown. Determine the current if the switch is closed at a time corresponding to $\phi = 90^\circ$. What is the current at time $t = 0$?



Solution By KVL, for $t > 0$,

$$40i(t) + \frac{1}{25\times 10^{-6}}\left[1.25\times 10^{-3} + \int_0^t i(t)dt\right] = 180\cos 2000t$$

Taking Laplace transform,

$$40I(s)+\frac{1.25\times10^{-3}}{25\times10^{-6}s}+\frac{4\times10^{4}}{s}I(s)=\frac{180s}{s^2+4\times10^6}$$

$$\Rightarrow \qquad I(s)=\frac{4.5s^2}{(s^2+4\times10^6)(s+10^3)}-\frac{1.25}{s+10^3}$$

Applying Heaviside expansion formula to find the first term on the right hand side, we have,

$$P(s) = 4.5s^2,$$

$$Q(s) = s^3 \times 10^3 s^2 + 4 \times 10^6 s + 4 \times 10^9,$$

$$Q'(s) = 3s^2 + 2 \times 10^3 s + 4 \times 10^6,$$

$$a_1 = -j2\times10^3;\ a_2 = j2\times10^3 \quad \text{and} \quad a_{31}=-10^3$$

Then,

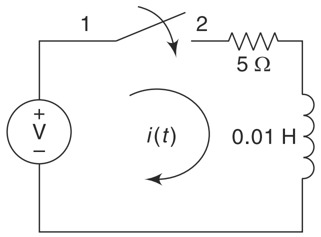
$$i(t) = \frac{P(-j2\times10^3)}{Q'(-j2\times10^3)}e^{-j2\times10^3t}+\frac{P(j2\times10^3)}{Q'(j2\times10^3)}e^{j2\times10^3t}+\frac{P(-10^3)}{Q'(-10^3)}e^{-10^3t}-1.25e^{-10^3t}$$

$$= (1.8-j0.9)\,e^{-j2\times10^3t}+(1.8+j0.9)\,e^{j2\times10^3t}-0.35\,e^{-10^3t}$$

$$= -1.8\sin 2000t+3.6\cos 2000t-0.35\,e^{-10^3t}$$

$$= 4.02\sin(2000t+116.6^\circ)-0.35\,e^{-10^3t}\ \text{(A)};\quad t>0$$

5.16 In the RL circuit of Figure, the source is $v=100\sin(500t+\phi)$. Determine the resulting current if the switch is closed at a time corresponding to $\phi = 0$.



Solution By KVL,

$$RI(s)+sLI(s)-Li(0-)=V(s)$$

or

$$5I(s)+0.01sI(s)=\frac{100\times500}{s^2+25\times10^4} \quad [\because i(0-)]=0$$

or

$$I(s)=\frac{5\times10^6}{(s^2+25\times10^4)(s+500)}$$

By Partial Fraction Expansion,

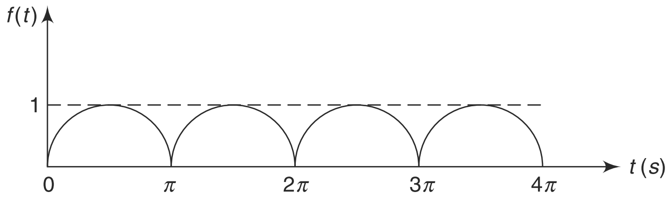
$$I(s) = 5\left(\frac{-1+j}{s+j500}\right)+5\left(\frac{-1-j}{s-j500}\right)+\frac{10}{s+500}$$

Taking inverse Laplace transform,

$$i(t) = 10\sin 500t-10\cos 500t+10e^{-500t}$$

$$= 14.14\sin(500t-45^\circ)+10\,e^{-500t}\ \text{(A)};\quad t>0$$

5.17 Determine the Laplace transform of the following periodic waveform.



Solution Let, for the first half sine wave, the transform is $F_1(s)$.

Now, $f_1(t) = \sin tu(t) + \sin(t - \pi)\, u(t - \pi)$

Taking Laplace transform,

$$F_1(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

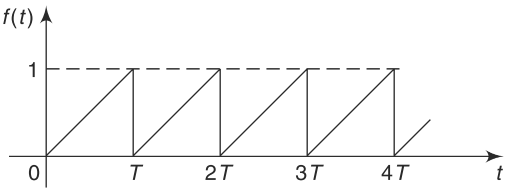
By the theory of periodicity of Laplace transform, the Laplace transform of the full periodic waveform will be,

$$F(s) = F_1(s) \times \frac{1}{1-e^{-Ts}} = \frac{1+e^{-\pi s}}{s^2+1} \times \frac{1}{1-e^{-\pi s}} \quad [\because T = \pi \text{ for the waveform given}]$$

$$= \left(\frac{1+e^{-\pi s}}{1-e^{-\pi s}}\right)\frac{1}{s^2+1}$$

$$= \frac{1}{s^2+1}\coth\left(\frac{\pi s}{2}\right) \qquad \textit{Ans.}$$

5.18 Determine the Laplace transform of the sawtooth waveform as shown below.



Solution For the first cycle,

$$f_1(t) = \frac{1}{T}r(t) - u(t-T) - \frac{1}{T}r(t-T)$$

Taking Laplace transform,

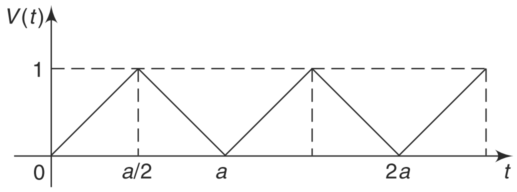
$$F_1(s) = \frac{1}{T}\frac{1}{s^2} - \frac{1}{s}e^{-Ts} - \frac{1}{T}\frac{1}{s^2}e^{-Ts} = \frac{1}{Ts^2}[1-(1+Ts)e^{-Ts}]$$

By Scalling Theorem (the theory of periodicity), Laplace transform of the given periodic function is,

$$F(s) = F_1(s) \times \frac{1}{1-e^{-Ts}} = \frac{1}{Ts^2}[1-(1+Ts)e^{-Ts}] \times \frac{1}{1-e^{-Ts}}$$

$$= \frac{1}{Ts^2} - \frac{e^{-Ts}}{s(1-e^{-Ts})} \qquad \textit{Ans.}$$

5.19 Find the Laplace transform of the waveform shown in figure.



Solution Here, $v_1(t) = \frac{2}{a}r(t) - \frac{4}{a}r\left(t - \frac{a}{2}\right) + \frac{2}{a}r(t-a)$

Taking Laplace transform,

$$V_1(s) = \frac{2}{a}\frac{1}{s^2} - \frac{4}{a}\frac{e^{-as/2}}{s^2} + \frac{2}{a}\frac{e^{-as}}{s^2}$$

$$= \frac{2}{as^2}(1 - 2e^{-as/2} + e^{-as})$$

$$= \frac{2}{as^2}(1 - e^{-as/2})^2$$

By Scalling Theorem (the theory of periodicity), Laplace transform of the given periodic function is,

$$V(s) = V_1(s) \times \frac{1}{1-e^{-Ts}} = \frac{2}{as^2}(1 - e^{-as/2})^2 \times \frac{1}{1-e^{-as}}$$

$$= \frac{2}{as^2}\left(\frac{1-e^{-as/2}}{1+e^{-as/2}}\right)$$

$$= \frac{2}{as^2}\tanh\left(\frac{as}{4}\right) \qquad \textit{Ans.}$$

5.20 The unit step response of a network is given by $(1 - e^{-bt})$. Determine the unit impulse response *h(t)* of this network.

Solution Here, the input is, $w(t) = u(t) \Rightarrow W(s) = \frac{1}{s}$

and the output is $y(t) = (1 - e^{-bt}) \Rightarrow Y(s) = \frac{1}{s} - \frac{1}{s+b} = \frac{b}{s(s+b)}$

By convolution theorem,

$$Y(s) = H(s)W(s)$$

$$\Rightarrow \qquad \frac{b}{s(s+b)} = H(s)\frac{1}{s}$$

$$\Rightarrow \qquad H(s) = \frac{b}{(s+b)}$$

Taking inverse Laplace transform, the impulse response is,

$$h(t) = be^{-bt} \qquad \textit{Ans.}$$

5.21 The unit impulse response of current of a circuit having R =1 ohm and C = 1 F in series is given by $[\delta(t) - \exp(-t)u(t)]$. Find the current expression when the circuit is driven by the voltage given as, $[1 - \exp(-2t)]u(t)$.

Solution Here, the impulse response is, $h(t) = [\delta(t) - \exp(-t)\, u(t)] \;\Rightarrow\; H(s) = 1 - \dfrac{1}{s+1} = \dfrac{s}{s+1}$.

The input is, $w(t) = [1 - \exp(-2t)]u(t) \;\Rightarrow\; W(s) = \dfrac{1}{s} - \dfrac{1}{s+2} = \dfrac{2}{s(s+2)}$

By convolution theorem, the output is given by,

$$Y(s) = H(s)W(s) = \frac{s}{s+1} \times \frac{2}{s(s+2)} = \frac{2}{(s+1)\,(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$

Taking inverse Laplace transform,

$$y(t) = (2e^{-t} - 2e^{-2t}) \qquad \textit{Ans.}$$

5.22 The response of a network to an impulse is $h(t) = 0.18(e^{-0.32t} - e^{-2.1t})$. Find the response of the network to a step function using convolution theorem.

Solution By convolution theorem,

$$Y(s) = H(s)W(s) = 0.18\left[\frac{1}{s+0.32} - \frac{1}{s+2.1}\right] \times \frac{1}{s}$$

$$= \frac{0.32}{s(s+0.32)\,(s+2.1)}$$

$$= \frac{A_1}{s} + \frac{A_2}{s+0.32} + \frac{A_3}{s+2.1}$$

$$\therefore \qquad A_1 = \left.\frac{0.32}{(s+0.32)\,(s+2.1)}\right|_{s=0} = 0.477$$

$$\therefore \qquad A_2 = \left.\frac{0.32}{s(s+2.1)}\right|_{s=-0.32} = -0.562$$

$$\therefore \qquad A_3 = \left.\frac{0.32}{s(s+0.32)}\right|_{s=-2.1} = 0.0856$$

Putting these values,

$$Y(s) = \frac{0.477}{s} - \frac{0.562}{s+0.32} + \frac{0.0856}{s+2.1}$$

Taking inverse Laplace transform,

$$y(t) = 0.477 - 0.562e^{-0.32t} + 0.0856e^{-2.1t} \qquad \textit{Ans.}$$

MULTIPLE-CHOICE QUESTIONS

5.1 The condition for over-damped response of an RLC series circuit is

(a) $\frac{R^2}{4L^2} = \frac{1}{LC}$ (b) $\frac{R^2}{4L^2} > \frac{1}{LC}$ (c) $\frac{R^2}{4L^2} < \frac{1}{LC}$ (d) $\frac{R^2}{4L^2} \le \frac{1}{LC}$

5.2 Transient current in an RLC circuit is oscillatory when

(a) $R = 2\sqrt{\frac{L}{C}}$ (b) $R > 2\sqrt{\frac{L}{C}}$ (c) $R < 2\sqrt{\frac{L}{C}}$ (d) $R = 0$.

5.3 Laplace transform analysis gives
(a) time domain response only (b) frequency domain response only
(c) both (a) and (b) (d) None of these.

5.4 A function $f(t)$ is shifted by a then it is correctly represented as
(a) $f(t - a)u(t)$ (b) $f(t)u(t - a)$ (c) $f(t - a)u(t - a)$ (d) $f(t - a)(t - a)$

5.5 Laplace transform of a delayed unit impulse function $\delta_s(t) = \delta(t - 1)$ is
(a) unity. (b) zero. (c) e^{-s}. (d) s.

5.6 The condition for under damped response of an RLC series circuit is

(a) $\frac{R^2}{4L^2} = \frac{1}{LC}$ (b) $\frac{R^2}{4L^2} > \frac{1}{LC}$ (c) $\frac{R^2}{4L^2} < \frac{1}{LC}$ (d) $\frac{R^2}{4L^2} \le \frac{1}{LC}$

5.7 The value of the impulse function $\delta(t)$ at $t = 0$ is
(a) 0 (b) $\propto$ (c) 1 (d) indeterminate

5.8 The value of the ramp function at $t = +\infty$ is
(a) infinity (b) unity (c) zero (d) indeterminate

5.9 The value of the ramp function at $t = -\infty$ is
(a) 0 (b) ∞ (c) $-\infty$ (d) 1

5.10 The value of the impulse function $\delta(t)$ for $t > 0$ is
(a) zero (b) unity
(c) k, where k is a constant (d) infinity.

5.11 The free response of RL and RC series networks having a time constant τ is of the form

(a) $A + Be^{-\frac{t}{\tau}}$ (b) $Ae^{-\frac{t}{\tau}}$ (c) $Ae^{-\frac{t}{\tau}} + Be^{-\frac{t}{\tau}}$ (d) $(A + Bt)e^{-\frac{t}{\tau}}$

5.12 In the complex frequency $s = \sigma + j\omega$, ω has the units of rad/s and σ has the units of
(a) Hz (b) neper/s (c) rad/s (d) rad

5.13 Time constant of a series RC circuit is
(a) C/R (b) R/C (c) RC (d) $1/RC$

5.14 Time constant of a series RL circuit is
(a) L/R (b) R/L (c) LR (d) $1/LR$

5.15 A coil with a certain number of turns has a specified time constant. If the number of turns is doubled, its time constant would
(a) remain unaffected (b) become doubled (c) become four-fold (d) get halved.

5.16 An RLC series circuit has $R = 1\Omega$, L = 1 H and $C = 1$F. Damping ratio of the circuit will be
(a) more than unity (b) unity (c) 0.5 (d) zero

5.17 A step function voltage is applied to an RLC series circuit having $R = 2\Omega$, $L = 1$H and C = 1F. The transient current response of the circuit would be
(a) over-damped
(b) critically damped
(c) under damped
(d) over, under or critically damped depending upon magnitude of the step voltage.

5.18 For an RC circuit comprising a capacitor $C = 2\ \mu$F in series with a resistance $R = 1$ MΩ period 6 seconds will be equal to
(a) one time constant (b) two time constants
(c) three time constants (d) four time constants

5.19 A series RL circuit with $R = 100$ ohm; $L = 50$H, is supplied to a d.c. source of 100V. The time taken for the current to rise 70% of its steady state value is
(a) 0.3s (b) 0.6s
(c) 2.4s (d) 70% of time required to reach steady state.

5.20 If *f(t)* and its first derivative are Laplace transformable then the initial value of *f(t)* is given by

(a) $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ sF(s)$ (b) $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to \infty}{\text{Lt}}\ \dfrac{F(s)}{s}$

(c) $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ \dfrac{F(s)}{s}$ (d) $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to \infty}{\text{Lt}}\ sF(s)$

5.21 If *f(t)* and its first derivative are Laplace transformable then the final value of *f(t)* is given by

(a) $\underset{t\to \infty}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ sF(s)$ (b) $\underset{t\to \infty}{\text{Lt}}\ f(t) = \underset{s\to \infty}{\text{Lt}}\ \dfrac{F(s)}{s}$

(c) $\underset{t\to \infty}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ \dfrac{F(s)}{s}$ (d) $\underset{t\to \infty}{\text{Lt}}\ f(t) = \underset{s\to \infty}{\text{Lim}}\ sF(s)$

5.22 At $t = 0^+$ with zero initial condition which of the following will act as short circuit?
(a) Inductor (b) Capacitor (c) Resistor (d) None of these

5.23 At $t = 0^+$ with zero initial condition which of the following will act as open circuit?
(a) Inductor (b) Capacitor (c) Resistor (d) None of these

5.24 A capacitor at time $t = 0^+$ with zero initial charge acts as a
(a) short circuit (b) open circuit (c) current source (d) voltage source.

5.25 A series RC circuit is suddenly connected to a dc voltage of V volt. The current in the series circuit just after the switch is closed is equal to

(a) zero (b) $\dfrac{V}{RC}$ (c) $\dfrac{VC}{R}$ (d) $\dfrac{V}{R}$

5.26 A series LC circuit is suddenly connected to a dc voltage of V volt. The current in the series circuit just after the switch is closed is equal to

(a) $\dfrac{V}{L}$ (b) $\dfrac{V}{C}$ (c) zero (d) $\dfrac{V}{LC}$

5.27 The steady state current in the RC series circuit, on the application of step voltage of magnitude E will be

(a) zero (b) $\dfrac{E}{R}$ (c) $\dfrac{E}{R}e^{-t/CR}$ (d) $\dfrac{E}{RC}e^{-t}$

5.28 A 10 Ω resistor, a 1H inductor and 1F capacitor are connected in parallel. The combination is driven by a unit step current. Under steady state conditions, the source current flows through
(a) resistor (b) inductor (c) capacitor only (d) All of three elements.

5.29 When a unit impulse voltage is applied to an inductor of 1H, the energy supplied by the source is
(a) ∞ (b) 1 Joule (c) $\frac{1}{2}$ Joule (d) 0.

5.30 Which of the following conditions are necessary for validity of Initial Value Theorem: $\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$?
(a) $f(t)$ and its derivative $f'(t)$ must have Laplace transform.
(b) If the Laplace transform of $f(t)$ is $F(s)$, then Lim $sF(s)$ must exist.
(c) Only $f(t)$ must have Laplace transform.
(d) (a) and (b) both.

5.31 Inverse Laplace transform of $\frac{1}{s-a}$ is
(a) sin at (b) cos at (c) e^{at} (d) e^{-at}

5.32 The impulse response of an RL circuit is a
(a) rising exponential function. (b) decaying exponential function.
(c) step function. (d) parabolic function.

5.33. Laplace transform of the output response of a linear system is the system transfer function when the input is
(a) a step signal. (b) a ramp signal. (c) an impulse signal. (d) a sinusoidal signal.

5.34 An initially relaxed RC series network with $R = 2M\Omega$ and $C = 1\mu$F is switched on to a 10V step input. The voltage across the capacitor after 2 seconds will be
(a) zero (b) 3.68 V (c) 6.32 V (d) 10 V

5.35 For $V(s) = \frac{(s+2)}{s(s+1)}$, the initial and final values of $v(t)$ will be respectively
(a) 1 and 1 (b) 2 and 2 (c) 2 and 1 (d) 1 and 2.

5.36 The Laplace transform of the function $i(t)$ is: $I(s)$ $\frac{10s+4}{s(s+1)(s^2+4s+5)}$. Its final value will be
(a) 4/5 (b) 5/4 (c) 4 (d) 5

5.37 An initially relaxed 100 mH inductor is switched 'ON' at $t = 1$ second to an ideal 2A dc current source. The voltage across the inductor would be
(a) zero (b) $0.2\delta(t)$ V (c) $0.2\delta(t-1)$ V (d) $0.2tu(t-1)$ V

5.38 If the unit step response of a network is $(1 - e^{-\alpha t})$, then its unit impulse response will be
(a) $\alpha e^{-\alpha t}$ (b) $\frac{1}{\alpha}e^{-t/\alpha}$ (c) $\frac{1}{\alpha}e^{-t/\alpha}$ (d) $(1-\alpha)\, e^{-\alpha t}$

5.39 The response of an initially relaxed system to a unit ramp excitation is $(1 + e^{-t})$. Its step response will be
(a) $\frac{1}{2}t^2 - e^{-t}$ (b) $1 - e^{-t}$ (c) $-e^{-t}$ (d) t.

5.40 A series circuit containing R, L and C is excited by a step voltage input. The voltage across the capacitance exhibits oscillations. Damping coefficient (ratio) of this circuit is given by

(a) $\xi = \frac{R}{2\sqrt{LC}}$ (b) $\xi = \frac{R}{LC}$ (c) $\xi = \frac{R}{2\sqrt{C/L}}$ (d) $\xi = \frac{R}{2\sqrt{L/C}}$

5.41 Consider the following statements:
A unit impulse $\delta(t)$ is mathematically defined as

1. $\delta(t) = 0, t \neq 0$ 2. $\int_{0+}^{\infty} \delta(t)\,dt = 1$ 3. $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

Of these statements
(a) 1, 2 and 3 are correct. (b) 1 and 2 are correct.
(c) 2 and 3 are correct. (d) 1 and 3 are correct.

5.42. With symbols having their usual meanings, the Laplace transform of $u(t - a)$ is

(a) $\frac{1}{s}$ (b) $\frac{1}{s-a}$ (c) $\frac{e^{-as}}{s}$ (d) $\frac{e^{as}}{s}$

5.43 Two coils having equal resistances but different inductances are connected in series. The time constant of the series combination is the
(a) sum of the time constants of the individual coils.
(b) average of the time constants of the individual coils.
(c) geometric mean of the time constants of the individual coils.
(d) product of the time constants of the individual coils.

5.44 If the step response of an initially relaxed circuit is known then the ramp response can be obtained by
(a) integrating the step response. (b) differentiating the step response.
(c) integrating the step response twice. (d) differentiating the step response twice.

5.45 If a capacitor is energized by a symmetrical square wave current source, then the steady state voltage across the capacitor will be a
(a) square wave (b) triangular wave (c) step function (d) impulse function.

5.46 A square wave is fed to an RC circuit, then
(a) voltage across R is square and across C is not square.
(b) voltage across C is not square and across R is not square.
(c) voltage across both R and C is square.
(d) voltage across both R and C is not square.

5.47 A step voltage is applied to an under-damped series RLC circuit with variable R. Which of the following statements correctly describe the behaviour of the circuit?
1. If R is increased, the steady state voltage across C will be reduced
2. If R is increased, the frequency of transient oscillation across C will be reduced.
3. If R is reduced, the transient oscillation will die down faster.
4. If R is reduced to zero, the peak amplitude of the voltage across C will be double the input step voltage.

Select the correct answer using the codes given below.
Codes: (a) 1 and 2 (b) 2 and 3 (c) 2 and 4 (d) 1, 3 and 4.

5.48 The number of turns of a coil having a time constant T is doubled. Then the new time constant will be
(i) T (b) $2T$ (c) $4T$ (d) $T/2$

5.49 The response of a network is of the form ke^{st}, where $s = \sigma + j\omega$, then σ is known as
(a) radian frequency (b) neper frequency
(c) complex frequency (d) None of these.

5.50 In Laplace transform the variable 's' equals $(\sigma + j\omega)$. Which of the following represent the true nature of σ?
1. σ has a damping effect.
2. σ is responsible for convergence of integral $\int_0^\infty f(t)e^{-st}\,dt$.
3. σ has a value less than zero.

Select the correct answer using the coeds given below.
Codes: (a) 1, 2 and 3 (b) 1 and 2 (c) 2 and 3 (d) 1 and 3.

5.51 Laplace transform of $t^n e^{-at}$ is
(a) $\dfrac{n}{(s-a)^{n+1}}$ (b) $\dfrac{n!}{(s+a)^{n+1}}$ (c) $\dfrac{n!}{(s-a)^{n}}$ (d) $\dfrac{n!}{(s-a)^{n+1}}$

5.52 $\dfrac{s}{(s^2+\omega^2)}$ is the Laplace transform of
(a) $\sin \omega t$ (b) $\cos \omega t$ (c) $\cosh \omega t$ (d) $\sinh \omega t$

5.53 Consider the following statements regarding an RC differentiating network.
1. For an applied rectangular pulse, the output is spiky in nature for RC << pulse duration.
2. The output is a ramp for rectangular input pulse.
3. The output has zero average for all inputs.

Of these statements:
(a) 1, 2 and 3 are correct. (b) 1 and 2 are correct.
(c) 2 and 3 are correct (d) 1 and 3 are correct.

5.54 The Laplace transform method enables one to find the response in
(a) the transient state only.
(b) the steady state only.
(c) both transient and steady states.
(d) the transient state provided sinusoidal forcing functions do not exist.

5.55 The convolution of a function $f(t)$ with the unit impulse function $\delta(t)$ is
(a) $\delta(t)$ (b) $f(t)\delta(t)$ (c) $f(t)$ (d) $f(\tau)\delta(t)$

5.56 The d.c. gain of a system represented by the transfer function $\dfrac{25}{(s+2)(s+3)}$ is
(a) 25 (b) 25/6 (c) 5 (d) 10

5.57 Consider the following statements
The impulse response of a linear network can be used to determine the
1. step response. 2. response of the sinusoidal input.
3. elements of the network uniquely. 4. interconnection of network elements.
5. Which of these statements are correct?
(a) 1 and 2 (b) 2 and 3 (c) 3 and 4 (d) 1 and 4.

5.58 Double integration of a unit step function would lead to
(a) an impulse (b) a parabola (c) a ramp (d) zero.

5.59 Which of the following integrals represents the convolution of two functions $f_1(t)$ and $f_2(t)$?

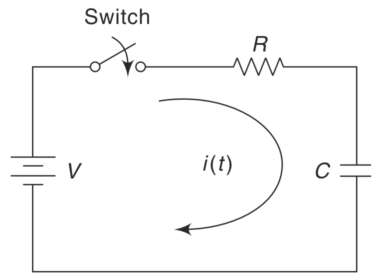
(a) $\int_0^t f_1(t) f_2(\tau - t) d\tau$ (b) $\int_0^t f_1(t-\tau) f_2(\tau) d\tau$

(c) $\int_0^t f_1(t-\tau) f_2(t) dt$ (d) $\int_0^t f_1(\tau - t) f_2(\tau) dt$

5.60 If $F(s) = \frac{1}{s}\frac{(s+1)}{(s+k)}$ and $f(t)$ as $t \to \infty$ is $\frac{1}{2}$, then the value of k is

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) ∞

5.61 The transient response of the initially relaxed network shown in the figure is



(a) $i(t) = \frac{V}{R} e^{-t/RC}$ (b) $i(t) = \frac{V}{R} e^{t/RC}$

(c) $i(t) = \frac{V}{R}(1 - e^{-t/RC})$ (d) $i(t) = \frac{V}{R}(1 + e^{-t/RC})$

5.62 A first order linear system is initially relaxed. For a unit step signal $u(t)$, the response is $v_1(t) = (1 - e^{-3t})$ for $t > 0$. If a signal $3u(t) + \delta(t)$ is applied to the same initially relaxed system, the response will be

(a) $(3 - 6e^{-3t})u(t)$ (b) $(3 - 3e^{-3t})u(t)$ (c) $3u(t)$ (d) $(3 + 3e^{-3t})\, u(t)$

5.63 A unit impulse input to a linear network has a response $R(t)$ and a unit step input to the same network has response $S(t)$. The response $R(t)$

(a) equals $\frac{dS(t)}{dt}$ (b) equals the integral of $S(t)$

(c) is the reciprocal of $S(t)$ (d) has no relation with $S(t)$

5.64 The response of an initially relaxed linear circuit to a signal V_S is $e^{-2t}u(t)$. If the signal is changed to $\left(V_S + 2\frac{dV_S}{dt}\right)$, the response would be

(a) $-4e^{-2t}\, u(t)$ (b) $-3e^{-2t}\, u(t)$ (c) $4e^{-2t}\, u(t)$ (d) $5e^{-2t}\, u(t)$

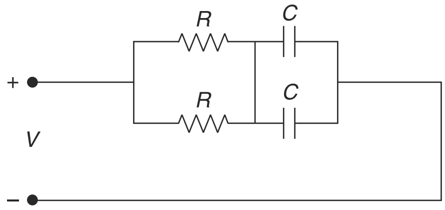
5.65 The impulse response of a circuit is given by $h(t) = \frac{1}{L} e^{-\frac{R}{L}t} u(t)$. Its step response is given as

(a) $\left(1 - e^{-\frac{R}{L}t}\right) u(t)$ (b) $\frac{1}{R}\left(1 - e^{-\frac{R}{L}t}\right) u(t)$ (c) $\frac{L}{R}\left(1 - e^{-\frac{R}{L}t}\right) u(t)$ (d) None of these.

5.66 The time constant of the network shown in the figure is

(a) CR (b) $2CR$

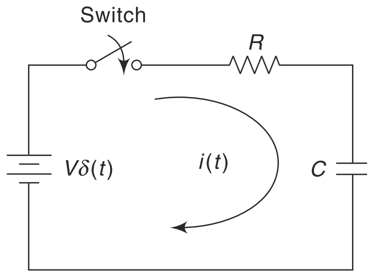
(c) $\frac{CR}{4}$ (d) $\frac{CR}{2}$



5.67 Non-linear system cannot be analyzed by Laplace transform because

(a) it has no zero initial conditions.
(b) superposition law cannot be applied.
(c) non-linearity is generally not well defined.
(d) All of the above.

5.68 In the circuit shown in figure, the response *i(t)* is



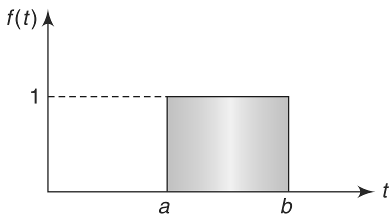
(a) $\frac{V}{R} \exp\left(-\frac{t}{RC}\right)$ (b) $\frac{V}{R}\delta(t)$

(c) $\frac{V}{R}\left[\delta(t) - \frac{1}{RC}\exp\left(-\frac{t}{RC}\right)\right]$ (d) $\frac{V}{R}\left[\delta(t) - \exp\left(-\frac{t}{RC}\right)\right]$

5.69 A voltage $v(t) = 6e^{-2t}$ is applied at $t = 0$ to a series RL circuit with $L = 1$H. If $i(t) = 6[e^{-2t} - e^{-3t}]$, then R will have a value of

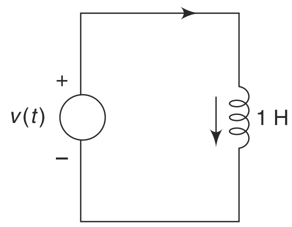
(a) $\frac{2}{3}\ \Omega$ (b) $1\ \Omega$ (c) $3\ \Omega$ (d) $\frac{1}{3}\Omega$

5.70 The Laplace transform of the signal described in figure is



(a) $e^{-as/s}$ (b) e^{-bs}/s^2 (c) $(e^{-as} + e^{-bs})/s$ (d) $(e^{-as} - e^{-bs})/s$

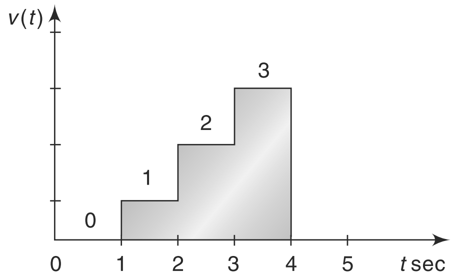
5.71 If a pulse voltage $v(t)$ of 4V magnitude and 2 second duration is applied to a pure inductor of 1H, with zero initial current, the current (in A) drawn at $t = 3$ second, will be



(a) zero (b) 2
(c) 4 (d) 8.

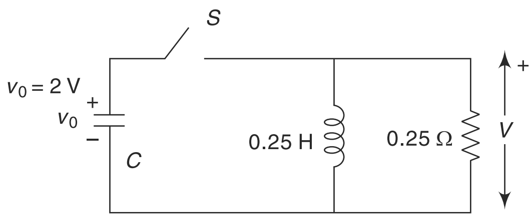
5.72 At certain current, the energy stored in an iron-cored coil is 1000J and its copper loss is 2000W. The time constant (in second) of the coil is
(a) 0.25 (b) 0.5
(c) 1.0 (d) 2.0.

5.73 Consider the voltage waveform shown in the given figure.



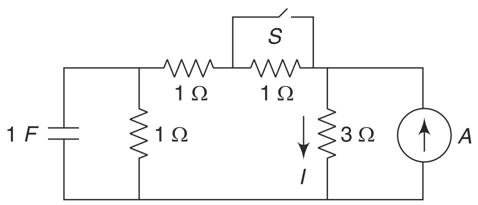
The equation for $v(t)$ is
(a) $u(t-1)+u(t-2)+u(t-3)$ (b) $u(t-1)+2u(t-2)+3u(t-3)$
(c) $u(t)+u(t-1)+u(t-2)+u(t-4)$ (d) $u(t-1)+u(t-2)+u(t-3)-3u(t-4)$

5.74 For the circuit given in the figure $V_0 = 2$ V and inductor is initially relaxed. The switch S is closed at $t = 0$. The value of v at $t = 0+$ is



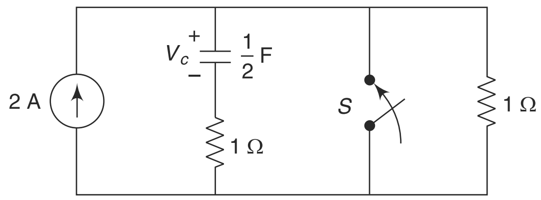
(a) 3 V (b) 2 V (c) 0.5 V (d) 0.25 V

5.75 In the circuit shown in the given figure, S is open for a long time and steady state is reached. S is closed at $t = 0$. The current I at $t = 0+$ is



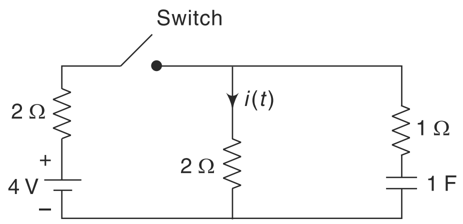
(a) 4A (b) 3A (c) 2A (d) 2A

5.76 The circuit shown in the given figure is in steady state with switch S open. The switch is closed at $t = 0$. The values of $V_C(0+)$ and $V_C(\infty)$ will be respectively



(a) 2 V, 0 V (b) 0 V, 2 V (c) 2 V, 2 V (d) 0 V, 0 V

5.77 In the circuit shown, the switch is opened at $t = 0$. Prior to that switch was closed, $i(t)$ at $t = 0+$ is



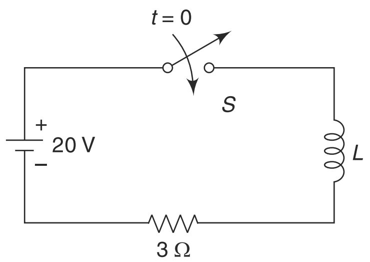
(a) $\frac{2}{3}$ A (b) $\frac{3}{2}$ A (c) $\frac{1}{3}$ A (d) 1 A.

5.78 Given the Laplace transform $\mathcal{L}[v(t)] = \int_0^\infty e^{-st} v(t)dt$, the inverse transform $v(t)$ is

(a) $\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st} V(s)ds$ (b) $\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} e^{st} V(s)ds$

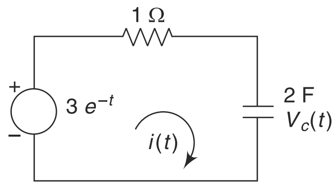
(c) $\frac{1}{2\pi j} \int_0^\infty e^{st} V(s)ds$ (d) $\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} e^{-st} V(s)ds$

5.79 In the circuit shown in the given figure, switch S is closed at $t = 0$. After some time when the current in the inductor was 6A, the rate of change of current through it was 4 A/s. The value of the inductor is



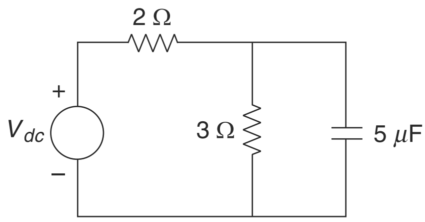
(a) indeterminate (b) 1.5 H (c) 1.0 H (d) 0.5 H

5.80 A circuit consisting of a 1Ω resistor and a 2F capacitor in series is excited from a voltage source with the voltage expressed as $3e^{-t}$, as shown in the given figure. If the $i(0-)$ and $v_c(0-)$ are both zero, then the values of $i(0+)$ and $i(\infty)$ will be respectively



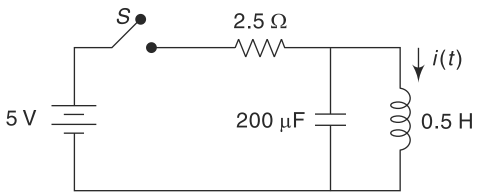
(a) 3 A and 1.5 A (b) 1.5 A and zero (c) 3 A and zero (d) 1.5 A and 3 A

5.81 The time constant associated with the capacitor charging in the circuit shown in the given figure is



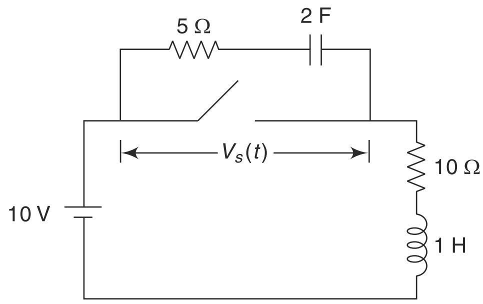
(a) 6 μs (b) 10 μs (c) 15 μs (d) 25 μs

5.82 In the network shown in the figure, the switch S is closed and a steady state is attained. If the switch is opened at $t = 0$, then the current $i(t)$ through the inductor will be



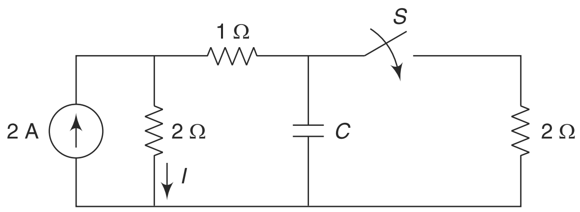
(a) cos $50t$ A (b) 2 A (c) 2 cos$100t$ A (d) 2 sin $50t$ A

5.83 In the network shown, the switch is opened at $t = 0$. Prior to that, the network was in the steady state. $V_s(t)$ at $t = 0^+$ is



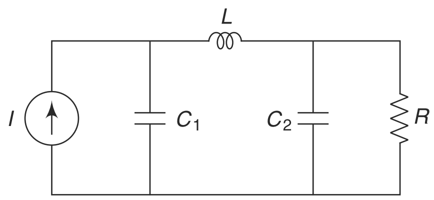
(a) 0 (b) 5 V (c) 10 V (d) 15 V.

5.84 The steady state in the circuit, shown in the given figure is reached with S open. S is closed at $t = 0$. The current I at $t = 0^+$ is



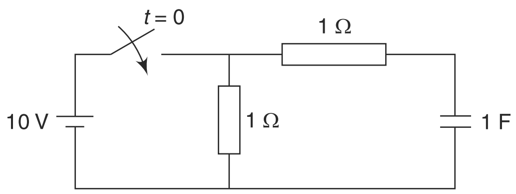
(a) 1 A (b) 2 A (c) 3 A (d) 4 A

5.85 For the circuit shown in the given figure, the current through L and the voltage across C_2 are respectively



(a) zero and RI (b) I and zero (c) Zero and zero (d) I and RI

5.86 In the circuit shown in the given figure, the switch is closed at $t = 0$. The current through the capacitor will decrease exponentially with a time constant

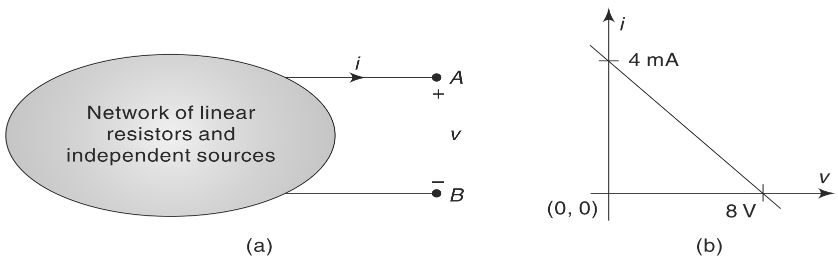


(a) 0.5 s (b) 1 s (c) 2 s (d) 10 s

5.87 The Laplace transformation of $f(t)$ is $F(s)$. Given $F(s) = \dfrac{\omega}{s^2 + \omega^2}$, the final value of $f(t)$ is

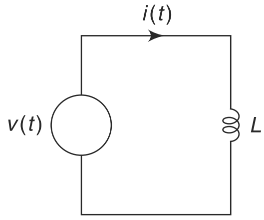
(a) infinity (b) zero (c) one (d) None of the above

5.88 The v–i characteristics as seen from the terminal-pair (A, B) of the network of Figure (a) is shown in Figure (b). If an inductance of value 6 mH is connected across the terminal-pair (A, B), the time constant of the system will be



(a) 3 μs (b) 12 s
(c) 32 s (d) unknown, unless the actual network is specified.

5.89 In the circuit shown in figure, it is desired to have a constant direct current $i(t)$ through the ideal inductor L. The nature of the voltage source $v(t)$ must be



(a) constant voltage (b) linearly increasing voltage
(c) an ideal impulse (d) exponentially increasing voltage.

5.90 The value of the integral $\int_{-\infty}^{\infty} e^{5t}\delta(t-5)dt$ is

(a) 1 (b) $(e^5 - 1)$ (c) e^{25} (d) zero.

5.91 An inductor at $t = 0+$ with initial current I_0 acts as

(a) voltage source (b) current source (c) open-circuit (d) short-circuit

5.92 A capacitor at $t = 0+$ with initial charge Q_0 acts as

(a) voltage source (b) current source (c) open-circuit (d) short-circuit

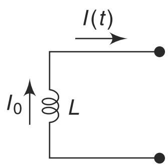
5.93 Consider the following statements

1. Current through an inductor cannot change abruptly.
2. Voltage across the capacitor cannot change abruptly.
3. Initial value of a function *f(t)* is $\lim_{s\to 0} sF(s)$
4. Final value of a function *f(t)* is $\lim_{s\to\infty} sF(s)$

Of these statements

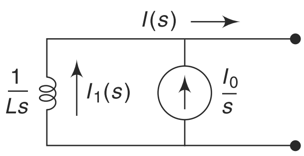
(a) 3 and 4 are correct (b) 1 and 4 are correct
(c) 1 and 2 are correct (d) 2 and 3 are correct.

5.94 An inductor with inductance L and initial current I_0 is shown as

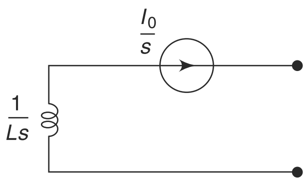


The correct admittance diagram for it is

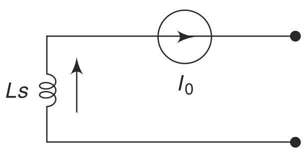
(a)



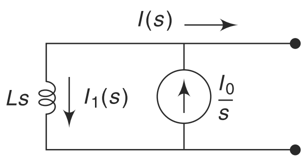
(b)



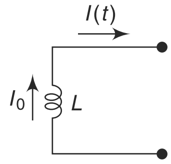
(c)



(d)

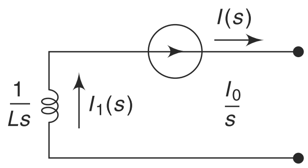


5.95 An inductor with inductance L and initial current I_0 is shown as

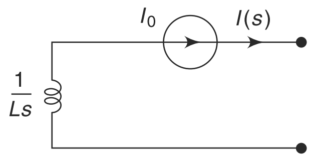


The correct impedance diagram for it is

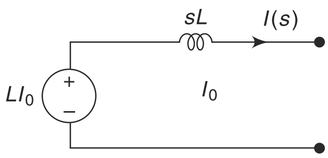
(a)



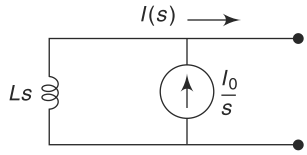
(b)



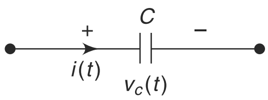
(c)



(d)

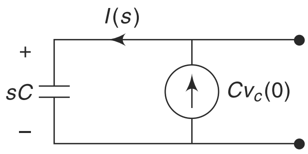


5.96 A capacitor with capacitance C and initial voltage $v_c(t)$ is shown here

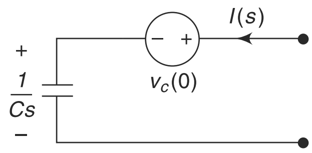


The correct admittance diagram for this circuit is

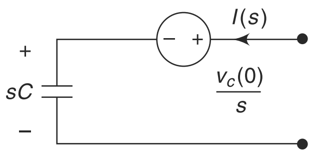
(a)



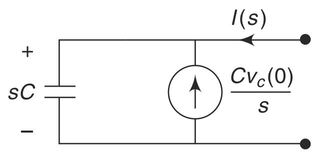
(b)



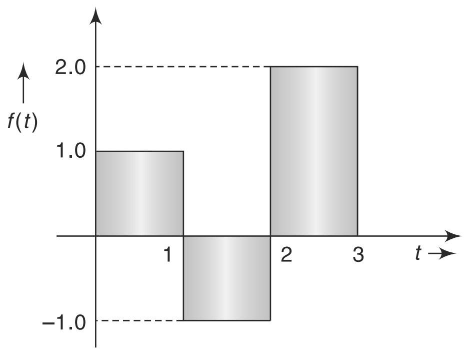
(c)



(d)



5.97

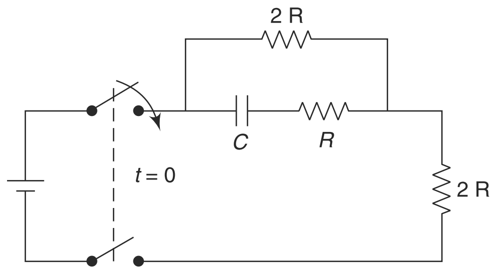


Laplace transform of $f(t)$ shown in the given figure is

(a) $F(s) = \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{3}{s}e^{-s}$ (b) $F(s) = \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{3}{s}e^{-2s} - \frac{2}{s}e^{-3s}$

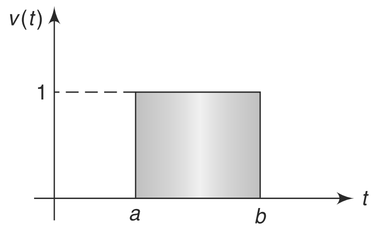
(c) $F(s) = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{2}{s}e^{-2s} - \frac{2}{s}e^{-3s}$ (d) $F(s) = \frac{1}{s} + \frac{2}{s}e^{-s} - \frac{3}{s}e^{-s}$

5.98 The time constant of the circuit shown in the given figure is



(a) R C (b) R C (c) R C (d) R C

5.99 Consider the following functions for the rectangular voltage pulse shown in the given figure



(a) $v(t) = u(t-a) - u(t-b)$ (b) $v(t) = u(b-t) - u(a-t)$

(c) $v(t) = u(b-t).u(t-a)$ (d) $v(t) = u(a-t).u(t-b)$

EXERCISES

5.1 (a) Find the initial values of the functions

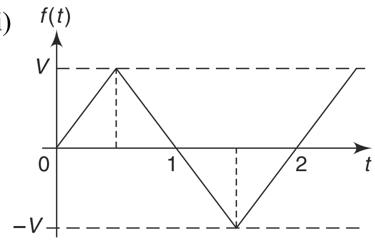
(i) $f(t) = e^{-at}\cos\omega t u(t)$ (ii) $F(s) = \frac{2(s+1)}{s^2+2s+5}$ [(i) 1, (ii) 2]

(b) Find the final value of the functions

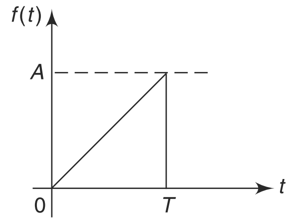
(i) $F(s) = \frac{7}{s(s+3)^2}$ (ii) $F(s) = \frac{s-1}{(s+1)(s+2)}$ $\left[\text{(i) } \frac{7}{9}, \text{(ii) } 0\right]$

5.2 Obtain the Laplace Transform of the following functions

(i)

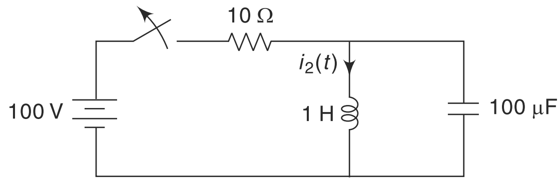


(ii)



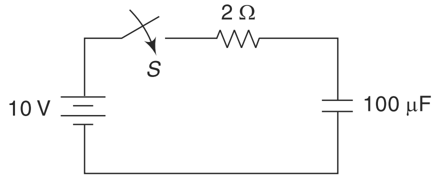
$$\left[\text{(i) } F(s) = \frac{A}{Ts^2}(1 - e^{-Ts} - se^{-Ts}) \text{ (ii) } \frac{A}{Ts^2}(1 - e^{-Ts} - Ts^{-Ts})\right]$$

5.3 In the network shown, the switch is closed and a steady state is reached in the network. At time $t = 0$, the switch is opened. Find an expression for the current through the inductor $i_2(t)$.



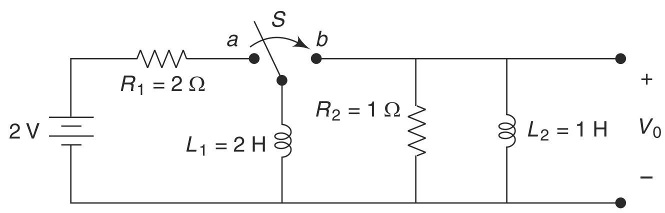
[10 cos100t (A)]

5.4 Find for the circuit shown, the current through C using Laplace transform. The switch is closed at $t = 0$ and the initial charge in the capacitor, i.e., at $t = 0$ is zero.



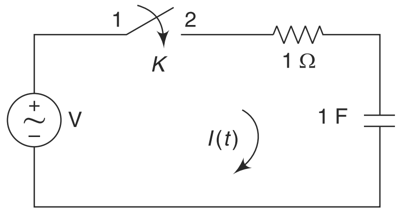
[10 sin100t (A)]

5.5 The circuit of figure was initially in the steady state with the switch S in position a. At $t = 0$, the switch goes from a to b. Find an expression for the voltage $v_0(t)$ for $t > 0$. Take the initial current in the inductor L_2 to be zero.



$$\left[v_0(t) = \frac{1}{2}e^{-\frac{3}{2}t} \text{ (V)}\right]$$

5.6 In the circuit of given figure, the applied voltage is $v(t)$= 10 sin $(10t + \pi/6)$, $R = 1\ \Omega$, $C = 1$ F. Using Laplace Transformation, find complete solution for current $i(t)$. Switch K is closed at time $t = 0$. Assume zero charge across the capacitor before switching.

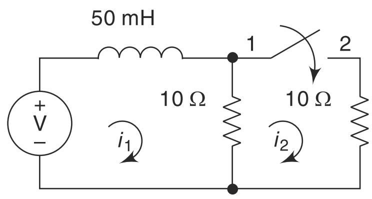


$$\left[i(t) = \frac{5}{101}(1 - 10\sqrt{3})e^{-t} + \frac{100}{\sqrt{101}}\cos(10t - 54°8')(\mathrm{A})\right]$$

5.7 A series RLC circuit, with $R = 5\ \Omega$, $L = 0.1$ H and $C = 500\ \mu$F, has a sinusoidal voltage source, $v = 1000 \sin 250t$. Find the resulting current if the switch is closed at $t = 0$.

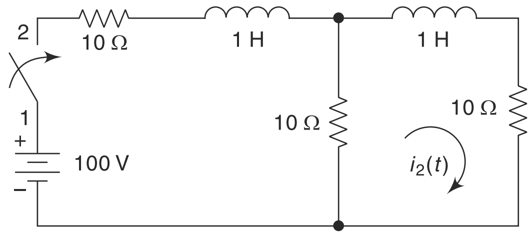
[$i(t) = e^{-25t}$ (5.42 cos 139t + 1.89 sin 139t) + 5.65 sin (250t – 73.6°)(A)]]

5.8 The two-mesh network shown in figure contains a sinusoidal voltage source, $v = 100 \sin (200t + \phi)$ (V). The switch is closed at an instant when the voltage is increasing at its maximum rate. Find the resulting mesh currents, with directions as shown in figure.



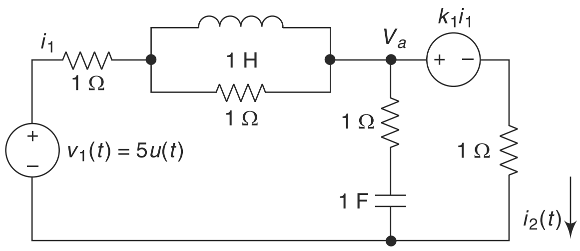
[$i_1(t) = 3.01^{e-100t} + 8.96 \sin (200t - 63.4°)$
$i_2(t) = 1.505e^{-100t} + 4.48 \sin (200t - 63.4°)$]

5.9 Find $i_2(t)$ for $t > 0$; assume all the initial conditions to be zero.



$$\left[i_2(t) = \frac{10}{3} + \frac{5}{3}e^{-30t} - 5e^{-10t}, \text{ for } t > 0\right]$$

5.10 In the network shown,

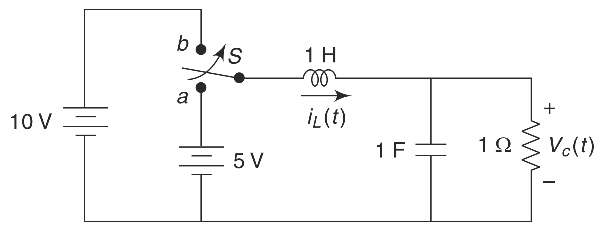


(a) determine $V_a(t)$, using Laplace transform method if $k_1 = -3$.
(b) determine $i_2(t)$, using Laplace transform method if $k_1 = 3$

[(a) $v_a(t) = 4 - e^{-0.75t}(1.5 \cos 0.25t - 0.5 \sin 0.25t)$
(b) $i_2(t) = -5 + 16.3375e^{-0.707t} - 1.3375e^{-0.707t}$ (A)]

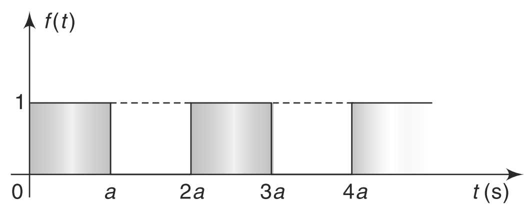
5.11 The network shown in figure, has reached steady state when the switch S moves from a to b.
(i) Determine initial values for $i_L(t)$ and $V_c(t)$ with switch in position b.
(ii) Determine $V_c(t)$ for $t > 0$. Sketch $V_c(t)$ as a function of time.
(iii) Determine damping ratio, undamped and damped natural frequencies.

$$\left[\text{(i) 5 A, 5 V; (ii) } 15 - 10e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)\frac{-20}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$



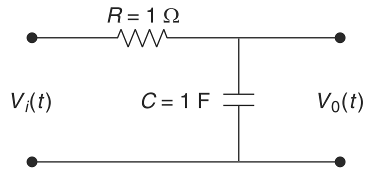
5.12 Show that the Laplace transform of the square wave is,

$$F(s) = \frac{1}{s(1+e^{-as})}$$



5.13 Verify that the convolution between two functions $f_1(t) = 2u(t)$ and $f_2(t) = \exp(-3t)u(t)$ is $\frac{2}{3}[1-\exp(-3t)];\ t > 0$ where $u(t)$ is the unit step function.

5.14 Find the response of the network shown in figure when the input voltage is: (a) unit impulse, and (b) $v_i(t) = e^{-2t}$.



[(a) e^{-t}; (b) $(e^{-t} - e^{-2t})$]

SHORT-ANSWER TYPE QUESTIONS

5.1 (a) What do you understand by Complex Frequency? Give its physical significance.
(b) Define Laplace transform of a function $f(t)$. What are the advantages of Laplace transform?

or

Explain clearly the advantages of Laplace transform method over classical method of solving differential equation with constant co-efficient describing electrical network.
(c) State and deduce initial-value and final value theorems.
(d) Write notes on: Application of Laplace transform to network analysis.

5.2 Define unit-step, unit ramp and unit impulse functions and derive their Laplace transform from first principles.

5.3 (a) Find the current $i(t)$ if unit step voltage is applied to an RL circuit.

or

Derive an expression for the current response in an R–L series circuit excited with constant voltage source.
(b) Define the term 'time-constant' of a circuit. What is the physical significance of time-constant of a circuit? Find its value for R–L series circuit.

5.4 (a) Derive an expression for the decay current in an RC circuit excited by a unit step voltage. What is the time-constant of the circuit?
Also, determine the nature of the voltage response across the capacitor.
(b) Under what conditions an RC series circuit will act as (i) a Differentiator? (ii) an Integrator?

5.5 (a) Explain the terms critical resistance, damping ratio and frequency as applied to the study of RLC series circuit. How they help in simplifying the analysis of the circuit?
(b) Derive an expression for the current $i(t)$ flowing through an RLC series circuit. Explain with suitable sketches the variation of current with time under three conditions:
(I) Under damped,
(II) Critically damped,
(III) Over damped.

5.6 What do you understand by the impulse response of a network? Briefly explain its importance in network analysis.

5.7 What do you understand by transient and steady state parts of response? How can they be identified in a general solution?

or

Discuss the natural and steady state response of an electrical circuit with illustrative examples.

or

Write notes on: (a) Transient and steady state response (b) Free and forced response.

5.8 State and prove Convolution Theorem. What is the necessity of convolution theorem in circuit analysis?

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

5.1 (b)	5.2 (c)	5.3 (a)	5.4 (c)	5.5 (c)	5.6 (c)	5.7 (c)
5.8 (a)	5.9 (a)	5.10 (a)	5.11 (a)	5.12 (b)	5.13 (c)	5.14 (a)
5.15 (b)	5.16 (c)	5.17 (b)	5.18 (c)	5.19 (b)	5.20 (d)	5.21 (a)
5.22 (b)	5.23 (a)	5.24 (a)	5.25 (d)	5.26 (c)	5.27 (a)	5.28 (b)
5.29 (c)	5.30 (d)	5.31 (c)	5.32 (b)	5.33 (c)	5.34 (c)	5.35 (d)
5.36 (a)	5.37 (a)	5.38 (a)	5.39 (c)	5.40 (d)	5.41 (d)	5.42 (d)
5.43 (b)	5.44 (a)	5.45 (b)	5.46 (d)	5.47 (c)	5.48 (b)	5.49 (b)
5.50 (b)	5.51 (b)	5.52 (b)	5.53 (d)	5.54 (c)	5.55 (c)	5.56 (b)
5.57 (a)	5.58 (b)	5.59 (b)	5.60 (c)	5.61 (a)	562 (c)	5.63 (a)
5.64 (b)	5.65 (b)	5.66 (a)	5.67 (a)	5.68 (c)	5.69 (c)	5.70 (d)
5.71 (d)	5.72 (b)	5.73 (d)	5.74 (b)	5.75 (a)	5.76 (b)	5.77 (d)
5.78 (b)	5.79 (d)	5.80 (c)	5.81 (a)	5.82 (c)	5.83 (b)	5.84 (b)
5.85 (d)	5.86 (b)	5.87 (d)	5.88 (a)	5.89 (c)	5.90 (c)	5.91 (b)
5.92 (a)	5.93 (c)	5.94 (a)	5.95 (c)	5.96 (a)	5.97 (b)	5.98 (c)
5.99 (a)						

CHAPTER

6

Two-port Network

6.1 INTRODUCTION

It is convenient to develop special methods for the systematic treatment of networks. In the case of a single port linear active network, we obtained the Thevenin's equivalent circuit and the Norton's equivalent circuit. When a linear passive network is considered, it is convenient to study its behaviour relative to a pair of designated nodes.

To represent the general nature of a network, it is normally represented by a rectangular box. If a conductor is fastened to a node in the network and brought for access, the end of the conductor is called a terminal. The minimum number of terminals that is useful are two.

A *Port* is a pair of nodes across which a device can be connected. The voltage is measured across the pair of nodes and the current going into one node is the same as the current coming out of the other node in the pair. These pairs are entry (or exit) points of the network.

So, a network with two input terminals and two output terminals is called a *four-terminal network* or a *two-port network.*

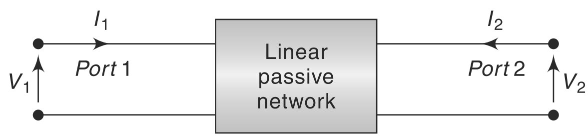


Figure 6.1 *Block diagram of a two-port network*

6.2 RELATIONSHIPS OF TWO-PORT VARIABLES

In order to describe the relationships among the port voltages and currents of an n-port network, 'n' number of linear equations is required. However, the choice of two independent and two dependent variables is dependent on the particular application.

For n-port network, the number of voltage and current variables is $2n$. The number of ways in which these $2n$ variables can be arranged in two groups of n each is $\frac{2n!}{n! \times n!} = \frac{2n!}{(n!)^2}$. So, there will be $\frac{2n!}{(n!)^2}$ types of port parameters.

For a two-port network ($n = 2$), there are six types of parameters as mentioned below:—

1. Open-Circuit Impedance Parameters (z-parameters),
2. Short-Circuit Admittance Parameters (y-parameters),
3. Transmission or Chain Parameters (T- parameters or $ABCD$ – parameters),
4. Inverse Transmission Parameters (T'-parameters),
5. Hybrid Parameters (h-parameters), and
6. Inverse Hybrid Parameters (g-parameters).

Note: Inverse parameters (T' & g) are not included in WBUT syllabus.

6.2.1 Open-Circuit Impedance Parameters (*z*-parameters)

The impedance parameters represent the relation between the voltages and the currents in the two-port network.

The impedance parameter matrix may be written as,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

In this matrix equation, it is easily seen without even expanding the individual equations, that

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \text{Driving Point Impedance at Port-1.}$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \text{Transfer Impedance}$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \text{Transfer Impedance}$$

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \text{Driving Point Impedance at Port-2}$$

It can be seen that the z-parameters correspond to the *driving point* and *transfer* impedances at each port with the other port having zero current (i.e. open circuit). Thus these parameters are also referred to as the open circuit parameters.

6.2.2 Short-Circuit Admittance Parameters (*y*-parameters)

The admittance parameters represent the relation between the currents and the voltages in the two-port network.

The admittance parameter matrix may be written as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The parameters y_{11}, y_{12}, y_{21}, y_{22} can be defined in a similar manner, with either V_1 or V_2 on short circuit.

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \text{Driving Point Admittance at Port-1}$$

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \text{Transfer Admittance}$$

$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \text{Transfer Admittance}$$

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \text{Driving Point Admittance at Port-2}$$

It can be seen that the y-parameters correspond to the *driving point* and *transfer* admittances at each port with the other port having zero voltage (i.e., short circuit). Thus these parameters are also referred to as the short circuit parameters.

6.2.3 Transmission Line Parameters (ABCD-parameters)

The *ABCD* parameters represent the relation between the input quantities and the output quantities in the two-port network. They are thus voltage-current pairs.

However, as the quantities are defined as an input-output relation, the output current is marked as going out rather than as coming into the port.

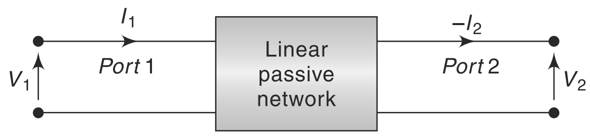


Figure 6.2 *Two-port current and voltage variables for calculation of transmission line parameters*

The transmission parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The parameters A, B, C, D can be defined in a similar manner with either port 2 on short circuit or port 2 on open circuit.

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \text{Open Circuit Reverse Voltage Gain}$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = \text{Short Circuit Transfer Impedance}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \text{Open Circuit Transfer Admittance}$$

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = \text{Short Circuit Reverse Current Gain}$$

These parameters are known as transmission parameters as in a transmission line, the currents enter at one end and leaves at the other end, and we need to know a relation between the sending end quantities and the receiving end quantities.

6.2.4 Hybrid Parameters (*h*-parameters)

The hybrid parameters represent a mixed or hybrid relation between the voltages and the currents in the two-port network.

The hybrid parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

The h-parameters can be defined in a similar manner and are commonly used in some electronic circuit analysis.

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \text{Short Circuit Impedance at Port-1}$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \text{Open Circuit Reverse Voltage Gain}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \text{Short Circuit Current Gain}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \text{Open Circuit Output Admittance}$$

As the h-parameters are dimensionally mixed, they are also named mixed parameters. Transistor circuit models are generally represented by these parameters as the input impedance (h_{11}) and the short-circuit current gain (h_{21}) can be easily measured by making the output short-circuited.

6.3 CONDITIONS FOR RECIPROCITY AND SYMMETRY

A network is said to be reciprocal if the ratio of the response transform to the excitation transform is invariant to an interchange of the positions of the excitation and response of the network.

A two-port network will be reciprocal if the interchange of an ideal voltage source at one port with an ideal current source at the other port does not alter the ammeter reading.

A two-port network is said to be symmetrical if the input and output ports can be interchanged without altering the port voltages and currents.

1. Conditions in terms of z-parameters ***Condition for Reciprocity*** We short circuit port $2 - 2'$ and apply a voltage source V_s at port $1 - 1'$.

Therefore, $V_1 = V_s$, $V_2 = 0$, $I_2 = -I_2'$

Writing the equations of z-parameters,

$$V_s = z_{11}I_1 - z_{12}I_2'$$

$$0 = z_{21}I_1 - z_{22}I_2'$$

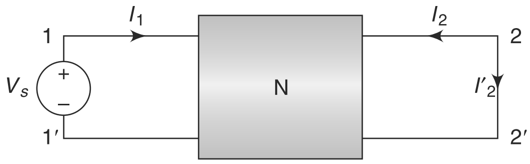


Fig. 6.3(a) *Reciprocal network*

Solving these two equations for I_2',

$$I_2' = V_s \frac{z_{21}}{z_{11}z_{22} - z_{12}z_{21}} \tag{6.1}$$

Now, interchanging the positions of response and excitations, i.e., shorting port $1 - 1'$ and applying V_s at port $2 - 2'$; $V_1 = 0$, $V_2 = V_s$, $I_1 = I_1'$

Writing the equations of z-parameters,

$$0 = -z_{11}I_1' + z_{12}I_2$$

$$V_s = -z_{21}I_1' + z_{22}I_2$$

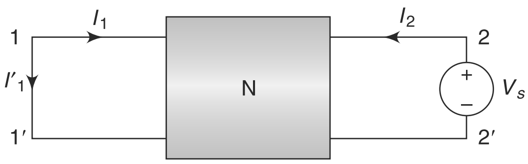


Fig. 6.3(b) *Reciprocal network*

Solving these two equations for I_1',

$$I_1' = V_s \frac{z_{12}}{z_{11}z_{22} - z_{12}z_{21}} \tag{6.2}$$

For the two-port network to be reciprocal, from Eq. (6.1) and Eq. (6.2), we have the condition as,

$$\boxed{\mathbf{z_{12} = z_{21}}}$$

Condition for symmetry

Applying a voltage V_s at port $1 - 1'$ with port $2 - 2'$ open, we have the equation,

$$V_s = z_{11}I_1 - z_{12} \cdot 0 = z_{11}I_1 \quad \Rightarrow \quad \left.\frac{V_s}{I_1}\right|_{I_2=0} = z_{11} \tag{6.3}$$

Now, applying a voltage V_s at port $2 - 2'$ with port $1 - 1'$ open, we have the equation,

$$V_s = z_{21} \cdot 0 + z_{22}I_2 = z_{22}I_2 \quad \Rightarrow \quad \left.\frac{V_s}{I_2}\right|_{I_1=0} = z_{22} \tag{6.4}$$

For the network to be symmetrical, the voltages and currents should be same. From Eq. (6.3) and Eq. (6.4), we have the condition for symmetry as,

$$\boxed{\mathbf{z_{11} = z_{22}}}$$

2. Conditions in terms of y-parameters *Condition for Reciprocity*

From Fig. 6.3(a), writing the y-parameter equations,

$$\begin{aligned} I_1 &= y_{11}V_s \\ -I_2' &= y_{21}V_s \end{aligned} \quad \Rightarrow \quad -\frac{I_2'}{V_s} = y_{21} \tag{6.5}$$

From Fig. 6.3(b), writing the y-parameter equations,

$$\begin{aligned} -I_1' &= y_{12}V_s \\ I_2 &= y_{22}V_s \end{aligned} \quad \Rightarrow \quad -\frac{I_1'}{V_s} = y_{12} \tag{6.6}$$

From the principle of reciprocity, the condition for reciprocity is,

$$\boxed{\mathbf{y_{12} = y_{21}}}$$

Condition for symmetry

As already stated, a two-port network is said to be symmetric if the ports can be interchanged without changing the port voltages and currents and thus the condition of symmetry becomes,

$$\boxed{\mathbf{y_{11} = y_{22}}}$$

3. Conditions in terms of ABCD-parameters *Condition for Reciprocity*

From Fig. 6.3(a), writing the $ABCD$-parameter equations,

$$\begin{aligned} V_s &= A \cdot 0 - B(-I_2') = BI_2' \\ I_1 &= C \cdot 0 - D(-I_2') = DI_2' \end{aligned} \quad \Rightarrow \quad \frac{I_2'}{V_s} = \frac{1}{B} \tag{6.7}$$

From Fig. 6.3(b), writing the $ABCD$-parameter equations,

$$\begin{aligned} 0 &= AV_s - BI_2 \\ -I_1' &= CV_s - DI_2 \end{aligned} \quad \Rightarrow \quad \frac{I_1'}{V_s} = \frac{AD - BC}{B} \tag{6.8}$$

From the principle of reciprocity, the condition for reciprocity is, $\dfrac{1}{B} = \dfrac{(AD - BC)}{B}$

$$\boxed{\mathbf{(AD - BC) = 1}}$$

Condition for symmetry

From Eq. (6.7), $I_1 = DI_2' = D\dfrac{V_s}{B}$ (6.9)

From Eq. (6.8), $I_2 = \dfrac{I_1' + CV_s}{D} = \dfrac{1}{D}\left\{V_s\left(\dfrac{AD - BC}{B}\right) + CV_s\right\} = V_s\dfrac{A}{B}$ (6.10)

From Eq. (6.9) and Eq. (6.10), we have the condition for symmetry as,

$$\boxed{\mathbf{A = D}}$$

4. Conditions in terms of h-parameters *Condition for Reciprocity*

From Fig. 6.3(a), writing the h-parameter equations,

$$\begin{aligned} V_s &= h_{11}I_1 + h_{12}\cdot 0 = h_{11}I_1 \\ -I_2' &= h_{21}I_1 + h_{22}\cdot 0 = h_{21}I_1 \end{aligned} \quad \Rightarrow \quad \frac{I_2'}{V_s} = -\frac{h_{21}}{h_{11}} \tag{6.11}$$

From Fig. 6.3(b), writing the h-parameter equations,

$$\begin{aligned} 0 &= -h_{11}I_1' + h_{12}V_s \\ I_2 &= -h_{21}I_1' + h_{22}V_s \end{aligned} \quad \Rightarrow \quad \frac{I_1'}{V_s} = \frac{h_{12}}{h_{11}} \tag{6.12 }$$

From the principle of reciprocity, the condition for reciprocity is,

$$\boxed{\mathbf{h_{12} = -h_{21}}}$$

From Eq. (6.11), $I_1 = \dfrac{V_s}{h_{11}}$ (6.13)

From Eq. (6.12), $I_2 = -h_{21}\left(\dfrac{h_{12}}{h_{11}}V_s\right) + h_{22}V_s = V_s\dfrac{h_{11}h_{22} - h_{12}h_{21}}{h_{11}}$ (6.14)

From Eq. (6.13) and Eq. (6.14), we have the condition for symmetry as,

$$\boxed{(h_{11}h_{22} - h_{12}h_{21}) = 1}$$

6.4 INTERRELATIONSHIPS BETWEEN TWO-PORT PARAMETERS

Each type of two-port parameter has its own utility and is suited for certain specific applications. However, it is sometime necessary to convert one set of parameters to another. It is possible through simple mathematical manipulations to convert one set to any of the remaining sets. It is discussed below.

1. z-parameters in Terms of Other Parameters

(a) in terms of y-parameters

The z-parameter equations are,

$$\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned} \tag{6.15}$$

The y-parameter equations are,

$$\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned} \tag{6.16}$$

From Eq. (6.16), $V_2 = \dfrac{I_2}{y_{22}} - \dfrac{y_{21}}{y_{22}}V_1$; substituting this in first equation,

$$I_1 = y_{11}V_1 + y_{12}\left(\frac{I_2}{y_{22}} - \frac{y_{21}}{y_{22}}V_1\right)$$

or $$V_1 = \frac{y_{22}}{\Delta y} I_1 - \frac{y_{12}}{\Delta y} I_2 \quad (6.17)$$

where, $\Delta y = (y_{11}y_{22} - y_{12}y_{21})$
Substituting this value in second equation of Eq. 6.16

$$I_2 = y_{21}\left(\frac{y_{22}}{\Delta y} I_1 - \frac{y_{12}}{\Delta y} I_2\right) + y_{22}V_2$$

or, $$V_2 = -\frac{y_{21}}{\Delta y} I_1 + \frac{y_{11}}{\Delta y} I_2 \quad (6.18)$$

Comparing Eqs (6.15), (6.17) and (6.18), we get,

$$\boxed{z_{11} = \frac{y_{22}}{\Delta y};\ z_{12} = -\frac{y_{12}}{\Delta y};\ z_{21} = -\frac{y_{21}}{\Delta y};\ z_{22} = \frac{y_{11}}{\Delta y}}$$

(b) in terms of transmission parameters
The Transmission parameter equations are,

$$\begin{aligned} V_1 &= AV_2 - BI_2 \\ I_1 &= CV_2 - DI_2 \end{aligned} \quad (6.19)$$

From second equation of Eq. (6.19),

$$V_2 = \left(\frac{1}{C}\right) I_1 + \left(\frac{D}{C}\right) I_2 \quad (6.20)$$

From first equation of Eq. (6.19),

$$V_1 = A\left[\left(\frac{1}{C}\right) I_1 + \left(\frac{D}{C}\right) I_2\right] - BI_2$$

$$= \left(\frac{A}{C}\right) I_1 + \left(\frac{AD - BC}{C}\right) I_2 \quad (6.21)$$

Comparing Eq. (6.20) and (6.21) with Eq. (6.15), we get,

$$\boxed{z_{11} = \frac{A}{C};\ z_{12} = \frac{AD - BC}{C} = \frac{\Delta T}{C};\ z_{21} = \frac{1}{C};\ z_{22} = \frac{D}{C}}$$

(c) in terms of hybrid parameters
The hybrid parameter equations are,

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ I_2 &= h_{21}I_1 + h_{22}V_2 \end{aligned} \quad (6.22)$$

From second equation, $$V_2 = \left(-\frac{h_{21}}{h_{22}}\right) I_1 + \left(\frac{1}{h_{22}}\right) I_2 \quad (6.23)$$

From first equation, $$V_1 = h_{11}I_1 + h_{12}\left[\left(-\frac{h_{21}}{h_{22}}\right) I_1 + \left(\frac{1}{h_{22}}\right) I_2\right] = \left(\frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}\right) I_1 + \left(\frac{h_{12}}{h_{22}}\right) I_2 \quad (6.24)$$

Comparing Eqs (6.23) and (6.24) with Eq. (6.15), we get,

$$\boxed{z_{11}=\frac{h_{11}h_{22}-h_{12}h_{21}}{h_{22}}=\frac{\Delta h}{h_{22}};\, z_{12}=\frac{h_{12}}{h_{22}};\, z_{21}=-\frac{h_{21}}{h_{22}};\, z_{22}=\frac{1}{h_{22}}}$$

Similarly, the inter-relation of the other parameter in terms of the remaining parameters is obtained by writing the remaining parameter equations in the same format as those of the other parameter; and comparing the co-efficients of the two sets of equations, a relation is obtained.

A summary of the relationships between impedance z-parameters, admittance y-parameters, hybrid h-parameters, and transmission $ABCD$-parameters is shown in Table where $\Delta z = (z_{11}z_{22} - z_{12}z_{21})$, $\Delta h = (h_{11}h_{22} - h_{12}h_{21})$, $\Delta T = (AD - BC)$, $\Delta T' = (A'D' - B'C')$, and $\Delta g = (g_{11}g_{22} - g_{12}g_{21})$.

Table 6.1 *Interrelationships between Two-Port Parameters*

	[z]	[y]	[ABCD]	[A′B′C′D′]	[h]	[g]
[z]	$\begin{matrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{matrix}$	$\begin{matrix} \frac{y_{22}}{\Delta y} & -\frac{y_{12}}{\Delta y} \\ \frac{-y_{21}}{\Delta y} & \frac{y_{11}}{\Delta y} \end{matrix}$	$\begin{matrix} \frac{A}{C} & \frac{\Delta T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$	$\begin{matrix} \frac{D'}{C'} & \frac{1}{C'} \\ \frac{\Delta T'}{C'} & \frac{A'}{C'} \end{matrix}$	$\begin{matrix} \frac{\Delta h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ -\frac{h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{11}} & -\frac{g_{12}}{g_{11}} \\ \frac{g_{21}}{h_{11}} & \frac{\Delta g}{g_{11}} \end{matrix}$
[y]	$\begin{matrix} \frac{z_{22}}{\Delta z} & -\frac{z_{12}}{\Delta z} \\ -\frac{z_{21}}{\Delta z} & \frac{z_{11}}{\Delta z} \end{matrix}$	$\begin{matrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{matrix}$	$\begin{matrix} \frac{D}{B} & -\frac{\Delta T}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{matrix}$	$\begin{matrix} \frac{A'}{B'} & -\frac{1}{B'} \\ -\frac{\Delta T'}{B'} & \frac{D'}{B'} \end{matrix}$	$\begin{matrix} \frac{1}{h_{11}} & -\frac{h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta h}{h_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta g}{g_{22}} & \frac{g_{12}}{g_{22}} \\ -\frac{g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{matrix}$
[$ABCD$]	$\begin{matrix} \frac{z_{11}}{z_{21}} & \frac{\Delta z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{matrix}$	$\begin{matrix} -\frac{y_{22}}{y_{21}} & -\frac{1}{y_{21}} \\ -\frac{\Delta y}{y_{21}} & -\frac{y_{11}}{y_{21}} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} \frac{D'}{\Delta T'} & \frac{B'}{\Delta T'} \\ \frac{C'}{\Delta T'} & \frac{A'}{\Delta T'} \end{matrix}$	$\begin{matrix} -\frac{\Delta h}{h_{21}} & -\frac{h_{11}}{h_{21}} \\ -\frac{h_{22}}{h_{21}} & -\frac{1}{h_{21}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{21}} & -\frac{g_{22}}{g_{21}} \\ \frac{g_{11}}{g_{21}} & \frac{\Delta g}{g_{21}} \end{matrix}$
[$A'B'C'D'$]	$\begin{matrix} \frac{z_{22}}{z_{12}} & \frac{\Delta z}{z_{12}} \\ \frac{1}{z_{12}} & \frac{z_{11}}{z_{12}} \end{matrix}$	$\begin{matrix} -\frac{y_{11}}{y_{12}} & -\frac{1}{y_{12}} \\ -\frac{\Delta y}{y_{12}} & -\frac{y_{22}}{y_{12}} \end{matrix}$	$\begin{matrix} \frac{D}{\Delta T} & \frac{B}{\Delta T} \\ \frac{C}{\Delta T} & \frac{A}{\Delta T} \end{matrix}$	$\begin{matrix} A' & B' \\ C' & D' \end{matrix}$	$\begin{matrix} \frac{1}{h_{22}} & \frac{h_{11}}{h_{12}} \\ \frac{h_{22}}{h_{12}} & \frac{\Delta h}{h_{12}} \end{matrix}$	$\begin{matrix} -\frac{\Delta g}{g_{12}} & -\frac{g_{22}}{g_{12}} \\ -\frac{g_{11}}{g_{12}} & -\frac{1}{g_{12}} \end{matrix}$
[h]	$\begin{matrix} \frac{\Delta z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ -\frac{z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{y_{11}} & -\frac{y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta y}{y_{11}} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta T}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{matrix}$	$\begin{matrix} \frac{B'}{A'} & \frac{1}{A'} \\ -\frac{\Delta T'}{A'} & \frac{D'}{B'} \end{matrix}$	$\begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix}$	$\begin{matrix} \frac{g_{22}}{\Delta g} & -\frac{g_{12}}{\Delta g} \\ -\frac{g_{211}}{\Delta g} & \frac{g_{11}}{\Delta g} \end{matrix}$
[g]	$\begin{matrix} \frac{1}{z_{11}} & -\frac{z_{12}}{z_{11}} \\ \frac{z_{21}}{z_{11}} & \frac{\Delta z}{z_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta y}{y_{22}} & \frac{y_{12}}{y_{22}} \\ -\frac{y_{21}}{y_{22}} & -\frac{1}{y_{22}} \end{matrix}$	$\begin{matrix} \frac{C}{A} & -\frac{\Delta T}{A} \\ \frac{1}{A} & \frac{B}{A} \end{matrix}$	$\begin{matrix} \frac{C'}{D'} & -\frac{1}{D'} \\ \frac{\Delta T'}{D'} & \frac{B'}{D'} \end{matrix}$	$\begin{matrix} \frac{h_{22}}{\Delta h} & -\frac{h_{12}}{\Delta h} \\ -\frac{h_{21}}{\Delta h} & \frac{h_{11}}{\Delta h} \end{matrix}$	$\begin{matrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{matrix}$

6.5 INTERCONNECTION OF TWO-PORT NETWORKS

In certain applications, it becomes necessary to connect the two-port networks together. The common connections are (a) series, (b) parallel and (c) cascade.

(a) Series connection of two-port networks

As in the case of elements, a series connection is defined when the currents in the series elements are equal and the voltages add up to give the resultant voltage.

In the case of two-port networks, this property must be applied individually to each of the ports. Thus, if we consider 2 networks ***r*** and ***s*** connected in series

At port 1,

$$I_{r1} = I_{s1} = I_1, \text{ and } V_{r1} + V_{s1} = V_1$$

Similarly, at port 2,

$$I_{r2} = I_{s2} = I_2 \text{ and } V_{r2} + V_{s2} = V_2$$

The two networks, ***r*** and ***s*** can be connected in the following manner to be in series with each other.

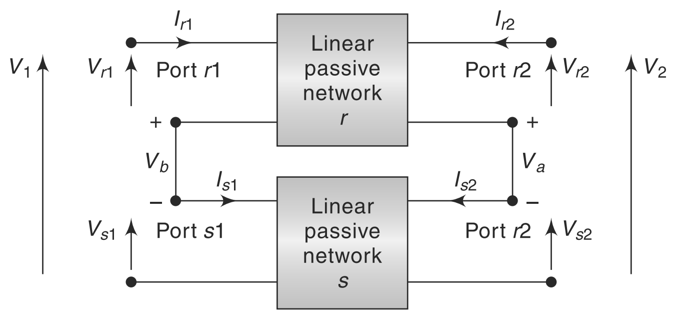


Figure 6.4 *Series connection of two-port networks*

Under these conditions,

$$V_1 = (V_{r1} + V_{s1}) = (z_{11r} + z_{11s})I_1 + (z_{12r} + z_{12s})I_2$$

$$V_2 = (V_{r2} + V_{s2}) = (z_{21r} + z_{21s})I_1 + (z_{22r} + z_{22s})I_2$$

It is seen that, the ***resultant impedance parameter matrix for the series combination is the addition of the two individual impedance matrices.***

$$[Z] = [Zr] + [Zs]$$

Note: In the interconnection of series networks, there is a strong requirement of isolation, since the ground node of upper network form the non-ground node of the lower network. For the port properties to be valid, the voltages V_a and V_b must be identically zero for the two networks *r* and *s* to be connected in series. If V_a and V_b are not zero, then by connecting the two ports there will be a circulating current and port property of the individual networks *r* and *s* will be violated.

(b) Parallel connection of two-port networks

As in the case of elements, a parallel connection is defined when the voltages in the parallel elements are equal and the currents add up to give the resultant current.

In the case of two-port networks, this property must be applied individually to each of the ports. Thus, if we consider 2 networks ***r*** and ***s*** connected in parallel,

At port 1,

$$I_{r1} + I_{s1} = I_1, \text{ and } V_{r1} = V_{s1} = V_1$$

Similarly, at port 2,

$$I_{r2} + I_{s2} = I_2 \text{ and } V_{r1} = V_{s1} = V_1$$

The two networks, ***r*** and ***s*** can be connected in following manner to be in parallel with each other.

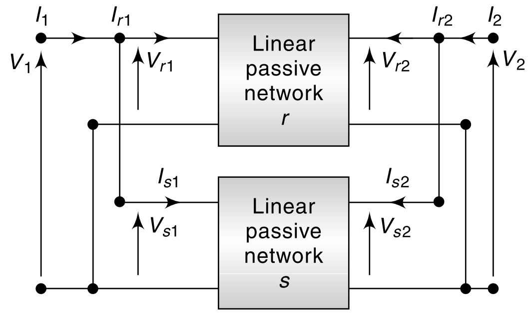


Figure 6.5 *Parallel connection of two-port networks*

Under these conditions,

$$I_1 = (I_{r1} + I_{s1}) = (y_{11r} + y_{11s})V_1 + (y_{12r} + y_{12s})V_2$$

$$I_2 = (I_{2r} + I_{2s}) = (y_{21r} + y_{21s})V_1 + (y_{22r} + y_{22s})V_2$$

It is seen that, the *resultant admittance parameter matrix for the parallel combination is the addition of the two individual admittance matrices.*

$$[Y] = [Yr] + [Ys]$$

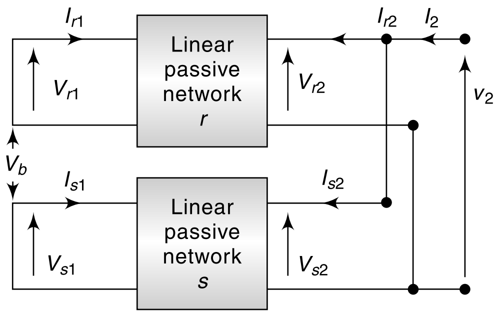


Figure 6.6(a) $V_b = 0$

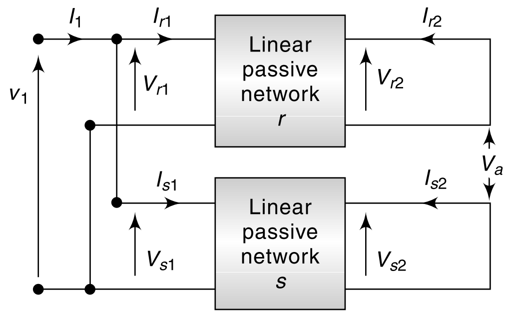


Figure 6.6(b) $V_a = 0$

Note: As in series connection, parallel connection is also possible under the condition that $V_a = V_b = 0$; otherwise they cannot be connected in parallel as that will violate the port properties.

(c) Cascade connection of two-port networks

A cascade connection is defined when the output of one network becomes the input to the next network.

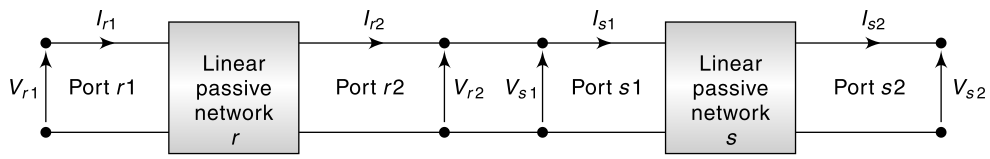


Figure 6.7 *Cascade connection of two-port network*

It can be easily seen that $I_{r2} = I_{s1}$ and $V_{r2} = V_{s1}$.

Therefore it can easily be seen that the *ABCD* parameters are the most suitable to be used for this connection.

$$\begin{bmatrix} V_{r1} \\ I_{r1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} V_{r2} \\ I_{r2} \end{bmatrix}, \quad \begin{bmatrix} V_{s1} \\ I_{s1} \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} V_{s2} \\ I_{s2} \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{r1} \\ I_{r1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} V_{r2} \\ I_{r2} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} V_{s1} \\ I_{s1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}\begin{bmatrix} V_{s2} \\ I_{s2} \end{bmatrix}$$

$$= \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}\begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Thus it is seen that the *overall ABCD matrix is the product of the two individual ABCD matrices*. This is a very useful property in practice, especially when analyzing transmission lines.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}$$

6.6 NETWORK FUNCTIONS

As already discussed in this chapter, any electrical network may be represented in the form of a black-box showing a two port network structure as shown in Fig. 6.1. The passive elements of this network transform a given excitation function at the input port into another response function at the output port. Thus, the two port network may also be viewed as a control block shown in Fig. 6.8 having a gain of $G(s)$, which modifies any excitation function $R(s)$ into a response function $C(s)$. In mathematical form this may be expressed as;

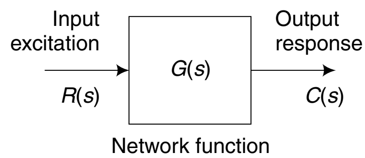


Figure 6.8 *A control block representing a network function*

$$C(s) = G(s).R(s) \tag{6.25}$$

or

$$G(s) = \frac{C(s)}{R(s)} \tag{6.26}$$

Network functions are used to describe networks which have two ports at the least. These functions may be broadly classified as (i) driving point functions and (ii) transfer functions in view of the two port network structure. In general, if a function relates the transform of a quantity at one port to the transform of another quantity at the same port it may be regarded as a driving point function. On the contrary, if a function relates the transform of a quantity at one port to the transform of another quantity at the other port it may be regarded as a transfer function. In view of Eqn. (6.26), it may be possible to express the transformed network function as the ratio of transformed response function to the transformed excitation function. According to the degree of complexity of the passive elements in the network, the general form of the network function $F(s)$ may be expressed as the ratio of two transform polynomials as shown in Eqn. (6.27).

$$F(s) = \frac{F(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots\dots\dots + a_2 s^2 + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \dots\dots\dots + b_2 s^2 + b_1 s + b_0} \tag{6.27}$$

Where,

$a_0, a_1, \ldots \ldots, a_n$ represent the coefficients of the numerator polynomial $N(s)$,

$b_0, b_1, \ldots \ldots, b_m$ represent the coefficients of the denominator polynomial $D(s)$,

n represents the highest order or degree of the numerator polynomial $N(s)$,

m represents the highest order or degree of the denominator polynomial $D(s)$.

A careful observation of Eqn. (6.27) would reveal the following characteristics in favour of the network function, $F(s)$.

(1) Where, m and n are integers, the network function $F(s)$ is a rational function of s.

(2) If the network contains only passive elements and no controlled sources, the coefficients of numerator and denominator polynomials must be positive real numbers.

(3) As the numerator polynomial is of order n, the expression $N(s) = 0$ should have n roots $(z_1, z_2, \ldots \ldots, z_n)$, usually called the zeros of the network function. Hence it may be possible to express the numerator polynomial as,

$$N(s) = A_0(s - z_1)(s - z_2) \ldots \ldots \ldots (s - z_n)$$

(4) As the denominator polynomial is of order m, the expression $D(s) = 0$ should have m roots $(p_1, p_2, \ldots \ldots, p_m)$, usually called the poles of the network function. Hence it may be possible to express the denominator polynomial as,

$$D(s) = B_0(s - p_1)(s - p_2) \ldots \ldots \ldots (s - p_m)$$

(5) If a pole or a zero is never repeated in a network function, it is treated as simple. If repeated twice/thrice, they may be called as double/triple, as the case may be.

(6) In any case, the number of poles and zeros of a network function must be same. Thus, the highest degree of numerator polynomial (n) should be numerically equal to highest degree of denominator polynomial (m). However, if the network function does not show this directly, then some poles and zeros may be considered to be located at infinity in order to satisfy this criterion. This condition emphasizes that, if, $n > m$, then the pole at infinity is of degree $(n - m)$, or otherwise, if, if $m > n$, then the zero at infinity is of degree $(m - n)$.

(7) The roots of the network function representing the poles and zeros are called critical complex frequencies for which the network function may be critical (zero or infinite).

(8) For any other complex frequency ($\omega \neq z$ and $\neq p$) the network function would have a non zero finite value.

In view of these observations, it may be possible to express Eqn. (6.27) in a new form as shown in Eqn. (6.28).

$$F(s) = \frac{N(s)}{D(s)} = \frac{A_0(s - z_1)(s - z_2)\ldots\ldots\ldots(s - z_n)}{B_0(s - p_1)(s - p_2)\ldots\ldots\ldots(s - p_m)} = H\frac{(s - z_1)(s - z_2)\ldots\ldots\ldots(s - z_n)}{(s - p_1)(s - p_2)\ldots\ldots\ldots(s - p_m)} \qquad (6.28)$$

Where, $H = \dfrac{A_0}{B_0}$ is called the scale factor. A pictorial view of a zero and a pole of a network function in a complex plane are shown in Fig. 6.9 for the sake of understanding only.

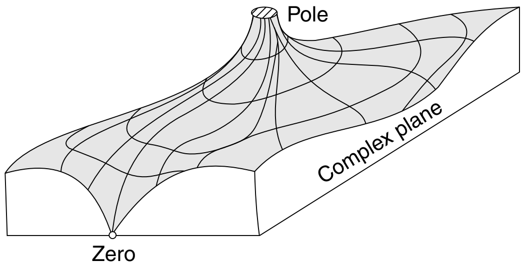


Figure 6.9 *Location of poles and zeros in a complex plane*

6.7 SIGNIFICANCE OF POLES AND ZEROS

Poles and zeros play an important role in any network function. Since these are the roots of a network function, knowing them it may be possible to derive the structure of a network and speculate the response for a deterministic excitation. Also, the poles and zeros contain some useful information regarding the criticality of a network function and they may predict/regulate the stability of a network too, as far as performance is concerned. The significance may be more appealing to the reader through the following case studies.

Case-I: While considering the network function as driving point impedance, given by $Z_{1_1}(s) = \dfrac{V_1(s)}{I_1(s)}$,

(i) current becomes zero for a finite value of driving voltage at the location of a pole, thus representing an open circuit condition.

(ii) voltage becomes zero for a finite value of driving point current at the location of a zero, thus representing a short circuit condition.

Case-II: While considering the network function as driving point admittance, given by $Y_{1_1}(s) = \dfrac{I_1(s)}{V_1(s)}$,

(i) voltage becomes zero for a finite value of driving current at the location of a pole, thus representing a short circuit condition.

(ii) current becomes zero for a finite value of driving voltage at the location of a zero, thus representing an open circuit condition.

Case-III: While considering the network function as voltage transfer ratio, given by $G_{2_1}(s) = \dfrac{V_2(s)}{V_1(s)}$,

(i) poles determine the time variation of the response, whereas,

(ii) zeros determine the magnitude variation of the response.

6.8 RESTRICTION ON LOCATION OF POLES AND ZEROS

As we find, the network functions may be clubbed into two distinct categories, such as; (i) driving point immittance (impedance or admittance) functions, and (ii) transfer functions. Therefore, it would be better if we consider the restrictions imposed on the location of poles and zeros for each case separately.

Case-I: While considering the network function as driving point immittance, given by $Z_{1_1}(s) = \dfrac{V_1(s)}{I_1(s)}$, or $Y_{1_1}(s) = \dfrac{I_1(s)}{V_1(s)}$, the following restrictions may be imposed to satisfy the necessary condition that the network driving point function is positive real;

(i) In order to ensure that the driving point impedance function is a positive real function, the coefficients of the numerator polynomial and denominator polynomial must be positive real numbers.

(ii) All the poles and zeros must lie on the left half of the s plane. Hence all poles and zeros should have negative real part or zero real part, so that the network function remains positive real. This would ensure that the response would be bounded for a bounded excitation over any stretch of time, hence is crucial from stability point of view.

(iii) If any pole or zero happens to be a complex number, it must occur in conjugate pairs, so that the network function remains positive real. This would ensure that the response would be bounded for a bounded excitation over any stretch of time, hence is crucial from stability point of view.

(iv) If any pole or zero of a network function has a zero real part, then the said pole or zero must be simple. Hence, multiple poles and zeros are neither permitted to be located on the $j\omega$ axis nor at the origin of the s plane.

(v) Neither the numerator polynomial nor the denominator polynomial should have missing terms in between the highest and lowest order of s, unless otherwise all the even or odd terms are missing.

(vi) The highest order of numerator polynomial may differ from the highest order of denominator polynomial by a margin of unity at the most.

(vii) The lowest order of numerator polynomial may differ from the lowest order of denominator polynomial by a margin of unity at the most.

Case-II: While considering the network function as transfer functions, given by $G_{2_1}(s)$, $\alpha_{2_1}(s)$, $Z_{2_1}(s)$, and $Y_{2_1}(s)$ the following restrictions may be imposed to satisfy the necessary condition that the network transfer function is positive real;

(i) In order to ensure that the driving point impedance function is a positive real function, the coefficients of the denominator polynomial must be positive real numbers, however some of the coefficients of the numerator polynomial may be negative.

(ii) All the poles must lie on the left half of the s plane. Hence all poles should have negative real part or zero real part, so that the network function remains positive real. This would ensure that the response would be bounded for a bounded excitation over any stretch of time, hence is crucial from stability point of view.

(iii) If any pole or zero happens to be a complex number, it must occur in conjugate pairs, so that the network function remains positive real. This would ensure that the response would be bounded for a bounded excitation over any stretch of time, hence is crucial from stability point of view.

(iv) If any pole of a network function has a zero real part, then the said pole must be simple. Hence, multiple poles are neither permitted to be located on the $j\omega$ axis nor at the origin of the s plane.

(v) The denominator polynomial should not have missing terms in between the highest and lowest order of s, unless otherwise all the even or odd terms are missing. However, the numerator polynomial may have missing terms in between the highest and lowest order of s.

(vi) The order of numerator polynomial may be as small as zero, and is independent of the order of the denominator polynomial.

(vii) The highest order of numerator polynomial should be same as the highest order of the denominator polynomial for transfer functions like $G_{2_1}(s)$, and $\alpha_{2_1}(s)$.

(viii) The highest order of numerator polynomial should be more than the highest order of the denominator polynomial by a margin of unity at the most for transfer functions like $Z_{2_1}(s)$, and $Y_{2_1}(s)$.

6.9 TIME DOMAIN BEHAVIOUR FROM POLE-ZERO PLOTS

Pole-zero plots indicate the location of poles with the symbol of a cross (×) and location of zeros with the symbol of a circle (O) in the complex s plane with the help of the information obtained about the roots of the network function described in Eqn. (6.28). A particular case of such a plot is shown in Fig. 6.10. for the network function given by $F(s) = \dfrac{s(s+3)}{(s+4)(s+2+j_1)(s+2-j_1)}$. The information contained in this plot may be useful for obtaining the time domain behavior of the network function subject to deterministic inputs. In the previous section it is highlighted that poles determine the time variation of the response, whereas, zeros determine the magnitude variation of the response.

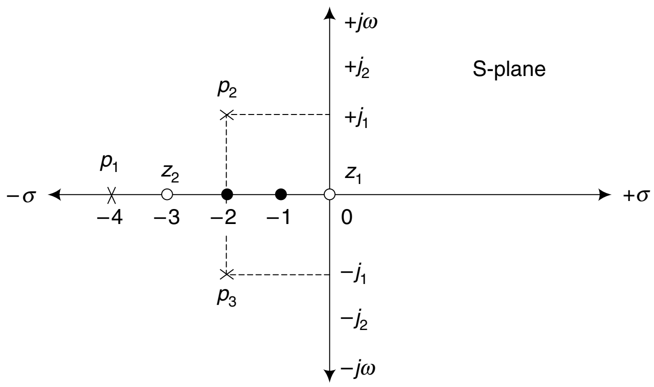


Figure 6.10 *Location of poles and zeros in s-plane*

Therefore, it may be possible to find out both the variations in time domain with the help of pole-zero plots. In addition to the information about the pole-zero location, information about the following parameters of the network may also be necessary to obtain the time response. These additional parameters are;

(a) Scale factor, (H),

(b) Damping ratio or damping coefficient, (ξ),

(c) Undamped natural frequency of the network, (ω_n).

The following cases may be possible regarding the pole-zero location, which depends on the damping ratio.

Case-I: If, $\xi = 0$, then the two roots may have values, $s_1, s_2 = \pm j\omega_n$.

Case-II: If, $\xi = 1$, then the two roots may have values, $s_1, s_2 = -\omega_n$.

Case-III: If, $\xi < 1$, then the two roots may have values,

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{\xi^2 - 1}.$$

Case-IV: If, $\xi > 1$, then the two roots may have values,

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}.$$

It would be worth assimilating the properties of these roots as contours in the s plane. If the roots have constant natural frequency (ω_n), the corresponding contours would be concentric semicircles on the left half of the s plane, which leads to undamped oscillatory response in the time domain. If the roots have constant damping ratio (ξ), the corresponding contours would be rays passing through the origin of the s plane. If the roots have constant damping ($\xi\omega_n$), the corresponding contours would be straight lines parallel to imaginary axis on the left half of the s plane. If the roots have constant complex frequency ($\omega = \omega_n\sqrt{1 - \xi^2}$ or $\omega = \omega_n\sqrt{\xi^2 - 1}$), the corresponding contours would be straight lines parallel to the real axis on either side of the real axis of the s plane, which leads to damped oscillatory response in the time domain.

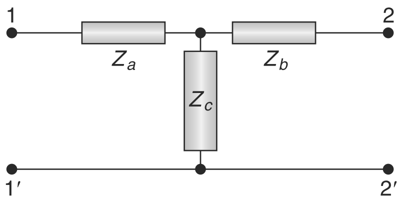
If any of the poles in the pole-zero plots resemble with the contours described above, the nature of the network function towards time domain response may be predictable. For the case of n-poles, the total time domain response may be described as an exponential function as shown in Eqn. (6.29).

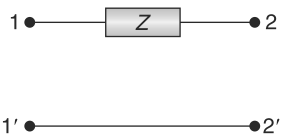
$$f(t) = K_1 e^{p_1 t} + K_2 e^{p_2 t} + \dots \dots \dots + K_n e^{p_n t} \tag{6.29}$$

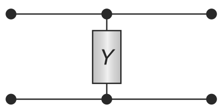
It may be understood that the involvement of zeros has been overlooked for the sake of clarity. Hence the total time domain response is the superposition of individual responses due to each pole or a pole pair. The effect of real poles contribute towards exponential damping behavior in the time domain whereas the effect of conjugate pole pairs contribute towards damped sinusoidal oscillations in the time domain.

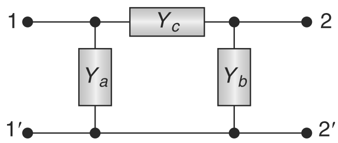
SOLVED PROBLEMS

6.1 Find the Z and Y parameter for the networks shown in figure.

(a)


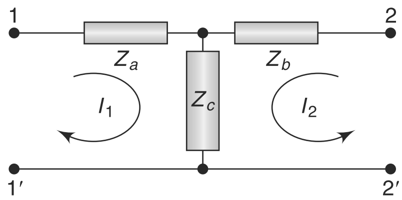
(b)


(c)


(d)


Solution

(a) By KVL, $(Z_a + Z_c)I_1 + Z_c I_2 = V_1$

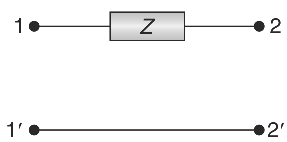


and $\quad Z_c I_1 + (Z_b + Z_c)I_2 = V_2$

Thus, the Z-parameters are:

$$z_{11} = (Z_a + Z_c),\ z_{12} = z_{21} = Z_c,\quad z_{22} = (Z_b + Z_c)$$

(b) By KCL,



$$I_1 = \frac{V_1 - V_2}{Z} = \frac{1}{Z}V_1 - \frac{1}{Z}V_2$$

and
$$I_2 = \frac{V_2 - V_1}{Z} = -\frac{1}{Z}V_1 + \frac{1}{Z}V_2$$

Thus, the y-parameters are,

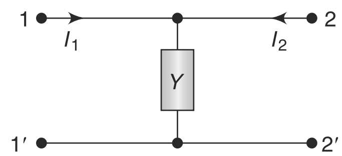
$$y_{11} = \frac{1}{Z} = y_{22} \quad y_{12} = y_{21} = -\frac{1}{Z}$$

Since, $\Delta y = y_{11}y_{22} - y_{12}y_{21} = 0$, the z-parameters do not exist for this network.

(c) By KVL,

$$V_1 = \frac{I_1 + I_2}{Y} = V_2 \quad \text{or,}\ V_1 = \left(\frac{1}{Y}\right)I_1 + \left(\frac{1}{Y}\right)I_2 \quad \text{and} \quad V_2 = \left(\frac{1}{Y}\right)I_1 + \left(\frac{1}{Y}\right)I_2$$

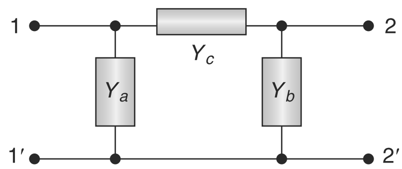
Thus, the z-parameters are,



$$z_{11} = z_{22} = \frac{1}{Y} = z_{12} = z_{21}$$

Since, $\Delta z = z_{11}z_{22} - z_{12}z_{21} = 0$, the y-parameters do not exist for this network.

(d) By KCL,



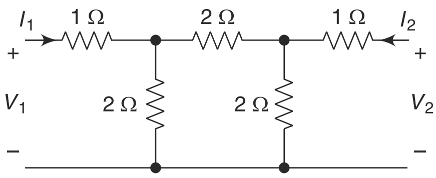
$$I_1 = Y_aV_1 + (V_1 - V_2)Y_c = V_1(Y_a + Y_c) - V_2Y_c$$
$$I_2 = Y_bV_2 + (V_2 - V_1)Y_c = -V_1Y_c + V_2(Y_b + Y_c)$$

Thus, the y-parameters are:

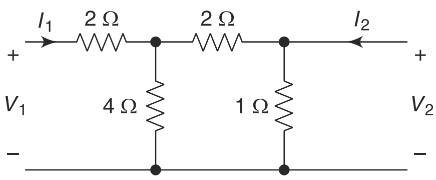
$$y_{11} = Y_a + Y_c;\ y_{12} = y_{21} = -Y_c;\ y_{22} = Y_b + Y_c$$

6.2 Obtain the z-parameters for the circuit shown in figure.

(a)

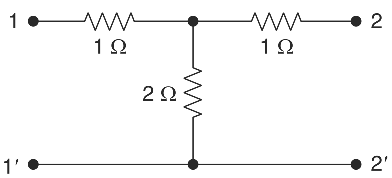


(b)

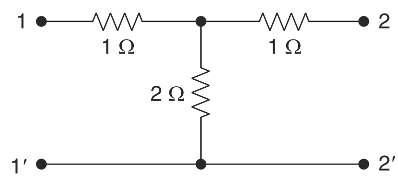


Solution

(a) The given circuit can be considered as the cascade connection of the following two networks:



(a)



(b)

From Prob. 6.1(a), $z_{11a} = z_{11b} = z_{22a} = z_{22b} = 3\Omega$
$z_{12a} = z_{21a} = z_{12b} = z_{21b} = 2\Omega$

So, the transmission parameters are,

$$\therefore \qquad A_a = A_b = \frac{z_{11}}{z_{21}} = \frac{3}{2}$$

$$\therefore \quad B_a = B_b = \frac{\Delta z}{z_{21}} = \frac{9-4}{2} = \frac{5}{2}\ \Omega$$

$$\therefore \quad C_a = C_b = \frac{1}{z_{21}} = \frac{1}{2}\ \mho$$

$$\therefore \quad D_a = D_b = \frac{z_{22}}{z_{21}} = \frac{3}{2}$$

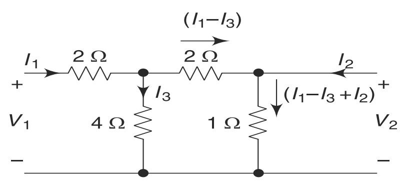
So, the transmission parameters of the resulting network are:

$$T = T_a \times T_b = \begin{bmatrix} 3/2 & 5/2 \\ 1/2 & 3/2 \end{bmatrix}\begin{bmatrix} 3/2 & 5/2 \\ 1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 7/2 & 15/2 \\ 3/2 & 7/2 \end{bmatrix}$$

So, the z-parameters are:

$$\left.\begin{aligned} z_{11} &= \frac{A}{C} = \frac{7}{3}\ \Omega \\ z_{12} &= \frac{\Delta T}{C} = \frac{2}{3}\ \Omega \\ z_{21} &= \frac{1}{C} = \frac{2}{3}\ \Omega \\ z_{22} &= \frac{D}{C} = \frac{7}{3}\ \Omega \end{aligned}\right\}$$

(b) By KVL,



$$V_1 = 2I_1 + 4I_3$$

$$V_2 = I_1 + I_2 - I_3$$

and $\quad 2(I_1 - I_3) + I_1 + I_2 - I_3 - 4I_3 = 0$

Eliminating I_3 from above equations,

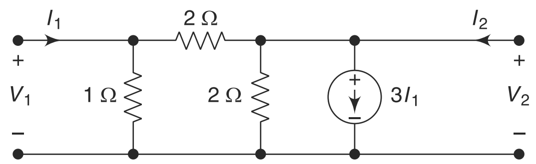
$$V_1 = \frac{26}{7} I_1 + \frac{4}{7} I_2$$

$$V_2 = \frac{4}{7} I_1 + \frac{6}{7} I_2$$

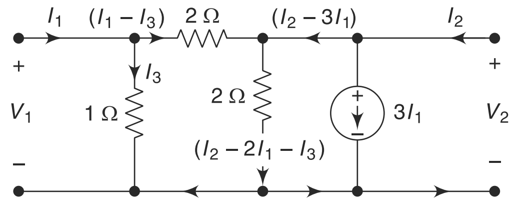
Thus, the z-parameters are:

$$[z] = \begin{bmatrix} 26/7 & 4/7 \\ 4/7 & 6/7 \end{bmatrix} \Omega$$

6.3 For the network shown, find z and y-parameters.



Solution From the figure, we can write the KVL equations,



$$V_1 = I_3 \tag{i}$$
$$V_2 = 2I_2 - 4I_1 - 2I_3 \tag{ii}$$

and, $2I_1 - 2I_3 + 2I_2 - 4I_1 - 2I_3 - I_3 = 0 \quad \Rightarrow \quad I_3 = \frac{2}{5}(I_2 - I_1)$

From (i), $V_1 = -\frac{2}{5}I_1 + \frac{2}{5}I_2 = -0.4I_1 + 0.4I_2$

From (ii), $V_2 = 2I_2 - 4I_1 - \frac{4}{5}I_2 + \frac{4}{5}I_1 = -3.2I_1 + 1.2I_2$

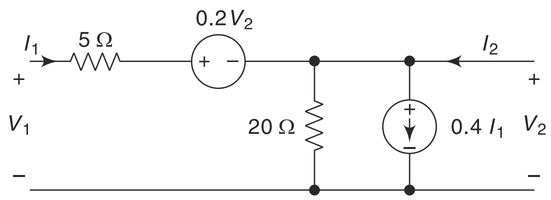
$\therefore$ $$[z] = \begin{bmatrix} -0.4 & 0.4 \\ -3.2 & 1.2 \end{bmatrix} \Omega$$

$$\Delta z = (-0.4 \times 1.2) - 0(0.4) \times (-3.2) = 0.8$$

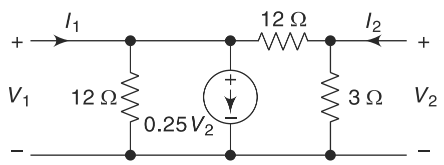
$\therefore$ $$[y] = \begin{bmatrix} 1.2/0.8 & -\dfrac{0.4}{0.8} \\ 3.2/0.8 & -\dfrac{0.4}{0.8} \end{bmatrix} ℧ = \begin{bmatrix} 1.5 & -0.5 \\ 4 & -0.5 \end{bmatrix} ℧$$

6.4 Find the y-parameters for the 2-port networks shown.

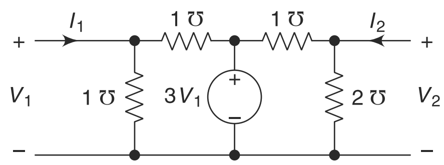
(a)



(b)



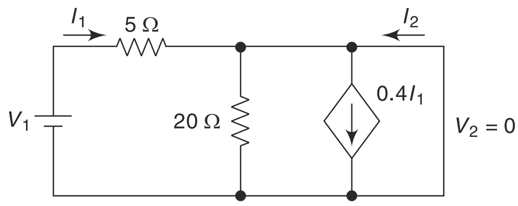
(c)

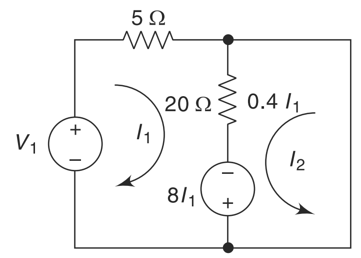


Solution

(a) We consider two cases to find out the y-parameters.

Case (I) Making port- 2 shorted and applying a voltage of V_1 at port- 1





By KVL,

$$17I_1 + 20I_2 = V_1$$

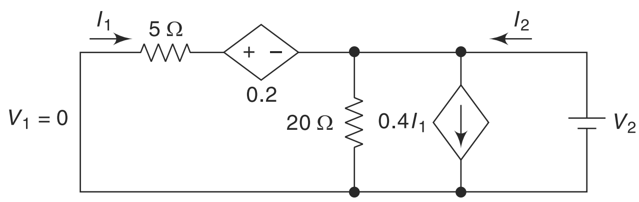
and $$12I_1 + 20I_2 = 0$$

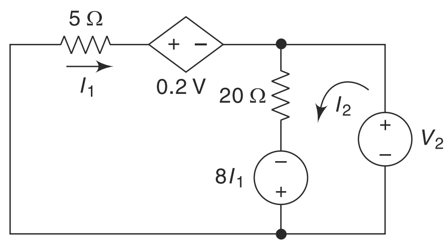
$$I_1 = \frac{\begin{vmatrix} V_1 & 20 \\ 0 & 20 \end{vmatrix}}{\begin{vmatrix} 17 & 20 \\ 12 & 20 \end{vmatrix}} = 0.2V_1 \Rightarrow y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 0.2\ \mho$$

Solving,

$$I_2 = \frac{\begin{vmatrix} 17 & V_1 \\ 12 & 0 \end{vmatrix}}{\begin{vmatrix} 17 & 20 \\ 12 & 20 \end{vmatrix}} = -0.12V_1 \Rightarrow y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -0.12\ \mho$$

Case (II) Making port-1 shorted and applying a voltage of V_2 at port- 2





By KVL,

$$17I_1 + 20I_2 = -0.2V_2$$

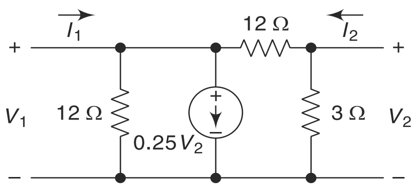
and $$12I_1 + 20I_2 = V_2$$

$$I_1 = \frac{\begin{vmatrix} -0.2V_2 & 20 \\ V_2 & 20 \end{vmatrix}}{\begin{vmatrix} 17 & 20 \\ 12 & 20 \end{vmatrix}} = -0.24V_2 \Rightarrow y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -0.24\ \mho$$

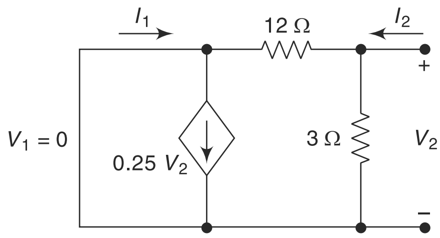
Solving,

$$I_2 = \frac{\begin{vmatrix} 17 & -0.2V_2 \\ 12 & V_2 \end{vmatrix}}{\begin{vmatrix} 17 & 20 \\ 12 & 20 \end{vmatrix}} = 0.194V_2 \quad \Rightarrow \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 0.194 \ \mho$$

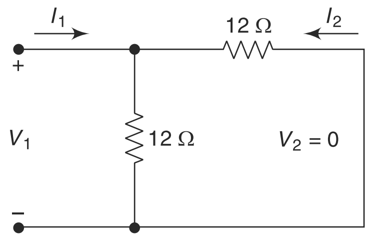
Thus, $$[y] = \begin{bmatrix} 0.2 & -0.24 \\ -0.12 & 0.194 \end{bmatrix} \mho$$



(b) We consider two cases.
 Case (I) $V_1 = 0$
 Case (II) $V_2 = 0$



Case (I): $V_1 = 0$



Case (II): $V_2 = 0$

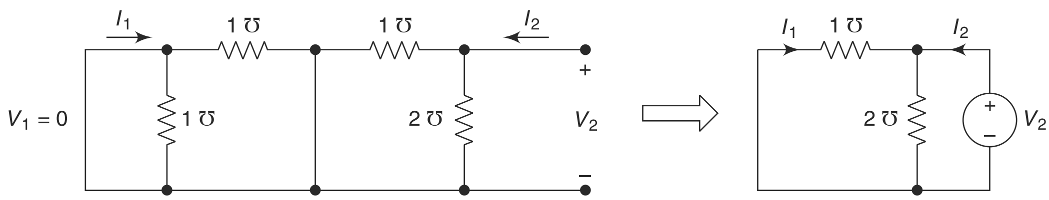
By KCL,

$$\left.\begin{aligned}
I_1 &= y_{12}V_2|_{V_1=0} = \frac{V_2}{4} + \left(\frac{0-V_2}{12}\right) \Rightarrow y_{12} = \frac{1}{6}\ \mho \\
I_2 &= y_{22}V_2|_{V_1=0} = \frac{V_2}{3} + \frac{V_2}{12} \Rightarrow y_{22} = \frac{5}{12}\ \mho \\
I_1 &= y_{11}V_1|_{V_2=0} = \left(\frac{1}{12} + \frac{1}{12}\right)V_1 \Rightarrow y_{11} = \frac{1}{6}\ \mho \\
I_2 &= y_{21}V_1|_{V_2=0} = -\frac{V_1}{12} \Rightarrow y_{21} = -\frac{1}{12}\ \mho
\end{aligned}\right\}$$

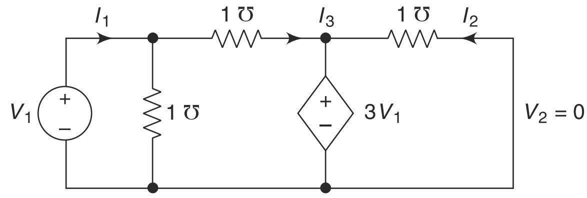
(c) For $V_1 = 0$, the circuit becomes as shown.

$\therefore$ $$I_2 = y_{22}V_2 = (1+2)V_2 = 3V_2 \Rightarrow y_{22} = 3\ \mho$$

Also, $$-\frac{I_1}{1} = V_2 \Rightarrow y_{12} = -1\ \mho$$



For $V_2 = 0$, the circuit becomes as shown.



$\therefore \quad -\frac{I_2}{1} = 3V_1$ (i)

$\frac{I_3}{1} + 3V_1 = V_1 \Rightarrow 2V_1 = -I_3$ (ii)

$I_1 = I_3 + I_4$ (iii)

and $V_1 = \frac{I_4}{1}$ (iv)

From (i) to (iv),

$$I_1 = V_1 + I_3 = V_1 - 2V_1 = -V_1 \quad \Rightarrow \quad y_{11} = -1 \text{ ℧}$$

From (i), $y_{21} = -3$ ℧

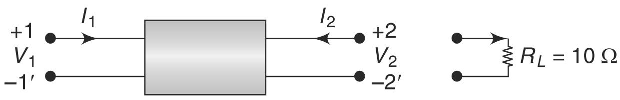
Thus, the y-parameters are:

$$[y] = \begin{bmatrix} -1 & -1 \\ -3 & 3 \end{bmatrix} \text{℧}$$

From the interrelationship, we get the z-parameters as:

$$[z] = \begin{bmatrix} -1 & 0 \\ -1 & 1/3 \end{bmatrix} (\Omega)$$

6.5 Measurements were made on a two-port network shown in the figure.



(i) With port-2 open, a voltage of $100\angle0°$ volt is applied to port-1, resulted in, $I_1 = 10\angle0°$ amp and $V_2 = 25\angle0°$ volt.

(ii) With port-1 open, a voltage of $100\angle0°$ volt is applied to port-2, resulted in, $I_2 = 20\angle0°$ amp and $V_1 = 50\angle0°$ volt.

(a) Write the loop equations for the network and also find the driving point and transfer impedance.

(b) What will be the voltage across a 10 Ω resistor connected across port-2 if a $100\angle0°$ volt source is connected across port-1.

Solution

(a) From the given data, we get the *z*-parameters as:

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{100\angle0°}{10\angle0°} = 10\ \Omega$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{25\angle0°}{10\angle0°} = 2.5\ \Omega$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{50\angle0°}{20\angle0°} = 2.5\ \Omega$$

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{100\angle 0^\circ}{20\angle 0^\circ} = 5\ \Omega$$

So, the loop equations are:

$$\left.\begin{aligned} V_1 &= 10I_1 + 2.5I_2 \\ V_2 &= 2.5I_1 + 5I_2 \end{aligned}\right\}$$

(b) Here, $V_1 = 100\angle 0^\circ$ and $V_2 = -I_2R_L = -10I_2$

Putting these values in loop equations,

$$100 = 10I_1 + 2.5I_2 \quad \Rightarrow \quad I_1 = 10 - 0.25I_2$$

and $\quad -10I_2 = 2.5I_1 + 5I_2$

or, $\quad -10I_2 = 2.5(10 - 0.25I_2) + 5I_2$

or, $\quad -15I_2 = 25 - 0.625I_2$

or, $\quad I_2 = \dfrac{-25}{14.375} = -1.74$ A

$\therefore$ Voltage across the resistor $= -I_2R_L = 17.4$ V

6.6 (a) The following equations give the voltages V_1 and V_2 at the two ports of a two port network, $V_1 = 5I_1 + 2I_2$, $\quad V_2 = 2I_1 + I_2$;

A load resistance of 3 Ω is connected across port-2. Calculate the input impedance.

(b) The z-parameters of a two port network are $z_{11} = 5\ \Omega$, $z_{22} = 2\ \Omega$, $z_{12} = z_{21} = 3\ \Omega$. Load resistance of 4 Ω is connected across the output port. Calculate the input impedance.

Solution

(a) From the given equations,

$$V_1 = 5I_1 + 2I_2 \qquad \text{(i)}$$

$$V_2 = 2I_1 + I_2 \qquad \text{(ii)}$$

At the output, $V_2 = -I_2R_L = -3I_2$

Putting this value in (ii),

$$-3I_2 = 2I_1 + I_2 \quad \Rightarrow \quad I_2 = -I_1/2$$

Putting in (i), $V_1 = 5I_1 + \left(\dfrac{-I_1}{2}\right) = 4I_1$

$\therefore$ Input impedance, $Z_{\text{in}} = \dfrac{V_1}{I_1} = 4\Omega$

(b) [Same as Prob. (a)] $Z_{\text{in}} = \dfrac{V_1}{I_1} = 3.5\Omega$

6.7 Determine the h-parameter with the following data:

(i) with the output terminals short circuited, $V_1 = 25$ V, $I_1 = 1$ A, $I_2 = 2$ A

(ii) with the input terminals open circuited, $V_1 = 10$ V, $V_2 = 50$ V, $I_2 = 2$ A

Solution The h-parameter equations are,

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

(a) With output short-circuited, $V_2 = 0$, given: $V_1 = 25$ V, $I_1 = 1$ A and $I_2 = 2$ A.

$$\left.\begin{aligned} \therefore \quad 25 &= h_{11} \times 1 \\ \text{and} \quad 2 &= h_{21} \times 1 \end{aligned}\right\} \Rightarrow h_{11} = 25\ \Omega, \text{ and } h_{21} = 2$$

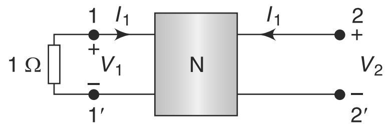
(b) With input open-circuited, $I_1 = 0$, given: $V_1 = 10$ V, $V_2 = 50$ V and $I_2 = 2$ A.

$$\left.\begin{aligned} \therefore \quad 10 &= h_{12} \times 50 \\ \text{and} \quad 2 &= h_{22} \times 50 \end{aligned}\right\} \Rightarrow h_{12} = \frac{1}{5} = 0.2 \text{ and } h_{23} = \frac{1}{25}\ \mho = 0.04\ \mho$$

Thus, the h-parameters are:

$$[h] = \begin{bmatrix} 25\ \Omega & 0.2 \\ 2 & 0.04\ \Omega^{-1} \end{bmatrix}$$

6.8 The y-parameters for a two-port network N are given as, $[y_{11} = 4\ \mho, y_{22} = 5\ \mho, y_{12} = y_{21} = 4\ \mho]$
If a resistor of 1 ohm is connected across port-1 of N, then find out the output impedance.



Solution Output impedance is given as,

$$Z_{\text{out}} = \frac{z_{11}z_{22} - z_{12}z_{21}}{z_{11} + Z_L}$$

Here, $y_{11} = 4\ \Omega^{-1},\ y_{12} = y_{21} = 4\ \Omega^{-1},\ y_{22} = 5\ \Omega^{-1}$

$$\therefore \quad z_{11} = \frac{y_{22}}{\Delta y} = \frac{5}{20-16} = \frac{5}{4}\ \Omega$$

$$z_{12} = z_{21} = -\frac{y_{12}}{\Delta y} = -\frac{4}{4} = -1\ \Omega$$

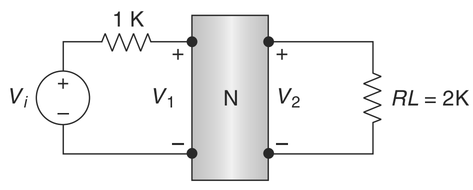
and
$$z_{22} = \frac{y_{11}}{\Delta y} = \frac{4}{4} = 1\ \Omega$$

Putting these values,

$$Z_{\text{out}} = \frac{z_{11}z_{22} - z_{12}z_{21}}{z_{11} + Z_L} = \frac{\frac{5}{4} \times 1 - (-1) \times (-1) + 1 \times 1}{5/4 + 1} = \frac{5}{9}\ \Omega$$

6.9 (a) The h-parameters of a two-port network are $h_{11} = 100\ \Omega$, $h_{12} = 0.0025$, $h_{21} = 20$ and $h_{22} = 1\ \text{m}\mho$. Find V_2/V_1.

(b) The h-parameters of a two-port network are $h_{11} = 1\ \Omega$, $h_{12} = -h_{21} = 2$, $h_{22} = 1\ \mho$. The power absorbed by a load resistance of 1 Ω connected across port-2 is 100 W. The network is excited by a voltage source of generated voltage V_s and internal resistance 2 Ω. Calculate the value of V_s.



Solution

(a) The h-parameter equations are:

$$V_1 = 100I_1 + 0.0025V_2 \qquad \text{(i)}$$

$I_2 = 20I_1 + 0.001V_2$ (ii)

By KVL at the output mesh, $V_2 = -2000I_2$ (iii)

$$V_1 = 100\left[\frac{I_2 - 0.001V_2}{20}\right] + 0.0025V_2 = 5\left(-\frac{V_2}{2000}\right) - 0.005V_2 + 0.0025V_2$$

From (i),

or $\dfrac{V_2}{V_1} = -200$

(b) The h-parameter equations are:

$V_1 = I_1 + 2V_2$ (i)

$I_2 = -2I_1 + V_2$ (ii)

Since the load resistance of 1 Ω is connected across port-2,

$\therefore \quad \dfrac{V_2^2}{1} = 100 \Rightarrow V_2 = 10$ V

By KVL, $V_2 = -I_2R_L = -I_2 \Rightarrow I_2 = -10$ A

and $2I_1 + V_1 = V_s$ (iii)

From (ii), putting the values of I_2 and V_2,

$-10 = -2I_1 + 10 \Rightarrow I_1 = 10$ A

From (iii),

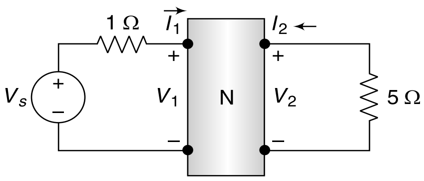
$V_s = 2 \times 10 + V_1 = 20 + I_1 + 2V_2$ {by (i)}

$= 20 + 10 + 2 \times 10$

or, $V_s = 50$ V

6.10 The z-parameters for a network N are:

$$\begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}$$



The terminal connections for the network are shown in the adjacent figure. Calculate the voltage ratio V_2/V_s, current ratio $-I_2/I_1$ and input resistance V_1/I_1.

Solution The *z*-parameter equations are:

$V_1 = 2I_1 + I_2$ (i)

$V_2 = 2I_1 + 5I_2$ (ii)

By KVL at the input and output circuits,

$I_1 + V_1 = V_s \Rightarrow 3I_1 + I_2 = V_s$ (iii) {by (i)}

and $5I_2 + V_2 = 0 \Rightarrow 2I_1 + 10I_2 = 0$ (iv) {by(ii)}

Solving (iii) and (iv),

$$I_1 = \frac{\begin{vmatrix} V_s & 1 \\ 0 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = \frac{10}{28}V_s \quad \text{and} \quad I_2 = \frac{\begin{vmatrix} 3 & V_s \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = -\frac{2}{28}V_s$$

$\therefore \qquad -\dfrac{I_2}{I_1} = \dfrac{1}{5}$

Now, $\qquad V_2 = (2I_1 + 5I_2) = \left(\dfrac{20}{28} - \dfrac{10}{28}\right)V_s = \dfrac{10}{28}V_s$

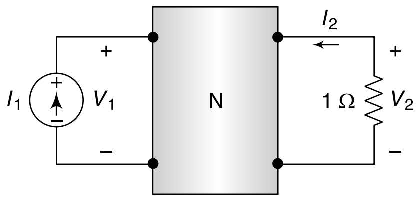
$\therefore \qquad \dfrac{V_2}{V_s} = \dfrac{5}{14}$

Again,

$$V_1 = (2I_1 + I_2) = \left(\frac{20}{28} - \frac{2}{28}\right)V_s = \frac{18}{28}V_s$$

$\therefore \qquad \dfrac{V_1}{I_1} = \dfrac{9}{14}\Omega$

6.11 For the two-port network in figure, terminated in a 1 Ω resistance, show that, $\dfrac{V_2}{I_1} = \dfrac{z_{21}}{1+z_{22}}$ and $\dfrac{V_1}{I_1} = \dfrac{z_{11}+\Delta z}{1+z_{22}}$



Solution The z-parameter equations are:

$$V_1 = z_{11}I_1 + z_{12}I_2 \qquad \text{(i)}$$

$$V_2 = z_{21}I_1 + z_{22}I_2 \qquad \text{(ii)}$$

By KVL at the output, $V_2 = -I_2 \times 1 \Rightarrow I_2 = -V_2$

$$V_2 = z_{21}I_1 + z_{22}I_2 = z_{21}I_1 + z_{22}(-V_2)$$

From (ii), or, $V_2(1+z_{22}) = z_{21}I_1$ (iii)

or $\qquad \dfrac{V_2}{I_1} = \dfrac{z_{21}}{1+z_{22}}$ (Proved)

From (i),

$$V_1 = z_{11}\left[\frac{V_2(1+z_{22})}{z_{21}}\right] + z_{12}(-V_2) \quad \{\text{by (iii)}\}$$

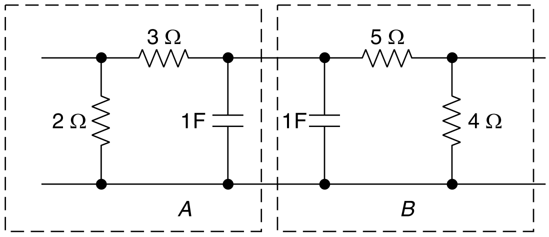
$$= V_2\left[\frac{z_{11} + z_{11}z_{22} - z_{12}z_{21}}{z_{21}}\right]$$

$$= V_2\left[\frac{z_{11}+\Delta z}{z_{21}}\right]$$

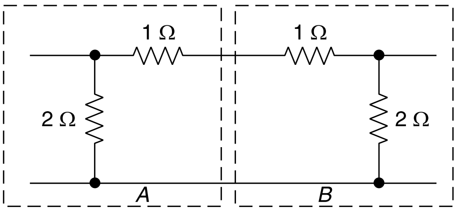
$\therefore \qquad \dfrac{V_1}{I_1} = \dfrac{V_1}{V_2} \times \dfrac{V_2}{I_1} = \dfrac{z_{11}+\Delta z}{z_{21}} \times \dfrac{z_{21}}{1+z_{22}} = \dfrac{z_{11}+\Delta z}{1+z_{22}}$ (Proved)

6.12 Calculate the T-parameters for the block A and B separately and then using these results, calculate the T-parameters of the whole circuit shown in the figure. Prove any formula used.

(a)



(b)



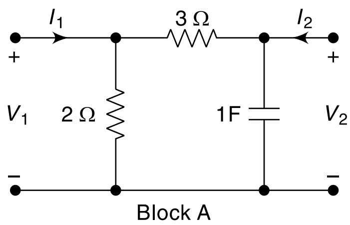
Solution

(a) We consider the given network as a cascade connection of two networks as shown.

For Block A:

Opening the port-2,

By KCL,



$$\left(\frac{1}{2}+\frac{1}{3}\right)V_1-\frac{1}{3}V_2=I_1$$

and $$-\frac{1}{3}V_1+\left(\frac{1}{3}+s\right)V_2=0$$

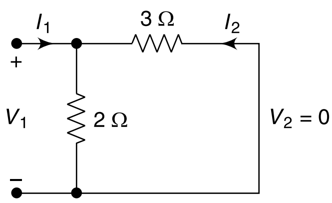
Solving for V_1 and V_2,

$$V_1=\frac{2I_1(1+3s)}{(1+5s)} \quad \text{and} \quad V_2=\frac{2I_1}{(1+5s)}$$

$$\left.\begin{aligned} \therefore \quad A_a &= \left.\frac{V_1}{V_2}\right|_{I_2=0}=(1+3s) \\ \text{and} \quad C_a &= \left.\frac{I_1}{V_2}\right|_{I_2=0}=\frac{(1+5s)}{2} \end{aligned}\right\}$$

Short-circuiting port-2,

$$\therefore \quad I_1=\frac{V_1}{2}+\frac{V_1}{3}=\frac{5}{6}V_1$$



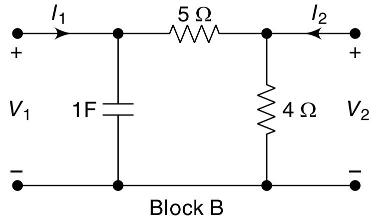
and $$V_1=-3I_2 \;\Rightarrow\; B_a=-\left.\frac{V_1}{I_2}\right|_{V_2=0}=3\Omega$$

and $$D_a=-\left.\frac{I_1}{I_2}\right|_{V_2=0}=\frac{5V_1}{6}\times\frac{3}{V_1}=\frac{5}{2}$$

For Block B:

Opening the port-2,

By KCL,



$$\left(\frac{1}{5}+s\right)V_1-\frac{1}{5}V_2=I_1$$

and
$$-\frac{1}{5}V_1+\left(\frac{1}{5}+\frac{1}{4}\right)V_2=0$$

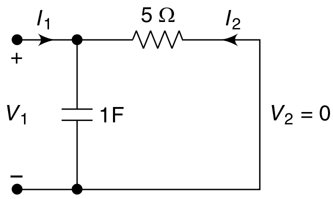
Solving for V_1 and V_2,

$$V_1=\frac{9I_1}{(1+9s)} \quad \text{and} \quad V_2=\frac{4I_1}{(1+9s)}$$

$$\left.\begin{aligned}\therefore \quad A_b&=\left.\frac{V_1}{V_2}\right|_{I_2=0}=\frac{9}{4}\\ \text{and} \quad C_b&=\left.\frac{I_1}{V_2}\right|_{I_2=0}=\frac{(1+9s)}{4}\end{aligned}\right\}$$

Short-circuiting port-2,

$$\therefore \quad I_1=\left(\frac{1}{5}+s\right)V_1$$



and
$$V_1=-5I_2 \;\Rightarrow\; B_b=-\left.\frac{V_1}{I_2}\right|_{V_2=0}=5\ \Omega$$

and
$$D_b=-\left.\frac{I_1}{I_2}\right|_{V_2=0}=(5s+1)$$

Since the two networks are connected in cascade, the overall transmission parameter matrix is obtained as,

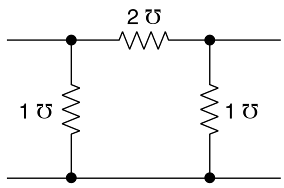
$$[T]=[T_a]\times[T_b]=\begin{bmatrix}(3s+1) & 3\\ \left(\dfrac{5s+1}{2}\right) & 5/2\end{bmatrix}\times\begin{bmatrix}9/4 & 5\\ \left(\dfrac{1+9s}{4}\right) & (5s+1)\end{bmatrix}=\begin{bmatrix}(13.5s+3) & (30s+8)\\ (11.25s+1.75) & (25s+5)\end{bmatrix}$$

(b) [Same as Prob. (a)]

Here,
$$[T_a]=\begin{bmatrix}1 & 1\\ 1/2 & 3/2\end{bmatrix} \quad \text{and} \quad [T_b]=\begin{bmatrix}3/2 & 1\\ 3/2 & 1\end{bmatrix}$$

$$\therefore \quad [T]=[T_a]\times[T_b]=\begin{bmatrix}3 & 2\\ 3 & 2\end{bmatrix}$$

6.13 Two identical sections of the network shown in the figure are connected in parallel. Obtain the y-parameters of the resulting network and verify by direct calculation.



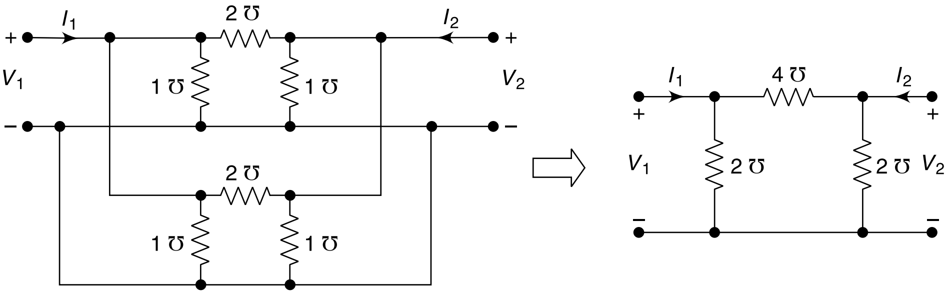
Solution For the circuit, $y_{11} = 3\ \Omega^{-1}$, $y_{12} = y_{21} = -2\ \Omega^{-1}$ and $y_{22} = 3\Omega^{-1}$

The y-parameters for the combination will be,

$$\left.\begin{aligned} y_{11} &= (y'_{11} + y''_{11}) = 6\ \Omega^{-1} \\ y_{12} &= y_{21} = (y'_{12} + y''_{12}) = -4\ \Omega^{-1} \\ y_{22} &= (y'_{22} + y''_{22}) = 6\ \Omega^{-1} \end{aligned}\right\}$$

To find the y-parameters by direct calculation, we consider the resulting network as shown.

For the entire network, $y_{11} = 4 + 2 = 6\ \Omega^{-1}$; $y_{12} = y_{21} = -4\ \Omega^{-1}$; $y_{22} = 4 + 2 = 6\ \Omega^{-1}$ (Proved)



6.14 Two networks have general *ABCD* parameters as shown below:

Parameter	*Network-1*	*Network-2*
A	1.50	5/3
B	11Ω	4Ω
C	0.25 siemens	1 siemens
D	2.5	3.0

If the two networks are connected with their inputs and outputs in parallel, obtain the admittance matrix of the resulting network.

Solution *For network-1:*

$$y_{11} = \frac{D}{B} = \frac{2.5}{11} = \frac{5}{22}\ \Omega^{-1}$$

$$y_{12} = -\frac{AD - BC}{B} = -\frac{1.5 \times 2.5 - 11 \times 0.25}{11} = -\frac{1}{11}\ \Omega^{-1}$$

$$y_{21} = -\frac{1}{B} = -\frac{1}{11}\ \Omega^{-1}$$

$$y_{22} = \frac{A}{B} = \frac{1.5}{11} = \frac{3}{22}\ \Omega^{-1}$$

For network-2:

$$y_{11} = \frac{D}{B} = \frac{3}{4}\ \Omega^{-1}$$

$$y_{12} = -\frac{AD - BC}{B} = -\frac{1}{4}\ \Omega^{-1}$$

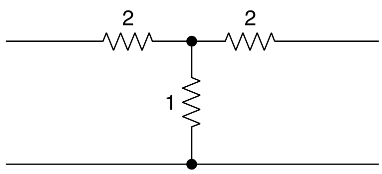
$$y_{21} = -\frac{1}{B} = -\frac{1}{4}\ \Omega^{-1}$$

$$y_{22} = \frac{A}{B} = \frac{5}{3 \times 4} = \frac{5}{12}\ \Omega^{-1}$$

So, the admittance matrix of the resulting network is:

$$[y] = \begin{bmatrix} 5/22 & -1/11 \\ -1/11 & 3/22 \end{bmatrix} + \begin{bmatrix} 3/4 & -1/4 \\ -1/4 & 5/12 \end{bmatrix} = \begin{bmatrix} 43/44 & -15/44 \\ -15/44 & 73/132 \end{bmatrix} \Omega^{-1}$$

6.15 Two identical sections of figure are connected in series. Obtain the z-parameters of the resulting network and verify by direct calculation. All values are in ohm.



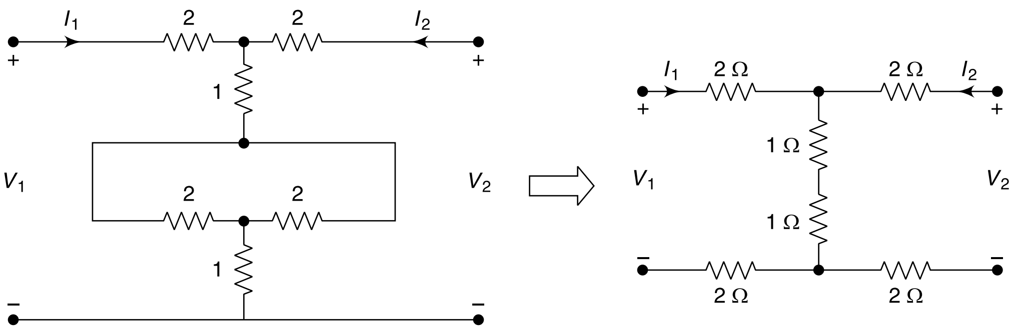
Solution The z-parameters of each section:

$$z_{11} = 3\ \Omega,\ z_{12} = z_{21} = 1\ \Omega,\ z_{22} = 3\ \Omega$$

So, the z-parameters of the combined series network are:

$$z_{11} = (3 + 3) = 6\ \Omega,\ z_{12} = z_{21} = (1 + 1) = 2\ \Omega,\ z_{22} = (3 + 3) = 6\ \Omega$$

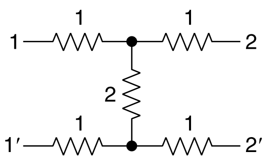
To find the z-parameters by direct calculation, we consider the resulting network as shown.



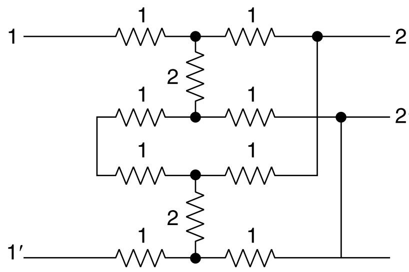
For the resulting network,

$$\left.\begin{aligned} z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} = 6\ \Omega \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 2\ \Omega \\ z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} = 6\ \Omega \quad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = 2\ \Omega \end{aligned}\right\}$$

6.16 (a) Find out the z- and h-parameters for the circuit shown in Fig. (a). All values are in ohm.
(b) Hence, obtain the hybrid parameters for the two-port network of Fig. (b).



(a)



(b)

Solution

(a) For Fig. (a), the z-parameters are:

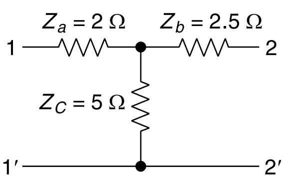
$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 4\ \Omega,\ z_{12} = z_{21} = 2\ \Omega,\ z_{11} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 4\ \Omega$$

$$\therefore \quad \left.\begin{aligned} h_{11} &= \frac{\Delta z}{z_{12}} = \frac{16-4}{4} = 3\ \Omega \\ h_{12} &= \frac{z_{12}}{z_{22}} = \frac{2}{4} = 0.5 \\ h_{21} &= -\frac{z_{21}}{z_{22}} = -\frac{2}{4} = -0.5 \\ h_{22} &= \frac{1}{z_{12}} = \frac{1}{4} = 0.25\ \Omega^{-1} \end{aligned}\right\}$$

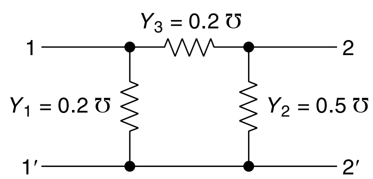
(b) The connection is series-parallel connection. For this connection, the overall h-parameters will be the sum of individual h-parameters.

$$\therefore \quad \left.\begin{aligned} h_{11} &= (3+3) = 6\Omega \\ h_{12} &= (0.5+0.5) = 1 \\ h_{21} &= (-0.5-0.5) = -1 \\ h_{22} &= (0.25+0.25) = 0.5\Omega^{-1} \end{aligned}\right\}$$

6.17 (a) Find the equivalent π-network for the T-network shown in the figure (a).
(b) Find the equivalent T-network for the π-network shown in the figure (b).



(a)



(b)

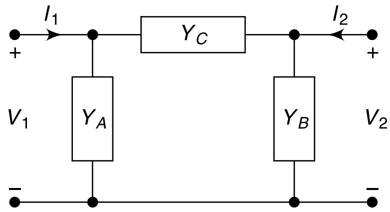
Solution

(a) Let the equivalent π-network have Y_C as the series admittance and Y_A and Y_B as the shunt admittances at port-1 and port-2, respectively.

Now, the z-parameters are given as:

$$z_{11} = (Z_A + Z_C) = 7\ \Omega,\ z_{12} = z_{21} = Z_C = 5\ \Omega,\ z_{22} = (Z_B + Z_C) = 7.5\ \Omega$$

$$\therefore \quad \Delta z = (7 \times 7.5 - 5 \times 5) = 27.5\ \Omega^2$$



$$\therefore \quad y_{11} = \frac{z_{22}}{\Delta z} = \frac{7.5}{27.5}\ \mho$$

$$y_{12} = y_{21} = -\frac{z_C}{\Delta z} = -\frac{5}{27.5}\ \mho$$

$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{7}{27.5}\ \mho$$

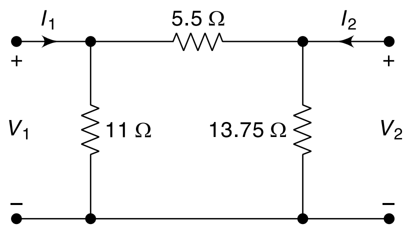
$$\therefore \quad Y_A = (y_{11} + y_{12}) = \frac{2.5}{27.5} = \frac{1}{11}\ \mho$$

$$\therefore \quad Y_B = (y_{22} + y_{12}) = \frac{2}{27.5}\ \mho$$

$$\text{and} \quad Y_C = -y_{21} = \frac{5}{27.5} = \frac{2}{11}\ \mho$$

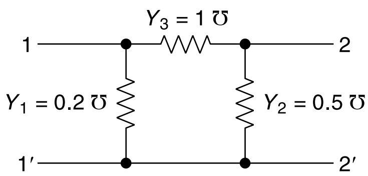
Thus, the impedances of the equivalent π-networks are:

$$\left.\begin{aligned} Z_A &= \frac{1}{Y_A} = 11\ \Omega, \\ Z_B &= \frac{1}{Y_B} = 13.75\ \Omega, \\ Z_C &= \frac{1}{Y_C} = 5.5\ \Omega \end{aligned}\right\}$$

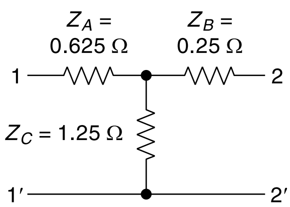


Equivalent π-network

(b)



π-network



Equivalent T-network

The y-parameters,

$$y_{11} = 1.2\ \mho,\ y_{12} = y_{21} = -1\ \mho, \text{ and } y_{22} = 1.5\ \mho$$

$$\therefore \quad \Delta y = (1.2 \times 1.5 - 1) = 0.8$$

$$\therefore \qquad z_{11} = \frac{y_{22}}{\Delta y} = \frac{1.5}{0.8}\ \Omega,\ z_{12} = z_{21} = -\frac{y_{12}}{\Delta y} = \frac{1}{0.8}\ \Omega,\ z_{22} = \frac{y_{11}}{\Delta y} = \frac{1.2}{0.8}\ \Omega$$

$$\therefore \qquad \left.\begin{aligned} Z_A &= (z_{11} - z_{12}) = \frac{0.5}{0.8} = 0.625\ \Omega \\ Z_B &= (z_{22} - z_{12}) = \frac{0.2}{0.8} = 0.25\ \Omega \\ Z_C &= z_{12} = \frac{1}{0.8} = 1.25\ \Omega \end{aligned}\right\}$$

6.18 The z-parameter of a 2-port network are:

$$z_{11} = 10\ \Omega,\ z_{22} = 20\ \Omega,\ z_{12} = z_{21} = 5\ \Omega.$$

Find the *ABCD*-parameters. Also find the equivalent *T*-network.

Solution

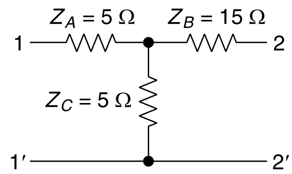
From the inter-relationship, we get the *ABCD* parameters as:

$$A = \frac{z_{11}}{z_{21}} = \frac{10}{5} = 2$$

$$B = \frac{z_{11}Z_{22} - Z_{12}Z_{21}}{z_{21}} = \frac{10 \times 20 - 5 \times 5}{5} = 35\ \Omega$$

$$C = \frac{1}{z_{21}} = \frac{1}{5} = 0.2\ \mho$$

$$D = \frac{z_{22}}{z_{21}} = \frac{20}{5} = 4$$



Equivalent T-network

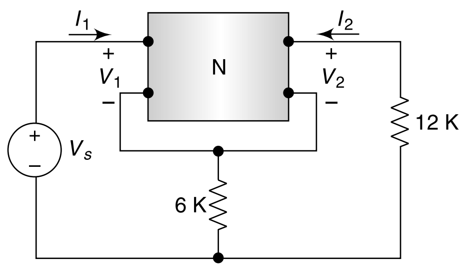
To find the equivalent *T*-network, we have the relations,

$$\left.\begin{aligned} z_{11} &= (Z_A + Z_C) = 10\ \Omega \\ z_{12} &= z_{21} = Z_C = 5\ \Omega \\ \text{and} \qquad z_{22} &= (Z_B + Z_C) = 20\ \Omega \end{aligned}\right\} \Rightarrow\ Z_A = 5\ \Omega,\ Z_B = 15\ \Omega,\ Z_C = 5\ \Omega$$

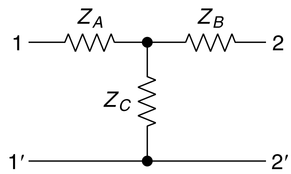
6.19 *Z*-parameters of the two-port network *N* in figure. are, $z_{11} = 4s$, $z_{12} = z_{21} = 3s$, $z_{22} = 9s$.

(a) Replace *N* by its *T*-equivalent.

(b) Use part (a) to find the input current I_1 for $V_s = \cos 1000t$.



Solution



Equivalent T-network

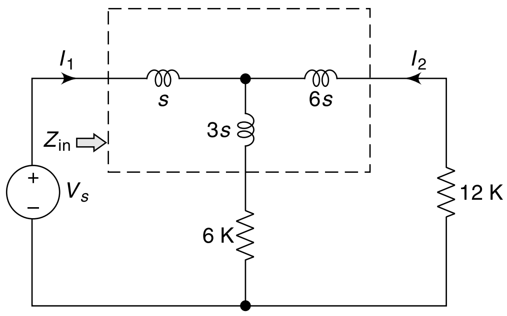
(a) The z-parameters are: $[z]=\begin{bmatrix}4s & 3s\\ 3s & 9s\end{bmatrix}(\Omega)$

Since the network is reciprocal, its T-equivalent exists. Its elements are:

$$Z_A = (z_{11}-z_{12}) = s,\ Z_B = (z_{22}-z_{21}) = 6s,$$

and $\quad Z_C = z_{21} = z_{12} = 3s$

So, the equivalent circuit is shown in figure.



(b) We repeatedly combine the series and parallel elements of above figure, with resistors in kΩ and s in Krad/s to find the input impedance, Z_{in} in kΩ.

$$\therefore \qquad Z_{in} = \frac{V_s}{I_1} = s + \frac{(6s+12)\,(3s+6)}{(6s+12)+(3s+6)} = (3s+4)$$

or $\quad Z_{in}(j) = (3j+4) = 5\angle 36.9^\circ$ kΩ

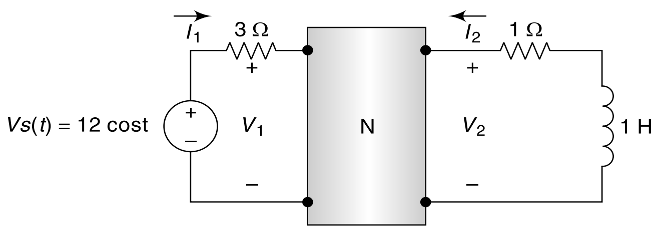
So, the current,

$$i(t) = \frac{v_s(t)}{Z_{in}(j)} = \frac{1}{5}\cos(1000t - 36.9^\circ)\ (\text{mA})$$

6.20 The z-parameters of a two-port network N are given by, $z_{11} = (2s + 1/s)$, $z_{12} = z_{21} = 2s$, $z_{22} = (2s + 4)$.

(a) Find the T-equivalent of N.

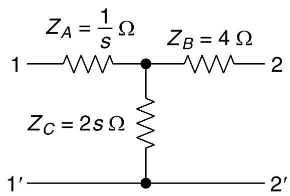
(b) The network N is connected to a source and a load as shown in figure. Replace N by its T-equivalent and then find I_1, I_2, V_1, and V_2.



Solution

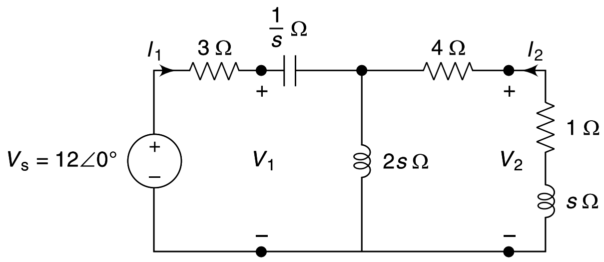
(a) To find the equivalent T-network, we have the relations,

$$\left.\begin{aligned} z_{11} &= (Z_A + Z_C) = \left(2s + \frac{1}{s}\right)\Omega \\ z_{12} &= z_{21} = Z_C = 2s\ \Omega \\ \text{and} \quad z_{22} &= (Z_B + Z_C) = (2s+4)\ \Omega \end{aligned}\right\} \Rightarrow Z_A = \frac{1}{s}\Omega,\ Z_B = 4\Omega,\ Z_C = 2s\ \Omega$$



Equivalent T-network

(b) The equivalent circuit is shown below.



By KVL, $I_1(3 + j) + I_2(j2) = 12\angle 0°$

$I_1(j2) + I_2(5 + j3) = 0$

$$I_1 = \frac{\begin{vmatrix} 12\angle 0° & j2 \\ 0 & (5+j3) \end{vmatrix}}{\begin{vmatrix} (3+j) & j2 \\ j2 & (5+j3) \end{vmatrix}} = \frac{\begin{vmatrix} 12\angle 0° & 2\angle 90° \\ 0 & 5.831\angle 30.96° \end{vmatrix}}{16 + j14} = 3.29\angle -10.22°\,(\text{A})$$

Solving,

and $$I_2 = \frac{\begin{vmatrix} (3+j) & 12\angle 0° \\ j2 & 0 \end{vmatrix}}{\begin{vmatrix} (3+j) & j2 \\ j2 & (5+j3) \end{vmatrix}} = 1.13\angle -131.19°\,(\text{A})$$

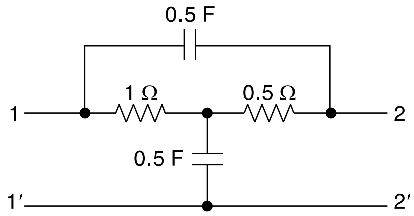
$\therefore$ $V_1 = 12\angle 0° - I_1 \times 3 = 12 - 3.29 \times 3\angle -10.22° = 2.28 + j1.75 = 2.88\angle 37.504°$ (V)

and $V_2 = -I_2(1 + j) = -1.13(1 + j)\angle -131.186° = 1.59\angle 93.81°$

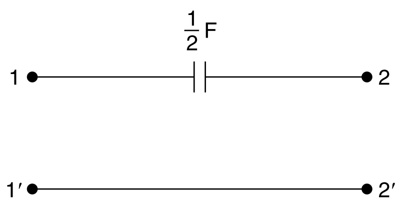
So, the currents and voltages are:

$$\left.\begin{aligned} i_1(t) &= 3.29 \cos (t - 10.2°) \text{ (A)} \\ i_2(t) &= 1.13 \cos (t - 131.2°) \text{ (A)} \\ v_1(t) &= 2.88 \cos (t + 37.5°) \text{ (A)} \\ v_2(t) &= 1.6 \cos (t + 93.8°) \text{ (A)} \end{aligned}\right\}$$

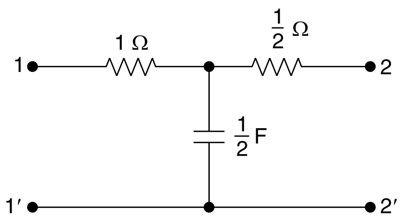
6.21 For the bridge-TRC network, find the y-parameters and its equivalent π-network.



Solution The given network is the parallel combination of the two networks:



(a) Network



(b) Network

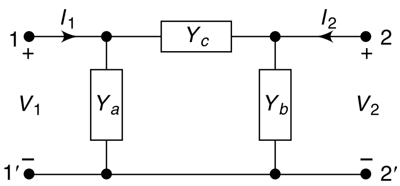
For network (a), the y-parameters are: $[y_a] = \begin{bmatrix} s/2 & -s/2 \\ -s/2 & s/2 \end{bmatrix}$ ℧

For network (b), the z-parameters are: $[z_b] = \begin{bmatrix} (1+2/s) & 2/s \\ 2/s & (1/2+2/s) \end{bmatrix}$ Ω

$$\therefore \qquad y_{11b} = \frac{z_{22b}}{\Delta z_b} = \frac{(1/2+2/s)}{(1+2/s)\,(1/2+2/s) - 4/s^2} = \frac{s+4}{s+6}$$

$$\therefore \qquad y_{12b} = y_{21b} = -\frac{z_{12b}}{\Delta z_b} = \frac{2/s}{(s+6)/2s} = \frac{4}{s+6}$$

$$\therefore \qquad y_{22b} = \frac{z_{11b}}{\Delta z_b} = \frac{(s+2)/2}{(s+6)/2s} = \frac{2(s+2)}{s+6}$$



Equivalent π network

For network (b), the y-parameters are: $[y_b] = \begin{bmatrix} \dfrac{s+4}{s+6} & \dfrac{4}{s+6} \\ \dfrac{4}{s+6} & \dfrac{2(s+2)}{s+6} \end{bmatrix}$

Thus, the overall y-parameters are:

$$[y] = [y_a] + [y_b] = \begin{bmatrix} s/2 & -s/2 \\ -s/2 & s/2 \end{bmatrix} + \begin{bmatrix} \dfrac{s+4}{s+6} & \dfrac{4}{s+6} \\ \dfrac{4}{s+6} & \dfrac{2(s+2)}{s+6} \end{bmatrix}$$

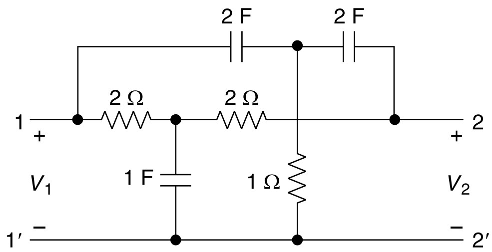
$$= \begin{bmatrix} \dfrac{s^2+8s+8}{2(s+6)} & -\dfrac{s^2+6s+8}{2(s+6)} \\ -\dfrac{s^2+6s+8}{2(s+6)} & \dfrac{s^2+10s+8}{2(s+6)} \end{bmatrix}$$

Equivalent π network can be found out from the relations:

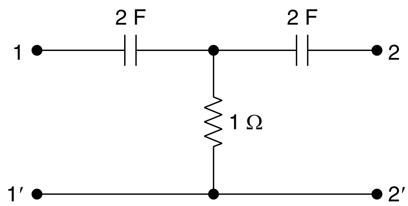
$$Y_a = (y_{11} + y_{12}) = \frac{s}{(s+6)}; Y_b = (y_{22} + y_{12})$$

$$= \frac{2s}{(s+6)}; Y_c = -y_{12} = -y_{21} = \frac{s^2+6s+8}{2(s+6)}$$

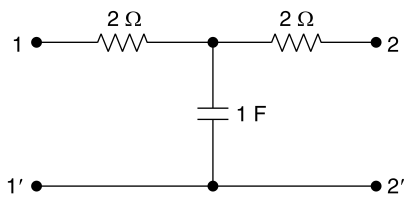
6.22 For the notch-filter network, determine the y-parameters.



Solution The given network is the parallel combination of the two networks:



(a) Network



(b) Network

For network (a), $z_{11a} = \left(\dfrac{1}{2s} + 1\right) = \dfrac{1+2s}{2s}; z_{12a} = z_{21a} = 1; z_{22a} = \left(\dfrac{1}{2s} + 1\right) = \dfrac{1+2s}{2s}$

$$\therefore \quad \Delta z_a = \frac{1+4s}{4s^2}$$

$$\therefore \quad y_{11a} = \frac{z_{22a}}{\Delta z_a} = \frac{2s(1+2s)}{(1+4s)}; y_{12a} = y_{21a} = -\frac{z_{12a}}{\Delta z_a}$$

$$= -\frac{4s^2}{(1+4s)}; y_{22a} = \frac{z_{11a}}{\Delta z_a} = \frac{2s(1+2s)}{(1+4s)}$$

For network (b), $z_{11b} = (1/s + 2) = \dfrac{1+2s}{s}$; $z_{12b} = z_{21b} = \dfrac{1}{s}$; $z_{22b} = (1/s + 2) = \dfrac{1+2s}{s}$

$$\therefore \quad \Delta z_b = \frac{4(s+1)}{s}$$

$$\therefore \quad y_{11b} = \frac{z_{22b}}{\Delta z_b} = \frac{(1+2s)}{4(s+1)};\ y_{12b} = y_{21b} = -\frac{z_{12b}}{\Delta z_b} = -\frac{1}{4(s+1)};\ y_{22b} = \frac{z_{11b}}{\Delta z_b} = \frac{(1+2s)}{4(s+1)}$$

Thus, the overall y-parameters are,

$$y_{11} = y_{22} = (y_{11a} + y_{11b}) = \frac{2s(1+2s)}{1+4s} + \frac{(1+2s)}{4+4s} = \frac{(1+2s)\,(8s^2+12s+1)}{4(s+1)\,(4s+1)}$$

and
$$y_{12} = y_{21} = (y_{12a} + y_{12b}) = -\frac{4s^2}{1+4s} - \frac{1}{4(s+1)} = -\frac{16s^3+16s^2+4s+1}{4(4s+1)\,(s+1)}$$

6.23 A network has two input terminals a, b and two output terminals c, d. The input impedance with c-d open-circuited is $(250 + j100)$ ohm and with c-d short-circuited is $(400 + j300)$ ohm. The impedance across c-d with a-b open-circuited is 200 ohm. Determine the equivalent T-network parameters.

Solution For c-d Terminals opened,

$$(Z_A + Z_B) = (250 + j100) \tag{i}$$

But, for c-d terminals shorted,

$$Z_A + \frac{Z_B Z_C}{Z_B + Z_C} = (400 + j300) \tag{ii}$$

Again, with a-b terminals opened,

$$(Z_B + Z_C) = 200 \tag{iii}$$

From (ii) and (i), we get,

$$\frac{Z_B Z_C}{Z_B + Z_C} - Z_B = 150 + j200$$

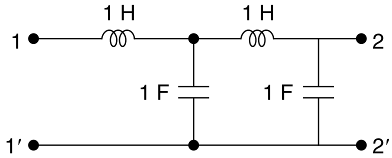
or
$$Z_B Z_C - Z_B^2 - Z_B Z_C = 200(150 + j200) \qquad \text{\{by (iii)}$$

or
$$Z_B^2 = 200(-150 - j200) = 10^4(1 - j2)^2$$

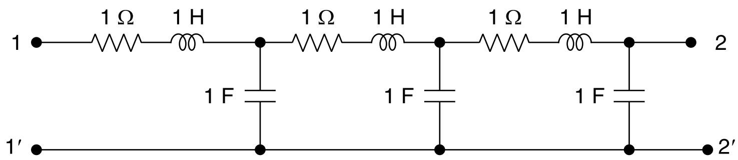
$$\left.\begin{aligned} \therefore \quad & Z_B = (100 - j200)\Omega \\ \therefore \quad & Z_A = (150 + j300)\Omega \\ \textit{and} \quad & Z_C = (100 + j200)\Omega \end{aligned}\right\}$$

6.24 Find the driving point impedance at the terminals 1-1′ of the ladder network shown in figure.

(a)



(b)



Solution

(a) The driving point impedance at 1-1′ is

$$Z_{11} = s + \cfrac{1}{s + \cfrac{1}{s + \cfrac{1}{s}}} = \frac{s^4 + 3s^2 + 1}{s^2 + 2s}$$

(b) The driving point impedance at 1-1′ is,

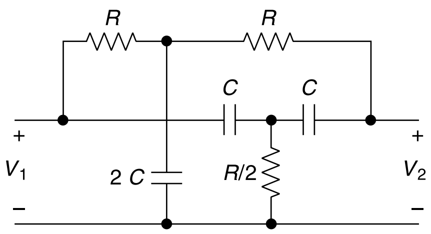
$$Z_{11} = (s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s}}}}} = \frac{s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1}{s^5 + 2s^4 + 5s^3 + 4s^2 + 3s}$$

6.25 For the Notch-filter (Twin-T) network, determine:

(a) y-parameters,

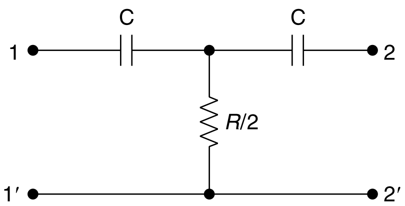
(b) the voltage ratio transfer function V_2/V_1 when no-load impedance is present, and

(c) the value of the frequency at which the output voltage is zero.

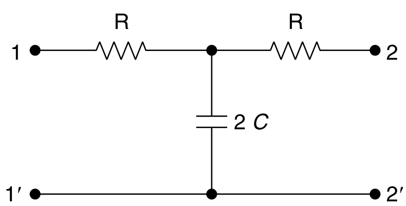


Solution

(a) The given network is the parallel combination of the two networks:



(a) Network



(b) Network

For network (a),

$$z_{11a} = \left(\frac{1}{Cs} + \frac{R}{2}\right) = \frac{2 + RCs}{2Cs};\; z_{12a} = z_{21a} = \frac{R}{2};\; z_{22a} = \left(\frac{1}{Cs} + \frac{R}{2}\right) = \frac{2 + RCs}{2Cs}$$

$$\therefore \qquad \Delta z_a = \frac{1 + RCs}{C^2 s^2}$$

$$\therefore \quad y_{11a} = \frac{z_{22a}}{\Delta z_a} = \frac{RCs(2+RCs)}{2R(1+RCs)}; \qquad y_{12a} = y_{21a} = -\frac{z_{12a}}{\Delta z_a} = -\frac{R^2C^2s^2}{2R(1+RCs)};$$

$$y_{22a} = \frac{z_{11a}}{\Delta z_a} = \frac{Cs\left(1+\frac{1}{2}Cs\right)}{(1+RCs)}$$

For network (b),

$$z_{11b} = \left(\frac{1}{2Cs} + R\right) = \frac{1+2RCs}{2Cs}; \; z_{12b} = z_{21b} = \frac{1}{2Cs}; \; z_{22b} = \left(\frac{1}{s} + 2\right) = \frac{1+2RCs}{2Cs}$$

$$\therefore \quad \Delta z_b = \frac{1+RCs}{C^2s^2}$$

$$\therefore \quad y_{11b} = \frac{z_{22b}}{\Delta z_b} = \frac{(1+2RCs)}{2R(RCs+1)}; \qquad y_{12b} = y_{21b} = -\frac{z_{12b}}{\Delta z_b} = -\frac{1}{2R(RCs+1)};$$

$$y_{22b} = \frac{z_{11b}}{\Delta z_b} = \frac{(1+2RCs)}{2R(RCs+1)}$$

Thus, the overall y-parameters are,

$$y_{11} = y_{22} = (y_{11a} + y_{11b}) = \frac{RCs(2+RCs)}{2R(1+RCs)} + \frac{(1+2RCs)}{2R(RCs+1)} = \frac{(R^2C^2s^2 + 4RCs + 1)}{2R(RCs+1)}$$

and $$y_{12} = y_{21} = (y_{12a} + y_{12b}) = -\frac{R^2C^2s^2}{2R(1+RCs)} - \frac{1}{2R(RCs+1)} = -\frac{R^2C^2s^2+1}{2R\,(RCs+1)}$$

(b) Now,
$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

When no-load impedance is present, $I_2 = 0$,

$$\therefore \quad \frac{V_2}{V_1} = -\frac{y_{21}}{y_{22}} = \frac{R^2C^2s^2+1}{2R(RCs+1)} \times \frac{2R(RCs+1)}{(R^2C^2s^2+4RCs+1)} = \frac{R^2C^2s^2+1}{(R^2C^2s^2+4RCs+1)}$$

(c) For $V_2 = 0 \Rightarrow 1 + R^2C^2s^2 = 0$

Putting $s = j\omega$, $1 - \omega^2R^2C^2 = 0$

$$\therefore \quad \omega = \frac{1}{RC}$$

Thus, the notch frequency is given by, $f_N = \dfrac{1}{2\pi RC}$

MULTIPLE-CHOICE QUESTIONS

6.1 Which one of the following pairs is correctly matched?

(a) Symmetrical two-port network: $AD - BC = 1$

(b) Reciprocal two-port network: $z_{11} = z_{22}$.

(c) Inverse hybrid parameters: A, B, C, D

(d) Hybrid parameters: $(V_1, I_2) = f(I_1, V_2)$

6.2 What is the condition for reciprocity in terms of h-parameters?
(a) $h_{11} = h_{22}$ (b) $h_{12}h_{21} = h_{11}h_{22}$ (c) $h_{12} + h_{21} = 0$ (d) $h_{12} = h_{21}$

6.3 For a reciprocal network, the two-port $ABCD$ parameters are related as follows
(a) $AD - BC = 1$ (b) $AD - BC = 0$ (c) $AC - BD = 0$ (d) $AC - BD = 1$

6.4 For a symmetrical two port network
(a) $z_{11} = z_{22}$ (b) $z_{12} = z_{21}$ (c) $z_{11}z_{22} - z_{12}^2 = 0$ (d) $z_{11} = z_{22}$ and $z_{12} = z_{21}$

6.5 For a two port network to be reciprocal, it is necessary that
(a) $z_{11} = z_{22}$ and $y_{12} = y_{21}$ (b) $z_{11} = z_{22}$ and $AD - BC = 0$.
(c) $h_{21} = -h_{12}$ and $AD - BC = 0$ (d) $y_{12} = y_{21}$ and $h_{21} = -h_{12}$

6.6 A two port network is symmetrical if
(a) $z_{11}z_{22} - z_{12}z_{21} = 1$ (b) $AD - BC = 1$ (c) $h_{11}h_{22} - h_{12}h_{21} = 1$ (d) $y_{11}y_{22} - y_{12}y_{21} = 1$

6.7 A two port network is reciprocal if and only if
(a) $z_{11} = z_{22}$ (b) $BC - AD = -1$ (c) $y_{12} = -y_{21}$ (d) $h_{12} = h_{21}$

6.8 In terms of ABCD parameters, a two port network is symmetrical if and only if:
(a) $A = B$ (b) $B = C$ (c) $C = D$ (d) $D = A$

6.9 The condition for reciprocity of a two port network having different parameters are:
1. $h_{12} = -h_{21}$ 2. $g_{12} = -g_{21}$ 3. $A = D$
Choose the correct combination.
(a) 1 and 2 (b) 1 and 3 (c) 2 and 3 (d) 1, 2 and 3.

6.10 Two two-port networks with transmission parameters A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 respectively are cascaded. The transmission parameter matrix of the cascaded network will be

(a) $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} + \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$ (b) $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$

(c) $\begin{bmatrix} A_1A_2 & B_1B_2 \\ C_1C_2 & D_1D_2 \end{bmatrix}$ (d) $\begin{bmatrix} (A_1A_2 + C_1C_2) & (A_1A_2 - B_1D_2) \\ (C_1A_2 - D_1C_2) & (C_1C_2 + D_1D_2) \end{bmatrix}$

6.11 Consider the following statements.
For a bilateral network,
1. $A = D$ 2. $z_{12} = z_{21}$ 3. $h_{12} = -h_{21}$
Of these statements.
(a) 1, 2 and 3 are correct (b) 1 and 2 are correct
(c) 1 and 3 are correct (d) 2 and 3 are correct.

6.12 In a two port network containing linear bilateral passive circuit elements, which one of the following conditions for z parameters would hold?
(a) $z_{11} = z_{22}$ (b) $z_{12}z_{21} = z_{11}z_{22}$ (c) $z_{11}z_{12} = z_{22}z_{21}$ (d) $z_{12} = z_{21}$

6.13 The relation $AD - BC = 1$, where A, B, C and D are the elements of a transmission matrix of a network, is valid for
(a) any type of network. (b) passive but not reciprocal network.
(c) passive and reciprocal network. (d) both active and passive network.

6.14 When a number of 2-port networks are connected in cascade, the individual:
(a) Z_{oc} matrices are added. (b) Y_{sc} matrices are added.
(c) chain matrices are multiplied. (d) H-matrices are multiplied.

6.15 The h parameters h_{11} and h_{22} are related to z and y parameters as
(a) $h_{11} = z_{11}$ and $h_{22} = \dfrac{1}{z_{22}}$ (b) $h_{11} = z_{11}$ and $h_{22} = y_{22}$
(c) $h_{11} = \dfrac{\Delta z}{z_{22}}$ and $h_{22} = \dfrac{1}{z_{22}}$ (d) $h_{11} = \dfrac{1}{y_{11}}$ and $h_{22} = y_{22}$

6.16 Two two-port networks α and β having A B C D parameters as

$A_\alpha = 4 = D_\alpha \quad A_\beta = 3 = D_\beta \quad B_\alpha = 5,\ C_\alpha = 3 \quad$ and $\quad B_\beta = 4,\ C_\beta = 2$

are connected in cascade in the order of α, β. The equivalent A parameters of the combination is

(a) 17 (b) 22 (c) 24 (d) 31.

6.17 With the usual notation, a two-port resistive network satisfies the condition $A = D = \frac{3}{2}B = \frac{4}{3}C$

The z_{11} of the network is

(a) $\frac{5}{3}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

6.18 The reciprocal of a network function is

(a) an immittance function, if the original function is an immittance function.
(b) a transfer function, if the original function is a transfer function.
(c) never an immittance function.
(d) never a transfer function.

6.19 A two-port network is defined by the relations $I_1 = 2V_1 + V_2,\ I_2 = 2V_1 + 3V_2$. Then z_{12} is

(a) $-2\ \Omega$ (b) $-1\ \Omega$ (c) $-\frac{1}{2}\ \Omega$ (d) $-\frac{1}{4}\ \Omega$

6.20 Consider the following statements

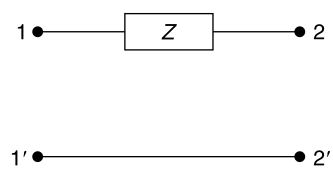
1. Transfer impedance is the reciprocal of transfer admittance.
2. One can derive transfer impedance of a network if its driving-point impedance and admittance are known.
3. Driving-point impedance is the ratio of the Laplace transform of voltage and current functions at the input.

Of these statements:

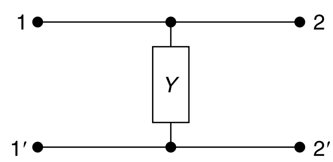
(a) 1, 2 and 3 are correct (b) 1 and 2 are correct
(c) 2 and 3 are correct (d) 3 alone is correct.

6.21 Consider the following statements

1. The two-port network shown below does NOT have an impedance matrix representation.



2. The two-port network shown below does NOT have an admittance matrix representation.



3. A two-port network is said to be reciprocal if it satisfies $z_{12} = z_{21}$ or an equivalent relationship. Of these statements:

(a) 1 and 2 are correct (b) 1 and 3 are correct
(c) 1 and 3 are correct (d) None is correct.

6.22 If two two-port networks are connected in series, and if the port current requirement is satisfied, which of the following is true?

(a) The z-parameter matrices add (b) The y-parameter matrices add.
(c) The $ABCD$-parameter matrices add. (d) None of these.

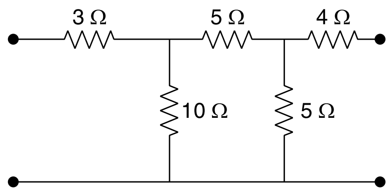
6.23 If two two-port networks are connected in parallel, and if the port current requirement is satisfied, which of the following is true?

(a) The z-parameter matrices add (b) The y-parameter matrices add.
(c) The $ABCD$-parameter matrices add (d) None of these.

6.24 If two two-port networks are connected in cascade, and if the port current requirement is satisfied, which of the following is true?

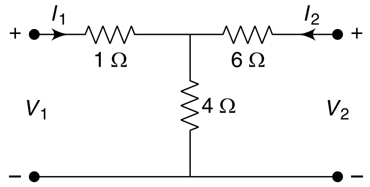
(a) The z-parameter matrices add (b) The y-parameter matrices add.
(c) The $ABCD$-parameter matrices add (d) None of these.

6.25 The z_{11} and z_{22} parameters of the given network are



(a) 8 Ω, 7.75 Ω
(b) 13 Ω, 9 Ω
(c) 12 Ω, 8.5 Ω
(d) None of the above.

6.26 For the network shown, the parameters h_{11} and h_{21} are



(a) 5 Ω and –2/3 Ω (b) 3.4 Ω and –2/5 Ω
(c) 3.4 Ω and –3/5 Ω (d) None of the above.

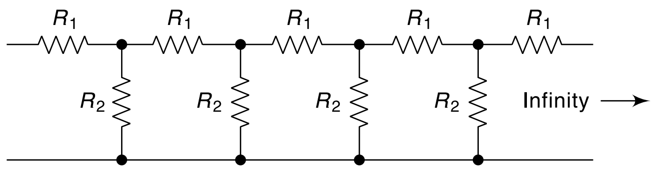
6.27 The maximum value of the transmission parameter A for a passive, reciprocal, linear two-port network is

(a) 1 (b) 2
(c) 3 (d) none of the above.

6.28. The unique feature of $ABCD$ parameters as compared to x, y and h parameters is

(a) none (b) short-circuit functions
(c) open-circuit functions (d) reverse transverse functions

6.29. The driving point impedance of the infinite ladder network shown in the given figure is



(given $R_1 = 2\ \Omega$ and $R_2 = 1.5\ \Omega$)

(a) 3 Ω (b) 3.5 Ω (c) $\frac{3}{3.5}\ \Omega$ (d) $\ln\left(1+\frac{3}{3.5}\right)\Omega$

EXERCISES

6.1 Current I_1 and I_2 entering at ports 1 and 2 respectively of a two-port network are given by the following equations:

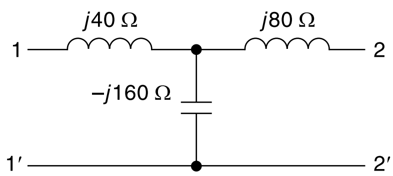
$$I_1 = 0.5V_1 - 0.2V_2$$
$$I_2 = -0.2V_1 + V_2$$

where V_1 and V_2 are the voltages at ports 1 and 2 respectively. Find the y, z and $ABCD$ parameters for the network. Also find its equivalent π-network.

[$y_{11} = 0.5\ \mho$; $y_{12} = -0.2\ \mho$; $y_{21} = -0.2\ \mho$; $y_{22} = 1\ \mho$;
$z_{11} = 2.174\ \Omega$; $z_{12} = z_{21} = -0.435\ \Omega$; $z_{22} = 1.086\ \Omega$;
$A = 5$, $B = 5\ \Omega$, $C = 2.3\ \mho$, $D = 2.5$; $Y_1 = 0.3\ \mho$; $Y_2 = 0.2\ \mho$; $Y_3 = 0.8\ \mho$]

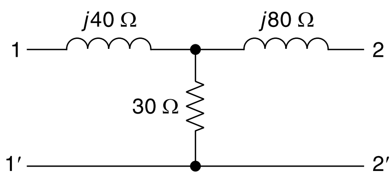
6.2 Determine the z-and y-parameters of the networks shown in figure.

(a)



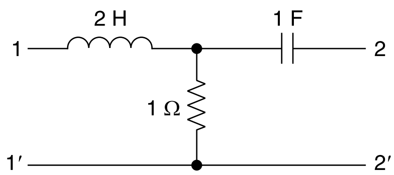
$$\left\{ z = \begin{bmatrix} -j120 & -j160 \\ -j160 & -j80 \end{bmatrix} (\Omega);\ y = \begin{bmatrix} -j/120 & j110 \\ j/110 & -j\dfrac{3}{4} \end{bmatrix} (\Omega^{-1}) \right\}$$

(b)



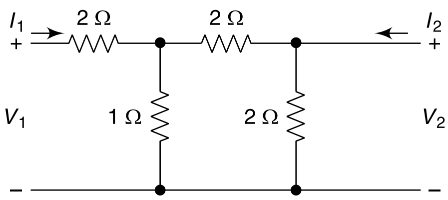
$$\left\{ z = \begin{bmatrix} (30 + j40) & j40 \\ j40 & (30 + j80) \end{bmatrix} (\Omega) \right\}$$

(c)



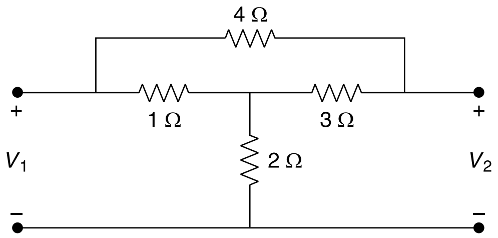
$$\left\{ z = \begin{bmatrix} (1 + 25) & 1 \\ 1 & \left(1 + \dfrac{1}{5}\right) \end{bmatrix} (\Omega) \right\}$$

6.3 Obtain the z-parameters for the circuit shown in figure and hence draw the z-parameter equivalent circuit.



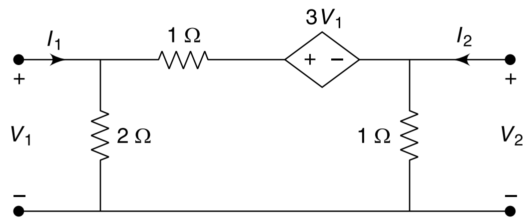
$$\left\{ z = \begin{bmatrix} \dfrac{14}{5} & \dfrac{2}{5} \\ \dfrac{2}{5} & \dfrac{6}{5} \end{bmatrix} (\Omega) \right\}$$

6.4 Find the open-circuit and short-circuit impedances of the network shown in figure.



$$\left[y = \begin{bmatrix} \frac{31}{44} & -\frac{19}{44} \\ -\frac{19}{44} & \frac{23}{44} \end{bmatrix}; z\text{-parameters do not exist} \right]$$

6.5 Find the z-parameters for the 2-port networks shown in figure containing a controlled source.



$$\left\{ z = \begin{bmatrix} -2 & -1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} (\Omega) \right\}$$

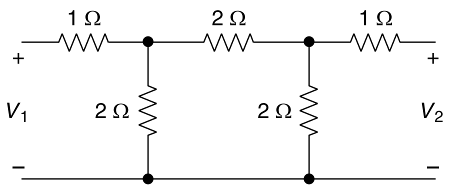
6.6 A 2-port network made up of passive linear resistors is fed at port 1 by an ideal voltage source of V volt. It is loaded at port 2 by a resistor R.

(i) With V = 10 volt and R = 6 Ω currents at ports 1 and 2 were 1.44 A and 0.2 A respectively.

(ii) With V = 15 volt and R = 8 Ω current at port 2 was 0.25 A.

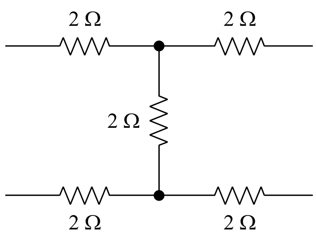
Determine the π-equivalent circuit of the 2-port network. $\{Y_A = 0.2;\ Y_B = 0.3;\ Y_C = 0.5 \text{ (mho)}\}$

6.7 Calculate the T-parameters for the block A and B separately and then using these results calculate the T-parameters of the whole circuit shown in figure. Prove any formula used.



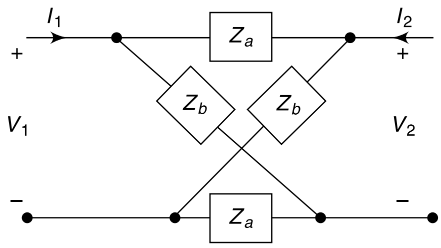
$$\left\{ A_a = A_b = D_a = D_b = \frac{3}{2} \right.$$
$$C_a = C_b = \frac{1}{2};\ B_a = B_b = \frac{5}{2};$$
$$\left. T = \begin{bmatrix} 7/2 & 15/2 \\ 3/2 & 7/2 \end{bmatrix} \right\}$$

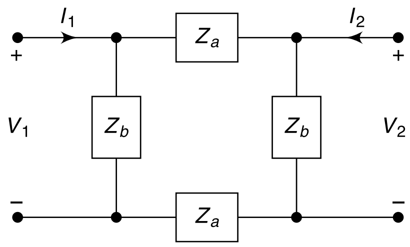
6.8 Find out the z-parameters of the two-port network shown in figure.



$$\left\{ z = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} (\Omega) \right\}$$

6.9 Find the z-parameters for the lattice network shown in Figure.





$$\left\{ z_{11} = z_{22} = \left(\frac{Z_b + Z_a}{2}\right);\ z_{12} = z_{21} = \left(\frac{Z_b - Z_a}{2}\right);\right.$$

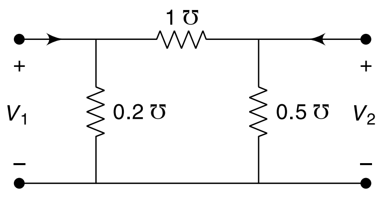
$$\left. z = \frac{1}{2Z_a + Z_b}\begin{bmatrix} 2(Z_a + Z_b) & Z_b^2 \\ Z_b^2 & 2Z_aZ_b \end{bmatrix}\right\}$$

6.10 Current I_1 and I_2 entering at port-1 and port-2 respectively of a two port network are given by the following equations: I_1=0.5V_1-0.2V_2, I_2=-0.2V_1+V_2, where V_1 and V_2 are the voltages at port-1 and port-2 respectively. Find the y, z and $ABCD$ parameters for the network. Also find the equivalent π-network.

$$\left\{ y = \begin{bmatrix} 0.5 & -0.2 \\ -0.2 & 1 \end{bmatrix}(\Omega^{-1});\ Z = \begin{bmatrix} 2.174 & 0.435 \\ 0.435 & 1.087 \end{bmatrix}(\Omega),\right.$$

$$\left. T = \begin{bmatrix} 5 & 5\,\Omega \\ 2.3\,\mho & 2.5 \end{bmatrix};\ Y_a = 0.3\,\mho,\ Y_b = 0.8\,\mho,\ Y_c = 0.2\,\mho \right\}$$

6.11 Two identical sections of the circuit shown in figure are connected in series. Obtain the z-parameters of the combination and verify by direct calculation. [$z_{11} = z_{22} = 6\ \Omega$; $z_{12} = z_{21} = 4\ \Omega$]



SHORT-ANSWER TYPE QUESTIONS

6.1 (a) Consider a linear passive two-port network and explain what are meant by (i) open-circuit impedance parameters and (ii) short-circuit admittance parameters.

(b) What are the open-circuit impedance parameters of a two-port network? How can the transmission parameters be obtained from open-circuit impedance parameters?

(c) Establish, for two-port networks, the relationship between the transmission parameters and the open-circuit parameters.

(d) Define z- and y-parameters of a typical four terminal network. Determine the relationship between the z and y parameters.

(e) Express h-parameters in terms of z-parameters for a two-port network.
(f) Derive expressions for the y-parameters in terms of $ABCD$ parameters of a two-port network.

6.2 (a) What do you understand by a reciprocal network? What is a symmetrical network?
(b) Write technical note on derivation of short-circuit admittance parameter y_{12} of a symmetrical and reciprocal two-port lattice network.
(c) How will you find the π-equivalent of a given network when its y-parameters are known?

6.3 (a) Explain what are meant by the transmission ($ABCD$) parameters of a two-port network. Derive the conditions necessary to be satisfied for the network to be (i) reciprocal and (ii) symmetrical.

Or,

Prove that for a reciprocal two-port network,

$$\Delta T = (AD - BC) = 1$$

(b) Prove that for a symmetrical two-port network,

$$\Delta h = (h_{11}h_{22} - h_{12}h_{21}) = 1$$

6.4 (a) Two two-port networks are connected in parallel. Prove that the overall y-parameters are the sum of corresponding individual y-parameters.
(b) Two two-port networks are connected in cascade. Prove that the overall transmission parameter matrix is the product of individual transmission parameter matrices.
(c) Two two-port networks are connected in series. Prove that the overall z-parameters are the sum of corresponding individual z-parameters.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

6.1 (d)	6.2 (c)	6.3 (a)	6.4 (a)	6.5 (d)	6.6 (c)	6.7 (b)
6.8 (d)	6.9 (a)	6.10 (b)	6.11 (d)	6.12 (d)	6.13 (c)	6.14 (c)
6.15 (c)	6.16 (b)	6.17 (b)	6.18 (a)	6.19 (d)	6.20 (d)	6.21 (b)
6.22 (a)	6.23 (b)	6.24 (d)	6.25 (a)	6.26 (b)	6.27 (d)	6.28 (d)
6.29 (a)						

CHAPTER 7
Fourier Series and Fourier Transform

PART I: FOURIER SERIES

7.1 INTRODUCTION

In 1807, the French mathematician Joseph Fourier (1768–1830) submitted a paper to the Academy of Sciences in Paris. In it, he presented a mathematical description of problems involving heat conduction. Although the paper was at first rejected, it contained ideas that would develop into an important area of mathematics named in honour, *Fourier analysis*. One surprising ramification of Fourier's work was that many familiar functions can be expanded in infinite series and integrals involving trigonometric functions. The idea today is important in modeling many phenomena in physics and engineering.

Fourier Analysis is a method for analysis of steady–state response of a network subject to a periodic input.

Periodic function A function of time $f(t)$ is said to be periodic if $f(t) = f(t \pm nT)$; where, n is a positive integer and 'T' is the period. Thus, a periodic function repeats itself every T second.

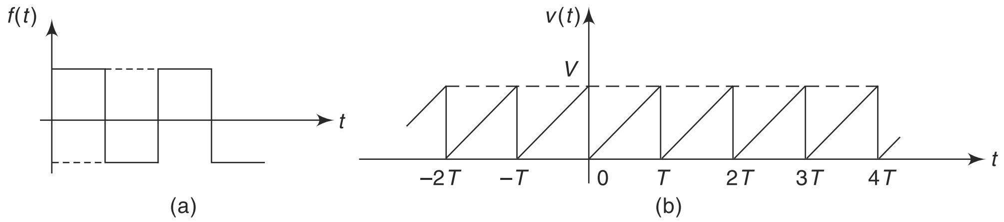


Figure 7.1 *Periodic function*

7.2 DEFINITION OF FOURIER SERIES

French mathematician J.B.J. Fourier first studied the periodic function in 1822 and published his theorem which states that,

"Any arbitrary periodic function can be represented by an infinite series of sinusoids of harmonically related frequencies." This infinite series is known as Fourier series.

Thus, if $f(t)$ is a periodic function, then the Fourier series is,

$$\begin{aligned} f(t) &= a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + a_n \cos n\omega t + \ldots \\ &\quad + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots + b_n \sin n\omega t + \ldots \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \end{aligned}$$

where, ω is the fundamental frequency = $\dfrac{2\pi}{T}$

$n\omega$ is the n^{th} harmonic of fundamental frequency

a_0, a_n, b_n are the Fourier Co-efficients

7.3 DIRICHLET'S CONDITIONS

The conditions, under which a periodic function $f(t)$ can be expanded in a convergent Fourier series, are known as Dirichlet's conditions.

These are as follows:

(i) $f(t)$ is a single valued function.

(ii) $f(t)$ has a finite number of discontinuities in each period, T.

(iii) $f(t)$ has a finite number of maxima and minima in each period, T.

(iv) The integral, $\int_0^T |f(t)|\,dt$ exists and is finite or in other way, $\int_0^T [f(t)]^2 dt < \infty$.

Note: If $f(t)$ is current or voltage, $\int_0^T [f(t)]^2 dt$ represents energy which would be supplied by the source in one cycle. That means the energy in the waveform for each cycle must be finite. All physical waveforms would, of course, satisfy this criterion.

Therefore, in practical engineering problems, it is not necessary to check whether a function satisfies Dirichlet condition.

7.4 FOURIER ANALYSIS

This involves two operations:

1. The evaluation of the co-efficient a_0, a_n and b_n.
2. Truncation of the infinite series after a finite number of terms so that $f(t)$ is represented within allowable error (-Done later).

7.4.1 Evaluation of Fourier Co-efficients

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{7.1}$$

From (7.1),

$$\int_0^T f(t)dt = a_0 \int_0^T dt + \sum_{n=1}^{\infty} \int_0^T (a_n \cos n\omega t + b_n \sin n\omega t)dt = a_0 T$$

$$\left\{ \because \int_{t_0}^{t_0+T} \sin m\omega t dt = 0 \text{ for all } m; \quad \text{and} \quad \int_{t_0}^{t_0+T} \cos n\omega t dt = 0 \text{ for all } n; \right\}$$

$$\boxed{\therefore a_0 = \frac{1}{T}\int_0^T f(t)dt}$$

This shows hat a_0 is the average value of *f(t)* over a period; therefore, called dc value of the signal. Now from equation (7.1),

$$\int_0^T f(t) \cos k\omega t dt = \int_0^T a_0 \cos k\omega t dt + \sum_{n=1}^{\infty} \int_0^T (a_n \cos k\omega t \cos n\omega t + b_n \cos k\omega t \sin n\omega t) dt$$

$$= 0 + a_k \frac{T}{2} + 0$$

$$\left\{ \because \int_{t_0}^{t_0+T} \sin n\omega t \sin m\omega t dt = 0 \text{ for } m \neq n \quad \text{and} \quad \int_{t_0}^{t_0+T} \cos n\omega t \cos m\omega t dt = 0 \text{ for } n \neq m \right.$$

$$\left. = \frac{T}{2} \text{ for } n = m \qquad\qquad = \frac{T}{2} \text{ for } n = m \right\}$$

$$\therefore \quad a_k = \frac{2}{T}\int_0^T f(t) \cos k\omega t dt$$

Again from equation (7.1),

$$\int_0^T f(t) \sin k\omega t dt = \int_0^T a_0 \sin k\omega t dt + \sum_{n=1}^{\infty} \int_0^T (a_n \sin k\omega t \cos n\omega t + b_n \sin k\omega t \sin n\omega t) dt$$

$$= 0 + 0 + b_k \frac{T}{2}$$

$$\therefore \quad b_k = \frac{2}{T}\int_0^T f(t) \sin k\omega t dt$$

Example 7.1 For the periodic waveform shown in figure, find the Fourier series expansion.

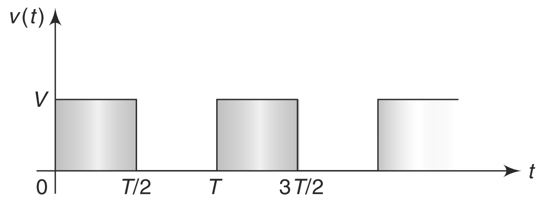


Figure 7.2 *Periodic waveform of Example (7.1)*

Solution Here, $v(t) = V$, for $0 < t < T/2$

$= 0$, for $T/2 < t < T$

$$a_0 = \frac{1}{T}\int_0^T v(t)dt = \frac{1}{T}\int_0^{T/2} Vdt = \frac{V}{2}$$

$$a_n = \frac{2}{T}\int_0^T v(t)\cos n\omega t dt = \frac{2}{T}\int_0^{T/2} V\cos\left(n\frac{2\pi}{T}\right)dt = 0$$

and,
$$b_n = \frac{2}{T}\int_0^T v(t)\sin n\omega t dt = \frac{2}{T}\int_0^{T/2} V\sin\left(n\frac{2\pi}{T}\right)dt$$

$$= \frac{V}{n\pi}(1-\cos n\pi);\ n = \pm 1, \pm 2, \pm 3, \ldots$$

$= 0$; for even n

$= \frac{V}{n\pi}$; for odd n

So, the Fourier series of the square wave is given as,

$$v(t) = V\left[\frac{1}{2} + \frac{2}{\pi}\sin\omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \ldots\right]$$

Exponential Form of Fourier Series We have the trigonometric Fourier series,

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t)$$

We know that, $\sin n\omega t = \dfrac{e^{jn\omega t} - e^{-jn\omega t}}{2j}$ and $\cos n\omega t = \dfrac{e^{jn\omega t} + e^{-jn\omega t}}{2}$

Thus,

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\frac{(e^{jn\omega t} + e^{-jn\omega t})}{2} + b_n\frac{(e^{jn\omega t} - e^{-jn\omega t})}{2j}\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\frac{1}{2}\left[\left(a_n + \frac{b_n}{j}\right)e^{jn\omega t} + \left(a_n - \frac{b_n}{j}\right)e^{-jn\omega t}\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - jb_n}{2}\right)e^{jn\omega t} + \left(\frac{a_n + jb_n}{2}\right)e^{-jn\omega t}\right]$$

Let, $C_0 = a_0, \quad C_n = \left(\dfrac{a_n - jb_n}{2}\right)$ and $C_n^* \text{ (or } C_{-n}) = \left(\dfrac{a_n + jb_n}{2}\right)$

Thus the series becomes,

$$f(t) = C_0 + \sum_{n=1}^{\infty} [C_n e^{jn\omega t} + C_{-n} e^{-jn\omega t}]$$

or $\boxed{f(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}}$ This is the exponential form of the Fourier series.

Now, $$C_n = \frac{a_n - jb_n}{2} = \frac{1}{2}\left[\frac{2}{T}\int_0^T f(t)\cos n\omega t dt - j\frac{2}{T}\int_0^T f(t)\sin n\omega t dt\right]$$

$$= \frac{1}{T}\int_0^T f(t)(\cos n\omega t - j\sin n\omega t)\,dt$$

Thus, $\boxed{C_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt}$ This equation is valid for both positive, negative and zero values of n.

Example 7.2 For the square wave shown in Example 7.1, find the exponential Fourier series.

Solution $f(t) = v(t) = V$, for $0 < t < T/2$
$= 0$, for $T/2 < t < T$

So, $$C_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt = \frac{1}{T}\int_0^{T/2} Ve^{-jn\omega t}dt$$

For $n = 0$, $$C_0 = \frac{1}{T}\int_0^{T/2} Vdt = \frac{V}{2}$$

For $n \neq 0$ $$C_n = \frac{1}{T}\int_0^{T/2} Ve^{-jn\omega t}dt = \frac{V}{T}\frac{1}{-jn\omega}[e^{-jn\omega T/2} - 1] = \frac{jV}{2\pi n}[e^{-jn\pi} - 1]$$

(since $\omega T = 2\pi$)

or $C_n = 0$ for even n

$$= -\frac{jV}{\pi n} \text{ for odd } n$$

Thus, the exponential Fourier series becomes,

$$v(t) = \ldots + \frac{jV}{5\pi}e^{j5\omega t} + \frac{jV}{3\pi}e^{j3\omega t} + \frac{jV}{\pi}e^{j\omega t} + \frac{V}{2} - \frac{jV}{\pi}e^{-j\omega t}$$

$$- \frac{jV}{3\pi}e^{-j3\omega t} - \frac{jV}{5\pi}e^{-j5\omega t} - \ldots \text{ for odd } n$$

Amplitude and Phase Spectrum From the trigonometric Fourier series,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$
$$= A_0 + \sum_{n=1}^{\infty} A_n \cos (n\omega t - \phi_n)$$

where, $A_0 = a_0$, $A_n = \sqrt{a_n^2 + b_n^2}$; $\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

Also, for exponential form, C_n is complex and we may write it as,

$$C_n = |C_n| e^{j\phi_n} \text{ and } |C_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = \frac{A_n}{2} \text{ and } \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

The quantities A_n and ϕ_n are called *the amplitude and the phase of the n^{th} harmonic*, respectively.

- Variation of A_n with n (or $n\omega$) is known as *the amplitude spectrum or Frequency – spectrum.*
- Variation of ϕ_n with n (or $n\omega$) is known as *the phase- spectrum* of the signal.

As both A_n and ϕ_n occurs at discrete values of the frequency, i.e., n = 1, 2, 3, etc. these spectra are called *Line spectra*.

Since $|C_n| = \frac{A_n}{2}$; there is a scale factor of ½ for the amplitude spectrum for exponential form for the Fourier series compared to the trigonometric form for all lines except the one for $n = 0$. Also, in the case of exponential form spectral lines are drawn for both for positive and negative values of n.

Example 7.3 For the square wave shown in Example 7.1, draw the amplitude and phase spectra.

Solution From the results of Example 7.1, we have,

$$v(t) = V\left[\frac{1}{2} + \frac{2}{\pi}\sin \omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \ldots\right]$$

Magnitudes: $V_0 = \frac{V}{2}\angle 0°$; $V_1 = \frac{2V}{\pi}\angle 90°$; $V_2 = 0$; $V_3 = \frac{2V}{3\pi}\angle 90°$ [since the cosine components are all zero, the phase angle will be $\tan^{-1}\left(\frac{b_n}{0}\right) = \tan^{-1}(\infty) = 90°$]

So, the line spectra become,

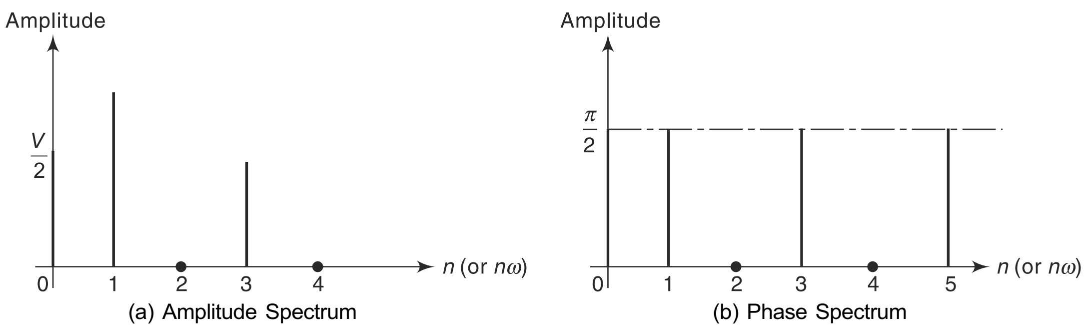


Figure 7.3 *Amplitude and phase spectra of Example 7.3*

Significance for Line Spectra: The amplitude- spectrum renders valuable information as to where to truncate the infinite series and yet maintain a good approximation to the original waveform.

Effective Value of a Periodic Function The effective (or R.M.S.) value of a periodic function $f(t)$ is defined as,

$$F_{eff}\,(F_{rms}) = \sqrt{\frac{1}{T}\int_0^T [f(t)]^2\,dt} = \sqrt{\frac{1}{T}\int_0^T \left[A_0 + \sum_{n=1}^{\infty} A_n \cos\,(n\omega t - \phi_n)\right]^2 dt}$$

$$= \sqrt{\frac{1}{T}\left[A_0^2 T + \sum_{n=1}^{\infty} A_n^2 \frac{T}{2}\right]}$$

$$\boxed{F_{eff}\,(F_{rms}) = \sqrt{A_0^2 + \sum_{n=1}^{\infty}\left(\frac{A_n}{\sqrt{2}}\right)^2}}$$

This shows that the effective value of a periodic function is the square root of the effective values of the harmonic components and the square of the d. c. value.

Waveform Symmetry There are few methods by which the evaluation of Fourier co-efficients is simplified by symmetry consideration.

These methods reduce the amount of labour involved in finding out the co-efficients.

Now, $$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_0^{T/2} f(t)dt\right]$$

Putting $t = -x$ in the first integrand and $t = x$ in the second integrand, we get

$$a_0 = \frac{1}{T}\left[\int_0^{T/2} [f(x) + f(-x)]dx\right]$$

Now, $$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt = \frac{2}{T}\left[\int_0^{T/2} f(t)\cos n\omega t dt + \int_{-T/2}^{0} f(t)\cos n\omega t dt\right]$$

$$= \frac{2}{T}[I_1 + I_2]$$

Since the variable t in I_1 and I_2 integrals is dummy variable, let $x = t$ in I_1 and $x = -t$ in I_2.

$\therefore$ $$a_n = \frac{2}{T}\left[\int_0^{T/2} f(x)\cos n\omega x dx - \int_0^{T/2} f(-x)\cos n\omega x(-dx)\right]$$

Thus, $$\boxed{a_n = \frac{2}{T}\int_0^{T/2} [f(x) + f(-x)]\cos n\omega x dx}$$

Similarly, $$\boxed{b_n = \frac{2}{T}\int_0^{T/2} [f(x) - f(-x)] \sin n\omega x dx}$$

Following symmetries are considered:

1. Odd or Rotation Symmetry,
2. Even or Mirror Symmetry,
3. Half-Wave or, Alternation Symmetry, and
4. Quarter-Wave Symmetry.

1. Odd Symmetry

A function $f(x)$ is said to be odd if,

$$f(x) = -f(-x)$$

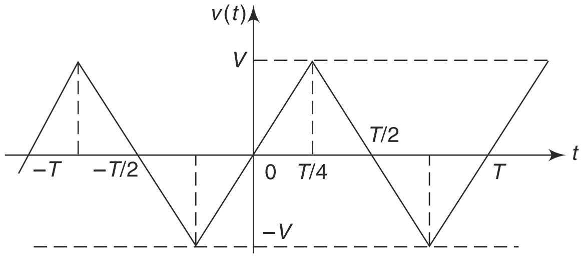


Figure 7.4 *Odd function*

Hence, for odd functions $a_0 = 0$ and $a_n = 0$ and $b_n = \frac{1}{T}\int_0^{T/2} f(x) \sin n\omega x \, dx$

Thus, the *Fourier series expansion of an odd function contains only the sine terms, the constant and the cosine terms being zero.*

2. Even Symmetry

A function $f(x)$ is said to be even, if

$$f(x) = f(-x)$$

$$\therefore \qquad a_0 = \frac{2}{T}\int_0^{T/2} f(x) dx$$

$$\therefore \qquad a_n = \frac{4}{T}\int_0^{T/2} f(x) \cos n\omega x dx$$

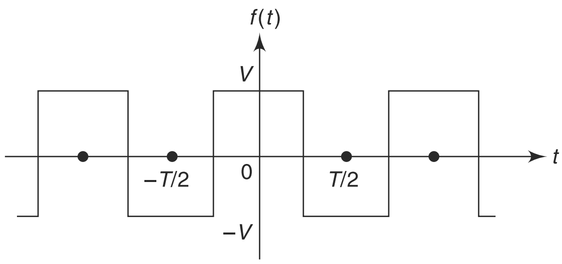


Figure 7.5 *Even function*

and $b_n = 0$

Thus, the *Fourier series expansion of an even periodic function contains only the cosine terms plus a constant, all sine terms being zero.*

3. Half –Wave or Alternation Symmetry

A periodic function $f(t)$ is said to have half wave symmetry if it satisfies the condition,

$$f(t) = -f(t \pm T/2), \text{ where } T - \text{time period of the function}$$

$$\therefore \qquad a_0 = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_{0}^{T/2} f(t)dt\right] = \frac{1}{T}[I_1 + I_2]$$

x	$T/2$	0
t	0	$-T/2$

For I_1, let $x = (t + T/2)$; so, $f(t) = f(x - T/2) = -f(x)$ and $dt = dx$

$$\therefore \qquad I_1 = \int_{-T/2}^{0} f(t)dt = \int_{0}^{T/2} -f(x)dx = -\int_{0}^{T/2} f(x)dx$$

$$\therefore \qquad a_0 = \frac{1}{T}\left[-\int_{0}^{T/2} f(x)dx + \int_{0}^{T/2} f(t)dt = \right] = \frac{1}{T}\left[\int_{0}^{T/2} f(x)dx - \int_{0}^{T/2} f(x)dx\right] = 0$$

$$\therefore \qquad a_n = \frac{2}{T}\left[\int_{-T/2}^{T/2} f(t)\cos n\omega t dt\right] = \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\cos n\omega t dt + \int_{0}^{T/2} f(t)\cos n\omega t dt\right] = \frac{2}{T}[I_1 + I_2]$$

Again putting $x = (t + T/2)$ and following the same procedure,

$$I_1 = \int_{-T/2}^{0} f(t)\cos n\omega t dt = \int_{0}^{T/2} -f(x)\cos n\omega(x - T/2)dx = \int_{0}^{T/2} -f(x)\cos(n\omega x - n\pi)dx$$

$$= \int_{0}^{T/2} -f(x)\cos n\pi \cos n\omega x dx = \int_{0}^{T/2} -f(t)\cos n\pi \cos n\omega t dt$$

$$a_n = \frac{2}{T}(1 - \cos n\pi)\int_{0}^{T/2} f(t)\cos n\omega t dt$$

$= 0$; for even n, and

$= \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega t dt$, for odd n.

Similarly, $\quad b_n = 0$, for even n; and

$= \frac{4}{T}\int_{0}^{T/2} f(t)\sin n\omega t dt$, for odd n.

Thus, the Fourier series expansion of a periodic function having half-wave symmetry contains only odd harmonics, the constant term being zero.

4. Quarter–Wave Symmetry

The symmetry may be regarded as a combination of first three kinds of symmetry provided that the origin is properly chosen. For Figure 7.6(a), the wave has alternation and odd symmetry; thus the Fourier series consists of odd sine terms only.

$\therefore \quad a_0 = 0$, $a_n = 0$ and $b_n = \frac{8}{T}\int_{0}^{T/4} f(t)\sin n\omega t\, dt$, n being odd only.

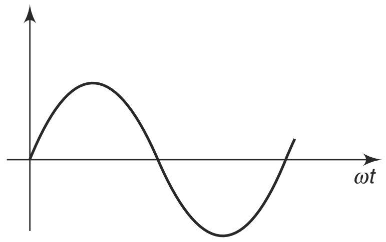


Figure 7.6(a) *Sin ωt: combination of half-wave and odd symmtery*

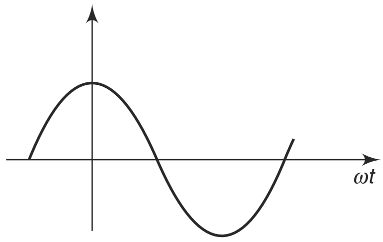


Figure 7.6(b) *Cos ωt: combination of half-wave and even symmetry*

For Figure 7.6(b), the origin, having chosen one quarter cycle away, as in Figure 7.6(a), the wave has alternation and even symmetry; thus the Fourier series consists of odd cosine terms only.

$\therefore$ $a_0 = 0$; $b_n = 0$; and $a_n = \dfrac{8}{T}\int_0^{T/4} f(t)\cos n\omega t\,dt$, n being odd only.

Note:

(i) The sum or product of two or more even functions is an even function, and with the addition of a constant, the even nature of the function is still preserved.

(ii) The sum of two or more odd functions is an odd function, but the addition of a constant removes the odd nature of the function. The product of two odd functions is an even function.

7.4.2 Truncating Fourier Series

When a periodic function is represented by a Fourier series, the series is truncated after a finite number of terms.

So, the periodic function is approximated by a trigonometric series of $(2N + 1)$ terms as,

$$S_N(t) = a_0 + \sum_{n=1}^{N}(a_n \cos n\omega t + b_n \sin n\omega t) \tag{7.2}$$

such that the co-efficients a_0, a_n and b_n are chosen to give the least mean square error.

The truncation error is,

$$e_N(t) = f(t) - S_N(t) \tag{7.3}$$

So, the mean square error/figure of merit/the cost criterion for optimal minimal error is,

$$E_N = \overline{e_N^2(t)} = \frac{1}{T}\int_0^T [e_N(t)]^2\,dt \tag{7.4}$$

where, E_N is a function of a_0 , a_n and b_n, but not of t.

Example 7.4 Show that the mean square error is a minimum if the co- efficients in the approximated trigonometric series $S_N(t)$ are the Fourier co- efficients.

Solution In order to make 'E_N' minimum, the necessary conditions are,

$$\frac{\partial E_N}{\partial a_n} = 0, \text{ for } n = 0, 1, 2, \ldots \tag{7.5a}$$

and $\quad \dfrac{\partial E_N}{\partial b_n} = 0$, for $n = 0, 1, 2, \ldots$ (7.5b)

These two equations give $(2N+1)$ equations from which $(N+1)$ number of a_n for $n=1, 2, \ldots, N$ and N number of b_n for $n=1, 2, \ldots$, N can be determined.

From Equations 7.4 and 7.5

$$\frac{\partial E_N}{\partial a_n} = \frac{2}{T}\int_0^T e_N(t)\frac{\partial e_N(t)}{\partial a_n}dt = \frac{2}{T}\int_0^T [f(t) - S_N(t)]\cos n\omega t dt = 0$$

or
$$\int_0^T f(t)\cos n\omega t dt = \int_0^T S_N(t)\cos n\omega t dt$$

$$= \int_0^T \left[a_0 + \sum_{n=1}^{N}(a_n \cos n\omega t + b_n \sin n\omega t)\right]\cos n\omega t dt$$

$$= \int_0^T a_n \cos^2 n\omega t dt$$

$$= a_n \frac{T}{2}$$

or
$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt \quad (n = 0, 1, 2, \ldots, N)$$

Similarly, from equation 7.5(b), we get,

$$\therefore \quad b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t dt \quad (n = 0, 1, 2, \ldots, N)$$

Therefore, it is proved that a Fourier series with a finite number of terms represents the best approximation for a given periodic function by any trigonometric series with the same number of terms.

However, there is no analytical method for the evaluation of estimation of error due to truncation of infinite series; i.e., we can not predict the number of minimum terms to be retained in the series within a prescribed accuracy. The minimisation of error is done by trial and error method, using more terms until specifications are met.

Example 7.6 If f(t) is approximated by $\dfrac{8}{\pi^2}\sin \omega t$, i.e., the first term in the Fourier Series, find the mean square error.

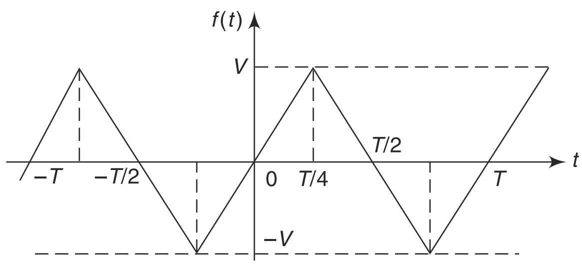


Figure 7.7 *Waveform of Example 7.6*

Solution Truncation Error, $e_N = f(t) - \frac{8}{\pi^2}\sin \omega t$

Mean Square Error, $E_N = \overline{e_N^2} = \frac{1}{T}\int_0^T e_N^2(t)dt = \frac{4}{T}\int_0^{T/4}\left[f(t) - \frac{8}{\pi^2}\sin\omega t\right]^2 dt$

(from symmetry consideration)

$$= \frac{4}{T}\int_0^{T/4}\left[\frac{4t}{T} - \frac{8}{\pi^2}\sin\omega t\right]^2 dt$$

$$= 0.0047$$

7.5 STEADY- STATE RESPONSE OF NETWORK TO PERIODIC SIGNALS

The voltage (periodic) is,

$$v(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t - \phi_n)$$

We want to find out the steady state current, $i(t)$. Phasors corresponding to terms in right hand side are,

$$\mathbf{V}_0 = A_0 e^{j0} \quad \text{and} \quad \mathbf{V}_n = A_n e^{-j\phi_n}$$

Let, $\boldsymbol{Z}(j\omega)$ = Impedance phasor of the network at any frequency ω.

So, the current phasors are,

$$\mathbf{I}_0 = \frac{\mathbf{V}_0}{\mathbf{Z}(j0)} = \frac{A_0 e^{j0}}{Z(j0)} = |I_0| e^{j0}$$

$$\mathbf{I}_n = \frac{\mathbf{V}_n}{\mathbf{Z}(j\omega)} = \frac{A_0 e^{-j\phi_n}}{Z(j\omega)} = |I_n| e^{-j\alpha_n}$$

By superposition principle, the net current phasor is,

$$i(t) = I_0 + I_1 + I_2 + \ldots$$

So, transforming from frequency domain to time domain,

$$i(t) = I_0 + \sum_{n=1}^{\infty} |I_n| \cos(n\omega t - \alpha_n)$$

7.5.1 Average Power Calculation

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega t - \phi_n)$$

$$i(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega t - \alpha_n)$$

Here, V_0 = DC voltage component

V_n = the amplitude of the n^{th} harmonic voltage

ϕ_n = the phase angle of the n^{th} harmonic voltage

I_0 = DC current component
I_n = the amplitude of the n^{th} harmonic current
α_n = the phase angle of the n^{th} harmonic current

Instantaneous power,

$$P(t) = v(t)\, i(t)$$

Average power,

$$P_{av} = \frac{1}{T}\int_0^T v(t)i(t) = \frac{1}{T}\int_0^T \left[\left(V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega t - \phi_n)\right)\left(I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega t - \alpha_n)\right)\right] dt$$

or
$$P_{av} = V_0 I_0 + \sum_{n=1}^{\infty}\int_0^T V_n I_n \cos(n\omega t - \phi_n)\cos(n\omega t - \alpha_n)\, dt$$

or
$$P_{av} = V_0 I_0 + \sum_{n=1}^{\infty} \frac{V_n I_n}{2}\cos(\phi_n - \alpha_n)$$

7.5.2 Steps for Application of Fourier Series to Circuit Analysis

1. Fourier series of the given periodic excitation function is obtained.
2. The circuit elements are transformed from time domain to frequency domain (i.e., $R \to R$, $L \to j\omega nL$, $C \to \dfrac{1}{j\omega nC}$ for n^{th} harmonic).
3. The Fourier series of the DC and AC components of the response are calculated.
4. Using Superposition, the Fourier series of the response is obtained by summing up the individual DC and AC response components.

7.5.3 Power Spectrum

It is the distribution of the average power over the different frequency components.
Let, P_n be the average power for the n^{th} harmonic component.
Note: P_n is always positive so that only a magnitude spectrum is possible.
Another form of line spectrum for power is also possible [Fig. 7.8(ii)]; obtained by assuming half of P_n to the positive frequency $n\omega$ and half to the negative frequency.

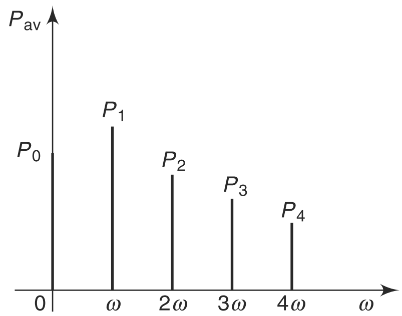


Figure 7.8(i) *Power Spectrum for positive ω*

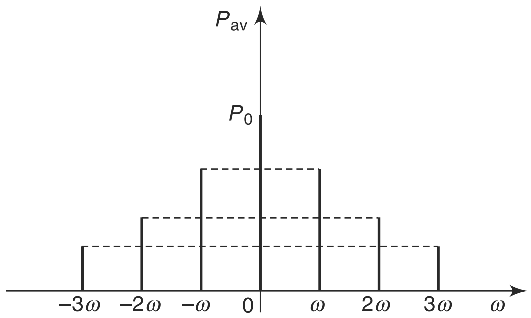


Figure 7.8(ii) *Power Spectrum for both positive and negative ω*

PART II: FOURIER TRANSFORM

7.6 INTRODUCTION

The Fourier series representation of a period function describes the function in the frequency domain in terms of amplitude and phase spectra. The Fourier transform extends this frequency domain description to functions that are not periodic.

Fourier transform is a powerful tool in the study of power spectra, correlation functions, noise and other advanced problems.

7.7 DEFINITION OF FOURIER TRANSFORM

The Fourier Transform or the Fourier integral of a function f(t) is denoted by $F(j\omega)$ and is defined by,

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{7.5}$$

and the inverse Fourier transform is defined by,

$$f(t) = \mathcal{F}^{-1}[F(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty} F(j2\pi f)\,e^{j2\pi f}\,df \tag{7.6}$$

Equations 7.5 and 7.6 form the Fourier transform pair.

Explanation

Consider the exponential Fourier Series,

$$f(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega t} \tag{7.7}$$

where,
$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega t}\,dt \tag{7.8}$$

If the period T becomes infinite, the function does not repeat itself and becomes aperiodic or non-periodic.

So, the interval between adjacent harmonic frequencies is,

$$\Delta\omega = (n+1) - n\omega = \omega = \frac{2\pi}{T}$$

or
$$\frac{1}{T} = \frac{\omega}{2\pi} = \frac{\Delta\omega}{2T} \tag{7.9}$$

As $T \to \infty$, $\Delta\omega \to d\omega$ and the frequency goes from a discrete variable over to a continuous variable.

$$\frac{1}{T} \to \frac{d\omega}{2\pi} \quad \text{and} \quad n\omega \to \omega \tag{7.10}$$

From 7.6 and 7.10,

$$C_nT \to \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$. This is the Fourier Transform of $f(t)$ i.e., $F(j\omega)$.

$$\boxed{F(j\omega) = F[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$$

So, from equation (7.7),

$$f(t) = \sum_{-\infty}^{\infty} (C_nT)e^{jn\omega t}\left(\frac{1}{T}\right) \tag{7.11}$$

As $T \to \infty$, $C_nT \to F(j\omega)$, $n\omega \to \omega$ and $\frac{1}{T} \to \frac{d\omega}{2\pi}$ and $\sum \to \int$ (summation approaches integration). Thus, from (7.11),

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)e^{j\omega t}d\omega}$$

Spectra Let, $F(j\omega) = |F(j\omega)|e^{j\phi(\omega)}$
The variation of $|F(j\omega)|$ with ω is referred to as the amplitude spectrum.
The variation of $\phi(\omega)$ with ω is referred to as the phase–spectrum.
Since $F(j\omega)$ is a continuous function, the corresponding amplitude and phase spectra are continuous spectra.

7.8 CONVERGENCE OF FOURIER TRANSFORM

When $f(t)$ is a single-valued function and is different from zero over an infinite interval of time, the behavior of $f(t)$ as $t \to \pm\infty$ determines the convergence of the Fourier transform.
The Fourier transform will exist, if

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

7.9 FOURIER TRANSFORM OF SOME FUNCTIONS

1. $f(t) = Ae^{-at}u(t)$, $a > 0$
Fourier transform will exist, if $a > 0$

$$\therefore \qquad F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = A\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = A\left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty} = \frac{A}{a+j\omega}$$

Amplitude, $|F(j\omega)| = \dfrac{A}{\sqrt{a^2+\omega^2}}$

Phase, $\phi(j\omega) = -\tan^{-1}\left(\dfrac{\omega}{a}\right)$

2. $f(t) = Ke^{-a|t|}$, for all values of t

$$F(j\omega) = \mathcal{F}[\mathbf{Ke}^{-\mathbf{a}|\mathbf{t}|}] = \int_{-\infty}^{\infty} \mathbf{Ke}^{-\mathbf{a}|\mathbf{t}|} e^{-j\omega t} dt = \int_{-\infty}^{0} Ke^{(a-j\omega)t} dt + \int_{0}^{\infty} Ke^{-(a+j\omega)t} dt$$

$$= \frac{K}{a-j\omega} + \frac{K}{a+j\omega} = \frac{2Ka}{a^2+\omega^2}$$

Note: There are some important functions which do not have Fourier transforms in a strict sense; because they do not satisfy the Dirichlet's condition, i.e., $\int_{-\infty}^{\infty} |f(t)| dt$ is infinite (such as, the step function and sinusoidal function). However, the Fourier transforms of these function are evaluated by approximating these functions in time domain as the limiting value of another function which possesses Fourier transform.

3. Fourier transform of some constant, *K*, for all values of *t*

Here, we can approximate the constant as,

$$f(t) = \operatorname*{Lt}_{a\to 0} [Ke^{-a|t|}]$$

$$\therefore \qquad \mathcal{F}[K] = \operatorname*{Lt}_{a\to 0} \int_{-\infty}^{\infty} Ke^{-a|t|} e^{-j\omega t} dt = \operatorname*{Lt}_{a\to 0} \frac{2Ka}{a^2+\omega^2}$$

$$\therefore \qquad \mathcal{F}[K] = 0;\ \text{for } \omega \neq 0$$
$$= \infty;\ \text{for } \omega = 0$$

[by L Hospital's rule, i.e. differentiating both numerates and denominator with respect to a]

Thus, $\mathcal{F}[K]$ is an impulse function at $\omega = 0$. The strength (amplitude) of the impulse function is obtained as,

$$\int_{-\infty}^{\infty} \mathcal{F}[K] d\omega = \int_{-\infty}^{\infty} \frac{2Ka}{a^2+\omega^2} d\omega = 2\pi K$$

$$\therefore \qquad \boxed{\mathcal{F}[K] = 2\pi K\delta(\omega)}$$

4. Unit impulse function or Dirac Delta Function, $\delta(t)$

Some problems involve the concept of an impulse, which may be intuitively thought of as a force of very large magnitude impacting just for an instant.

$$\therefore \qquad F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{0} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

5. Fourier transform of Signum Function, Sgn(t)

A signum function is defined as,

$$\begin{aligned} \text{Sgn}(t) &= +1 \text{ for } \quad t > 0 \\ &= 0 \text{ for } \quad t = 0 \\ &= -1 \text{ for } \quad t < 0 \end{aligned}$$

$\therefore \int_{-\infty}^{\infty} \text{Sgn}(t)dt$ is infinite, direct evaluation of Fourier transform is not possible.

Therefore, the given function has to be expressed as limiting case of some other function and then the Fourier Transform is computed. Let, the Sgn(t) be multiplied by $e^{-a|t|}$ and $a \to 0$.

$$F[\text{Sgn}(t)] = \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-a|t|}\,\text{Sgn}(t)e^{-j\omega t}dt = \lim_{a\to 0}\left[-\int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt\right]$$

$$= \lim_{a\to 0}\left[\frac{-1}{a-j\omega}+\frac{1}{a+j\omega}\right]$$

or $\boxed{F[\text{Sgn}(t)] = \frac{2}{j\omega}}$

6. Fourier transform of Unit Step Function, *u*(*t*)

$$\begin{aligned} u(t) &= 1 \text{ for } t > 0 \\ &= 0 \text{ for } t < 0 \end{aligned}$$

Since $\int_{-\infty}^{\infty} u(t)dt$ is infinite, direct evaluation of Fourier Transform is impossible.

Let, $u(t) = \frac{1}{2}+\frac{1}{2}\text{Sgn}(t)$

$$\therefore \quad \mathcal{F}[u(t)] = \mathcal{F}\left[\frac{1}{2}\right]+\mathcal{F}\left[\frac{1}{2}\text{Sgn}(t)\right] = 2\pi\times\frac{1}{2}\delta(\omega)+\frac{1}{2}\times\frac{2}{j\omega}$$

or $\boxed{\mathcal{F}[u(t)] = \pi\delta(\omega)+\frac{1}{j\omega}}$

Thus, the amplitude of unit step function $u(t)$ in Frequency domain will be a combination of rectangular hyperbola and impulse function (of strength π at $\omega = 0$).

7.10 PROPERTIES OF FOURIER TRANSFORMS

1. Linearity

If a, b, $\in C$, then

$$F\{\alpha f(t)+\beta g(t)\} = \alpha F\{f(t)\}+\beta F\{g(t)\} = \alpha F(\omega)+\beta G(\omega)$$

provided the Fourier transforms of $f(t)$ and $g(t)$ exist.

2. Scaling

If $F\{f(t)\} = F(\omega)$ and $c \in R$, then

$$F\{cf(t)\} = \frac{1}{|c|} F\left(\frac{\omega}{c}\right)$$

3. Time shifting

If $F\{f(t)\} = F(\omega)$ and $t_0 \in R$, then

$$F\{f(t - t_0)\} = e^{-j\omega t_0} F(\omega)$$

4. Frequency shifting

If $F\{f(t)\} = F(\omega)$ and $\omega \in R$, then

$$F(\omega - \omega_0) = F\{e^{j\omega_0} f(t)\}$$

5. Symmetry

If $F\{f(t)\} = F(\omega)$, then

$$F\{F(t)\} = 2\pi f(-\omega)$$

6. Modulation

If $F\{f(t)\} = F(\omega)$ and $\omega_0 \in R$, then

$$F\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$$

$$F\{f(t)\sin(\omega_0 t)\} = \frac{1}{2}[F(\omega + \omega_0) - F(\omega - \omega_0)]$$

7. Differentiation in time

Let $n \in N$ and suppose that $f^{(n)}$ is piecewise continuous. Assume that $\lim\limits_{x \to \infty} f^{(k)}(t) = 0$, then

$$F\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$$

In particular

$$F\{f'(t)\} = j\omega F(\omega)$$

and

$$F\{f''(t)\} = -\omega^2 F(\omega)$$

8. Frequency differentiation

Let $n \in N$ and suppose that f is piecewise continuous. Then

$$F\{t^n f(t)\} = j^n F^{(n)}(\omega)$$

In particular

$$F\{tf(t)\} = jF'(\omega)$$

and

$$F\{t^2 f(t)\} = -F''(\omega)$$

These properties can be tabulated as follows (Table 7.1).

Table 7.1 *Properties of Fourier Transforms*

Sl No.	*Time Domain* $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}dt$	*Frequency Domain* $F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
1	$f(t)$ real	$F(j\omega) = F^*(-j\omega)$
2	$f(t)$ even, $f(t) = f(-t)$	$F(j\omega) = F(-j\omega)$, $F(j\omega)$ is real
3	$f(t)$ odd, $f(t) = -f(-t)$	$F(j\omega) = -F(-j\omega)$, $F(j\omega)$ is imaginary,
4	$y(t) = t^n f(t)$	$Y(j\omega) = (j)^n \frac{d^n F(j\omega)}{d\omega^n}$
5	$y(t) = f(at)$	$F(j\omega) = \frac{1}{a}F\left(\frac{j\omega}{a}\right), a>0$
6	$y(t) = f(t-t_0)$	$Y(j\omega) = e^{-j\omega t_0}F(j\omega)$
7	$y(t) = \frac{d^n f(t)}{dt^n}$	$Y(j\omega) = (j\omega)^n F(j\omega)$
8	$y(t) = \int_{-\infty}^{\infty} f(t)dt$	$Y(j\omega) = \frac{F(j\omega)}{j\omega}$
9	$y(t) = f(t)e^{j\omega_0 t}$	$Y(j\omega) = F[j(\omega-\omega_0)]$

Example 7.7 Show that when $f(t)$ is an even function of t, its Fourier transform $F(j\omega)$ is a function of ω and is real; while when $f(t)$ is an odd function of t, its Fourier transform $F(j\omega)$ is an odd function of ω and is imaginary.

Solution From the definition,

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)(\cos\omega t - j\sin\omega t)dt$$

$$= \int_{-\infty}^{\infty} f(t)\cos\omega t dt - j\int_{-\infty}^{\infty} f(t)\sin\omega t dt = P(\omega) + jQ(\omega)$$

where, $$P(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t dt = \text{Even function of } \omega, \text{i.e., } P(\omega) = P(-\omega)$$

and $$Q(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t dt = \text{Odd function of } \omega, \text{i.e., } Q(\omega) = -Q(-\omega)$$

Now, $$F(j\omega) = |F(j\omega)|\, e^{j\phi(\omega)}$$

$$|F(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)} = \text{Even function of } \omega$$

and $$F(j\omega) = \tan^{-1}\left[\frac{Q(\omega)}{P(\omega)}\right] = \text{Odd function of } \omega$$

- *When f(t) is an even function*

  $f(t)$ cos ωt is an even function

  $f(t)$ sin ωt is odd function.

$$\therefore \quad P(\omega) = 2\int_{0}^{\infty} f(t)\cos \omega t dt$$

$$Q(\omega) = 0$$

so, $F(j\omega) = P(\omega)$ = Even and Real (Proved)

- *When f(t) is an odd function*

  $f(t)$ cos ωt is an odd function

  $f(t)$ sin ωt is an even function

$$\therefore \quad P(\omega) = 0$$

$$\text{and } \therefore Q(\omega) = -2\int_{0}^{\infty} f(t)\sin \omega t dt$$

so, $F(j\omega) = jQ(\omega)$ = Odd and Imaginary (Proved)

7.11 ENERGY DENSITY AND PERSEVAL'S THEOREM

This theorem states that the energy content (W) of a waveform (periodic or non-periodic) over the whole frequency band is,

$$W = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega$$

Proof: We have,

$$W = \int_{-\infty}^{\infty} f^2(t)dt = \int_{-\infty}^{\infty} f(t)\cdot[f(t)dt]$$

$$= \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,dt\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\left[\int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\cdot F(-j\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega$$

or
$$W = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega \qquad \textit{Proved}$$

Note

(i) Since $|F(j\omega)|$ is an even function of ω,

$$W = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{\pi}\int_0^{\infty} |F(j\omega)|^2 \, d\omega$$

(ii) Since $\omega = 2\pi f$, where f is the frequency,

$$W = \int_{-\infty}^{\infty} f^2(t)dt = \int_{-\infty}^{\infty} |F(j2\pi f)|^2 \, df = 2\int_0^{\infty} |F(j2\pi f)|^2 \, df$$

The quantity $|F(j2\pi f)|^2 \, df$ is the energy in an infinitesimal band of frequency df. It represents the energy density in the frequency domain and has unit of Joule/Hertz.
Total energy content within the frequency band f_1 and f_2 is,

$$W_b = 2\int_{f_1}^{f_2} F(j2\pi f)|^2 \, df$$

For the integration range $-\infty$ to $+\infty$, the total energy is,

$$W_b = \int_{-f_1}^{-f_2} |F(j2\pi f)|^2 \, df + \int_{f_1}^{f_2} |F(j2\pi f)|^2 \, df$$

(iii) If $f(t)$ is the voltage across a 1 Ω resistance or current through the same resistance, then W_b is known as 1 Ω energy.

Example 7.8 The current in a 10 Ω resistor is $i(t) = 10e^{-2t}u(t)\,(A)$. What is the energy associated with the frequency band $0 \le \omega \le 2$ rad/s?

Solution Here, $f(t) = i(t) = 10e^{-2t}u(t)$

$$\therefore \qquad F(j\omega) = \frac{10}{2 + j\omega}$$

So, the energy associated with the given frequency band is,

$$W = \frac{10}{\pi}\int_0^2 |F(j\omega)|^2 \, d\omega = \frac{10}{\pi}\int_0^2 \frac{100\,d\omega}{4+\omega^2} = \frac{10^3}{\pi}\left[\frac{1}{2}\tan^{-1}\left(\frac{\omega}{2}\right)\right]\Bigg|_0^2$$

$$= \frac{10^3}{\pi}\left[\frac{\pi}{8}\right]$$

$= 125$ Joule *Ans.*

7.12 COMPARISON BETWEEN FOURIER TRANSFORM AND LAPLACE TRANSFORM

The defining equations are,

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \quad \text{and} \quad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Followings are some differences and similarities

1. Laplace Transform is one-sided in the interval $0 < t < \infty$ and Fourier Transform is double-sided in the interval $-\infty < t < \infty$. Thus, Laplace Transform is applicable for positive time function, $f(t)$, $t > 0$; while Fourier Transform is applicable for functions defined for all times.
2. Laplace Transform includes the initial conditions and is applicable for transient analysis; while Fourier Transform is only applicable for steady-state analysis.
3. For functions $f(t) = 0$ for $t < 0$ and $\int_0^{\infty} |f(t)|\,dt < \infty$, the two transforms are related as, $F(j\omega) = F(s)|_{s=j\omega}$. Thus, Laplace Transform is associated with entire s-plane, while, Fourier Transform is restricted to the imaginary ($j\omega$) axis.
4. Laplace Transform is applicable to a wider range of functions than the Fourier Transform. On the other hand, Fourier Transforms exist for signals that are not physically realizable and have no Laplace Transform.

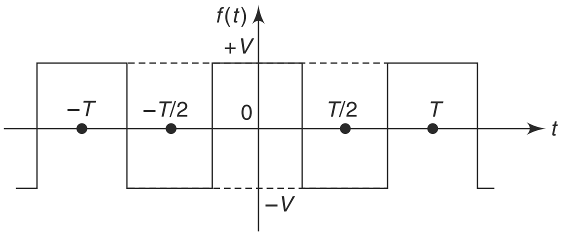
7.13 STEPS FOR APPLICATION OF FOURIER TRANSFORM TO CIRCUIT ANALYSIS

By Fourier Transform, we can find the response of a circuit due to non-periodic functions. The general procedure is described below.

1. Fourier Transform of the given excitation function is obtained.
2. Fourier Transform of the circuit elements is obtained $\left(\text{i.e., } R \to R, L \to j\omega L, C \to \dfrac{1}{j\omega C}\right)$.
3. The transfer function in Fourier Transform Domain is defined as, $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$ or $Y(j\omega) = H(j\omega) \cdot X(j\omega)$; where, $Y(j\omega)$ is the response transform and $X(j\omega)$ is the excitation transform.
4. Taking the inverse Fourier Transform of the product $H(j\omega) \cdot X(j\omega)$, we get the response $y(t)$.

SOLVED PROBLEMS

7.1 Determine the Fourier series for the square waveform shown below and plot the magnitude and the phase spectra.



Solution The waveform, $f(t) = V;\ 0 < t < T/4$

$= -V;\ T/4 < t < 3T/4$

$= V;\ 3T/4 < t < T$

Obviously, the given function is an even function.

$\therefore \quad b_n = 0$

Now, $$a_0 = \frac{2}{T}\int_0^{T/2} f(t)dt = \frac{2}{T}\int_0^{T/4} Vdt = -\frac{2}{T}\int_{T/4}^{T/2} Vdt = 0$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t dt$$

$$= \frac{4}{T}\left[\int_0^{T/4} V\cos n\omega t dt - \int_{T/4}^{T/2} V\cos n\omega t dt\right]$$

$$= \frac{4V}{n\omega T}\left[\sin\left(\frac{n\omega T}{4}\right) - \sin\left(\frac{n\omega T}{2}\right) + \sin\left(\frac{n\omega T}{4}\right)\right]$$

$$= \frac{4V}{n2\pi}\left[2\sin\left(\frac{n\pi}{2}\right)\right] \qquad [\because \omega T = 2\pi]$$

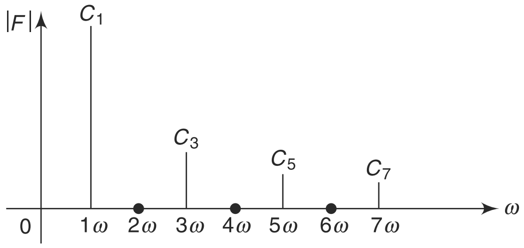
$$= \frac{4V}{n\pi}\sin\frac{n\pi}{2} = \frac{4V}{n\pi};\ \text{for } n = 1, 5, 9, \ldots\ldots\ldots$$

$$= -\frac{4V}{n\pi};\ \text{for } n = 3, 7, 11, \ldots\ldots\ldots$$

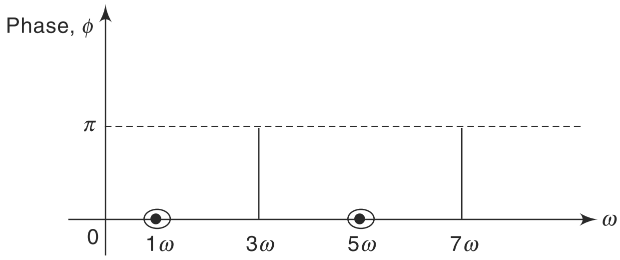
$= 0$; for $n = 2, 4, 6, \ldots\ldots\ldots$

So, $$f(t) = \frac{4V}{\pi}\left(\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \frac{1}{7}\cos 7\omega t + \frac{1}{9}\cos 9\omega t \ldots\ldots\ldots\right)$$ *Ans.*

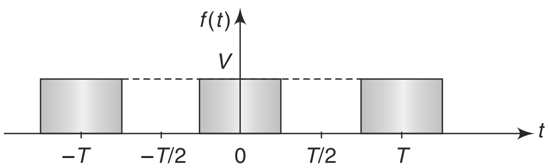
Magnitude Spectra



Phase Spectra



7.2 Find the Fourier series of the function whose periodic waveform is shown in the below figure and plot its frequency spectra.



Solution The function is even

$\therefore \quad b_n = 0$

$$\therefore \quad a_0 = \frac{2}{T}\int_0^{T/2} f(t)dt = \frac{2}{T}\int_0^{T/4} Vdt = \frac{2V}{T}\times\frac{T}{4} = \frac{V}{2}$$

$$\therefore \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t dt$$

$$= \frac{4V}{T}\int_0^{T/4} f(t)\cos n\omega t dt$$

$$= \frac{4V}{T}\left[\left(\frac{\sin n\omega t}{n\omega}\right)_0^{T/4}\right] \qquad [\because \omega T = 2\pi]$$

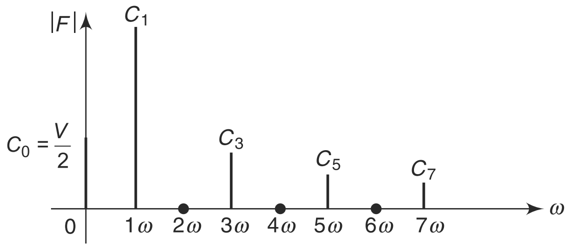
$$= \frac{4V}{n\omega T}\left[\sin\left(\frac{n\omega T}{4}\right)\right]$$

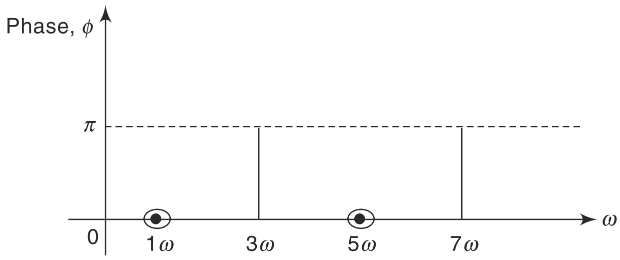
$$= \frac{4V}{2n\pi}.\sin\frac{n\pi}{2} = \frac{2V}{n\pi};\ n = 1, 5, 9 \ldots$$

$$= -\frac{2V}{n\pi};\ n = 3, 7, 11, \ldots$$

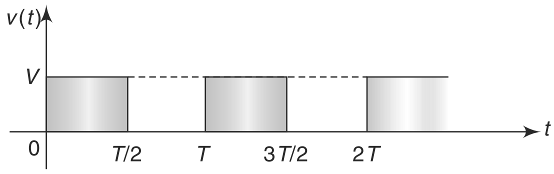
$$\therefore \qquad f(t) = \frac{V}{2} + \frac{2V}{\pi}\left(\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \frac{1}{7}\cos 7\omega t + \ldots\right) \qquad \textit{Ans.}$$

Line Spectra





7.3 Find the Fourier series for the train of pulses shown in the below figure and draw the amplitude and the phase spectra.



Solution Here, $v(t) = V$; for $0 < t < T/2$

$= 0$; for $\frac{T}{2} < t < T$

$$\therefore \qquad a_0 = \frac{1}{T}\int_0^T V(t)dt = \frac{1}{T}\int_0^{T/2} Vdt = \frac{V}{2}$$

and $$a_n = \frac{2}{T}\int_0^T V(t)\cos n\omega t dt = \frac{2}{T}\int_0^{T/2} V\cos n\omega t dt$$

$$= \frac{2V}{n\omega T}\left[\sin\left(\frac{n\omega T}{2}\right)\right] = 0 \quad [\because\ \omega T = 2\pi]$$

and $$b_n = \frac{2}{T}\int_0^T V(t)\sin n\omega t dt = \frac{2V}{T}\int_0^{T/2} \sin n\omega t dt$$

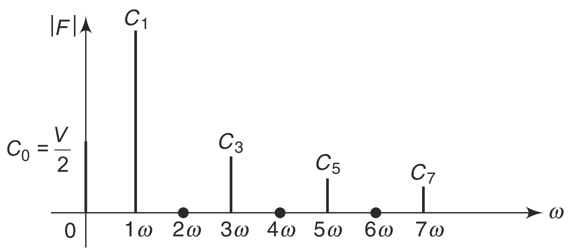
$$= \frac{2V}{n\omega T}\left[1 - \cos\left(\frac{n\omega T}{2}\right)\right] = \frac{V}{n\pi}(1 - \cos n\pi),\ [\because\ \omega T = 2\pi]$$

$$= \frac{2V}{n\pi},\ \text{for } n \text{ odd.}$$

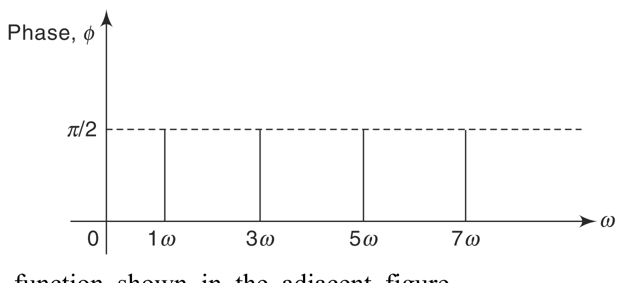
$= 0$, for n even.

$\therefore$ $$V(t) = V\left[\frac{1}{2} + \frac{2}{\pi}\sin n\omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \ldots\right]$$

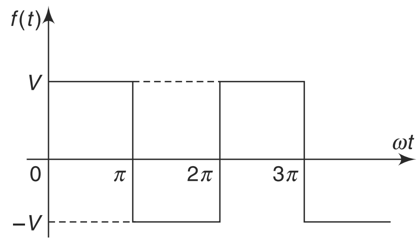
Amplitude Spectra



Phase Spectra



7.4 For the periodic function shown in the adjacent figure determine the exponential form of Fourier series and show the line spectra. Also find its trigonometric form.



Solution The function is defined as,

$f(t) = V, \quad 0 < t < \pi, \quad [T = 2\pi]$

$= -V, \quad \pi < t < 2\pi$

Since the function is odd, the co-efficients $\bar{C}_n$ will be purely imaginary.

$\therefore \quad \bar{C}_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-jn\omega t}\, dt$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} V e^{-jn\omega t} dt - V\int_{\pi}^{2\pi} e^{-jn\omega t} dt\right]; \quad \text{for} \quad n \neq 0$$

$$= \frac{1}{2\pi}\left[\frac{V}{-jn\omega} e^{-jn\omega t}\right]_0^{\pi} - \frac{V}{2\pi}\left[\frac{1}{-jn\omega} e^{-jn\omega t}\right]_{\pi}^{2\pi}$$

$$= \frac{V}{j2\pi n\omega}(1 - e^{-jn\omega\pi}) + \frac{V}{j2\pi n\omega}(e^{-jn\omega 2\pi} - e^{jn\omega\pi}) \qquad (\because\ T = 2\pi,\ \therefore\ \omega = 1)$$

$$= \frac{V}{j2\pi n}(1 - e^{-jn\pi}) + \frac{V}{j2\pi n}(e^{-jn2\pi} - e^{jn\pi})$$

Now, $\quad e^{-jn\pi} = \cos n\pi - j\sin n\pi = (-1)^n$

and $\quad e^{-j2n\pi} = \cos 2n\pi - j\sin 2n\pi = 1$

$\therefore \quad \bar{C}_n = \frac{2V}{j2n\pi}[1-(-1)^n]; \ n \neq 0$

$$= \frac{2V}{jn\pi}; \quad \text{for } n \text{ odd;}$$

$= 0$; for n even.

$$C_{-n} = -\frac{2V}{jn\pi}$$

For $n = 0$, $\bar{C}_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = \frac{1}{2\pi}\left[\int_0^{\pi} V dt - \int_{\pi}^{2\pi} V dt\right] = 0$

$\therefore$ Exponential form of Fourier series is,

$$f(t) = \frac{2V}{j\pi}\sum_{n-1}^{\infty}\frac{1}{n} e^{jn\omega t} \ ; \ n \text{ odd only}$$

$$= \frac{2V}{j\pi}\left[e^{j\omega t} + \frac{1}{3}e^{3j\omega t} + \frac{1}{5}e^{j5\omega t} + \frac{1}{7}e^{j7\omega t} + \ldots\right] \qquad \textit{Ans.}$$

To find Trigonometric form,

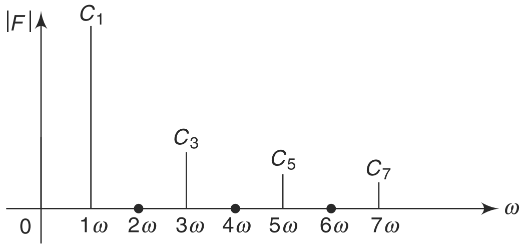
$a_0 = 0$,

$$a_n = (C_n + C_{-n}) = \frac{2V}{jn\pi} - \frac{2V}{jn\pi} = 0$$

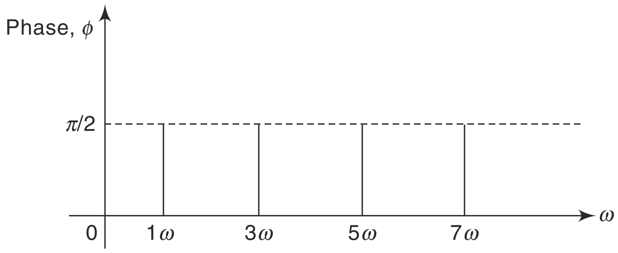
$$b_n = j(C_n - C_{-n}) = j\left[\frac{2V}{jn\pi} + \frac{2V}{jn\pi}\right] = \frac{4V}{n\pi} \qquad \text{for } n \text{ odd.}$$

$\therefore \quad f(t) = \frac{4V}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right] \qquad \textit{Ans.}$

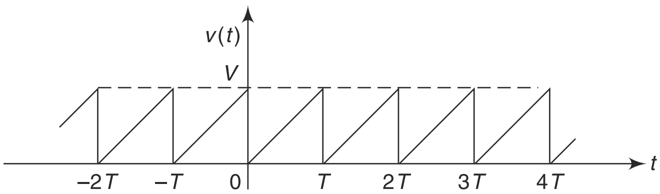
Amplitude Spectra



Phase Spectra



7.5 The waveform shown in the following figure is used as 'sweep' in radar and television circuits. Find the Fourier series and plot the line spectra.



Solution The function, $V(t) = \frac{V}{T}t\ ;\ 0 < t < T$

$$\therefore \qquad \bar{C}_n = \frac{1}{T}\int_0^T \frac{V}{T} t e^{-jn\omega t} dt\ ;\ n \neq 0$$

$$= \frac{V}{T^2}\left[\frac{te^{jn\omega t}}{-jn\omega} + \int \frac{e^{-jn\omega t}}{jn\omega}\right]_0^T$$

$$= \frac{V}{T^2}\left[\frac{Te^{-jn\omega T}}{-jn\omega} - \frac{e^{-jn\omega T}}{(jn\omega)^2}\Bigg|_0^T\right]$$

$$= \frac{V}{T^2}\left[\frac{T^2 e^{-j2\pi n}}{-j2n\pi} + \frac{(e^{-j2n\pi}-1)}{n^2\omega^2}\right] \qquad [\because\ \omega T = 2\pi]$$

$$= \frac{jV}{2n\pi} e^{-j2n\pi} + \frac{V}{4n^2\pi^2} e^{-j2n\pi} - \frac{V}{4n^2\pi^2}$$

Since, $\quad e^{-j2n\pi} = (\cos 2n\pi - j\sin 2n\pi) = 1$

$\therefore \quad \bar{C}_n = \frac{jV}{2n\pi}; \quad \text{for} \quad n \neq 0$

For $n = 0$, $\bar{C}_0 = \frac{V}{T^2}\int_0^T t\,dt = \frac{V}{2}$

$\therefore$ Exponential form,

$$v(t) = \ldots - \frac{jV}{6\pi} e^{-j3\omega t} - \frac{jV}{4\pi} e^{-j2\omega t} - \frac{jV}{2\pi} e^{-j\omega t} + \frac{V}{2} + \frac{jV}{2\pi} e^{j\omega t} + \frac{jV}{4\pi} e^{j2\omega t} + \frac{jV}{6\pi} e^{j3\omega t} + \ldots$$

To convert into Trigonometric form

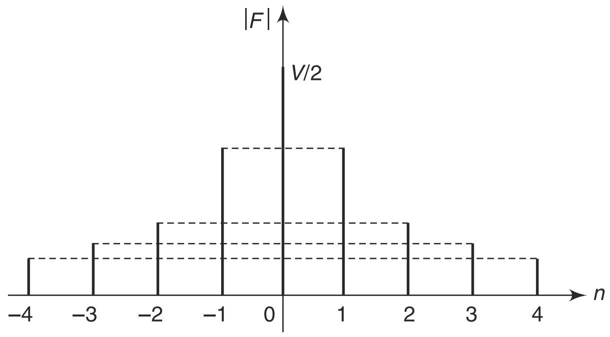
Here, $\quad \bar{C}_n = \frac{jV}{2n\pi}, \; \bar{C}_{-n} = -\frac{jV}{2n\pi}$

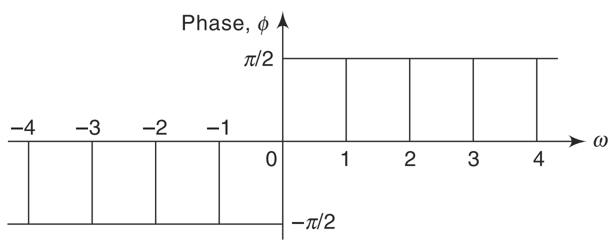
$\therefore \quad a_0 = C_0 = \frac{V}{2}, \quad a_n = (\bar{C}_n + \bar{C}_{-n}) = 0$

and $\quad b_n = j\left(\bar{C}_n - \bar{C}_{-n}\right) = -\frac{V}{n\pi}$

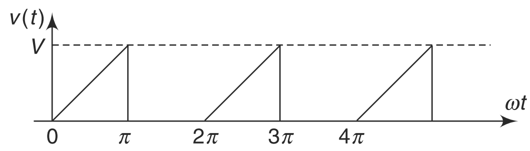
$\therefore \quad V(t) = \frac{V}{2} - \frac{V}{\pi}\left[\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \ldots\right] \qquad$ *Ans.*

Line Spectra





7.6 Find the trigonometric Fourier series for the waveform shown in Figure and sketch the spectra.



Solution Here, $f(t) = \dfrac{V}{\pi}\omega t$; for $0 < \omega t < \pi$ and

$= 0$; for $\pi < \omega t < 2\pi$

$$\Rightarrow \quad a_0 = \frac{V}{4}$$

$$\Rightarrow \quad a_n = \frac{1}{\pi}\int_0^{\pi} \frac{V}{\pi}\omega t \cos n\omega t\, d(\omega t)$$

$$= -\frac{2V}{n^2\pi^2}\text{; for } n \text{ odd.}$$

$$= 0\text{ ; for } n \text{ even.}$$

$$\Rightarrow \quad b_n = \frac{1}{\pi}\int_0^{\pi} \frac{V}{\pi}\omega t \sin n\omega t\, d(\omega t)$$

$$= -\frac{V}{n\pi}\text{; for } n \text{ even.}$$

$$= \frac{V}{n\pi}\text{; for } n \text{ odd.}$$

$$\therefore \quad f(t) = \frac{V}{4} - \frac{2V}{\pi^2}\left[\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \ldots\right] + \frac{V}{\pi}\left[\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \ldots\right]$$

Ans.

Line Spectra

The even harmonic amplitudes are given directly by b_n coefficients, since there are no even cosine terms.

But, the odd harmonic amplitudes are given by computation

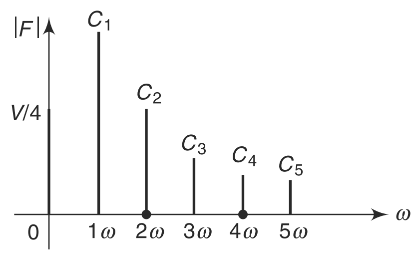
$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\therefore \quad C_1 = \sqrt{\left(\frac{2V}{\pi^2}\right)^2 + \left(\frac{V}{\pi}\right)^2} = (0.377)\text{V}.$$

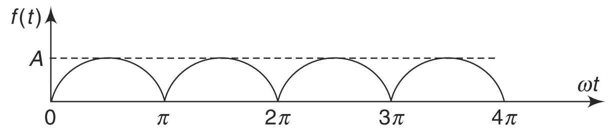
$$C_3 = (0.109)\,V,\ C_5 = (0.064)V$$

and $C_2 = -\dfrac{V}{2\pi}$, $C_4 = -\dfrac{V}{4\pi} = -0.0795$ V

$= -0.159$ V



7.7 Find the Fourier series expansion of the rectified sine waveforms shown in the followig figure.



Solution Here, $f(t) = A \sin \omega t$; for $0 < \omega t < \pi$

$= -A \sin \omega t$; for $\pi < \omega t < 2\pi$

Since, $f(t) = f(-t) \Rightarrow$ The function is even.

$\therefore \quad b_n = 0$

$$\therefore \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\, d(\omega t)$$

$$= \frac{4}{2\pi}\int_0^{\pi} A\sin \omega t \cos n\omega t\, d(\omega t)$$

$$= \frac{A}{\pi}\int_0^{\pi} 2\sin \omega t \cos n\omega t\, d(\omega t)$$

$$= \frac{A}{\pi}\int_0^{\pi}\left[\sin(n+1)\omega t - \sin(n-1)\omega t\right] d(\omega t)$$

$$= \frac{A}{\pi}\left[\frac{-\cos(n+1)\omega t}{n+1} + \frac{\cos(n-1)\omega t}{n-1}\right]_0^{\pi}; \text{ for } n \neq 1$$

For odd n; $a_n = \frac{A}{\pi}\left[\left(-\frac{1}{n+1} + \frac{1}{n+1}\right) + \left(\frac{1}{n-1} - \frac{1}{n-1}\right)\right]; \; n \neq 1$

$= 0$

For even n; $a_n = \frac{A}{\pi}\left[\left(\frac{2}{n+1}\right) + \left(\frac{-2}{n-1}\right)\right]$

$$= \frac{A}{\pi}\left[\frac{2n - 2 - 2n - 2}{(n+1)(n+1)}\right] = -\frac{4A}{\pi(n^2-1)}$$

For $n = 1$, $a_1 = \frac{4}{T}\int_0^{T/2} f(t)\cos \omega t\, d(\omega t)$

$$= \frac{4}{2\pi}\int_0^{\pi} A\sin \omega t \cos \omega t\, d(\omega t)$$

$$= \frac{A}{\pi}\int_0^{\pi} \sin 2\omega t\, d(\omega t)$$

$$= -\frac{A}{2\pi}[\cos 2\omega t]_0^{\pi}$$

$$= -\frac{A}{2\pi}[\cos 2\pi - 1] = 0$$

Also, $$a_0 = \frac{2}{T}\int_0^{T/2} f(t)dt = \frac{2}{2\pi}\int_0^{\pi} A\sin \omega t d(\omega t)$$

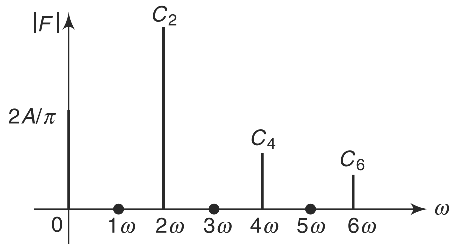
$$= -\frac{A}{\pi}[\cos \omega t]_0^{\pi} = \frac{2A}{\pi}$$

So, the Fourier series is,

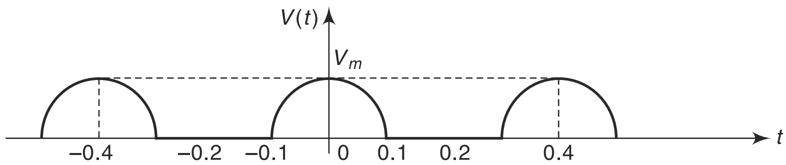
$$f(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=2,4,6}^{\alpha} \frac{\cos n\omega t}{(n^2-1)} = \frac{2A}{\pi} - \frac{4A}{\pi}\left(\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \frac{1}{35}\cos 6\omega t + ...\right)$$

Ans.

Spectra



7.8 Determine the Fourier series of voltage response obtained at the output of a half-wave rectifier shown in the figure. Plot the discrete spectrum of the waveform.



Solution Here, time period $T = 0.4$ second;

$$f = \frac{1}{T} = 2.5 \text{ Hz};$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.4} = 5\pi \text{ rad/s}$$

The function, $v(t) = V_m \cos 5\pi t;\ 0 \le t \le 0.1$

$= 0;\quad 0.1 \le t \le 0.3$

$= V_m \cos 5\pi t;\ 0.3 \le t \le 0.4$

If the period extending from $t = -0.1$ to $t = 0.3$ is taken, it will result in fewer equations and hence, fewer integrals.

$\therefore$ $v(t) = V_m \cos 5\pi t;\ -0.1 \le t \le 0.1$

$= 0;\ 0.1 \le t \le 0.3$

$$\therefore \quad a_0 = \frac{1}{0.4}\int_{-0.1}^{0.3} v(t)dt = \frac{1}{0.4}\left[\int_{-0.1}^{0.1} V_m \cos 5\pi dt + \int_{0.1}^{0.3}(0)dt\right] = \frac{V_m}{\pi}$$

$$\therefore \quad a_n = \frac{2}{0.4}\int_{-0.1}^{0.3} V_m \cos 5\pi nt dt\ ;\ n \neq 1$$

$$= 5V_m \int_{-0.1}^{0.1} \cos 5\pi t \cos 5\pi nt dt$$

$$= 5V_m \int_{-0.1}^{0.1} \frac{1}{2}\left[\cos 5\pi(1+n)t + \cos 5\pi(1-n)t\right]dt$$

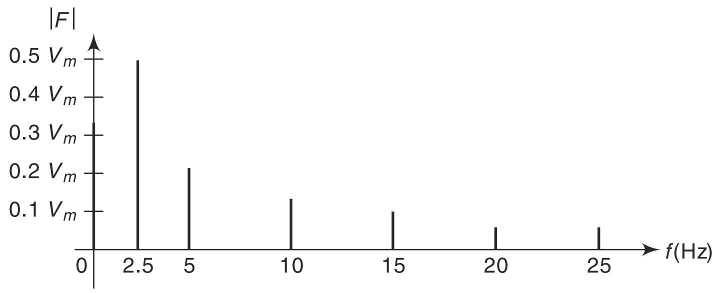
$$= \frac{2V_m}{\pi}\frac{\cos(\pi n/2)}{1-n^2}\ ;\ n \neq 1$$

For, a =1, $a_1 = 5V_m \int_{-0.1}^{0.1} \cos^2 5\pi t dt = \frac{V_m}{2}$

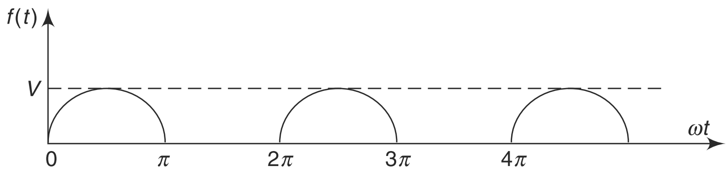
Similarly, $b_n = 0$ for any value of n, and the Fourier series thus contains no sine terms.

$$\therefore \quad v(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\cos 5\pi t + \frac{2V_m}{3\pi}\cos 10\pi t - \frac{2V_m}{15\pi}\cos 20\pi t + \frac{2V_m}{35\pi}\cos 30\pi t - \ldots$$

Spectra



7.9 Find the trigonometric Fourier series for the half-wave rectified sine-wave shown in the following figure. and sketch the spectrum.



Solution Here, the wave is, $f(t) = V \sin \omega t$; $0 < \omega t < \pi$

$= 0$; $\pi < \omega t < 2\pi$

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{\pi} V \sin \omega t\, d(\omega t) = \frac{V}{\pi}$$

$$\therefore \qquad a_n = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \cos n\omega t d(\omega t)\ ;\ n \neq 1$$

$$= \frac{V}{2\pi}\int_0^{\pi}\left[\sin(1+n)\ \omega t + \sin(1-n)\omega t\right] d\omega t$$

$$= \frac{V}{2\pi}\left[\frac{-\cos(1+n)\omega t}{1+n} - \frac{\cos(1-n)\omega t}{1-n}\right]_0^{\pi}$$

$$= \frac{V}{\pi(1-n^2)}(1+\cos n\pi)\ ;\ n \neq 1$$

$= 0$; for n odd

$$= \frac{2V}{\pi(1-n^2)}\ ;\ \text{for } n \text{ even.}$$

For $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \cos \omega t d(\omega t) = 0$

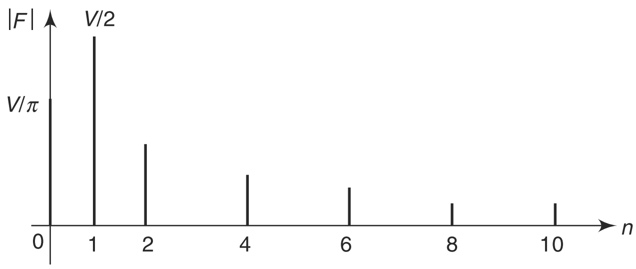
Similarly, $b_n = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \sin n\omega t d(\omega t)\ ;\ n \neq 1$

$= 0$

For $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{\pi} V \sin^2 \omega t d(\omega t) = \frac{V}{2}$

So the series is,

$$f(t) = \frac{V}{\pi} + \frac{V}{2}\sin \omega t - \frac{V}{\pi}\left(\frac{2}{3}\cos 2\omega t + \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t + \ldots\right)$$



7.10 State and prove Parseval's theorem useful in computing the effective value of a given periodic function, $f(t)$.

Or,

A periodic function $f(\theta)$ with period 2π is expressed in Fourier series as follows:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

Prove that,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2\,d\theta = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Solution

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

Since, $\displaystyle\int_{-\pi}^{\pi}\cos n\theta \sin n\theta d\theta = \int_{-\pi}^{\pi}\cos n\theta d\theta = \int_{-\pi}^{\pi}\sin n\theta d\theta = 0$

$$\therefore \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}a_n^2\cos^2 n\theta + \sum_{n=1}^{\infty}b_n^2\sin^2\theta\right]d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{a_0}{2}\right)^2 d\theta + \frac{1}{2\pi}\sum_{n=1}^{\alpha}\frac{a_n^2}{2}\int_{-\pi}^{\pi}2\cos^2 n\theta d\theta + \frac{1}{2\pi}\sum_{n=1}^{\alpha}\frac{b_n^2}{2}\int_{-\pi}^{\pi}2\sin^2 n\theta d\theta$$

$$= \frac{1}{\pi}\left(\frac{a_0}{2}\right)^2 .2\pi + \frac{a_n^2}{2\pi}.\frac{1}{2}\sum_{n=1}^{\alpha}\int_{-\pi}^{\pi}(1+\cos 2n\theta d\theta) + \frac{b_n^2}{2\pi}.\frac{1}{2}.\sum_{n=1}^{\alpha}\int_{-\pi}^{\pi}(1-\cos 2n\theta)d\theta$$

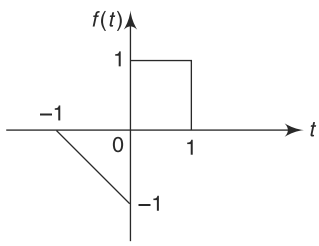
$$= \left(\frac{a_0}{2}\right)^2 + \frac{1}{4\pi}\sum_{n=1}^{\alpha}\left[a_n^2\left\{\theta + \frac{\sin 2n\theta}{2n}\right\}_{-\pi}^{\pi} + b_n^2\left\{\theta - \frac{\sin 2n\theta}{2n}\right\}_{-\pi}^{\pi}\right]$$

$$= \left(\frac{a_0}{2}\right)^2 + \frac{1}{4\pi}\sum_{n=1}^{\alpha}[a_n^2(2\pi) + b_n^2(2\pi)]$$

$$\therefore \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2 d\theta = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad \text{[Proved]}$$

Note: *For statement and proof of this theorem consult the text earlier.*

7.11 Resolve the waveform of the adjacent figure. into even and odd components and plot the two components.



Solution Let, $f_0(t)$ and $f_e(t)$ be respectively the odd and even parts of $f(t)$

$$\therefore \quad f(t) = f_e(t) + f_0(t) \qquad \text{(i)}$$

$$\therefore \quad f(-t) = f_e(-t) + f_0(-t) = f_e(t) - f_0(t) \qquad \text{(ii)}$$

Solving (i) and (ii); $f_e(t) = \dfrac{1}{2}[f(t) + f(-t)]$

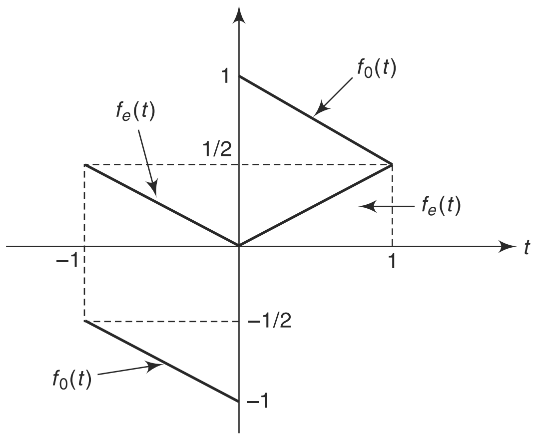
and $$f_0(t) = \frac{1}{2}[f(t) - f(-t)]$$

For the given waveform,

$$f(t) = 1;\ 0 < t < 1 \qquad \therefore\ f_e(t) = \frac{t}{2}$$

and $\quad f(-t) = (t - 1);\ 0 < t < 1 \qquad$ and $\ f_0(t) = (1 - t/2)$

Thus, the components are



7.12 If $v(t) = 10 + 6\cos(t + 45°) + 1.8\cos(2t - 10°)$ volt and $i(t) = 3 + 1.4\cos(t + 20°) + 0.5\cos 2t$ mA, calculate the average power in Watt. Determine also the effective voltage and effective current.

Solution Average Power $= \dfrac{V_{M1}I_{M1}}{2}\cos\phi_1 + \dfrac{V_{M2}I_{M2}}{2}\cos\phi_2 + \dfrac{V_{M3}I_{M3}}{2}\cos\phi_3$

$$= 10 \times 3 + \frac{6 \times 1.4}{2}\cos(45° - 20°) + \frac{1.8 \times 0.5}{2}\cos 10°$$

$$= 34.25 \text{ W}$$

$$\text{Effective Voltage} = \sqrt{10^2 + \frac{6^2 + (1.8)^2}{2}} = 12.58 \text{ V}$$

$$\text{Effective current} = \sqrt{3^2 + \frac{1}{2}(1.4^2 + 0.5^2)} = 3.178 \text{ A}$$

7.13 Determine the effective voltage, effective current, and average power supplied to a passive network if the supplied voltage is,

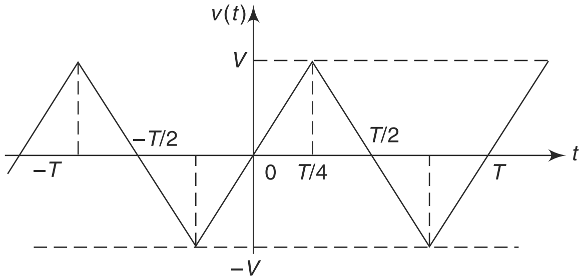
$$v(t) = 100 + 50\cos(10t + 30°) + 25\cos(30t + 60°) \text{ V}$$

and the resulting current is,

$$i(t) = 2\cos(10t + 75°) + 3\cos(30t + 78°) \text{ A.}$$

Solution Same as Prob. 7.12.

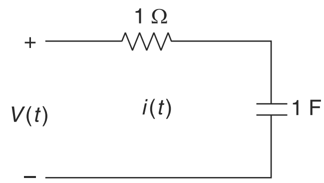
7.14 (a) Find the trigonometric Fourier series of the triangular waveform shown in the following figure.



(b) If this voltage is approximated by

$$\frac{8V}{\pi^2}\sin \omega t, \text{ find the mean-square error.}$$

(c) If this voltage waveform is applied to the network in the below figure, then find the current i(t) and draw the magnitude and phase spectra of $i(t)$. Take $\omega_0 = 1$ radian/second for the waveform.



Solution

(a) The wave is an odd function and is having half wave symmetry.

$\therefore \qquad a_n = 0 \quad \text{and} \quad a_0 = 0$

Now, $$V(t) = \frac{4V}{T}t;\ 0 < t < T/4$$

$$= -\frac{4V}{T}t + 2V;\ T/4 < t < \frac{3T}{4}$$

$$\therefore \qquad b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega t\, dt; \ n \text{ is odd only.}$$

$$= \frac{8}{T}\int_0^{T/4} \frac{4V}{T} t \sin n\omega t\, dt$$

$$= \frac{32V}{T^2}\left[\frac{-t\cos n\omega t}{n\omega} + \int \frac{\cos n\omega t}{n\omega}\, dt\right]_0^{T/4}$$

$$= \frac{16V}{n\pi T}\left[-\frac{T}{4}\cos\frac{n\pi}{2} + \frac{\sin n\omega t}{n\omega}\bigg|_0^{T/4}\right]$$

$$= \frac{16}{n\pi T}\left[-\frac{T}{4}\times 0 + \frac{T}{2n\pi}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8V}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \{\because\ \omega T = 2\pi\}$$

$\therefore \qquad b_n = \frac{8V}{n^2\pi^2}$, $n = 1, 5, 9, \ldots$

$$= -\frac{8V}{n^2\pi^2}, \; n = 3, 7, 11, \ldots$$

Hence,

$$V(t) = \frac{8V}{\pi^2}\left(\sin\omega t - \frac{1}{3^2}\sin 3\omega t + \frac{1}{5^2}\sin 5\omega t - \frac{1}{7^2}\sin 7\omega t + \ldots\right) \qquad \textit{Ans.}$$

(b) The error is, $\varepsilon(t) = v(t) - \frac{8V}{\pi^2}\sin\omega t$

The main square error is,

$$E_N = \frac{1}{T}\int_0^T \varepsilon^2(t)dt$$

Since, the wave is having half–wave symmetry,

$$\therefore \qquad E_N = \frac{4}{T}\int_0^{T/4} \varepsilon^2(t)dt$$

Now, $\qquad v(t) = \frac{4V}{T}t$; for $0 < t < T/4$

$$\therefore \qquad E_N = \frac{4}{T}\int_0^{T/4}\left[\frac{4V}{T}t - \frac{8V}{\pi^2}\sin\omega t\right]^2 dt = 0.0047 \text{ V}^2 \qquad \textit{Ans.}$$

(c) Here,

$$i(n\theta) = \frac{V_i(n\theta)}{Z(n\theta)} = \frac{V_i(n\theta)}{1 - j/n}$$

$$= \frac{nV_i(n\theta)}{\sqrt{1+n^2}} \Big| \tan^{-1}(1/n)$$

$$\therefore \qquad i(n\theta) = \frac{n}{\sqrt{1+n^2}} \times \frac{8V}{n^2\pi^2}\sin[nt + \tan^{-1}(1/n)] \; ; \quad \text{for } n = 1, 5, 9, \ldots$$

$$= \frac{8V}{\pi^2 n\sqrt{1+n^2}}\sin[nt + \tan^{-1}(1/n)]$$

and

$$= \frac{8V}{\pi^2 n\sqrt{1+n^2}}\sin[nt + \pi + \tan^{-1}(1/n)]; \text{ for } n = 3, 7, 11, \ldots$$

$$\therefore \qquad i_1 = \frac{8V}{\pi^2\sqrt{2}}\sin(t + 45°) = 0.707\frac{8V}{\pi^2}\sin(t + 45°)$$

$$\therefore \qquad i_3 = \frac{8V}{\pi^2 3\sqrt{10}} \sin(3t + 180° + 18\cdot 44°) = 0.949\frac{8V}{\pi^2 3^2}\sin(3t + 198\cdot 44°)$$

$$\therefore \qquad i_5 = \frac{8V}{\pi^2 5\sqrt{26}} \sin(5t + 11\cdot 31°) = 0.98\frac{8V}{\pi^2 5^2}\sin(5t + 11\cdot 31°)$$

$$\therefore \quad i(t) = \frac{8V}{\pi^2}[0.707\sin(t + 45°) + 0.105\sin(3t + 198.44°) + 0.039\sin(5t + 11.31°) + ...\} \qquad \textit{Ans.}$$

7.15 A series RL circuit with R = 10 ohm and L = 5 H contains a current

$$i(t) = 10 \sin 1000\,t + 5 \sin 3000t + 3 \sin 5000t \text{ A}$$

Find the effective voltage and the average power.

Solution Here, ω = 1000 rad/s and it contains three harmonics:

For fundamental harmonic

$$R_1 = 10\,\Omega, X_{L1} = \omega L = 1000 \times 5 = 5000\,\Omega$$

$$\therefore \qquad Z_1 = R_1 + j\omega L = (10 + j5000) = 5000\angle 89.88°$$

For third harmonic

$$R_3 = 10\,\Omega,\ X_{L3} = 3\omega L = 15000\,\Omega$$

$$\therefore \qquad Z_3 = (10 + j15000) = 15000\cdot 003\angle 89.96°$$

For fifth harmonic

$$R_5 = 10\,\Omega, X_{L5} = 5\omega L = 25000\,\Omega$$

$$\therefore \qquad Z_5 = (10 + j25000) = 25000\cdot 001\angle 89.977°$$

$$\therefore \qquad v(t) = 10\,|Z_1|\sin(1000t - 89.88°) + 5\,|Z_3|\sin(3000t - 89.96°) + 3\,|Z_5|\sin(5000t - 89.977°)$$

$$= 5000\cdot 01\sin(1000t - 89\cdot 88°) + 75000\cdot 015\sin(3000t - 89\cdot 96°)$$

$$+ 75000\cdot 003\sin(5000t - 89\cdot 977°)$$

$$\therefore \qquad \text{Effective Voltage, } V = \frac{1}{\sqrt{2}}[(5000\cdot 01)^2 + (75000\cdot 015)^2 + (75000\cdot 003)^2]^{\frac{1}{2}}$$

$$= 8\cdot 291 \times 10^4 \text{ volt}$$

$$= 82\cdot 91\,\text{kV} \qquad \textit{Ans.}$$

Average power

$$P_{av} = \frac{V_{m1}I_{m1}}{2}\cos\phi_1 + \frac{V_{mL}I_{m2}}{2}\cos\phi_2 + \frac{V_{m3}I_{m3}}{2}\cos\phi_3$$

$$= \frac{5000\cdot 01 \times 10}{2}\cos 89\cdot 88° + \frac{75000\cdot 015 \times 5}{2}\cos 89\cdot 96° + \frac{75000\cdot 003 \times 3}{2}\cos 89\cdot 977°$$

$= 691\cdot6595$ Watt *Ans.*

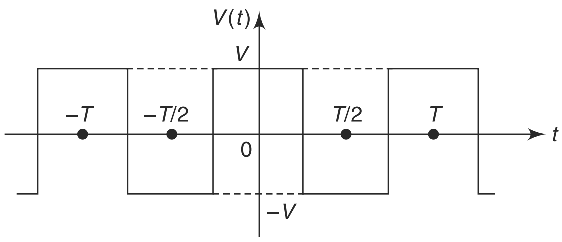
7.16 A periodic current source, $i(t) = 10 + 6\cos(100t + 45°) + 3\cos(200t - 10°) + 2.1\cos(300t + 35°)$ is the input to a parallel RC circuit with $R = 0.5$ ohm and $C = 0.02$ F. Calculate the steady-state response $v(t)$ of the circuit.

Solution {same as Prob. 7.15}

$$Z_1 = 0\cdot35\angle-45°;\ Z_2 = 0\cdot22\angle-63.43°;\ Z_3 = 0\cdot158\angle-71.56° \qquad \textit{Ans.}$$

$$\therefore \quad v(t) = 5 + 2\cdot121\cos 100t + 0\cdot671\cos(200t - 73\cdot43°) + 0\cdot332\cos(300t - 36\cdot56°)$$

7.17 The square wave source, v(t) shown in figure excites a series RL circuit with $R = 2$ ohm and $L = 2$ H. Determine the current response i(t), taking $\omega = 1$ radian/second and $V = \frac{\pi}{4}$ volt.



Solution [same as Prob 7.14]

Here, from Prob 7.1

$$v(t) = \frac{4V}{\pi}\left(\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \frac{1}{7}\cos 7\omega t + \ldots\right)$$

Here, $\quad V = \frac{\pi}{4}$ volt

$$\therefore \quad v(t) = \cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \ldots$$

$$Y(jn) = \frac{1}{2 + j2n}$$

$$\Rightarrow \quad Y_1 = 0\cdot353\angle-45°;\ V_1 = 1\angle0°$$

$$Y_3 = 0\cdot158\angle-71.565°;\ V_3 = \frac{1}{3}\angle-180°$$

$$Y_5 = 0\cdot098\angle-78.69°;\ V_5 = \frac{1}{5}\angle0°$$

$$\therefore \quad I_1 = V_1Y_1 = 0\cdot353\angle-45°$$

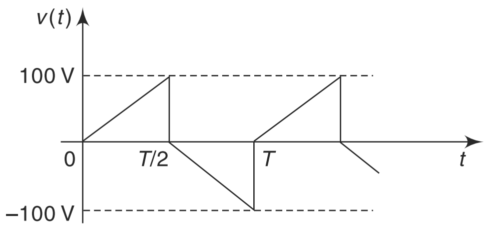
$$\therefore \quad I_3 = V_3Y_3 = 0\cdot0527\angle108.435°$$

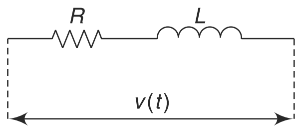
and $\quad I_5 = V_5Y_5 = 0\cdot0196\angle-78.69°$

$$i(t) = 0\cdot353\cos(t - 45°) + 0\cdot0527\cos(3t - 251\cdot6°) + 0\cdot0196\cos(5t - 78\cdot69°) + \ldots \qquad \textit{Ans.}$$

7.18 Determine the Fourier series of repetitive waveform of figure upto 5^{th} harmonic, when time of repetition, $T = 20$ ms.

Calculate the fundamental frequency current in the circuit of figure, where $R = 10$ ohm and $L = 0.0318$H with voltage transform of the waveform.





Solution The wave is having half wave symmetry.

$a_n = b_n = 0$; for n even ; and

For n odd,

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$$

and $a_0 = 0$

Now, $v(t) = \dfrac{200}{T}t;\ 0 \le t \le \dfrac{T}{2}$

$$\therefore \quad a_n = \frac{4}{T}\int_0^{T/t} \frac{200}{T} t\cos n\omega t\, dt$$

$$= \frac{800}{T^2}\left[\frac{t\sin nwt}{n\omega} - \int \frac{\sin n\omega t}{n\omega}dt\right]$$

$$= \frac{800}{T^2}\left[\frac{T}{2}\times\frac{\sin n\pi}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\Big|_0^{T/2}\right]$$

$$= \frac{800}{n^2\omega^2T^2}\left[\cos n\pi - 1\right]$$

$$= \frac{800}{n^2 4\pi^2}(-2)$$

$$= -\frac{400}{n^2\pi^2}$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} \frac{200}{T} t\sin n\omega t dt = \frac{200}{n\pi}$$

$$\therefore \quad v(t) = -\frac{400}{\pi^2}\left(\cos\omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\omega t + ...\right) + \frac{200}{\pi}\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + ...\right)$$

The fundamental frequency voltage is

$$V_f = \left(\frac{200}{\pi}\sin\omega t - \frac{400}{\pi^2}\cos\omega t\right) = \frac{400}{\pi^2}\sqrt{\left(\frac{\pi}{2}\right)^2 + 1} \qquad \textit{Ans.}$$

Impedance, $Z = (R + j\omega L) = 10 + j\omega(0\cdot 0318)$

Current due to fundamental frequency,

$$I_f = \frac{V_f}{Z} = \frac{400}{\pi^2(10 + j0\cdot 0318\omega)}\left(\frac{\pi}{2}\sin\omega t - \cos\omega t\right)$$

or $$I_f = \frac{400}{\pi^2}\sqrt{\left(\frac{\pi}{2}\right)^2 + 1} \times \frac{1}{\sqrt{(10)^2 + (0\cdot 0318\omega)^2}} \angle \tan^{-1}\frac{0.0318\omega}{10}$$

Here, $$\omega = \frac{2\pi}{T} = \frac{2\pi}{20\times 10^{-3}} = 100\pi \text{ rad/s}$$

Patting this value,

$\therefore$ $$I_f = 5\cdot 33\angle -44.9°$$

$\therefore$ $$I_{f\,(\text{rms})} = \frac{5\cdot 33}{\sqrt{2}}\text{ A} = 3\cdot 76\text{ A}$$

7.19 An RLC series circuit with R = 25 ohm, L = 1 H, and C = 10 microfarad is energized with a voltage source,

$$V(t) = 15 \sin 100t + 10 \sin 200t + 5 \sin 300t \text{ (V)}$$

Find the expression for the current i(t). Determine the effective value of the current, and the average power consumed by the circuit.

Solution [Same as Prob. 7.16]

$$Z_1 = R + j\left(\omega L - \frac{1}{\omega C}\right) = 900\cdot 3\angle +88\cdot 4°$$

$$Z_2 = R + j\left(2\omega L - \frac{1}{2\omega C}\right) = 301.04\angle 85.2°$$

$$Z_3 = R + j\left(3\omega L - \frac{1}{3\omega C}\right) = 41\cdot 62\angle 53\cdot 1°$$

$\therefore$ $$i(t) = \frac{15}{Z_1}\sin 100t + \frac{10}{Z_2}\sin 200t + \frac{5}{Z_3}\sin 300t$$

$$= 0\cdot 0167\sin(100t + 88\cdot 4°) + 0\cdot 0332\sin(200t + 85\cdot 2°) + 0\cdot 12\sin(300t + 53\cdot 1°) + \ldots$$

Ans.

$\therefore$ $$I_{\text{rms}} = \frac{1}{\sqrt{2}}[I_1^2 + I_2^2 + I_3^2]^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}[(0\cdot 0167)^2 + (0\cdot 0332)^2 + (0\cdot 12)^2]^{\frac{1}{2}}$$

$$= 0\cdot 088\text{ A} = 88\text{ mA} \qquad \textit{Ans.}$$

$$\therefore \qquad P_{av} = \frac{15 \times 0\cdot 0167}{2}\cos 88\cdot 4° + \frac{10 \times 0\cdot 0332}{2}\cos 85\cdot 2° + \frac{5 \times 0\cdot 12}{2}\cos 53\cdot 1° = 0.197 \text{ W}$$

Ans.

7.20 Determine the expression for current in an impedance of $R = 10$ ohm, $L = 0.0318$ H with applied emf,

$$e(t) = 200\sin 314t + 40\sin(942t + 30°) + 10 \text{ V}$$

Also, calculate the rms value of voltage and current as well as the power factor of the circuit.

Solution [Same as Prob. 7.19]

$$i_1 = \frac{200\sin 314t}{10 + j314 \times 0\cdot 0318} = 14\cdot 14\sin 314t\angle -44.95°$$

$$i_2 = \frac{40\sin(942t + 30°)}{10 + j942 \times 0\cdot 0318} = 1\cdot 28\sin(942t + 30°)\angle -71\cdot 54°$$

$$i_0 = \frac{10}{10} = 1$$

$$i(t) = 14\cdot 14\sin(314t - 44\cdot 95°) + 1\cdot 28\sin(942t - 41.54°)$$

$$\therefore \qquad V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_1^2 + V_2^2}{2}}$$

$$= \sqrt{10^2 + \frac{200^2 + 40^2}{2}} = 144\cdot 568 \text{ V} \qquad \textit{Ans.}$$

$$\therefore \qquad I_{\text{rms}} = \sqrt{I_0^2 + \frac{I_1^2 + I_2^2}{2}}$$

$$= \sqrt{1^2 + \frac{14\cdot 14^2 + 1\cdot 28^2}{2}} = 10.089 \text{ A} \qquad \textit{Ans.}$$

$$\therefore \qquad \text{Power factor} = \frac{\text{Average Power}}{\text{Apparent Power}}$$

$$= \frac{V_0 I_0 + \frac{V_1 I_1}{2}\cos\phi_1 + \frac{V_2 I_2}{2}\cos\phi_2}{V_{\text{rms}} \times I_{\text{rms}}}$$

$$= \frac{10 \times 1 + \frac{200 \times 14\cdot 14}{2}\cos 44\cdot 95° + \frac{40 \times 1\cdot 28}{2}\cos 71\cdot 54°}{144\cdot 568 \times 10\cdot 089}$$

$= 0.69$ *Ans.*

7.21 In a two-element series network, voltage v(t) is applied, which is given by,

$$v(t) = 50 + 50\sin 5000t + 30\sin 10000t + 20\sin 20000t \text{ (V)}$$

The resulting current is given as,

$$i(t) = 11.2\sin(5000t + 63.4°) + 10.6\sin(10000t + 45°) + 8.97\sin(20000t + 26.6°) \text{ (A)}$$

Determine the network elements and the power dissipated in the circuit.

Solution Power dissipated,

$$P_{av} = 50 \times 0 + \frac{50 \times 11 \cdot 2}{2} \cos 63 \cdot 4^\circ + \frac{30 \times 10 \cdot 6}{2} \cos 45^\circ + \frac{8 \cdot 97 \times 20}{2} \cos 26 \cdot 6^\circ = 318 \text{ W} \qquad \textit{Ans.}$$

In the expression of current *i(t)*, the d.c. term is missing though it is present in the applied voltage, $v(t)$. Hence, in the series network, there must be a capacitor which blocks d.c. components. Again from the expression of *i(t),* we see that the current is leading by an angle less than 90°. Hence, the conclusion is the presence of a resistive element in series with the capacitor (RC).

Now, $$I_{eff} = \sqrt{\frac{11 \cdot 2^2 + 10 \cdot 6^2 + 8 \cdot 97^2}{2}} = 12 \cdot 6 \text{ A}$$

$\therefore$ $$P_{av} = I_{eff}^2 R \Rightarrow R = \frac{318}{(12 \cdot 6)^2} = 2\,\Omega \qquad \textit{Ans.}$$

Again, $at\ \omega = 10{,}000\ rad/s,\ \phi = 45^\circ = \tan^{-1}\left(\dfrac{1}{\omega CR}\right)$

$\Rightarrow$ $$C = \frac{1}{\omega R} = \frac{1}{20{,}000} = 50\ \mu\text{F} \qquad \textit{Ans.}$$

7.22 Calculate the impedance consisting of R and L and the power factor of a circuit whose expression for voltage and current are,

$$v(t) = 250 \sin 314t + 50 \sin(942t + 30^\circ) \text{ (V)}$$

$$i(t) = 17.7 \sin(314t - 45^\circ) + 1.583 \sin(942t - 41.6^\circ) \text{ (A)}$$

Solution The fundamental frequency current,

$$I_1 = \frac{250 \sin 314t}{R + j\omega L} = 17 \cdot 7 \sin(314t - 45^\circ) \qquad \text{(i)}$$

The third harmonic current,

$$I_3 = \frac{50 \sin(942t + 30^\circ)}{R + j3\omega L} = 1 \cdot 583 \sin(942t - 41 \cdot 6^\circ) \qquad \text{(ii)}$$

Equating the magnitudes of (i),

$$\frac{250}{\sqrt{R^2 + \omega^2 L^2}} = 17.7$$

$\Rightarrow$ $$R^2 + \omega^2 L^2 = 199.495 \qquad \text{(iii)}$$

Equating the angles of (i)

$$314t - \tan^{-1} \frac{\omega L}{R} = 314t - 45^\circ$$

$\Rightarrow$ $$\tan^{-1} \frac{\omega L}{R} = 45^\circ \Rightarrow \frac{\omega L}{R} = 1 \Rightarrow \omega L = R$$

Putting in (iii), $\Rightarrow (\omega L)^2 = 99 \cdot 747 \Rightarrow \omega L = 9 \cdot 987 = R$

$\therefore \qquad L = \dfrac{9\cdot 987}{314} = 0\cdot 0318$

$\therefore \qquad R = 9\cdot 987\ \Omega$

$L = 0\cdot 0318\ \text{H} \qquad Ans$

$$\text{Power factor} = \frac{\text{Average Power}}{\text{Apparent Power}} = \frac{\frac{V_1 I_1}{2}\cos\phi_1 + \frac{V_3 I_3}{2}\cos\phi_3}{\sqrt{\frac{V_1^2}{2}+\frac{V_3^2}{2}} \times \sqrt{\frac{I_1^2}{2}+\frac{I_3^2}{2}}}$$

$$= \frac{\frac{250\times 17\cdot 7}{2}\cos 45° + \frac{50\times 1\cdot 583}{2}\cos 71\cdot 6°}{\sqrt{\frac{250^2+50^0}{2}} \times \sqrt{\frac{17\cdot 7^2 + 1\cdot 583^2}{2}}}$$

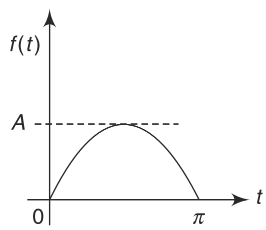
$= 0.69 \qquad Ans.$

Fourier Transform

7.23 Determine the Fourier transform of one cycle of sine wave, $f(t) = A\sin\omega_0 t$.

Solution $\quad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$

$$= A\int_0^T \sin\omega_0 t e^{-j\omega t}\,dt = I \text{ (say)}$$



$$= A\left[e^{-j\omega t}\left(-\frac{\cos\omega_0 t}{\omega_0}\right)\Bigg|_0^T - \int_0^T \frac{\cos\omega_0 t}{\omega_0}(j\omega e^{-j\omega t})\,dt\right]$$

$$= A\left[-\frac{1}{\omega_0}(e^{-j\omega t}\cos\omega_0 T - 1) - \frac{j\omega}{\omega_0}\left\{\int_0^T \cos\omega_0 t e^{-j\omega t}\,dt\right\}\right]$$

$$= A\left[\frac{1}{\omega_0}(e^{-j\omega t}+1) - \frac{j\omega}{\omega_0}\left[\left\{e^{-j\omega t}\left(\frac{\sin\omega_0 t}{\omega_0}\right)\right\}\Bigg|_0^T - \int_0^T\left(\frac{\sin\omega_0 t}{\omega_0}\right)(-j\omega)\,e^{-j\omega t}\,dt\right]\right]$$

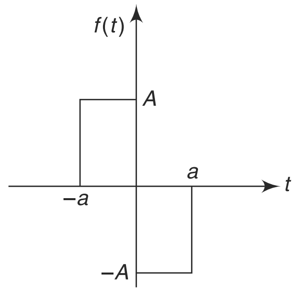
or, $$= \frac{A}{\omega_0}(e^{-j\omega t}+1) + j\frac{A\omega}{\omega_0}\left[0 + \frac{j\omega}{\omega_0}\int_0^T \sin\omega_0 t\, e^{-j\omega t}\,dt\right] \quad [\because \cos\omega_0 T = \cos\pi = -1]$$

$$= \frac{A}{\omega_0}(e^{-j\omega t}+1) + I\frac{\omega^2}{\omega_0^2}$$

or, $$I\left[1 - \frac{\omega^2}{\omega_0^2}\right] = A\omega_0(e^{-j\omega T}+1)$$

$\Rightarrow$ $$I = \frac{A\omega_0}{\omega_0^2 - \omega^2}(e^{-j\omega T}+1) \qquad Ans.$$

7.24 Find the Fourier transform of the single pulse shown in the figure. Draw the continuous magnitude and phase spectra.



Solution Here, $f(t) = A\ ;\ -a \le t \le 0;$

$= -A\ ;\ 0 \le t \le a$

$= 0$; for all other values of t

$$\therefore \qquad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

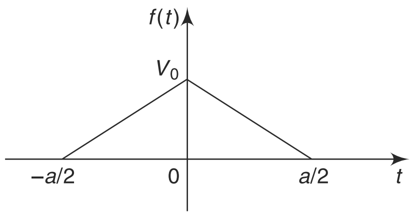
$$= \int_{-a}^{0} Ae^{-j\omega t}\,dt + \int_{0}^{a} -Ae^{-j\omega t}\,dt$$

$$= A\left[\left.\frac{je^{-j\omega t}}{\omega}\right|_{-a}^{0} - \left.\frac{je^{-j\omega t}}{\omega}\right|_{0}^{a}\right]$$

$$= \frac{jA}{\omega}[1 - e^{+j\omega a} - e^{-j\omega a} + 1]$$

$$\Rightarrow \qquad F(j\omega) = j\frac{2A}{\omega}(1 - \cos\omega a) \qquad \textit{Ans.}$$

7.25 Find the Fourier transform of the single triangular pulse shown in the adjacent figure and draw the continuous spectra.



Solution The wave is, $f(t) = V_0\left[1 - \frac{2}{a}|t|\right]$

i.e., $$f(t) = V_0\left[1 - \frac{2}{a}t\right]; \quad \text{for} \quad t > 0$$

and $$f(t) = V_0\left[1 + \frac{2}{a}t\right]; \quad \text{for} \quad t < 0$$

$$\therefore \qquad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} V_0\left[1 - \frac{2}{a}|t|\right]e^{-j\omega t}\,dt$$

$$= V_0 \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j\omega t}\,dt - \frac{2V_0}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} |t|\,e^{-j\omega t}\,dt$$

$$= \frac{V_0}{-j\omega}\left[\left.e^{-j\omega t}\right|_{-\frac{a}{2}}^{\frac{a}{2}} - \frac{2V_0}{a}\left\{\int_{-\frac{a}{2}}^{0} -te^{-j\omega t}\,dt + \int_{0}^{\frac{a}{2}} te^{j\omega t}\,dt\right\}\right]$$

$$= \frac{V_0}{-j\omega}\left(e^{-j\omega\frac{a}{2}} - e^{+j\omega\frac{a}{2}}\right) + \frac{2V_0}{a}\int_{-\frac{a}{2}}^{0} te^{-j\omega t}\,dt - \frac{2V_0}{a}\int_{0}^{\frac{a}{2}} te^{-j\omega t}\,dt$$

$$= \frac{2V_0}{\omega}\left(\frac{e^{j\omega\frac{a}{2}} - e^{-j\omega\frac{a}{2}}}{2j}\right) + \frac{2V_0}{a}\left[\frac{te^{-j\omega t}}{-j\omega}\bigg|_{-\frac{a}{2}}^{\frac{a}{2}} - \int_{-\frac{a}{2}}^{0}\frac{e^{-j\omega t}}{-j\omega}\,dt\right] - \frac{2V_0}{a}\left[\frac{te^{-j\omega t}}{-j\omega}\bigg|_{0}^{\frac{a}{2}} - \int_{0}^{\frac{a}{2}}\frac{e^{-j\omega t}}{-j\omega}\,dt\right]$$

$$= \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) + \frac{2V_0}{2}\left[\left\{0 + \frac{a}{2}\,\frac{e^{+j\omega a l_2}}{-j\omega}\right\} + \frac{e^{-j\omega t}}{\omega^2}\bigg|_{-\frac{a}{2}}^{0}\right] - \frac{2V_0}{a}\left[\left\{\frac{a}{2}\,\frac{e^{-j\omega a l_2}}{-j\omega} - 0\right\} + \frac{e^{-j\omega t}}{\omega^2}\bigg|_{0}^{\frac{a}{2}}\right]$$

$$= \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) - \frac{V_0}{j\omega}e^{+j\omega\frac{a}{2}} + \frac{2V_0}{a\omega^2}\left(1 - e^{+j\omega\frac{a}{2}}\right) + \frac{V_0}{j\omega}e^{-j\omega a/2} - \frac{2V_0}{a\omega^2}\left(e^{-j\omega a/2} - 1\right)$$

$$= \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) + \frac{2V_0}{\omega}\left(\frac{e^{-j\omega a/2} - e^{+j\omega a/2}}{2j}\right) + \frac{2V_0}{a\omega^2}\left(1 - e^{+j\omega a/2} - e^{-j\omega a/2} + 1\right)$$

$$= \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) - \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) + \frac{2V_0}{a\omega^2}\left(2 - e^{-j\omega a/2} - e^{j\omega a/2}\right)$$

$$= \frac{4V_0}{a\omega^2}\left[1 - 2\left(\frac{e^{+j\omega a/2} - e^{-j\omega a/2}}{2}\right)\right]$$

$$= \frac{4V_0}{a\omega^2}\left[1 - \cos\left(\frac{\omega a}{2}\right)\right]$$

$$= \frac{4V_0}{a\omega^2} \times 2\sin^2\left(\frac{\omega a}{4}\right)$$

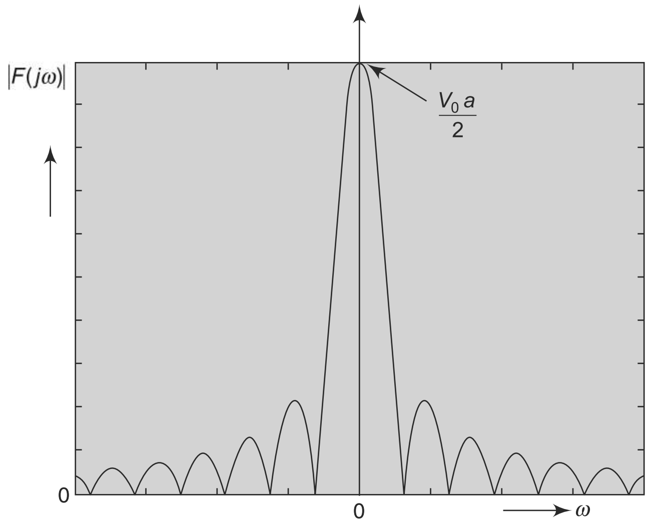
$$\therefore \quad \boxed{F(j\omega) = \frac{8V_0}{a\omega^2}\sin^2\left(\frac{\omega a}{4}\right)}$$

Bringing it into standard form,

$$\boxed{F(j\omega) = \frac{V_0 a}{2}\,\frac{\sin^2\left(\frac{\omega a}{4}\right)}{\left(\frac{\omega a}{4}\right)^2}} \qquad \textit{Ans.}$$

Its continuous amplitude spectrum is shown. The first zero occurs when, $\frac{\omega a}{4} = \pi$ i.e. $\omega = \frac{4\pi}{a}$.

Spectra



7.26 Find the Fourier transform of the existing voltage,

$$v(t) = V_0 e^{-t}, \quad t \geq 0$$
$$= 0, \quad t \leq 0$$

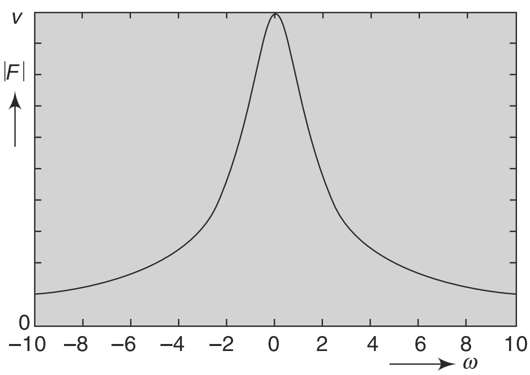
and sketch approximately its amplitude and phase spectrum.

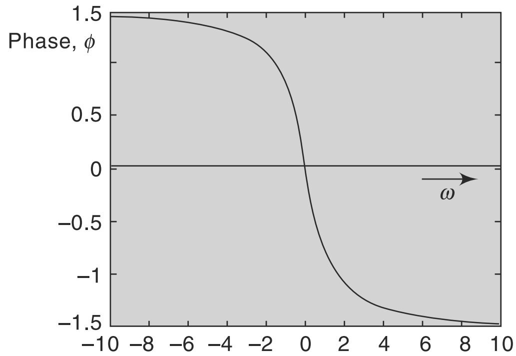
Solution $F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} V_0^{-t} e^{-j\omega t} dt = V_0 \int_{-\infty}^{\infty} e^{-(1+j\omega)t} dt = \frac{V_0}{(1+j\omega)} [e^{-(1+j\omega)t}]_{-\infty}^{\infty}$

$$= \frac{V_0}{1+j\omega}$$

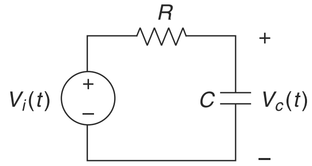
The amplitude and phase are $|F(j\omega)| = \frac{V_0}{\sqrt{1+\omega^2}}$ and $\phi(j\omega) = -\tan^{-1}(\omega)$

Spectra





7.27 In the figure, $V_i(t)$ = 10 sgn(t) volt. Using the Fourier transform method, find $V_c(t)$ and sketch $V_c(t)$ versus time, t. Given: R = 5 ohm, C = 1F.



Solution $v_i(t)$ = 10 sgn(t)

$$V_i(j\omega) = 10 \times \frac{2}{j\omega} = \frac{20}{j\omega} \qquad V_e(j\omega) = \frac{V_i(j\omega)}{z(j\omega)} \times X_c$$

Transfer function of the circuit

$$H(j\omega) = \frac{V_c(j\omega)}{V_i(j\omega)} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

where, $V_c(j\omega)$ is the Fourier transform of $V_c(t)$

$$\therefore \qquad V_c(j\omega) = H(j\omega) \times V_i(j\omega) = \frac{V_i(j\omega)}{Z(j\omega)} \times X_C = \frac{20}{j\omega(1 + j\omega RC)} = \frac{20}{j\omega(1 + j\omega 5)}$$

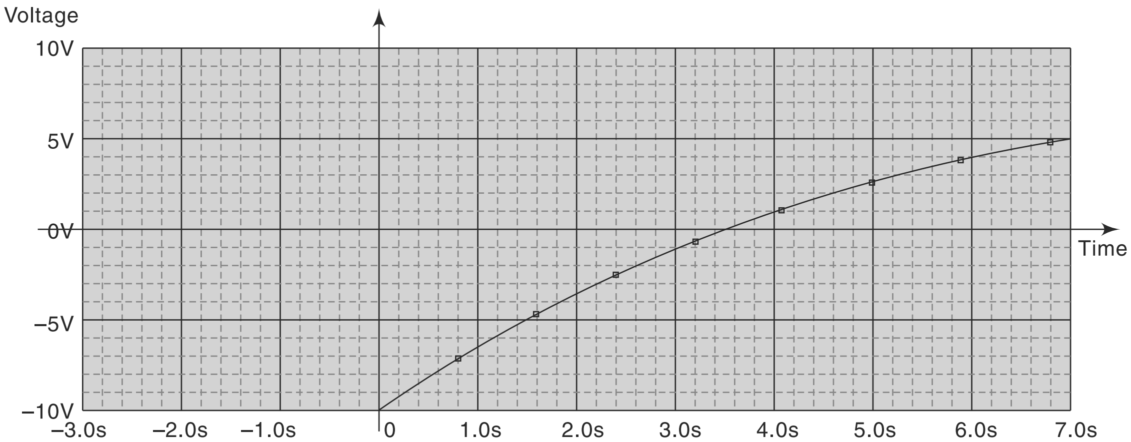
$$= \frac{20}{j\omega} - \frac{100}{1 + (j\omega 5)} = 10\left(\frac{2}{j\omega}\right) - 20\frac{1}{(1/5) + j\omega}$$

Taking inverse Laplace transform,

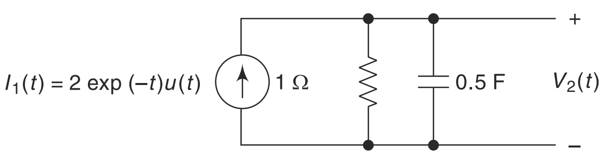
$$v_C(t) = 10\,\text{sgn}(t) - 20e^{-t/5}u(t) \text{ V}$$

To plot this curve, we follow the following steps:

- From $-\infty < t < 0$, $v_i(t)$ = –10V, $v_C(t)$ = –10V;
- At t = 0, $v_i(t)$ jumps from –10V to 10V and thus, $v_C(t)$ approaches its final value of 10V exponentially with time-constant of 5 second.



7.28 Find the response voltage in the network shown in the below figure. Use Fourier transform method.



Solution By KCL, $i_1(t) = \dfrac{v_2(t)}{1} + \dfrac{v_2(t)}{1/j\omega\frac{1}{2}} = v_2(t) + \dfrac{1}{2}j\omega v_2(t)$

Given: $i_1(t) = 2e^{-t}u(t)$

Taking Fourier transform,

$$I_1(j\omega) = V_2(j\omega) + \frac{1}{2}j\omega V_2(j\omega)$$

or $$\frac{2}{1+j\omega} = V_2(j\omega)\left[1 + \frac{1}{2}j\omega\right]$$

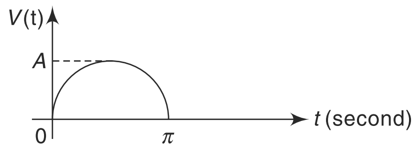
$$V_2(j\omega) = \frac{4}{(1+j\omega)(2+j\omega)} = 4\left[\frac{1}{1+j\omega} - \frac{1}{2+j\omega}\right]$$

Taking inverse Fourier transform,

$$v_2(t) = (4e^{-t} - 4e^{-2t})u(t) \qquad \textit{Ans.}$$

7.29 Find the Fourier transform of the sine pulse shown in the adjacent figure and sketch the amplitude and phase spectra.

This voltage is applied to a series RL circuit with R = 1 ohm and L = 1.0H. Determine the amplitude and phase spectra for the resulting current, $i(t)$.



Solution [from Prob. 7.23] $\Rightarrow V(j\omega) = A\left[\dfrac{1+e^{-j\omega\pi}}{1-\omega^2}\right] = \dfrac{A(1+\cos\omega\pi)}{1-\omega^2} - j\dfrac{A\sin\omega\pi}{1-\omega^2}$

$$\therefore \qquad |V(j\omega)| = A\left|\frac{\sqrt{(1+\cos\omega\pi)^2 + \sin^2\omega\pi}}{1-\omega^2}\right| = A\left|\frac{\sqrt{2(1+\cos\omega\pi)}}{1-\omega^2}\right| = 2A\left|\frac{\cos\left(\frac{\omega\pi}{2}\right)}{1-\omega^2}\right|$$

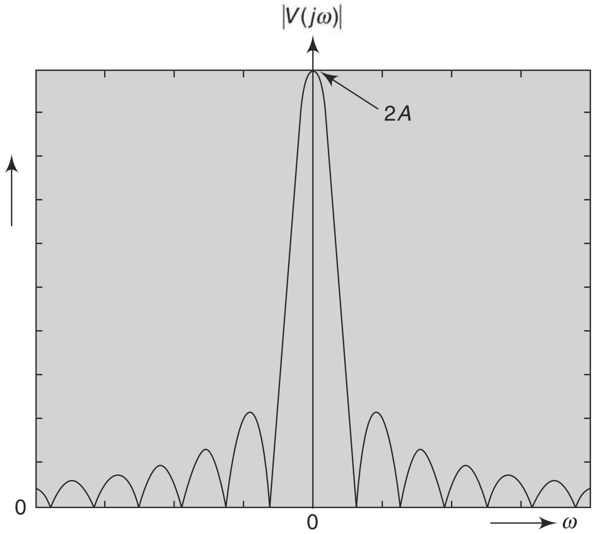
$$\therefore \qquad \phi(j\omega)\ \left[\text{Angle of } V(j\omega)\right] = \tan^{-1}\left(\frac{-\sin\omega\pi}{1+\cos\omega\pi}\right) = \tan^{-1}\left(\tan\frac{-\omega\pi}{2}\right) = \frac{-\omega\pi}{2}$$

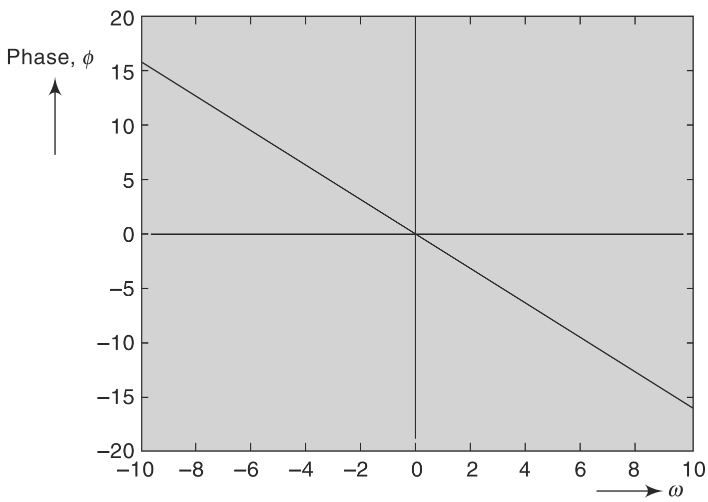
The amplitude and phase spectra are shown.

The current in the RL series circuit,

$$I(j\omega) = \frac{V(j\omega)}{R+j\omega L} = \frac{|V(j\omega)|\langle\phi(j\omega)}{\sqrt{R^2+\omega^2L^2}\angle\tan^{-1}\frac{\omega L}{R}}$$

$$= \frac{|V(j\omega)|\angle\phi(j\omega)}{\sqrt{1+\omega^2}\angle\tan^{-1}\omega}$$





$$= 2A\left|\frac{\cos \omega\pi/2}{1-\omega^2}\right| \frac{1}{\sqrt{1+\omega^2}} \angle -\frac{\omega\pi}{2} - \tan^{-1}\omega$$

$$= |I(j\omega)| \langle\ \theta(j\omega)$$

$\therefore$ $$|I(j\omega)| = 2A\left|\frac{\cos\left(\frac{\omega\pi}{2}\right)}{1-\omega^2}\right| \frac{1}{\sqrt{1+\omega^2}}$$

and $$\theta(j\omega) = -\frac{\omega\pi}{2} - \tan^{-1}\omega$$

7.30 The current in a 10 ohm resistor is, $i(t) = 10\,e^{-2t}u(t)$ A. Calculate the total energy W dissipated in the resistor during the time interval $t = 0$ to ∞. What is the energy W1 associated with the frequency band $0 \le \omega \le 2$ rad/s?

Solution The instantaneous power, $p(t) = i^2(t).R = 10 \times 100e^{-4t};\ t > 0$

Total energy dissipated

$$W = \int_{-\infty}^{\infty} p(t)dt = \int_0^{\infty} 1000e^{-4t}\,dt = 1000\left[\frac{e^{-4t}}{-4}\right]_0^{\infty} = -\frac{1000}{4}[\,0-1] = 250 \text{ Joule} \qquad \textit{Ans.}$$

The Fourier transform of $i(t)$ is,

$$I(j\omega) = \frac{10}{2 + j\omega}$$

The Energy associated,

$$W_1 = \frac{10}{\pi}\int_0^2 |I(j\omega)|^2\,d\omega \qquad \left\{\because\ 1\Omega \text{ Energy is, } W_{1\Omega} = \frac{1}{\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\,d\omega\right\}$$

$$= \frac{10}{\pi}\int_0^2 \frac{100}{4+\omega^2}\,d\omega$$

$$= \frac{1000}{\pi}\left[\frac{1}{2}\tan^{-1}\frac{\omega}{2}\right]_0^2$$

$$= \frac{500}{\pi}[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{500}{\pi} \times \frac{\pi}{4} = 125 \text{ Joule} \qquad \textit{Ans.}$$

7.31 A voltage, $v(t) = 100e^{-25t}\,u(t)$ volt is applied to the input of an ideal low-pass filter having a cut-off frequency of 25 rad/s. Calculate the percentage of the total energy transmitted through the filter.

Solution Fourier transform of $v(t)$

$$V(j\omega) = \frac{100}{25 + j\omega}$$

$$\therefore \qquad |V(j\omega)|^2 = \frac{10^4}{625+\omega^2}$$

Total 1 Ω energy available at the filter input is,

$$W_{i1\Omega} = \frac{1}{\pi}\int_0^{\infty} \frac{10^4\,d\omega}{625+\omega^2}$$

$$= \frac{10^4}{\pi}\int_0^{\infty} \frac{d\omega}{625+\omega^2}\,d\omega$$

$$= \frac{10^4}{\pi}\left[\frac{1}{25}\tan^{-1}\frac{\omega}{25}\right]_0^\infty$$

$$= \frac{10^4}{\pi} \times \frac{1}{25} \times \frac{\pi}{2} = 200 \text{ Joule} \qquad \textit{Ans.}$$

The 1 Ω energy available at the filter output is,

$$W_{01\Omega} = \frac{1}{\pi}\int_0^{25} |V(j\omega)|^2 \, d\omega$$

$$= \frac{10^4}{\pi}\int_0^{25}\frac{d\omega}{625+\omega^2} = \frac{10^4}{\pi}\left[\frac{1}{25}\times\tan^{-1}\frac{\omega}{25}\right]_0^{25}$$

$$= \frac{10^4}{\pi}\times\frac{1}{25}\times\frac{\pi}{4} = 100 \text{ Joule}$$

∴ Percentage of the input energy appearing at the output,

$$\frac{W_{01\Omega}}{W_{i1\Omega}}\times 100 = \frac{100}{200}\times 100\% = 50\%$$

7.32 A voltage, $v(t) = 4e^{-3t}\,u(t)$ volt is applied to the input of an ideal band-pass filter having a pass-band defined by $1 < f < 2$ Hz. Calculate the total 1 Ω energy available at the output of the filter.

Solution Let the output voltage is $v_0(t)$. The energy in $v_0(t)$ will be equal to the energy of that part of $v(t)$, having frequency components in the intervals, $1 < f < 2$ and $-2 < f < -1$.

Fourier transform of input,

$$V(j\omega) = 4\int_{-\infty}^{\infty} e^{-3t}u(t)e^{-j\omega t}\,dt = 4\int_{-\infty}^{\infty} e^{-(3+j\omega)t}u(t)\,dt = \frac{4}{3+j\omega}$$

So, the total 1 Ω energy in the input signal is,

$$W_{1\Omega} = \int_{-\infty}^{\infty} v^2(t)\,dt = 16\int_0^{\infty} e^{-6t}\,dt = \frac{8}{3} \text{ Joule}$$

or, $$W_{i1\Omega} = \frac{16}{\pi}\int_0^{\infty}\frac{d\omega}{9+\omega^2} = \frac{16}{\pi}\int_0^{\infty}\frac{d\omega}{9+\omega^2} = \frac{16}{\pi}\left[\frac{1}{3}\tan^{-1}\frac{\omega}{3}\right]_0^{\infty} = \frac{16}{\pi}\times\frac{1}{3}\times\frac{\pi}{2} = \frac{8}{3} \text{ Joule}$$

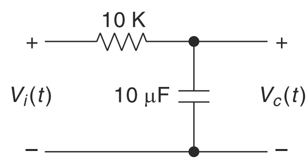
Total energy in the output is,

$$W_0 = \frac{1}{2\pi}\int_{-4\pi}^{-2\pi}\frac{16d\omega}{9+\omega^2} = \frac{16}{2\pi}\int_{0-4\pi}^{-2\pi}\frac{d\omega}{9+\omega^2} = \frac{16}{\pi}\left[\frac{1}{3}\tan^{-1}\frac{\omega}{3}\right]_{-4\pi}^{-2\pi}$$

$$= \frac{16}{\pi}\times\frac{1}{3}\times\left[\tan^{-1}\left(\frac{4\pi}{3}\right) - \tan^{-1}\left(\frac{2\pi}{3}\right)\right]$$

= 0.358 Joule

7.33. The voltage, $V_i(t) = 5e^{-5t}\, u(t)$ volt is applied to the input of the RC circuit shown in Figure. Determine the percentage of the 1 Ω energy that is transmitted to the output.



Solution Here, the cut-off frequency,

$$\omega_c = \frac{1}{RC} = \frac{1}{10^4 \times 10 \times 10^{-6}} = 10 \text{ rad/s}$$

Fourier transform of $v_i(t)$

$$V_i(j\omega) = \frac{5}{5 + j\omega}$$

$$\therefore \qquad |V_i(j\omega)|^2 = \frac{25}{25 + \omega^2}$$

Total 1 Ω energy available at the filter input is,

$$W_{i1\Omega} = \frac{1}{\pi}\int_0^\infty \frac{25d\omega}{25+\omega^2}\, d\omega$$

$$= \frac{25}{\pi}\int_0^\infty \frac{d\omega}{25+\omega^2}\, d\omega$$

$$= \frac{25}{\pi}\left[\frac{1}{5}\tan^{-1}\frac{\omega}{5}\right]_0^\infty$$

$$= \frac{25}{\pi} \times \frac{1}{5} \times \frac{\pi}{2} = 2.5 \text{ Joule} \qquad \textit{Ans.}$$

The 1 Ω energy available at the filter output is,

$$W_{01\Omega} = \frac{1}{\pi}\int_0^{10} |V_i(j\omega)|^2\, d\omega$$

$$= \frac{25}{\pi}\int_0^{10} \frac{d\omega}{25+\omega^2} = \frac{25}{\pi}\left[\frac{1}{5}\times\tan^{-1}\frac{\omega}{5}\right]_0^{10}$$

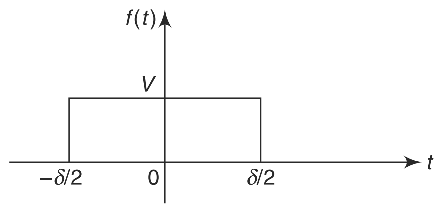
$$= \frac{25}{\pi} \times \frac{1}{5} \times 1.107 = 1.762 \text{ Joule}$$

∴ Percentage of the input energy appearing at the output,

$$\frac{W_{01\Omega}}{W_{i1\Omega}} \times 100 = \frac{1.762}{2.5} \times 100\% = 70.48\% \qquad \textit{Ans.}$$

7.34 (a) For the pulse shown in Figure prove that,

$$F(j\omega) = V\delta \frac{\sin\dfrac{\omega\delta}{2}}{\dfrac{\omega\delta}{2}}$$



(b) Draw the frequency spectra of this waveform and explain how you would use this result to estimate the bandwidth required for the transmission of such a signal.

(c) Calculate the percentage of energy associated with this pulse that lies in the dominant portion of the amplitude spectrum.

Solution

(a) The pulse is, $f(t) = V, -\delta/2 < t < \delta/2$

So, the Fourier transform,

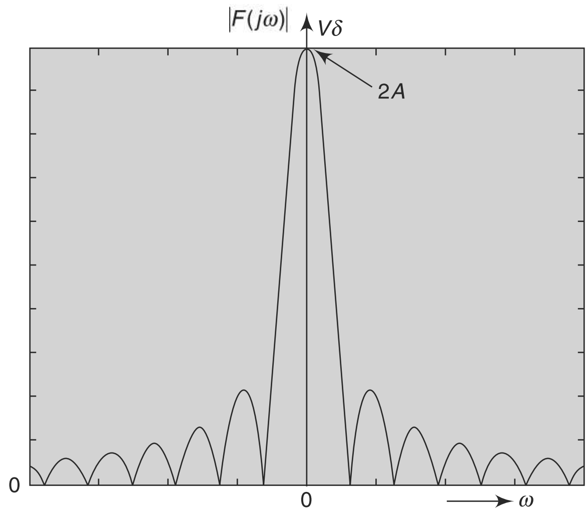
$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} Ve^{-j\omega t}\,dt = V\frac{e^{j\omega\delta/2} - e^{-j\omega\delta/2}}{j\omega}$$

$$= 2V\frac{\sin\left(\dfrac{\omega\delta}{2}\right)}{\omega}$$

$$= 2V\frac{\sin\left(\dfrac{\omega\delta}{2}\right)}{\left(\dfrac{\omega\delta}{2}\right)} \times \frac{\delta}{2}$$

$$\therefore \quad F(j\omega) = V\delta\frac{\sin\left(\dfrac{\omega\delta}{2}\right)}{\left(\dfrac{\omega\delta}{2}\right)}$$

The plot of $\left|\dfrac{\sin x}{x}\right|$ versus x $\left(\text{here, } x = \dfrac{\omega\delta}{2}\right)$ is shown in the below figure.



(b) The function goes through zero when $x = \frac{\omega\delta}{2}$ is an integral multiple of π. The function is unity at $x = 0$. This form is called sampling function, and it occurs frequently in modern communication theory.

From the figure, we see that the major portion of the amplitude spectrum of the rectangular pulse spreads over the frequency range from $-\frac{2\pi}{\delta}$ to $\frac{2\pi}{\delta}$. If the pulse is carried through a transmission system, the bandwidth (BW) of the system must accommodate the major portion of the amplitude spectrum for reasonable fidelity in transmission; i.e. the cut-off frequency of the system must be at least, $\omega_C = \frac{2\pi}{\delta}$.

Thus, $\omega_C \times \delta = 2\pi \left[\text{BW} = \frac{2\pi}{\delta} \right]$

$\therefore$ Product of the bandwidth and pulse width is a constant.

(c) We know that the dominant portion of the amplitude spectrum lies in the frequency range $0 \le \omega \le \frac{2\pi}{\delta}$. The Fourier transform of the rectangular voltage pulse is,

$$V(j\omega) = V\delta \frac{\sin\left(\frac{\omega\delta}{2}\right)}{\left(\frac{\omega\delta}{2}\right)}$$

The portion of the total 1Ω energy associated with $v(t)$ that lies in the dominant portion of the amplitude spectrum is,

$$W'_{1\Omega} = \frac{1}{\pi} \int_0^{2\pi/\delta} V^2 \delta^2 \frac{\sin^2\left(\frac{\omega\delta}{2}\right)}{\left(\frac{\omega\delta}{2}\right)^2} d\omega$$

$$= \frac{2V^2\delta}{\pi} \int_0^{\pi} \frac{\sin^2 x}{x^2} dx \quad \left\{ let,\ x = \frac{\omega\delta}{2},\ \therefore dx = \frac{\delta}{2} d\omega \right\}$$

$$= \frac{2V^2\delta}{\pi} \left[\sin^2 x \left\{ -\frac{1}{x} \right\}_0^{\pi} + \int_0^{\pi} \frac{\sin 2x}{x} dx \right]$$

$$= \frac{2V^2\delta}{\pi} \left[0 + \int_0^{\pi} \frac{\sin 2x}{x} dx \right]$$

$$= \frac{4V^2\delta}{\pi} \int_0^{\pi} \frac{\sin 2x}{2x} dx$$

$$= \frac{2V^2\delta}{\pi}\int_0^{\pi}\frac{\sin\theta}{\theta}\,d\theta \quad \left[Let,\ \theta = 2x, \therefore\ dx = \frac{1}{2}d\theta\right]$$

$$= \frac{2V^2\delta}{\pi}\times 1.418$$

[The value of the integral as found from the table of sine integrals is 1.418]

$$\therefore \qquad W'_{1\Omega} = \frac{2V^2\delta}{\pi}\times 1.418$$

Total 1Ω energy for *v(t)* is,

$$W_{1\Omega} = \int_0^{\delta} V^2 d\delta = V^2\delta$$

Hence the percentage of total energy contained in the dominant portion of the amplitude spectrum is,

$$\frac{W'_{1\Omega}}{W_{1\Omega}}\times 100 = \frac{2\times 1.418}{\pi}\times 100 = 90.2\% \qquad Ans.$$

MULTIPLE-CHOICE QUESTIONS

7.1 A current consists of a fundamental component of amplitude I_1, and a third harmonic of amplitude I_3. The rms value of current will be

(a) $(I_1 + I_3)/\sqrt{2}$ (b) $(I_1 + I_3)/2\sqrt{2}$ (c) $\sqrt{I_1^2 + I_3^2}$ (d) $\sqrt{(I_1^2 + I_3^2)/2}$

7.2 The Fourier series expansion of a periodic function with half wave symmetry contains only

(a) sine terms (b) cosine terms (c) odd harmonics (d) even harmonics

7.3 A periodic function $f(t)$ is said to have a quarter wave symmetry, if it possesses

(a) even symmetry at an interval of quarter of a wave.
(b) even symmetry and half wave symmetry only
(c) even or odd symmetry without the half wave symmetry
(d) even or odd symmetry with the half wave symmetry.

7.4 If *f(t)* is a periodic waveform with even symmetry, then its Fourier series expansion does not contain

(a) sine terms (b) cosine terms (c) odd harmonics (d) even harmonics

7.5 Periodic signal that obeys Dirichlet's condition can be represented by

(a) Fourier series (b) Fourier transform
(c) Inverse Fourier transform (d) None of these

7.6 Which of the following conditions is true for even function?

(i) $f(t) = -f(t \pm T/2)$ (b) $f(t) = -f(-t)$ (c) $f(t) = f(-t)$ (d) $f(t) = f(T)$

7.7 Which of the following conditions is true for odd function?

(a) $f(t) = -f(t \pm T/2)$ (b) $f(t) = -f(-t)$ (c) $f(t) = f(-t)$ (d) $f(t) = f(T)$

7.8 A periodic function $f(t)$ having a time period T, repeats itself after half time period $T/2$. The Fourier series of $f(t)$ would contain.

(a) cosine terms only (b) sine terms only
(c) odd harmonic terms only (d) even harmonic terms only

7.9 Which of the following statements is true for a delayed step function $u(t - T)$?
(a) It has an infinite Fourier series (b) It has no Fourier series
(c) It has a finite Fourier series (d) Its Laplace transform is $1/s$.

7.10 Which one of the following is the correct Fourier transform of the unit step signal $u(t)$?
(a) $\pi\delta(\omega)$ (b) $\frac{1}{j\omega}$ (c) $\frac{1}{j\omega} + \pi\delta(\omega)$ (d) $\frac{1}{j\omega} + 2\pi\delta(\omega)$

7.11 If $f(t) = -f(-t)$ and $f(t)$ satisfy the Dirichlet's conditions, then $f(t)$ can be expanded in a Fourier series containing
(a) only sine terms (b) only cosine terms
(c) cosine terms and a constant term (d) sine terms and a constant term.

7.12 Fourier transform $F(j\omega)$ of an arbitrary signal has the property:
(a) $F(j\omega) = F(-j\omega)$ (b) $F(j\omega) = -F(-j\omega)$ (c) $F(j\omega) = F^*(-j\omega)$ (d) $F(j\omega) = -F^*(-j\omega)$

7.13 The Fourier series expansion of an odd periodic function contains
(a) cosine terms (b) constant terms only (c) sine terms.

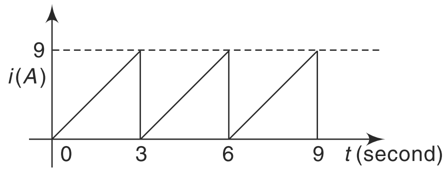
7.14 For the expansion of $f(\omega t)$ in Fourier series $a_0 + a_1 \cos \omega t + \ldots + a_n \cos n\omega t + \ldots + b_1 \sin \omega t + \ldots + b_a \sin n\omega t$ if $f(\omega t) = f(-\omega t)$ then:
(a) $a_n = 0$ (b) $b_n = 0$ for all n
(c) $a_0 = 0$ (d) $a_n = 0$ for all n except $n = 0$.

7.15 Two complex waves will have the same waveform if
(a) they contain the same harmonics.
(b) harmonics are similarly spaced with respect to the fundamental.
(c) the ratio of corresponding harmonics to their respective fundamentals is the same.
(d) all of the above.

7.16 The complex wave is symmetrical when
(a) it contains only even harmonics.
(b) it contains only odd harmonics.
(c) it contains both odd and even harmonics.
(d) phase difference between even harmonics and fundamental is either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

7.17 An even waveform when expressed in exponential Fourier series will contain:
(a) only imaginary coefficient (b) only real coefficient
(c) both (a) and (b) (d) None of these.

7.18 The current waveform in a pure resistor of 10 Ω is shown in the given figure. Power dissipated in the resistor is
(a) 7.29 W (b) 52.4 W
(c) 135 W (d) 270 W.



7.19 The inverse Fourier transform of

$$F(j\omega) = \int_{-\infty}^{\infty} \exp(-j\omega t) f(t) dt \text{ is:}$$

(a) $f(t) = \int_{-\infty}^{\infty} \exp(+j\omega t) F(j\omega) d\omega$ (b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(+j\omega t) F(j\omega) d\omega$

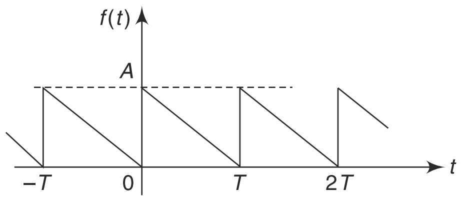
(c) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-j\omega t) F(+j\omega) d\omega$ (d) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-j\omega t) F(-j\omega) d\omega$

EXERCISES

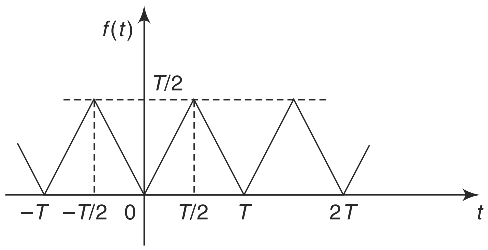
Fourier Series

7.1 Find the Fourier series expansion for the following functions and sketch the frequency spectrum.

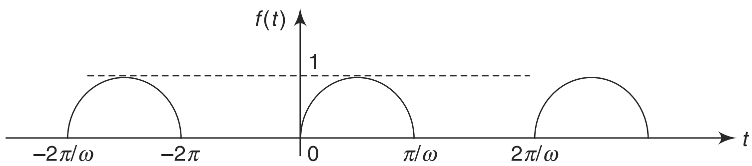
(a)



(b)



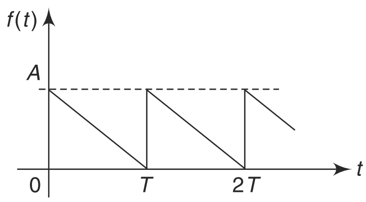
(c)



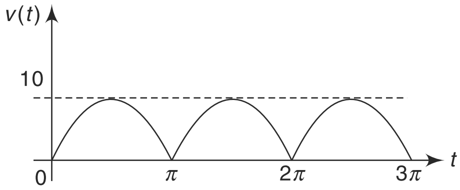
$$\left[Ans. \text{ (a) } f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{n\pi} \sin n\omega t \text{ (b) } f(t) = \frac{T}{4} - \frac{2T}{\pi^2}\left[\cos \omega t + \frac{1}{3^2}\cos 3\omega t + .. \right] \right.$$

$$\left. \text{(c) } f(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}\cos 2n\omega t \right]$$

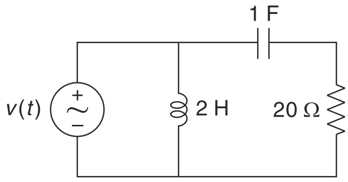
7.2 A periodic waveform as shown in the below figure feeds an RL load with $R = 10$ ohm and $L = \frac{1}{2\pi}$ H. Calculate the power at the fundamental frequency supplied to the load.



7.3 A waveform of the shape shown in the below figure (i) is applied to the network shown in the below figure (ii). Calculate the power dissipated in a 20 Ω resistor. Take $\omega = 1$ rad/s. [1.17 W]



(i)



(ii)

7.4 A series RLC circuit with $R = 5\ \Omega$, $L = 5$ mH, $C = 50\ \mu$F has an applied voltage

$$v(t) = 150 \sin 1000\,t + 100 \sin 2000\,t + 75 \sin 3000\,t\ (V)$$

Determine the effective current and average power. [16.58 A; 1374 W]

Fourier Transform

7.5 Find the Fourier transform of the following functions:

(i) $f(t) = e^{-at}\,u(t)$, $a > 0$.

(ii) $f(t) = e^{-a|t|}$, for all values of t.

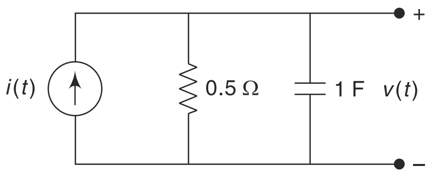
(iii) $f(t) = 1$

(iv) Unit impulse function, $\delta(t)$.

(v) Signum function, sgn(t).

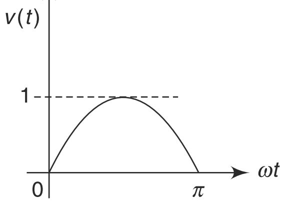
(vi) Unit step function, $u(t)$.

7.6 Determine the output voltage response across the capacitor to a current source excitation $i(t) = e^{-t}u(t)$, as shown in the below figure.

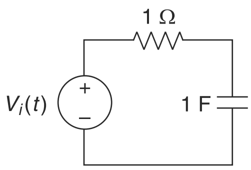


[$v(t) = e^{-t} - e^{-2t}$ (V)]

7.7 Determine the response of the network shown in the below figure when a voltage having the waveform shown in figure is applied to it., by using Fourier transform method.



(i)



(ii)

SHORT-ANSWER TYPE QUESTIONS

7.1 (a) What are the conditions which a periodic function must satisfy to have its Fourier series expansion?

(b) Write the trigonometric form of the Fourier series for a function $f(t)$ and explain, by deriving necessary relations, how the values of various co-efficients are obtained.

or

What do you understand by Fourier series? Outline the general procedure of determining Fourier series of periodic waveform.

(c) Give the exponential form of Fourier series for a periodic function.

7.2 Derive an expression for the effective value of a non-sinusoidal periodic waveform

or

Discuss the method of computing the effective value of a non-sinusoidal periodic waveform.

7.3 (a) Explain clearly the significance of the following terms used in determining Fourier series of a given waveform:

(i) Odd symmetry or Rotation symmetry,
(ii) Even symmetry or Mirror symmetry,
(iii) Half-wave symmetry or Alternation symmetry,
(iv) Quarter-wave symmetry.

(b) Show that the Fourier series expansion of a periodic function with odd (rotation) symmetry contains only the sine terms.

(c) Show that the Fourier series expansion of a periodic function with even (mirror) symmetry contains only the cosine terms plus a constant.

(d) Show that the Fourier series expansion of a periodic function with half-wave symmetry contains only the odd harmonics.

7.4 Discuss in brief the following:

(i) Fourier series and its applications to network analysis,
(ii) Method of analyzing the complex waveform by Fourier series,
(iii) Frequency and phase spectra of periodic waveform.
(iv) Truncating Fourier series.
(v) Gibb's phenomenon.

7.5 (a) Give the definitions of a Fourier transform pair and illustrate its use in network analysis with one example.

(b) Explain clearly the difference between Fourier transform and Laplace transform and discuss briefly their importance in analyzing electrical network.

or

Define Fourier's transform. How does Fourier transform differ from (i) Fourier integral and (ii) Laplace transform?

(c) Write a brief note on the use of Fourier transform and Fourier integrals in the analysis of circuits excited by ideal sources of non-sinusoidal waveforms.

(d) Discuss the important properties of Fourier transforms.

7.6 When do we use Fourier transform?

Discuss that Fourier integral is the limit of Fourier series, as time period T of a repetitive wave approaches infinity as the limit.

or

How would you obtain Fourier integral from Fourier series?

7.7 Find the amplitude-frequency distribution of a single non-repetitive voltage pulse of duration one microsecond and explain how its frequency-bandwidth is estimated.

or

Consider a periodic voltage pulse waveform of period T (second) and width T_0 (second). Find an expression for the frequency-spectra of this waveform and explain how you would use this result to estimate the bandwidth required for the transmission of such a signal.

7.8 State and prove Parseval's theorem for a periodic function.

7.9 Show that when $f(t)$ is an even function of t, its Fourier transform $F(j\omega)$ is an even function of ω and is real; while when $f(t)$ is an odd function of t, its Fourier transform $F(j\omega)$ is an odd function of ω and is imaginary.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

7.1 (d)	7.2 (c)	7.3 (d)	7.4 (a)	7.5 (a)	7.6 (c)	7.7 (b)
7.8 (c)	7.9 (c)	7.10 (c)	7.11 (a)	7.12 (c)	7.13 (c)	7.14 (b)
7.15 (d)	7.16 (b)	7.17 (b)	7.18 (d)	7.19 (b)		

CHAPTER
8
Active Filter

8.1 INTRODUCTION

Passive filters are built from passive components; resistors, capacitors, and inductors. **Active filters** also use resistors and capacitors, but the inductors are replaced by active devices capable of producing power gain. These devices can range from single transistor to integrated circuit (IC)—controlled sources such as the operational amplifier (op amp), and more exotic devices, such as the operational transconductance amplifier (OTA), the generalized impedance converter (GIC), and the frequency-dependent negative resistor (FDNR).

In this chapter, active filters with op-amp have been discussed.

8.1.1 Operational Amplifier (Op Amp)

An operational amplifier is a direct- coupled high gain, differential-input amplifier.

With the addition of suitable external feedback components, an op- amp can be used for a variety of application, such as ac and dc signal amplification, active filters, oscilators, comparators, regulators, and others.

8.1.2 Operational Amplifier Terminals

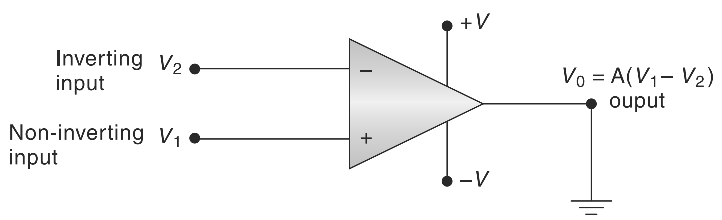


Figure 8.1 *Operational amplifier*

Op-amp has five basic terminals—

(i) Two for input signals, V_1 and V_2 – differential input terminals.
(ii) One for output signal, V_0 single-ended output.
(iii) Two for power supply, $+V$ and $-V$. (Maximum $V = \pm$ 18 V)

Note The power supply has three terminals: positive, negative and power supply common. The common terminal may or may not be wired to earth ground via the third wire of line cord. However, it has become standard practice to show power common as a ground symbol.

Use of the term 'ground' on the ground symbol is a convention which indicates that all voltage measurements are with respect to 'ground'.

8.1.3 Op- Amp Characteristics

Ideal Characteristics

(i) An infinite voltage gain
(ii) An infinite bandwidth
(iii) An infinite input impedance
(iv) Zero output impedance
(v) Perfect balance, i.e., the output is zero when equal voltages are present at the two input terminals; and
(vi) The characteristics do not change with temperature

Practical (Actual) Characteristics

(i) The gain at low- frequency is finite and very high (of the order of 10^3 to 10^6). The gain is constant upto a few hundred kHz and then decreases monotonically with the increase in frequency.
(ii) The bandwidth is finite and very high.
(iii) The input impedance lies in the range of 150 kΩ to a few hundred M Ω.
(iv) The output impedance of a practical op-amp lies between 0.75 to 100 Ω.
(v) Perfect balance is not achieved with practical op-amps.

8.2 FILTER

An electric filter is a four-terminal frequency-selective network designed generally with reactive elements to transmit freely a specified band of frequency and block or attenuate signals of frequency outside this band.

- The band of frequency transmitted through the filter is called the Pass-band.
- The band of frequency which is severely attenuated by the filter is called the attenuated on stop-band.

8.2.1 Classification of Filter

(i) Analog or Digital Filters,
(ii) Active or Passive Filters,

Analog filters are designed to process analog signals while digital filters process analog signals using digital techniques.

Passive filters consist of passive elements, i.e., *R*, *L* and *C*. On the other hand, active filters consist of active components such as op-amp, transistors, in addition to *R* and *C*.

8.3 ADVANTAGES OF ACTIVE FILTERS OVER PASSIVE FILTERS

1. **Less Cost** Active filters are very much inexpensive than passive filters due to the variety of cheaper op-amp and the absence of costly inductors.
2. **Gain and Frequency Adjustment Flexibility** Since the op-amp is capable of providing a gain (which may also be variable), the input signal is not attenuated as it is in a passive filter. In addition, the active filter is easier to tune or adjust.
3. **No Loading Problem** Active filters provide an excellent isolation between the individual stages due to the high input impedence (ranging from a few kΩ to a several thousand MΩ) and low output impedance (ranging from less than 1 Ω to a few hundred Ω). So, the active filter does not cause loading of the source or load.
4. **Size and Weight** Active filters are small in size and less bulky (due to the absence of bulky '*L*') and are rugged.
5. **Non-floating Input and Output** Active filters generally have single ended inputs and outputs which do not 'float' with respect to the system power supply or common. This property is different from that of the passive filters.

8.4 APPLICATION OF ACTIVE FILTERS

Application of active filters is given below. They are used
(i) in the field of communication and signal processing
(ii) in almost all sophisticated electronic systems, such as radio, television, telephone, radar, space satellites, biomedical equipments, and so on.

8.5 TYPES OF ACTIVE FILTERS

1. **Low-Pass Filter** It is a circuit that has a constant output (or gain) from zero to a cut-off frequency, f_c and attenuation of all frequencies above f_c.

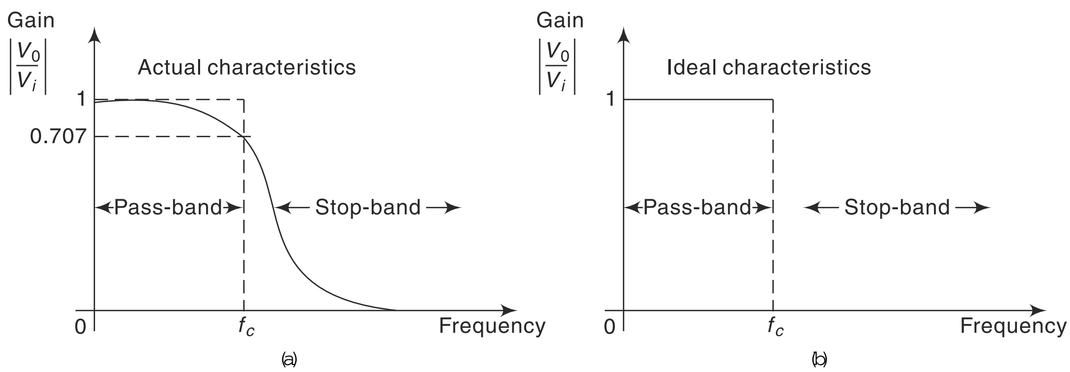


Figure 8.2(a) *Low-pass filter characteristics (a) Actual (b) Ideal*

2. **High-Pass Filter** It is a circuit that attenuates all signals of frequency below the cut-off frequency and has a constant output (or gain) above this frequency.

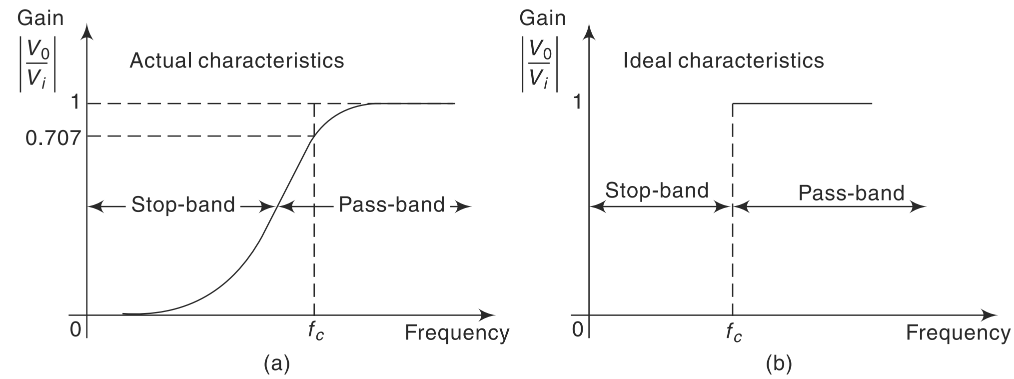


Figure 8.3 *High pass filter characteristics (a) Actual (b) Ideal*

3. **Band–Pass Filter** It is a circuit that passes a band of frequencies and attenuates all frequencies outside the band.

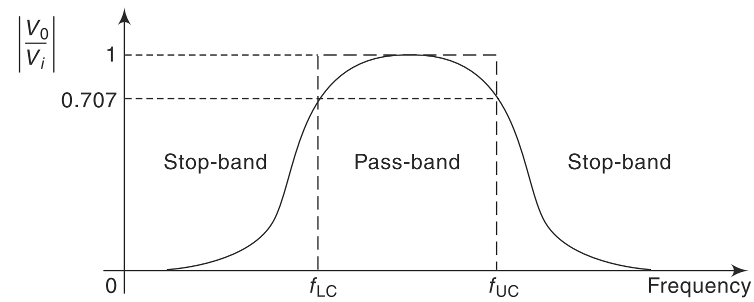


Figure 8.4 *Band pass filter characteristics*

4. **Band-Rejection/Elimination Filter or Band Stop Filter or Notch Filter** It rejects a specified Band of frequencies while passing all other frequencies outside the band.

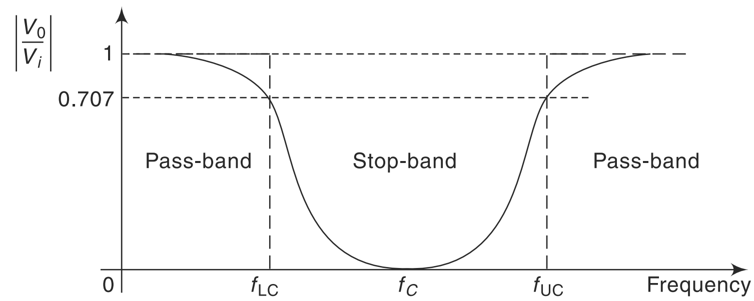


Figure 8.5 *Band reject filter characteristics*

5. **All–Pass Filter** It passes all frequencies equally well, i.e., output and input voltages are equal in magnitude for all frequency; with the phase–shift between the two a function of frequency.

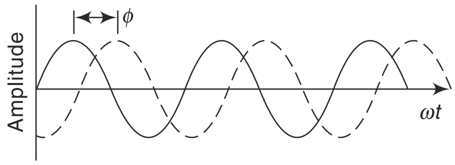


Figure 8.6 *All pass filters characteristics*

This filter is also known as a **phase-shift filter**, **time-delay filter,** or simply the **delay equalizer.** One major application of an all-pass filter is the simulation of a lossless transmission line. The magnitude of the output voltage is the same as the input voltage but the output voltage is shifted in phase with respect to the input voltage.

The highest frequency up to which the input and output amplitudes remain equal is dependent on the unity-gain bandwidth of the op-amp. At this frequency, however, the phase-shift between the input and output is maximum.

8.6 LOW-PASS ACTIVE FILTER

The circuit of Figure 8.7 is a commonly used low-pass active filter.

The filtering is done by the *RC* network, and the op-amp is used as a unity-gain amplifier. The resistor R_f (= R) is included for DC offset.

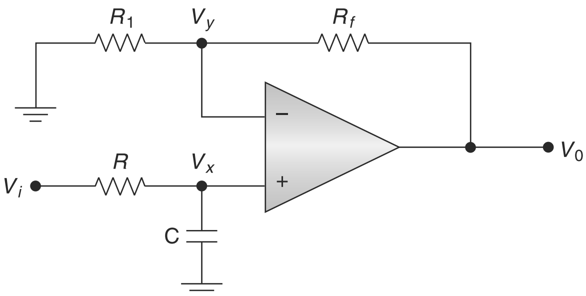


Figure 8.7 *First order low-pass active filter circuit*

[At DC, the capacitive reactance is infinite and the dc resistive path to ground for both terminals should be equal.]

Here, all the voltages V_i, V_x, V_y, V_o are measured with respect to ground.

Since the input impedance of the op-amp is infinite, no current will flow into the input terminals.

$$V_y = \frac{V_0}{R_1 + R_f} \times R_1 \tag{8.1}$$

According to the voltage divider – rule, the voltage across the capacitor,

$$V_x = \frac{X_c}{R + X_c} V_{\mathrm{i}}\,; \quad X_c = \frac{1}{j\omega C} = \frac{1}{j2\pi f C}$$

$$= \frac{1/j2\pi f C}{R + \dfrac{1}{j2\pi f C}} V_i$$

$$= \frac{V_i}{1 + j2\pi f RC} \tag{8.2}$$

Since the op-amp gain is infinite,

$$\therefore \qquad V_x = V_y$$

or, $$\frac{V_0 R_1}{R_1 + R_f} = \frac{V_i}{1 + j2\pi f RC}$$

$$\Rightarrow \qquad \frac{V_0}{V_i} = \frac{(1 + R_f / R_1)}{1 + j2\pi fRC}$$

or, $$\boxed{\frac{V_0}{V_i} = \frac{A_F}{1 + j(f/f_c)} = A_{cL}}$$

where, $A_F = \left(1 + \dfrac{R_f}{R_1}\right)$ = pass-band gain of the filter.

f = frequency of the input signal.

$f_c = \dfrac{1}{2\pi RC}$ = cut-off frequency of the filter.

A_{cL} = Closed- loop gain of the filter as a function of frequency.

The gain magnitude,

$$|A_{cL}| = \left|\frac{V_0}{V_i}\right| = \frac{A_F}{\sqrt{1 + (f/f_c)^2}} = \frac{A_F}{\sqrt{1 + \omega^2 R^2 C^2}}$$

and phase angle (in degree),

$$\phi = -\tan^{-1}\left(\frac{f}{f_c}\right) = -\tan^{-1}(\omega RC)$$

8.6.1 Operation of the Filter

The operation of the low-pass filter can be verified from the gain magnitude equation as follows:

1. At very low frequencies , i.e., $f << f_c$,

$$|A_{CL}| \cong A_F$$

2. At $f = f_C$, $|A_{CL}| = \dfrac{A_F}{\sqrt{2}}$ = 0.707 A_F = –3dB A_F, ϕ = 45°

3. At $f > f_C$, $|A_{cL}| < A_F$

Thus, the filter has a constant gain of A_F from 0 Hz to the cut –off frequency f_c. At f_c, the gain is 0.707A_F and after f_c, it decreases at a constant rate with an increase in frequency.

Figure 8.8 shows that the actual response deviates from the straight dashed-line approximation at the vicinity of 'f_c'.

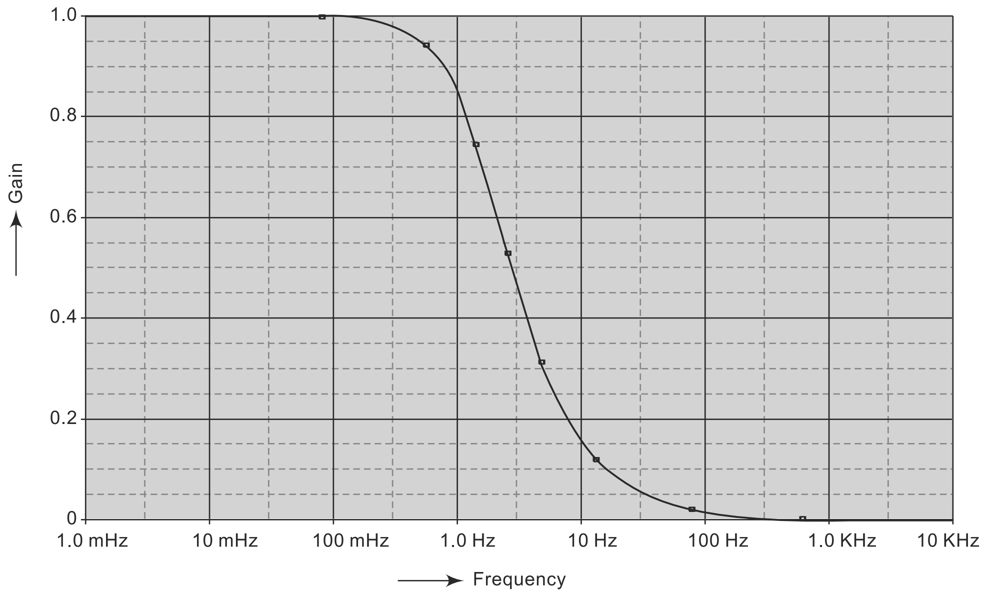


Figure 8.1 *Low pass filter characteristics*

At $\omega = 0.1\ \omega_C$, $|A_{CL}| \cong 1(0\text{dB})$

At $\omega = 10\ \omega_C$, $|A_{CL}| \cong 0.1(-20\text{dB})$

The table below gives the magnitude and phase angle for different values of ω between $0.1\omega_c$ and $10\omega_c$.

ω	A_{cL}	*Phase-angle (degree)*
$0.1\omega_c$	1.0	–6
$0.25\omega_c$	0.97	–14
$0.5\omega_c$	0.89	–27
ω_c	0.707	–45
$2\omega_c$	0.445	–63
$4\omega_c$	0.25	–76
$10\omega_c$	0.1	–84

8.6.2 Filter Design

A low-pass active filter can be designed by implementing the following steps:-

1. A value of the cut-off frequency ω_c (or, f_c) is chosen.
2. A value of the capacitance C is selected; usually the value is between 0.001 and 0.1μF. Mylar or tantalum capacitors are recommended for better performance.
3. The value of the resistance R is calculated from the relation,

$$R(\text{in }\Omega) = \frac{1}{\omega_C C} = \frac{1}{2\pi f_C C}$$

f_c = cut-off frequency in hertz
ω_c = cut-off frequency radian/second
C = in farad

4. Finally, the values of R_1 and R_f are selected depending on the desired pass band gain by using the relation $A_F = \left(1 + \dfrac{R_f}{R_1}\right)$.

8.6.3 Frequency Scaling

Once a filter is designed, there may be a need to change its cut-off frequency. The procedure used to convert an original cut-off frequency f_c to a new cut-off frequency f_c' is called '*frequency- scaling*'.

It is accomplished as follows:-

To change a cut-off frequency, multiply R or C, but not both by the ratio

$$\left(\frac{\text{Old Cut-off Frequency, } f_{\text{cold}}}{\text{New Cut-off Frequnecy, } f_{\text{cnew}}}\right)$$

Example 8.1 (a) Design a low-pass active filter at a cut-off frequency of 1kHz with a pass band gain of 2. Using the frequency scaling technique, convert this filter to a low-pass filter of cut-off frequency 1.6 kHz.
(b) Plot the frequency response of this low-pass active filter.

Solution (a) Here, f_c = 1 kHz, A_F = 2; Let, C = 0.01 μF.

$$R = \frac{1}{2\pi f_c C} = \frac{1}{2\pi \times 10^3 \times 0.01 \times 10^{-6}} = 15.9\ \text{k}\Omega$$

$$\because \quad A_F = 2 = \left(1 + \frac{R_f}{R_1}\right) \quad \Rightarrow \quad R_f = R_1 = 10\ \text{k}\Omega$$

So, the complete circuit is shown in Figure 8.9(a).

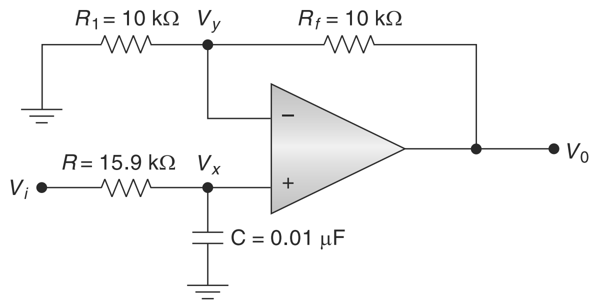


Figure 8.9(a) *Circuit of Example 8.1*

To change the cut-off frequency from 1 kHz to 1.6 kHz, we multiply the 15.9 kΩ resistor by

$$\frac{\text{Original Cut-off frequency}}{\text{New Cut-off frequency}} = \frac{1}{1.6} = 0.625$$

$\because$ New resistor, R = 15.9 × 0.625 = 9.94 kΩ

(b) To plot the frequency–response, the data are obtained from the equation,

$$\left|\frac{V_0}{V_{\text{in}}}\right| = \frac{A_F}{\sqrt{1+(f/f_c)^2}}$$

Frequency (Hz)	*Gain*	*Gain (in dB)*
10	2	6.02
100	1.99	5.98
200	1.96	5.85
700	1.64	4.29
1,000	1.41	3.01
3,000	0.63	–3.98
7,000	0.28	–10.97
10,000	0.20	–14.02
30,000	0.07	–23.53
100,000	0.02	–33.98

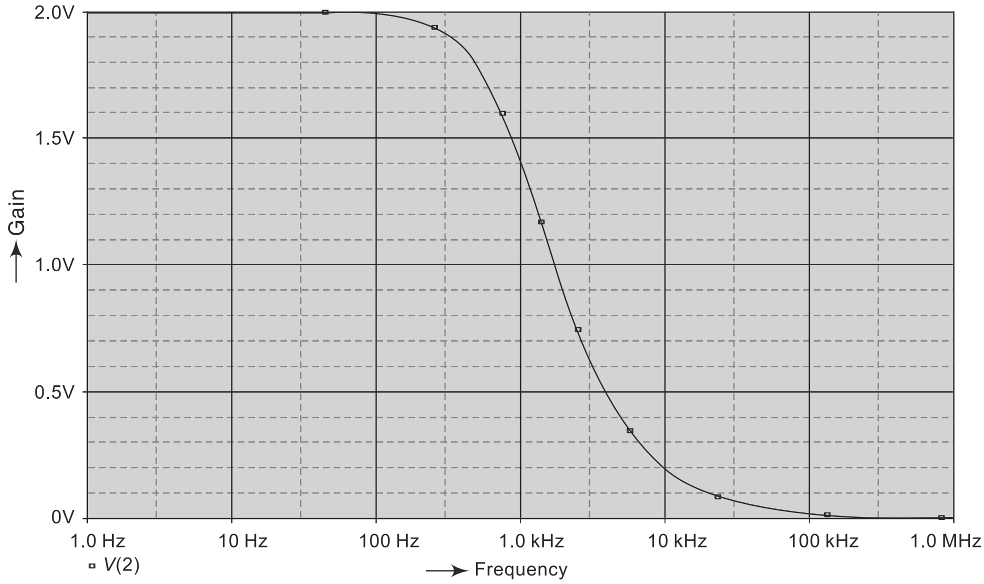


Figure 8.9(b) *Filter characteristics of Example 8.1*

8.7 HIGH-PASS ACTIVE FILTER

The circuit is shown in Fig. 8.10.

The filtering is done by the CR network and the op-amp is connected as a unity – gain follower. The feedback resistor, R_f is included to minimize dc off-set.

Here,

$$V_y = V_0 \frac{R_1}{R_1 + R_f} \tag{8.3}$$

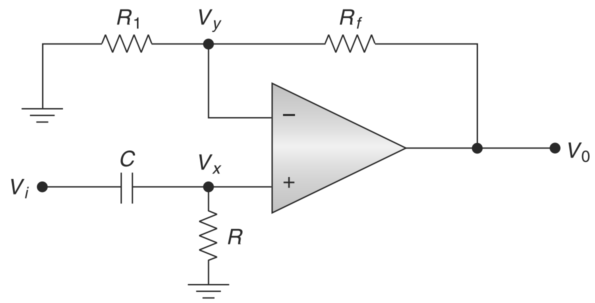


Figure 8.10 *First order high pass active filter circuit*

Voltage across the resistor R,

$$V_x = \frac{R}{R+X_c}V_i = \frac{R}{R+\frac{1}{j\omega C}}V_i = \frac{j\omega RC}{1+j\omega RC}V_i \tag{8.4}$$

Since op-amp gain is infinite,

$$V_x = V_y$$

$$\Rightarrow \quad \frac{V_0 R_1}{R_f + R_1} = \frac{j\omega RC}{1+j\omega RC}V_i$$

$$\Rightarrow \quad \frac{V_0}{V_i} = \left(\frac{R_f + R_1}{R_1}\right)\left(\frac{j\omega RC}{1+j\omega RC}\right) = A_F \times \frac{j2\pi fRC}{1+j2\pi fRC}$$

$$\boxed{\frac{\mathbf{V_0}}{\mathbf{V_i}} = \mathbf{A_F}\left[\frac{\mathbf{j(f/f_c)}}{\mathbf{1+j(f/f_c)}}\right]}$$

where, $A_F = (1 + R_f/R_1)$ = Pass-band Gain of the filter,
f = frequency of the input signal (Hz),
$f_c = \frac{1}{2\pi RC}$ cut-off frequency of the filter (Hz).

The gain- magnitude,

$$\left|\frac{V_0}{V_i}\right| = \frac{A_F(f/f_c)}{\sqrt{1+(f/f_c)^2}} = A_F \cdot \frac{\omega RC}{\sqrt{1+\omega^2R^2C^2}}$$

and phase-angle (in degree), $\phi = 90° - \tan^{-1}(f/f_c) = 90° - \tan^{-1}(\omega RC)$

8.7.1 Operation of the Filter

The operation of the high-pass filter can be verified from the gain–magnitude equation as follows:

1. At very low frequencies, i.e., $f < f_c$, $\left|\frac{V_0}{V_i}\right| < A_F$

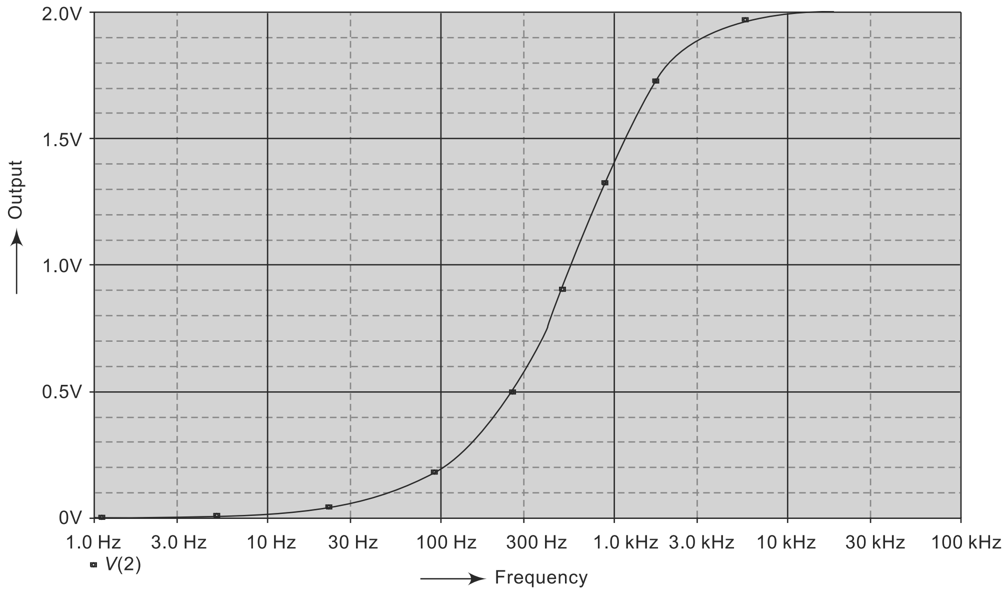


Figure 8.11 *High pass filter characteristics*

2. At $f = f_c$, $\left|\frac{V_0}{V_i}\right| = \frac{A_F}{\sqrt{2}} = 0.707\ A_F = -3$ dB, $\phi = 45°$

3. At $f >> f_c$, $\left|\frac{V_0}{V_i}\right| \cong A_F$

8.7.2 Filter Design

A high-pass active filter can be designed by implementing the following steps:

1. A value of the cut-off frequency, ω_c (or f_c) is chosen.
2. A value of the capacitance C, usually between 0.001 and 0.1 μF, is selected.
3. The value of the resistance R is calculated using the relation,

$$R = \frac{1}{\omega_c C} = \frac{1}{2\pi f_c C}$$

4. Finally, the values of R_1 and R_f are selected depending on the desired pass-band gain, using, the relation, $A_F = \left(1 + \frac{R_f}{R_1}\right)$.

Example 8.2 (a) Design a high-pass active filter of cut-off frequency 1 kHz with a pass-band gain of 2.
(b) Plot the frequency response of the filter.

Solution (a) Here, f_c = 1kHz, A_F = 2

Let, C = 0.01 μF.

$$\therefore \quad R = \frac{1}{2\pi f_c C} = \frac{1}{2\pi \times 10^3 \times 0.01 \times 10^{-6}} = 15.9 \text{ k } \Omega$$

$$\therefore \quad A_F = 2 = \left(1 + \frac{R_f}{R_1}\right) \Rightarrow R_f = R_1 = 10 \text{ k}\Omega$$

So, the complete circuit is shown in Figure 8.12(a).

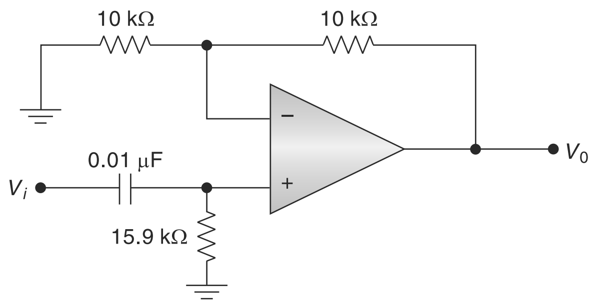


Figure 8.12(a) *Circuit of Example (8.2)*

(b) The data for the frequency response plot can be obtained by substituting the input frequency (f) values from 100 Hz to 100 kHz in the equation.

$$\frac{V_0}{V_i} = \frac{A_F (f / f_c)}{\sqrt{1 + (f/f_c)^2}}$$

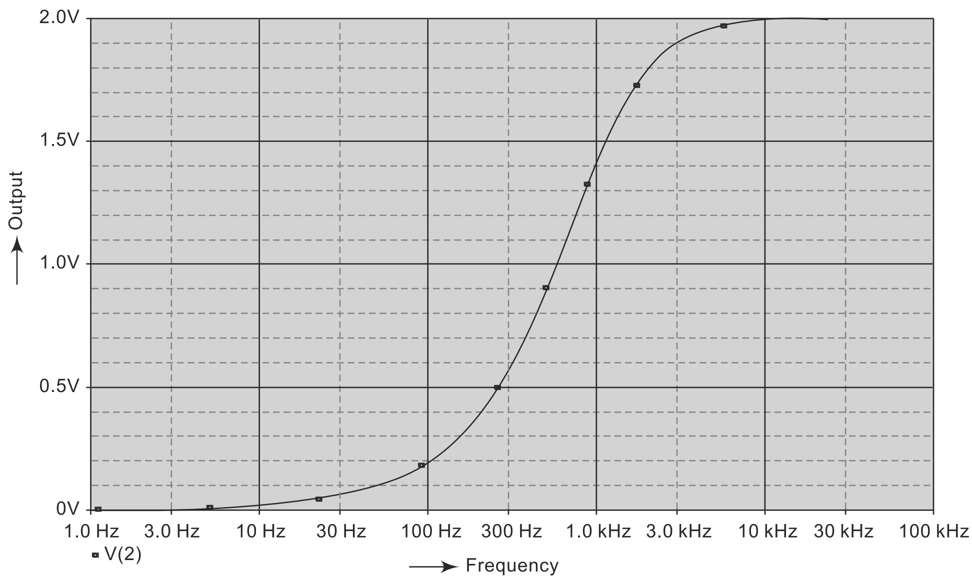


Figure 8.12(b) *Filter characteristics of Example (8.2)*

Frequency (Hz)	*Gain*	*Gain (in dB)*
100	0.20	–14.02
200	0.39	–8.13
400	0.74	–2.58
700	1.15	1.19
1,000	1.41	3.01
3,000	1.90	5.56
7,000	1.98	5.93
10,000	1.99	5.98
30,000	2	6.02
100,000	2	6.02

8.8 BAND–PASS ACTIVE FILTER

A band-pass filter has a pass-band between two cut-off frequencies f_{ce} (lower cut-off frequency) and f_{cu} (upper cut-off frequency) such that $f_{cu} > f_{cl.}$ Any input frequency outside this pass-band is attenuated.

8.8.1 Bandwidth (BW)

The range of frequency between f_{CL} and f_{CU} is called the bandwidth.

$$\text{BW} = (f_{CU} - f_{CL})$$

The bandwidth is not exactly centered on the resonant frequency (f_r).
If f_{CU} and f_{CL} are known, the resonant frequency can be found from,

$$f_r = \sqrt{f_{CL} \cdot f_{CU}}$$

If 'f_r' and BW are known, cut-off frequencies are found from,

$$f_{cl} = \left(\sqrt{\left(\frac{\text{BW}}{2}\right)^2 - f_r^2}\right) - \left(\frac{\text{BW}}{2}\right)$$

$$f_{cu} = (f_{cl} + \text{BW})$$

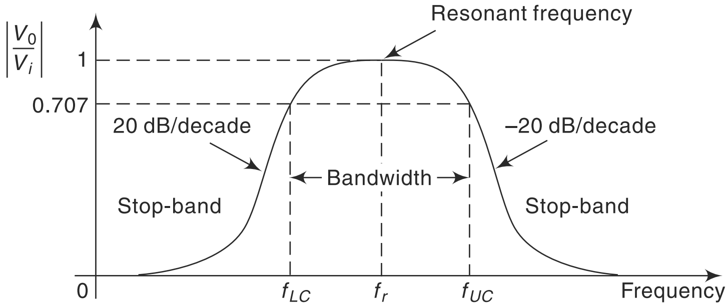


Figure 8.13 *Band pass filter characteristics*

8.8.2 Quality Factor (Q)

It is defined as the ratio of resonant frequency to bandwidth, i.e., $Q = \dfrac{f_r}{\text{BW}}$

Q is a measure of the selectivity. Higher the value of Q, the more selective is the filter, i.e., narrower is the bandwidth.

Example 8.3 A band-pass voice filter has lower and upper cut-off frequencies of 300 and 3000 Hz, respectively. Find (a) Bandwidth, (b) The resonant frequency, (c) The quality factor.

Solution

(a) $\text{BW} = (f_{CU} - f_{CL}) = (3000 - 300) = 2700$ Hz *Ans.*

(b) $f_r = \sqrt{f_{CL} f_{CU}} = \sqrt{300 \times 3000} = 950$ Hz *Ans.*

(c) $\text{Q} = \dfrac{f_r}{\text{BW}} = \dfrac{950}{2700} = 0.35$ *Ans.*

[**Note** f_r is below the centre frequency $\dfrac{300+3000}{2} = 1650$ Hz]

Example 8.4 A band-pass filter has a resonant frequency of 950 Hz and a bandwidth of 2700 Hz. Find its lower and upper cut-off frequencies.

Solution

$$f_{CL} = \left(\sqrt{\left(\frac{\text{BW}}{2}\right)^2 + f_r^2}\right) - \left(\frac{\text{BW}}{2}\right)$$

$$= \left(\sqrt{\left(\frac{2700}{2}\right)^2 + (950)^2}\right) - \left(\frac{3700}{2}\right) = (1650 - 1350)$$

$= 300$ Hz *Ans.*

$\therefore \quad f_{cu} = (300 + 2700) = 3000$ Hz

8.8.3 Types of Band–Pass Filters

1. **Wide Band Pass Filter** wide-band filter has a bandwidth that is two or more times the resonant frequency; i.e., $Q \leq 0.5$.
   It is made by cascading a low-pass and a high-pass filter circuit.
2. **Narrow Band Pass Filter** A narrow band filter has a quality factor, $Q > 0.5$.
   It is made by using a single op-amp and multiple feed back circuits.

Wide Band- Pass Active Filter In general, a wide-band filter ($Q \leq 0.5$) is made by cascading a low-and a high-pass filter, provided the cut-off frequency of the low-pass section is greater than that for the high-pass section.

Characteristics

(i) The cut-off frequency of low-pass filter should be 10 or more times the cut-off frequency of the high-pass filter.
(ii) Each section should have the same pass band gain.
(iii) The lower cut-off frequency, f_{cl}, will be determined only by the high-pass filter.
(iv) The higher cut-off frequency, f_{cu}, will be determined only by the low-pass filter.
(v) Gain will be maximum at the resonant frequency, f_r, and equal to the pass-band gain of either filter.

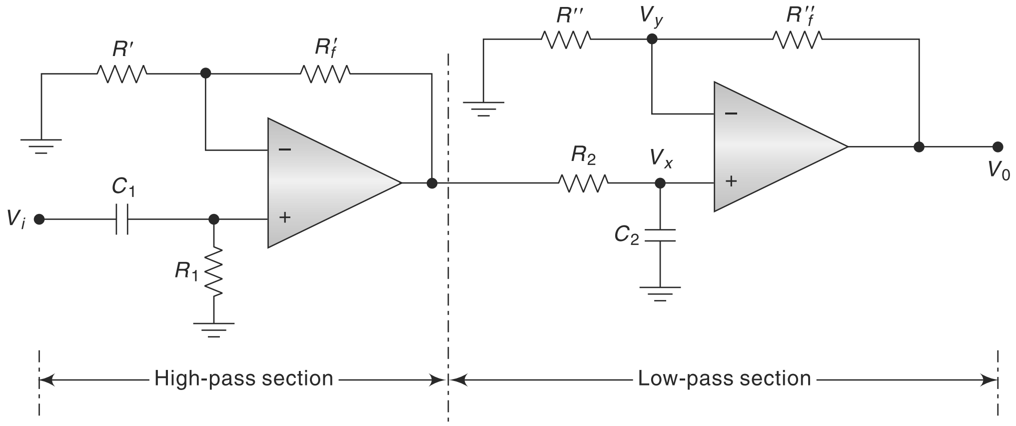


Figure 8.14(a) *Wide band-pass active filter circuit*

Frequency Response

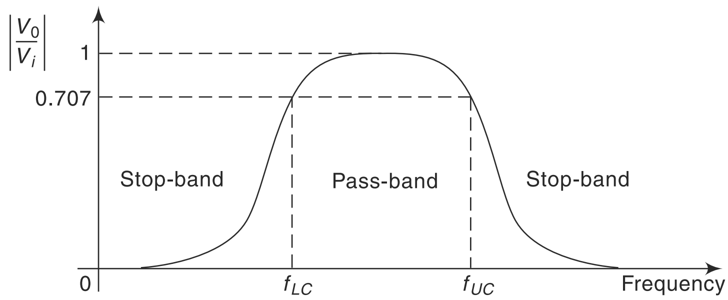


Figure 8.14(b) *Frequency response of wide band-pass active filter circuit*

Here, $f_{CL} = \dfrac{1}{2\pi R_1 C_1}$, $f_{CU} = \dfrac{1}{2\pi R_2 C_2}$

The Voltage gain magnitude of the band–pass filter is equal to the product of the voltage gain magnitudes of the high-pass and the low-pass filters.

$$\boxed{\therefore \quad \left|\frac{V_0}{V_i}\right| = \frac{A_{FL} A_{FH} (f/f_{CL})}{\sqrt{[1+(f/f_{CL})^2]} \cdot \sqrt{[1+(f/f_{CU})^2]}}}$$

Where, A_{FL}, A_{FH} = Pass-band gain of low-pass and high-pass filter;

f = frequency of input signal (Hz);

f_{cl} = lower cut-off frequency (Hz);

f_{cu} = higher cut-off frequency (Hz);

At the centre frequency, $f_r \left(= \sqrt{f_{CL} f_{CU}}\right)$, the Gain is, $\left|\frac{V_0}{V_i}\right| = K = A_{FL} A_{FH} \frac{f_{CU}}{f_{CL} + f_{CU}}$

At $f = f_{CL}$, $$\left|\frac{V_0}{V_i}\right| = \frac{A_{FL} A_{FH} (f_{CL}/f_{CL})}{\sqrt{[1+(f_{CL}/f_{CL})^2][1+(f_{CL}/f_{CU})^2]}} = \frac{A_{FL} A_{FH}}{\sqrt{(2)[1+(f_{CL}/f_{CU})^2]}}$$

$$\left|\frac{V_0}{V_i}\right| = \frac{A_{FL} A_{FH} f_{CU}}{\sqrt{2}\sqrt{f_{CL}^2 + f_{CU}^2}}$$

At $f = f_{CU}$, $$\left|\frac{V_0}{V_i}\right| = \frac{A_{FL} A_{FH} (f_{CU}/f_{CL})}{\sqrt{(2)[1+(f_{CL}/f_{CL})^2]}} = \frac{A_{FL} A_{FH} f_{CU}}{\sqrt{2}\sqrt{f_{CL}^2 + f_{CU}^2}}$$

At $f = f_{CL} = f_{CU}$, Gain, $$\left|\frac{V_0}{V_i}\right| = \frac{A_{FL} A_{FH}}{\sqrt{2}} \left[\frac{f_{CU}}{\sqrt{f_{CL}^2 + f_{CU}^2}}\right]$$

$$\Rightarrow \qquad \frac{V_0}{V_i} = \frac{A_{FL} A_{FH}}{2}$$

Narrow Band-pass Active Filter In general, a narrow band-pass filter is made by using multiple feedback circuit with a single op-amp.

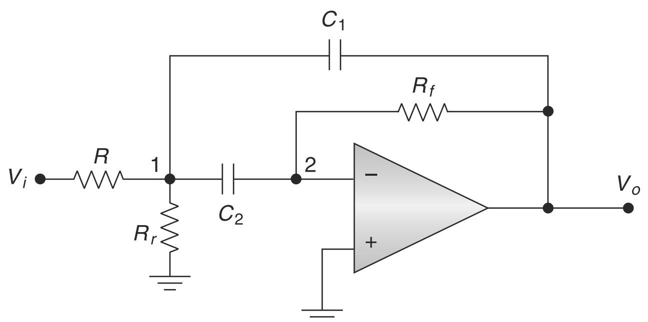


Figure 8.15 *Multiple feedback narrow BP active filter*

Compared to all other filters, it has some unique features, as given below.

(i) It has two feedback paths, hence the name 'multiple feedback filter'.

(ii) The op-amp is used in the inverting mode.

(iii) Its centre frequency can be changed without changing the gain or bandwidth.

Performance Equations

Writing KCL at (1)

$$\frac{(V_1 - V_i)}{R} + \frac{V_1 - V_0}{1/sC_1} + \frac{V_1 - 0}{1/sC_2} + \frac{V_1}{R_r} = 0$$

or $\quad (V_1 - V_i)\, R_r + (V_1 - V_0)\, sR_rRC_1 + V_1sRRrC_2 + V_1R = 0$

or $$V_1 = \frac{V_iR_r + V_0sRR_rC_1}{R + R_r + sRR_r(C_1 + C_2)} \tag{8.5}$$

Again, writing KCL at (2),

$$\frac{0 - V_0}{R_f} + \frac{0 - V_1}{1/sC_2} = 0$$

or $\quad V_0 = -V_1\, sR_f\, C_2$

$$= -\left[\frac{V_iR_r + V_0sRR_rC_1}{R + R_r + sRR_r(C_1 + C_2)}\right] sR_fC_2 \text{ \{by the value of } V_1 \text{ from (8.5)\}}$$

or $\quad V_0\, [R + R_r + sRR_r(C_1 + C_2) + s_2RR_rR_f\, C_1C_2] = -VisRrR_f\, C_2$

$\therefore$ $$\frac{V_0}{V_i} = -\frac{sR_rR_fC_2}{s^2RR_rR_fC_1C_2 + sRR_r(C_1 + C_2) + R + R_r}$$

So, the gain,

$$\boxed{\frac{V_0}{V_i} = -\frac{(s/RC_1)}{s^2 + s\left(\dfrac{C_1 + C_2}{R_fC_1C_2}\right) + \dfrac{R + R_r}{RR_rR_fC_1C_2}}}$$

The general transfer function is of the form,

$$\frac{V_0}{V_i} = -\frac{s\left(\dfrac{\bar{\omega}_r}{Q}\right)}{s^2 + s\left(\dfrac{\omega_r}{Q}\right) + \omega_r^2} = -\frac{s(\text{BW})A_F}{s^2 + s(\text{BW}) + \omega_r^2}, \text{ where, } A_F = \text{Gain}$$

So, here, $\text{BW} = \left(\dfrac{C_1 + C_2}{R_fC_1C_2}\right) \times \dfrac{1}{2\pi} \text{(in Hz)} \;\{\because \omega = 2\pi f\}$

With matched capacitor, i.e., $C_1 = C_2 = C$

$$\text{BW} = \frac{1}{\pi R_fC} \quad \Rightarrow \quad R_f = \frac{Q}{\pi f_rC}$$

Also, $$(\text{BW})A_F = \frac{1}{RC_1} = \frac{1}{RC} \mid \text{with } C_1 = C_2 = C$$

$\therefore \quad R = \dfrac{1}{(BW)CA_F} = \dfrac{Q}{\omega_r CA_F} \Rightarrow R = \dfrac{Q}{2\pi f_r CA_F}$

$$\text{BW} = \frac{1}{2\pi RCA_F} \text{ Hz}$$

Similarly, $\omega_r^2 = \dfrac{R + R_r}{RR_r R_f C^2}$ | with $C_1 = C_2 = C$

or, $4\pi^2 f_r^2 \times RR_r R_f C^2 = R + R_r$

or $\quad 4\pi^2 f_r^2 \times \dfrac{Q}{2\pi f_r CA_F} \times R_r R_f C^2 = \dfrac{Q}{2\pi f_r CA_F} + R_r$ [Putting the value of R_f]

or $\quad 2\pi f_r Q R_r R_f C = \dfrac{Q}{2\pi f_r C} + R_r A_F$

or $\quad 2\pi Q f_r R_r \times \dfrac{Q}{\pi f_r C} \times C = \dfrac{Q}{2\pi f_r C} + R_r A_F$ [Putting the value of R_f]

or $\quad R_r\left[2Q^2 - A_F\right] = \dfrac{Q}{2\pi f_r C}$

$\therefore \quad R_r = \dfrac{Q}{2\pi f_r C(2Q^2 - A_F)} = R\left(\dfrac{A_F}{2Q^2 - A_F}\right)$

Also, $\quad \dfrac{R_f}{R} = \dfrac{Q}{\pi f_r C} \times \dfrac{2\pi frCA_F}{Q} = 2A_F$

$\therefore \quad A_F = \dfrac{R_f}{2R}$. So, the gain is a maximum of 1 at f_r if $R_f = 2R$

However, the gain must satisfy the condition, $A_F < 2Q^2$.

So, the narrow-band-pass active filter is designed for specific values of resonant frequency f_r and Q (or, f_r and BW) by using the relations.

$$R = \frac{Q}{2\pi f_r CA_F}, \quad R_f = \frac{Q}{\pi f_r C}, \quad R_r = \frac{Q}{2\pi f_r C(2Q^2 - A_F)}, \quad A_F = \frac{R_f}{2R} \tag{8.6}$$

$$\text{BW} = \frac{f_r}{Q} = \frac{1}{2\pi RCA_F}(\text{Hz}) = \frac{0.1591}{A_F RC}(\text{H}_\text{Z}) \text{ and} \tag{8.7}$$

$$f_r = \frac{1}{2\sqrt{2}RC}\sqrt{\frac{1}{A_F}\left(1 + \frac{R}{R_r}\right)} = \frac{0.1125}{RC}\sqrt{\frac{1}{A_F}\left(1 + \frac{R}{R_r}\right)} \tag{8.8}$$

Note: The resonant frequency can be changed to a frequency f_r' without changing the gain or BW, by, changing R_r to a new value R_r' = so that, $R_r' = R_r\left(\dfrac{f_r}{f_r'}\right)^2$.

Example 8.5 (a) Design a wide band-pass filter with f_{CL} = 200 Hz and f_{CU} = 1kHz, and a pass-band gain = 4.

(b) Draw the frequency response plot of this filter.

(c) Calculate the value of Q for the filter.

Solution (a) To design the low-pass section:

$$f_c = 1 \text{ kHz}$$

Let, C_2 = 0.01 μF, $R_2 = \dfrac{1}{2\pi f_c C_2} = 15.9\ \text{k}\Omega$

To design the high-pass section:

$$f_c = 200 \text{ Hz}$$

Let, C_1 = 0.05 μF, $R_1 = \dfrac{1}{2\pi f_c C_1}$ = 15.9 kΩ

Since the band-pass gain is 4, the gain of both HP and LP sections could be set equal to 2.

$$\therefore \qquad 2 = \left(1 + \frac{R_f'}{R'}\right) = \left(1 + \frac{R_f''}{R''}\right) \quad \Rightarrow \quad R_f' = R_f'' = R' = R'' = 10\ \text{k}\Omega$$

(b) The frequency response will be as shown below.

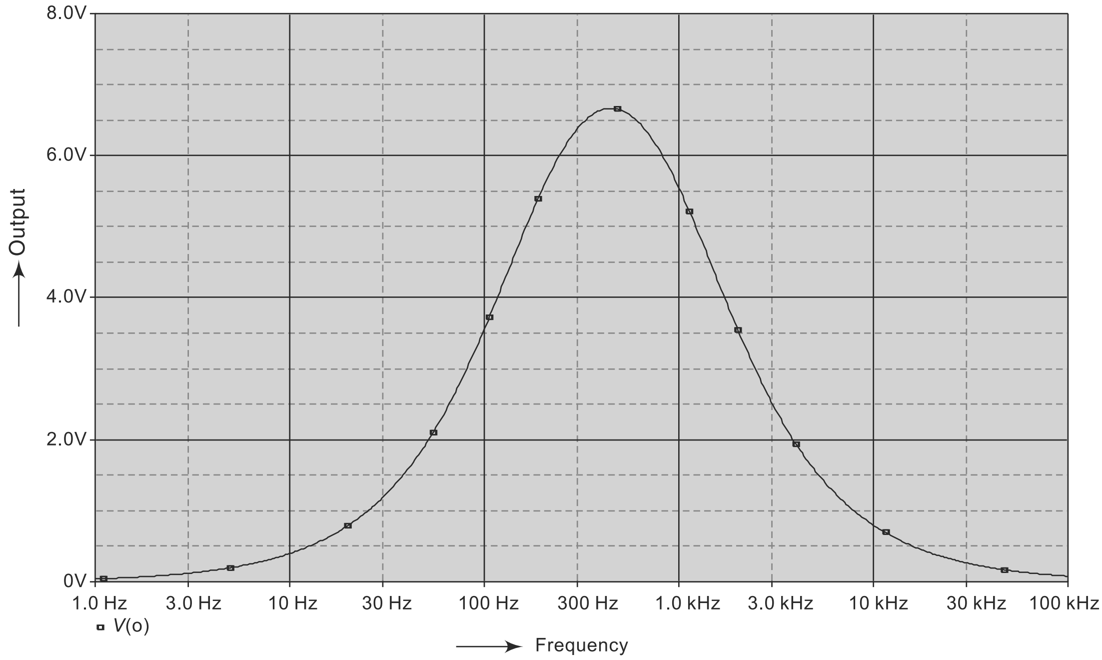


Figure 8.16 *Frequency response of Example (8.5)*

(c) Resonant frequency, $f_r = \sqrt{200 \times 1000} = 447.2$ Hz

So, the quality factor, $Q = \dfrac{f_r}{\text{BW}} = \dfrac{447.2}{(1000 - 200)} = 0.56$

Example 8.6 (a) Design a narrow band-pass filter with resonant frequency f_r = 1 kHz, Q = 3, and A_F = 10.

(b) Change the resonant frequency to 1.5 kHz, keeping A_F and the bandwidth constant.

Solution (a) Let, $C_1 = C_2 = 0.01$ μF

$$R_f = \frac{Q}{\pi f_r C} = \frac{3}{\pi \times 10^3 \times 10^{-8}} = 95.5 \text{ k}\Omega\,;$$

$$R = \frac{Q}{2\pi f_r C A_F} = \frac{3}{2\pi \times 10^3 \times 10^{-8} \times 10} = 4.77 \text{ k}\Omega$$

$$R_r = \frac{Q}{2\pi f_r C(2Q^2 - A_F)} = \frac{4.77 \times 10}{(2 \times 9 - 10)} = 5.97 \text{ k}\Omega$$

(b) To change the resonant frequency, the resistance value will be,

$$R' = 5.97 \times 10^3 \times \left(\frac{1}{1.5}\right) = 3.98 \text{ k}\Omega$$

The frequency response is shown below.

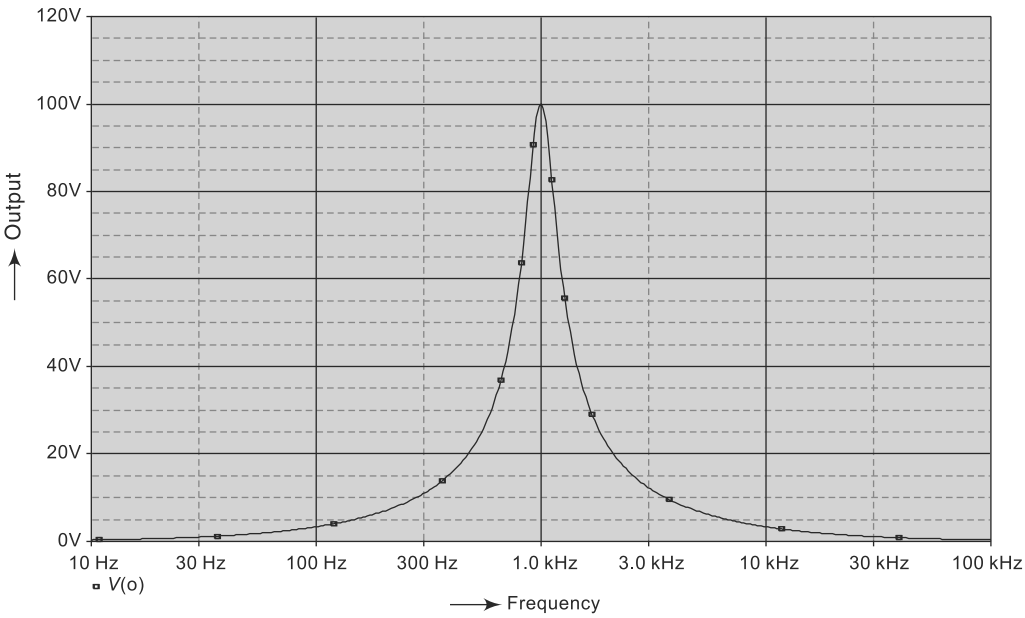


Figure 8.17 *Frequency response of Example (8.6)*

Example 8.7 A band-pass filter has the component values, $R = 21.12$ kΩ, $R_f = 42.42$ kΩ, $R_r = 3.03$ kΩ and $C_1 = C_2 = 0.015$ μF. Find the resonant frequency and the bandwidth.

Solution Here, since $R_f = 2R$, so, $A_F = 1$.

$$\therefore \quad f_r = \frac{0.1125}{RC}\sqrt{\frac{1}{A_F}\left(1+\frac{R}{R_r}\right)} = \frac{0.1125}{21.21\times 10^3 \times 0.015\times 10^{-6}}\sqrt{1+\frac{21.21}{3.03}} \cong 1000 \text{ Hz}$$

$$\text{BW} = \frac{0.1591}{A_F RC} = \frac{0.1591}{1\times 21.21\times 10^3\times 0.015\times 10^{-6}} \cong 500 \text{ Hz}$$

8.9 BAND- REJECT (NOTCH) ACTIVE FILTER

1. It may be obtained by the parallel connection of a high-pass section with a low-pass section. The cut-off frequency of the high-pass section must be greater than that of the low-pass section.

   The outputs of HP and LP sections are fed to an adder whose output voltage V_0 will have the notch filter characteristics.

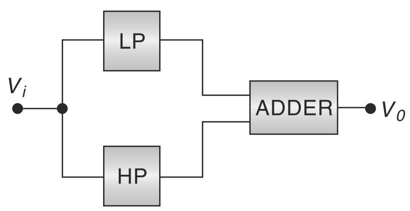


Figure 8.18(a) *Block diagram of BR filter*

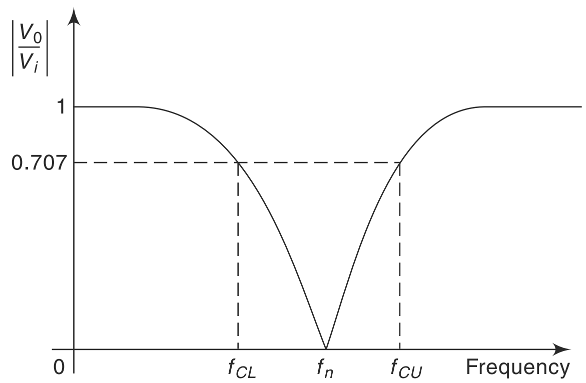


Figure 8.18(b) *Frequency response of band reject filter*

The circuit of the BR filter is shown in Fig. 8.19.

Obviously, the gain of the adder is set at unity; and thus,

$$V_0 = \left(\frac{V_0'}{R_2}+\frac{V_0''}{R_3}\right)R_4 \quad \Rightarrow \quad R_2 = R_3 = R_4$$

and $$R_{OM} = R_2 \| R_3 \| R_4$$

So, $$V_0 = A_{FH}\left[\frac{j(f/f_{CH})}{1+j(f/f_{CH})}\right] + A_{FL}\left[\frac{1}{1+j(f/f_{CL})}\right]$$

If $A_{FL} = A_{FH} = A$, then at the center frequency, $f_r = \sqrt{f_{CL} f_{CH}}$, the Gain is $K = A\cdot\dfrac{2f_{CL}}{f_{CL}+f_{CH}}$

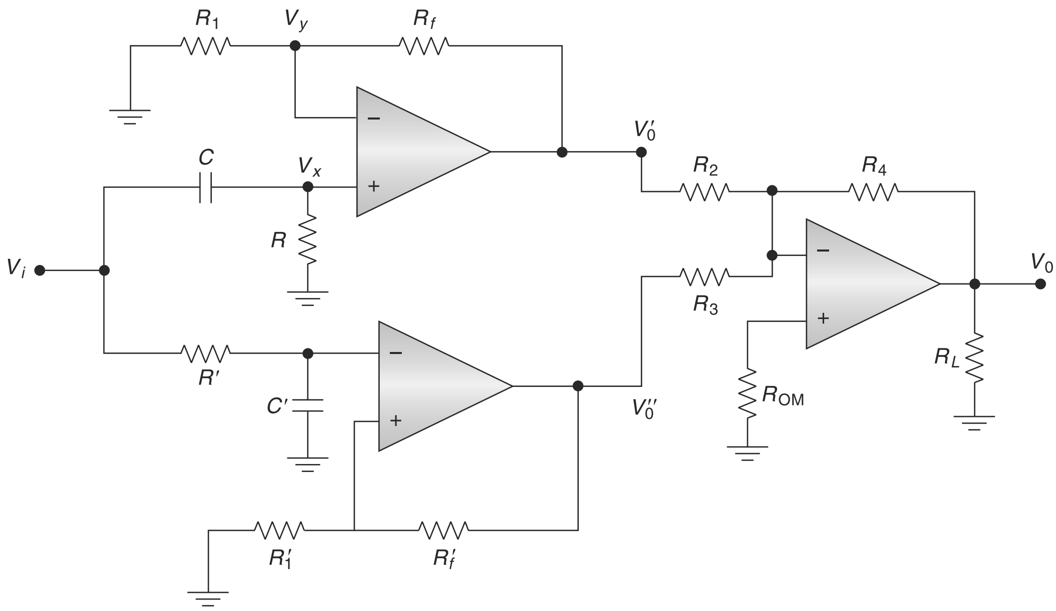


Figure 8.19 *Band reject active filter circuit using parallel connection of high pass and low pass filters*

2. Band-reject filter may also be obtained by using the multiple-feedback band-pass filter circuit with an adder. That is, the notch filter is made by a circuit that subtracts the output of a band pass filter from the original signal.

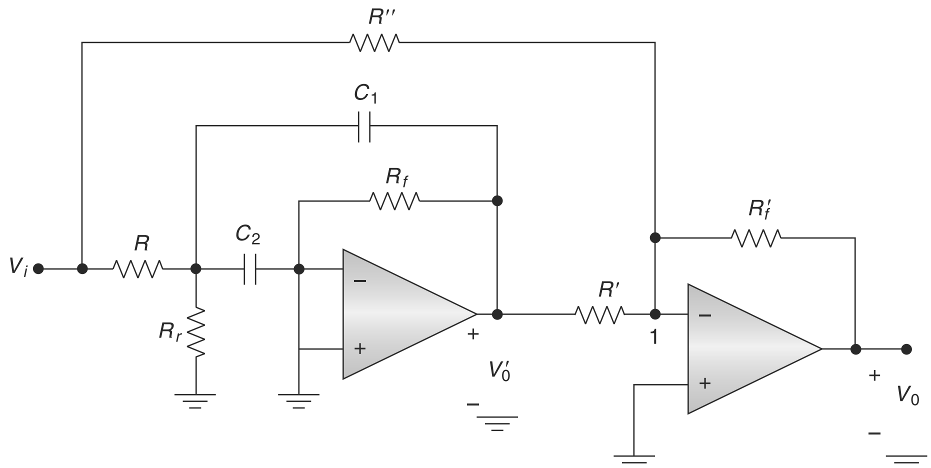


Figure 8.20 *Band reject active filter circuit using multiple feedback band pass filter with an adder*

So, $$\frac{V_0'}{V_i} = -\frac{(s/RC_1)}{s^2 + s\left(\dfrac{C_1 + C_2}{R_f C_1 C_2}\right) + \dfrac{R + R_r}{RR_r R_f C_1 C_2}} = T(s)$$

Now, writing KCL at (1),

$$\frac{V_0'}{R'} + \frac{V_0}{R_f'} + \frac{V_i}{R''} = 0$$

$$\Rightarrow \qquad V_0 = -R_f'\left(\frac{V_1'}{R''} + \frac{V_0'}{R'}\right)$$

$$= -R_f' V_i \left[\frac{1}{R''} + \frac{T(s)}{R'}\right]$$

At notch frequency, the output is zero (ideally).

So, $\qquad T(s) = -\dfrac{R'}{R''}$

But, at ω_n (or f_n), $T(s) = -A_F$ (A_F = gain of the BP section)

With $C_1 = C_2$, Gain for BP section, $A_F = \dfrac{R_f}{2R}$

$$\boxed{\therefore A_F = \frac{R_f}{2R} = \frac{R'}{R''}}$$

So, the design equations are all those of BP section and this one.

Example 8.8 Design a notch filter having a resonant frequency, $f_r = 400$ Hz and $Q = 10$. Make the resonant frequency gain, $A_F = 2$.

Solution Here, $f_r = 400$ Hz, $Q = 10$, $A_F = 2$

Let, $C = 0.1$ µF

$$\therefore \qquad R = \frac{Q}{2\pi frCA_F} = \frac{10}{2\pi \times 400 \times 0.1 \times 10^{-6} \times 2} = 19.89\text{ k}\Omega \qquad \textit{Ans.}$$

$$\therefore \qquad R_f = \frac{Q}{\pi frC} = \frac{10}{\pi \times 400 \times 0.1 \times 10^{-6}} = 79.58\text{ k}\Omega$$

$$\therefore \qquad R_r = \frac{RA_F}{2Q^2 - A_F} = \frac{19.89 \times 2 \times 10^3}{200 - 2} = 202\ \Omega$$

Let, $R' = 1$ kΩ (arbitrary) $= R_f'$

$$R'' = \frac{R'}{A_F} = 500\ \Omega \qquad \textit{Ans.}$$

8.9.1 Applications of Notch Filters

Notch filter is used where unwanted frequencies are to be attenuated while permitting the other signal frequencies to pass through.

For examples, 50 Hz, 60 Hz, or 400 Hz frequencies from power lines, ripple from a full-wave rectifiers, etc.

Example 8.9 Design an active notch filter to eliminate 120 Hz hum (noise). Take the bandwidth, BW = 12 Hz.

Solution Hare, f_r = 120 Hz, BW = 12Hz, $Q = \frac{120}{12} = 10$

The gain of the filter in the pass-band will be maximum of 1,

$AF = 1.$

Let , $C_1 = C_2 = 0.1\ \mu F$

$$R = \frac{10}{2\pi \times 120 \times 0.1 \times 10^{-6} \times 1} = 132.66\ k\Omega$$

$$R_f = 2R = 265.32\ k\Omega$$

$$R_r = \frac{R}{200 - 1} = 663.3\ k\Omega$$

Now, let $R' = R'_f = 1\ k\Omega$ (arbitrary)

So, $$R'' = \frac{R'}{A_F} = 1k\ \Omega$$

Thus the filter will pass all frequencies from (0 – 114) Hz and 126 Hz onwards.

8.10 FILTER APPROXIMATION

In the earlier sections, we saw several examples of amplitude response curves for various filter types. These always included an "ideal" curve with a rectangular shape, indicating that the boundary between the pass-band and the stop-band was abrupt and that the roll-off slope was infinitely steep. This type of response would be ideal because it would allow us to completely separate signals at different frequencies from one another. Unfortunately, such an amplitude response curve is not physically realizable. We will have to settle for the best approximation that will still meet our requirements for a given application. Deciding on the best approximation involves making a compromise between various properties of the filter's transfer function, such as, filter order, ultimate roll-off rate, attenuation rate near the cut-off frequency, transient response, ripples, etc.

If we can define our filter requirements in terms of these parameters, we will be able to design an acceptable filter using standard design methods.

8.10.1 Butterworth Filters

The first and probably best-known filter approximation is the Butterworth or maximally-flat response. It exhibits a nearly flat pass-band with no ripple. The roll-off is smooth and monotonic, with a low-pass or high-pass roll-off rate of 20 dB/decade (6 dB/octave) for every pole. Thus, a 5^{th}-order

Butterworth low-pass filter would have an attenuation rate of 100 dB for every factor of ten increase in frequency beyond the cutoff frequency.

The general equation for a Butterworth filter's amplitude response is,

$$H(\omega) = \frac{1}{1 + \left(\dfrac{\omega}{\omega_0}\right)^{2n}} \tag{8.9}$$

where n is the order of the filter, and can be any positive whole number (1, 2, 3,…), and ω_0 is the -3 dB frequency of the filter.

Figure 8.21 shows the amplitude response curves for Butterworth low-pass filters of various orders.

The coefficients for the denominators of Butterworth filters of various orders are shown in Table.

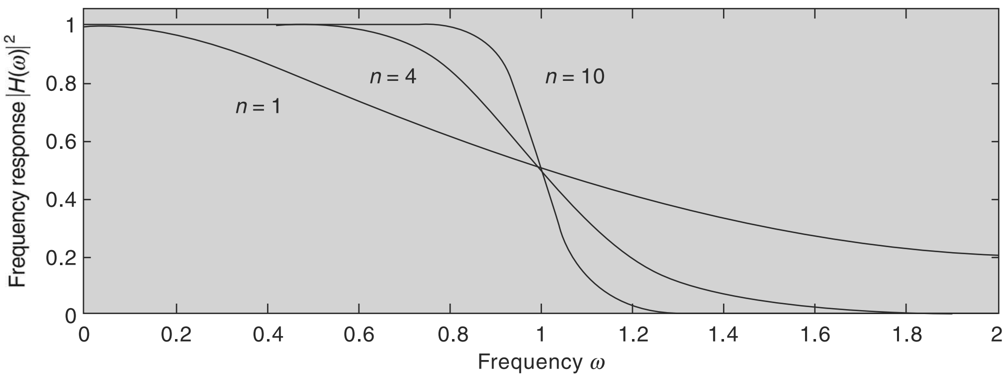


Figure 8.21 *Amplitude response curves for butterworth low-pass filters of different orders*

Table shows the denominators factored in terms of second-order polynomials. Again, all of the coefficients correspond to a corner frequency of 1 radian/s

Table *Butterworth Polynomials*

Denominator coefficients for polynomials of the form $s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1	1									
2	1	1.414								
3	1	2.000	2.000							
4	1	2.613	3.414	2.613						
5	1	3.236	5.236	5.236	3.236					
6	1	3.864	7.464	9.142	7.464	3.864				
7	1	4.494	10.098	14.592	14.592	10.098	4.494			
8	1	5.126	13.137	21.846	25.688	21.846	13.137	5.126		
9	1	5.759	16.582	31.163	41.986	41.986	31.163	16.582	5.759	
10	1	6.392	20.432	42.802	64.882	74.233	64.882	42.802	20.432	6.392

Butterworth Quadratic Factors

n	
1	$(s-1)$
2	$(s^2 - 1.4142s - 1)$
3	$(s-1)(s^2 - s - 1)$
4	$(s^2 - 0.7654s - 1)(s^2 - 1.8478s - 1)$
5	$(s-1)(s^2 - 0.6180s - 1)(s^2 - 1.6180s - 1)$
6	$(s^2 - 0.5176s - 1)(s^2 - 1.4142s - 1)(s^2 - 1.9319)$
7	$(s-1)(s^2 - 0.4450s - 1)(s^2 - 1.2470s - 1)(s^2 - 1.8019s - 1)$
8	$(s^2 - 0.3902s - 1)(s^2 - 1.1111s - 1)(s^2 - 1.6629s - 1)(s^2 - 1.9616s - 1)$
9	$(s-1)(s^2 - 0.3479s - 1)(s^2 - 1.0000s - 1)(s^2 - 1.5321s - 1)(s^2 - 1.8794s - 1)$
10	$(s^2 - 0.3129s - 1)(s^2 - 0.9080s - 1)(s^2 - 1.4142s - 1)(s^2 - 1.7820s - 1)(s^2 - 1.9754s - 1)$

8.10.2 Second Order Low-pass Active Filter

The circuit is shown in Figure 8.22.

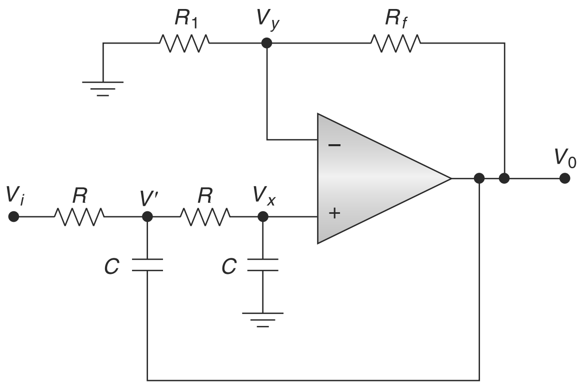


Figure 8.22 *Second order low-pass active filter*

Here, $V_y = \dfrac{V_0}{R_1 + R_f} R_1$ and $V_x = V_y$

Writing KCL at node V',

$$\frac{V' - V_i}{R} + \frac{V' - V_0}{1/sC} + \frac{V' - V_x}{R} = 0$$

or $$(V' - V_i) + (V' - V_0)sRC + (V' - V_x) = 0$$

or $$(-1)V_x + (2 + sRC)V' + (-sRC)V_0 = V_i \quad (8.10)$$

Writing KCL at node x,

$$\frac{V_x - V'}{R} + \frac{V_x}{1/sC} = 0$$

or $$(1 + sRC)V_x + (-1)V' + (0)V_0 = 0 \quad (8.11)$$

Writing KCL at node y,

$$\frac{V_x}{R_1} + \frac{V_x - V_0}{R_f} = 0$$

or $$(R_1 + R_f)V_x + (0)V' + (-R_1)V_0 = 0 \tag{8.12}$$

Solving for V_0 from equations (8.10), (8.11), and (8.12), we get,

$$V_0 = \frac{\begin{vmatrix} -1 & (2+sRC) & V_i \\ (1+sRC) & -1 & 0 \\ (R_1+R_f) & 0 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & (2+sRC) & -sRC \\ (1+sRC) & -1 & 0 \\ (R_1+R_f) & 0 & -R_1 \end{vmatrix}} = V_i \frac{\dfrac{(R_1+R_f)}{R_1}}{s^2 + 3sRC - sRC\left(\dfrac{(R_1+R_f)}{R_1}\right) + 1}$$

or, $$\frac{V_0(s)}{V_i(s)} = \frac{K}{s^2 + s\left(\dfrac{3-K}{RC}\right) + \left(\dfrac{1}{RC}\right)^2} \tag{8.13}$$

where, $K = \dfrac{R_1 + R_f}{R_1}$ = DC gain of the amplifier.

Substituting $s = j\omega$, the transfer function is,

$$H(j\omega) = \frac{V_0(j\omega)}{V_i(j\omega)} = \frac{K}{1 + j(3-K)RC\omega - R^2C^2\omega^2}$$

The magnitude of the transfer function is,

$$|H(j\omega)| = \frac{K}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_c}\right)^2\right]^2 + [3-K]^2\left(\dfrac{\omega}{\omega_c}\right)^2}}\;; \text{ where, } \omega_c = \frac{1}{RC}$$

In the above equation, when $\omega \to 0$, $|H(j\omega)| = K$. Thus, the low frequency gain of the filter is K and when $\omega \to \infty$, $|H(j\omega)| = 0$, i.e., high frequency gain is zero.

From the Table of the Butterworth Filter, the transfer function for second order ($n = 2$) filter is,

$$T(s) = \frac{K}{\left(\dfrac{s}{\omega_c}\right)^2 + 1.414\left(\dfrac{s}{\omega_c}\right) + 1} = \frac{K\omega_c^2}{s^2 + 1.414\omega_c s + \omega_c^2} \tag{8.14}$$

where, ω_c is the cut-off frequency. Comparing equations (8.13) and (8.14), we get,

$$\boxed{\omega_c = \frac{1}{RC} \quad \text{or,} \quad f_c = \frac{1}{2\pi RC}} \quad \text{and,} \quad \boxed{K = (3 - 1.414) = 1.586}$$

The frequency response of a second order low-pass active filter is shown in Figure 8.23. It is noted that the filter has very sharp roll-off response.

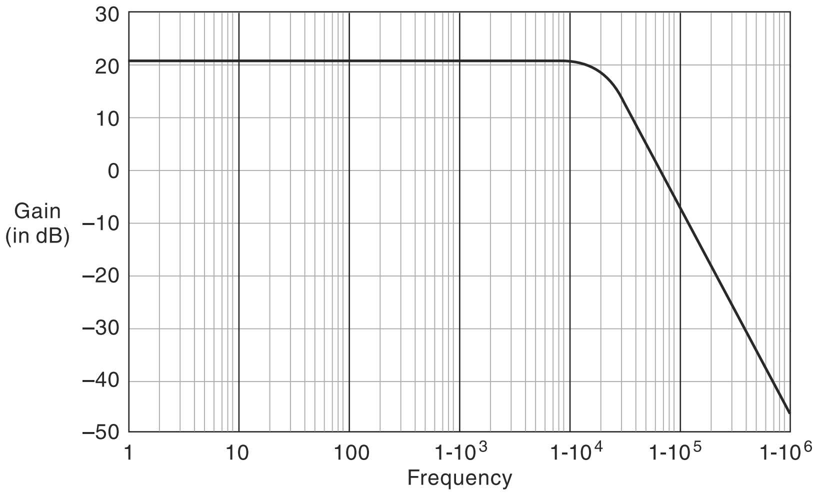


Figure 8.23 *Frequency response of the second order low-pass filter*

Filter Design

1. Choose a value of the cut-off frequency, ω_c (or f_c).
2. Select a convenient value for the capacitors C, between 100pF and 0.1µF.
3. Calculate the value of the resistors R from the relation,
$$R = \frac{1}{2\pi f_c C}$$
4. For minimization of dc offset, the feedback resistor is calculated from the relation, $R_f = K\,(2R) = 3.172R$.
5. Calculate the value of the resistor R_1 for the value of the gain $K = 1.586$ from the relation, $K = \frac{R_1 + R_f}{R_1}$, i.e., $R_1 = \frac{R_f}{0.586}$.

Example 8.10 Design a second-order low-pass filter with a gain of 11 and cut-off frequency of 20 kHz.

Solution Let us arbitrarily select $C = 200$ pF.

For a cut-off frequency of 20 kHz, we need $R = \frac{1}{2\pi f_c C} = \frac{1}{2\pi \times 20 \times 10^3 \times 200 \times 10^{-12}}$

$= 39.789$ kΩ

If we select a standard resistor of 39 kΩ for R, then the cut-off frequency is about 20.4 kHz.

The dc gain for this filter cannot be anything other than K where $K = 1.586$.

Thus, for a dc gain of 1.586, $K = 1 + R_f/R1 = 1.586$.

This in turn implies that $R_f = 0.586\,R_1$.

Imposing the dc bias-current balance condition, we obtain

$0.586\,R_1 = 1.586\,(2\,R) = 123.708$ kΩ.

Consequently, $R_1 = 211.11$ kΩ and $R_f = 123.708$ kΩ.

Let us select a standard value of 130 kΩ for R_f. Then R_1 should be about 221.8 kΩ.

We need another amplifying stage to obtain the needed gain of 11. The gain of this stage should be $11/K = 6.936$. We have chosen to use non-inverting amplifier for this stage. The output amplifier resistors are calculated as,

$$6.936 = \left(1 + \frac{R_2}{R_A}\right) \text{ and for } R_A = 100 \text{ k}\Omega., R_2 = 593.6 \text{ k}\Omega.$$

Thus, the final circuit for the second order low-pass active filter becomes as shown below.

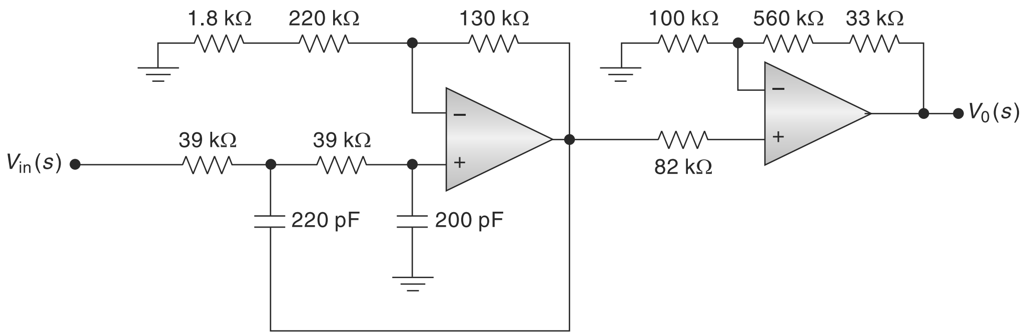


Figure 8.24 *Circuit of Example (8.10)*

8.10.3 Second Order High Pass Active Filter

The circuit is shown in Figure 8.25.

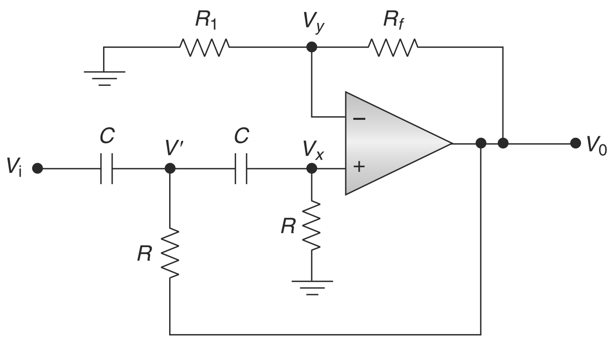


Figure 8.25 *Second order high-pass filter*

Here, $V_y = \dfrac{V_0}{R_1 + R_f} R_1$ and $V_x = V_y$

Writing KCL at node V',

$$\frac{V' - V_i}{1/sC} + \frac{V' - V_0}{R} + \frac{V' - V_x}{1/sC} = 0 \tag{8.15}$$

Writing KCL at node x,

$$\frac{V_x - V'}{1/sC} + \frac{V_x}{R} = 0 \tag{8.16}$$

Writing KCL at node y,

$$\frac{V_x}{R_1} + \frac{V_x - V_0}{R_f} = 0 \tag{8.17}$$

Solving for V_0 from equations (8.15), (8.16), and (8.17), we get,

$$\text{or,}\ \frac{V_0(s)}{V_i(s)} = \frac{Ks^2}{s^2 + s\left(\frac{3-K}{RC}\right) + \left(\frac{1}{RC}\right)^2} \tag{8.18}$$

where, $K = \dfrac{R_1 + R_f}{R_1}$ = DC gain of the amplifier.

Note The transfer function of the high-pass filter can also be obtained from the transfer function of the low-pass filter by the transformation $\left.\left(\dfrac{s}{\omega_c}\right)\right|_{LP} \to \left.\left(\dfrac{\omega_c}{s}\right)\right|_{HP}$

Substituting $s = j\omega$, the transfer function is,

$$H(j\omega) = \frac{V_0(j\omega)}{V_i(j\omega)} = -\frac{KR^2C^2\omega^2}{1 + j(3-K)RC\omega - R^2C^2\omega^2}$$

The magnitude of the transfer function is,

$$|H(j\omega)| = \frac{K\left(\dfrac{\omega}{\omega_c}\right)^2}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_c}\right)^2\right]^2 + [3-K]^2\left(\dfrac{\omega}{\omega_c}\right)^2}}\ ;\ \text{where,}\ \omega_c = \frac{1}{RC}$$

In the above equation, when $\omega \to 0$, $|H(j\omega)| = 0$. Thus, the low frequency gain of the filter is zero. When $\omega \to \infty$, $|H(j\omega)| = K$, i.e., high frequency gain is K.

Here, again, comparing with Butterworth Transfer function, we get,

$$\boxed{\omega_c = \frac{1}{RC} \quad \text{or,} \quad f_c = \frac{1}{2\pi RC}}$$

$$\boxed{K = (3 - 1.414) = 1.586}$$

The frequency response of a second order low-pass active filter is shown below. It is noted that the filter has very sharp roll-off response.

The design procedure for high-pass will be same as low-pass.

The frequency response will be a maximally flat one, i.e., having a very sharp roll-off response.

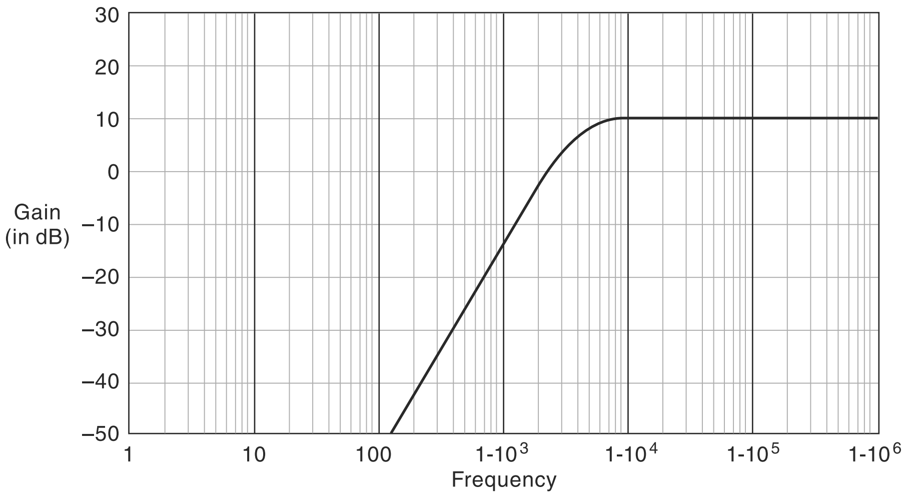


Figure 8.26 *Gain vs. frequency plot of a second-order high-pass filter*

Example 8.11 A second-order high-pass filter is given in Figure 8.27. Determine its cut-off frequency and high frequency gain. Sketch its gain vs. frequency response.

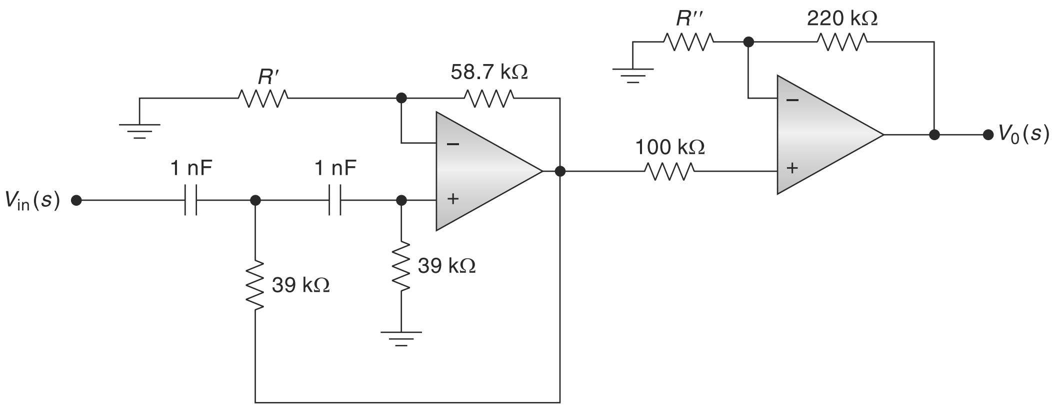


Figure 8.27 *Circuit of Example (8.11)*

Solution In the second-order filter on the left side of the figure, the gain $\boldsymbol{K} = \left(1 + \frac{58.7}{100}\right) = 1.587.$

Since it is very close to 1.586, we can assume that the filter is maximally flat and its transfer function is as given for Butterworth filters. From the given values of R and C, the cut-off frequency is,

$$\omega_c = \frac{1}{RC} = \frac{1}{39 \times 10^3 \times 1 \times 10^{-9}} = 25{,}641 \text{ rad/s}$$

The cut-off frequency in Hz, $f_c = \dfrac{25,641}{2\pi} = 4081$ Hz

The gain of the non-inverting amplifier, $A = \left(1 + \dfrac{220}{220}\right) = 2$

Hence, the overall gain of the high-pass filter is,

$$A_H = 1.587 \times 2 = 3.174 \text{ or approximately 10 dB.}$$

The gain vs. frequency will be as shown in Figure 8.26.

8.10.4 Second Order Band-Pass Active Filter

It can be built by the cascade connection of a second order high-pass and a second order low-pass filter.

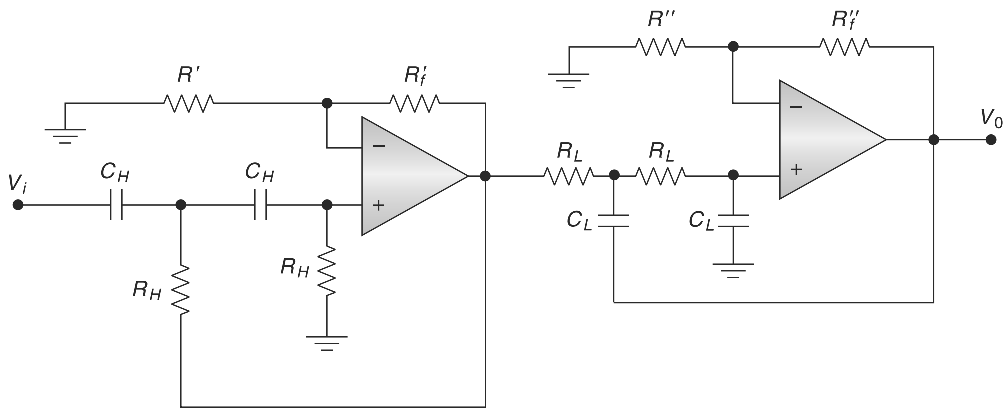


Figure 8.28 *Second-order band-pass active filter circuit*

Lower cut-off frequency, $\omega_1 = \dfrac{1}{R_H C_H}$

Upper cut-off frequency, $\omega_2 = \dfrac{1}{R_L C_L}$

Voltage gains, $K_H = \left[1 + \dfrac{R'_f}{R'}\right]$ and $K_L = \left[1 + \dfrac{R''_f}{R''}\right]$

For maximally flat response (or, Butterworth) filter, $K_H = K_L = 1.586$.

$$\therefore \qquad \frac{R'_f}{R'} = \frac{R''_f}{R''} = 0.586$$

The overall transfer function is the product of the transfer function of the high-pass and low-pass filters.

$$\therefore \qquad H(s) = \frac{K_H\left(\dfrac{s}{\omega_1}\right)}{1 + \left(\dfrac{s}{\omega_1}\right)^2 + (3 - K_H)\left(\dfrac{s}{\omega_1}\right)} \times \frac{K_L}{1 + \left(\dfrac{s}{\omega_2}\right)^2 + (3 - K_L)\left(\dfrac{s}{\omega_2}\right)}$$

Substituting the values of K_H and K_L, magnitude of the gain is,

$$|H(j\omega)| = \frac{2.5154\left(\frac{\omega}{\omega_1}\right)^2}{\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^4}\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^4}}$$

Note In the pass-band, the gain is 2.5154.
The frequency response is more flat near the cut-off frequencies.

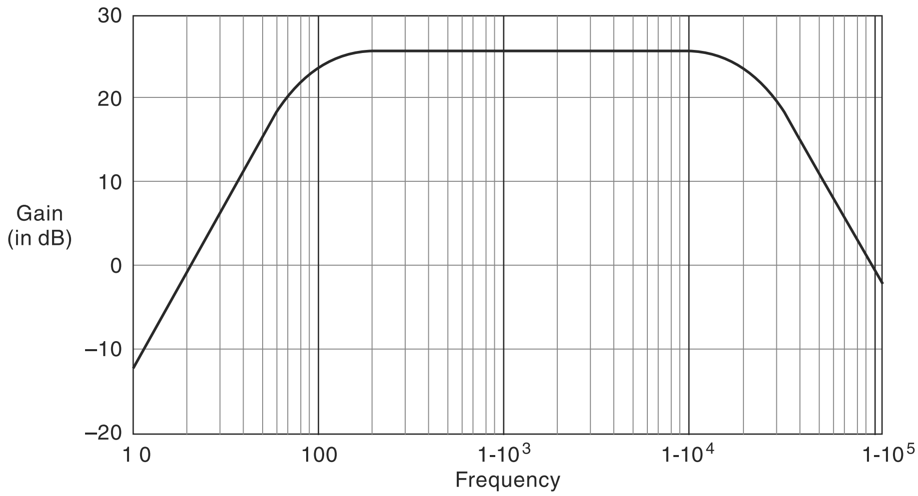


Figure 8.29 *Frequency response of second order band-pass filter*

8.10.5 Second Order Band-Reject Active Filter

It can be built by the summation of a second order high-pass and a second order low-pass filter.

The cut-off frequency of LPF, $\omega_1 = \frac{1}{R_L C_L}$ and the cut-off frequency of HPF, $\omega_2 = \frac{1}{R_H C_H}$.

The magnitude of the overall transfer function is the sum of the transfer function of the high-pass and low-pass filters,

$$|H(j\omega)| = \frac{1}{2}\left[1+\frac{R_2}{R_1}\right]\left[\frac{K_H\left(\frac{\omega}{\omega_2}\right)^2}{\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^4}} + \frac{K_L}{\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^4}}\right]$$

where, $K_H = \left(1+\frac{R'_f}{R'}\right)$ and, $K_L = \left(1+\frac{R''_f}{R''}\right)$ and for Butterworth filters, $K_H = K_L = 1.586$.

The roll-off frequency response will be very smooth as shown.

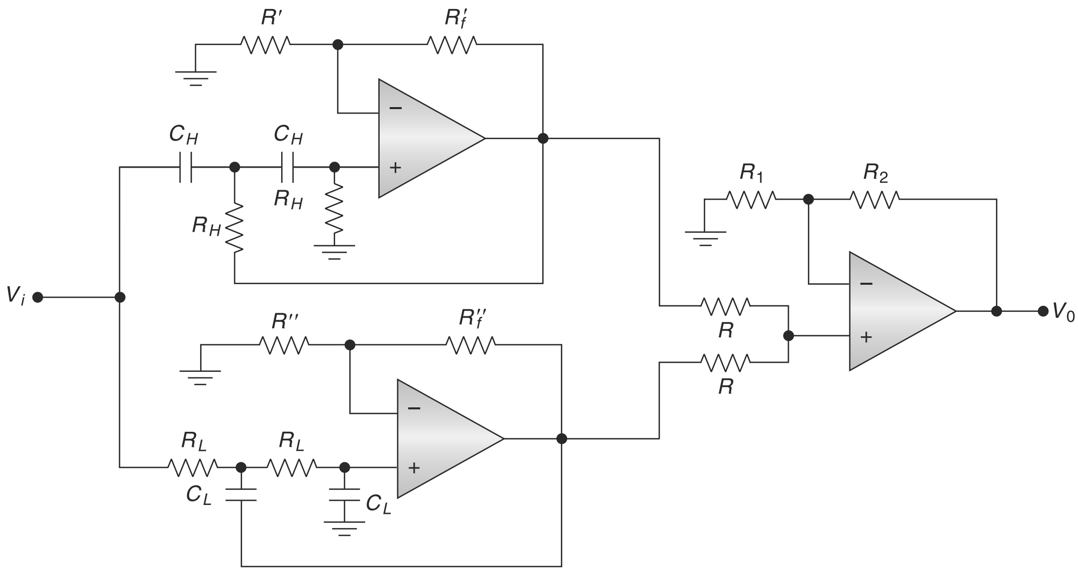


Figure 8.30 *Second-order band-reject active filter circuit*

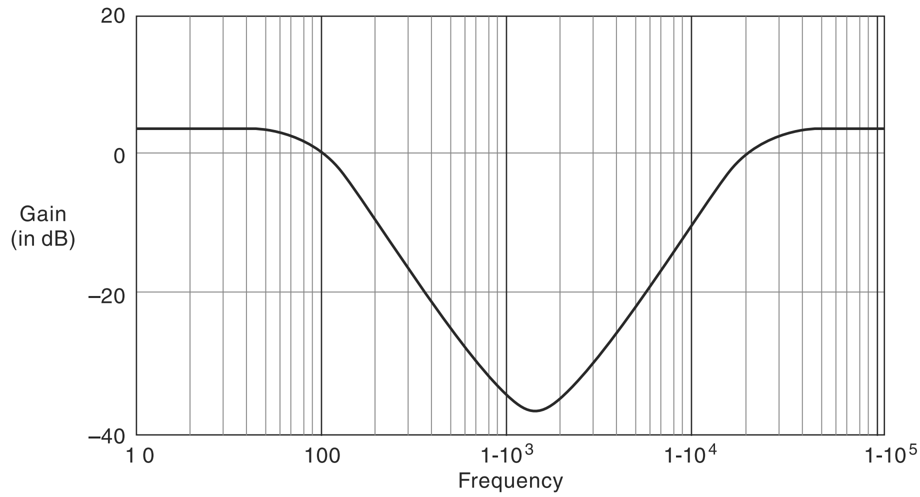


Figure 8.31 *Frequency response of second order band-reject filter*

8.11 ALL-PASS ACTIVE FILTER

This filter passes all frequency component of the input signal without attenuation and provides some phase shifts between the input and output signals.

The circuit of an active all-pass active filter with lagging output is shown in Figure 8.32.

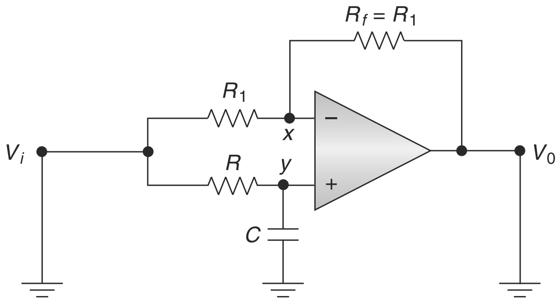


Figure 8.32 *Circuit of an all-pass active filter with lagging output*

For the circuit, by KCL at node x,

$$\frac{V_x - V_i}{R_1} + \frac{V_x - V_0}{R_1} = 0 \quad \Rightarrow \quad V_x = \frac{V_i + V_0}{2} \tag{8.19}$$

By KCL at node y,

$$\frac{V_y - V_i}{R} + \frac{V_y}{1/j\omega C} = 0 \quad \Rightarrow \quad V_y = \frac{V_i}{1 + j\omega RC} \tag{8.20}$$

Also, from Op-Amp property,

$$V_x = V_y$$

$$\Rightarrow \quad \left(\frac{V_i + V_0}{2}\right) = \left(\frac{V_i}{1 + j\omega RC}\right)$$

$$\Rightarrow \quad (V_i + V_0)(1 + j\omega RC) = 2V_i$$

$$\Rightarrow \quad V_0(1 + j\omega RC) = V_i[2 - (1 + j\omega RC)] = V_i(1 - j\omega RC)$$

$$\boxed{\therefore \frac{V_0}{V_i} = \frac{1 - j\omega RC}{1 + j\omega RC}}$$

Thus, the amplitude of the gain,

$$\left|\frac{V_0}{V_i}\right| = 1 \quad \text{i.e.,} \quad |V_{\text{out}}| = |V_{\text{in}}| \text{ throughout the entire frequency range}$$

Also, the phase shift between the input and the output voltages is,

$\phi = -2 \tan^{-1} (\omega RC)$ i.e., phase-shift is a function of frequency

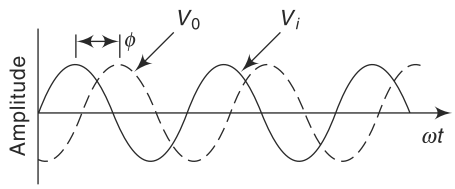


Figure 8.33 *Characteristics of all-pass filter*

By interchanging the positions of R and C in the circuit, the output can be made leading the input.

MULTIPLE-CHOICE QUESTIONS

8.1 The two input terminals of an op-amp are labeled as
(a) high and low (b) positive and negative
(c) inverting and non-inverting (d) differential and non-differential

8.2 Consider the following statements for an ideal op-amp.
1. The differential voltage across the input terminals is zero.
2. The current into the input terminals is zero.
3. The current from the output terminals is zero.
4. The input resistance is zero.
5. The output resistance is zero.

Of these statements, those which are not true are
(a) 1 and 5 (b) 3 and 4 (c) 2 and 4 (d) 1 and 4

8.3 In a series resonant circuit, to obtain a low-pass characteristic, across which element should the output voltage be taken?
(a) Resistor (b) Inductor (c) Capacitor

8.4 In a series resonant circuit, to obtain a high-pass characteristic, across which element should the output voltage be taken?
(a) Resistor (b) Inductor (c) Capacitor

8.5 In a series resonant circuit, to obtain a band-pass characteristic, across which element should the output voltage be taken?
(a) Resistor (b) Inductor (c) Capacitor

8.6 A high-pass filter circuit is basically
(a) a differentiating circuit with low time constant.
(b) a differentiating circuit with large time constant.
(c) an integrating circuit with low time constant.
(d) an integrating circuit with large time constant.

8.7 The transfer function of an electrical low-pass RC network is
(a) $\dfrac{RCs}{1+RCs}$ (b) $\dfrac{1}{1+RCs}$ (c) $\dfrac{RC}{1+RCs}$ (d) $\dfrac{s}{1+RCs}$

8.8 For a high-pass RC circuit, when subjected to a unit step input voltage, the voltage across the capacitor will be
(a) $1-e^{-t/RC}$ (b) $e^{-t/RC}$ (c) $e^{t/RC}$ (d) 1

8.9 In the magnitude plot of a low-pass filter, at what frequency does the peak of the magnitude characteristic occur?
(a) At resonant frequency (b) Below resonant frequency
(c) Above resonant frequency (d) At any frequency.

8.10 In the magnitude plot of a high-pass filter, at what frequency does the peak of the magnitude characteristic occur?
(a) At resonant frequency (b) Below resonant frequency
(c) Above resonant frequency (d) At any frequency.

8.11 In the magnitude plot of a band-pass filter, at what frequency does the peak of the magnitude characteristic occur?
(a) At resonant frequency (b) Below resonant frequency
(c) Above resonant frequency (d) At any frequency.

8.12 If a filter is de-normalized to a higher frequency, which of the following occurs?
(a) Inductors increase in value while capacitors decrease.
(b) Inductors decrease in value while capacitors increase.
(c) Inductors and capacitors increase in value.
(d) Inductors and capacitors decrease in value.

8.13 The transfer function $\frac{V_2(s)}{V_1(s)} = \frac{10s}{s^2 + 10s + 100}$ is for an active
(a) low pass filter (b) band pass filter (c) high pass filter (d) all pass filter.

8.14 The transfer function $T(s) = \frac{s^2}{s^2 + as + b}$ belongs to an active
(a) low pass filter (b) high pass filter (c) band pass filter (d) band reject filter.

8.15 The voltage-ratio transfer function of an active filter is given by $\frac{V_2(s)}{V_1(s)} = \frac{s^2 + \delta}{s^2 + \alpha s + \delta}$. The circuit in question is a
(a) low pass filter (b) high pass filter (c) band pass filter (d) band reject filter.

EXERCISES

8.1 Design a second order low pass active filter having a cut-off frequency of 5 kHz.
[C = 0.03 mF; R = 1 kΩ; R_1 = 10 kΩ; R_2 = 5.86 kΩ]

8.2 Design a second order band pass active filter that has a centre frequency of 1 kHz and a bandwidth of 100 Hz. Take the centre frequency gain to be 2.
[$C_1 = C_2$ = 0.02 mF; R_1 = 40 kΩ; R_3 = 160 kΩ; R_2 = 400 Ω]

8.3 Design a second order high pass Butterworth filter with a cut-off frequency of 200 Hz.
[C = 0.053 mF; R = 1.5 kΩ; R_1 = 10 kΩ; R_2 = 5.86 kΩ]

8.4 Design a second order band pass active filter with a centre frequency gain A_0 = 50. Given: f_0 = 160 Hz and Q = 10. [assuming $C_1 = C_2$ = 0.1 mF; R_1 = 2 kΩ; R_3 = 200 kΩ; R_2 = 667 Ω]

SHORT-ANSWER TYPE QUESTIONS

8.1 (a) What is an operational-amplifier? State the characteristics of an op-amp.
(b) What is filter? Classify them.
(c) Discuss the advantages of an active filter over a passive filter.

8.2 (a) Briefly discuss the operating principle of an active low-pass filter and derive its gain-frequency characteristics. Explain the design procedure of a low-pass active filter.
(b) Briefly discuss the operating principle of an active high-pass filter and derive its gain-frequency characteristics. Explain the design procedure of a high-pass active filter.

8.3 (a) Define the following terms with reference to a band-pass active filter: -
(i) Bandwidth,
(ii) Cut-off frequency,
(iii) Quality factor.

(b) What are the different types of band-pass filters? Give the salient features and performance equations for the following filters: -
  (i) Wide Band-Pass Active Filter,
  (ii) Narrow Band-Pass Active Filter.

8.4 Define Notch-frequency. Explain the operational characteristics of an active Notch filter. Where are these filters used?

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

8.1 (c)	8.2 (b)	8.3 (c)	8.4 (b)	8.5 (a)	8.6 (a)	8.7 (b)
8.8 (a)	8.9 (b)	8.10 (c)	8.11 (a)	8.12 (d)	8.13 (c)	8.14 (b)
8.15 (c)						

CHAPTER 9

Network Synthesis

9.1 INTRODUCTION

While switching over from the spectrum of network analysis that has been carried out satisfactorily in the preceding chapters to the sphere of network synthesis to be dealt in this chapter, one has to be very assertive about the facts and findings as the principles undergo a paradigm shift in the process. Yet, one thing that remains common to both 'network analysis' and 'network synthesis' is the network. As regards to the most striking difference,

- In case of network analysis, the network remains defined and the response characteristic is determined (as an output parameter) subject to an impressed excitation function that remains predefined and is better known as the input parameter.
- On the contrary, in that of the case of network synthesis, the input parameter (that is the impressed excitation function) and the output parameter (that is the response characteristic) remain declared and accordingly a quest for a possible network formulation is performed with help of various mathematical tools.

The following table would serve as a supplement for a quick reference for the notable disclosures of the preceding paragraphs.

Table 9.1 *Highlights of Network Analysis and Network Synthesis.*

Item under Study	*Known Parameters*	*Parameters to be Evaluated*
Network Analysis	• Network structure or Network Function • Input parameter or Excitation Function	Output parameter or Response Function
Network Synthesis	• Output parameter or Response Function • Input parameter or Excitation Function	Network structure or Network Function

The lexicon meaning of synthesis is synonymous to production, creation, making, manufacturing or formulation of a system to meet a specific purpose in a desirable manner. In view of this, network synthesis also simply and directly mean the formulation of a possible electrical network by realizing the most competitive passive elements, which would satisfy the specific correspondence between the given set of excitation and response functions in a desirable and conducive manner. There could be lots of logistics involved in reaching the desired goal (starting from the hard core mathematical approach through a series of practice involving hit and trial to the soft core computer simulation techniques available as on date), which poses a real challenge for the designer or synthesizer in order to establish the findings as optimal.

9.2 ELEMENTS OF NETWORK SYNTHESIS

By this point the meaning and objectives of network synthesis might have created some impression in the mind of the reader in a broad sense. However, it requires furthermore explanation in the true light of the necessity and requirements associated in this process, which make network synthesis so appealing in the area of network theory. In addition to the information supplied in the preceding section (Table 9.1) it is mandatory in network synthesis that while formulating or synthesizing a possible electrical network through realization of the passive elements (R, L and C) and establishment of a suitable connectivity (series or parallel connection) for them the following requirements must not be ignored. This requirement emphasizes that (i) the correspondence between input excitation and output response should be satisfactory as desirable and (ii) the system performance should remain stable. This fact explores the introduction to three most important elements of network synthesis, which are described as (i) Causality, (ii) Stability, and (iii) Realizability. While, causality describes the possibility of a cause for the process under study, stability ensures systematic and bound performance of the process. In as much so realizability describes the possibility of finding a set of passive network elements and a possible connectivity for synthesizing the required, may be complex network.

Sometimes, it may so happen that, a number of possibilities get explored in the process of network synthesis for a given and specific set of input excitation function and output response function and each of them might show a justifiable stance for selection. Therefore, the synthesizer has to be often judicious and apply his/her intuition and experience in selecting the optimal result of network synthesis.

Causality is merely a certification of the cause of a fact in the merit of the findings, which is of course the foundation of the studies related to science. As we better know, no current would ever flow between a pair of terminals/nodes of an electrical network even having well established electrical connectivity between them without mere existence of a potential difference across the said pair of nodes/terminals. This fact verifies the causality and certifies that potential difference or electric pressure is causal for flow of electric current in a network. A dual of this statement is also true and may state that without a flow of electric current in a conductor or through any part of an electrical network, no electric potential difference can exist across the ends of the said conductor or across any pair of points of the said network. This fact too verifies the causality and certifies that flow of electric current is causal for existence of potential difference across the ends of a conductor or a pair of points of a network.

By principle, a causal function is realizable if it satisfies that the amplitude function maintains at least a non zero magnitude over a finite band of frequencies in the frequency domain operation. It is also possible that this criterion may be interpreted in a different perspective as regards to stability of the system operation. The well known stability criteria states that, for a stable network, a bounded excitation as input function must produce a bounded response as output function. These facts are thus considered as the foundation of network synthesis as they equally highlight the importance of causality, stability and realizability.

Figure 9.1 shows the block diagrammatic representation of a two port network having a network function of $G(s)$ in the s plane. Assuming that an input excitation of $R(s)$ impressed on the network results in an output response function represented as $C(s)$, the relationship that holds good for the said network in the frequency domain is given in Laplace transformation form as shown in Eqn. (9.1). In order to have a time domain analysis of the network, the inverse of Laplace transform may be performed over the excitation function and response function, which would result in $r(t)$ and $c(t)$ respectively.

Figure 9.1 *A two port network block showing excitation r(t) and response c(t)*

$$G(s) = \frac{C(s)}{R(s)} \tag{9.1}$$

The necessary and sufficient conditions of stability of the network should ensure that for a bounded excitation $r(t)$, there must exist a well bounded response $c(t)$ for all times. Mathematically, this may be interpreted as, "Given a stable network $G(s)$, a bounded excitation function expressed as $|r(t)| < A_1$ for $0 \le t \le \infty$, when impressed on the network at the input should produce a bounded response at the output which may be expressed as $|c(t)| < A_2$ for $0 \le t \le \infty$, where A_1 and A_2 represent positive, real and finite numbers".

Thus, the figure of merit as observed from the stability criterion of the above paragraph rests over a quest for a positive and real function (p.r. function) for both excitation and response functions of the network which remains finite over a considerable stretch of time frame of operation, so that these functions remain bounded thereby yielding stability for the network. The stability criterion for electrical networks can also be examined in view of the location of the poles of the network function $G(s)$ in the s plane. If the electrical network happens to be complex in form it may have higher order terms in the numerator polynomial and denominator polynomial as well. In order to justify this let us express the network function of Eqn. (9.1) as the ratio of two polynomials as shown in Eqn. (9.2) being represented through $N(s)$, the numerator polynomial and $D(s)$, the denominator polynomial.

$$G(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots\ldots\ldots + a_2 s^2 + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \ldots\ldots\ldots + b_2 s^2 + b_1 s + b_0} \tag{9.2}$$

Clearly, the terms in the above expression have their usual meaning;

a_i : the ith coefficients of the numerator polynomial $N(s)$,

n : highest order or degree of the numerator polynomial $N(s)$,

b_i : the ith coefficients of the denominator polynomial $D(s)$,

m : highest order or degree of the denominator polynomial $D(s)$.

Now, the necessary and sufficient condition for the network to remain stable can be laid down in view of the characteristics of the rational network function $G(s)$ in the light of the statements given below;

1. All the poles of the network function $G(s)$ must be restricted to the left half of the s plane as shown clearly by the shaded region of Fig. 9.2.

2. Poles of the network function $G(s)$, if any, present on the $j\omega$ axis must be simple poles, or otherwise, no multiple poles be placed on the $j\omega$ axis.
3. The order of numerator polynomial should be higher than the order of denominator polynomial just by unity.

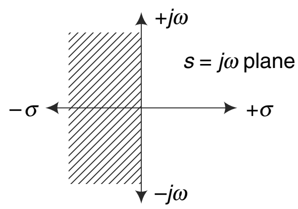


Figure 9.2 *Left half of the s plane shown shaded*

Since these conditions are mostly governed by the nature and location of the poles of the network function and poles being the characteristic roots of the denominator polynomial of the network function, it becomes mandatory to examine the poles thoroughly. The most convenient way of determining the poles of the network function $G(s)$, it is highly desirable that factorization of the denominator polynomial $D(s)$ be performed and it should be expressed as shown in Eqn. (9.3).

$$\begin{aligned} D(s) &= b_m s^m + b_{m-1} s^{m-1} + \dots \dots \dots \dots + b_1 s + b_0 \\ &= (s + p_1)(s + p_2)(s + p_2) \dots \dots \dots \dots \dots (s + p_m) \end{aligned} \tag{9.3}$$

It is also possible to establish another set of conditions for the parameters of Eqn. (9.3) in order to satisfy the stability criteria as detailed earlier. A careful observation of Eqn. (9.3) would reveal that the conditions laid down under stability criteria are also contained in the coefficients and roots of the denominator polynomial, which may be expressed as;

(i) The coefficients of the denominator polynomial should have non-zero value.
(ii) The roots of the denominator polynomial should have negative real parts.
(iii) Only one root of the denominator polynomial may have a zero value.
(iv) If a root of the denominator polynomial is found to be complex, it must have a conjugate with a negative real part.

A polynomial satisfying these four conditions (i) through (iv) is called a Hurwitz polynomial.

9.3 HURWITZ POLYNOMIAL

Hurwitz polynomial is a special type of polynomial that has numerous applications in solving problems related to network analysis, network synthesis, control systems, stability analysis, optimization related studies and many more. Hence it is presented in this section with some more details. Let us now declare a polynomial of n^{th} order as shown in Eqn. (9.4), which on factorization takes the shape of Eqn. (9.5).

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots \dots + a_i s^i + \dots \dots + a_1 s + a_0 \tag{9.4}$$

$$P(s) = (s_1 + p_1)(s_2 + p_2)(s_3 + p_3) \dots \dots (s_i + p_i) \dots \dots (s_n + p_n) \tag{9.5}$$

The four requirements as indicated through points (i)-(iv) of the preceding section need to be satisfied for any polynomial to be Hurwitz. These requirements may be compressed and expressed as a set of two basic requirements for any polynomial to qualify as Hurwitz polynomial as may be seen in this section. Therefore, a polynomial $P(s)$ is said to be Hurwitz, if it satisfies the following two basic requirements.

I. $P(s)$ must be real and positive for all real values of s. This criterion satisfies the requirement of item (i) of the four conditions indicated in the preceding section.

II. The roots of $P(s)$ must have real parts, which are either zero or negative. If any complex root is present, it should occur as a conjugate pair. In view of this, the probable form of the roots may be given as;

$s_i = -\alpha_i$ (root having negative real part and zero imaginary part), or

$s_i = +j\beta_i$ (root having zero real part and conjugate imaginary part), or

$s_i = -\alpha_i + j\beta_i$ (root having negative real part and conjugate imaginary part).

This criterion satisfies the requirements of items (ii), (iii) and (iv) of the four conditions indicated in the preceding section.

9.4 HURWITZ TEST ON POLYNOMIALS

In this section we would discuss about two practical approaches, which are very useful in performing Hurwitz test on polynomials in order to access whether a given polynomial qualifies to be considered as Hurwitz by satisfying the two basic conditions (I & II) presented in the preceding section. These approaches are named as (i) Continued fraction expansion method, and (ii) Routh-Hurwitz array method. Now we present them in order of their sequence.

9.4.1 Hurwitz Test by Continued Fraction Expansion

It would be interesting to observe one more property of a Hurwitz polynomial, which demonstrates that while taking the ratio of terms of the said polynomial figuring under the odd series to that of the terms under the even series (or vice versa), by a long division method or continued fraction expansion method, it yields only positive quotient terms. This property may be utilized properly in examining whether a given polynomial qualifies to be Hurwitz. Application of this method for the purpose of conducting the Hurwitz test on polynomials may be explained as presented below.

Step-1: This step helps in separating the odd series from the even series of the given polynomial $P(s)$ and expressing them as an odd series $x(s)$, and an even series $y(s)$. Odd series means the series of all the terms having odd powers of s, and even series means the series of all the terms having even powers of s. In view of this separation, Eqn. (9.4) would result in Eqn. (9.6) and Eqn. (9.7), which represent the odd and even series of terms respectively. Since $P(s) = x(s) + y(s)$, therefore we may separately write them as;

$$x(s) = a_1 s + a_2 s^2 + a_5 s^5 + a_7 s^7 + \ldots \ldots \ldots \ldots \tag{9.6}$$

$$y(s) = a_0 + a_2 s_2 + a_4 s^4 + a_6 s^6 + \ldots \ldots \ldots \ldots \tag{9.7}$$

In this step one has to ensure that there should not be any missing terms in any of the polynomials namely, $P(s)$, $x(s)$, and $y(s)$. If this goes true, then the variation in the highest order of s between the odd series and the even series should differ by unity only.

Step-2: Here we take the ratio of the both series of terms by placing the series with highest power of s in the numerator. In case of power of s being highest in the odd series it would be proper to take the ratio of odd series to that of the even series by placing $x(s)$ in the numerator and $y(s)$ in the denominator as shown in Eqn. (9.8).

$$w(s) = \frac{x(s)}{y(s)} \tag{9.8}$$

On the other hand, if the power of s is observed to be highest in the even series it would be proper to take the ratio of even series to that of the odd series by placing $y(s)$ in the numerator and $x(s)$ in the denominator as shown in Eqn. (9.9).

$$w(s) = \frac{y(s)}{x(s)} \tag{9.8}$$

Assuming that the polynomial satisfies the expression of Eqn. (9.8), the ratio may be taken between $x(s)$ and $y(s)$ as indicated below resulting in a quotient term of $q_1(s)$ and a remainder $r_1(s)$ in the first stage.

$$y(s)\left]\begin{array}{c}\overline{x(s)}\\ \text{----}\\ r_1(s)\end{array}\right[\,q_1(s)$$

The mathematical expression of this division process may be represented as;

$$x(s) = q_1(s),\ y(s) + r_1(s) \tag{9.9}$$

Step-3: In the previous step it may be observed that the order of s in $y(s)$ and $r_1(s)$ differ by unity and the order is higher with $y(s)$. Hence the next stage of division may be carried out between $y(s)$ and $r_1(s)$, which requires an inversion first and division next to it. This has been demonstrated in this step and the division results in a quotient term of $q_2(s)$ and a remainder $r_2(s)$ in the second stage.

$$r_1(s)\left]\begin{array}{c}\overline{y(s)}\\ \text{----}\\ r_2(s)\end{array}\right[\,q_2(s),\quad \text{which leads to Eqn. (9.10).}$$

$$y(s) = q_2(s),\ r_1(s) + r_2(s) \tag{9.10}$$

The process of inversion first and division next to it may be continued in a repeated manner until finally the long division converges. This method is also called as continued fraction expansion. Third stage division results are given in Eqn. (9.11).

$$r_2(s)\left]\begin{array}{c}\overline{r_1(s)}\\ \text{----}\\ r_2(s)\end{array}\right[\,q_2(s),\quad \text{which leads to Eqn. (9.11).}$$

$$r_1(s) = q_2(s),\ r_2(s) + r_2(s) \tag{9.11}$$

n^{th} stage division results:

$$r_{n-1}(s)\left]\frac{r_{n-2}(s)}{\frac{----}{r_n(s)}}\right[q_n(s),\quad \text{which leads to Eqn. (9.12).}$$

$$r_{n-2}(s) = q_n(s),\ r_{n-1}(s) + r_n(s) \tag{9.12}$$

Step-4: In this step the summary of observations may be presented in a mathematical form in order to express the ratio of the long division process in the form of continued fraction expansion as indicated in Eqn. (9.12).

$$w(s) = \frac{x(s)}{y(s)} = q_1(s) + \cfrac{1}{q_2(s) + \cfrac{1}{q_3(s) + \cfrac{1}{q_4(s) + \ldots\ldots + \cfrac{1}{q_n(s)}}}} \tag{9.13}$$

The final inference of this method may be drawn as this;

"In order that a given polynomial $P(s) = x(s) + y(s)$ qualifies as being Hurwitz, it must produce only positive quotient terms [i.e. $q_i(s) > 0$] out of the long division through the process of continued fraction expansion of the ratio $x(s)$: $y(s)$ or $y(s)$: $x(s)$, as the case may be, where $x(s)$ and $y(s)$ represent respectively the odd and even series of terms of the given polynomial $P(s)$".

Example 9.1 Examine whether the polynomial given by $P(s) = s^2 + 2s^2 + 4s + 3$, is Hurwitz by applying continued fraction expansion method.

Solution Given that the polynomial has four terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = s^3 + 4s \quad \text{and} \quad y(s) = 2s^2 + 3$$

Clearly, the highest order of s in $x(s)$ being 3 and the highest order of s in $y(s)$ being 2, they differ by unity and hence there is no missing term in each series. The order of s in $x(s)$ being higher than the order of s in $y(s)$, it is recommended that $x(s)$ be put in the numerator and $y(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{x(s)}{y(s)} = \frac{s^2 + 4s}{2s^2 + 3}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{x(s)}{y(s)} = \left(\frac{1}{2}\right)s + \cfrac{1}{\left(\frac{4}{5}\right)s + \cfrac{1}{\left(\frac{5}{6}\right)s}}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = \frac{1}{2}$$

$$q_2 = \frac{4}{5}$$

$$q_3 = \frac{5}{6}$$

Since all the quotient terms obtained in this process are found to be positive, it may suffice that the given polynomial $P(s) = s^2 + 2s^2 + 4s + 3$, is a Hurwitz polynomial.

Example 9.2 By applying continued fraction expansion method, examine whether the polynomial, $P(s) = s^6 + 2s^5 + 3s^4 + s^3 + 5s^2 + s + 2$, is Hurwitz.

Solution Given that the polynomial has seven terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = 2s^5 + s^3 + s \quad \text{and} \quad y(s) = s^6 + 3s^4 + 5s^3 + 2$$

Clearly, the highest order of s in $y(s)$ being 6 and the highest order of s in $x(s)$ being 5, they differ by unity and hence there is no missing term in each series. The order of s in $y(s)$ being higher than the order of s in $x(s)$, it is recommended that $y(s)$ be put in the numerator and $x(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{y(s)}{x(s)} = \frac{s^6 + 3s^4 + 5s^2 + 2}{2s^5 + s^3 + s}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{y(s)}{x(s)} = \left(\frac{1}{2}\right)s + \cfrac{1}{\left(\frac{4}{5}\right)s + \cfrac{1}{\left(-\frac{25}{26}\right)s + \cfrac{1}{\left(-\frac{169}{255}\right)s + \cfrac{1}{\left(\frac{200}{37}\right)s + \cfrac{1}{\left(\frac{37}{102}\right)s}}}}}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +\frac{1}{2}$$

$$q_2 = +\frac{4}{5}$$

$$q_3 = -\left(\frac{25}{26}\right)$$

$$q_4 = -\left(\frac{169}{255}\right)$$

$$q_5 = +\frac{200}{37}$$

$$q_6 = +\frac{37}{102}$$

Since all the quotient terms obtained in this process are not positive, it may be inferred that the given polynomial $P(s) = s^6 + 2s^5 + 3s^4 + s^3 + 5s^2 + s + 2$, is not a Hurwitz polynomial.

Example 9.3 By applying continued fraction expansion method, examine whether the polynomial, $P(s) = 7s^6 + s^5 + 8s^4 + 2s^3 + 3s^2 + 4s + 5$, is Hurwitz.

Solution Given that the polynomial has seven terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = s^5 + 2s^3 + 4s \quad \text{and} \quad y(s) = 7s^6 + 8s^4 + 3s^3 + 5$$

Clearly, the highest order of s in $y(s)$ being 6 and the highest order of s in $x(s)$ being 5, they differ by unity and hence there is no missing term in each series. The order of s in $y(s)$ being higher than the order of s in $x(s)$, it is recommended that $y(s)$ be put in the numerator and $x(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{y(s)}{x(s)} = \frac{7s^6 + 8s^4 + 3s^2 + 5}{s^5 + 2s^3 + 4s}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{y(s)}{x(s)} = (7)s + \cfrac{1}{(-0.166)s + \cfrac{1}{(2.769)s + \cfrac{1}{(0.186)s + \cfrac{1}{(-2.977)s \cfrac{1}{(0.78)s}}}}}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +7$$
$$q_2 = -0.166$$
$$q_3 = +2.769$$
$$q_4 = +0.186$$
$$q_5 = -2.977$$
$$q_6 = +0.78$$

Since all the quotient terms obtained in this process are not positive, it may be inferred that the given polynomial $P(s) = 7s^6 + s^5 + 8s^4 + 2s^3 + 3s^2 + 4s + 5$, is not a Hurwitz polynomial.

Example 9.4 Examine whether the polynomial given by $P(s) = 3s^3 + s^2 + 4s + 3$, is Hurwitz by applying continued fraction expansion method.

Solution Given that the polynomial has four terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = 3s^2 + 4s \quad \text{and} \quad y(s) = s^2 + 3$$

Clearly, the highest order of s in $x(s)$ being 3 and the highest order of s in $y(s)$ being 2, they differ by unity and hence there is no missing term in each series. The order of s in $x(s)$ being higher than the order of s in $y(s)$, it is recommended that $x(s)$ be put in the numerator and $y(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{x(s)}{y(s)} = \frac{3s^3 + 4s}{s^2 + 3}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{x(s)}{y(s)} = (3)s + \cfrac{1}{\left(-\frac{1}{5}\right)s + \cfrac{1}{\left(-\frac{5}{3}\right)s}}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +3$$

$$q_2 = -\frac{1}{5}$$

$$q_3 = -\frac{5}{3}$$

Since all the quotient terms obtained in this process are not positive, it may suffice that the given polynomial $P(s) = 3s^3 + s^2 + 4s + 3$, is not a Hurwitz polynomial.

Example 9.5 Examine whether the polynomial given by $P(s) = s^3 + 8s^2 + s + 6$, is Hurwitz by applying continued fraction expansion method.

Solution Given that the polynomial has four terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = s^3 + s \quad \text{and} \quad y(s) = 8s^2 + 6$$

Clearly, the highest order of s in $x(s)$ being 3 and the highest order of s in $y(s)$ being 2, they differ by unity and hence there is no missing term in each series. The order of s in $x(s)$ being higher than the order of s in $y(s)$, it is recommended that $x(s)$ be put in the numerator and $y(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{x(s)}{y(s)} = \frac{s^3 + s}{8s^2 + 6}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{x(s)}{y(s)} = (0.125)s + \cfrac{1}{(32)s + \cfrac{1}{(0.0416)s}}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +0.125$$

$$q_2 = -32$$

$$q_3 = +0.0416$$

Since all the quotient terms obtained in this process are positive, it may suffice that the given polynomial $P(s) = s^3 + 8s^2 + s + 6$, is a Hurwitz polynomial.

Example 9.6 Examine whether the polynomial given by $P(s) = 5s^2 + 4s + 3$, is Hurwitz by applying continued fraction expansion method.

Solution Given that the polynomial has three terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = 4s \quad \text{and} \quad y(s) = 5s^2 + 3$$

Clearly, the highest order of s in $x(s)$ being 1 and the highest order of s in $y(s)$ being 2, they differ by unity and hence there is no missing term in each series. The order of s in $y(s)$ being higher than the order of s in $x(s)$, it is recommended that $y(s)$ be put in the numerator and $x(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{y(s)}{x(s)} = \frac{5s^3 + 3}{4s}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{y(s)}{x(s)} = (1.25)s + \frac{1}{(1.33)s}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +1.25$$

$$q_2 = +1.33$$

Since all the quotient terms obtained in this process are positive, it may suffice that the given polynomial $P(s) = 5s^2 + 4s + 3$, is a Hurwitz polynomial.

Example 9.7 Examine whether the polynomial given by $P(s) = 2s^4 + 3s^3 + 5s^2 + 4s + 3$, is Hurwitz by applying continued fraction expansion method.

Solution Given that the polynomial has five terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = 3s^3 + 4s \quad \text{and} \quad y(s) = 2s^4 + 5s^2 + 3$$

Clearly, the highest order of s in $x(s)$ being 3 and the highest order of s in $y(s)$ being 4, they differ by unity and hence there is no missing term in each series. The order of s in $y(s)$ being higher than the order of s in $x(s)$, it is recommended that $y(s)$ be put in the numerator and $x(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{y(s)}{x(s)} = \frac{2s^4 + 5s^2 + 3}{3s^3 + 4s}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{y(s)}{x(s)} = (0.666)s + \cfrac{1}{(1.285)s + \cfrac{1}{(16.33)s + \cfrac{1}{(0.0476)s}}}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +0.666$$
$$q_2 = +1.285$$
$$q_3 = +16.33$$
$$q_4 = +0.0476$$

Since all the quotient terms obtained in this process are positive, it may suffice that the given polynomial $P(s) = 2s^4 + 3s^3 + 5s^2 + 4s + 3$, is a Hurwitz polynomial.

Example 9.8 Examine whether the polynomial given by $P(s) = s^4 + 2s^3 + 4s^2 + s + 3$, is Hurwitz by applying continued fraction expansion method.

Solution Given that the polynomial has five terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by $x(s) = 2s^3 + s$ and $y(s) = s^4 + 4s^2 + 3$.

Clearly, the highest order of s in $x(s)$ being 3 and the highest order of s in $y(s)$ being 4, they differ by unity and hence there is no missing term in each series. The order of s in $y(s)$ being higher than the order of s in $x(s)$, it is recommended that $y(s)$ be put in the numerator and $x(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{y(s)}{x(s)} = \frac{s^4 + 4s^2 + 3}{2s^3 + s}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{y(s)}{x(s)} = (0.5)s + \cfrac{1}{(0.571)s + \cfrac{1}{(-4.93)s + \cfrac{1}{(-0.236)s}}}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +0.5$$
$$q_2 = +0.571$$
$$q_3 = -4.93$$
$$q_4 = -0.236$$

Since all the quotient terms obtained in this process are not positive, it may suffice that the given polynomial $P(s) = s^4 + 2s^3 + 4s^2 + s + 3$, is not a Hurwitz polynomial.

Example 9.9 Examine whether the polynomial given by $P(s) = s^2 + 9s + 3$, is Hurwitz by applying continued fraction expansion method.

Solution Given that the polynomial has three terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = 9s \quad \text{and} \quad y(s) = s^2 + 3$$

Clearly, the highest order of s in $x(s)$ being 1 and the highest order of s in $y(s)$ being 2, they differ by unity and hence there is no missing term in each series. The order of s in $y(s)$ being higher than the order of s in $x(s)$, it is recommended that $y(s)$ be put in the numerator and $x(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{y(s)}{x(s)} = \frac{s^2 + 3}{9s}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{y(s)}{x(s)} = (0.111)s + \frac{1}{(3)s}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +0.111$$
$$q_2 = +3$$

Since all the quotient terms obtained in this process are positive, it may suffice that the given polynomial $P(s) = s^2 + 9s + 3$, is a Hurwitz polynomial.

Example 9.10 Examine whether the polynomial given by $P(s) = s^2 + 8s^2 + 6s + 2$, is Hurwitz by applying continued fraction expansion method.

Solution Given that the polynomial has four terms, it may be rewritten as a combination of two groups, one odd group $x(s)$ and the other being an even group $y(s)$. When separated, the individual groups are represented by

$$x(s) = s^3 + 6s \quad \text{and} \quad y(s) = 8s^2 + 2$$

Clearly, the highest order of s in $x(s)$ being 3 and the highest order of s in $y(s)$ being 2, they differ by unity and hence there is no missing term in each series. The order of s in $x(s)$ being higher than the order of s in $y(s)$, it is recommended that $x(s)$ be put in the numerator and $y(s)$ be put in the denominator while taking the ratio. This is shown as;

$$w(s) = \frac{x(s)}{y(s)} = \frac{s^2 + 6s}{8s^2 + 2}$$

The quotient terms of this ratio may be determined by the repeated process of inversion and division. Finally, the continued fraction expansion form of representation for the quotient terms result in the following expression.

$$w(s) = \frac{x(s)}{y(s)} = (0.125)s + \cfrac{1}{(1.391)s + \cfrac{1}{(2.875)s}}$$

From the above expression, the quotient terms of the ratio may be obtained in respect of the terms shown inside the parentheses and are noted down as;

$$q_1 = +0.125, \quad q_2 = +1.391, \quad q_2 = +2.875$$

Since all the quotient terms obtained in this process are positive, it may suffice that the given polynomial $P(s) = s^2 + 8s^2 + 6s + 2$, is a Hurwitz polynomial.

9.4.2 Hurwitz Test by Routh-Hurwitz Array Formation

In the previous section we have verified the possibility of conducting Hurwitz test for polynomials by observing the sign of the quotient terms obtained from the ratio of even terms to odd terms (or vice versa) of the said polynomial by the application of continued fraction expansion approach. However, in this section we would see yet another simpler technique for performing the Hurwitz test on polynomials that is based on formulation of an array of coefficients. Such an array is known as Routh-Hurwitz array. Formation of the Routh-Hurwitz array also requires the two (odd and even) groups of terms of the given polynomial in the form of $x(s)$ and $y(s)$.

Sometimes we may come across situations where a polynomial may contain a particular type or series of terms (either even terms only or odd terms only) and still satisfying all the requirements of being Hurwitz. As a particular series is missing completely it becomes really difficult in such situations to apply the continued fraction approach that determines the quotient terms through the ratio of the even to odd terms or vice versa. However, it may be possible to generate a fictitious series by differentiating the given polynomial with respect to s in order to find a replacement or a substitute of the missing series. If the polynomial in question is given by $P(s)$ that exhibits a particular series (either even or odd), then the missing series may be generated in the form of a derivative of the polynomial with respect to s. This is shown in Eqn. (9.14).

$$P(s) = \frac{dP(s)}{ds} \tag{9.14}$$

From the above discussion, it is learnt that, both even and odd terms are essentially required for formation of the Routh-Hurwitz array. However, in the event of a particular series missing completely it is also possible to form the array by considering the polynomial and its derivative with respect to s. The following steps may be followed as a guideline for the formation of Routh-Hurwitz array.

Step-1: This step helps in separating the odd series from the even series of the given polynomial $P(s)$ and expressing them as an odd series $x(s)$, and an even series $y(s)$. Odd series means the series of all the terms having odd powers of s, and even series means the series of all the terms having even powers of s. In this step one has to ensure that there should not be any missing terms in any of the polynomials namely, $P(s)$, $x(s)$, and $y(s)$. If this goes true, then the variation in the highest order of s between the odd series and the even series should differ by unity only. However, in the event of a particular series missing completely, it may be desirable to find the derivative of the polynomial with respect to s, as shown in Eqn. (9.14).

Step-2: In this step the coefficients of the polynomial need to be arranged in two rows with the nomenclature of s^n and s^{n-1}, where n represents the highest order present in the polynomial. This arrangement serves as the preliminary structure of the Routh-Hurwitz array, which may be represented as;

$$\begin{array}{r|llllll} s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & a_{n-8} & \cdots\cdots\cdots\cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & a_{n-9} & \cdots\cdots\cdots\cdots \end{array}$$

It may be observed that the coefficients of a particular row are alternately picked up from the polynomial. Therefore, a particular row contains only odd coefficients and the other row contains even coefficients only. The uppermost row would contain even (odd) coefficients if the highest order of the polynomial is an even (odd) number, which is primarily decided by 'n'.

Step-3: The main objective of this step is to complete the Routh-Hurwitz array by filling the elements in the blank spaces of the array. The first two rows of the array are formed by picking up the appropriate coefficients from the given polynomial as already discussed in the previous step. These two rows have the nomenclature of s^n and s^{n-1}. The remaining rows of the array would have their nomenclature according to the descending order of n as regards to the first two rows. The skeleton structure of the array is shown in Table 9.2 for the sake of visual concept. The symbol for the coefficients of the remaining rows may be selected according to the convenience of the reader.

In this book we make use of the alphabets for this purpose, such as 'b' for the coefficients of third row, 'c' for the coefficients of fourth row, 'd' for the coefficients of fifth row, and so on until the rows get exhausted. Therefore the complete structure of the Routh-Hurwitz array would look something like the one shown in Table 9.3.

Table 9.2 *Skeleton structure of Routh-Hurwitz array.*

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	a_{n-8}	$\cdots$	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	a_{n-9}	$\cdots$	$\cdots$
s^{n-2}							
s^{n-3}							
s^{n-4}							
$\vdots$							
s^1							
s^0							

Table 9.3 *Complete structure of Routh-Hurwitz array.*

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	a_{n-8}	$\cdots$	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	a_{n-9}	$\cdots$	$\cdots$
s^{n-2}	b_n	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	$\cdots$	$\cdots$
s^{n-3}	c_n	c_{n-1}	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	$\cdots$
s^{n-4}	d_n	d_{n-1}	d_{n-2}	d_{n-3}	d_{n-4}	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1	i_n	i_{n-1}	i_{n-2}	i_{n-3}	i_{n-4}	$\vdots$	$\vdots$
s^0	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Step-4: In this step, we would present the procedure for obtaining the remaining elements of the Routh-Hurwitz array beyond the second row, which corresponds to all the elements in the rows with nomenclature in between s^{n-2} and s^0. The general formula for finding any element in this cluster may be outlined as the one shown in the expression of Eqn. (9.15). In order to simplify the understanding behind this general formula, let us represent the total structure of the Routh-Hurwitz array in the form of a general $n \times n$ matrix structure as shown below in Table 9.4 with elements represented as A.

Table 9.4 *General Matrix structure of Routh-Hurwitz array.*

row-1	$A_{1,1}$	$A_{1,2}$	$\cdots$	$A_{1,m}$	$A_{1,m+1}$	$\cdots$	$A_{1,n}$
row-2	$A_{2,1}$	$A_{2,2}$	$\cdots$	$A_{2,m}$	$A_{2,m+1}$	$\cdots$	$A_{2,n}$
row-3	$A_{3,1}$	$A_{3,2}$	$\cdots$	$A_{3,m}$	$A_{3,m+1}$	$\cdots$	$A_{3,n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i^{th} row	$A_{i,1}$	$A_{i,2}$	$\cdots$	$A_{i,m}$	$A_{i,m+1}$	$\cdots$	$A_{i,n}$
j^{th} row	$A_{j,1}$	$A_{j,2}$	$\cdots$	$A_{j,m}$	$A_{j,m+1}$	$\cdots$	$A_{j,n}$
k^{th} row	$A_{k,1}$	$A_{k,2}$	$\cdots$	$A_{k,m}$	$A_{k,m+1}$	$\cdots$	$A_{k,n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n^{th} row	$A_{n,1}$	$A_{n,2}$	$\cdots$	$A_{n,m}$	$A_{n,m+1}$	$\cdots$	$A_{n,n}$

While computing the element $A_{k,m}$ in the k^{th} row and m^{th} column, the formula presented in Eqn. (9.15) may be used. The other elements, which have been used in the said expression for obtaining the required element $A_{k,m}$ correspond to the elements of the same matrix in the two immediately preceding rows (i.e. the i^{th} row and j^{th} row) and two columns corresponding to first column and $(m+1)^{\text{th}}$ column as shown in different shades.

$$A_{k,m} = \frac{A_{j,1}\,A_{i,m+1} - A_{j,m+1}\,A_{i,1}}{A_{j,1}} \tag{9.15}$$

Thus, we may now apply the general formula presented in Eqn. (9.15) for finding the unknown elements of the Routh-Hurwitz array corresponding to Table 9.3. Since the elements in the first two rows of Table 9.3 are known elements, we now present the expressions for the elements in the rows following the second row, as shown below.

Elements of third row:

$$b_n = \frac{a_{n-1}\,a_{n-2} - a_n\,a_{n-2}}{a_{n-1}} \tag{9.16}$$

$$b_{n-1} = \frac{a_{n-1}\,a_{n-4} - a_n\,a_{n-5}}{a_{n-1}} \tag{9.17}$$

$$b_{n-2} = \frac{a_{n-1}\,a_{n-6} - a_n\,a_{n-7}}{a_{n-1}} \tag{9.18}$$

Elements of the fourth row:

$$c_n = \frac{b_n\,a_{n-2} - a_{n-1}\,b_{n-1}}{b_n} \tag{9.19}$$

$$c_{n-1} = \frac{b_n\,a_{n-5} - a_{n-1}\,b_{n-2}}{b_n} \tag{9.20}$$

Elements of the fifth row:

$$d_n = \frac{c_n\,b_{n-1} - b_n\,c_{n-1}}{c_n} \tag{9.21}$$

$$d_{n-1} = \frac{c_n\,b_{n-2} - b_n\,c_{n-2}}{c_n} \tag{9.22}$$

Step-5: This is the last stage of this method, in which any one of the following inferences may be drawn from the observations of the previous steps.

I. While observing the elements present in the first column of the final Routh-Hurwitz array as obtained in the previous step, if there is no sign change at any stage, then the given polynomial qualifies to be Hurwitz. Otherwise,

II. While observing the elements present in the first column of the final Routh-Hurwitz array as obtained in the previous step, if there is one or multiple sign changes at any stage, then the given polynomial does not qualify to be Hurwitz.

Example 9.11 Form the Routh-Hurwitz array for the polynomial given by, $P(s) = 4s^5 + 3s^4 + 2s^3 + s^2 + 5s + 3$, and hence examine whether the polynomial is Hurwitz?

Solution For the given polynomial $P(s) = 4s^5 + 3s^4 + 2s^3 + s^2 + 5s + 3$, the highest order of s is an odd number, i.e. $n = 5$. Hence, there will be $(n + 1 = 6)$ six rows in the Routh-Hurwitz array. On separation of odd and even terms of the polynomial, we get

Odd series: $x(s) = 4s^5 + 2s^3 + 5s$, and Even series: $y(s) = 3s^4 + s^2 + 3$

Since the odd series contains the highest order of s, hence the coefficients of the odd series form the elements of row-1 followed by the coefficients of the even series that forms the elements of the row-2. This gives

Elements of row-1: 4, 2, 5

Elements of row-2: 3, 1, 3

In the next step, we may compute the remaining elements of row-3 to row-6 with the help of Eqn. (9.15), which gives

Elements of row-3

$$b_n = \frac{a_{n-1}\,a_{n-2} - a_n\,a_{n-2}}{a_{n-1}} = \frac{(3\times 2)-(4\times 1)}{3} = 0.666$$

$$b_{n-1} = \frac{a_{n-1}\,a_{n-4} - a_n\,a_{n-5}}{a_{n-1}} = \frac{(5\times 3)-(4\times 3)}{3} = 1$$

$$b_{n-2} = \frac{a_{n-1}\,a_{n-6} - a_n\,a_{n-7}}{a_{n-1}} = \frac{(3\times 0)-(4\times 0)}{3} = 0$$

Elements of row-4

$$c_n = \frac{b_n\,a_{n-3} - a_{n-1}\,b_{n-1}}{b_n} = \frac{(0.666\times 1)-(3\times 1)}{0.666} = -3.5$$

$$c_{n-1} = \frac{b_n\,a_{n-5} - a_{n-1}\,b_{n-2}}{b_n} = \frac{(0.666\times 3)-(3\times 0)}{0.666} = 3$$

$$c_{n-2} = \frac{b_n\,a_{n-7} - a_{n-1}\,b_{n-3}}{b_n} = \frac{(0.666\times 0)-(3\times 0)}{0.666} = 0$$

Elements of row-5

$$d_n = \frac{c_n\, b_{n-1} - b_n\, c_{n-1}}{c_n} = \frac{(-3.5\times 1)-(3\times 1)}{-3.5} = 1.57$$

$$d_{n-1} = \frac{c_n\, b_{n-2} - b_n\, c_{n-2}}{c_n} = \frac{(-3.5\times 0)-(3\times 0)}{-3.5} = 0$$

$$d_{n-2} = \frac{c_n\, b_{n-3} - b_n\, c_{n-3}}{c_n} = \frac{(-3.5\times 0)-(3\times 0)}{-3.5} = 0$$

Elements of row-6

$$e_n = \frac{d_n\, c_{n-1} - c_n\, d_{n-1}}{d_n} = \frac{(1.57\times 3)-(-3.5\times 0)}{1.57} = 3$$

$$e_{n-1} = \frac{d_n\, c_{n-2} - c_n\, d_{n-2}}{d_n} = \frac{(1.57\times 0)-(-3.5\times 0)}{1.57} = 0$$

$$e_{n-2} = \frac{d_n\, c_{n-3} - c_n\, d_{n-3}}{d_n} = \frac{(1.57\times 0)-(-3.5\times 0)}{1.57} = 0$$

In view of the above findings, we may now construct the complete Routh-Hurwitz array as indicated below.

s^5	4	2	5
s^4	3	1	3
s^3	0.666	1	0
s^2	–3.500	3	0
s^1	1.570	1	0
s^0	3	0	0

It may be observed from the elements of first column that there is a sign change at one place as shown shaded. Hence, the given polynomial $P(s) = 4s^5 + 3s^4 + 2s^3 + s^2 + 5s + 3$, does not qualify as Hurwitz.

Example 9.12 Verify that the polynomial given by, $P(s) = s^4 + 12s^4 + 6s^3 + 3s^2 + 10$, is not Hurwitz.

Solution For the given polynomial $P(s) = s^4 + 12s^3 + 6s^2 + 3s + 10$, the highest order of s is an even number, i.e. $n = 4$. Hence, there will be $(n + 1 = 5)$ five rows in the Routh-Hurwitz array. On separation of odd and even terms of the polynomial, we get

Even series: $y(s) = s^4 + 6s^2 + 10$, and Odd series: $x(s) = 12s^3 + 3s$

Since the even series contains the highest order of s, hence the coefficients of the even series form the elements of row-1 followed by the coefficients of the odd series that forms the elements of the row-2. This gives

Elements of row-1: 1, 6, 10

Elements of row-2: 12, 3, 0

In the next step, we may compute the remaining elements of row-3 to row-5 with the help of Eqn. (9.15), which gives

Elements of row-3

$$b_n = \frac{a_{n-1}\,a_{n-2} - a_n\,a_{n-2}}{a_{n-1}} = \frac{(12\times 6)-(3\times 1)}{12} = 5.75$$

$$b_{n-1} = \frac{a_{n-1}\,a_{n-4} - a_n\,a_{n-5}}{a_{n-1}} = \frac{(12\times 10)-(0\times 1)}{12} = 10$$

$$b_{n-2} = \frac{a_{n-1}\,a_{n-6} - a_n\,a_{n-7}}{a_{n-1}} = \frac{(12\times 0)-(0\times 1)}{12} = 0$$

Elements of row-4

$$c_n = \frac{b_n\,a_{n-3} - a_{n-1}\,b_{n-1}}{b_n} = \frac{(5.75\times 3)-(10\times 12)}{5.75} = -17.87$$

$$c_{n-1} = \frac{b_n\,a_{n-5} - a_{n-1}\,b_{n-2}}{b_n} = \frac{(5.75\times 0)-(0\times 12)}{5.75} = 0$$

$$c_{n-2} = \frac{b_n\,a_{n-7} - a_{n-1}\,b_{n-3}}{b_n} = \frac{(5.75\times 0)-(0\times 12)}{5.75} = 0$$

Elements of row-5

$$d_n = \frac{c_n\,b_{n-1} - b_n\,c_{n-1}}{c_n} = \frac{(-17.87\times 10)-(0\times 5.75)}{-17.87} = 10$$

$$d_{n-1} = \frac{c_n\,b_{n-2} - b_n\,c_{n-2}}{c_n} = \frac{(-17.87\times 0)-(0\times 5.75)}{-17.87} = 0$$

$$d_{n-2} = \frac{c_n\,b_{n-3} - b_n\,c_{n-3}}{c_n} = \frac{(-17.87\times 0)-(0\times 5.75)}{-17.87} = 0$$

In view of the above findings, we may now construct the complete Routh-Hurwitz array as indicated below.

s^4	1	6	10
s^3	12	3	0
s^2	5.75	10	0
s^1	–17.87	0	0
s^0	10	0	0

It may be observed from the elements of first column that there is a sign change at one place. Hence, the given polynomial $P(s) = s^4 + 12s^3 + 6s^2 + 3s + 10$, does not qualify as Hurwitz.

Example 9.13 Examine whether the polynomial given by, $P(s) = s^4 + 5s^3 + 5s^2 + 10s + 3$, is Hurwitz?

Solution For the given polynomial $P(s) = s^4 + 5s^3 + 5s^2 + 10s + 3$, the highest order of s is an even number, i.e. $n = 4$. Hence, there will be $(n + 1 = 5)$ five rows in the Routh-Hurwitz array. On separation of odd and even terms of the polynomial, we get

Even series: $y(s) = s^4 + 5s^2 + 3$, and Odd series: $x(s) = 5s^3 + 10s$

Since the even series contains the highest order of s, hence the coefficients of the even series form the elements of row-1 followed by the coefficients of the odd series that forms the elements of the row-2. This gives

Elements of row-1: 1, 5, 3

Elements of row-2: 5, 10, 0

In the next step, we may compute the remaining elements of row-3 to row-5 with the help of Eqn. (9.15), which gives

Elements of row-3

$$b_n = \frac{a_{n-1}\,a_{n-2} - a_n\,a_{n-2}}{a_{n-1}} = \frac{(5\times 5)-(10\times 1)}{5} = 3$$

$$b_{n-1} = \frac{a_{n-1}\,a_{n-4} - a_n\,a_{n-5}}{a_{n-1}} = \frac{(5\times 3)-(0\times 1)}{5} = 3$$

$$b_{n-2} = \frac{a_{n-1}\,a_{n-6} - a_n\,a_{n-7}}{a_{n-1}} = \frac{(5\times 0)-(0\times 1)}{5} = 0$$

Elements of row-4

$$c_n = \frac{b_n\,a_{n-3} - a_{n-1}\,b_{n-1}}{b_n} = \frac{(3\times 10)-(3\times 5)}{3} = 5$$

$$c_{n-1} = \frac{b_n\,a_{n-5} - a_{n-1}\,b_{n-3}}{b_n} = \frac{(3\times 0)-(0\times 5)}{3} = 0$$

$$c_{n-2} = \frac{b_n a_{n-7} - a_{n-1} b_{n-3}}{b_n} = \frac{(3 \times 0) - (0 \times 5)}{3} = 0$$

Elements of row-5

$$d_n = \frac{c_n b_{n-1} - b_n c_{n-1}}{c_n} = \frac{(5 \times 3) - (0 \times 3)}{5} = 10$$

$$d_{n-1} = \frac{c_n b_{n-2} - b_n c_{n-2}}{c_n} = \frac{(5 \times 0) - (0 \times 3)}{5} = 0$$

$$d_{n-2} = \frac{c_n b_{n-3} - b_n c_{n-3}}{c_n} = \frac{(5 \times 0) - (0 \times 3)}{5} = 0$$

In view of the above findings, we may now construct the complete Routh-Hurwitz array as indicated below.

s^4	1	5	3
s^3	5	10	0
s^2	3	3	0
s^1	5	0	0
s^0	3	0	0

It may be observed from the elements of first column that there is no sign change at any place. Hence, the given polynomial $P(s) = s^4 + 5s^3 + 5s^2 + 10s + 3$, qualifies as being Hurwitz.

Example 9.14 Examine whether the polynomial given by, $P(s) = 4s^4 + 3s^3 + 2$, is Hurwitz?

Solution For the given polynomial $P(s) = 4s^4 + 3s^3 + 2$, the highest order of s is an even number, i.e. $n = 4$. Hence, there will be ($n + 1 = 5$) five rows in the Routh-Hurwitz array. On separation of odd and even terms of the polynomial, we find that the polynomial contains only even terms. Hence the derivative of the polynomial with respect to s is essential for formation of the Routh-Hurwitz array. This results in the following expressions.

$$P(s) = 4s^4 + 3s^2 + 2$$

$$P'(s) = \frac{dP(s)}{ds} = \frac{d(4s^4 + 3s^2 + 2)}{ds} = 16s^3 + 6s$$

Since the polynomial contains the highest order of s, in respect of its derivative, hence the coefficients of the polynomial form the elements of row-1 followed by the coefficients of the derivative polynomial that forms the elements of the row-2. This gives

Elements of row-1: 4, 3, 2

Elements of row-2: 16, 6, 0

In the next step, we may compute the remaining elements of row-3 to row-5 with the help of Eqn. (9.15), which gives the following

Elements of row-3

$$b_n = \frac{a_{n-1}\,a_{n-2} - a_n\,a_{n-3}}{a_{n-1}} = \frac{(16\times 3)-(6\times 4)}{16} = 1.5$$

$$b_{n-1} = \frac{a_{n-1}\,a_{n-4} - a_n\,a_{n-5}}{a_{n-1}} = \frac{(16\times 2)-(0\times 4)}{16} = 2$$

$$b_{n-2} = \frac{a_{n-1}\,a_{n-6} - a_n\,a_{n-7}}{a_{n-1}} = \frac{(16\times 0)-(0\times 4)}{16} = 0$$

Elements of row-4

$$c_n = \frac{b_n\,a_{n-3} - a_{n-1}\,b_{n-1}}{b_n} = \frac{(1.5\times 6)-(2\times 16)}{1.5} = -15.33$$

$$c_{n-1} = \frac{b_n\,a_{n-5} - a_{n-1}\,b_{n-2}}{b_n} = \frac{(1.5\times 0)-(0\times 16)}{1.5} = 0$$

$$c_{n-2} = \frac{b_n\,a_{n-7} - a_{n-1}\,b_{n-3}}{b_n} = \frac{(1.5\times 0)-(0\times 16)}{1.5} = 0$$

Elements of row-5

$$d_n = \frac{c_n\,b_{n-1} - b_n\,c_{n-1}}{c_n} = \frac{(-15.33\times 2)-(0\times 1.5)}{-15.33} = 2$$

$$d_{n-1} = \frac{c_n\,b_{n-2} - b_n\,c_{n-2}}{c_n} = \frac{(-15.33\times 0)-(0\times 1.5)}{-15.33} = 0$$

$$d_{n-2} = \frac{c_n\,b_{n-3} - b_n\,c_{n-3}}{c_n} = \frac{(-15.33\times 0)-(0\times 1.5)}{-15.33} = 0$$

In view of the above findings, we may now construct the complete Routh-Hurwitz array as indicated below.

s^4	4	3	2
s^3	16	6	0
s^2	1.5	2	0
s^1	–15.33	0	0
s^0	2	0	0

It may be observed from the elements of the first column that there is a sign change at one place as shown shaded. Hence, the given polynomial $P(s) = 4s^4 + 3s^3 + 2$ is not Hurwitz.

Example 9.15 Examine if the polynomial given by, $P(s) = s^5 + s^3 + 5s$, is Hurwitz?

Solution For the given polynomial $P(s) = s^5 + s^3 + 5s$, the highest order of s is an odd number, i.e. $n = 5$. Hence, there will be $(n + 1 = 6)$ six rows in the Routh-Hurwitz array. On separation of odd and even terms of the polynomial, we find that the polynomial contains odd terms only. Hence the derivative of the polynomial with respect to s is essential for the formation of the Routh-Hurwitz array. This results in the following expressions.

$$P(s) = s^5 + s^3 + 5s \quad \text{and} \quad P'(s) = \frac{dP(s)}{ds} = \frac{d(s^5 + s^3 + 5s)}{ds} = 5s^4 + 3s^2 + 5$$

Since the polynomial contains the highest order of s, in respect of its derivative, hence the coefficients of the polynomial form the elements of row-1 followed by the coefficients of the derivative polynomial that forms the elements of the row-2. This gives the following:

Elements of row-1: 1, 1, 5

Elements of row-2: 5, 3, 5

In the next step, we may compute the remaining elements of row-3 to row-5 with the help of Eqn. (9.15), which gives the following:

Elements of row-3

$$b_n = \frac{a_{n-1}\,a_{n-2} - a_n\,a_{n-3}}{a_{n-1}} = \frac{(5\times 1)-(3\times 1)}{5} = 0.4$$

$$b_{n-1} = \frac{a_{n-1}\,a_{n-4} - a_n\,a_{n-5}}{a_{n-1}} = \frac{(5\times 5)-(5\times 1)}{5} = 4$$

$$b_{n-2} = \frac{a_{n-1}\,a_{n-6} - a_n\,a_{n-7}}{a_{n-1}} = \frac{(5\times 0)-(0\times 1)}{5} = 0$$

Elements of row-4

$$c_n = \frac{b_n\,a_{n-3} - a_{n-1}\,b_{n-1}}{b_n} = \frac{(0.4\times 3)-(4\times 5)}{0.4} = -47$$

$$c_{n-1} = \frac{b_n\,a_{n-5} - a_{n-1}\,b_{n-2}}{b_n} = \frac{(0.4\times 5)-(0\times 5)}{0.4} = 5$$

$$c_{n-2} = \frac{b_n\,a_{n-7} - a_{n-1}\,b_{n-3}}{b_n} = \frac{(0.4\times 0)-(0\times 5)}{0.4} = 0$$

Elements of row-5

$$d_n = \frac{c_n\,b_{n-1} - b_n\,c_{n-1}}{c_n} = \frac{(-47\times 4)-(5\times 0.4)}{-47} = 4.04$$

$$d_{n-1} = \frac{c_n b_{n-2} - b_n c_{n-2}}{c_n} = \frac{(-47 \times 0) - (0 \times 0.4)}{-47} = 0$$

$$d_{n-2} = \frac{c_n b_{n-3} - b_n c_{n-3}}{c_n} = \frac{(-47 \times 0) - (0 \times 0.4)}{-47} = 0$$

Elements of row-6

$$e_n = \frac{d_n c_{n-1} - c_n d_{n-1}}{d_n} = \frac{(4.04 \times 5) - (-47 \times 0)}{4.04} = 5$$

$$e_{n-1} = \frac{d_n c_{n-2} - c_n d_{n-2}}{d_n} = \frac{(4.04 \times 0) - (-47 \times 0)}{4.04} = 0$$

$$e_{n-2} = \frac{d_n c_{n-3} - c_n d_{n-3}}{d_n} = \frac{(4.04 \times 0) - (-47 \times 0)}{4.04} = 0$$

In view of the above findings, we may now construct the complete Routh-Hurwitz array as indicated below.

s^5	1	1	5
s^4	5	3	5
s^3	0.4	4	0
s^2	−47	5	0
s^1	4.04	0	0
s^0	5	0	0

It may be observed from the elements of first column that there is a sign change at one place as shown shaded. Hence, the given polynomial $P(s) = s^5 + s^3 + 5s$ does not qualify as Hurwitz.

9.5 POSITIVE REAL FUNCTION

In the preceding sections of this chapter, the significance of positive real functions (often expressed as **p.r. functions**) is discussed at many places. Most importantly, it is used as a stability indicator of network response subject to excitation. According to the stability criteria and realizability criteria, the following two points clearly emphasize this requirement.

1. If the excitation function and response function need to be bounded over a considerable stretch of time, both of them must be positive real functions.
2. The scale factor (if any) must be a positive real function.
3. The passive elements of a network representing the impedances and admittances may be realized if the network function is a positive real function.

Therefore, it becomes essential to discuss the properties of positive real functions in detail. A polynomial function $P(s)$ is said to be positive and real if it satisfies the following criteria.

Conditions for Positive Real Functions

I. In order to be a positive real function, $P(s)$ must be real for all real values of s.

II. The real part of $P(s)$ must be greater than or equal to zero for all values of the real part of s being equal to or greater than zero. Stated mathematically, it must satisfy, $\mathrm{Re}|P(s)| \geq 0$ for all values of $\mathrm{Re}|(s)| \geq 0$.

Properties of Positive Real Functions

i. The sum of two positive real functions is also a positive real function.

ii. The reciprocal of a positive real function is also a positive real function.

iii. The roots of a positive real function must lie on the left half of the s plane. Hence, the roots of a positive real function should have negative real parts.

iv. Some roots of positive real functions may also lie on the $j\omega$ axis if a conjugate pair for the same pair of roots exists at all.

v. The highest order of the numerator (denominator) polynomial of a positive real function may differ from the highest order of the denominator (numerator) polynomial at best by unity. This property rules out the occurrence of multiple poles at infinity.

vi. The lowest order of the numerator (denominator) polynomial of a positive real function may differ from the lowest order of the denominator (numerator) polynomial at best by unity. This property rules out the occurrence of multiple poles at the origin of the s plane.

9.6 APPLICATION OF P.R. FUNCTIONS IN NETWORK SYNTHESIS

In Section 9.4, it is mentioned that realization of the passive network elements may be possible if the network function happens to be a positive real function. In this section let us first focus the attention in justifying the veracity of this statement. In Chapter-5, Article 5.10, transfer impedances of networks have been discussed, which covers the highlights of three basic elements (R, L and C). In case of a resistive network as shown in Fig. 9.3, the network equation is expressed by

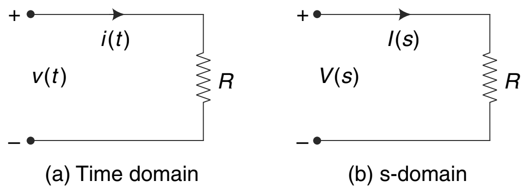


Figure 9.3 *Impedance of a purely resistive network*

$$V(s) = R.I(s) \tag{9.23}$$

Eqnuation (9.23) can also be represented in the form of a transfer function, which is shown in Eqn. (9.24).

$$\frac{V(s)}{I(s)} = R \tag{9.24}$$

However, it is obvious that the transfer function on the left-hand side of Eqn. (9.24) describes an impedance function $Z(s)$ which may be expressed in the manner of Eqn. (9.25).

$$\frac{V(s)}{I(s)} = Z(s) \tag{9.25}$$

Analysis of these two equations, Eqn. (9.24) and Eqn. (9.25), may reveal that the existence of the transfer function $F(s) = V(s)/I(s)$ necessarily means one thing, i.e., $F(s)$ must be a positive real function. A corollary of this statement may be put up as follows:

(i) If $F(s)$ is a positive real function representing an impedance function then the impedance function is positive and real too.

(ii) If $F(s)$ is a positive real function representing an impedance function of a purely resistive network, it is implied that the impedance would be numerically same as the resistance (R) of the network, and the resistance would be a positive real number.

Therefore, in order to justify that the passive resistance of a network is a positive real number, it is mandatory to see that the concerned impedance function or the network transfer function $F(s) = V(s)/I(s)$ is a positive real function. The above justification can possibly be extended for networks consisting of other two types of network elements, such as inductance and capacitance.

In case of a purely inductive network as shown in Fig. 9.4, the network equation may be expressed by

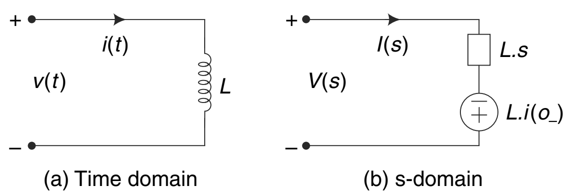


Figure 9.4 *Impedance of purely inductive network*

$$V(s) = L[s.I(s) - i(0\_)] \tag{9.26}$$

Where, the initial conditions happen to be zero, Eqn. (9.26) may be reduced to the form of Eqn. (9.27).

$$V(s) = L[s.I(s)] \tag{9.27}$$

Eqnuation (9.27) can also be represented in the form of a transfer function, which is shown in Eqn. (9.28).

$$\frac{V(s)}{I(s)} = L.s \tag{9.28}$$

However, it is obvious that the transfer function on the left hand side of Eqn. (9.28) describes an impedance function Z(s) which may be expressed in the manner of Eqn. (9.29).

$$\frac{V(s)}{I(s)} = Z(s) \tag{9.29}$$

Analysis of these two equations, (9.28) and (9.29) may reveal that the existence of the transfer function $F(s) = V(s)/I(s)$ necessarily means one thing, i.e., $F(s)$ must be a positive real function. A corollary of this statement may be put up as follows:

(i) If $F(s)$ is a positive real function representing an impedance function, then the impedance function is positive and real too.

(ii) If $F(s)$ is a positive real function representing an impedance function of a purely inductive network, it must be implied that for all positive and real values of s, the inductance (L) must be a positive real number.

Therefore, in order to justify that the passive inductance of a network is a positive real number, it is mandatory to see that the concerned impedance function or the network transfer function $F(s) = V(s)/I(s)$ is a positive real function.

In case of a purely capacitive network as shown in Fig. 9.5, the network equation may be expressed by

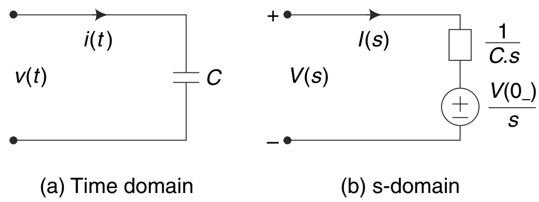


Figure 9.5 *Impedance of a purely capacitive network*

$$V(s) = \frac{I(s)}{C.s} + \frac{v(0\_)}{s} \tag{9.30}$$

Where, the initial conditions happen to be zero, Eqn. (9.30) may be reduced to the form of Eqn. (9.31).

$$V(s) = \frac{I(s)}{C.s} \tag{9.31}$$

Eqnuation (9.31) can also be represented in the form of a transfer function, which is shown in Eqn. (9.32).

$$\frac{V(s)}{I(s)} = \frac{1}{C.s} \tag{9.32}$$

However, it is obvious that the transfer function on the left-hand side of Eqn. (9.32) describes an impedance function Z(s) which may be expressed in the manner of Eqn. (9.33).

$$\frac{V(s)}{I(s)} = Z(s) \tag{9.33}$$

Analysis of these two equations, Eqn. (9.32) and Eqn. (9.33), may reveal that the existence of the transfer function $F(s) = V(s)/I(s)$ necessarily means one thing, i.e., $F(s)$ must be a positive real function. A corollary of this statement may be put up as follows:

(i) If $F(s)$ is a positive real function representing an impedance function then the impedance function is positive and real too.

(ii) If $F(s)$ is a positive real function representing an impedance function of a purely capacitive network, it must be implied that for all positive and real values of s, the capacitance (C) must be a positive real number.

Therefore, in order to justify that the passive capacitance of a network is a positive real number, it is mandatory to see that the concerned impedance function or the network transfer function $F(s) = V(s)/I(s)$ is a positive real function.

So far, the discussion was centered on one thing, i.e., the justification of passive network elements for positive real numbers and it is conditionally dependent on the positive realness of the concerned network transfer function given by $F(s) = V(s)/I(s)$. However, another way of interpreting the facts and findings of the above discussion may be very useful in the perspective of network synthesis through realization of these elements. This may be as stated below.

I. If $F(s) = V(s)/I(s)$, is a positive real function representing the impedance of a network and bears an expression of the form $F(s) = R$ then it is obvious and true that R is a positive real number such that the passive impedance element of the network becomes $Z(s) = R$; hence realizable in form of a resistance.

II. If $F(s) = V(s)/I(s)$ is a positive real function representing the impedance of a network and bears an expression of the form $F(s) = L.s$ then it is obvious and true that L is a positive real number such that the passive impedance element of the network becomes $Z(s) = L.s$; hence realizable in form of an inductance.

III. If $F(s) = V(s)/I(s)$ is a positive real function representing the impedance of a network and bears an expression of the form $F(s) = 1/C.s$ then it is obvious and true that C is a positive real number such that the passive impedance element of the network becomes $Z(s) = 1/C.s$; hence realizable in form of a capacitance.

IV. A dual statement of condition-I would mean the same thing in respect of an admittance function. Hence, if $F(s) = I(s)/V(s)$ is a positive real function representing the admittance of a network and bears an expression of the form $F(s) = 1/R$ then it is obvious and true that R is a positive real number such that the passive admittance element of the network becomes $Y(s) = 1/R$; hence realizable.

V. A dual statement of condition-II would mean the same thing in respect of an admittance function. Hence, if $F(s) = I(s)/V(s)$ is a positive real function representing the admittance of a network and bears an expression of the form $F(s) = 1/L.s$ then it is obvious and true that L is a positive real number such that the passive admittance element of the network becomes $Y(s) = 1/L.s$; hence realizable.

VI. A dual statement of condition-III would mean the same thing in respect of an admittance function. Hence, if $F(s) = I(s)/V(s)$ is a positive real function representing the admittance of a network and bears an expression of the form $F(s) = C.s$ then it is obvious and true that C is a positive real number such that the passive admittance element of the network becomes $Y(s) = C.s$; hence realizable.

Thus, it is verified that both impedance and admittance elements (imittances) of electrical networks are positive real functions and may be realizable for network synthesis.

Example 9.16 Prove that the equivalent impedance resulting out of the series combination of two impedances gives a positive real function.

Solution Let the two impedances which are to be connected in series are given as $Z_1(s)$ and $Z_2(s)$. The equivalent impedance resulting out of their series combination may be given by $Z(s)$, such that $Z(s) = Z_1(s) + Z_2(s)$. This is also shown in Fig. 9.6.

According to the property the sum of two positive real functions must be positive real too. Since basic impedances would be positive real, hence their sum (series equivalent) should also qualify to be positive real.

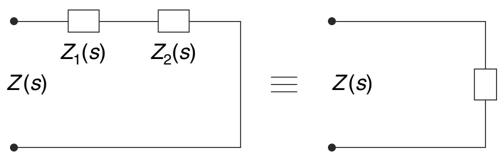


Figure 9.6 *Series combination of impedances*

Example 9.17 Prove that the equivalent impedance resulting out of the parallel combination of two impedances gives a positive real function.

Solution Let the two impedances which are to be connected in parallel are given as $Z_1(s)$ and $Z_2(s)$. The equivalent impedance resulting out of their parallel combination may be given by $Z(s)$, such that $\frac{1}{Z(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}$. This may also have an admittance form of representation such that $Y(s) = Y_1(s) + Y_2(s)$. This is also shown in Fig. 9.7.

According to the property the sum of two positive real functions must be positive real too. Since basic admittances would be positive real, hence their sum (parallel equivalent) should also qualify to be positive real.

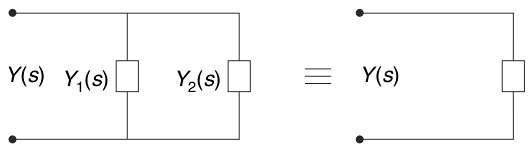


Figure 9.7 *Parallel combination of impedances*

Example 9.18 According to the property, reciprocal and sum of positive real functions must be positive real. Does this also satisfy for the difference of two positive real numbers? Give one example in support of your justification.

Solution Let a positive real function be taken as s. Its reciprocal $1/s$ is also positive real. While taking the sum of the two, it would also give a positive real function. However the difference of the two may be analyzed further before giving any justification in support of positive realness.

$$F(s) = s - \frac{1}{s} = \frac{s^2 - 1}{s} = \frac{(s+1)(s-1)}{s}$$

From the above it may be observed that all the roots of the polynomial are not strictly on the left half of the s plane. Hence, it may not be always true that the difference of two positive real functions would result in a positive real function.

9.7 TESTING A GIVEN FUNCTION FOR POSITIVE REALNESS

The necessary conditions for a function to be positive real and the inherent properties of positive real functions have been enumerated in the previous section precisely. However, it is often desirable on the face of network synthesis to examine the given network function for positive realness, as it forms one of the necessary criterion of realizability. Therefore, it would be proper to throw some light on the testing procedures available for verifying the behavior of functions for positive realness. In this section, we will first see the necessary and sufficient conditions for a function with real coefficients to be positive real as outlined below.

1. For a function $F(s)$ with real coefficients to be positive real, it is necessary that all the poles and zeros of the said function should lie on the left half of the s plane. The left half of this plane is shown shaded in Fig. 9.8.
2. If some poles/zeros of the function are found on the $j\omega$ axis or at the origin of the s plane, the necessary condition as indicated in item-1 above may not be sufficient to guarantee the positive realness of the function. Under such circumstances, the sufficient condition for positive realness may be stated as follows:
   (a) The poles of the function located on the $j\omega$ axis must be simple and must have real and positive residues.
   (b) For all values of ω, the real part of the function $F(j\omega)$ must be equal to or greater than zero. This implies $Re\,|F(j\omega)| \geq 0$ for all values of ω.

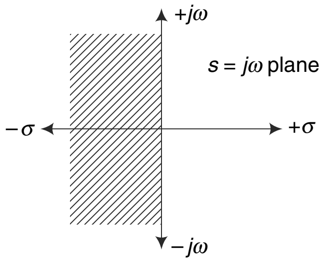


Figure 9.8 *Shaded region suitable for locating poles and zeros.*

Given a function $F(s) = \dfrac{N(s)}{D(s)}$, with numerator polynomial $N(s)$ and denominator polynomial $D(s)$, the necessary criterion for positive realness (item-1) may be tested by identifying and locating all possible roots of numerator and denominator functions in the s plane. Alternatively, a simple Hurwitz test may be conducted on the given function either by the continued fraction expansion method or by the formation of Routh–Hurwitz array. This has been discussed earlier in this section.

In order to check the sufficient criterion as indicated in item-2(a) of this section, let us consider a situation where a function has a pole on the $j\omega$ axis. In this event the poles must be simple and should occur in conjugate pairs. Assuming that such a pole is located at $s = \pm j\omega_0$, the transfer function of the network would look like the expression in Eqn. (9.34).

$$F(s) = \frac{N(s)}{(s + j\omega_0)(s - j\omega_0)} \tag{9.34}$$

The partial fraction expansion of the function would result in the following expression.

$$F(s) = \frac{K_0}{(s + j\omega_0)} + \frac{K_0^*}{(s - j\omega_0)} \tag{9.35}$$

In the above equation, K_0^* is the conjugate of K_0. This also satisfies that the residues of complex conjugate poles are conjugate themselves. However, for the function $F(s)$ to be positive real, the residues must be positive real too. This condition demands that K_0^* and K_0 must be equal to each other (say $K_0^* = K_0 = K$). Replacing these values in Eqn. (9.35) we may get Eqn. (9.36).

$$F(s) = \frac{K_0}{(s + j\omega_0)} + \frac{K_0^*}{(s - j\omega_0)} = \frac{2Ks}{s^2 - j^2\omega_0^2} = \frac{2Ks}{s^2 + \omega_0^2} \tag{9.36}$$

The above equation may be treated as an alternate proof for the sufficiency condition mentioned in item-2(a). Next to this, let us examine the requirements for satisfying the sufficiency condition mentioned in item-2(b). This would require the computation of $Re\,|F(j\omega)|$ from the given network function $F(s)$. For this purpose, we may have to express the given function as a separate combination of odd terms (N_x, D_x) and even terms (N_y, D_y) as shown in Eqn. (9.37).

$$F(s) = \frac{N(s)}{D(s)} = \frac{N_x(s)+N_y(s)}{D_x(s)+D_y(s)} \tag{9.37}$$

Now multiplying a term $\{D_y(s) - D_x(s)\}$ both in the numerator and denominator of Eqn. (9.37), the parent equation does not change, which results in

$$F(s) = \frac{\{N_x(s)+N_y(s)\}\{D_y(s)-D_x(s)\}}{D_y^2(s)-D_x^2(s)} \tag{9.38}$$

In order to maintain simplicity in the derivations, let us temporarily remain silent about the mention of s within parentheses of each term on the right-hand side of expression, which would result in

$$F(s) = \frac{(N_x+N_y)(D_y-D_x)}{(D_y^2-D_x^2)} = \frac{N_yD_y - N_xD_x + N_xD_y - N_yD_x}{(D_y^2-D_x^2)} \tag{9.39}$$

Now let us apply the following basic properties of odd and even functions to the above expression.

(i) Multiplication of two even terms results in an even function

(ii) Multiplication of two odd terms results in an even function

(iii) Multiplication of an even term with an odd term results in an odd function.

Application of these thumb rules in Eqn. (9.39) results in two distinct terms, one being the even part of $F(s)$ and the other being the odd part of $F(s)$. Thus, we get

$$F(s) = \frac{N_yD_y - N_xD_x}{(D_y^2-D_x^2)} + \frac{N_xD_y - N_yD_x}{(D_y^2-D_x^2)} = \text{Even } F(s) + \text{Odd } F(s) \tag{9.40}$$

where, $$\text{Even } F(s) = \frac{N_yD_y - N_xD_x}{(D_y^2-D_x^2)} \tag{9.41}$$

and, $$\text{Odd } F(s) = \frac{N_xD_y - N_yD_x}{(D_y^2-D_x^2)} \tag{9.42}$$

Now, it would be appropriate to replace the term s, which was kept silent for a while. This would result in obtaining the real part of $F(s)$ from the even part and the imaginary part of $F(s)$ from the odd part. Thus we may write the following terms as

$$\text{Re } |F(j\omega)| = \text{Even } F(s)_{s=j\omega} = \left.\frac{N_y(s)D_y(s) - N_x(s)D_x(s)}{(D_y^2(s)-D_x^2(s))}\right|_{s=j\omega} \tag{9.43}$$

$$\text{Im } |F(j\omega)| = \text{Odd } F(s)_{s=j\omega} = \left.\frac{N_x(s)D_y(s) - N_y(s)D_x(s)}{(D_y^2(s)-D_x^2(s))}\right|_{s=j\omega} \tag{9.44}$$

Due to squaring effect, the denominator would be always positive. Hence, for positive realness of $F(s)$, the least requirement to be fulfilled reduces to the criterion that the numerator of Eqn. (9.43) should be equal to or greater than zero for all values of ω. This may be written as a new function $P(\omega^2)$;

$$P(\omega^2) = N_y(s)D_y(s) - N_x(s)D_x(s)\Big|_{s=j\omega} \geq 0 \text{ for all values of } \omega. \tag{9.45}$$

Equation (9.45) may be used as the governing expression for satisfying the sufficient condition of item-2(b).

Example 9.19 Examine whether the network function given by $F(s) = \dfrac{s^2+4s+3}{s^2+6s+8}$ is positive real.

Solution The given network function may be written as

$$F(s) = \frac{s^2+4s+3}{s^2+6s+8} = \frac{(s+3)(s+1)}{(s+4)(s+2)}$$

By observing individual factor terms of the numerator and denominator, we may find that the poles of the function are located at $s = -4$, $s = -2$, and the zeros of the function are located at $s = -3$, $s = -1$ respectively. The following points need to be examined for positive realness.

i. All the roots have negative real parts. Hence all are located on the left half of the s plane.

ii. As there is no pole on the $j\omega$ axis, no residue is to be calculated.

iii. In order to check that Re $|F(j\omega)| \geq 0$, for all values of ω, we may verify for $P(\omega^2) = N_x(s)D_x(s) - N_y(s)D_y(s)\Big|_{s=j\omega}$ for all values of ω.

For the given polynomial, $N(s) = s^2 + 4s + 3$ and $D(s) = s^2 + 6s + 8$

where, $N_y(s) = s^2 + 3$, $N_x(s) = 4s$, $D_y(s) = s^2 + 8$, $D_x(s) = 6s$

So,
$$\begin{aligned}
P(\omega^2) &= N_y(s)D_y(s) - N_x(s)D_x(s)\Big|_{s=j\omega} \\
&= \{(s^2+3)(s^2+8) - (4s)(6s)\}\Big|_{s=j\omega} \\
&= s^4 - 13s^2 + 24\Big|_{s=j\omega} \\
&= \omega^4 + 13\omega^2 + 24 \geq 0 \text{ for all values of } \omega
\end{aligned}$$

So, this condition is satisfied.

Since, all the above conditions are satisfied, the function is positive real.

Example 9.20 Examine whether the network function given by $F(s) = \dfrac{s+3}{s+8}$ is positive real.

Solution By observing individual terms of the numerator and denominator, we may find that the pole of the function is located at $s = -8$, and the zero of the function is located at $s = -3$ respectively. The following points need to be examined for positive realness.

i. All the roots have negative real parts. Hence all are located on the left half of the s plane.

ii. As there is no pole on the $j\omega$ axis, no residue is to be calculated.

iii. In order to check that Re $|F(j\omega)| \geq 0$, for all values of ω, we may verify for $P(\omega^2) = N_x(s)\,D_x(s) - N_y(s)\,D_y(s)\big|_{s=j\omega} \geq 0$ for all values of ω.

For the given polynomial, $N(s) = s + 3$ and $D(s) = s + 8$

where, $N_y(s) = 3$, $N_x(s) = s$, $D_y(s) = 8$, $D_x(s) = s$

So,
$$P(\omega^2) = N_y(s)\,D_y(s) - N_x(s)\,D_x(s)\big|_{s=j\omega}$$
$$= \{(3)(8) - (s)(s)\}\big|_{s=j\omega}$$
$$= 24 - s^2\big|_{s=j\omega}$$
$$= 24 + \omega^2 \geq 0 \text{ for all values of } \omega$$

So, this condition is satisfied.

Since all the above conditions are satisfied, the function is positive real

9.8 CONCEPTS OF NETWORK SYNTHESIS

Fundamentally speaking, network synthesis is the art of finding the passive elements (impedances and admittances) of a network from the given network function satisfying the required excitation–response relationship, and to represent them with proper connectivity (series or parallel) by realizing the basic rules covered so far in this chapter. The following few guidelines may be useful for the reader in this regard.

- It may be noted that a network function $F(s)$ forms the basic requirement for network synthesis, which may be decomposed by partial fraction expansion to obtain the component functions.
- In most of the cases, $F(s)$ may be composed of passive functions (driving point impedances and admittances).
- Since the impedances and admittances are inherently positive and real, hence the network function should be a positive real function. Therefore, a formal test for positive realness of $F(s)$ may be viewed as a mandatory requirement before decomposing the function into component terms.
- If $F(s)$ is taken as an impedance function then a series connection of component elements is preferable, i.e., $Z(s) = Z_1(s) + Z_2(s)$. Thus, removal of one component from the parent function would result in the other component.
- If, $F(s)$ is taken as an admittance function then a parallel connection of component elements is preferable, i.e., $Y(s) = Y_1(s) + Y_2(s)$. Thus, removal of one component from the parent function would result in the other component.
- While removing one positive real function from the parent function which is also positive real, it should be borne in mind that the poles and zeros of the component are completely removed

from the overall pole-zero listing of the parent function. Removal of these poles and zeros require careful analysis of the situation. This will be dealt subsequently.

9.8.1 Removal of a Pole at Infinity

When a network function is said to have pole at infinity (∞) then it is obvious that the highest degree of s in the numerator polynomial exceeds the highest degree of s in the denominator polynomial by a margin of unity. Such a function is shown in the following expression.

$$F(s) = \frac{a_{n+1}s^{n+1} + a_n s^n + a_{n-1}s^{n-1} + \dots\dots\dots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \dots\dots\dots + b_1 s + b_0} \tag{9.46}$$

By partial fraction expansion this may be decomposed into

$$F(s) = \frac{a_{n+1}}{b_n} = \frac{c_n s^n + c_{n-1}s^{n-1} + \dots\dots\dots + c_1 s + c_0}{d_n s^n + d_{n-1}s^{n-1} + \dots\dots\dots + d_1 s + d_0} \tag{9.47}$$

or,

$$F(s) = K.s + \frac{c_n s^n + c_{n-1}s^{n-1} + \dots\dots\dots + c_1 s + c_0}{d_n s^n + d_{n-1}s^{n-1} + \dots\dots\dots + d_1 s + d_0} \tag{9.48}$$

Two possible cases may arise for interpretation of Eqn. (9.48) in this process of network synthesis. In the first case, $F(s)$ may be treated as an impedance function and would give a particular network structure. In the second case, $F(s)$ may be treated as an admittance function which would give another network structure. These two cases have been dealt here separately.

Case-I If $F(s)$ represents an impedance function then it may be possible to represent Eqn. (9.48) as a series combination of two impedances, $Z(s) = Z_1(s) + Z_{11}(s)$, such that

$$Z_1(s) = K.s = \frac{a_{n+1}}{b_n} \tag{9.49}$$

and,

$$Z_{11}(s) = \frac{c_n s^n + c_{n-1}s^{n-1} + \dots\dots\dots + c_1 s + c_0}{d_n s^n + d_{n-1}s^{n-1} + \dots\dots\dots + d_1 s + d_0} \tag{9.50}$$

This would give the scope for removal of $Z_1(s)$ from $Z(s)$, thereby leaving $Z_{11}(s)$ for further decomposition. Since $Z_1(s)$ is in the form of $K.s$, as an impedance it would represent an inductor of value $L = K$. Network structure for the expressions represented by Eqns. (9.48) to (9.50) may be shown by drawing a series circuit of the two impedances as shown in Fig. 9.9. However, this may not be the final result of network synthesis.

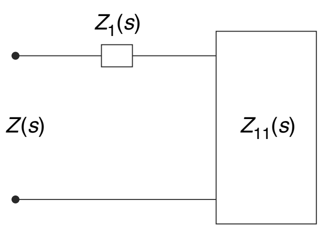


Figure 9.9 *Network structure for impedance function*

Then $Z_{11}(s)$ may be decomposed further through partial fraction expansion in order to split it into a series combination of another two impedances given by $Z_{11}(s) = Z_2(s) + Z_{22}(s)$, thereby enabling the removal of another series element $Z_2(s)$ from $Z_{11}(s)$. This process would continue until the last series element $Z_n(s)$ is obtained. A circuit diagram containing all the impedances in series, i.e., $Z_1(s)$, $Z_2(s)$,

$Z_3(s)$, ..., and $Z_n(s)$ may be drawn for representing the result of network synthesis. This is shown in Fig. 9.10.

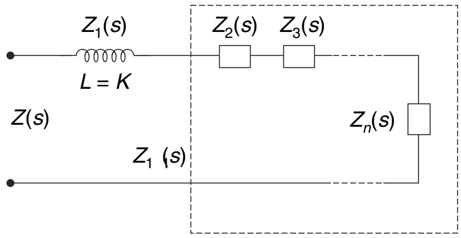


Figure 9.10 *Final network structure for impedance function*

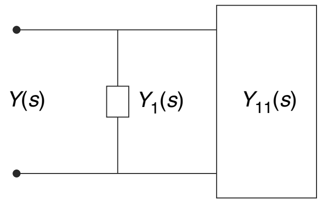


Figure 9.11 *Network structure for admittance function*

Case-II If $F(s)$ represents an admittance function, then it may be possible to represent Eqn. (9.48) as a parallel combination of two admittances, $Y(s) = Y_1(s) + Y_{11}(s)$, such that

$$Y_1(s) = K.s = \frac{a_{n+1}}{b_n} \tag{9.51}$$

and,

$$Y_{11}(s) = \frac{c_n s^n + c_{n-1}s^{n-1} + \ldots\ldots\ldots + c_1 s + c_0}{d_n s^n + d_{n-1}s^{n-1} + \ldots\ldots\ldots + d_1 s + d_0} \tag{9.52}$$

This would give the scope for removal of $Y_1(s)$ from $Y(s)$, thereby leaving $Y_{11}(s)$ for further decomposition. Since $Y_1(s)$ is in the form of $K.s$, as an admittance it would represent a capacitor of value $C = K$. Network structure for the expressions represented by Eqns. (9.48), (9.51) and (9.52) may be shown by drawing a parallel circuit of the two admittances as shown in Fig. 9.11. However, this may not be the final result of network synthesis.

Then $Y_{11}(s)$ may be decomposed further through partial fraction expansion in order to split it into a parallel combination of two admittances, $Y_{11}(s) = Y_2(s) + Y_{22}(s)$, thereby enabling the removal of another parallel element $Y_2(s)$ from $Y_{11}(s)$. This process would continue until the last parallel element $Y_n(s)$ is obtained. A circuit diagram containing all the admittances in parallel, i.e., $Y_1(s)$, $Y_2(s)$, $Y_3(s)$, ..., and $Y_n(s)$ may be drawn for representing the result of network synthesis. This is shown in Fig. 9.12.

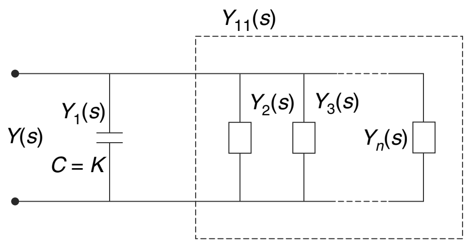


Figure 9.12 *Final network structure for admittance function*

9.8.2 Removal of a Pole at the Origin

When a network function is said to have a pole at the origin (0,0) of the s plane then it is obvious that the lowest degree of s in the numerator polynomial exceeds the lowest degree of s in the denominator polynomial by a margin of unity. Such a function is shown in the following expression.

$$F(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots\ldots\ldots + a_{n-1}s^{n-1} + a_n s^n}{b_1 s + b_2 s^2 + \ldots\ldots\ldots + b_{n-1}s^{n-1} + b_n s^n} \tag{9.53}$$

By partial fraction expansion this may be decomposed into

$$F(s) = \frac{a_0}{b_1} + \frac{c_1 + c_2 s + \ldots\ldots\ldots + c_{n-1}s^{n-2} + c_n s^{n-1}}{d_1 + d_2 s + \ldots\ldots\ldots + d_{n-1}s^{n-2} + d_n s^{n-1}} \tag{9.54}$$

or,

$$F(s) = \frac{K}{s} + \frac{c_1 + c_2 s + \ldots\ldots\ldots + c_{n-1}s^{n-2} + c_n s^{n-1}}{d_1 + d_2 s + \ldots\ldots\ldots + d_{n-1}s^{n-2} + d_n s^{n-1}} \tag{9.55}$$

Two possible cases may arise for interpretation of Eqn. (9.55) in this process of network synthesis. In the first case, $F(s)$ may be treated as an impedance function and would give a particular network structure. In the second case, $F(s)$ may be treated as an admittance function which would give another network structure. These two cases have been dealt here separately.

Case-I If $F(s)$ represents an impedance function, then it may be possible to represent Eqn. (9.55) as a series combination of two impedances, $Z(s) = Z_1(s) + Z_{11}(s)$, such that

$$Z_1(s) = \frac{K}{s} = \frac{a_0}{b_1} \tag{9.56}$$

and,

$$Z_{11}(s) = \frac{c_1 + c_2 s + \ldots\ldots\ldots + c_{n-1}s^{n-2} + c_n s^{n-1}}{d_1 + d_2 s + \ldots\ldots\ldots + d_{n-1}s^{n-2} + d_n s^{n-1}} \tag{9.57}$$

This would give the scope for removal of $Z_1(s)$ from $Z(s)$, thereby leaving $Z_{11}(s)$ for further decomposition. Since $Z_1(s)$ is in the form of K/s, as an impedance it would represent a capacitor of value $C = 1/K$. Network structure for the expressions represented by Eqns. (9.55) to (9.57) may be shown by drawing a series circuit of the two impedances as shown in Fig. 9.13. However, this may not be the final result of network synthesis.

Then $Z_{11}(s)$ may be decomposed further through partial fraction expansion in order to split it into a series combination of another two impedances given by $Z_{11}(s) = Z_2(s) + Z_{22}(s)$, thereby enabling the removal of another series element $Z_2(s)$ from $Z_{11}(s)$. This process would continue until the last series element $Z_n(s)$ is obtained. A circuit diagram containing all the impedances in series, i.e., $Z_1(s)$, $Z_2(s)$, $Z_3(s)$, ..., and $Z_n(s)$ may be drawn for representing the result of network synthesis. This is shown in Fig. 9.14.

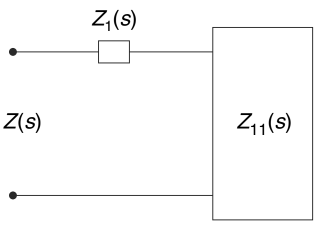


Figure 9.13 *Network structure for impedance function*

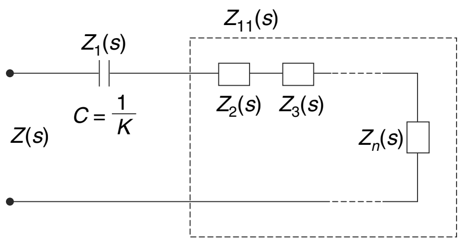


Figure 9.14 *Final network structure for impedance function*

Case-II If $F(s)$ represents an admittance function then it may be possible to represent Eqn. (9.55) as a parallel combination of two admittances, $Y(s) = Y_1(s) + Y_{11}(s)$, such that

$$Y_1(s) = \frac{K}{s} = \frac{a_0}{b_1} \tag{9.58}$$

And,
$$Y_{11}(s) = \frac{c_1 + c_2 s + \dots\dots\dots + c_{n-1}s^{n-2} + c_n s^{n-1}}{d_1 + d_2 s + \dots\dots\dots + d_{n-1}s^{n-2} + d_n s^{n-1}} \tag{9.59}$$

This would give the scope for removal of $Y_1(s)$ from $Y(s)$, thereby leaving $Y_{11}(s)$ for further decomposition. Since $Y_1(s)$ is in the form of K/s, as an admittances, it would represent an inductor of value $L = 1/K$. Network structure for the expressions represented by Eqns. (9.55), (9.58) and (9.59) may be shown by drawing a parallel circuit of the two admittances as shown in Fig. 9.15. However, this may not be the final result of network synthesis.

Then $Y_{11}(s)$ may be decomposed further through partial fraction expansion in order to split it into a parallel combination of two admittances, $Y_{11}(s) = Y_2(s) + Y_{22}(s)$, thereby enabling the removal of another parallel element $Y_2(s)$ from $Y_{11}(s)$. This process would continue until the last parallel element $Y_n(s)$ is obtained. A circuit diagram containing all the admittances in parallel, i.e., $Y_1(s)$, $Y_2(s)$, $Y_3(s)$, ..., and $Y_n(s)$ may be drawn for representing the result of network synthesis. This is shown in Fig. 9.16.

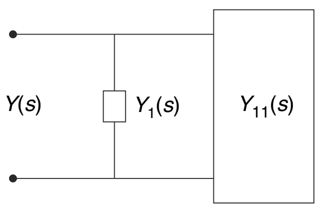


Figure 9.15 *Network structure for admittance function*

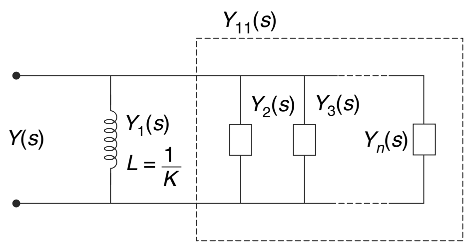


Figure 9.16 *Final network structure for admittance function*

9.8.3 Removal of Conjugate Imaginary Poles from $j\omega$ Axis

When a network function is said to have pairs of poles on the $j\omega$ axis of the s plane then it is obvious that the roots may have a value in the form of $s = \pm j\omega$. Assuming that the exact location of these roots have a value $s = \pm j\omega_0$, then the denominator polynomial must contain two factors in the form $(s - j\omega_0)$ and $(s + j\omega_0)$. Such a network function may be represented as the one shown in Eqn. (9.60).

$$F(s) = \frac{N(s)}{D(s)} = \frac{p(s)}{(s - j\omega_0)(s + j\omega_0)q(s)} \tag{9.60}$$

By partial fraction expansion this may be decomposed into

$$F(s) = \frac{K_0}{(s + j\omega_0)} + \frac{K_0^*}{(s - j\omega_0)} + F_2(s) \tag{9.61}$$

where, K_0 and K_0^* are the residues and are conjugate of each other. A similar analysis is available in Section 9.6, which substantiates the condition that K_0 and K_0^* must be identical and be real, i.e., $K_0 = K_0^* = K$, in order to give positive real residues. In view of this, Eqn. (9.61) reduces to

$$F(s) = \frac{2Ks}{s^2 + \omega_0^2} + F_2(s) \tag{9.62}$$

Two possible cases may arise for interpretation of Eqn. (9.62) in this process of network synthesis. In the first case, $F(s)$ may be treated as an impedance function and would give a particular network structure. In the second case, $F(s)$ may be treated as an admittance function which would give another network structure. These two cases have been dealt here separately.

Case-I If $F(s)$ represents an impedance function then it may be possible to represent Eqn. (9.62) as a series combination of two impedances, $Z(s) = Z_1(s) + Z_{11}(s)$, such that

$$Z_1(s) = \frac{2Ks}{s^2 + \omega_0^2} = \frac{1}{\dfrac{s}{2K} + \dfrac{\omega_0^2}{2Ks}} \tag{9.63}$$

and, $$Z_{11}(s) = F_2(s) \tag{9.64}$$

This would give the scope for removal of $Z_1(s)$ from $Z(s)$, thereby leaving $Z_{11}(s)$ for further decomposition. Since, $Z_1(s)$ is in the form of $1/Y_1(s)$, as an impedance, it would represent a parallel combination of an inductor of value $L = \dfrac{2Ks}{\omega_0^2}$ and a capacitor of value $C = \dfrac{1}{2K}$. Network structure for the expressions represented by Eqns. (9.62) to (9.64) may be shown by drawing a series circuit of the two impedances as shown in Fig. 9.17. However, this may not be the final result of network synthesis.

Then $Z_{11}(s)$ may be decomposed further through partial fraction expansion in order to split it into a series combination of another two impedances given by $Z_{11}(s) = Z_2(s) + Z_{22}$, thereby enabling the removal of another series element $Z_2(s)$ from $Z_{11}(s)$. This process would continue until the last series element $Z_n(s)$ is obtained. A circuit diagram containing all the impedances in series, i.e., $Z_1(s)$, $Z_2(s)$, $Z_3(s)$, ..., and $Z_n(s)$ may be drawn for representing the result of network synthesis. This is shown in Fig. 9.18.

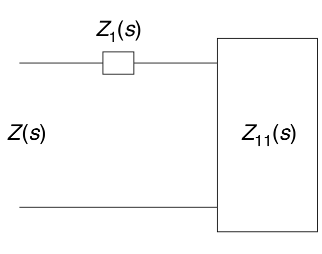


Figure 9.17 *Network structure for impedance function*

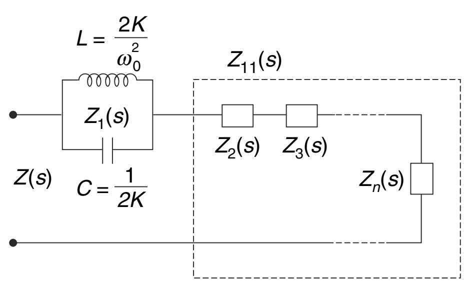


Figure 9.18 *Final network structure for impedance function*

Case-II If $F(s)$ represents an admittance function then it may be possible to represent Eqn. (9.62) as a parallel combination of two admittances, $Y(s) = Y_1(s) + Y_{11}(s)$, such that

$$Y_1(s) = \frac{2Ks}{s^2+\omega_0^2} = \frac{1}{\frac{s}{2K}+\frac{\omega_0^2}{2Ks}} \tag{9.65}$$

And, $$Y_{11}(s) = F_2(s) \tag{9.66}$$

This would give the scope for removal of $Y_1(s)$ from $Y(s)$, thereby leaving $Y_{11}(s)$ for further decomposition. Since $Y_1(s)$ is in the form of $1/Z_1(s)$, as an admittance it would represent a series combination of a capacitor of value $C = \frac{2Ks}{\omega_0^2}$ and an inductor of value $L = \frac{1}{2K}$. Network structure for the expressions represented by Eqns. (9.62), (9.65) and (9.66) may be shown by drawing a parallel circuit of the two admittances as shown in Fig. 9.19. However, this may not be the final result of network synthesis.

Then $Y_{11}(s)$ may be decomposed further through partial fraction expansion in order to split it into a parallel combination of two admittances, $Y_{11}(s) = Y_2(s) + Y_{22}(s)$, thereby enabling the removal of another parallel element $Y_2(s)$ from $Y_{11}(s)$. This process would continue until the last parallel element $Y_n(s)$ is obtained. A circuit diagram containing all the admittances in parallel, i.e., $Y_1(s)$, $Y_2(s)$, $Y_3(s)$, …….., and $Y_n(s)$ may be drawn for representing the result of network synthesis. This is shown in Fig. 9.20.

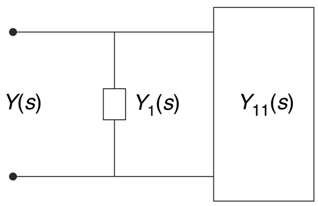


Figure 9.19 *Network structure for admittance function*

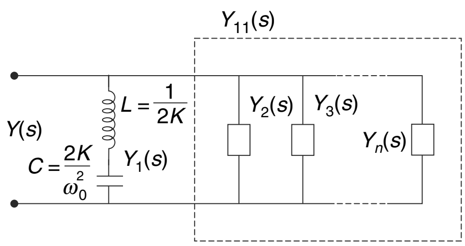


Figure 9.20 *Final network structure for admittance function*

It may be inferred from the above case that removal of a pair of complex conjugate poles from the network function, which are situated on the $j\omega$ axis of the s plane, is equivalent to a

(i) serial removal of an L–C parallel combination from the parent network if the network represents an impedance function, or otherwise

(ii) parallel removal of an L–C series combination from the parent network if the network represents an admittance function.

9.8.4 Removal of a Constant

When a network function contains a constant term K, it may be expressed as

$$F(s) = K + F_2(s) \tag{9.67}$$

Two possible cases may arise for interpretation of Eqn. (9.67) in this process of network synthesis. In the first case, $F(s)$ may be treated as an impedance function and would give a particular network structure. In the second case, $F(s)$ may be treated as an admittance function which would give another network structure. These two cases have been dealt here separately.

Case-I If $F(s)$ represents an impedance function then it may be possible to represent Eqn. (9.67) as a series combination of two impedances, $Z(s) = Z_1(s) + Z_{11}(s)$, such that

$$Z_1(s) = K \tag{9.68}$$

and, $$Z_{11}(s) = F_2(s) \tag{9.69}$$

This would give the scope for removal of $Z_1(s)$ from $Z(s)$, thereby leaving $Z_{11}(s)$ for further decomposition. Since $Z_1(s)$ is in the form of a positive real constant K, as an impedance it would represent a pure resistance of value $R = K$. Network structure for the expressions represented by Eqns. (9.67) to (9.69) may be shown by drawing a series circuit of the two impedances as shown in Fig. 9.21. However, this may not be the final result of network synthesis.

Then $Z_{11}(s)$ may be decomposed further through partial fraction expansion in order to split it into a series combination of another two impedances given by $Z_{11}(s) = Z_2(s) + Z_{22}(s)$, thereby enabling the removal of another series element $Z_2(s)$ from $Z_{11}(s)$. This process would continue until the last series element $Z_n(s)$ is obtained. A circuit diagram containing all the impedances in series, i.e., $Z_1(s)$, $Z_2(s)$, $Z_3(s)$, ..., and $Z_n(s)$ may be drawn for representing the result of network synthesis. This is shown in Fig. 9.22.

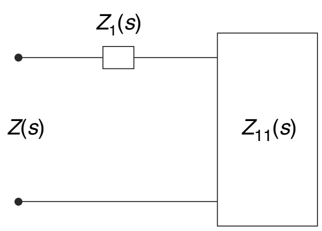


Figure 9.21 *Network structure for impedance function*

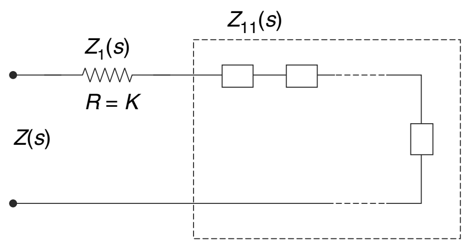


Figure 9.22 *Final network structure for impedance function*

Case-II If $F(s)$ represents an admittance function then it may be possible to represent Eqn. (9.67) as a parallel combination of two admittances, $Y(s) = Y_1(s) + Y_{11}(s)$, such that

$$Y_1(s) = K \tag{9.70}$$

and, $$Y_{11}(s) = F_2(s) \tag{9.71}$$

This would give the scope for removal of $Y_1(s)$ from $Y(s)$, thereby leaving $Y_{11}(s)$ for further decomposition. Since $Y_1(s)$ is in the form of a positive real constant K, as an admittance, it would represent a pure conductance of value $G = K$. Network structure for the expressions represented by Eqns. (9.67), (9.70) and (9.71) may be shown by drawing a parallel circuit of the two admittances as shown in Fig. 9.23. However, this may not be the final result of network synthesis.

Then $Y_{11}(s)$ may be decomposed further through partial fraction expansion in order to split it into a parallel combination of two admittances, $Y_{11}(s) = Y_2(s) + Y_{22}(s)$, thereby enabling the removal of another parallel element $Y_2(s)$ from $Y_{11}(s)$. This process would continue until the last parallel element $Y_n(s)$ is obtained. A circuit diagram containing all the admittances in parallel, i.e., $Y_1(s)$, $Y_2(s)$, $Y_3(s)$, ..., and $Y_n(s)$ may be drawn for representing the result of network synthesis. This is shown in Fig. 9.24.

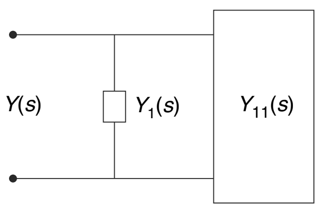


Figure 9.23 *Network structure for admittance function*

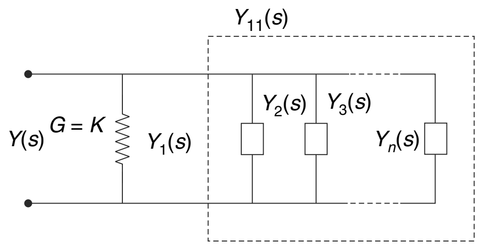


Figure 9.24 *Final network structure for admittance function*

9.9 REALIZATION OF ELEMENTS BY CAUER AND FOSTER FORMS

In the preceding section we have come across various schemes for conducting a step-by-step removal of passive network elements from the original network, which is based on removal of network poles from various locations of the *s* plane. Two basic mathematical tools involved in decomposing the network function into component functions are (i) continued fraction expansion method, and (ii) partial fraction expansion method.

In this section, we would see similar workouts in a different nomenclature, i.e.,

(i) *Cauer form* or *Ladder form* of representation of results of network synthesis, which has two classifications:
- Cauer-I form
- Cauer-II form

(ii) *Foster form* of representation of results of network synthesis, which has two classifications:
- Foster-I form
- Foster-II form

A systematic approach for realization of the passive network elements with various possible combinations or grouping (i.e., R–L, R–C and L–C) based on the Cauer and Foster forms is presented below.

9.9.1 Realization of Elements by Cauer-I Form

Cauer-I form of network realization is primarily based on the continued fraction expansion technique, which is applied to decompose a given network function by adopting repeated inversion and division about the pole located at infinity. In order to understand the mechanism followed in this method, let us assume a network function in the form of Eqn. (9.72).

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \dots\dots\dots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \dots\dots\dots + b_1 s + b_0} \quad (9.72)$$

The quotient terms may be found from division with repeat inversion by applying the continued fraction expansion, which has been explained earlier in this chapter. The quotient terms resulting in this process are $Q_1(s)$, $Q_2(s)$, $Q_3(s)$, ... so on. The following few guidelines may be helpful while dealing with the continued fraction expansion and analyzing the results of this process.

1. $F(s)$ may be treated as an impedance function or an admittance function,
2. The numerator and denominator polynomials described by $N(s)$ and $D(s)$ must be arranged in order of decreasing power of s.
3. Continued fraction expansion should be performed with $N(s)$ treated as dividend and $D(s)$ as the divisor.
4. If $F(s)$ happens to be an impedance function then the first quotient term, i.e., $Q_1(s)$ will be having the properties of impedance with subsequent quotient terms having alternate properties of admittance and impedances.
5. If $F(s)$ happens to be an admittance function then the first quotient term, i.e., $Q_1(s)$ will be having the properties of admittance with subsequent quotient terms having alternate properties of impedance and admittances.
6. In the course of continued fraction expansion, if any remainder term acquires negative sign of computation then the process should not be carried forward any further. Rather, the whole process of continued fraction expansion should be restarted by mutually interchanging the positions of numerator and denominator polynomials, as shown below. In this case $D(s)$ should be treated as dividend and $N(s)$ as the divisor.
7. While showing the connectivity of elements in the circuit diagram, impedances should be connected in series branch, and admittances should be connected in parallel branches.

Example 9.21 Realize the network function given by $Z(s) = \dfrac{6s^4 + 42s^2 + 48}{s^2 + 18s^3 + 48s}$ in Cauer-I form.

Solution In Cauer-I form, the function may be treated as an impedance or admittance as well. However, in this case, since the denominator has a higher order than the numerator, we have to find the quotient terms from the continued fraction expansion of $1/Z(s)$ and may treat it as an admittance.

$$Y(s) = \frac{1}{F(s)} = \frac{s^2 + 18s^3 + 48s}{6s^4 + 42s^2 + 48}$$

which gives,

$$Y(s) = (0.166)s + \cfrac{1}{(0.545)s + \cfrac{1}{(0.545)s + \cfrac{1}{(1.458)s + \cfrac{1}{(0.288)s}}}}$$

Realization of quotient terms

- The first quotient term is $0.166s$, which as an admittance represents a capacitance in a parallel branch having a value $C = 0.166$ F.
- The second quotient term is $0.545s$, which as an impedance represents an inductance in a series branch having a value $L = 0.545$ H.
- The third quotient term is $0.545s$, which as an admittance represents a capacitance in a parallel branch having a value $C = 0.545$ F.
- The fourth quotient term is $1.458s$, which as an impedance represents an inductance in a series branch having a value $L = 1.458$ H.
- The fifth quotient term is $0.288s$, which as an admittance represents a capacitance in a parallel branch having a value $C = 0.288$ F.

The circuit diagram and connectivity for these elements is shown in Fig. 9.25.

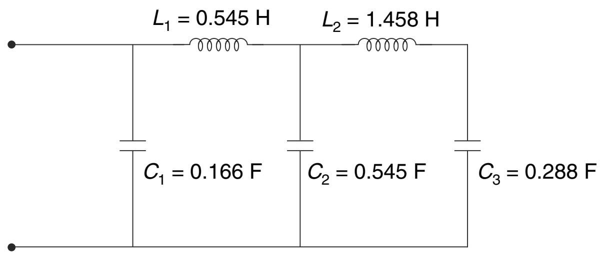


Figure 9.25

9.9.2 Realization of Elements by Cauer-II Form

The Cauer-II form of network realization is primarily based on continued fraction expansion technique, which is applied to decompose a given network function by adopting repeated inversion and division about the pole located at the origin. In order to understand the mechanism followed in this method, let us assume a network function in the form of Eqn. (9.72).

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \dots\dots\dots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \dots\dots\dots + b_1 s + b_0} \tag{9.72}$$

The quotient terms may be found from division with repeat inversion by applying the continued fraction expansion, which has been explained earlier in this chapter. The quotient terms resulting in this process are $Q_1(s)$, $Q_2(s)$, $Q_3(s)$, ... so on. The following few guidelines may be helpful while dealing with the continued fraction expansion and analyzing the results of this process.

1. $F(s)$ may be treated as an impedance function or an admittance function.
2. The numerator and denominator polynomials described by $N(s)$ and $D(s)$ must be arranged in order of increasing power of s.
3. Continued fraction expansion should be performed with $N(s)$ treated as dividend and $D(s)$ as the divisor.

4. If $F(s)$ happens to be an impedance function then the first quotient term, i.e., $Q_1(s)$ will be having the properties of impedance with subsequent quotient terms having alternate properties of admittance and impedances.
5. If $F(s)$ happens to be an admittance function then the first quotient term, i.e., $Q_1(s)$ will be having the properties of admittance with subsequent quotient terms having alternate properties of impedance and admittances.
6. In the course of continued fraction expansion, if any remainder term acquires negative sign of computation then the process should not be carried forward any further. Rather, the whole process of continued fraction expansion should be restarted by mutually interchanging the positions of numerator and denominator polynomials, as shown below. In this case $D(s)$ should be treated as dividend and $N(s)$ as the divisor.
7. While showing the connectivity of elements in the circuit diagram, impedances should be connected in series branch and admittances should be connected in parallel branches.

Example 9.22 Realize the network function given by $Z(s) = \dfrac{s^5 + 3s^3 + s}{3s^4 + 4s^2 + 1}$ in Cauer-II form as an admittance.

Solution In the Cauer-II form, the function may be treated as an impedance or admittance as well. According to the question, we have to realize the network considering the given function as an admittance. Hence, the quotient terms may be obtained from the continued fraction expansion of $Y(s) = 1/Z(s)$. By arranging the order of polynomials in increasing order of s, we get

$$Y(s) = \frac{1}{Z(s)} = \frac{1 + 4s^2 + 3s^4}{s + 3s^3 + s^5}$$

Continued fraction expansion of this gives,

$$Y(s) = 1/s + \cfrac{1}{1/s + \cfrac{1}{1/s + \cfrac{1}{1/s + \cfrac{1}{1/s}}}}$$

Realization of quotient terms

- The first quotient term is $1/s$, which as an admittance represents an inductance in a parallel branch having a value $L = 1$ H.
- The second quotient term is $1/s$, which as an impedance represents a capacitance in a series branch having a value $C = 1$ F.
- The third quotient term is $1/s$, which as an admittance represents an inductance in a parallel branch having a value $L = 1$ H.
- The fourth quotient term is $1/s$, which as an impedance represents a capacitance in a series branch having a value $C = 1$ F.

- The fifth quotient term is $1/s$, which as an admittance represents an inductance in a parallel branch having a value $L = 1$ H.

The circuit diagram and connectivity for these elements is shown in Fig. 9.26.

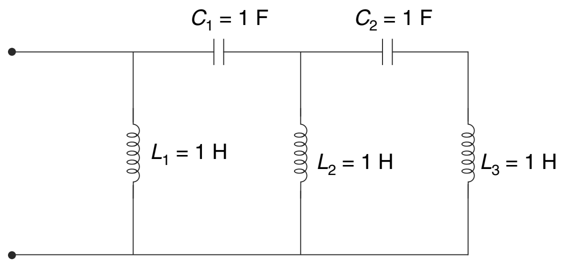


Figure 9.26

9.9.3 Realization of Elements by Foster-I Form

The Foster-I form of network realization is primarily based on the partial fraction expansion technique, which is applied to decompose network functions $F(s)$ representing impedance functions $Z(s)$ only into component functions. In this process the main objective is to obtain the residues of partial fraction expansion. If, at any stage, a residue acquires a negative sign of computation, then the process should be terminated at once and another partial fraction expansion should be performed for the function, $F'(s) = \frac{F(s)}{s}$. Finally, s may be multiplied to each term of the result so obtained in order to compensate for the factor s in $F'(s)$. Each component term of the partial fraction expansion represents a particular type of network element and may be shown in the circuit diagram with proper connectivity in order to realize the said network. While showing the connectivity of elements in the circuit diagram, impedances should be connected in a series branch and admittances should be connected in parallel branches.

Example 9.23 Realize the network function given by $Y(s) = \frac{s^2 + 8s + 12}{s + 4}$ in Foster-I form.

Solution In the Foster-I form, the network function is to be treated as an impedance. Since the given function is in the form of an admittance, we better take the inverse and express it as an impedance for application of partial fraction expansion. So,

$Z(s) = \frac{1}{Y(s)} = \frac{s+4}{s^2+8s+12} = \frac{(s+4)}{(s+2)(s+6)} = \frac{A_1}{(s+2)} + \frac{A_2}{(s+6)}$, On solving this, we may get, $A_1 = A_2 = 1/2$. On substitution of the values, we find that

$$Z(s) = \frac{\frac{1}{2}}{(s+2)} + \frac{\frac{1}{2}}{(s+6)} = \frac{1}{2s+4} + \frac{1}{2s+12} = Z_1 + Z_2 \; (say)$$

This would lead to series combination of two impedances $Z_1 = \dfrac{1}{2s+4}$ and $Z_2 = \dfrac{1}{2s+12}$. When viewed separately, each impedance may be realized as a parallel combination of two elements.

When Z_1 is viewed as $1/(Y_a + Y_b)$ then the first admittance element corresponding to $Y_a = 2s$ would represent a capacitance of value $C_1 = 2$ F, and the second element corresponding to $Y_b = 4$ would mean a conductance with a value of 4 mho or a resistance of value $R_1 = 1/4\ \Omega$.

In a similar way, while viewing Z_2 as $1/(Y_c + Y_d)$, the first admittance element corresponding to $Y_c = 2s$ would represent a capacitance of value $C_2 = 2$ F, and the second element corresponding to $Y_d = 12$ would mean a conductance with a value of 12 mho or a resistance of value $R_2 = 1/12\ \Omega$.

The above break up indicates the series combination of two parallel branches, each parallel branch having an R–C combination. The network diagram for this realization is shown in Fig. 9.27.

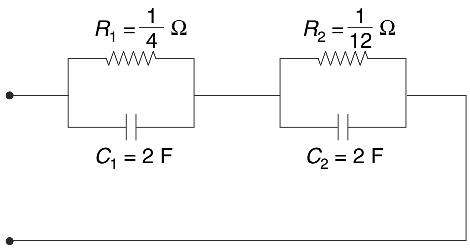


Figure 9.27

9.9.4 Realization of Elements by Foster-II Form

The Foster-II form of network realization is primarily based on partial fraction expansion technique, which is applied to decompose network functions $F(s)$ representing admittance functions $Y(s)$ only into component functions. In this process the main objective is to obtain the residues of partial fraction expansion. If, at any stage, a residue acquires a negative sign of computation then the process should be terminated at once and another partial fraction expansion should be performed for the function, $F'(s) = \dfrac{F(s)}{s}$. Finally, s may be multiplied to each term of the result so obtained in order to compensate for the factor s in $F'(s)$. Each component term of the partial fraction expansion represents a particular type of network element and may be shown in the circuit diagram with proper connectivity in order to realize the said network. While showing the connectivity of elements in the circuit diagram, impedances should be connected in a series branch and admittances should be connected in parallel branches.

Example 9.24 Realize the network function given by $Y(s) = \dfrac{(s+1)(s+3)}{2(s+2)(s+4)}$ in the Foster-II form.

Solution In the Foster-II form, the network function is to be treated as an admittance and the function is to be decomposed by partial fraction expansion. So, by taking the partial fraction of $Y(s)/s$, we may represent the function as

$$\frac{Y(s)}{s} = \frac{A_0}{s} + \frac{A_1}{(s+2)} + \frac{A_2}{(s+4)}$$

The coefficients A_0, A_1, and A_2 are found to be 3/16, 1/8, and 3/16 respectively. On substitution of these values and multiplying s on both sides of the function, we get

$$Y(s) = \frac{3}{16} + \frac{\frac{s}{8}}{(s+2)} + \frac{\frac{3s}{16}}{(s+4)} = \frac{3}{16} + \frac{1}{\left(8 + \frac{16}{s}\right)} + \frac{1}{\left(\frac{16}{3} + \frac{64}{3s}\right)}$$

$$= Y_1 + Y_2 + Y_3 \ (say)$$

The break-up in the last expression indicates the parallel combination of three branches having admittances Y_1, Y_2, and Y_3 respectively. When viewed separately, each admittance may be realized as some combination of passive elements.

When Y_1 is viewed as a constant of value 3/16, it would represent a conductance of 3/16 mho or a resistance of R_1 = 16/3 Ω.

In a similar way, while viewing Y_2 as $1/(Z_a + Z_b)$, the first impedance element corresponding to Z_a = 8, would mean a resistance of value R_2 = 8 Ω in series with the second element corresponding to Z_b = 16/s that would represent a capacitance of value C_2 = 1/16 F.

In a similar way, while viewing Y_3 as $1/(Z_c + Z_d)$, the first impedance element corresponding to Z_c = 16/3, would mean a resistance of value R_3 = 16/3 Ω in series with the second element corresponding to Z_d = 64/3s that would represent a capacitance of value C_3 = 3/64 F. The network diagram for this realization is shown in Fig. 9.28.

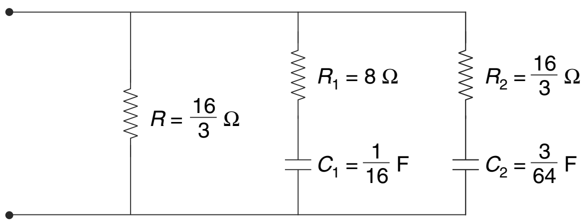


Figure 9.28

SOLVED PROBLEMS

9.1 Realize the network function given by $F(s) = \frac{(s+1)(s+3)}{s(s+2)}$ in the Foster-I form.

Solution In the Foster-I form, the network function is to be treated as an impedance. So, the function may be written as

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)} = A_0 + \frac{A_1}{s} + \frac{A_2}{(s+2)} \ (say)$$

On solving this, we may get, A_0 = 1, A_1 = 3/2, and A_2 = 1/2. On substitution of the values, we find that

$$Z(s) = 1 + \frac{3/2}{s} + \frac{1/2}{(s+2)} = 1 + \frac{1}{\frac{2s}{3}} + \frac{1}{(2s+4)} = Z_1 + Z_2 + Z_3 \ (say)$$

This would lead to a series combination of three impedances; $Z_1 = 1$, $Z_2 = \frac{1}{\frac{2s}{3}}$ and $Z_3 = \frac{1}{(2s+4)}$. When viewed separately, each impedance may be realized as some combination of passive elements.

When Z_1 is viewed as a constant, it would represent a resistance of value $R_1 = 1\ \Omega$.

In a similar way, while viewing Z_2 as $1/C.s$, it would represent a capacitance of value $C_2 = 2/3$ F.

In a similar way, while viewing Z_3 as $1/(Y_a + Y_b)$, the first admittance element corresponding to $Y_a = 2s$ would represent a capacitance of value $C_3 = 2$ F, and the second element corresponding to $Y_b = 4$ would mean a conductance with a value of 4 mho or a resistance of value $R_3 = 1/4\ \Omega$. The network diagram for this realization is shown in Fig. 9.29.

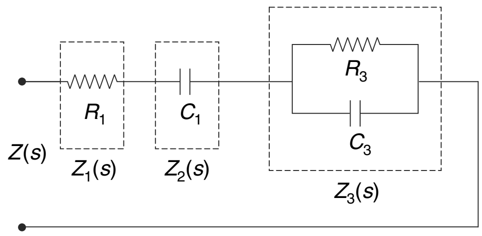


Figure 9.29

9.2 Realize the network function given by $F(s) = \frac{(s+1)(s+3)}{s(s+2)}$ in the Foster-II form.

Solution In the Foster-II form, the network function is to be treated as an admittance. So, the function may be written as

$$Y(s) = \frac{(s+1)(s+3)}{s(s+2)} = A_0 + \frac{A_1}{s} + \frac{A_2}{(s+2)} \ (say).$$

On solving this, we may get, $A_0 = 1$, $A_1 = 3/2$, and $A_2 = 1/2$. On substitution of the values, we find that

$$Y(s) = 1 + \frac{3/2}{s} + \frac{1/2}{(s+2)} = 1 + \frac{1}{\frac{2s}{3}} + \frac{1}{(2s+4)} = Y_1 + Y_2 + Y_3 \ (say)$$

This would lead to parallel combination of three admittances; $Y_1 = 1$, $Y_2 = \dfrac{1}{\dfrac{2s}{3}}$ and

$Y_3 = \dfrac{1}{(2s+4)}$. When viewed separately, each admittance may be realized as some combination of passive elements.

When Y_1 is viewed as a constant, it would represent a conductance with a value of 1 mho, which is equivalent to a resistance of value $R_1 = 1\ \Omega$.

In a similar way, while viewing Y_2 as $1/L.s$, it would represent an inductance of value L_2 = 2/3 H.

In a similar way, while viewing Y_3 as $1/(Z_a + Z_b)$, the first impedance element corresponding to $Z_a = 2s$ would represent an inductance of value $L_3 = 2$ H, and the second element corresponding to $Z_b = 4$ would mean a resistance of value $R_3 = 4\ \Omega$. The network diagram for this realization is shown in Fig. 9.30.

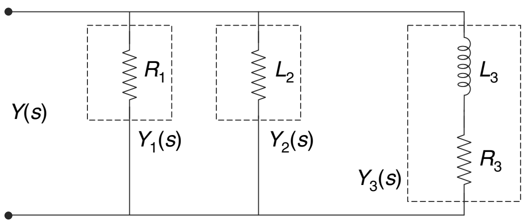


Figure 9.30

9.3 Realize the network function given by $F(s) = \dfrac{(s+2)}{(s+1)(s+3)}$ in the Foster-I form.

Solution In the Foster-I form, the network function is to be treated as an impedance. So, the function may be written as

$$Z(s) = \frac{(s+2)}{(s+1)(s+3)} = \frac{A_1}{(s+1)} + \frac{A_2}{(s+3)} \ (say).$$

On solving this, we may get, $A_1 = 1/2$, and $A_2 = 1/2$. On substitution of the values, we find that

$$Z(s) = \frac{1/2}{(s+1)} + \frac{1/2}{(s+3)} = \frac{1}{(2s+2)} + \frac{1}{(2s+6)} = Z_1 + Z_2 \ (say)$$

This would lead to a series combination of two impedances; $Z_1 = \dfrac{1}{(2s+2)}$, and $Z_2 = \dfrac{1}{(2s+6)}$.

When viewed separately, each impedance may be realized as some combination of passive elements.

While viewing Z_1 as $1/(Y_a + Y_b)$, the first admittance element corresponding to $Y_a = 2s$ would represent a capacitance of value $C_1 = 2$ F, and the second element corresponding to $Y_b = 2$ would mean a conductance with a value of 2 mho or a resistance of value $R_1 = 1/2\ \Omega$.

In a similar way, while viewing Z_2 as $1/(Y_c + Y_d)$, the first admittance element corresponding to $Y_c = 2s$ would represent a capacitance of value $C_2 = 2$ F, and the second element corresponding to $Y_b = 6$ would mean a conductance value of 6 mho or a resistance of value $R_2 = 1/6\ \Omega$. The network diagram for this realization is shown in Fig. 9.31.

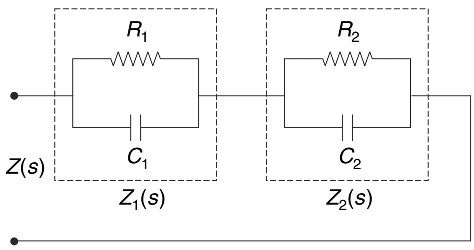


Figure 9.31

9.4 Realize the network function given by $\dfrac{F(s)}{s} = \dfrac{(s+2)}{(s+1)(s+3)}$ in the Foster-II form.

Solution In the Foster-II form, the network function is to be treated as an admittance. So, the function may be written as

$$\frac{Y(s)}{s} = \frac{(s+2)}{(s+1)(s+3)} = \frac{A_1}{(s+1)} + \frac{A_2}{(s+3)}\ (say).$$

On solving this, we may get, A_1=1/2, and A_2=1/2. On substitution of the values, we find that

$$\frac{Y(s)}{s} = \frac{1/2}{(s+1)} + \frac{1/2}{(s+3)} = \frac{1}{(2s+2)} + \frac{1}{(2s+6)}$$

$$\Rightarrow \qquad Y(s) = \frac{s}{(2s+2)} + \frac{s}{(2s+6)} = \frac{1}{\left(2+\dfrac{2}{s}\right)} + \frac{1}{\left(2+\dfrac{6}{s}\right)} = Y_1(s) + Y_2(s)\ (say)$$

This would lead to parallel combination of two admittances; $Y_1 = \dfrac{1}{\left(2+\dfrac{2}{s}\right)} = \dfrac{1}{Z_a + Z_b}$, and

$Y_2 = \dfrac{1}{\left(2+\dfrac{6}{s}\right)} = \dfrac{1}{Z_c + Z_d}$. When viewed separately, each admittance may be realized as some combination of passive elements.

When Y_1 is viewed as $\dfrac{1}{Z_a + Z_b}$, it would represent a series combination of a resistance of value R_1 = 2 Ω and a capacitance of value C_1 = 1/2 F.

In a similar way, while viewing Y_2 as $\dfrac{1}{Z_c + Z_d}$, it would represent a series combination of a resistance of value R_2 = 2 Ω and a capacitance of value C_2 = 1/6 F. The network diagram for this realization is shown in Fig. 9.32.

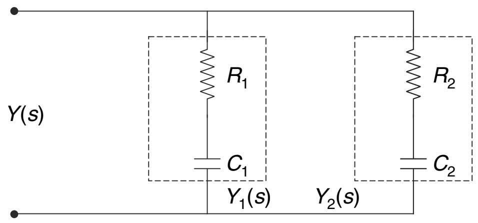


Figure 9.32

9.5 Realize the network function given by $F(s) = \dfrac{s(s+1)(s+3)}{s(s+2)}$ in the Foster-II form.

Solution In the Foster-II form, the network function is to be treated as an admittance. So, the function may be written as

$$\frac{Y(s)}{s} = \frac{s(s+1)(s+3)}{s(s+2)} = A_0 + \frac{A_1}{s} + \frac{A_2}{(s+2)} \ (say).$$

On solving this, we may get, $A_0 = 1$, $A_1 = 3/2$, and $A_2 = 1/2$. On substitution of the values, we find that

$$\frac{Y(s)}{s} = 1 + \frac{3/2}{s} + \frac{1/2}{(s+2)} = 1 + \frac{1}{\frac{2s}{3}} + \frac{1}{(2s+4)}$$

$$\Rightarrow \qquad Y(s) = s + \frac{s}{\frac{2s}{3}} + \frac{s}{(2s+4)} = s + \frac{3}{2} + \frac{1}{\left(2+\frac{4}{s}\right)} = Y_1(s) + Y_2(s) + Y_3(s) \ (say)$$

This would lead to parallel combination of three admittances; $Y_1 = S$, $Y_2 = \dfrac{3}{2}$ and $Y_3 = \dfrac{1}{\left(2+\frac{4}{s}\right)}$. When viewed separately, each admittance may be realized as some combination of passive elements.

When Y_1 is viewed as $C.s$, it would represent a capacitance of value C_1 = 1 F. When Y_2 is viewed as a constant, it would represent a conductance value of 3/2 mho, which is equivalent to a resistance of value R_2 = 2/3 Ω. In a similar way, while viewing Y_3 as $1/(Z_a + Z_b)$, the first impedance element corresponding to Z_a = 2 would represent a resistance of value R_3 = 2 Ω, and the second element corresponding to Z_b = 4/s would mean a capacitance of value C_3 = 1/4 F. The network diagram for this realization is shown in Fig. 9.33.

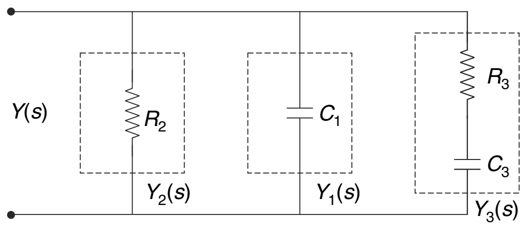


Figure 9.33

9.6 Realize the network function given by $F(s) = \dfrac{s^2 + 4s + 3}{s^2 + 2s}$ in the Cauer-I form by assuming that $F(s)$ represents an impedance function.

Solution In the Cauer-I form, the function is expanded through continued fraction expansion by performing alternate division and inversion for obtaining the quotient terms. Since, the function is to be treated as an impedance function, hence the first quotient term must represent an impedance followed by alternate terms of admittance and impedance in the subsequent quotient terms. While performing the continued fraction expansion in the Cauer-I form, the polynomials in the numerator and denominator must be arranged in an increasing order of s. Thus we may get the quotient terms of the division as

$$Z(s) = Z_1(s) + \cfrac{1}{Y_1(s) + \cfrac{1}{Z_2(s) + \cfrac{1}{Y_2(s) + \ldots}}}$$

The general circuit for this network may be as shown in Fig. 9.34.

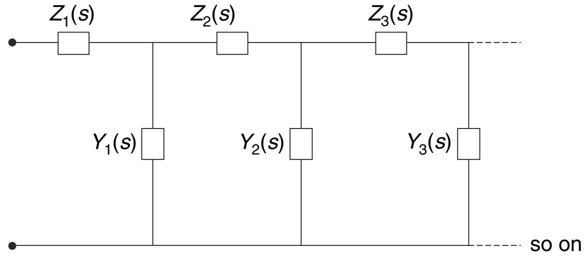


Figure 9.34

The actual result of continued fraction expansion would give the following form.

$$Z(s) = \frac{s^2+4s+3}{s^2+2s} = 1 + \cfrac{1}{\cfrac{s}{2} + \cfrac{1}{4 + \cfrac{1}{\cfrac{s}{6}}}}$$

On comparison of the result with the general form indicated above, we may find that the first series element represents a resistance of value R_1 = 1 Ω. The second element is connected in a parallel branch and happens to be a capacitance of value C_1 = 1/2 F. The third element represents a resistance in the series branch of value R_2 = 4 Ω, and the fourth element connected in a parallel branch happens to be a capacitance of value C_2 = 1/6 F. The circuit model for this network may be as shown in Fig. 9.35.

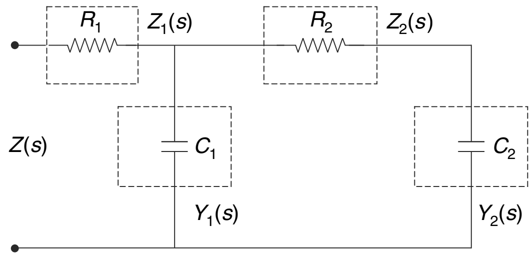


Figure 9.35

9.7 Realize the network function given by $F(s) = \frac{s^2+4s+3}{s^2+2s}$ in the Cauer-I form by assuming that $F(s)$ represents an admittance function.

Solution In the Cauer-I form, the function is expanded through continued fraction expansion by performing alternate division and inversion for obtaining the quotient terms. Since, the function is to be treated as an admittance function, hence the first quotient term must represent an admittance followed by alternate terms of impedance and admittance in the subsequent quotient terms. While performing the continued fraction expansion in the Cauer-I form, the polynomials in the numerator and denominator must be arranged in an increasing order of s. Thus we may get the quotient terms of the division as

$$F(s) = Y_1(s) + \cfrac{1}{Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_2(s) + \ldots}}}$$

The general circuit for this network may be as shown in Fig. 9.36.

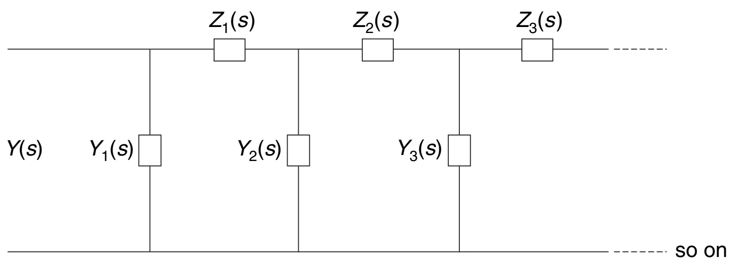


Figure 9.36

The actual result of continued fraction expansion would give the following form.

$$Y(s) = \frac{s^2+4s+3}{s^2+2s} = 1 + \cfrac{1}{\cfrac{s}{2} + \cfrac{1}{4 + \cfrac{1}{\cfrac{s}{6}}}}$$

On comparison of the result with the general form indicated above, we may find that the first parallel element represents a conductance value of 1 mho, which is equivalent to a resistance of value R_1 = 1 Ω. The second element is connected in a series branch and happens to be an inductance of value L_1 = 1/2 H. The third element represents a conductance value of 4 mho, which is equivalent to a resistance in the parallel branch of value R_2 = 1/4 Ω, and the fourth element connected in a series branch happens to be an inductor of value L_2 = 1/6 H. The circuit model for this network may be as shown in Fig. 9.37.

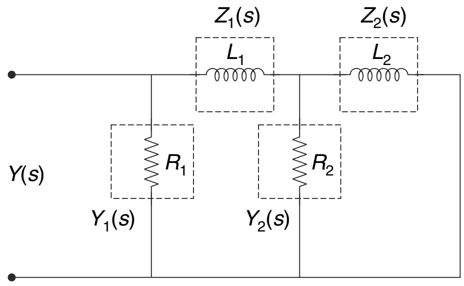


Figure 9.37

9.8 Realize the network function given by $F(s) = \dfrac{s^2+4s+3}{s^2+2s}$ in the Cauer-II form by assuming that $F(s)$ represents an impedance function.

Solution In the Cauer-II form, the function is expanded through continued fraction expansion by performing alternate division and inversion for obtaining the quotient terms. Since the function is to be treated as an impedance function hence the first quotient term must represent an impedance followed by alternate terms of admittance and impedance in the subsequent quotient terms. While performing the continued fraction expansion in the Cauer-II form, the polynomials in the numerator and denominator must be arranged in a decreasing order of s. Thus we may get the quotient terms of the division as;

$$Z(s) = Z_1(s) + \cfrac{1}{Y_1(s) + \cfrac{1}{Z_2(s) + \cfrac{1}{Y_2(s) + \ldots}}}$$

The general circuit for this network may be as shown in Fig. 9.38.

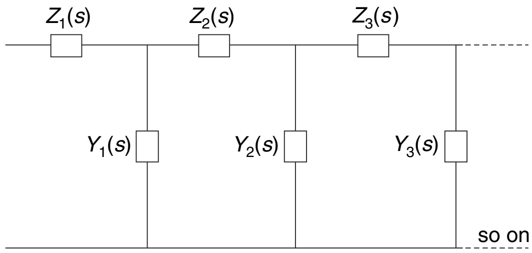


Figure 9.38

The actual result of continued fraction expansion would give the following form.

$$Z(s) = \frac{3 + 4s + s^2}{2s + s^2} = \frac{3s}{2} + \cfrac{1}{\cfrac{4}{5} + \cfrac{1}{\cfrac{25s}{2} + \cfrac{1}{\cfrac{1}{5}}}}$$

On comparison of the result with the general form indicated above, we may find that the first series element represents an inductance of value L_1 = 3/2 H. The second element is connected in a parallel branch and happens to be a conductance value of 4/5 mho, which is equivalent to a resistance of value R_1 = 5/4 Ω. The third element represents an inductance in the series branch of value L_2 = 25/2 H, and the fourth element connected in a parallel branch happens to be a conductance value of 1/5 mho, which is equivalent to a resistance of value R_2 = 5 Ω. The circuit model for this network may be as shown in Fig. 9.39.

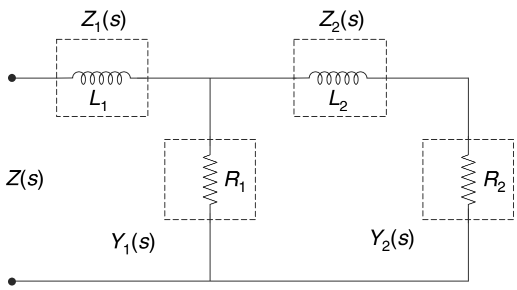


Figure 9.39

9.9 Realize the network function given by $F(s) = \dfrac{s^2+4s+3}{s^2+2s}$ in the Cauer-II form by assuming that $F(s)$ represents an admittance function.

Solution In the Cauer-II form, the function is expanded through continued fraction expansion by performing alternate division and inversion for obtaining the quotient terms. Since the function is to be treated as an admittance function, hence the first quotient term must represent an admittance followed by alternate terms of impedance and admittance in the subsequent quotient terms. While performing the continued fraction expansion in the Cauer-II form, the polynomials in the numerator and denominator must be arranged in an decreasing order of s. Thus we may get the quotient terms of the division as

$$Y(s) = Y_1(s) + \cfrac{1}{Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_2(s) + \ldots}}}$$

The general circuit for this network may be as shown in Fig. 9.40.

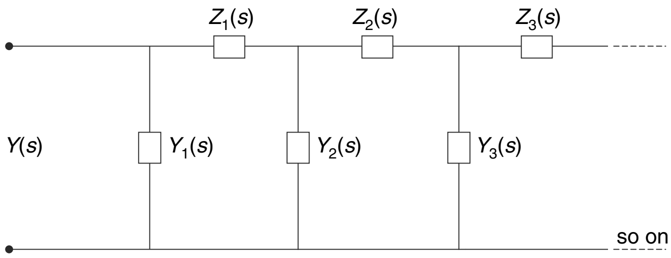


Figure 9.40

The actual result of continued fraction expansion would give the following form.

$$Y(s) = \frac{3+4s+s^2}{2s+s^2} = \frac{3s}{2} + \cfrac{1}{\cfrac{4}{5} + \cfrac{1}{\cfrac{25s}{2} + \cfrac{1}{\cfrac{1}{5}}}}$$

On comparison of the result with the general form indicated above, we may find that the first parallel element represents a capacitance of value $C_1 = 3/2$ F. The second element is connected in a series branch and happens to be a resistance of value $R_1 = 4/5\ \Omega$. The third element represents a capacitance of value $C_2 = 25/2$ F, and the fourth element connected in a series branch happens to be a resistor of value $R_2 = 1/5\ \Omega$. The circuit model for this network may be as shown in Fig. 9.41.

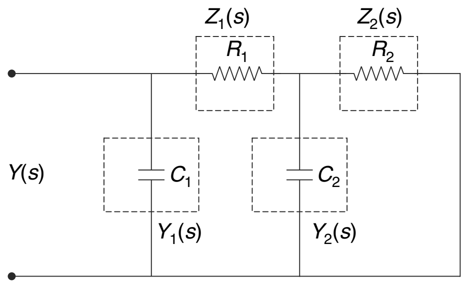


Figure 9.41

MULTIPLE-CHOICE QUESTIONS

9.1 A root of a polynomial has a negative real part of 7 and zero imaginary part. What should be the value of s for this root?

(a) –7 (b) 7 (c) $+j7$ (d) $-j7$

9.2 A root of a polynomial has a zero real part and a conjugate imaginary part of 5. What should be the value of s for this root?

(a) –5 (b) 5 (c) $+j5$ (d) $\pm j5$

9.3 For the function $F(s) = s^3 + 5s^2 + 7s + 3$, how many roots possibly lie on the left half of s-plane?

(a) 0 (b) 1 (c) 2 (d) 3

9.4 Which test can confirm a polynomial to be Hurwitz?

(a) Cauer-I (b) Cauer-II (c) Foster-I (d) none

9.5 Which test can confirm a polynomial to be positive real?

(a) Cauer-I (b) Cauer-II (c) Foster-I (d) Hurwitz

9.6 If a polynomial is positive real then one of its roots on $j\omega$ axis must be,

(a) real (b) imaginary (c) simple (d) complex

9.7 In network synthesis, what is the unknown?

(a) excitation (b) response (c) network (d) none

9.8 In network analysis, what is the unknown?

(a) excitation (b) response (c) network (d) none

9.9 In network synthesis, passive elements can be realized if, transfer function is

(a) Hurwitz (b) positive (c) negative (d) positive real

EXERCISES

9.1 A network function is given as $F(s) = s^7 + 4s^5 + 3s^3 + s$. Check this function for Hurwitz polynomial.

9.2 Examine whether the network function $F(s) = s^7 + 3s^6 + 4s^5 + 2s^4 + 3s^3 + 5s^2 + s + 3$ is Hurwitz.

9.3 A network function is given as $F(s) = 10s^5 + 7s^4 + 2s^3 + s^2 + 8s + 6$. Verify this function for Hurwitz polynomial.

9.4 A network function is given as $F(s) = 2s^6 + 3s^5 + 5s^4 + 3s^3 + 7s^2 + 9s + 4$. Check this function for Hurwitz polynomial.

9.5 Examine whether the network function $F(s) = 4s^6 + 5s^5 + 3s^4 + 8s^3 + 3s^2 + 2s + 7$ is Hurwitz.

9.6 Examine whether the network function $F(s) = \dfrac{s^2 + 2s + 4}{(s+2)(s+4)}$ is positive real.

9.7 Examine whether the network function $F(s) = \dfrac{s + 4}{(s+3)(s+1)}$ is positive real.

9.8 Examine whether the network function $F(s) = \dfrac{3s^3 + s^2 + 2s + 4}{(s+2)(s+4)}$ is positive real.

9.9 Realize the functions in Cauer-I form:

(i) $Z(s) = \dfrac{s^3 + 2s}{s^4 + 4s^2 + 3}$ (ii) $Y(s) = \dfrac{s(s+3)}{6(s+2)(s+4)}$

9.10 Realize the functions in Cauer-II form:

(i) $Z(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$ (ii) $Y(s) = \dfrac{(s+2)(s+6)}{2(s+1)(s+3)}$

9.11 Realize the function in Foster-I form: $Z(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$

9.12 Realize the function in Foster-II form: $Y(s) = \dfrac{(s+2)(s+6)}{2(s+1)(s+3)}$

SHORT ANSWER TYPE QUESTIONS

9.1 Give a comparison between network analysis and network synthesis.

9.2 What is the significance of Hurwitz test on polynomials?

9.3 Describe the necessary condition for positive realness.

9.4 Describe the sufficient condition for positive realness.

9.5 What should be the properties of a positive real function?

9.6 What do you mean by removal of poles from a function in a physical sense?

9.7 What are the important elements of network synthesis?

9.8 How is stability related to network synthesis?

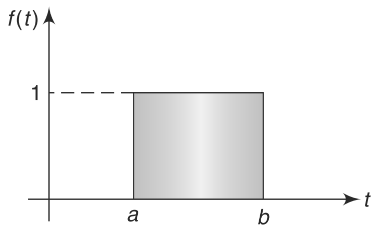
9.9 What is the correspondence between causality and realizability?
9.10 What does removal of a constant from a function mean?
9.11 What do you mean by continued fraction expansion method?
9.12 Why can a function with missing terms fail to be positive real?
9.13 Explain the Cauer-I form of network realization.
9.14 Explain the Cauer-II form of network realization.
9.15 Explain the Foster-I form of network realization.
9.16 Explain the Foster-II form of network realization.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

9.1 (a)	9.2 (d)	9.3 (a)	9.4 (d)	9.5 (d)	9.6 (c)	9.7 (c)
9.8 (b)	9.9 (d)					

Model Question Paper I

1. Choose the correct answer.
   (a) The parameter A of a two-port network is equal to
   (i) V_2/V_1 with $I_1 = 0$ (ii) V_1/I_2 with $V_2 = 0$
   (iii) I_1/I_2 with $V_2 = 0$ (iv) V_1/V_2 with $I_2 = 0$
   (b) An R-L series circuit has a time constant given by
   (i) RL (ii) L/R (iii) R/L (iv) $1/RL$
   (c) In a linear network, when the a.c. input is doubled, the a.c. output becomes
   (i) one fourth (ii) half
   (iii) two times (iv) four times
   (d) The Laplace transform of the signal described in figure below is



   (i) e^{-as}/s (ii) e^{-bs}/s^2 (iii) $(e^{-as} + e^{-bs})/s$ (iv) $(e^{-as} - e^{-bs})/s$
   (e) The step response of a series R-C circuit with applied voltage V is
   (i) $i(t) = \frac{V}{R}e^{-t/RC}$ (ii) $i(t) = \frac{V}{R}(1 - e^{=t/RC})$
   (iii) $i(t) = \left(1 - \frac{V}{R}\right)e^{-t/RC}$ (iv) $i(t) = 1 - \frac{V}{R}e^{-t/RC}$
   (f) Norton's equivalent circuit consists of
   (i) voltage source in parallel with impedance
   (ii) voltage source in series with impedance

(iii) current source in parallel with impedance
(iv) current source in series with impedance

(g) The number of links for a graph having 'n' nodes and V branches are
(i) $b - n + 1$ (ii) $n - b + 1$ (iii) $b + n - l$ (iv) $b + n$

(h) For an R-C circuit comprising a capacitor $C = 2\ uF$ in series with a resistance $R = 1\ M\ \Omega$, the period 6 sec will be equal to
(i) one time constant (ii) two time constants
(iii) three time constants (iv) four time constants.

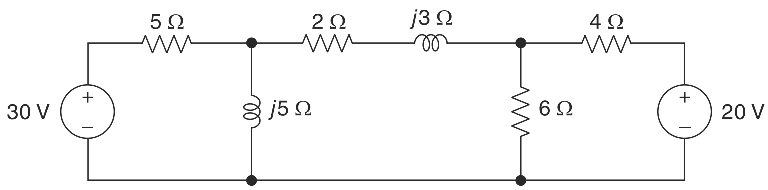
(i) For a symmetrical network
(i) $Z_{11} = Z_{22}$ (ii) $Z_{12} = Z_{21}$
(iii) $Z_{11}Z_{12} - Z_{12}Z_{21} = 0$ (iv) $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$

(j) The impedance of a 1 Henry inductor at 50 Hz is
(i) 1 Ω (ii) 31.4 Ω (iii) 50 Ω (iv) 314 Ω

Solution

(a) (iv) (b) (ii) (c) (iii) (d) (iv) (e) (i) (f) (iii) (g) (i) (h) (iii)
(i) (i) (j) (iv)

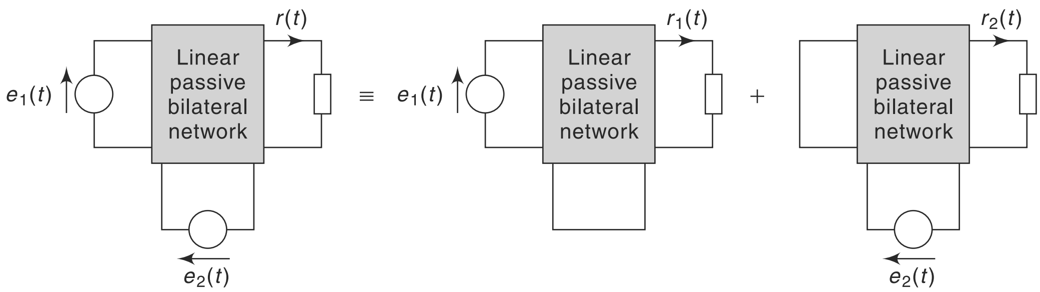
2. (a) State Superposition Theorem.
(b) Using superposition theorem, calculate the current through the $(2 + j3)\ \Omega$ impedance branch of the circuit shown in the figure below.



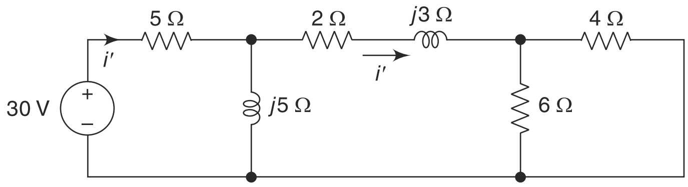
Solution

(a) Superposition Theorem

This theorem states that in a linear bilateral network, the current at any point (or voltage between any two points) due to the simultaneous action of a number of independent sources in the network is equal to the summation of the component currents (or voltages). A component current (or voltage) is defined as that due to one source acting alone in the network with all the remaining sources removed.



(b) **Case (I)** 30 V source is acting alone

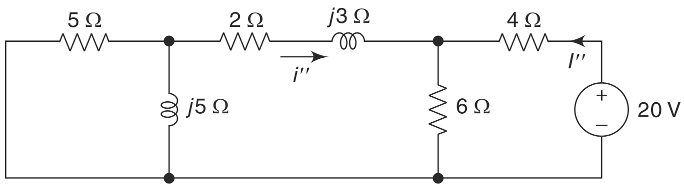


Impedance, $Z = 5 + \dfrac{(4.4 + j3) \times j5}{4.4 + j3 + j5} = (6.32 + j2.6)\ \Omega$

$$\therefore \qquad I' = \frac{30}{Z} = \frac{30}{6.32 + j2.6} = (4.06 - j1.67)\ \text{A}$$

$$i' = I' \times \frac{j5}{4.4 + j3 + j5} = (2.39 + j0.27)\ \text{A}$$

Case (II) 20 *V* source is acting alone



Impedance, $Z = 4 + \dfrac{(4.5 + j5.5) \times 6}{4.5 + j5.5 + 6} = (7.31 + j1.41)\ \Omega$

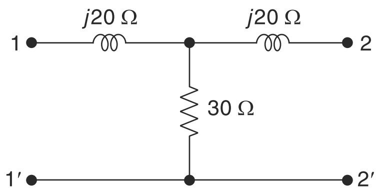
$$\therefore \qquad I'' = \frac{20}{Z} = \frac{20}{7.31 + j1.41} = (2.64 - 0.509)\ \text{A}$$

$$i'' = -I'' \times \frac{6}{4.5 + j5.5 + 6} = -(1.064 - j0.848)\ \text{A}$$

By superposition theorem, total current flowing through the (2 + *j*3) impedance is,

$$i = (i' + i'') = (2.39 + j0.27) - (1.064 - j0.848) = (1.325 + j1.117)\ \text{A} = 1.733\angle 40.14^\circ\ \text{A}$$

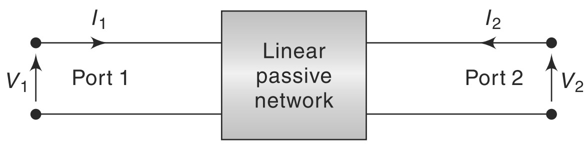
3. (a) Define the ABCD parameters of a 4-terminal network.
   (b) Calculate the ABCD parameters of the network shown in the figure below.



Solution

(a) ABCD Parameters

The *ABCD* parameters represent the relation between the input quantities and the output quantities in the two-port network. They are thus voltage-current pairs.



However, as the quantities are defined as an input-output relation, the output current is marked as going out rather than as coming into the port.

The impedance parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The parameters A, B, C, D can be defined in a similar manner with either port 2 on short circuit or port 2 on open circuit.

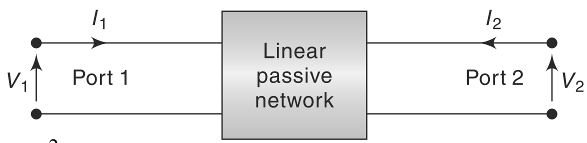
$$A = \left.\frac{V_1}{V_2}\right|_{I_2 = 0} = \text{Open Circuit Reverse Voltage Gain}$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2 = 0} = \text{Short Circuit Transfer Impedance}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2 = 0} = \text{Open Circuit Transfer Admittance}$$

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2 = 0} = \text{Short Circuit Reverse Current Gain}$$

(b) For this *T*-circuit, the *z*-parameters are given as,



$z_{11} = z_{22} = (30 + j20)\ \Omega$

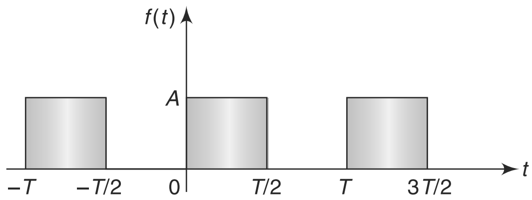
$z_{12} = z_{21} = 30\ \Omega$

$\therefore\ \Delta z = (z_{11}\, z_{22} - z_{12}\, z_{21}) = (30 + j20)^2 - 30^2$

$= (60 + j20)j20 = (-400 + j1200)$

$$\left.\begin{aligned}
\therefore\quad A &= \frac{z_{11}}{\Delta z} = \frac{30 + j20}{(60 + j20)\, j20} = \left(1 + j\frac{2}{3}\right) \\
\therefore\quad B &= \frac{\Delta z}{z_{21}} = \frac{(60 + j20)\, j20}{30} = \left(-\frac{40}{3} + j40\right)\Omega \\
\therefore\quad C &= \frac{1}{z_{12}} = \frac{1}{30}\ \mho \\
\therefore\quad D &= \frac{z_{22}}{z_{12}} = \frac{30 + j20}{30} = \left(1 + j\frac{2}{3}\right)
\end{aligned}\right\} \quad \textit{Ans.}$$

4. Determine the Fourier series of the wave form shown in the figure below.



Solution

Here, $f(t) = A;$ for $0 < t < T/2$

$= 0;$ for $T/2 < t < T$

$$\therefore \quad a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^{T/2} A dt = \frac{A}{2}$$

and
$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt = \frac{2}{T}\int_0^{T/2} A\cos n\omega t dt$$

$$= \frac{2A}{n\omega T}\left[\sin\left(\frac{n\omega T}{2}\right)\right] = 0 \quad [\because \ \omega T = 2\pi]$$

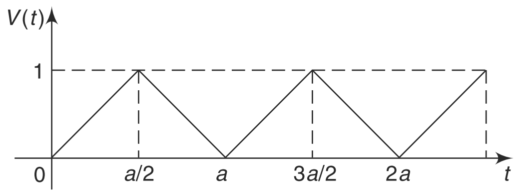
and
$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t dt = \frac{2A}{T}\int_0^{T/2} \sin n\omega t dt$$

$$= \frac{2A}{n\omega T}\left[1 - \cos\left(\frac{n\omega T}{2}\right)\right] = \frac{A}{n\pi}(1 - \cos n\pi), \text{ for } n \text{ odd.}$$

$= 0$, for n even.

$$\therefore \quad f(t) = A\left[\frac{1}{2} + \frac{2}{\pi}\sin n\omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \dots\right]$$ *Ans.*

5. (a) Find the Laplace transform of the triangular waveform shown in figure below:



(b) Find the inverse Laplace transform of the function

$$V(s) = \frac{10(s+4)}{s(s+3)(s+1)^2}$$

Solution

(a) Here, $v_1(t) = \frac{2}{a} r(t) - \frac{4}{a} r(t - a/2) + \frac{2}{a} r(t - a)$

Taking Laplace transform,

$$V_1(s) = \frac{2}{a}\frac{1}{s^2} - \frac{4}{a}\frac{e^{-as/2}}{s^2} + \frac{2}{a}\frac{e^{-as}}{s^2}$$

$$= \frac{2}{as^2}(1 - 2e^{-as/2} + e^{-as})$$

$$= \frac{2}{as^2}(1 - e^{-as/2})^2$$

By Scalling Theorem (the theory of periodicity), Laplace transform of the given periodic function is,

$$V(s) = V_1(s) \times \frac{1}{1 - e^{-Ts}} = \frac{2}{as^2}(1 - e^{-as/2})^2 \times \frac{1}{1 - e^{-as}}$$

$$= \frac{2}{as^2}\left(\frac{1 - e^{-as/2}}{1 + e^{-as/2}}\right)$$

$$= \frac{2}{as^2}\tanh\left(\frac{as}{4}\right) \qquad \textit{Ans.}$$

(b) $V(s) = \dfrac{10(s+4)}{s(s+3)(s+1)^2}$

We have to use Heaviside theorem,

$$V(s) = \frac{P(s)}{Q(s)} = \frac{10(s+4)}{s(s+3)(s+1)^2} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)}$$

By Heaviside theorem,

$$A = \left[s\frac{P(s)}{Q(s)}\right]_{s=0} = \left.\frac{10(s+4)}{(s+3)(s+1)^2}\right|_{s=0} = \frac{40}{3}$$

$$B = \left[(s+3)\frac{P(s)}{Q(s)}\right]_{s=-3} = \left.\frac{10(s+4)}{s(s+1)^2}\right|_{s=-3} = -\frac{5}{6}$$

$$C = \left[(s+1)^2\frac{P(s)}{Q(s)}\right]_{s=1} = \left.\frac{10(s+4)}{s(s+3)}\right|_{s=-1} = -15$$

$$D = \left[\frac{d}{ds}\left\{(s+1)^2\frac{P(s)}{Q(s)}\right\}\right]_{s=-1} = \left.\frac{d}{ds}\left\{\frac{10(s+4)}{s(s+3)}\right\}\right|_{s=-1} = -\frac{25}{2}$$

So, the partial fraction expansion of the given function $V(s)$ will be,

$$V(s) = \frac{40/3}{s} - \frac{5/6}{s+3} - \frac{15}{(s+1)^2} - \frac{25/2}{(s+1)}$$

Taking inverse Laplace transform,

$$v(t) = \frac{40}{3}u(t) - \frac{5}{6}e^{-3t} - 15te^{-t} - \frac{25}{2}e^{-t} \qquad \textit{Ans.}$$

6. (a) What is an oriented graph?
   (b) What is a sub-graph? Give example.
   (c) The reduced incidence matrix of a network is given below.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

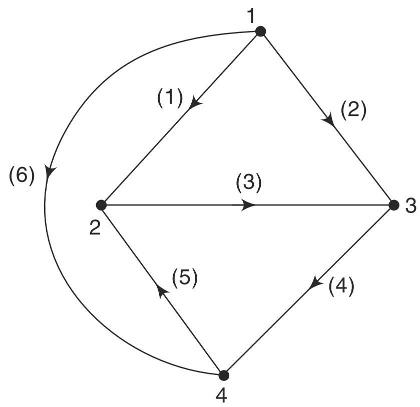
Draw the graph of the network.

Solution

(a) Oriented Graph

A graph is defined as a collection of points called nodes, and line segment called branches, the nodes being joined together by the branches. A graph whose branches are oriented is called a directed or oriented graph. The orientation is indicated by an arrow head in each of the branch representing the direction of current flow in the branch.

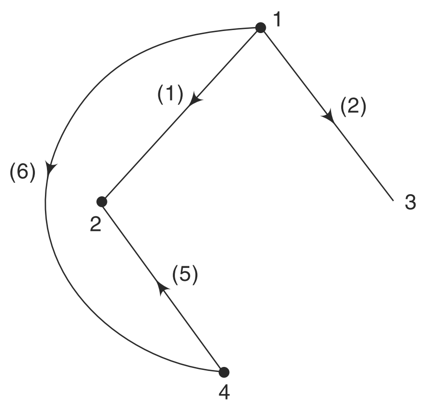
Figure shows an oriented graph.



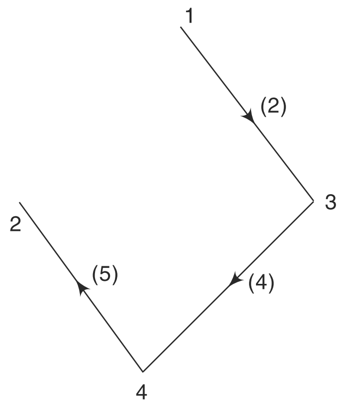
(b) Sub-graph

A subgraph is a subset of the branches and nodes of a graph. For example, for the graph shown in figure, some subgraphs are shown below.

The subgraph is said to be proper if it consists of strictly less than all the branches and nodes of the graph.



Subgraph 1



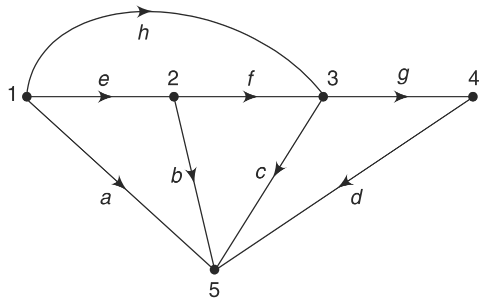
Subgraph 2

(c) From the reduced incidence matrix, we obtain the complete incidence matrix as,

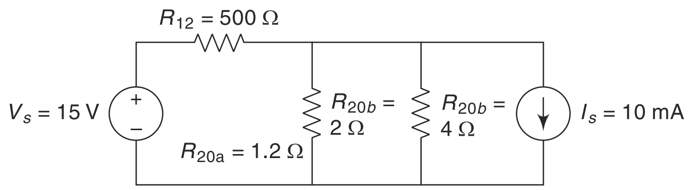
Nodes ↓ Branches →

$$\begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{array}\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

So, the graph has 5 nodes and 8 branches. The oriented graph is shown below.



7. (a) Differentiate between active and passive filters.
   (b) Write the SPICE input file for the circuit shown in the figure below.



Solution

(a) Difference between Active and Passive Filter

(1) Active filters are very much inexpensive than passive filters due to the variety of cheaper op-amp and the absence of costly inductors.

(2) Since the op-amp is capable of providing a gain (which may also be variable), the input signal is not attenuated as it is in a passive filter. In addition, the active filter is easier to tune or adjust.

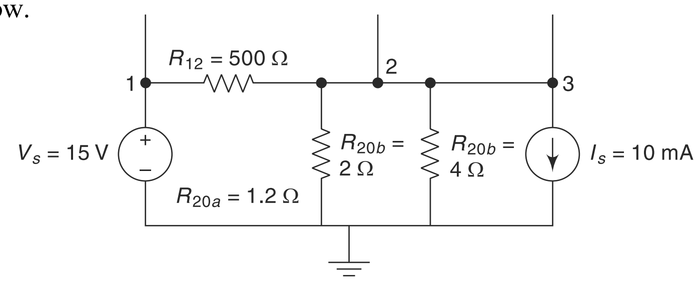
(3) Active filters provide an excellent isolation between the individual stages due to the high input impedence (ranging from a few kΩ to a several thousand MΩ) and low output impedance (ranging from less than 1Ω to a few hundred Ω). So, the active filter does not cause loading of the source or load.

(4) Active filters are small in size and less bulky (due to the absence of bulky 'L') and are rugged.

(5) Active filters generally have single ended inputs and outputs which do not 'float' with respect to the system power supply or common. This property is different from that of the passive filters.

(b) The Input File is written below.

```
*Title*
VS     0  1  DC   15
IS     0  3  DC   10M
R12    1  2  500
R20A   0  2  1.2K
R20B   0  2  2K
R5     0  2  4K
END
```



8. (a) Define Fourier transform of a function $f(t)$.

(b) Find the Fourier transform of a voltage waveform as defined below:

$$\begin{aligned} v(t) &= 0, && \text{for} \quad t < -r \\ &= A, && \text{for} \quad -r \geq t < 0 \\ &= 0, && \text{for} \quad t > r \end{aligned}$$

Solution

(a) Definition of Fourier Transform

The Fourier Transform of a function $f(t)$ is denoted by $F(j\omega)$ and is defined by,

$$F(j\omega) = \mathcal{F}\,[f(t)] = = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{i}$$

and the inverse Fourier transform is defined by,

$$f(t) = \mathcal{F}^{-1}\,[F(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty} F(j2\pi f)e^{j2\pi f}\,df \tag{ii}$$

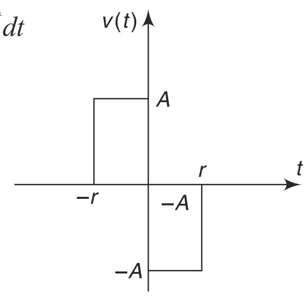
Equations (i) and (ii) form the Fourier transform pair.

(b) The voltage waveform is shown in figure.

$$\therefore \qquad F(j\omega) = \int_{-r}^{r} v(t)e^{-j\omega t}\,dt = \int_{-r}^{0} Ae^{-j\omega t}\,dt + \int_{0}^{r} -Ae^{-j\omega t}\,dt$$

$$= A\left[\left.\frac{je^{-j\omega t}}{\omega}\right|_{-r}^{0} - \left.\frac{je^{-j\omega t}}{\omega}\right|_{0}^{r}\right]$$

$$= \frac{jA}{\omega}[1 - e^{+j\omega r} - e^{-j\omega r} + 1]$$



$$\Rightarrow \qquad F(j\omega) = j\frac{2A}{\omega}(1 - \cos\omega r) \qquad \textit{Ans.}$$

9. Explain clearly with the help of suitable examples the following terms used in network analysis:

(a) Cut set matrix.

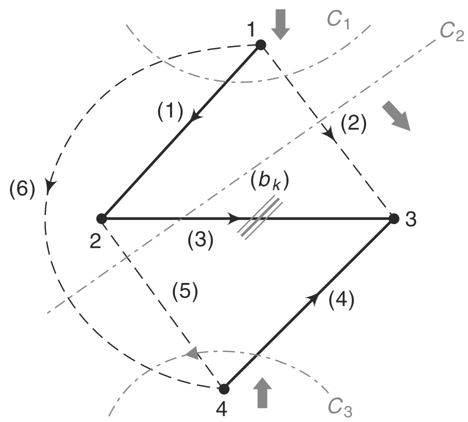
(b) Tie set matrix
(c) Incidence matrix and its properties.

Solution

(a) Cut Set Matrix

For a given graph, a cut-set matrix (Q_C) is defined as a rectangular matrix whose rows correspond to cut-sets and columns correspond to the branches of the graph. Its elements have the following values:

Q_{ij} = 1, if branch j is in the cut-set i and the orientations coincide.
= –1, if branch j is in the cut-set i and the orientations do not coincide.
= 0, if branch j is not in the cut-set i.



Example A cut-set is a minimum set of elements that when cut, or removed, separates the graph into two groups of nodes. For the graph shown in figure, for the particular tree considered (tree-branches: 1, 2, 3), fundamental cut-sets have been identified as:

f-cut-set – 1: [1, 2, 6];
f-cut-set – 2: [2, 3, 5, 6];
f-cut-set – 3: [4, 5, 6]

So, the cut-set matrix is written as,

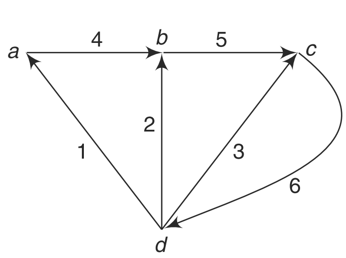
	Branch no.					
f-cut-sets	1	2	3	4	5	6
1	1	1	0	0	0	1
2	0	1	1	0	1	1
3	0	0	0	1	–1	–1

(b) **Tie Set Matrix**

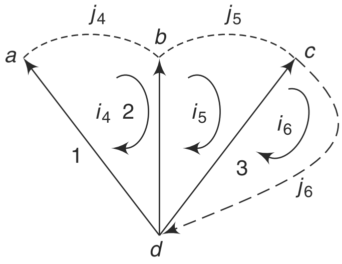
For a given graph having n nodes and b branches, tie-set matrix is a rectangular matrix with b columns and as many rows as there are loops. Its elements have the following values:

B_{ij} = 1, if branch j is in loop i and their orientations coincide (i.e. loop current and branch current flows in the same direction);
= –1, if branch j is in loop i and their orientations do not coincide;
= 0, if branch j is not in loop i.

Example For the graph shown in Fig. (a) and tree selected in Fig. (b), the tie-set matrix is written as follows. The entries in the Tie-set schedule are given as +1 or –1 if the branch current is in the same direction as the link current or not. If the branch current does not depend on the link current, then entry is zero.



(a) Graph



(b) Formation of Loops

	Branch no. (i)					
Links (j)	1	2	3	4	5	6
4	1	–1	0	1	0	0
4	0	1	–1	0	1	0
6	0	0	1	0	0	1

(c) **Incidence Matrix and its Properties**

The incidence matrix symbolically describes a network. It also facilitates the testing and identification of the independent variables. Incidence matrix is a matrix which represents a graph **uniquely**.

For a given graph with n nodes and b branches, the complete incidence matrix A_a is a rectangular matrix of order $n \times b$, whose elements have the following values:

Number of columns in [A] = Number of branches = b

Number of rows in [A] = Number of nodes = n

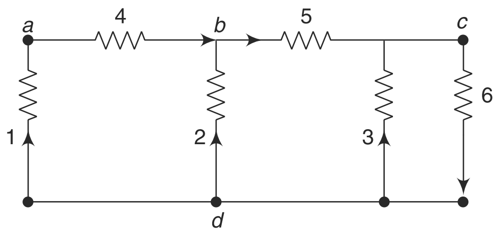
A_{ij} = 1, if branch j is associated with node i and oriented away from node j.

= –1, if branch j is associated with node i and oriented towards node j.

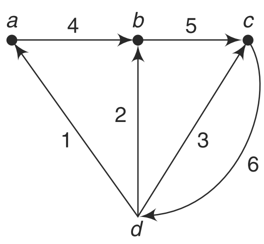
= 0, if branch j is not associated with node i.

This matrix tells us which branches are incident at which nodes and what orientations relative to the nodes are.

Example



(a) Network



(b) Graph of Network

Incidence matrix A

		Branches					
		1	2	3	4	5	6
	a	–1	0	0	1	0	0
Nodes	*b*	0	–1	0	–1	1	0
	c	0	0	–1	0	–1	1
Reference Node	*d*	1	1	1	0	0	–1

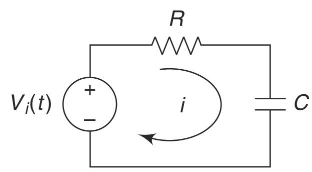
- Properties of Incidence Matrix
  (i) The sum of the entries in any column is zero.
  (ii) The determinant of the incidence matrix of a closed loop is zero.
  (iii) The rank of incidence matrix of a connected graph is $(n - 1)$.

10. Explain under what condition, an RC series circuit behaves as
    (a) differentiator
    (b) low pass filter
    (c) coupling network
    (d) integrator.

Solution

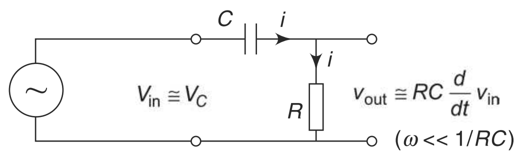
We consider the RC series circuit.

(a) RC Series Circuit as Differentiator



We have an AC source with voltage $v_{in}(t)$, input to an RC series circuit. This time the output is the voltage across the resistor.

We consider only **low** frequencies $\omega << 1/RC$, so that the capacitor has time to charge up until its voltage almost equals that of the source.



$$V_{in} = IZ = I\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

But, $\quad R << \frac{1}{\omega C}$, so $V_{in} = \frac{I}{\omega C}$

For frequencies, $\quad \omega << \frac{1}{RC}, V_{in} \cong V_C$

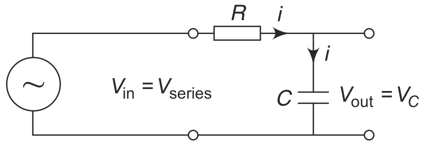
$$\therefore \quad V_{out} = V_R = iR = R\frac{dq}{dt} = R\frac{d}{dt}CV_C$$

$$\therefore \quad \boxed{V_{out} \cong RC\frac{d}{dt}V_{in}}$$

Thus, the output is the differentiation of the input and the **RC series circuit acts as** ***Differentiator***.

(b) RC Series Circuit as Low Pass Filter

If the RC series circuit is supplied with a frequency varying source, then it will act as a low pass filter if the output is taken as the voltage across the capacitor.

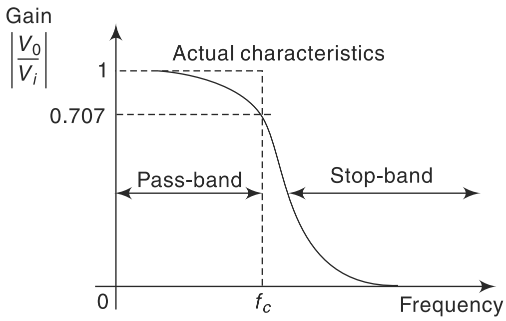


The voltage across the capacitor is $IX_C = 1/\omega C$. The voltage across the series combination is:

$IZ = I\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$, so the gain is:

$$g \equiv \frac{V_{out}}{V_{in}} = \frac{IX_C}{IZ} = \frac{1/\omega C}{\sqrt{R^2 + \left(1/\omega C\right)^2}}$$

$$\therefore \quad \boxed{g = \frac{1}{\sqrt{1 + (\omega RC)^2}}}$$



Here, at low frequencies capacitive reactance ($Xc = 1/2\pi fC$) is very high and therefore the circuit can be considered as an open circuit. Under these conditions, input signal is equal to output signal. At very high frequencies, the capacitive reactance ($Xc = 1/2\pi fC$) is very low and therefore the output signal is very small as compared with the input signal. Thus, the circuit acts as low pass filter with the frequency characteristics as shown in figure.

(c) RC Series Circuit as Coupling Network

A coupling network is used for coupling a signal at a frequency from a voltage source to a load. The voltage source has a source resistance. The load has a load resistance and a load reactance. The ratio of the load reactance to the load resistance is greater than 100. The coupling network includes a reactive element and a delay circuit. The reactive element is arranged in series with the load to resonate with the load reactance at the frequency. The delay circuit is between the reactive element and the source, has a delay equivalent to a quarter wave length transmission line at the frequency and has a characteristic impedance equal to the square root of the product of the values of the load resistance and the source required resistance.

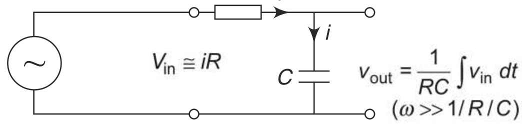
Thus, an RC series circuit will act as coupling network only when the ratio of load resistance to load reactance is greater than 100 and the suply frequency is such that the capacitor resonates at that frequency.

(d) RC Series Circuit as Integrator

We have an AC source with voltage $v_{in}(t)$, input to an *RC* series circuit. The output is the voltage across the capacitor.

We consider only **high frequencies** $\omega >> 1/RC$, so that the capacitor has insufficient time to charge up, its voltage is small, so the input voltage approximately equals the voltage across the resistor.

$$V_{in} = IZ = I\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$



But $\quad \omega C >> 1/R, \;$ so $\; V_{in} \cong IR$

For frequencies, $\omega >> \dfrac{1}{RC}, V_{\text{in}} \cong V_R$

$$\therefore \qquad V_{\text{out}} = V_C = \frac{1}{C}\int i dt \cong \frac{1}{C}\int \frac{V_{\text{in}}}{R} dt$$

$$\therefore \qquad \boxed{V_{\text{out}} \cong \frac{1}{RC}\int V_{in} dt}$$

Thus, the voltage v_C is the integration of the input voltage and hence the **RC series circuit acts as an *Integrator***.

Model Question Paper II

1. Answer any five questions by choosing the right answer from the following: 2×5

(a) The reduced incidence matrix of a circuit is given by

$$A_i = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The set of branches forming a tree are

(i) 1, 2 and 3 (ii) 2, 3 and 5

(iii) 1, 2 and 4 (iv) 1, 2 and 6.

(b) In the circuit shown in Fig. 1, the potential between points P and Q is

(i) 12 V (ii) 10 V

(iii) –6 V (iv) 8 V

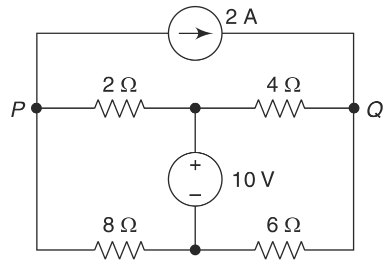


Figure 1

(c) If $f(t)$ and its first derivative is Laplace transformable then the initial value of $f(t)$ is given by

(i) $\underset{t \to 0}{\text{Lt}}\, f(t) = \underset{s \to 0}{\text{Lt}}\, sF(s)$ (ii) $\underset{t \to 0}{\text{Lt}}\, f(t) = \underset{s \to \infty}{\text{Lt}}\, \dfrac{F(s)}{s}$

(iii) $\underset{t \to 0}{\text{Lt}}\, f(t) = \underset{s \to 0}{\text{Lt}}\, \dfrac{F(s)}{s}$ (iv) $\underset{t \to 0}{\text{Lt}}\, f(t) = \underset{s \to \infty}{\text{Lt}}\, sF(s)$

(d) In a four terminal network containing linear, bilateral, passive elements the following condition for Z parameters generally holds.

(i) $Z_{11} = Z_{22}$ (ii) $Z_{12} = Z_{21}$

(iii) $Z_{11}Z_{22} - Z_{12}Z_{21} = 0$ (iv) $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$

(e) The value of the impulse function $\delta(t)$ at $t = 0$ is

(i) 0 (ii) ∞ (iii) 1 (iv) Indeterminate.

(f) A network has seven nodes and five independent loops, the number of branches in the network is:

(i) 7 (ii) 5 (iii) 11 (iv) 12

(g) The dc gain of a system having the transfer function $H(s) = \dfrac{12}{(s+2)(s+3)}$ is:

(i) 1 (ii) 12 (iii) 3 (iv) 2

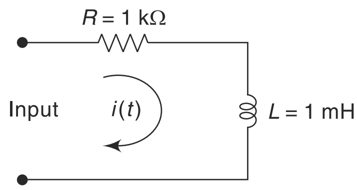
Solution

(a) (i) 1, 2 and 3 (b) (iii) –6 V (c) (iv) $\underset{t\to 0}{\text{Lt}}\, f(t) = \underset{s\to\infty}{\text{Lt}}\,[sF(s)]$

(d) (ii) $z_{12} = z_{21}$ (e) (ii) $\propto$ (f) (iii) 11

(g) (iv)

2. (a) The circuit given in figure is excited by a unit step voltage. Find the expression for the current *i(t)* if the circuit is initially relaxed. What is the steady state value of the current? Find the rise time and the time constant. What is their significance?



(b) Find the expression of steady state current $I(P)$ of a series R-L-C circuit., when it is excited by a sinusoidal voltage $V(P)$. Under what condition the system will behave as resistive?

Solution

(a) Here,
$$v(t) = 0, \quad \text{for} \quad t \le 0$$
$$= 1, \quad \text{for} \quad t > 0$$

Thus the differential equation governing the behaviour of the circuit would be

$$Ri(t) + L\frac{di(t)}{dt} = u(t)$$

Taking Laplace transform, we get

$$RI(s) + L\left[sI(s) - i(0-)\right] = \frac{1}{s}$$

or
$$I(s) = \frac{1/L}{s\left(s + R/L\right)} = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s + R/L}\right) \quad \{\text{since } i(0\_) = 0\}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}(1 - e^{-(R/L)t})$$

Putting the values of R =1 kΩ and L =1 mH, we get,

$$i(t) = \frac{1}{1\times 10^3}(1 - e^{-(1\times 10^3/1\times 10^{-3})t}) = (1 - e^{-10^6 t})\text{ mA} \qquad \textit{Ans.}$$

The steady-state value of the current, $i_s = \dfrac{1}{R} = \dfrac{1}{1\times 10^3} = 1\text{ mA}$ *Ans.*

Rise time is the time taken by the current to reach 90% of the steady-state value starting from 10%. It is given as,

$$T_r = \frac{L}{R}\ln 9 = 2.2\frac{L}{R} = 2.2 \times \frac{1\times 10^{-3}}{1\times 10^{3}} = 2.2\ \mu\text{s} \qquad \textit{Ans.}$$

Time constant, $\tau = \dfrac{L}{R} = \dfrac{1\times 10^{-3}}{1\times 10^{3}} = 1\ \mu\text{s}$ *Ans.*

Significance of Rise Time and Time Constant

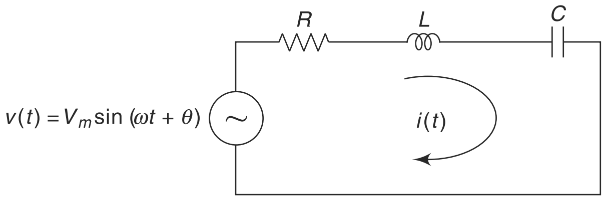
Time constant is the time taken for the current to reach 63% of its final value. Thus, it is a measure of the rapidity with which the steady state is reached.

Also, at $t = 5\tau$, $i = 0.993i$; the transient is therefore, said to be practically disappeared in five time constants.

Physically, time-constant represents the speed of the response of a circuit. A low value of time-constant represents a fast response and a high value of time-constant represents a sluggish response.

Similarly, rise time also gives indication of the system speed of operation.

(b) Steady-state Current in RLC Series Circuit with Sinusoidal Input



Sinusoidal voltage $v(t) = V_m \sin(\omega t + \theta)$ is applied to a series *RLC* circuit at time $t = 0$. We want to find the steady-state part for the current *i(t)* using Laplace transform method. By KVL,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t)dt = V_m \sin(\omega t + \theta)$$

Taking Laplace transform with zero initial conditions,

$$I(s)\left[R + sL + \frac{1}{Cs}\right] = V_m \frac{(s\sin\theta + \omega\cos\theta)}{s^2 + \omega^2}$$

or

$$I(s) = \frac{V_m s(s\sin\theta + \omega\cos\theta)}{L(s^2+\omega^2)\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$$= \frac{V_m}{L}\frac{s(s\sin\theta + \omega\cos\theta)}{(s+j\omega)(s-j\omega)(s-s_1)(s-s_2)}$$

where, s_1, s_2 are the roots of the quadratic equation:

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

Thus, $s_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ and, $s_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

Now, let $\frac{s(s\sin\theta + \omega\cos\theta)}{(s+j\omega)(s-j\omega)(s-s_1)(s-s_2)} = \frac{K_1}{s-s_1} + \frac{K_2}{s-s_2} + \frac{K_3}{s+j\omega} + \frac{K_4}{s-j\omega}$

So, by residue method, multiplying by $(s - s_1)$ and putting $s = s_1$,

$$K_1 = \frac{s_1(s_1\sin\theta + \omega\cos\theta)}{(s_1+j\omega)(s_1-j\omega)(s_1-s_2)} \quad \text{and} \quad K_2 = \frac{s_2(s_2\sin\theta + \omega\cos\theta)}{(s_2+j\omega)(s_2-j\omega)(s_2-s_1)}$$

Similarly, multiplying by $(s + j\omega)$ and putting $s = -j\omega$,

$$K_3 = \frac{-j\omega(-j\omega\sin\theta + \omega\cos\theta)}{(-j\omega-j\omega)(-j\omega-s_1)(-j\omega-s_2)} = \frac{\omega(\cos\theta - j\sin\theta)}{2(s_1+j\omega)(s_2+j\omega)}$$

and, $$K_4 = \frac{j\omega(-\omega\sin\theta + \omega\cos\theta)}{(j\omega+j\omega)(j\omega-s_1)(j\omega-s_2)} = \frac{\omega(\cos\theta + j\sin\theta)}{2(s_1-j\omega)(s_2-j\omega)}$$

Hence the current response becomes,

$$i(t) = \frac{V_m}{L}[K_1e^{s_1t} + K_2e^{s_2t}] + \frac{V}{L}[K_3e^{-j\omega t} + K_4e^{j\omega t}] = I_{tr} + I_{ss}$$

Thus, steady-state part of the total current is obtained as follows.

$$I_{ss} = \frac{V_m}{2L}\left[\frac{\omega e^{-j\theta}e^{-j\omega t}}{(s_1+j\omega)(s_2+j\omega)} + \frac{\omega e^{j\theta}e^{j\omega t}}{(s_1-j\omega)(s_2-j\omega)}\right]$$

$$= \frac{V_m\omega}{2L}\left[\frac{e^{-j(\omega t+\theta)}}{(s_1+j\omega)(s_2+j\omega)} + \frac{e^{j(\omega t+\theta)}}{(s_1-j\omega)(s_2-j\omega)}\right]$$

$$I_{ss} = \frac{V_m\omega}{2L(s_1^2+\omega^2)(s_2^2+\omega^2)}\left[e^{-j(\omega t+\theta)}(s_1s_2 - \omega^2 - j\omega s_1 - j\omega s_2)\right]$$

or $$= \frac{V_m\omega}{2L(s_1^2+\omega^2)(s_2^2+\omega^2)}\left[(s_1s_2-\omega^2)2\cos(\omega t+\theta) - (\omega s_1 + \omega s_2)2\sin(\omega t+\theta)\right]$$

$$= \frac{V_m\omega}{L}\frac{1}{(s_1^2+\omega^2)(s_2^2+\omega^2)}\left[\left(\frac{1}{LC} - \omega^2\right)\cos(\omega t+\theta) - \left(-\frac{\omega R}{L}\right)\sin(\omega t+\theta)\right]$$

or $$I_{ss} = \frac{V_m\omega}{L}\frac{\left[\frac{\omega R}{L}\sin(\omega t+\theta) - \left(\omega^2 - \frac{1}{LC}\right)\cos(\omega t+\theta)\right]}{(s_1^2+\omega^2)(s_2^2+\omega^2)}$$

$$= \frac{V_m \omega}{L} \frac{1}{(s_1^2+\omega^2)(s_2^2+\omega^2)} \sin\left\{\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right\}$$

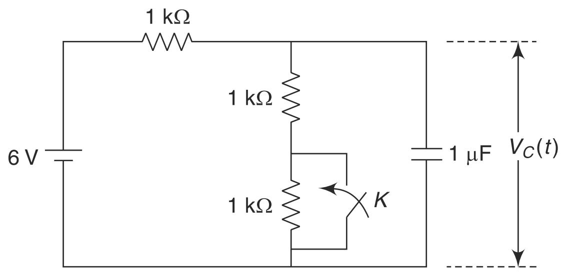
$$\times \frac{\omega}{L}\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

or
$$\boxed{I_{ss} = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left\{\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right\}}$$

This gives the steady-state current of the series *RLC* circuit to a sinusoidal voltage. If $\omega L = \frac{1}{\omega L}$, then the system will behave as resistive and under this condition, the current will become,

$$I_{SS} = \frac{V}{R} \sin(\omega t + \theta)$$

3. (a) The circuit given in the figure is initially at steady state with the switch K open. If the switch is closed at time $t = 0$, find the voltage $V_C(t)$ across the capacitor.



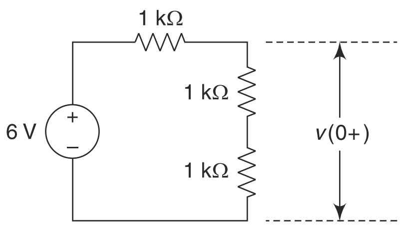
(b) How does the Fourier Transform differ from the Laplace Transform? Find the Fourier transform of the current,

$$i(t) = \begin{cases} I_0 e^{-at} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

and $a > 0$. Sketch the magnitude spectrum $|I(j\omega)|$

Solution

(a) At steady-state before closing the switch, the capacitor becomes open-circuited. So, the circuit becomes as shown above.



$$v(0-) = \frac{2}{3} \times 6 = 4 \text{ V}$$

For $t > 0$, by KVL,

$$1\times10^3 \times I_1 + 1\times10^3 \times (I_1 - I_2) = \frac{6}{s} \Rightarrow 2000I_1 - 1000I_2 = \frac{6}{s} \quad \text{(i)}$$

and
$$\frac{10^6}{s} I_2 + 1\times10^3 \times (I_2 - I_1) = -\frac{4}{s} \Rightarrow -1000I_1 + \left(1000 + \frac{10^6}{s}\right) I_2 = -\frac{4}{s} \quad \text{(ii)}$$

Solving equations (i) and (ii),

$$I_2 = \frac{\begin{vmatrix} 2000 & 6/s \\ -1000 & -4/s \end{vmatrix}}{\begin{vmatrix} 2000 & -1000 \\ -1000 & \left(1000 + 10^6/Cs\right) \end{vmatrix}} = -\frac{2}{1000}\left(\frac{1}{s+2000}\right)$$

$$\therefore \qquad V_c(s) = I_2 \times \frac{10^6}{s} + \frac{4}{s} = -\frac{2000}{s(s+2000)} + \frac{4}{s}$$

$$= \frac{3}{s} + \frac{1}{s+2000}$$

Taking inverse Laplace transform,

$$v_c(t) = 3 + e^{-2000t} \quad \text{(V)}, t > 0$$

(b) Difference between Fourier Transform and Laplace Transform
The defining equations are,

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \quad \text{and} \quad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Following are some differences and similarities:

1. Laplace Transform is one-sided in the interval $0 < t < \infty$ and Fourier Transform is double-sided in the interval $-\infty < t < \infty$. Thus, Laplace Transform is applicable for positive time function, $f(t)$, $t > 0$; while Fourier Transform is applicable for functions defined for all times.
2. Laplace Transform includes the initial conditions and is applicable for transient analysis, while Fourier Transform is only applicable for steady-state analysis.
3. For functions $f(t) = 0$ for $t < 0$ and $\int_0^{\infty} |f(t)|\,dt < \infty$, the two transforms are related as, $F(j\omega) = F(s)\big|_{s=j\omega}$. Thus, Laplace Transform is associated with entire s-plane, while Fourier Transform is restricted to the imaginary ($j\omega$) axis.
4. Laplace Transform is applicable to a wider range of functions than the Fourier Transform. On the other hand, Fourier Transforms exist for signals that are not physically realisable and have no Laplace Transform.

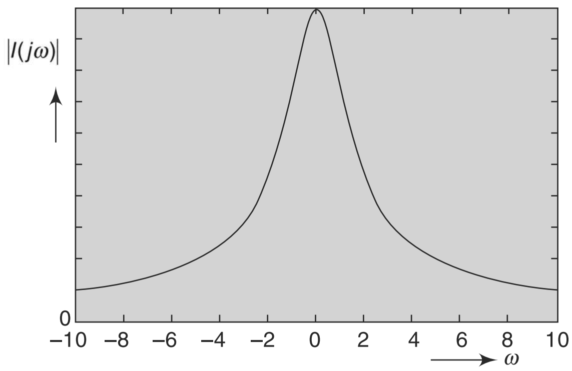
- **Solution to the numerical problem**

Here, $$i(t) = \begin{cases} I_0 e^{-at} & \text{for} \quad t \ge 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \quad \text{with a} > 0$$

$$F(j\omega) = \int_{\infty}^{\infty} i(t)e^{-j\omega t}dt = \int_{\infty}^{\infty} I_0 e^{-at}e^{-j\omega t}dt = I_0 \int_{\infty}^{\infty} e^{-(a+j\omega)t}dt = \frac{I_0}{a + j\omega}$$

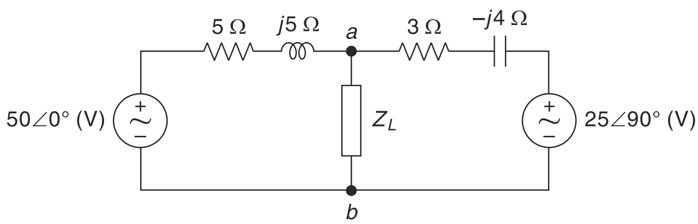
The amplitude is, $|I(j\omega)| = \dfrac{I_0}{\sqrt{a^2 + \omega^2}}$

The magnitude spectrum is shown below.

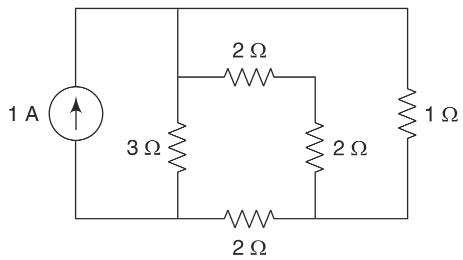


4. (a) Find the Thevenin's equivalent between the points a and b for the circuit given in the figure.

What should be the value of impedance connected between a and b for maximum power to be transferred from the sources?



(b) Find the current through the 1 Ω resistor in the circuit in figure using Tellegen's Theorem.



Solution

(a) Here,

$$V_1 = 50\angle 0° = 50 \text{ V}; \quad \text{and} \quad V_2 = 25\angle 90° = j25 \text{ V}$$

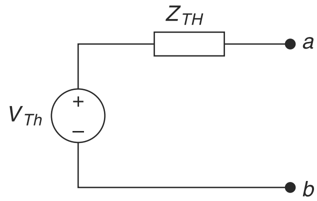
Current in the circuit, $I = \dfrac{50 - j25}{5 + j5 + 3 - j4} = \dfrac{50 - j25}{8 + j1}$ A

Thevenin voltage,

$$V_{\text{Th}} = 50 - I \times (5 + j5) = 50 - \left(\frac{50 - j25}{8 + j1}\right) \times (5 + j5) = \frac{25 - j75}{8 + j1}$$
$$= 9.8\angle -78.7^\circ$$
$$= (1.923 - j9.615)\ \text{V} \qquad \textit{Ans.}$$

Thevenin impedance,

$$Z_{\text{Th}} = \frac{(5 + j5) \times (3 - j4)}{(5 + j5) + (3 - j4)} = \frac{35 - j5}{8 + j1} = (4.23 - j1.154)\ \Omega$$

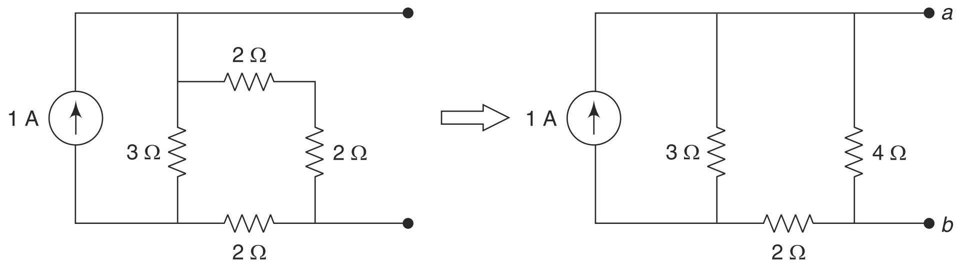


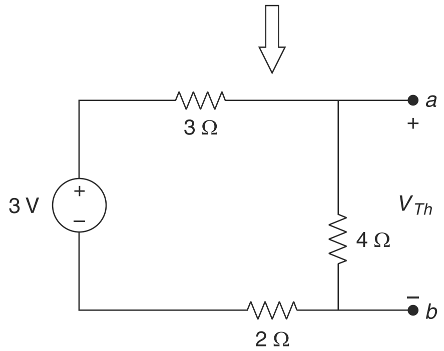
Thus, the Thevenin's equivalent circuit is shown in the figure.

For maximum power transfer to the load,

$$Z_L = Z_{\text{Th}}^* = (4.23 + j1.154)\ \Omega \qquad \textit{Ans.}$$

(b) To find the current, using Tellegen's theorem, we first find the Thevenin's equivalent circuit with respect to terminals a and b.





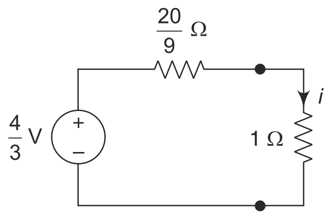
Thevenin voltage,

$$V_{\text{Th}} = \frac{3}{3 + 4 + 2} \times 4 = \frac{4}{3}\ \text{V}$$

Thevenin impedance,

$$Z_{\text{Th}} = \frac{5 \times 4}{5 + 4} = \frac{20}{9}\ \Omega$$

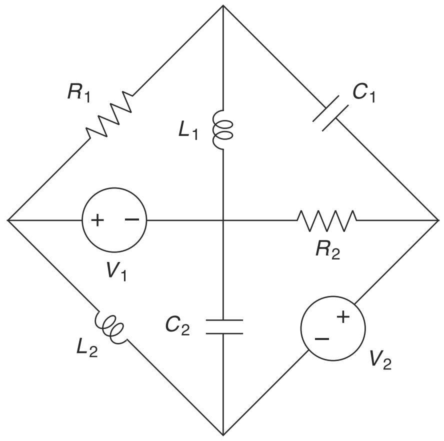
Thus, the equivalent circuit is shown.



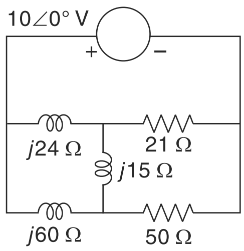
Now, applying Tellegen's Theorem,

$$-\frac{4}{3}\times I+\frac{20}{9}\times I\times I+1\times I\times I=0 \quad\Rightarrow\quad I=0.414\ \text{A} \qquad \textit{Ans.}$$

5. (a) Write the loop equations and the node equations for the circuit in the below figure.

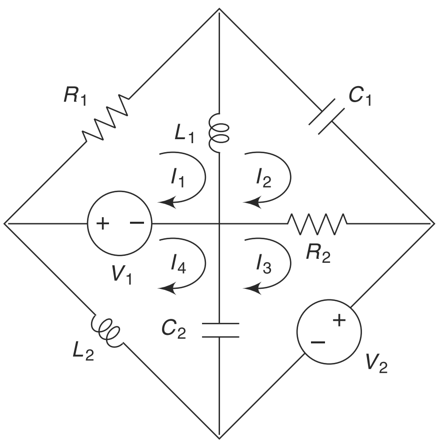


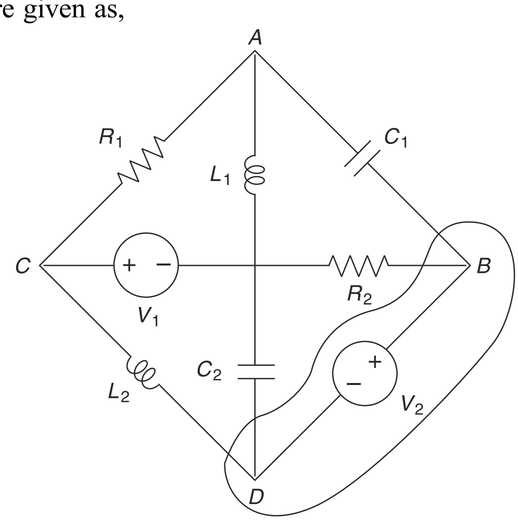
(b) Find all the node voltages and the loop currents in the circuit as given in the below figure.



Solution

(a) Let the four mesh currents be I_1, I_2, I_3, and I_4, and the node voltages be V_A, V_B, V_C, V_D. The loop equations in matrix form are given as,





$$\begin{bmatrix} (R_1+sL_1) & (-sL_1) & 0 & 0 \\ (-sL_1) & \left(R_2+sL_1+\dfrac{1}{sC_1}\right) & (-R_2) & 0 \\ 0 & (-R_2) & \left(R_2+\dfrac{1}{sC_2}\right) & \left(-\dfrac{1}{sC_2}\right) \\ 0 & 0 & \left(-\dfrac{1}{sC_2}\right) & \left(sL_2+\dfrac{1}{sC_2}\right) \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ -V_2 \\ 0 \end{bmatrix}$$

Similarly, the node equations are given in matrix form as,

KCL at node A: $V_A\left(\dfrac{1}{R_1}+\dfrac{1}{sL_1}+sC_1\right)+V_B(-sC_1)+V_C\left(-\dfrac{1}{R_1}\right)=0$

Constraints equations are: $V_C=V_1$ and $(V_B-V_D)=V_2$

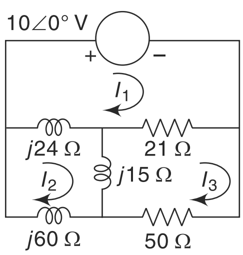
KCL for the supernode is

$$\frac{V_D-V_C}{sL_2}+V_DsC_2+\frac{V_B}{R_2}+(V_B-V_A)sC_1=0$$

$$\Rightarrow \qquad V_A(-sC_1)+V_B\left(\frac{1}{R_2}+sC_1\right)+V_C\left(-\frac{1}{sL_2}\right)+V_D\left(\frac{1}{sL_2}+sC_2\right)=0$$

$$\begin{bmatrix} \left(\dfrac{1}{R_1}+\dfrac{1}{SL_1}+SC_1\right) & -SC_1 & -\dfrac{1}{R_1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -SC_1 & \left(\dfrac{1}{R_2}+SC_1\right) & -\dfrac{1}{SL_2} & \left(\dfrac{1}{SL_2}+SC_2\right) \end{bmatrix} \times \begin{bmatrix} V_A \\ V_B \\ V_C \\ V_D \end{bmatrix} = \begin{bmatrix} 0 \\ V_1 \\ V_2 \\ 0 \end{bmatrix}$$

(b)



By KVL for the three meshes,

$$(21+j24)I_1+(-j24)I_2-21I_3=-10 \qquad \text{(i)}$$

$$-j24I_1+j99I_2-j15I_3=0 \qquad \text{(ii)}$$

$$-21I_1-j15I_2+(71+j15)I_3=0 \qquad \text{(iii)}$$

Solving Eqs (i) to (iii),

$$I_1 = \frac{\begin{vmatrix} -10 & -j24 & -21 \\ 0 & j99 & -j15 \\ 0 & -j15 & (71+j15) \end{vmatrix}}{\begin{vmatrix} (21+j24) & -j24 & -21 \\ -j24 & j99 & -j15 \\ -21 & -j15 & (71+j15) \end{vmatrix}}$$

$$I_2 = \frac{\begin{vmatrix} (21+j24) & -10 & -21 \\ -j24 & 0 & -j15 \\ -21 & 0 & (71+j15) \end{vmatrix}}{\begin{vmatrix} (21+j24) & -j24 & -21 \\ -j24 & j99 & -j15 \\ -21 & -j15 & (71+j15) \end{vmatrix}}$$

$$I_3 = \frac{\begin{vmatrix} (21+j24) & -j24 & -10 \\ -j24 & j99 & 0 \\ -21 & -j15 & 0 \end{vmatrix}}{\begin{vmatrix} (21+j24) & -j24 & -21 \\ -j24 & j99 & -j15 \\ -21 & -j15 & (71+j15) \end{vmatrix}}$$

From these equations all loop currents and node voltages can be evaluated.

6. (a) Consider the circuit shown in Q. 5(a). Draw the corresponding graph. Find the complete incidence matrix and the reduced incidence matrix. Find the possible number of trees.

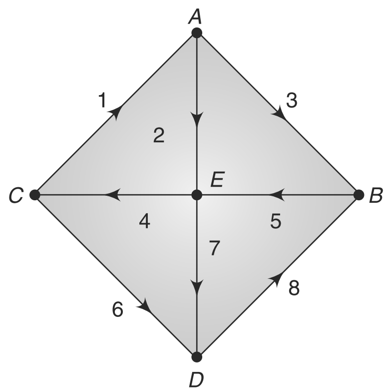
(b) Consider the same circuit. Indicate the branch currents by i_j, $j = 1, 2, \ldots$ and Loop currents by I_j , $j = 1, 2, \ldots$ Find the tie-set matrix and express the Loop currents in terms of branch currents.

(c) Find the *f*-cutset matrix for the same circuit.

Solution

(a) The graph of the network is shown.

The complete incidence matrix is obtained as,



$$A_a = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline A & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ B & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ C & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ E & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}$$

Taking E as the datum node, the reduced incidence matrix is,

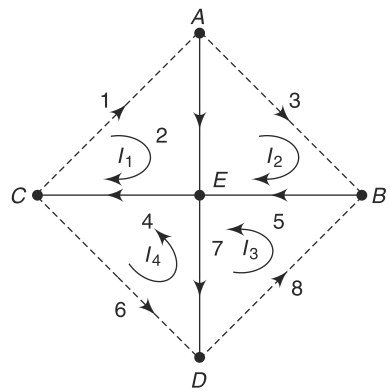
$$A = \begin{matrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{matrix}$$

Number of possible tree is,

$$N = \det\{[A]\times[A^T]\} = \det\left\{\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right\}$$

$$= \det\left\{\begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}\right\} = 45 \qquad \textit{Ans.}$$

(b) We consider a tree as shown.
Here, branch currents are: i_1, i_2, i_3, …, i_8.
Loop currents are: I_1, I_2, I_3 and I_4.
Tie set matrix is given as,

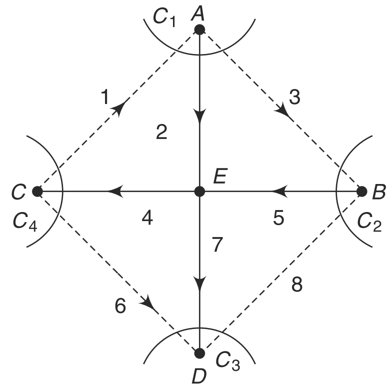


$$B = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline L_1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ L_2 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ L_3 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ L_4 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{array}$$

Thus, the branch currents are given in terms of loop current as,

$$[I_b] = [B]^T \times [I_L]$$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}$$



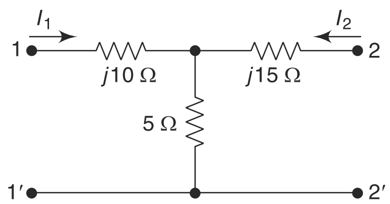
(c) We consider the same tree as in part (b),
The fundamental cut sets are:

$$C_1: [1, 2, 3] \quad C_2: [3, 5, 8] \quad C_3: [6, 7, 8] \quad C_4: [1, 4, 6]$$

Thus, the cut set matrix is given as,

$$Q = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline C_1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ C_2 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ C_3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ C_4 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{array}$$

7. (a) Find the open-circuit impedance parameters of the circuit given in the figure.
(b) Find the h-parameters of the circuit.
(c) Indicate how these parameters (obtained in parts (a) and (b)) are related.



Solution

(a) By KVL,

$$(j10 + 5)I_1 + 5I_2 = V_1 \tag{i}$$

and $$5I_1 + (j15 + 5)I_2 = V_2 \tag{ii}$$

Thus, the z-parameters are:

$$z_{11} = (5 + j10)\ \Omega \quad z_{12} = z_{21} = 5\ \Omega \quad Z_{22} = (5 + j15)\ \Omega \qquad \textit{Ans.}$$

(b) The hybrid parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

From Eq (ii), we get,

$$I_2 = -\frac{5}{5+j15}I_1 + \frac{V_2}{5+j15}$$

$$= -\frac{1}{1+j3}I_1 + \frac{1}{5+j15}V_2 \qquad \text{(iii)}$$

Putting this value of I_2 in Eq. (i), we get,

$$(5+j10)I_1 + 5\left[-\frac{5}{5+j15}I_1 + \frac{V_2}{5+j15}\right] = V_1$$

$$\Rightarrow \qquad V_1 = \frac{(5+j10)\times(5+j15)-25}{(5+j15)}I_1 + \frac{5}{5+j15}V_2$$

$$= \frac{30+j25}{1+j3}I_1 + \frac{1}{1+j3}V_2 \qquad \text{(iv)}$$

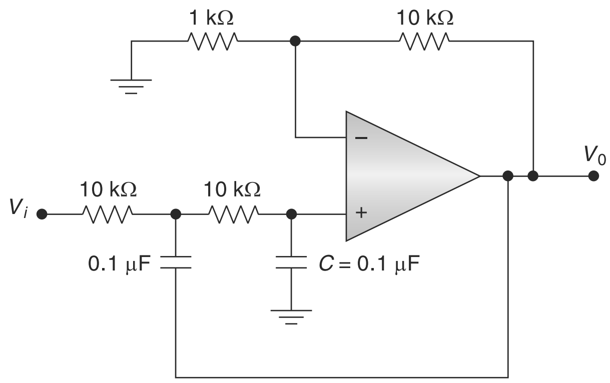
Comparing Eqs (iii) and (iv) with the standard equations of h-parameters, we get,

$$h_{11} = \frac{30+j25}{1+j3}; \quad h_{12} = \frac{1}{1+j3}; \quad h_{21} = -\frac{1}{1+j3}; \quad h_{22} = \frac{1}{5+j15} \qquad \textit{Ans.}$$

(c) Relation between z-parameters and h-parameters
The relationship between them is given in matrix form as,

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\Delta z}{z_{22}} & \dfrac{z_{12}}{z_{22}} \\ -\dfrac{z_{21}}{z_{22}} & \dfrac{1}{z_{22}} \end{bmatrix}$$

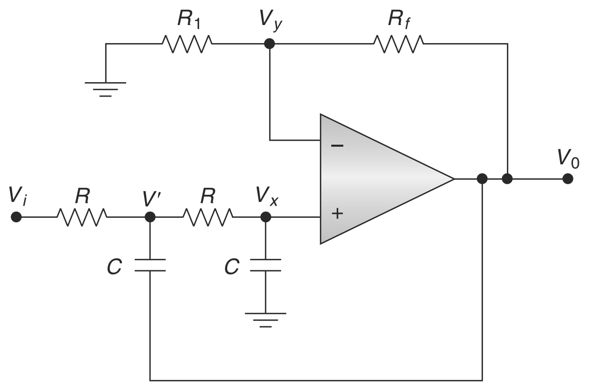
8. (a) The circuit in the figure is a low-pass second order active filter. Analyze the circuit and find the cut-off frequency.



(b) Draw the circuit diagram of a first order high-pass active filter and find out the expression of the cut-off frequency.

Solution

(a) The circuit is shown below.



Here, $V_y = \dfrac{V_0}{R_1 + R_f} R_1$ and $V_x = V_y$

Writing KCL at node V',

$$\frac{V' - V_i}{R} + \frac{V' - V_0}{1/sC} + \frac{V' - V_x}{R} = 0$$

or $$(V' - V_i) + (V' - V_0)sRC + (V' - V_0) = 0$$

or $$(-1)V_x + (2 + sRC)V' + (-sRC)V_0 = V_i \tag{1}$$

Writing KCL at node x,

$$\frac{V_x - V'}{R} + \frac{V_x}{1/sC} = 0$$

or $$(1 + sRC)V_x + (-1)V' + (0)V_0 = 0 \tag{2}$$

Writing KCL at node y,

$$\frac{V_x}{R_1} + \frac{V_x - V_0}{R_f} = 0$$

or $$(R_1 + R_f)V_x + (0)V' + (-R_1)V_0 = 0 \tag{3}$$

Solving for V_0 from equations (1), (2), and (3), we get,

$$V_0 = \frac{\begin{vmatrix} -1 & (2+sRC) & V_i \\ (1+sRC) & -1 & 0 \\ (R_1+R_f) & 0 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & (2+sRC) & -sRC \\ (1+sRC) & -1 & 0 \\ (R_1+R_f) & 0 & -R_1 \end{vmatrix}} = V_i \frac{\dfrac{(R_1+R_f)}{R_1}}{s^2 + 3sRC - sRC\left(\dfrac{(R_1+R_f)}{R_1}\right) + 1}$$

or $$\frac{V_0(s)}{V_i(s)} = \frac{K}{s^2 + s\left(\frac{3-K}{RC}\right) + \left(\frac{1}{RC}\right)^2} \tag{4}$$

where $K = \frac{R_1 + R_f}{R_1}$ = D. C. gain of the amplifier.

Substituting $s = j\omega$, the transfer function is,

$$H(j\omega) = \frac{V_0(j\omega)}{V_i(j\omega)} = \frac{K}{1 + j(3-K)RC\omega - R^2C^2\omega^2}$$

The magnitude of the transfer function is,

$$|H(j\omega)| = \frac{K}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]^2 + [3-K]^2\left(\frac{\omega}{\omega_c}\right)^2}}\text{; where, } \omega_c = \frac{1}{RC}$$

In the above equation, when $\omega \to 0$, $|H(j\omega)| = K$. Thus, the low frequency gain of the filter is K and when $\omega \to \infty$, $|H(j\omega)| = 0$, i.e., high frequency gain is zero.

From the Table of the Butterworth Filter, the transfer function for second order (n = 2) filter is,

$$T(s) = \frac{K}{\left(\frac{s}{\omega_c}\right)^2 + 1.414\left(\frac{s}{\omega_c}\right) + 1} = \frac{K\omega_c^2}{s^2 + 1.414\omega_c s + \omega_c^2} \tag{5}$$

where, ω_c is the cut-off frequency. Comparing equations (4) and (5), we get,

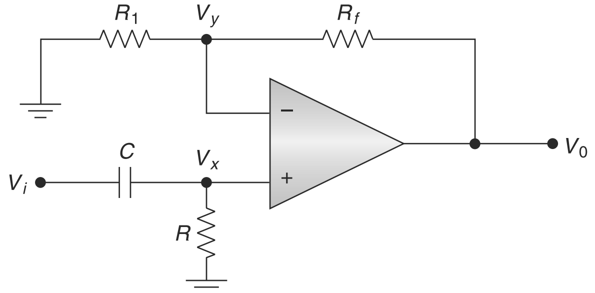
$$\boxed{\omega_c = \frac{1}{RC} \quad \text{or,} \quad f_c = \frac{1}{2\pi RC}}$$

Putting the value R = 10 kΩ and C = 0.1 μF,

$$\omega_c = 10 \text{ rad/s} \quad \text{or,} \quad f_c = 1.59 \text{ Hz}$$

(b) First Order High-Pass Active Filter

The circuit is shown in figure.



The filtering is done by the *CR* network and the op-amp is connected as a unity – gain follower. The feedback resistor, R_f is included to minimize dc off-set.
Here,

$$V_y = V_0 \frac{R_1}{R_1 + R_f} \tag{1}$$

Voltage across the resistor *R*,

$$V_x = \frac{R}{R + X_c} V_i = \frac{R}{R + \frac{1}{j\omega C}} V_i = \frac{j\omega RC}{1 + j\omega RC} V_i \tag{2}$$

Since op-amp gain is infinite,

$$V_x = V_y$$

$$\Rightarrow \quad \frac{V_0 R_1}{R_f + R_1} = \frac{j\omega RC}{1 + j\omega RC} V_i$$

$$\Rightarrow \quad \frac{V_0}{V_i} = \left(\frac{R_f + R_1}{R_1}\right)\left(\frac{j\omega RC}{1 + j\omega RC}\right) = A_F \times \frac{j2\pi f RC}{1 + j2\pi f RC}$$

where, $A_F = (1 + R_f/R_1)$ = Pass-band Gain of the filter,

f = frequency of the input signal (Hz),

$f_c = \frac{1}{2\pi RC}$ cut-off frequency of the filter (Hz).

The gain-magnitude,

$$\left|\frac{V_0}{V_i}\right| = \frac{A_F (2\pi f RC)}{\sqrt{1 + (2\pi f RC)^2}} = A_F \cdot \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$

For this magnitude to be $\frac{A_F}{\sqrt{2}}$ at $f = f_c$, we have,

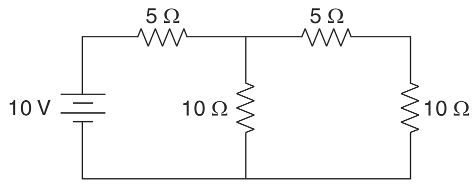
$$\frac{A_F}{\sqrt{2}} = \frac{A_F (2\pi_c f RC)}{\sqrt{1 + (2\pi_c f RC)^2}}$$

or

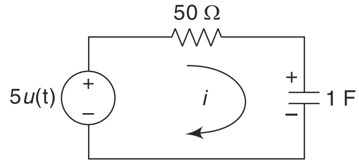
$$\boxed{f_c = \frac{1}{2\pi RC}}$$

This is the cut-off frequency of the high-pass filter.

9. (a) Write the input file in SPICE to find the node voltages in the circuit in figure.



(b) Write the input file in SPICE to plot the capacitor voltage and capacitor current (initial voltage for the capacitor is 1 volt) in the circuit given in figure.



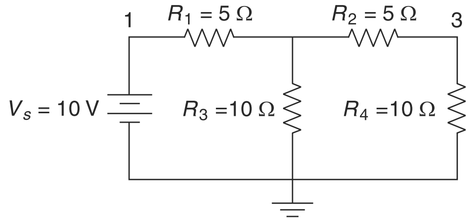
(c) What is the statement in SPICE for a damped sine wave $V(t)$ with a delay and offset as given below:

$$V(t) = 5 \text{ volt} \quad \text{for } 0 < t < 5 \text{ second}$$
$$= 5 + 10\exp[-0.1(t-5)]\sin(2\pi 100 t) \text{ volt} \quad \text{for 5 second} < t < T$$

Solution

(a) To write the input file, we first label the nodes and give the reference name for the resistors.

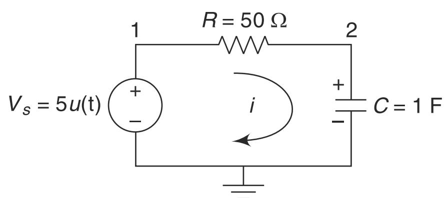
The input file is shown below:




```
*DC Analysis*
VS      1     0     DC     10V
R1      1     2     5
R2      2     3     5
R3      0     2     10
R4      0     3     10
.DC     VS    10    10     1
.PRINT  DC    V(1)  V(2)   V(3)
.END
```

(b) To write the input file, we first label the nodes and give the reference name for the resistors.

The input file is shown below:




```
*DC Analysis*
VS      1     0     DC     5V
R       1     2     50
C       0     2     1F     IC = 1V
.DC     VS    5     5      1
.PRINT  DC    V(2)  I(C)
.PLOT   DC    V(2)  I(C)
.END
```

(c) SPICE Statement of Damped Sine Wave

The statement is,

$$\boxed{\text{SIN} \quad (\text{PO} \quad \text{PA} \quad \text{FREQ} \quad \text{TD} \quad \text{ALPHA} \quad \text{THETA})}$$

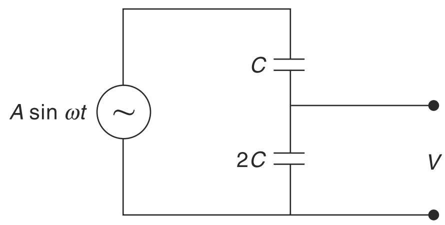
where , PO – dc offset value for the sinusoid;
PA – peak amplitude (taken from the dc offset value);

FREQ – frequency in Hz;
TD – delay time before the start of the sinusoid;
ALPHA – damping factor, it is the inverse of time–constant;
THETA – phase delay in degree.

The mathematical description of the function defined by this function is,

$$f(t) = PO + PAe^{-ALPHA(t-TD)}\sin[\{2\pi FEREQ(t-TD)\} - THETA]u(t-TD)$$

10. Answer any four.
   (a) Compare lumped and distributed networks.
   (b) $f(t)$ has the Fourier Transform $F(j\omega)$. What will be the Fourier Transform of time-scaled function $f(at)$, $a > 0$?
   (c) What is the value of the voltage V in the given circuit in the figure?



   (d) What are transmission parameters? Where are they most effectively used?
   (e) What is Compensation Theorem?
   (f) Convert a voltage source V with internal resistance R to a corresponding current source. Can you convert a voltage source V with zero internal resistance to a corresponding current source?

Solution

(a) Lumped and Distributed Networks

All physical systems contain distributed parameters because of the physical size of the system components. For example, the resistance of a resistor is distributed throughout its volume.

However, if the ***size of the system components is very small with respect to the wave-length*** of the highest frequency present in the signals associated with it, then the system components behave as if it all were occurring at a point. This system is said to be ***lumped-parameter system.***

For Example, we consider an electric power system of frequency 50 Hz. The wavelength of the signal is obtained as,

$$n\lambda = C \quad \Rightarrow \quad \lambda = \frac{C}{n} = \frac{3\times 10^5}{50}\text{ km} = 6000\text{ km}$$

Thus, the **electrical network inside a room can be treated as a lumped-parameter network,** but will be treated as **distributed network for a long-distance transmission lines**.

Therefore, if the size of all the components of any network is very small compared to the wavelength of the highest frequency of the source, then the network is said to be lumped, otherwise it will be a distributed network.

(b) If $F\{f(t)\} = F(j\omega)$, then $F\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{j\omega}{a}\right), \quad a > 0$

Proof Let, $at = x \quad \Rightarrow t = \dfrac{x}{a} \quad \therefore dt = \dfrac{dx}{a}$

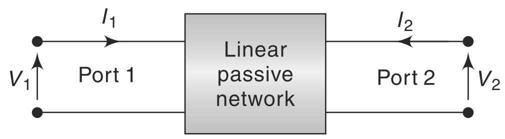
$$F\{f(at)\} = \int_{-\infty}^{\infty} f(at)e^{-j\omega at}dt = \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\frac{dx}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx = \frac{1}{a}F\left(\frac{j\omega}{a}\right), \quad a > 0$$

(c) By the voltage division principle for capacitor, the voltage across the capacitor $2C$ is,

$$V = \frac{C}{C+2C}A\sin\omega t = \frac{1}{3}A\sin\omega t$$

(d) Transmission Parameters

The *ABCD* parameters represent the relation between the input quantities and the output quantities in the two-port network. They are thus voltage-current pairs.



However, as the quantities are defined as an input-output relation, the output current is marked as going out rather than as coming into the port.

The transmission parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The parameters A, B, C, D can be defined in a similar manner with either port 2 on short circuit or port 2 on open circuit.

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \text{Open Circuit Reverse Voltage Gain}$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = \text{Short Circuit Transfer Impedance}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \text{Open Circuit Transfer Admittance}$$

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = \text{Short Circuit Reverse Current Gain}$$

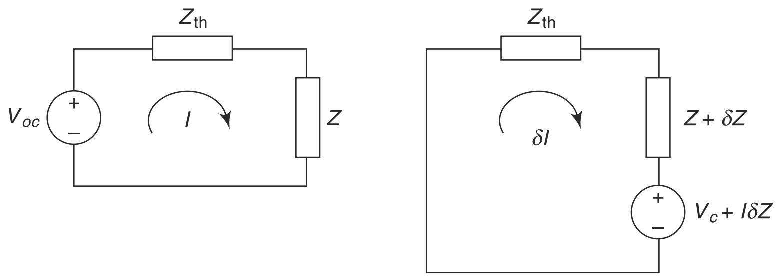
These parameters are **most effectively used in transmission lines**. In a transmission line, the currents enter at one end and leaves at the other end, and we need to know a relation between the sending end quantities and the receiving end quantities.

(e) Compensation Theorem

In any linear bilateral active network, if any branch carrying a current I has its impedance Z changed by an amount δZ, the resulting changes that occur in the other branches are the same as those which would have been caused by the injection of a voltage source of $(-I\delta Z)$ in the modified branch.

We consider the network N, having branch impedance Z.

Let the current through Z be I and its voltage be V.



Let δZ be the change in Z.
Then, I' can be written as,

$$I' = \frac{V_{oc}}{Z+\delta Z+Z_{th}}; \quad \delta I = I' - I = \frac{V_{oc}}{Z+\delta Z+Z_{th}} - \frac{V_{oc}}{Z+Z_{th}} = -\left(\frac{V_{oc}}{Z+Z_{th}}\right)\left(\frac{\delta Z}{Z+\delta Z+Z_{th}}\right)$$

$$= -\frac{I\delta Z}{Z+\delta Z+Z_{th}} = -\frac{V_c}{Z+\delta Z+Z_{th}} \quad \text{where} \quad V_c = I\delta Z$$

(f) Conversion of Voltage Source into Current Source

A voltage source $V(t)$ with an internal resistance R can be converted into a current source $I(t)$ in parallel with the same resistance R, where, $I(t) = \dfrac{V(t)}{R}$.

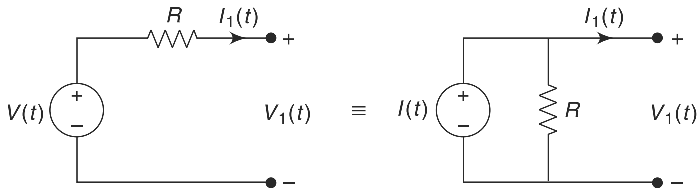
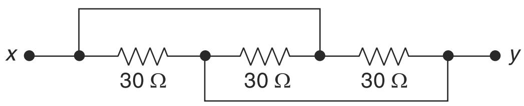


Figure *Conversion of Voltage Source into Current Source*

A voltage source can be converted into a current source and vise-versa if and only if their respective open circuit voltage and short circuit current are same. However, an ideal voltage source can never be open-circuited and an ideal current source can never be short-circuited, as this is in contrary to the definitions of ideal voltage and current sources. Thus, we cannot convert a voltage source V with zero internal resistance to a corresponding current source.

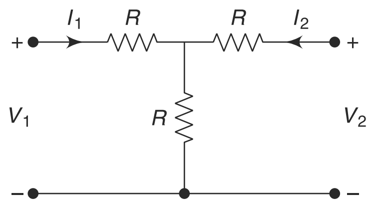
Model Question Paper III

1. Choose the correct answer.
   (a) Which of the following constitutes a bilateral element?
   (i) MOSFET (ii) metal rectifier (iii) a resistor (iv) diode
   (b) Two resistors are connected in parallel and each dissipates 20 watt. What is the total power dissipated across the two resistors:
   (i) 40 Watt (ii) 80 Watt
   (iii) 100 Watt (iv) none of the above
   (c) The equivalent resistance of the Figure between x and y is



   (i) 30 Ω (ii) 50 Ω (iii) 60 Ω (iv) 10 Ω
   (d) A dc voltage V is applied to a series R-L circuit. The steady state current is
   (i) $\dfrac{V}{R^2 + L^2}$ (ii) 0 (iii) $\dfrac{V}{L}$ (iv) $\dfrac{V}{R}$
   (e) If $f(t)$ and its first derivative are Laplace transformable, then final value theorem is
   (i) $\underset{t\to\infty}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ F(s)$ (ii) $\underset{t\to\infty}{\text{Lt}}\ f(t) = \underset{s\to\infty}{\text{Lt}}\ sF(s)$
   (iii) $\underset{s\to 0}{\text{Lt}}\ f(t) = \underset{s\to\infty}{\text{Lt}}\ sF(s)$ (iv) none of the above.
   (f) A network has 7 nodes and five independent loops. The number of branches is:
   (i) 11 (ii) 10 (iii) 13 (iv) 19
   (g) Superposition theorem is not applicable to networks having
   (i) transformers (ii) dependent voltage sources
   (iii) non-linear elements (iv) dependent current sources

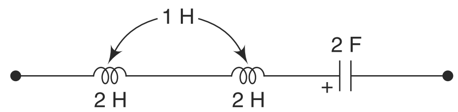
(h) A 2-port network is shown in the figure. The parameter h_{21} for this network can be given by



(i) $-\frac{1}{2}$ (ii) $+\frac{1}{2}$

(iii) $+\frac{3}{2}$ (iv) $-\frac{3}{2}$

(i) The resonant frequency of the series circuit shown in the figure. is



(i) $\frac{1}{4\pi\sqrt{3}}$ Hz (ii) $\frac{1}{4\pi}$ Hz

(iii) $\frac{1}{4\pi\sqrt{2}}$ Hz (iv) $\frac{1}{2\pi\sqrt{5}}$ Hz

(j) A ramp function
   (i) has both Laplace and Fourier transforms.
   (ii) has Laplace transform but not Fourier transform.
   (iii) has Fourier transform but not Laplace transform.
   (iv) none of these.

Solution

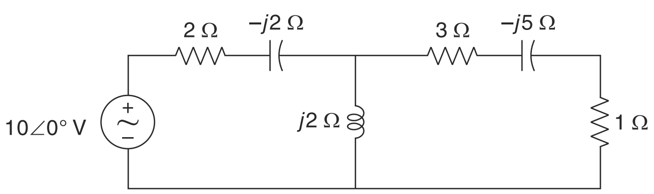
(a) (iii) resistor (b) (iii) 40 Watt (c) (iv) 10 Ω

(d) (ii) V/R (e) (ii) $\underset{t\to\infty}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ [sF(s)]$ (f) (iii) 11

(g) (i) non-linear (h) $-\frac{1}{2}$ (i) both (iii) $\frac{1}{4\pi\sqrt{3}}$ and (i) $\frac{1}{4\pi}$

(j) (ii) has Laplace transform but not Fourier transform

2. (a) State KCL and KVL 3
   (b) For the circuit shown in figure, 7
      (i) Determine the KVL equations.
      (ii) Find the two loop currents I_1 and I_2.
      (iii) Find the power supplied by the source and the power dissipated in each resistor.



Solution

(a) Kirchhoff's Current Law (KCL)

Kirchhoff's current law is based on the principle of conservation of charge. This requires that the algebraic sum of the charges within a system cannot change. Thus the total rate of change of charge must add up to zero. Rate of change of charge is current.

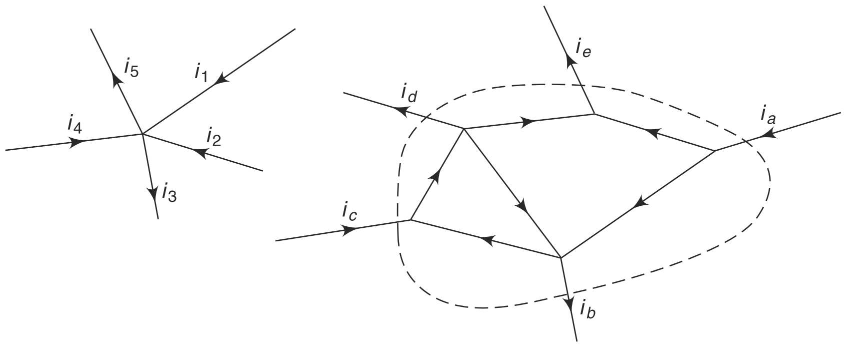


Figure *Illustration of KCL*

This gives us our basic Kirchhoff's current law as the algebraic sum of the currents meeting at a point is zero i.e., at a node, $\Sigma Ir = 0$, where Ir are the currents in the branches meeting at the node

This is also sometimes stated as the sum of the currents entering a node is equal to the sum of the current leaving the node.

The theorem is applicable not only to a node, but to a closed system.

$$i_1 + i_2 - i_4 - i_5 = 0; \text{ Also for the closed boundary, } i_a - i_b + i_c - i_d - i_e = 0$$

Kirchhoff's Voltage Law (KVL)

Kirchhoff's voltage law is based on the principle of conservation of energy. This requires that the total work done in taking a unit positive charge around a closed path and ending up at the original point is zero.

This gives us our basic Kirchhoff's law as the algebraic sum of the potential differences taken around a closed loop is zero.

i.e., around a loop, $\Sigma Vr = 0$, where Vr are the voltages across the branches in the loop.

$$v_a + v_b + v_c + v_d - v_e = 0$$

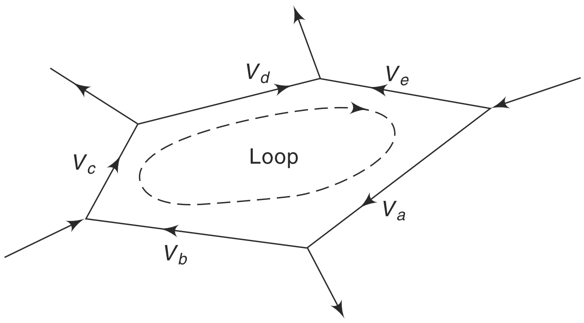
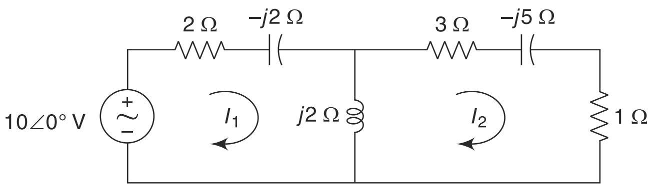


Figure *Illustration of KVL*

This is also sometimes stated as the sum of the emfs taken around a closed loop is equal to the sum of the voltage drops around the loop.

(b) (i) KVL Equations

$$\left.\begin{aligned} 2I_1 - j2I_2 &= 10 \\ \text{and } -j2I_1 + (4 - j3)I_2 &= 0 \end{aligned}\right\} \quad \textit{Ans.}$$



(ii) Solving for the currents,

$$I_1 = \frac{\begin{vmatrix} 10 & -j2 \\ 0 & (4-j3) \end{vmatrix}}{\begin{vmatrix} 2 & -j2 \\ -j2 & (4-j3) \end{vmatrix}} = \frac{40-j30}{4-j6} = 6.933\angle 19.44°\text{ A} \qquad \textit{Ans.}$$

and
$$I_2 = \frac{\begin{vmatrix} 2 & 10 \\ -j2 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -j2 \\ -j2 & (4-j3) \end{vmatrix}} = \frac{j20}{4-j6} = 2.773\angle 143.31°\text{ A} \qquad \textit{Ans.}$$

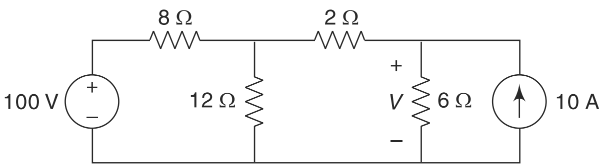
(iii) Power supplied by the source,

$$P_s = VI_1 \cos\phi_1 = 10 \times 6.933 \cos(19.44°) = 65.28\text{ Watt}$$

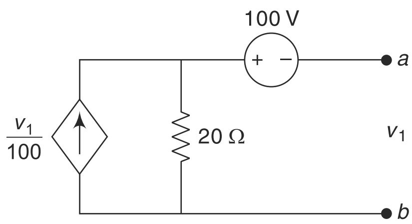
Power dissipated in resistors,

$$\left.\begin{aligned} P_{2\Omega} &= |I_1|^2 \times 2 = 96.15\text{ W} \\ P_{3\Omega} &= |I_2|^2 \times 3 = 23.08\text{ W} \\ P_{1\Omega} &= |I_2|^2 \times 1 = 7.69\text{ W} \end{aligned}\right\} \qquad \textit{Ans.}$$

3. (a) For the circuit shown below, determine the voltage v using nodal analysis. 5

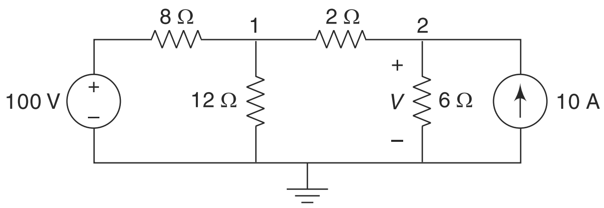


(b) The following circuit has a dependent current source and an independent voltage source. Find the Thevenin equivalent network of the circuit across the terminals a and b. 5



Solution

(a)



Let the node voltages be V_1 and V_2. Here, $V_2 = v$

By KCL,

$$\frac{V_1 - 100}{8} + \frac{V_1}{12} + \frac{V_1 - V_2}{2} = 0 \quad \Rightarrow \quad 17V_1 - 12v = 300 \tag{i}$$

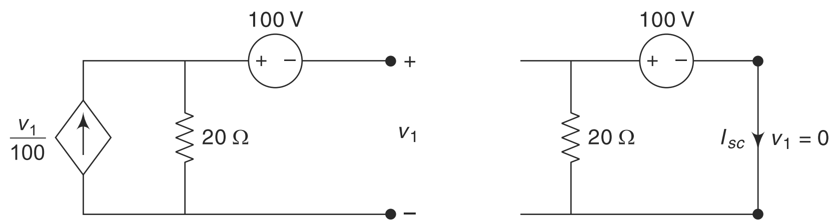
and

$$\frac{V_2 - V_1}{2} + \frac{V_2}{6} - 10 = 0 \quad \Rightarrow \quad -3V_1 + 4v = 60 \tag{ii}$$

Solving Eqs (i) and (ii), we get,

$$v = \frac{\begin{vmatrix} 17 & 300 \\ -3 & 60 \end{vmatrix}}{\begin{vmatrix} 17 & -12 \\ -3 & 4 \end{vmatrix}} = \frac{1920}{32} = 60 \text{ Volt} \qquad \textit{Ans.}$$

(b)



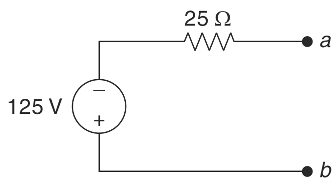
With open circuit, $v_1 = v_{oc}$. By KCL,

$$-\frac{v_{oc}}{100} + \frac{100 + v_{oc}}{20} = 0 \quad \Rightarrow \quad -v_{oc} + 500 + 5v_{oc} = 0 \quad \Rightarrow \quad v_{oc} = -125 \text{ Volt}$$

With short-circuit, $v_1 = 0$ and the dependent current source is open, so that, $I_{sc} = -5$ A

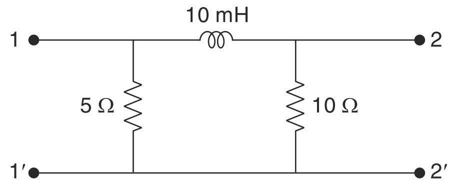
Thus, Thevenin impedance, $R_{Th} = \dfrac{v_{oc}}{I_{sc}} = \dfrac{-125}{-5} = 25\ \Omega$

So, the Thevenin's equivalent circuit is shown in figure below.



4. (a) Find the z-parameters in terms of h-parameters. 5

(b) Find the open circuit impedance parameters for the two-port network shown in figure below. 5



Solution

(a) z-parameters in terms of h-parameters

The z-parameter equations are,

$$\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned} \tag{i}$$

The hybrid parameter equations are,

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ I_2 &= h_{21}I_1 + h_{22}V_2 \end{aligned} \tag{ii}$$

From second equation of (ii), $V_2 = \left(-\dfrac{h_{21}}{h_{22}}\right)I_1 + \left(\dfrac{1}{h_{22}}\right)I_2$ (iii)

From first equation of (ii),

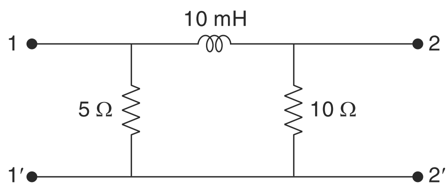
$$V_1 = h_{11}I_1 + h_{12}\left[\left(-\frac{h_{21}}{h_{22}}\right)I_1 + \left(\frac{1}{h_{22}}\right)I_2\right]$$

$$= \left(\frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}\right)I_1 + \left(\frac{h_{12}}{h_{22}}\right)I_2 \tag{iv}$$

Comparing Eq. (iii) and (iv) with Eq. (i), we get,

$$z_{11} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}};\ z_{12} = \frac{h_{12}}{h_{22}};\ z_{21} = -\frac{h_{21}}{h_{22}};\ z_{22} = \frac{1}{h_{22}}$$

(b)



For this π-network, the y-parameters are given as,

$$y_{11} = \left(\frac{1}{5} + \frac{1}{0.01s}\right) = \left(0.2 + \frac{100}{s}\right);$$

$$y_{12} = y_{21} = -\frac{1}{0.01s} = -\frac{100}{s};$$

$$y_{22} = \left(\frac{1}{10} + \frac{1}{0.01s}\right) = \left(0.1 + \frac{100}{s}\right)$$

$$\therefore \qquad \Delta y = (y_{11}y_{22} - y_{12}y_{21}) = \left(0.2 + \frac{100}{s}\right) \times \left(0.1 + \frac{100}{s}\right) - \left(-\frac{100}{s}\right)^2$$

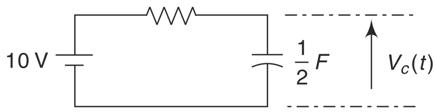
$$= 0.02 + \frac{30}{s} + \left(\frac{100}{s}\right)^2 - \left(-\frac{100}{s}\right)^2$$

$$= \left(0.02 + \frac{30}{s}\right)$$

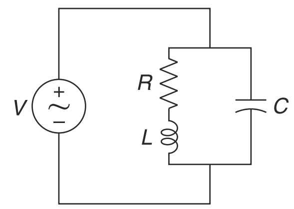
Thus, the z-parameters are,

$$\left.\begin{aligned} z_{11} &= \frac{y_{22}}{\Delta y} = \frac{0.1 + 100/s}{0.02 + 30/s} = \frac{0.1s + 100}{0.02s + 30} = \frac{5s + 5000}{s + 1500}\ \Omega \\ z_{12} &= z_{21} = -\frac{y_{12}}{\Delta y} = -\frac{-100/s}{0.02 + 30/s} = \frac{100}{0.02s + 30} = \frac{5000}{s + 1500}\ \Omega \\ z_{22} &= \frac{y_{11}}{\Delta y} = \frac{0.2 + 100/s}{0.02 + 30/s} = \frac{0.2s + 100}{0.02s + 30} = \frac{10s + 5000}{s + 1500}\ \Omega \end{aligned}\right\} \quad \textit{Ans.}$$

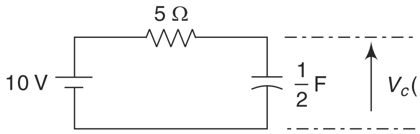
5. (a) Find the current $i(t)$ flowing through the circuit if the circuit is initially relaxed. Find the voltage across the capacitor $v_c(t)$ also. What is the value of the steady state current? 6



(b) Derive an expression of the resonant frequency ω_o for the circuit shown in figure below. 4



Solution

(a) 

By KVL,

$$\left(5 + \frac{1}{s/2}\right) I(s) = \frac{10}{s} \quad \Rightarrow \quad I(s)(5s + 2) = 10$$

$$\therefore \qquad I(s) = \frac{10}{5s + 2} = \frac{2}{s + 2/5}$$

Taking inverse Laplace transform, the current in the circuit,

$$i(t) = 2e^{-2t/5} \text{ (A)} \qquad \textit{Ans.}$$

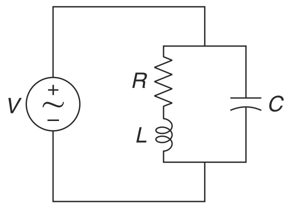
Voltage across the capacitor is,

$$V_C(s) = I(s) \times \frac{1}{\frac{1}{2}s} = \frac{2}{s} \times \frac{2}{s+2/5} = \frac{4}{s(s+2/5)} = 10\left(\frac{1}{s} - \frac{1}{s+2/5}\right)$$

Taking inverse Laplace transform,

$$V_C(t) = 10\,[1 - e^{-2t/5}] \text{ (V)} \qquad \textit{Ans.}$$

From the current expression, as $t \to \propto$, $i(t) \to 0$. So, the steady state value of the current is, $I_{ss} = 0$ *Ans.*

(b) 

Here,

$$Z_1 = (R + j\omega L)$$

$$Z_2 = \frac{1}{j\omega C}$$

$$Y = \frac{1}{R + j\omega L} + j\omega C = \left(\frac{R}{R^2 + \omega^2 L^2}\right) + j\omega\left(C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)$$

For resonance to occur, the imaginary part of the admittance should be zero.

$$\omega_0\left(C - \frac{\omega L}{R^2 + \omega^2 L^2}\right) = 0$$

$$\Rightarrow \qquad \omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

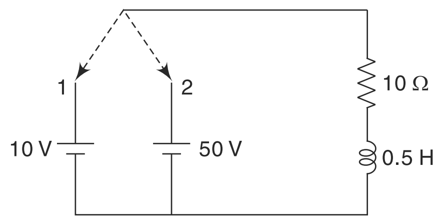
Thus, the resonant frequency is,

$$\boxed{\omega_0 = \sqrt{\frac{1}{LC}\left(1 - \frac{C{R_L}^2}{L}\right)}}$$

6. (a) Find the inverse Laplace transform of the function given below.

$$\frac{s+2}{s(s+1)^2} \qquad 4$$

(b) The circuit was in steady state with switch in position 1. Find the current $i(t)$ for $t > 0$ if the switch is moved from position 1 to 2 at $t = 0$. 6



Solution

(a) Let, $F(s) = \dfrac{s+2}{s(s+1)^2} = \dfrac{A}{s} + \dfrac{B}{(s+1)^2} + \dfrac{C}{(s+1)}$

By residue method,

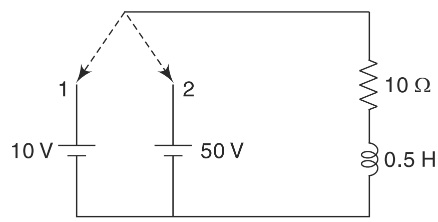
$$A = \left.\frac{s+2}{(s+1)^2}\right|_{s=0} = 2;\ B = \left.\frac{s+2}{s}\right|_{s=-1} = -1;\ C = \left.\frac{d}{ds}\left[\frac{s+2}{s}\right]\right|_{s=-1} = -\left.\frac{2}{s^2}\right|_{s=-1} = -2$$

$$\therefore \qquad F(s) = \frac{s+2}{s(s+1)^2} = \frac{2}{s} - \frac{1}{(s+1)^2} - \frac{2}{s+1}$$

Taking inverse Laplace transform,

$$f(t) = [2u(t) - 2e^{-t} - te^{-t}]$$ *Ans.*

(b) When the switch is in position 1, steady-state exists and the initial current through the inductor is,



$$i(0-) = \frac{10}{10} = 1\text{ A}$$

After the switch is moved to position 2, the KVL gives, in Laplace transform,

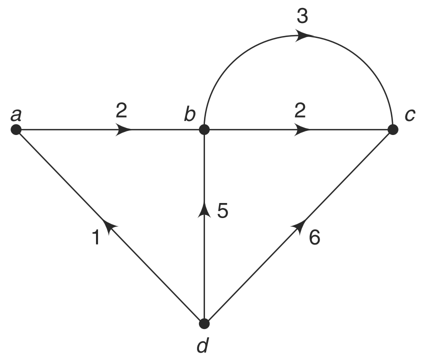
$$10I(s) + 0.5sI(s) - 0.5 \times 1 = \frac{50}{s}$$

or, $$I(s) = \frac{100}{s(s+20)} + \frac{1}{s+20} = 5\left[\frac{1}{s} - \frac{1}{s+20}\right] + \frac{1}{s+20}$$

Taking inverse Laplace transform,

$$i(t) = 5 - 4e^{-20t}\ (\text{A});\ t > 0$$ *Ans.*

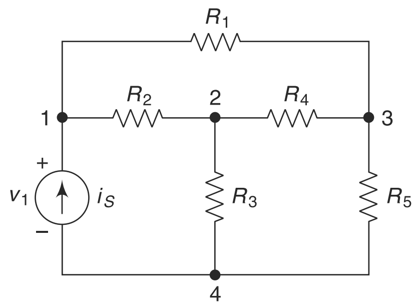
7. (a) Define *tree* and *cotree* of a graph. 3
   (b) Explain the *complete* and *reduced incidence matrix* with suitable example. 3
   (c) The following graph has 4 nodes and 6 branches. Find the reduced incidence matrix taking d as *datum node*. 4



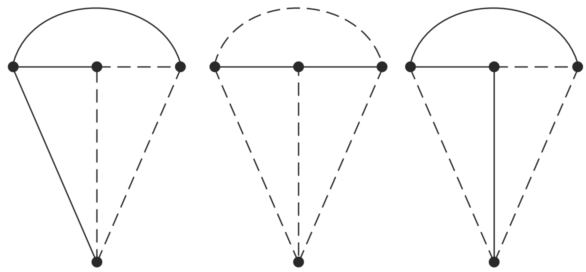
Solution

(a) **Tree of a Graph**

For a given connected graph of a network, a connected subgraph is known as a tree of the graph if the subgraph has all the nodes of the graph without containing any loop.



(a) Circuit



(b) Trees and Links of Circuit of Figure (a)

The branches of tree are called twigs or tree-branches. These are shown by solid lines in Fig. (b).

Cotree of a Graph

If a graph for a network is known and a particular tree is specified, ***the remaining branches*** are referred to as the ***links***. The ***collection of links*** is called a ***co-tree***. So, co-tree is the complement of a tree. These are shown by dotted lines in Fig. (b).

The branches of a co-tree may or may not be connected, whereas the branches of a tree are always connected.

(b) **Complete Incidence Matrix**

The incidence matrix symbolically describes a network. It also facilitates the testing and identification of the independent variables. Incidence matrix is a matrix which represents a graph **uniquely**.

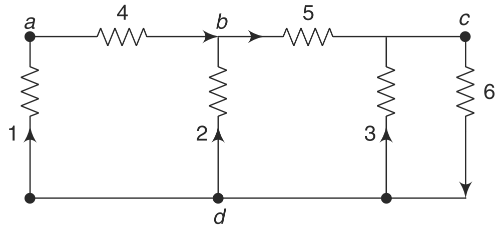
For a given graph with n nodes and b branches, the complete incidence matrix A_a is a rectangular matrix of order $n \times b$, whose elements have the following values:

Number of columns in [A] = Number of branches = b

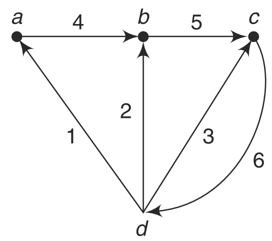
Number of rows in [A] = Number of nodes = n

A_{ij} = 1, if branch j is associated with node i and oriented away from node j.
= –1, if branch j is associated with node i and oriented towards node j.
= 0, if branch j is not associated with node i.

Example



(a) Incidence matrix network



(b) Graph of Network

		Branches						
		1	2	3	4	5	6	
	a	1	0	0	–1	0	0	Reduced
Nodes	b	0	1	0	1	–1	0	Incidence
	c	0	0	1	0	1	–1	Matrix A_I
Reference Node	d	–1	–1	–1	0	0	1	

Reduced incidence matrix [A]

The matrix obtained from A_a by eliminating one of the rows is called Reduced Incidence Matrix. In other words, suppression of the datum node (reference node) from the incidence matrix results in reduced incidence matrix.

For the graph shown above, reduced incidence matrix is given as,

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

(c) For the graph shown, the complete incidence matrix is given as,

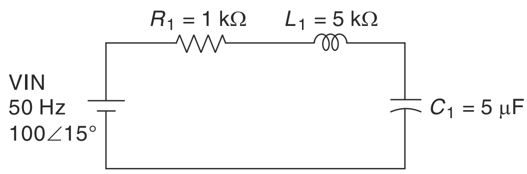
$$A_a = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Taking d as datum node, the reduced incidence matrix is given as,

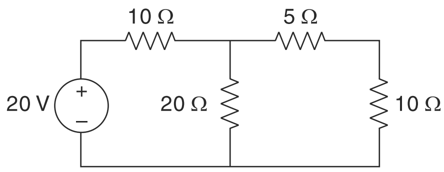
$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 \end{bmatrix}$$ *Ans.*

8. (a) What is SPICE? 2

(b) Write SPICE input file for the circuit shown below. 4



(c) Write SPICE input file for the circuit shown below to find the node voltage. 4



Solution

(a) SPICE

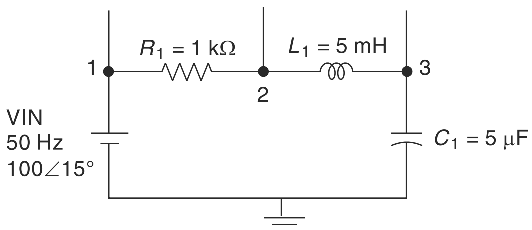
Spice (*Simulation Program with Integrated Circuit Emphasis*) is a software package developed in the 1970 at the University of California at Berkeley for simulating electronic circuits. It is used as a tool for analysis, design and testing of integrated circuits as well as a wide range of other electronic and electrical circuits.

The commercially supported versions of SPICE can be divided into two types:

(I) Mainframe versions, and

(II) PC – based versions.

(b)



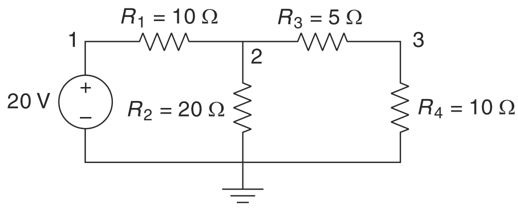
For this circuit, we first label the nodes.

The source file is,

```
*AC     Analysis*
VIN     1    0    SIN (0    100    50    0    0    15)
R1      1    2    1K
L1      2    3    5M
C1      0    3    5UF
.END
```

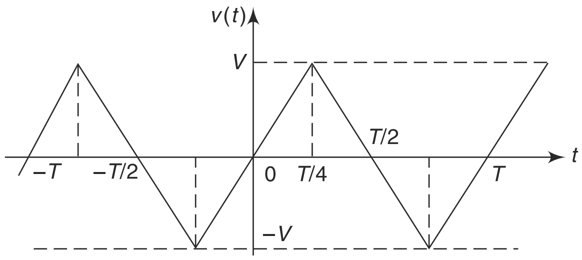
(c) Labeling the nodes and names of the elements, the input file is written as follows.

*DC			ANALYSIS*	
V1	1	2	20	
R1	1	2	10	
R2	0	2	20	
R3	2	3	5	
R4	0	3	10	
.DC	*V*1	20	20	1
.PRINT	DC	*V*(1)	*V*(2)	*V*(3)
.END				



9 (a) Find the Fourier transform of unit impulse function. 4

(b) Find the Fourier series expansion of the triangular wave shown below. 6



Solution

(a) Fourier Transform of Unit Impulse Function, $\delta(t)$

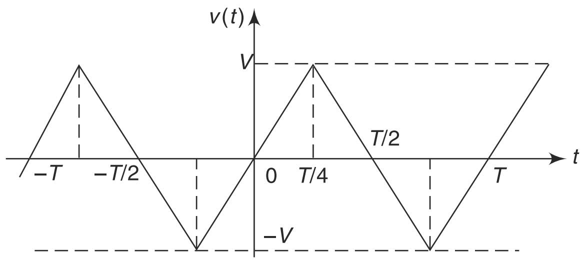
We know that, impulse function is defined as,

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0$$

and
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$\therefore$
$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t)e^{0}dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

(b)



The wave is an odd function and is having half wave symmetry.

$\therefore$ $a_n = 0$ and $a_0 = 0$

Now, $V(t) = \frac{4}{T}t\ ;\ 0 < t < T/4$

$$= -\frac{4}{T}t + 2\ ;\ T/4 < t < \frac{3T}{4}$$

$\therefore$ $b_n = \frac{8}{T}\int_0^{T/4} f(t)\ \sin n\omega t dt$; n is odd only.

$$= \frac{8}{T}\int_0^{T/4} \frac{4}{T} t \sin n\omega t dt$$

$$= \frac{32}{T^2}\left[\frac{-t \cos n\omega t}{n\omega} + \int \frac{\cos n\omega t}{n\omega} dt\right]_0^{T/4}$$

$$= \frac{16}{n\pi T}\left[-\frac{T}{4}\cos\frac{n\pi}{2} + \frac{\sin n\omega t}{n\omega}\Big|_0^{T/4}\right]$$

$$= \frac{16}{n\pi T}\left[-\frac{T}{4} \times 0 + \frac{T}{2n\pi}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \{\because\ \omega T = 2\pi\}$$

$\therefore$ $b_n = \frac{8}{n^2\pi^2}$, $n = 1, 5, 9, \ldots$

$$= -\frac{8}{n^2\pi^2},\ n = 3, 7, 11, \ldots$$

Hence,

$$V(t) = \frac{8}{\pi^2}\left(\sin \omega t - \frac{1}{3^2}\sin 3\omega t + \frac{1}{5^2}\sin 5\omega t - \frac{1}{7^2}\sin 7\omega t + \ldots\right)$$ *Ans.*

Model Question Paper IV

GROUP–A

(Multiple-Choice Questions)

1. Choose the most appropriate answers for any *ten* of the following: $10 \times 1 = 10$

(i) A capacitor of 0.01 farad has a leakage resistance 100 ohm across its terminals. The quality factor of it at 10 rad/s should be

(a) $\frac{1}{10}$ (b) 1 (c) 10 (d) 100

(ii) An RLC series circuit consists of a resistance of 1 kilo-ohm, an inductance of 0.1H and capacitance of 10 micro micro-farad. The Q-factor of the circuit will be

(a) 100 (b) 50 (c) 10 (d) $\frac{1}{100}$

(iii) The superposition theorem is applicable to

(a) linear responses only

(b) linear and non-linear responses

(c) linear, non-linear and time-variant responses.

(iv) A circuit having neither an e.m.f. source or any energy source is

(a) active circuit (b) passive circuit (c) unilateral circuit (d) bilateral circuit

(v) What should be the internal impedance of an ideal current source?

(a) Zero (b) Infinite (c) Both (a) and (b) (d) None of these

(vi) A two port network is reciprocal if and only if

(a) $z_{11} = z_{22}$ (b) $BC - AD = -1$ (c) $y_{12} = -y_{21}$ (d) $h_{12} = h_{21}$

(vii) A series RLC circuit is over-damped when

(a) $\left[\frac{R^2}{(4L^2)}\right] > \left[\frac{1}{(LC)}\right]$ (b) $\left[\frac{R^2}{(4L^2)}\right] = \left[\frac{1}{(LC)}\right]$ (c) $\left[\frac{R^2}{(4L^2)}\right] < \left[\frac{1}{(LC)}\right]$ (d) none of these.

(viii) Laplace transform analysis gives
(a) time domain response only (b) frequency domain response only
(c) both (a) and (b) (d) None of these

(ix) The value of the ramp function $t.u(t)$ at $t = -\infty$ is
(a) 0 (b) ∞ (c) $-\infty$ (d) 1

(x) The Fourier series expansion of a periodic function having half wave symmetry contains only
(a) cosine terms (b) sine terms (c) even harmonics (d) odd harmonics

(xi) The number of links for a graph having n nodes and b branches are
(a) $b - n + 1$ (b) $n - b + 1$ (c) $b + n - 1$ (d) $b + n$

(xii) The *dc* gain of a system having the transfer function $H(s) = \dfrac{12}{(s+2)(s+3)}$ is
(a) 2 (b) 1 (c) 12 (d) 3
(e) 0.

(xiii) A R-C series circuit has a time constant given by
(a) R/C (b) C/R (c) 1/(RC) (d) RC.

(xiv) If a function *f(t)* is shifted by *a* then it is correctly represented as
(a) $f(t-a)u(t)$ (b) $f(t)u(t-a)$ (c) $f(t-a)u(t-a)$ (d) $f(t-a)(t-a)$

(xv) In a four terminal network containing linear bilateral passive circuit elements, which one of the following conditions for z parameters generally holds?
(a) $z_{11} = z_{22}$ (b) $z_{12} = z_{21}$ (c) $z_{12}z_{21} = z_{11}z_{22}$ (d) $z_{11}^2 = z_{21}z_{22}$

(xvi) Two networks can be dual when
(a) their nodal equations are the same.
(b) the loop equations of one network are the nodal equations of the other.
(c) their loop equations are the same.
(d) none of these.

Solution

(i) (a) $\dfrac{1}{10}$ (ii) (a) 100
(iii) (a) linear responses only (iv) (b) passive circuit
(v) (b) Infinite (vi) both (b) $BC - AD = -1$ and (c) $y_{12} = -y_{21}$
(vii) (a) $\left[\dfrac{R^2}{(4L^2)}\right] > \left[\dfrac{1}{(LC)}\right]$ (viii) (c) both (a) and (b)
(ix) (a) 0 (x) (d) odd harmonics
(xi) (a) $b - n + 1$ (xii) (a) 2
(xiii) (d) RC (xiv) (c) $f(t-a)u(t-a)$
(xv) (b) $z_{12} = z_{21}$
(xvi) (b) the loop equations of one network are the nodal equations of the other.

GROUP–B

Answer any *three* questions $3 \times 5 = 15$

2. (a) What is compensation theorem? 2½

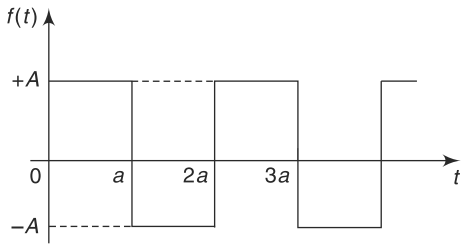
   (b) Convert a voltage source V with internal resistance R to a corresponding current source. Can you convert a voltage source V with zero internal resistance to a corresponding current source? 2½

   Solution

   (a) Consult WBUT 2004, Q. 10 (e).

   (b) Consult WBUT 2004, Q. 10 (f)

3. Find the Laplace transform of the square wave shown in the below figure. 5



Solution

The equation of the square wave is,

$$\begin{aligned} f(t) &= u(t) - u(t-a) - u(t-a) + u(t-2a) + u(t-2a) - u(t-3a) - \ldots \\ &= u(t) - 2u(t-a) + 2u(t-2a) - 2u(t-3a) + \ldots \end{aligned}$$

Taking Laplace transform,

$$\begin{aligned} F(s) &= \frac{1}{s} - \frac{2e^{-as}}{s} + \frac{2e^{-2as}}{s} - \frac{2e^{-3as}}{s} + \ldots \\ &= \frac{1}{s}[1 - 2e^{-as}(1 - e^{-as} + e^{-2as} - e^{-3as} + \ldots)] \\ &= \frac{1}{s}\left[1 - \frac{2e^{-as}}{1+e^{-as}}\right] \quad \left\{\because \quad \text{sum of G.P. series} = \frac{1}{1+e^{-as}}\right\} \\ &= \frac{1}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right] \end{aligned}$$

$$F(s) = \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$ *Ans.*

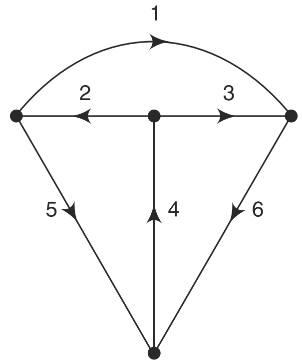
4 Define tie-set. With the help of a suitable example, explain the term 'tie-set matrix' used in network analysis. 5

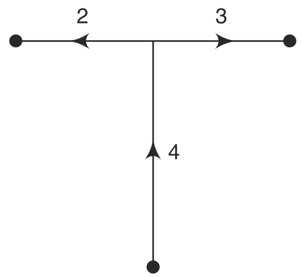
Solution

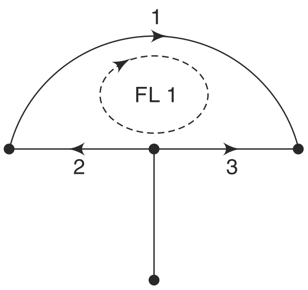
Tie-set

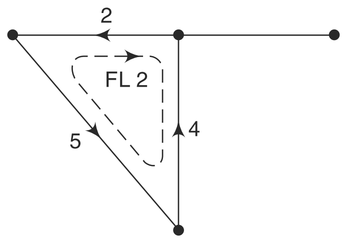
A ***tie-set*** is a set of branches contained in a loop such that each loop contains one link or chord and the remainder are tree branches.

Consider the graph and the tree as shown. This selected tree will result in **three fundamental loops** as we connect each link, in turn to the tree.

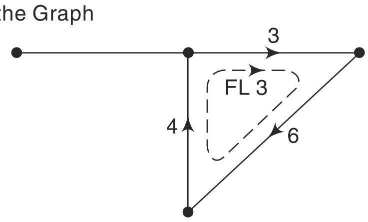








the Graph



Fundamental Loop 1 (FL1): Connecting link 1 to the tree.
Fundamental Loop 2 (FL2): Connecting link 5 to the tree.
Fundamental Loop 3 (FL3): Connecting link 6 to the tree.
These sets of branches (1, 2, 3), (2, 4, 5) and (3, 4, 6) form three tie-sets.

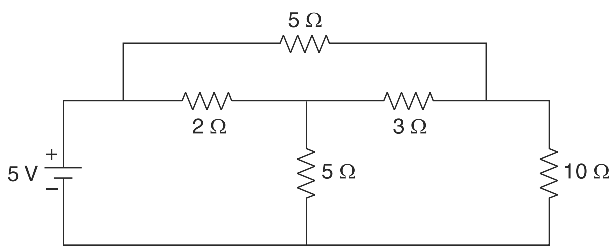
Tie-Set Matrix: Consult WBUT 2003, Q. 9 (b)

5. Explain under what condition, an RC series circuit behaves as
   (i) low pass filter
   (ii) integrator. 5

Solution

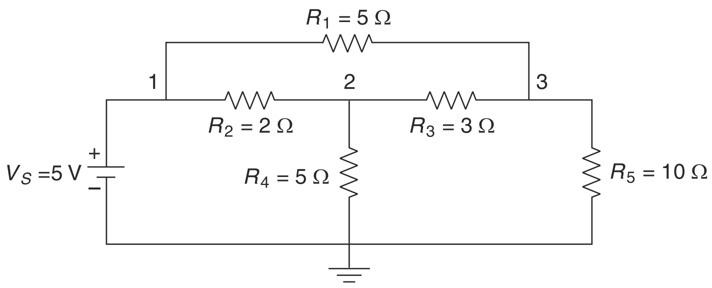
Consult WBUT 2003, Q. 10 (b) and (d).

6. Write the input file in SPICE to find the node voltages in the circuit in the below figure. 5



Solution

To write the SPICE input file, we label the nodes and give the reference nodes of the elements.



The input file is written below.

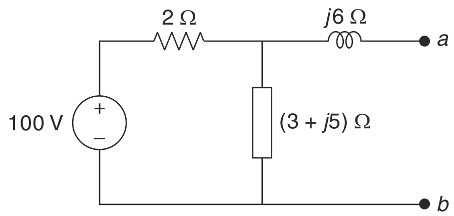
* INPUT		FILE*		
VS	1	0	DC	5V
R1	1	3	5	
R2	1	2	2	
R3	2	3	3	
R4	0	2	5	
R5	0	3	10	
.END				

GROUP-C

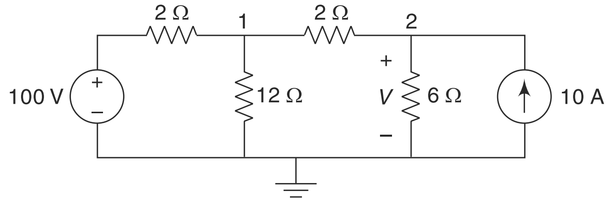
Answer any *three* questions $3 \times 15 = 45$

7. (a) State maximum power transfer theorem. 2

(b) Find the Thevenin's equivalent between the points *a* and *b* for the circuit given in figure. What should be the value of impedance connected between *a* and *b* for maximum power to be transferred from the sources? Obtain the amount of the maximum power. 8

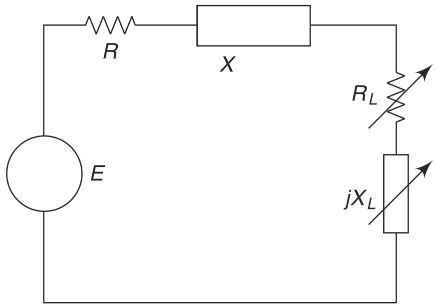


(c) Determine the voltage *v* in the network in the below figure using nodal analysis. 5



Solution

(a) Maximum Power Transfer Theorem



Maximum power is absorbed by one network from another connected to it at two terminals, when the impedance of one is the complex conjugate of the other.

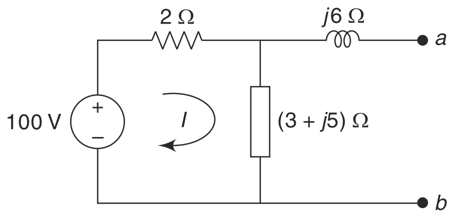
This means that for maximum active power to be delivered to the load, load impedance must correspond to the conjugate of the source impedance (or in the case of direct quantities, be equal to the source impedance).

Let E be the voltage source, $(R + jX)$ the internal impedance of the source and $(R_L + jX_L)$ the load impedance.

Then, according to Maximum Power Transfer Theorem maximum power will be transferred to the load if:

$$R = R_L \quad \text{and} \quad X = -X_L$$

(b) Here the current I is,



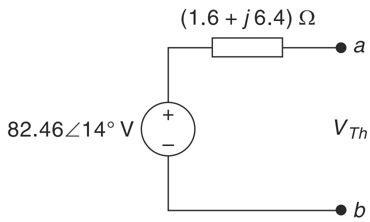
$$I = \frac{100}{2+3+j5} = \frac{100}{5+j5} = (10 - j10)\text{ A}$$

$$\therefore \; V_{\text{Th}} = I \times (3+j5) = (10-j10)\times(3+j5)$$

$$= (80 + j20)$$

$$= 82.46\angle 14^\circ \text{ V}$$

$$Z_{\text{Th}} = j6 + \frac{2\times(3+j5)}{2+3+j5} = (1.6 + j6.4)\ \Omega$$



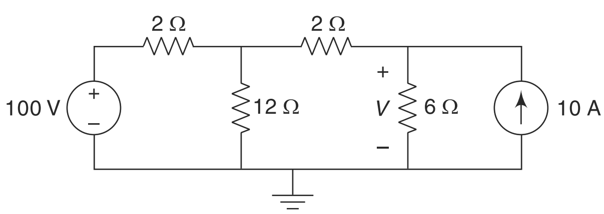
Thevenin's equivalent circuit is shown.

For maximum power transfer, the impedance should be complex conjugate of Thevenin Impedance.

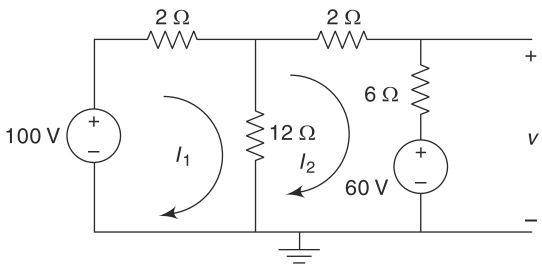
$$\therefore \qquad Z_L = (1.6 - j6.4)\ \Omega \qquad \textit{Ans.}$$

Amount of the maximum power is, $P_{\max} = \dfrac{V_{\text{Th}}^2}{4R} = \dfrac{(82.46)^2}{4\times 1.6} = 1062.5\text{ W}$ *Ans.*

(c)



Converting the current source into equivalent voltage source, we get the following circuit.



By KVL,

$$14I_1 - 12I_2 = 100$$

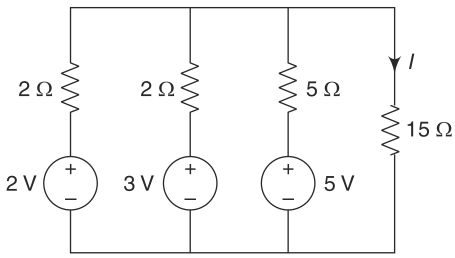
$$-12I_1 + 20I_2 = -60$$

Solving for I_2,

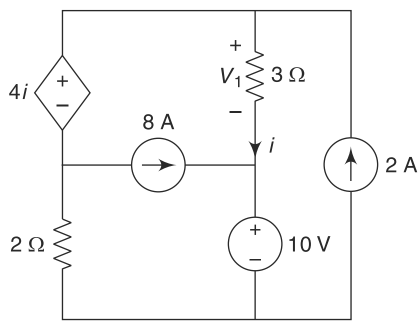
$$I_2 = \frac{\begin{vmatrix} 14 & 100 \\ -12 & -60 \end{vmatrix}}{\begin{vmatrix} 14 & -12 \\ -12 & 20 \end{vmatrix}} = \frac{-840 + 1200}{280 - 144} = \frac{360}{136} = 2.64 \text{ A}$$

$\therefore \quad v = (6I_2 + 60) = 75.88 \text{ V}$

8. (a) State and explain Millman's theorem. Calculate the load current I in the circuit in figure by Millman's theorem. 2 + 6



(b) Using Superposition theorem determine V_1, the voltage across the 3 ohm resistor in the below figure. 7



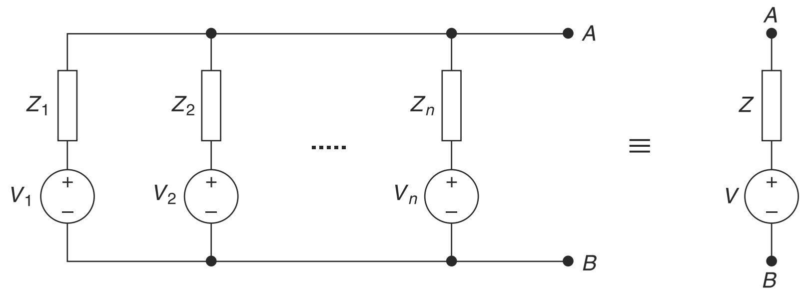
Solution

(a) Millman's Theorem

Statement

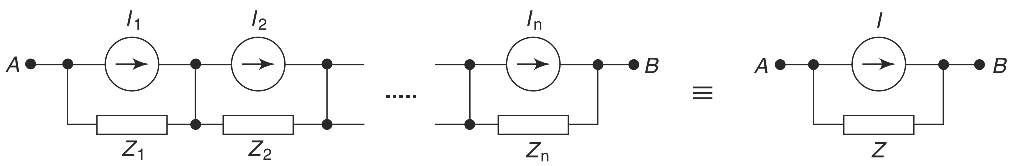
(I) This theorem states that if several ideal voltage sources (V_1, V_2, …) in series with impedances (Z_1, Z_2,…) are connected in parallel, then the circuit may be replaced by a single ideal voltage source (V) in series with an impedance (Z); where,

$$V = \frac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i} \quad \text{and,} \quad Z = \frac{1}{\sum_{i=1}^{n} Y_i}$$



(II) If several ideal current sources (I_1, I_2,…) in parallel with impedances (Z_1, Z_2, …) are connected in series, then the circuit may be replaced by a single ideal current source (I) in parallel with an impedance (Z); where,

$$I = \frac{\sum_{i=1}^{n} \frac{I_i}{Y_i}}{\sum_{i=1}^{n} \frac{1}{Y_i}} \quad \text{and,} \quad Y = \frac{1}{\sum_{i=1}^{n} \frac{1}{Y_i}} \quad \text{or,} \quad Z = \sum_{i=1}^{n} Z_i$$



Solution to Numerical Problem

By Millman's Theorem,

$$V = \frac{\sum EY}{\sum Y} = \frac{\frac{2}{2} + \frac{3}{2} + \frac{5}{5}}{\frac{1}{2} + \frac{1}{2} + \frac{1}{5}} = \frac{35}{12} = 2.91667 \text{ V}$$

$$Z = \frac{1}{\sum Y} = \frac{1}{\frac{1}{2}+\frac{1}{2}+\frac{1}{5}} = \frac{10}{12} = 0.833\ \Omega$$

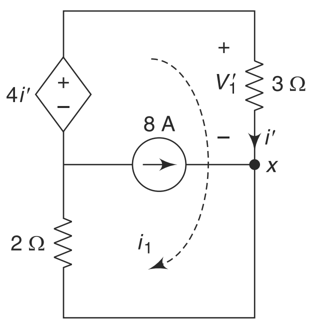
$\therefore$ $$I = \frac{V}{Z+15} = \frac{2.91667}{0.833+15} = 0.184\ \text{A} \qquad \textit{Ans.}$$

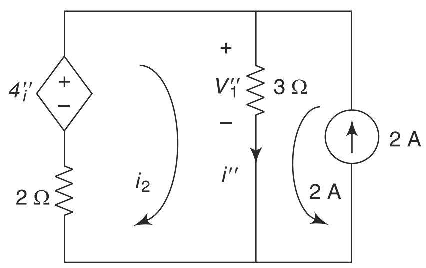
(b) **Case (I)** When 8 A current source is acting alone

By KVL for the Super mesh, $3i' + 2i_1 - 4i' = 0 \quad \Rightarrow \quad i_1 = \frac{1}{2}i'$

By KCL at node x, $i_1 = (8 + i') \quad \Rightarrow \quad \frac{1}{2}i' = 8 + i' \quad \Rightarrow \quad i' = -16$ A

$\therefore$ $$V_1' = 3i' = 3 \times (-16) = -48\ \text{V}$$





Case (II) When 2 A current source is acting alone
By KVL,

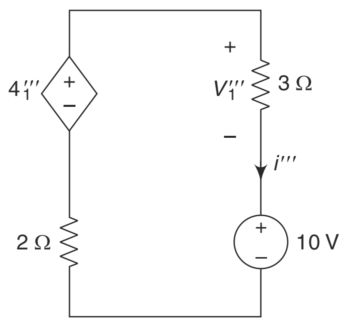
$$3(i_2 + 2) + 2i_2 - 4i'' = 0 \quad \Rightarrow \quad 5i_2 + 6 - 4i'' = 0$$

Now, $\quad i'' = (i_2 + 2)$

$\therefore \quad 5i_2 + 6 - 4(i_2 + 2) = 0 \quad \Rightarrow \quad i_2 = 2$ A

$\therefore \quad i'' = (i_2 + 2) = (2 + 2) = 4$ A

$\therefore \quad V_1'' = 3i'' = 3 \times 4 = 12$ V



Case (III) When 10 V voltage source is acting alone
By KVL,

$$3i''' - 10 + 2i''' - 4i''' = 0 \quad \Rightarrow \quad i''' = 10\ \text{A}$$

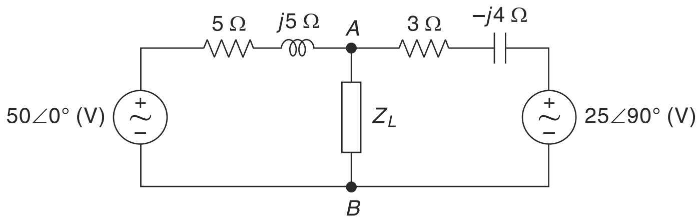
$\therefore$ $$V_1''' = 10 \times 3 = 30\ \text{V}$$

When all the sources are acting simultaneously, by superposition theorem the voltage is given as,

$$V_1 = (V_1' + V_1'' + V_1''') = (-48 + 12 + 30) = -6\ \text{V}$$

9. (a) A potential difference of $200 + 200\sqrt{2}\sin 314t$ is applied to a circuit having a resistance of 15 ohm in series with a reactance of 20 ohm. Find the power consumed in the circuit and the impedance and p. f. of the circuit. 8

(b) In the network in the figure two voltage sources act on the load impedance connected to the terminal *A* and *B*. If the load is variable in both reactance and resistance, for what load, Z_L will receive maximum power? What is the value of maximum power? 7



Solution

(a) Applied voltage, $v(t) = 200 + 200\sqrt{2}\sin 314t$,

Impedance of the circuit, $Z = (15 + j20) = 25\angle 53.13°\ \Omega$

Power Factor of the circuit, $=\cos(53.13°) = 0.6$ (lagging)

For the D.C. component of the voltage, the current, $I_0 = \dfrac{200}{15} = 13.33$ A

For the sinusoidal component of the voltage, the current is,

$$I_1 = \frac{200\sqrt{2}\angle 0°}{15 + j20} = \frac{200\sqrt{2}\angle 0°}{25\angle 53.13°} = 8\sqrt{2}\angle -53.13°\ \text{A}$$

Thus, the power consumed in the circuit,

$$P = I_0^2 \times R + I_{1\text{rms}}^2 \times R = (13.33^2 + 8^2) \times 15 = 3626.67\ \text{W}$$

(b) Consult WBUT 2004, Q. 4 (a).

The value of the maximum power is, $P_{\max} = \dfrac{V_{\text{Th}}^2}{4R} = \dfrac{9.8^2}{4 \times 4.23} = 5.676$ W

10. (a) 4 wires are joined at a node. The current entering this node through 3 of them are $5\cos\omega t$, $6\sin(\omega t + \pi/6)$ and $2\cos(\omega t + \pi/3)$. Using the phasor method, determine the current leaving this junction through the 4 wire. 8

(b) A 3 ohm resistor and a 4 ohm (at a frequency *f*) capacitor are in parallel. This combination is in series with a pure inductor. An alternating voltage at a frequency *f* when impressed in this combination delivers current at unity power factor. Keeping the same voltage but doubling the frequency, what will be the percentage change in the current drawn by the circuit based on the current at lower frequency? 7

Solution

(a) Here,

$$I_1 = 5\cos\omega t = 5\angle 0°;\quad I_2 = 6\sin\left(\omega t + \frac{\pi}{6}\right) = 6\angle -60°;\quad I_3 = 2\cos\left(\omega t + \frac{\pi}{3}\right) = 2\angle 60°$$

$\therefore$ Current leaving this node is,

$$I_4 = 5\angle 0° + 6\angle -60° + 2\angle 60° = 5 + 3 - j5.196 + 1 + j1.732 = (9 - j3.464) = 9.643\angle -21°$$

$\therefore$ $i_4(t) = 9.643\cos(\omega t - 21°)$ A *Ans.*

(b) As the current at lower frequency, (f) is at unity power factor, the circuit is at resonance at this frequency.

Given: $R = 3\,\Omega;\quad X_C\big|_{\text{at resonance}} = \frac{1}{\omega_r C} = 4\,\Omega$

For this circuit, the impedance at resonance,

$$Z_1 = j\omega_r L + \frac{1}{\frac{1}{R} + j\omega_r C} = j\omega_r L + \frac{R}{1 + j\omega_r CR} = \frac{R}{1 + \omega_r^2 C^2 R^2} + j\left(\omega_r L - \frac{\omega_r CR^2}{1 + \omega_r^2 C^2 R^2}\right)$$

At resonance, the inductive reactance is obtained as,

$$\left(\omega_r L - \frac{\omega_r CR^2}{1 + \omega_r^2 C^2 R^2}\right) = 0 \;\Rightarrow\; \omega_r L = \frac{\omega_r CR^2}{1 + \omega_r^2 C^2 R^2} = \frac{3^2/4}{1 + 3^2/4^2} = \frac{36}{25}\,\Omega$$

$$\therefore\; Z_1 = \frac{R}{1 + \omega_r^2 C^2 R^2} = \frac{3}{1 + 3^2/4^2} = \frac{48}{25}\,\Omega = 1.92\,\Omega$$

If the frequency is doubled the capacitive reactance will become half and the inductive reactance will be twice. So, the impedance at the higher frequency is,

$$Z_2 = \frac{R}{1 + \omega_r^2 C^2 R^2} + j\left(\omega_r L - \frac{\omega_r CR^2}{1 + \omega_r^2 C^2 R^2}\right) = \frac{3}{1 + 3^2/2^2} + j\left(\frac{72}{25} - \frac{3^2/2}{1 + 3^2/2^2}\right)$$

$$= (0.923 + j1.49)\,\Omega$$

$$= 3.088\angle 58.22°\,\Omega$$

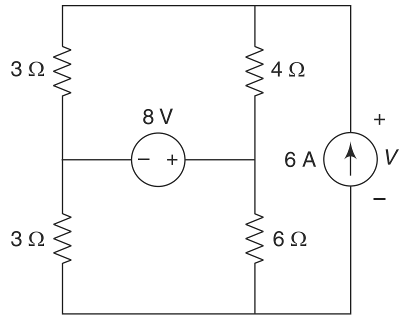
Since the voltage is kept constant the current is inversely proportional to the impedance.

$$\therefore \qquad \frac{I_2}{I_1} = \frac{Z_1}{Z_2}$$

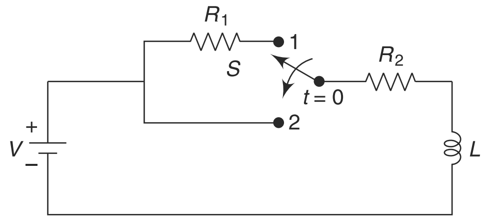
$\therefore$ The percentage change in the current drawn by the circuit based on the current at lower frequency is,

$$\frac{I_2 - I_1}{I_1} = \frac{Z_1 - Z_2}{Z_2} \times 100 = \frac{1.92 - 3.088}{3.088} \times 100 = -37.82\%$$ *Ans.*

11. (a) Determine the voltage V using source transformation and simplification in the figure. 6



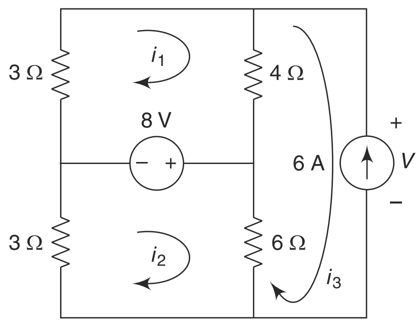
(b) In the circuit shown in figure. The switch S has been thrown to position 1 for a long period of time. Find the complete expression for the current after throwing the switch S to 2 which removes R_1 from the circuit. 5



(c) If the values of V, R_1, R_2 and L be 10 V, 1 ohm, 2 ohm and 1 H respectively, calculate
  (i) steady state current
  (ii) the energy stored in the inductance at steady state period
  (iii) time constant of the circuit for both the positions of the switch S.
  Also calculate the voltage across the resistor R_2 and inductor L, at 0.05 second after the switch S has been thrown to position 2. 4

Solution

(a) By KVL,



$$4(i_1 + 6) + 8 + 3_1 = 0 \quad \Rightarrow \quad i_1 = -\frac{32}{7}\text{ A}$$

and $$9i_2 + 36 - 8 = 0 \quad \Rightarrow \quad i_2 = -\frac{28}{9}\text{ A}$$

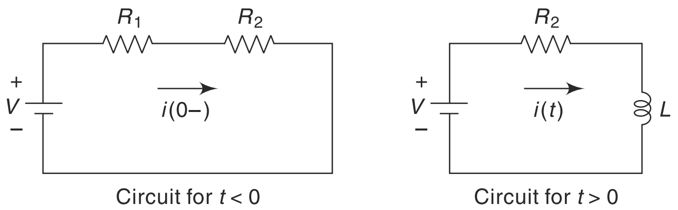
Thus, the voltage is,

$$V = 4(i_1 + 6) + 6(i_2 + 6)$$

$$= 4\left(-\frac{32}{7} + 6\right) + 6\left(-\frac{28}{9} + 6\right) = 23.05\text{ V}$$

(b) For $t < 0$, as the circuit was in steady state with the switch in position 1, the circuit becomes as shown below.

$$\therefore \qquad i(0-) = \frac{V}{R_1 + R_2}$$



For $t > 0$, the circuit becomes as shown.

By KVL,

$$R_2 I(s) + sLI(s) - Li(0-) = \frac{V}{s}$$

$$\Rightarrow \qquad [R_2 + sL]I(s) = \frac{V}{s} + \frac{VL}{R_1 + R_2}$$

$$\Rightarrow \qquad I(s) = \frac{V}{R_2}\left[\frac{1}{s(s + R_2/L)}\right] + \frac{V}{R_1 + R_2}\left(\frac{1}{(s + R_2/L)}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R_2}(1 - e^{-(R_2/L)t}) + \frac{V}{R_1 + R_2} e^{-(R_2/L)t}(A), \; t > 0 \qquad \textit{Ans.}$$

(c) $V = 10$ V, $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$ and $L = 1$ H

(i) Steady state current, $I_{ss} = \frac{V}{R_2} = \frac{10}{2} = 5$ A *Ans.*

(ii) Energy stored in the inductance at steady state period,

$$W = \frac{1}{2}LI^2 = \frac{1}{2} \times 1 \times 5^2 = 12.5 \text{ W} \qquad \textit{Ans.}$$

(iii) Time constant of the circuit for switch in position 1 is,

$$\tau_1 = \frac{L}{R_1 + R_2} = \frac{1}{1 + 2} = 0.33 \text{ s} \qquad \textit{Ans.}$$

Time constant of the circuit for switch in position 2 is,

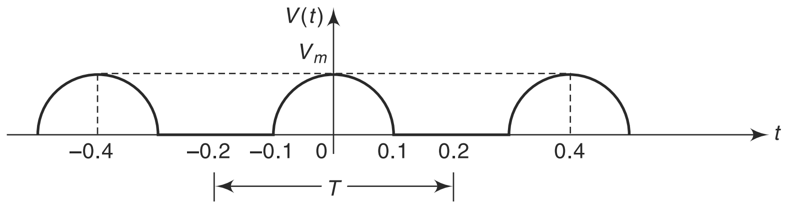
$$\tau_2 = \frac{L}{R_2} = \frac{1}{2} = 0.5 \text{ s} \qquad \textit{Ans.}$$

For $t = 0.05$, voltage across the resistor, $V_{R_2} = i \times R_2 = 5(1 - e^{-2t})|_{t=0.05} \times 2$

$$+ \frac{10}{3} e^{-2t}|_{t=0.05} \times 2 \; = 7 \text{ V}$$

and voltage across the inductor, $V_L = (10 - 7) = 3$ V

12. (a) Find the Fourier series of the voltage response at the output of a half-wave rectifier shown in the below figure. Plot the discrete spectrum of the waveform. 7



(b) Define Fourier transform of an aperiodic function $f(t)$. Obtain the Fourier transform of a single pulse of magnitude V and duration T. Show that as $f(t)$ changes from periodic to aperiodic, the amplitude spectrum changes from a line spectrum to a continuous spectrum, keeping their envelopes of the same shape. 8

Solution

(a) Here, time period $T = 0.4$ s;

$$f = \frac{1}{T} = 2.5 \text{ Hz};$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.4} = 5\pi \text{ rad/s}$$

The function, $v(t) = V_m \cos 5\pi t; \quad 0 \le t \le 0.1$

$$= 0; \quad 0.1 \le t \le 0.3$$

$$= V_m \cos 5\pi t; \quad 0.3 \le t \le 0.4$$

If the period extending from $t = -0.1$ to $t = 0.3$ is taken, it will result in fewer equations and hence, fewer integrals.

$\therefore$ $v(t) = V_m \cos 5\pi t; \quad -0.1 \le t \le 0.1$

$$= 0 \quad ; \quad 0.1 \le t \le 0.3$$

$\therefore$ $$a_0 = \frac{1}{0.4}\int_{-0.1}^{0.3} v(t)\,dt = \frac{1}{0.4}\left[\int_{-0.1}^{0.1} V_m \cos 5\pi dt + \int_{0.1}^{0.3} (0)\,dt\right] = \frac{V_m}{\pi}$$

$\therefore$ $$a_n = \frac{2}{0.4}\int_{-0.1}^{0.3} V_m \cos 5\pi n t dt; \quad n \ne 1$$

$$= 5V_m \int_{-0.1}^{0.1} \cos 5\pi t \cos 5\pi n t dt$$

$$= 5V_m \int_{-0.1}^{0.1} \frac{1}{2}[\cos 5\pi(1+n)t + \cos 5\pi(1-n)xt]\,dt$$

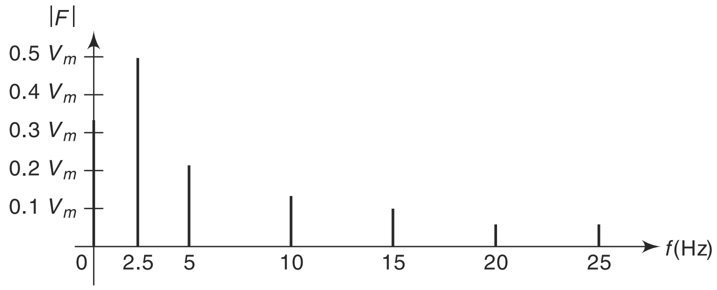
$$= \frac{2V_m}{\pi}\frac{\cos(\pi n/2)}{1-n^2}; \quad n \ne 1$$

For, $a = 1$, $a_1 = 5V_m \int_{-0.1}^{0.1} \cos^2 5\pi t dt = \frac{V_m}{2}$

Similarly, $b_n = 0$ for any value of n, and the Fourier series thus contains no sine terms.

$\therefore$ $$v(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\cos 5\pi t + \frac{2V_m}{3\pi}\cos 10\pi t - \frac{2V_m}{15\pi}\cos 20\pi t + \frac{2V_m}{35\pi}\cos 30\pi t - \ldots$$

Spectra:



(b) Definition of Fourier Transform

The Fourier Transform or the Fourier integral of a function $f(t)$ is denoted by $F(j\omega)$ and is defined by,

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \qquad \text{(i)}$$

and the inverse Fourier transform is defined by,

$$f(t) = \mathcal{F}^{-1}[F(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty} F(j2\pi f)e^{j2\pi f}\,df \qquad \text{(ii)}$$

Equations (i) and (ii) form the Fourier transform pair.

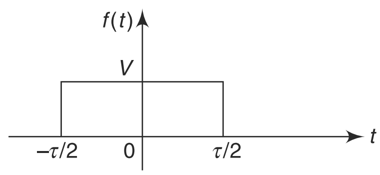
Fourier Transform of Single Pulse

The pulse is, $f(t) = V, -\frac{\tau}{2} < t < \frac{\tau}{2}$

So, the Fourier transform,

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\tau/2}^{\tau/2} Ve^{-j\omega t}\,dt = V\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega}$$

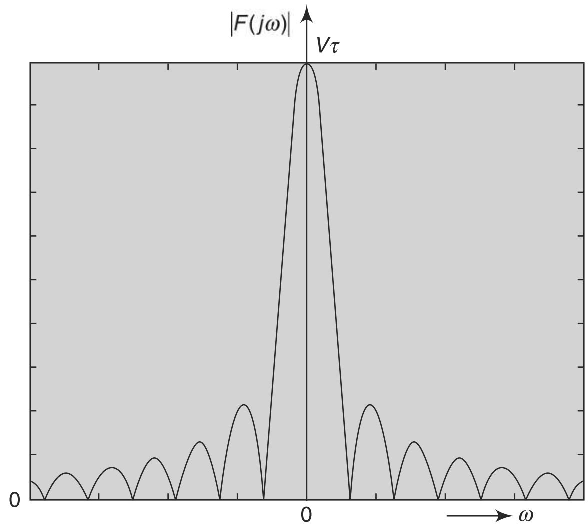
$$= 2V\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$



$$= 2V\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} \times \frac{\tau}{2}$$

$$\therefore \qquad F(j\omega) = V\tau\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} \qquad \text{(iii)} \qquad \textit{Ans.}$$

The plot of $\left|\frac{\sin x}{x}\right|$ versus x $\left(\text{here, } x = \frac{\omega\tau}{2}\right)$ is shown in the following figure.

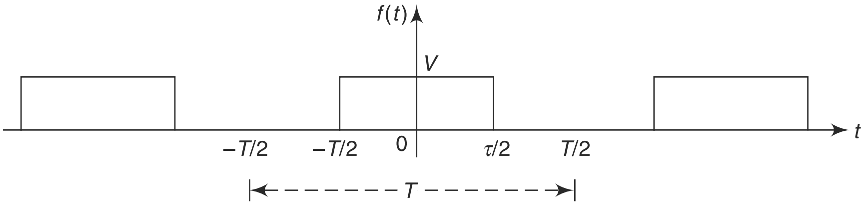


We consider the periodic function, $f(t)$ as shown below, consisting of a train of pulses. Here,

$$f(t) = V; \quad \text{for } -\frac{\tau}{2} < t < \frac{\tau}{2}$$

$$= 0, \quad \text{for } -\frac{T}{2} < t < -\frac{\tau}{2} \quad \text{and} \quad \frac{\tau}{2} < t < \frac{T}{2}$$

where T is the period of the periodic function.



The Fourier co-efficients of $f(t)$ is given by,

$$C_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ve^{-jn\omega_0 t}dt = \frac{V}{n\pi}\frac{(e^{jn\omega_0\,\tau/2} - e^{-jn\omega_0\,\tau/2})}{2j} = \frac{V\tau}{T}\frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)}$$

This C_n has values only at discrete frequencies, $n\omega_0$ so that the plot of the magnitude of C_n versus n (or $n\omega_0$) will be a line spectrum. The envelope of C_n is a continuous function of frequency given by,

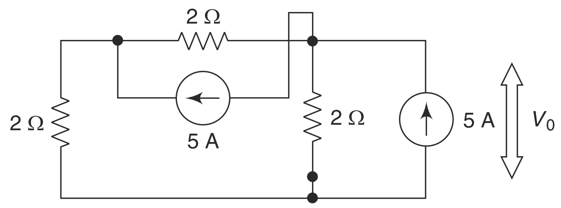
Envelope of $$C_n = \frac{V\tau}{T}\frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)} \qquad \text{(iv)}$$

However, when T approaches infinity, the train of pulses will become a single pulse. The Fourier transform of that single pulse is given in Eq (iii).

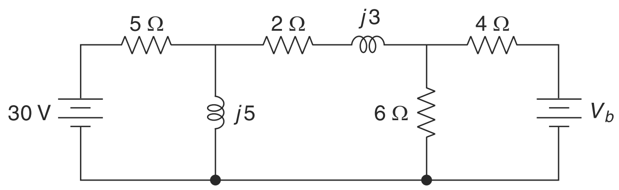
Comparing Equations (iii) and (iv), we conclude that as $f(t)$ changes from periodic to aperiodic, the amplitude spectrum changes from a line spectrum to a continuous spectrum. The envelope of the continuous spectrum is of the same shape as that of the line spectrum.

13. (a) A voltage source has a generated voltage V and an internal resistance R. How can it be converted into a current source? 3

(b) Convert the current sources into the equivalent voltage source given in figure. And hence find the voltage V_0. 4

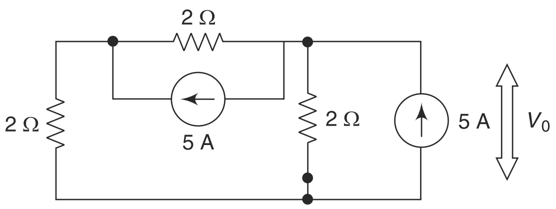


(c) In the network shown in the below figure determine the voltage V_b which result in a zero current through the $(2 + j3)$ Ω impedance in a branch. 8

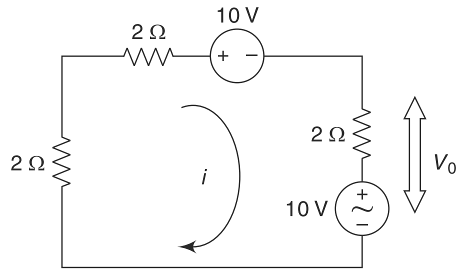


Solution

(a) Consult WBUT 2004, Q. 10 (f).

(b) 

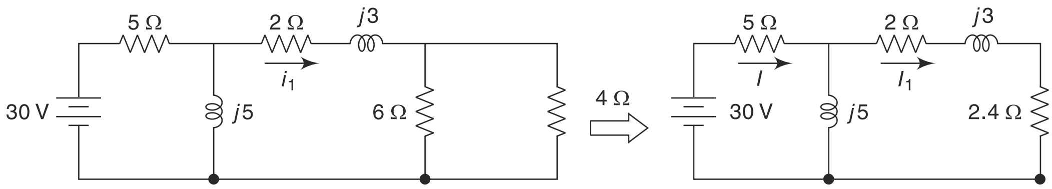
Converting the current sources into voltage sources, we get the following circuit.



$\therefore \quad i = -\frac{20}{6} = -\frac{10}{3}$ A

$\therefore \quad V_0 = 2i + 10 = 2 \times \left(-\frac{10}{3} + 10\right) = \frac{10}{3} = 3.33$ V *Ans.*

(c) When the 30 V source is acting alone, let the current through the branch $(2 + j3)\,\Omega$ be I_1.



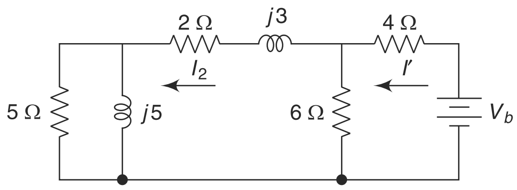
Impedance,

$$Z = 5 + \frac{j5 \times (4.4 + j3)}{4.4 + j8} = \left(\frac{7 + j62}{4.4 + j8}\right)\Omega$$

$$I = \frac{30}{Z} = \frac{30(4.4 + j8)}{7 + j62}$$

$$I_1 = I \times \frac{j5}{4.4 + j8} = \frac{30(4.4 + j8)}{7 + j62} \times \left(\frac{j5}{4.4 + j8}\right) = \frac{j150}{7 + j62}\text{ A}$$

When V_b source is acting alone, let the current through the branch $(2 + j3)\,\Omega$ be I_2.



Impedance,

$$Z = 4 + \frac{6 \times (4.5 + j5.5)}{10.5 + j5.5} = \left(\frac{69 + j55}{10.5 + j5.5}\right)\Omega$$

$$\therefore \quad I' = \frac{V_b}{Z} = \frac{V_b(10.5 + j5.5)}{69 + j55}$$

$$\therefore \quad I_2 = I' \times \frac{6}{10.5 + j5.5} = \frac{V_b(10.5 + j5.5)}{69 + j55} \times \left(\frac{6}{10.5 + j5.5}\right) = \frac{6V_b}{69 + j55}\text{ A}$$

Current through the branch $(2 + j3)\,\Omega$ will be zero, if

$$I_1 = I_2$$

$$\Rightarrow \quad \frac{j150}{7 + j62} = \frac{6V_b}{69 + j55}$$

$$\Rightarrow \quad V_b = (25 + j25)\text{ V} = 35.35\angle 45^\circ\text{ V}$$

Model Question Paper V

GROUP–A
(Multiple Choice Questions)

1. Choose the correct alternatives for any ten of the following: $10 \times 1 = 10$

(i) A 1 μF capacitor is connected across a 4 V battery. The steady state current will be:

(a) 4×10^{-6} Amp (b) $10^6/4$ Amp (c) zero (d) 4 Amp

(ii) The internal impedance of a dependent voltage source is

(a) zero (b) infinity
(c) fraction of ohms (d) any unknown value.

(iii) Periodic signal that obeys Dirichlet's condition can be represented by

(a) Fourier series (b) Fourier transform
(c) Inverse Fourier transform (d) none of these.

(iv) An initially relaxed *RC* series network with $R = 2$ MΩ and $C = 1$ μF is switched to a 10 *V* step input. The voltage across the capacitor after 2 seconds will be

(a) zero (b) 3.68 V (c) 6.32 V (d) 10 V.

(v) When a source is delivering maximum power to a load, the efficiency of the circuit:

(a) is always 50% (b) depends on the circuit parameters
(c) is always 75% (d) none of these.

(vi) The output y and the input x of a system is related by the relation $y = ax + b$ where a and b are constants. The system is

(a) linear (b) non-linear (c) bilateral (d) none of these.

(vii) A periodic function $f(t)$ of time period T repeats itself after $T/2$. The Fourier series of $f(t)$ will posses only

(a) sine terms (b) cosine terms (c) even harmonics (d) odd harmonics

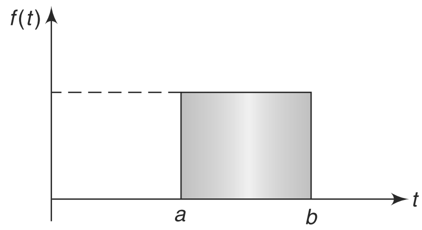
(viii) When a unit impulse voltage is applied to an inductor of 1 H, the energy supplied by the source is

(a) $\propto$ (b) 1 J (c) 1/2 J (d) 0.

(ix) The Tie-Set matrix gives the relation between
(a) branch currents and link currents (b) branch voltages and link currents
(c) branch currents and link voltages (d) none of these.

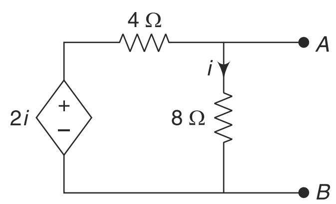
(x) Transient current in an RLC circuit is oscillatory when
(a) $R = 2\sqrt{\frac{L}{C}}$ (b) $R > 2\sqrt{\frac{L}{C}}$ (c) $R < 2\sqrt{\frac{L}{C}}$ (d) $R = 0$

(xi) The Laplace transform of the signal described in figure is



(a) e^{-as}/s (b) e^{-bs}/s^2
(c) $(e^{-as} + e^{-bs})/s$ (d) $(e^{-as} - e^{-bs})/s$

(xii) The Thevenin's equivalent with respect to the terminals A and B would be only a resistance R_{th} equal to



(a) 2.66 Ω (b) 3.2 Ω (c) 8 Ω (d) 12 Ω.

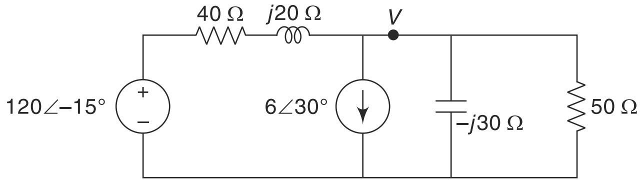
Solution

(i) (c) zero (ii) (d) any unknown value
(iii) (a) Fourier series (iv) (c) 6.32 V
(v) (a) is always 50% (vi) (b) non-linear
(vii) (d) odd harmonics (viii) (b) 1 J
(ix) (a) branch currents and link currents (x) (c) $R < 2\sqrt{\frac{L}{C}}$
(xi) (d) $(e^{-as} - e^{-bs})/s$ (xii) (b) 3.2 Ω

GROUP–B
(Short Answer Type Questions)

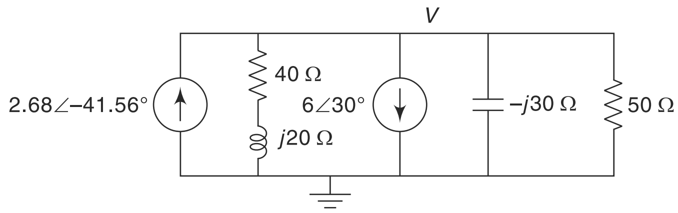
Answer any *three* of the following questions. $3 \times 5 = 15$

2. Use Node voltage method to find V in the circuit.



Solution

Converting the voltage source into current source, we get the circuit shown in the figure given below.



By KCL,

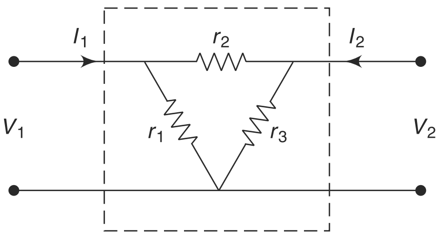
$$\frac{V}{40+j20}+\frac{V}{-j30}+\frac{V}{50}=2.68\angle-41.56°-6\angle30°$$

$$\Rightarrow \quad V[0.022\angle26.56°+j0.033+0.02]=2-j1.78-5.196-j3$$

$$\Rightarrow \quad V=\frac{-3.196-j4.78}{0.02+j0.01+j0.033+0.02}=\frac{-3.196-j4.78}{0.04+j0.043}$$

$$\Rightarrow \quad V=-97.62\angle8.94°\ \text{V}$$

3. Determine the hybrid parameters for the network in the figure shown below.



Solution

for this π-network, the y-parameters are given as,

$$y_{11}=\left(\frac{1}{r_1}+\frac{1}{r_2}\right)=\left(\frac{r_1+r_2}{r_1r_2}\right);\quad y_{12}=y_{21}=-\frac{1}{r_2};\quad y_{22}=\left(\frac{1}{r_2}+\frac{1}{r_3}\right)=\left(\frac{r_2+r_3}{r_2r_3}\right)$$

By inter-relationship, the h-parameters are obtained as,

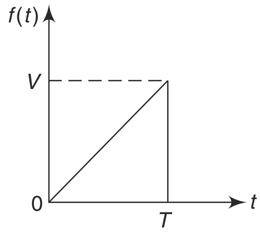
$$h_{11}=\frac{1}{y_{11}}=\left(\frac{r_1r_2}{r_1+r_2}\right)$$

$$h_{12}=-\frac{y_{12}}{y_{11}}=-\frac{-\frac{1}{r_2}}{\left(\frac{r_1+r_2}{r_1r_2}\right)}=\frac{r_1}{r_1+r_2}$$

$$h_{21}=\frac{y_{21}}{y_{11}}=\frac{-\frac{1}{r_2}}{\left(\frac{r_1+r_2}{r_1r_2}\right)}=-\frac{r_1}{r_1+r_2}$$

$$h_{22} = \frac{\Delta y}{y_{11}} = \left\{ \frac{(r_1 + r_2)(r_2 + r_3) - r_1 r_3}{r_1 r_2^2 r_3} \right\} \times \left(\frac{r_1 r_2}{r_1 + r_2} \right) = \frac{(r_1 + r_2)(r_2 + r_3) - r_1 r_3}{r_2 (r_1 + r_2)}$$

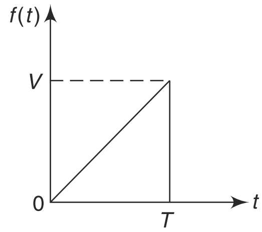
4. (a) What are the advantages of an active filter? 2
   (b) Determine the Laplace transform of the sawtooth waveform in the figure shown below. 3



Solution

(a) Advantages of Active Filter

1. **Less cost** Active filters are inexpensive as compared to passive filters, due to the variety of cheaper op-amp and the absence of costly inductors.
2. **Gain and frequency adjustment flexibility** Since op-amp is capable of providing a gain (which may also be variable), the input signal is not attenuated as it is in a passive filter. In addition, the active filter is easier to tune or adjust.
3. **No loading problem** Active filters provide an excellent isolation between the individual stages due to the high input impendence (ranging from a few kΩ to a several thousand MΩ) and low output impedance (ranging from less than 1Ω to a few hundred Ω). So, the active filter does not cause loading of the source or load.
4. **Size and Weight** Active filters are small in size and less bulky (due to the absence of bulky 'L') and are rugged.
5. **Non-floating Input and Output** Active filters generally have single ended inputs and outputs which do not 'float' with respect to the system power supply or common. This property is different from that of the passive filters.



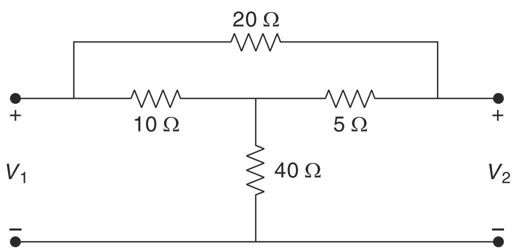
(b) Here, the function can be written as,

$$f(t) = \frac{V}{T} r(t) - \frac{V}{T} r(t - T) - Vu(t - T)$$

Taking Laplace transform, we get,

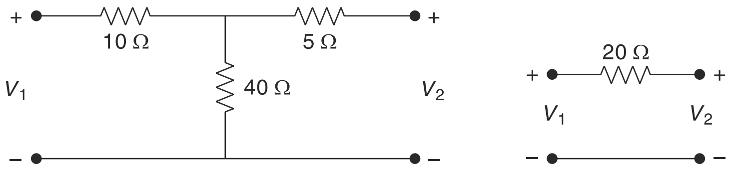
$$F(s) = \frac{V}{Ts^2} - \frac{V}{Ts^2} e^{-Ts} - \frac{V}{s} e^{-Ts} = \frac{V}{Ts^2}(1 - e^{-Ts}) - \frac{V}{s} e^{-Ts} \qquad Ans.$$

5. Find the y-parameters for the following network.



Solution

This two-port network can be considered as the parallel connection of two two-port networks as shown below.



(a) (b)

For network (a), the z-parameters are

$$z_{11a} = 50\ \Omega;\quad z_{12a} = z_{21a} = 40\ \Omega;\quad z_{22a} = 45\ \Omega;\quad \therefore \Delta z = (50 \times 45 - 40^2) = 650$$

Thus, the y-parameters are

$$y_{11a} = \frac{z_{22a}}{\Delta z} = \frac{45}{650} = \frac{9}{130}\ \mho$$

$$y_{12a} = y_{21a} = -\frac{z_{12}}{\Delta z} = -\frac{40}{650} = -\frac{4}{65}\ \mho$$

$$y_{22a} = \frac{z_{11a}}{\Delta z} = \frac{50}{650} = \frac{1}{13}\ \mho$$

For network (b), the y-parameters are

$$y_{11b} = y_{22b} = \frac{1}{20}\ \mho;\quad y_{12b} = y_{21b} = -\frac{1}{20}\ \mho$$

We know that for parallel connection of two two-port networks the over all y-parameters are the summation of individual y-parameters. Thus,

$$\left.\begin{aligned} y_{11} &= (y_{11a} + y_{11b}) = \left(\frac{9}{130} + \frac{1}{20}\right) = 0.119\ \mho \\ y_{12} = y_{21} &= (y_{12a} + y_{12b}) = \left(-\frac{4}{65} - \frac{1}{20}\right) = -0.111\ \mho \\ y_{22} &= (y_{22a} + y_{22b}) = \left(\frac{1}{13} + \frac{1}{20}\right) = 0.127\ \mho \end{aligned}\right\} \quad \textit{Ans.}$$

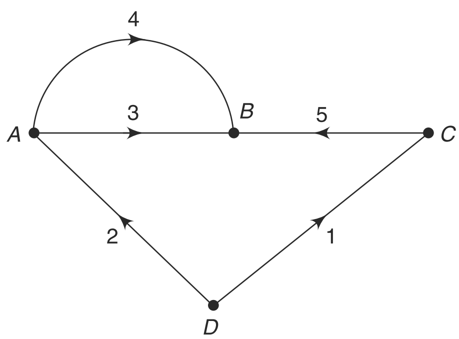
6. Define incidence matrix. The reduced incidence matrix of an oriented graph is

$$[A] = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Draw the graph.

Solution

- **Incidence Matrix: Consult WBUT 2005 Q. 7(b).**
- **Solution to Numerical Problem:**



From the property that for complete incidence matrix, the summation of all entries in any column must be zero, the complete incidence is obtained as,

Some Typical Short Answer Type Questions

1. Explain the limitations of ohm's law.
   Answer: Limitations of Ohm's Law
   (a) It is not applicable to non-linear circuits, like circuits with powdered carbon, thyrite, etc.
   (b) It is not applicable to unilateral circuits, like circuits with electron tubes, transistors, etc.
2. How ideal voltage sources can be converted into ideal current sources and vise-versa?
   Answer: A voltage source $V(t)$ with an internal resistance R can be converted into a current source $I(t)$ in parallel with the same resistance R, where, $I(t) = \dfrac{V(t)}{R}$.

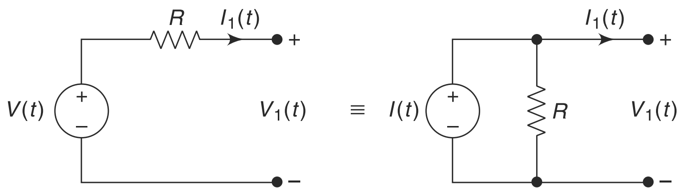


Figure *Conversion of Voltage Source into Current Source*

   A voltage source can be converted into a current source and vise-versa if and only if their respective open-circuit voltage and short circuit current are same. However, an ideal voltage source can never be open-circuited and an ideal current source can never be short-circuited, as this is in contrary to the definitions of ideal voltage and current sources.

   Thus, we cannot convert a voltage source V with zero internal resistance to a corresponding current source.
3. What is the difference between circuits and networks?
   Answer: Any combination and interconnection of network elements like resistor or inductor or capacitor or electrical energy sources are known as 'networks'. However, a closed energised network is known as 'circuit'. A network need not contain an energy source; but a circuit must contain energy source.

4. What is the difference between loop and mesh?

Answer: A loop or mesh denote a closed path obtained by starting at a node and returning back to the same node through a set of connected circuit elements without passing through any intermediate node more than once. However, the difference between mesh and loop is that a mesh does not contain any other loop within it, i.e., mesh is the smallest loop.

5. What do you understand by transient and steady-state response? How can they be identified in a general solution?

Answer: In electrical engineering, a **transient response** or **natural response** is the electrical response of a system to a change from equilibrium.

The condition prevailing in an electric circuit between two steady-state conditions is known as the *transient state*; it lasts for a very short time. The currents and voltages during the transient state are called *transients*.

In general, transient phenomena occur whenever

(i) a circuit is suddenly connected or disconnected to/from the supply,

(ii) there is a sudden change in the applied voltage from one finite value to another,

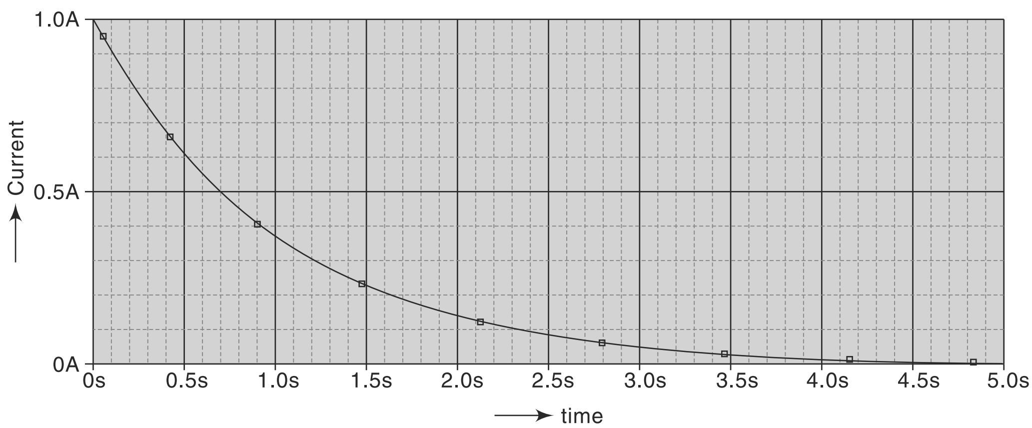
(iii) a circuit is short-circuited.

The transient currents are not caused by any part of the supply voltage, but are entirely associated with the changes in the stored energy in capacitor and inductors. As there is no energy stored in resistors, there are *no transients in purely resistive circuits*.

When the transient phenomena die out the circuit becomes steady and the state of the circuit is called '*steady state'*.

In electrical engineering, a simple example would be the output of a 5 volt DC power supply when it is turned on: the transient response is from the time the switch is turned on and the output is a steady 5 volt. At this point, the power supply reaches its steady-state response of a constant 5 volt.

Another practical example will be an RC series circuit. When it is suddenly switched to a d. c. supply, the transient current through the circuit is the maximum and it gradually decreases so that the steady state current in the circuit becomes zero.



In a general solution, the part of the solution that diminishes with time is identified as the transient part and the part that exists with time is identified as the steady state part. For example, for the general solution, $f(t) = A + Be^{-t}$, the transient response is Be^{-t} and steady state response is A.

6. What do you understand by initial conditions before and after switching.
   Answer: It is possible that a capacitor or an inductor might have been used in some other circuit earlier, where it absorbed some energy and then it was disconnected. Because of its non-dissipative nature, the energy was stored within the capacitor (or the inductor). Now, as this capacitor (or inductor) is connected to a circuit, it gets some path to release its stored energy. This stored energy is represented by the initial voltage $V_C(0)$ or initial current $I_L(0)$.
7. Explain the following
   (a) The current through an inductor cannot change instantaneously.
   (b) The voltage across a capacitor cannot change instantaneously.
   Answer:
   (a) The equation relating inductance and flux linkages can be rearranged as follows:

$$\lambda = Li \tag{1}$$

Taking the time derivative of both sides of the equation yields

$$\frac{d\lambda}{dt} = L\frac{di}{dt} + i\frac{dL}{dt}$$

In most physical cases, the inductance is constant with time and so

$$\frac{d\lambda}{dt} = L\frac{di}{dt}$$

By Faraday's Law of Induction, we have

$$\frac{d\lambda}{dt} = -E = v$$

where E is the electromotive force (emf) and v is the induced voltage. Note that the emf is opposite to the induced voltage. Thus

$$v = L\frac{di}{dt} \tag{2}$$

or $$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + i(0)$$

where $i(0)$ is the initial curent. When initial current is zero,

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt \tag{3}$$

These equations together state that, for a steady applied voltage v, the current changes in a linear manner, at a *rate* proportional to the applied voltage, but inversely proportional to the inductance. Conversely, if the current through the inductor is changing at a constant rate, the induced voltage is constant.

From equation (2), it is clear that for an abrupt change in current, the voltage across the inductor becomes infinite. Also, from equation (3), it is observed that for a finite change in voltage in zero time the integral must be zero.

Therefore, *the curerent through an inductor cannot change instantaneously.*

   (b) The relation between charge and voltage in a capacitor is written as,

$$Q = CV \tag{4}$$

The current, $i = \frac{dQ}{dt} = C\frac{dV}{dt} + V\frac{dC}{dt}$

In most physical cases, the capacitance is constant with time

$$\therefore \qquad i = C\frac{dV}{dt} \qquad (5)$$

$$\therefore \qquad dV = \frac{1}{C} i\, dt$$

Taking integration on both sides,

$$\int_0^{v_c} dV = \frac{1}{C}\int_0^t i\, dt$$

or

$$v_c(t) = \frac{1}{C}\int_0^t i(t)\, dt + v_c(0)$$

where, $v_c(0)$ is the initial voltage across the capacitor. For zero initial voltage,

$$v_c = \frac{1}{C}\int_0^t i\, dt \qquad (6)$$

From equation (5), it is clear that for an abrupt change of voltage across the capacitor, the current becomes infinite. Also, from equation (6), it is observed that for a finite change of current in zero time the integral must be zero.

Therefore, *the voltage acorss a capacitor cannot change instantaneously.*

8. What is the difference between a Fourier series and Fourier integral?

Answer: Differences between Fourier Series and Fourier Integral

(a) Fourier Series is applicable for periodic functions, whereas Fourier Integral (transform) is applicable for non-periodic functions.

(b) Amplitude spectrum in case of Fourier series is line spectrum, whereas in case of Fourier transform, the amplitude spectrum is a continuous spectrum.

9. How does Fourier transform differ from Laplace transform?

Answer: Differences between Fourier Transform and Laplace Transform are given below.

The defining equations are,

$$F(s) = \int_0^{\infty} f(t)e^{-st}\, dt \quad \text{and} \quad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

Followings are some differences and similarities:

(a) Laplace Transform is one-sided in the interval $0 < t < \infty$ and Fourier Transform is double-sided in the interval $-\infty < t < \infty$. Thus, Laplace Transform is applicable for positive time function, $f(t)$, $t > 0$; while Fourier Transform is applicable for functions defined for all times.

(b) Laplace Transform includes the initial conditions and is applicable for transient analysis; while Fourier Transform is only applicable for steady-state analysis.

(c) For functions $f(t) = 0$ for $t < 0$ and $\int_0^\infty |f(t)|\, dt < \infty$, the two transforms are related as, $F(j\omega) = F(s)\big|_{s=j\omega}$. Thus, Laplace Transform is associated with entire s-plane, while, Fourier Transform is restricted to the imaginary ($j\omega$) axis.

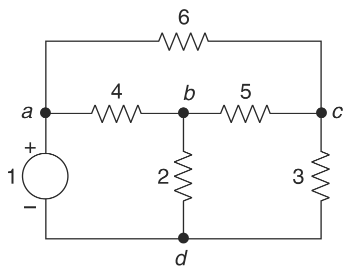
(d) Laplace Transform is applicable to a wider range of functions than the Fourier Transform. On the other hand, Fourier Transforms exist for signals that are not physically realizable and have no Laplace Transform.

10. Explain 'Network Topology' and 'Graph' of a network.

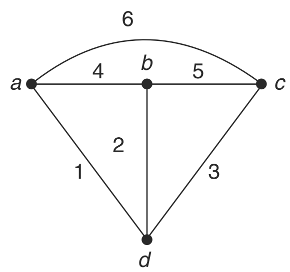
Answer:

Network Topology The word *topology* refers to the science of place. In mathematics, *topology is a branch of geometry in which figures are considered perfectly elastic*. Therefoere, *Network Topology refers to* the properties that relate to the *geometry of the network* (circuit). These properties remain unchanged even if the circuit is bent into any other shape provided that *no parts are cut* and *no new connections* are made.

Graph of Network A linear graph (or simply a graph) is defined as a collection of points called nodes, and line segment called branches, the nodes being joined together by the branches.



Circuit



Graph of the Circuit

11. Mention some examples where Thevenin's theorem cannot be applied.

Answer: Limitations of Thevenin's Theorem:

(i) This theorem is inapplicable to magnetically coupled circuits.

(ii) This theorem is inapplicable for non-linear and unilateral networks.

12. Why are the *ABCD* parameters termed as 'Transmission Parameters'?

Answer: The *ABCD* parameters represent the relation between the input quantities and the output quantities in the two-port network. They are thus voltage-current pairs.

These parameters are known as transmission parameters as in a transmission line, the currents enter at one end and leaves at the other end, and we need to know a relation between the sending end quantities and the receiving end quantities.

13. Show that under the condition of maximum power transfer, the efficiency of a circuit is 50%.

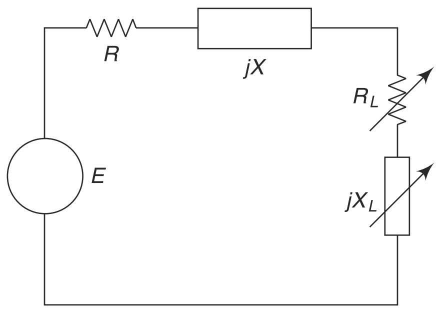
Answer: Let E be the voltage source, $(R + jX)$ the internal impedance of the source and $(R_L + jX_L)$ the load impedance.

$$E = \frac{E}{Z + Z_L} = \frac{E}{(R + R_L) + j(X + X_L)} \tag{1}$$

Power delivered to the load is,

$$P = |I|^2 R_L = \frac{E^2 R_L}{(R+R_L)^2 + (X+X_L)^2} \qquad (2)$$

where, $Z = R + jX$, $Z_L = R_L + jX_L$



For maximum power, $\dfrac{\partial P}{\partial X_L}$ must be zero.

Now, $$\frac{\partial P}{\partial X_L} = \frac{-2(E)^2 R_L (X_L + X)}{[(R_L+R)^2 + (X_L+X)^2]^2} = 0$$

from which, $X_L + X = 0$ or $X_L = -X$

i.e., the reactance of the load impedance is of opposite sign to the reactance of the source impedance.

Putting $X_L = -X$ in equation no. (2) $P = \dfrac{E^2 R_L}{(R_L+R)^2}$

For maximum power, $$\frac{\partial P}{\partial R_L} = \frac{E^2(R_L+R)^2 - 2E^2 R_L (R_L+R)}{(R_L+R)^4} = 0$$

or, $E^2(R_L + R) - 2E^2 R_L = 0$ or $R_L = R$

The maximum power transferred will be $P_{\max} = \dfrac{E^2}{4R_L} = \dfrac{(E/2)^2}{R_L}$

Thus, the efficiency of the circuit is be 50%.

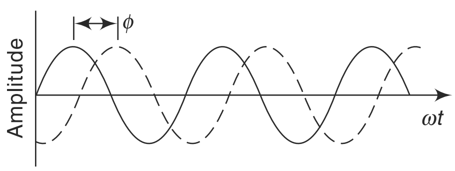
14. Explain why the lower limit of the Laplace transform integral $\left(\int_{0-}^{\infty} f(t) e^{-st}\, dt \right)$ is taken as 0- instead of 0+.

Answer: The *lower limit of the integration should be* $0-$ *instead of* 0_+ *or simple 0*. If $f(t)$ is continuous at $t = 0$, then the value of $f(0)$ is well-defined. But, if $f(t)$ is not continuous at $t = 0$, then the meaning of $f(0)$ becomes ambiguous. To consider the effect of "instantaneous energy transfer", we must use 0- as the lower limit to include the impulses at $t = 0$. The use of 0_+ will exclude the existence of any impulses at the origin.

So, we use 0- as the lower limit.

15. All pass filters pass all the frequencies; still it is termed as 'filter' why?

Answer: All pass filter passes all frequencies equally well, i.e., output and input voltages are equal in magnitude for all frequency; but the output voltage is shifted in phase with respect to the input voltage, with the phase–shift between the two being a function of frequency.



This filter is also known as a **phase-shift filter**, **time-delay filter,** or simply the **delay equalizer.** One major application of an all-pass filter is the simulation of a lossless transmission line. The magnitude of the output voltage is the same as the input voltage but the output voltage is shifted in phase with respect to the input voltage.

The highest frequency up to which the input and output amplitudes remain equal is dependent on the unity-gain bandwidth of the op-amp. At this frequency, however, the phase-shift between the input and output is maximum.

16. Explain why a capacitor is considered as a linear circuit element.

Answer: Let V_{C_1} and V_{C_2} individually excite a relaxed capacitor, producing the respective currents,

$$i_{C_1} = C\frac{dV_{C_1}}{dt} \text{ and } i_{C_2} = C\frac{dV_{C_2}}{dt}$$

Let i_C be the current induced by a voltage $(V_{C_1} + V_{C_2})$

$$\therefore \qquad i_C = C\frac{d}{dt}(V_{C_1} + V_{C_2}) = C\frac{dV_{C_1}}{dt} + C\frac{dV_{C_2}}{dt} = (i_{C_1} + i_{C_2})$$

This shows that the *v*-*i* characteristic of a capacitor obeys the superposition principles. Therefore, capacitor is considered as a linear element.

17. Explain why an inductor is considered as a linear circuit element.

Answer: Let V_{L_1} and V_{L_2} individually excite a relaxed inductor, producing the respective currents,

$$i_{L_1} = \frac{1}{L}\int V_{L_1}\, dt \text{ and } i_{L_2} = \frac{1}{L}\int V_{L_2}\, dt$$

Let i_L be the current induced by a voltage $(V_{L_1} + V_{L_2})$

$$\therefore \qquad i_L = \frac{1}{L}\int V_L\, dt = \frac{1}{L}\int (V_{L_1} + V_{L_2})\, dt = (i_{L_1} + i_{L_2})$$

This shows that the *v*-*i* characteristic of an inductor obeys the superposition principles. Therefore, inductor is considered as a linear element.

18. What is the Laplace transform of a function which is nonzero for $t < 0$?

Answer: As the lower limit of integration of Laplace transform is 0–, the Laplace transform does not distinguish between functions that are different for $t < 0$ bur identical for $t \geq 0$.

For example, the Laplace transforms of $u(t)$ and $u(t + 1)$ will be same.

However, $t = 0$ is physically the starting time of a circuit or system and all the signals considered are usually zero for $t < 0$. For this reason, all will have a unique (one-sided) Laplace transform. Conversely, all Laplace transform $F(s)$ will have a unique time function, such that $f(t) = 0$ for $t = 0$.

19. Does every signal $f(t)$, such that $f(t) = 0$ for $t < 0$, have a Laplace transform?

Answer: The existence of Laplace transform $X(s)$ of a given $x(t)$ depends on whether the transform integral converges

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\, dt = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\, dt < \infty$$

which in turn depends on the duration and magnitude of $x(t)$ as well as the real part of s, $\mathrm{Re}[s] = \sigma$ (the imaginary part of s $\mathrm{Im}[s] = j\omega$ determines the frequency of a sinusoid which is bounded and has no effect on the convergence of the integral).

This limits the variable $s = (\sigma + j\omega)$ to a part of the complex plane. The subset of values of s for which the Laplace transform exists is called the *region of convergence* (ROC) or the *domain of convergence.*

Thus, the Laplace transform $F(s)$ typically exists for all complex numbers such that $\mathrm{Re}\{s\} > a$, where a is a real constant which depends on the growth behavior of $f(t)$ or precisely the condition is given as,

$$|f(t)| = k_1 e^{k_2 t} \text{ where, } k_1 \text{ and } k_2 \text{ are some constants}$$

For example, for the function, $f(t) = e^{t^2} u(t)$, Laplace transform integral becomes,

$$\int_{0-}^{\infty} e^{t^2} e^{-st}\, dt = \int_{0-}^{\infty} e^{t^2 - st}\, dt = \int_{0-}^{\infty} e^{t^2 - \sigma t - j\omega t}\, dt$$

As t approaches infinity, the area under the curve $(t^2 - \sigma t)$ goes to infinity. Thus, the Laplace transform of this function does not exist.

20. Discuss the advantages of Laplace transform method over the conventional classical methods of solving the linear differential equations with constant coefficients.

Answer: Advantages of Laplace Transform Method are given below.
1. It gives complete solution.
2. Initial conditions are automatically considered in the transformed equations.
3. Much less time involved in solving differential equations.
4. It gives systematic and routine solutions for differential equations.

21. What do you understand by 'Complex Frequency'? Give its physical significance.

Answer: Complex Frequency

The complex frequency (s) is the sum of two frequencies the real and imaginary.

$$s = \text{Complex frequency} = (\sigma + j\omega)$$

Where, σ = Real part of s = neper frequency
ω = Imaginary part of s = radian frequency.

The general solution of the differential equation in time-domain is,

$$i(t) = I_0 e^{st}, \text{ where } s = (\sigma + j\omega)$$

Since e^{st} is a dimensionless quantity and so, also, the product st a dimensionless quantity, the unit of s must be $(\text{time})^{-1}$ or Hz.

Here, ω is interpreted as radian frequency; as radian is a ratio of two lengths, ω is effectively $(\text{time})^{-1}$, i.e. frequency in Hz.

Also, as σ and ω must have the same dimension, i.e., the dimension of σ should be $(\text{time})^{-1}$. Also, with $\omega = 0$,

$$i(t) = I_0 e^{\sigma t} \qquad \Rightarrow \qquad \sigma = \frac{1}{t} \ln\left[\frac{i(t)}{I_0}\right]$$

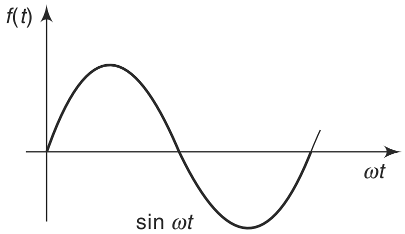
Since the unit of ln of some number is neper, the unit of σ is neper per second.

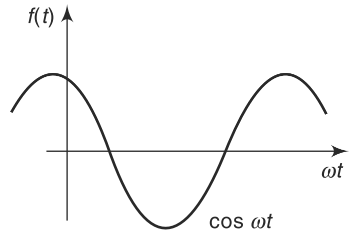
Physical significance of Complex Frequency

We have,

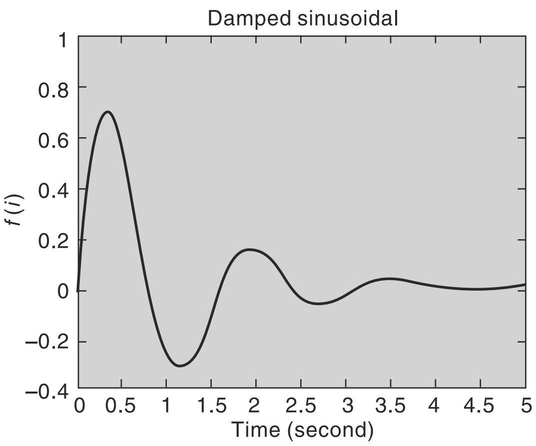
$$i(t) = I_0 e^{st} = I_0 e^{(\sigma + j\omega)t} = I_0 e^{\sigma t}[\cos \omega t + j \sin \omega t]$$

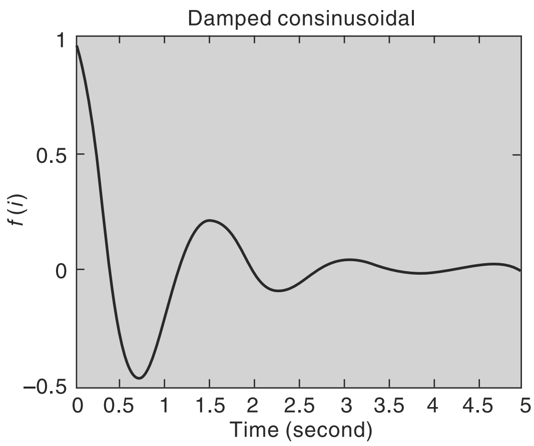
If $\sigma = 0$, then the variation of the real and imaginary parts of the function is shown below.



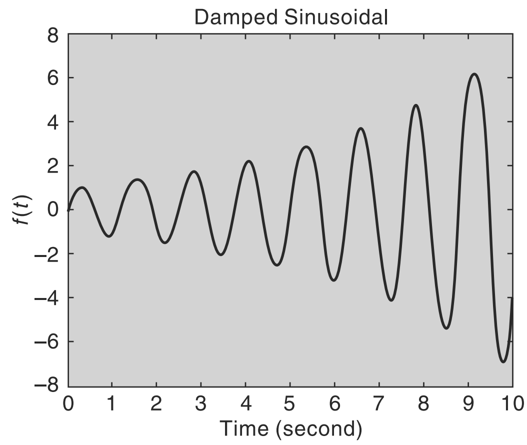


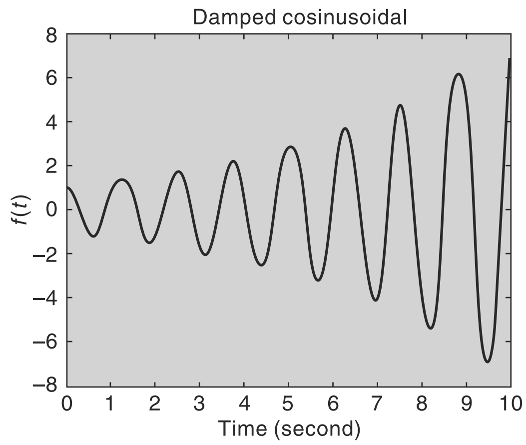
If $\sigma < 0$, then the variation of the real and imaginary parts of the function is shown below.





If $\sigma > 0$, then the variation of the real and imaginary parts of the function is shown below.





From these figures, it is clear that,

ω decides the number of oscillations per second,

σ decides the magnitude of these oscillations.